

CHERN CHARACTER FOR GLOBAL MATRIX FACTORIZATIONS

by

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DISSERTATION ABSTRACT

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Title: Chern Character for Global Matrix Factorizations

We give a formula for the Chern character on the DG category of global matrix factorizations on a smooth scheme X with superpotential $w \in \Gamma(\mathcal{O}_X)$. Our formula takes values in a Čech model for Hochschild homology. Our methods may also be adapted to get an explicit formula for the Chern character for perfect complexes of sheaves on X taking values in right derived global sections of the De-Rham algebra. Along the way we prove that the DG version of the Chern Character coincides with the classical one for perfect complexes.

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CHAPTER I

INTRODUCTION

Shklyarov in [Shk] gives a beautiful interpretation of the Chern Character and Riemann-Roch theorem in the context of DG-categories over a field k . In his treatment, he uses functoriality of Hochschild homology and the canonical functor $k_E : k \rightarrow \mathfrak{C}$, which simply sends the DG-algebra k to the object $E \in \mathfrak{C}$, to get the Chern character of E ,

$$ch(E) = HH(k_E) : k = HH(k) \rightarrow HH(\mathfrak{C}).$$

In the case when \mathfrak{C} is a proper DG-category, i.e. the diagonal bimodule, Δ , takes values in perfect k -modules ($\text{Perf } k$), we use the Kunnetth isomorphism and the isomorphism $HH(\mathfrak{C}^{op}) \cong HH(\mathfrak{C})$ to obtain a pairing on homology:

$$\langle -, - \rangle_{\mathfrak{C}} : HH(\mathfrak{C}) \otimes HH(\mathfrak{C}) \cong HH(\mathfrak{C} \otimes \mathfrak{C}^{op}) \xrightarrow{HH(\Delta)} HH(\text{Perf } k) = k$$

With this pairing and definition of the Chern character, the Riemann-Roch theorem,

$$\langle ch(E), ch(F) \rangle_{\mathfrak{C}} = \text{str Hom}_{\mathfrak{C}}(E, F),$$

then becomes almost tautological, following easily from functoriality.

As with all beautiful things, the hard part is in the application. That is, for a particular DG-category, \mathfrak{C} , the difficulty is to get a meaningful handle on the Chern character and the pairing on Hochschild homology. The DG-categories of interest to us presently are certain categories of (global) matrix factorizations. We also only focus

on the first half of the problem, i.e. to compute the Chern Character, taking values in some reasonable model for Hochschild homology. We, in fact, concern ourselves with a mildly more general problem: to compute the so called *boundary bulk map*. This is a map from the endomorphism DG-algebra of an object to Hochschild homology. Our formula for this map is rather involved, too much so to reproduce here (the impatient reader may thumb to theorem 5.13), however in the case when our matrix factorization, \mathcal{E} , admits a global connection, ∇ , i.e. global connections on graded components $\nabla_i : \mathcal{E}_i \rightarrow \Omega \otimes \mathcal{E}_i$, $i = 0, 1$, we obtain the following formula for the Chern Character:

$$ch(\mathcal{E}) = str \left(\sum_{i=0}^{\dim X} \frac{[\nabla, e]^i}{i!} \right)$$

where str denotes the super-trace, e is the curved differential on \mathcal{E} , and

$$[\nabla, e] = \nabla_{i+1}e_i - 1 \otimes e_i \nabla_i \quad i = 0, 1.$$

We save understanding the pairing for a later work.

This thesis is organized as follows. Chapter II contains the general background information for this work. Specifically we highlight the pertinent information on DG categories, Hochschild homology, the Chern character and boundary bulk map, and our particular version of matrix factorizations (taken from [PV1]). Section III contains results on the homotopy theory of matrix factorizations. Much of the work therein is adapted from [Pos2], some of results therein have not appeared in the generality in which we state them, but for the most part are not new. The promotion of an equivalence between modules over matrix factorizations on one hand and quasi-coherent curved modules (more or less proven in [Dyc]) to a Quillen equivalence appears to be new, however. In chapter IV, we carry out the computation

of Hochschild Homology for our categories of matrix factorizations. The method for this computation is suggested in [PL] and the analogous computation is carried out for Hochschild Cohomology therein. We give the details for homology. This result is also known by other methods from [Pre].

Chapter V forms the heart of the thesis, culminating in the a formula for the boundary-bulk map which takes values in a Čech model for Hochschild homology of matrix factorizations. This formula makes use of a choice of local connections on a Čech cover of the scheme X . In our opinion, more interesting than the formula, is the observation that the boundary-bulk map, which is a map in the derived category of complexes of vector spaces, may be promoted to the derived category of sheaves on our space X . Section 5.1. is concerned with understanding this promotion. Section 5.2. is concerned with what then happens upon applying right-derived global sections.

CHAPTER II

BACKGROUND

2.1. DG Categories

Recall that the category, $C(k)$, of chain complexes over the commutative ring k forms a symmetric monoidal category under tensor product. A Differential Graded (DG) category is a category enriched in $C(k)$. Specifically, this means that a DG category, \mathfrak{A} , consists of the following data:

- A class of objects, $ob(\mathfrak{A})$.
- For each pair of objects $A, B \in ob(\mathfrak{A})$, there is a chain complex $\text{Hom}_{\mathfrak{A}}(A, B)$.
- For each triple of objects $A, B, C \in ob(\mathfrak{A})$, there is a composition morphism of chain complexes

$$\circ_{A,B,C} : \text{Hom}_{\mathfrak{A}}(A, B) \otimes \text{Hom}_{\mathfrak{A}}(B, C) \rightarrow \text{Hom}_{\mathfrak{A}}(A, C)$$

- For each $A \in ob(\mathfrak{A})$ we have a morphism

$$\mu_A : k \rightarrow \text{Hom}_{\mathfrak{A}}(A, A)$$

such that both

$$\text{Hom}_{\mathfrak{A}}(A, B) \cong k \otimes \text{Hom}_{\mathfrak{A}}(A, B) \xrightarrow{\mu_A \otimes id} \text{Hom}_{\mathfrak{A}}(A, A) \otimes \text{Hom}_{\mathfrak{A}}(A, B) \xrightarrow{\circ_{A,A,B}} \text{Hom}_{\mathfrak{A}}(A, B)$$

and

$$\mathrm{Hom}_{\mathfrak{A}}(B, A) \cong \mathrm{Hom}_{\mathfrak{A}}(B, A) \otimes k \xrightarrow{id \otimes \mu_A} \mathrm{Hom}_{\mathfrak{A}}(B, A) \otimes \mathrm{Hom}_{\mathfrak{A}}(A, A) \xrightarrow{\circ_{B,A,A}} \mathrm{Hom}_{\mathfrak{A}}(B, A)$$

are the identity

– Composition is associative, i.e the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathfrak{A}}(A, B) \otimes (\mathrm{Hom}_{\mathfrak{A}}(B, C) \otimes \mathrm{Hom}_{\mathfrak{A}}(C, D)) & \xrightarrow{id \otimes \circ_{B,C,D}} & \mathrm{Hom}_{\mathfrak{A}}(A, B) \otimes \mathrm{Hom}_{\mathfrak{A}}(B, D) \\ \downarrow \alpha & & \searrow \circ_{A,B,D} \\ & & \mathrm{Hom}_{\mathfrak{A}}(A, D) \\ & & \nearrow \circ_{A,C,D} \\ (\mathrm{Hom}_{\mathfrak{A}}(A, B) \otimes \mathrm{Hom}_{\mathfrak{A}}(B, C)) \otimes \mathrm{Hom}_{\mathfrak{A}}(C, D) & \xrightarrow{\circ_{A,B,C} \otimes id} & \mathrm{Hom}_{\mathfrak{A}}(A, C) \otimes \mathrm{Hom}_{\mathfrak{A}}(C, D) \end{array}$$

where α is the associativity isomorphism associated to the monoidal category $C(k)$.

A DG category with one object is simply a differential graded algebra over k . In this way, we think of DG categories as “differential graded algebras with many objects.”

For a DG category, \mathfrak{A} , we may form a new DG category \mathfrak{A}^{op} with

$$\mathrm{Hom}_{\mathfrak{A}^{op}}(A, B) := \mathrm{Hom}_{\mathfrak{A}}(B, A)$$

and the composition $f \circ g$ replaced by $(-1)^{|f||g|} g \circ f$. We will use the notation A^\vee to mean the object $A \in \mathfrak{A}$ viewed as an object of \mathfrak{A}^{op} .

For a pair of DG categories, \mathfrak{A} and \mathfrak{B} , we may form the tensor product $\mathfrak{A} \otimes \mathfrak{B}$ whose objects are formal tensors $A \otimes B$ with $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$ with morphisms

$$\mathrm{Hom}_{\mathfrak{A} \otimes \mathfrak{B}}(A \otimes B, C \otimes D) := \mathrm{Hom}_{\mathfrak{A}}(A, C) \otimes_k \mathrm{Hom}_{\mathfrak{B}}(B, D).$$

A covariant (resp. contravariant) functor of DG categories is a covariant (resp. contravariant) $C(k)$ enriched functor of $C(k)$ enriched categories. Specifically, a covariant functor $F : \mathfrak{A} \rightarrow \mathfrak{B}$ consists of the data:

- For each object $A \in ob(\mathfrak{A})$, we have an object $F(A) \in ob(\mathfrak{B})$.
- For each pair $A, B \in ob(\mathfrak{A})$, we have a morphism of chain complexes

$$F_{A,B} : \text{Hom}_{\mathfrak{A}}(A, B) \rightarrow \text{Hom}_{\mathfrak{B}}(F(A), F(B))$$

which make the following diagram commute:

$$\begin{array}{ccc}
 \text{Hom}_{\mathfrak{A}}(A, B) \otimes \text{Hom}_{\mathfrak{A}}(B, C) & \xrightarrow{\circ_{A,B,C}} & \text{Hom}_{\mathfrak{A}}(A, C) \\
 \downarrow F_{A,B} \otimes F_{B,C} & & \downarrow F_{A,C} \\
 \text{Hom}_{\mathfrak{B}}(F(A), F(B)) \otimes \text{Hom}_{\mathfrak{B}}(F(B), F(C)) & \xrightarrow{\circ_{F(A),F(B),F(C)}} & \text{Hom}_{\mathfrak{B}}(F(A), F(C))
 \end{array}$$

A contravariant DG functor is similar only we have morphisms

$$F_{A,B} : \text{Hom}_{\mathfrak{A}}(A, B) \rightarrow \text{Hom}_{\mathfrak{B}}(F(B), F(A))$$

satisfying an analogous commutativity condition. We will often shorten covariant DG functor to DG functor or even to functor.

Morphisms between DG functors are enriched natural transformations. Specifically a natural transformation $\tau : F \rightarrow G$ between functors $F : \mathfrak{A} \rightarrow \mathfrak{B}$ and $G : \mathfrak{A} \rightarrow \mathfrak{B}$ consists of the data of a closed degree zero morphism of complexes

$$\tau_A : k \rightarrow \text{Hom}_{\mathfrak{B}}(F(A), G(A))$$

for each $A \in \mathfrak{A}$ satisfying the commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathfrak{A}}(A, B) & \xrightarrow{G} & k \otimes \mathrm{Hom}_{\mathfrak{B}}(G(A), G(B)) & \xrightarrow{\tau_A \otimes id} & \mathrm{Hom}_{\mathfrak{B}}(F(A), G(A)) \otimes \mathrm{Hom}_{\mathfrak{B}}(G(A), G(B)) \\
\downarrow F & & & & \downarrow \circ \\
\mathrm{Hom}_{\mathfrak{B}}(F(A), F(B)) \otimes k & \xrightarrow{id \otimes \tau_B} & \mathrm{Hom}_{\mathfrak{B}}(F(A), F(B)) \otimes \mathrm{Hom}_{\mathfrak{B}}(F(B), G(B)) & \xrightarrow{\circ} & \mathrm{Hom}_{\mathfrak{B}}(F(A), G(B))
\end{array}$$

The category of DG-categories, $DG - Cat$, comes equipped with two functors to the category of k -linear categories, $Z^0 : DG - Cat \rightarrow k - Cat$ and $H^0 : DG - Cat \rightarrow k - Cat$, the “Zero cycles” functor and the “Zero Homology functor.” For a DG category \mathfrak{A} , $Z^0\mathfrak{A}$ is the category with the same objects as \mathfrak{A} but whose morphisms are given by

$$\mathrm{Hom}_{Z^0\mathfrak{A}}(A, B) := Z^0\mathrm{Hom}_{\mathfrak{A}}(A, B)$$

where $Z^0\mathrm{Hom}_{\mathfrak{A}}(A, B)$ is the usual zero cycles of the complex $\mathrm{Hom}_{\mathfrak{A}}(A, B)$. For a DG category \mathfrak{A} , $H^0\mathfrak{A}$ is the category with the same objects as \mathfrak{A} but whose morphisms are given by

$$\mathrm{Hom}_{H^0\mathfrak{A}}(A, B) = H^0\mathrm{Hom}_{\mathfrak{A}}(A, B)$$

where $H^0\mathrm{Hom}_{\mathfrak{A}}(A, B)$ is the usual zero homology of the complex $\mathrm{Hom}_{\mathfrak{A}}(A, B)$.

A DG functor $F : \mathfrak{A} \rightarrow \mathfrak{B}$ is a *quasi-equivalence* if the map

$$F_{A,B} : \mathrm{Hom}_{\mathfrak{A}}(A, B) \rightarrow \mathrm{Hom}_{\mathfrak{B}}(F(A), F(B))$$

is a quasi-isomorphism of complexes for all $A, B \in \mathfrak{A}$ and for each $B \in \mathfrak{B}$ there is $A \in \mathfrak{A}$ along with closed degree zero morphisms

$$f \in Z^0\mathrm{Hom}_{\mathfrak{B}}(F(A), B)$$

and

$$g \in Z^0 \text{Hom}_{\mathfrak{B}}(B, F(A))$$

such that fg is homologous to id_B and gf is homologous to $id_{F(A)}$.

A left \mathfrak{A} module is a covariant DG functor $M : \mathfrak{A} \rightarrow C(k)$. A right \mathfrak{A} module is a contravariant DG functor $M : \mathfrak{A} \rightarrow C(k)$. A \mathfrak{A} - \mathfrak{B} bimodule is a left $\mathfrak{A} \otimes \mathfrak{B}^{op}$ module or equivalently a right $\mathfrak{A}^{op} \otimes \mathfrak{B}$ module. When dealing with specific bimodules we will always make the first variable the contravariant one and the second variable covariant. We will denote the category of right \mathfrak{A} modules $\mathfrak{A} - Mod$ and by “module”, we will always mean right module.

Given a \mathfrak{B} - \mathfrak{A} bimodule, F , and a \mathfrak{C} - \mathfrak{A} bimodule, G , we may form an \mathfrak{C} - \mathfrak{B} bimodule,

$$\text{Hom}_{\mathfrak{A} - Mod}(F, G)$$

via the end:

$$\begin{aligned} \text{Hom}_{\mathfrak{A} - Mod}(F, G)(B, C) &= \int_{A \in \mathfrak{A}} \text{Hom}_{C(k)}(F(A, B), G(A, C)) \\ &= \ker \left(\prod_{A, A' \in \mathfrak{A}} \text{Hom}_{C(k)}(F(A, B), G(A', C)) \otimes \text{Hom}_{\mathfrak{A}}(A, A') \xrightarrow{\nu} \prod_{A \in \mathfrak{A}} \text{Hom}_{C(k)}(F(A, B), G(A, C)) \right) \end{aligned}$$

where $\nu(f \otimes g) = f \circ F(g) - G(g) \circ f$. In particular taking $\mathfrak{B} = k$ and $\mathfrak{C} = k$, we obtain a k - k bimodule (i.e. a complex) of homs between F and G making $\mathfrak{A} - Mod$ into a DG category. Alternatively, one may describe the complex

$$\text{Hom}_{\mathfrak{A} - Mod}(F, G)$$

as the collection of graded natural transformations from F to G viewed as graded objects with the differential given by

$$\partial(\tau) = d_G\tau - (-1)^{|\tau|}d_F.$$

Observe for this definition, that d_F (resp. d_G) is a degree 1 natural transformation of F (resp. G) viewed as a graded functor.

Given a \mathfrak{A} - \mathfrak{B} bimodule F and a \mathfrak{B} - \mathfrak{C} bimodule G we may form the \mathfrak{A} - \mathfrak{C} bimodule as the tensor product over \mathfrak{B} . The tensor product is defined by the coend

$$\begin{aligned} F \otimes_{\mathfrak{B}} G(A, C) &= \int^{B \in \mathfrak{B}} F(A, B) \otimes G(B, C) \\ &= \text{coker} \left(\bigoplus_{B, B' \in \mathfrak{B}} F(A, B) \otimes \text{Hom}_{\mathfrak{B}}(B, B') \otimes G(B', C) \xrightarrow{\nu} \bigoplus_{B \in \mathfrak{B}} F(A, B) \otimes G(B, C) \right) \end{aligned}$$

where, in this case $\nu(x \otimes f \otimes y) = F(f)(x) \otimes y - x \otimes G(f)(y)$. In the case when $\mathfrak{A} = k$ (resp. $\mathfrak{C} = k$) we obtain a k - \mathfrak{C} (resp. \mathfrak{A} - k) bimodule, i.e. we obtain a right \mathfrak{C} module (resp. left \mathfrak{A} module). In particular when $\mathfrak{A} = \mathfrak{C} = k$ we obtain a k - k bimodule, i.e. a complex.

Given a \mathfrak{A} - \mathfrak{B} bimodule, F , a \mathfrak{B} - \mathfrak{C} bimodule G and a \mathfrak{D} - \mathfrak{C} bimodule H we have the following isomorphism of \mathfrak{D} - \mathfrak{A} bimodules:

$$\begin{aligned}
\mathrm{Hom}_{\mathfrak{C}\text{-mod}}(F \otimes_{\mathfrak{B}} G, H) &= \int_{C \in \mathfrak{C}} \mathrm{Hom}_{C(k)}(F \otimes_{\mathfrak{B}} G(C, -), H(C, -)) \\
&= \int_{C \in \mathfrak{C}} \mathrm{Hom}_{C(k)} \left(\int_{B \in \mathfrak{B}} F(B, -) \otimes G(C, B), H(C, -) \right) \\
&= \int_{C \in \mathfrak{C}} \int_{B \in \mathfrak{B}} \mathrm{Hom}_{C(k)}(F(B, -) \otimes G(C, B), H(C, -)) \\
&= \int_{C \in \mathfrak{C}} \int_{B \in \mathfrak{B}} \mathrm{Hom}_{C(k)}(F(B, -), \mathrm{Hom}_{C(k)}(G(C, B), H(C, -))) \\
&= \int_{B \in \mathfrak{B}} \int_{C \in \mathfrak{C}} \mathrm{Hom}_{C(k)}(F(B, -), \mathrm{Hom}_{C(k)}(G(C, B), H(C, -))) \\
&= \int_{B \in \mathfrak{B}} \mathrm{Hom}_{C(k)} \left(F(B, -), \int_{C \in \mathfrak{C}} \mathrm{Hom}_{C(k)}(G(C, B), H(C, -)) \right) \\
&= \int_{B \in \mathfrak{B}} \mathrm{Hom}_{C(k)}(F(B, -), \mathrm{Hom}_{\mathfrak{C}\text{-Mod}}(G(-, B), H)) \\
&= \mathrm{Hom}_{\mathfrak{B}\text{-Mod}}(F, \mathrm{Hom}_{\mathfrak{C}\text{-Mod}}(G, H)).
\end{aligned}$$

In particular when $\mathfrak{A} = \mathfrak{D} = k$ and taking zero cycles we get the standard adjunction between tensor and hom.

The category $Z^0\mathfrak{A} - \mathrm{Mod}$ admits arbitrary limits and colimits computed object-wise, i.e for a diagram, \mathcal{F} , in $\mathfrak{A} - \mathrm{Mod}$ and $A \in \mathfrak{A}$ we have

$$(\lim_{F \in \mathcal{F}} F)(A) := \lim_{F \in \mathcal{F}} F(A)$$

and

$$(\operatorname{colim}_{F \in \mathcal{F}} F)(A) := \operatorname{colim}_{F \in \mathcal{F}} F(A).$$

For $F \in \mathfrak{A} - Mod$, we may also form the shift $F[1]$ defined object-wise:

$$F[1](A) := F(A)[1].$$

For a morphism of modules $\tau : F \rightarrow G$ we may form the DG cone, $Cone(\tau)$ which is given object-wise by taking the usual cone of the morphism $\tau_A : F(A) \rightarrow G(A)$ in $C(k)$ for each $A \in \mathfrak{A}$. Similarly we may form the DG cylinder of a morphism, $Cyl(\tau)$, object-wise by taking the usual cylinder of $\tau_A : F(A) \rightarrow G(A)$ in $C(k)$ for each $A \in \mathfrak{A}$.

The category $Z^0\mathfrak{A} - Mod$ and admits a closed, $C(k)$ enriched model structure¹ where the weak equivalences are object-wise quasi-isomorphisms, i.e. a weak equivalence between two modules F and G is a DG natural transformation $\tau : F \rightarrow G$, such that the induced map

$$\tau_A : F(A) \rightarrow G(A)$$

is a quasi-isomorphism for all $A \in \mathfrak{A}$, fibrations are object-wise surjections, i.e.

$$\tau : F \rightarrow G$$

is a fibration provided

$$\tau_A : F(A) \rightarrow G(A)$$

¹We recommend [Hov] for background on model categories.

is surjective for all A (see [Kel2] Theorem 3.2). We remark that under this model structure every module is fibrant. The derived category of the DG category \mathfrak{A} , $D(\mathfrak{A})$, is the localization of $Z^0\mathfrak{A} - Mod$ with respect to these weak equivalences. One may check that $Cyl(id_F)$ for a cylinder object (in the model category sense) for the module F , as such one may check that the left homotopy relation (in the model category sense) is simply given by $\tau \sim \sigma$ for $\tau, \sigma : F \rightarrow G$, if there is $\epsilon : F[1] \rightarrow G$ such that

$$\partial\epsilon_A = \tau_A - \sigma_A$$

for all $A \in \mathfrak{A}$. We may then deduce from [Hov] Theorem 1.2.10 that

$$\mathrm{Hom}_{D(\mathfrak{A})}(F, G) = H^0\mathrm{Hom}_{\mathfrak{A} - Mod}(Q(F), G) = \mathrm{Hom}_{H^0\mathfrak{A} - Mod}(Q(F), G) \quad (\text{Equation 2.1.})$$

where $Q(F)$ is a cofibrant replacement for F .

From the universal property of localizations, the projection $Z^0\mathfrak{A} - Mod \rightarrow D(\mathfrak{A})$ factors as

$$\begin{array}{ccc} \mathfrak{A} - Mod & & \\ \downarrow & \searrow & \\ H^0(\mathfrak{A} - Mod) & \longrightarrow & D(\mathfrak{A}) \end{array}$$

If we apply the cofibrant replacement functor followed by the projection

$$Z^0\mathfrak{A} - Mod \rightarrow H^0\mathfrak{A} - Mod,$$

we get a functor

$$Z^0\mathfrak{A} - Mod \xrightarrow{Q} Z^0\mathfrak{A} - Mod \rightarrow H^0\mathfrak{A} - Mod$$

which sends weak equivalences to isomorphisms. Hence we obtain a map

$$D(\mathfrak{A}) \rightarrow H^0\mathfrak{A} - Mod.$$

which sends F to $Q(F)$. We will by abuse call this functor Q . From (Equation 2.1.) we deduce that Q is a fully faithful left adjoint to the projection $H^0\mathfrak{A} - Mod \rightarrow D(\mathfrak{A})$. The category $H^0(\mathfrak{A} - Mod)$ admits the structure of a triangulated category using the shift functor and cone from $\mathfrak{A} - Mod$. The (essential image) of $D(\mathfrak{A})$ forms a full subcategory of $H^0(\mathfrak{A} - Mod)$, stable under shifts and cones and therefore is a triangulated category in its own right. One sees immediately that $D(\mathfrak{A})$ admits arbitrary direct sums.

We may form the derived tensor product of a \mathfrak{A} - \mathfrak{B} bimodule with a \mathfrak{B} - \mathfrak{C} bimodule G by

$$F \underset{\mathfrak{B}}{\overset{\mathbb{L}}{\otimes}} G := Q(F) \underset{\mathfrak{B}}{\overset{\mathbb{L}}{\otimes}} G \cong F \underset{\mathfrak{B}}{\overset{\mathbb{L}}{\otimes}} Q(G)$$

where $Q(F)$ (resp. $Q(G)$) is a cofibrant replacement for F (resp. G) in $\mathfrak{A}^{op} \otimes \mathfrak{B} - Mod$ (resp. $\mathfrak{B}^{op} \otimes \mathfrak{C} - Mod$).

We define $\text{Perf}D(\mathfrak{A})$ to be the smallest idempotent complete triangulated containing the (images of) the representable modules in $D(\mathfrak{A})$. By [Kel2] Corollary 3.7, the elements of $\text{Perf}D(\mathfrak{A})$ are precisely the compact objects of $D(\mathfrak{A})$, i.e. those objects, F , for which the natural map

$$\bigoplus_i \text{Hom}_{D(\mathfrak{A})}(F, G_i) \rightarrow \text{Hom}_{D(\mathfrak{A})}(F, \bigoplus_i G_i)$$

is an isomorphism. We define the category $\text{Perf}\mathfrak{A}$ to be the full DG subcategory of $\mathfrak{A} - \text{Mod}$ generated by objects which are isomorphic to an object of $\text{Perf}D(\mathfrak{A})$ in $D(\mathfrak{A})$.

Definition 2.1. A DG functor $F : \mathfrak{A} \rightarrow \mathfrak{B}$ is a Morita equivalence if the induced map

$$\mathfrak{A} - \text{Mod} \rightarrow \mathfrak{B} - \text{Mod}$$

is a quasi-equivalence of DG categories. \mathfrak{A} and \mathfrak{B} are Morita equivalent if there exists a zig-zag of Morita equivalences connected \mathfrak{A} to \mathfrak{B} .

Definition 2.2. There is a distinguished object $\Delta_{\mathfrak{A}}$ in the category of $\mathfrak{A}^{op} \otimes \mathfrak{A}$ modules given by

$$A^{\vee} \otimes B \mapsto \text{Hom}_{\mathfrak{A}}(B, A).$$

We call \mathfrak{A} *smooth* if $\Delta_{\mathfrak{A}}$ has finite projective dimension in the abelian category $Z^0(\mathfrak{A}^{op} \otimes \mathfrak{A} - \text{Mod})$

Definition 2.3. A DG category \mathfrak{A} is called *proper* if the by module $\Delta_{\mathfrak{A}}$ (see 2.2) factors through $\text{Perf}k$, i.e. the complex

$$\text{Hom}_{\mathfrak{A}}(A, B)$$

has finitely generated homology.

Definition 2.4. A DG category, \mathfrak{A} , is *saturated* if it is Morita equivalent to a smooth proper DG algebra (i.e. DG category with one object).

If \mathfrak{A} and \mathfrak{B} are saturated DG categories, then every functor $\text{Perf}\mathfrak{A} \rightarrow \text{Perf}\mathfrak{B}$ is weakly equivalent to a kernel in $\text{Perf}(\mathfrak{A}^{op} \otimes \mathfrak{B})$, the correspondence given by

$$F \mapsto \text{Hom}_{\mathfrak{B}}(-, F(-)),$$

see [TV].

2.2. Hochschild Homology and Categorical Hirzebruch-Riemann-Roch

As in 2.2, we have a distinguished object $\Delta_{\mathfrak{A}}$ in the category of $\mathfrak{A}^{op} \otimes \mathfrak{A}$ modules given by

$$A^{\vee} \otimes B \mapsto \text{Hom}_{\mathfrak{A}}(B, A).$$

Left derived tensor product with $\Delta_{\mathfrak{A}}$ then gives a functor

$$Tr : \mathfrak{A} \otimes \mathfrak{A}^{op} - Mod \rightarrow C(k)$$

given by

$$Tr(M) = M \underset{\mathfrak{A} \otimes \mathfrak{A}^{op}}{\overset{\mathbb{L}}{\otimes}} \Delta_{\mathfrak{A}}.$$

Definition 2.5. We define the *Hochschild homology* of the DG category \mathfrak{A} , denoted $HH(\mathfrak{A})$ by

$$HH(\mathfrak{A}) := Tr(\Delta_{\mathfrak{A}^{op}})$$

Remark 2.6. In the case when \mathfrak{A} has one object, we may set $A = \text{Hom}_{\mathfrak{A}}(*, *)$. Then the category of $\mathfrak{A} - \mathfrak{A}$ bimodules, is naturally equivalent to the category of $A - A$ bimodules and this equivalence realizes the bimodule A as the diagonal bimodule.

Then one sees that

$$HH(\mathfrak{A}) = A \underset{A \otimes A^{op}}{\overset{\mathbb{L}}{\otimes}} A$$

which is the usual Hochschild homology of the DGA A .

There is a canonical complex for computing $HH(\mathfrak{A})$ obtained as follows. We have the *Bar Resolution* of $\Delta_{\mathfrak{A}}$ in the abelian category of $\mathfrak{A} - \mathfrak{A}$ bimodules with closed degree 0 morphisms:

$$\begin{array}{c}
\vdots \\
\downarrow \\
\bigoplus_{A_0, A_1, A_2} \text{Hom}_{\mathfrak{A}}(-, A_0) \otimes \text{Hom}_{\mathfrak{A}}(A_0, A_1) \otimes \text{Hom}_{\mathfrak{A}}(A_1, A_2) \otimes \text{Hom}_{\mathfrak{A}}(A_2, -) \\
\downarrow \\
\bigoplus_{A_0, A_1} \text{Hom}_{\mathfrak{A}}(-, A_0) \otimes \text{Hom}_{\mathfrak{A}}(A_0, A_1) \otimes \text{Hom}_{\mathfrak{A}}(A_1, -) \\
\downarrow \\
\bigoplus_{A_0} \text{Hom}_{\mathfrak{A}}(-, A_0) \otimes \text{Hom}_{\mathfrak{A}}(A_0, -) \\
\downarrow \\
\text{Hom}_{\mathfrak{A}}(-, -)
\end{array}$$

(Equation 2.2.)

Where the morphism

$$b : \bigoplus_{A_0, \dots, A_n} \text{Hom}_{\mathfrak{A}}(-, A_0) \otimes \dots \otimes \text{Hom}_{\mathfrak{A}}(A_n, -) \rightarrow \bigoplus_{A_0, \dots, A_{n-1}} \text{Hom}_{\mathfrak{A}}(-, A_0) \otimes \dots \otimes \text{Hom}_{\mathfrak{A}}(A_{n-1}, -)$$

is given by the formula

$$b(f_0 \otimes f_1 \otimes \cdots \otimes f_n) = \sum_{i=0}^{n-1} (-1)^i f_0 \otimes f_1 \otimes \cdots \otimes f_i \circ f_{i+1} \otimes \cdots \otimes f_n.$$

We will denote the complex Equation 2.2. by $Bar_{\Delta_{\mathfrak{A}}}$. By passing to the direct sum total complex $Tot^{\oplus}(Bar_{\Delta_{\mathfrak{A}}})$ we obtain a by module along with a morphism

$$Tot^{\oplus}(Bar_{\Delta_{\mathfrak{A}}}) \rightarrow \Delta$$

Lemma 2.7. $Tot^{\oplus}(Bar_{\Delta_{\mathfrak{A}}}) \rightarrow \Delta$ is a cofibrant replacement for $\Delta_{\mathfrak{A}}$ in the model category of $\mathfrak{A} - \mathfrak{A}$ bimodules.

Proof. First one can verify that $Tot^{\oplus}(Bar_{\Delta_{\mathfrak{A}}}) \rightarrow \Delta$ is an object-wise quasi-isomorphism by explicitly constructing a homotopy analogous to that of the standard bar complex. Then we have a natural isomorphism

$$\begin{aligned} & \text{Hom}_{\mathfrak{A}}(-, A_0) \otimes \text{Hom}_{\mathfrak{A}}(A_0, A_1) \otimes \cdots \otimes \text{Hom}_{\mathfrak{A}}(A_n, -) \\ & \cong \text{Hom}_{\mathfrak{A}}(-, A_0) \otimes \text{Hom}_{\mathfrak{A}}(A_n, -) \otimes \text{Hom}_{\mathfrak{A}}(A_0, A_1) \otimes \cdots \otimes \text{Hom}_{\mathfrak{A}}(A_{n-1}, A_n) \\ & = \text{Hom}_{\mathfrak{A} \otimes \mathfrak{A}^{op}}(-, A_0 \otimes A_n^{\vee}) \otimes \text{Hom}_{\mathfrak{A}}(A_0, A_1) \otimes \cdots \otimes \text{Hom}_{\mathfrak{A}}(A_{n-1}, A_n) \end{aligned}$$

If C is any fixed complex of k vector spaces then one easily checks that

$$\text{Hom}_{\mathfrak{A} \otimes \mathfrak{A}^{op}}(-, A_0 \otimes A_n^{\vee}) \otimes C$$

is cofibrant using the adjunction between tensor and Hom and the enriched Yoneda lemma. This implies

$$\mathrm{Hom}_{\mathfrak{A}}(-, A_0) \otimes \mathrm{Hom}_{\mathfrak{A}}(A_0, A_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathfrak{A}}(A_n, -)$$

is cofibrant and therefore

$$\bigoplus_{A_0, \dots, A_n} \mathrm{Hom}_{\mathfrak{A}}(-, A_0) \otimes \mathrm{Hom}_{\mathfrak{A}}(A_0, A_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathfrak{A}}(A_n, -)$$

is cofibrant as well since cofibrant objects are closed under direct sum. The proof will be finished once we prove the claim that if

$$\cdots \rightarrow T^{-n} \rightarrow T^{-n+1} \rightarrow \cdots \rightarrow T^0$$

is a complex of cofibrant bimodules then $Tot^{\oplus}(T^{\bullet})$ is cofibrant as well. To prove the claim, observe that

$$Tot^{\oplus}(T^{\bullet}) = \mathrm{colim}_n Tot^{\oplus}(\sigma^{-n}T^{\bullet})$$

where $\sigma^{-n}T^{\bullet}$ is the stupid truncation

$$\sigma^{-n}T^{\bullet} = (T^{-n} \rightarrow T^{-n+1} \rightarrow \cdots \rightarrow T^0).$$

Since cofibrant objects are closed under directed colimits, it will suffice to show that $Tot(\sigma^{-n}T^{\bullet})$ is cofibrant for each n . This follows from the observation that

$$Tot(\sigma^{-n-1}T^{\bullet}) = Cone(T^{-n-1}[n] \rightarrow Tot(\sigma^{-n}T^{\bullet}))$$

and that the subcategory of cofibrant bimodules is pretriangulated, and therefore closed under taking (DG) cones. \square

The bar resolution of $\Delta_{\mathfrak{A}}$ gives us an explicit complex for computing $HH(\mathfrak{A})$. One may check that upon tensoring $Bar_{\Delta_{\mathfrak{A}}}$ with $\Delta_{\mathfrak{A}}$ we obtain the bi-complex

$$\begin{array}{c}
\vdots \\
\downarrow \\
\bigoplus_{A_0, A_1, A_2, A_3} \text{Hom}_{\mathfrak{A}}(A_0, A_1) \otimes \text{Hom}_{\mathfrak{A}}(A_1, A_2) \otimes \text{Hom}_{\mathfrak{A}}(A_2, A_3) \otimes \text{Hom}_{\mathfrak{A}}(A_3, A_0) \\
\downarrow \\
\bigoplus_{A_0, A_1, A_2} \text{Hom}_{\mathfrak{A}}(A_0, A_1) \otimes \text{Hom}_{\mathfrak{A}}(A_1, A_2) \otimes \text{Hom}_{\mathfrak{A}}(A_2, A_0) \\
\downarrow \\
\bigoplus_{A_0, A_1} \text{Hom}_{\mathfrak{A}}(A_0, A_1) \otimes \text{Hom}_{\mathfrak{A}}(A_1, A_0) \\
\downarrow \\
\bigoplus_{A_0} \text{Hom}_{\mathfrak{A}}(A_0, A_0)
\end{array}$$

(Equation 2.3.)

Where the vertical maps are given by the equation

$$b(f_0 \otimes f_1 \otimes \cdots \otimes f_n) = \sum_{i=0}^{n-1} (-1)^i f_0 \otimes f_1 \otimes \cdots \otimes f_i \circ f_{i+1} \otimes \cdots \otimes f_n + (-1)^n f_n \circ f_0 \otimes f_1 \otimes \cdots \otimes f_{n-1}.$$

The cohomology of the direct sum total complex of the bicomplex (Equation 2.3.) computes the Hochschild homology of the category \mathfrak{A} .

Remark 2.8. Observe again, that when \mathfrak{A} has only one object, and setting $A = \text{Hom}_{\mathfrak{A}}(*, *)$ we see the bicomplex (Equation 2.3.), becomes the bicomplex

$$\cdots \rightarrow A^{\otimes 3} \rightarrow A^{\otimes 2} \rightarrow A$$

where the horizontal maps are given by

$$b(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}.$$

The reader will recognize that the total complex of this bicomplex is the standard one for computing the Hochschild homology of the DG algebra, A .

From (Equation 2.3.), functoriality of Hochschild homology is clear. Applying the isomorphism

$$\mathfrak{A} \otimes \mathfrak{A}^{op} \cong \mathfrak{A}^{op} \otimes \mathfrak{A}$$

obtained by switching factors we get a natural isomorphism

$$HH(\mathfrak{A}) \cong HH(\mathfrak{A}^{op}). \quad (\text{Equation 2.4.})$$

Observing that $\Delta_{\mathfrak{A} \otimes \mathfrak{B}} = \Delta_{\mathfrak{A}} \otimes_k \Delta_{\mathfrak{B}}$ we get the following Künneth isomorphism

$$\begin{aligned} \Delta_{\mathfrak{A} \otimes \mathfrak{B}} \overset{\mathbb{L}}{\otimes}_{\mathfrak{A} \otimes \mathfrak{B} \otimes \mathfrak{A}^{op} \otimes \mathfrak{B}^{op}} \Delta_{\mathfrak{A} \otimes \mathfrak{B}} &= \Delta_{\mathfrak{A}} \otimes_k \Delta_{\mathfrak{B}} \overset{\mathbb{L}}{\otimes}_{\mathfrak{A} \otimes \mathfrak{B} \otimes \mathfrak{A}^{op} \otimes \mathfrak{B}^{op}} \Delta_{\mathfrak{A}} \otimes_k \Delta_{\mathfrak{B}} \\ &= \Delta_{\mathfrak{A}} \overset{\mathbb{L}}{\otimes}_{\mathfrak{A} \otimes \mathfrak{A}^{op}} \Delta_{\mathfrak{A}} \otimes_k \Delta_{\mathfrak{B}} \overset{\mathbb{L}}{\otimes}_{\mathfrak{B} \otimes \mathfrak{B}^{op}} \Delta_{\mathfrak{B}} \end{aligned} \quad (\text{Equation 2.5.})$$

We will also require the following fact proved in [Kel1]

Theorem 2.9. For any DG algebra, A , the inclusion $A \rightarrow \text{Perf}A$ induces an isomorphism $HH(A) \cong HH(\text{Perf}A)$.

we will also require the following result (see [Toë])

Theorem 2.10. Hochschild homology is a Morita invariant. In particular Hochschild homology respects quasi-equivalences.

Now let us recall Shklyerov's categorical Hirzebruch-Riemann-Roch theorem from [Shk].

Definition 2.11. Fix a commutative ring K . A *homology theory* for DG-cat is a covariant tensor functor

$$\mathbf{H} : DG - cat \rightarrow K - mod$$

satisfying the axioms

HT1.) \mathbf{H} respects quasi-equivalences.

HT2.) For any DG algebra A the canonical embedding $A \rightarrow \text{Perf}A$ induces an isomorphism

$$\mathbf{H}(A) \cong \mathbf{H}(\text{Perf}A).$$

HT3.) $\mathbf{H}(k) = K$.

HT4.) There is a functorial isomorphism

$$\vee : \mathbf{H}(\mathfrak{A}) \cong \mathbf{H}(\mathfrak{A}^{op})$$

which equals the identity when $\mathfrak{A} = k$.

Remark 2.12. As pointed out in [Shk] we may replace the target category, $K\text{-mod}$, with other flavors of K -modules. In particular we will be interested in the case when $K = k$ and we take our modules to be differential graded modules over k .

Remark 2.13. (Equation 2.4.), (Equation 2.5.), and Theorem 2.9, Theorem 2.10, along with the well-known fact that $HH(k) = k$, imply that Hochschild homology is one such homology theory when we take $K = k$.

After fixing a homology theory \mathbf{H} we may define the *Chern character* with values in \mathbf{H} . For each DG category \mathfrak{A} we obtain a function

$$Ch_{\mathbf{H}}^{\mathfrak{A}} : ob(\mathfrak{A}) \rightarrow \mathbf{H}(\mathfrak{A})$$

defined by

$$Ch_{\mathbf{H}}^{\mathfrak{A}}(N) = \mathbf{H}(T_N)(1_K),$$

where $T_N : k \rightarrow \mathfrak{A}$ is the functor sending the unique object of k to $N \in \mathfrak{A}$. It is easy to see that for two DG categories \mathfrak{A} and \mathfrak{B} and a DG functor $F : \mathfrak{A} \rightarrow \mathfrak{B}$ we have

$$Ch_{\mathbf{H}}^{\mathfrak{B}} \circ F = \mathbf{H}(F) \circ Ch_{\mathbf{H}}^{\mathfrak{A}},$$

i.e. the chern character is natural.

Now we restrict our attention the *proper* DG categories, i.e. categories \mathfrak{A} that $\text{Hom}_{\mathfrak{A}}(M, N)$ is a perfect DG k -module for all $M, N \in \mathfrak{A}$. In this case the DG functor

$$\Delta_{\mathfrak{A}^{op}} : \mathfrak{A} \otimes \mathfrak{A}^{op} \rightarrow C(k), \quad N \otimes M^{\vee} \mapsto \text{Hom}_{\mathfrak{A}}(M, N)$$

factors through the inclusion $\text{Perf}k \hookrightarrow C(k)$. Then by **HT2** and **HT3** we obtain a K linear morphism

$$\mathbf{H}(\Delta_{\mathfrak{A}}) : \mathbf{H}(\mathfrak{A} \otimes \mathfrak{A}^{op}) \rightarrow K \quad (\text{Equation 2.6.})$$

By assumption \mathbf{H} is a tensor functor so there is a natural the Künneth isomorphism

$$\mathbf{H}(\mathfrak{A} \otimes \mathfrak{B}) \cong \mathbf{H}(\mathfrak{A}) \otimes \mathbf{H}(\mathfrak{B}).$$

Using the inverse to this isomorphism along with **HT 4** and the morphism (Equation 2.6.), we obtain a K -bilinear pairing

$$\langle \ , \ \rangle_{\mathfrak{A}} : \mathbf{H}(\mathfrak{A}) \otimes_K \mathbf{H}(\mathfrak{A}) \cong \mathbf{H}(\mathfrak{A}) \otimes_K \mathbf{H}(\mathfrak{A}^{op}) \cong \mathbf{H}(\mathfrak{A} \otimes \mathfrak{A}^{op}) \rightarrow K.$$

Theorem 2.14 (Categorical HRR). For any proper DG category \mathfrak{A} and two objects $N, M \in \mathfrak{A}$ we have

$$Ch_{\mathbf{H}}^{\text{Perf}k}(\text{Hom}_{\mathfrak{A}}(N, M)) = \langle Ch_{\mathbf{H}}^{\mathfrak{A}}(M), Ch_{\mathbf{H}}^{\mathfrak{A}}(N)^{\vee} \rangle_{\mathfrak{A}}.$$

Proof. We begin by noting that the functoriality of ${}^{\vee}$ and the requirement that

$$\mathbf{H}(k) \cong \mathbf{H}(k^{op})$$

is the identity imply that

$$\mathbf{H}(T_N)(1_K)^{\vee} = \mathbf{H}(T_{N^{\vee}})(1_K),$$

where M^\vee denotes the object M viewed as an element of \mathfrak{A}^{op} . Then we have

$$\begin{aligned}
\langle Ch_{\mathbf{H}}^{\mathfrak{A}}(M), Ch_{\mathbf{H}}^{\mathfrak{A}}(N)^\vee \rangle_{\mathfrak{A}} &= \mathbf{H}(\Delta_{\mathfrak{A}^{op}})(\mathbf{H}(T_M)(1_K) \otimes \mathbf{H}(T_N)(1_K)^\vee) \\
&= \mathbf{H}(\Delta_{\mathfrak{A}^{op}})(\mathbf{H}(T_M)(1_K) \otimes \mathbf{H}(T_{N^\vee})(1_K)) \\
&= \mathbf{H}(\Delta_{\mathfrak{A}^{op}})(\mathbf{H}(T_{M \otimes N^\vee})(1_K)) \\
&= \mathbf{H}(\Delta_{\mathfrak{A}^{op}} \circ T_{M \otimes N^\vee})(1_K) \\
&= \mathbf{H}(T_{\text{Hom}_{\mathfrak{A}}(N, M)})(1_K) \\
&= Ch_{\mathbf{H}}^{\text{Perfk}}(\text{Hom}_{\mathfrak{A}}(N, M))
\end{aligned}$$

□

Applying this when $\mathbf{H} = HH$ (see remark 2.13) and abbreviating $Ch_{HH}^{\mathfrak{A}}$ simply to Ch we have the following Hirzebruch-Riemann-Roch formula:

$$\langle Ch(M), Ch(N)^\vee \rangle = Ch^{\text{Perfk}}(\text{Hom}_{\mathfrak{A}}(N, M)) \quad (\text{Equation 2.7.})$$

Theorems 3.1 and 3.2 from [Shk] combine to give for any DG algebra, A , the Chern character Ch^A descends to a group homomorphism from the Grothendieck group of $H^0(\text{Perf}A)$ to $HH(A)$. This implies that Ch^{Perfk} is in fact none other than the euler characteristic on the category Perfk . So we may rewrite the Hirzebruch-Riemann-Roch formula as

$$\langle Ch(M), Ch(N)^\vee \rangle = \chi(\text{Hom}_{\mathfrak{A}}(N, M)) \quad (\text{Equation 2.8.})$$

2.3. Boundary-Bulk

If we restrict our attention to saturated (recall definition 2.4) DG categories we have a second, “coordinate free”, description of the the functoriality of Hochschild homology. This description allows us to define the so called *Boundary-Bulk* map and prove a generalized Hirzebruch-Riemann-Roch theorem. Throughout this section we assume that all DG categories are saturated and proper (recall 2.3)

Given a functor $F : \mathfrak{A} \rightarrow \mathfrak{B}$ we obtain a $\mathfrak{A} - \mathfrak{B}$ bimodule

$$\hat{F} : A^\vee \otimes B \mapsto \text{Hom}_{\mathfrak{B}}(B, F(A))$$

One may check that for a pair of functors $F : \mathfrak{A} \rightarrow \mathfrak{B}$ and $G : \mathfrak{B} \rightarrow \mathfrak{C}$ we have the following equality of $\mathfrak{A} - \mathfrak{C}$ bimodules

$$\text{Hom}_{\mathfrak{C}}(-, G \circ F(-)) = \text{Hom}_{\mathfrak{B}}(-, F(-)) \otimes_{\mathfrak{B}} \text{Hom}_{\mathfrak{C}}(-, G(-)).$$

Lemma 2.15. Let F be an $\mathfrak{A} - \mathfrak{B}$ bimodule and G be an $\mathfrak{B} - \mathfrak{A}$ bimodule, then we have a canonical isomorphism

$$\text{Tr}_{\mathfrak{A}}(F \otimes_{\mathfrak{B}} G) \cong \text{Tr}_{\mathfrak{B}}(G \otimes_{\mathfrak{A}} F)$$

Proof. We consider the object $F \otimes_k G$ in $\mathfrak{A} \otimes \mathfrak{B}^{op} \otimes \mathfrak{B} \otimes \mathfrak{A}^{op} - Mod$. Commuting tensor factors, we have a canonical isomorphism

$$\text{Tr}_{\mathfrak{A}}(\text{Tr}_{\mathfrak{B}}(F \otimes_k G)) \cong \text{Tr}_{\mathfrak{B}}(\text{Tr}_{\mathfrak{A}}(F \otimes_k G)).$$

Then one verifies that

$$Tr_{\mathfrak{B}}(F \otimes_k G) \cong F \otimes_{\mathfrak{B}} G$$

and

$$Tr_{\mathfrak{A}}(F \otimes_k G) \cong G \otimes_{\mathfrak{A}} F.$$

□

Let $F : \text{Perf}\mathfrak{A} \rightarrow \text{Perf}\mathfrak{B}$ be a DG functor. For each $B \in \mathfrak{B}$ we obtain a functor

$$\mathfrak{A}^{op} \rightarrow \text{Perf}k : A \mapsto \text{Hom}_{\mathfrak{B}}(F(A), B).$$

By the saturated assumption on \mathfrak{A} this functor is represented by $G(B) \in \text{Perf}\mathfrak{A}$. This gives a functor $G : \text{Perf}\mathfrak{B} \rightarrow \text{Perf}\mathfrak{A}$ which is right adjoint to F . Then using the canonical adjunction maps

$$\epsilon : id \rightarrow G \circ F \quad \text{and} \quad \eta : F \circ G \rightarrow id$$

we obtain morphisms

$$\hat{\epsilon} : \Delta_{\mathfrak{A}} \rightarrow \hat{F} \otimes_{\mathfrak{B}} \hat{G} \quad \text{and} \quad \hat{\eta} : \hat{G} \otimes_{\mathfrak{A}} \hat{F} \rightarrow \Delta_{\mathfrak{B}}.$$

Applying traces and using Lemma 2.15 we get a morphism

$$F_* : Tr_{\mathfrak{A}}(\Delta_{\mathfrak{A}}) \rightarrow Tr_{\mathfrak{A}}(\hat{F} \otimes_{\mathfrak{B}} \hat{G}) \cong Tr_{\mathfrak{B}}(\hat{G} \otimes_{\mathfrak{A}} \hat{F}) \rightarrow Tr_{\mathfrak{B}}(\Delta_{\mathfrak{B}}).$$

The association $F \mapsto F_*$ realizes the functoriality of Hochschild homology without appeal to explicit complexes.

Definition 2.16. For each $E \in \text{Perf}\mathfrak{A}$ we have a canonical morphism in $\text{Perf}\mathfrak{A}^{\text{op}} \otimes \mathfrak{A}$

$$c_E : E^\vee \otimes E \rightarrow \Delta_{\mathfrak{A}}$$

corresponding the morphism of functors

$$\text{Hom}_{\text{Perf}\mathfrak{A}}(E, -) \otimes E \rightarrow id_{\text{Perf}A}.$$

Remark 2.17. In the case when $E = \text{Hom}(-, A)$ for $A \in \mathfrak{A}$, we have

$$E^\vee = \text{Hom}(A, -),$$

and c_E corresponds to the natural map

$$\text{Hom}(A, -) \otimes \text{Hom}(-, A) \rightarrow \text{Hom}(-, -).$$

Definition 2.18. Let $E \in \mathfrak{A}$ we have a map

$$\tau^E : \text{Hom}_{\mathfrak{A}}(E, E) = \text{Tr}_{\mathfrak{A}}(\text{Hom}_{\mathfrak{A}}(E, -) \otimes \text{Hom}_{\mathfrak{A}}(-, E)) \xrightarrow{\text{Tr}(c_E)} \text{Tr}_{\mathfrak{A}}(\Delta_{\mathfrak{A}}) = HH(\mathfrak{A})$$

Lemma 2.19. For $E \in \mathfrak{A}$ we have $Ch(E) = \tau^E(id_E)$.

Proof. Recall that $Ch(E)$ is computed by applying Hochschild homology to the functor

$$T_E : k \rightarrow \mathfrak{A} : * \mapsto E.$$

We may extend T_E , to a functor $T_E : \text{Perf}k \rightarrow \text{Perf}\mathfrak{A}$. Then T_E admits a right adjoint, namely

$$\text{Hom}_{\text{Perf}\mathfrak{A}}(\text{Hom}_{\mathfrak{A}}(-, E), -).$$

Then one computes $(T_E)_*$ as

$$k \rightarrow \text{Hom}_{\mathfrak{A}}(E, E) = \text{Tr}_{\mathfrak{A}}(\text{Hom}_{\text{Perf}\mathfrak{A}}(\text{Hom}_{\mathfrak{A}}(-, E), -) \otimes \text{Hom}_{\mathfrak{A}}(-, E))(-, E) \xrightarrow{\text{Tr}(c_E)} \text{Tr}_{\mathfrak{A}}(\Delta_{\mathfrak{A}})$$

where the first map is $k \mapsto id_E$, which gives the result. \square

2.4. Matrix Factorizations

We work with categories of matrix factorizations as in [PV1] and [Pos1]. Specifically we let X be a noetherian k -scheme, \mathcal{L} a line bundle on X and $w \in \mathcal{L}(X)$ a global section. A *matrix factorization*, denoted

$$\mathcal{E} = \mathcal{E}_0 \begin{array}{c} \xrightarrow{e_0} \\ \xleftarrow{e_1} \end{array} \mathcal{E}_1$$

on X with *potential* $w \in \mathcal{L}(X)$ consists of the data of two vector-bundles \mathcal{E}_0 and \mathcal{E}_1 on X and maps

$$e_1 : \mathcal{E}_1 \rightarrow \mathcal{E}_0 \quad \text{and} \quad e_0 : \mathcal{E}_0 \rightarrow \mathcal{E}_1 \otimes \mathcal{L}$$

such that $e_0 e_1 = id_{\mathcal{E}_1} \otimes w$ and $(e_1 \otimes id_{\mathcal{L}}) e_0 = id_{\mathcal{E}_0} \otimes w$. Twisting by \mathcal{L} and expanding 2-periodically we may view a matrix factorization as a “complex” of sheaves except the differential, e , has e^2 is multiplication by w :

$$\cdots \rightarrow \mathcal{E}_1 \otimes \mathcal{L}^{-1} \rightarrow \mathcal{E}_0 \otimes \mathcal{L}^{-1} \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_0 \rightarrow \mathcal{E}_1 \otimes \mathcal{L} \rightarrow \mathcal{E}_0 \otimes \mathcal{L} \rightarrow \cdots \quad (\text{Equation 2.9.})$$

Here the term $\mathcal{E}_i \otimes \mathcal{L}^k$ lives in degree $2k - i$. Given two matrix factorizations on X with potential w ,

$$\mathcal{E} = \mathcal{E}_0 \begin{array}{c} \xrightarrow{e_0} \\ \xleftarrow{e_1} \end{array} \mathcal{E}_1, \quad \mathcal{D} = \mathcal{D}_0 \begin{array}{c} \xrightarrow{d_0} \\ \xleftarrow{d_1} \end{array} \mathcal{D}_1$$

we may define a complex of morphisms $\mathrm{Hom}(\mathcal{E}, \mathcal{D})$ whose underlying graded components are

$$\mathrm{Hom}^{2k}(\mathcal{E}, \mathcal{F}) := \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_0, \mathcal{F}_0 \otimes \mathcal{L}^k) \oplus \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_1, \mathcal{F}_1 \otimes \mathcal{L}^k)$$

and

$$\mathrm{Hom}^{2k+1}(\mathcal{E}, \mathcal{F}) := \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_1, \mathcal{F}_0 \otimes \mathcal{L}^k) \oplus \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_0, \mathcal{F}_1 \otimes \mathcal{L}^{k+1}).$$

The differential on $\mathrm{Hom}(\mathcal{E}, \mathcal{D})$ is given by $\partial(f) = df - (-1)^{|f|}fe$. One easily verifies that $\partial^2 = 0$ so $\mathrm{Hom}(\mathcal{E}, \mathcal{D})$ is indeed an honest complex, even though \mathcal{E} and \mathcal{D} are not.

The DG-category of matrix factorizations defined above is not “correct” in the global setting. It contains objects which “should be” 0 in the DG-derived category but are not, i.e. there are locally contractible matrix factorizations which are not globally contractible. There are several ways of dealing with this. In [Orl] Orlov defines the derived category of matrix factorizations to be the Verdier quotient of the derived category of matrix factorizations in standard DG sense by the thick subcategory of locally contractible objects. Alternatively one can form a new DG-category $\mathfrak{MF}_{loc}(X, \mathcal{L}, w)$, in which we localize with respect to the spacial variable. The objects of $\mathfrak{MF}_{loc}(X, \mathcal{L}, w)$ are the same as in $\mathfrak{MF}(X, \mathcal{L}, w)$ and morphisms are given by a suitably functorial models (so that composition is well-defined) for the complexes $\mathbb{R}\mathrm{Hom}(\mathcal{E}_i, \mathcal{F}_j \otimes \mathcal{L}^n)$ for $i, j = 0, 1$ and all n , then defining the morphism

complex in the “corrected” category to be

$$\mathrm{Hom}_{\mathfrak{M}\mathfrak{F}_{loc}(X, \mathcal{L}, w)}(\mathcal{E}, \mathcal{F}) := \mathrm{Tot}(\mathbb{R}\mathrm{Hom}(\mathcal{E}, \mathcal{F}))$$

This can be done using a Cech model as in [PL] and [Shi] or by choosing functorial injective resolutions which we explain below.

In what follows we will want to consider a slightly larger class of objects obtained by dropping the restriction that \mathcal{E}_0 and \mathcal{E}_1 be vector bundles and allowing the graded components of \mathcal{E} to arbitrary quasi-coherent sheaves on X . We will refer to such an object as a *curved quasi-coherent \mathcal{O}_X module*. The DG-category of curved modules with Hom complexes defined above will be denoted by $\mathfrak{Qcoh}(X, \mathcal{L}, w)$, the full subcategory of matrix factorizations will be denoted by $\mathfrak{MF}(X, \mathcal{L}, w)$ and the full subcategory formed by considering curved sheaves with coherent graded components will be $\mathfrak{Coh}(X, \mathcal{L}, w)$. We will often drop X and or \mathcal{L} from the notation, when they are clear from context. We will keep w , to distinguish $\mathfrak{Coh}(w)$ (resp. $\mathfrak{Qcoh}(w)$) from the categories $\mathbf{Coh}(X)$ (resp. $\mathbf{Qcoh}(X)$) of ordinary (quasi-)coherent sheaves on X .

In [Pos2] Positselski defines the the notion of a *curved differential graded ring* (CDG-ring) as a graded ring $B = \bigoplus B_i$ along with a degree 1 endomorphism d and an element $w \in B_2$ such that $\delta^2 = [w, -]$. A B -module is a graded (left) $B^\#$ -module M endowed with its own differential d_M satisfying the compatibility identity

$$d_M(am) = d(a)m - (-1)^{|a|}ad_M(m).$$

Morphisms between curved modules are $B^\#$ module morphisms and are endowed with a differential in the standard way. As with matrix factorizations this differential produces a complex.

As in [Pos1] we may use a sheafified version of curved modules to describe the category $\mathfrak{Qcoh}(w)$. We define a sheaf of CDG-rings $S(\mathcal{L}) = \bigoplus_{i \in \mathbb{Z}} \mathcal{L}^i$ as the “free algebra” on \mathcal{L} , graded such that \mathcal{L} lives in degree 2 and we endow $S(\mathcal{L})$ with the trivial differential. Then a curved \mathcal{O}_X module with potential $w \in \mathcal{L}(X)$ is a Quasi-coherent CDG $S(\mathcal{L})$ -module, i.e. a curved quasi-coherent module as defined above gives rise to a \mathbb{Z} -graded $S(\mathcal{L})$ -module

$$\dots \xrightarrow{e_1} \mathcal{E}_0 \otimes \mathcal{L}^{-1} \xrightarrow{e_0} \mathcal{E}_1 \rightarrow \mathcal{E}_0 \xrightarrow{e_1} \mathcal{E}_1 \otimes \mathcal{L}^1 \xrightarrow{e_0} \mathcal{E}_0 \otimes \mathcal{L}^2 \xrightarrow{e_1} \dots$$

with differential e such that e^2 is multiplication by w . One can check that the morphisms in $\mathfrak{Qcoh}(w)$ are precisely the morphisms of CDG $S(\mathcal{L})$ -modules, i.e. graded-morphisms on the underlying \mathcal{O}_X modules which commute with the $S(\mathcal{L})$ action.

Conversely given a curved $S(\mathcal{L})$ -module $(\mathcal{M}, d_{\mathcal{M}})$, the natural isomorphisms

$$\mathcal{L}^{-1} \otimes \mathcal{L} \cong \mathcal{O}_X \cong \mathcal{L} \otimes \mathcal{L}^{-1}$$

and the associativity of multiplication imply that any $S(\mathcal{L})$ -module, \mathcal{M} , must have isomorphisms

$$\mathcal{M}_i \otimes \mathcal{L} \cong \mathcal{M}_{i+2}$$

for all i . This gives an equivalence of categories between the $\mathfrak{Qcoh}(w)$ and the category of CDG curved $S(\mathcal{L})$ modules with curvature w . We will use both interpretations of $\mathfrak{Qcoh}(X, \mathcal{L}, w)$ interchangeably. We will continue to use the notation $\mathfrak{Qcoh}(X, \mathcal{L}, w)$ for both.

For curved modules $\mathcal{M} \in \mathfrak{Coh}(X, \mathcal{L}, w)$ and $\mathcal{N} \in \mathfrak{Qcoh}(X, \mathcal{L}, w')$ we may form the curved module $\mathcal{H}om_{S(\mathcal{L})}(\mathcal{M}, \mathcal{N}) \in \mathfrak{Qcoh}(X, \mathcal{L}, w' - w)$ defined by

$$\mathcal{H}om_{S(\mathcal{L})}(\mathcal{M}, \mathcal{N})^\# = \mathcal{H}om_{S(\mathcal{L})^\#}(\mathcal{M}^\#, \mathcal{N}^\#)$$

and whose differential is given by $\partial(f) = d_{\mathcal{M}'}f - (-1)^{|f|}fd_{\mathcal{N}'}$. In particular for

$$\mathcal{M} \in \mathfrak{Coh}(X, \mathcal{L}, w)$$

we have the dual module

$$\mathcal{M}^\vee = \mathcal{H}om_{S(\mathcal{L})}(\mathcal{M}, S(\mathcal{L})) \in \mathfrak{Coh}(X, \mathcal{L}, -w).$$

For $\mathcal{M} \in \mathfrak{Qcoh}(X, \mathcal{L}, w)$ and $\mathcal{N} \in \mathfrak{Qcoh}(X, \mathcal{L}, w')$ we may form the tensor product $\mathcal{M} \otimes_{S(\mathcal{L})} \mathcal{N} \in \mathfrak{Qcoh}(X, \mathcal{L}, w + w')$ defined by

$$(\mathcal{M} \otimes_{S(\mathcal{L})} \mathcal{N})^\# = \mathcal{M}^\# \otimes_{S(\mathcal{L})^\#} \mathcal{N}^\#$$

and whose differential is given by $d_{\mathcal{M} \otimes \mathcal{N}} = d_{\mathcal{M}} \otimes 1 + 1 \otimes d_{\mathcal{N}}$. More explicitly for

$$\mathcal{M} = \mathcal{M}_0 \begin{array}{c} \xrightarrow{m_0} \\ \xleftarrow{m_1} \end{array} \mathcal{M}_1, \quad \mathcal{N} = \mathcal{N}_0 \begin{array}{c} \xrightarrow{n_0} \\ \xleftarrow{n_1} \end{array} \mathcal{N}_1$$

we have

$$\mathcal{H}om_{S(\mathcal{L})}(\mathcal{M}, \mathcal{N}) =$$

$$\mathcal{H}om(\mathcal{M}_0, \mathcal{N}_0) \oplus \mathcal{H}om(\mathcal{M}_1, \mathcal{N}_1) \begin{array}{c} \xrightarrow{\begin{pmatrix} -n_{0*} & m_{0*} \\ m_{1*} & -n_{1*} \end{pmatrix}} \\ \xleftarrow{\begin{pmatrix} n_{1*} & m_{1*} \\ m_{0*} & n_{0*} \end{pmatrix}} \end{array} \mathcal{H}om(\mathcal{M}_0, \mathcal{N}_1) \oplus \mathcal{H}om(\mathcal{M}_1, \mathcal{N}_0)$$

and

$$\mathcal{M} \otimes_{S(\mathcal{L})} \mathcal{N} =$$

$$\mathcal{M}_0 \otimes \mathcal{N}_0 \oplus \mathcal{M}_1 \otimes \mathcal{N}_1 \begin{array}{c} \xrightarrow{\begin{pmatrix} n_0 \otimes 1 & 1 \otimes m_0 \\ 1 \otimes m_1 & n_1 \otimes 1 \end{pmatrix}} \\ \xleftarrow{\begin{pmatrix} n_1 \otimes 1 & 1 \otimes m_1 \\ 1 \otimes m_0 & n_0 \otimes 1 \end{pmatrix}} \end{array} \mathcal{M}_0 \otimes \mathcal{N}_1 \oplus \mathcal{M}_1 \otimes \mathcal{N}_0$$

The following facts about the $\mathcal{H}om_{S(\mathcal{L})}$ and $\otimes_{S(\mathcal{L})}$ functors are easily verified by sheafifying the natural isomorphisms that arise when X is affine.

Proposition 2.20. Let $\mathcal{M} \in \mathfrak{Coh}(X, \mathcal{L}, w)$, $\mathcal{N} \in \mathfrak{Coh}(X, \mathcal{L}, v)$, $\mathcal{P} \in \mathfrak{Qcoh}(X, \mathcal{L}, u)$ and $\mathcal{E} \in \mathfrak{MF}(X, \mathcal{L}, t)$ and $\mathcal{D} \in \mathfrak{MF}(X, \mathcal{L}, s)$ then

1. $\mathcal{H}om_{S(\mathcal{L})}(\mathcal{M} \otimes_{S(\mathcal{L})} \mathcal{N}, \mathcal{P}) \cong \mathcal{H}om_{S(\mathcal{L})}(\mathcal{M}, \mathcal{H}om_{S(\mathcal{L})}(\mathcal{N}, \mathcal{P}))$ naturally as objects of $Z^0\mathfrak{Qcoh}(X, \mathcal{L}, u - v - w)$.
2. $\mathcal{H}om_{S(\mathcal{L})}(\mathcal{M} \otimes_{S(\mathcal{L})} \mathcal{E}, \mathcal{N}) \cong \mathcal{H}om_{S(\mathcal{L})}(\mathcal{M}, \mathcal{E}^\vee \otimes_{S(\mathcal{L})} \mathcal{N}) \cong \mathcal{H}om_{\mathfrak{S}(\mathcal{L})}(\mathcal{M}, \mathcal{N}) \otimes_{S(\mathcal{L})} \mathcal{E}^\vee$ naturally as objects of $Z^0\mathfrak{Qcoh}(X, \mathcal{L}, v - w - t)$.
3. $\mathcal{E}^\vee \otimes_{S(\mathcal{L})} \mathcal{D} \cong \mathcal{H}om_{S(\mathcal{L})}(\mathcal{E}, \mathcal{D})$ as objects of $Z^0\mathfrak{Qcoh}(X, \mathcal{L}, s - t)$.

4. $(\mathcal{E}^\vee)^\vee \cong \mathcal{E}$ naturally in $Z^0\mathfrak{MF}(X, \mathcal{L}, t)$ and the functor

$$\vee : \mathfrak{MF}(X, \mathcal{L}, t)^{op} \rightarrow \mathfrak{MF}(X, \mathcal{L}, -t)$$

is an equivalence of DG-categories.

If

$$\dots \rightarrow \mathcal{M}^{-1} \rightarrow \mathcal{M}^0 \rightarrow \mathcal{M}^1 \rightarrow \dots$$

is a complex of curved modules (where the curved modules are viewed in the abelian category $Z^0\mathfrak{Qcoh}(w)$) we may form the direct sum total curved module $\text{Tot}(\mathcal{M}^\bullet)$ whose graded components are $\text{Tot}(\mathcal{M}^\bullet)^n = \bigoplus_{p+q=n} \mathcal{M}_q^p$ and whose curved differential is given by the formula analogous to forming the total complex for complexes of sheaves.

The functor $\# : Z^0\mathfrak{Qcoh}(w) \rightarrow S(\mathcal{L})^\# - \text{Mod}_0$, where $S(\mathcal{L})^\# - \text{Mod}_0$ denotes the category of $S(\mathcal{L})^\#$ modules with degree 0 morphisms, admits left and right adjoints $+$ and $-$ defined by

$$\mathcal{M}^+ = \mathcal{M}_0 \oplus \mathcal{M}_1 \begin{array}{c} \xrightarrow{\begin{pmatrix} 0 & 1 \\ w & 0 \end{pmatrix}} \\ \xleftarrow{\begin{pmatrix} 0 & w \\ 1 & 0 \end{pmatrix}} \end{array} \mathcal{M}_1 \otimes \mathcal{L}^{-1} \oplus \mathcal{M}_0$$

and

$$\mathcal{M}^- = \mathcal{M}_0 \oplus \mathcal{M}_1 \otimes \mathcal{L} \begin{array}{c} \xrightarrow{\begin{pmatrix} 0 & w \\ 1 & 0 \end{pmatrix}} \\ \xleftarrow{\begin{pmatrix} 0 & 1 \\ w & 0 \end{pmatrix}} \end{array} \mathcal{M}_1 \oplus \mathcal{M}_0$$

Evidently the functors $+$ and $-$ are exact.

Remark 2.21. One easily sees that the functors $+$ and $-$ always produces contractible modules.

We may use these adjoints to construct right and left resolutions in the abelian $Z^0\mathfrak{Qcoh}(w)$, by first resolving as graded sheaves of $S(\mathcal{L})^\#$ -modules and then applying either $+$ or $-$ appropriately. Specifically when

$$(\mathcal{F}^\bullet)^\# \rightarrow \mathcal{E}^\#$$

is a resolution of \mathcal{E} as a graded $S(\mathcal{L})$ module then

$$((\mathcal{F}^\bullet)^\#)^+ \rightarrow \mathcal{E}$$

resolves \mathcal{E} as a w -curved $S(\mathcal{L})$ modules and similarly when

$$\mathcal{E}^\# \rightarrow (\mathcal{I}^\bullet)^\#$$

resolves \mathcal{E} then

$$\mathcal{E} \rightarrow ((\mathcal{I}^\bullet)^\#)^-$$

is a resolution as w -curved modules. We will be particularly interested in the cases when $(\mathcal{F}^\bullet)^\#$ consists of flat sheaves, vector bundles, or locally free sheaves and when $(\mathcal{I}^\bullet)^\#$ consists of injective sheaves.

The category $\mathfrak{Qcoh}(w)$ is tensored over $C(k)$ by defining $A \cdot \mathcal{M}$ for $A \in C(k)$ and $\mathcal{M} \in \mathfrak{Qcoh}(w)$ to have underlying graded sheaf $(A \cdot \mathcal{M})(U) = A \otimes \mathcal{M}(U)$ with restriction functions $1_A \otimes \rho_U^V : A \otimes \mathcal{M}(U) \rightarrow A \otimes \mathcal{M}(V)$, where ρ_U^V is the restriction

function on \mathcal{M} . The differential on $A \cdot \mathcal{M}$ is given by the formula

$$d_{A \cdot \mathcal{M}} = 1_A \otimes d_M + d_A \otimes j,$$

where j is the grading operator. One easily checks that

$$\mathrm{Hom}_{\mathfrak{Q}\mathrm{coh}(w)}(A \cdot \mathcal{M}, \mathcal{N}) = \mathrm{Hom}_{\mathfrak{C}(k)}(A, \mathrm{Hom}_{\mathfrak{Q}\mathrm{coh}(w)}(\mathcal{M}, \mathcal{N})).$$

$\mathfrak{Q}\mathrm{coh}(w)$ is also cotensored over $\mathfrak{C}(k)$ by defining for a finitely generated complex, A ,

$$\mathcal{M}^A(U) = \mathrm{Hom}(A, \mathcal{M}(U))$$

where here Hom is graded k -module Homs. We give \mathcal{M}^A the differential

$$d_{\mathcal{M}^A}(f) = f d_A - (-1)^{|f|} d_{\mathcal{M}} f.$$

One easily checks that

$$\mathrm{Hom}_{\mathfrak{Q}\mathrm{coh}(w)}(A \cdot \mathcal{M}, \mathcal{N}) = \mathrm{Hom}_{\mathfrak{Q}\mathrm{coh}(w)}(\mathcal{M}, \mathcal{N}^A).$$

For a general complex B we set

$$M^B = \lim_{A \subset B} \mathcal{M}^A,$$

where the limit is taken over all finitely generated subcomplexes A of B . Again we have the following relation

$$\mathrm{Hom}_{\Omega\mathrm{co}\mathfrak{h}(w)}(B \cdot \mathcal{M}, \mathcal{N}) = \mathrm{Hom}_{\Omega\mathrm{co}\mathfrak{h}(w)}(\mathrm{colim}_{A \subset B} A \cdot \mathcal{M}, \mathcal{N}) = \mathrm{Hom}_{\Omega\mathrm{co}\mathfrak{h}(w)}(\mathcal{M}, \mathcal{N}^B),$$

where the first equality follows from the observation that \cdot commutes with colimits.

CHAPTER III

HOMOTOPY THEORY OF MATRIX FACTORIZATIONS

3.1. The Triangulated Structure of Curved Modules

The ideas, definitions and results in this section come entirely from [Pos2]. Positselski works in the affine situation with CDG modules over a CDG algebra, but the arguments carry over without any modification. Basically the point is that what is necessary in *loc. cit.* is that we are working with graded objects whose underlying graded pieces lie in an abelian category with enough injectives.

The category $Z^0\mathfrak{Qcoh}(X, \mathcal{L}, w)$ is an abelian category with arbitrary direct sums. One easily checks the standard constructions for chain complexes carry through without modification to this category. Specifically, if $\mathcal{M} \xrightarrow{f} \mathcal{N}$ is a closed degree 0 morphism of w -curved quasi-coherent $S(\mathcal{L})$ modules then $Cone(f)$ defined in the obvious way is still a w -curved module and if

$$\dots \xrightarrow{d} \mathcal{M}^{-1} \xrightarrow{d} \mathcal{M}^0 \xrightarrow{d} \mathcal{M}^1 \rightarrow \dots$$

is a complex ($d^2 = 0$) of w -curved modules then $\text{Tot}^\oplus(\mathcal{M}^\bullet)$ is in $\mathfrak{Qcoh}(w)$. Then we may imbue $H^0\mathfrak{Qcoh}(X, \mathcal{L}, w)$ with the structure of a triangulated category in the standard way by declaring $\mathcal{M} \xrightarrow{f} \mathcal{N} \rightarrow Cone(f) \rightarrow \mathcal{M}[1]$ to be a distinguished triangle.

Definition 3.1. Let $H^0\mathfrak{Qcoh}(w)_{abs} \subset H^0\mathfrak{Qcoh}(w)$ be smallest thick triangulated subcategory containing the total complexes of short exact sequences. We call objects of $H^0\mathfrak{Qcoh}(w)_{abs}$ absolutely acyclic. Let $H^0\mathfrak{Qcoh}(w)_{co} \subset H^0\mathfrak{Qcoh}(w)$ be the smallest thick subcategory containing $H^0\mathfrak{Qcoh}(w)_{abs}$ and which is closed under arbitrary direct

sums. Objects of $H^0\mathfrak{Qcoh}(w)_{co}$ are called *coacyclic*. An object of $Z^0\mathfrak{Qcoh}(w)$ will be called absolutely acyclic (resp. coacyclic) if its image in $H^0\mathfrak{Qcoh}(w)$ is absolutely acyclic (resp. coacyclic). We may define $H^0\mathfrak{Coh}(w)_{abs}$ analogously.

Theorem 3.2. For each $\mathcal{M} \in \mathfrak{Qcoh}(X, w)$ there is a (closed) morphism $\mathcal{M} \rightarrow K(\mathcal{M})$ such that $K(\mathcal{M})$ is graded-injective and whose cone is coacyclic. The assignment $\mathcal{M} \mapsto K(\mathcal{M})$ is functorial.

Proof. The category of quasi-coherent sheaves on X has an injective generator and therefore has functorial injective resolutions. We may extend these resolutions to the entire category of \mathbb{Z} -graded $S(\mathcal{L})$ modules to obtain functorial resolutions $\mathcal{I}^\bullet(-)^\sharp$ of the graded module \mathcal{M}^\sharp . Lastly we apply the functor $-$ to get a resolution of \mathcal{M} , $\mathcal{I}^\bullet(\mathcal{M})$, whose underlying graded components are injective \mathcal{O}_X modules. For each curved module \mathcal{M} we define $K(\mathcal{M}) = \text{Tot}^\oplus(\mathcal{I}^\bullet(\mathcal{M}))$. Then we have a closed inclusion $\mathcal{M} \xrightarrow{i} K(\mathcal{M})$ and we claim that the cone of this inclusion is coacyclic. Indeed, let $T(n)$ denote the total curved module of the complex of curved modules

$$0 \rightarrow \mathcal{M} \rightarrow \mathcal{I}^1(\mathcal{M}) \rightarrow \cdots \rightarrow \mathcal{I}^n(\mathcal{M}) \rightarrow \mathcal{K}^n \rightarrow 0$$

obtained by applying the n -th canonical truncation to $\mathcal{M} \rightarrow \mathcal{I}^\bullet(\mathcal{M})$. Then we have then $T(n)$ is coacyclic for each n and we may compute $Cone(i) = \text{Tot}^\oplus(\mathcal{M} \rightarrow \mathcal{I}^\bullet(\mathcal{M}))$ as the cokernel

$$0 \rightarrow \oplus_n T(n) \rightarrow \oplus_n T(n) \rightarrow Cone(i) \rightarrow 0.$$

This shows $Cone(i)$ is coacyclic.

Lastly it only remains to note that on a noetherian scheme, X , the direct sum of injective quasi-coherent sheaves is injective, hence $K(\mathcal{M})$ is graded-injective. \square

Corollary 3.3. The composition

$$H^0\mathfrak{Qcoh}(X, w)_{inj} \rightarrow H^0\mathfrak{Qcoh}(X, w) \rightarrow D^{co}\mathfrak{Qcoh}(X, w)$$

is a triangulated equivalence of categories and $\text{Hom}(\mathcal{M}, \mathcal{I})$ is acyclic for all graded-injective \mathcal{I} if and only if \mathcal{M} is coacyclic.

Proof. This will follow immediately from Lemma 1.3 [Pos2] once we show that that $\text{Hom}(\mathcal{M}, \mathcal{I})$ is acyclic whenever \mathcal{M} is coacyclic and \mathcal{I} is graded injective. This argument comes directly from *loc. cit.*. We note that since $\text{Hom}(-, \mathcal{I})$ turns direct sums, shifts and cones into direct products shifts and cones, it will suffice to consider the total complex of the and exact sequence

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow 0.$$

But since \mathcal{I} is graded injective the sequence

$$0 \rightarrow \text{Hom}(\mathcal{C}, \mathcal{I}) \rightarrow \text{Hom}(\mathcal{B}, \mathcal{I}) \rightarrow \text{Hom}(\mathcal{A}, \mathcal{I}) \rightarrow 0$$

is also exact. Then the observation

$$\text{Hom}(\text{Tot}(\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}), \mathcal{I}) = \text{Tot}(\text{Hom}(\mathcal{C}, \mathcal{I}) \rightarrow \text{Hom}(\mathcal{B}, \mathcal{I}) \rightarrow \text{Hom}(\mathcal{A}, \mathcal{I})).$$

proves the result. □

Corollary 3.3 tells us how to compute the homs in the coderived category: we pick some graded-injective replacements \mathcal{I} of \mathcal{M} and \mathcal{J} of \mathcal{N} then

$$\mathrm{Hom}_{D^{co}\Omega\mathrm{coh}(X,\mathcal{L},w)}(\mathcal{M},\mathcal{N}) = H^0\mathrm{Hom}_{\Omega\mathrm{coh}(X,\mathcal{L},w)}(\mathcal{I},\mathcal{J}).$$

Definition 3.4. The category $\mathfrak{M}\mathfrak{F}_{loc}(X,\mathcal{L},w)$ has the same objects as $\mathfrak{M}\mathfrak{F}(X,\mathcal{L},w)$ and has

$$\mathrm{Hom}_{\mathfrak{M}\mathfrak{F}_{loc}(w)}(\mathcal{E},\mathcal{D}) = \mathrm{Hom}_{\mathfrak{M}\mathfrak{F}(w)}(K(\mathcal{E}),K(\mathcal{D}))$$

where K is the functor constructed in Theorem 3.2. Equivalently $\mathfrak{M}\mathfrak{F}_{loc}(X,\mathcal{L},w)$ is the full subcategory of $\Omega\mathrm{coh}(X,\mathcal{L},w)_{inj}$ generated by the images of matrix factorizations under the functor K .

Lemma 3.5. Let \mathcal{I} be a graded-injective curved module. If $\mathrm{Hom}(\mathcal{M},\mathcal{I})$ is acyclic for every graded coherent curved module \mathcal{M} then \mathcal{I} is contractible.

Proof. Let Λ be the set of pairs (\mathcal{J},h) where \mathcal{J} is a curved submodule in \mathcal{I} and h is a contracting homotopy for the inclusion $j : \mathcal{J} \hookrightarrow \mathcal{I}$. Order Λ by $(\mathcal{J},h) \leq (\mathcal{J}',h')$ if $\mathcal{J} \subset \mathcal{J}'$ and $h'|_{\mathcal{J}} = h$. Apply Zorn's lemma to get a maximal (\mathcal{J},h) . Then the result will follow if we can show $\mathcal{J} = \mathcal{I}$.

Suppose instead that $\mathcal{J} \neq \mathcal{I}$. In this case we may find $\mathcal{J} \subsetneq \mathcal{J}' \subset \mathcal{I}$ such that \mathcal{J}'/\mathcal{J} is graded-coherent. Since \mathcal{I} is graded-injective the degree -1 map h may be extended to a degree -1 map $g' : \mathcal{J}' \rightarrow \mathcal{I}$. The map $j' - \partial(g') : \mathcal{J}' \rightarrow \mathcal{I}$ is a closed morphism of curved modules, where $j' : \mathcal{J}' \rightarrow \mathcal{I}$ is the inclusion and ∂ is the differential on the complex $\mathrm{Hom}(\mathcal{J}',\mathcal{I})$. Restricted to \mathcal{J} the map $j' - \partial(g')$ vanishes and therefore descends to a closed map

$$g : \mathcal{J}'/\mathcal{J} \rightarrow \mathcal{I}$$

which has a contracting homotopy c since $\text{Hom}(\mathcal{J}'/\mathcal{J}, \mathcal{I})$ is acyclic. Then $h' = g' + c$ has

$$\partial(h') = \partial(g') + \partial(c) = j'$$

and therefore is a contracting homotopy for j' . But then (\mathcal{J}', h') is strictly larger than our maximal element (\mathcal{J}, h) , which is the requisite contradiction, so we must have $\mathcal{J} = \mathcal{I}$.

□

Corollary 3.6. The category $D^{abs}\mathfrak{Coh}(w)$ forms a set of compact generators in $D^{co}\mathfrak{Qcoh}(w)$.

Proof. Suppose $\text{Hom}_{D^{co}\mathfrak{Qcoh}(w)}(\mathcal{M}, \mathcal{N}) = 0$ for all $\mathcal{M} \in D^{abs}\mathfrak{Coh}(w)$. By corollary 3.3, the proof of 3.5 and Lemma 1.3 of [Pos2] we have

$$H^0\text{Hom}_{\mathfrak{Qcoh}(w)}(\mathcal{M}, K(\mathcal{N})) = H^0\text{Hom}_{\mathfrak{Qcoh}(w)}(K(\mathcal{M}), K(\mathcal{N})) = \text{Hom}_{D^{co}\mathfrak{Qcoh}(w)}(\mathcal{M}, \mathcal{N}).$$

By taking shifts of \mathcal{M} this implies $\text{Hom}_{\mathfrak{Qcoh}(w)}(\mathcal{M}, K(\mathcal{N}))$ is acyclic. So $K(\mathcal{N})$ is contractible by 3.5 and then \mathcal{N} is coacyclic.

Compactness of objects follows from similar considerations. By 3.3 we have an triangulated equivalence

$$H^0\mathfrak{Qcoh}(w)_{inj} = D^{co}\mathfrak{Qcoh}(w). \quad (\text{Equation 3.1.})$$

By Lemma 1.3 of [BN], the localization functor

$$H^0\mathfrak{Qcoh}(w) \rightarrow D^{co}\mathfrak{Qcoh}(w)$$

commutes with direct sums so the equivalence of Equation 3.1. commutes with arbitrary direct sums. \mathcal{I}_λ is a collection of graded injective curved modules and \mathcal{J} is a graded injective sheaf which is weakly equivalent to a graded coherent sheaf \mathcal{C} then we have quasi-isomorphisms of complexes

$$\mathrm{Hom}(\mathcal{J}, \bigoplus_{\lambda} \mathcal{I}_{\lambda}) \cong \mathrm{Hom}(\mathcal{C}, \bigoplus_{\lambda} \mathcal{I}_{\lambda}) \cong \bigoplus_{\lambda} \mathrm{Hom}(\mathcal{C}, \mathcal{I}_{\lambda}) \cong \bigoplus_{\lambda} \mathrm{Hom}(\mathcal{J}, \mathcal{I}_{\lambda}).$$

Note that the middle isomorphism comes from the the fact that graded coherent $S(\mathcal{L})$ modules are compact in $\mathfrak{Qcoh}(w)$. \square

Corollary 3.7. Suppose X is smooth. The image of $\mathfrak{MF}(X, \mathcal{L}, w)$ forms a set of compact generators in $D^{co}\mathfrak{Qcoh}(X, \mathcal{L}, w)$.

Proof. We need only to show that matrix factorizations generate since corollary 3.6 already implies that they are compact. Since X is smooth, for a coherent curved module \mathcal{M} we may form a finite resolution of $\mathcal{M}^\#$ by a complex of sheaves whose graded components are vector bundles. Applying the $+$ functor yields a resolution of \mathcal{M} by matrix factorizations. Then \mathcal{M} is weakly equivalent to the matrix factorization obtained by taking the total curved module of this resolution. \square

The following theorem is announced in [PL], in the case when $\mathcal{L} = \mathcal{O}_X$. We include a few details to the proof.

Theorem 3.8. Assume that w is not a zero divisor, i.e. the map $\mathcal{O}_X \xrightarrow{w} \mathcal{L}$ is injective and that X is smooth, then the category $\mathfrak{MF}_{loc}(X, \mathcal{L}, w)$ has a compact generator.

Proof. We use the global version of Orlov's theorem given as the Main Theorem (2.7) from [Pos1] to get an equivalence

$$\mathfrak{Coh}(X, \mathcal{L}, w)/\mathrm{Coac}(X, \mathcal{L}, w) \cap \mathfrak{Coh}(X, \mathcal{L}, w) \cong D_{Sing}^b(X_0/X)$$

where X_0 is closed subscheme defined by $w = 0$ and $D_{sing}^b(X_0/X)$ is the relative singularity category defined in *loc. cit.* As a piece of notation we will set

$$D^{abs} \mathfrak{Coh}(X, \mathcal{L}, w) := \mathfrak{Coh}(X, \mathcal{L}, w) / \text{Coac}(X, \mathcal{L}, w) \cap \mathfrak{Coh}(X, \mathcal{L}, w).$$

By Rouquier Theorem 7.39 [Rou] the bounded derived category of coherent sheaves on X_0 has a classical generator, \mathcal{G} . This classical generator then descends to a classical generator for the quotient $\mathcal{D}_{Sing}^b(X_0/X)$ and therefore gives a classical generator (which we will also call \mathcal{G}) for the category $D^{abs}(\mathfrak{Coh}(X, \mathcal{L}, w))$. By corollary 3.7, since \mathcal{G} is coherent, there is a weak equivalence between \mathcal{G} and some matrix factorization $\mathcal{E}_{\mathcal{G}}$. By corollary 3.6, $D^{abs} \mathfrak{Coh}(X, \mathcal{L}, w)$ generates $D^{co} \mathfrak{Qcoh}(X, \mathcal{L}, w)$, and therefore $\mathcal{E}_{\mathcal{G}}$ also generates $D^{co} \mathfrak{Qcoh}(X, \mathcal{L}, w)$. Applying the injective replacement functor $K : \mathfrak{Qcoh}(X, \mathcal{L}, w) \rightarrow \mathfrak{Qcoh}(X, \mathcal{L}, w)_{inj}$, and using Corollary 3.3 we get $K(\mathcal{E}_{\mathcal{G}})$ is a compact generator for $\mathfrak{Qcoh}(X, \mathcal{L}, w)_{inj}$. By definition $K(\mathcal{E}_{\mathcal{G}})$ lies in $\mathfrak{MF}_{loc}(X, \mathcal{L}, w)$ therefore is a compact generator for $\mathfrak{MF}_{loc}(X, \mathcal{L}, w)$ as well. \square

Lemma 3.9. Assume X is a smooth k -scheme, with k a perfect field. Let $w \in \mathcal{O}_X$ and let \tilde{w} denote the doubled potential $\tilde{w} = p_1^*(w) - p_2^*(w)$ on $X \times X$. Suppose in addition that the singular locus of w is contained in the zero locus of w . Then, the exterior product induces an equivalence of derived categories

$$D(\mathfrak{MF}_{loc}(X, \mathcal{O}_X, w) \otimes \mathfrak{MF}_{loc}(X, \mathcal{O}_X, w)^{op}) \cong D(\mathfrak{MF}_{loc}(X \times X, \mathcal{O}_{X \times X}, \tilde{w}))$$

and under this isomorphism the diagonal bimodule corresponds to the diagonal curved module $\Delta_* S(\mathcal{O}_X)$.

Proof. The first claim follows from the proof of Corollary 4.8.A in [PP]. The second is the calculation

$$\mathbb{R}\mathrm{Hom}(\mathcal{E} \boxtimes \mathcal{F}^\vee, \Delta_* S(\mathcal{O}_X)) = \mathbb{R}\mathrm{Hom}(\mathcal{E} \otimes \mathcal{F}^\vee, S(\mathcal{O}_X)) = \mathbb{R}\mathrm{Hom}(\mathcal{E}, \mathcal{F}).$$

□

3.2. Model Structure for Curved Modules

Theorem 3.10. The category $Z^0\mathcal{Q}\mathrm{coh}(X, \mathcal{L}, w)$ has the structure of a model category where the weak equivalences are the morphisms whose cone is coacyclic, the cofibrations are injections and the fibrations are surjections whose kernel is injective as a \mathbb{Z} graded module of Quasi-coherent \mathcal{O}_X modules.

Proof. The proof follows from the following series of lemmas. □

Lemma 3.11. The collection of morphisms with coacyclic cone is closed under composition and has the 2 out three property and is closed under retracts.

Proof. Let $\mathcal{E} \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{G}$ be morphisms in $Z^0\mathcal{Q}\mathrm{coh}(w)$. Then we have induced morphisms

$$\mathrm{Cone}(f) \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}} \mathrm{Cone}(gf) \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}} \mathrm{Cone}(g) \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \mathrm{Cone}(f)[1]$$

The homotopy

$$\begin{array}{ccccccc}
Cone(f) & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}} & Cone(gf) & \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}} & Cone(g) & \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} & Cone(f)[1] \\
& & \swarrow & & \swarrow & & \swarrow \\
& & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
& & \swarrow & & \swarrow & & \swarrow \\
Cone(f) & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}} & Cone(gf) & \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}} & Cone(g) & \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} & Cone(f)[1]
\end{array} \quad (\text{Equation 3.2.})$$

provides a contraction of the diagram Equation 3.2., hence this candidate triangle is distinguished. Then since $H^0\mathfrak{Qcoh}(w)_{co}$ is a triangulated subcategory of $H^0\mathfrak{Qcoh}(w)$ each of $Cone(f)$, $Cone(g)$ and $Cone(gf)$ are in $H^0\mathfrak{Qcoh}(w)_{co}$ whenever the other two are.

If f is a retract of g then there are induced maps

$$Cone(f) \rightarrow Cone(g) \rightarrow Cone(f)$$

whose composition is the identity. Then if $Cone(g)$ is in $H^0\mathfrak{Qcoh}(X, w)_{co}$ so is $Cone(f)$ by thickness. \square

Lemma 3.12. The classes of cofibrations and fibrations are closed under retracts.

Proof. For cofibrations one simply checks that a retract of an injection is an injection. Similarly a retract of a surjection is a surjection. A retraction induces a retract in kernels, so the result follows from the fact that injective objects are closed under direct summands. \square

Lemma 3.13. Given a solid arrow diagram of the form

$$\begin{array}{ccc}
\mathcal{A} & \xrightarrow{f} & \mathcal{E} \\
\downarrow i & \nearrow & \downarrow p \\
\mathcal{X} & \xrightarrow{g} & \mathcal{B}
\end{array}$$

with p a surjection with graded injective kernel and i a injection, then the dotted arrow exist making the diagram commute whenever either the cone of i or the cone of p is coacyclic.

Proof. We let \mathcal{C} be the cokernel of i and \mathcal{J} be the kernel of p . We claim first that it suffices to show that $Ext^1(\mathcal{C}, \mathcal{J}) = 0$, where Ext^1 is computed in the abelian category $Z^0\mathbf{Qcoh}(w)$. Indeed the obstruction to extending $f : \mathcal{A} \rightarrow \mathcal{E}$ to a morphism $f' : \mathcal{X} \rightarrow \mathcal{E}$ is $o(f) \in Ext^1(\mathcal{C}, \mathcal{E})$. Since $pf = gi \in Hom(\mathcal{A}, \mathcal{B})$, we find

$$p_*(o(f)) = o(p_*(g)) = 0 \in Ext^1(\mathcal{C}, \mathcal{B})$$

and therefore $o(f)$ comes from $Ext^1(\mathcal{C}, \mathcal{J}) = 0$ and therefore vanishes. Now we have $(pf' - g)i = 0$, thus there exists $h : \mathcal{C} \rightarrow \mathcal{B}$ such that $pf' - g = h\pi$, where $\pi : \mathcal{X} \rightarrow \mathcal{C}$ is the projection. The obstruction to extending h to a morphism $h' : \mathcal{C} \rightarrow \mathcal{E}$ lies in $Ext^1(\mathcal{C}, \mathcal{J})$. Now $f' - h'\pi : \mathcal{X} \rightarrow \mathcal{E}$ is the requisite morphism.

Now we need only show that $Ext^1(\mathcal{C}, \mathcal{J}) = 0$. Suppose that

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{X} \rightarrow \mathcal{C} \rightarrow 0$$

is such an extension. Since \mathcal{J} is graded injective, this extension is split as graded modules and therefore it is given by a closed morphism $\mathcal{C} \rightarrow \mathcal{J}[1]$. Then the result will follow if we can show that $Hom(\mathcal{C}, \mathcal{J}[1])$ is acyclic whenever either \mathcal{C} or \mathcal{J} is

coacyclic, since this will clearly imply that $\mathcal{C} \rightarrow \mathcal{J}[1]$ is homotopic to 0. This is the content of corollary 3.3. \square

Lemma 3.14. The category $Z^0\mathcal{Q}\text{coh}(w)$ with the above proposed model structure has functorial factorizations.

Proof. Let $f : \mathcal{M} \rightarrow \mathcal{N}$ be a morphism. We may factor f as a cofibration followed by a trivial fibration as follows. We choose a functorial inclusion graded modules $\mathcal{M}^\# \rightarrow i(\mathcal{M}^\#)$, where $i(\mathcal{M}^\#)$ is graded injective. Then by adjunction we have an inclusion $\mathcal{M} \rightarrow i(\mathcal{M}^\#)^-$. $i(\mathcal{M}^\#)^-$ is contractible by the construction of the $-$ functor (see remark 2.21). Then

$$\mathcal{M} \rightarrow i(\mathcal{M}^\#)^- \oplus \mathcal{N} \rightarrow \mathcal{N}$$

is the requisite factorization.

For the other factorization take let $\mathcal{M} \xrightarrow{i} \mathcal{E} \xrightarrow{p} \mathcal{N}$ be a functorial factorization with i a cofibration and p a fibration (for example take the one just constructed). Then set $\mathcal{M}_1 = \text{coker}(i)$ and $\mathcal{E}_1 = \text{Cone}(\mathcal{E} \rightarrow K(\mathcal{M}_1))$. We have an inclusion $\mathcal{M} \rightarrow \mathcal{E}_1[-1]$ and may factor the map $\mathcal{M} \rightarrow \mathcal{N}$ as $\mathcal{M} \rightarrow \mathcal{E}_1[-1] \rightarrow \mathcal{E} \rightarrow \mathcal{N}$ the map $\mathcal{E}_1[-1] \rightarrow \mathcal{E}$ is surjective and has kernel $K(\mathcal{M}_1)[-1]$ hence is a fibration, so it only remains to show that the inclusion $\mathcal{M} \rightarrow \mathcal{E}_1[-1]$ is a weak equivalence. To see this we need only look at the following commutative diagram

$$\begin{array}{ccccccc} \mathcal{M} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{N} & \longrightarrow & \mathcal{M}[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{E}_1[-1] & \longrightarrow & \mathcal{E} & \longrightarrow & K(\mathcal{M}_1) & \longrightarrow & \mathcal{E}_1 \end{array}$$

the rows form distinguished triangles and the middle two maps have coacyclic cones so the map $\mathcal{M} \rightarrow \mathcal{E}_1[-1]$ also has a coacyclic cone. \square

3.3. Homotopy Theory of Matrix Factorizations

Our goal in this section is to compare the category of modules over matrix factorizations to the category $\mathfrak{Qcoh}(w)$. Given a curved quasi-coherent $S(\mathcal{L})$ module, \mathcal{M} we may obtain a (right) module over matrix factorizations by simply considering the functor $\mathrm{Hom}_{\mathfrak{Qcoh}(w)}(-, \mathcal{M})$ restricted to $\mathfrak{MF}_{loc}(w)$. We will see that the assignment $\mathcal{M} \rightarrow \mathrm{Hom}_{\mathfrak{Qcoh}(w)}(-, \mathcal{M})$ is the right adjoint in a Quillen pair between $\mathfrak{Qcoh}(w)$ and $\mathfrak{MF}_{loc}(w) - Mod$. We will develop conditions for which this Quillen adjunction is a Quillen equivalence. In particular we will see, using corollary 3.7 that it suffices for X to be smooth.

The proof of the following proposition is identical to the affine version proven in [Pos2].

Lemma 3.15. The model structure on $Z^0\mathfrak{Qcoh}(w)$ gives rise to a $C(k)$ -model structure on $\mathfrak{Qcoh}(w)$.

Proof. We need to check that the map

$$Z^0\mathfrak{Qcoh}(w) \times Z^0C(k) \rightarrow Z^0\mathfrak{Qcoh}(w) : (\mathcal{M}, A) \mapsto A \cdot \mathcal{M}.^1$$

is an quillen bifunctor (see [Hov]). By *loc. cit.* it is sufficient to check that for cofibration $\mathcal{M} \xrightarrow{i} \mathcal{N}$ and a fibration $\mathcal{P} \xrightarrow{p} \mathcal{Q}$, the map of complexes

$$n : \mathrm{Hom}(\mathcal{N}, \mathcal{P}) \rightarrow \mathrm{Hom}(\mathcal{M}, \mathcal{P}) \times_{\mathrm{Hom}(\mathcal{M}, \mathcal{Q})} \mathrm{Hom}(\mathcal{N}, \mathcal{Q})$$

¹Recall that $A \cdot \mathcal{M}$ denotes the action of $A \in C(k)$ on $\mathcal{M} \in \mathfrak{Qcoh}(w)$ from section 2.4.

is a fibration (i.e. surjection) in $Z^0C(k)$ which is a quasi-isomorphism if either $\mathcal{M} \xrightarrow{i} \mathcal{N}$ or $\mathcal{P} \xrightarrow{p} \mathcal{Q}$ is a weak equivalence. We let \mathcal{I} be the graded-injective kernel of $\mathcal{P} \rightarrow \mathcal{Q}$, then we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}(\mathcal{N}, \mathcal{I}) & \longrightarrow & \mathrm{Hom}(\mathcal{N}, \mathcal{P}) & \longrightarrow & \mathrm{Hom}(\mathcal{N}, \mathcal{Q}) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow n & & \downarrow \parallel & & \\
0 & \longrightarrow & \mathrm{Hom}(\mathcal{M}, \mathcal{I}) & \longrightarrow & \mathrm{Hom}(\mathcal{M}, \mathcal{P}) \times_{\mathrm{Hom}(\mathcal{M}, \mathcal{Q})} \mathrm{Hom}(\mathcal{N}, \mathcal{Q}) & \longrightarrow & \mathrm{Hom}(\mathcal{N}, \mathcal{Q}) & \longrightarrow & 0 \\
& & \downarrow \parallel & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(\mathcal{M}, \mathcal{I}) & \longrightarrow & \mathrm{Hom}(\mathcal{M}, \mathcal{P}) & \longrightarrow & \mathrm{Hom}(\mathcal{M}, \mathcal{Q}) & \longrightarrow & 0
\end{array}$$

Note that the rows are exact because

$$0 \rightarrow \mathcal{I}^\# \rightarrow \mathcal{P}^\# \rightarrow \mathcal{Q}^\# \rightarrow 0$$

is split exact in the category of graded \mathcal{O}_X modules. The map

$$\mathrm{Hom}(\mathcal{N}, \mathcal{I}) \rightarrow \mathrm{Hom}(\mathcal{M}, \mathcal{I})$$

is surjective since $\mathcal{M} \rightarrow \mathcal{N}$ is an injection and \mathcal{I} is graded-injective. Then by the 5 lemma n is surjective.

If $p : \mathcal{P} \rightarrow \mathcal{Q}$ is a weak equivalence then \mathcal{I} is coacyclic and hence contractible by corollary 3.3. Therefore $\mathrm{Hom}(\mathcal{M}, \mathcal{I})$ and $\mathrm{Hom}(\mathcal{N}, \mathcal{I})$ are both acyclic. So in this case, or in the case when $\mathcal{M} \rightarrow \mathcal{N}$ is a weak equivalence we have that the map

$$\mathrm{Hom}(\mathcal{M}, \mathcal{I}) \rightarrow \mathrm{Hom}(\mathcal{N}, \mathcal{I})$$

is a quasi-isomorphism, which then implies n , too, is a quasi-isomorphism. \square

There is an obvious functor $\eta : \mathfrak{Qcoh}(w) \rightarrow \mathfrak{MF}_{loc}(w) - Mod$

$$\eta(\mathcal{M}) = \text{Hom}_{\mathfrak{Qcoh}(w)}(-, \mathcal{M}),$$

which admits a left adjoint $|-|$ defined by

$$|M| = \int^{\mathcal{E} \in \mathfrak{MF}_{loc}(w)} M(\mathcal{E}) \cdot \mathcal{E}.$$

It is an easy calculation to see that $|-|$ is indeed left adjoint to η , keeping in mind that

$$\text{Hom}_{\mathfrak{Qcoh}(w)}(M(\mathcal{E}) \cdot \mathcal{E}, \mathcal{D}) = \text{Hom}_{C(k)}(M(\mathcal{E}), \text{Hom}_{\mathfrak{Qcoh}(w)}(\mathcal{E}, \mathcal{D})).$$

Lemma 3.16. $|-| : \mathfrak{MF}_{loc}(w) - Mod \xleftarrow{\eta} \mathfrak{Qcoh}(w) : \eta$ is a Quillen adjunction.

Proof. It will suffice to show that η sends (trivial) fibrations in $\mathfrak{Qcoh}(w)$ to (trivial) fibrations in $\mathfrak{MF}_{loc}(w) - Mod$.

Suppose $\mathcal{M} \xrightarrow{f} \mathcal{N}$ is a fibration in $\mathfrak{Qcoh}(w)$, i.e. f is surjective and its kernel \mathcal{I} is graded-injective. Then the sequence

$$0 \rightarrow \mathcal{I}^\# \rightarrow \mathcal{M}^\# \xrightarrow{f} \mathcal{N}^\# \rightarrow 0$$

is split exact and therefore

$$0 \rightarrow \text{Hom}(\mathcal{E}, \mathcal{I}) \rightarrow \text{Hom}(\mathcal{E}, \mathcal{M}) \rightarrow \text{Hom}(\mathcal{E}, \mathcal{N}) \rightarrow 0$$

is exact for any matrix factorization \mathcal{E} , which shows $\eta(f)$ is an objectwise surjection.

If in addition $Cone(f)$ is coacyclic, then \mathcal{I} is also coacyclic and hence contractible by 3.3. Then $\text{Hom}(\mathcal{E}, \mathcal{I})$ is acyclic for all \mathcal{E} which shows

$$\text{Hom}(\mathcal{E}, \mathcal{M}) \rightarrow \text{Hom}(\mathcal{E}, \mathcal{N})$$

is a surjective quasi-isomorphism. □

Lemma 3.17. The natural map $M \rightarrow \text{Hom}_{\Omega\text{cof}(w)}(-, K(|M|))$ is a weak equivalence for all cofibrant M .

Proof. We let Ω denote the class of objects in $\mathfrak{MF}_{loc}(w)$ for which the lemma holds.

We compute

$$\begin{aligned} \text{Hom}_{\Omega\text{cof}(w)}(|\text{Hom}(-, \mathcal{E})|, \mathcal{M}) &= \text{Hom}_{\mathfrak{MF}_{loc}(w) - Mod}(\text{Hom}(-, \mathcal{E}), \text{Hom}(-, \mathcal{M})) \\ &= \text{Hom}_{\Omega\text{cof}(w)}(\mathcal{E}, \mathcal{M}) \end{aligned}$$

for any $\mathcal{M} \in \mathfrak{MF}(w) - Mod$, so $|\text{Hom}(-, \mathcal{E})| = \mathcal{E}$. The map

$$\text{Hom}(-, \mathcal{E}) \rightarrow \text{Hom}(-, K(\mathcal{E}))$$

is an objectwise quasi isomorphism since \mathcal{E} and $K(\mathcal{E})$ are both graded injective and are weakly equivalent. Then Ω contains all representable modules.

Now we show Ω is closed under direct sums. Indeed take a collection \mathcal{M}^i of objects in Ω . Since the functor $|-|$ is a left adjoint, we have

$$|\bigoplus M^i| = \bigoplus |M^i|.$$

The maps $|M^i| \rightarrow \bigoplus |M^i|$ induce a map $c : \bigoplus K(|M^i|) \rightarrow K(\bigoplus |M^i|)$. We then have a commutative triangle

$$\begin{array}{ccc}
\bigoplus |M^i| & & \\
\bigoplus K \downarrow & \searrow K & \\
\bigoplus K(|M^i|) & \xrightarrow{c} & K(\bigoplus |M^i|)
\end{array}$$

The map K is a weak equivalence by construction, the map $\bigoplus K$ is the direct sum of weak equivalences, hence is a weak equivalence itself. This implies c is a weak equivalence as well. Moreover since both $\bigoplus K(|M^i|)$ and $K(\bigoplus |M^i|)$ are graded-injective, the induced map

$$c_* : \text{Hom}(-, \bigoplus K(|M^i|)) \rightarrow \text{Hom}(-, K(\bigoplus |M^i|))$$

is a weak equivalence of modules. Now since every object of $\mathfrak{MF}_{loc}(w)$ is of the form $K(\mathcal{E})$ for $\mathcal{E} \in \mathfrak{MF}(w)$, and \mathcal{E} is compact in $\mathfrak{Qcoh}(w)$ we have that the map

$$\bigoplus \text{Hom}(K(\mathcal{E}), K(|M^i|)) \rightarrow \text{Hom}(K(\mathcal{E}), \bigoplus K(|M^i|))$$

is a quasi-isomorphism since it fits into a commutative where all three other maps are quasi-isomorphisms:

$$\begin{array}{ccc}
\bigoplus \text{Hom}(\mathcal{E}, K(|M^i|)) & \longrightarrow & \text{Hom}(\mathcal{E}, \bigoplus K(|M^i|)) \\
\downarrow & & \downarrow \\
\bigoplus \text{Hom}(K(\mathcal{E}), K(|M^i|)) & \longrightarrow & \text{Hom}(K(\mathcal{E}), \bigoplus K(|M^i|))
\end{array}$$

The map

$$\bigoplus M^i \rightarrow \text{Hom}(-, K(\bigoplus M^i))$$

factors as the composition of three weak equivalences

$$\bigoplus M^i \rightarrow \bigoplus \mathrm{Hom}(-, K(|M^i|)) \rightarrow \mathrm{Hom}(-, \bigoplus K(|M^i|)) \rightarrow \mathrm{Hom}(-, K(\bigoplus |M^i|))$$

and hence is a weak equivalence.

We observe that Ω is closed under taking cones. This follows from the fact that K commutes with taking cones by construction and that $|-|$ commutes with cones. This second fact follows from the computation

$$\begin{aligned} \mathrm{Hom}(|\mathrm{Cone}(f)|, \mathcal{M}) &= \mathrm{Hom}(\mathrm{Cone}(f), \mathrm{Hom}(-, \mathcal{M})) \\ &= \mathrm{Cone}(\mathrm{Hom}(f, \mathrm{Hom}(-, \mathcal{M}))[-1]) \\ &= \mathrm{Cone}(\mathrm{Hom}(|f|, \mathcal{M}))[-1] \\ &= \mathrm{Hom}(\mathrm{Cone}(|f|), \mathcal{M}). \end{aligned}$$

Now we show that Ω is closed under taking directed colimits. We let

$$M^0 \rightarrow M^1 \rightarrow M^2 \rightarrow \dots$$

be a directed system of objects in Ω . $\mathrm{colim}_i M^i$ is computed via the exact sequence

$$0 \rightarrow \bigoplus M^i \xrightarrow{j} \bigoplus M^i \rightarrow \mathrm{colim}_i M^i \rightarrow 0.$$

The map $\mathrm{Cone}(j) \rightarrow \mathrm{colim}_i M^i$ is a weak equivalence in $Z^0\mathfrak{M}\mathfrak{F}_{loc}(w) - \mathrm{Mod}$. The functor $|-|$, being a left adjoint, commutes with colimits, so we may compute

$|\operatorname{colim}_i M^i| = \operatorname{colim}_i |M^i|$ via the exact sequence

$$0 \rightarrow \bigoplus |M^i| \xrightarrow{|j|} \bigoplus |M^i| \rightarrow |\operatorname{colim}_i M^i| \rightarrow 0.$$

Once again we have a weak equivalence $\operatorname{Cone}(|j|) \rightarrow \operatorname{colim}_i |M^i|$ in $Z^0\mathfrak{Qcoh}(w)$, applying K to this weak equivalence gives a weak equivalence

$$K(\operatorname{Cone}(|j|)) \rightarrow K(|\operatorname{colim}_i M^i|)$$

with contractible cone. And therefore the map

$$\operatorname{Hom}(-, K(\operatorname{Cone}(|j|))) \rightarrow \operatorname{Hom}(-, K(|\operatorname{colim}_i M^i|))$$

is an object-wise quasi-isomorphism. We observe that $\operatorname{Cone}(j) \in \Omega$ since $\bigoplus M^i \in \Omega$ and Ω is closed under cones. Then we have a commutative square

$$\begin{array}{ccc} \operatorname{Cone}(j) & \longrightarrow & \operatorname{Hom}(-, K(|\operatorname{Cone}(j)|)) \\ \downarrow & & \downarrow \\ \operatorname{colim}_i M^i & \longrightarrow & \operatorname{Hom}(-, K(|\operatorname{colim}_i M^i|)) \end{array}$$

The top map and the vertical map are weak equivalences so then the map

$$\operatorname{colim}_i M^i \rightarrow \operatorname{Hom}(-, K(|\operatorname{colim}_i M^i|)).$$

is a weak equivalence.

Now we show Ω closed under push-out by direct sums of generating cofibrations. First recall ([Hov]) that $\mathfrak{M}\mathfrak{F}_{loc}(w) - Mod$ has generating cofibrations given by the set

$$I = \{S^n \otimes \text{Hom}(-, \mathcal{E}) \rightarrow D^n \otimes \text{Hom}(-, \mathcal{E}) \mid \mathcal{E} \in \mathfrak{M}\mathfrak{F}_{loc}(w)\},$$

where S^n is the complex with k in degree n and D^n is the cone on the identity map of S^n . Since

$$S^n \otimes \text{Hom}(-, \mathcal{E}) = \text{Hom}(-, S^n \cdot \mathcal{E})$$

and

$$D^n \otimes \text{Hom}(-, \mathcal{E}) = \text{Hom}(-, D^n \cdot \mathcal{E}),$$

and $S^n \cdot \mathcal{E} = \mathcal{E}[n]$ and $D^n \cdot \mathcal{E} = D^0 \cdot \mathcal{E}[n]$ we have that

$$I = \{\text{Hom}(-, \mathcal{E}) \rightarrow \text{Hom}(-, D^0 \cdot \mathcal{E}) \mid \mathcal{E} \in \mathfrak{M}\mathfrak{F}_{loc}(w)\}.$$

Now suppose M is formed via the pushout

$$\begin{array}{ccc} \bigoplus \text{Hom}(-, \mathcal{E}) & \longrightarrow & M' \\ \downarrow & \lrcorner & \downarrow \\ \bigoplus \text{Hom}(-, D^0 \cdot \mathcal{E}) & \longrightarrow & M \end{array}$$

with $M' \in \Omega$.

Since each map $\text{Hom}(-, \mathcal{E}) \rightarrow \text{Hom}(-, D^0 \cdot \mathcal{E})$ is an inclusion, the pushout is computed via the exact sequence

$$0 \rightarrow \bigoplus \text{Hom}(-, \mathcal{E}) \rightarrow \bigoplus \text{Hom}(-, D^0 \cdot \mathcal{E}) \oplus M' \rightarrow M \rightarrow 0.$$

Moreover, by the argument at the beginning of this proof we have that $|-|$ takes the morphism

$$\mathrm{Hom}(-, \mathcal{E}) \rightarrow \mathrm{Hom}(-, D^0 \cdot \mathcal{E})$$

to the injective map $\mathcal{E} \rightarrow D^0 \cdot \mathcal{E}$, therefore we may compute $|M|$ via the exact sequence

$$0 \rightarrow \bigoplus \mathcal{E} \rightarrow \bigoplus D^0 \cdot \mathcal{E} \oplus |M'| \rightarrow |M| \rightarrow 0.$$

Then a similar argument as the one used to show that Ω is closed under directed colimits gives that $M \in \Omega$.

Now to finish the proof, we observe that the “small object argument” (see [Hov]) gives us a means of constructing a cofibrant replacement, $Q(M)$ for the module M , by taking iterated pushouts along generating cofibrations and directed colimits, so then $Q(M) \in \Omega$. Then if M is cofibrant then since $p : Q(M) \rightarrow M$ is a trivial fibration, the identity map of M admits a lift to $Q(M)$, i.e. we have a retraction $M \rightarrow Q(M) \rightarrow M$. This induces a retraction

$$\begin{array}{ccccc}
 & & \textit{id} & & \\
 & & \curvearrowright & & \\
 M & \xrightarrow{\quad} & Q(M) & \xrightarrow{\quad} & M \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathrm{Hom}(-, K(|M|)) & \longrightarrow & \mathrm{Hom}(-, K(|Q(M)|)) & \longrightarrow & \mathrm{Hom}(-, K(|M|)) \\
 & & \curvearrowleft & & \\
 & & \textit{id} & &
 \end{array}$$

which shows that the map $M \rightarrow \mathrm{Hom}(-, K(|M|))$ is a weak equivalence.

□

Theorem 3.18. The following are equivalent

1. $|-| : \mathfrak{MF}_{loc}(w) - Mod \xleftrightarrow{\eta} \mathfrak{QCoh}(w) : \eta$ is a quillen equivalence.
2. The image of $\mathfrak{MF}(w)$ forms a set of compact generators for $D^{co}\mathfrak{QCoh}(w)$.
3. Every coherent curved module is weakly equivalent to a matrix factorization.

Proof. Suppose that

$$|-| : \mathfrak{MF}_{loc}(w) - Mod \xleftrightarrow{\eta} \mathfrak{QCoh}(w) : \eta$$

is a Quillen equivalence. Then the induced map

$$\mathbb{R}\eta : D^{co}\mathfrak{QCoh}(w) \rightarrow D(\mathfrak{MF}_{loc}(w))$$

is an equivalence of categories which sends (the image of) matrix factorizations to the (image of) representable modules. Since representable modules compactly generate $D(\mathfrak{MF}_{loc}(w))$, we obtain that $\mathfrak{MF}(w)$ compactly generates $D^{co}\mathfrak{QCoh}(w)$.

Now suppose that $\mathfrak{MF}(w)$ forms a set of generators for $\mathfrak{QCoh}(w)$. Since, by corollary 3.6, $D^{abs}\mathfrak{Coh}(w)$ forms a set of compact generators for $D^{co}\mathfrak{QCoh}(w)$, we may apply [BvdB] Theorem 2.1.2 to get that $\mathfrak{MF}(w)$ classically generates $D^{abs}\mathfrak{Coh}(w)$, i.e. every object of $D^{abs}\mathfrak{Coh}(w)$ can be obtained from $\mathfrak{MF}(w)$ by a finite number of shifts, finite sums and cones (in $D^{co}\mathfrak{QCoh}(w)$). Since $\mathfrak{MF}(w)$ is closed under shifts, finite sums and cones in $Z^0\mathfrak{QCoh}(w)$, we obtain that every coherent curved module is weakly equivalent a matrix factorization.

By [Hov] Corollary 1.3.16 and Lemma 3.17, to show that

$$|-| : \mathfrak{MF}_{loc}(w) - Mod \xleftrightarrow{\eta} \mathfrak{QCoh}(w) : \eta$$

is a Quillen equivalence, it suffices to show that η reflects weak equivalences between fibrant objects. Unpacking the definitions, one sees that it is then sufficient to show that if $\text{Hom}(-, \mathcal{I})$ is acyclic for all matrix factorizations then \mathcal{I} is contractible. Assuming that every coherent module is weakly equivalent to a matrix factorization gives us that, for any coherent \mathcal{C} ,

$$\text{Hom}(\mathcal{C}, \mathcal{I}) \cong \text{Hom}(\mathcal{E}, \mathcal{I})$$

for some matrix factorization \mathcal{E} . Thus if \mathcal{I} is right orthogonal to every matrix factorization, \mathcal{I} is also right orthogonal to every coherent curved module and therefore contractible by Corollary 3.6. \square

Using Corollary 3.7 we see immediately

Corollary 3.19. When X is smooth

$$|-| : \mathfrak{MF}_{loc}(w) - Mod \xleftrightarrow{\eta} \mathfrak{QCoh}(w) : \eta$$

is a Quillen equivalence.

This corollary has an immediate consequence:

Corollary 3.20. If X is smooth, there is a triangulated equivalence of categories

$$D(\mathfrak{MF}_{loc}(w)) \cong D^{co}(\mathfrak{QCoh}(w))$$

induced by $\mathcal{M} \mapsto \text{Hom}(-, \mathcal{M})$.

Remark 3.21. Corollary 3.20 is more or less well known to the experts. It follows easily from Theorem 5.1 of [Dyc] and the existence of a compact generator for the category of matrix factorizations (Theorem 3.8).

CHAPTER IV

HOCHSCHILD HOMOLOGY

In this section we compute the Hochschild homology of the category of matrix factorizations in the case when $\mathcal{L} = \mathcal{O}_X$. In fact, from now on all of our results will apply only to the case $\mathcal{L} = \mathcal{O}_X$, we save the more general case for later work. We will also assume now on that X is smooth. We follow very closely the computation of Hochschild cohomology which appears in [PL]. An alternative computation appears in [Pre] and at this point this result is well-known to the experts. We include our computation for completeness and since we will later have use to examine more closely the particular isomorphisms needed to compare the Hochschild homology to a certain complex involving forms on X .

Following [PL], we define the *complete bar complex*, $\widehat{\mathcal{B}ar}$. This complex has graded components $\widehat{\mathcal{B}ar}_{-q} = \mathcal{O}_{\mathfrak{X}^{q+2}}$ for $q \geq 0$, where \mathfrak{X}^k is the completion of

$$X^k = X \times \cdots \times X$$

along the diagonal and $p_{1,q+2} : X \times \cdots \times X \rightarrow X \times X$ projects to the first and last factor in the obvious way. To reduce clutter with our notation, we will hence forth simply write $\mathcal{O}_{\mathfrak{X}^k}$, rather than the push forward onto the first and last factor. The reader hopefully will keep in mind that $\mathcal{O}_{\mathfrak{X}^k}$ is actually viewed as a sheaf on $X \times X$.

The differential,

$$b : \widehat{\mathcal{B}ar}_{-q} \rightarrow \widehat{\mathcal{B}ar}_{-q+1}$$

is given locally by the standard formula for the bar differential:

$$b(a_0 \boxtimes \cdots \boxtimes a_{q+1}) = \sum_{i=0}^q (-1)^i a_0 \boxtimes \cdots \boxtimes a_i a_{i+1} \boxtimes \cdots \boxtimes a_{q+1}.$$

Here (and elsewhere) we use \boxtimes to emphasize that this is an external tensor (i.e) only scalars commute with it as opposed to a tensor over \mathcal{O}_X . We introduce a new “differential” of degree -1, B_w , defined locally by the equation

$$B_w(a_0 \boxtimes \cdots \boxtimes a_{q+1}) = \sum_{i=0}^q (-1)^i a_0 \boxtimes \cdots \boxtimes a_i \boxtimes w \boxtimes a_{i+1} \boxtimes \cdots \boxtimes a_{q+1}.$$

We now define the *curved complete bar complex*, $\widehat{\mathcal{B}ar}_{\tilde{w}}$, as follows. This will be an object of $\mathfrak{Qcoh}(X \times X, \mathcal{O}_{X \times X}, \tilde{w})$, where once again $\tilde{w} = p_1^*(w) - p_2^*(w)$ and $p_i : X \times X \rightarrow X$ are the standard projections. Again this follows [PL].

We put

$$(\widehat{\mathcal{B}ar}_{\tilde{w}})_q = \bigoplus_{p \equiv q \pmod{2}} \widehat{\mathcal{B}ar}_{-p}.$$

The map B_w may be viewed as a map of degree 1 in $\widehat{\mathcal{B}ar}_{\tilde{w}}$ by mapping the factor $(\widehat{\mathcal{B}ar}_{\tilde{w}})_p$ in $(\widehat{\mathcal{B}ar}_{\tilde{w}})_q$ to $(\widehat{\mathcal{B}ar}_{\tilde{w}})_{-(p+1)}$ in $(\widehat{\mathcal{B}ar}_{\tilde{w}})_{q+1}$. We imbue $\widehat{\mathcal{B}ar}_{\tilde{w}}$ with the curved differential $\partial = b + B_w$, then one checks that $B_w^2 = 0$ and then that

$$\partial^2 = bB_w + B_w b = \tilde{w}$$

so $\widehat{\mathcal{B}ar}_{\tilde{w}}$ is indeed a \tilde{w} -curved module.

It is helpful to view $\widehat{\mathcal{B}ar}_{\tilde{w}}$ as the total complex (perhaps modulo some signs) of the following “bi-complex”:

$$\begin{array}{ccccccc}
& & \vdots & & \vdots & & \vdots \\
& & \swarrow B_w & & \swarrow B_w & & \swarrow B_w \\
\dots & \dots & \rightarrow 0 & \rightarrow 0 & \rightarrow 0 & \rightarrow 0 & \dots \\
& & \swarrow B_w & & \swarrow B_w & & \swarrow B_w \\
\dots & \xrightarrow{b} & \mathcal{O}_{\mathfrak{X}^4} & \xrightarrow{b} & \mathcal{O}_{\mathfrak{X}^3} & \xrightarrow{b} & \mathcal{O}_{\mathfrak{X}^2} \\
& & \swarrow B_w & & \swarrow B_w & & \swarrow B_w \\
\dots & \dots & \rightarrow 0 & \rightarrow 0 & \rightarrow 0 & \rightarrow 0 & \dots \\
& & \swarrow B_w & & \swarrow B_w & & \swarrow B_w \\
\dots & \xrightarrow{b} & \mathcal{O}_{\mathfrak{X}^4} & \xrightarrow{b} & \mathcal{O}_{\mathfrak{X}^3} & \xrightarrow{b} & \mathcal{O}_{\mathfrak{X}^2} \\
& & \swarrow B_w & & \swarrow B_w & & \swarrow B_w \\
\dots & \dots & \rightarrow 0 & \rightarrow 0 & \rightarrow 0 & \rightarrow 0 & \dots \\
& & \vdots & & \vdots & & \vdots
\end{array}$$

There is a map $\widehat{\mathcal{B}ar}_{\tilde{w}}(X) \xrightarrow{\epsilon} \Delta_* S(\mathcal{O}_X)$ given by projecting the even components onto $\mathcal{O}_{\mathfrak{X}^2}$ then using the multiplication map

$$\mathcal{O}_{\mathfrak{X}^2} \rightarrow \mathcal{O}_{\Delta}$$

and sending the odd components to 0. It is easy to check that this defines a closed degree 0 morphism of curved modules.

Lemma 4.1. $\widehat{\mathcal{B}ar}_{\tilde{w}} \otimes_{S(\mathcal{O}_{X \times X})} \mathcal{M}$ is isomorphic to $\Delta \otimes_{S(\mathcal{O}_{X \times X})}^{\mathbb{L}} \mathcal{M}$ in $D(X)$ for any $-\tilde{w}$ curved module \mathcal{M} .

Proof. We let \mathcal{W} be the cone of the morphism $\epsilon : \widehat{\mathcal{B}ar}_{\tilde{w}}(X) \rightarrow \Delta_* S(\mathcal{O}_X)$. Then

$$\mathcal{W}_n = \bigoplus_{k \equiv n \pmod{2}} \mathcal{O}_{\mathfrak{X}^k}$$

where we consider $\mathfrak{X}^1 = \Delta$.

Consider first the case when \mathcal{M} is graded-flat. The n -th graded component of the complex $\mathcal{W} \otimes_{S(\mathcal{O}_{X \times X})} \mathcal{M}$ is

$$\begin{aligned} (\mathcal{W} \otimes_{S(\mathcal{O}_{X \times X})} \mathcal{M})_n &= \bigoplus_{k \equiv n \pmod{2}} \mathcal{O}_{\mathfrak{X}^k} \otimes \mathcal{M}_0 \oplus \bigoplus_{k \equiv n+1 \pmod{2}} \mathcal{O}_{\mathfrak{X}^k} \otimes \mathcal{M}_1 \\ &= \bigoplus_k \mathcal{O}_{\mathfrak{X}^k} \otimes \mathcal{M}_{n-k} \end{aligned}$$

Taking the differential into account, may view $\mathcal{W} \otimes_{S(\mathcal{O}_{X \times X})} \mathcal{M}$ as the total complex of the “bi-complex”

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \uparrow & & \uparrow & & \uparrow \\ \dots & \longrightarrow & \mathcal{O}_{\mathfrak{X}^4} \otimes \mathcal{M}_1 & \longrightarrow & \mathcal{O}_{\mathfrak{X}^3} \otimes \mathcal{M}_1 & \longrightarrow & \mathcal{O}_{\mathfrak{X}^2} \otimes \mathcal{M}_1 & \xrightarrow{\epsilon} & \mathcal{O}_{\Delta} \otimes \mathcal{M}_1 \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \dots & \longrightarrow & \mathcal{O}_{\mathfrak{X}^4} \otimes \mathcal{M}_0 & \longrightarrow & \mathcal{O}_{\mathfrak{X}^3} \otimes \mathcal{M}_0 & \longrightarrow & \mathcal{O}_{\mathfrak{X}^2} \otimes \mathcal{M}_0 & \xrightarrow{\epsilon} & \mathcal{O}_{\Delta} \otimes \mathcal{M}_0 \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \dots & \longrightarrow & \mathcal{O}_{\mathfrak{X}^4} \otimes \mathcal{M}_1 & \longrightarrow & \mathcal{O}_{\mathfrak{X}^3} \otimes \mathcal{M}_1 & \longrightarrow & \mathcal{O}_{\mathfrak{X}^2} \otimes \mathcal{M}_1 & \xrightarrow{\epsilon} & \mathcal{O}_{\Delta} \otimes \mathcal{M}_1 \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \dots & \longrightarrow & \mathcal{O}_{\mathfrak{X}^4} \otimes \mathcal{M}_0 & \longrightarrow & \mathcal{O}_{\mathfrak{X}^3} \otimes \mathcal{M}_0 & \longrightarrow & \mathcal{O}_{\mathfrak{X}^2} \otimes \mathcal{M}_0 & \xrightarrow{\epsilon} & \mathcal{O}_{\Delta} \otimes \mathcal{M}_0 \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \dots & \longrightarrow & \mathcal{O}_{\mathfrak{X}^4} \otimes \mathcal{M}_1 & \longrightarrow & \mathcal{O}_{\mathfrak{X}^3} \otimes \mathcal{M}_1 & \longrightarrow & \mathcal{O}_{\mathfrak{X}^2} \otimes \mathcal{M}_1 & \xrightarrow{\epsilon} & \mathcal{O}_{\Delta} \otimes \mathcal{M}_1 \\ & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

where the horizontal maps are induced by b , diagonal maps induced by B_w and the vertical maps by the differential on \mathcal{M} . This “bi-complex” is of course just a

mnemonic, but it gives us insight into how to deal with the complex

$$\mathcal{W} \otimes_{S(\mathcal{O}_{X \times X})} \mathcal{M}.$$

In particular, we may “filter the bi-complex by rows” to get a filtration on

$\mathcal{W} \otimes_{S(\mathcal{O}_{X \times X})} \mathcal{M}$. One should convince oneself that this indeed a filtration by subcomplexes. This filtration is bounded below and exhaustive, therefore the associated spectral sequence converges. Already on the E_1 page all of the groups are 0 since the rows of the “bi-complex” associated to $\mathcal{W} \otimes_{S(\mathcal{O}_{X \times X})} \mathcal{M}$ are exact. This gives us that the map

$$\widehat{\mathcal{B}ar}_{\tilde{w}}(X) \otimes_{S(\mathcal{O}_{X \times X})} \mathcal{M} \rightarrow \Delta_* S(\mathcal{O}_X) \otimes_{S(\mathcal{O}_{X \times X})} \mathcal{M}$$

is a quasi-isomorphism.

Now for general \mathcal{M} , let $\underline{\mathcal{M}} = \text{Tot}(\mathcal{F}^\bullet)$ be a flat replacement of \mathcal{M} , where \mathcal{F}^\bullet is a (finite) resolution of \mathcal{M} by flat $-\tilde{w}$ -curved modules. This can be done as in corollary 3.6; finiteness is possible since X is smooth. It is well known (see for example [Yek]) that $(p_{1,k})_* \mathcal{O}_{\tilde{x}^k}$ is flat as an \mathcal{O}_{X^2} -module and therefore the graded components of $\widehat{\mathcal{B}ar}_{\tilde{w}}$ are flat. This implies the morphism

$$\widehat{\mathcal{B}ar}_{\tilde{w}} \otimes \underline{\mathcal{M}} \rightarrow \widehat{\mathcal{B}ar}_{\tilde{w}} \otimes \mathcal{M}$$

is a quasi-isomorphism.

We have

$$\widehat{\mathcal{B}ar}_{\tilde{w}} \otimes_{S(\mathcal{O}_{X \times X})} \underline{\mathcal{M}} = \widehat{\mathcal{B}ar}_{\tilde{w}} \otimes_{S(\mathcal{O}_{X \times X})} \text{Tot}(\mathcal{F}^\bullet) = \text{Tot}(\widehat{\mathcal{B}ar}_{\tilde{w}} \otimes_{S(\mathcal{O}_{X \times X})} \mathcal{F}^\bullet)$$

The cone of the morphism

$$\mathrm{Tot}(\widehat{\mathcal{B}ar}_{\tilde{w}} \otimes \mathcal{F}^\bullet) \rightarrow \mathrm{Tot}(\Delta_* S(\mathcal{O}_X) \otimes \mathcal{F}^\bullet)$$

is given by $\mathrm{Tot}(\mathcal{W} \otimes \mathcal{F}^\bullet)$, which is the total complex of a bicomplex with exact columns (by the above argument) and uniformly bounded rows and therefore is acyclic.

Therefore we obtain a zig-zag of quasi-isomorphisms

$$\widehat{\mathcal{B}ar}_{\tilde{w}} \otimes_{S(\mathcal{O}_{X \times X})} \mathcal{M} \leftarrow \widehat{\mathcal{B}ar}_{\tilde{w}} \otimes_{S(\mathcal{O}_{X \times X})} \underline{\mathcal{M}} \rightarrow \Delta_*(S(\mathcal{O}_X)) \otimes_{S(\mathcal{O}_{X \times X})} \underline{\mathcal{M}}$$

Since $\Delta_*(S(\mathcal{O}_X)) \otimes_{S(\mathcal{O}_{X \times X})} \underline{\mathcal{M}}$ computes $\Delta_* S(\mathcal{O}_X) \overset{\mathbb{L}}{\otimes}_{S(\mathcal{O}_{X \times X})} \mathcal{M}$, we are done. \square

Lemma 4.2. The map $\widehat{\mathcal{B}ar}_{\tilde{w}} \rightarrow \Delta_* S(\mathcal{O}_X)$ is a weak equivalence in $Z^0 \mathfrak{Qcoh}(X, \mathcal{O}_{X \times X}, \tilde{w})$

Proof. Again we use \mathcal{W} for the cone of the map $\widehat{\mathcal{B}ar}_{\tilde{w}} \rightarrow \Delta_* S(\mathcal{O}_X)$. Let \mathcal{G} be a compact generator for $\mathfrak{Qcoh}(X \times X, \mathcal{O}_{X \times X}, \tilde{w})$ and by 3.6 we can take \mathcal{G} to be a matrix factorization. By the previous lemma

$$\mathcal{G}^\vee \otimes_{S(\mathcal{O}_{X \times X})} \mathcal{W} = \mathcal{H}om_{S(\mathcal{O}_{X \times X})}(\mathcal{G}, \mathcal{W})$$

is acyclic, and since \mathcal{G} is locally free we have a quasi-isomorphism

$$\mathcal{H}om_{S(\mathcal{O}_{X \times X})}(\mathcal{G}, \mathcal{W}) \cong \mathcal{H}om_{S(\mathcal{O}_{X \times X})}(\mathcal{G}, K(\mathcal{W})),$$

where $K : \mathfrak{Qcoh}(X \times X, \mathcal{O}_{X \times X}, \tilde{w}) \rightarrow \mathfrak{Qcoh}(X \times X, \mathcal{O}_{X \times X}, \tilde{w})_{inj}$ is our chosen functorial injective replacement (see Theorem 3.2). By adjunction, the complex of sheaves $\mathcal{H}om_{S(\mathcal{O}_X)}(\mathcal{G}, K(\mathcal{W}))$ has injective graded components. We are want to say

that that having injective graded components is sufficient for $\mathcal{H}om(\mathcal{G}, K(\mathcal{W}))$ to be adapted to the global sections functor. If we could the proof would be done. However, this complex is unbounded in both directions, so care must be taken.

Since X is smooth, and thus has finite homological dimension, each of the cokernels of the differentials are injective. Then by exactness, the kernels of the differentials are also injective. Using these facts one can easily verify directly that the global sections functor is exact by checking at any particular spot and truncating appropriately, so that the truncated sequence is a bounded exact sequence of injective sheaves. Finally we can conclude that the complex of vector spaces

$$\mathrm{Hom}(\mathcal{G}, K(\mathcal{W})) = \Gamma(\mathcal{H}om(\mathcal{G}, K(\mathcal{W})))$$

is exact. Since \mathcal{G} is a generator this implies that $K(\mathcal{W})$ is coacyclic and therefore \mathcal{W} is as well. □

Lemma 4.3. The isomorphism

$$D(\mathfrak{M}\mathfrak{F}_{loc}(X \times X, \tilde{w})) \cong D(\mathfrak{M}\mathfrak{F}_{loc}(x, w) \otimes \mathfrak{M}\mathfrak{F}_{loc}(X, w)^{op})$$

from Lemma 3.9 followed by the trace functor is quasi-isomorphic to the functor

$$\mathbb{R}\Gamma(\mathbb{L}\Delta^* -).$$

Proof. Both Tr and $\mathbb{R}\Gamma(\mathbb{L}\Delta^* -)$ are triangulated functors from

$$D(\mathfrak{M}\mathfrak{F}_{loc}(X \times X, \tilde{w}))$$

to $C(k)$ that commute with arbitrary direct sums, so it will suffice to check that they give the same result at the compact generator of $D(\mathfrak{MF}_{loc}(X \times X, \tilde{w}))$. For this we compute

$$\mathbb{R}\Gamma(\mathbb{L}\Delta^*\mathcal{E} \boxtimes \mathcal{F}^\vee) = \mathbb{R}\Gamma(\mathcal{E} \otimes \mathcal{F}^\vee) = \mathbb{R}\mathrm{Hom}(\mathcal{F}, \mathcal{E}).$$

□

Theorem 4.4. The Hochschild homology of $\mathfrak{MF}(X, \mathcal{O}_X, w)$ is $\mathbb{R}\Gamma(\Omega_{dw})$, where Ω_{dw} is the two periodic complex of sheaves

$$\dots \xrightarrow{dw \wedge} \bigoplus_{i \text{ odd}} \Omega^i \xrightarrow{dw \wedge} \bigoplus_{i \text{ even}} \Omega^i \xrightarrow{dw \wedge} \bigoplus_{i \text{ odd}} \Omega^i \xrightarrow{dw \wedge} \bigoplus_{i \text{ even}} \Omega^i \xrightarrow{dw \wedge} \dots$$

with $\bigoplus_{i \text{ even}} \Omega^i$ in even degrees.

Proof. By Lemmas 4.3 and 3.9 we compute the hochschild homology of $\mathfrak{MF}(X, \mathcal{O}_X, w)$ as $\mathbb{R}\Gamma(\mathbb{L}\Delta^*\Delta_*S(\mathcal{O}_X))$. By Lemma 4.1, we may compute $\mathbb{L}\Delta^*\Delta_*S(\mathcal{O}_X)$ as $\Delta^*\widehat{\mathcal{B}ar}_{\tilde{w}}$.

Now, $\Delta^* \widehat{\mathcal{B}ar}_{\bar{w}}$ is given as the total complex of the “bi-complex”

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \swarrow B_w & & \swarrow B_w & & \swarrow B_w \\
 \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & \searrow B_w & & \searrow B_w & & \searrow B_w & \\
 \dots & \xrightarrow{b} & \Delta^* \mathcal{O}_{\mathfrak{x}^4} & \xrightarrow{b} & \Delta^* \mathcal{O}_{\mathfrak{x}^3} & \xrightarrow{b} & \Delta^* \mathcal{O}_{\mathfrak{x}^2} \\
 & \swarrow B_w & & \swarrow B_w & & \swarrow B_w & \\
 \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & \searrow B_w & & \searrow B_w & & \searrow B_w & \\
 \dots & \xrightarrow{b} & \Delta^* \mathcal{O}_{\mathfrak{x}^4} & \xrightarrow{b} & \Delta^* \mathcal{O}_{\mathfrak{x}^3} & \xrightarrow{b} & \Delta^* \mathcal{O}_{\mathfrak{x}^2} \\
 & \swarrow B_w & & \swarrow B_w & & \swarrow B_w & \\
 \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & \searrow B_w & & \searrow B_w & & \searrow B_w & \\
 \dots & & \Delta^* \mathcal{O}_{\mathfrak{x}^4} & & \Delta^* \mathcal{O}_{\mathfrak{x}^3} & & \Delta^* \mathcal{O}_{\mathfrak{x}^2} \\
 & & \vdots & & \vdots & & \vdots
 \end{array}
 \tag{Equation 4.1.}$$

Applying the Hochschild-Kostant-Rosenburg (HKR) quasi-isomorphism, which is given locally by

$$a_0 \boxtimes \cdots \boxtimes a_q \mapsto \frac{1}{q!} a_0 a_q da_1 \wedge \cdots \wedge da_{q-1}$$

(see [Yek] Theorem 4.8) along the rows we obtain a quasi-isomorphism between (Equation 4.1.) and the bicomplex

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \swarrow \wedge dw & & \swarrow \wedge dw & & \swarrow \wedge dw \\
 \dots & \dots & \rightarrow 0 & \dots & \rightarrow 0 & \dots & \rightarrow 0 \\
 & & \swarrow \wedge dw & & \swarrow \wedge dw & & \swarrow \wedge dw \\
 \dots & \xrightarrow{0} & \Omega^2 & \xrightarrow{0} & \Omega & \xrightarrow{0} & \mathcal{O}_X \\
 & & \swarrow \wedge dw & & \swarrow \wedge dw & & \swarrow \wedge dw \\
 \dots & \dots & \rightarrow 0 & \dots & \rightarrow 0 & \dots & \rightarrow 0 \\
 & & \swarrow \wedge dw & & \swarrow \wedge dw & & \swarrow \wedge dw \\
 \dots & \xrightarrow{0} & \Omega^2 & \xrightarrow{0} & \Omega & \xrightarrow{0} & \mathcal{O}_X \\
 & & \swarrow \wedge dw & & \swarrow \wedge dw & & \swarrow \wedge dw \\
 \dots & \dots & \rightarrow 0 & \dots & \rightarrow 0 & \dots & \rightarrow 0 \\
 & & \swarrow \wedge dw & & \swarrow \wedge dw & & \swarrow \wedge dw \\
 \dots & \xrightarrow{0} & \Omega^2 & \xrightarrow{0} & \Omega & \xrightarrow{0} & \mathcal{O}_X \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

(Equation 4.2.)

Under the HKR quasi-isomorphism the map B_w does indeed become $\wedge dw$: locally we have

$$dw \wedge HKR((1 \boxtimes a_1 \boxtimes a_2 \cdots \boxtimes a_q \boxtimes 1) \boxtimes 1) = \frac{1}{q!} dw \wedge da_1 \wedge \cdots \wedge da_q$$

and

$$\begin{aligned}
 HKR(B_w(1 \boxtimes a_1 \boxtimes \cdots \boxtimes a_q \boxtimes 1)) &= \frac{1}{q+1!} \sum_{i=0}^{k+1} (-1)^i da_1 \wedge \cdots \wedge da_i \wedge dw \wedge da_{i+1} \wedge \cdots \wedge da_q \\
 &= \frac{1}{q!} dw \wedge da_1 \wedge \cdots \wedge da_q.
 \end{aligned}$$

This gives that the Hochschild homology of $\mathfrak{M}\mathfrak{F}_{loc}(X, \mathcal{O}_X, w)$ is given as the hypercohomology of the complex

$$\dots \xrightarrow{\wedge dw} \bigoplus_{i \text{ even}} \Omega^i \xrightarrow{dw \wedge} \bigoplus_{i \text{ odd}} \Omega^i \xrightarrow{dw \wedge} \bigoplus_{i \text{ even}} \Omega^i \xrightarrow{dw \wedge} \bigoplus_{i \text{ odd}} \Omega^i \xrightarrow{dw \wedge} \dots$$

where $\bigoplus_{i \text{ even}} \Omega^i$ is in even degrees and $dw \wedge$ wedges dw in the first slot. □

CHAPTER V

BOUNDARY-BULK AND CHERN CHARACTER

5.1. Sheafified Boundary-Bulk

From corollary 3.20 we have an equivalence of categories

$$H^0\mathfrak{Qcoh}(X \times X, \mathcal{O}_{X \times X}, w)_{inj} \cong D(\mathfrak{M}\mathfrak{F}_{loc}(X \times X, \mathcal{O}_{X \times X}, w))$$

which sends $M \rightarrow \text{Hom}(-, M)$. In for a matrix factorization $\mathcal{E} \in Z^0\mathfrak{Qcoh}(w)$, there is a map

$$eval : \mathcal{E} \boxtimes \mathcal{E}^\vee \rightarrow \Delta_*S(\mathcal{O}_X).$$

Since

$$\begin{aligned} \text{Hom}_{\mathfrak{M}\mathfrak{F}_{loc}(w)}(\mathcal{E}, -) \otimes \text{Hom}_{\mathfrak{M}\mathfrak{F}_{loc}(w)}(-, \mathcal{E}) &= \text{Hom}_{\mathfrak{M}\mathfrak{F}_{loc}(-w)}(-, \mathcal{E}^\vee) \otimes \text{Hom}_{\mathfrak{M}\mathfrak{F}_{loc}(w)}(-, \mathcal{E}) \\ &= \text{Hom}_{\mathfrak{M}\mathfrak{F}_{loc}(\tilde{w})}(-, \mathcal{E} \boxtimes \mathcal{E}^\vee) \end{aligned}$$

and using remark 2.17, we see that the map $eval : \mathcal{E} \boxtimes \mathcal{E}^\vee \rightarrow \Delta_*S(\mathcal{O}_X)$ corresponds to the map c_E of perfect bimodules from 2.16. We have a quasi-isomorphism

$$\text{Hom}_{\mathfrak{M}\mathfrak{F}_{loc}(w)}(\mathcal{E}, \mathcal{E}) \cong Tr(\mathcal{E} \boxtimes \mathcal{E}^\vee)$$

Then we apply the trace functor the evaluation map $\mathcal{E} \boxtimes \mathcal{E}^\vee \rightarrow \Delta_*S(\mathcal{O}_X)$ to get a map

$$\tau^\mathcal{E} : \text{Hom}_{\mathfrak{M}\mathfrak{F}_{loc}(X, \mathcal{L}, w)}(\mathcal{E}, \mathcal{E}) \cong Tr(\mathcal{E} \boxtimes \mathcal{E}^\vee) \rightarrow Tr(\Delta_*S(\mathcal{O}_X)) = HH(\mathfrak{M}\mathfrak{F}_{loc}(X, \mathcal{L}, w)).$$

By the discussion above and lemma 4.3 the map $\tau^\mathcal{E}$ just defined corresponds to the categorical boundary bulk map from Definition 2.18.

Having computed the Hochschild homology for the category $\mathfrak{MF}_{loc}(X, \mathcal{O}_X, w)$ as $\mathbb{R}\Gamma(\Omega_{dw})$ and now making the trivial observation that since $\mathcal{E} \in \mathfrak{MF}(w)$ is graded locally free we have

$$\mathbb{R}\mathrm{Hom}(\mathcal{E}, \mathcal{F}) = \mathbb{R}\Gamma\mathrm{Hom}(\mathcal{E}, \mathcal{F}),$$

we may promote the boundary bulk-map to a map in the derived category of sheaves on X :

$$\mathcal{T}_\mathcal{E} : \mathrm{Hom}(\mathcal{E}, \mathcal{E}) \rightarrow \Omega_{dw},$$

and thereby understand the particular invariants we wish to compute in two steps, first to get an explicit representative for $\mathcal{T}_\mathcal{E}$ and then to understand the more classical problem of deducing the induced map on cohomology.

Lemma 5.1. Define a map $\mathcal{T}_\mathcal{E} : \mathrm{Hom}(\mathcal{E}, \mathcal{E}) \rightarrow \Omega_{dw}$ in $D(X)$ by

$$\mathrm{Hom}(\mathcal{E}, \mathcal{E}) = \mathcal{E} \otimes \mathcal{E}^\vee \cong \mathbb{L}\Delta^*(\mathcal{E} \boxtimes \mathcal{E}^\vee) \xrightarrow{eval} \mathbb{L}\Delta^*(\Delta_*S(\mathcal{O}_X)) \cong \Omega_{dw}$$

Then $\tau_\mathcal{E} = \mathbb{R}\Gamma(\mathcal{T}_\mathcal{E})$.

Proof. This is clear. □

We wish now to get a better handle on this map $\mathcal{T}_\mathcal{E}$. We may resolve a matrix factorization \mathcal{E} by $\epsilon \otimes 1 : \widehat{\mathcal{B}ar}_{\tilde{w}} \otimes_{S(\mathcal{O}_X)} \mathcal{E} \rightarrow \Delta_*S(\mathcal{O}_X) \otimes_{S(\mathcal{O}_X)} \mathcal{E} = \mathcal{E}$. Here we use the short hand $\otimes_{S(\mathcal{O}_X)}$ between an \tilde{w} curved module on $X \times X$ and a w -curved module on X to mean

$$\widehat{\mathcal{B}ar}_{\tilde{w}} \otimes_{S(\mathcal{O}_X)} \mathcal{E} := (p_1)_*(\widehat{\mathcal{B}ar}_{\tilde{w}} \otimes_{S(\mathcal{O}_{X^2})} p_2^*\mathcal{E})$$

where p_1 and p_2 are the natural projections from $X \times X$ to X . Since matrix factorizations are flat, lemma 4.1 implies that this map is a weak equivalence in $Z^0\mathbf{Qcoh}(w)$. Then the map

$$\mathcal{E}^\vee \otimes (\widehat{\mathcal{B}ar}_{\bar{w}} \otimes_{S(\mathcal{O}_X)} \mathcal{E}) \rightarrow \mathcal{E}^\vee \otimes \mathcal{E} = \mathcal{H}om(\mathcal{E}, \mathcal{E})$$

is a quasi-isomorphism of complexes of sheaves.

This gives us an explicit representative for $\mathcal{T}_{\mathcal{E}}$ given by the roof

$$\begin{array}{ccc} \mathcal{E}^\vee \otimes (\hat{B}_{\bar{w}} \otimes_{S(\mathcal{O}_X)} \mathcal{E}) & \xrightarrow{1 \otimes \sigma} & \mathcal{E}^\vee \otimes \mathcal{E} \otimes_{S(\mathcal{O}_X)} \widehat{\mathcal{B}ar}_{\bar{w}} \\ \downarrow 1 \otimes \epsilon \otimes 1 \wr & & \downarrow ev \otimes 1 \otimes 1 \\ \mathcal{E}^\vee \otimes \mathcal{E} & & \Delta^* \widehat{\mathcal{B}ar}_{\bar{w}} \\ & & \downarrow HKR \\ & & \Omega_{dw} \end{array}$$

where ev is the evaluation map of \mathcal{E}^\vee on \mathcal{E} , σ is switching the factors in the tensor product and $\bar{\otimes}$ is contraction of tensor.

The goal now is to construct a natural morphism

$$\mathcal{E}xp(at(\mathcal{E})) : \mathcal{E} \rightarrow \Omega_{dw} \otimes_{S(\mathcal{O}_X)} \mathcal{E}$$

in the coderived category of w -curved modules, such that

$$\mathcal{T}_{\mathcal{E}} = str(- \circ \mathcal{E}xp(at(\mathcal{E})) : \mathcal{H}om(\mathcal{E}, \mathcal{E}) \rightarrow \mathcal{H}om(\mathcal{E}, \Omega_{dw} \otimes_{S(\mathcal{O}_X)} \mathcal{E}) = \mathcal{H}om(\mathcal{E}, \mathcal{E}) \otimes_{S(\mathcal{O}_X)} \Omega_{dw} \rightarrow \Omega_{dw}$$

This morphism will then be a sort of internal Chern character for the category of matrix factorization. In what follows we will want to fix $n = dim(X)$.

Before we proceed we wish take take a motivational digression and consider the category of complexes of coherent sheaves on X . We will follow very closely the treatment from [Mar]. The idea is that in *loc. cit*, Markarian constructs an internal Chern character by exponentiating the Atiyah class map and which takes values in Hochschild homology sheaves. We wish to mimic this construction. The main technical problem, as we will see, is that there is no obvious analog to the Atiyah class for matrix factorizations. But, oddly enough, even though the class $at(\mathcal{E})$ does not seem to exist, its exponential does.

We have the exact sequence of \mathcal{O}_{X^2} modules

$$0 \rightarrow \mathcal{I}/\mathcal{I}^2 \rightarrow \mathcal{O}_{X^2}/\mathcal{I}^2 \rightarrow \mathcal{O}_\Delta \rightarrow 0$$

where \mathcal{I} is the kernel of the multiplication map $\mathcal{O}_{X^2} \rightarrow \mathcal{O}_\Delta$. We will write Ω_Δ for $\mathcal{I}/\mathcal{I}^2$ and \mathcal{J}_Δ^1 for $\mathcal{O}_{X^2}/\mathcal{I}^2$. Given an honest complex ($d^2 = 0$) of sheaves, \mathcal{E} , we may “tensor on the right” by \mathcal{E} to get an exact sequence of \mathcal{O}_X -complexes

$$0 \rightarrow \Omega^1 \otimes_{\mathcal{O}_X} \mathcal{E} \rightarrow \mathcal{J}^1 \otimes_{\mathcal{O}_X} \mathcal{E} \rightarrow \mathcal{E} \rightarrow 0. \quad (\text{Equation 5.1.})$$

where for an \mathcal{O}_{X^2} module \mathcal{M} and an \mathcal{O}_X -module \mathcal{F}

$$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{F} := (p_1)_*(\mathcal{M} \otimes_{\mathcal{O}_{X^2}} p_2^*(\mathcal{F}))$$

where $p_i : X \times X \rightarrow X$ are the standard projections. The extension in (Equation 5.1.) gives an element of

$$Ext^1(\mathcal{E}, \Omega^1 \otimes_{\mathcal{O}_X} \mathcal{E}) = Hom_{D(X)}(\mathcal{E}, \Omega^1 \otimes_{\mathcal{O}_X} \mathcal{E}[1]).$$

This element, $at(\mathcal{E}) : \mathcal{E} \rightarrow \Omega \otimes_{\mathcal{O}_X} \mathcal{E}[1]$, is called the Atiyah class of \mathcal{E} .

Composing the morphism $at(\mathcal{E})$ with itself i times and then wedging forms we obtain a map

$$\wedge at(\mathcal{E})^i : \mathcal{E} \rightarrow \Omega^i \otimes_{\mathcal{O}_X} \mathcal{E}[i].$$

Using the isomorphism $\mathcal{O}_\Delta \otimes_{\mathcal{O}_X} \Omega^1 \cong \Omega_\Delta$, get a long exact sequence

$$0 \rightarrow \Omega^{\otimes i-1} \otimes_{\mathcal{O}_X} \Omega_\Delta \rightarrow \Omega^{\otimes i-1} \otimes_{\mathcal{O}_X} \mathcal{J}_\Delta^1 \rightarrow \cdots \rightarrow \Omega^1 \otimes_{\mathcal{O}_X} \mathcal{J}_\Delta^1 \rightarrow \mathcal{J}_\Delta^1 \rightarrow \mathcal{O}_\Delta$$

Tensoring this sequence on the right with \mathcal{E} we get a long exact sequence

$$0 \rightarrow \Omega^{\otimes i} \otimes_{\mathcal{O}_X} \mathcal{E} \rightarrow \Omega^{\otimes i-1} \otimes_{\mathcal{O}_X} \mathcal{J}^1(\mathcal{E}) \rightarrow \cdots \rightarrow \Omega^1 \otimes_{\mathcal{O}_X} \mathcal{J}^1(\mathcal{E}) \rightarrow \mathcal{J}^1(\mathcal{E}) \rightarrow \mathcal{E}. \quad (\text{Equation 5.2.})$$

Here we denote by $\mathcal{J}^1(\mathcal{E})$ the tensor product $\mathcal{J}_\Delta^1 \otimes_{\mathcal{O}_X} \mathcal{E}$. One sees easily that this exact sequence represents $\wedge at(\mathcal{E})^i$ as a Yoneda extension and so the map $\wedge at(\mathcal{E})^i$ is given as the zig-zag

$$\mathcal{E} \leftarrow (\Omega^{\otimes i} \otimes_{\mathcal{O}_X} \mathcal{E} \rightarrow \Omega^{\otimes i-1} \otimes_{\mathcal{O}_X} \mathcal{J}^1(\mathcal{E}) \rightarrow \cdots \rightarrow \Omega^1 \otimes_{\mathcal{O}_X} \mathcal{J}^1(\mathcal{E}) \rightarrow \mathcal{J}^1(\mathcal{E})) \rightarrow \Omega^i \otimes \mathcal{E}[i]$$

where the last map is simply projection onto the last factor followed by wedging forms.

When we try to mimic this construction for curved ($w \neq 0$) modules, the projection onto the last factor is no longer a map in the category we care about. Or more accurately the inclusion of graded \mathcal{O}_X modules $\Omega^i \otimes \mathcal{E}[i] \rightarrow \Omega_{dw} \otimes \mathcal{E}[i]$ is not a map of curved modules, unless $i = n$ or $dw = 0$. Our first observation is that we can view the exponential of the Atiyah class as a map from the total complex of the

resolution,

$$\Omega^{\otimes n} \otimes_{\mathcal{O}_X} \mathcal{E} \rightarrow \Omega^{\otimes n-1} \otimes_{\mathcal{O}_X} \mathcal{J}^1(\mathcal{E}) \rightarrow \cdots \rightarrow \Omega^1 \otimes_{\mathcal{O}_X} \mathcal{J}^1(\mathcal{E}) \rightarrow \mathcal{J}^1(\mathcal{E})$$

of \mathcal{E} , to $\Omega^\bullet \otimes \mathcal{E}$, by using the various projections onto $\Omega^{\otimes i} \otimes \mathcal{E}$, for $i \leq n$, where again $n = \dim(X)$. The second observation is that we still can in $\mathfrak{M}\mathfrak{F}_{loc}(X, \mathcal{L}, w)$ construct appropriate analogs of this resolution of \mathcal{E} . We do this now.

As with the curved bar complex we may use the resolution

$$0 \rightarrow \Omega^{\otimes n-1} \otimes_{\mathcal{O}_X} \Omega_\Delta \rightarrow \Omega^{\otimes n-1} \otimes_{\mathcal{O}_X} \mathcal{J}_\Delta^1 \rightarrow \cdots \rightarrow \Omega^1 \otimes_{\mathcal{O}_X} \mathcal{J}_\Delta^1 \rightarrow \mathcal{J}_\Delta^1 \rightarrow \mathcal{O}_\Delta \quad (\text{Equation 5.3.})$$

to build a \tilde{w} curved complex $\mathcal{A}t$ which resolves $\Delta_*S(\mathcal{O}_X)$. Set

$$\mathcal{A}_i = \begin{cases} \Omega^{\otimes i} \otimes_{\mathcal{O}_X} \mathcal{J}_\Delta^1 & \text{if } 0 \leq i < n \\ \Omega^{\otimes(n-1)} \otimes_{\mathcal{O}_X} \Omega_\Delta & \text{if } i = n \\ 0 & \text{else} \end{cases}$$

Then define the graded components of $\mathcal{A}t$ by folding 2-periodically:

$$\mathcal{A}t_i = \bigoplus_{j \equiv i \pmod{2}} \mathcal{A}_j.$$

We have the differential, $m : \mathcal{A}t_i \rightarrow \mathcal{A}t_{i+1}$ coming from the resolution (Equation 5.3.)

which (locally) is given by the equation

$$\begin{aligned} m(da_1 \otimes da_2 \otimes \cdots \otimes da_n \otimes a_0 \boxtimes a_{n+1}) &= a_0 a_{n+1} da_1 \otimes \cdots \otimes da_{n-1} \otimes a_n \boxtimes 1 \\ &\quad - a_0 a_{n+1} da_1 \otimes \cdots \otimes da_{n-1} \otimes 1 \boxtimes a_{n-1}. \end{aligned}$$

Here we have chosen indices in preparation for certain morphisms involving the curved bar complex. Again we use \boxtimes to emphasize external tensor. Depending on our purposes, i.e. whether we want to emphasize or deemphasize the role of J_Δ^1 in the tensor $\Omega^{\otimes q} \otimes J_\Delta^1$, we will alternatively simply write

$$a_0 da_1 \otimes \cdots \otimes da_q \boxtimes a_{q+1} = a_1 \otimes \cdots \otimes a_q \otimes a_0 \boxtimes a_{q+1}$$

Coordinate free, this map m is simply induced by the multiplication map $\mathcal{J}_\Delta^1 \rightarrow \mathcal{O}_\Delta$ followed by the isomorphism $\Omega^{\otimes i} \otimes_{\mathcal{O}_X} \mathcal{O}_\Delta \cong \Omega^{\otimes(i-1)} \otimes_{\mathcal{O}_X} \Omega_\Delta$ and then the inclusion

$$\Omega^{\otimes(i-1)} \otimes \Omega_\Delta \rightarrow \Omega^{\otimes(i-1)} \otimes \mathcal{J}^1.$$

And, of course, on the summand $\mathcal{A}_n = \Omega^{\otimes(n-1)} \otimes_{\mathcal{O}_X} \Omega_\Delta$, m is simply the inclusion of $\Omega^{\otimes(n-1)} \otimes_{\mathcal{O}_X} \Omega_\Delta$ into $\Omega^{\otimes(n-1)} \otimes_{\mathcal{O}_X} \mathcal{J}_\Delta^1$. To curve $\mathcal{A}t$ by \tilde{w} we add a second differential B_{dw} given by the formula

$$B_{dw}(\omega_1 \otimes \cdots \otimes \omega_n \otimes a_0 \boxtimes a_{q+1}) = \sum_{i=0}^q (-1)^i \omega_1 \otimes \cdots \otimes \omega_i \otimes dw \otimes \omega_{i+1} \otimes \cdots \otimes \omega_q \otimes a_0 \boxtimes a_{q+1}.$$

As with $\widehat{\mathcal{B}ar}_{\tilde{w}}$, we may picture $\mathcal{A}t$ as the total complex of the bicomplex:

$$\begin{array}{ccccccc}
& & \vdots & & \vdots & & \vdots \\
& & \swarrow B_{dw} & & \swarrow B_{dw} & & \swarrow B_{dw} \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\Omega^{\otimes n} \otimes \mathcal{J}_\Delta^1 & \xrightarrow{b} & \cdots & \xrightarrow{b} & \Omega^{\otimes 2} \otimes \mathcal{J}_\Delta^1 & \xrightarrow{b} & \Omega \otimes \mathcal{J}_\Delta^1 & \xrightarrow{b} & \mathcal{J}_\Delta^1 \\
& & \swarrow B_{dw} & & \swarrow B_{dw} & & \swarrow B_{dw} & & \swarrow B_{dw} \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\Omega^{\otimes n} \otimes \mathcal{J}_\Delta^1 & \xrightarrow{b} & \cdots & \xrightarrow{b} & \Omega^{\otimes 2} \otimes \mathcal{J}_\Delta^1 & \xrightarrow{b} & \Omega \otimes \mathcal{J}_\Delta^1 & \xrightarrow{b} & \mathcal{J}_\Delta^1 \\
& & \swarrow B_{dw} & & \swarrow B_{dw} & & \swarrow B_{dw} & & \swarrow B_{dw} \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
& & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

Now we claim that $\mathcal{A}t$ imbued with the differential $B_{dw} - m\gamma$ is a \tilde{w} curved module, where γ is the grading operator with respect to forms, i.e. $\gamma|_{\Omega^q \otimes \mathcal{J}_\Delta^1} = (-1)^q$. Indeed the computations

$$\begin{aligned}
& m\gamma B_{dw}(a_0 da_1 \otimes \cdots \otimes da_q \boxtimes a_{q+1}) \\
&= (-1)^{q+1} \left[\sum_{i=0}^q (-1)^i m(a_0 da_1 \otimes \cdots \otimes dw \otimes \cdots \otimes da_q \boxtimes a_{q+1}) \right] \\
&= (-1)^{q+1} \left[(-1)^q (a_0 a_{q+1} da_1 \otimes \cdots \otimes da_q \otimes w \boxtimes 1 - a_0 a_{q+1} da_1 \otimes \cdots \otimes da_q \otimes 1 \boxtimes w) \right. \\
&\quad + \sum_{i=0}^{q-1} (-1)^i a_0 a_{q+1} da_1 \otimes \cdots \otimes da_i \otimes dw \otimes da_{i+1} \otimes \cdots \otimes da_{q-1} \otimes a_q \boxtimes 1 \\
&\quad \left. - \sum_{i=0}^{q-1} (-1)^i a_0 a_{q+1} da_1 \otimes \cdots \otimes da_i \otimes dw \otimes da_{i+1} \otimes \cdots \otimes da_{q-1} \otimes 1 \boxtimes a_q \right]
\end{aligned}$$

and

$$\begin{aligned}
& B_{dw}m\gamma(da_1 \otimes \cdots \otimes da_q \otimes a_0 \boxtimes a_{q+1}) \\
&= (-1)^q \left[\sum_{i=0}^{q-1} (-1)^i a_0 a_{q+1} da_1 \otimes \cdots \otimes da_i \otimes dw \otimes da_{i+1} \otimes \cdots \otimes da_{q-1} \otimes a_q \boxtimes 1 \right. \\
&\quad \left. - \sum_{i=0}^{q-1} (-1)^i a_0 a_{q+1} da_1 \otimes \cdots \otimes da_i \otimes dw \otimes da_{i+1} \otimes \cdots \otimes da_{q-1} \otimes 1 \boxtimes a_q \right]
\end{aligned}$$

show that

$$\begin{aligned}
(B_{dw} - m\gamma)^2 &= -B_{dw}m\gamma - m\gamma B_{dw}(da_1 \otimes \cdots \otimes da_q \otimes a_0 \boxtimes a_{q+1}) \\
&= a_0 a_{q+1} da_1 \otimes \cdots \otimes da_q \otimes (w \boxtimes 1 - 1 \boxtimes w) \\
&= da_1 \otimes \cdots \otimes da_q \otimes (a_0 a_{q+1} w \boxtimes 1 - a_0 a_{q+1} \boxtimes w)
\end{aligned}$$

The final observations are that \tilde{w} acts on $\Omega^{\otimes q} \otimes_{\mathcal{O}_X} \mathcal{J}_\Delta^1$ by

$$\tilde{w} \cdot \omega_1 \otimes \cdots \otimes \omega_q \otimes a_0 \boxtimes a_{q+1} = w\omega_1 \otimes \cdots \otimes \omega_q \otimes a_0 \boxtimes a_{q+1}w = \omega_1 \otimes \cdots \otimes \omega_q \otimes wa_0 \boxtimes a_{q+1}w$$

and the difference between this action and the above computation for $(B_{dx} - m\gamma)^2$ is

$$a_0 a_{q+1} w \boxtimes 1 - a_0 a_{q+1} \boxtimes w - wa_0 \boxtimes a_{q+1} + a_0 \boxtimes wq = (a_0 \boxtimes 1)(w \boxtimes 1 - 1 \boxtimes w)(a_{q+1} \boxtimes 1 - 1 \boxtimes a_{q+1})$$

which is 0 in \mathcal{J}_Δ^1 . Therefore the map $(B_{dw} - m\gamma)^2 = -B_{dw}m\gamma - m\gamma B_{dw}$ is indeed multiplication by \tilde{w} .

Now there are maps $\pi : \mathcal{O}_{\mathbb{A}^{q+2}} \rightarrow \Omega^{\otimes q} \otimes_{\mathcal{O}_X} \mathcal{J}_\Delta^1$ given by

$$\pi(a_0 \boxtimes a_1 \boxtimes \cdots \boxtimes a_q \boxtimes a_{q+1}) = a_0 da_1 \otimes \cdots \otimes da_q \boxtimes a_{q+1}$$

It is easy to see $\pi B_w = B_{dw} \pi$. We observe that for an elements of $\mathcal{O}_{\mathbb{X}^{q+1}}$ of the form $a_0 \boxtimes \cdots \boxtimes a_i a_{i+1} \boxtimes \cdots \boxtimes a_{q+1}$, with $0 < i < q + 1$ we have

$$\begin{aligned}
\pi(a_0 \boxtimes \cdots \boxtimes a_i a_{i+1} \boxtimes \cdots \boxtimes a_{q+1}) &= da_1 \otimes \cdots \otimes d(a_i a_{i+1}) \otimes da_q \otimes a_0 \boxtimes a_{q+1} \\
&= a_0 a_{i+1} da_1 \otimes \cdots \otimes da_i \otimes da_{i+2} \otimes \cdots \otimes da_q \boxtimes a_{q+1} \\
&\quad + a_0 a_i da_1 \otimes \cdots \otimes da_{i-1} \otimes da_{i+1} \otimes \cdots \otimes da_q \boxtimes a_{q+1} \\
&= a_0 a_i da_1 \otimes \cdots \otimes \widehat{da_i} \otimes \cdots \otimes da_q \boxtimes a_{q+1} \\
&\quad + a_0 a_i da_1 \otimes \cdots \otimes \widehat{da_{i+1}} \otimes \cdots \otimes da_q \boxtimes a_{q+1}
\end{aligned}$$

where $\widehat{da_i}$ indicates to omit this tensor. We also have

$$\begin{aligned}
\pi(a_0 a_1 \boxtimes a_2 \boxtimes \cdots \boxtimes a_{q+1}) &= a_0 a_1 \otimes a_2 \otimes \cdots \otimes da_q \boxtimes a_{q+1} \\
&= a_0 a_1 \otimes \widehat{da_1} \otimes da_2 \otimes \cdots \otimes da_q \boxtimes a_{q+1}
\end{aligned}$$

So, again using b for the Hochschild differential, we have

$$\begin{aligned}
\pi b(a_0 \boxtimes \cdots \boxtimes a_{q+1}) &= \sum_{i=0}^q (-1)^i \pi(a_0 \boxtimes \cdots \boxtimes a_i a_{i+1} \boxtimes \cdots \boxtimes a_{q+1}) \\
&= (-1)^q a_0 da_1 \otimes \cdots \otimes da_{q-1} \boxtimes a_q a_{q+1} + a_0 a_1 da_2 \otimes \cdots \otimes da_n \boxtimes a_{n+1} \\
&\quad + \sum_{i=1}^{q-1} (-1)^i a_0 a_i \otimes da_1 \otimes \cdots \otimes \widehat{da_i} \otimes \cdots \otimes da_q \boxtimes da_{q+1} \\
&\quad + \sum_{i=1}^{q-1} (-1)^i a_0 a_{i+1} \otimes da_1 \otimes \cdots \otimes \widehat{da_{i+1}} \otimes \cdots \otimes da_q \boxtimes da_{q+1} \\
&= (-1)^q a_0 da_1 \otimes \cdots \otimes da_{q-1} \boxtimes a_q a_{q+1} + a_0 a_1 da_2 \otimes \cdots \otimes da_n \boxtimes a_{n+1} \\
&\quad + \sum_{i=1}^{q-1} (-1)^i a_0 a_i \otimes da_1 \otimes \cdots \otimes \widehat{da_i} \otimes \cdots \otimes da_q \boxtimes da_{q+1} \\
&\quad + \sum_{i=2}^q (-1)^{i-1} a_0 a_i \otimes da_1 \otimes \cdots \otimes \widehat{da_i} \otimes \cdots \otimes da_q \boxtimes da_{q+1} \\
&= (-1)^q a_0 da_1 \otimes \cdots \otimes da_{q-1} \boxtimes a_q a_{q+1} + a_0 a_1 da_2 \otimes \cdots \otimes da_q \boxtimes a_{q+1} \\
&\quad - a_0 a_1 da_2 \otimes \cdots \otimes \widehat{da_i} \otimes \cdots \otimes da_q \boxtimes da_{q+1} \\
&\quad + (-1)^{q-1} a_0 da_q da_1 \otimes \cdots \otimes da_{q-1} \boxtimes a_{q+1} \\
&= (-1)^{q+1} m \pi(a_0 \boxtimes \cdots \boxtimes a_{q+1})
\end{aligned}$$

The above discussion proves the following lemma:

Lemma 5.2. The map $\pi : \widehat{\mathcal{B}ar}_{\tilde{w}} \rightarrow \mathcal{A}t$ is a closed morphism of \tilde{w} -curved modules.

Incidentally this discussion also explains the appearance of the grading operator in the horizontal direction.

Remark 5.3. It is clear that $\pi : \widehat{\mathcal{B}ar}_{\tilde{w}} \rightarrow \mathcal{A}t$ is a weak equivalence of \tilde{w} curved modules on $X \times X$, since both $\widehat{\mathcal{B}ar}_{\tilde{w}}$ and $\mathcal{A}t$ are weakly equivalent to $\Delta_* S(\mathcal{O}_X)$ via projection.

As a piece of notation, for $\mathcal{E} \in \mathfrak{Qcoh}(X, \mathcal{O}_X, w)$, we define

$$\mathcal{A}t(\mathcal{E}) := \mathcal{A}t \otimes_{S(\mathcal{O}_X)} \mathcal{E} := (p_1)_*(\mathcal{A}t \otimes_{S(\mathcal{O}_{X^2})} p_2^* \mathcal{E}).$$

Lemma 5.4. Let $\wedge : \Omega^{\otimes q} \otimes \mathcal{J}^1(\mathcal{E}) \rightarrow \Omega^q \otimes \mathcal{E}$ denote the anti-symmetrization map:

$$\wedge(a_0 da_1 \otimes \cdots \otimes da_q \boxtimes e) = a_0 da_1 \wedge \cdots \wedge da_q \otimes e$$

Then the map

$$\sum_{i=0}^n \frac{\wedge}{i!} : \mathcal{A}t(\mathcal{E}) \rightarrow \Omega_{dw} \otimes_{S(\mathcal{O}_X)} \mathcal{E}$$

gives a closed degree 0 morphism of w -curved modules.

Proof. This follows from the calculations

$$\begin{aligned} & \frac{\wedge}{(q+1)!} B_{dw}(a_0 da_1 \otimes \cdots \otimes da_q \boxtimes e) \\ &= \frac{1}{(q+1)!} \sum_{i=0}^q (-1)^i a_0 da_1 \wedge \cdots \wedge da_i \wedge dw \wedge da_{i+1} \wedge \cdots \wedge da_q \otimes e \\ &= \frac{1}{(q+1)!} \sum_{i=0}^q a_0 dw \wedge da_1 \wedge \cdots \wedge da_q \otimes e \\ &= \frac{1}{q!} dw \wedge a_0 da_1 \wedge \cdots \wedge da_q \otimes e \\ &= dw \wedge \left(\frac{\wedge}{q!} (a_0 a_1 \otimes \cdots \otimes da_n \boxtimes e) \right) \end{aligned}$$

and

$$\begin{aligned}
\frac{\wedge}{(q-1)!} m(a_0 da_1 \otimes \cdots \otimes da_q \boxtimes e) &= \frac{1}{(q-1)!} (a_0 a_q da_1 \wedge \cdots \wedge da_{q-1} \otimes e) \\
&\quad - \frac{1}{(q-1)!} (a_0 da_1 \wedge \cdots \wedge da_{q-1} \otimes a_q e) \\
&= 0
\end{aligned}$$

and the observation that the differential on \mathcal{E} obviously commutes with the map $\sum_i \wedge_i$. \square

Definition 5.5. Define the map $\mathcal{E}xp(at(\mathcal{E})) : \mathcal{E} \rightarrow \Omega_{dw} \otimes \mathcal{E}$ in the category $D^{co}\mathcal{Q}coh(X, \mathcal{O}_X, w)$ by the roof

$$\begin{array}{ccc}
& \mathcal{A}t(\mathcal{E}) & \\
\swarrow \wr & & \searrow \Sigma \frac{\wedge}{i!} \\
\mathcal{E} & & \Omega_{dw} \otimes \mathcal{E}
\end{array}$$

Lemma 5.6. The sheafified boundary bulk map $\mathcal{T}_{\mathcal{E}} : \mathcal{H}om_{S(\mathcal{O}_X)}(\mathcal{E}, \mathcal{E}) \rightarrow \Omega_{dw}$ is given by $str(- \circ \mathcal{E}xp(at(\mathcal{E})))$, where $str : \mathcal{H}om_{S(\mathcal{O}_X)}(\mathcal{E}, \mathcal{E}) \rightarrow S(\mathcal{O}_X)$ is the super-trace map.

Proof. Recall from the discussion at the beginning of this section that we have the following representative for $\mathcal{T}_{\mathcal{E}}$

$$\begin{array}{ccc}
\mathcal{E}^{\vee} \otimes (\hat{B}_{\bar{w}} \otimes_{S(\mathcal{O}_X)} \mathcal{E}) & \xrightarrow{1 \otimes \sigma} & \mathcal{E}^{\vee} \otimes \mathcal{E} \otimes_{S(\mathcal{O}_X)} \widehat{\mathcal{B}ar}_{\bar{w}} \\
\downarrow 1 \otimes \rho \otimes 1 \wr & & \downarrow ev_{\bar{w}} \otimes 1 \\
\mathcal{E}^{\vee} \otimes \mathcal{E} & & \Delta^* \widehat{\mathcal{B}ar}_{\bar{w}} \\
& & \downarrow HKR \\
& & \Omega_{dw}
\end{array}$$

From lemmas 5.2 and 5.4 we may complete this diagram to the following picture

$$\begin{array}{ccccc}
\mathcal{E}^\vee \otimes (\widehat{B}_{\bar{w}} \otimes_{S(\mathcal{O}_X)} \mathcal{E}) & \xrightarrow{1 \boxtimes \sigma \otimes 1} & \mathcal{E}^\vee \otimes \mathcal{E} \otimes_{S(\mathcal{O}_X)} \widehat{\mathcal{B}ar}_{\bar{w}} & & \\
\downarrow 1 \boxtimes \rho \otimes 1 & \searrow 1 \boxtimes \pi \otimes 1 & \downarrow ev \otimes 1 \otimes 1 & & \\
\mathcal{E}^\vee \otimes \mathcal{E} & \xleftarrow{\sim} & \mathcal{E}^\vee \otimes \mathcal{A}t(\mathcal{E}) & & \Delta^* \widehat{\mathcal{B}ar}_w \\
\downarrow & & \downarrow & & \downarrow HKR \\
\mathcal{H}om(\mathcal{E}, \mathcal{E}) & & \mathcal{E}^\vee \otimes (\Omega_{dw} \otimes \mathcal{E}) & \xrightarrow{\quad} & \Omega_{dw} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{H}om(\mathcal{E}, \Omega_{dw} \otimes \mathcal{E}) & \xrightarrow{str} & \Omega_{dw} & &
\end{array}$$

It is easy to check that everything commutes, at least perhaps with the added observation that the bottom arrow, which we have by abuse simply called *str*, first commutes the tensors under the isomorphism

$$\mathcal{H}om(\mathcal{E}, \Omega_{dw} \otimes \mathcal{E}) \cong \mathcal{H}om(\mathcal{E}, \mathcal{E}) \otimes \Omega_{dw}$$

before applying the super-trace. □

Remark 5.7. Taking $w = 0$, our results give us directly information about \mathbb{Z}_2 complexes of vector bundles on X .¹ Moreover, one checks (essentially by taking a standard tensor product on complexes, rather than the folded tensor we use for matrix factorizations) that all of the above constructions and theorem go through. This gives the following result, which follows formally from [Că] and [Ram]. However, there seems to be a problem in Ramadoss's proof in [Ram]: in the proof of Proposition

¹There is a mild issue here with respect to 3.8 which requires that w not be a zero divisor, however the conclusion of this theorem is well-know to still hold when $w = 0$, so there is no problem.

2 he uses without explaining the coincidence of the two versions of the Chern character of \mathcal{O}_Δ , one defined in [Căl] and the one one coming from DG theory.

Theorem 5.8. The DG Chern Character map for perfect complexes on smooth X in the sense of [Shk] coincides with the classical Chern Character.

Proof. We apply our remark 5.7 to lemma 5.6 and 5.1 to find that the DG Chern Character of a bounded complex of locally free sheaves \mathcal{E} is given by

$$\mathbb{R}\Gamma(\text{str}(\mathcal{E}xp(at(E)))) \in \mathbb{R}\Gamma(\bigoplus \Omega^i) = HH(X).$$

This according to [Mar] is exactly the classical Chern Character. □

5.2. A Formula for The Boundary-Bulk Map

In this section we wish to develop a global analog of the Chern character formula for global matrix factorizations computed for a formal disk in [PV2]. There should be some question about what such an analog could be since globality generally prohibits formulas, at least formulas involving coordinates. Another option would be to relate to Chern character to certain classes which exist globally, e.g. Chern classes or the Atiyah class. At some level we have already done this and at another we have already discussed the obstruction to doing so. We should probably also point out here that we do not know what Chern classes are for matrix factorizations.

We have taken the task of finding a global Chern character formula and more generally the boundary bulk map to mean the following: understand the image of the boundary bulk map in some computable model for $\mathbb{R}\Gamma(\Omega_{dw})$. This will be a Cech model and we will give our formula in terms of local connections on a Cech cover.

Lemma 5.9. Let \mathcal{E} be a matrix factorization with curved differential e . Suppose ∇ is a connection on \mathcal{E} , i.e. ∇ consists of standard connections on underlying graded components $\nabla_i : \mathcal{E}_0 \rightarrow \Omega \otimes \mathcal{E}_i$ for $i = 0, 1$. Then the morphism

$$\begin{array}{ccc} & \mathcal{A}t(\mathcal{E}) & \\ \swarrow & & \searrow \\ \mathcal{E} & \xrightarrow{\sim} & \Omega_{dw} \otimes \mathcal{E} \end{array}$$

which represents the map $\mathcal{E}xp(at(\mathcal{E}))$ in the coderived category from definition 5.5 is given by the map of w -curved complexes

$$\mathcal{E}xp(at(\mathcal{E})) = \sum_{i=0}^n \frac{\wedge[\nabla, e]^i}{i!}.$$

Proof. Given a connection ∇ on \mathcal{E} we obtain splittings $\mathcal{J}^1(\mathcal{E}_i) = \Omega^1 \otimes \mathcal{E}_i \oplus \mathcal{E}_i$ under these splittings the induced map by e on $\mathcal{J}^1(\mathcal{E})$ becomes

$$\begin{pmatrix} 1 \otimes e & [\nabla, e] \\ 0 & e \end{pmatrix},$$

The map

$$m : \Omega^{\otimes q+1} \otimes \mathcal{E}_i \oplus \Omega^q \otimes \mathcal{E}_i \rightarrow \Omega^q \otimes \mathcal{E}_i \oplus \Omega^{q-1} \mathcal{E}_i$$

is simply given by the projection

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and the map B_{dw} splits as

$$\begin{pmatrix} B'_{dw} & 0 \\ 0 & B''_{dw} \end{pmatrix}$$

where $B'_{dw} : \Omega^{\otimes q+1} \otimes \mathcal{E}_i \mapsto \Omega^{\otimes(q+2)} \otimes \mathcal{E}_i$ is given by

$$B'_{dw}(a_0 da_1 \otimes \cdots \otimes da_{q+1} \otimes e) = \sum_{j=0}^q (-1)^j a_0 da_1 \otimes \cdots \otimes da_j \otimes dw \otimes da_{j+1} \otimes \cdots \otimes da_{q+1} \otimes e$$

and $B''_{dw} : \Omega^{\otimes q} \otimes \mathcal{E}_i \mapsto \Omega^{\otimes(q+1)} \otimes \mathcal{E}_i$ is given by

$$B''_{dw}(a_0 da_1 \otimes \cdots \otimes a_q \otimes e) = \sum_{j=0}^q (-1)^j a_0 da_1 \otimes \cdots \otimes da_j \otimes dw \otimes da_{j+1} \otimes \cdots \otimes da_q \otimes e.$$

Using these splittings we may view $\sum_{i=0}^n [\nabla, e]^i$ as a degree 0 map $\mathcal{E} \rightarrow \mathcal{A}t(\mathcal{E})$ then it will suffice to show that that map is a closed morphism of w -curved modules since it obviously splits the weak-equivalence $\mathcal{A}t(\mathcal{E}) \rightarrow \mathcal{E}$ induced by projecting and we also have

$$\left(\sum_i \frac{\wedge}{i!} \right) \circ \left(\sum_i [\nabla, e]^i \right) = \exp(at(\mathcal{E})).$$

We first need to make the simple calculation:

$$e[\nabla, e] + [\nabla, e]e = \nabla w - w\nabla = dw.$$

Now

$$\begin{aligned} e\gamma[\nabla, e]^q - [\nabla, e]^q e &= (-1)^q e[\nabla, e]^q - [\nabla, e]^q e \\ &= (-1)^q \sum_{i=0}^{q-1} (-1)^i [\nabla, e]^i (e[\nabla, e] + [\nabla, e]e) [\nabla, e]^{q-i-1} \\ &= - \sum_{i=0}^{q-1} (-1)^{q-i-1} [\nabla, e]^i \otimes dw \otimes [\nabla, e]^{q-i-1} \\ &= -B''_{dw}[\nabla, e]^{q-1} \end{aligned}$$

The equality going from line 3 to 4 above is a bit tricky: The map $dw \otimes -$ sends

$$a_0 da_1 \otimes da_{q-i-1} \otimes e \mapsto a_0 da_1 \otimes da_{q-i-1} \otimes dw \otimes e$$

and then for the composition we have

$$\begin{array}{ccc} \mathcal{E}_j \xrightarrow{[d, \nabla]^{q-i-1}} \Omega^{\otimes q-i-1} \otimes \mathcal{E}_{j+q-i-1} & & \\ & \downarrow dw \otimes - & \\ \Omega^{\otimes q-i-1} \otimes \Omega^1 \otimes \mathcal{E}_{j+q-i-1} & \xrightarrow{[d, \nabla]^i} & \Omega^{\otimes q-i-1} \otimes \Omega^1 \otimes \Omega^i \otimes \mathcal{E}_{j+q-1} \end{array}$$

so computing $[\nabla, e]^i \otimes dw \otimes [e, \nabla]^{q-i-1}$ is the same as computing $[\nabla, e]^{q-1}$ and then inserting dw in the $q - i - 1$ st slot. Now we may compute

$$\begin{aligned} & \left[\begin{pmatrix} B'_{dw} & 0 \\ 0 & B''_{dw} \end{pmatrix} - \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} d\varepsilon\gamma & [\nabla, e]\gamma \\ 0 & e\gamma \end{pmatrix} \right] \begin{pmatrix} 0 \\ \sum_{i=0}^n [\nabla, e]^i \end{pmatrix} \\ &= \sum_{i=0}^n \begin{pmatrix} 0 \\ B''_{dw} [\nabla, e]^i \end{pmatrix} - \sum_{i=1}^n \begin{pmatrix} (-1)^i [\nabla, e]^i \\ 0 \end{pmatrix} \\ & \quad + \sum_{i=0}^n \begin{pmatrix} 0 \\ (-1)^i e [\nabla, e]^i \end{pmatrix} + \sum_{i=0}^n \begin{pmatrix} (-1)^i [\nabla, e]^{i+1} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \sum_{i=0}^n [\nabla, e]^i e \end{pmatrix} \end{aligned}$$

which finishes the proof. Note that the second sum on the second line starts at $i = 1$ because the component of the differential on $\mathcal{A}t(E)$ coming from m is 0 on $\mathcal{J}^1(\mathcal{E})$. \square

Remark 5.10. In the case when X is a formal disk, we recover the formula for the Chern Character from [PV2] by identifying the cohomology $\mathbb{R}\Gamma(\Omega_X)$ with the Tyurina algebra .

Naturality of the Chern character (or more generally the boundary bulk map) implies that it commutes with restriction to open subschemes. The above lemma tells

us what happens to the Chern Character upon restriction to open affine subschemes. Of course, upon restricting we loose information. The following lemmas describe how we can go the other direction. That is, they give us a method to take this local data (along with an appropriate collection of homotopies) to a global a global morphism to Cech Cohomology.

Lemma 5.11. Let $(\mathcal{F}, d_{\mathcal{F}})$ and $(\mathcal{G}, d_{\mathcal{G}})$ be complexes of sheaves (or w -curved $S(\mathcal{O}_X)$ modules) on X and U_1, \dots, U_n be a Cech cover of X . Denote

$$\mathcal{G}_{i_0 \dots i_p} = (j_{i_0 \dots i_p})_* \mathcal{G}|_{U_{i_0} \cap \dots \cap U_{i_p}},$$

where $j_{i_0 \dots i_p}$ is the inclusion of $U_{i_0} \cap \dots \cap U_{i_p}$ into X . Suppose we are given the following data: for each $0 \leq p \leq n$ and each tuple $i_0 i_1 \dots i_p$ with $1 \leq i_0 < i_1 < \dots < i_p \leq n$ we have a map

$$f_{i_0 \dots i_p} : \mathcal{F} \rightarrow \mathcal{G}_{i_0 \dots i_p}[p]$$

such that

$$d_{\mathcal{G}} f_{i_0 \dots i_p} - (-1)^p f_{i_0 \dots i_p} d_{\mathcal{F}} = \sum_{j=0}^p (-1)^k f_{i_0 \dots \widehat{i}_k \dots i_p} |_{U_{i_0 \dots i_p}},$$

then the map $f : \mathcal{F} \rightarrow \text{Cech}(\mathcal{G})$ defined on \mathcal{F}^q by

$$f = \sum_p \sum_{i_0 \dots i_p} (-1)^{\frac{p(p-1)}{2}} f_{i_0 \dots i_p}$$

is a closed degree 0 map of complexes.

Proof. First observe that $f_{i_0 \dots i_p}$ takes \mathcal{F}^q to $\mathcal{G}_{i_0 \dots i_p}^{q-p}$ and $\mathcal{G}_{i_0 \dots i_p}^{q-p}$ lives in degree q of $\text{Cech}(\mathcal{G})$, therefore the map f is indeed degree 0.

If we consider the composition cf where c is the Čech differential, we get

$$\begin{aligned} cf &= \sum_{p=0}^n \sum_{i_0 \dots i_p} (-1)^{\frac{p(p-1)}{2}} cf_{i_0 \dots i_p} \\ &= \sum_{p=0}^n \sum_{\substack{\mathbf{i}=i_0 \dots i_p \\ \mathbf{j}=j_0 \dots j_{p+1}}} (-1)^{\frac{p(p-1)}{2}} \sigma(\mathbf{i}, \mathbf{j}) f_{i_0 \dots i_p} |_{U_{j_0 \dots j_{p+1}}} \end{aligned}$$

where

$$\sigma(\mathbf{i}, \mathbf{j}) = \begin{cases} 1 & \text{if } i_0 \dots i_p = j_0 \dots \hat{j}_k \dots j_{p+1}, k \text{ even} \\ -1 & \text{if } i_0 \dots i_p = j_0 \dots \hat{j}_k \dots j_{p+1}, k \text{ odd} \\ 0 & \text{else} \end{cases}$$

On the other hand, by our assumption on $f_{i_0 \dots i_p}$, we have

$$\begin{aligned} \gamma d_{\mathcal{G}} f - f d_{\mathcal{F}} &= \sum_{p=0}^n \sum_{i_0 \dots i_p} (-1)^{\frac{p(p-1)}{2} + p} (d_{\mathcal{G}} f_{i_0 \dots i_p} - (-1)^p f_{i_0 \dots i_p} d_{\mathcal{F}}) \\ &= \sum_{p=0}^n \sum_{i_0 \dots i_p} (-1)^{\frac{p(p-1)}{2} + p} \sum_{k=0}^p (-1)^k f_{i_0 \dots \hat{i}_k \dots i_p} |_{U_{i_0 \dots i_p}} \\ &= \sum_{p=0}^n \sum_{\substack{\mathbf{i}=i_0 \dots i_p \\ \mathbf{j}=j_0 \dots j_{p+1}}} (-1)^{\frac{(p+1)(p)}{2} + p + 1} \sigma(\mathbf{i}, \mathbf{j}) f_{i_0 \dots i_p} |_{U_{j_0 \dots j_{p+1}}} \\ &= - \sum_{p=0}^n \sum_{\substack{\mathbf{i}=i_0 \dots i_p \\ \mathbf{j}=j_0 \dots j_{p+1}}} (-1)^{\frac{p(p-1)}{2}} \sigma(\mathbf{i}, \mathbf{j}) f_{i_0 \dots i_p} |_{U_{j_0 \dots j_{p+1}}} \\ &= -cf \end{aligned}$$

where γ is the grading operator on the Čech complex: $\gamma |_{\mathcal{G}_{i_0 \dots i_p}} = (-1)^p$. □

Lemma 5.12. Let \mathcal{E} be a matrix factorization with curved differential e . Let ∇_j be a choice of connection on U_j , where $\{U_j\}_{j=1}^N$ is a Cech cover of X . The collection of maps

$$f_{i_0 \dots i_p} : \mathcal{E} \rightarrow \mathcal{A}t(\mathcal{E})_{i_0 \dots i_p}[p],$$

$$f_{i_0 \dots i_p} = \sigma_{i_0} \sum_{k_0, k_1, \dots, k_p} \tau_p(k_0, \dots, k_p) [e, \nabla_{i_0}]^{k_0} (\nabla_{i_0} - \nabla_{i_1}) [e, \nabla_{i_1}]^{k_1} (\nabla_{i_1} - \nabla_{i_2}) \dots [e, \nabla_{i_p}]^{k_p}$$

$\tau_p(k_0, \dots, k_p) = (-1)^{\sum_{j=0}^p j(k_j+1)}$ and $\sigma_{i_0} : \Omega^{\otimes q} \otimes \mathcal{E} \rightarrow \Omega^{\otimes q} \otimes \mathcal{J}^1(\mathcal{E})$ is the splitting induced by ∇_{i_0} satisfies the hypothesis of lemma 5.11.

Proof. We know from the proof of lemma 5.9 that

$$(-1)^k e[\nabla_{i_0}, e] - [\nabla_{i_0}, e]^k e = -B_{dw}[\nabla_{i_0}, e]^{k-1}$$

so that

$$\gamma e f_{i_0} - f_{i_0} e = -B_{dw} f_{i_0}.$$

We claim that

$$e f_{i_0 \dots i_p} - (-1)^p f_{i_0 \dots i_p} e = -B_{dw} + \sum_{j=0}^p (-1)^j f_{i_0 \dots \widehat{i}_j \dots i_p}.$$

We will prove this by induction, but before we do, let us see how this proves the lemma.

Recall from the proof of lemma 5.9 that after splitting $\mathcal{A}t(\mathcal{E})$ with respect to ∇_{i_0} the differential is given as the sum of three components

$$\gamma e = (-1)^q \begin{pmatrix} 1 \otimes e & [\nabla_{i_0}, e] \\ 0 & e \end{pmatrix} : \Omega^{\otimes q+1} \otimes \mathcal{E}_i \oplus \Omega^q \otimes \mathcal{E}_i \rightarrow \Omega^{q+1} \otimes \mathcal{E}_{i+1} \oplus \Omega^q \otimes \mathcal{E}_{i+1}$$

$$-\gamma m = (-1)^q \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} : \Omega^{q+1} \otimes \mathcal{E}_i \oplus \Omega^q \otimes \mathcal{E}_i \rightarrow \Omega^q \otimes \mathcal{E}_i \oplus \Omega^{q-1} \otimes \mathcal{E}_i$$

and

$$B_{dw} = \begin{pmatrix} B'_{dw} & 0 \\ 0 & B''_{dw} \end{pmatrix} : \Omega^{q+1} \otimes \mathcal{E}_i \oplus \Omega^q \otimes \mathcal{E}_i \rightarrow \Omega^{q+2} \otimes \mathcal{E}_i \oplus \Omega^{q+1} \otimes \mathcal{E}_i.$$

We have the relation

$$\begin{aligned} & [\nabla_{i_0}, e] f_{i_0 \dots i_p} - f_{i_0 \dots i_p} \\ &= - \sum_{k_1 \dots k_p} (-1)^{\sum_{j=1}^p j(k_j+1)} (\nabla_{i_0} - \nabla_{i_1}) [\nabla_{i_1}, e]^{k_1} \dots [\nabla_{i_p}, e]^{k_p} \\ &= (\nabla_{i_1} - \nabla_{i_0}) \sum_{k_1 \dots k_p} (-1)^{k_1 + \dots + k_p + p - 1} (-1)^{\sum_{j=0}^p j(k_j+1)} (\nabla_{i_0} - \nabla_{i_1}) [\nabla_{i_1}, e]^{k_1} \dots [\nabla_{i_p}, e]^{k_p} \end{aligned}$$

So when we take into account the grading operator γ we get

$$(\gamma[\nabla_{i_0}, e] - \gamma) f_{i_0 \dots i_p} = (\nabla_{i_1} - \nabla_{i_0}) f_{i_1 \dots i_p}.$$

Now the observation is that $(\nabla_{i_1} - \nabla_{i_0}) f_{i_1 \dots i_p}$ is exactly the difference between splitting with respect to ∇_{i_1} and splitting with respect to ∇_{i_0} i.e.

$$\sigma_{i_0} f_{i_1 \dots i_p} + (\nabla_{i_1} - \nabla_{i_0}) f_{i_1 \dots i_p} = \sigma_{i_1} f_{i_1 \dots i_p}$$

This combined with the claim gives us the lemma.

To prove the claim, notice first that we can write

$$f_{i_0 \dots i_p} = \sum_{k=0}^n (-1)^{kp+p} f_{i_0 \dots i_{p-1}} (\nabla_{i_{p-1}} - \nabla_{i_p}) [\nabla_{i_p}, d]^k.$$

Then we make the computation:

$$\begin{aligned}
\gamma e f_{i_0 \dots i_p} - (-1)^p f_{i_0 \dots i_p} e &= \sum_k (-1)^{kp+p+k+1} (\gamma e f_{i_0 \dots i_{p-1}} - (-1)^{p-1} f_{i_0 \dots i_{p-1}}) (\nabla_{i_{p-1}} - \nabla_{i_p}) [\nabla_{i_p}, e]^k \\
&+ \sum_k (-1)^{kp+k} f_{i_0 \dots i_{p-1}} ([\nabla_{i_p}, e] - [\nabla_{i_{p-1}}, e]) [\nabla_{i_p}, e]^k \\
&+ \sum_k (-1)^{kp} f_{i_0 \dots i_{p-1}} (\nabla_{i_{p-1}} - \nabla_{i_p}) ((-1)^k e [\nabla_{i_p}, e]^k - [\nabla_{i_p}, e]^k e) \\
&= \sum_k (-1)^{kp+k+p+1} (-B_{dw} f_{i_0 \dots i_{p-1}}) (\nabla_{i_{p-1}} - \nabla_{i_p}) [\nabla_{i_p}, e]^k
\end{aligned} \tag{Equation 5.4.}$$

$$+ \sum_k (-1)^{(k+1)(p-1)} \left(\sum_{j=0}^{p-1} (-1)^j f_{i_0 \dots \widehat{i_j} \dots i_{p-1}} \right) (\nabla_{i_{p-1}} - \nabla_{i_p}) [\nabla_{i_p}, e]^k \tag{Equation 5.5.}$$

$$+ \sum_k (-1)^{kp+k} f_{i_0 \dots i_{p-1}} ([\nabla_{i_p}, e] - [\nabla_{i_{p-1}}, e]) [\nabla_{i_p}, e]^k \tag{Equation 5.6.}$$

$$+ \sum_k (-1)^{kp} f_{i_0 \dots i_{p-1}} (\nabla_{i_{p-1}} - \nabla_{i_p}) (-B_{dw} [\nabla_{i_p}, e]^{k-1}) \tag{Equation 5.7.}$$

Note that the sign $(-1)^{kp+k+p+1}$ on the first line appears because applying $(\nabla_{i_{p-1}} - \nabla_{i_p}) [\nabla_{i_p}, e]^k$ before $f_{i_0 \dots i_{p-1}}$ introduces an extra $k + 1$ tensor factors of Ω^1 . So what we will need to show is that lines (Equation 5.4.), (Equation 5.5.), (Equation 5.6.) and (Equation 5.7.) sum to give

$$-B_{dw} f_{i_0 \dots i_p} + \sum_{j=0}^p (-1)^j f_{i_0 \dots \widehat{i_j} \dots i_p}.$$

Now we note that

$$\begin{aligned}
&(-1)^{k+1} (-B_{dw} f_{i_0 \dots i_{p-1}}) (\nabla_{i_{p-1}} - \nabla_{i_p}) [\nabla_{i_p}, e]^k \\
&+ f_{i_0 \dots i_{p-1}} (\nabla_{i_{p-1}} - \nabla_{i_p}) (-B_{dw} [\nabla_{i_p}, e]^{k-1}) \\
&= B_{dw} f_{i_0 \dots i_{p-1}} (\nabla_{i_{p-1}} - \nabla_{i_p}) [\nabla_{i_p}, e]^k
\end{aligned}$$

so the sums

$$\sum_k (-1)^{kp} f_{i_0 \dots i_{p-1}} (\nabla_{i_{p-1}} - \nabla_{i_p}) (-B_{dw} [\nabla_{i_p}, e]^{k-1})$$

from (Equation 5.7.) and

$$\sum_{k=0}^{\infty} (-1)^{kp+k+p+1} (-B_{dw} f_{i_0 \dots i_{p-1}}) (\nabla_{i_{p-1}} - \nabla_{i_p}) [\nabla_{i_p}, e]^k$$

from (Equation 5.4.) add to give $-B_{dw} f_{i_0 \dots i_p}$ as needed.

We have the following relation

$$f_{i_0 \dots i_{p-1}} [\nabla_{i_{p-1}}, e] = (-1)^{p-1} f_{i_0 \dots i_{p-1}} - f_{i_0 \dots i_{p-2}} (\nabla_{i_{p-2}} - \nabla_{i_{p-1}})$$

so then for the sum from (Equation 5.6.) we have

$$\begin{aligned} & \sum_k (-1)^{kp+k} f_{i_0 \dots i_{p-1}} ([\nabla_{i_p}, e] - [\nabla_{i_{p-1}}, e]) [\nabla_{i_p}, e]^k \\ &= \sum_k (-1)^{kp+k} f_{i_0 \dots i_{p-1}} [\nabla_{i_p}, e]^{k+1} \\ & \quad - \sum_k (-1)^{kp+k} f_{i_0 \dots i_{p-1}} [\nabla_{i_{p-1}}, e] [\nabla_{i_p}, e]^k \\ &= \sum_k (-1)^{kp+k} f_{i_0 \dots i_{p-1}} [\nabla_{i_p}, e]^{k+1} \\ & \quad + \sum_k (-1)^{kp+k+p} f_{i_0 \dots i_{p-1}} [\nabla_{i_p}, e]^k \\ & \quad + \sum_k (-1)^{kp+k+1} f_{i_0 \dots i_{p-2}} (\nabla_{i_{p-2}} - \nabla_{i_{p-1}}) [\nabla_{i_p}, e]^k \\ &= (-1)^p f_{i_0 \dots i_{p-1}} \tag{Equation 5.8.} \\ & \quad + \sum_k (-1)^{kp+k+1} f_{i_0 \dots i_{p-2}} (\nabla_{i_{p-2}} - \nabla_{i_{p-1}}) [\nabla_{i_p}, e]^k \tag{Equation 5.9.} \end{aligned}$$

Now we turn our attention to the sum

$$\sum_{k=0}^n (-1)^{(k+1)(p-1)} \left(\sum_{j=0}^{p-1} (-1)^j f_{i_0 \dots \widehat{i_j} \dots i_{p-1}} \right) (\nabla_{i_{p-1}} - \nabla_{i_p}) [\nabla_{i_p}, e]^k$$

from (Equation 5.5.) In the case when $j \neq p-1$ we have

$$\sum_{k=0}^{\infty} (-1)^{(k+1)(p-1)} f_{i_0 \dots \widehat{i_j} \dots i_{p-1}} (\nabla_{i_{p-1}} - \nabla_{i_p}) [\nabla_{i_p}, e]^k = f_{i_0 \dots \widehat{i_j} \dots i_p} \quad (\text{Equation 5.10.})$$

When $j = p-1$ we may add

$$\sum_{k=0}^n (-1)^{(k+1)(p-1)+p-1} f_{i_0 \dots i_{p-2}} (\nabla_{i_{p-1}-\nabla_{i_p}}) [\nabla_{i_p}, e]^k$$

to

$$\sum_{k=0}^n (-1)^{kp+k+1} f_{i_0 \dots i_{p-2}} (\nabla_{i_{p-2}} - \nabla_{i_{p-1}}) [\nabla_{i_p}, e]^k$$

from (Equation 5.9.) to get

$$(-1)^{p-1} \sum_{k=0}^{\infty} (-1)^{k(p-1)} f_{i_0 \dots i_{p-2}} (\nabla_{i_{p-2}} - \nabla_{i_p}) [\nabla_{i_p}, e]^k = (-1)^{p-1} f_{i_0 \dots \widehat{i_{p-1}} i_p}. \quad (\text{Equation 5.11.})$$

Adding the sums from (Equation 5.8.), (Equation 5.10.) and (Equation 5.11.) gives

$$\sum_{j=0}^p (-1)^j f_{i_0 \dots \widehat{i_j} \dots i_p}$$

which finishes the claim and thus the lemma. □

Theorem 5.13. Let $f = \{f_{i_0 \dots i_p}\} \in Cech(\mathcal{H}om(\mathcal{E}, \mathcal{E}))$. The boundary bulk map $\tau_{\mathcal{E}}$ is computed on f as

$$f_{i'_0 \dots i'_q} \mapsto str \left(\sum_p \sum_{i_0 \dots i_p} \sum_{k_1 \dots k_p} (-1)^{p + \sum_{j=0}^p j k_j} \frac{[\nabla_{i_0}, e]^{k_0} (\nabla_{i_0} - \nabla_{i_1}) \dots (\nabla_{i_{p-1}} - \nabla_{i_p}) [\nabla_{i_p}, e]^{k_p}}{(k_0 + \dots + k_p + p)!} \circ f_{i'_0 \dots i'_q} \right)$$

Proof. This is mostly an amalgamation of lemmas 5.11, 5.12 and 5.6. The division by $(k_0 + \dots + k_p + p)!$ comes from applying the map

$$\sum \frac{\wedge}{i!} : Cech(\mathcal{A}t(\mathcal{E})) \rightarrow Cech(\Omega_{dw} \otimes \mathcal{E})$$

(see lemma 5.4). The sign comes from the fact that

$$\frac{p(p-1)}{2} + \sum_{j=0}^p j(k_j + 1) = \frac{p(p-1)}{2} + \frac{p(p+1)}{2} + \sum_{j=0}^p j k_j = p^2 + \sum_{j=0}^p j k_j$$

and p is congruent to p^2 modulo 2. We need only check that this map we have constructed actually computes the boundary bulk-map.

We have the following diagram

$$\begin{array}{ccc} \mathcal{A}t(\mathcal{E}) & \longrightarrow & Cech(\mathcal{A}t(\mathcal{E})) \\ \pi \downarrow & \nearrow & \downarrow Cech(\pi) \\ \mathcal{E} & \longrightarrow & Cech(\mathcal{E}) \end{array}$$

where the diagonal map is given by

$$\sum_p \sum_{i_0 \dots i_p} \sum_{k_1 \dots k_p} (-1)^{p^2 + \sum_{j=0}^p j k_j} [\nabla_{i_0}, e]^{k_0} (\nabla_{i_0} - \nabla_{i_1}) \dots (\nabla_{i_{p-1}} - \nabla_{i_p}) [\nabla_{i_p}, e]^{k_p}. \quad (\text{Equation 5.12.})$$

Now the outside square commutes as well as the bottom right triangle. And then, since $Cech(\pi)$ is a weak equivalence, the upper left triangle commutes in the coderived

category. It follows then that composition of the diagonal map, (Equation 5.12.), with the map $\sum_i \frac{\Delta}{i!} : Cech(\mathcal{A}t(\mathcal{E})) \rightarrow Cech(\Omega_{dw})$ computes $\iota \mathcal{E}xp(\mathcal{A}t(\mathcal{E}))$, where $\iota : \Omega_{dw} \rightarrow Cech(\Omega_{dw})$ is the inclusion. \square

Corollary 5.14. The Chern Character \mathcal{E} is given by the cocycle

$$ch(\mathcal{E}) = str \left(\sum_p \sum_{i_0 \dots i_p} \sum_{k_1 \dots k_p} (-1)^{p + \sum_{j=0}^p j k_j} \frac{[\nabla_{i_0}, e]^{k_0} (\nabla_{i_0} - \nabla_{i_1}) \dots (\nabla_{i_{p-1}} - \nabla_{i_p}) [\nabla_{i_p}, e]^{k_p}}{(k_0 + \dots + k_p + p)!} \right)$$

in a Cech model for $\mathbb{R}\Gamma(\Omega_{dw})$.

Remark 5.15. In light of Remark 5.7 and Theorem 5.8, Corollary 5.14 translates directly to give a formula for the Chern character of complexes of vector bundles.

REFERENCES CITED

- [BN] Marcel Bökstedt and Amnon Neeman. Homotopy limits in triangulated categories. *Compositio Math.*, 86(2):209–234, 1993.
- [BvdB] A. Bondal and M. van den Bergh. Generators and representability of functors in commutative and noncommutative geometry. *Mosc. Math. J.*, 3(1):1–36, 258, 2003.
- [Căl] Andrei Căldăraru. The Mukai pairing. II. The Hochschild-Kostant-Rosenberg isomorphism. *Adv. Math.*, 194(1):34–66, 2005.
- [Dyc] Tobias Dyckerhoff. Compact generators in categories of matrix factorizations. *Duke Math. J.*, 159:223–274, 2011.
- [Hov] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.
- [Kel1] Bernhard Keller. On the cyclic homology of exact categories. *J. Pure Appl. Algebra*, 136(1):1–56, 1999.
- [Kel2] Bernhard Keller. On differential graded categories. In *International Congress of Mathematicians. Vol. II*, pages 151–190. Eur. Math. Soc., Zürich, 2006.
- [Mar] Nikita Markarian. The Atiyah class, Hochschild cohomology and the Riemann-Roch theorem. *J. Lond. Math. Soc. (2)*, 79(1):129–143, 2009.
- [Orl] Dmitri Orlov. Matrix factorizations for nonaffine LG-models. *Math. Ann.*, 353(1):95–108, 2012.
- [PL] Daniel Pomerlenano and Kevin Lin. Global matrix factorizations. *Preprint math.AG/1011.5847v1*, 2011.
- [Pos1] Leonid Positselski. Coherent analogs of matrix factorizations and relative singularity categories. *Preprint math.CT/1102.0261v4*, 2011.
- [Pos2] Leonid Positselski. Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence. *Mem. Amer. Math. Soc.*, 212(996):vi+133, 2011.
- [PP] A. Polishchuk and L. Positselski. Hochschild (co)homology of the second kind i. *Trans. Amer. Math. Soc.*, 364(10):5311–5368, 258, 2012.
- [Pre] Anatoly Preygel. Thom-Sebastiani & duality for matrix factorizations. *Preprint, math.AG/1101.5834v1*, 2011.

- [PV1] Alexander Polishchuk and Arkady Vaintrob. Matrix factorizations and singularity categories for stacks. *Preprint math.AG/1011.4544v1*, 2010.
- [PV2] Alexander Polishchuk and Arkady Vaintrob. Chern characters and Hirzebruch-Riemann-Roch formula for matrix factorizations. *Duke Math. J.*, 161:1863–1926, 2012.
- [Ram] Ajay C. Ramadoss. The Mukai pairing and integral transforms in Hochschild homology. *Mosc. Math. J.*, 10(3):629–645, 662–663, 2010.
- [Rou] Raphael Rouquier. Dimensions of triangulated categories. *Journal of K-Theory. K-Theory and its Applications in Algebra, Geometry, Analysis & Topology*, 1:193–256, 2008.
- [Shi] Ian Shipman. *A geometric approach to Orlov’s theorem*. ProQuest LLC, Ann Arbor, MI, 2011. Thesis (Ph.D.)—The University of Chicago.
- [Shk] Dmytro Shklyarov. *Hirzebruch-Riemann-Roch theorem for differential graded algebras*. ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)—Kansas State University.
- [Toë] Bertrand Toën. Lectures on DG-categories. In *Topics in algebraic and topological K-theory*, volume 2008 of *Lecture Notes in Math.*, pages 243–302. Springer, Berlin, 2011.
- [TV] Bertrand Toën and Michel Vaquié. Moduli of objects in dg-categories. *Ann. Sci. École Norm. Sup. (4)*, 40(3):387–444, 2007.
- [Yek] Amnon Yekutieli. The continuous Hochschild cochain complex of a scheme. *Canadian Journal of Math*, 54(6):1319–1337, 2002.