The E-Correspondence Principle*

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Abstract

We introduce the E-correspondence principle for stochastic dynamic expectations models as a tool for comparative dynamics analysis. The principle is applicable to equilibria that are stable under least squares and closely related learning rules. With this technique it is possible to study, without explicit solving for the equilibrium, how properties of the equilibrium are affected by changes in the structural parameters of the model. Even if qualitative comparative dynamics results are not obtainable, a quantitative version of the principle can be applied.

1 Introduction

The Correspondence Principle introduced by Paul Samuelson over 60 years ago, see (Samuelson 1941) and (Samuelson 1942), became a standard tool in the 1950's and 60's in comparative statics analysis in both micro- and macroeconomic theory. The applications have ranged from Walrasian stability analysis in general equilibrium theory, stability of macroeconomic sys-

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tems, deterministic dynamic optimization models and international trade.[1] Samuelson suggested that there is a mutually supportive relationship between the stability of an equilibrium and its comparative statics. The form of the relationship is open to varying interpretations and consequently the usefulness of Samuelson's correspondence principle has been widely debated, see e.g. the sceptical conclusions in the context of general equilibrium theory and stability of Walrasian tatonnêment in (Quirk and Saposnik 1968) and (Arrow and Hahn 1971).

With the advent of stochastic rational expectations (RE) models this classic technique fell into disuse in dynamic equilibrium theory. Our goal in this paper is to resuscitate key ideas behind Samuelson's principle in a way that makes them applicable to dynamic stochastic expectations models.

For stochastic expectations models the appropriate concept of equilibrium is usually taken to be the rational expectations equilibrium (REE). We will mainly focus on REE, but our argument applies also to "restricted perceptions equilibria," a weakening of rational expectations to set-ups in which agents use misspecified models.[2] We introduce the E- (or Expectational) correspondence principle, by which it is possible to study, without explicit solution of the equilibrium, how properties of a stable equilibrium are affected by changes in the structural parameters of the model. The E-correspondence principle can be applied to analyze comparative dynamic properties of REE that are stable under adaptive learning.

The theory of stability under adaptive learning is a concept of stability for REE that operates in real time and has been widely studied in the recent literature.[3] A basic result in this literature is the E- (or Expectational) stability principle according to which, for a very wide variety of models, stability of an equilibrium, under least squares and closely related learning schemes, is equivalent to E-stability of the equilibrium. See (Evans and Honkapohja 2001) for a detailed discussion of the E-stability concept and the models and adaptive learning rules to which it applies. E-stability is defined

[1] There is a very large literature in these different areas see e.g. (Samuelson 1947), (Patinkin 1965) and (Quirk and Saposnik 1968) for general equilibrium theory and macroeconomics, (Mortensen 1973), (Burmeister and Long 1977) and (Brock and Malliaris 1989) for dynamic optimization models and (Neary 1978) for international trade.

[2] See the example at the end of Section 3.1.

[3] (Evans and Honkapohja 2001) is a treatise on the analysis of adaptive learning and its implications in macroeconomics. (Evans and Honkapohja 1999), (Evans and Honkapohja 1995), (Marimon 1997), (Sargent 1993) and (Sargent 1999) provide surveys of the field.

in virtual or notional time using an ordinary differential equation that is associated with the stochastic dynamics of learning. The use of a differential equation system describing adjustment dynamics out of equilibrium was a central characteristic in the stability analysis of Walrasian tatonnêment in classic general equilibrium theory and in the dynamics of traditional macroeconomic models, see the references in footnote 1. This feature makes it possible to develop the E-correspondence principle for stochastic models.

Broadly speaking, the E-correspondence principle states that E-stability of a dynamic equilibrium implies useful sign restrictions when local comparative dynamic properties of the equilibrium are studied. This study is based on the implicit function theorem operating in the parameter spaces of the model and the parameters of the equilibrium stochastic process. This formulation is analogous to a form of Samuelson's classic Correspondence Principle, which derived sign restrictions from stability under Walrasian tatonnêment or related dynamics of model variables. We remark that, in specific contexts, there have been attempts to derive converse results, i.e. to show that comparative static properties imply stability of equilibrium, but this aspect does not seem useful in our context.

The paper is organized as follows. Section 2 first develops a simple example of a standard rational expectations model and REE. This facilitates the formulation of an abstract framework for the E-correspondence principle operating in the spaces of the structural model parameters and the parameters of the equilibrium process. Section 3 contains several economic applications. Section 4 develops a quantitative version of the E-correspondence principle that can be applied when qualitative results are ambiguous. Section 5 concludes.

2 The General Framework

We explain and illustrate the concept of E-stability using an example of standard linear expectations model, and then give a general formulation.

2.1 Example 1: A Linear RE Model

There is a single endogenous variable y_t that depends linearly on the expectations of its value next period $E_t^* y_{t+1}$ and it lagged value y_{t-1} and an exogenous shock v_t that is taken to be *iid* with zero mean for simplicity. Thus the model is

$$y_t = \alpha + \beta E_t^* y_{t+1} + \delta y_{t-1} + \gamma v_t. \quad (1)$$

We introduce a multiplying constant γ to the shock, which is convenient for studying the effects of an increase in the variance of the shock. Here $E_t^*(.)$ denotes possibly nonrational expectations, and RE are denoted by $E_t(.)$. We need to use $E_t^*(.)$ because the E-stability concept is defined in terms of disequilibrium learning dynamics. This also allows us to consider restricted perceptions equilibria as well as REE. To compute REE, conjecture that they take the $AR(1)$ form

$$y_t = a + by_{t-1} + cv_t, \quad (2)$$

where a, b and c are unknown constants that can be determined as solutions to a set of equations given below. As is usual, we restrict attention to stationary stochastic solutions, i.e. solutions with $|b| < 1$.

In adaptive learning it is assumed that agents guess that the stochastic process for the endogenous variables has the form (2) but do not know the equilibrium values of the parameters a, b and c. Agents try to estimate the values of a, b and c using past data and a standard statistical technique such as least squares. At each moment of time the agents are assumed to forecast the value of y_{t+1} using their estimated model. These parameter estimates together with y_{t-1} and v_t are used to calculate $E_t^* y_{t+1}$ and (1) then determines the "temporary equilibrium" value of y_t. An REE is said to stable under learning if agents' estimates of a, b, c converge over time to the REE values.

The key result for least squares learning is that estimated values of the parameters a, b and c locally converge to REE values if and only if the REE is E-stable. The definition of E-stability is based on a mapping from the (possibly non-rational) perceived law of motion (PLM) parameters to the actual law of motion (ALM) parameters that these perceptions.

The formal definition of E-stability for model (1) is as follows. Assuming that the current value of the endogenous variable is not known, while the current value of the exogenous variable is known, at the time when agents make forecasts,[4] we have

$$E_t^* y_{t+1} = a(1 + b) + b^2 y_{t-1} + bcv_t$$

[4]This assumption is often made in the literature. It is also possible to allow for full contemporaneous data or only lagged data in the formation of expectations $E_t^*(.)$.

for any given values a, b, c. Inserting this forecast into (1) yields the ALM, which is thus given by

$$y_t = \alpha + \beta a(1+b) + (\beta b^2 + \delta)y_{t-1} + (\beta bc + \gamma)v_t.$$

This specifies the mapping $(a, b, c) \to T(a, b, c)$ from the PLM to the ALM, where

$$T(a, b, c) = (\alpha + \beta a(1+b), \beta b^2 + \delta, \beta bc + \gamma).$$

An REE of the form (2) is a fixed point (a, b, c) of T, i.e. a solution to the equation

$$(a, b, c) = T(a, b, c),$$

and it is easily seen that there are two (or zero) REE of this form. A fixed point is said to be E-stable if it is a locally asymptotically stable equilibrium point of the ordinary differential equation

$$\frac{d(a, b, c)}{d\tau} = T(a, b, c) - (a, b, c),$$

in which a, b, c are now treated as functions of τ, denoting notional or virtual time. By differentiation it is easily seen that a fixed point (a, b, c) is E-stable if and only if all the eigenvalues of $DT - I$ have negative real parts.[5]

Assuming the parameters β, δ are such that real solutions exist of the form (2), either one or both of the solutions can be stochastically stationary. However, if both solutions are stationary then only one of the solutions will be E-stable, i.e. stable under least squares learning.[6] The E-correspondence principle provides information on the comparative dynamics of solutions that are stable under least-squares learning (or any learning rule whose stability properties are governed by E-stability).

2.2 General Formulation

Example 1 illustrates that for many expectations models relevant equilibria can be computed from an equation in the space of parameters characterizing the equilibrium process, which is the space of (a, b, c) in Example 1. The model itself is usually dependent on the finite set of structural parameters,

[5] Throughout we assume that the relevant eigenvalues do not have zero real parts.

[6] However, in more elaborate linear models there can be multiple stationary E-stable solutions.

which in Example 1 are $(\alpha, \beta, \delta, \gamma)$. We now develop this in an abstract setting.

Denote the equilibrium process parameters by Θ and the structural parameters by Φ. Θ is assumed to be an $n-$dimensional vector $\Theta \in \mathbb{R}^n$ and Φ is $m-$dimensional, i.e. $\Phi \in \mathbb{R}^m$. We also assume that there is a mapping from the PLM, which is parametrized by Θ, to the ALM and we denote this mapping by $T(\Theta)$. For given values of Φ, the relevant equilibria are given by the equation

$$T(\Theta, \Phi) = \Theta, \tag{3}$$

where we have made explicit the dependence of the T mapping on the structural parameters Φ. E-stability is defined by the local asymptotic dynamics of the differential equation

$$\frac{d\Theta}{d\tau} = T(\Theta, \Phi) - \Theta,$$

where Φ is kept fixed.

The E-stability condition is that all n eigenvalues of the matrix $D_1 T(\Theta, \Phi) - I$ have negative real parts at the equilibrium of interest. When $D_1 T(\Theta, \Phi) - I$ is non-singular we can use the implicit function theorem. A small change in Φ defines through the equation (3) a function $\Theta = F(\Phi)$ describing in the parameter space $\mathbb{R}^n$ how the equilibrium shifts as a result of the change in Φ. Moreover, this function is differentiable when $T(\Theta, \Phi)$ is continuously differentiable, see e.g. (Simon and Blume 1994). Taking differentials of (3), we have

$$(D_1 T - I) d\Theta + D_2 T d\Phi = 0 \tag{4}$$

and the partial derivatives $\frac{\partial F_i}{\partial \Phi_j}$ can be obtained from (4) using Cramer's rule. We get

$$\frac{\partial F_i}{\partial \Phi_j} = -\frac{\Delta_{ij}}{\Delta}, \tag{5}$$

where

$$\Delta = \det(D_1 T - I)$$

and Δ_{ij} is obtained from Δ by replacing its i'th column by the j'th column of $D_2 T$, see e.g. (Simon and Blume 1994).

The following lemma is the key to the E-correspondence principle:

Lemma 1 *If the equilibrium of interest is E-stable, then* $\operatorname{sgn}(\Delta) = \operatorname{sgn}(-1)^n$*, where* n *is the dimension of* Θ*.*

Proof. Δ is equal to the product of the eigenvalues of $D_1T - I$. If the eigenvalues of $D_1T - I$ are all real, the result follows at once. If $D_1T - I$ has any complex eigenvalues, they appear in conjugate pairs and their product is positive. ■

On the basis of Lemma 1 we can always sign the denominator in (5) when the equilibrium of interest is E-stable. This allows us to state:

Theorem 2 *(The E-correspondence principle) If the equilibrium defined by the equation $T(\Theta, \Phi) = \Theta$ is E-stable and the parameter Φ_j undergoes a small change, then the direction of the change in the equilibrium value of $\Theta = F(\Phi)$ is given by*

$$\operatorname{sgn}\left(\frac{\partial F_i}{\partial \Phi_j}\right) = -\operatorname{sgn}(-1)^n \operatorname{sgn}(\Delta_{ij}),\ i = 1, \ldots, n. \tag{6}$$

The proof is immediate from the preceding considerations. Note that a similar result fails for equilibria that are not E-stable since, for example, the determinant Δ cannot then in general be signed.

3 Economic Applications

We first continue the analysis of comparative dynamics of the linear model in Example 1 and then develop two further examples.

3.1 Example 1, Continued

For the linear model (1) we have $\Theta = (a, b, c)$, $\Phi = (\alpha, \beta, \delta, \gamma)$ and the equations defining the equilibrium are

$$(a, b, c) = (\alpha + \beta a(1 + b), \beta b^2 + \delta, \beta bc + \gamma).$$

In this case

$$D_1T - I = \begin{pmatrix} \beta(1+b) - 1 & \beta a & 0 \\ 0 & 2\beta b - 1 & 0 \\ 0 & \beta c & \beta b - 1 \end{pmatrix},$$

and by E-stability $\operatorname{sgn}(\Delta) = \operatorname{sgn}(\det(D_1T - I)) = -1$. Furthermore, E-stability also implies that $\beta(1 + b) - 1$, $2\beta b - 1$ and $\beta b - 1$ are all negative at an E-stable REE.

For the effects of changes in the structural parameters one can now easily compute the following results.

(1) Effect of a change in α:

$$\begin{aligned}\frac{\partial a}{\partial \alpha} &= -\Delta^{-1}\begin{vmatrix} 1 & \beta a & 0 \\ 0 & 2\beta b-1 & 0 \\ 0 & \beta c & \beta b-1 \end{vmatrix} \\ &= -\Delta^{-1}(2\beta b-1)(\beta b-1) > 0,\end{aligned}$$

in other words a shift in the constant α shifts up the constant of an E-stable REE process. Similarly, we have

$$\frac{\partial b}{\partial \alpha} = \frac{\partial c}{\partial \alpha} = 0.$$

Unsurprisingly, a shift in α does not influence the coefficients of y_{t-1} and v_t of the (E-stable) REE process.

(2) A change in the coefficient of the expectations term β in (1):

$$\begin{aligned}\left(\frac{\partial a}{\partial \beta}\right) &= -\Delta^{-1}a(2\beta b+\beta b^2-1-b) \gtrless 0, \\ \frac{\partial b}{\partial \beta} &= -\Delta^{-1}(\beta(1+b)-1)b^2(\beta b-1) > 0, \\ \operatorname{sgn}\left(\frac{\partial c}{\partial \beta}\right) &= \operatorname{sgn}(bc).\end{aligned}$$

In particular, for the case $\beta > 0$ and $b > 0$ we have the important result that a higher weight on future expectations in the structural model leads to higher persistence in an E-stable REE. The other effects are ambiguous unless further assumptions are made.

(3) A change in coefficient of the lagged endogenous variable δ in (1):

$$\begin{aligned}\operatorname{sgn}\left(\frac{\partial a}{\partial \delta}\right) &= \operatorname{sgn}(\beta a), \\ \frac{\partial b}{\partial \delta} &= -\Delta^{-1}(\beta(1+b)-1)(\beta b-1) > 0, \\ \operatorname{sgn}\left(\frac{\partial c}{\partial \delta}\right) &= \operatorname{sgn}(\beta c).\end{aligned}$$

For $\delta > 0$ and $b > 0$, a higher weight on the lagged value of the endogenous variable in the structural model leads to higher persistence in an E-stable REE. The other effects are in general ambiguous.

(4) A change in the coefficient of the shock term γ in (1):

$$\begin{aligned} \frac{\partial a}{\partial \gamma} &= \frac{\partial b}{\partial \gamma} = 0, \\ \frac{\partial c}{\partial \gamma} &= -\Delta^{-1}(\beta(1+b)-1)(2\beta b-1) > 0. \end{aligned}$$

Note that $var(y_t) = \frac{c^2 var(v_t)}{1-b^2}$, so that the last result implies that an increase in γ increases $var(y_t)$ when $c > 0$.

The above analysis has restricted attention to REE that are stable under the learning dynamics. However it is also possible to use the E-correspondence principle to examine the comparative dynamics of "restricted perceptions equilibria" (RPE) in which agents are boundedly rational in the sense that they use the optimal choice of models from within a misspecified class.[7] A very simple example is based on Example 1. Suppose that agents underparameterize the law of motion in (1) by omitting the lagged dependent variable. Their PLM then takes the very simple form $y_t = a + \varepsilon_t$, where agents are (incorrectly) assuming that ε_t is white noise. Under this PLM expectations are given by $E_t^* y_{t+1} = a$ and the ALM is given by $y_t = \alpha + \beta a + \delta y_{t-1} + v_t$. In the case of RPE, E-stability is defined in terms of the mapping from the PLM to the "projected ALM," i.e. to the corresponding minimum mean square error forecasting model within the class of PLMs. Provided $|\delta| < 1$ so that the implied ALM is stochastically stationary, this mapping is well defined and given by

$$T(a) = \frac{\alpha + \beta a}{1 - \delta}$$

The RPE is determined by the fixed point of the T-map,

$$a = \frac{\alpha}{1 - \beta - \delta},$$

[7]RPE are discussed in Chapter 13 of (Evans and Honkapohja 2001). (Sargent 1999) uses the closely related concept of a self confirming equilibrium.

so that in the RPE

$$y_t = (1-\beta-\delta)^{-1}\alpha(1-\delta) + \delta y_{t-1} + v_t.$$

The corresponding E-stability condition $T'(a) < 1$, evaluated at the fixed point, is given by $(1-\delta)^{-1}\beta < 1$, which, using $|\delta| < 1$, is equivalent to $\beta + \delta < 1$. This has immediate implications for the effect on the sample mean of the y_t process in the RPE. Since $E(y_t) = (1-\beta-\delta)^{-1}\alpha$ the E-correspondence principle implies

$$\frac{\partial E(y_t)}{\partial \alpha} > 0.$$

We have developed this example, of the application of the E-correspondence principle to an RPE, for a very simple case, but it would be easy to develop a more elaborate example. For example, suppose (1) were augmented to include a vector of observable exogenous shocks following a stationary first-order vector autoregression. In the REE the PLM would include all of these variables as well as the intercept and y_{t-1}. Boundedly rational agents might omit either y_{t-1} or some of the exogenous observables and the E-correspondence principle could be used to investigate the resulting RPE.

The findings of our first example are three-fold. First, we see that the E-correspondence principle in many cases yields important qualitative results on the comparative dynamics of E-stable REE. These include effects of parameter changes on the mean, variance and persistence properties of the REE. Second, E-stability is not always sufficient to give unambiguous qualitative results. This is a point to which we will return in Section 4. Third, the use of the E-correspondence principle is not restricted to examination of the comparative dynamics properties of REE. If agents in the model are boundedly rational and use the best model within a misspecified class of PLMs, the E-correspondence principle can provide qualitative information on the comparative dynamics properties of the RPE.

3.2 Example 2: Sunspot Equilibria

We consider a standard one-step forward looking nonlinear model

$$x_t = E_t F(x_{t+1}), \tag{7}$$

which is known to have different types of REE, depending on the shape of the $F(.)$ function. x_t is a scalar endogenous variable and its value in period t

depends on the forecasts of a nonlinear function of its value next period. The equilibria for the model can include steady states, perfect foresight cycles and sunspot equilibria.

A widely studied case of a sunspot solution has the form of 2−state Markov chain. Suppose $s_t \in \{1,2\}$ is a two-state Markov chain with time-invariant transition probabilities $\pi_{ij} = \Pr\{s_{t+1} = j \,|\, s_t = i\}$. The Markov chain induces, via expectations, an REE $\{x_1, x_2\}$ that is also a Markov chain with the same transition probabilities. Thus, for all t, the REE satisfies

$$x_t = x_i = \pi_{ii}F(x_i) + (1-\pi_{ij})F(x_j) \text{ if } s_t = i.$$

A (2−state) Markov sunspot equilibrium (SSE) with transition probabilities $\{\pi_{11}, \pi_{22}\}$ is thus a pair $\{x_1, x_2\}$ of distinct values for the state variable that satisfy the equations

$$\begin{aligned} \pi_{11}F(x_1) + (1-\pi_{11})F(x_2) - x_1 &= 0, \quad (8) \\ (1-\pi_{22})F(x_1) + \pi_{22}F(x_2) - x_2 &= 0. \quad (9) \end{aligned}$$

We will assume that $x_1 > x_2$, without loss of generality.

Using an overlapping generations model, (Woodford 1990) showed that the economy can in some cases converge to an SSE through adaptive learning. (Evans and Honkapohja 1994) and (Evans and Honkapohja 2003b) derived local stability conditions for adaptive learning for model (7) and showed that these conditions are in turn given by E-stability arguments. Moreover, they showed how stability of SSEs sufficiently near non-stochastic solutions can be obtained from the stability properties of the non-stochastic equilibria.

In this paper we employ the E-correspondence principle to derive the following comparative dynamics result for Markov SSE's with respect to a variation in the transition probabilities of the sunspot:

Proposition 3 *Suppose $x_1 > x_2$ and that $F'(x_i) < 1, i = 1,2$. Consider an increase in π_{11}. If the SSE is E-stable, then* $\operatorname{sgn}\left(\frac{\partial(x_1-x_2)}{\partial\pi_{11}}\right) = \operatorname{sgn}[F(x_1) - F(x_2)]$.

Proof. The E-stability condition is obtained from the mapping from the PLM to the ALM

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \pi_{11}F(x_1) + (1-\pi_{11})F(x_2) \\ (1-\pi_{22})F(x_1) + \pi_{22}F(x_2) \end{pmatrix}$$

and the stability condition is that the eigenvalues of the matrix

$$DT - I = \begin{pmatrix} \pi_{11}F'(x_1) - 1 & (1-\pi_{11})F'(x_2) \\ (1-\pi_{22})F'(x_1) & \pi_{22}F'(x_2) - 1 \end{pmatrix}$$

have negative real parts.

Turning to the comparative dynamics, we differentiate (8)-(9) and obtain the system

$$\begin{aligned} & \begin{pmatrix} \pi_{11}F'(x_1) - 1 & (1-\pi_{11})F'(x_2) \\ (1-\pi_{22})F'(x_1) & \pi_{22}F'(x_2) - 1 \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} \\ = & -\begin{pmatrix} F(x_1) - F(x_2) & 0 \\ 0 & F(x_2) - F(x_1) \end{pmatrix} \begin{pmatrix} d\pi_{11} \\ d\pi_{22} \end{pmatrix}. \end{aligned}$$

Using (6) we have

$$\begin{aligned} \operatorname{sgn}\left(\frac{\partial x_1}{\partial \pi_{11}}\right) &= -\operatorname{sgn}[(F(x_1) - F(x_2))(\pi_{22}F'(x_2) - 1)] \\ &= \operatorname{sgn}[F(x_1) - F(x_2)], \end{aligned}$$

and

$$\begin{aligned} \operatorname{sgn}\left(\frac{\partial x_2}{\partial \pi_{11}}\right) &= -\operatorname{sgn}[(F(x_2) - F(x_1))(\pi_{11}F'(x_1) - 1)] = \\ & -\operatorname{sgn}[F(x_1) - F(x_2)] \end{aligned}$$

since the system is two-dimensional and thus

$$\det \begin{pmatrix} \pi_{11}F'(x_1) - 1 & (1-\pi_{11})F'(x_2) \\ (1-\pi_{22})F'(x_1) & \pi_{22}F'(x_2) - 1 \end{pmatrix} > 0$$

if the SSE is E-stable. ■

This result can be given a precise interpretation in two specific cases.

(1) If the mapping $F(x)$ is strictly decreasing, then the amplitude of the sunspot fluctuations is reduced by an increase in the probability of the economy of staying in the high state. This case arises, for example, in the case of E-stable SSE's sufficiently near a single steady state. (Evans and Honkapohja 2003b) show that E-stable SSE's exist near a single steady state $\hat{x} = F(\hat{x})$ provided $F'(\hat{x}) < -1$ at the steady state $\hat{x}$.

(2) If the mapping $F(x)$ is strictly increasing, there may exist two distinct steady states $\hat{x}_1$, $\hat{x}_2$ and if $F'(\hat{x}_1), F'(\hat{x}_2) < 1$, then there exist E-stable SSE's for which sunspot states are near $\hat{x}_1$, $\hat{x}_2$, see (Evans and Honkapohja 1994) and Section 4.6 of (Evans and Honkapohja 2001). Two-state Markov SSE's sufficiently near $\hat{x}_1$, $\hat{x}_2$ must satisfy the derivative condition of Proposition 3 and $F(x_1) > F(x_2)$, so that in this case the amplitude of the sunspot fluctuation is increased by an increase in the probability of the economy staying at the high state x_1.

3.3 Example 3: Monetary Policy in the New Keynesian Model

As a third example of the usefulness of the E-correspondence principle we consider a bivariate linear model, the New Keynesian model of monetary policy, which takes the form

$$z_t = c_z + E_t^* z_{t+1} - \sigma^{-1}(r_t - E_t^* \pi_{t+1}) + g_t, \tag{10}$$

$$\pi_t = c_\pi + \kappa z_t + \mathcal{B} E_t^* \pi_{t+1}. \tag{11}$$

Here z_t is the output gap, π_t is the inflation rate and r_t is the nominal interest rate. The parameters $\sigma, \kappa > 0$ and $0 < \mathcal{B} < 1$. c_z and c_π are intercepts, which are from the log-linearization of the exact model. g_t is an observable shock to the output gap. The stochastic process for g_t will be specified below. (Here we focus on the one-shock case that is often employed.) The first equation is the IS curve that comes from the Euler equation for consumer optimality and the second equation is the forward-looking Phillips curve based on Calvo price stickiness. This model is widely used in current discussions of monetary policy. See e.g. (Clarida, Gali, and Gertler 1999), (Svensson 2003) and (Woodford 2003) for details and analysis.

The model is completed by specification of an interest rate rule. A wide variety of different rules have been studied in the literature. The issue of stability under learning has been examined by (Bullard and Mitra 2002), (Evans and Honkapohja 2003c) and several other papers. For a review of the literature see (Evans and Honkapohja 2003a). For concreteness, we consider interest rate setting by a forward-looking Taylor rule

$$r_t = c_r + \varphi_\pi E_t^* \pi_{t+1} + \varphi_z E_t^* z_{t+1}, \tag{12}$$

where c_r denotes an intercept. The parameters satisfy $\varphi_\pi, \varphi_z > 0$. We are interested in examining how changes in these policy parameters affect the volatilities of the output gap and inflation, as measured by the variances.

Introducing the notation $y_t = (z_t, \pi_t)'$, equations (10), (11) and (12) can be combined to yield the bivariate system

$$\begin{aligned} y_t &= A + ME_t^* y_{t+1} + Pg_t, \\ g_t &= \rho g_{t-1} + \varepsilon_t, \end{aligned}$$

where $|\rho| < 1$ and ε_t is white noise with variance σ_ε^2. The coefficient matrices are

$$M = \begin{pmatrix} 1 - \sigma^{-1}\varphi_z & \sigma^{-1}(1-\varphi_\pi) \\ \kappa(1-\sigma^{-1}\varphi_z) & \mathcal{B} + \kappa\sigma^{-1}(1-\varphi_\pi) \end{pmatrix}, \; P = \begin{pmatrix} \sigma^{-1} \\ \kappa\sigma^{-1} \end{pmatrix}.$$

Section 3.3 of (Bullard and Mitra 2002) discusses in detail the determinacy and E-stability conditions. There is a unique solution of the form

$$y_t = a + hg_t,$$

where a and h are 2×1 vectors. The mapping from the PLM to the ALM is

$$\begin{aligned} T_a(a) &= A + Ma, \\ T_h(h) &= \rho Mh + P \end{aligned}$$

and the fixed point $a = T_a(a)$, $h = T_h(h)$ defines the REE values of the coefficients a and h. The E-stability conditions for this solution are that the real parts of the eigenvalues of the matrices $M - I$ and $\rho M - I$ are negative.

We now turn to the comparative dynamics, focusing on the derivatives $\frac{\partial |h_i|}{\partial \varphi_j}, i = 1, 2; j = z, \pi$, i.e. whether more aggressive policy response to either output gap or inflation increases or decreases the magnitude of the response of output gap and inflation to the shock g_t. Taking differentials of the equation $h = \rho Mh + P$ we get

$$dh = \rho(Mdh + (dM)h)$$

or

$$(I - \rho M)dh = \rho \frac{\partial M}{\partial \varphi_j} h d\varphi_j, \; j = z, \pi,$$

where

$$
\begin{aligned}
I-\rho M &= \begin{pmatrix} \sigma^{-1}\varphi_z\rho+1-\rho & -\rho\sigma^{-1}(1-\varphi_\pi) \\ -\rho\kappa(1-\sigma^{-1}\varphi_z) & 1-\rho(\mathcal{B}+\kappa\sigma^{-1}(1-\varphi_\pi)) \end{pmatrix}, \\
\frac{\partial M}{\partial \varphi_z} &= \begin{pmatrix} -\sigma^{-1} & 0 \\ -\kappa\sigma^{-1} & 0 \end{pmatrix}, \frac{\partial M}{\partial \varphi_\pi} = \begin{pmatrix} 0 & -\sigma^{-1} \\ 0 & -\kappa\sigma^{-1} \end{pmatrix}.
\end{aligned}
$$

Consider the effect of a change in φ_z. We get

$$
(I-\rho M)\frac{\partial h}{\partial \varphi_z} = -\rho\sigma^{-1}h_1 \begin{pmatrix} 1 \\ \kappa \end{pmatrix}.
$$

By E-stability $\Delta = \det(I-\rho M) > 0$ and so

$$
\begin{aligned}
\frac{\partial h_1}{\partial \varphi_z} &= -\Delta\rho\sigma^{-1}h_1 \begin{vmatrix} 1 & -\rho\sigma^{-1}(1-\varphi_\pi) \\ \kappa & 1-\rho(\mathcal{B}+\kappa\sigma^{-1}(1-\varphi_\pi)) \end{vmatrix} \\
&= -\Delta\rho\sigma^{-1}h_1(1-\mathcal{B}\rho),
\end{aligned}
$$

so that

$$
\operatorname{sgn}\left(\frac{\partial h_1}{\partial \varphi_z}\right) = -\operatorname{sgn}(h_1) \text{ and hence } \frac{\partial |h_1|}{\partial \varphi_z} < 0.
$$

An analogous argument establishes that

$$
\frac{\partial |h_2|}{\partial \varphi_z} < 0
$$

and also that

$$
\frac{\partial |h_1|}{\partial \varphi_\pi} < 0 \text{ and } \frac{\partial |h_2|}{\partial \varphi_\pi} < 0.
$$

The variances of output gap and inflation are given by $h_1^2\sigma_g^2$ and $h_2^2\sigma_g^2$, respectively, and thus the formal results can be summarized as:

Proposition 4 *More aggressive interest rate policy $d\varphi_\pi > 0, d\varphi_z > 0$ decreases the variances of the output gap and inflation.*

4 A Quantitative Version

The preceding examples have shown that the E-correspondence principle can be used to obtain useful results for comparative dynamics. However, even in Example 1 we noted that the qualitative effects were ambiguous for some aspects of comparative dynamics. In other words, E-stability conditions are not always sufficient to pin down the signs of parameter changes. We will discuss this limitation further in the concluding section. In this section we consider a more complicated example to illustrate that E-stability can be used numerically to derive quantitative comparative dynamics results in cases in which qualitative results are not available.

Example 4:

Consider the monetary model of Example 3 but with a different interest rate rule, known as the lagged-data Taylor rule

$$r_t = a_r + \varphi_\pi \pi_{t-1} + \varphi_z z_{t-1}.$$

The reduced form is now

$$\begin{aligned} y_t &= A + ME_t^* y_{t+1} + Ny_{t-1} + Pg_t, \\ g_t &= \rho g_{t-1} + \varepsilon_t, \end{aligned}$$

where

$$M = \begin{pmatrix} 1 & \sigma^{-1} \\ \kappa & \mathcal{B} + \kappa\sigma^{-1} \end{pmatrix}, N = \begin{pmatrix} -\sigma^{-1}\varphi_z & -\sigma^{-1}\varphi_\pi \\ -\kappa\sigma^{-1}\varphi_z & -\kappa\sigma^{-1}\varphi_\pi \end{pmatrix}, P = \begin{pmatrix} \sigma^{-1} \\ \kappa\sigma^{-1} \end{pmatrix}.$$

We consider the REEs of the form

$$y_t = a + by_{t-1} + cg_t,$$

and the mapping from the PLM to the ALM is

$$\begin{aligned} T_a(a,b) &= A + M(I+b)a, \\ T_b(b) &= Mb^2 + N, \\ T_c(b,c) &= Mbc + \rho Mc. \end{aligned}$$

E-stability is, of course, defined by local stability of the differential equation

$$\frac{d}{d\tau}(a,b,c) = T(a,b,c) - (a,b,c),$$

where $T = (T_a, T_b, T_c)$.

It can be verified that E-stability conditions are insufficient to provide unambiguous qualitative comparative dynamics. An example will be given below. However, it is still possible to use E-stability differential equations to compute numerically comparative dynamics results.

We introduce two calibrations for the New Keynesian model suggested by (Clarida, Gali, and Gertler 2000) and (Woodford 1999), respectively.

CGG: $\mathcal{B} = 0.99$, $\sigma = 1$, $\kappa = 0.3$;
W: $\mathcal{B} = 0.99$, $\sigma = 0.157$, $\kappa = 0.024$.

We also set $\rho = 0.35$ and $\sigma_g^2 = 0.02$. Our interest is in the effects of changes in the policy rule parameters φ_π and φ_z on the asymptotic variances of the output gap and inflation, which are affected by b and c. We calculate the values of b and c using the E-stability differential equations. For example, for $\varphi_\pi = 1.5$ and $\varphi_z = 0.15$ and the CGG calibration we calculate the equilibrium values of b and c using a numerical differential equation solver in Mathematica starting from arbitrary initial conditions.[8] This yields

$$b = \begin{pmatrix} -0.105692 & -1.05692 \\ -0.0236791 & -0.236791 \end{pmatrix}, \; c = \begin{pmatrix} 0.9317 \\ 0.290193 \end{pmatrix}.$$

The variances of z_t and π_t can be computed from the linear equations

$$Var(y_t) = b(Var(y_t))b' + \sigma_g^2 cc'.$$

This provides an efficient way of computing the solution and its properties and can obviously be used to compute comparative dynamics numerically for global as well as local changes in exogenous parameters. As an illustration we consider the effects of changes in φ_π and φ_z on $var(z_t)$ and $var(\pi_t)$ for the CGG and W calibrations of the model with lagged Taylor rule. Table 1 gives the results when either φ_π or φ_z is increased from the base line.

	CGG		
φ_π, φ_z	$1.5, 0.15$	$1.5, 0.20$	$1.75, 0.15$
$var(z_t)$	0.02108	0.02053	0.02014
$var(\pi_t)$	0.001871	0.001751	0.001670

[8] The REE is only locally stable, but the basin of attraction appears to be quite large.

	W		
φ_π, φ_z	$1.5, 0.15$	$1.5, 0.20$	$1.75, 0.15$
$var(z_t)$	0.80125	0.9515	0.8050
$var(\pi_t)$	0.000284	0.0002597	0.0002771

Table 1: Output gap and inflation variances

We note that, in the case of the CGG calibration, increases in either φ_π or φ_z reduce both variances, whereas in the W calibration the variance of the output gap increases while the variance of inflation decreases. This shows that the qualitative comparative dynamics are ambiguous.

Clearly, there exist alternative methods of computing how the equilibrium shifts as a result of a parameter change. However, the application of the E-stability differential equation ensures that attention is directed only at E-stable REE. In this sense the computation relies on the E-correspondence principle.

5 Concluding Remarks

We have introduced the E-correspondence principle and shown that it can be exploited to obtain useful comparative dynamic results for stable rational expectations solutions in dynamic stochastic expectations models. The E-correspondence principle can also be applied to stable restricted perceptions equilibria in which the rationality concept is weakened. In the E-correspondence principle the stability criterion is that of real-time least squares learning and is governed by associated E-stability conditions. Our principle is motivated by Samuelson's classic correspondence principle, which was applied to comparative statics in nonstochastic models.

Samuelson's principle was criticized in the subsequent literature. For example, in the context of classic general equilibrium theory (Arrow and Hahn 1971), p.321 conclude that "the necessary conditions for local stability are too weak for the comparison task." However, our analysis demonstrates that the E-correspondence principle can indeed provide important qualitative comparative dynamic results in a number of concrete models. Example 2 shows that how the amplitude of sunspot fluctuations depends on transition probabilities for E-stable sunspot equilibria. In Example 3 we derive qualitative results on how output gap and inflation volatilities depend on the parameters of the interest rate rule.

Clearly, the E-correspondence principle does not always yield unambiguous qualitative comparative dynamic results. This was illustrated in parts of Example 1 as well as in Example 4. At the same time, Example 4 illustrates how a quantitative version of the E-correspondence principle can be applied using a computer and standard numerical techniques for ordinary differential equations that were unavailable at the time Samuelson introduced his classic concept.

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