# UNIQUENESS OF CONFORMAL RICCI FLOW AND BACKWARD RICCI FLOW ON HOMOGENEOUS 4-MANIFOLDS

by

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DISSERTATION ABSTRACT

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In the first chapter we consider the question of uniqueness of conformal Ricci flow. We use an energy functional associated with this flow along closed manifolds with a metric of constant negative scalar curvature. Given initial conditions we use this functional to demonstrate the uniqueness of the solution to both the metric and the pressure function along conformal Ricci flow.

In the next chapter we study backward Ricci flow of locally homogeneous geometries of 4-manifolds which admit compact quotients. We describe the long-term behavior of each class and show that many of the classes exhibit the same behavior near the singular time. In most cases, these manifolds converge to a sub-Riemannian geometry after suitable rescaling.

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TABLE OF CONTENTS

Chapter Page

## LIST OF TABLES

Table Page

# CHAPTER I

## UNIQUENESS OF CONFORMAL RICCI FLOW

The uniqueness of Ricci Flow on closed manifolds was originally proven by Hamilton [9]. Later on, Chen and Zhu proved the uniqueness on complete noncompact manifolds with bounded curvature [5]. The method employed in [5] utilizes DeTurck Ricci Flow. Recently Kotschwar used energy techniques to give another proof of the uniqueness on complete manifolds [12]. Kotchwar's proof does not rely on DeTurck Ricci Flow. A natural question is whether similar techniques can be applied to demonstrate the uniqueness of other geometric flows. One of these flows we have in mind is Conformal Ricci Flow, introduced by Fischer [8]. Ricci Flow preserves many important properties of metrics, but it generally does not preserve the property of constant scalar curvature. Conformal Ricci Flow is a modification of Ricci Flow which is intended for this purpose and for this reason it is restricted to the class of metrics of constant scalar curvature. Conformal Ricci Flow is, like Ricci Flow, a weakly parabolic flow of the metric on manifolds, except that Conformal Ricci Flow is coupled with an elliptic equation. Unlike Ricci Flow, Conformal Ricci Flow is restricted to the class of metrics of constant scalar curvature.

Let $(M^n, g_0)$ be a smooth n-dimensional Riemannian manifold with a metric $g_0$

of constant scalar curvature $s_0$. Conformal Ricci Flow on $M$ is defined as follows:

$$\begin{cases} \dfrac{\partial g}{\partial t} = -2\mathrm{Ric}_{g(t)} + 2\dfrac{s_0}{n}g(t) - 2p(t)g(t) \\ s\big(g(t)\big) = s_0 \end{cases} \text{on } M \times [0,T]. \tag{I.1}$$

Here $g(t),\ t \in [0,T]$, is a family of metrics on $M$ with $g(0) = g_0$, $s\big(g(t)\big)$ is the scalar curvature of $g(t)$, and $p(t),\ t \in [0,T]$, is a family of functions on $M$. In [8] and [13] we see that (I.1) is equivalent to the following system:

$$\begin{cases} \dfrac{\partial g}{\partial t} = -2\mathrm{Ric}_{g(t)} + 2\dfrac{s_0}{n}g(t) - 2p(t)g(t) \\ \big((n-1)\Delta_{g(t)} + s_0\big)p(t) = -\left\langle \mathrm{Ric}_{g(t)} - \dfrac{s_0}{n}g(t), \mathrm{Ric}_{g(t)} - \dfrac{s_0}{n}g(t) \right\rangle \end{cases} \tag{I.2}$$

Throughout this chapter we will use $V$ to denote the following symmetric 2-tensor:

$$V(t) = \mathrm{Ric}_{g(t)} - \frac{s_0}{n}g(t) + p(t)g(t) \tag{I.3}$$

In this chapter we use Kotchwar's energy techniques to give a proof of the uniqueness of Conformal Ricci Flow for closed manifolds with metrics of constant negative scalar curvature. It is worth noting similarities to the study of certain elliptic-hyperbolic systems done by Andersson and Moncrief in [1]. The existence of solutions to Conformal Ricci Flow has been shown by Fischer /citeFi and by Lu, Qing and Zheng, [13], the latter paper using DeTurck Conformal Ricci Flow. More precisely we will prove the following uniqueness theorem of Conformal Ricci Flow:

**Theorem I.1.** *Let* $(M^n, g_0)$ *be a closed manifold with constant negative scalar curvature* $s_0$. *Suppose* $\big(g(t), p(t)\big)$ *and* $\big(\tilde{g}(t), \tilde{p}(t)\big)$ *are two solutions of (I.1) on* $M \times [0, T]$ *with* $\tilde{g}(0) = g(0)$. *Then* $\big(\tilde{g}(t), \tilde{p}(t)\big) = \big(g(t), p(t)\big)$ *for* $0 \leq t \leq T$.

## The Differences Between $g(t)$ and $\tilde{g}(t)$

Let $g(t)$ and $\tilde{g}(t)$ be as in Theorem I.1. We will treat $g$ as our background metric and $\tilde{g}$ as our alternative metric. Let $\nabla, \tilde{\nabla}$ be the Riemannian connections of $g$ and $\tilde{g}$ respectively. Similarly, let $R, \tilde{R}$ represent the full Riemannian curvature tensors of $g$ and $\tilde{g}$ respectively.

Let $h = g - \tilde{g}$. Let $A = \nabla - \tilde{\nabla}$. Explicitly, $A^i_{jk} = \Gamma^i_{jk} - \tilde{\Gamma}^i_{jk}$ where $\Gamma^i_{jk}$ and $\tilde{\Gamma}^i_{jk}$ are the Christoffel symbols of $\nabla$ and $\tilde{\nabla}$ respectively. Also let $S = R - \tilde{R}$, $q = p - \tilde{p}$.

In this section we find bounds on $h$, $A$, $S$, $q$, $\nabla q$ and $\nabla\nabla q$ (see Propositions I.2 and I.4). Throughout this chapter we will use the convention $X * Y$ to denote any finite sum of tensors of the form $X \cdot Y$. We use C(X) to denote a finite sum of tensors of the form $X$.

### Preliminary Calculations

First we calculate some useful expressions for quantities which will arise in the proofs of Propositions I.2 and I.4. We calculate

$$g^{ij} - \tilde{g}^{ij} = g^{ik}(\tilde{g}^{j\ell}\tilde{g}_{k\ell}) - \tilde{g}^{j\ell}(g^{ik}g_{k\ell}) = -g^{ik}\tilde{g}^{j\ell}h_{k\ell},$$

i.e.

$$g^{-1} - \tilde{g}^{-1} = \tilde{g}^{-1} * h.$$

If $X$ is any tensor which is not a function we have

$$\big(\nabla - \tilde{\nabla}\big)X = A * X.$$

We check this when $X$ is a $(1,1)$-tensor. Calculating in local coordinates we see

$$\begin{aligned}\big(\nabla_i - \tilde{\nabla}_i\big)X_j^k &= \partial_i X_j^k - \Gamma_{ij}^\ell X_\ell^k + \Gamma_{i\ell}^k X_j^\ell - \partial_i X_j^k + \tilde{\Gamma}_{ij}^\ell X_\ell^k - \tilde{\Gamma}_{i\ell}^k X_j^\ell \\ &= A_{i\ell}^k X_j^\ell - A_{ij}^\ell X_\ell^k = A * X.\end{aligned}$$

If $f$ is a function however, then we have the following:

$$\big(\nabla^i - \tilde{\nabla}^i\big)f = \big(g^{ij} - \tilde{g}^{ij}\big)\partial_j f = -g^{ik}\tilde{g}^{j\ell}h_{k\ell}\partial_j f = -g^{ik}h_{k\ell}\tilde{\nabla}^\ell f,$$

or in other words

$$\big(\nabla - \tilde{\nabla}\big)f = h * \tilde{\nabla} f.$$

We now calculate

$$\nabla \tilde{g}^{-1} = \big(\nabla - \tilde{\nabla}\big)\tilde{g}^{-1} = \tilde{g}^{-1} * A.$$

The following calculation will also be important.

$$\nabla_i h_{jk} = \nabla_i g_{jk} - \nabla_i \tilde{g}_{jk} = -\big(\nabla_i - \tilde{\nabla}_i\big)\tilde{g}_{jk}.$$

Thus we have

$$\nabla h = \tilde{g} * A.$$

Now we are able to calculate the following for a function $f$.

$$\begin{aligned}\nabla\big(\nabla - \tilde{\nabla}\big)f &= \nabla\big(h * \tilde{\nabla} f\big) \\ &= \nabla h * \tilde{\nabla} f + h * \big(\nabla - \tilde{\nabla}\big)\tilde{\nabla} f + h * \tilde{\nabla}\tilde{\nabla} f \\ &= \tilde{g} * A * \tilde{\nabla} f + h * A * \tilde{\nabla} f + h * \tilde{\nabla}\tilde{\nabla} f.\end{aligned}$$

Now let

$$\begin{aligned}U^a_{ijk\ell} &= g^{ab}\nabla_b \tilde{R}_{ijk\ell} - \tilde{g}^{ab}\tilde{\nabla}_b \tilde{R}_{ijk\ell} \\ &= g^{ab}(\nabla_b - \tilde{\nabla}_b)\tilde{R}_{ijk\ell} + \big(g^{ab} - \tilde{g}^{ab}\big)\tilde{\nabla}_b \tilde{R}_{ijk\ell} \\ &= A * \tilde{R} + \tilde{g}^{-1} * h * \tilde{\nabla}\tilde{R},\end{aligned} \tag{I.4}$$

and we may calculate

$$\begin{aligned}\nabla_a\big(g^{ab}\nabla_b R - \tilde{g}^{ab}\tilde{\nabla}_b \tilde{R}\big) &= \nabla_a\big(g^{ab}\nabla_b \tilde{R} - \tilde{g}^{ab}\tilde{\nabla}_b \tilde{R}\big) + g^{ab}\nabla_a\nabla_b\big(R - \tilde{R}\big) \\ &= \operatorname{div} U + \Delta S.\end{aligned}$$

We summarize the above calculations in the following Lemma:

**Lemma I.1.** *Using the notation defined at the beginning of this section,*

$$g^{-1} - \tilde{g}^{-1} = \tilde{g}^{-1} * h \tag{I.5}$$

$$\big(\nabla - \tilde{\nabla}\big)X = A * X \tag{I.6}$$

$$\big(\nabla - \tilde{\nabla}\big)f = h * \tilde{\nabla} f \tag{I.7}$$

$$\nabla \tilde{g}^{-1} = \tilde{g}^{-1} * A \tag{I.8}$$

$$\nabla h = \tilde{g} * A \tag{I.9}$$

$$\nabla\big(\nabla - \tilde{\nabla}\big)f = \tilde{g} * A * \tilde{\nabla} f + h * A * \tilde{\nabla} f + h * \tilde{\nabla}\tilde{\nabla} f \tag{I.10}$$

$$U = A * \tilde{R} + \tilde{g}^{-1} * h * \tilde{\nabla}\tilde{R} \tag{I.11}$$

$$\nabla_a\big(g^{ab}\nabla_b R - \tilde{g}^{ab}\tilde{\nabla}_b \tilde{R}\big) = \operatorname{div} U + \Delta S \tag{I.12}$$

*where $U$ is defined in (I.4).*

Bounds on Time Derivatives of $h$, $A$ and $S$

In this subsection we derive bounds on the time derivatives of $h$, $A$ and $S$. In particular we will prove the following proposition. Here, as well as throughout this chapter, $C$ will denote a constant dependent only upon $n$ while $N$ will denote a constant with further dependencies.

**Proposition I.2.** *Let $\big(g(t), p(t)\big)$ and $\big(\tilde{g}(t), \tilde{p}(t)\big)$ be two solutions of (I.1) on $M \times [0, T]$. Using the notation defined at the beginning of this section, there exist* *constants $N_h$, $N_A$ and $N_S$ such that*

$$\left|\frac{\partial}{\partial t}h\right| \leq N_h|h| + C\big(|S| + |q|\big) \tag{I.13}$$

$$\left|\frac{\partial}{\partial t}A\right| \leq N_A\big(|h| + |A|\big) + C\big(|\nabla S| + |\nabla q|\big) \tag{I.14}$$

$$\left|\frac{\partial}{\partial t}S - \Delta S - \operatorname{div} U\right| \leq N_S\big(|h| + |A| + |S| + |q|\big) + C|\nabla\nabla q| \tag{I.15}$$

*where $U$ is defined in (I.4).*

*Proof.* We start with the time derivative of $h$. By (I.1) we have

$$\begin{aligned}
\frac{\partial}{\partial t}h_{ij} &= -2(R_{ij} - \tilde{R}_{ij}) + 2\frac{s_0}{n}(g_{ij} - \tilde{g}_{ij}) - 2(p\, g_{ij} - \tilde{p}\, \tilde{g}_{ij}) \\
&= -2S^k_{kij} + 2\frac{s_0}{n}h_{ij} - 2\big[(p - \tilde{p})g_{ij} + \tilde{p}(g_{ij} - \tilde{g}_{ij})\big] \\
&= -2S^k_{kij} + 2\frac{s_0}{n}h_{ij} - 2q\, g_{ij} - 2\tilde{p}\, h_{ij}.
\end{aligned}$$

Hence

$$\frac{\partial}{\partial t}h = C(S) + C(s_0 h) + C(q) + \tilde{p} * h$$

and

$$\left|\frac{\partial}{\partial t}h\right| \leq C\Big(\big(|s_0| + |\tilde{p}|\big)|h| + |S| + |q|\Big). \tag{I.16}$$

This proves (I.13).

Recall the definition of $V$ from (I.3):

$$V(t) = \mathrm{Ric}_{g(t)} - \frac{s_0}{n}g(t) + p(t)g(t). \tag{I.17}$$

We may define $\tilde{V}$ similarly using our alternate metric $\tilde{g}$. Since $V$ and $\tilde{V}$ are symmetric 2-tensors, then by [7, p. 108] we may calculate

$$\frac{\partial}{\partial t}A^k_{ij} = \tilde{g}^{k\ell}\big(\tilde{\nabla}_i\tilde{V}_{j\ell} + \tilde{\nabla}_j\tilde{V}_{i\ell} - \tilde{\nabla}_\ell\tilde{V}_{ij}\big) - g^{k\ell}\big(\nabla_i V_{j\ell} + \nabla_j V_{i\ell} - \nabla_\ell V_{ij}\big). \tag{I.18}$$

We proceed to calculate

$$
\begin{aligned}
&\tilde{g}^{k\ell}\tilde{\nabla}_i\tilde{V}_{j\ell} - g^{k\ell}\nabla_i V_{j\ell} \\
=&\tilde{g}^{k\ell}(\tilde{\nabla}_i\tilde{R}_{j\ell}) - g^{k\ell}(\nabla_i R_{j\ell}) + \tilde{g}^{k\ell}\tilde{\nabla}_i(\tilde{p}\ \tilde{g}_{j\ell}) - g^{k\ell}\nabla_i(p\ g_{j\ell}) \\
=&\big(\tilde{g}^{k\ell} - g^{k\ell}\big)\tilde{\nabla}_i\tilde{R}_{j\ell} + g^{k\ell}(\tilde{\nabla}_i - \nabla_i)\tilde{R}_{j\ell} - g^{k\ell}\nabla_i(S^m_{mj\ell}) + \delta^k_j\tilde{\nabla}_i\tilde{p} - \delta^k_j\nabla_i p \\
=&\tilde{g}^{-1} * h * \tilde{\nabla}\tilde{R} + A * \tilde{R} + C(\nabla S) + h * \tilde{\nabla}\tilde{p} + C(\nabla q),
\end{aligned}
\tag{I.19}
$$

where we have used (I.7) to get the last equality. Similarly we find

$$
\begin{aligned}
&\tilde{g}^{k\ell}\tilde{\nabla}_j\tilde{V}_{i\ell} - g^{k\ell}\nabla_j V_{i\ell} \\
=&\tilde{g}^{-1} * h * \tilde{\nabla}\tilde{R} + A * \tilde{R} + C(\nabla S) + h * \tilde{\nabla}\tilde{p} + C(\nabla q).
\end{aligned}
\tag{I.20}
$$

Now we consider

$$
\begin{aligned}
&- \tilde{g}^{k\ell}\tilde{\nabla}_\ell\tilde{V}_{ij} + g^{k\ell}\nabla_\ell V_{ij} \\
=&\tilde{g}^{-1} * h * \tilde{\nabla}\tilde{R} + A * \tilde{R} + C(\nabla S) + \tilde{g}^{k\ell}\tilde{g}_{ij}\tilde{\nabla}_\ell\tilde{p} - g^{k\ell}g_{ij}\nabla_\ell p \\
=&\tilde{g}^{-1} * h * \tilde{\nabla}\tilde{R} + A * \tilde{R} + C(\nabla S) + \big(\tilde{g}^{k\ell} - g^{k\ell}\big)\tilde{g}_{ij}\tilde{\nabla}_\ell\tilde{p} + g^{k\ell}\big(\tilde{g}_{ij} - g_{ij}\big)\tilde{\nabla}_\ell\tilde{p} \\
&+ g^{k\ell}g_{ij}(\tilde{\nabla}_\ell - \nabla_\ell)\tilde{p} + g^{k\ell}g_{ij}\nabla_\ell(\tilde{p} - p) \\
=&\tilde{g}^{-1} * h * \tilde{\nabla}\tilde{R} + A * \tilde{R} + C(\nabla S) + \tilde{g}^{-1} * h * \tilde{g} * \tilde{\nabla}\tilde{p} + h * \tilde{\nabla}\tilde{p} + C(\nabla q).
\end{aligned}
\tag{I.21}
$$

Hence by (I.18), (I.19), (I.20) and (I.21),

$$
\frac{\partial}{\partial t}A = \tilde{g}^{-1} * h * \tilde{\nabla}\tilde{R} + A * \tilde{R} + C(\nabla S) + h * \tilde{\nabla}\tilde{p} + C(\nabla q) + \tilde{g}^{-1} * h * \tilde{g} * \tilde{\nabla}\tilde{p}
$$

and

$$
\left|\frac{\partial}{\partial t}A\right| \le C\Big(\big(|\tilde{g}^{-1}||\tilde{\nabla}\tilde{R}| + |\tilde{\nabla}\tilde{p}| + |\tilde{g}^{-1}||\tilde{g}||\tilde{\nabla}\tilde{p}|\big)|h| + |\tilde{R}||A| + |\nabla S| + |\nabla q|\Big). \tag{I.22}
$$

This proves (I.14).

By [7, eqn. (2.67)] we have

$$
\begin{aligned}
\frac{\partial}{\partial t} R_{ijk}^{\ell} &= g^{\ell m}\big(\nabla_i\nabla_k V_{jm} - \nabla_i\nabla_m V_{jk} - \nabla_j\nabla_k V_{im} + \nabla_j\nabla_m V_{ik}\big) \\
&\qquad - g^{\ell m}\big(R_{ijk}^{r} V_{rm} + R_{ijm}^{q} V_{kq}\big) \\
&= g^{\ell m}\big(-\nabla_i\nabla_k R_{jm} + \nabla_i\nabla_m R_{jk} + \nabla_j\nabla_k R_{im} - \nabla_j\nabla_m R_{ik}\big) \\
&\qquad + g^{\ell m}\big(-g_{jm}\nabla_i\nabla_k p + g_{jk}\nabla_i\nabla_m p + g_{im}\nabla_j\nabla_k p - g_{ik}\nabla_j\nabla_m p\big) \\
&\qquad + g^{\ell m}\big(R_{ijk}^{r} R_{rm} + R_{ijm}^{r} R_{kr}\big) - \frac{s_0}{n} g^{\ell m}\big(R_{ijk}^{r} g_{rm} + R_{ijm}^{r} g_{kr}\big) p \\
&\qquad + g^{\ell m}\big(R_{ijk}^{r} g_{rm} + R_{ijm}^{r} g_{kr}\big) p.
\end{aligned} \tag{I.23}
$$

Following the calculations in [7, p. 119-120] we have

$$
\begin{aligned}
\Delta R_{ijk}^{\ell} &= g^{ab}\nabla_a\nabla_b R_{ijk}^{\ell} = g^{ab}\big(-\nabla_a\nabla_i R_{jbk}^{\ell} - \nabla_a\nabla_j R_{bik}^{\ell}\big) \\
&= g^{ab}\big(-\nabla_i\nabla_a R_{jbk}^{\ell} + R_{aij}^{m} R_{mbk}^{\ell} + R_{aib}^{m} R_{jmk}^{\ell} + R_{aik}^{m} R_{jbm}^{\ell} - R_{aim}^{\ell} R_{jbk}^{m} \\
&\qquad - \nabla_j\nabla_a R_{bik}^{\ell} + R_{ajb}^{m} R_{mik}^{\ell} + R_{aji}^{m} R_{bmk}^{\ell} + R_{ajk}^{m} R_{bim}^{\ell} - R_{ajm}^{\ell} R_{bik}^{m}\big) \\
&= g^{\ell m}\big(-\nabla_i\nabla_k R_{jm} + \nabla_i\nabla_m R_{jk} + \nabla_j\nabla_k R_{im} - \nabla_j\nabla_m R_{ik}\big) \\
&\qquad + g^{mr}\big(-R_{ir} R_{jmk}^{\ell} - R_{jr} R_{mik}^{\ell}\big) \\
&\qquad + g^{ab}\big(R_{aij}^{m} R_{mbk}^{\ell} + R_{aik}^{m} R_{jbm}^{\ell} - R_{aim}^{\ell} R_{jbk}^{m} \\
&\qquad\qquad + R_{aji}^{m} R_{bmk}^{\ell} + R_{ajk}^{m} R_{bim}^{\ell} - R_{ajm}^{\ell} R_{bik}^{m}\big).
\end{aligned} \tag{I.24}
$$

Combining (I.23) and (I.24) we have

$$\begin{aligned}
\frac{\partial}{\partial t} R^{\ell}_{ijk} &= \Delta R^{\ell}_{ijk} + g^{mr}\big(R_{ir} R^{\ell}_{jmk} + R_{jr} R^{\ell}_{mik}\big) \\
&\quad + g^{ab}\big(-R^{m}_{aij} R^{\ell}_{mbk} - R^{m}_{aik} R^{\ell}_{jbm} + R^{\ell}_{aim} R^{m}_{jbk} \\
&\qquad - R^{m}_{aji} R^{\ell}_{bmk} - R^{m}_{ajk} R^{\ell}_{bim} + R^{\ell}_{ajm} R^{m}_{bik}\big) \\
&\quad + g^{\ell m}\big(-g_{jm} \nabla_i \nabla_k p + g_{jk} \nabla_i \nabla_m p + g_{im} \nabla_j \nabla_k p - g_{ik} \nabla_j \nabla_m p\big) \\
&\quad + g^{\ell m}\big(R^{r}_{ijk} R_{rm} + R^{r}_{ijm} R_{kr}\big) - \frac{s_0}{n} g^{\ell m}\big(R^{r}_{ijk} g_{rm} + R^{r}_{ijm} g_{kr}\big) \\
&\quad + g^{\ell m}\big(R^{r}_{ijk} g_{rm} + R^{r}_{ijm} g_{kr}\big) p. \qquad \text{(I.25)}
\end{aligned}$$

Hence the evolution of $S$ is

$$\begin{aligned}
\frac{\partial}{\partial t} S^{\ell}_{ijk} &= \Delta R^{\ell}_{ijk} - \tilde{\Delta} \tilde{R}^{\ell}_{ijk} \\
&\quad + g^{mr}\big(R_{ir} R^{\ell}_{jmk} + R_{jr} R^{\ell}_{jmk}\big) - \tilde{g}^{mr}\big(\tilde{R}_{ir} \tilde{R}^{\ell}_{jmk} + \tilde{R}_{jr} \tilde{R}^{\ell}_{mik}\big) \\
&\quad + g^{ab}\big(-R^{m}_{aij} R^{\ell}_{mbk} - R^{m}_{aik} R^{\ell}_{jbm} + R^{\ell}_{aim} R^{m}_{jbk} \\
&\qquad - R^{m}_{aji} R^{\ell}_{bmk} - R^{m}_{ajk} R^{\ell}_{bim} + R^{\ell}_{ajm} R^{m}_{bik}\big) \\
&\quad - \tilde{g}^{ab}\big(-\tilde{R}^{m}_{aij} \tilde{R}^{\ell}_{mbk} - \tilde{R}^{m}_{aik} \tilde{R}^{\ell}_{jbm} + \tilde{R}^{\ell}_{aim} \tilde{R}^{m}_{jbk} \\
&\qquad - \tilde{R}^{m}_{aji} \tilde{R}^{\ell}_{bmk} - \tilde{R}^{m}_{ajk} \tilde{R}^{\ell}_{bim} + \tilde{R}^{\ell}_{ajm} \tilde{R}^{m}_{bik}\big) \\
&\quad + g^{\ell m}\big(-g_{jm} \nabla_i \nabla_k p + g_{jk} \nabla_i \nabla_m p + g_{im} \nabla_j \nabla_k p - g_{ik} \nabla_j \nabla_m p\big) \\
&\quad - \tilde{g}^{\ell m}\big(-\tilde{g}_{jm} \tilde{\nabla}_i \tilde{\nabla}_k \tilde{p} + \tilde{g}_{jk} \tilde{\nabla}_i \tilde{\nabla}_m \tilde{p} + \tilde{g}_{im} \tilde{\nabla}_j \tilde{\nabla}_k \tilde{p} - \tilde{g}_{ik} \tilde{\nabla}_j \tilde{\nabla}_m \tilde{p}\big) \\
&\quad + g^{\ell m}\big(R^{r}_{ijk} R_{rm} + R^{r}_{ijm} R_{kr}\big) - \tilde{g}^{\ell m}\big(\tilde{R}^{r}_{ijk} \tilde{R}_{rm} + \tilde{R}^{r}_{ijm} \tilde{R}_{kr}\big) \\
&\quad - \frac{s_0}{n} g^{\ell m}\big(R^{r}_{ijk} g_{rm} + R^{r}_{ijm} g_{kr}\big) + \frac{s_0}{n} \tilde{g}^{\ell m}\big(\tilde{R}^{r}_{ijk} \tilde{g}_{rm} + \tilde{R}^{r}_{ijm} \tilde{g}_{kr}\big) \\
&\quad + g^{\ell m}\big(R^{r}_{ijk} g_{rm} + R^{r}_{ijm} g_{kr}\big) p - \tilde{g}^{\ell m}\big(\tilde{R}^{r}_{ijk} \tilde{g}_{rm} + \tilde{R}^{r}_{ijm} \tilde{g}_{kr}\big) \tilde{p}. \qquad \text{(I.26)}
\end{aligned}$$

Looking at the individual components, we see

$$
\begin{aligned}
&\Delta R - \tilde{\Delta}\tilde{R} \\
=&g^{ab}\nabla_a\nabla_b R - \tilde{g}^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\tilde{R} \\
=&\nabla_a(g^{ab}\nabla_b R) - \nabla_a(\tilde{g}^{ab}\tilde{\nabla}_b\tilde{R}) + (\nabla_a - \tilde{\nabla}_a)(\tilde{g}^{ab}\tilde{\nabla}_b\tilde{R}) \\
=&\nabla_a\big(g^{ab}\nabla_b R - \tilde{g}^{ab}\tilde{\nabla}_b\tilde{R}\big) + \tilde{g}^{-1} * A * \tilde{\nabla}\tilde{R},
\end{aligned} \tag{I.27}
$$

while

$$
\begin{aligned}
&g^{-1}RR - \tilde{g}^{-1}\tilde{R}\tilde{R} \\
=&(g^{-1} - \tilde{g}^{-1})(\tilde{R}\tilde{R}) + g^{-1}(RR - \tilde{R}\tilde{R}) \\
=&\tilde{g}^{-1} * h * \tilde{R} * \tilde{R} + g^{-1}(R - \tilde{R})\tilde{R} + g^{-1}(RR - R\tilde{R}) \\
=&\tilde{g}^{-1} * h * \tilde{R} * \tilde{R} + S * \tilde{R} + S * R,
\end{aligned} \tag{I.28}
$$

and

$$
\begin{aligned}
&g^{-1}g\nabla\nabla p - \tilde{g}^{-1}\tilde{g}\tilde{\nabla}\tilde{\nabla}\tilde{p} \\
=&(g^{-1} - \tilde{g}^{-1})\tilde{g}\tilde{\nabla}\tilde{\nabla}\tilde{p} + g^{-1}(g - \tilde{g})\tilde{\nabla}\tilde{\nabla}\tilde{p} + g^{-1}g(\nabla\nabla p - \tilde{\nabla}\tilde{\nabla}\tilde{p}) \\
=&\tilde{g}^{-1} * h * \tilde{g} * \tilde{\nabla}\tilde{\nabla}\tilde{p} + h * \tilde{\nabla}\tilde{\nabla}\tilde{p} + g^{-1}g(\nabla - \tilde{\nabla})(\tilde{\nabla}\tilde{p}) + g^{-1}g(\nabla\nabla p - \nabla\tilde{\nabla}\tilde{p}) \\
=&\tilde{g}^{-1} * h * \tilde{g} * \tilde{\nabla}\tilde{\nabla}\tilde{p} + h * \tilde{\nabla}\tilde{\nabla}\tilde{p} + A * \tilde{\nabla}\tilde{p} + g^{-1}g\nabla(\nabla - \tilde{\nabla})\tilde{p} + g^{-1}g\nabla\nabla(p - \tilde{p}) \\
=&\tilde{g}^{-1} * h * \tilde{g} * \tilde{\nabla}\tilde{\nabla}\tilde{p} + h * \tilde{\nabla}\tilde{\nabla}\tilde{p} + A * \tilde{\nabla}\tilde{p} + h * A * \tilde{\nabla}\tilde{p} + C(\nabla\nabla q),
\end{aligned} \tag{I.29}
$$

where in the last equality we used (I.10). We also have

$$\begin{aligned}
&g^{-1}gR - \tilde{g}^{-1}\tilde{g}\tilde{R}\\
=&(g^{-1} - \tilde{g}^{-1})\tilde{g}\tilde{R} + g^{-1}(g - \tilde{g})\tilde{R} + g^{-1}g(R - \tilde{R})\\
=&\tilde{g}^{-1} * h * \tilde{g} * \tilde{R} + h * \tilde{R} + C(S),
\end{aligned} \tag{I.30}$$

and lastly

$$\begin{aligned}
&g^{-1}gRp - \tilde{g}^{-1}\tilde{g}\tilde{R}\tilde{p}\\
=&(g^{-1} - \tilde{g}^{-1})\tilde{g}\tilde{R}\tilde{p} + g^{-1}(g - \tilde{g})\tilde{R}\tilde{p} + g^{-1}g(R - \tilde{R})\tilde{p} + g^{-1}gR(p - \tilde{p})\\
=&\tilde{g}^{-1} * h * \tilde{g} * \tilde{R} * \tilde{p} + h * \tilde{R} * \tilde{p} + S * \tilde{p} + R * q.
\end{aligned} \tag{I.31}$$

Now by (I.26), (I.27), (I.28), (I.29), (I.30) and (I.31) we see

$$\begin{aligned}
\frac{\partial}{\partial t}S = &\nabla_a\big(g^{ab}\nabla_b R - \tilde{g}^{ab}\tilde{\nabla}_b\tilde{R}\big) + \tilde{g}^{-1} * A * \tilde{\nabla}\tilde{R} + \tilde{g}^{-1} * h * \tilde{R} * \tilde{R}\\
&+ S * \tilde{R} + S * R + \tilde{g}^{-1} * h * \tilde{g} * \tilde{\nabla}\tilde{\nabla}\tilde{p} + h * \tilde{\nabla}\tilde{\nabla}\tilde{p} + A * \tilde{\nabla}\tilde{p}\\
&+ h * A * \tilde{\nabla}\tilde{p} + C(\nabla\nabla q) + \tilde{g}^{-1} * h * \tilde{g} * \tilde{R} + h * \tilde{R} + C(S)\\
&+ \tilde{g}^{-1} * h * \tilde{g} * \tilde{R} * \tilde{p} + h * \tilde{R} * \tilde{p} + S * \tilde{p} + R * q.
\end{aligned}$$

Hence by (I.12) we have

$$\begin{aligned}
&\left|\frac{\partial}{\partial t}S - \Delta S - \text{div } U\right| \\
\leq &C\bigg(\Big(|\tilde{g}^{-1}||\tilde{R}|^2 + |\tilde{g}^{-1}||\tilde{g}||\tilde{\nabla}\tilde{\nabla}\tilde{p}| + |\tilde{\nabla}\tilde{\nabla}\tilde{p}| \\
&\quad + |\tilde{g}^{-1}||\tilde{g}||\tilde{R}| + |\tilde{R}| + |\tilde{g}^{-1}||\tilde{g}||\tilde{R}||\tilde{p}| + |\tilde{R}||\tilde{p}|\Big)|h| \\
&\quad + \Big(|\tilde{g}^{-1}||\tilde{\nabla}\tilde{R}| + |\tilde{\nabla}\tilde{p}| + |h||\tilde{\nabla}\tilde{p}|\Big)|A| \\
&\quad + \Big(|\tilde{R}| + |R| + 1 + |\tilde{p}|\Big)|S| + |R||q| + |\nabla\nabla q|\bigg). \qquad \text{(I.32)}
\end{aligned}$$

This proves (I.15). □

**Remark I.3.** Upon closer observation we notice the following dependencies:

$$N_h = N_h\big(n, s_0, |\tilde{p}|\big),$$

$$N_A = N_A\big(n, s_0, |\tilde{g}|, |\tilde{g}^{-1}|, |\tilde{R}|, |\tilde{\nabla}\tilde{R}|, |\tilde{\nabla}\tilde{p}|\big),$$

$$N_S = N_S\big(n, s_0, |\tilde{g}|, |\tilde{g}^{-1}|, |h|, |R|, |\tilde{R}|, |\tilde{\nabla}\tilde{R}|, |\tilde{p}|, |\tilde{\nabla}\tilde{p}|, |\tilde{\nabla}\tilde{\nabla}\tilde{p}|\big).$$

$M$ is closed, so $M \times [0, T]$ is compact. Thus, given two metrics $g$ and $\tilde{g}$, all of these quantities will be bounded.

Bounds on $q$ and Its Spatial Derivatives

We turn our attention now to finding bounds on the differences between our pressure functions $p$ and $\tilde{p}$. We have the following proposition:

**Proposition I.4.** *Let $\big(g(t), p(t)\big)$ and $\big(\tilde{g}(t), \tilde{p}(t)\big)$ be two solutions of (I.1) on $M \times$*

$[0,T]$. *Then there exist constants $N_q$ and $\hat{N}_q$ such that*

$$\int_M |q|^2 d\mu \leq N_q \int_M \big(|h|^2 + |A|^2 + |S|^2\big) d\mu \tag{I.33}$$

$$\int_M |\nabla q|^2 d\mu \leq N_q \int_M \big(|h|^2 + |A|^2 + |S|^2\big) d\mu \tag{I.34}$$

$$\int_M |\nabla\nabla q|^2 d\mu \leq \hat{N}_q \int_M \big(|h|^2 + |A|^2 + |S|^2\big) d\mu \tag{I.35}$$

*Proof.* We let $f$ represent any smooth function or tensor. In particular we will let $f$ be represented by the function $q$, the difference of the pressure functions. Since $M$ is compact we have

$$\begin{aligned}
&\int_M \big((n-1)\Delta + s_0\big)(f) \cdot f \ d\mu \\
=& s_0 \int_M |f|^2 d\mu - (n-1) \int_M \big\langle \nabla f, \nabla f \big\rangle d\mu.
\end{aligned}$$

Since $s_0 < 0$, taking the absolute value gives

$$\left| \int_M \big((n-1)\Delta + s_0)(f) \cdot f d\mu \right| = |s_0| \int_M |f|^2 d\mu + (n-1) \int_M |\nabla f|^2 d\mu \tag{I.36}$$

Now we deal specifically with $p$, $\tilde{p}$ and $q$. By (I.2) we have the following equations for the pressure functions $p$ and $\tilde{p}$:

$$\big((n-1)\Delta + s_0\big)p = -\left\langle \mathrm{Ric} - \frac{s_0}{n} g, \mathrm{Ric} - \frac{s_0}{n} g \right\rangle \tag{I.37}$$

$$\big((n-1)\tilde{\Delta} + s_0\big)\tilde{p} = -\left\langle \tilde{\mathrm{Ric}} - \frac{s_0}{n} \tilde{g}, \tilde{\mathrm{Ric}} - \frac{s_0}{n} \tilde{g} \right\rangle. \tag{I.38}$$

Now we calculate

$$\begin{aligned}\Delta p - \tilde{\Delta}\tilde{p} &= g^{ab}\nabla_a\nabla_b p - \tilde{g}^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\tilde{p}\\ &= (g^{-1}-\tilde{g}^{-1})\tilde{\nabla}\tilde{\nabla}\tilde{p} + g^{-1}(\nabla-\tilde{\nabla})\tilde{\nabla}\tilde{p} + g^{-1}\nabla(\nabla-\tilde{\nabla})\tilde{p} + \Delta(p-\tilde{p})\\ &= \tilde{g}^{-1} * h * \tilde{\nabla}\tilde{\nabla}\tilde{p} + A * \tilde{\nabla}\tilde{p} + h * A * \tilde{\nabla}\tilde{p} + \Delta q. \end{aligned} \tag{I.39}$$

We also compute

$$\begin{aligned} &-\left\langle \mathrm{Ric} - \frac{s_0}{n}g, \mathrm{Ric} - \frac{s_0}{n}g\right\rangle + \left\langle \tilde{\mathrm{Ric}} - \frac{s_0}{n}\tilde{g}, \tilde{\mathrm{Ric}} - \frac{s_0}{n}\tilde{g}\right\rangle\\ =& -\left(g^{ik}g^{j\ell}R_{ij}R_{k\ell} - \tilde{g}^{ik}\tilde{g}^{j\ell}\tilde{R}_{ij}\tilde{R}_{k\ell}\right) + 2\frac{s_0}{n}\left(g^{ij}R_{ij} - \tilde{g}^{ij}\tilde{R}_{ij}\right)\\ =& -(g^{-1}-\tilde{g}^{-1})\tilde{g}^{-1}\tilde{R}\tilde{R} - g^{-1}(g^{-1}-\tilde{g}^{-1})\tilde{R}\tilde{R} - g^{-1}g^{-1}(R-\tilde{R})\tilde{R}\\ &\quad - g^{-1}g^{-1}R(R-\tilde{R}) + 2\frac{s_0}{n}(g^{-1}-\tilde{g}^{-1})\tilde{R} + 2\frac{s_0}{n}g^{-1}(R-\tilde{R})\\ =&\tilde{g}^{-1} * \tilde{g}^{-1} * h * \tilde{R} * \tilde{R} + \tilde{g}^{-1} * h * \tilde{R} * \tilde{R}\\ &\quad + S * \tilde{R} + S * R + \tilde{g}^{-1} * h * \tilde{R} + C(S). \end{aligned} \tag{I.40}$$

Combining (I.37), (I.38), (I.39) and (I.40), we see that $q$ satisfies the following elliptic equation at each time $t \in [0,T]$:

$$\begin{aligned} Lq &= \big((n-1)\Delta + s_0\big)(q)\\ &= \tilde{g}^{-1} * h * \tilde{\nabla}\tilde{\nabla}\tilde{p} + A * \tilde{\nabla}\tilde{p} + h * A * \tilde{\nabla}\tilde{p} + \tilde{g}^{-1} * \tilde{g}^{-1} * h * \tilde{R} * \tilde{R}\\ &\quad + \tilde{g}^{-1} * h * \tilde{R} * \tilde{R} + S * \tilde{R} + S * R + \tilde{g}^{-1} * h * \tilde{R} + C(S) \end{aligned} \tag{I.41}$$

Hence

$$|Lq| = \left|\big((n-1)\Delta + s_0\big)(q)\right| \le N\big(|h| + |A| + |S|\big). \tag{I.42}$$

To find estimates for $q$ and $\nabla q$, we combine (I.36) and (I.42):

$$\begin{aligned}
&|s_0|\int_M |q|^2 d\mu + (n-1)\int_M |\nabla q|^2 d\mu \\
=&\left|\int_M \big((n-1)\Delta + s_0\big)(q)\cdot q\ d\mu\right| \\
\leq&\int_M N\big(|h|+|A|+|S|\big)|q|\ d\mu \\
\leq&\frac{|s_0|}{2}\int_M |q|^2 d\mu + N\int_M \big(|h|^2+|A|^2+|S|^2\big)d\mu.
\end{aligned}$$

Thus

$$\frac{|s_0|}{2}\int_M |q|^2 d\mu + (n-1)\int_M |\nabla q|^2 d\mu \leq N\int_M \big(|h|^2+|A|^2+|S|^2\big)d\mu,$$

and we proved (I.33) and (I.34).

To find an appropriate bound for $|\nabla\nabla q|$ we must turn to Interior Regularity Theory for Elliptic PDE. From (I.41) we see that $Lq = f$ is an Elliptic Equation. We then have the following estimate from [15, p. 229].

$$|q|_{H^2(W)} \leq K\big(|Lq|_{L^2(M)} + |q|_{H^1(M)}\big),$$

where $W$ is any compactly supported open subset of $M$ and $K$ depends only upon the coefficients of the operator $L$, the subset $W$ and the manifold $M$. Since $M$ is a closed manifold we may in fact choose $W = M$. Thus we have

$$|q|_{H^2(M)} \leq K\big(|Lq|_{L^2(M)} + |q|_{H^1(M)}\big). \tag{I.43}$$

Upon squaring both sides we observe

$$\int_M |\nabla\nabla q|^2 d\mu \leq |q|^2_{H^2(M)} \leq K^2\left(\int_M |Lq|^2 d\mu + |q|^2_{H^1(M)}\right). \tag{I.44}$$

Now (I.33) and (I.34) imply that

$$|q|^2_{H^1(M)} \leq N \int_M \big(|h|^2 + |A|^2 + |S|^2\big)d\mu. \tag{I.45}$$

Combining (I.42), (I.44) and (I.45) we have

$$\int_M |\nabla\nabla q|^2 d\mu \leq N \int_M \big(|h|^2 + |A|^2 + |S|^2\big)d\mu,$$

and we proved (I.35).

$\square$

**Remark I.5.** We observe the following dependencies:

$$N_q = N_q\big(n, s_0, |\tilde{g}^{-1}|, |h|, |R|, |\tilde{R}|, |\tilde{\nabla}\tilde{p}|, |\tilde{\nabla}\tilde{\nabla}\tilde{p}|\big)$$

$$\hat{N}_q = \hat{N}_q\big(n, s_0, |\tilde{g}^{-1}|, |h|, |R|, |\tilde{R}|, |\tilde{\nabla}\tilde{p}|, |\tilde{\nabla}\tilde{\nabla}\tilde{p}|, K\big)$$

where $K$ is from (I.43).

## Energy Estimates

Now we shall approximate the energy

$$\mathcal{E}(t) = \int_M \big(|h|^2 + |A|^2 + |S|^2\big)d\mu. \tag{I.46}$$

We also define the following:

$$\mathcal{H}(t) = \int_M |h|^2 d\mu \tag{I.47}$$

$$\mathcal{A}(t) = \int_M |A|^2 d\mu \tag{I.48}$$

$$\mathcal{S}(t) = \int_M |S|^2 d\mu \tag{I.49}$$

$$\mathcal{D}(t) = \int_M |\nabla S|^2 d\mu \tag{I.50}$$

Note that $\mathcal{E}(t) = \mathcal{H}(t) + \mathcal{A}(t) + \mathcal{S}(t)$. We now estimate the evolution of the energy functional under Conformal Ricci Flow, $\mathcal{E}'(t)$, by first estimating the evolutions of $\mathcal{H}$, $\mathcal{A}$ and $\mathcal{S}$.

Evolution of $\mathcal{H}(t)$

In [13], Lu, Qing and Zheng give the evolution of the volume element under Conformal Ricci Flow:

$$\frac{\partial}{\partial t} d\mu_{g(t)} = -np(t) d\mu_{g(t)} \tag{I.51}$$

Hence by (I.13) and (I.47) we have

$$\begin{aligned}
\mathcal{H}'(t) &\leq N \int_M |h|^2 d\mu + \int_M 2 \left\langle \frac{\partial h}{\partial t}, h \right\rangle d\mu \\
&\leq N\mathcal{H}(t) + \int_M 2|h| \left| \frac{\partial h}{\partial t} \right| d\mu \\
&\leq N\mathcal{H}(t) + N \int_M \big( |S||h| + |h|^2 + |q||h| \big) d\mu.
\end{aligned}$$

Now we know that $N\big(|S||h|+|q||h|\big)\leq N\big(|h|^2+|S|^2+|q|^2\big)$. Hence

$$\begin{aligned}\mathcal{H}'(t)&\leq N\mathcal{H}(t)+N\int_M\big(|S|^2+|q|^2\big)d\mu\\&\leq N\mathcal{H}(t)+N\int_M\big(|S|^2+|h|^2+|A|^2\big)d\mu\\&\leq N\mathcal{H}(t)+N\mathcal{S}(t)+N\mathcal{A}(t)=N\mathcal{E}(t).\end{aligned}\tag{I.52}$$

Evolution of $\mathcal{A}(t)$

By (I.14), (I.48) and (I.51) we have

$$\begin{aligned}\mathcal{A}'(t)&\leq N\mathcal{A}(t)+\int_M 2|A|\left|\frac{\partial A}{\partial t}\right|d\mu\\&\leq N\mathcal{A}(t)+\int_M\Big(N|h||A|+N|A|^2+C|\nabla S||A|+C|\nabla q||A|\Big)d\mu.\end{aligned}$$

Now

$$N|h||A|+C|\nabla S||A|+C|\nabla q||A|\leq N|h|^2+N|A|^2+|\nabla S|^2+|\nabla q|^2.$$

hence we have that

$$\begin{aligned}\mathcal{A}'(t)&\leq N\mathcal{A}(t)+\int_M\Big(N|h|^2+N|A|^2+|\nabla S|^2+|\nabla q|^2\Big)d\mu\\&\leq N\mathcal{A}(t)+N\mathcal{H}(t)+\mathcal{D}(t)+N\int_M\big(|h|^2+|A|^2+|S|^2\big)d\mu\\&\leq N\mathcal{A}(t)+N\mathcal{H}(t)+N\mathcal{S}(t)+\mathcal{D}(t)=N\mathcal{E}(t)+\mathcal{D}(t).\end{aligned}\tag{I.53}$$

Evolution of $\mathcal{S}(t)$

By (I.15), (I.49) and (I.51) we have

$$\begin{aligned}\mathcal{S}'(t) &\le N\int_M |S|^2 d\mu + \int_M 2\Big\langle \frac{\partial S}{\partial t}, S\Big\rangle d\mu \\ &\le N\mathcal{S}(t) + \int_M \Big(2\big\langle \Delta S + \text{div } V, S\big\rangle \\ &\qquad + N\big(|h| + |A| + |S| + |q|\big)|S| + C|\nabla\nabla q||S|\Big) d\mu \\ &\le N\mathcal{S}(t) + \int_M \Big(2\big\langle \Delta S + \text{div } V, S\big\rangle \\ &\qquad + N\big(|h|^2 + |A|^2 + |S|^2 + |q|^2 + |\nabla\nabla q|^2\big)\Big) d\mu.\end{aligned}$$

Now by (I.33) and (I.35) we have

$$\begin{aligned}\mathcal{S}'(t) &\le N\mathcal{S}(t) + N\mathcal{H}(t) + N\mathcal{A}(t) \\ &\qquad + \int_M \Big(2\big\langle \Delta S + \text{div } V, S\big\rangle + N\big(|A|^2 + |S|^2 + |h|^2\big)\Big) d\mu \\ &\le N\mathcal{S}(t) + N\mathcal{H}(t) + N\mathcal{A}(t) + \int_M 2\big\langle \Delta S + \text{div } V, S\big\rangle d\mu.\end{aligned}$$

Upon integrating by parts we get

$$\begin{aligned}\mathcal{S}'(t) &\le N\mathcal{E}(t) - 2\int_M \big\langle \nabla S + V, \nabla S\big\rangle d\mu \\ &\le N\mathcal{E}(t) - 2\int_M |\nabla S|^2 d\mu + \int_M 2|V||\nabla S| d\mu.\end{aligned}$$

Now we know that

$$2|V||\nabla S| \le |\nabla S|^2 + |V|^2 \le |\nabla S|^2 + N\big(|h|^2 + |A|^2\big),$$

hence

$$\mathcal{S}'(t) \le N\mathcal{E}(t) + N\int_M \big(|h|^2 + |A|^2\big) d\mu - \int_M |\nabla S|^2 d\mu \le N\mathcal{E}(t) - \mathcal{D}(t). \qquad \text{(I.54)}$$

## Proof of Main Theorem

Now we are ready to prove Theorem 1:

*Proof.* By (I.54), (I.52) and (I.53) we know that

$$\mathcal{H}'(t) \leq N\mathcal{E}(t),\ \mathcal{A}'(t) \leq N\mathcal{E}(t) + \mathcal{D}(t),\ \mathcal{S}'(t) \leq N\mathcal{E}(t) - \mathcal{D}(t),$$

so

$$\mathcal{E}'(t) \leq N\mathcal{E}(t).$$

Our initial condition $\tilde{g}(0) = g(0)$ tells us that at $t = 0$ we have $|h| = |A| = |S| = 0$. Therefore by the smoothness and integrability of our solutions we know

$$\lim_{t \to 0^+} \mathcal{E}(t) = 0,$$

so by Gronwall's Inequality we know that $\mathcal{E} \equiv 0$ on $[0, T]$. Thus for $t \in [0, T]$ we have that $h \equiv 0$ and $g(t) \equiv \tilde{g}(t)$. Also, $\mathcal{E} \equiv 0$ implies $A \equiv 0$ and $S \equiv 0$, so (I.33) forces $q \equiv 0$. Thus $p(t) \equiv \tilde{p}(t)$. Therefore $\big(\tilde{g}(t), \tilde{p}(t)\big) = \big(g(t), p(t)\big),\ t \in [0, T]$.

$\square$

CHAPTER II

# BACKWARD RICCI FLOW OF HOMOGENEOUS 4-GEOMETRIES

Ricci flow on a manifold, $(M, g_0)$ is an evolution equation on the metric tensor given by

$$\frac{\partial g}{\partial t} = -2\text{Ric}_{g(t)}. \tag{II.1}$$

This equation was first introduced by Richard Hamilton in [9], where he demonstrated that one can always expect short time existence as long as $(M, g_0)$ is a smooth manifold. Backward Ricci flow is described by

$$\frac{\partial g}{\partial t} = 2\text{Ric}_{g(t)}. \tag{II.2}$$

In general, we cannot expect short time existence of solutions to this equation. However, in the case of locally homogeneous manifolds, Ricci flow reduces to a system of Ordinary Differential Equations. As mentioned in [4], this eliminates all barriers to backward short-time existence of Ricci flow in these manifolds.

In [10], Isenberg and Jackson studied the behavior of solutions to Ricci flow along locally homogeneous 3-manifolds. Later, in [11], Isenberg Jackson and Lu studied Ricci flow along locally homogeneous 4-manifolds which admit compact quotients. Subsequently, in [4], Cao and Saloff-Coste used calculations from [6] and [10] to study backward Ricci flow on the homogeneous 3-manifolds.

In this chapter we use many of the calculations in [11] to examine backward Ricci flow of compact locally homogeneous geometries on 4-manifolds. The analysis in this chapter is often very similar to that in [11], and we will use many of the same calculations, some of which are included for completeness.

The classes of locally homogeneous manifolds are described in [11] and [14]. For the geometries which are also Lie Groups we will choose a particular basis for the Lie Algebra, $\{X_1, X_2, X_3, X_4\}$, that satisfies certain bracket relations. More detail on these classes can be found in [14]. Letting $\{\phi_i\}_{i=1}^4$ be the frame of 1-forms dual to $\{X_i\}$ we can form a metric $g_0 = A^{ij}\phi_i \otimes \phi_j$.

Our solutions will represent diagonalized Riemannian metrics. Thus they only exist as long as all unknowns remain positive and finite. We denote by $T_0$ the postiive time at which the various solutions to (II.2) fail to exist.

We divide this chapter into two main sections. First is the non-trivial section where we describe the Bianchi cases. The Lie Group structure has trivial Isotopy group, so the manifold is in itself a Lie Group. In the next section we give a quick description of each of the 4-dimensional non-Bianchi cases. These are all metrics of constant sectional or holomorphic bi-sectional curvature or products of such metrics. The evolution of these metrics is well understood, but we include it in this thesis for completeness. In the last section we summarize and compare the behaviors under backward Ricci flow of the various geometries under.

## The Bianchi Cases

The reduction of Ricci flow, equation (II.1), on the Bianchi classes of 4-manifolds, to a system of ODE was done in [11] using the following Ricci curvature formula for unimodular Lie groups from [2, p. 184]. Recall that the elemenets $X, Y, Z, W$ come from the Lie Algebra $\mathfrak{g}$ of our Lie Group $G$ which represents our manifold.

$$Ric(W,W) = -\frac{1}{2}\sum_i \big|[W,Y_i]\big|^2 - \frac{1}{2}\sum_i \big\langle \big[W,[W,Y_i]\big], Y_i \big\rangle + \frac{1}{2}\sum_{i<j} \big\langle [Y_i,Y_j], W \big\rangle^2. \tag{II.3}$$

The only difference in calculating the system of ODE for backwards Ricci flow, equation (II.2), is that the evolutions of the various metrics are negative of those found in [11]. In this section I use these systems of ODE without further explanation.

A formula to calculate the sectional curvature on Lie Groups, with corresponding

Lie Algebras, is found in [2, p. 183]:

$$\left\langle R(X,Y)X,Y\right\rangle = -\frac{3}{4}\left|[X,Y]\right|^2 - \frac{1}{2}\left\langle \left[X,[X,Y]\right],Y\right\rangle - \frac{1}{2}\left\langle \left[Y,[Y,X]\right],X\right\rangle + \left|U(X,Y)\right|^2 - \left\langle U(X,X),U(Y,Y)\right\rangle, \qquad \text{(II.4)}$$

where $U$ is defined by

$$\left\langle U(X,Y),Z\right\rangle = \frac{1}{2}\left\langle [Z,X],Y\right\rangle + \frac{1}{2}\left\langle X,[Z,Y]\right\rangle \text{ for all } Z \in \mathfrak{g}. \qquad \text{(II.5)}$$

The classification notation I use comes from [14] and [11]. See [14] for more details on this classification.

## A1. Class $U1\lfloor(1,1,1)\rfloor$

Here we may choose a basis for the Lie Albegra $\{X_1, X_2, X_3, X_4\}$ such that the Lie bracket is of the form

$$[X_1,X_2]=0 \qquad [X_1,X_3]=0 \qquad [X_1,X_4]=0$$
$$[X_2,X_3]=0 \qquad [X_2,X_4]=0 \qquad [X_3,X_4]=0.$$

Our Lie Group structure is $(M,G) = (\mathbb{R}^4, \mathbb{R}^4)$. The metric $g$ is flat, hence also Ricci flat and thus remains constant.

## A2. Class $U1\lfloor 1,1,1\rfloor$

Here we may choose a basis for the Lie Albegra $\{X_1, X_2, X_3, X_4\}$ such that the

Lie bracket is of the form

$$[X_1, X_2] = 0 \qquad [X_1, X_3] = 0 \qquad [X_1, X_4] = X_1$$

$$[X_2, X_3] = 0 \qquad [X_2, X_4] = kX_2 \qquad [X_3, X_4] = -(k+1)X_3.$$

When $k = 0$ this corresponds to the geometry $(M, G) = \big(\tilde{Sol^3} \times \mathbb{R}, \tilde{Sol^3} \times \mathbb{R}\big)$.

When $k = 1$ this corresponds to the geometry $(M, G) = \big(Sol_0^4, Sol_0^4\big)$.

If $k \neq 0, 1$ and there is some number $\alpha > 0$ with $\beta = k\alpha,\ \gamma = -(k+1)\alpha$ such that $e^\alpha,\ e^\beta$ and $e^\gamma$ are roots of $\lambda^3 - m\lambda^2 + n\lambda - 1 = 0$, then this corresponds to the geometry $(M, G) = \big(Sol_{m,n}^4, Sol_{m,n}^4\big)$.

We diagonalize our initial metric $g_0$ by letting $Y_i = Upsilon_i^k X_k$ for constants $\Upsilon_i^k$. Now letting $\{\theta_i\}$ be the frame of 1-forms dual to $\{Y_i\}$ we may write the metric as

$$g_0 = \lambda_1\theta_1^2 + \lambda_2\theta_2^2 + \lambda_3\theta_3^2 + \lambda_4\theta_4^2.$$

In [11] we find outlines as to when exactly the Ricci tensor is also diagonal under these same coordinates. This is exactly when the metric will remain diagonal under Ricci flow and also backward Ricci flow. The property of a metric to remain diagonal under this flow is essential to our calculations, and we will only consider those families which satisfy this property. In each remaining subsection we will describe a flow which is of the form

$$g(t) = A(t)\theta_1^2 + B(t)\theta_2^2 + C(t)\theta_3^2 + D(t)\theta_4^2,$$

where

$$A(0) = \lambda_1,\ B(0) = \lambda_2,\ C(0) = \lambda_3,\ \text{and } D(0) = \lambda_4.$$

The systems of ODE governing the evolution of the quantities $A,\ B,\ C$ and $D$ were calculated in [11].

In the class $U1[(1,1,1)]$ we may diagonalize the metric by letting $Y_i = \Upsilon_i^k X_k$ with

$$\Upsilon = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ a_2 & a_3 & 1 & 0 \\ a_4 & a_5 & a_6 & 1 \end{bmatrix}.$$

By Proposition 1 in [11], if $k = 1$ then the metric $g(t)$ remains diagonal in the basis $Y_i$ if and only if $a_1 = a_2 = a_3 = 0$, and if $k \neq 1$ then $g(t)$ remains diagonal in the basis $Y_i$ if and only if $a_2 = a_3 = 0$. In either of these cases we find that $\{Y_i\}$ satisfies the following bracket relations:

$$[Y_1, Y_2] = 0 \qquad [Y_1, Y_3] = 0 \qquad [Y_1, Y_4] = Y_1$$
$$[Y_2, Y_3] = 0 \qquad [Y_2, Y_4] = kY_2 \qquad [Y_3, Y_4] = -(k+1)Y_3.$$

Let $g(t) = A(t)\theta_1^2 + B(t)\theta_2^2 + C(t)\theta_3^2 + D(t)\theta_4^2$ where $A(0) = \lambda_1,\ B(0) = \lambda_2,\ C(0) = \lambda_3$ and $D(0) = \lambda_4$. Then backward Ricci flow (II.2) reduces to the following system of equations by [11]:

$$\begin{aligned} \frac{dA}{dt} &= \frac{dB}{dt} = \frac{dC}{dt} = 0, \\ \frac{dD}{dt} &= -4\big(k^2 + k + 1\big). \end{aligned} \tag{II.6}$$

The solution is

$$\begin{aligned} A(t) &= \lambda_1 \\ B(t) &= \lambda_2 \\ C(t) &= \lambda_3 \\ D(t) &= \lambda_4 - 4\big(k^2 + k + 1\big)t. \end{aligned} \tag{II.7}$$

It is clear that

$$T_0 = \frac{\lambda_4}{4}\big(k^2 + k + 1\big)^{-1}, \tag{II.8}$$

and that as $t \to T_0$ the volume normalized flow will approach the hyperplane, $\mathbb{R}^3$.

The sectional curvatures, also calculated in [11], are as follows:

$$\begin{aligned} K(Y_1, Y_2) &= -\frac{k}{D} \qquad & K(Y_1, Y_3) &= \frac{k+1}{D} \qquad & K(Y_1, Y_4) &= \frac{k(k+1)}{D} \\ K(Y_2, Y_3) &= -\frac{1}{D} \qquad & K(Y_2, Y_4) &= -\frac{k^2}{D} \qquad & K(Y_3, Y_4) &= -\frac{(k+1)^2}{D}. \end{aligned}$$

Thus, as $D$ approaches 0 linearly in $t$ we see that the non-zero curvatures approach infinity with a singularity of the form $(T_0 - t)^{-1}$.

A3. Class $U1\lfloor \mathbb{Z}, \bar{\mathbb{Z}}, 1 \rfloor$

Here we may choose a basis for the Lie Albegra $\{X_1, X_2, X_3, X_4\}$ such that the Lie bracket is of the form

$$\begin{aligned} [X_1, X_2] &= 0 \qquad & [X_1, X_3] &= 0 \qquad & [X_1, X_4] &= kX_1 + X_2 \\ [X_2, X_3] &= 0 \qquad & [X_2, X_4] &= -X_1 + kX_2c \qquad & [X_3, X_4] &= -2kX_3. \end{aligned}$$

This corresponds to the geometry $(M, G) = (\mathbb{R}^4, E(2) \times \mathbb{R}^2)$. Here we diagonalize the metric by letting $Y_i = \Upsilon_i^k X_k$ with

$$\Upsilon = \begin{bmatrix} 1 & a_2 & a_3 & 0 \\ 0 & 1 & a_1 & 0 \\ 0 & 0 & 1 & 0 \\ a_4 & a_5 & a_6 & 1 \end{bmatrix}.$$

By Proposition 2 in [11], the metric $g(t)$ remains diagonal in the basis $Y_i$ if and only if $a_1 = a_2 = a_3 = 0$. We find that $\{Y_i\}$ satisfies the following bracket relations:

$$[Y_1, Y_2] = 0 \qquad [Y_1, Y_3] = 0 \qquad [Y_1, Y_4] = kY_1 + Y_2$$

$$[Y_2, Y_3] = 0 \qquad [Y_2, Y_4] = -Y_1 + kY_2 \qquad [Y_3, Y_4] = -2kY_3.$$

Let $g(t) = A(t)\theta_1^2 + B(t)\theta_2^2 + C(t)\theta_3^2 + D(t)\theta_4^2$ where $A(0) = \lambda_1,\ B(0) = \lambda_2,\ C(0) = \lambda_3$ and $D(0) = \lambda_4$. Then backward Ricci flow reduces to the following system of equations:

$$\begin{aligned} \frac{dA}{dt} &= \frac{A^2 - B^2}{BD} \\ \frac{dB}{dt} &= \frac{B^2 - A^2}{AD} \\ \frac{dC}{dt} &= 0 \\ \frac{dD}{dt} &= -\frac{(A-B)^2 + 12k^2 AB}{AB}. \end{aligned} \tag{II.9}$$

Clearly, $C(t) = \lambda_3$. Also,

$$\frac{d}{dt}\big(A - B\big) = \frac{(A+B)^2}{ABD}(A - B),$$

so the conditions $A = B,\ A > B$ and $A < B$ are all preserved.

First assume $\lambda_1 = \lambda_2$. Then $A = B$ for all time $t$, and our solution is

$$\begin{aligned} A(t) &= \lambda_1 \\ B(t) &= \lambda_1 \\ C(t) &= \lambda_3 \\ D(t) &= \lambda_4 - 12k^2 t. \end{aligned} \tag{II.10}$$

Thus

$$T_0 = \frac{\lambda_4}{12k^2}, \tag{II.11}$$

and it is clear that as $t \to T_0$, the volume normalized flow will approach the hyperplane, $\mathbb{R}^3$.

Now assume $\lambda_1 \neq \lambda_2$. By the symmetry of (II.9), we may assume that $\lambda_1 > \lambda_2$. Then $A(t) \geq B(t)$ for all time $t$. Now,

$$\frac{d}{dt}\big(AB\big) = A\frac{B^2 - A^2}{AD} + B\frac{A^2 - B^2}{BD} = 0,$$

so $AB = \lambda_1\lambda_2$ is constant. It is also clear that $T_0 < \infty$, because $\frac{dD}{dt} < -12k^2$ implies $T_0 < \frac{\lambda_4}{12k^2}$. With $AB = \lambda_1\lambda_2$, we see that

$$\frac{dA}{dt} = \frac{A^4 - \lambda_1^2\lambda_2^2}{\lambda_1\lambda_2 AD}, \quad \frac{dD}{dt} = -\frac{\big(A - B\big)^2}{\lambda_1\lambda_2} - 12k^2,$$

hence

$$\frac{dD}{dA} = -\frac{\big(A - B\big)^2 + 12\lambda_1\lambda_2 k^2}{\lambda_1\lambda_2} \cdot \frac{\lambda_1\lambda_2 AD}{A^4 - \lambda_1^2\lambda_2^2},$$

so

$$\frac{1}{D}\frac{dD}{dA} = -\frac{\left[\left(A - \frac{\lambda_1\lambda_2}{A}\right)^2 + 12\lambda_1\lambda_2 k^2\right] A}{A^4 - \lambda_1^2\lambda_2^2}$$
$$= \frac{-A^4 + (2 - 12k^2)\lambda_1\lambda_2 A^2 - \lambda_1^2\lambda_2^2}{A(A + \sqrt{\lambda_1\lambda_2})(A - \sqrt{\lambda_1\lambda_2})(A^2 + \lambda_1\lambda_2)}. \tag{II.12}$$

Under the substitution $\tilde{A} = \sqrt{\lambda_1\lambda_2}A$, we have the following:

$$\frac{1}{D}\frac{dD}{d\tilde{A}} = \frac{-\tilde{A}^4 + (2 - 12k^2)\tilde{A}^2 - 1}{\tilde{A}(\tilde{A}+1)(\tilde{A}-1)(\tilde{A}^2+1)}$$
$$= \frac{1}{\tilde{A}} - \frac{3k^2}{\tilde{A}+1} - \frac{3k^2}{\tilde{A}-1} - \frac{2(1-3k^2)\tilde{A}}{\tilde{A}^2+1}.$$

Solving this equation we find

$$D = \Lambda\left(\frac{\tilde{A}}{\tilde{A}^2+1}\right)\left(\frac{\tilde{A}^2+1}{\tilde{A}^2-1}\right)^{3k^2}$$
$$= \Lambda\left(\frac{\sqrt{\lambda_1\lambda_2}A}{A^2+\lambda_1\lambda_2}\right)\left(\frac{A^2+\lambda_1\lambda_2}{A^2-\lambda_1\lambda_2}\right)^{3k^2}, \tag{II.13}$$

where

$$\Lambda = \lambda_4\left(\frac{\lambda_1+\lambda_2}{\sqrt{\lambda_1\lambda_2}}\right)\left(\frac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2}\right)^{3k^2}. \tag{II.14}$$

We see that $D \to 0$ as $t \to T_0$ if and only if $A \to \infty$ as $t \to T_0$. But since $AB$ is constant, we know that $B \to 0$ if and only if $A \to \infty$. Thus we know that $D \to 0,\ B \to 0$ and $A \to \infty$ at the same time, $T_0 < \frac{\lambda_4}{12k^2} < \infty$.

As $A \to \infty$, we see that

$$D \approx \frac{\Lambda\sqrt{\lambda_1\lambda_2}}{A} = \frac{\Lambda}{\sqrt{\lambda_1\lambda_2}}B. \tag{II.15}$$

Now, as $t \to T_0$, we may approximate

$$\frac{dA}{dt} = \frac{A^2 - B^2}{BD} \sim \frac{A^4}{\Lambda(\lambda_1\lambda_2)^{3/2}}. \tag{II.16}$$

Thus

$$\frac{d}{dt}\left(\frac{1}{A^3}\right) \to -\frac{3}{\Lambda(\lambda_1\lambda_2)^{3/2}}$$

as $t \to T_0$. Since $A \to \infty$ as $t \to T_0$, we see that

$$\frac{1}{A^3} = \frac{3}{\Lambda(\lambda_1\lambda_2)^{3/2}}(T_0 - t)\big(1 + o(T_0 - t)\big).$$

Thus

$$\begin{aligned} A &= \left(\frac{3}{\Lambda(\lambda_1\lambda_2)^{3/2}}(T_0 - t)\right)^{-1/3} \big(1 + o(T_0 - t)\big)^{-1/3} \\ &= \sqrt{\lambda_1\lambda_2}\left(\frac{3}{\Lambda}(T_0 - t)\right)^{1/3} \big(1 + o(T_0 - t)\big) \end{aligned}$$

Now by (II.15) we have the following behavior as $t \to T_0$:

$$\begin{aligned} A &\approx \sqrt{\lambda_1\lambda_2}\left(\frac{3}{\Lambda}(T_0 - t)\right)^{-1/3} \\ B &\approx \sqrt{\lambda_1\lambda_2}\left(\frac{3}{\Lambda}(T_0 - t)\right)^{1/3} \\ C &= \lambda_3 \\ D &\approx \Lambda\left(\frac{3}{\Lambda}(T_0 - t)\right)^{1/3}. \end{aligned} \tag{II.17}$$

where $\Lambda$ is given by (II.14).

The volume normalized solution will converge to the plane $\mathbb{R}^2$.

The sectional curvatures are as follows:

$$K(Y_1,Y_2) = -\frac{\frac{A}{B}+\frac{B}{A}-2-4k^2}{4D} \qquad K(Y_1,Y_3) = \frac{2k^2}{D}$$
$$K(Y_1,Y_4) = \frac{\frac{A}{B}-3\frac{B}{A}+2-4k^2}{D} \qquad K(Y_2,Y_3) = \frac{2k^2}{D}$$
$$K(Y_2,Y_4) = \frac{-3\frac{A}{B}+\frac{B}{A}+2-4k^2}{D} \qquad K(Y_3,Y_4) = -\frac{4k^2}{D}.$$

Thus we see that the sectional curvatures perpendicular to $Y_3$ approach infinity at a rate of $(T_0-t)^{-1}$. If $k=0$ the sectional curvatures parallel to $Y_3$ remain 0, while if $k \neq 0$ then these curvatures approach infinity at a rate of $(T_0-t)^{-1/3}$.

## A4. Class $U1\lfloor 2,1 \rfloor,\ \mu = 0$

Here we may choose a basis for the Lie Albegra $\{X_1, X_2, X_3, X_4\}$ such that the Lie bracket is of the form

$$[X_1,X_2]=0 \qquad [X_1,X_3]=0 \qquad [X_1,X_4]=X_2$$
$$[X_2,X_3]=0 \qquad [X_2,X_4]=0 \qquad [X_3,X_4]=0.$$

This corresponds to the geometry $(M,G) = (Nil^3 \times \mathbb{R}, Nil^3 \times \mathbb{R})$. Here we diagonalize the metric by letting $Y_i = \Upsilon_i^k X_k$ with

$$\Upsilon = \begin{bmatrix} 1 & a_2 & a_3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a_1 & 1 & 0 \\ a_4 & a_5 & a_6 & 1 \end{bmatrix}.$$

By Proposition 3 in [11], the metric $g(t)$ remains diagonal in the basis $Y_i$ for all $t$.

We find that $\{Y_i\}$ satisfies the following bracket relations:

$$[Y_1, Y_2] = 0 \qquad [Y_1, Y_3] = 0 \qquad [Y_1, Y_4] = Y_2$$
$$[Y_2, Y_3] = 0 \qquad [Y_2, Y_4] = -Y_1 + kY_2 \qquad [Y_3, Y_4] = -2kY_3.$$

Let $g(t) = A(t)\theta_1^2 + B(t)\theta_2^2 + C(t)\theta_3^2 + D(t)\theta_4^2$ where $A(0) = \lambda_1,\ B(0) = \lambda_2,\ C(0) = \lambda_3$ and $D(0) = \lambda_4$. Then backward Ricci flow reduces to the following system of equations:

$$\begin{aligned} \frac{dA}{dt} &= -\frac{B}{D} \\ \frac{dB}{dt} &= \frac{B^2}{AD} \\ \frac{dC}{dt} &= 0 \\ \frac{dD}{dt} &= -\frac{B}{A}. \end{aligned} \tag{II.18}$$

We calculate

$$\frac{d}{dt}(AB) = -B\frac{B}{D} + A\frac{B^2}{AD} = 0$$

and

$$\frac{d}{dt}\left(\frac{A}{D}\right) = \frac{1}{D^2}\left(-D\frac{B}{D} + A\frac{B}{A}\right) = 0.$$

Thus we have

$$B = \frac{\lambda_1\lambda_2}{A}, \text{ and } D = \frac{\lambda_4}{\lambda_1}A, \tag{II.19}$$

so

$$A^2\frac{dA}{dt} = -AB\frac{A}{D} = -\frac{\lambda_1^2\lambda_2}{\lambda_4}, \tag{II.20}$$

hence

$$A^3 = \lambda_1^3 - \frac{3\lambda_1^2\lambda_2}{\lambda_4}t. \tag{II.21}$$

Thus by (II.18), (II.19) and (II.21), we have the following solution to (II.18):

$$\begin{aligned} A &= \lambda_1 \left(1 - \frac{3\lambda_2}{\lambda_1\lambda_4} t\right)^{1/3} \\ B &= \lambda_2 \left(1 - \frac{3\lambda_2}{\lambda_1\lambda_4} t\right)^{-1/3} \\ C &= \lambda_3 \\ D &= \lambda_4 \left(1 - \frac{3\lambda_2}{\lambda_1\lambda_4} t\right)^{1/3}. \end{aligned} \tag{II.22}$$

We see that

$$T_0 = \frac{\lambda_1\lambda_4}{3\lambda_2}, \tag{II.23}$$

and that the volume normalized solution will converge to the plane $\mathbb{R}^2$.

The sectional curvatures are as follows:

$$\begin{aligned} &K(Y_1, Y_2) = \frac{B}{4AD} \qquad && K(Y_1, Y_3) = 0 \qquad && K(Y_1, Y_4) = -\frac{3B}{4AD} \\ &K(Y_2, Y_3) = 0 && K(Y_2, Y_4) = \frac{B}{4AD} && K(Y_3, Y_4) = 0. \end{aligned}$$

Thus all non-zero curvatures will approach infinity near $t = T_0$ at a rate of $(T_0 - t)^{-1}$.

## A5. Class $U1\lfloor 2, 1\rfloor$, $\mu = 1$

Here we may choose a basis for the Lie Albegra $\{X_1, X_2, X_3, X_4\}$ such that the Lie bracket is of the form

$$\begin{aligned} &[X_1, X_2] = 0 \qquad && [X_1, X_3] = 0 \qquad && [X_1, X_4] = -\frac{1}{2}X_1 + X_2 \\ &[X_2, X_3] = 0 && [X_2, X_4] = -\frac{1}{2}X_2 && [X_3, X_4] = X_3. \end{aligned}$$

This does not correspond to any of the compact homogeneous geometries. Here we diagonalize the metric by letting $Y_i = \Upsilon_i^k X_k$ with

$$\Upsilon = \begin{bmatrix} 1 & a_2 & a_3 & 0 \\ 0 & 1 & a_1 & 0 \\ 0 & 0 & 1 & 0 \\ a_4 & a_5 & a_6 & 1 \end{bmatrix}.$$

By Proposition 4 in [11], the metric $g(t)$ remains diagonal in the basis $Y_i$ if and only if $a_1 = a_3 = 0$. We find that $\{Y_i\}$ satisfies the following bracket relations:

$$[Y_1, Y_2] = 0 \qquad [Y_1, Y_3] = 0 \qquad [Y_1, Y_4] = -\frac{1}{2}Y_1 + Y_2$$
$$[Y_2, Y_3] = 0 \qquad [Y_2, Y_4] = -\frac{1}{2}Y_2 \qquad [Y_3, Y_4] = Y_3.$$

Let $g(t) = A(t)\theta_1^2 + B(t)\theta_2^2 + C(t)\theta_3^2 + D(t)\theta_4^2$ where $A(0) = \lambda_1$, $B(0) = \lambda_2$, $C(0) = \lambda_3$ and $D(0) = \lambda_4$. Then backward Ricci flow reduces to the following system of equations:

$$\begin{aligned} \frac{dA}{dt} &= -\frac{B}{D} \\ \frac{dB}{dt} &= \frac{B^2}{AD} \\ \frac{dC}{dt} &= 0 \\ \frac{dD}{dt} &= -3 - \frac{B}{A}. \end{aligned} \tag{II.24}$$

Since $\frac{dD}{dt} < -3$ we see that there is a maximal time $T_0 < \frac{\lambda_4}{3}$. Also,

$$\frac{d}{dt}(AB) = A\left(\frac{B^2}{AD}\right) + B\left(-\frac{B}{D}\right) = 0, \tag{II.25}$$

so

$$AB = \lambda_1\lambda_2. \tag{II.26}$$

Thus (II.24) reduces to

$$\begin{aligned} \frac{dA}{dt} &= -\frac{\lambda_1\lambda_2}{AD} \\ B &= \frac{\lambda_2}{\lambda_1}A \\ C &= \lambda_3 \\ \frac{dD}{dt} &= \frac{-3A^2-\lambda_1\lambda_2}{A^2}. \end{aligned} \tag{II.27}$$

Now we calculate

$$\begin{aligned} \frac{dD}{dA} &= \frac{3A^2+\lambda_1\lambda_2}{A^2}\cdot\frac{AD}{\lambda_1\lambda_2}, \\ \frac{1}{D}\cdot\frac{dD}{dA} &= \frac{3}{\lambda_1\lambda_2}A+\frac{1}{A}. \end{aligned}$$

Solving gives us

$$D = \Lambda\cdot Ae^{\left(\frac{3A^2}{2\lambda_1\lambda_2}\right)}, \tag{II.28}$$

where

$$\Lambda = \frac{\lambda_4}{\lambda_1}e^{\left(-\frac{3\lambda_1}{2\lambda_2}\right)}. \tag{II.29}$$

By (II.26) and (II.28) we see that $A\to 0,\ B\to\infty$ and $D\to 0$ all together as $t\to T_0$.

To describe the behavior near $t=T_0$ we observe that as $A$ approaches 0, (II.24), (II.26) and (II.28) tell us

$$A^2\frac{dA}{dt}\to -\frac{\lambda_1\lambda_2}{\Lambda},$$

hence

$$A^3 = \frac{3\lambda_1\lambda_2}{\Lambda}(T_0 - t)\big(1 + o(T_0 - t)\big). \tag{II.30}$$

Thus, by (II.24), (II.26) and (II.28), we have the following solutions to (II.24):

$$\begin{aligned} A &\approx \left(\frac{3\lambda_1\lambda_2}{\Lambda}(T_0 - t)\right)^{1/3} \\ B &\approx \lambda_1\lambda_2\left(\frac{3\lambda_1\lambda_2}{\Lambda}(T_0 - t)\right)^{-1/3} \\ C &= \lambda_3 \\ D &\approx \Lambda\left(\frac{3\lambda_1\lambda_2}{\Lambda}(T_0 - t)\right)^{1/3}. \end{aligned} \tag{II.31}$$

where $\Lambda$ is given in (II.29). Again we notice that as $t$ approaches $T_0$ the renormalized flow approaches the plane $\mathbb{R}^2$.

The sectional curvatures are as follows:

$$\begin{aligned} K(Y_1, Y_2) &= \frac{-1 + \frac{B}{A}}{4D} & K(Y_1, Y_3) &= \frac{1}{2D} & K(Y_1, Y_4) &= -\frac{1 + 3\frac{B}{A}}{4D} \\ K(Y_2, Y_3) &= \frac{1}{2D} & K(Y_2, Y_4) &= \frac{-1 + \frac{B}{A}}{4D} & K(Y_3, Y_4) &= -\frac{1}{D}. \end{aligned}$$

Thus curvatures perpendicular to $Y_3$ will have a singularity $t = T_0$ of the form $(T_0 - t)^{-1}$, while those parallel to $Y_3$ will have a singularity of the form $(T_0 - t)^{-1/3}$.

## A6. Class $U1\lfloor 3 \rfloor$

Here we may choose a basis for the Lie Albegra $\{X_1, X_2, X_3, X_4\}$ such that the

Lie bracket is of the form

$$[X_1, X_2] = 0 \qquad [X_1, X_3] = 0 \qquad [X_1, X_4] = X_2$$

$$[X_2, X_3] = 0 \qquad [X_2, X_4] = X_3 \qquad [X_3, X_4] = 0.$$

This corresponds to the geometry $(M, G) = (Nil^4, Nil^4)$.

Here we diagonalize the metric by letting $Y_i = \Upsilon_i^k X_k$ with

$$\Upsilon = \begin{bmatrix} 1 & a_2 & a_3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a_1 & 1 & 0 \\ a_4 & a_5 & a_6 & 1 \end{bmatrix}.$$

By Proposition 5 in [11], the metric $g(t)$ remains diagonal in the basis $Y_i$ if and only if $a_1 = a_2$. We find in this case that $\{Y_i\}$ satisfies the following bracket relations:

$$[Y_1, Y_2] = 0 \qquad [Y_1, Y_3] = 0 \qquad [Y_1, Y_4] = Y_2$$

$$[Y_2, Y_3] = 0 \qquad [Y_2, Y_4] = Y_3 \qquad [Y_3, Y_4] = 0.$$

Let $g(t) = A(t)\theta_1^2 + B(t)\theta_2^2 + C(t)\theta_3^2 + D(t)\theta_4^2$ where $A(0) = \lambda_1,\ B(0) = \lambda_2,\ C(0) = \lambda_3$ and $D(0) = \lambda_4$. Then backward Ricci flow reduces to the following system of equations:

$$\begin{aligned} \frac{dA}{dt} &= -\frac{B}{D} \\ \frac{dB}{dt} &= \frac{B^2 - AC}{AD} \\ \frac{dC}{dt} &= \frac{C^2}{BD} \\ \frac{dD}{dt} &= -\frac{B}{A} - \frac{C}{B}. \end{aligned} \tag{II.32}$$

We calculate

$$\frac{d}{dt}\big(ABC\big) = AB \cdot \frac{C^2}{BD} + AC \cdot \frac{B^2 - AC}{AD} - BC \cdot \frac{B}{D} = 0$$

and

$$\frac{d}{dt}\left(\frac{A}{CD}\right) = \frac{-CD \cdot \frac{B}{D} + AC\left(\frac{B}{A} + \frac{C}{B}\right) - AD \cdot \frac{C^2}{BD}}{C^2D^2} = 0,$$

so we have

$$ABC = \lambda_1\lambda_2\lambda_3, \tag{II.33}$$

$$\frac{A}{CD} = \frac{\lambda_1}{\lambda_3\lambda_4}. \tag{II.34}$$

Now we define

$$E = \frac{B}{AD} \;\; F = \frac{C}{BD}. \tag{II.35}$$

Then

$$\frac{dE}{dt} = \frac{AD \cdot \frac{B^2-AC}{AD} + AB\left(\frac{B}{A} + \frac{C}{B}\right) + BD \cdot \frac{B}{D}}{(AD)^2} = \frac{3B^2}{(AD)^2} = 3E^2, \tag{II.36}$$

and

$$\frac{dF}{dt} = \frac{BD \cdot \frac{C^2}{BD} + BC\left(\frac{B}{A} + \frac{C}{B}\right) - CD \cdot \frac{B^2-AC}{AD}}{(BD)^2} = \frac{3C^2}{(BD)^2} = 3F^2. \tag{II.37}$$

Solving (II.36) and (II.37) gives us

$$E(t) = \frac{E(0)}{1 - 3E(0)t} \tag{II.38}$$

and

$$F(t) = \frac{F(0)}{1 - 3F(0)t}. \tag{II.39}$$

Now by (II.32), (II.35) and (II.38) we have

$$\frac{1}{A}\frac{dA}{dt} = -E(t) = \frac{E(0)}{3E(0)t - 1},$$

so

$$\ln(A) = \frac{1}{3}\ln\left|3E(0)t - 1\right| + \ln(\lambda_1)$$

and

$$A = \lambda_1\left(1 - 3E(0)t\right)^{1/3}. \tag{II.40}$$

Similarly, by (II.32), (II.35) and (II.39), we have

$$\frac{1}{C}\frac{dC}{dt} = F(t) = \frac{F(0)}{1 - 3F(0)t}, \tag{II.41}$$

so

$$C = \lambda_3\left(1 - 3F(0)t\right)^{-1/3}. \tag{II.42}$$

Now using (II.33), (II.34) and (II.35) we have the following solution to (II.32):

$$\begin{aligned} A &= \lambda_1 \left(1 - \frac{3\lambda_2}{\lambda_3\lambda_4}t\right)^{1/3} \\ B &= \lambda_2 \left(1 - \frac{3\lambda_2}{\lambda_3\lambda_4}t\right)^{-1/3} \left(1 - \frac{3\lambda_3}{\lambda_2\lambda_4}t\right)^{1/3} \\ C &= \lambda_3 \left(1 - \frac{3\lambda_3}{\lambda_2\lambda_4}t\right)^{-1/3} \\ D &= \lambda_4 \left(1 - \frac{3\lambda_2}{\lambda_3\lambda_4}t\right)^{1/3} \left(1 - \frac{3\lambda_3}{\lambda_2\lambda_4}t\right)^{1/3}. \end{aligned} \tag{II.43}$$

where

$$T_0 = \min\left\{\frac{\lambda_3\lambda_4}{3\lambda_2}, \frac{\lambda_2\lambda_4}{3\lambda_3}\right\}. \tag{II.44}$$

The sectional curvatures are as follows:

$$\begin{aligned} K(Y_1,Y_2) &= \frac{B}{4AD} & K(Y_1,Y_3) &= 0 & K(Y_1,Y_4) &= \frac{-3B}{4AD} \\ K(Y_2,Y_3) &= \frac{C}{4BD} & K(Y_2,Y_4) &= \frac{\frac{B}{A} - 3\frac{C}{B}}{4D} & K(Y_3,Y_4) &= \frac{C}{4BD}. \end{aligned}$$

If $\lambda_2 < \lambda_3$, then $T_0 = \dfrac{\lambda_2\lambda_4}{3\lambda_3}$, and as $t \to T_0$ we have

$$\begin{aligned} A &\approx k_1 \\ B &\approx k_2(T_0 - t)^{1/3} \\ C &= k_3(T_0 - t)^{-1/3} \\ D &\approx k_4(T_0 - t)^{1/3}. \end{aligned} \tag{II.45}$$

Also, $K(Y_1, Y_2)$ will approach a positive constant, $K(Y_1, Y_4)$ will approach a negative constant, and all curvatures perpendicular to $Y_1$ will have singularities of the form $(T_0 - t)^{-1}$.

If $\lambda_2 > \lambda_3$, then $T_0 = \dfrac{\lambda_3\lambda_4}{3\lambda_2}$, and as $t \to T_0$ we have

$$\begin{aligned} A &= k_1(T_0 - t)^{1/3} \\ B &\approx k_2(T_0 - t)^{-1/3} \\ C &\approx k_3 \\ D &\approx k_4(T_0 - t)^{1/3}. \end{aligned} \tag{II.46}$$

Here $K(Y_2, Y_3)$ and $K(Y_3, Y_4)$ will approach positive constants while each curvature perpendicular to $Y_3$ will have a singularity of the form $(T_0 - t)^{-1}$.

If $\lambda_2 = \lambda_3$, then $T_0 = \dfrac{\lambda_4}{3}$, and as $t \to T_0$ we have

$$
\begin{aligned}
A &= k_1 (T_0 - t)^{1/3} \\
B &= \lambda_2 \\
C &= k_3 (T_0 - t)^{-1/3} \\
D &= k_4 (T_0 - t)^{2/3}.
\end{aligned}
\tag{II.47}
$$

Here all non-zero curvatures will have a singularity of the form $(T_0 - t)^{-1}$. It is interesting to note that if $3\lambda_1 = \lambda_2 = \lambda_3$, then $K(Y_2, Y_4) = 0$ for all $t$, while in all other cases it explodes near the singular time.

In all three cases the volume normalized solution approaches the plane $\mathbb{R}^2$.

## A7. Class $U3I0$

Here we may choose a basis for the Lie Albegra $\{X_1, X_2, X_3, X_4\}$ such that the Lie bracket is of the form

$$
\begin{array}{lll}
[X_1, X_2] = -X_3 & [X_1, X_3] = -X_2 & [X_1, X_4] = 0 \\
[X_2, X_3] = X_4 & [X_2, X_4] = 0 & [X_3, X_4] = 0.
\end{array}
$$

This corresponds to the geometry $(M, G) = (Sol^4, Sol^4)$. Here we diagonalize the metric by letting $Y_i = \Upsilon_i^k X_k$ with

$$\Upsilon = \begin{bmatrix} 1 & a_4 & a_5 & a_6 \\ 0 & 1 & a_2 & a_3 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let $\alpha = a_2,\ \beta = a_1a_2 -_3 -a_4$ and $\gamma = a_1 - a_1a_2^2 + a_2a_3 + a_2a_4 - a_5$. Proposition 6 in [11], the metric $g(t)$ remains diagonal in the basis $Y_i$ if and only if one of the following hold:

$$\begin{array}{ll} \text{(i)} & \alpha = \beta = \gamma = 0 \\ \text{(ii)} & \beta = \gamma = 0 \text{ and } \lambda_2 = (1-\alpha^2)\lambda_3 \end{array}$$

We analyze these cases separately.

A7(i)

Here we consider the case where $\alpha = \beta = \gamma = 0$. We find that $\{Y_i\}$ satisfies the following bracket relations:

$$[Y_1, Y_2] = -Y_3 \qquad [Y_1, Y_3] = -Y_2 \qquad [Y_1, Y_4] = 0$$

$$[Y_2, Y_3] = Y_4 \qquad [Y_2, Y_4] = 0 \qquad [Y_3, Y_4] = 0.$$

Let $g(t) = A(t)\theta_1^2 + B(t)\theta_2^2 + C(t)\theta_3^2 + D(t)\theta_4^2$ where $A(0) = \lambda_1,\ B(0) = \lambda_2,\ C(0) = \lambda_3$ and $D(0) = \lambda_4$. Then backward Ricci flow reduces to the following system of equations:

$$\begin{aligned}
\frac{dA}{dt} &= -\frac{B}{C} - \frac{C}{B} - 2 \\
\frac{dB}{dt} &= -\frac{C}{A} - \frac{D}{C} + \frac{B^2}{AC} \\
\frac{dC}{dt} &= -\frac{B}{A} - \frac{D}{B} + \frac{C^2}{AB} \\
\frac{dD}{dt} &= \frac{D^2}{BC}.
\end{aligned} \tag{II.48}$$

By the symmetry of $B$ and $C$ in (II.48), we may assume that $\lambda_2 \geq \lambda_3$. We calculate

$$\begin{aligned}
\frac{d}{dt}\big(BCD^2\big) = 2BCD \cdot \frac{D^2}{BC} &+ CD^2\left(\frac{B^2}{AC} - \frac{C}{A} - \frac{D}{C}\right) \\
&+ BD^2\left(\frac{C^2}{AB} - \frac{B}{A} - \frac{D}{B}\right) \\
= 0,&
\end{aligned}$$

so

$$BCD^2 = \lambda_2\lambda_3\lambda_4^2. \tag{II.49}$$

Now we observe

$$\frac{dD}{dt} = \frac{D^2}{BC} = \frac{D^4}{\lambda_2\lambda_3\lambda_4^2},$$

so

$$\frac{1}{D^4} \cdot \frac{dD}{dt} = \frac{1}{\lambda_2\lambda_3\lambda_4^2},$$

and

$$D = \lambda_4\left(1 - \frac{3\lambda_4}{\lambda_2\lambda_3}t\right)^{-1/3}. \tag{II.50}$$

We may also calculate

$$\frac{d}{dt}\big(AD(B-C)\big) = AD\left(\frac{B^2}{AC} - \frac{C}{A} - \frac{D}{C} + \frac{B}{A} + \frac{D}{B} - \frac{C^2}{AB}\right) + A(B-C)\cdot\frac{D^2}{BC} - D(B-C)\left(\frac{B}{C} + \frac{C}{B} + 2\right) = 0,$$

so

$$AD(B-C) = \lambda_1\lambda_4(\lambda_2 - \lambda_3). \tag{II.51}$$

Now by (II.49) and (II.51) we have

$$\frac{(B-C)^2}{BC}A^2 = \frac{\big(AD(B-C)\big)^2}{BCD^2} = \frac{\lambda_1^2(\lambda_2-\lambda_3)^2}{\lambda_2\lambda_3} = 4k^2. \tag{II.52}$$

Now

$$\frac{dA}{dt} = -\frac{(B+C)^2}{BC} = -\left(\frac{(B-C)^2}{BC}A^2\right)\cdot\frac{1}{A^2} - 4 = \frac{-4\big(k^2 + A^2\big)}{A^2}, \tag{II.53}$$

hence

$$\left(1 - \frac{k^2}{A^2+k^2}\right)\frac{dA}{dt} = -4, \tag{II.54}$$

so

$$A - k\tan^{-1}\left(\frac{A}{k}\right) = -4t + \lambda_1 - k\tan^{-1}\left(\frac{\lambda_1}{k}\right), \tag{II.55}$$

where $k$ is given in (II.52). Now by (II.50) and (II.55) we have

$$T_0 = \min\left\{\frac{\lambda_2\lambda_3}{3\lambda_4},\ \frac{\lambda_1}{4} - \frac{\lambda_1(\lambda_2-\lambda_3)}{2\sqrt{\lambda_2\lambda_3}}\tan^{-1}\left(\frac{2\sqrt{\lambda_2\lambda_3}}{\lambda_2-\lambda_3}\right)\right\} = \min\big\{T_1, T_2\big\}. \tag{II.56}$$

Using (II.49) and (II.51) we can calculate

$$B = \frac{\lambda_4}{2AD}\left(\lambda_1(\lambda_2 - \lambda_3) + \sqrt{\lambda_1^2(\lambda_2 - \lambda_3)^2 + 4A^2\lambda_2\lambda_3}\right), \tag{II.57}$$

and

$$C = \frac{\lambda_4}{2AD}\left(\lambda_1(\lambda_3 - \lambda_2) + \sqrt{\lambda_1^2(\lambda_2 - \lambda_3)^2 + 4A^2\lambda_2\lambda_3}\right). \tag{II.58}$$

Now we calculate

$$\lim_{A\to 0}\left(\frac{A - k\tan^{-1}\left(\frac{A}{k}\right)}{A^3}\right) = \frac{1}{3k^2}.$$

Thus by (II.55), near $A = 0$ we have

$$A \approx \left(12k^2(T_2 - t)\right)^{1/3}, \tag{II.59}$$

where $k$ is given in (II.52).

The sectional curvatures are as follows:

$$K(Y_1, Y_2) = \frac{\frac{B}{C} - 3\frac{C}{B} - 2}{4A} \qquad K(Y_1, Y_3) = \frac{\frac{C}{B} - 3\frac{B}{C} - 2}{4A}$$

$$K(Y_1, Y_4) = 0 \qquad K(Y_2, Y_3) = \frac{B^2 + C^2 + 2BC - 3AD}{4ABC}$$

$$K(Y_2, Y_4) = \frac{D}{4BC} \qquad K(Y_3, Y_4) = \frac{D}{4BC}.$$

If $T_0 = T_1 < T_2$, then we have the following behavior as $t \to T_0$:

$$\begin{aligned} A &\approx k_1 \\ B &\approx k_2(T_0 - t)^{1/3} \\ C &\approx k_3(T_0 - t)^{1/3} \\ D &= k_4(T_0 - t)^{-1/3}. \end{aligned} \tag{II.60}$$

In this case the curvatures parallel to $Y_1$ will approach constants while those perpendicular to $Y_1$ will have a singularity of the form $(T_0 - t)^{-1}$.

If $T_0 = T_2 < T_1$ and $\lambda_2 = \lambda_3$, then as $t \to T_0$,

$$\begin{aligned} A &\approx k_1(T_0 - t)^{1/3} \\ B = C &\approx k_2 \\ D &\approx k_4. \end{aligned} \tag{II.61}$$

Here the curvatures perpendicular to $Y_4$ will approach infinity at a rate of $(T_0 - t)^{-1/3}$ while those curvatures parallel to $Y_4$ will approach constants.

If $T_0 = T_2 < T_1$ and $\lambda_2 > \lambda_3$, then as $t \to T_0$,

$$\begin{aligned} A &\approx k_1(T_0 - t)^{1/3} \\ B &\approx k_2(T_0 - t)^{-1/3} \\ C &\approx k_3(T_0 - t)^{1/3} \\ D &\approx k_4. \end{aligned} \tag{II.62}$$

In this case the curvatures perpendicular to $Y_4$ will approach infinity at a rate of $(T_0 - t)^{-1}$ while those curvatures parallel to $Y_4$ will approach constants.

If $T_0 = T_1 = T_2$ and $\lambda_2 = \lambda_3$, then as $t \to T_0, AD \to$ constant, so

$$\begin{aligned} A &\approx k_1(T_0 - t)^{1/3} \\ B = C &\approx k_2(T_0 - t)^{1/3} \\ D &= k_4(T_0 - t)^{-1/3}. \end{aligned} \tag{II.63}$$

Here, $K(Y_1, Y_2)$ and $K(Y_1, Y_3)$ will have singularities at $t = T_0$ of the form $(T_0 - t)^{-1/3}$. All other non-zero sectional curvatures will have singularities of the form $(T_0 - t)^{-1}$.

If $T_0 = T_1 = T_2$ and $\lambda_2 > \lambda_3$, then as $t \to T_0, AD \to$ constant, so

$$\begin{aligned} A &\approx k_1(T_0 - t)^{1/3} \\ B &\approx k_2 \\ C &\approx k_3(T_0 - t)^{2/3} \\ D &= k_4(T_0 - t)^{-1/3}. \end{aligned} \tag{II.64}$$

Here, all non-zero curvatures will have singularities of the form $(T_0 - t)^{-1}$.

In the special case where both $T_0 = T_2 < T_1$ and $\lambda_1 = \lambda_2$, the normalized solution will approach the hyperplane $\mathbb{R}^3$. In all other cases the normalized solution will approach $\mathbb{R}^2$.

A7(ii)

Here we consider the case where $\beta = \gamma = 0$ and $\lambda_2 = (1-\alpha^2)\lambda_3$. We find that $\{Y_i\}$ satisfies the following bracket relations:

$$[Y_1, Y_2] = -\alpha Y_2 \qquad [Y_1, Y_3] = \alpha Y_3 - Y_2 \qquad [Y_1, Y_4] = 0$$

$$[Y_2, Y_3] = Y_4 \qquad [Y_2, Y_4] = 0 \qquad [Y_3, Y_4] = 0.$$

Let $g(t) = A(t)\theta_1^2 + B(t)\theta_2^2 + C(t)\theta_3^2 + D(t)\theta_4^2$ where $A(0) = \lambda_1,\ B(0) = \lambda_2,\ C(0) = \lambda_3$ and $D(0) = \lambda_4$. Then backward Ricci flow reduces to the following system of equations:

$$\begin{aligned}
\frac{dA}{dt} &= -\frac{B^2 + 2(1+\alpha^2)BC + (1-\alpha^2)^2C^2}{BC} \\
\frac{dB}{dt} &= \frac{-AD + B^2 - (1-\alpha^2)^2C^2}{AC} \\
\frac{dC}{dt} &= \frac{-AD - B^2 + (1-\alpha^2)^2C^2}{AB} \\
\frac{dD}{dt} &= -\frac{D^2}{BC} \\
\lambda_2 &= (1-\alpha^2)\lambda_3.
\end{aligned} \tag{II.65}$$

We observe

$$\frac{d}{dt}\big(B - (1-\alpha^2)C\big) = 0,$$

so $B = (1-\alpha^2)C$ is preserved under (II.65), which then reduces to

$$\begin{aligned}
\frac{dA}{dt} &= -4 \\
\frac{dB}{dt} &= -(1-\alpha^2)\frac{D}{B} \\
B &= (1-\alpha^2)C \\
\frac{dD}{dt} &= (1-\alpha^2)\frac{D^2}{B^2}.
\end{aligned} \tag{II.66}$$

Now

$$\frac{d}{dt}\big(BD\big) = 0,$$

so

$$BD = \lambda_2\lambda_4. \tag{II.67}$$

Thus

$$\frac{dB}{dt} = -(1-\alpha^2)\frac{\lambda_2\lambda_4}{B^2},$$

so

$$\frac{1}{3}B^3 = \frac{1}{3}\lambda_2^3 - (1-\alpha^2)\lambda_2\lambda_4 t,$$

and we have the following solution to (II.65):

$$\begin{aligned} A &= \lambda_1 - 4t \\ B &= \big(\lambda_2^3 - 3(1-\alpha^2)\lambda_2\lambda_4 t\big)^{1/3} \\ C &= \frac{1}{1-\alpha^2}\big(\lambda_2^3 - 3(1-\alpha^2)\lambda_2\lambda_4 t\big)^{1/3} \\ D &= \lambda_2\lambda_4\big(\lambda_2^3 - 3(1-\alpha^2)\lambda_2\lambda_4 t\big)^{-1/3}, \end{aligned} \tag{II.68}$$

where

$$T_0 = \min\left\{\frac{\lambda_1}{4},\ \frac{\lambda_2^2}{3(1-\alpha^2)\lambda_4}\right\} = \min\big\{T_1, T_2\big\}. \tag{II.69}$$

The sectional curvatures are as follows:

$$\begin{aligned} &K(Y_1,Y_2) = -\frac{1}{A} && K(Y_1,Y_3) = -\frac{1}{A} && K(Y_1,Y_4) = 0 \\ &K(Y_2,Y_3) = \frac{4BC-3AD}{4ABC} && K(Y_2,Y_4) = \frac{D}{4BC} && K(Y_3,Y_4) = \frac{D}{4BC}. \end{aligned}$$

If $T_0 = T_1 < T_2$, then near $t = T_0$ we have

$$\begin{aligned} A &= 4(T_0 - t) \\ B &\approx k_2 \\ C &\approx k_3 \\ D &\approx k_4. \end{aligned} \tag{II.70}$$

In this case all sectional curvatures perpendicular to $Y_4$ will have a singularity at $T_0$ of the form $(T_0 - t)^{-1}$ while those parallel to $Y_4$ will approach constants. The normalized solution will converge to $\mathbb{R}^3$.

If $T_0 = T_2 < T_1$, then near $t = T_0$ we have

$$\begin{aligned} A &\approx k_1 \\ B &\approx k_2(T_0 - t)^{1/3} \\ C &\approx k_3(T_0 - t)^{1/3} \\ D &\approx k_4(T_0 - t)^{-1/3}. \end{aligned} \tag{II.71}$$

Here all sectional curvatures perpendicular to $Y_1$ will have a singularity at $T_0$ of the form $(T_0 - t)^{-1}$ while those parallel to $Y_1$ will approach constants. The normalized solution will converge to $\mathbb{R}^2$.

If $T_0 = T_1 = T_2$, then near $t = T_0$ we have

$$\begin{aligned} A &= 4(T_0 - t) \\ B &\approx k_2(T_0 - t)^{1/3} \\ C &\approx k_3(T_0 - t)^{1/3} \\ D &\approx k_4(T_0 - t)^{-1/3}. \end{aligned} \tag{II.72}$$

Here all non-zero curvatures will have a singularity of the form $(T_0 - t)^{-1}$, and the normalized solution will converge to a product metric $M^2 \times \mathbb{R}$.

## A8. Class $U3I2$

Here we may choose a basis for the Lie Albegra $\{X_1, X_2, X_3, X_4\}$ such that the Lie bracket is of the form

$$\begin{array}{lll} [X_1, X_2] = X_3 & [X_1, X_3] = -X_2 & [X_1, X_4] = 0 \\ [X_2, X_3] = -X_4 & [X_2, X_4] = 0 & [X_3, X_4] = 0. \end{array}$$

This does not correspond to any of the compact homogeneous geometries.

Here we diagonalize the metric by letting $Y_i = \Upsilon_i^k X_k$ with

$$\Upsilon = \begin{bmatrix} 1 & a_4 & a_5 & a_6 \\ 0 & 1 & a_2 & a_3 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

By Proposition 7 in [11], the metric $g(t)$ remains diagonal in the basis $Y_i$ if and only if $a_2 = 0$, $a_1 = a_5$ and $a_3 = a_4$. We find in this case that $\{Y_i\}$ satisfies the following bracket relations:

$$[Y_1, Y_2] = Y_3 \qquad [Y_1, Y_3] = -Y_2 \qquad [Y_1, Y_4] = 0$$

$$[Y_2, Y_3] = -Y_4 \qquad [Y_2, Y_4] = 0 \qquad [Y_3, Y_4] = 0.$$

Let $g(t) = A(t)\theta_1^2 + B(t)\theta_2^2 + C(t)\theta_3^2 + D(t)\theta_4^2$ where $A(0) = \lambda_1,\ B(0) = \lambda_2,\ C(0) = \lambda_3$ and $D(0) = \lambda_4$. Then backward Ricci flow reduces to the following system of equations:

$$\begin{aligned} \frac{dA}{dt} &= -\frac{(B-C)^2}{BC} \\ \frac{dB}{dt} &= \frac{B^2 - C^2 - AD}{AC} \\ \frac{dC}{dt} &= \frac{C^2 - B^2 - AD}{AB} \\ \frac{dD}{dt} &= \frac{D^2}{BC}. \end{aligned} \tag{II.73}$$

By the symmetry of $B$ and $C$ in (II.73) we may assume that $\lambda_2 \geq \lambda_3$. Note also that the equations for $B, C$ and $D$ are identical to those in (II.48), so by (II.49) and (II.50) we have

$$BCD^2 = \lambda_2\lambda_3\lambda_4^2 \tag{II.74}$$

and

$$D = \lambda_4 \left(1 - \frac{3\lambda_4}{\lambda_2\lambda_3} t\right)^{-1/3}. \tag{II.75}$$

Similar calculations as those used to compute (II.51) show that

$$AD(B + C) = \lambda_1\lambda_4(\lambda_2 + \lambda_3). \tag{II.76}$$

Using (II.73) and (II.76), we can solve for $A$:

$$\frac{dA}{dt} = \frac{-4(k^2 + A^2)}{A^2},$$

and

$$\frac{A^2}{k^2 - A^2} \cdot \frac{dA}{dt} = -4,$$

where

$$\frac{\lambda_1^2(\lambda_2 + \lambda_3)^2}{\lambda_2\lambda_3} = 4k^2. \tag{II.77}$$

Now we integrate to find

$$k \tanh^{-1}\left(\frac{A}{k}\right) - A = -4t + k \tanh^{-1}\left(\frac{\lambda_1}{k}\right) - \lambda_1, \tag{II.78}$$

where $k$ is given in (II.77). We then calculate

$$\lim_{A \to 0} \frac{k \tanh^{-1}\left(\frac{A}{k}\right) - A}{A^3} = \frac{1}{3k^2}, \tag{II.79}$$

hence near $t = T_0$ we have

$$A \approx \left(12k^2(T_0 - t)\right)^{1/3}. \tag{II.80}$$

Using (II.74) and (II.76) we find

$$B = \frac{\lambda_4}{2AD}\left(\lambda_1(\lambda_2 + \lambda_3) + \sqrt{\lambda_1^2(\lambda_2 + \lambda_3)^2 - 4A^2\lambda_2\lambda_3}\right), \tag{II.81}$$

$$C = \frac{\lambda_4}{2AD}\left(\lambda_1(\lambda_2 + \lambda_3) - \sqrt{\lambda_1^2(\lambda_2 + \lambda_3)^2 - 4A^2\lambda_2\lambda_3}\right), \tag{II.82}$$

and

$$T_0 = \min\left\{\frac{\lambda_2\lambda_3}{3\lambda_4},\ \frac{1}{4}\left(k \tanh^{-1}\left(\frac{\lambda_1}{k}\right) - \lambda_1\right)\right\}. \tag{II.83}$$

The sectional curvatures are as follows:

$$K(Y_1, Y_2) = \frac{\frac{B}{C} - 3\frac{C}{B} + 2}{4A} \qquad K(Y_1, Y_3) = \frac{\frac{C}{B} - 3\frac{B}{C} + 2}{4A}$$

$$K(Y_1, Y_4) = 0 \qquad K(Y_2, Y_3) = \frac{B^2 + C^2 - 2BC - 3AD}{4ABC}$$

$$K(Y_2, Y_4) = \frac{D}{4BC} \qquad K(Y_3, Y_4) = \frac{D}{4BC}.$$

If $T_0 = T_1 < T_2$, then we have the following behavior as $t \to T_0$:

$$\begin{aligned} A &\approx k_1 \\ B &\approx k_2(T_0 - t)^{1/3} \\ C &\approx k_3(T_0 - t)^{1/3} \\ D &= k_4(T_0 - t)^{-1/3}. \end{aligned} \tag{II.84}$$

In this case the curvatures parallel to $Y_1$ will approach constants while those perpendicular to $Y_1$ will have a singularity of the form $(T_0 - t)^{-1}$.

If $T_0 = T_2 < T_1$, then as $t \to T_0$,

$$\begin{aligned} A &\approx k_1(T_0 - t)^{1/3} \\ B &\approx k_2(T_0 - t)^{-1/3} \\ C &\approx k_3(T_0 - t)^{1/3} \\ D &\approx k_4. \end{aligned} \tag{II.85}$$

In this case the curvatures perpendicular to $Y_4$ will approach infinity at a rate of $(T_0 - t)^{-1}$ while those curvatures parallel to $Y_4$ will approach constants.

If $T_0 = T_1 = T_2$, then as $t \to T_0, AD \to$ constant, so

$$\begin{aligned} A &\approx k_1(T_0 - t)^{1/3} \\ B &\approx k_2 \\ C &\approx k_3(T_0 - t)^{2/3} \\ D &= k_4(T_0 - t)^{-1/3}. \end{aligned} \tag{II.86}$$

Here, all non-zero curvatures will have singularities of the form $(T_0 - t)^{-1}$.

## A9. Class $U3S1$

Here we may choose a basis for the Lie Albegra $\{X_1, X_2, X_3, X_4\}$ such that the Lie bracket is of the form

$$\begin{array}{lll} [X_1, X_2] = -X_3 & [X_1, X_3] = -X_2 & [X_1, X_4] = 0 \\ [X_2, X_3] = X_1 & [X_2, X_4] = 0 & [X_3, X_4] = 0. \end{array}$$

This Lie Algebra structure is a direct sum $\mathfrak{sl}_2 \oplus \mathbb{R}$, and the Lie Group structure structure is $(M, G) = (\hat{SL}(2, \mathbb{R}) \times \mathbb{R}, \hat{SL}(2, \mathbb{R}) \times \mathbb{R})$.

Here we diagonalize the metric by letting $Y_i = \Upsilon_i^k X_k$ with

$$\Upsilon = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a_1 & a_2 & a_3 & 1 \end{bmatrix}.$$

**Remark II.1.** There are initial metrics in this class which cannot be diagonalized so easy. However, the situations presented in these cases provide us with very complicated situations which we will not address here. See [11, p. 376-377].

Here we calculate the curvatures of this diagonalized metric in general form so that we can use it to the specific cases outlined below. First we calculate the operator $U$ using (II.5):

$$
\begin{aligned}
U(Y_i, Y_i) &= 0 \text{ for all } i \\
U(Y_1, Y_2) &= \frac{B-A}{2C}Y_3 + \frac{a_3(B-A)}{2D}Y_4 \\
U(Y_1, Y_3) &= \frac{A+C}{2B}Y_2 + \frac{A_2(A+C)}{2D}Y_4 \\
U(Y_1, Y_4) &= \frac{a_3 A}{2B}Y_2 - \frac{a_2 A}{2C}Y_3 \\
U(Y_2, Y_3) &= -\frac{B+C}{2A}Y_1 - \frac{a_1(B+C)}{2D}Y_4 \\
U(Y_2, Y_4) &= -\frac{a_3 B}{2A}Y_1 + \frac{a_1 B}{2C}Y_3 \\
U(Y_3, Y_4) &= -\frac{a_2 C}{2A}Y_1 + \frac{a_1 C}{2B}Y_2.
\end{aligned}
$$

The sectional curvatures may then be calculated using (II.4):

$$
\begin{aligned}
K(Y_i, Y_i) &= 0 \text{ for all } i \\
K(Y_1, Y_2) &= \frac{1}{4AB}\left[-3C - 2B - 2A + (A-B)^2\left(\frac{1}{C} + \frac{a_3^2}{D}\right)\right] \\
K(Y_1, Y_3) &= \frac{1}{4AC}\left[-3B - 2C + 2A + (A+C)^2\left(\frac{1}{B} + \frac{a_2^2}{D}\right)\right] \\
K(Y_1, Y_4) &= \frac{1}{4AD}\left[-3(a_3^2 B + a_2^2 C) + 2A(a_3^2 - a_2^2) + A^2\left(\frac{a_3^2}{B} + \frac{a_2^2}{C}\right)\right] \qquad \text{(II.87)} \\
K(Y_2, Y_3) &= \frac{1}{4BC}\left[-3A - 2C + 2B + (B+C)^2\left(\frac{1}{A} + \frac{a_1^2}{D}\right)\right] \\
K(Y_2, Y_4) &= \frac{1}{4BD}\left[-3(a_3^2 A + a_1^2 C) + 2B(a_3^2 - a_1^2) + B^2\left(\frac{a_3^2}{A} + \frac{a_1^2}{C}\right)\right] \\
K(Y_3, Y_4) &= \frac{1}{4CD}\left[-3(a_2^2 A + a_1^2 B) - 2C(a_1^2 + a_2^2) + C^2\left(\frac{a_2^2}{A} + \frac{a_1^2}{B}\right)\right].
\end{aligned}
$$

Proposition 8 in [11] tells us that when $g_0$ can in fact be diagonalized as above:

(i) If $\lambda_1 \neq \lambda_2$, the metric remains diagonal if and only if $a_1 = a_2 = a_3 = 0$

(ii) If $\lambda_1 = \lambda_2$, the metric remains diagonal if and only if $a_1 = a_2 = 0$.

A9(i)

Here we consider the case where $\lambda_1 \neq \lambda_2$ and $a_1 = a_2 = a_3 = 0$. In this case, $\Upsilon = I$, and the Lie Bracket relations remain the same. It also happens that this metric is just the product metric $\tilde{SL}(2,\mathbb{R}) \times \mathbb{R}$, so backwards Ricci flow reduces to the case of the three dimensional flow on $\tilde{SL}(2,\mathbb{R})$. The volume-normalized version of this flow has been discussed in [4].

In this case the non-zero curvatures are given by

$$\begin{aligned}
K(Y_1, Y_2) &= \frac{1}{4ABC}\left[-3C^2 - 2BC - 2AC + A^2 - 2AB + B^2\right] \\
K(Y_1, Y_3) &= \frac{1}{4ABC}\left[-3B^2 - 2BC + 2AB + A^2 + 2AC + C^2\right] \\
K(Y_2, Y_3) &= \frac{1}{4ABC}\left[-3A^2 - 2AC + 2AB + B^2 + 2BC + C^2\right].
\end{aligned} \tag{II.88}$$

Let $Y_i = X_i$, and let $g(t) = A(t)\theta_1^2 + B(t)\theta_2^2 + C(t)\theta_3^2 + D(t)\theta_4^2$ where $A(0) = \lambda_1,\ B(0) = \lambda_2,\ C(0) = \lambda_3$ and $D(0) = \lambda_4$. Then backward Ricci flow reduces to the following system of equations:

$$\begin{aligned}
\frac{dA}{dt} &= \frac{A^2 - (B+C)^2}{BC} \\
\frac{dB}{dt} &= \frac{B^2 - (A+C)^2}{AC} \\
\frac{dC}{dt} &= \frac{C^2 - (A-B)^2}{AB} \\
\frac{dD}{dt} &= 0.
\end{aligned} \tag{II.89}$$

By the symmetry of the system we may assume that $\lambda_1 > \lambda_2$. We also have

$$\frac{d}{dt}(A - B) = \frac{2}{ABC}(A - B)(A + B + C),$$

so $A > B$ is preserved by (II.89).

We set

$$Q = \left\{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3 \middle| a \geq b > 0,\ c > 0\right\}. \tag{II.90}$$

The result is that there exists a partition of $Q$ into $Q_1,\ Q_2,\ S_0$, with $S_0$ a hypersurface in $\mathbb{R}^3$ and $Q_1,\ Q_2$ open and connected such that:

If $(\lambda_1, \lambda_2, \lambda_3) \in Q_1$, then

$$\begin{aligned} A(t) &\approx k_1(T_0 - t)^{1/3} \\ B(t) &\approx k_2(T_0 - t)^{1/3} \\ C(t) &\approx k_3(T_0 - t)^{-1/3} \\ D(t) &= \lambda_4. \end{aligned} \tag{II.91}$$

If $(\lambda_1, \lambda_2, \lambda_3) \in Q_2$, then

$$\begin{aligned} A(t) &\approx k_1(T_0 - t)^{-1/3} \\ B(t) &\approx k_2(T_0 - t)^{1/3} \\ C(t) &\approx k_3(T_0 - t)^{1/3} \\ D(t) &= \lambda_4. \end{aligned} \tag{II.92}$$

If $(\lambda_1, \lambda_2, \lambda_3) \in S_0$, then

$$\begin{aligned} A(t) &\approx k_1 \\ B(t) &\approx 4(T_0 - t) \\ C(t) &\approx k_1 \\ D(t) &= \lambda_4. \end{aligned} \tag{II.93}$$

In all three cases all the non-zero curvatures have a singularity at $T_0$ of the form $(T_0 - t)^{-1}$.

We first define

$$Q_1 = \left\{(\lambda_1, \lambda_2, \lambda_3) \in Q \middle| C(t_1) \geq A(t_1) \text{ for some time } t_1 \geq 0\right\} \tag{II.94}$$

$$Q_2 = \left\{(\lambda_1, \lambda_2, \lambda_3) \in Q \middle| C(t_1) \leq A(t_1) - B(t_1) \text{ for some time } t_1 \geq 0\right\} \tag{II.95}$$

$$S_0 = \left\{(\lambda_1, \lambda_2, \lambda_3) \in Q \middle| A(t) - B(t) \leq C(t) \leq A(t) \text{ for all time } t \geq 0\right\}. \tag{II.96}$$

Now, if $C(t) \geq A(t)$, then

$$\frac{d}{dt}\big(C - A\big) = \frac{1}{ABC}\big[(C - A)\big((C + A)^2 - B^2\big) + 4ABC\big] > 4,$$

so $C \geq A$ is preserved.

Similarly, if $C(t) \leq A(t) - B(t)$, then

$$\frac{d}{dt}\big(A - B - C\big) = \frac{1}{ABC}\big[(A - B - C)\big(A(A + 2B + 2C) + (B - C)^2\big) + 8ABC\big] > 8,$$

so $C \leq A - B$ is preserved.

Thus $Q_1$, $Q_2$ and $S_0$ are mutually exclusive sets whose union is all of $Q$. The facts that $Q_1$ and $Q_2$ are open and $S_0$ is a hypersurface in $Q$ are shown in [3]. These results are presented in [4] for the normalized Ricci flow, so we give an un-normalized version here to better describe the behavior.

We first consider the set $Q_1$, given by (II.94). Then $C(t) \geq A(t)$ for all $t \geq t_1$, so we have

$$\frac{dA}{dt} = \frac{A^2 - (B + C)^2}{BC} \leq \frac{-B - 2C}{C} < -2,$$

hence $A$ is decreasing, and $T_0 < \infty$. Since $B < A < C$, either $\lim_{t \to T_0} C(t) = \infty$ or $\lim_{t \to T_0} B(t) = 0$. We show that in fact both of these situations happen at the same time. Since $A$ and $B$ are decreasing, then for $C$ large enough we have

$$\frac{d}{dt}(AC) = 2(A + B - C) < 0. \tag{II.97}$$

Thus $AC$ is bounded above. Thus if $C \to \infty$ at $T_0$, then it must be the case that $A \to 0$ at $T_0$, hence also $B \to 0$ since $B < A$. Now we observe

$$\frac{d}{dt}\big(B(C - A)\big) = 4B > 0, \tag{II.98}$$

so $\big(B(C - A)\big)$ is bounded below. Thus if $B \to 0$ at $T_0$, then $C - A \to \infty$, hence also $C \to \infty$. Thus we see that $A \to 0$, $B \to 0$ and $C \to \infty$ all together at time $t = T_0$. Now,

$$\frac{d}{dt}(AC - AB) = 4A, \tag{II.99}$$

which is positive and approaches 0 at $t = T_0 < \infty$. Therefore we know that $AC - AB \to k_A > 0$ as $t \to T_0$. Since $AB \to 0$, we see that

$$AC \to k_A. \tag{II.100}$$

Similarly,

$$\frac{d}{dt}(BC - AB) = 4B \to 0 \tag{II.101}$$

implies that as $t \to T_0$,

$$BC \to k_B > 0. \tag{II.102}$$

Now by (II.89) we see that

$$\frac{dC}{dt} \sim \frac{C^2}{AB} \sim kC^4, \tag{II.103}$$

hence by the same argument used to solve (II.16), we see

$$C \approx k_3(T_0 - t)^{-1/3}. \tag{II.104}$$

Now combining (II.100), (II.102) and (II.104), we have the solution to (II.89):

$$\begin{aligned} A(t) &\approx k_1(T_0 - t)^{1/3} \\ B(t) &\approx k_2(T_0 - t)^{1/3} \\ C(t) &\approx k_3(T_0 - t)^{-1/3} \\ D(t) &= \lambda_4. \end{aligned} \tag{II.105}$$

The normalized solution will approach the plane $\mathbb{R}^2$.

Now we consider the set $Q_2$ given by (II.95). Here there is some time $t_1$ such that for $t \geq t_1 \geq 0$, $(A - B) < C$. We denote $A_1 = A(t_1)$, $B_1 = B(t_1)$ and $C_1 = C(t_1)$. We see that for $t \geq t_1$

$$\frac{dC}{dt} = \frac{C^2 - (A+B)^2}{AB} < 0,$$

and

$$\frac{dA}{dt} = \frac{A^2 - (B-C)^2}{BC} > 0,$$

so $C$ is decreasing and $A$ is increasing. Thus, as $t \to T_0$, either $A \to \infty,\ B \to 0$ or $C \to 0$. We calculate

$$\frac{d}{dt}\big(A(B+C)\big) = -4B - 4C < 0, \tag{II.106}$$

so $\big(A(B+C)\big)$ is bounded above. Thus if $A \to \infty$ as $t \to T_0$, then we know $(B+C) \to 0$ at $t = T_0$, hence $B \to 0$ and $C \to 0$. Now for $t \geq t_1$,

$$\frac{d}{dt}\left(\frac{B}{C}\right) = \frac{2}{A}(B+C)(B-C-A) < 0, \tag{II.107}$$

so $\dfrac{B}{C}$ is bounded above and if $C \to 0$ then $B \to 0$. Lastly,

$$\frac{d}{dt}\big(B(A-B-C)\big) = \frac{2B}{AC}(A^2 + C^2 - B^2) > 0, \tag{II.108}$$

so $\big(B(A-B-C\big)$ is bounded below, and if $B \to 0$ then $(A-B-C) \to \infty$, hence also $A \to \infty$. Thus we see that at $t = T_0$ we have $A \to \infty,\ B \to 0$ and $C \to 0$.

Now by (II.108) we see that $(AB - BC - B^2)$ is increasing for $t \geq t_1$.Thus we see that $AB > B_1(A_1 - B_1 - C_1)$. However,

$$\frac{d}{dt}(AB) = -2(A+B+C) < 0, \tag{II.109}$$

so $AB$ is decreasing and bounded below by a positive number. Thus at $t \to T_0$, we have

$$AB \to k_{AB} \tag{II.110}$$

for some positive constant $k_{AB}$. Now

$$\frac{d}{dt}(AC) = 2(A-B-C) > 0, \tag{II.111}$$

so $(AC)$ is increasing and bounded below. Thus, since $B$ and $C$ are approaching 0 and $A$ is approaching infinity at $T_0$, then by (II.89) and (II.110) there is some positive number $k$ such that

$$\frac{dA}{dt} = \frac{A^2 - (B+C)^2}{BC} \leq kA^4, \tag{II.112}$$

hence we know

$$A < \left(A_1^{-3} - 3kt\right)^{-1/3}. \tag{II.113}$$

By (II.111) we see that

$$\frac{d}{dt}(AC) < A < \left(A_1^{-3} - 3kt\right)^{-1/3}. \tag{II.114}$$

Upon integrating we discover

$$AC < A_1C_1 + \frac{1}{2A_1^2} - \frac{1}{2k}\left(A_1^{-3} - 3kt\right)^{-1/3} < \infty. \tag{II.115}$$

Since $AC$ is increasing, then we see that at $t \to T_0$, we have

$$AC \to k_{AC} \tag{II.116}$$

for some positive constant $k_{AC}$. Now by (II.89), (II.110) and (II.116),

$$A \sim kA^4. \tag{II.117}$$

Solving (II.117) the same way we solved (II.16), and then using (II.110) and

(II.116), gives the following end behavior of the solutions to (II.89):

$$\begin{aligned} A(t) &\approx k_1(T_0 - t)^{-1/3} \\ B(t) &\approx k_2(T_0 - t)^{1/3} \\ C(t) &\approx k_3(T_0 - t)^{1/3} \\ D(t) &= \lambda_4. \end{aligned} \tag{II.118}$$

The normalized solution will approach the plane $\mathbb{R}^2$.

Now we consider the set $S_0$ given by (II.96). Here we know

$$A(t) - B(t) < C(t) < A(t)$$

for all time $0 \leq t < T_0$. Thus we have

$$\frac{dA}{dt} = \frac{A^2 - (B + C)^2}{BC} < 0,$$

and

$$\frac{dC}{dt} = \frac{C^2 - (A - B)^2}{AB} > 0,$$

so $A$ is decreasing while $C$ is increasing.

However, $A(t) > C(t)$, so both $A$ and $C$ are approaching constants. Now

$$\frac{dB}{dt} = \frac{B^2 - (A + C)^2}{AC} < -2,$$

so $B$ approaches 0 in finite time $T_0 < \infty$. Since $A - B < C < A$, then in fact $A$ and $C$ approach the same constant, $k_1$, at time $t = T_0$. Thus near $t = T_0$ we have the approximation:

$$\frac{dB}{dt} \approx -\frac{(A+C)^2}{AC} \approx -4. \tag{II.119}$$

Thus near $t = T_0$ we have the following behavior:

$$\begin{aligned} A &\approx k_1 \\ B &\approx -4(t - T_0) \\ C &\approx k_1 \\ D &= \lambda_4. \end{aligned} \tag{II.120}$$

The normalized solution will approach $\mathbb{R}^3$.

A9(ii)

We find in this case that $\{Y_i\}$ satisfies the following bracket relations:

$$[Y_1, Y_2] = -Y_3 \qquad [Y_1, Y_3] = -Y_2 \qquad [Y_1, Y_4] = -a_3 Y_2$$

$$[Y_2, Y_3] = Y_1 \qquad [Y_2, Y_4] = a_3 Y_1 \qquad [Y_3, Y_4] = 0.$$

Let $g(t) = A(t)\theta_1^2 + B(t)\theta_2^2 + C(t)\theta_3^2 + D(t)\theta_4^2$ where $A(0) = \lambda_1,\ B(0) = \lambda_2,\ C(0) = \lambda_3$ and $D(0) = \lambda_4$. Then backward Ricci flow reduces to the following system of equations:

$$\begin{aligned}
\frac{dA}{dt} &= \frac{A^2-(B+C)^2}{BC} + \frac{A^2-B^2}{BD}a_3^2 \\
\frac{dB}{dt} &= \frac{B^2-(A+C)^2}{AC} + \frac{B^2-A^2}{AD}a_3^2 \\
\frac{dC}{dt} &= \frac{C^2-(A-B)^2}{AB} \\
\frac{dD}{dt} &= -\frac{(A+B)^2}{AB}a_3^2 \\
\lambda_1 &= \lambda_2.
\end{aligned} \tag{II.121}$$

We calculate

$$\begin{aligned}
\frac{d}{dt}\big(A-B\big) = \Big[&\frac{1}{ABC}\big(-C^2+A^2+2AB+B^2\big) \\
&+ \frac{a_3^2}{ABD}\big(A^2+2AB+B^2\big)\Big]\big(A-B\big),
\end{aligned}$$

so $A = B$ for all $t$.

Now (II.121) reduces to

$$\begin{aligned}
\frac{dA}{dt} &= -\frac{C}{A} - 2 \\
B &= A \\
\frac{dC}{dt} &= \frac{C^2}{A^2} \\
\frac{dD}{dt} &= -4a_3^2.
\end{aligned} \tag{II.122}$$

This equations for $A$, $B$ and $C$ in (II.122) are just a special case of the corresponding equations in (II.89). However, in this special case we may derive more explicit solutions.

It is clear that $D = \lambda_4 - 4a_3^2 t$, so $T_0 \leq \frac{\lambda_4}{4a_3^2}$. Also, letting $f'$ denote $\frac{df}{dt}$ we have

$$\begin{aligned} A'' &= \frac{-A\left(\frac{C^2}{A^2}\right) + C\left(-\frac{C}{A} - 2\right)}{A^2} \\ &= -\frac{2C^2}{A^3} - \frac{2C}{A} \\ &= -\frac{2}{A}\big((A'+2)^2 - A' + 2\big) \\ &= -\frac{2}{A}\big((A'+1)(A'+2)\big), \end{aligned}$$

hence

$$\frac{A'}{(A'+1)(A'+2)} A'' = -\frac{2}{A} A'.$$

Integrating gives

$$2\ln|A'+2| - \ln|A'+1| = -2\ln|A| + \ln(\Lambda),$$

where

$$\Lambda = \frac{\lambda_3^2 \lambda_1}{\lambda_3 + \lambda_1}. \tag{II.123}$$

Solving for $A'$ gives us

$$A' = \frac{-\Lambda - 4A^2 \pm \sqrt{\Lambda^2 + 4\Lambda A^2}}{2A^2}. \tag{II.124}$$

Comparing equations (II.122) and (II.124) at $t = 0$, we confirm in fact that

$$A' = -\frac{\Lambda + 4A^2 + \sqrt{\Lambda^2 + 4\Lambda A^2}}{2A^2},$$

hence

$$\left(\frac{2A^2}{\Lambda + 4A^2 + \sqrt{\Lambda^2 + 4\Lambda A^2}}\right) A' = -1. \tag{II.125}$$

Now we observe

$$\frac{2A^2}{\Lambda + 4A^2 + \sqrt{\Lambda^2 + 4\Lambda A^2}} = \frac{1}{2} - \frac{\sqrt{\Lambda}}{2\sqrt{\Lambda + 4A^2}},$$

so

$$\left(\frac{1}{2} - \frac{\sqrt{\Lambda}}{2\sqrt{\Lambda + 4A^2}}\right) A' = -1,$$

and

$$\frac{A}{2} - \frac{\sqrt{\Lambda}}{4} \sinh^{-1}\left(\frac{2A}{\sqrt{\Lambda}}\right) = T_2 - t. \tag{II.126}$$

Now we calculate that

$$\lim_{A \to 0} \frac{\frac{A}{2} - \frac{\sqrt{\Lambda}}{4} \sinh^{-1}\left(\frac{2A}{\sqrt{\Lambda}}\right)}{A^3} = \frac{1}{3e^{\Lambda}}. \tag{II.127}$$

Thus using (II.125) and (II.127) we find that near $t = T_2$,

$$A \approx \left(3e^{\Lambda}(T_2 - t)\right)^{1/3}. \tag{II.128}$$

By (II.122) we know that $C = -A\left(\frac{dA}{dt} + 2\right)$. Thus near $A = 0$ we know that

$$C \approx \Lambda \left(3\Lambda(T_2 - t)\right)^{-1/3}. \tag{II.129}$$

We observe that

$$T_0 = \min\left\{\frac{\lambda_4}{4a_3^2},\ \frac{\lambda_1}{2} - \frac{\lambda_3}{4}\sqrt{\frac{\lambda_1}{\lambda_1 + \lambda_3} \sinh^{-1}\left(\frac{2}{\lambda_3}\sqrt{\lambda_1(\lambda_1 + \lambda_3)}\right)}\right\} \tag{II.130}$$

$$= \min\left\{T_1, T_2\right\}. \tag{II.131}$$

We calculate the sectional curvatures using (II.87) with $a_1 = a_2 = 0$ and $A = B$:

$$K(Y_1, Y_2) = -\frac{4A + 3C}{4A^2} \qquad K(Y_1, Y_3) = \frac{C}{4A^2} \qquad K(Y_1, Y_4) = 0$$

$$K(Y_2, Y_3) = \frac{C}{4A^2} \qquad K(Y_2, Y_4) = 0 \qquad K(Y_3, Y_4) = 0.$$

If $T_0 = T_1 < T_2$, then near $t = T_0$ we have

$$\begin{aligned} A = B &\approx k_1 \\ C &\approx k_3 \\ D &= k_4(T_0 - t). \end{aligned} \tag{II.132}$$

Here all sectional curvatures will also approach constants as $t \to T_0$, and the volume normalized solution will approach the hyperplane $\mathbb{R}^3$.

If $T_0 = T_2 < T_1$, then near $t = T_0$ we have

$$\begin{aligned} A = B &\approx k_1(T_0 - t)^{1/3} \\ C &\approx k_3(T_0 - t)^{-1/3} \\ D &\approx k_4. \end{aligned} \tag{II.133}$$

Here all non-zero sectional curvatures will have a singularity of the form $(T_0 - t)^{-1}$. The volume normalized solution will approach the plane $\mathbb{R}^2$.

If $T_0 = T_1 = T_2$, then near $t = T_0$ we have

$$\begin{aligned} A = B &\approx k_1(T_0 - t)^{1/3} \\ C &\approx k_3(T_0 - t)^{-1/3} \\ D &= k_4(T_0 - t). \end{aligned} \tag{II.134}$$

Here all non-zero sectional curvatures will have a singularity of the form $(T_0 - t)^{-1}$, and the volume normalized solution will approach a product metric

$M^2 \times \mathbb{R}$.

## A10. Class $U3S3$

Here we may choose a basis for the Lie Albegra $\{X_1, X_2, X_3, X_4\}$ such that the Lie bracket is of the form

$$[X_1, X_2] = X_3 \qquad [X_1, X_3] = -X_2 \qquad [X_1, X_4] = 0$$

$$[X_2, X_3] = X_1 \qquad [X_2, X_4] = 0 \qquad [X_3, X_4] = 0.$$

This Lie Algebra structure is a direct sum $\mathfrak{su}(2) \oplus \mathbb{R}$, and the Lie Group structure structure is $(M, G) = (S^3 \times \mathbb{R}, SU(2) \times \mathbb{R})$.

Here we diagonalize the metric by letting $Y_i = \Upsilon_i^k X_k$ with

$$\Upsilon = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a_1 & a_2 & a_3 & 1 \end{bmatrix}.$$

Here I calculate the curvatures of this diagonalized metric in general form so that I can use it to the specific cases outlined below in Proposition 9. First I calculate the operator $U$ using (II.5):

$$
\begin{aligned}
U(Y_i, Y_i) &= 0 \text{ for all } i \\
U(Y_1, Y_2) &= \frac{B-A}{2C} Y_3 + \frac{a_3(B-A)}{2D} Y_4 \\
U(Y_1, Y_3) &= \frac{A-C}{2B} Y_2 + \frac{A_2(A-C)}{2D} Y_4 \\
U(Y_1, Y_4) &= \frac{a_3 A}{2B} Y_2 - \frac{a_2 A}{2C} Y_3 \\
U(Y_2, Y_3) &= \frac{C-B}{2A} Y_1 + \frac{a_1(C-B)}{2D} Y_4 \\
U(Y_2, Y_4) &= -\frac{a_3 B}{2A} Y_1 + \frac{a_1 B}{2C} Y_3 \\
U(Y_3, Y_4) &= \frac{a_2 C}{2A} Y_1 - \frac{a_1 C}{2B} Y_2.
\end{aligned}
$$

The sectional curvatures may then be calculated using (II.4):

$$
\begin{aligned}
K(Y_i, Y_i) &= 0 \text{ for all } i \\
K(Y_1, Y_2) &= \frac{1}{4AB} \left[ -3C + 2B + 2A + (A-B)^2 \left( \frac{1}{C} + \frac{a_3^2}{D} \right) \right] \\
K(Y_1, Y_3) &= \frac{1}{4AC} \left[ -3B + 2C + 2A + (A-C)^2 \left( \frac{1}{B} + \frac{a_2^2}{D} \right) \right] \\
K(Y_1, Y_4) &= \frac{1}{4AD} \left[ -3(a_3^2 B + a_2^2 C) + 2A(a_2^2 + a_3^2) + A^2 \left( \frac{a_3^2}{B} + \frac{a_2^2}{C} \right) \right] \\
K(Y_2, Y_3) &= \frac{1}{4BC} \left[ -3A + 2C + 2B + (B-C)^2 \left( \frac{1}{A} + \frac{a_1^2}{D} \right) \right] \\
K(Y_2, Y_4) &= \frac{1}{4BD} \left[ -3(a_3^2 A + a_1^2 C) + 2B(a_3^2 + a_1^2) + B^2 \left( \frac{a_3^2}{A} + \frac{a_1^2}{C} \right) \right] \\
K(Y_3, Y_4) &= \frac{1}{4CD} \left[ -3(a_2^2 A + a_1^2 B) + 2C(a_1^2 + a_2^2) + C^2 \left( \frac{a_2^2}{A} + \frac{a_1^2}{B} \right) \right].
\end{aligned} \tag{II.135}
$$

Proposition 9 in [11] breaks this down into three situations:

(i) If $\lambda_1 = \lambda_2 = \lambda_3$, them the metric remains diagonal for all choices of $a_1,\ a_2$ and $a_3$.

(ii) If $\lambda_i \neq \lambda_j = \lambda_k$ for some permutation $\{i, j, k\}$ of $\{1, 2, 3\}$, then the metric remains diagonal if and only if $a_j = a_k = 0$.

(iii) If $\lambda_1,\ \lambda_2$ and $\lambda_3$ are all different, then the metric will remain diagonal if and only if $a_1 = a_2 = a_3 = 0$.

In all three of these cases, we have that backward Ricci flow reduces to the system :

$$\begin{aligned}\frac{dA}{dt} &= \frac{A^2 - (B-C)^2}{BC}\\ \frac{dB}{dt} &= \frac{B^2 - (A-C)^2}{AC}\\ \frac{dC}{dt} &= \frac{C^2 - (A-B)^2}{AB}\\ \frac{dD}{dt} &= 0.\end{aligned} \tag{II.136}$$

Thus this reduces to 3 dimensions, and the normalized flow is analyzed in [4].

By symmetry, we may in fact assume $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Now

$$\frac{d}{dt}(A-B) = \frac{1}{ABC}(A-B)\big((A+B)^2 - C^2\big), \tag{II.137}$$

so the conditions $A(t) \geq B(t)$ and $A(t) = B(t)$ are preserved under backwards Ricci flow. Similarly, the conditions $B(t) \geq C(t)$ and $B(t) = C(t)$ are preserved.

A10(i)

If $\lambda_1 = \lambda_2 = \lambda_3$, then $A \equiv B \equiv C$. Then (II.136) reduces to

$$\begin{aligned} \frac{dA}{dt} = \frac{dB}{dt} = \frac{dC}{dt} &= 1 \\ \frac{dD}{dt} &= 0, \end{aligned} \tag{II.138}$$

and the solution, which exists for all time, is

$$\begin{aligned} A(t) = B(t) = C(t) &= \lambda_1 + t, \\ D(t) &= \lambda_4. \end{aligned} \tag{II.139}$$

Using (II.135) with $A = B = C$ we find the non-zero curvatures are

$$K(Y_1, Y_2) = K(Y_1, Y_3) = K(Y_2, Y_3) = \frac{1}{4A}. \tag{II.140}$$

Thus all non-zero curvatures approach 0 at a rate of $4t^{-1}$.

The solution here is actually just a product of an expanding 3-sphere with a line. The volume normalized solution converges to the line, $\mathbb{R}$.

A10(ii)

If $\lambda_1 = \lambda_2 > \lambda_3$, then $A(t) = B(t) > C(t)$ for all $0 < t < T_0$. Then (II.136) reduces to

$$\begin{aligned} \frac{dA}{dt} &= 2 - \frac{C}{A} \\ B &= A \\ \frac{dC}{dt} &= \frac{C^2}{A^2} \\ \frac{dD}{dt} &= 0. \end{aligned} \tag{II.141}$$

Denoting $\frac{df}{dt}$ by $f'$ we have

$$\begin{aligned}
A'' &= \frac{1}{A^2}\left(C\left(2-\frac{C}{A}\right)-\frac{C^2}{A}\right)\\
&= \frac{1}{A}\big((2-A')A'-(2-A')^2\big)\\
&= -\frac{2}{A}(A'-1)(A'-2),
\end{aligned}$$

hence

$$\frac{-A'}{(A'-1)(A'-2)}A'' = \frac{2}{A}A',$$

and thus

$$\ln\left|\frac{A'-1}{(A'-2)^2}\right| = \ln(A^2)+\ln(\Lambda), \tag{II.142}$$

where

$$\Lambda = \frac{\lambda_1-\lambda_3}{\lambda_1\lambda_3^2}. \tag{II.143}$$

Since $A' > 1$, (II.142) becomes

$$\frac{A'-1}{(A'-2)^2} = \Lambda A^2.$$

Solving we have

$$A' = \frac{1+4\Lambda A^2-\sqrt{1+4\Lambda A^2}}{2\Lambda A^2}. \tag{II.144}$$

Note here that since $A' = 2-\frac{C}{A}$ by (II.136), then

$$C = \frac{-1+\sqrt{1+4\Lambda A^2}}{2\Lambda A}. \tag{II.145}$$

Continuing on, (II.144) becomes

$$\frac{2\Lambda A^2}{1+4\Lambda A^2-\sqrt{1+4\Lambda A^2}}A' = 1.$$

Upon integrating we have

$$\frac{1}{2}A + \frac{1}{4\sqrt{\Lambda}}\sinh^{-1}\big(2\sqrt{\Lambda}A\big) = t + \frac{1}{2} + \frac{1}{4\sqrt{\Lambda}}\sinh^{-1}\big(2\sqrt{\Lambda}\lambda_1\big) \tag{II.146}$$

Now we observe that

$$\lim_{A\to\infty}\frac{\frac{1}{2}A + \frac{1}{4\sqrt{\Lambda}}\sinh^{-1}\big(2\sqrt{\Lambda}A\big)}{A} = \frac{1}{2}, \tag{II.147}$$

so as $A$ approaches infinity, $t \approx \frac{1}{2}A$.

Now by equations (II.136) and (II.145), we have the following behavior as $t \to \infty$:

$$\begin{aligned} A(t) &= B(t) \approx 2t \\ C(t) &\to \frac{1}{\sqrt{\Lambda}} \\ D(t) &= \lambda_4. \end{aligned} \tag{II.148}$$

where $\Lambda$ is given in (II.143).

Using (II.135) with $a_1 = a_2 = 0$ and $A = B$ we find the non-zero curvatures are given by

$$\begin{aligned} K(Y_1, Y_2) &= \frac{4A - 3C}{4A^2} \\ K(Y_1, Y_3) &= \frac{C}{4A^2} \\ K(Y_2, Y_3) &= \frac{C}{4A^2}. \end{aligned} \tag{II.149}$$

Thus all non-zero curvatures parallel to $Y_3$ approach 0 at a rate of $t^{-2}$ while the remaining nonzero curvatures approach 0 at a rate of $t^{-1}$. The normalized solution will converge to the plane $\mathbb{R}^2$.

If $\lambda_1 > \lambda_2 = \lambda_3$, then $A(t) > B(t) = C(t)$ for all $0 < t < T_0$, and (II.136) reduces to the following:

$$\begin{aligned} \frac{dA}{dt} &= \frac{A^2}{B^2} \\ \frac{dB}{dt} &= 2 - \frac{A}{B} \\ C &= B \\ \frac{dD}{dt} &= 0. \end{aligned} \tag{II.150}$$

Similarly to the case $A = B > C$ we calculate

$$\begin{aligned} B'' &= \frac{1}{B^2}\left(A\left(2 - \frac{A}{B}\right) - \frac{A^2}{B}\right) \\ &= \frac{1}{B}\big((2 - B')B' - (2 - B')^2\big) \\ &= -\frac{2}{B}(B' - 1)(B' - 2), \end{aligned}$$

hence

$$\frac{-B'}{(B'-1)(B'-2)}B'' = \frac{2}{B}B',$$

and thus

$$\ln\left|\frac{B'-1}{(B'-2)^2}\right| = \ln(B^2) + k, \tag{II.151}$$

where

$$\Lambda = \frac{\lambda_1 - \lambda_2}{\lambda_1^2 \lambda_2}. \tag{II.152}$$

Since $B' < 1$, (II.151) becomes

$$\frac{1 - B'}{(B'-2)^2} = \Lambda B^2.$$

Solving we have

$$B' = \frac{4\Lambda B^2 - 1 - \sqrt{1 - 4\Lambda B^2}}{2\Lambda B^2}. \tag{II.153}$$

Note here that since $B' = 2 - \dfrac{A}{B}$, then

$$A = \frac{1 + \sqrt{1 - 4\Lambda B^2}}{2\Lambda B}. \tag{II.154}$$

Now (II.153) becomes

$$\frac{2\Lambda B^2}{4\Lambda B^2 - 1 - \sqrt{1 - 4\Lambda B^2}} B' = 1.$$

Upon integrating we have

$$\frac{1}{2}B - \frac{1}{4\sqrt{\Lambda}} \sin^{-1}\big(2\sqrt{\Lambda}B\big) = t - T_0, \tag{II.155}$$

where

$$T_0 = \frac{\lambda_2}{2} - \frac{\lambda_1\sqrt{\lambda_2}}{4\sqrt{\lambda_1 - \lambda_2}} \sin^{-1}\left(\frac{2\sqrt{\lambda_2(\lambda_1 - \lambda_2)}}{\lambda_1}\right). \tag{II.156}$$

Now we calculate

$$\lim_{B \to 0} \frac{\frac{1}{2}B - \frac{1}{4\sqrt{\Lambda}} \sin^{-1}\big(2\sqrt{\Lambda}B\big)}{B^3} = -\frac{\Lambda}{3},$$

so as $B$ approaches 0, we have $t - T_0 \approx -\dfrac{\Lambda}{3}B^3$, so by equations (II.150) and (II.154), we have the following behavior as $t \to T_0$:

$$\begin{aligned} A(t) &\approx \frac{1}{\Lambda}\left(\frac{3}{\Lambda}(T_0 - t)\right)^{-1/3} \\ B(t) = C(t) &\approx \left(\frac{3}{\Lambda}(T_0 - t)\right)^{1/3} \\ D(t) &= \lambda_4. \end{aligned} \tag{II.157}$$

where $\Lambda$ is given in (II.152), and $T_0$ is given in (II.156).

With $a_2 = a_3 = 0$ and $B = C$, (II.135) tells us the non-zero curvatures are given by

$$\begin{aligned} K(Y_1, Y_2) &= \frac{A}{4B^2} \\ K(Y_1, Y_3) &= \frac{A}{4B^2} \\ K(Y_2, Y_3) &= \frac{4B - 3A}{4B^2}. \end{aligned} \tag{II.158}$$

Thus all non-zero curvatures are of the form $(T_0 - t)^{-1}$. The volume normalized solution will approach the plane $\mathbb{R}^2$.

A10(iii)

In the case $\lambda_1 > \lambda_2 > \lambda_3$, we have by [4] that the end behavior near $t = T_0$ is the same as when $\lambda_1 > \lambda_2 = \lambda_3$. However, we do not have more explicit solutions like we do in the special case. Since the solutions in [4] are using normalized backward Ricci flow, I shall present a slightly different argument here.

We know that for all $t$ we have $A(t) > B(t) > C(t)$. From (II.136) we calculate

$$\frac{dC}{dt} = \frac{C^2 - A^2 + 2AB - B^2}{AB} < 2 - \frac{A}{B} < 1, \tag{II.159}$$

so $C(t) < \lambda_3 + t$. Now we may calculate

$$\begin{aligned}
\frac{d}{dt}\left(\frac{A-B}{C}\right) &= \frac{2}{ABC}(A-B)(A^2+B^2-C^2) \\
&> \frac{2A^2(A-B)}{ABC^2} \\
&> 2\left(\frac{A-B}{C}\right)\cdot\frac{1}{C} \\
&> 2\left(\frac{A-B}{C}\right)\cdot\frac{1}{t+\lambda_3},
\end{aligned}$$

so

$$\ln\left(\frac{A-B}{C}\right) > 2\ln(t+\lambda_3) + \ln\left(\frac{\lambda_1-\lambda_2}{\lambda_3}\right),$$

and we have that

$$\left(\frac{A-B}{C}\right) > \left(\frac{\lambda_1-\lambda_2}{\lambda_3}\right)(t+\lambda_3)^2.$$

Therefore $\frac{A-B}{C}$ increases at least quadratically. Thus either $T_0 < \infty$ or $\frac{dC}{dt} = \frac{C^2-(A-B)^2}{AB} < 0$ for all $t$ large enough.

Either way, we know by (II.159) that $C$ is bounded, hence $C \le k_C$ for some $0 < k_C < \infty$. Now

$$\begin{aligned}
\frac{d}{dt}\left(\frac{A}{B}\right) &= \frac{2(A-B)}{B^2C}(A+B-C) \\
&> \frac{2A(A-B)}{k_C B^2} \\
&= \frac{2}{k_C}\left(\frac{A}{B}\right)\left(\frac{A}{B}-1\right).
\end{aligned} \tag{II.160}$$

Thus we see that $\frac{A}{B} \to \infty$ in finite time, so we know $T_0 < \infty$. Since for all $t$

we know $A > B > C$, then near $t = T_0$ either $A \to \infty$ or $C \to 0$. Now we may calculate

$$\frac{d}{dt}\left(\frac{B}{C}\right) = \frac{2}{AC^2}(B - C)(B + C - A),$$

which is negative for $t$ close enough to $T_0$. Thus we know by monotone convergence that

$$\lim_{t \to T_0} \frac{B}{C} = k_{BC} \tag{II.161}$$

for some $k_{BC} \geq 1$. Thus if $C \to 0$ then $B \to 0$ as well. Now we observe

$$\frac{d}{dt}(ABC) = -(A^2 + B^2 + C^2) + 2(AB + AC + BC), \tag{II.162}$$

which is negative for $A$ large enough and $B$ and $C$ bounded. Thus if $A \to \infty$ as $t \to T_0$, then $ABC$ is bounded, hence $C \to 0$. Similarly,

$$\frac{d}{dt}(AC) = 2(A + C - B), \tag{II.163}$$

which is positive for $B$ and $C$ close enough to 0 and $A$ bounded below. Hence if $C \to 0$ then also $A \to \infty$. Thus we know that as $t \to T_0$ we have $A \to \infty$, $B \to 0$ and $C \to 0$.

Now by (II.136) we observe

$$\frac{dA}{dt} = \frac{A^2 - (B - C)^2}{BC} < \frac{A^2}{BC}. \tag{II.164}$$

Also, by (II.163) we know that $AC$ is bounded from below. Similarly, $AB$ is bounded from below, so (II.164) becomes

$$\frac{dA}{dt} < kA^4,$$

hence

$$A(t) < \left(\lambda_1^{-3} - 3kt\right)^{-1/3}. \tag{II.165}$$

Now we have

$$\frac{d}{dt}(AB) = 2(A + B - C) < \tilde{k}A < \tilde{k}\left(\lambda_1^{-3} - 3kt\right)^{-1/3}. \tag{II.166}$$

Integrating (II.166) gives us

$$AB < \lambda_1\lambda_2 + \frac{\tilde{k}}{2\lambda_1^2} - \frac{\tilde{k}}{2}\left(\lambda_1^{-3} - 3kt\right)^{2/3}.$$

hence

$$\lim_{t\to T_0}(AB) = k_{AB}. \tag{II.167}$$

for some $0 < k_{AB} < \infty$. Combining (II.161) and (II.167) we conclude

$$\lim_{t\to T_0}(AC) = \frac{k_{AB}}{k_{BC}}. \tag{II.168}$$

Thus we see that near $t = T_0$,

$$BC \sim \frac{k_{AB}^2}{k_{BC}A^2},$$

hence (II.136) tells us

$$\frac{dA}{dt} \sim \frac{k_{BC}}{k_{AB}^2}A^4. \tag{II.169}$$

Solving as we did for equation $(II.16)$ gives

$$A \approx k_1(T_0 - t)^{-1/3}. \tag{II.170}$$

Now using (II.167) and (II.168) we have the end behavior of (II.136) near $t = T_0$:

$$\begin{aligned} A(t) &\approx k_1(T_0 - t)^{-1/3} \\ B(t) &\approx k_2(T_0 - t)^{1/3} \\ C(t) &\approx k_3(T_0 - t)^{1/3} \\ D(t) &= \lambda_4. \end{aligned} \tag{II.171}$$

From (II.135) with $a_1 = a_2 = a_3 = 0$ we have that the non-zero curvatures are

$$\begin{aligned} K(Y_1, Y_2) &= \frac{1}{4ABC}(-3C^2 + 2AC + 2BC + A^2 - 2AB + B^2) \\ K(Y_1, Y_3) &= \frac{1}{4ABC}(-3B^2 + 2AB + 2BC + A^2 - 2AC + C^2) \\ K(Y_2, Y_3) &= \frac{1}{4ABC}(-3A^2 + 2AB + 2BC + B^2 - 2BC + C^2). \end{aligned} \tag{II.172}$$

Thus all the non-zero curvatures have a singularity of the form $(T_0 - t)^{-1}$, and the volume-normalized solution approaches $\mathbb{R}^2$.

## The Non-Bianchi Cases

In this section we examine the compact locally homogeneous geometries whose isotropy group is not trivial, so the dimension of the Lie Group is higher than the dimension of the manifold. Again, these cases are well-understood, but I include them here for completion. The following can be found in [11].

B1. $H^3 \times \mathbb{R}$

Any initial metric can be written as

$$g_0 = R^2 g_{H^3} + du^2 \qquad \text{(II.173)}$$

for some $R > 0$. The solution to backward Ricci flow is given by

$$g(t) = (R^2 - 4t)g_{H^3} + du^2, \quad -\infty < t < \frac{R^2}{4}. \qquad \text{(II.174)}$$

B2. $S^2 \times \mathbb{R}^2$

Any initial metric can be written as

$$g_0 = R^2 g_{S^3} + du_1^2 + du_2^2 \qquad \text{(II.175)}$$

for some $R > 0$. The solution to backward Ricci flow is given by

$$g(t) = (R^2 + 2t)g_{S^2} + du_1^2 + du_2^2, \quad -\frac{R^2}{2} < t < \infty. \qquad \text{(II.176)}$$

B3. $H^2 \times \mathbb{R}^2$

Any initial metric can be written as

$$g_0 = R^2 g_{H^2} + du_1^2 + du_2^2 \qquad \text{(II.177)}$$

for some $R > 0$. The solution to backward Ricci flow is given by

$$g(t) = (R^2 - 2t)g_{H^2} + du_1^2 + du_2^2, \quad -\infty < t < \frac{R^2}{2}. \tag{II.178}$$

## B4. $S^2 \times S^2$

Any initial metric can be written as

$$g_0 = R_1^2 g_{S^2} + R_2^2 g_{S^2} \tag{II.179}$$

for some $R_1 > 0,\ R_2 > 0$. The solution to backward Ricci flow is given by

$$g(t) = (R_1^2 + 2t)g_{S^2} + (R_2^2 + 2t)g_{S^2}, \quad -\min\left\{\frac{R_1^2}{2}, \frac{R_2^2}{2}\right\} < t < \infty. \tag{II.180}$$

## B5. $S^2 \times H^2$

Any initial metric can be written as

$$g_0 = R_1^2 g_{S^2} + R_2^2 g_{H^2} \tag{II.181}$$

for some $R_1 > 0,\ R_2 > 0$. The solution to backward Ricci flow is given by

$$g(t) = (R_1^2 + 2t)g_{S^2} + (R_2^2 - 2t)g_{H^2}, \quad -\frac{R_1^2}{2} < t < \frac{R_2^2}{2}. \tag{II.182}$$

B6. $H^2 \times H^2$

Any initial metric can be written as

$$g_0 = R_1^2 g_{H^2} + R_2^2 g_{H^2} \tag{II.183}$$

for some $R_1, R_2 > 0,\ R_1 \neq R_2$. The solution to backward Ricci flow is given by

$$g(t) = (R_1^2 - 2t)g_{H^2} + (R_2^2 - 2t)g_{H^2}, \qquad -\infty < t < \min\left\{\frac{R_1^2}{2}, \frac{R_2^2}{2}\right\}. \tag{II.184}$$

B7. $\mathbb{C}P^2$

Any initial metric can be written as

$$g_0 = R^2 g_{FS} \tag{II.185}$$

for some $R > 0$, where $g_{FS}$ is the Fubini-Study metric on complex projective space, $\mathbb{C}P^2$, with constant holomorphic bisectional curvature 1. The solution to backward Ricci flow is given by

$$g(t) = (R^2 + 6t)g_{FS}, \qquad -\frac{R^2}{6} < t < \infty. \tag{II.186}$$

B8. $\mathbb{C}H^2$

Any initial metric can be written as

$$g_0 = R^2 g_{\mathbb{C}H^2} \tag{II.187}$$

for some $R > 0$, where $g_{\mathbb{C}H^2}$ is the Kähler metric on complex hyperbolic space, $\mathbb{C}H^2$, with constant holomorphic bisectional curvature $-1$. The solution to backward Ricci flow is given by

$$g(t) = (R^2 - 6t)g_{\mathbb{C}H^2}, \qquad -\infty < t < \frac{R^2}{6}. \tag{II.188}$$

## B9. $S^4$

Any initial metric can be written as

$$g_0 = R^2 g_{S^4}, \tag{II.189}$$

for some $R > 0$. The solution to backward Ricci flow is given by

$$g(t) = (R^2 + 6t)g_{S^4}, \qquad -\frac{R^2}{6} < t < \infty. \tag{II.190}$$

## B10. $H^4$

Any initial metric can be written as

$$g_0 = R^2 g_{H^4} \tag{II.191}$$

for some $R > 0$. The solution to backward Ricci flow is given by

$$g(t) = (R^2 - 4t)g_{H^4}, \qquad -\infty < t < \frac{R^2}{6}. \tag{II.192}$$

<u>Conclusions</u>

Our conclusion is that the end behavior of locally homogeneous manifolds under backward Ricci flow is fairly consistent. In general however, the Bianchi classes have very different, and more interesting, behavior than the non-Bianchi classes. Recall that all of our solutions are Riemannian metrics of the form

$$g(t) = A(t)\theta_1^2 + B(t)\theta_2^2 + C(t)\theta_3^2 + D(t)\theta_4^2$$

where we call $A, B, C$ and $D$ the metric coefficients. Each class of manifolds studied in this chapter has one of the following types of behavior near the end:

Expanding-1: All metric coefficients grow linearly for all time. The volume-normalize metric approaches an Einstein metric with positive Ricci curvature.

Expanding-2: One metric coefficient approaches a constant while the other three grow linearly for all time. The volume-normalized metric approaches the span of the three two-forms whose coefficients grow linearly.

Expanding-3: Two metric coefficients approach constants while the other two grow linearly for all time. The volume-normalized metric approaches the span of the two two-forms whose coefficients grow linearly.

Trivial: Each metric coefficient approaches a constant. Thus, the metric approaches an Einstein metric with constant 0. Among the classes considered, this only happens for trivial metrics which are just quotients of Euclidean Space.

Pancake-1: One metric coefficient approaches 0 linearly while the other three approach constants. The volume-normalized solution will approach the sub-riemannian geometry $\mathbb{R}^3$.

Pancake-2: One metric coefficient approaches 0 at a rate of $(T_0 - t)^{1/3}$, one approaches a constant, and the last two approach infinity at a rate of $(T_0 - t)^{-1/3}$. The volume-normalized solution will also approach $\mathbb{R}^3$.

Pancake-3: One metric coefficient approaches a constant, two approach 0 on the order of $(T_0 - t)^{1/3}$, and the final approaches infinity on the order of $(T_0 - t)^{-1/3}$. Here the volume-normalized solution will approach the plane $\mathbb{R}^2$.

Pancake-4: One metric coefficient approaches a constant, one approaches infinity on the order of $(T_0 - t)^{-1/3}$, and the last two approach 0 on the order of $(T_0 - t)^{1/3}$ and $(T_0 - t)^{2/3}$ respectively. The volume-normalized solution will approach the plane $\mathbb{R}^2$. Nearby initial conditions exhibit Pancake-3 behavior.

Pancake-5: Two metric coefficients approach 0 linearly while the other two approach constants. The volume-normalized solution will approach the plane $\mathbb{R}^2$.

Line: One metric coefficient approaches a constant while the others approach 0 linearly. The volume-normalized solution will approach the line $\mathbb{R}$.

Point: All metric coefficients approach 0 linearly. Any compact quotients will collapse to a point at $T_0$. Among the classes of manifolds we have considered, this happens only for the Einstein manifolds $\mathbb{C}H^2$ and $H^4$. Thus the volume-normalized solutions are constant solutions.

Tube-1: One metric coefficient approaches 0 linearly while two others approach 0 at the rate of $(T_0 - t)^{1/3}$. The final metric coefficient is of the form $(T_0 - t)^{-1/3}$. The volume-normalized solution will approach a product metric $g = M^2 \times \mathbb{R}$. Nearby initial conditions will exhibit Pancake-1 and Pancake-3 behavior.

Tube-2: Two metric coefficients approach constants, while the other two have behaviors $(T_0 - t)^{1/3} \to 0$ and $(T_0 - t)^{-1/3}$ respectively. The volume-normalized solution will approach a product metric $g = M^2 \times \mathbb{R}$. Nearby initial conditions exhibit Pancake-2 and Pancake-3 end behavior.

| Class | Lie Group Structure | End Behavior |
|---|---|---|
| A1 | $(\mathbb{R}^4, \mathbb{R}^4, \{0\})$ | Trivial |
| A2 | $(\tilde{Sol}^3 \times \mathbb{R}, \tilde{Sol}^3 \times \mathbb{R}, e)$<br>$(Sol_0^4, Sol_0^4, e)$<br>$(Sol_{m,n}^4, Sol_{m,n}^4, e)$ | Pancake-1 |
| A3 | $(\mathbb{R}^4, E(2) \times \mathbb{R}^2, e)$ | Pancake-1<br>Pancake-3 |
| A4 | $(Nil^3 \times \mathbb{R}, Nil^3 \times \mathbb{R}, e)$ | Pancake-3 |
| A5 | No Compact Geometries | Pancake-3 |
| A6 | $(Nil^4, Nil^4, e)$ | Pancake-3<br>Pancake-4 |
| A7 | $(Sol^4, Sol^4, e)$ | Pancake-1<br>Pancake-2<br>Pancake-3<br>Pancake-4<br>Tube-1<br>Tube-2 |
| A8 | No Compact Geometries | Pancake-2<br>Pancake-3<br>Tube-2 |
| A9 | $(\hat{SL}(2,\mathbb{R}) \times \mathbb{R}, \hat{SL}(2,\mathbb{R}) \times \mathbb{R}, e)$ | Pancake-1<br>Pancake-3<br>Tube-1 |
| A10 | $(S^3 \times \mathbb{R}, SU(2) \times \mathbb{R}, e)$ | Expanding-2<br>Expanding-3<br>Pancake-3 |
| B1 | $(H^3 \times \mathbb{R}, H(3) \times \mathbb{R}, SO(3) \times \{0\})$ | Line |
| B2 | $(S^2 \times \mathbb{R}^2, SO(3) \times \mathbb{R}^2, SO(2) \times \{0\})$ | Expanding-3 |
| B3 | $(H^2 \times \mathbb{R}^2, S0(3) \times \mathbb{R}^2, SO(2) \times \{0\})$ | Pancake-5 |
| B4 | $(S^2 \times S^2, SO(3) \times SO(3), SO(2) \times SO(2))$ | Expanading-1 |
| B5 | $(S^2 \times H^2, S0(3) \times H(2), SO(2) \times SO(2))$ | Pancake-5 |
| B6 | $(H^2 \times H^2, H(2) \times H(2), SO(2) \times SO(2))$ | Pancake-5 |
| B7 | $(\mathbb{C}P^2, SU(3), U(2))$ | Expanding-1 |
| B8 | $(\mathbb{C}H^2, SU(1,2), U(2))$ | Point |
| B9 | $(S^4, SO(5), SO(4))$ | Expanding-1 |
| B10 | $(H^4, H(4), SO(4))$ | Point |

Table 2.1. End Behavior of Backward Ricci Flow

In the non-Bianchi cases, it is clear that the shrinking behavior under backward Ricci flow of negatively curved spaces and the expanding behavior of positively curved spaces is exactly opposite of that seen in forward Ricci flow. Similarly, the behaviors in classes A10 and B1 are reversed.

In the Bianchi cases, we notice that in forward Ricci flow all solutions exist for all time except for a certain case of A9: $\hat{SL}(2,\mathbb{R})\times\mathbb{R}$, where the volume-normalized solution collapses to a plane and exhibits pancake-like behavior. In contrast, the backward Ricci flow produces finite-time singularities in all cases except for special cases of A10: $SU(2)\times\mathbb{R}$, where the solution increases linearly for all time.

It is also worth mentioning that Pancake-1 type behavior in backward Ricci flow occurs as a possibility in exactly the same classes of manifolds which exhibit linear growth in one or more metric coefficient in forward Ricci flow.

In generic cases, end-behavior of solutions to the differential equations in forward Ricci flow are of the forms $kt^{\pm 1/3}$, $k$ or $kt$. Conversely, end-behavior of solutions to the differential equations in backward Ricci flow are of the forms $k(T_0-t)^{\pm 1/3}$, $k$ or $k(T_0-t)$.

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