SOLVABLE PARTICLE MODELS RELATED TO THE β-ENSEMBLE

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DISSERTATION ABSTRACT

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For $\beta > 0$, the β-ensemble corresponds to the joint probability density on the real line proportional to

$$\prod_{n>m}^{N} |x_n - x_m|^\beta \prod_{n=1}^{N} w(x_n)$$

where w is the weight of the system. It has the application of being the Boltzmann factor for the configuration of N charge-one particles interacting logarithmically on an infinite wire inside an external field $Q = -\log w$ at inverse temperature β. Similarly, the circular β-ensemble has joint probability density proportional to

$$\prod_{n>m}^{N} \left|e^{i\theta_n} - e^{i\theta_m}\right|^\beta \prod_{n=1}^{N} w(x_n) \quad \text{for} \quad \theta_n \in [-\pi, \pi)$$

and can be interpreted as N charge-one particles on the unit circle interacting logarithmically with no external field. When $\beta = 1, 2,$ and 4, both ensembles are said to be solvable in that their correlation functions can be expressed in a form

which allows for asymptotic calculations. It is not known, however, whether the general β-ensemble is solvable.

We present four families of particle models which are solvable point processes related to the β-ensemble. Two of the examples interpolate between the circular β-ensembles for $\beta = 1, 2$, and 4. These give alternate ways of connecting the classical β-ensembles besides simply changing the values of β. The other two examples are "mirrored" particle models, where each particle has a paired particle reflected about some point or axis of symmetry.

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TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

INTRODUCTION

Notation I.1. The following notation will be used throughout:

$$\delta_{n,m} = \begin{cases} 1 & \text{if } n = m \\ \\ 0 & \text{if } n \neq m \end{cases}$$

$$\boldsymbol{x}_N = (x_1, \ldots, x_N)$$

$$\boldsymbol{x}_N \vee \boldsymbol{y}_M = (x_1, \ldots, x_N, y_1, \ldots, y_M)$$

$$d\boldsymbol{x}_N = dx_1 \cdots dx_N$$

$$\underline{N} = \{1, \ldots, N\}$$

If $N \geq M$, then $\mathfrak{t} : \underline{M} \nearrow \underline{N}$ is a strictly increasing function, and $\mathfrak{t}' : \underline{N - M} \nearrow \underline{N}$ is the unique strictly increasing function whose range is disjoint from $\mathfrak{t}$. Denote by $\operatorname{sgn} \mathfrak{t}$ the sign of the permutation necessary to put the sequence

$$\mathfrak{t}(1), \ldots, \mathfrak{t}(M), \mathfrak{t}'(1), \ldots, \mathfrak{t}'(N - M)$$

back in order.

The Gaussian and Circular Ensembles

In the 1950s, Eugene Wigner introduced random matrices to study the energy level behavior of highly excited nuclei. According to quantum mechanics, the energy levels are the spectrum of the nuclear system's Hamiltonian, which is a Hermitian operator on an infinite dimensional Hilbert space. While the Hamilton is unknown (and would be too complicated to write down if known), Wigner hypothesized that the Hamiltonian can instead be approximated by a large matrix selected randomly from a family of Hermitian matrices [44], and thus that the eigenvalues of the random matrix predict the average behavior of the energy levels. Depending on the physical specifications of the nuclear system, the matrix entries are real (time-reversal invariant), complex (not time-reversal invariant), or quaternion (time-reversal invariant but do not have spin-rotational symmetry). Because the space of Hermitian matrices is not compact, the entries are chosen to have Gaussian distribution to maximize entropy.

The random matrix ensemble corresponding to complex entries is called the Gaussian unitary ensemble (GUE). More precisely, let $\{X_{n,m}, Y_{n,m} : 1 \leq n < m \leq N\}$ be a set of independent normal random variables with mean 0 and variance $\frac{1}{2}$, and let $\{X_{n,n} : 1 \leq n \leq N\}$ be a set of independent normal random variables with mean 0 and variance 1, with independence between the two sets of random variables. Then the random points of the GUE are the N eigenvalues generated from the $N \times N$ random Hermitian matrix

$$\mathbf{X} = \begin{bmatrix} X_{1,1} & X_{1,2} + iY_{1,2} & \cdots & X_{1,N} + iY_{1,N} \\ X_{1,2} - iY_{1,2} & X_{2,2} & \cdots & X_{2,N} + iY_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ X_{1,N} - iY_{1,N} & X_{2,N} - iY_{2,N} & \cdots & X_{N,N} \end{bmatrix}.$$

When the matrix entries in the Gaussian ensembles are real and quaternion, the ensembles are called the Gaussian orthogonal ensemble (GOE) and the Gaussian sympletic ensemble (GSE), respectively. The joint distributions of the entries are

$$c \cdot \exp\left\{-\frac{\beta}{4}\mathrm{tr}(\mathbf{X}^2)\right\}$$

where the GOE corresponds to $\beta = 1$, GUE to $\beta = 2$, and GSE to $\beta = 4$, and c is some constant depending on N and β. These distributions are invariant under the actions by the orthogonal, unitary, and symplectic groups, respectively. By the spectral theorem for Hermitian matrices, we can decompose the matrix uniquely into a diagonal matrix of eigenvalues (in increasing order) and a unitary matrix of eigenvectors. After applying this change of variables and integrating out the eigenvector variables, for $w(x) = e^{-\frac{\beta}{4}x^2}$, the eigenvalue distribution is

$$\frac{1}{Z_{N;\beta} N!}\Omega_{N;\beta}(\boldsymbol{x}_N)$$

where

$$\Omega_{N;\beta}(\boldsymbol{x}_N) = \prod_{n>m}^{N} |x_n - x_m|^{\beta} \prod_{n=1}^{N} w(x_n), \tag{I.1}$$

and the normalization, called the *partition function*, is

$$Z_{N;\beta} = \frac{1}{N!}\int_{\mathbb{R}^N} \Omega_{N;\beta}(\boldsymbol{x}_N)\, d\boldsymbol{x}_N. \tag{I.2}$$

Freeman Dyson [10] later introduced the circular orthogonal (COE), unitary (CUE), and symplectic (CSE) ensembles corresponding to the distribution of points on the unit circle proportional to

$$\Omega_{N;\beta} = \prod_{n>m}^{N} \left| e^{i\theta_n} - e^{i\theta_m} \right|^{\beta} \tag{I.3}$$

for $\beta = 1$, $\beta = 2$, and $\beta = 4$. These are the joint distributions of eigenvalues from random unitary matrices chosen according to Haar measure with real, complex, and real quaternion entries. Dyson's motivation was to alter Wigner's energy level model so that each state could be selected uniformly at random. While the circular ensemble does not realistically depict the energy levels of a nucleus because the energy level range of a nucleus is infinite, the statistics in any small region of the circular ensemble match the statistics in any region in the *bulk* (inside the particle support) of the Gaussian ensemble [23].

This leads to the concept of *universality* in random matrix theory, which states that in certain classes of random matrices, local statistics of eigenvalues are independent of the matrix entry distributions as the dimension of the matrix increases. Much of today's study focuses on the description of these classes. An example is the *Wigner matrix*, which is a complex Hermitian random matrix such that

1. the upper triangular entries are identically distributed with finite moments and variance equal to one,
2. the diagonal entries are identically distributed,

3. all the entries have mean zero,

4. and all the entries are independent (besides the symmetry condition).

Tao and Vu's four moment theorem [37] states that the local statistics of eigenvalues of a Wigner matrix depend only on the first four moments of the matrix entries.

We will later show that for some point processes (like the Gaussian and circular ensembles), the local statistics are encoded in a single object called the *kernel*. As the number of points N increases to infinity, the three limiting kernels which commonly appear are the *sine kernel*, *Airy kernel*, and *Bessel kernel*; see (I.8), (I.9), and (I.10) below.

The statistics from random matrices have appeared in other areas of mathematics. For example, in [3], the authors derive the limiting distribution of the length of the longest increasing subsequence from a permutation in S_N selected uniformly at random as $N \to \infty$, where S_N is the permutation group on N elements. This distribution is the same as the limiting distribution of the largest eigenvalue of the GUE ensemble, called the *Tracy-Widom distribution*, which we mention again below.

The β-Ensemble and 2-Dimensional Electrostatics For $\beta > 0$, the point process with density of points (I.1) and general weight w is called the β-ensemble. This distribution has interpretations in 2-dimensional electrostatics. Consider N charge-one particles on the real line interacting logarithmically, i.e. the energy between two particles located at x and y is

$$-\log|x-y|.$$

The particles lie in an external field Q which keeps the particles from escaping to infinity. The total potential energy of the system is

$$E_N(\boldsymbol{x}_N) = -\sum_{n>m} \log|x_n - x_m| + \sum_{n=1}^{N} Q(x_n).$$

The *Boltzmann factor* with inverse temperature equal to β, which gives the relative density of states, is

$$e^{-\beta E_N(\boldsymbol{x}_N)} = \prod_{n>m}^{N} |x_n - x_m|^{\beta} \prod_{n=1}^{N} e^{-\beta Q(x_n)}.$$

With $w(x) = e^{-\beta Q(x)}$, the Boltzmann factor is the same as (I.1). For the circular ensemble, we have an analogous 2-dimensional electrostatic interpretation for particles on the unit circle in which the Boltzmann factor is (I.3).

Correlation Functions We present this section specifically for the β-ensemble on the real line, and any analogous definitions can be translated to the circular β-ensemble by changing the domain from $\mathbb{R}$ to $[-\pi,\pi)$. For a fixed N and $n \leq N$, let $D_1,\ldots,D_n$ be a family of mutually disjoint subsets of $\mathbb{R}$. Denote $\mathcal{X}(D_i)$ by the random number of points that land in D_i. The *correlation functions* are a family of N functions $R^N_{n;\beta} : \mathbb{R}^n \to [0,\infty)$ defined by

$$\mathbb{E}\left[\prod_{i=1}^n \mathcal{X}(D_i)\right] = \int_{D_1} \cdots \int_{D_n} R^N_{n;\beta}(\boldsymbol{x}_n)\, d\boldsymbol{x}_n.$$

We can write the correlation functions in terms of the joint density. For $x_1, \ldots, x_n \in \mathbb{R}$, let $\varepsilon > 0$ be such that $B_\varepsilon(x_1), \ldots, B_\varepsilon(x_n)$ are mutually disjoint. Let $D_\varepsilon = \mathbb{R} \backslash \bigcup B_\varepsilon(x_i)$. For ε small, we can approximate (I.4) by

$$\begin{aligned}
&\mathbb{P}[\mathcal{X}(D_1) = 1, \ldots, \mathcal{X}(D_n) = 1, \mathcal{X}(D) = N - n] \\
&\quad = \frac{1}{Z_{N;\beta} N!}\binom{N}{N-n, 1, \ldots, 1} \int_{B_\varepsilon(x_1)} \cdots \int_{B_\varepsilon(x_n)} \int_{D_\varepsilon^{N-n}} \Omega_{N;\beta}(\boldsymbol{x}_n \vee \boldsymbol{y}_{N-n})\, d\boldsymbol{y}_{N-n} d\boldsymbol{x}_n. \\
&\quad = \int_{B_\varepsilon(x_1)} \cdots \int_{B_\varepsilon(x_n)} \left[\frac{1}{Z_{N;\beta}(N-n)!} \int_{D_\varepsilon^{N-n}} \Omega_{N;\beta}(\boldsymbol{x}_n \vee \boldsymbol{y}_{N-n})\, d\boldsymbol{y}_{N-n}\right] d\boldsymbol{x}_n.
\end{aligned}$$

As $\varepsilon \to 0$,

$$\frac{1}{(\pi\varepsilon^2)^n} \int_{B_\varepsilon(x_1)} \cdots \int_{B_\varepsilon(x_n)} R^N_{n;\beta}(\boldsymbol{x}_n)\, d\boldsymbol{x}_n \to R^N_{n;\beta}(\boldsymbol{x}_n)$$

and

$$\begin{aligned}
&\frac{1}{(\pi\varepsilon^2)^n} \int_{B_\varepsilon(x_1)} \cdots \int_{B_\varepsilon(x_n)} \left[\frac{1}{Z_{N;\beta}(N-n)!} \int_{D_\varepsilon^{N-n}} \Omega_{N;\beta}(\boldsymbol{x}_n \vee \boldsymbol{y}_{N-n})\, d\boldsymbol{x}_n\right] d\boldsymbol{y}_{N-n} \\
&\quad \to \frac{1}{Z_{N;\beta}(N-n)!} \int_{\mathbb{R}^{N-n}} \Omega_{N;\beta}(\boldsymbol{x}_n \vee \boldsymbol{y}_{N-n})\, d\boldsymbol{y}_{N-n}
\end{aligned}$$

at all Lebesgue points. Because the correlation functions we are working with are continuous, we can write

$$R^N_{n;\beta}(\boldsymbol{x}_n) = \frac{1}{Z_{N;\beta}(N-n)!} \int_{\mathbb{R}^{N-n}} \Omega_{N;\beta}(\boldsymbol{x}_n \vee \boldsymbol{y}_{N-n})\, d\boldsymbol{y}_{N-n}. \tag{I.4}$$

Many important statistics of the ensemble are derived in terms of the correlation functions, like the gap probabilities described below. As mentioned above we are particularly interested in the situation where the dimension of the matrix N increases to infinity. Note that the correlation functions become difficult to work with since the number of iterated integrals increases with N. For certain values of β, correlation functions can be written in a more tractable form.

Determinantal Point Process

When $\beta = 2$ in (I.1), the corresponding point process is a *determinantal point process*, i.e. the nth correlation function can be expressed as a determinant of a scalar kernel. We follow the calculation in [41] for the β-ensemble in the real line, which we divide into two parts.

The Partition Function as a Determinant

The first step is to write the partition function as a determinant of a Gram matrix. The product

$$\prod_{n>m} (x_n - x_m)$$

can be expressed as the determinant of the *Vandermonde matrix*

$$\mathbf{V}_N(\boldsymbol{x}_N) = \begin{bmatrix} \rho_0(x_1) & \cdots & \rho_0(x_N) \\ \rho_1(x_1) & \cdots & \rho_1(x_N) \\ \vdots & \ddots & \vdots \\ \rho_{N-1}(x_1) & \cdots & \rho_{N-1}(x_N) \end{bmatrix}$$

where $\{\rho_n\}$ is any family of monic polynomials such that $\deg(\rho_n) = n$. Then

$$Z_{N;2} = \frac{1}{N!} \int_{\mathbb{R}^N} (\det \mathbf{V}_N(\boldsymbol{x}_N))^2 \prod_{n=1}^{N} w(x_n)\, d\boldsymbol{x}_N.$$

Next, using the Leibniz formula for the determinant,

$$\begin{aligned} Z_{N;2} &= \frac{1}{N!} \sum_{\tau \in S_N} \sum_{\sigma \in S_N} \operatorname{sgn}\tau \operatorname{sgn}\sigma \int_{\mathbb{R}^N} \prod_{n=1}^{N} \rho_{\sigma(n)}(x_n)\rho_{\tau(n)}(x_n) w(x_n)\, d\boldsymbol{x}_N \\ &= \frac{1}{N!} \sum_{\tau \in S_N} \sum_{\sigma \in S_N} \operatorname{sgn}\tau \operatorname{sgn}\sigma \prod_{n=1}^{N} \int_{\mathbb{R}} \rho_{\sigma(n)}(x)\rho_{\tau(n)}(x) w(x)\, dx. \end{aligned}$$

Now we eliminate one of the sums. Let

$$\langle f | g \rangle = \int_{\mathbb{R}} f(x) g(x) w(x)\, dx.$$

We change variables in the product by substituting $\tau^{-1}(n)$ for n to get

$$\begin{aligned} Z_{N;2} &= \frac{1}{N!} \sum_{\tau \in S_N} \sum_{\sigma \in S_N} \operatorname{sgn}\tau \operatorname{sgn}\sigma \prod_{n=1}^{N} \int_{\mathbb{R}} \rho_{\sigma(\tau^{-1}(n))}(x)\rho_{\tau(\tau^{-1}(n))}(x) w(x)\, dx \\ &= \frac{1}{N!} \sum_{\tau \in S_N} \sum_{\sigma\tau^{-1} \in S_N} \operatorname{sgn}\tau \operatorname{sgn}(\sigma\tau^{-1}) \prod_{n=1}^{N} \int_{\mathbb{R}} \rho_{\sigma(\tau^{-1}(n))}(x)\rho_n(x) w(x)\, dx \\ &= \frac{1}{N!} \sum_{\tau \in S_N} \sum_{\sigma \in S_N} \operatorname{sgn}\sigma \prod_{n=1}^{N} \int_{\mathbb{R}} \rho_{\sigma(n)}(x)\rho_n(x) w(x)\, dx \\ &= \sum_{\sigma \in S_N} \operatorname{sgn}\sigma \prod_{n=1}^{N} \int_{\mathbb{R}} \rho_{\sigma(n)}(x)\rho_n(x) w(x)\, dx \\ &= \det\left[\langle \rho_{n-1} | \rho_{m-1} \rangle\right]_{n,m=1}^{N}. \end{aligned} \tag{I.5}$$

This completes the first step.

Correlation Functions as a Determinant

Next, we create a generating function for the correlation functions. Let $\{c_n\}$ be indeterminants, $\{u_n\} \subset \mathbb{R}$, and δ be the Dirac delta function. Let

$$d\mu(x) = \sum_{n=1}^{N} (c_n d\delta_{u_n}(x) + dx).$$

Define

$$Z_{N;2}^{\mu} = \frac{1}{N!} \int_{\mathbb{R}^N} \Omega_{N;2}(\boldsymbol{x}_N)\, d\mu(x_1) \cdots d\mu(x_N).$$

Then

$$\frac{Z_{N;2}^{\mu}}{Z_{N;2}} = \frac{1}{Z_{N;2} N!} \int_{\mathbb{R}^N} \Omega_{N;2}(\boldsymbol{x}_N)\, d\mu(x_1) \cdots d\mu(x_N).$$

We expand the product $d\mu(x_1) \cdots d\mu(x_N)$, and simplify by using the symmetry of $\Omega_{N;2}$ and that $\Omega_{N;2}$ vanishes if two of the entries are equal. Collecting the indeterminants gives

$$\frac{Z_{N;2}^{\mu}}{Z_{N;2}} = \sum_{n=0}^{N} \sum_{\mathfrak{t}:\underline{n} \nearrow \underline{N}} c_{\mathfrak{t}(1)} \cdots c_{\mathfrak{t}(n)} R_{n;2}^{N}(u_{\mathfrak{t}(1)}, \ldots, u_{\mathfrak{t}(n)}). \tag{I.6}$$

We next use the determinant expression for the partition function (I.5). Let

$$\mathbf{C}_N = [\langle \rho_{n-1} | \rho_{m-1} \rangle]_{n,m=1}^{N}$$

and

$$\mathbf{A}_N = \left[\sqrt{c_m} \sqrt{w(u_m)} \rho_{n-1}(u_m) \right]_{n,m=1}^{N}.$$

Then

$$\frac{Z_{N;2}^{\mu}}{Z_{N;2}} = \frac{\det(\mathbf{C}_N + \mathbf{A}_N \mathbf{A}_N^T)}{\det \mathbf{C}_N}.$$

We factor out $\mathbf{C}_N$ and use Sylvester's determinant theorem to rewrite the above as

$$= \det(\mathbf{I}_N + \mathbf{C}_N^{-1} \mathbf{A}_N \mathbf{A}_N^T)$$

$$= \det(\mathbf{I}_N + \mathbf{A}_N^T \mathbf{C}_N^{-1} \mathbf{A}_N).$$

We are free to choose any family of monic polynomials, and the family which most simplifies $\mathbf{C}_N$ is the monic orthogonal polynomials $\{p_n\}$ with respect to the inner product. With this choice and

$$r_n = \int_{\mathbb{R}} (p_n(x))^2 w(x)\, dx,$$

$\mathbf{C}_N$ is diagonal, and

$$\mathbf{C}_N^{-1} = \left[\frac{1}{r_n}\delta_{n,m}\right]_{n,m=1}^{N}.$$

We introduce the weighted *Christoffel-Darboux* kernel for the orthogonal polynomials $\{p_n\}$

$$K_N(x,y) = \sqrt{w(x)w(y)} \sum_{n=0}^{N-1} \frac{p_n(x)p_n(y)}{r_n}.$$

Some matrix multiplication shows that

$$\mathbf{I}_N + \mathbf{A}_N^T \mathbf{C}_N^{-1} \mathbf{A}_N = \left[\delta_{n,m} + \sqrt{c_n c_m} K_N(u_n, u_m)\right]_{n,m=1}^{N}.$$

Thus, the coefficient of $c_1 \cdots c_n$ is

$$\det \left[K_N(u_j, u_k)\right]_{j,k=1}^{n}.$$

The Christoffel-Darboux formula simplifies the kernel K_N to

$$K_N(x,y) = \begin{cases} \sqrt{w(x)w(y)}\dfrac{p_N(x)p_{N-1}(y) - p_N(y)p_{N-1}(x)}{r_{N-1}(x-y)} & \text{if } x \neq y \\ w(x)\dfrac{p_N'(x)p_{N-1}(x) - p_N(x)p_{N-1}'(x)}{r_{N-1}} & \text{if } x = y. \end{cases}$$

Note that the parameter N only appears in the kernel, and studying the ensemble reduces to understanding the orthogonal polynomials and their asymptotics. Thus, this particular point process is more specifically called an *orthogonal polynomial ensemble*.

Pfaffian Point Process

When $\beta = 1$ or 4, the β-ensembles are *Pfaffian point processes*. That is, the correlation functions can be expressed as a Pfaffian of k^2 for $k \geq 1$ matrix kernels of size 2×2.

First we need a couple of definitions. The *Pfaffian* of a $2N \times 2N$ anti-symmetric matrix $\mathbf{A} = [a_{i,j}]_{i,j=1}^{2N}$ is given by

$$\operatorname{Pf} \mathbf{A} = \frac{1}{2^N N!} \sum_{\sigma \in S_{2N}} \operatorname{sgn} \sigma \prod_{n=1}^{N} a_{\sigma(2n-1),\sigma(2n)}.$$

A *skew-inner product* $\langle \cdot | \cdot \rangle$ is an inner product which is skew symmetric, or

$$\langle f | g \rangle = -\langle g | f \rangle.$$

Analogous to orthogonal polynomials, the *skew-orthogonal polynomials* are a family of polynomials $\{q_n\}$ which satisfy

$$\langle q_{2n} | q_{2m} \rangle = \langle q_{2n+1} | q_{2m+1} \rangle = 0$$

$$\langle q_{2n} | q_{2m+1} \rangle = r_n \delta_{n,m}.$$

The steps in deriving the correlation functions are the same as in the determinantal point process case. We outline the steps here. The details can be found in [41],

[35], [30], and [7].

With skew-inner products

$$\langle f|g\rangle_\beta = \begin{cases} \displaystyle\int_{\mathbb{R}} f(x)g(y)\,\mathrm{sgn}(y-x)w(x)\,dydx & \text{if } \beta = 1 \\ \displaystyle\int_{\mathbb{R}} (f(x)g'(x) - g(x)f'(x))w(x)\,dx & \text{if } \beta = 4, \end{cases}$$

the first step is to show that the partition functions can be written as a Gram matrices

$$\mathrm{Pf}\left[\langle \rho_n|\rho_m\rangle_1\right]_{n,m=0}^{N-1} \quad \text{and} \quad \mathrm{Pf}\left[\langle \rho_n|\rho_m\rangle_4\right]_{n,m=0}^{2N-1}.$$

When $\beta = 1$, we assume N is even. The matrix for N odd is slightly more complicated and was studied in [34].

Next, we create a generating function $\dfrac{Z_{N;\beta}^{\mu}}{Z_{N;\beta}}$ which can be expressed as (I.6). We then expand $\dfrac{Z_{N;\beta}^{\mu}}{Z_{N;\beta}}$ using the Pfaffian expression of the partition function and match coefficients to reveal that the correlation functions can be expressed as Pfaffians of a 2×2 matrix kernel. For $\beta = 1$, the matrix kernel entries are

$$S_{N;1}(x,y) = \sqrt{w(x)w(y)} \sum_{n=0}^{\frac{N}{2}-1} \frac{q_{2n;1}(x)\varepsilon q_{2n+1;1}(y) - q_{2n+1;1}(x)\varepsilon q_{2n;1}(y)}{r_{n;1}}$$

$$DS_{N;1}(x,y) = \sqrt{w(x)w(y)} \sum_{n=0}^{\frac{N}{2}-1} \frac{q_{2n;1}(x)q_{2n+1;1}(y) - q_{2n+1;1}(x)q_{2n;1}(y)}{r_{n;1}}$$

$$IS_{N;1}(x,y) = \sqrt{w(x)w(y)} \sum_{n=0}^{\frac{N}{2}-1} \frac{\varepsilon q_{2n;1}(x)\varepsilon q_{2n+1;1}(y) - \varepsilon q_{2n+1;1}(x)\varepsilon q_{2n;1}(y)}{r_{n;1}} + \mathrm{sgn}(y-x)$$

where

$$\varepsilon f(x) = \int_{\mathbb{R}} f(y)\,\mathrm{sgn}(y-x)\,dy.$$

When $\beta = 4$, the matrix kernel entries are

$$S_{N;4}(x,y) = \sqrt{w(x)w(y)} \sum_{n=0}^{N-1} \frac{q_{2n;4}(x)q'_{2n+1;4}(y) - q_{2n+1;4}(x)q'_{2n;4}(y)}{r_{n;4}}$$
$$DS_{N;4}(x,y) = \sqrt{w(x)w(y)} \sum_{n=0}^{N-1} \frac{q'_{2n;4}(x)q'_{2n+1;4}(y) - q'_{2n+1;4}(x)q'_{2n;4}(y)}{r_{n;4}}$$
$$IS_{N;4}(x,y) = \sqrt{w(x)w(y)} \sum_{n=0}^{N-1} \frac{q_{2n;4}(x)q_{2n+1;4}(y) - q_{2n+1;4}(x)q_{2n;4}(y)}{r_{n;4}}.$$

Then the 2×2 matrix kernel is

$$\mathbf{K}_{N;\beta}(x,y) = \begin{bmatrix} DS_{N;\beta}(x,y) & S_{N;\beta}(x,y) \\ -S_{N;\beta}(y,x) & IS_{N;\beta}(x,y) \end{bmatrix},$$

and the nth correlation function can be expressed as

$$R^N_{n;\beta}(\boldsymbol{x}_n) = \mathrm{Pf}\left[\mathbf{K}_{N;\beta}(x_j,x_k)\right]_{j,k=1}^{n}.$$

We shall occasionally refer to a point process which has the property of being determintantal or Pfaffian as *solvable.*

The circular β-ensembles have analogous results, i.e. $\beta = 2$ correspond to a determinantal point process while $\beta = 1$ and 4 corresponds to Pfaffian point processes. The kernels are listed in the Appendix.

Gap Probabilities

An example of the utility of the determinantal and Pfaffian expressions of the correlation functions is in calculating *gap probabilities*, or the probability that there are no eigenvalues in a given interval A. This can be expressed as

$$\begin{aligned}
&\frac{1}{Z_{N;\beta}N!}\int_{\mathbb{R}^N}\left\{\prod_{n=1}^{N}(1-\chi_A(x_n))\right\}\Omega_{N;\beta}(\boldsymbol{x}_N)\,d\boldsymbol{x}_N \\
&\quad=\frac{1}{Z_{N;\beta}N!}\sum_{n=0}^{N}(-1)^n\binom{N}{n}\int_{A^n}\int_{\mathbb{R}^{N-n}}\Omega_{N;\beta}(\boldsymbol{x}_N)\,d\boldsymbol{x}_N \\
&\quad=\sum_{n=0}^{N}\frac{(-1)^n}{n!}\int_{A^n}R^N_{n;\beta}(\boldsymbol{x}_n)\,d\boldsymbol{x}_n.
\end{aligned}$$

In the determinantal case, this is equal to

$$\sum_{n=0}^{N}\frac{(-1)^n}{n!}\int_{A^n}\det\left[K_N(x_j,x_k)\right]_{j,k=1}^{n}\,d\boldsymbol{x}_n. \tag{I.7}$$

The expression (I.7) is a *Fredholm determinant* of the kernel K_N restricted to $A\times A$. In the Pfaffian case, the same calculation holds with the Pfaffian in place of the determinant, and we have a *Fredholm Pfaffian.*

By continuity properties of the Fredholm determinant, if the kernel K_N (with perhaps some scaling so that the expected number of points in an interval is nonzero and finite) converges to K as $N\to\infty$, then the gap probability converges to

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{A^n}\det\left[K(x_i,x_j)\right]_{j,k=1}^{n}\,d\boldsymbol{x}_n.$$

For instance, by scaling $x\mapsto\dfrac{x}{\sqrt{2N}}$ in the GUE kernel,

$$\frac{1}{\sqrt{2N}}K_N\left(\frac{x}{\sqrt{2N}},\frac{y}{\sqrt{2N}}\right)\to\frac{\sin(x-y)}{\pi(x-y)} \tag{I.8}$$

known as the *sine kernel*. In [20], the authors Jimbo, Miwa, Mori, and Sato expressed the Fredholm determinant with the sine kernel when $A = (-t, t)$ in terms of an integrable system of partial differential equations whose solution is the Painleve V transcendent.

If $A = (t, \infty)$, the gap probability is the density of the largest eigenvalue. With Ai being the Airy function and scaling at the *soft edge* (the edge of the support without a boundary) by $x \mapsto 2\sqrt{N} + \frac{x}{N^{1/6}}$,

$$\frac{1}{N^{1/6}} K_N \left(2\sqrt{N} + \frac{x}{N^{1/6}}, 2\sqrt{N} + \frac{y}{N^{1/6}} \right) \to \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}(y)\,\mathrm{Ai}'(x)}{x - y}. \quad \text{(I.9)}$$

This limiting kernel is known as the *Airy kernel*. The law of the largest eigenvalue, called the Tracy-Widom law, is expressed in terms of the Painleve II transcendent [38].

Another reoccurring kernel is the *Bessel kernel*, which appears when scaling at a *hard edge*, or at a boundary of the support. Consider for example the Jacobi weight

$$w(x) = (1 - x)^{\alpha}(1 + x)^{\gamma}$$

for $-1 \leq x \leq 1$ and $\alpha, \gamma > -1$. The orthogonal polynomials with respect to this weight are the Jacobi polynomials, and the N points whose distribution is (I.1) are the Jacobi ensemble. With the proper scaling at 1 and J_α denoting the Bessel function of order α, the kernel converges to

$$\frac{\sqrt{y}J_\alpha(\sqrt{x})J'_\alpha(\sqrt{y}) - \sqrt{x}J_\alpha(\sqrt{y})J'_\alpha(\sqrt{x})}{2(x-y)}. \tag{I.10}$$

Existing Work on the β-Ensemble

While it is unknown whether other values of β correspond to determinantal or Pfaffian point processes besides $\beta = 1, 2$, or 4, different techniques have been used to study these ensembles. For $\beta = L^2$ when L is any positive integer, and for $\beta = L^2 + 1$ when L is an odd integer, Sinclair showed that the partition functions can be expressed as hyperpfaffians, although it is still not known if the correlation functions can be expressed as a hyperpfaffian [35]. The partition functions in the circular cases (I.3) and (I.1) for the case that w is the Jacobi weight, called the Mehta integral and Selberg integral, respectively, have been explicitly calculated in [11], and the correlation functions for certain values of β have been expressed in terms of hypergeometric functions and Jack polynomials [13]. The general β-ensembles has been found to correspond to the distributions of eigenvalues generated by random tridiagonal matrices [9], [21]. Using this random matrix model, Ramirez, Rider, and Virag discovered the law of the largest eigenvalue for the Gaussian β-ensemble in [29]. Their distributions interpolate between the laws of the largest eigenvalues for $\beta = 1, 2$ and 4, called the Tracy-Widom distributions [41].

While much progress has been made on the β-ensembles, other works have focused on finding solvable families of point processes which interpolate between the $\beta = 1, 2$, or 4 ensembles, besides simply changing the parameter β. In [30] (mentioned above), the authors Rider, Sinclair, and Xu study the ensemble of mixed charge-one and charge-two particles (a two species ensemble) on the real line with the harmonic oscillator potential (Gaussian weight). They introduce a fugacity parameter which controls the distributions of the number of each particle type, and show that each value of the parameter gives a Pfaffian point process. The solvability of the point process and the skew-orthogonal polynomials with respect to the skew-inner product allow the authors to study global statistics including the densities and distributions of the charge-one and charge-two particles. Furthermore, the collection of ensembles continuously interpolates between the $\beta = 1$ (all charge-one particles) and the $\beta = 4$ (all charge-two particles) Gaussian ensembles. Chapter II and Chapter III introduce two ensembles on the unit circle which interpolate between the $\beta = 1, 2$, and 4 circular ensembles.

In Chapter II, we study a solvable family of point processes which interpolates between the CUE and the CSE. Let N be a nonnegative integer and let $r > 0$. In the complex plane with no external field, consider N charge-one particles on the unit circle paired with N charge-one particles on the the circle of radius r with the same arguments, i.e. the locations of the particles are $\{e^{i\theta_1}, \ldots, e^{i\theta_N}\}$ and

$\{re^{i\theta_1}, \ldots, re^{i\theta_N}\}$. The particles interact logarithmically.

We first show that for any N and r, the ensemble generates a Pfaffian point process. For fixed N, the finite kernels converge to the CUE kernels as $r \to \infty$ and to the CSE kernels as $r \to 1$.

We next investigate the limiting kernels. We first fix the radius $r \neq 1$ and let $N \to \infty$ while scaling the particle locations by $\theta \mapsto \dfrac{2\pi\theta}{N}$ (here $\theta \in (-\infty, \infty)$, since for large enough N, $\dfrac{2\pi\theta}{N} \in (-\pi, \pi])$. With this scaling, we see that the kernels converge to the CUE kernel. Seen in a different perspective, the scaling magnifies the circumference of the two circles by a factor of $\dfrac{N}{2\pi}$ with the effect that the average distance between two consecutive particles is one unit. At the same time, this increases the distance between the two circles by a factor of N. Thus, the interaction between the particles on different circles is negligible compared to the interaction between the particles on the same circle.

To see significant interaction between different circles in the scaling, we must scale the distance between the two circles by $r \mapsto 1 + \dfrac{s}{N}$. With this scaling, the limiting kernels interpolate between the limiting scaled kernels of CUE ($s \to \infty$) and CSE ($s \to 0$).

In Chapter III, we consider the same model as [30] of mixed charge-one and charge-two particles with a fugacity parameter $X \geq 0$ controlling the random number of each particle type, with the modification that the particles live on the unit circle with no external field. In this case, we will show that the different values of the parameter give a collection of Pfaffian point processes which continuously interpolate between the $\beta = 1$ and the $\beta = 4$ circular ensembles and explicitly derive the interpolating kernels. By universality, the local statistics of our circular ensemble should translate to the bulk statistics of the ensemble studied in [30].

Similar to the setup in [30], we fix N to be a nonnegative integer representing the total charge of the particles in the system. For a two species ensemble, we need a more generalized definition for the partition function than (I.2). Denote (L, M) to be the set of all pair of nonnegative integers which satisfy the relation $L+2M = N$. We first consider the specified configuration of L charge-one particles and M charge-two particles interacting logarithmically and confined to the unit circle. The total potential energy of this particular configuration is

$$\begin{aligned} E(\boldsymbol{\theta}_L, \boldsymbol{\phi}_M) = &- \sum_{l_1 > l_2}^{L} \log \left| e^{i\theta_{l_1}} - e^{i\theta_{l_2}} \right| - 4 \sum_{m_1 > m_2}^{M} \log \left| e^{i\phi_{m_1}} - e^{i\phi_{m_2}} \right| \\ &- 2 \sum_{l=1}^{L} \sum_{m=1}^{M} \left| e^{i\theta_l} - e^{i\phi_m} \right|. \end{aligned}$$

Then the Boltzmann factor with inverse temperature equal to one is

$$e^{-E(\boldsymbol{\theta}_L, \boldsymbol{\phi}_M)} = \prod_{l_1 > l_2}^{L} \left| e^{i\theta_{l_1}} - e^{i\theta_{l_2}} \right| \prod_{m_1 > m_2}^{M} \left| e^{i\phi_{m_1}} - e^{i\phi_{m_2}} \right|^4 \prod_{l=1}^{L} \prod_{m=1}^{M} \left| e^{i\theta_l} - e^{i\phi_m} \right|^2,$$

and thus the (L,M) partial partition function is

$$Z_{L,M} = \frac{1}{L!M!}\int_{[-\pi,\pi)^L}\int_{[-\pi,\pi)^M} e^{-E(\boldsymbol{\theta}_L,\boldsymbol{\phi}_M)}\, d\boldsymbol{\theta}_L d\boldsymbol{\phi}_M.$$

We then allow (L,M) to vary and include a weight $X \geq 0$, called the *fugacity* of the system, and take the weighted union over all possible states (L,M). So the partition function is the weighted sum over all (L,M) partial partition functions

$$Z_N(X) = \sum_{(L,M)} X^L Z_{L,M}. \tag{I.11}$$

Our first task is to show that the partition function $Z_N(X)$ can be expressed as a Pfaffian of an $N \times N$ antisymmetric matrix if N is even, and an $(N+1)\times(N+1)$ antisymmetric matrix if N is odd.

Next, we study the properties of $L_N(X)$, the random number of charge-one particles for a given N and X. Our Pfaffian expression of $Z_N(X)$ and formula (I.11) give us nice expressions for the probability generating functions of $L_N(X)$. The distribution of $L_N(X)$ differs slightly when N is even or odd because L must match parity with N. For fixed X and as $N \to \infty$, we show that in the even and odd cases, $L_N(X)$ converges to even and odd analogues of the Poisson distribution with finite expectations and variances. In order for L and M to be of the same order in the limit, we will find that X must increase proportionally to N, i.e. $X = Nr$ for some $r > 0$. In this situation, we give a central limit theorem for the suitably normalized random variable $L_N(Nr)$.

In analyzing the local statistics of the particles, we will assume N is even for convenience. Let l and m be nonnegative integers such that $l + 2m \leq N$. Similar to the definition of the correlation functions for a one species ensemble, for $C_1, \ldots, C_l \subset [-\pi, \pi)$ mutually disjoint and $D_1, \ldots, D_m \subset [-\pi, \pi)$ mutually disjoint, let $\mathcal{X}_1(C_i)$ denote the number of charge-one particles in C_i and let $\mathcal{X}_2(D_i)$ denote the number of charge-two particles in D_i. The (l, m) correlation function $R^N_{l,m} : [-\pi, \pi)^l \times [-\pi, \pi)^m \to [0, \infty)$ is defined as

$$\mathbb{E}\left[\prod_{i=1}^{l} \mathcal{X}_1(C_i) \prod_{j=1}^{m} \mathcal{X}_2(D_j)\right] = \int_{C_1} \cdots \int_{C_l} \int_{D_1} \cdots \int_{D_m} R^N_{l,m}(\boldsymbol{x}_l, \boldsymbol{y}_m)\, d\boldsymbol{x}_l d\boldsymbol{y}_m. \tag{I.12}$$

Because the numbers of charge-one and charge-two particles vary, we need to first consider the situation when there are L charge-one particles and M charge-two particles such that $l \leq L$ and $m \leq M$. Then for small sets C_i and D_j, the expected value is approximately

$$\int_{C_1} \cdots \int_{C_l} \int_{D_1} \cdots \int_{D_m} \left[\frac{X^L}{Z_N(X)(L-l)!(M-m)!} \times \int_{[-\pi,\pi)^{L-l}} \int_{[-\pi,\pi)^{M-m}} e^{-E(\boldsymbol{x}_l \vee \boldsymbol{\theta}_{L-l}, \boldsymbol{y}_m \vee \boldsymbol{\phi}_{M-m})}\, d\boldsymbol{\theta}_{L-l} d\boldsymbol{\phi}_{M-m}\right] d\boldsymbol{x}_l d\boldsymbol{y}_m$$

We then take the weighted sum over all possible (L, M). Again under the assumption that the correlation functions are continuous, the correlations functions can be expressed as

$$
\begin{aligned}
&R^N_{l,m}(\boldsymbol{x}_l, \boldsymbol{y}_m) \\
&\quad = \frac{1}{Z_N(X)} \sum_{\substack{(L,M) \\ L \geq l, M \geq m}} \frac{X^L}{(L-l)!(M-m)!} \\
&\qquad \times \int_{[-\pi,\pi)^{L-l}} \int_{[-\pi,\pi)^{M-m}} e^{-E(\boldsymbol{x}_l \vee \boldsymbol{\theta}_{L-l}, \boldsymbol{y}_m \vee \boldsymbol{\phi}_{M-m})} \, d\boldsymbol{\theta}_{L-l} d\boldsymbol{\phi}_{M-m}.
\end{aligned}
$$

We sketch a proof that the correlation functions can be expressed as a Pfaffian of a $2(l+m) \times 2(l+m)$ matrix consisting of four 2×2 matrix block kernels. For finite N, the kernel entries interpolate between the (finite) kernels of $\beta = 1$ ($X \to \infty$) and $\beta = 4$ ($X \to 0^+$). Then, after scaling the fugacity by $X = Nr$ and letting $N \to \infty$, we obtain kernels which interpolate between the limiting kernels of $\beta = 1$ ($r \to \infty$) and $\beta = 4$ ($r \to 0^+$).

Other two species models, besides that of [30], have been studied (and solved) in the past. For example, the *Ginibre ensemble*, first studied by Jean Ginibre in the 1960s [16], is the random matrix ensemble of $N \times N$ real matrices (not necessarily symmetric) with independent standard Gaussian entries. This generates random real and complex eigenvalues. In [7], Alexi Borodin and Christopher Sinclair show that the ensemble is Pfaffian for N even, and later Sinclair did the N odd case. See [15] and [14] for other examples of two species ensembles. In [34], the partition function was calculated for the general system of multi-charged particles interacting logarithmically on the real line.

In Chapter IV, we study another two species ensemble. For a fixed N, consider $2N$ charge-one particles confined to the real and imaginary axes in a radial external field, where each particle has a reflected particle about the origin. As in Chapter III, we start with a pair (L, M) such that $L + M = N$ and consider the situation where L particles live on the positive real axis and M particles live on the positive imaginary axis. Then we take the weighted sum over all possible (L, M) partial partition functions $Z_{L,M}$ to get the total partition function

$$Z_N(X) = \sum_{(L,M)} X^M Z_{L,M}.$$

With $w(x) = e^{-Q(x)}$ defined on the positive half of the real line, we show that for all values of X the partition function is a Pfaffian of a Gram matrix with skew-inner product

$$\begin{aligned}\langle f|g\rangle = &\int_0^\infty (f(x)g(-x) - f(-x)g(x))w(x)\,dx \\ &- iX\int_0^\infty (f(ix)g(-ix) - f(-ix)g(ix))w(x)\,dx.\end{aligned}$$

The skew-orthogonal polynomials $\{q_n^X\}$ can be written in terms of orthogonal polynomials $\{p_n^X\}$ with respect to the weight on the real line

$$w_X(x) = \begin{cases} w(\sqrt{x}) & \text{if } x \geq 0 \\ X \cdot w(\sqrt{-x}) & \text{if } x < 0 \end{cases}$$

given by the relations

$$q^X_{2n}(x) = p^X_n(x^2)$$

$$q^X_{2n+1}(x) = xp^X_n(x^2).$$

The weight has a jump discontinuity at the origin controlled by the fugacity parameter when $X \neq 1$. These orthogonal polynomials have been studied when the weight $w(\sqrt{x})$ is Jacobi [25] and Gaussian [19].

We sidetrack here to remark that because orthogonal polynomials are well-studied, it is a common technique to use orthogonal polynomials to derive skew-orthogonal polynomials and facilitate the study of Pfaffian point processes. One example is the β-ensemble for $\beta = 1$ or 4 (see (I.1) for the joint density) when the weight is of the form $w(x) = e^{-V(x)}$ where $V'(x)$ is a rational function. Adler, Forrester, and Nagao derived an explicit relation between the skew-orthogonal and the orthogonal polynomials with respect to the weight w in [1]. Another example is the skew-orthogonal polynomials derived in the previously mentioned mixed charge ensemble on the real line studied in [30], which were expressed in terms of generalized Laguerre polynomials. In our ensemble, the orthogonal polynomials play a crucial role.

With the skew-orthogonal polynomials we show that the correlation functions can be expressed as a Pfaffian of four 2×2 matrix kernels, where each of the scalar kernel entries is expressed in terms of a single kernel S_N defined in Theorem IV.1 below. From the relationship between skew-orthogonal and orthogonal polynomials, S_N has a Christoffel-Darboux form: Given that K_N is the weighted Christoffel-Darboux kernel

$$K_N(x,y) = \begin{cases} \sqrt{w_X(x)w_X(y)}\dfrac{p_N^X(x)p_{N-1}^X(y) - p_N^X(y)p_{N-1}^X(x)}{r_{N-1}(x-y)} & \text{if } x \neq y \\ w_X(x)\dfrac{(p_N^X(x))'p_{N-1}^X(x) - p_N^X(x)(p_{N-1}^X(x))'}{r_{N-1}} & \text{if } x = y, \end{cases}$$

we have

$$S_N(x,y) = (x+y)K_N(x^2,y^2).$$

The relation between the scalar kernels K_N and S_N suggests that we can focus on K_N (since orthogonal polynomials are easier to work with), then translate the results to S_N. In fact, there is a direct correspondence between our two species model and the orthogonal polynomial ensemble with distribution of points (I.1) and weight w_X. To a given particle at $x > 0$ of the orthogonal polynomial ensemble corresponds a particle from our two species ensemble at $\sqrt{x}$. Likewise, a particle at $x < 0$ corresponds to a particle at $i\sqrt{-x}$. We derive the relationships between the limiting kernels (both at and away from the origin), global densities, and scalings of K_N and S_N. We use the results to record some properties when the external field corresponds to the Jacobi weight, and we derive new edge kernels for S_N when $X = 0$ and $X = 1$.

In Chapter V, we study a similar reflection ensemble on the unit circle, where the charge-one particles come in conjugate pairs. We place the external field

$$\alpha \log\left(2\sin^2\frac{\theta}{2}\right) + \gamma \log\left(2\cos^2\frac{\theta}{2}\right) \quad \text{for} \quad 0 \le \theta \le 2\pi$$

on the unit circle which corresponds to the Jacobi weight. This ensemble is a Pfaffian point process, and the skew orthogonal polynomials are written in terms of the Jacobi polynomials. We calculate the global density of the particles, and the limiting scaled kernels inside the bulk and at the edge.

The statistics near the hard edges (at $\theta = 0$ or $\theta = \pi$) should be similar to the statistics near the origin of the ensemble studied in Chapter IV in the case that $X = 0$, or when all the particles lie on the real line.

CHAPTER II

A SOLVABLE ENSEMBLE WHICH INTERPOLATES BETWEEN CUE AND CSE

Setup

In the complex plane with no external field, consider N charge-one particles on the unit circle paired with N charge-one particles on the the circle of radius r having the same arguments, i.e. the locations of the particles are $\{e^{i\theta_1}, \dots, e^{i\theta_N}\}$ and $\{re^{i\theta_1}, \dots, re^{i\theta_N}\}$. With the particles interacting logarithmically, the total energy of the system is

$$E_N(\boldsymbol{\theta}_N, r) = -\sum_{n>m}^{N} \log \left|e^{i\theta_n} - e^{i\theta_m}\right| - \sum_{n>m}^{N} \log \left|re^{i\theta_n} - re^{i\theta_m}\right| - \sum_{n,m=1}^{N} \log \left|re^{i\theta_n} - e^{i\theta_m}\right|.$$

The Boltzmann factor with inverse temperature equal to 1 is

$$\begin{aligned} e^{-E_N(\boldsymbol{\theta}_N, r)} &= \left(\prod_{n>m}^{N} \left|e^{i\theta_n} - e^{i\theta_m}\right| \left|re^{i\theta_n} - re^{i\theta_m}\right|\right) \prod_{n,m=1}^{N} \left|re^{i\theta_n} - e^{i\theta_m}\right| \\ &= r^{\frac{N(N-1)}{2}} \left(\prod_{n>m}^{N} \left|e^{i\theta_n} - e^{i\theta_m}\right|^2 \left|re^{i\theta_n} - e^{i\theta_m}\right| \left|e^{i\theta_n} - re^{i\theta_m}\right|\right) \\ &\quad \times \prod_{n=1}^{N} \left|re^{i\theta_n} - e^{i\theta_n}\right| \end{aligned} \tag{II.1}$$

and so the joint probability density is

$$\frac{1}{Z_N(r) N!} e^{-E_N(\boldsymbol{\theta}_N, r)}$$

with partition function

$$Z_N(r) = \frac{1}{N!} \int_{[-\pi,\pi)^N} e^{-E_N(\boldsymbol{\theta}_N, r)} \, d\boldsymbol{\theta}_N.$$

We first show that the partition function can be expressed as a Pfaffian, then show that the correlation functions can be expressed as Pfaffians of a 2×2 matrix kernel.

<u>Pfaffian Expression for the Partition Function</u>

Theorem II.1. *The partition function $Z_N(r)$ can be expressed as the Pfaffian of a $2N \times 2N$ antisymmetric matrix*

$$Z_N(r) = \operatorname{Pf} \boldsymbol{U}_N(r)$$

where

$$\boldsymbol{U}_N(r) = \Big[2\pi (r^{n-1} - r^{m-1}) \delta_{2N+1,n+m} \Big]_{n,m=1}^{2N}.$$

<u>Correlation Functions and the Kernel Entries</u>

Corollary II.1. *Let*

$$\widetilde{S}_N(\theta, \phi, r) = \frac{1}{2\pi} \sum_{n=1}^{2N} \frac{e^{i\left(-N-\frac{1}{2}+n\right)(\theta-\phi)}}{1 - r^{2N+1-2n}},$$

$$\widetilde{DS}_N(\theta, \phi, r) = \frac{1}{2\pi} \sum_{n=1}^{2N} \frac{e^{i\left(-N-\frac{1}{2}+n\right)(\theta-\phi)}}{r^{n-1} - r^{2N-n}},$$

$$\widetilde{IS}_N(\theta, \phi, r) = \frac{r^{2N-1}}{2\pi} \sum_{n=1}^{2N} \frac{e^{i\left(-N-\frac{1}{2}+n\right)(\theta-\phi)}}{r^{n-1} - r^{2N-n}}$$

and set

$$\widetilde{\boldsymbol{K}}_N(\theta,\phi,r) = \begin{bmatrix} \widetilde{DS}_N(\theta,\phi,r) & \widetilde{S}_N(\theta,\phi,r) \\ -\widetilde{S}_N(\phi,\theta,r) & \widetilde{IS}_N(\theta,\phi,r) \end{bmatrix}.$$

Then the nth correlation function R_n^N may be expressed as

$$R_n^N(\boldsymbol{\theta}_n, r) = \mathrm{Pf}\left[\widetilde{\boldsymbol{K}}_N(\theta_j,\theta_k,r)\right]_{j,k=1}^n.$$

We will see below that the correlation functions converge as $r \to \infty$ and as $r \to 1$. However, inspection of the kernel entries reveals that the entries diverge as $r \to 1$. Fortunately, we have the freedom to manipulate the kernel entries as long as the Pfaffian and the 2×2 matrix kernel block structure are unchanged. We multiply

$$\left[\widetilde{\mathbf{K}}_N(\theta_j,\theta_k,r)\right]_{j,k=1}^n$$

on the left by

$$\mathbf{I}_n \otimes \begin{bmatrix} \dfrac{r^{\frac{2N+1}{2}}}{r-1} & -\dfrac{r}{r-1} \\ 0 & 1 \end{bmatrix}$$

and on the right by

$$\mathbf{I}_n \otimes \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{r^{\frac{2N-1}{2}}} & \dfrac{r-1}{r^{\frac{2N+1}{2}}} \end{bmatrix}$$

where $\mathbf{I}_n$ is the $n \times n$ identity matrix and $\otimes$ is the Kronecker product. This alteration preserves the block structure and the antisymmetry of the matrix. Because the square of the Pfaffian is equal the determinant, it is easy to see that up to sign, the Pfaffians of the old and new matrices are equal. We arrive at the new scalar kernels

$$
\begin{aligned}
S_N(\theta, \phi, r) &= \frac{1}{2\pi} \sum_{n=1}^{2N} \left(\frac{1 - r^{N+\frac{1}{2}-n}}{1 - r^{2N+1-2n}} \right) e^{i(-N-\frac{1}{2}+n)(\theta-\phi)} && \text{(II.2)} \\
&= \frac{1}{2\pi} \sum_{n=1}^{N} \left(\cos\left[\left(n - \frac{1}{2}\right)(\theta - \phi) \right] \right. \\
&\quad \left. + \, i \left(\frac{r^{n-\frac{1}{2}} - 1}{r^{n-\frac{1}{2}} + 1} \right) \sin\left[\left(n - \frac{1}{2}\right)(\theta - \phi) \right] \right) \\
& && \text{(II.3)} \\
DS_N(\theta, \phi, r) &= \frac{(r-1)}{2\pi r} \sum_{n=1}^{2N} \left(\frac{r^{N+\frac{1}{2}-n}}{1 - r^{2N+1-2n}} \right) e^{i\left(-N-\frac{1}{2}+n\right)(\theta-\phi)} && \text{(II.4)} \\
&= -\frac{i(r-1)}{\pi r} \sum_{n=1}^{N} \left(\frac{r^{n-\frac{1}{2}}}{1 - r^{2n-1}} \right) \sin\left[\left(n - \frac{1}{2}\right)(\theta - \phi) \right] && \text{(II.5)} \\
IS_N(\theta, \phi, r) &= \frac{r}{2\pi(r-1)} \sum_{n=1}^{2N} \left(\frac{r^{N+\frac{1}{2}-n} - 1}{r^{N+\frac{1}{2}-n} + 1} \right) e^{i\left(-N-\frac{1}{2}+n\right)(\theta-\phi)} && \text{(II.6)} \\
&= -\frac{ir}{\pi(r-1)} \sum_{n=1}^{N} \left(\frac{r^{n-\frac{1}{2}} - 1}{r^{n-\frac{1}{2}} + 1} \right) \sin\left[\left(n - \frac{1}{2}\right)(\theta - \phi) \right] && \text{(II.7)}
\end{aligned}
$$

and the new matrix kernel

$$
\mathbf{K}_N(\theta, \phi, r) = \begin{bmatrix} IS_N(\theta, \phi, r) & S_N(\theta, \phi, r) \\ -S_N(\phi, \theta, r) & DS_N(\theta, \phi, r) \end{bmatrix}.
$$

The multiple representations for each scalar kernel will be useful when evaluating limits. The evaluations of the limits with respect to r now follow immediately from (II.3), (II.5), and (II.7). Indeed, with $S_{N;2}$ as the finite CUE scalar kernel defined in the Appendix (A.1) and as $r \to \infty$,

$$\begin{aligned} S_N(\theta,\phi,r) &\to \frac{1}{2\pi}\sum_{n=1}^{N}\left(\cos\left[\left(n-\frac{1}{2}\right)(\theta-\phi)\right] + i\sin\left[\left(n-\frac{1}{2}\right)(\theta-\phi)\right]\right) \\ &= e^{\frac{i}{2}(\theta-\phi)}S_{N;2}(\theta,\phi) \end{aligned}$$

$$\begin{aligned} DS_N(\theta,\phi,r) &\to 0 \\ IS_N(\theta,\phi,r) &\to -\frac{i}{\pi}\sum_{n=1}^{N}\sin\left[\left(n-\frac{1}{2}\right)(\theta-\phi)\right]. \end{aligned}$$

Now

$$\mathrm{Pf}\begin{bmatrix} * & e^{\frac{i}{2}(\theta_j-\theta_k)}S_{N;2}(\theta_j,\theta_k) \\ -e^{\frac{i}{2}(\theta_k-\theta_j)}S_{N;2}(\theta_k,\theta_j) & 0 \end{bmatrix}_{j,k=1}^{N}$$

is equal to

$$\mathrm{Pf}\begin{bmatrix} 0 & e^{\frac{i}{2}(\theta_j-\theta_k)}S_{N;2}(\theta_j,\theta_k) \\ -e^{\frac{i}{2}(\theta_k-\theta_j)}S_{N;2}(\theta_k,\theta_j) & 0 \end{bmatrix}_{j,k=1}^{N}$$

which is equal to

$$\mathrm{Pf}\begin{bmatrix} 0 & S_{N,2}(\theta_j,\theta_k) \\ S_{N,2}(\theta_k,\theta_j) & 0 \end{bmatrix}_{j,k=1}^{N}.$$

Comparing this matrix kernel to (A.3) shows that this limiting point process is equivalent to the CUE limiting point process. As $r \to 1$,

$$\begin{aligned} S_N(\theta,\phi,r) &\to \frac{1}{2\pi}\sum_{n=1}^{N}\cos\left[\left(\frac{2n-1}{2}\right)(\theta-\phi)\right] \\ DS_N(\theta,\phi,r) &\to \frac{i}{2\pi}\sum_{n=1}^{N}\left(\frac{2}{2n-1}\right)\sin\left[\left(\frac{2n-1}{2}\right)(\theta-\phi)\right] \\ IS_N(\theta,\phi,r) &\to -\frac{i}{2\pi}\sum_{n=1}^{N}\left(\frac{2n-1}{2}\right)\sin\left[\left(\frac{2n-1}{2}\right)(\theta-\phi)\right] \end{aligned}$$

which are the CSE kernels (C.1).

Remark II.2. As $r \to 0^+$, similar calculations as above show that the kernels converge to the CUE kernels, and we omit the proofs. Intuitively, this is analogous to the situation when $r \to \infty$, since we can rescale the sizes of the circles so that the shrinking circle now has constant radius one, which forces the other circle's radius to increase to infinity. Thus, we have two copies of the COE ensemble again.

Limiting Kernels while Scaling the Radius

We next investigate the behavior of the kernel entries when we fix the distance between the two circles and increase the number of particles to infinity. In order to calculate non-trivial probabilities in the limit, we scale the particle locations by $\theta \mapsto \dfrac{2\pi\theta}{N}$. As described in the Introduction, the scaling magnifies a region of the circle (in this case, at $\theta = 0$) so that on average, the particles are one unit apart. By the rotational symmetry of the kernels, the local statistics is independent of the location of the scaling.

Proposition II.1. *Let S_2 be the limiting scaled kernel for the CUE, as defined in (A.2). As $N \to \infty$,*

$$\begin{aligned}
\frac{2\pi}{N} S_N\left(\frac{2\pi\theta}{N}, \frac{2\pi\phi}{N}, r\right) &\to e^{\pi i(\theta-\phi)} \frac{\sin \pi(\theta-\phi)}{\pi(\theta-\phi)} \\
&= S_2(\theta, \phi) \\
\frac{2\pi}{N} IS_N\left(\frac{2\pi\theta}{N}, \frac{2\pi\phi}{N}, r\right) &\to \frac{ir}{(r-1)}\left(\frac{\cos\left[2\pi(\theta-\phi)\right]-1}{\pi(\theta-\phi)}\right) \\
\frac{2\pi}{N} DS_N\left(\frac{2\pi\theta}{N}, \frac{2\pi\phi}{N}, r\right) &\to 0.
\end{aligned}$$

Again we use

$$\mathrm{Pf}\begin{bmatrix} * & S_2(\theta_j,\theta_k) \\ -S_2(\theta_k,\theta_j) & 0 \end{bmatrix}_{j,k=1}^{n} = \mathrm{Pf}\begin{bmatrix} 0 & S_2(\theta_j,\theta_k) \\ -S_2(\theta_k,\theta_j) & 0 \end{bmatrix}_{j,k=1}^{n}$$

which shows that the limiting point process is independent of r. By (A.3), this is equivalent to the CUE limiting point process. As discussed in the Introduction, the particle interaction between the two circles becomes negligible in the scaling. So we must also scale the radius to decrease at a rate depending on N. Before the next theorem, we again alter the kernel entries so that they converge in the limit as $N \to \infty$. Multiply the left and right side of

$$\Big[\mathbf{K}_N(\theta_j,\theta_k,r)\Big]_{j,k=1}^{n}$$

by

$$\mathbf{I}_n \otimes \begin{bmatrix} \frac{1}{\sqrt{N}} & 0 \\ 0 & \sqrt{N} \end{bmatrix}.$$

This multiplies IS_N by $\frac{1}{N}$ and DS_N by N while preserving the Pfaffian and the block structure. Define

$$\begin{aligned} S(\theta,\phi,s) &= \lim_{N\to\infty} \frac{2\pi}{N} S_N\left(\frac{2\pi\theta}{N},\frac{2\pi\phi}{N},1+\frac{s}{N}\right) \\ DS(\theta,\phi,s) &= \lim_{N\to\infty} 2\pi DS_N\left(\frac{2\pi\theta}{N},\frac{2\pi\phi}{N},1+\frac{s}{N}\right) \\ IS(\theta,\phi,s) &= \lim_{N\to\infty} \frac{2\pi}{N^2} IS_N\left(\frac{2\pi\theta}{N},\frac{2\pi\phi}{N},1+\frac{s}{N}\right). \end{aligned}$$

Theorem II.2. *Let* $r = 1 + \dfrac{s}{N}$ *for* $0 < s < \infty$. *Then*

$$\begin{aligned}
S(\theta,\phi,s) &= \frac{1}{2}\int_0^2 \left(\cos[\pi(\theta-\phi)t] + i\tanh\left[\frac{st}{4}\right]\sin[\pi(\theta-\phi)t]\right)dt \\
DS(\theta,\phi,s) &= \frac{is}{2}\int_0^2 \operatorname{csch}\left[\frac{st}{2}\right]\sin[\pi(\theta-\phi)t]\,dt \\
IS(\theta,\phi,s) &= -\frac{i}{s}\int_0^2 \tanh\left[\frac{st}{4}\right]\sin[\pi(\theta-\phi)t]\,dt.
\end{aligned} \tag{II.8}$$

The limiting kernels with respect to s are now easily calculated. As $s \to \infty$,

$$\begin{aligned}
S(\theta,\phi,s) &\to \frac{1}{2}\int_0^2 e^{\pi i(\theta-\phi)}dt \\
&= S_2(\theta,\phi) \\
DS(\theta,\phi,s) &\to 0 \\
IS(\theta,\phi,s) &\to 0
\end{aligned}$$

and as $s \to 0$,

$$\begin{aligned}
S(\theta,\phi,s) &\to \frac{1}{2}\int_0^2 \cos[\pi(\theta-\phi)t]\,dt \\
&= \frac{\sin[2\pi(\theta-\phi)]}{2\pi(\theta-\phi)} \\
DS(\theta,\phi,s) &\to i\int_0^2 \frac{\sin[\pi(\theta-\phi)t]}{t}dt \\
IS(\theta,\phi,s) &\to -\frac{i}{4}\int_0^2 t\cdot\sin[\pi(\theta-\phi)t]\,dt \\
&= i\left(\frac{\cos[2\pi(\theta-\phi)]}{2\pi(\theta-\phi)} - \frac{\sin[2\pi(\theta-\phi)]}{4\pi^2(\theta-\phi)^2}\right).
\end{aligned}$$

Thus, we recover the $\beta = 2$ (A.2) and the $\beta = 4$ (C.2) asymptotic kernels. Figure 2.1. are the graphs of the second correlation functions as a function of the distance between two particles for various values of s. It represents the density of the

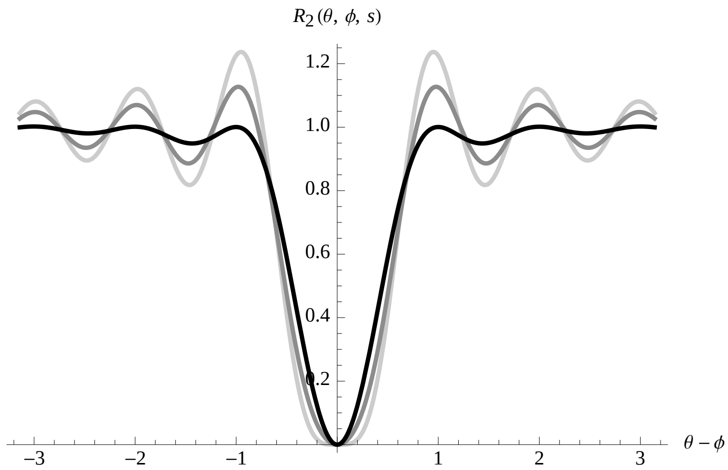

Figure 2.1.: Plots of $R_2(\theta, \phi, s)$ with kernel (II.8) for $s = \dfrac{1}{25}, 3, 25$ from lightest to darkest.

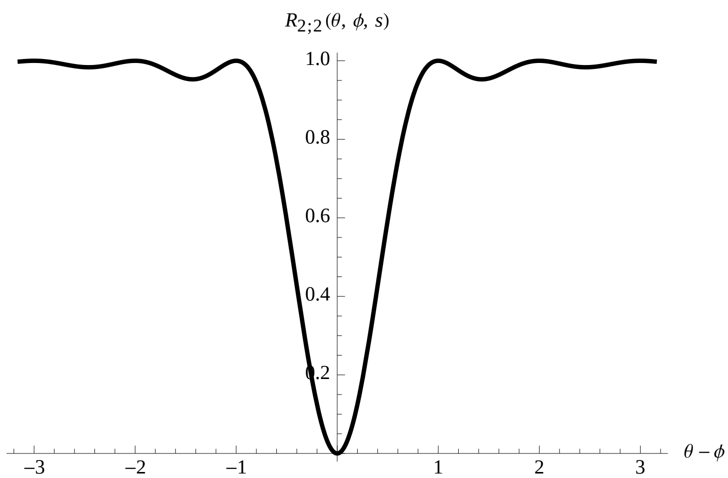

Figure 2.2.: Plot of $R_{2;2}(\theta, \phi)$ with kernel (A.2) (CUE).

particles given that there is a particle located at the origin. Figure 2.2. and 2.3. are the graphs of the second correlation functions of the CUE and CSE, respectively. The three figures exhibit the interpolation property.

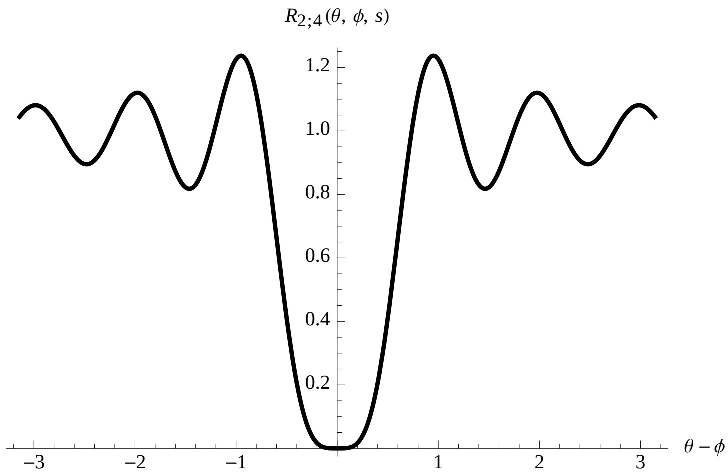

Figure 2.3.: Plot of $R_{2;4}(\theta, \phi)$ with kernel (C.2) (CSE).

Density of Points

Let $A \subset [0, 2\pi)$. By the definition of the correlation functions (I.4),

$$\int_A R_1^N(\theta)\, d\theta$$

is the expected number of particles in A. Thus, the density of a single point on the unit circle is

$$\frac{1}{N} R_1^N(\theta) = \frac{1}{N} S_N(\theta, \theta, r).$$

Using $(II.3)$, we see that density is independent of r and that for all N, the particles are uniformly distributed on the unit circle.

Proofs

Proof of Theorem II.1

We may express (II.1) without the absolute values using the following identities:

$$\left|e^{i\theta_n} - e^{i\theta_m}\right|^2 = -e^{-i(\theta_n+\theta_m)}(e^{i\theta_n} - e^{i\theta_m})^2,$$

$$\left|re^{i\theta_n} - e^{i\theta_m}\right|\left|e^{i\theta_n} - re^{i\theta_m}\right| = -e^{-i(\theta_n+\theta_m)}(re^{i\theta_n} - e^{i\theta_m})(e^{i\theta_n} - re^{i\theta_m}),$$

and

$$\left|re^{i\theta_n} - e^{i\theta_n}\right| = e^{-i\theta_n}(re^{i\theta_n} - e^{i\theta_n}).$$

Thus,

$$\begin{aligned} e^{-E_N(\boldsymbol{\theta}_N, r)} = &\left(\prod_{n>m}^{N} (e^{i\theta_n} - e^{i\theta_m})(re^{i\theta_n} - re^{i\theta_m})\right) \\ &\times \left(\prod_{n\neq m}^{N} (re^{i\theta_n} - e^{i\theta_m})\right)\left(\prod_{n=1}^{N} e^{-i(2N-1)\theta_n}\right). \end{aligned}$$

The first two products together make the Vandermonde product of

$$e^{i\theta_1}, \ldots, e^{i\theta_N}, re^{i\theta_1}, \ldots, re^{i\theta_N},$$

so the above is equal to

$$\begin{aligned} &\left(\prod_{n=1}^{N} e^{-i(2N-1)\theta_n}\right) \cdot \det\left[e^{i(n-1)\theta_m} \quad r^{n-1}e^{i(n-1)\theta_m}\right]_{n,m=1}^{2N;N} \\ &= \det\left[e^{i(-N-\frac{1}{2}+n)\theta_m} \quad r^{n-1}e^{i(-N-\frac{1}{2}+n)\theta_m}\right]_{n,m=1}^{2N;N}. \end{aligned}$$

Here the subscripts and superscripts of the matrices indicate that $n = 1, \ldots, N$ and

$m = 1, \ldots, 2N$. We express the determinant using the Laplace expansion:

$$\det\left[e^{i(-N-\frac{1}{2}+n)\theta_m} \quad r^{n-1}e^{i(-N-\frac{1}{2}+n)\theta_m}\right]_{n,m=1}^{2N;N}$$
$$= \sum_{\substack{\sigma\in S_{2N}\\ \sigma(2m-1)<\sigma(2m)\ \forall m}} \operatorname{sgn}\sigma \prod_{m=1}^{N}\left(r^{\sigma(2m)-1} - r^{\sigma(2m-1)-1}\right) e^{(-2N+\sigma(2m-1)+\sigma(2m)-1)\theta_m}.$$

So

$$\begin{aligned}
Z_N(r) &= \frac{1}{N!}\int_{[-\pi,\pi)^N} e^{-E_N(\boldsymbol{\theta}_N,r)}\, d\boldsymbol{\theta}_N \\
&= \frac{1}{N!}\int_{[-\pi,\pi)^N} \sum_{\substack{\sigma\in S_{2N}\\ \sigma(2m-1)<\sigma(2m)\forall m}} \operatorname{sgn}\sigma \\
&\quad \times \prod_{m=1}^{N}\left(r^{\sigma(2m)-1} - r^{\sigma(2m-1)-1}\right) e^{(-2N+\sigma(2m-1)+\sigma(2m)-1)\theta_m}\, d\boldsymbol{\theta}_N \\
&= \frac{1}{N!} \sum_{\substack{\sigma\in S_{2N}\\ \sigma(2m-1)<\sigma(2m)\forall m}} \operatorname{sgn}\sigma \\
&\quad \times \prod_{m=1}^{N}\left(r^{\sigma(2m)-1} - r^{\sigma(2m-1)-1}\right) \int_{-\pi}^{\pi} e^{(-2N+\sigma(2m-1)+\sigma(2m)-1)\theta}\, d\theta.
\end{aligned}$$

If we remove the condition on σ, each term will appear 2^N times because the factor of -1 introduced from swapping the difference $r^{\sigma(2m)-1} - r^{\sigma(2m-1)-1}$ will be negated by the factor of -1 introduced from the sign of a transposition composed with σ. Thus, without the condition and adding in a factor of $\frac{1}{2^N}$, the above is equal to

$$\begin{aligned}
&= \frac{1}{2^N N!}\sum_{\sigma\in S_{2N}} \operatorname{sgn}\sigma \prod_{m=1}^{N}\left(r^{\sigma(2m)-1} - r^{\sigma(2m-1)-1}\right) \int_{-\pi}^{\pi} e^{(-2N+\sigma(2m-1)+\sigma(2m)-1)\theta}\, d\theta \\
&= \operatorname{Pf}\left[\left(r^{m-1} - r^{n-1}\right)\int_{-\pi}^{\pi} e^{i(-2N+n+m-1)\theta}\, d\theta\right]_{n,m=1}^{2N}. \qquad \text{(II.9)}
\end{aligned}$$

Since

$$\int_{-\pi}^{\pi} e^{i(-2N+n+m-1)\theta}\,d\theta = \begin{cases} 2\pi & \text{if } n+m = 2N+1 \\ \\ 0 & \text{otherwise,} \end{cases}$$

the matrix in (II.9) is equal to $\mathbf{U}_N(r)$.

Proof of Corollary II.1

We sketch the proof whose missing details can be found in Section 7 of [7]. Let $\theta_1, \ldots, \theta_N \in [-\pi, \pi)$ and let $c_1, \ldots, c_N$ be indeterminants. Let

$$d\nu(\theta) = \sum_{n=1}^{N} c_k d\delta_{\theta_k}(\theta)$$

and

$$Z_N^{\nu}(r) = \frac{1}{N!}\int_{[-\pi,\pi)^N} e^{-E_N(\boldsymbol{\theta}_N, r)} \prod_{n=1}^{N}(d\theta_n + d\nu(\theta_n)).$$

By expanding the product and using the symmetry of the integrand,

$$\frac{Z_N^{\nu}(r)}{Z_N(r)} = \sum_{\substack{n \le N \\ k_1 < \cdots < k_n}} c_{k_1} \cdots c_{k_n} R_n^N(\theta_{k_1}, \ldots, \theta_{k_n})$$

i.e. $\dfrac{Z_N^{\nu}(r)}{Z_N(r)}$ is a generating function for the correlation functions. On the other hand, from Theorem II.1,

$$Z_N^{\nu}(r) = \mathrm{Pf}\left[(r^{m-1} - r^{n-1})\int_{-\pi}^{\pi} e^{i(-2N+n+m-1)\theta}(d\theta + d\nu(\theta))\right]_{n,m=1}^{2N}$$

and

$$(r^{m-1} - r^{n-1})\int_{-\pi}^{\pi} e^{i(-2N+n+m-1)\theta}\,d\nu(\theta) = \sum_{k=1}^{N} c_k (r^{m-1} - r^{n-1}) e^{i(-2N+m+n-1)\theta_k}.$$

Let

$$\mathbf{A}_N(\boldsymbol{\theta}_N, r) = \left[\sqrt{c_k} e^{i\left(-N-\frac{1}{2}+n\right)\theta_k} \quad \sqrt{c_k} r^{n-1} e^{i\left(-N-\frac{1}{2}+n\right)\theta_k}\right]_{n,k=1}^{2N;N}$$

and let $\mathbf{J}_{2N}$ be the $2N \times 2N$ antisymmetric matrix

$$\mathbf{J}_{2N} = \mathbf{I}_N \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then matrix multiplication gives

$$\frac{Z_N^{\nu}(r)}{Z_N(r)} = \frac{\mathrm{Pf}\left[\mathbf{U}_N(r) + \mathbf{A}_N(\boldsymbol{\theta}, r)\mathbf{J}_{2N}\mathbf{A}_N^T(\boldsymbol{\theta}, r)\right]}{\mathrm{Pf}\, \mathbf{U}_N(r)}.$$

Using the Pfaffian Cauchy-Binet formula in Appendix B of [7],

$$\begin{aligned} \frac{Z_N^{\nu}(r)}{Z_N(r)} &= \frac{\mathrm{Pf}\left[-\mathbf{J}_{2N}^{-T} - \mathbf{A}_N^T(\boldsymbol{\theta}, r)\mathbf{U}_N^{-T}(r)\mathbf{A}_N(\boldsymbol{\theta}, r)\right]}{\mathrm{Pf}\left[-\mathbf{J}_{2N}^{-T}\right]} \\ &= \mathrm{Pf}\left[\mathbf{J}_{2N} + \mathbf{A}_N^T(\boldsymbol{\theta}, r)\mathbf{U}_N^{-T}(r)\mathbf{A}_N(\boldsymbol{\theta}, r)\right]. \end{aligned}$$

We see that

$$\mathbf{U}_N^{-T}(r) = \left[\frac{1}{2\pi\left(r^{n-1} - r^{m-1}\right)}\delta_{2N+1,n+m}\right]_{n,m=1}^{2N},$$

so

$$\frac{Z_N^{\nu}(r)}{Z_N(r)} = \mathrm{Pf}\left[\mathbf{J}_{2N} + \left[\sqrt{c_n c_m}\mathbf{K}_N(\theta_n, \theta_m, r)\right]_{n,m=1}^{N}\right].$$

Finally using the addition formula for the Pfaffian which can be found in Lemma 7 of [33],

$$\frac{Z_N^{\nu}(r)}{Z_N(r)} = \sum_{\substack{n \le N \\ k_1 < \cdots < k_n}} c_{k_1} \cdots c_{k_n}\, \mathrm{Pf}\left[\mathbf{K}_N(\theta_{k_i}, \theta_{k_j}, r)\right]_{i,j=1}^{n}.$$

Matching coefficients gives the result.

Proof of Proposition II.1

We can rewrite (II.2) and (II.6) as

$$\begin{aligned}
S_N(\theta,\phi,r) &= \frac{1}{2\pi}\sum_{n=1}^{N} e^{i\left(n-\frac{1}{2}\right)(\theta-\phi)} - \frac{i}{\pi}\sum_{n=1}^{N}\frac{\sin\left[\left(n-\frac{1}{2}\right)(\theta-\phi)\right]}{r^{n-\frac{1}{2}}+1} \\
IS_N(\theta,\phi,r) &= -\frac{ir}{\pi(r-1)}\sum_{n=1}^{N}\sin\left[\left(n-\frac{1}{2}\right)(\theta-\phi)\right] \\
&\quad + \frac{2ir}{\pi(r-1)}\sum_{n=1}^{N}\frac{\sin\left[\left(n-\frac{1}{2}\right)(\theta-\phi)\right]}{r^{n-\frac{1}{2}}+1},
\end{aligned}$$

and we use the representation (II.5) for DS_N. Before evaluating the limit, we need the following estimate. Recalling that $r > 1$,

$$\begin{aligned}
\left|\frac{1}{N}\sum_{n=1}^{N}\frac{\sin\left[\frac{2\pi}{N}\left(n-\frac{1}{2}\right)(\theta-\phi)\right]}{r^{n-\frac{1}{2}}+1}\right| &\leq \frac{1}{N}\sum_{n=0}^{N-1}\frac{1}{r^{n+\frac{1}{2}}+1} \\
&\leq \frac{1}{Nr^{\frac{1}{2}}}\sum_{n=0}^{\infty}\frac{1}{r^n} \\
&= \frac{r^{\frac{1}{2}}}{N(r-1)}.
\end{aligned}$$

Thus, as $N \to \infty$,

$$\begin{aligned}
\frac{2\pi}{N}S_N\left(\frac{2\pi\theta}{N},\frac{2\pi\phi}{N},r\right) &= \frac{1}{2}\cdot\frac{2}{N}\sum_{n=1}^{N} e^{i\pi\left(\frac{2n-1}{N}\right)(\theta-\phi)} - \frac{2i}{N}\sum_{n=1}^{N}\frac{\sin\left[\frac{2\pi}{N}\left(n-\frac{1}{2}\right)(\theta-\phi)\right]}{r^{n-\frac{1}{2}}+1} \\
&\to \frac{1}{2}\int_0^2 e^{i\pi(\theta-\phi)t}\,dt \\
&= e^{\pi i(\theta-\phi)}\frac{\sin\pi(\theta-\phi)}{\pi(\theta-\phi)}
\end{aligned}$$

and

$$\begin{aligned}
\frac{2\pi}{N} IS_N\left(\frac{2\pi\theta}{N}, \frac{2\pi\phi}{N}, r\right) &= -\frac{2ir}{(r-1)N}\sum_{n=1}^{N}\sin\left[\pi\left(\frac{2n-1}{N}\right)(\theta-\phi)\right] \\
&\quad + \frac{4ir}{N(r-1)}\sum_{n=1}^{N}\frac{\sin\left[\frac{2\pi}{N}\left(n-\frac{1}{2}\right)(\theta-\phi)\right]}{r^{n-\frac{1}{2}}+1} \\
&\to \frac{ir}{(r-1)}\int_0^2 \cos\left[\pi(\theta-\phi)t\right]dt \\
&= \frac{ir}{(r-1)}\left(\frac{\cos\left[2\pi(\theta-\phi)\right]-1}{\pi(\theta-\phi)}\right).
\end{aligned}$$

Lastly,

$$\frac{2\pi}{N} DS_N\left(\frac{2\pi\theta}{N}, \frac{2\pi\phi}{N}, r\right) = -\frac{2i(r-1)}{Nr}\sum_{n=1}^{N}\left(\frac{r^{n-\frac{1}{2}}}{1-r^{2n-1}}\right)\sin\left[\frac{2\pi}{N}\left(n-\frac{1}{2}\right)(\theta-\phi)\right].$$

Let $k = \lceil \log_r(\sqrt{2}) - \dfrac{1}{2}\rceil$, where $\lceil\cdot\rceil$ is the ceiling function. Then for $n \geq k$,

$$1 - \frac{1}{r^{2n+1}} \geq \frac{1}{2}.$$

Thus,

$$\begin{aligned}
&\left|-\frac{2i(r-1)}{Nr}\sum_{n=1}^{N}\left(\frac{r^{n-\frac{1}{2}}}{1-r^{2n-1}}\right)\sin\left[\frac{2\pi}{N}\left(n-\frac{1}{2}\right)(\theta-\phi)\right]\right| \\
&\quad \leq \frac{2(r-1)}{Nr}\sum_{n=0}^{N-1}\frac{r^{n+\frac{1}{2}}}{r^{2n+1}-1} \\
&\quad \leq \frac{2(r-1)}{Nr}\sum_{n=0}^{\infty}\frac{1}{r^{n+\frac{1}{2}}\left(1-\frac{1}{r^{2n+1}}\right)} \\
&\quad = \frac{2(r-1)}{Nr}\sum_{n=0}^{k-1}\frac{1}{r^{n+\frac{1}{2}}\left(1-\frac{1}{r^{2n+1}}\right)} + \frac{2(r-1)}{Nr}\sum_{n=k}^{\infty}\frac{1}{r^{n+\frac{1}{2}}\left(1-\frac{1}{r^{2n+1}}\right)} \\
&\quad \leq \frac{2(r-1)}{Nr}\sum_{n=0}^{k-1}\frac{1}{r^{n+\frac{1}{2}}\left(1-\frac{1}{r^{2n+1}}\right)} + \frac{(r-1)}{Nr}\sum_{n=0}^{\infty}\frac{1}{r^{n}} \\
&\quad = \frac{2(r-1)}{Nr}\sum_{n=0}^{k-1}\frac{1}{r^{n+\frac{1}{2}}\left(1-\frac{1}{r^{2n+1}}\right)} + \frac{1}{N}
\end{aligned}$$

which shows that

$$\frac{2\pi}{N} DS_N\left(\frac{2\pi\theta}{N}, \frac{2\pi\phi}{N}, r\right) \to 0.$$

Proof of Theorem II.2

We use (II.3) to represent S_N. Then

$$\begin{aligned}
S(\theta,\phi,s) &= \lim_{N\to\infty} \frac{2\pi}{N} S_N\left(\frac{2\pi\theta}{N}, \frac{2\pi\phi}{N}, 1+\frac{s}{N}\right) \\
&= \lim_{N\to\infty} \left\{ \frac{1}{N}\sum_{n=1}^{N} \cos\left[\frac{2\pi}{N}\left(n-\frac{1}{2}\right)(\theta-\phi)\right]\right. \\
&\quad \left. + \frac{i}{N}\sum_{n=1}^{N}\left(\frac{\left(1+\frac{s}{N}\right)^{n-\frac{1}{2}}-1}{\left(1+\frac{s}{N}\right)^{n-\frac{1}{2}}+1}\right)\sin\left[\frac{2\pi}{N}\left(n-\frac{1}{2}\right)(\theta-\phi)\right]\right\} \\
&= \lim_{N\to\infty} \left\{ \frac{1}{2}\cdot\frac{2}{N}\sum_{n=1}^{N} \cos\left[\pi\left(\frac{2n-1}{N}\right)(\theta-\phi)\right]\right. \\
&\quad \left. + \frac{i}{2}\cdot\frac{2}{N}\sum_{n=1}^{N}\left(\frac{\left(1+\frac{s}{N}\right)^{N\left(\frac{2n-1}{2N}\right)}-1}{\left(1+\frac{s}{N}\right)^{N\left(\frac{2n-1}{2N}\right)}+1}\right)\sin\left[\pi\left(\frac{2n-1}{N}\right)(\theta-\phi)\right]\right\}.
\end{aligned}$$

Let

$$\begin{aligned}
t_N &= \frac{2}{N}\sum_{n=1}^{N}\frac{\left(1+\frac{s}{N}\right)^{N\left(\frac{2n-1}{2N}\right)}-1}{\left(1+\frac{s}{N}\right)^{N\left(\frac{2n-1}{2N}\right)}+1} \\
&= \frac{2}{N}\sum_{n=1}^{N}\frac{\left(1+s+\frac{s^2}{2!}\left(1-\frac{1}{N}\right)+\cdots+\frac{s^N}{N!}\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{N-1}{N}\right)\right)^{\frac{2n-1}{2N}}-1}{\left(1+s+\frac{s^2}{2!}\left(1-\frac{1}{N}\right)+\cdots+\frac{s^N}{N!}\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{N-1}{N}\right)\right)^{\frac{2n-1}{2N}}+1}.
\end{aligned}$$

For some number $A \geq 0$ depending $N, n,$ and s, each summand can be expressed as

$$\frac{\left(1+s+\frac{s^2}{2!}+\cdots+\frac{s^N}{N!}\right)^{\frac{2n-1}{2N}}-1-A}{\left(1+s+\frac{s^2}{2!}+\cdots+\frac{s^N}{N!}\right)^{\frac{2n-1}{2N}}+1-A}$$

Note that this fraction is less than or equal to one. So by adding A to the numerator and denominator, we get

$$\frac{\left(1+s+\frac{s^2}{2!}+\cdots+\frac{s^N}{N!}\right)^{\frac{2n-1}{2N}}-1-A}{\left(1+s+\frac{s^2}{2!}+\cdots+\frac{s^N}{N!}\right)^{\frac{2n-1}{2N}}+1-A} \leq \frac{\left(1+s+\frac{s^2}{2!}+\cdots+\frac{s^N}{N!}\right)^{\frac{2n-1}{2N}}-1}{\left(1+s+\frac{s^2}{2!}+\cdots+\frac{s^N}{N!}\right)^{\frac{2n-1}{2N}}+1}.$$

Then for all N,

$$\begin{aligned} t_N &\leq \frac{2}{N}\sum_{n=1}^{N} \frac{\left(1+s+\frac{s^2}{2!}+\cdots+\frac{s^N}{N!}\right)^{\frac{2n-1}{2N}}-1}{\left(1+s+\frac{s^2}{2!}+\cdots+\frac{s^N}{N!}\right)^{\frac{2n-1}{2N}}+1} \\ &\leq \frac{2}{N}\sum_{n=1}^{N}\left(\frac{(e^s)^{\frac{2n-1}{2N}}-1}{(e^s)^{\frac{2n-1}{2N}}+1}\right). \end{aligned}$$

Thus,

$$\limsup_{N\to\infty} t_N \leq \int_0^2 \frac{e^{\frac{st}{2}}-1}{e^{\frac{st}{2}}+1}\,dt.$$

On the other hand, for all $m \leq N$ and for some $B > 0$ depending on N, n, m, and s, we can write

$$\begin{aligned} &\frac{\left(1+s+\frac{s^2}{2!}\left(1-\frac{1}{N}\right)+\cdots+\frac{s^N}{N!}\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{N-1}{N}\right)\right)^{\frac{2n-1}{2N}}-1}{\left(1+s+\frac{s^2}{2!}\left(1-\frac{1}{N}\right)+\cdots+\frac{s^N}{N!}\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{N-1}{N}\right)\right)^{\frac{2n-1}{2N}}+1} \\ &= \frac{\left(1+s+\frac{s^2}{2!}\left(1-\frac{1}{N}\right)+\cdots+\frac{s^m}{m!}\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{m-1}{N}\right)\right)^{\frac{2n-1}{2N}}-1+B}{\left(1+s+\frac{s^2}{2!}\left(1-\frac{1}{N}\right)+\cdots+\frac{s^m}{m!}\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{m-1}{N}\right)\right)^{\frac{2n-1}{2N}}+1+B} \\ &\geq \frac{\left(1+s+\frac{s^2}{2!}\left(1-\frac{1}{N}\right)+\cdots+\frac{s^m}{m!}\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{m-1}{N}\right)\right)^{\frac{2n-1}{2N}}-1}{\left(1+s+\frac{s^2}{2!}\left(1-\frac{1}{N}\right)+\cdots+\frac{s^m}{m!}\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{m-1}{N}\right)\right)^{\frac{2n-1}{2N}}+1}. \end{aligned}$$

Thus,

$$\begin{aligned}
t_N &\geq \frac{2}{N}\sum_{n=1}^{N}\frac{\left(1+s+\frac{s^2}{2!}\left(1-\frac{1}{N}\right)+\cdots+\frac{s^m}{m!}\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{m-1}{N}\right)\right)^{\frac{2n-1}{2N}}-1}{\left(1+s+\frac{s^2}{2!}\left(1-\frac{1}{N}\right)+\cdots+\frac{s^m}{m!}\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{m-1}{N}\right)\right)^{\frac{2n-1}{2N}}+1} \\
&\geq \frac{2}{N}\sum_{n=1}^{N}\frac{\left(1+s+\frac{s^2}{2!}\left(1-\frac{1}{N}\right)+\cdots+\frac{s^m}{m!}\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{m-1}{N}\right)\right)^{\frac{2n-1}{2N}}-1}{\left(1+s+\frac{s^2}{2!}+\cdots+\frac{s^m}{m!}\right)^{\frac{2n-1}{2N}}+1}.
\end{aligned}$$

Now for any n, we give an estimate on the first part of the numerator:

$$\begin{aligned}
&\left(1+s+\frac{s^2}{2!}\left(1-\frac{1}{N}\right)+\cdots+\frac{s^m}{m!}\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{m-1}{N}\right)\right)^{\frac{2n-1}{2N}} \\
&\qquad\geq \left(\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{m-1}{N}\right)\right)^{\frac{2n-1}{2N}}\cdot\left(1+s+\frac{s^2}{2!}+\cdots+\frac{s^m}{m!}\right)^{\frac{2n-1}{2N}} \\
&\qquad\geq \left(1-\frac{1}{N}\right)\cdots\left(1-\frac{m-1}{N}\right)\left(1+s+\frac{s^2}{2!}+\cdots+\frac{s^m}{m!}\right)^{\frac{2n-1}{2N}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{2}{N}\sum_{n=1}^{N}\frac{\left(1+s+\frac{s^2}{2!}\left(1-\frac{1}{N}\right)+\cdots+\frac{s^m}{m!}\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{m-1}{N}\right)\right)^{\frac{2n-1}{2N}}}{\left(1+s+\frac{s^2}{2!}+\cdots+\frac{s^m}{m!}\right)^{\frac{2n-1}{2N}}+1} \\
&\quad\geq \frac{2}{N}\cdot\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{m-1}{N}\right)\sum_{n=1}^{N}\frac{\left(1+s+\frac{s^2}{2!}+\cdots+\frac{s^m}{m!}\right)^{\frac{2n-1}{2N}}}{\left(1+s+\frac{s^2}{2!}+\cdots+\frac{s^m}{m!}\right)^{\frac{2n-1}{2N}}+1} \\
&\quad\to \int_0^2\frac{\left(1+s+\frac{s^2}{2!}+\cdots+\frac{s}{m!}\right)^{\frac{t}{2}}}{\left(1+s+\frac{s^2}{2!}+\cdots+\frac{s}{m!}\right)^{\frac{t}{2}}+1}\,dt
\end{aligned}$$

as $N\to\infty$. The above calculation together with

$$\frac{2}{N}\sum_{n=1}^{N}\frac{1}{\left(1+s+\frac{s^2}{2!}+\cdots+\frac{s^m}{m!}\right)^{\frac{2n-1}{2N}}+1}\to\int_0^2\frac{1}{\left(1+s+\frac{s^2}{2!}+\cdots+\frac{s}{m!}\right)^{\frac{t}{2}}+1}\,dt$$

as $N\to\infty$ gives

$$\liminf_{N\to\infty} t_N \geq \int_0^2\frac{\left(1+s+\frac{s^2}{2!}+\cdots+\frac{s}{m!}\right)^{\frac{t}{2}}-1}{\left(1+s+\frac{s^2}{2!}+\cdots+\frac{s}{m!}\right)^{\frac{t}{2}}+1}\,dt$$

for all m. Therefore, letting $m \to \infty$,

$$\liminf_{N\to\infty} t_N \geq \int_0^2 \frac{e^{\frac{st}{2}}-1}{e^{\frac{st}{2}}+1}\,dt$$

This shows that

$$S(\theta,\phi,s) = \frac{1}{2}\int_0^2 \cos\left[\pi(\theta-\phi)t\right] + i\left(\frac{e^{\frac{st}{2}}-1}{e^{\frac{st}{2}}+1}\right)\sin\left[\pi(\theta-\phi)t\right]dt$$

We next use (II.7) to represent IS_N and (II.5) to represent DS_N. A similar argument shows that

$$\begin{aligned}
DS(\theta,\phi,s) &= \lim_{N\to\infty} 2\pi\cdot DS_N\left(\frac{2\pi\theta}{N},\frac{2\pi\phi}{N},1+\frac{s}{N}\right)\\
&= -\lim_{N\to\infty}\frac{2is}{\left(1+\frac{s}{N}\right)N}\sum_{n=1}^{N}\left(\frac{\left(1+\frac{s}{N}\right)^{n-\frac{1}{2}}}{1-\left(1+\frac{s}{N}\right)^{2n-1}}\right)\sin\left[\frac{2\pi}{N}\left(n-\frac{1}{2}\right)(\theta-\phi)\right]\\
&= -\lim_{N\to\infty}\frac{is}{1+\frac{s}{N}}\cdot\frac{2}{N}\sum_{n=1}^{N}\left(\frac{\left(1+\frac{s}{N}\right)^{N\left(\frac{2n-1}{2N}\right)}}{1-\left(1+\frac{s}{N}\right)^{N\left(\frac{2n-1}{N}\right)}}\right)\sin\left[\pi\left(\frac{2n-1}{N}\right)(\theta-\phi)\right]\\
&= -is\int_0^2\left(\frac{e^{\frac{st}{2}}}{1-e^{st}}\right)\sin\left[\pi(\theta-\phi)t\right]dt
\end{aligned}$$

and

$$\begin{aligned}
IS(\theta,\phi,s) &= \lim_{N\to\infty}\frac{2\pi}{N^2}IS_N\left(\frac{2\pi\theta}{N},\frac{2\pi\phi}{N},1+\frac{s}{N}\right)\\
&= -\lim_{N\to\infty}\frac{2i\left(1+\frac{s}{N}\right)}{Ns}\sum_{n=1}^{N}\left(\frac{\left(1+\frac{s}{N}\right)^{n-\frac{1}{2}}-1}{\left(1+\frac{s}{N}\right)^{n-\frac{1}{2}}+1}\right)\sin\left[\frac{2\pi}{N}\left(n-\frac{1}{2}\right)(\theta-\phi)\right]\\
&= -\lim_{N\to\infty}\frac{i\left(1+\frac{s}{N}\right)}{s}\cdot\frac{2}{N}\sum_{n=1}^{N}\left(\frac{\left(1+\frac{s}{N}\right)^{N\left(\frac{2n-1}{2N}\right)}-1}{\left(1+\frac{s}{N}\right)^{N\left(\frac{2n-1}{2N}\right)}+1}\right)\\
&\quad\times\ \sin\left[\pi\left(\frac{2n-1}{N}\right)(\theta-\phi)\right]\\
&= -\frac{i}{s}\int_0^2\left(\frac{e^{\frac{st}{2}}-1}{e^{\frac{st}{2}}+1}\right)\sin\left[\pi(\theta-\phi)t\right]dt.
\end{aligned}$$

CHAPTER III

A SOLVABLE MULTICHARGE ENSEMBLE THAT INTERPOLATES BETWEEN COE AND CSE

Setup

Fix N to be a nonnegative integer representing the total charge of the system. We study the ensemble of mixed charge-one and charge-two particles on the unit circle with no external field and interacting logarithmically. The numbers of each particle types are random, and a fugacity parameter X is included to control these distributions. The 2-dimensional electrostatics model and the construction of the partition function was discussed in the Introduction, and we summarize the setup here. For a given pair (L, M) such that $L+2M = N$, we first consider a system of L charge-one particles located at $\{e^{i\theta_1}, \ldots, e^{i\theta_L}\}$ and M charge-two particles located at $\{e^{i\phi_1}, \ldots, e^{i\phi_M}\}$. The energy of this particular configuration is

$$
\begin{aligned}
E(\boldsymbol{\theta}_L, \boldsymbol{\phi}_M) = &-\sum_{l_1>l_2}^{L} \log\left|e^{i\theta_{l_1}} - e^{i\theta_{l_2}}\right| - 4\sum_{m_1>m_2}^{M} \log\left|e^{i\phi_{m_1}} - e^{i\phi_{m_2}}\right| \\
&- 2\sum_{l=1}^{L}\sum_{m=1}^{M} \log\left|e^{i\theta_l} - e^{i\phi_m}\right|,
\end{aligned} \tag{III.1}
$$

and the partial partition function $Z_{L,M}$ is

$$
Z_{L,M} = \frac{1}{L!M!}\int_{[-\pi,\pi)^L}\int_{[-\pi,\pi)^M} e^{-E(\boldsymbol{\theta}_L,\boldsymbol{\phi}_M)}\, d\boldsymbol{\theta}_L d\boldsymbol{\phi}_M. \tag{III.2}
$$

The total partition function is the weighted sum of the partition functions over all possible pairs (L, M)

$$Z_N(X) = \sum_{(L,M)} X^L Z_{L,M}.$$

Pfaffian Expression for the Partition Function

We first give a Pfaffian expression for the partition function for both N even and N odd. Let

$$\mathbf{A}_N = \left[\frac{8\pi}{N+1-2n}\delta_{n+m,N+1}\right]_{n,m=1}^{N} \tag{III.3}$$

$$\mathbf{B}_N = \left[2\pi(N+1-2n)\delta_{n+m,N+1}\right]_{n,m=1}^{N} \tag{III.4}$$

with the exception that when N is odd, the $\left(\frac{N+1}{2}, \frac{N+1}{2}\right)$ entry in $\mathbf{A}_N$ is 0.

Theorem III.1. *If N is even, then*

$$\begin{aligned} Z_N(X) &= \mathrm{Pf}(X^2\boldsymbol{A}_N + \boldsymbol{B}_N) \\ &= (2\pi)^{\frac{N}{2}} \prod_{n=1}^{\frac{N}{2}} \frac{(2X)^2 + (2n-1)^2}{2n-1}. \end{aligned}$$

For N odd, first define $\mathbf{W}_N(X)$ to be the $(N+1)\times(N+1)$ matrix

$$\mathbf{W}_N(X) = \begin{bmatrix} & & & & 0 \\ & & & & \vdots \\ & & X^2\mathbf{A}_N + \mathbf{B}_N & & 2\pi e^{\frac{\pi i}{4}}X \\ & & & & \vdots \\ 0 & \cdots & -2\pi e^{\frac{\pi i}{4}}X & \cdots & 0 \end{bmatrix}$$

where the nonzero entry in the border is in the $\frac{N+1}{2}$ entry.

Theorem III.2. *If N is odd, then*

$$\begin{aligned} Z_N(X) &= e^{-\frac{\pi i}{4}} \operatorname{Pf} \boldsymbol{W}_N(X) \\ &= (2\pi)^{\frac{N+1}{2}} X \prod_{n=1}^{\frac{N-1}{2}} \frac{(2X)^2 + (2n)^2}{2n}. \end{aligned}$$

<u>Distribution of Charge-One Particles</u>

Let $L_N(X)$ be the random variable of the number of charge-one particles. From the formula (I.11), $\dfrac{Z_N(tX)}{Z_N(X)}$ is the probability generating function for $L_N(X)$, that is, the probability that there are L charge-one particles is the coefficient of t^L of $\dfrac{Z_N(tX)}{Z_N(X)}$. The distributions of $L_N(X)$ for N even and odd will have different but similar forms.

Corollary III.1. *The probability generating function of $L_N(X)$ is*

$$\frac{Z_N(tX)}{Z_N(X)} = \prod_{n=1}^{\frac{N}{2}} \frac{(2Xt)^2 + (2n-1)^2}{(2X)^2 + (2n-1)^2}$$

if N is even and

$$\frac{Z_N(tX)}{Z_N(X)} = t \prod_{n=1}^{\frac{N-1}{2}} \frac{(Xt)^2 + n^2}{X^2 + n^2}$$

if N is odd.

We next show that for fixed X and as $N \to \infty$, the random variables $L_{2N}(X)$ and $L_{2N-1}(X)$ converge to an analogue of the Poisson distribution restricted to even and odd values, respectively.

Corollary III.2. *The limiting probability generation function is*

$$\lim_{N\to\infty} \frac{Z_{2N}(tX)}{Z_{2N}(X)} = \frac{\cosh(\pi X t)}{\cosh(\pi X)}$$

in the even case, and is

$$\lim_{N\to\infty} \frac{Z_{2N+1}(tX)}{Z_{2N+1}(X)} = \frac{\sinh(\pi X t)}{\sinh(\pi X)}$$

in the odd case.

Proof. This follows from the infinite product representations of hyperbolic cosine and sine in Section 2.1 of [24]:

$$\cosh(\pi X) = \prod_{n=1}^{\infty} \frac{(2X)^2 + (2n-1)^2}{(2n-1)^2}$$

and

$$\sinh(\pi X) = \pi X \prod_{n=1}^{\infty} \frac{X^2 + n^2}{n^2}.$$

□

Define

$$L_e(X) = \lim_{N\to\infty} L_{2N}(X)$$

$$L_o(X) = \lim_{N\to\infty} L_{2N+1}(X).$$

Corollary III.2 shows that the moments of $L_{\mathrm{e}}(X)$ and $L_{\mathrm{o}}(X)$, given by

$$\mathbb{E}[(L_e(X))^k] = \left[\left(t\frac{d}{dt}\right)^k \left(\frac{\cosh(\pi X t)}{\cosh(\pi X)}\right)\right]\Bigg|_{t=1}$$
$$\mathbb{E}[(L_o(X))^k] = \left[\left(t\frac{d}{dt}\right)^k \left(\frac{\sinh(\pi X t)}{\sinh(\pi X)}\right)\right]\Bigg|_{t=1},$$

respectively, are finite.

Distribution of Charge-One Particles for Scaled Fugacity

We would like a scenario where the numbers of charge-one and charge-two particles have the same order. Larger values of X increase the expected value of L, and so to hope for L and M to be on the same order, we would need to increase X with N. With this in mind, fix $r > 0$ and let $X = Nr$.

Proposition III.1. *For $r > 0$, with the fugacity scaled to Nr, we have*

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left[L_N(Nr)\right] &= 2r\arctan\left(\frac{1}{2r}\right) \\ \lim_{N\to\infty} \frac{1}{N}\operatorname{Var}\left[L_N(Nr)\right] &= 2r\arctan\left(\frac{1}{2r}\right) - \frac{4r^2}{1+4r^2}. \end{aligned}$$

In particular, $\mathbb{E}[L_N(Nr)] = O(N)$ *and* $\operatorname{Var}[L_N(Nr)] = O(N)$.

We now give a central limit theorem for $L_N(Nr)$.

Theorem III.2. *For fixed $r > 0$, let*

$$\mu_N = 2Nr\arctan\left(\frac{1}{2r}\right),$$

and let σ_N be the positive number such that

$$\sigma_N^2 = \left(2r\arctan\left(\frac{1}{2r}\right) - \frac{4r^2}{1+4r^2}\right)N.$$

Then

$$\frac{L_N(Nr) - \mu_N}{\sigma_N}$$

converges weakly to the standard normal distribution.

Remark III.3. In statistical mechanics, *entropy* is a quantity which measures the uncertainty of the state a system [8]. By our central limit theorem, entropy will be maximized when the variance is maximized. Therefore the value of r that maximizes entropy will be the r that maximizes

$$2r \arctan\left(\frac{1}{2r}\right) - \frac{4r^2}{1+4r^2},$$

or $r \approx 0.273934$. The expected limiting proportion of charge-one particles with this value of r is approximately 0.585955.

Correlation Functions and Kernel Entries

The last half of this section is devoted to the calculation of the kernel entries. Recall the two species definition of the (l, m)-correlation function

$$R^N_{l,m}(\boldsymbol{x}_l, \boldsymbol{y}_m, X) = \frac{1}{Z_N(X)} \sum_{\substack{(L,M) \\ L \geq l, M \geq m}} \frac{X^L}{(L-l)!(M-m)!} \\ \times \int_{[-\pi,\pi)^{L-l}} \int_{[-\pi,\pi)^{M-m}} e^{-E(\boldsymbol{x}_l \vee \boldsymbol{\theta}_{L-l}, \boldsymbol{y}_m \vee \boldsymbol{\phi}_{M-m})}\, d\boldsymbol{\theta}_{L-l} d\boldsymbol{\phi}_{M-m}.$$

We assume N is even. We will not prove the odd case, which is slightly more complicated, but the same results hold. The following Corollary states that the correlation function $R^N_{l,m}(\boldsymbol{\theta}_l, \boldsymbol{\phi}_m, X)$ can be expressed as a Pfaffian of a $2(m+l) \times 2(m+l)$ antisymmetric matrix with four 2×2 matrix kernels.

Corollary III.1. *The matrix entries*

$$\widetilde{S}_N^{1,1}(\theta,\phi,X) = \frac{4X^2}{\pi}\sum_{n=1}^{\frac{N}{2}} \frac{\cos\left[\frac{1}{2}(2n-1)(\theta-\phi)\right]}{(2X)^2+(2n-1)^2}$$

$$\widetilde{DS}_N^{1,1}(\theta,\phi,X) = \frac{iX^2}{\pi}\sum_{n=1}^{\frac{N}{2}} \frac{(2n-1)\sin\left[\frac{1}{2}(2n-1)(\theta-\phi)\right]}{(2X)^2+(2n-1)^2}$$

$$\widetilde{IS}_N^{1,1}(\theta,\phi,X) = -\frac{16iX^2}{\pi}\sum_{n=1}^{\frac{N}{2}} \frac{\sin\left[\frac{1}{2}(2n-1)(\theta-\phi)\right]}{(2n-1)((2X)^2+(2n-1)^2)} + \mathrm{sgn}(\phi-\theta)$$

encode information about the interaction between charge-one particles in the $(1,1)$ *matrix kernel*

$$\boldsymbol{K}_N^{1,1}(\theta,\phi,X) = \begin{bmatrix} \widetilde{DS}_N^{1,1}(\theta,\phi,X) & \widetilde{S}_N^{1,1}(\theta,\phi,X) \\ -\widetilde{S}_N^{1,1}(\phi,\theta,X) & \widetilde{IS}_N^{1,1}(\theta,\phi,X) \end{bmatrix}.$$

The matrix entries

$$\widetilde{S}_N^{2,2}(\theta,\phi,X) = \frac{1}{2\pi}\sum_{n=1}^{\frac{N}{2}} \frac{(2n-1)^2\cos\left[\frac{1}{2}(2n-1)(\theta-\phi)\right]}{(2X)^2+(2n-1)^2}$$

$$\widetilde{DS}_N^{2,2}(\theta,\phi,X) = \frac{i}{\pi}\sum_{n=1}^{\frac{N}{2}} \frac{(2n-1)\sin\left[\frac{1}{2}(2n-1)(\theta-\phi)\right]}{(2X)^2+(2n-1)^2}$$

$$\widetilde{IS}_N^{2,2}(\theta,\phi,X) = -\frac{i}{4\pi}\sum_{n=1}^{\frac{N}{2}} \frac{(2n-1)^3\sin\left[\frac{1}{2}(2n-1)(\theta-\phi)\right]}{(2X)^2+(2n-1)^2}$$

encode information about the interaction between charge-two particles in the $(2,2)$ *matrix kernel*

$$\boldsymbol{K}_N^{2,2}(\theta,\phi,X) = \begin{bmatrix} \widetilde{DS}_N^{2,2}(\theta,\phi,X) & \widetilde{S}_N^{2,2}(\theta,\phi,X) \\ -\widetilde{S}_N^{2,2}(\phi,\theta,X) & \widetilde{IS}_N^{2,2}(\theta,\phi,X) \end{bmatrix}.$$

The matrix entries

$$\widetilde{S}_N^{1,2}(\theta,\phi,X) = \frac{X}{2\pi}\sum_{n=1}^{\frac{N}{2}} \frac{(2n-1)^2\cos\left[\frac{1}{2}(2n-1)(\theta-\phi)\right]}{(2X)^2+(2n-1)^2}$$

$$\widetilde{DS}_N^{1,2}(\theta,\phi,X) = \frac{iX}{\pi}\sum_{n=1}^{\frac{N}{2}} \frac{(2n-1)\sin\left[\frac{1}{2}(2n-1)(\theta-\phi)\right]}{(2X)^2+(2n-1)^2}$$

$$-\widetilde{S}_N^{2,1}(\phi,\theta,X) = -\frac{4X}{\pi}\sum_{n=1}^{\frac{N}{2}} \frac{\cos\left[\frac{1}{2}(2n-1)(\theta-\phi)\right]}{(2X)^2+(2n-1)^2}$$

$$\widetilde{IS}_N^{1,2}(\theta,\phi,X) = -\frac{2iX}{\pi}\sum_{n=1}^{\frac{N}{2}} \frac{(2n-1)\sin\left[\frac{1}{2}(2n-1)(\theta-\phi)\right]}{(2X)^2+(2n-1)^2}$$

encode information about the interaction between charge-one and charge-two particles in the $(1,2)$ *matrix kernel*

$$\boldsymbol{K}_N^{1,2}(\theta,\phi,X) = \begin{bmatrix} \widetilde{DS}_N^{1,2}(\theta,\phi,X) & \widetilde{S}_N^{1,2}(\theta,\phi,X) \\ -\widetilde{S}_N^{2,1}(\phi,\theta,X) & \widetilde{IS}_N^{1,2}(\theta,\phi,X) \end{bmatrix}.$$

The $(2,1)$ *matrix kernel also encodes mixed interaction and is determined by antisymmetry. Let*

$$\boldsymbol{K}_{l,m}(\boldsymbol{\theta}_l,\boldsymbol{\phi}_m,X) = \begin{bmatrix} \boldsymbol{K}_N^{1,1}(\theta_j,\theta_{j'},X) & \boldsymbol{K}_N^{1,2}(\theta_j,\phi_{k'},X) \\ \boldsymbol{K}_N^{2,1}(\phi_k,\theta_{j'},X) & \boldsymbol{K}_N^{2,2}(\phi_k,\phi_{k'},X) \end{bmatrix}$$

for $j, j' = 1,\ldots,l$ *and* $k, k' = 1,\ldots,m$. *Then*

$$R_{l,m}^N(\boldsymbol{\theta}_l,\boldsymbol{\phi}_m,X) = \operatorname{Pf} \boldsymbol{K}_{l,m}(\boldsymbol{\theta}_l,\boldsymbol{\phi}_m,X).$$

Remark III.2. Comparing to (C.3) and (B.1) in the Appendix, note that when $X=0$, we recover the kernel for $\beta=4$, and as $X\to\infty$, we recover the kernels for $\beta=1$.

Limiting Kernels as $N \to \infty$ for Scaled Fugacity

Our next goal is to find the limiting kernel entries as $N \to \infty$ in the situation when $r > 0$, $X = Nr$. Recall that in this scenario, the orders of the number of charge-one and charge-two particles are the same. Because some of the current kernel entries will diverge in the limit, we need to first alter the kernel entries in a manner which does not change the Pfaffian or the block structure of the matrix kernels. Let $\mathbf{D}$ be the $N \times N$ diagonal matrix

$$\mathbf{D} = \operatorname{diag}\left(\underbrace{\sqrt{\frac{r}{X}}, \sqrt{\frac{X}{r}}, \ldots, \sqrt{\frac{r}{X}}, \sqrt{\frac{X}{r}}}_{2l}, \underbrace{\sqrt{\frac{X}{r}}, \sqrt{\frac{r}{X}}, \ldots, \sqrt{\frac{X}{r}}, \sqrt{\frac{r}{X}}}_{2m}\right).$$

Then

$$\operatorname{Pf} \mathbf{K}_{l,m}(\boldsymbol{\theta}_l, \boldsymbol{\phi}_m) = \operatorname{Pf}(\mathbf{D} \cdot \mathbf{K}_{l,m}(\boldsymbol{\theta}_l, \boldsymbol{\phi}_m) \cdot \mathbf{D}^T),$$

where we use $\det \mathbf{D} = 1$ and the well-known identity

$$\operatorname{Pf}(\mathbf{B}\mathbf{A}\mathbf{B}^T) = \operatorname{Pf} \mathbf{A} \det \mathbf{B}$$

for any antisymmetric matrix $\mathbf{A}$ and any matrix $\mathbf{B}$ of the same size. Also, because

$\mathbf{D}$ is diagonal, the block structure is preserved. The new kernel entries are

$$\begin{aligned}
S_N^{1,1}(\theta,\phi,X) &= \frac{4X^2}{\pi}\sum_{n=1}^{\frac{N}{2}} \frac{\cos\left[\frac{1}{2}(2n-1)(\theta-\phi)\right]}{(2X)^2+(2n-1)^2} \\
DS_N^{1,1}(\theta,\phi,X) &= \frac{iXr}{\pi}\sum_{n=1}^{\frac{N}{2}} \frac{(2n-1)\sin\left[\frac{1}{2}(2n-1)(\theta-\phi)\right]}{(2X)^2+(2n-1)^2} \\
IS_N^{1,1}(\theta,\phi,X) &= -\frac{16iX^3}{\pi r}\sum_{n=1}^{\frac{N}{2}} \frac{\sin\left[\frac{1}{2}(2n-1)(\theta-\phi)\right]}{(2n-1)((2X)^2+(2n-1)^2)} + \frac{X}{r}\cdot\operatorname{sgn}(\phi-\theta) \\
S_N^{2,2}(\theta,\phi,X) &= \frac{1}{2\pi}\sum_{n=1}^{\frac{N}{2}} \frac{(2n-1)^2\cos\left[\frac{1}{2}(2n-1)(\theta-\phi)\right]}{(2X)^2+(2n-1)^2} \\
IS_N^{2,2}(\theta,\phi,X) &= -\frac{ir}{4X\pi}\sum_{n=1}^{\frac{N}{2}} \frac{(2n-1)^3\sin\left[\frac{1}{2}(2n-1)(\theta-\phi)\right]}{(2X)^2+(2n-1)^2}. \\
DS_N^{2,2}(\theta,\phi,X) &= \frac{1}{r^2}DS_N^{1,1}(\theta,\phi,X) \\
S_N^{1,2}(\theta,\phi,X) &= rS_N^{2,2}(\theta,\phi,X) \\
-S_N^{2,1}(\phi,\theta,X) &= -\frac{1}{r}S_N^{1,1}(\theta,\phi,X) \\
DS_N^{1,2}(\theta,\phi,X) &= \frac{1}{r}DS_N^{1,1}(\theta,\phi,X) \\
IS_N^{1,2}(\theta,\phi,X) &= -\frac{2}{r}DS_N^{1,1}(\theta,\phi,X).
\end{aligned} \tag{III.5}$$

Now let $X = Nr$. Since there will be order N particles on an interval of length 2π, the average distance between two consecutive particles is $\frac{2\pi}{N}$ units. In order to calculate nontrivial probabilities, we must rescale the length of the interval to be of order N. Then we study the behavior of those particles with arguments near 0.

Define

$$
\begin{aligned}
S^{1,1}(\theta,\phi,r) &= \lim_{N\to\infty} \frac{2\pi}{N} S_N^{1,1}\left(\frac{2\pi\theta}{N}, \frac{2\pi\phi}{N}, Nr\right) \\
DS^{1,1}(\theta,\phi,r) &= \lim_{N\to\infty} \frac{2\pi}{N} DS_N^{1,1}\left(\frac{2\pi\theta}{N}, \frac{2\pi\phi}{N}, Nr\right) \\
IS^{1,1}(\theta,\phi,r) &= \lim_{N\to\infty} \frac{2\pi}{N} IS_N^{1,1}\left(\frac{2\pi\theta}{N}, \frac{2\pi\phi}{N}, Nr\right) \\
S^{2,2}(\theta,\phi,r) &= \lim_{N\to\infty} \frac{2\pi}{N} S_N^{2,2}\left(\frac{2\pi\theta}{N}, \frac{2\pi\phi}{N}, Nr\right) \\
IS^{2,2}(\theta,\phi,r) &= \lim_{N\to\infty} \frac{2\pi}{N} IS_N^{2,2}\left(\frac{2\pi\theta}{N}, \frac{2\pi\phi}{N}, Nr\right).
\end{aligned}
$$

Theorem III.1. *Rescale the position of the particles by* $\theta \mapsto \frac{2\pi\theta}{N}$*. Then the limiting kernels are*

$$
\begin{aligned}
S^{1,1}(\theta,\phi,r) &= 4r^2 \int_0^1 \frac{\cos[\pi(\theta-\phi)t]}{(2r)^2+t^2} dt \\
DS^{1,1}(\theta,\phi,r) &= ir^2 \int_0^1 \frac{t\cdot\sin[\pi(\theta-\phi)t]}{(2r)^2+t^2} dt \\
IS^{1,1}(\theta,\phi,r) &= -16ir^2 \int_0^1 \frac{\sin[\pi(\theta-\phi)t]}{(2r)^2 t+t^3} dt + 2\pi\cdot\operatorname{sgn}(\phi-\theta) \\
S^{2,2}(\theta,\phi,r) &= \frac{1}{2}\int_0^1 \frac{t^2\cos[\pi(\theta-\phi)t]}{(2r)^2+t^2} dt \\
IS^{2,2}(\theta,\phi,r) &= -\frac{i}{4}\int_0^1 \frac{t^3\sin[\pi(\theta-\phi)t]}{(2r)^2+t^2} dt,
\end{aligned}
\tag{III.6}
$$

and the remaining limiting kernels are given by the relations (III.5).

We now let $r \to \infty$ and $r \to 0^+$ to confirm that we recover the $\beta = 1$ (B.2) and

$\beta = 4$ (C.4) circular ensemble kernels in the limits. First, for $r \to \infty$,

$$\begin{aligned}
\lim_{r\to\infty} S^{1,1}(\theta,\phi,r) &= \int_0^1 \cos[\pi(\theta-\phi)t]\,dt \\
&= \frac{\sin[\pi(\theta-\phi)]}{\pi(\theta-\phi)} \\
\lim_{r\to\infty} DS^{1,1}(\theta,\phi,r) &= \frac{i}{4}\int_0^1 t\cdot\sin[\pi(\theta-\phi)t]\,dt \\
&= \frac{i}{4}\left(\frac{\sin[\pi(\theta-\phi)]}{\pi^2(\theta-\phi)^2} - \frac{\cos[\pi(\theta-\phi)]}{\pi(\theta-\phi)}\right) \\
\lim_{r\to\infty} IS^{1,1}(\theta,\phi,r) &= -4i\int_0^1 \frac{\sin[\pi(\theta-\phi)t]}{t}dt + 2\pi\cdot\operatorname{sgn}(\phi-\theta),
\end{aligned}$$

and it is clear that the remaining kernel entries converge to 0. Finally, for $r \to 0^+$,

$$\begin{aligned}
\lim_{r\to0^+} S^{2,2}(\theta,\phi,r) &= \frac{1}{2}\int_0^1 \cos[\pi(\theta-\phi)t]\,dt \\
&= \frac{\sin[\pi(\theta-\phi)]}{2\pi(\theta-\phi)}, \\
\lim_{r\to0^+} DS^{2,2}(\theta,\phi,r) &= i\int_0^1 \frac{\sin[\pi(\theta-\phi)t]}{t}dt \\
\lim_{r\to0^+} IS^{2,2}(\theta,\phi,r) &= -\frac{i}{4}\int_0^1 t\cdot\sin[\pi(\theta-\phi)t]\,dt \\
&= -\frac{i}{4}\left(\frac{\sin[\pi(\theta-\phi)]}{\pi^2(\theta-\phi)^2} - \frac{\cos[\pi(\theta-\phi)]}{\pi(\theta-\phi)}\right) \\
\lim_{r\to0^+} IS^{1,1}(\theta,\phi,r) &= 2\pi\cdot\operatorname{sgn}(\phi-\theta),
\end{aligned}$$

and it is clear that the remaining kernel entries converge to 0.

Remark III.2. By the rotational symmetry of the kernels, the limiting kernels (III.6) are independent of the location of the scaling in that if we scale anywhere else on the unit circle, we would get the same kernels.

Remark III.3. The definition of the correlation functions of a two species ensemble (I.12) shows that the densities of each particles are expressed in terms of $R_{1,0}$ and

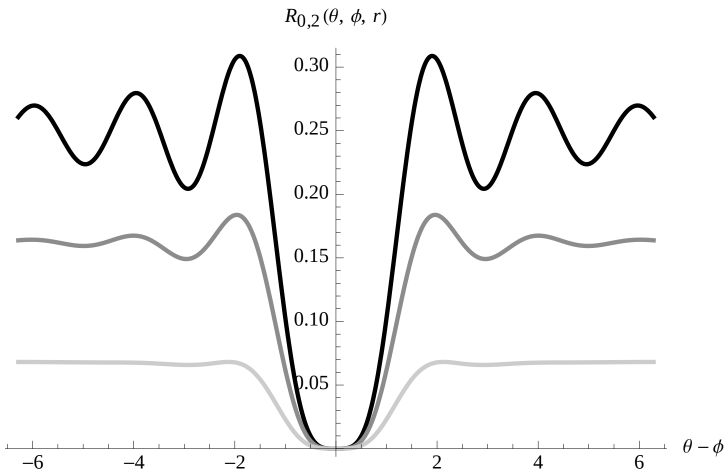

Figure 3.1.: Plots of $R_{0,2}(\theta, \phi, r)$ with kernel (III.6) for $r = \frac{1}{5}, \frac{1}{15}, \frac{1}{5000}$ from lightest to darkest.

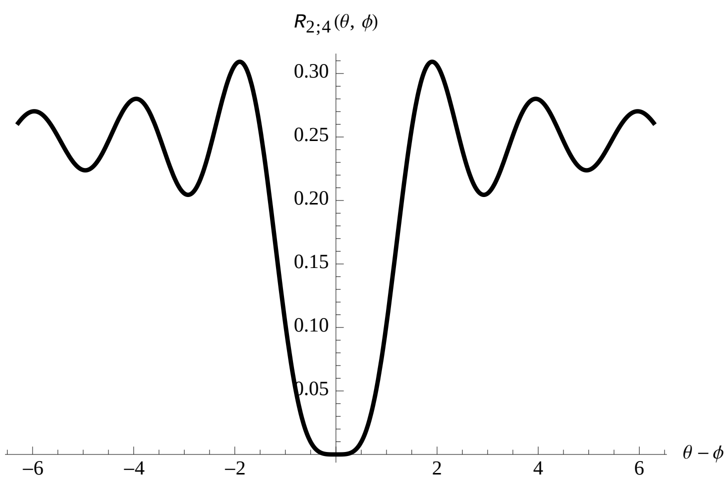

Figure 3.2.: Plot of $R_{2;4}(\theta, \phi)$ with kernel (C.4) (CSE).

$R_{0,1}$, i.e. $S_N^{1,1}(\theta, \theta)$ and $S_N^{2,2}(\theta, \theta)$ (III.5). These quantities are independent of θ, so the density must be invariant under rotations. Thus the densities are uniform.

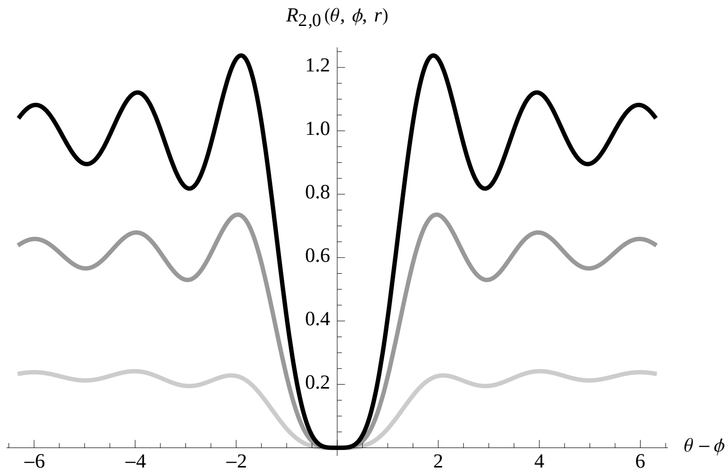

Figure 3.3.: Plots of $R_{2,0}(\theta, \phi, r)$ with kernel (III.6) for $r = \frac{1}{5}, \frac{1}{2}, 20$ from lightest to darkest.

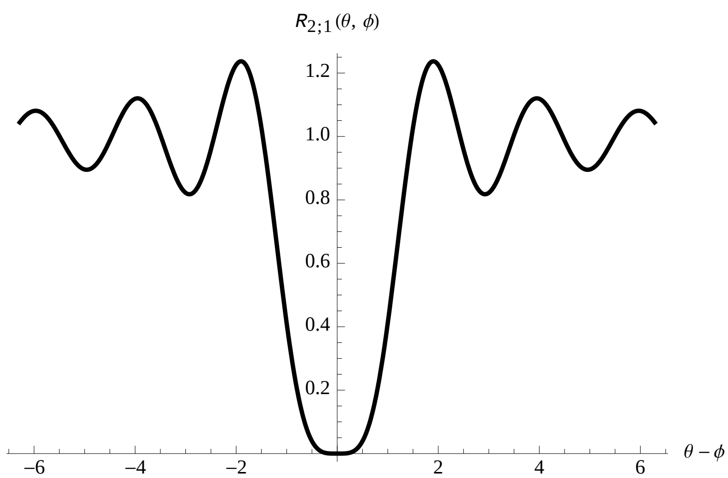

Figure 3.4.: Plot of $R_{2;1}(\theta, \phi)$ with kernel (B.2) (COE).

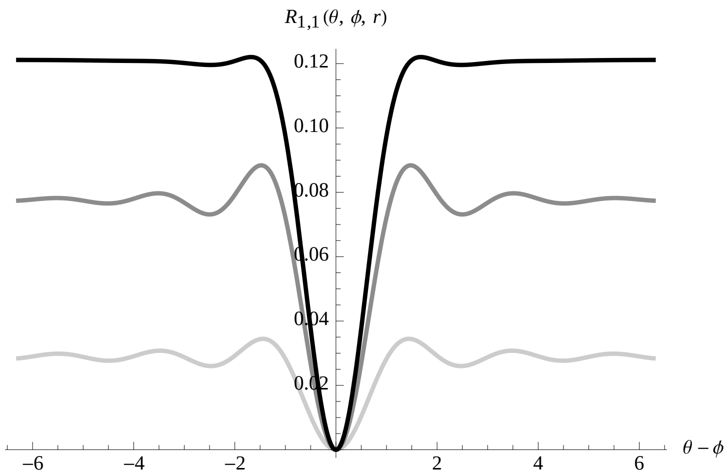

Figure 3.5.: Plots of $R_{1,1}(\theta, \phi, r)$ with kernel (III.6) for $r = \dfrac{1}{50}, \dfrac{1}{15}, 0.274$ from lightest to darkest.

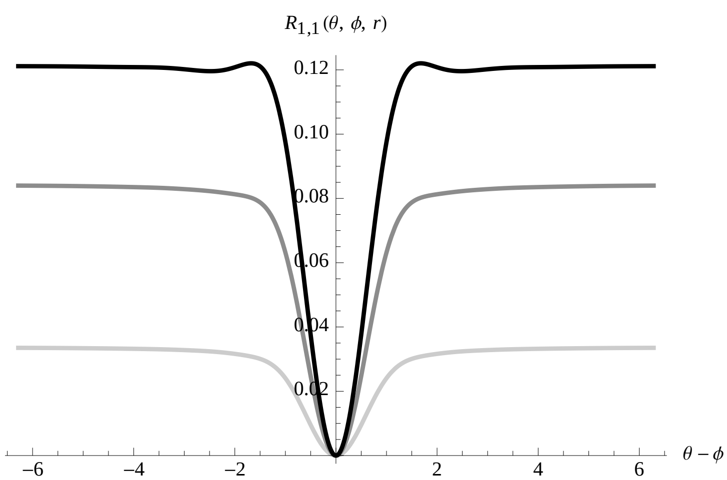

Figure 3.6.: Plots of $R_{1,1}(\theta, \phi, r)$ with kernel (III.6) for $r = 1, \dfrac{1}{2}, 0.274$ from lightest to darkest.

Proofs

Proof of Theorem III.1

We follow closely the proof in [30]. The first step is to rewrite $Z_{L,M}$ in terms of Pfaffians. We eliminate the absolute values by using

$$\left|e^{i\theta} - e^{i\phi}\right| = -i\left(e^{i\theta} - e^{i\phi}\right) e^{-i\left(\frac{\theta+\phi}{2}\right)} \operatorname{sgn}(\theta - \phi)$$

to write

$$\begin{aligned}
e^{-E(\boldsymbol{\theta}_L,\boldsymbol{\phi}_M)} &= \prod_{l_1>l_2}^{L} (-i)\left(e^{i\theta_{l_1}} - e^{i\theta_{l_2}}\right) e^{-\frac{i}{2}\left(\theta_{l_1}+\theta_{l_2}\right)} \operatorname{sgn}(\theta_{l_1} - \theta_{l_2}) \\
&\quad \times \prod_{m_1>m_2}^{M} \left(-i\left(e^{i\phi_{m_1}} - e^{i\phi_{m_2}}\right) e^{-\frac{i}{2}\left(\phi_{m_1}+\phi_{m_2}\right)} \operatorname{sgn}(\phi_{m_1} - \phi_{m_2})\right)^4 \\
&\quad \times \prod_{l=1}^{L}\prod_{m=1}^{M} \left(-i\left(e^{i\theta_l} - e^{i\phi_m}\right) e^{-\frac{i}{2}(\theta_l+\phi_m)} \operatorname{sgn}(\theta_l - \phi_m)\right)^2 \\
&= \prod_{l_1>l_2}^{L} (-i)\left(e^{i\theta_{l_1}} - e^{i\theta_{l_2}}\right) \operatorname{sgn}(\theta_{l_1} - \theta_{l_2}) \prod_{m_1>m_2}^{M} \left(e^{i\phi_{m_1}} - e^{i\phi_{m_2}}\right)^4 \\
&\quad \times \prod_{l=1}^{L}\prod_{m=1}^{M} \left(e^{i\theta_l} - e^{i\phi_m}\right)^2 \prod_{l=1}^{L} e^{-\frac{i}{2}(L+2M-1)\theta_l} \prod_{m=1}^{M} e^{-i(L+2M-2)\phi_m} \\
&= \prod_{l_1>l_2}^{L} \left(e^{i\theta_{l_1}} - e^{i\theta_{l_2}}\right) \operatorname{sgn}(\theta_{l_1} - \theta_{l_2}) \prod_{m_1>m_2}^{M} \left(e^{i\phi_{m_1}} - e^{i\phi_{m_2}}\right)^4 \\
&\quad \times \prod_{l=1}^{L}\prod_{m=1}^{M} \left(e^{i\theta_l} - e^{i\phi_m}\right)^2 \prod_{l=1}^{L} e^{\frac{3\pi i}{4}} \cdot e^{-\frac{i}{2}(N-1)\theta_l} \prod_{m=1}^{M} e^{-i(N-2)\phi_m}.
\end{aligned} \tag{III.7}$$

Using the confluent Vandermonde determinant identity, we can rewrite part of

(III.7) as

$$\left(\prod_{l_1>l_2}^{L}\left(e^{i\theta_{l_1}}-e^{i\theta_{l_2}}\right)\right)\left(\prod_{m_1>m_2}^{M}\left(e^{i\phi_{m_1}}-e^{i\phi_{m_2}}\right)^4\right)\prod_{l=1}^{L}\prod_{m=1}^{M}\left(e^{i\theta_l}-e^{i\phi_m}\right)^2$$
$$=\det\begin{bmatrix} 1 & \cdots & 1 & 1 & 0 & \cdots & 1 & 0 \\ e^{i\theta_1} & \cdots & e^{i\theta_L} & e^{i\phi_1} & 1 & \cdots & e^{i\phi_M} & 1 \\ e^{2i\theta_1} & \cdots & e^{2i\theta_L} & e^{2i\phi_1} & 2e^{i\phi_1} & \cdots & e^{2i\phi_M} & 2e^{i\phi_M} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ e^{i(N-1)\theta_1} & \cdots & e^{i(N-1)\theta_L} & e^{i(N-1)\phi_1} & (N-1)e^{i(N-2)\phi_1} & \cdots & e^{i(N-1)\phi_M} & (N-1)e^{i(N-2)\phi_M} \end{bmatrix}$$

We can take

$$\prod_{l=1}^{L} e^{\frac{3\pi i}{4}}\cdot e^{-\frac{i}{2}(N-1)\theta_l}$$

from (III.7) and push the factor inside the matrix by multiplying the lth columns by $e^{\frac{3\pi i}{4}}\cdot e^{-\frac{i}{2}(N-1)\theta_l}$ for $l=1,\dots,L$. Similarly, we can push

$$\prod_{m=1}^{M} e^{-i(N-2)\phi_m}$$

inside the matrix by multiplying the $L+2m-1$th column by $e^{\frac{i}{2}(-N+1)\phi_m}$ and multiply the $L+2m$th column by $e^{\frac{i}{2}(-N+3)\phi_m}$ for $m=1,\dots,M$. We arrive at

$$e^{-E(\boldsymbol{\theta}_L,\boldsymbol{\phi}_M)}=\det\left[\begin{matrix} e^{\frac{3\pi i}{4}} & \cdots & e^{\frac{3\pi i}{4}} \\ e^{\frac{3\pi i}{4}}\cdot e^{i\theta_1} & \cdots & e^{\frac{3\pi i}{4}}\cdot e^{i\theta_L} \\ e^{\frac{3\pi i}{4}}\cdot e^{2i\theta_1} & \cdots & e^{\frac{3\pi i}{4}}\cdot e^{2i\theta_L} \\ \vdots & \ddots & \vdots \\ e^{\frac{3\pi i}{4}}\cdot e^{i(N-1)\theta_1} & \cdots & e^{\frac{3\pi i}{4}}\cdot e^{i(N-1)\theta_L} \end{matrix}\right.$$
$$\left.\begin{matrix} e^{\frac{i}{2}(-N+1)\phi_1} & 0 & \cdots & e^{\frac{i}{2}(-N+1)\phi_M} & 0 \\ e^{\frac{i}{2}(-N+3)\phi_1} & e^{\frac{i}{2}(-N+3)\phi_1} & \cdots & e^{\frac{i}{2}(-N+3)\phi_M} & e^{\frac{i}{2}(-N+3)\phi_M} \\ e^{\frac{i}{2}(-N+5)\phi_1} & 2e^{\frac{i}{2}(-N+5)\phi_1} & \cdots & e^{\frac{i}{2}(-N+5)\phi_M} & 2e^{\frac{i}{2}(-N+5)\phi_M} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ e^{\frac{i}{2}(N-1)\phi_1} & (N-1)e^{\frac{i}{2}(N-1)\phi_1} & \cdots & e^{\frac{i}{2}(N-1)\phi_M} & (N-1)e^{\frac{i}{2}(N-1)\phi_M} \end{matrix}\right]$$
$$\times\prod_{l_1>l_2}^{L}\operatorname{sgn}(\theta_{l_1}-\theta_{l_2}).$$

Finally, after some elementary column operations on the last $2M$ columns, we can express the above expression as

$$
\begin{aligned}
&e^{-E(\boldsymbol{\theta}_L,\boldsymbol{\phi}_M)}\\
&\quad = \det \left[\begin{matrix}
e^{\frac{3\pi i}{4}} & \cdots & e^{\frac{3\pi i}{4}} \\
e^{\frac{3\pi i}{4}} \cdot e^{i\theta_1} & \cdots & e^{\frac{3\pi i}{4}} \cdot e^{i\theta_L} \\
e^{\frac{3\pi i}{4}} \cdot e^{2i\theta_1} & \cdots & e^{\frac{3\pi i}{4}} \cdot e^{2i\theta_L} \\
\vdots & \ddots & \vdots \\
e^{\frac{3\pi i}{4}} \cdot e^{i(N-1)\theta_1} & \cdots & e^{\frac{3\pi i}{4}} \cdot e^{i(N-1)\theta_L}
\end{matrix}\right.\\
&\qquad \left.\begin{matrix}
e^{\frac{i}{2}(-N+1)\phi_1} & (-N+1)/2 \cdot e^{\frac{i}{2}(-N+1)\phi_1} & \cdots & e^{\frac{i}{2}(-N+1)\phi_M} & (-N+1)/2 \cdot e^{\frac{i}{2}(-N+1)\phi_M} \\
e^{\frac{i}{2}(-N+3)\phi_1} & (-N+3)/2 \cdot e^{\frac{i}{2}(-N+3)\phi_1} & \cdots & e^{\frac{i}{2}(-N+3)\phi_M} & (-N+3)/2 \cdot e^{\frac{i}{2}(-N+3)\phi_M} \\
e^{\frac{i}{2}(-N+5)\phi_1} & (-N+5)/2 \cdot e^{\frac{i}{2}(-N+5)\phi_1} & \cdots & e^{\frac{i}{2}(-N+5)\phi_M} & (-N+5)/2 \cdot e^{\frac{i}{2}(-N+5)\phi_M} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
e^{\frac{i}{2}(N-1)\phi_1} & (N-1)/2 \cdot e^{\frac{i}{2}(N-1)\phi_1} & \cdots & e^{\frac{i}{2}(N-1)\phi_M} & (N-1)/2 \cdot e^{\frac{i}{2}(N-1)\phi_M}
\end{matrix}\right]\\
&\qquad \times \prod_{l_1>l_2}^{L} \operatorname{sgn}(\theta_{l_1}-\theta_{l_2}).
\end{aligned}
$$

Applying the Laplace expansion to the determinant, we obtain

$$
\begin{aligned}
e^{-E(\boldsymbol{\theta}_L,\boldsymbol{\phi}_M)} = \sum_{\mathfrak{t}:\underline{L}\nearrow\underline{N}} &\operatorname{sgn}\mathfrak{t} \cdot \prod_{l_1>l_2}^{L} \operatorname{sgn}(\theta_{l_1}-\theta_{l_2}) \det\left[e^{\frac{3\pi i}{4}} \cdot e^{\frac{i}{2}(-N-1+2\mathfrak{t}(m))\theta_n}\right]_{m,n=1}^{L}\\
&\times \det\left[e^{\frac{i}{2}(-N-1+2\mathfrak{t}'(m))\phi_n} \quad \frac{1}{2}\left(-N-1+2\mathfrak{t}'(m)\right) e^{\frac{i}{2}(-N-1+2\mathfrak{t}'(m))\phi_n}\right]_{m,n=1}^{2M;M}
\end{aligned}
$$

Let

$$
A_{\mathfrak{t}} = \frac{1}{L!}\int_{[-\pi,\pi)^L} \prod_{l_1>l_2}^{L} \operatorname{sgn}(\theta_{l_1}-\theta_{l_2}) \det\left[e^{\frac{3\pi i}{4}} \cdot e^{\frac{i}{2}(-N-1+2\mathfrak{t}(m))\theta_n}\right]_{m,n=1}^{L} d\boldsymbol{\theta}_L
$$

and

$$
\begin{aligned}
B_{\mathfrak{t}'} = \frac{1}{M!}\int_{[-\pi,\pi)^M} \det\Big[&e^{\frac{i}{2}(-N-1+2\mathfrak{t}'(m))\phi_n}\\
&\frac{1}{2}\left(-N-1+2\mathfrak{t}'(m)\right) e^{\frac{i}{2}(-N-1+2\mathfrak{t}'(m))\phi_n}\Big]_{m,n=1}^{2M;M} d\boldsymbol{\phi}_M
\end{aligned}
$$

Thus,

$$
\begin{aligned}
X^L Z_{L,M} &= \frac{X^L}{L!M!} \int_{[-\pi,\pi)^L} \int_{[-\pi,\pi)^M} e^{-E(\boldsymbol{\theta},\boldsymbol{\phi})}\, d\boldsymbol{\theta}_L d\boldsymbol{\phi}_M \\
&= X^L \sum_{\mathfrak{t}:\underline{L}\nearrow\underline{N}} \operatorname{sgn}\mathfrak{t}\cdot A_{\mathfrak{t}} B_{\mathfrak{t}'}.
\end{aligned}
$$

We now express $A_{\mathfrak{t}}$ as a Pfaffian. Note that the possible nonzero values of L must be even because it must match the parity of N. Therefore, defining

$$
\mathbf{T}_L(\boldsymbol{\theta}_L) = \Big[\operatorname{sgn}(\theta_k - \theta_j)\Big]_{j,k=1}^{L},
$$

we may express

$$
\prod_{l_1>l_2}^{L} \operatorname{sgn}(\theta_{l_1} - \theta_{l_2}) = \operatorname{Pf}\mathbf{T}_L(\boldsymbol{\theta}_L).
$$

Expanding the determinant gives

$$
\det\left[e^{\frac{3\pi i}{4}} \cdot e^{\frac{i}{2}(-N-1+2\mathfrak{t}(m))\theta_n}\right]_{m,n=1}^{L} = \sum_{\sigma\in S_L} \operatorname{sgn}\sigma \prod_{l=1}^{L} e^{\frac{3\pi i}{4}} \cdot e^{\frac{i}{2}(-N-1+2\mathfrak{t}(l))\theta_{\sigma(l)}}
$$

and so

$$
A_{\mathfrak{t}} = \frac{1}{L!} \sum_{\sigma\in S_L} \operatorname{sgn}\sigma \int_{[-\pi,\pi)^L} \prod_{l=1}^{L} \left(e^{\frac{3\pi i}{4}} \cdot e^{\frac{i}{2}(-N-1+2\mathfrak{t}(l))\theta_{\sigma(l)}}\right) \cdot \operatorname{Pf}\mathbf{T}_L(\boldsymbol{\theta}_L)\, d\boldsymbol{\theta}_L.
$$

Denote $\sigma(\boldsymbol{\theta}_L)$ by the action of σ permuting on the indices of $\boldsymbol{\theta}_L$. Then

$$
\operatorname{Pf}\mathbf{T}_L(\sigma(\boldsymbol{\theta}_L)) = \operatorname{sgn}\sigma \cdot \operatorname{Pf}\mathbf{T}_L(\boldsymbol{\theta}_L).
$$

This allows us to eliminate the sum by reindexing $\boldsymbol{\theta} \mapsto \sigma^{-1}(\boldsymbol{\theta}_L)$:

$$\begin{aligned}
A_{\mathfrak{t}} &= \frac{1}{L!} \sum_{\sigma \in S_L} \operatorname{sgn} \sigma \int_{[-\pi,\pi)^L} \prod_{l=1}^{L} \left(e^{\frac{3\pi i}{4}} \cdot e^{\frac{i}{2}(-N-1+2\mathfrak{t}(l))\theta_{\sigma(l)}} \right) \cdot \operatorname{Pf} \mathbf{T}_L(\sigma(\boldsymbol{\theta}_L))\, d\boldsymbol{\theta}_L \\
&= \frac{1}{L!} \sum_{\sigma \in S_L} \int_{[-\pi,\pi)^L} \prod_{l=1}^{L} \left(e^{\frac{3\pi i}{4}} \cdot e^{\frac{i}{2}(-N-1+2\mathfrak{t}(l))\theta_l} \right) \cdot \operatorname{Pf} \mathbf{T}_L(\boldsymbol{\theta}_L)\, d\boldsymbol{\theta}_L \\
&= \int_{[-\pi,\pi)^L} \prod_{l=1}^{L} \left(e^{\frac{3\pi i}{4}} \cdot e^{\frac{i}{2}(-N-1+2\mathfrak{t}(l))\theta_l} \right) \cdot \operatorname{Pf} \mathbf{T}_L(\boldsymbol{\theta}_L)\, d\boldsymbol{\theta}_L.
\end{aligned}$$

Next, for $2K = L$, we expand the Pfaffian

$$\operatorname{Pf} \mathbf{T}_L(\sigma(\boldsymbol{\theta}_L)) = \frac{1}{2^K K!} \sum_{\tau \in S_{2K}} \operatorname{sgn} \tau \cdot \prod_{k=1}^{K} \operatorname{sgn}(\theta_{\tau(2k)} - \theta_{\tau(2k-1)}).$$

So

$$A_{\mathfrak{t}} = \frac{1}{2^K K!} \sum_{\tau \in S_{2K}} \operatorname{sgn} \tau \int_{[-\pi,\pi)^L} \prod_{l=1}^{L} e^{\frac{3\pi i}{4}} \cdot e^{\frac{i}{2}(-N-1+2\mathfrak{t}(l))\theta_l} \prod_{k=1}^{K} \operatorname{sgn}(\theta_{\tau(2k)} - \theta_{\tau(2k-1)})\, d\boldsymbol{\theta}_L.$$

Reindexing $l \mapsto \tau(l)$, we get

$$\begin{aligned}
A_{\mathfrak{t}} = \frac{1}{2^K K!} \sum_{\tau \in S_{2K}} \operatorname{sgn} \tau \int_{[-\pi,\pi)^L} \prod_{k=1}^{K} (-i) e^{\frac{i}{2}(-N-1+2(\mathfrak{t}\circ\tau)(2k))\theta_{\tau(2k)}} e^{\frac{i}{2}(-N-1+2(\mathfrak{t}\circ\tau)(2k-1))\theta_{\tau(2k-1)}} \\
\times \ \operatorname{sgn}(\theta_{\tau(2k)} - \theta_{\tau(2k-1)})\, d\boldsymbol{\theta}_L.
\end{aligned}$$

Now, letting

$$\mathbf{A}_{\mathfrak{t}} = \left[-i \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{\frac{i}{2}(-N-1+2\mathfrak{t}(m))\phi} e^{\frac{i}{2}(-N-1+2\mathfrak{t}(n))\theta} \operatorname{sgn}(\phi - \theta)\, d\phi d\theta \right]_{n,m=1}^{L},$$

we apply Fubini's theorem and relabel the variables of integration to arrive at

$$
\begin{aligned}
A_{\mathfrak{t}} = \frac{1}{2^K K!} \sum_{\tau \in S_{2K}} \operatorname{sgn} \tau \prod_{k=1}^{K} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (-i) e^{\frac{i}{2}(-N-1+2(\mathfrak{t}\circ\tau)(2k))\theta_{\tau(2k)}} e^{\frac{i}{2}(-N-1+2(\mathfrak{t}\circ\tau)(2k-1))\theta_{\tau(2k-1)}} \\
\times \ \operatorname{sgn}(\theta_{\tau(2k)} - \theta_{\tau(2k-1)}) \, d\theta_{\tau(2k)} d\theta_{\tau(2k-1)} \\
= \frac{1}{2^K K!} \sum_{\tau \in S_{2K}} \operatorname{sgn} \tau \\
\times \prod_{k=1}^{K} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (-i) e^{\frac{i}{2}(-N-1+2(\mathfrak{t}\circ\tau)(2k))\phi} e^{\frac{i}{2}(-N-1+2(\mathfrak{t}\circ\tau)(2k-1))\theta} \operatorname{sgn}(\phi - \theta) \, d\phi d\theta \\
= \operatorname{Pf} \mathbf{A}_{\mathfrak{t}}.
\end{aligned}
$$

Next, we express $B_{\mathfrak{t}'}$ as a Pfaffian. We first use the Leibniz formula to express the determinant as

$$
\begin{aligned}
\det \left[e^{\frac{i}{2}(-N-1+2\mathfrak{t}'(m))\beta_n} \quad \frac{1}{2} (-N-1+2\mathfrak{t}'(m)) \, e^{\frac{i}{2}(-N-1+2\mathfrak{t}'(m))\beta_n} \right]_{m,n=1}^{2M;M} \\
= \sum_{\sigma \in S_{2M}} \operatorname{sgn} \sigma \\
\times \prod_{m=1}^{M} \frac{1}{2}(-N-1+2(\mathfrak{t}' \circ \sigma)(2m)) e^{\frac{i}{2}(-N-1+2(\mathfrak{t}'\circ\sigma)(2m-1))\beta_m} e^{\frac{i}{2}(-N-1+2(\mathfrak{t}'\circ\sigma)(2m))\beta_m} \\
= \sum_{\sigma \in S_{2M}} \operatorname{sgn} \sigma \prod_{m=1}^{M} \frac{1}{2}(-N-1+2(\mathfrak{t}' \circ \sigma)(2m)) e^{i(-N-1+(\mathfrak{t}'\circ\sigma)(2m-1)+(\mathfrak{t}'\circ\sigma)(2m)\beta_m}.
\end{aligned}
$$

Letting

$$
\mathbf{B}_{\mathfrak{t}'} = \left[(\mathfrak{t}'(m) - \mathfrak{t}'(n)) \int_{-\pi}^{\pi} e^{i(-N-1+\mathfrak{t}'(n)+\mathfrak{t}'(m))\theta} \, d\theta \right]_{m,n=1}^{N},
$$

we use Fubini's Theorem and relabel the variables of integration to get

$$\begin{aligned}
B_{\mathfrak{t}'} = &\frac{1}{M!} \sum_{\sigma \in S_{2M}} \operatorname{sgn} \sigma \\
&\times \int_{[-\pi,\pi)^M} \prod_{m=1}^{M} \frac{1}{2}(-N-1+2(\mathfrak{t}' \circ \sigma)(2m)) e^{i(-N-1+(\mathfrak{t}'\circ\sigma)(2m-1)+(\mathfrak{t}'\circ\sigma)(2m))\beta_m} \, d\boldsymbol{\beta}_M \\
= &\frac{1}{2^M M!} \sum_{\sigma \in S_{2M}} \operatorname{sgn} \sigma \prod_{m=1}^{M} (-N-1+2(\mathfrak{t}' \circ \sigma)(2m)) \\
&\times \int_{-\pi}^{\pi} e^{i(-N-1+(\mathfrak{t}'\circ\sigma)(2m-1)+(\mathfrak{t}'\circ\sigma)(2m))\theta} \, d\theta \\
= &\operatorname{Pf} \mathbf{B}_{\mathfrak{t}'}.
\end{aligned}$$

We finally arrive at

$$X^L Z_{L,M} = X^L \sum_{\mathfrak{t}:\underline{L}\nearrow\underline{N}} \operatorname{sgn} \mathfrak{t} \cdot \operatorname{Pf} \mathbf{A}_{\mathfrak{t}} \cdot \operatorname{Pf} \mathbf{B}_{\mathfrak{t}'},$$

and so

$$Z_N(X) = \sum_{(L,M)} X^L \sum_{\mathfrak{t}:\underline{L}\nearrow\underline{N}} \operatorname{sgn} \mathfrak{t} \cdot \operatorname{Pf} \mathbf{A}_{\mathfrak{t}} \cdot \operatorname{Pf} \mathbf{B}_{\mathfrak{t}'}.$$

Let

$$\mathbf{X}_L = X \mathbf{I}_L$$

where $\mathbf{I}_L$ is the $L \times L$ identity matrix. Then

$$\begin{aligned}
X^L \operatorname{Pf} \mathbf{A}_{\mathfrak{t}} &= \det \mathbf{X}_L \cdot \operatorname{Pf} \mathbf{A}_{\mathfrak{t}} \\
&= \operatorname{Pf}(\mathbf{X}_L \mathbf{A}_{\mathfrak{t}} \mathbf{X}_L^T) \\
&= \operatorname{Pf}(X^2 \mathbf{A}_{\mathfrak{t}}).
\end{aligned}$$

Therefore,

$$\begin{aligned} Z_N(X) &= \sum_{(L,M)} \sum_{\mathfrak{t}:\underline{L}\nearrow\underline{N}} \operatorname{sgn}\mathfrak{t}\cdot \operatorname{Pf}(X^2\mathbf{A}_{\mathfrak{t}})\cdot \operatorname{Pf}\mathbf{B}_{\mathfrak{t}'} \\ &= \operatorname{Pf}(X^2\mathbf{A}_N + \mathbf{B}_N). \end{aligned}$$

Proof of Theorem III.2

For N odd, we have

$$\begin{aligned} e^{-E(\boldsymbol{\theta}_L,\boldsymbol{\phi}_M)} = e^{-\frac{\pi i}{4}} \prod_{l_1>l_2}^{L} \left(e^{i\theta_{l_1}} - e^{i\theta_{l_2}}\right) \operatorname{sgn}(\theta_{l_1}-\theta_{l_2}) \prod_{m_1>m_2}^{M} \left(e^{i\phi_{m_1}} - e^{i\phi_{m_2}}\right)^4 \\ \times \prod_{l=1}^{L}\prod_{m=1}^{M}\left(e^{i\theta_l} - e^{i\phi_m}\right)^2 \prod_{l=1}^{L} e^{\frac{\pi i}{4}} \cdot e^{-\frac{i}{2}(N-1)\theta_l} \prod_{m=1}^{M} e^{-i(N-2)\phi_m}. \end{aligned}$$

We use the same notation as in the even case. Then

$$Z_N(X) = e^{-\frac{\pi i}{4}} \sum_{(L,M)} X^L \sum_{\mathfrak{t}:\underline{L}\nearrow\underline{N}} \operatorname{sgn}\mathfrak{t}\cdot A_{\mathfrak{t}} B_{\mathfrak{t}'}.$$

As above, we express $A_{\mathfrak{t}}$ as a Pfaffian. Note that L must also be odd, so

$$\prod_{l_1>l_2}^{L} \operatorname{sgn}(\theta_{l_2}-\theta_{l_2}) = \operatorname{Pf}\begin{bmatrix} & & & 1 \\ & \mathbf{T}_L(\boldsymbol{\theta}_L) & & \vdots \\ & & & 1 \\ -1 & \cdots & -1 & 0 \end{bmatrix}.$$

Let

$$\widetilde{\mathbf{A}}_{\mathfrak{t}} = \begin{bmatrix} & & & e^{\frac{\pi i}{4}}\int_{-\pi}^{\pi} e^{\frac{i}{2}(-N-1+2\mathfrak{t}(1))\theta}d\theta \\ & \mathbf{A}_{\mathfrak{t}} & & \vdots \\ & & & e^{\frac{\pi i}{4}}\int_{-\pi}^{\pi} e^{\frac{i}{2}(-N-1+2\mathfrak{t}(L))\theta}d\theta \\ -e^{\frac{\pi i}{4}}\int_{-\pi}^{\pi} e^{\frac{i}{2}(-N-1+2\mathfrak{t}(1))\theta}d\theta & \cdots & -e^{\frac{\pi i}{4}}\int_{-\pi}^{\pi} e^{\frac{i}{2}(-N-1+2\mathfrak{t}(L))\theta}d\theta & 0 \end{bmatrix}$$

Then

$$A_{\mathfrak{t}} = \operatorname{Pf}\widetilde{\mathbf{A}}_{\mathfrak{t}}.$$

Now

$$B_{\mathfrak{t}'} = \operatorname{Pf} \mathbf{B}_{\mathfrak{t}'}$$

so that

$$Z_N(X) = e^{-\frac{\pi i}{4}} \sum_{(L,M)} X^L \sum_{\mathfrak{t}:\underline{L} \nearrow \underline{N}} \operatorname{sgn} \mathfrak{t} \cdot \operatorname{Pf} \widetilde{\mathbf{A}}_{\mathfrak{t}} \cdot \operatorname{Pf} \mathbf{B}_{\mathfrak{t}'}.$$

Let $\widetilde{\mathbf{X}}_L$ be the $(L+1) \times (L+1)$ matrix

$$\widetilde{\mathbf{X}}_L = \begin{bmatrix} X & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & X & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Then

$$\begin{aligned} X^L \operatorname{Pf} \widetilde{\mathbf{A}}_{\mathfrak{t}} &= \det \widetilde{\mathbf{X}}_L \cdot \operatorname{Pf} \widetilde{\mathbf{A}}_{\mathfrak{t}} \\ &= \operatorname{Pf}(\widetilde{\mathbf{X}}_L \widetilde{\mathbf{A}}_{\mathfrak{t}} \widetilde{\mathbf{X}}_L^T). \end{aligned}$$

Let

$$\widehat{\mathbf{A}}_N = \begin{bmatrix} & & & e^{\frac{\pi i}{4}} \int_{-\pi}^{\pi} e^{\frac{i}{2}(-N+1)\theta} d\theta \\ & \mathbf{A}_N & & \vdots \\ & & & e^{\frac{\pi i}{4}} \int_{-\pi}^{\pi} e^{\frac{i}{2}(N-1)\theta} d\theta \\ -e^{\frac{\pi i}{4}} \int_{-\pi}^{\pi} e^{\frac{i}{2}(-N+1)\theta} d\theta & \cdots & -e^{\frac{\pi i}{4}} \int_{-\pi}^{\pi} e^{\frac{i}{2}(N-1)\theta} d\theta & 0 \end{bmatrix}$$

and

$$\widehat{\mathbf{B}}_N = \begin{bmatrix} & & & 0 \\ & \mathbf{B}_N & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Then

$$
\begin{aligned}
Z_N(X) &= e^{-\frac{\pi i}{4}} \sum_{(L,M)} \sum_{\mathfrak{t}:\underline{L}\nearrow\underline{N}} \operatorname{sgn}\mathfrak{t}\cdot \operatorname{Pf}(\widetilde{\mathbf{X}}_L \widetilde{\mathbf{A}}_{\mathfrak{t}} \widetilde{\mathbf{X}}_L^T)\cdot \operatorname{Pf}\mathbf{B}_{\mathfrak{t}'} \\
&= e^{-\frac{\pi i}{4}} \sum_{(L,M)} \sum_{\substack{\mathfrak{u}:\underline{L+1}\nearrow\underline{N+1}\\ \mathfrak{u}(L+1)=N+1}} \operatorname{sgn}\mathfrak{u}\cdot \operatorname{Pf}(\widetilde{\mathbf{X}}_N \widehat{\mathbf{A}}_N \widetilde{\mathbf{X}}_N^T)_{\mathfrak{u}}\cdot \operatorname{Pf}\widehat{\mathbf{B}}_{\mathfrak{u}'}.
\end{aligned}
$$

Note that with the extra condition on $\mathfrak{u}$ in the second sum, $\operatorname{sgn}\mathfrak{u} = \operatorname{sgn}\mathfrak{t}$. Furthermore, $\operatorname{Pf}\widehat{\mathbf{B}}_{\mathfrak{u}'} = 0$ if $\mathfrak{u}'(N-L) = N+1$. Thus, the extra condition is redundant:

$$
\begin{aligned}
Z_N(X) &= e^{-\frac{\pi i}{4}} \sum_{(L,M)} \sum_{\mathfrak{u}:\underline{L+1}\nearrow\underline{N+1}} \operatorname{sgn}\mathfrak{u}\cdot \operatorname{Pf}(\widetilde{\mathbf{X}}_N \widehat{\mathbf{A}}_N \widetilde{\mathbf{X}}_N^T)_{\mathfrak{u}}\cdot \operatorname{Pf}(\widehat{\mathbf{B}}_N)_{\mathfrak{u}'} \\
&= e^{-\frac{\pi i}{4}} \operatorname{Pf}\left(\widetilde{\mathbf{X}}_N \widehat{\mathbf{A}}_N \widetilde{\mathbf{X}}_N^T + \operatorname{Pf}\widehat{\mathbf{B}}_N\right) \\
&= e^{-\frac{\pi i}{4}} \operatorname{Pf}\mathbf{W}_N(X).
\end{aligned}
$$

Proof of Proposition III.1

First, we calculate the characteristic function $\varphi_{L_N(X)}(t)$ and the cumulant generating function $K_N(X,t)$ to get the first cumulants:

$$
\begin{aligned}
\varphi_{L_N(X)}(t) &= \mathbb{E}\left[e^{itL_N(X)}\right] \\
&= \frac{Z_N(Xe^{it})}{Z_N(X)} \\
&= \prod_{n=1}^{\frac{N}{2}} \frac{(2Xe^{it})^2 + (2n-1)^2}{(2X)^2 + (2n-1)^2},
\end{aligned}
$$

and

$$
\begin{aligned}
K_N(X,t) &= \log\varphi_{L_N(X)}(t) \\
&= \sum_{n=1}^{\frac{N}{2}} \log\left[(2Xe^{it})^2 + (2n-1)^2\right] - \sum_{n=1}^{\frac{N}{2}} \log\left[(2X)^2 + (2n-1)^2\right].
\end{aligned}
$$

In particular,

$$\begin{aligned}\mathbb{E}\left[L_N(X)\right] &= \frac{1}{i}K_N'(X,0)\\ &= \sum_{n=1}^{\frac{N}{2}} \frac{8X^2}{4X^2+(2n-1)^2}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Var}\left[L_N(X)\right] &= -K_N''(X,0)\\ &= \sum_{n=1}^{\frac{N}{2}} \left(\frac{4X(2n-1)}{4X^2+(2n-1)^2}\right)^2.\end{aligned}$$

We let $X = Nr$ and calculate

$$\begin{aligned}\frac{1}{N}\mathbb{E}\left[L_N(Nr)\right] &= \frac{2}{N}\sum_{n=1}^{\frac{N}{2}} \frac{(2Nr)^2}{(2Nr)^2+(2n-1)^2}\\ &= \frac{2}{N}\sum_{n=1}^{\frac{N}{2}} \frac{(2r)^2}{(2r)^2+\left(\frac{2n-1}{N}\right)^2}\\ &\to \int_0^1 \frac{(2r)^2}{(2r)^2+t^2}\,dt\\ &= 2r\arctan\left(\frac{1}{2r}\right)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{N}\mathrm{Var}\left[L_N(Nr)\right] &= \frac{4}{N}\sum_{n=1}^{\frac{N}{2}} \frac{(2Nr)^2(2n-1)^2}{((2Nr)^2+(2n-1)^2)^2}\\ &= 2\cdot\frac{2}{N}\sum_{n=1}^{\frac{N}{2}} \frac{(2r)^2\left(\frac{2n-1}{N}\right)^2}{\left((2r)^2+\left(\frac{2n-1}{4r}\right)^2\right)^2}\\ &\to 2\int_0^1 \frac{(2rt)^2}{((2r)^2+t^2)^2}\,dt\\ &= 2r\arctan\left(\frac{1}{2r}\right)-\frac{4r^2}{1+4r^2}.\end{aligned}$$

The same calculations for N odd gives the same limiting expectation and variance.

Proof of Theorem III.2

We take the logarithm of the characteristic function

$$\mathbb{E}\left[\exp\left(it\frac{L_N(Nr)-\mu_N}{\sigma_N}\right)\right],$$

but first we must define a branch of the logarithm. Because $\dfrac{\mu_N}{\sigma_N} = O(N^{-\frac{1}{2}})$, for fixed $t \in \mathbb{R}$, there exists N large enough such that

$$-\pi < t\frac{\mu_N}{\sigma_N} < \pi.$$

So take the branch to be the negative real line. Then for this N,

$$\begin{aligned}
&\log\left[\mathbb{E}\left[\exp\left(it\frac{L_N(Nr)-\mu_N}{\sigma_N}\right)\right]\right] \\
&= \log\left[\mathbb{E}\left[\exp\left(it\frac{L_N(Nr)}{\sigma_N}\right)\right]\right] + \log\left[\mathbb{E}\left[\exp\left(-it\frac{\mu_N}{\sigma_N}\right)\right]\right] \\
&= \log\left[\prod_{n=1}^{\frac{N}{2}} \frac{\left(2Nr\exp\left(\frac{it}{\sigma_N}\right)\right)^2 + (2n-1)^2}{(2Nr)^2+(2n-1)^2}\right] - it\frac{\mu_N}{\sigma_N} \\
&= \sum_{n=1}^{\frac{N}{2}} \log\left[\frac{\left(2Nr\exp\left(\frac{it}{\sigma_N}\right)\right)^2 + (2n-1)^2}{(2Nr)^2+(2n-1)^2}\right] - it\frac{\mu_N}{\sigma_N} \\
&= \sum_{n=1}^{\frac{N}{2}} \log\left[\frac{\exp\left(\frac{2it}{\sigma_N}\right) + \left(\frac{2n-1}{2Nr}\right)^2}{1+\left(\frac{2n-1}{2Nr}\right)^2}\right] - it\frac{\mu_N}{\sigma_N}.
\end{aligned} \tag{III.8}$$

We claim that the above expression (III.8) may be replaced by

$$\sum_{n=1}^{\frac{N}{2}} \left[\left(\frac{\frac{2it}{\sigma_N} + \frac{1}{2!}\left(\frac{2it}{\sigma_N}\right)^2}{1+\left(\frac{2n-1}{2Nr}\right)^2}\right) - \frac{1}{2}\left(\frac{\frac{2it}{\sigma_N} + \frac{1}{2!}\left(\frac{2it}{\sigma_N}\right)^2}{1+\left(\frac{2n-1}{2Nr}\right)^2}\right)^2\right] - it\frac{\mu_N}{\sigma_N}, \tag{III.9}$$

i.e. the (III.8) and (III.9) converge to the same limit as $N \to \infty$. To see this, first take the Taylor expansion in (III.8) for the logarithm, which is valid since for large enough N, the value inside the logarithm is less than 2. So (III.8) is equal to

$$\begin{aligned}
&\sum_{n=1}^{\frac{N}{2}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left(\frac{\exp\left(\frac{2it}{\sigma_N}\right) + \left(\frac{2n-1}{2Nr}\right)^2}{1+\left(\frac{2n-1}{2Nr}\right)^2} - 1 \right)^k - it\frac{\mu_N}{\sigma_N} \\
&\quad = \sum_{n=1}^{\frac{N}{2}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left(\frac{\exp\left(\frac{2it}{\sigma_N}\right) - 1}{1+\left(\frac{2n-1}{2Nr}\right)^2} \right)^k - it\frac{\mu_N}{\sigma_N}. \qquad \text{(III.10)}
\end{aligned}$$

So the absolute value of the difference between (III.10) and (III.9) is

$$\begin{aligned}
&\left| \sum_{n=1}^{\frac{N}{2}} \left[\left(\frac{\frac{2it}{\sigma_N} + \frac{1}{2!}\left(\frac{2it}{\sigma_N}\right)^2}{1+\left(\frac{2n-1}{2Nr}\right)^2} \right) - \left(\frac{\exp\left(\frac{2it}{\sigma_N}\right) - 1}{1+\left(\frac{2n-1}{2Nr}\right)^2} \right) \right] \right. \\
&\quad - \sum_{n=1}^{\frac{N}{2}} \left[\frac{1}{2} \left(\frac{\frac{2it}{\sigma_N} + \frac{1}{2!}\left(\frac{2it}{\sigma_N}\right)^2}{1+\left(\frac{2n-1}{2Nr}\right)^2} \right)^2 - \frac{1}{2} \left(\frac{\exp\left(\frac{2it}{\sigma_N}\right) - 1}{1+\left(\frac{2n-1}{2Nr}\right)^2} \right)^2 \right] \\
&\quad \left. + \sum_{n=1}^{\frac{N}{2}} \sum_{k=3}^{\infty} \frac{(-1)^{k+1}}{k} \left(\frac{\exp\left(\frac{2it}{\sigma_N}\right) - 1}{1+\left(\frac{2n-1}{2Nr}\right)^2} \right)^k \right|. \qquad \text{(III.11)}
\end{aligned}$$

We show that the absolute values of the three terms converges to 0 as $N \to \infty$. Now the third term of (III.11) is

$$\begin{aligned}
&\left| \sum_{n=1}^{\frac{N}{2}} \sum_{k=3}^{\infty} \frac{(-1)^{k+1}}{k} \left(\frac{\exp\left(\frac{2it}{\sigma_N}\right) - 1}{1+\left(\frac{2n-1}{2Nr}\right)^2} \right)^k \right| \\
&\quad \leq \sum_{n=1}^{\frac{N}{2}} \left(\left| \frac{\exp\left(\frac{2it}{\sigma_N}\right) - 1}{1+\left(\frac{2n-1}{2Nr}\right)^2} \right|^3 \left| \sum_{k=1}^{\infty} \frac{(-1)^{k+4}}{k+3} \left(\frac{\exp\left(\frac{2it}{\sigma_N}\right) - 1}{1+\left(\frac{2n-1}{2Nr}\right)^2} \right)^k \right| \right).
\end{aligned}$$

Using the Taylor expansion for the exponential function, there exists $A > 0$ such that

$$\left|\frac{\exp\left(\frac{2it}{\sigma_N}\right)-1}{1+\left(\frac{2n-1}{2Nr}\right)^2}\right|^3 \leq \frac{A\left|t\right|^3}{\sigma_N^3\left(1+\left(\frac{2n-1}{2Nr}\right)^2\right)^3} \leq \frac{A\left|t\right|^3}{\sigma_N^3} = \frac{A\left|t\right|^3}{\sigma^3 N^3}. \tag{III.12}$$

Next,

$$\begin{aligned}\left|\sum_{k=1}^{\infty}\frac{(-1)^{k+4}}{k+3}\left(\frac{\exp\left(\frac{2it}{\sigma_N}\right)-1}{1+\left(\frac{2n-1}{2Nr}\right)^2}\right)^k\right| &\leq \left|\log\left[\frac{\exp\left(\frac{2it}{\sigma_N}\right)+\left(\frac{2n-1}{2Nr}\right)^2}{1+\left(\frac{2n-1}{2Nr}\right)^2}\right]\right| \\ &\leq \left|\log\left[\exp\left(\frac{2it}{\sigma_N}\right)+\left(\frac{2N-1}{2Nr}\right)^2\right]\right|.\end{aligned} \tag{III.13}$$

Thus (III.12) and (III.13) show that

$$\begin{aligned}\sum_{n=1}^{\frac{N}{2}}&\left(\left|\frac{\exp\left(\frac{2it}{\sigma_N}\right)-1}{1+\left(\frac{2n-1}{2Nr}\right)^2}\right|^3\left|\sum_{k=1}^{\infty}\frac{(-1)^{k+4}}{k+3}\left(\frac{\exp\left(\frac{2it}{\sigma_N}\right)-1}{1+\left(\frac{2n-1}{2Nr}\right)^2}\right)^k\right|\right) \\ &\leq \sum_{n=1}^{\frac{N}{2}}\frac{A\left|t\right|^3}{\sigma^3N^3}\cdot\left|\log\left[\exp\left(\frac{2it}{\sigma_N}\right)+\left(\frac{2N-1}{2Nr}\right)^2\right]\right| \\ &= \frac{A\left|t\right|^3}{2\sigma^3N^2}\cdot\left|\log\left[\exp\left(\frac{2it}{\sigma_N}\right)+\left(\frac{2N-1}{2Nr}\right)^2\right]\right| \\ &\to 0\end{aligned}$$

as $N \to \infty$. Using the same argument from (III.12), we have the estimate on the first term of (III.11)

$$\begin{aligned}\left|\sum_{n=1}^{\frac{N}{2}}\left[\left(\frac{\frac{2it}{\sigma_N}+\frac{1}{2!}\left(\frac{2it}{\sigma_N}\right)^2}{1+\left(\frac{2n-1}{2Nr}\right)^2}\right)-\left(\frac{\exp\left(\frac{2it}{\sigma_N}\right)-1}{1+\left(\frac{2n-1}{2Nr}\right)^2}\right)\right]\right| &\leq \sum_{n=1}^{\frac{N}{2}}\frac{A\left|t\right|^3}{\sigma^3N^3} \\ &= \frac{A\left|t\right|^3}{2\sigma^3N^2} \\ &\to 0\end{aligned}$$

as $N \to \infty$. Finally, the summands of the second term in (III.11) can be combined as

$$\frac{1}{2}\left[\frac{\left(\frac{2it}{\sigma_N}+\frac{1}{2!}\left(\frac{2it}{\sigma_N}\right)^2-\exp\left(\frac{2it}{\sigma_N}\right)+1\right)^2+2\left(\frac{2it}{\sigma_N}+\frac{1}{2!}\left(\frac{2it}{\sigma_N}\right)^2\right)\left(\exp\left(\frac{2it}{\sigma_N}\right)-1\right)}{\left(1+\left(\frac{2n-1}{2Nr}\right)^2\right)^2}\right].$$

Again, using the same argument as in (III.12), there exists $A > 0$ such that the absolute value of the first part of the sum is

$$\left|\frac{\left(\frac{2it}{\sigma_N}+\frac{1}{2!}\left(\frac{2it}{\sigma_N}\right)^2-\exp\left(\frac{2it}{\sigma_N}\right)+1\right)^2}{\left(1+\left(\frac{2n-1}{2Nr}\right)^2\right)^2}\right| \le \frac{A^2t^6}{\sigma^6N^6\left(1+\left(\frac{2n-1}{2Nr}\right)^2\right)^2} \le \frac{A^2t^6}{\sigma^6N^6}, \tag{III.14}$$

and there exists $B > 0$ such that the absolute value of the second part of the sum is

$$\begin{aligned}\left|\frac{2\left(\frac{2it}{\sigma_N}+\frac{1}{2!}\left(\frac{2it}{\sigma_N}\right)^2\right)\left(\exp\left(\frac{2it}{\sigma_N}\right)-1\right)}{\left(1+\left(\frac{2n-1}{2Nr}\right)^2\right)^2}\right| &\le \left|\frac{2\left(\frac{2it}{\sigma N}+\frac{1}{2!}\left(\frac{2it}{\sigma N}\right)^2\right)\left(\frac{2Bit}{\sigma N}\right)}{\left(1+\left(\frac{2n-1}{2Nr}\right)^2\right)^2}\right| \\ &= \left|\frac{4Bt^2\left(1+\frac{1}{2!}\left(\frac{2it}{\sigma N}\right)\right)}{\sigma^2N^2\left(1+\left(\frac{2n-1}{2Nr}\right)^2\right)^2}\right| \\ &\le \left|\frac{4Bt^2\left(1+\frac{it}{\sigma N}\right)}{\sigma^2N^2}\right|.\end{aligned} \tag{III.15}$$

Thus, using (III.14) and (III.15) the absolute value of the second summand

$$\begin{aligned}
&\left|\sum_{n=1}^{\frac{N}{2}}\left[\frac{1}{2}\left(\frac{\frac{2it}{\sigma_N}+\frac{1}{2!}\left(\frac{2it}{\sigma_N}\right)^2}{1+\left(\frac{2n-1}{2Nr}\right)^2}\right)^2-\frac{1}{2}\left(\frac{\exp\left(\frac{2it}{\sigma_N}\right)-1}{1+\left(\frac{2n-1}{2Nr}\right)^2}\right)^2\right]\right| \\
&\qquad\qquad\qquad\leq \frac{1}{2}\sum_{n=1}^{\frac{N}{2}}\left(\frac{A^2t^6}{\sigma^6N^6}+\left|\frac{4Bt^2\left(1+\frac{it}{\sigma N}\right)}{\sigma^2N^2}\right|\right) \\
&\qquad\qquad\qquad= \frac{1}{2\sigma^2N^2}\sum_{n=1}^{\frac{N}{2}}\left(\frac{A^2t^6}{\sigma^4N^4}+\left|4Bt^2\left(1+\frac{it}{\sigma N}\right)\right|\right) \\
&\qquad\qquad\qquad= \frac{1}{4\sigma^2N}\left(\frac{A^2t^6}{\sigma^4N^4}+\left|4Bt^2\left(1+\frac{it}{\sigma N}\right)\right|\right) \\
&\qquad\qquad\qquad\to 0
\end{aligned}$$

as $N\to\infty$. Thus, (III.11) converges to 0, so (III.8) and (III.9) converge to the same limit. We rewrite (III.9) as

$$\begin{aligned}
&\sum_{n=1}^{\frac{N}{2}}\left[\left(\frac{\frac{2it}{\sigma_N}+\frac{1}{2!}\left(\frac{2it}{\sigma_N}\right)^2}{1+\left(\frac{2n-1}{2Nr}\right)^2}\right)-\frac{1}{2}\left(\frac{\frac{2it}{\sigma_N}+\frac{1}{2!}\left(\frac{2it}{\sigma_N}\right)^2}{1+\left(\frac{2n-1}{2Nr}\right)^2}\right)^2\right]-it\frac{\mu_N}{\sigma_N} \\
&\quad=\underbrace{\sum_{n=1}^{\frac{N}{2}}\left[-\frac{2t^2}{\sigma_N^2\left(1+\left(\frac{2n-1}{2Nr}\right)^2\right)}+\frac{2t^2}{\sigma_N^2\left(1+\left(\frac{2n-1}{2Nr}\right)^2\right)^2}\right]}_{(*)} \\
&\qquad\underbrace{+\sum_{n=1}^{\frac{N}{2}}\left[\frac{2it}{\sigma_N\left(1+\left(\frac{2n-1}{2Nr}\right)^2\right)}+\frac{4it^3}{\sigma_N^3\left(1+\left(\frac{2n-1}{2Nr}\right)^2\right)^2}-\frac{2t^4}{\sigma_N^4\left(1+\left(\frac{2n-1}{2Nr}\right)^2\right)^2}\right]-it\frac{\mu_N}{\sigma_N}}_{(**)}.
\end{aligned}$$

We show that the expressions $(*)$ and $(**)$ converge to $-\dfrac{t^2}{2}$ and 0, respectively. First we define

$$\mu=\frac{\mu_N}{N}=2r\arctan\left(\frac{1}{2r}\right),$$

and let σ be the positive number such that

$$\sigma^2 = \frac{\sigma_N^2}{N} = \left(2r\arctan\left(\frac{1}{2r}\right) - \frac{4r^2}{1+4r^2}\right).$$

Then

$$\begin{aligned}
-\frac{2t^2}{\sigma_N^2}\sum_{n=1}^{\frac{N}{2}}\frac{1}{1+\left(\frac{2n-1}{2Nr}\right)^2} &= -\frac{t^2}{\sigma^2}\cdot\frac{2}{N}\sum_{n=1}^{\frac{N}{2}}\frac{1}{1+\left(\frac{2n-1}{2Nr}\right)^2}\\
&\to -\frac{t^2}{\sigma^2}\int_0^1\frac{1}{1+\left(\frac{t}{2r}\right)^2}\,dt\\
&= -2r\arctan\left(\frac{1}{2r}\right)\cdot\frac{t^2}{\sigma^2}
\end{aligned}$$

and

$$\begin{aligned}
\frac{2t^2}{\sigma_N^2}\sum_{n=1}^{\frac{N}{2}}\frac{1}{\left(1+\left(\frac{2n-1}{2Nr}\right)^2\right)^2} &= \frac{t^2}{\sigma^2}\cdot\frac{2}{N}\sum_{n=1}^{\frac{N}{2}}\frac{1}{\left(1+\left(\frac{2n-1}{2Nr}\right)^2\right)^2}\\
&\to \frac{t^2}{\sigma^2}\int_0^1\frac{1}{\left(1+\left(\frac{t}{2r}\right)^2\right)^2}\,dt\\
&= \left(2r\arctan\left(\frac{1}{2r}\right)+\frac{4r^2}{1+4r^2}\right)\frac{t^2}{2\sigma^2}.
\end{aligned}$$

Thus, changing $\frac{N}{2}\mapsto N$,

$$\begin{aligned}
&\sum_{n=1}^{N}\left[-\frac{2t^2}{\sigma_{2N}^2\left(1+\left(\frac{2n-1}{4Nr}\right)^2\right)}+\frac{2t^2}{\sigma_{2N}^2\left(1+\left(\frac{2n-1}{4Nr}\right)^2\right)^2}\right]\\
&\quad\to\left(2r\arctan\left(\frac{1}{2r}\right)+\frac{4r^2}{1+4r^2}\right)\frac{t^2}{2\sigma^2}-2r\arctan\left(\frac{1}{2r}\right)\cdot\frac{t^2}{\sigma^2}\\
&\quad= -\frac{t^2}{2}.
\end{aligned}$$

Next,

$$\frac{2it}{\sigma_N}\sum_{n=1}^{\frac{N}{2}}\frac{1}{1+\left(\frac{2n-1}{2Nr}\right)^2}-it\frac{\mu_N}{\sigma_N}=\frac{\sqrt{N}it}{\sigma}\left(\frac{2}{N}\sum_{n=1}^{\frac{N}{2}}\frac{1}{1+\left(\frac{2n-1}{2Nr}\right)^2}-\mu\right)$$
$$=\frac{\sqrt{N}it}{\sigma}\left(\frac{2}{N}\sum_{n=1}^{\frac{N}{2}}\frac{1}{1+\left(\frac{2n-1}{2Nr}\right)^2}-\int_0^1\frac{1}{1+\left(\frac{t}{2r}\right)^2}\,dt\right).$$

It is a well-known calculus fact that the error term of the Riemann sum using the midpoint rule is $O(N^{-2})$ if the integrand is twice differentiable. That is,

$$\frac{2it}{\sigma_N}\sum_{n=1}^{\frac{N}{2}}\left[\frac{1}{1+\left(\frac{2n-1}{2Nr}\right)^2}\right]-it\frac{\mu_N}{\sigma_N}\to 0.$$

Similarly,

$$-\frac{2t^4}{\sigma_N^4}\sum_{n=1}^{\frac{N}{2}}\frac{1}{\left(1+\left(\frac{2n-1}{2Nr}\right)^2\right)^2}=-\frac{t^4}{N\sigma^4}\cdot\frac{2}{N}\sum_{n=1}^{\frac{N}{2}}\frac{1}{\left(1+\left(\frac{2n-1}{2Nr}\right)^2\right)^2},$$
$$\to 0$$

and since the sums converge to finite integrals,

$$\frac{4it^3}{\sigma_N^3}\sum_{n=1}^{\frac{N}{2}}\frac{1}{\left(1+\left(\frac{2n-1}{2Nr}\right)^2\right)^2}=\frac{2it^3}{\sqrt{N}\sigma^3}\cdot\frac{2}{N}\sum_{n=1}^{\frac{N}{2}}\frac{1}{\left(1+\left(\frac{2n-1}{2Nr}\right)^2\right)^2}$$
$$\to 0.$$

This shows that

$$\log\left[\mathbb{E}\left[\exp\left(it\frac{L_N(Nr)-\mu_N}{\sigma_N}\right)\right]\right]\to-\frac{t^2}{2},$$

proving that the random variable converges weakly to the standard normal variable.

Proof of Corollary III.1

We sketch the main idea of this calculation and refer the reader to Section 4.5 of [30] for the details. Let $x_1,\dots,x_N,y_1,\dots,y_N\in[-\pi,\pi)$, and let $a_1,\dots,a_N,b_1,\dots,b_N$

be indeterminants. Define measures η_1 and η_2 on $[-\pi, \pi)$ by

$$d\eta_1 = \sum_{n=1}^{N} a_n d\delta_{x_n} \quad \text{and} \quad d\eta_2 = \sum_{n=1}^{N} b_n d\delta_{y_n}.$$

Let

$$Z_{L,M}^{\eta_1,\eta_2} = \frac{1}{L!M!} \int_{[-\pi,\pi)^L} \int_{[-\pi,\pi)^M} e^{-E(\boldsymbol{\theta},\boldsymbol{\phi})} (d\theta + d\eta_1)^L (d\phi + d\eta_2)^M$$

and

$$Z_N^{\eta_1,\eta_2}(X) = \sum_{(L,M)} X^L Z_{L,M}^{\eta_1,\eta_2}.$$

Then expanding the measure we get

$$\frac{Z_N^{\eta_1,\eta_2}(X)}{Z_N(X)} = \sum_{n=1}^{N} \sum_{\substack{(l,m) \\ l+2m=n}} a_{j_1} \cdots a_{j_l} b_{k_1} \cdots b_{k_m} R_{l,m}(\theta_{j_1}, \ldots, \theta_{j_l}, \phi_{k_1}, \ldots, \phi_{k_m}).$$

On the other hand, we may use Theorem III.1 to express

$$\frac{Z_N^{\eta_1,\eta_2}(X)}{Z_N(X)} = \sum_{n=1}^{N} \sum_{\substack{(l,m) \\ l+2m=n}} a_{j_1} \cdots a_{j_l} b_{k_1} \cdots b_{k_m} \operatorname{Pf} \mathbf{K}_{l,m}(\theta_{j_1}, \ldots, \theta_{j_l}, \phi_{k_1}, \ldots, \phi_{k_m}).$$

Matching coefficients will give the result.

Proof of Theorem III.1

We calculate

$$\begin{aligned}
S^{1,1}(r,\theta,\phi) &= \lim_{N\to\infty} \frac{2\pi}{N} S_N^{1,1}\left(Nr, \frac{2\pi\theta}{N}, \frac{2\pi\phi}{N}\right) \\
&= \lim_{N\to\infty} 8Nr^2 \sum_{n=1}^{\frac{N}{2}} \frac{\cos\left[\frac{\pi}{N}(2n-1)(\theta-\phi)\right]}{(2Nr)^2 + (2n-1)^2} \\
&= \lim_{N\to\infty} 4r^2 \cdot \frac{2}{N} \sum_{n=1}^{\frac{N}{2}} \frac{\cos\left[\pi\left(\frac{2n-1}{N}\right)(\theta-\phi)\right]}{(2r)^2 + \left(\frac{2n-1}{N}\right)^2} \\
&= 4r^2 \int_0^1 \frac{\cos\left[\pi(\theta-\phi)t\right]}{(2r)^2 + t^2}\, dt,
\end{aligned}$$

$$
\begin{aligned}
DS^{1,1}\left(r,\theta,\phi\right) &= \lim_{N\to\infty}\frac{2\pi}{N}DS_N^{1,1}\left(Nr,\frac{2\pi\theta}{N},\frac{2\pi\phi}{N}\right)\\
&= \lim_{N\to\infty} 2ir^2\sum_{n=1}^{\frac{N}{2}}\frac{(2n-1)\sin\left[\frac{\pi}{N}(2n-1)(\theta-\phi)\right]}{(2Nr)^2+(2n-1)^2}\\
&= \lim_{N\to\infty} ir^2\cdot\frac{2}{N}\sum_{n=1}^{\frac{N}{2}}\frac{\left(\frac{2n-1}{N}\right)\sin\left[\pi\left(\frac{2n-1}{N}\right)(\theta-\phi)\right]}{(2r)^2+\left(\frac{2n-1}{N}\right)^2}\\
&= ir^2\int_0^1\frac{t\cdot\sin\left[\pi(\theta-\phi)t\right]}{(2r)^2+t^2}dt,
\end{aligned}
$$

$$
\begin{aligned}
IS^{1,1}\left(r,\theta,\phi\right) &= \lim_{N\to\infty}\frac{2\pi}{N}IS_N^{1,1}\left(Nr,\frac{2\pi\theta}{N},\frac{2\pi\phi}{N}\right)\\
&= -\lim_{N\to\infty} 32iN^2r^2\sum_{n=1}^{\frac{N}{2}}\frac{\sin\left[\frac{\pi}{N}(2n-1)(\theta-\phi)\right]}{(2n-1)((2Nr)^2+(2n-1)^2)}+2\pi\cdot\operatorname{sgn}(\phi-\theta)\\
&= -\lim_{N\to\infty} 16ir^2\cdot\frac{2}{N}\sum_{n=1}^{\frac{N}{2}}\frac{\sin\left[\pi\left(\frac{2n-1}{N}\right)(\theta-\phi)\right]}{\left(\frac{2n-1}{N}\right)\left((2r)^2+\left(\frac{2n-1}{N}\right)^2\right)}+2\pi\cdot\operatorname{sgn}(\phi-\theta)\\
&= -16ir^2\int_0^1\frac{\sin\left[\pi(\theta-\phi)t\right]}{(2r)^2t+t^3}dt+2\pi\cdot\operatorname{sgn}(\phi-\theta),
\end{aligned}
$$

and

$$
\begin{aligned}
S^{2,2}\left(r,\theta,\phi\right) &= \lim_{N\to\infty}\frac{2\pi}{N}S_N^{2,2}\left(Nr,\frac{2\pi\theta}{N},\frac{2\pi\phi}{N}\right)\\
&= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{\frac{N}{2}}\frac{(2n-1)^2\cos\left[\frac{\pi}{N}(2n-1)(\theta-\phi)\right]}{(2Nr)^2+(2n-1)^2}\\
&= \lim_{N\to\infty}\frac{1}{2}\cdot\frac{2}{N}\sum_{n=1}^{\frac{N}{2}}\frac{\left(\frac{2n-1}{N}\right)^2\cos\left[\pi\left(\frac{2n-1}{N}\right)(\theta-\phi)\right]}{(2r)^2+\left(\frac{2n-1}{N}\right)^2}\\
&= \frac{1}{2}\int_0^1\frac{t^2\cos\left[\pi(\theta-\phi)t\right]}{(2r)^2+t^2}dt,
\end{aligned}
$$

$$
\begin{aligned}
IS^{2,2}\left(r,\theta,\phi\right) &= \lim_{N\to\infty}\frac{2\pi}{N}IS_N^{2,2}\left(Nr,\frac{2\pi\theta}{N},\frac{2\pi\phi}{N}\right)\\
&= \lim_{N\to\infty}-\frac{i}{2N^2}\sum_{n=1}^{\frac{N}{2}}\frac{(2n-1)^3\sin\left[\frac{\pi}{N}(2n-1)(\theta-\phi)\right]}{(2Nr)^2+(2n-1)^2}\\
&= \lim_{N\to\infty}-\frac{i}{4}\cdot\frac{2}{N}\sum_{n=1}^{\frac{N}{2}}\frac{\left(\frac{2n-1}{N}\right)^3\sin\left[\pi\left(\frac{2n-1}{N}\right)(\theta-\phi)\right]}{(2r)^2+\left(\frac{2n-1}{N}\right)^2}\\
&= -\frac{i}{4}\int_0^1\frac{t^3\sin\left[\pi(\theta-\phi)t\right]}{(2r)^2+t^2}dt.
\end{aligned}
$$

CHAPTER IV

TWO SPECIES OF REFLECTED PARTICLES ON THE REAL AND IMAGINARY AXES

Setup

Fix N and consider $2N$ particles confined to the real and imaginary axes inside a radial external field Q where each particle has a corresponding particle reflected about the origin. The number of particles on each axis is random and the distributions are controlled by a fugacity parameter X. We first consider the configuration of L particles on the positive real axis and M particles on the positive imaginary axis such that $L + M = N$. Denote the particle locations by

$$(x_1, \dots, x_L, -x_1, \dots, -x_L, iy_1, \dots, iy_M, -iy_1, \dots, -iy_M)$$

where $x_1, \dots, x_L, y_1, \dots, y_M \geq 0$. The energy of the (L, M) configuration is given by

$$\begin{aligned} E(\boldsymbol{x}_L, \boldsymbol{y}_M) = 2 \sum_{l_1>l_2}^{L} (\log|x_{l_1} - x_{l_2}| + \log|x_{l_1} + x_{l_2}|) \\ - 2 \sum_{m_1>m_2}^{M} (\log|iy_{m_1} - iy_{m_2}| + \log|iy_{m_1} + iy_{m_2})|) - \sum_{l=1}^{L} \log|2x_l| \\ - \sum_{m=1}^{M} \log|2iy_m| - 2 \sum_{l=1}^{L} \sum_{m=1}^{M} (\log|x_l - iy_m| + \log|x_l + iy_m|) \\ +2 \sum_{l=1}^{L} Q(x_l) + 2 \sum_{m=1}^{M} Q(y_m). \end{aligned}$$

Let $w(x) = e^{-2Q(x)}$ be a weight on the positive half of the real line, and denote Δ to be the Vandermonde product

$$\Delta(\boldsymbol{x}_N) = \prod_{n>m} (x_n - x_m).$$

The Boltzmann factor with inverse temperature equal to 1 is

$$\begin{aligned} e^{-E(\boldsymbol{x}_L, \boldsymbol{y}_M)} &= |\Delta(\boldsymbol{x}_L \vee -\boldsymbol{x}_L \vee i\boldsymbol{y}_M \vee -i\boldsymbol{y}_M)| \prod_{l=1}^{L} w(x_l) \prod_{m=1}^{M} w(y_m) \\ &= (-i)^M \Delta(\boldsymbol{x}_L \vee -\boldsymbol{x}_L \vee i\boldsymbol{y}_M \vee -i\boldsymbol{y}_M) \prod_{l=1}^{L} w(x_l) \prod_{m=1}^{M} w(y_m). \end{aligned} \tag{IV.1}$$

As in the setup of the two-species ensemble in Chapter III, the total partition function is the weighted sum over all possible (L, M) partition functions

$$Z_N(X) = \sum_{(L,M)} X^M Z_{L,M},$$

where

$$Z_{L,M} = \frac{1}{L!M!} \int_{[0,\infty)^L} \int_{[0,\infty)^M} e^{-E(\boldsymbol{x}_L, \boldsymbol{y}_M)}\, d\boldsymbol{x}_L d\boldsymbol{y}_M. \tag{IV.2}$$

The Partition Function as a Pfaffian

The partition function is the Pfaffian of a Gram matrix, where the skew-inner product can be decomposed as a sum of two skew-inner products.

Theorem IV.1. *For $f, g \in L^2$, let $\langle \cdot | \cdot \rangle_1$ and $\langle \cdot | \cdot \rangle_2$ be skew-inner products defined by*

$$\langle f|g\rangle_1 = \int_0^\infty (f(x)g(-x) - f(-x)g(x))w(x)\,dx$$

and

$$\langle f|g\rangle_2 = -i\int_0^\infty (f(ix)g(-ix) - f(-ix)g(ix))w(x)\,dx.$$

Then for any family of monic polynomials $\{\rho_n\}$ where $\deg(\rho_n) = n$,

$$Z_N(X) = \mathrm{Pf}\left[\langle \rho_{n-1}|\rho_{m-1}\rangle_1 + X\langle \rho_{n-1}|\rho_{m-1}\rangle_2\right]_{n,m=1}^{2N}. \tag{IV.3}$$

Skew-Orthogonal Polynomials

Choosing the monic skew-orthogonal polynomials $\{q_n^X\}$ with respect to the skew-inner product $\langle \cdot | \cdot \rangle_1 + \langle \cdot | \cdot \rangle_2$, and letting

$$r_n(X) = \langle q_{2n}^X | q_{2n+1}^X\rangle_1 + X\langle q_{2n}^X | q_{2n+1}^X\rangle_2,$$

the partition function is

$$Z_N(X) = \prod_{n=0}^{N-1} r_n(X).$$

From here on, we suppress the dependence of the skew-orthogonal polynomials on X. Next we give a concrete description of the polynomials. Because the external field is radial, the even and odd degree polynomials are even and odd functions, respectively, and both are expressed in terms of orthogonal polynomials whose weight has a jump at the origin.

Proposition IV.1. *Let*

$$w_X(x) = \begin{cases} w(\sqrt{x}) & \text{if } x \geq 0 \\ X \cdot w(\sqrt{-x}) & \text{if } x < 0 \end{cases} \tag{IV.4}$$

and let $\{p_n\}$ *be the monic orthogonal polynomials with respect to the inner product*

$$\langle p_n | p_m \rangle := \int_{-\infty}^{\infty} p_n(x) p_m(x) w_X(x)\, dx. \tag{IV.5}$$

The skew-orthogonal polynomials with respect to the skew-inner product $\langle \cdot | \cdot \rangle_1 + \langle \cdot | \cdot \rangle_2$ *are defined by*

$$\begin{aligned} q_{2n}(x) &= p_n(x^2) \\ q_{2n+1}(x) &= x p_n(x^2). \end{aligned} \tag{IV.6}$$

Finally, the normalization of the skew-orthogonal polynomials is equal to the normalization of the orthogonal polynomials:

$$r_n(X) = \langle p_n | p_n \rangle.$$

The weight w_X has a jump discontinuity at the origin when $X \neq 1$. We exploit the relationship between the skew-orthogonal and orthogonal polynomials.

Distribution of Each Species

For fixed N, let $M_N(X)$ be the random variable of the number of particles on the positive imaginary axis. Note that $M_N(X)$ also depends on Q and X, but we suppress this dependence in the notation.

Proposition IV.1. *Let*

$$\lambda_n = \frac{\int_{-\infty}^{0} (p_n(x))^2 w_X(x)\, dx}{\int_{-\infty}^{\infty} (p_n(x))^2 w_X(x)\, dx}.$$

Then the distribution of $M_N(X)$ is

$$M_N(X) \sim \sum_{n=0}^{N-1} \text{Bernoulli}(\lambda_n),$$

and each Bernoulli random variable is independent.

Proof. The probability generation function for $M_N(X)$ is

$$\frac{Z_N(tX)}{Z_N(X)} = \prod_{n=0}^{N-1} \frac{r_n(tX)}{r_n(X)},$$

that is, the probability that $M_N(X) = k$ is the coefficient of t^k. From (IV.5),

$$\frac{r_n(tX)}{r_n(X)} = (1 - \lambda_n) + t\lambda_n,$$

which is the probability generating function for the sum of independent Bernoulli random variables with parameters λ_n. □

Corollary IV.2. *When $X = 1$, $M_N(X)$ has the distribution of a binomial random variable with N trials and probability $\frac{1}{2}$.*

Proof. In this scenario, the weight is even, so p_n^2 is even for all n. Then

$$\begin{aligned} r_n(t) &= \int_0^\infty \left((p_n(x))^2 + t(p_n(-x))^2\right) w(\sqrt{x})\, dx \\ &= (t+1)\int_0^\infty (p_n(x))^2 w(\sqrt{x})\, dx \\ &= (t+1)\frac{1}{2}\int_{-\infty}^\infty (p_n(x))^2 w_X(x)\, dx \\ &= (t+1)\frac{r_n(1)}{2}. \end{aligned}$$

Thus,

$$\frac{Z_N(t)}{Z_N(1)} = \prod_{n=0}^{N-1} \frac{r_n(t)}{r_n(1)} = \frac{(t+1)^N}{2^N}.$$

□

Under certain conditions, $M_N(X)$ has a central limit theorem. Denote $N(0,1)$ to be the standard normal distribution.

Corollary IV.3. *If*

$$0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1$$

then

$$\frac{M_N(X) - \mathbb{E}[M_N(X)]}{\sqrt{\mathrm{Var}[M_N(X)]}} \to N(0,1)$$

where the convergence is weak.

Correlation Functions

Recall for the two species particle case, because the numbers of each type of particle are allowed to vary, we must alter the definition of the correlation functions from (I.4). Fix l particles on the positive real axis and m particles on the positive imaginary axis. For a given (L, M) such that $l \leq L$ and $m \leq M$, the (l, m, L, M) partial correlation function is

$$\frac{1}{Z_N(X)(L-l)!(M-m)!} \int_{[0,\infty)^{L-l}} \int_{[0,\infty)^{M-m}} e^{-E(\boldsymbol{x}_l \vee \boldsymbol{u}_{L-l}, \boldsymbol{y}_m \vee \boldsymbol{v}_{M-m})} \, d\boldsymbol{u}_{L-l} d\boldsymbol{v}_{M-m}.$$

We then take the weighted sum over all possible (L, M). So the l, m correlation function $R^N_{l,m} : [0,\infty)^l \times [0,\infty)^m \to [0,\infty)$ is defined as

$$\begin{aligned} &R^N_{l,m}(\boldsymbol{x}_l, \boldsymbol{y}_m) \\ &\quad = \frac{1}{Z_N(X)} \sum_{\substack{(L,M) \\ L \geq l, M \geq m}} \frac{X^M}{(L-l)!(M-m)!} \\ &\quad \times \int_{[0,\infty)^{L-l}} \int_{[0,\infty)^{M-m}} e^{-E(\boldsymbol{x}_l \vee \boldsymbol{u}_{L-l}, \boldsymbol{y}_m \vee \boldsymbol{v}_{M-m})} \, d\boldsymbol{u}_{L-l} d\boldsymbol{v}_{M-m}. \end{aligned}$$

The next theorem states that the correlation functions can be expressed as a Pfaffian of four 2×2 matrix kernels.

Theorem IV.1. *For fixed X, let $\{q_n\}$ be the set of monic skew-orthogonal polynomials as defined in Proposition IV.1. Let*

$$S_N(x, y) = \begin{cases} \sqrt{w(x)w(y)} \dfrac{p_N(x^2)p_{N-1}(y^2) - p_N(y^2)p_{N-1}(x^2)}{r_{N-1}(x-y)} & \text{if } x \neq y \\ 2x \cdot w(x) \dfrac{p'_N(x^2)p_{N-1}(x^2) - p_N(x^2)p'_{N-1}(x^2)}{r_{N-1}} & \text{if } x = y, \end{cases} \tag{IV.7}$$

let

$$\begin{aligned}
\boldsymbol{K}_N^{1,1}(x,y) &= \begin{bmatrix} -S_N(-x,y) & S_N(x,y) \\ -S_N(x,y) & S_N(-x,y) \end{bmatrix} \\
\boldsymbol{K}_N^{2,2}(x,y) &= -iX\boldsymbol{K}_N^{1,1}(ix,iy) \\
\boldsymbol{K}_N^{1,2}(x,y) &= e^{\frac{3\pi i}{4}}\sqrt{X}\boldsymbol{K}_N^{1,1}(x,iy) \\
\boldsymbol{K}_N^{2,1}(x,y) &= e^{\frac{3\pi i}{4}}\sqrt{X}\boldsymbol{K}_N^{1,1}(ix,y),
\end{aligned} \tag{IV.8}$$

and let $K_{l,m}^N(\boldsymbol{x}_L, \boldsymbol{y}_M)$ *be the* $2(l+m) \times 2(l+m)$ *defined by*

$$\boldsymbol{K}_{l,m}^N(\boldsymbol{x}_l, \boldsymbol{y}_m) = \begin{bmatrix} \boldsymbol{K}_N^{1,1}(x_j, x_{j'}) & \boldsymbol{K}_N^{1,2}(x_j, y_{k'}) \\ \boldsymbol{K}_N^{2,1}(y_k, x_{j'}) & \boldsymbol{K}_N^{2,2}(y_k, y_{k'}) \end{bmatrix}$$

for $j, j' = 1, \ldots, l$ *and* $k, k' = 1, \ldots, m$. *Then*

$$R_{l,m}^N(\boldsymbol{x}_l, \boldsymbol{y}_m) = \mathrm{Pf}\, \boldsymbol{K}_{l,m}^N(\boldsymbol{x}_l, \boldsymbol{y}_m). \tag{IV.9}$$

Remark IV.2. The $(1,1)$ matrix kernel encodes the interaction between the particles on the real axis, the $(2,2)$ matrix kernel encodes the interaction between the particles on the imaginary axis, and the $(1,2)$ and $(2,1)$ matrix kernels encode the mixed interactions. Within each matrix kernel, the scalar kernels encode the interactions between particles on the various rays. For example, in the $(1,1)$ matrix kernel, $S_N(x,y)$ describes the relation between particles on the positive real axis, while $S_N(x,-y)$ describes the relation between particles on the positive and negative real axis, and so on.

Reduction to an Orthogonal Polynomial Ensemble

We introduce the following orthogonal polynomial ensemble which will play an important role. Consider the jump weight w_X on $\mathbb{R}$ defined by (IV.4). By (IV.5), the orthogonal polynomials are $\{p_n\}$, so the weighted Christoffel-Darboux kernel corresponding to the weight w_X is

$$K_N(x,y) = \begin{cases} \sqrt{w_X(x)w_X(y)}\dfrac{p_N(x)p_{N-1}(y) - p_N(y)p_{N-1}(x)}{r_{N-1}(x-y)} & \text{if } x \neq y \\ w_X(x)\dfrac{p_N'(x)p_{N-1}(x) - p_N(x)p_{N-1}'(x)}{r_{N-1}} & \text{if } x = y \end{cases} \tag{IV.10}$$

for $x, y \in \mathbb{R}$. Note that

$$(x+y)K_N(x^2,y^2) = S_N(x,y) \quad \text{for} \quad x,y \geq 0$$

$$(x+y)K_N(x^2,y^2) = -XS_N(-x,-y) \quad \text{for} \quad x,y \leq 0.$$

This relation between the two kernels suggests that the study of our two species ensemble reduces to the study of the orthogonal polynomial ensemble with kernel (IV.10). In fact, there is a direct correspondence between our two species model and the orthogonal polynomial ensemble. Recall from (I.1) that the joint density of the orthogonal polynomial ensemble with weight w_X (which is a determinantal point process with kernel (IV.10)) is proportional to

$$\Omega_{N;2}(\boldsymbol{x}_N) = \prod_{n>m}^{N} |x_n - x_m|^2 \prod_{n=1}^{N} w_X(x_n).$$

We can rewrite the (unnormalized) density (IV.1) as

$$
\begin{aligned}
&X^M e^{-E(\boldsymbol{x}_L,\boldsymbol{y}_M)} \\
&\quad = \prod_{l_1>l_2}^{L} (x_{l_1}^2 - x_{l_2}^2)^2 \prod_{m_1>m_2}^{M} (y_{m_1}^2 - y_{m_2}^2)^2 \prod_{l\neq m} (x_l^2 + y_m^2)^2 \\
&\qquad \times \prod_{l=1}^{L} 2x_l w_X(x_l^2) \prod_{m=1}^{M} 2y_m w_X(-y_m^2) \\
&\quad = \Omega_{N;2}(\boldsymbol{x}_L^2 \vee -\boldsymbol{y}_M^2) \prod_{l=1}^{L} 2x_l \prod_{m=1}^{M} 2y_m.
\end{aligned}
$$

Then by a change of variables,

$$
\begin{aligned}
X^M Z_{L,M} &= \frac{1}{L!M!} \int_{[0,\infty)^L} \int_{[0,\infty)^M} \Omega_{N;2}(\boldsymbol{x}_L^2 \vee -\boldsymbol{y}_M^2) \prod_{l=1}^{L} 2x_l \prod_{m=1}^{M} 2y_m \, d\boldsymbol{x}_L d\boldsymbol{y}_M \\
&= \frac{1}{L!M!} \int_{[0,\infty)^L} \int_{[0,\infty)^M} \Omega_{N;2}(\boldsymbol{u}_L \vee -\boldsymbol{v}_M) \, d\boldsymbol{u}_L d\boldsymbol{v}_M. \qquad \text{(IV.11)}
\end{aligned}
$$

Adding up all the partial partition functions, we get

$$
\begin{aligned}
Z_N(X) &= \sum_{(L,M)} X^M Z_{L,M} \\
&= \sum_{(L,M)} \frac{1}{L!M!} \int_{[0,\infty)^L} \int_{[0,\infty)^M} \Omega_{N;2}(\boldsymbol{u}_L \vee -\boldsymbol{v}_M) \, d\boldsymbol{u}_L d\boldsymbol{v}_M \\
&= \frac{1}{N!} \sum_{M=0}^{N} \binom{N}{M} \int_{[0,\infty)^{N-M}} \int_{[0,\infty)^M} \Omega_{N;2}(\boldsymbol{u}_L \vee -\boldsymbol{v}_M) \, d\boldsymbol{u}_L d\boldsymbol{v}_M \\
&= \frac{1}{N!} \int_{\mathbb{R}^N} \Omega_{N;2}(\boldsymbol{u}_N) \, d\boldsymbol{u}_N \\
&= Z_{N;2}. \qquad \text{(IV.12)}
\end{aligned}
$$

The calculations (IV.11) and (IV.12) show that the marginal joint density $X^M Z_{L,M}$ corresponds to the point process with density $\Omega_{N;2}$ restricted to L positive and M negative points located at $\{u_1, \ldots, u_L, -v_1, \ldots, -v_M\}$, then relocated to $\{\sqrt{u_1}, \ldots, \sqrt{u_L}, -\sqrt{-v_1}, \ldots, -\sqrt{-v_M}\}$. Our current ensemble can now be interpreted by the points selected in the following manner: Pick N points from the orthogonal polynomial ensemble characterized as $\Omega_{N;2}$, shift the particle locations by the square root, then add in the reflected points.

The next sections describe this relation between the local global statistics of the orthogonal polynomial ensemble and the current two species ensemble. We compare the limiting global densities and the limiting kernels for K_N and S_N. We then use these relationships to investigate an example with the Jacobi weight.

Density of Points

Let $A \subset [0, \infty)$. By (I.12), the expected number of particles in A is $\displaystyle\int_A R_{1,0}^N(x)dx$. Thus, the density of a single point on the positive real axis is

$$\frac{1}{N} R_{1,0}^N(x) = \frac{1}{N} S_N(x, x) = \frac{2x}{N} K_N(x^2, x^2) \quad \text{for} \quad x \geq 0.$$

Similarly, the density of a single point of the positive imaginary axis is

$$\frac{1}{N} R_{0,1}^N(x) = \frac{-iX}{N} S_N(ix, ix) = \frac{2x}{N} K_N(-x^2, -x^2) \quad \text{for} \quad x \geq 0.$$

This indicates that we can derive the density of the real and imaginary particles in terms of the density $\dfrac{1}{N} K_N(x, x)$ in the orthogonal polynomial ensemble.

Proposition IV.1. *Suppose that the limiting density of points of the orthogonal* *polynomial ensemble with scaling $a_1 N^{\alpha_1}$ is f, i.e.*

$$\frac{a_1 N^{\alpha_1}}{N} K_N \left(a_1 N^{\alpha_1} x, a_1 N^{\alpha_1} x\right) \to f(x)$$

weakly. Then the proper scaling in the current ensemble is $\sqrt{a_1 N_1} x$ and the weak limiting density of points on the positive real axis is

$$\frac{\sqrt{a_1 N^{\alpha_1}}}{N} S_N \left(\sqrt{a_1 N^{\alpha_1}} x, \sqrt{a_1 N^{\alpha_1}} x\right) \to 2x f(x^2) \quad \textit{for} \quad x \geq 0. \tag{IV.13}$$

Llikewise, the weak limiting density of points on the positive imaginary axis is

$$-iX \frac{\sqrt{a_1 N^{\alpha_1}}}{N} S_N \left(i\sqrt{a_1 N^{\alpha_1}} x, i\sqrt{a_1 N^{\alpha_1}} x\right) \to 2x f(-x^2) \quad \textit{for} \quad x \geq 0. \tag{IV.14}$$

By our central limit theorem, if the condition of Theorem IV.3 is satisfied, we expect there to be roughly the same number of particles on either side of the origin. In fact, the next theorem states that if $0 < X < \infty$, the limiting distribution behaves as if the fugacity parameter X is set equal to one. In other words, the point process behaves globally as if the weight (IV.4) has no jump and is symmetric.

Theorem IV.2. *If $0 < X < \infty$, as $N \to \infty$, the limiting density (and the scaling) is independent of X.*

Later, we will show that near the origin, the jump does not affect the local statistics.

Scaled Kernel away from the Origin

We calculate the kernels away from the origin. This includes scaling in the bulk or at an edge. As expected, the mixed kernels vanish because the interactions between particles on different rays become insignificant relative to the nearby particle interactions as the number of particles increases.

Theorem IV.1. *For $a_1, a_2 > 0$, suppose by scaling $x \mapsto a_1 N^{\alpha_1} + \dfrac{x}{a_2 N^{\alpha_2}}$,*

$$\frac{1}{a_2 N^{\alpha_2}} K_N\left(a_1 N^{\alpha_1} + \frac{x}{a_2 N^{\alpha_2}}, a_1 N^{\alpha_1} + \frac{y}{a_2 N^{\alpha_2}}\right) \to A(x,y)$$

uniformly on compact subsets for $\alpha_1, \alpha_2 \geq 0$ and A is continuous. Then when scaling $x \mapsto \sqrt{a_1 N^{\alpha_1}} + \dfrac{x}{2\sqrt{a_1} a_2 N^{\frac{\alpha_1}{2}+\alpha_2}}$, $S_N(x,y)$ and $S_N(ix,iy)$ also converge uniformly on compact sets to $A(x,y)$ while the remaining scalar kernels converge uniformly on compact sets to 0.

Scaled Kernel at 0

Scaled at the origin, all the kernels converge to nonzero quantities since the particle interactions between the two axes do not vanish in the limit.

Theorem IV.1. *Suppose the scaling at 0 in the orthogonal polynomial ensemble is $\dfrac{x}{bN^\beta}$, and*

$$\frac{1}{bN^\beta} K_N\left(\frac{x}{bN^\beta}, \frac{y}{bN^\beta}\right) \to B(x,y) \quad \textit{for} \quad x, y \in \mathbb{R}.$$

Then scaling S_N by $x \mapsto \dfrac{x}{\sqrt{bN^\beta}}$, the scaled kernel converges to

$$(x+y)B(x^2,y^2) \quad \textit{for} \quad x,y \in [0,\infty). \tag{IV.15}$$

The remaining scalar kernels at the origin all converge to similar versions of the kernel (IV.15), with the variations corresponding to (IV.8). For example,

$$-\frac{e^{\frac{3\pi i}{4}}\sqrt{X}}{\sqrt{bN^\beta}} S_N\left(-\frac{x}{\sqrt{bN^\beta}}, \frac{iy}{\sqrt{bN^\beta}}\right) \to -e^{\frac{3\pi i}{4}}\sqrt{X}(-x+iy)B(x^2,-y^2)$$
$$\textit{for} \quad x,y \in [0,\infty).$$

Proof of Theorem IV.1. A straightforward calculation shows that

$$\frac{1}{\sqrt{bN^\beta}} S_N\left(\frac{x}{\sqrt{bN^\beta}}, \frac{y}{\sqrt{bN^\beta}}\right) = \frac{x+y}{bN^\beta} K_N\left(\frac{x^2}{bN^\beta}, \frac{y^2}{bN^\beta}\right)$$
$$\to (x+y)B(x^2,y^2) \quad \text{for} \quad x,y \in [0,\infty).$$

The proofs for the other kernels are exactly the same. □

Law of the Point Closest to the Origin

By the correspondence of points, between our current ensemble and the orthogonal polynomial ensemble, the gap probability for the ball of radius $t > 0$ centered at the origin (the probability that there is no particle in this set) is equal to the gap probability of the interval $(-t^2, t^2)$ in the orthogonal polynomial ensemble.

Similarly, the gap probability for the interval $(-t, t)$ is equal to the gap probability of the interval $(0, t^2)$ in the orthogonal polynomial ensemble.

Example: Jacobi

When $X = 1$ (the weight is symmetric about the origin) or $X = 0$ (the support is in the positive half of the real line), the corresponding orthogonal polynomials are classical for particular weights, and the orthogonal polynomial ensembles are well-studied. For example, consider the weight

$$w_X(x) = \begin{cases} (1-x)^\alpha & \text{if } 0 < x \leq 1 \\ X(1+x)^\alpha & \text{if } -1 \leq x \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

When $X = 1$, the weight is the Jacobi symmetric weight $w_1(x) = (1-x^2)^\alpha$, and the orthogonal polynomials are monic symmetric Jacobi polynomials $\{P_n^{(\alpha,\alpha)}\}$. When $X = 0$, the weight is again a Jacobi weight (although no longer symmetric), with the support shifted to $[0, 1]$, and the orthogonal polynomials are

$$\left\{ \frac{1}{2^n} P_n^{(\alpha,0)}(2x-1) \right\}.$$

Results for the Jacobi orthogonal ensemble can be found in [28].

In this section we list known results for the Jacobi ensemble ($\beta = 2$ ensemble with Jacobi weight), and the results regarding our current ensemble can be translated using the results above.

Density of Points

From Theorem IV.2, when $0 < X < \infty$, the density of particles of the orthogonal

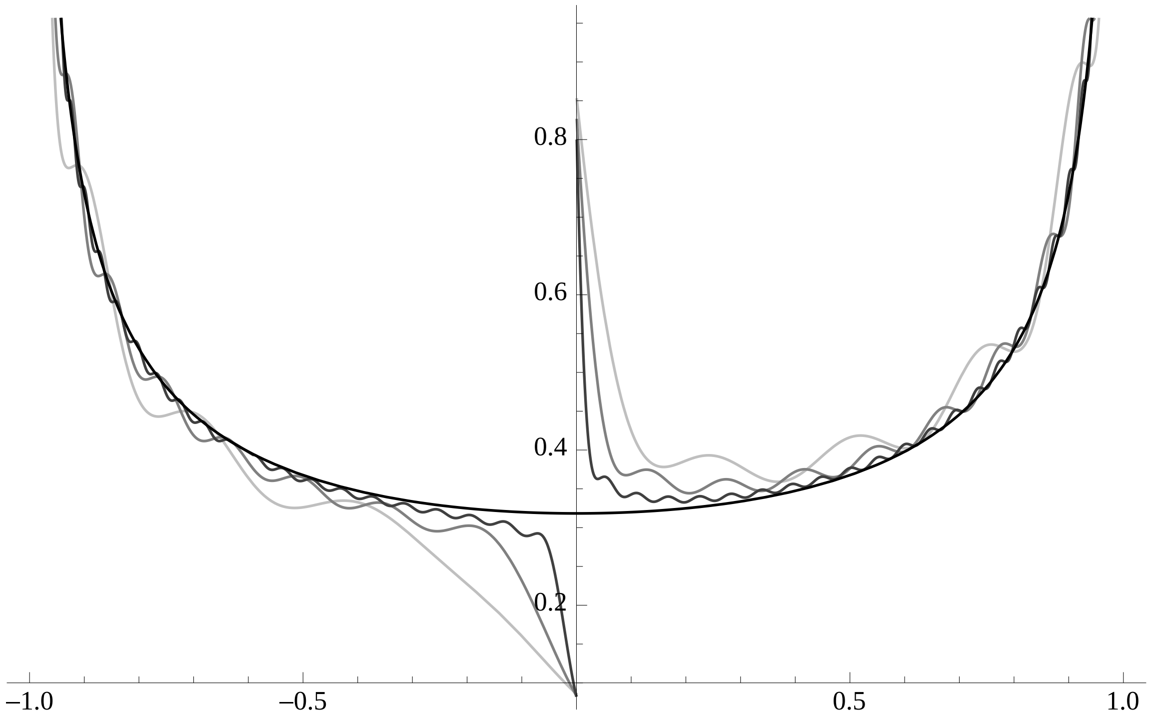

Figure 4.1.: Plots of density for $\alpha = 0$ (Legendre weight), $X = \dfrac{1}{10}$ and $N = 10, 20, 52$ from lightest to darkest, and the arcsine density.

polynomial ensemble converges weakly to the equilibrium measure on $[-1, 1]$, which is the arcsine density

$$\frac{1}{\pi\sqrt{1-x^2}}dx \quad \text{for} \quad x \in [-1, 1].$$

Figure 4.1. is the densities when $X = \dfrac{1}{10}$ with varying values of N, along with the arcsine density.

On the other hand, if we first let $X \to 0$, the particle support is $[0, 1]$ and the equilibrium measure is

$$\frac{2}{\pi\sqrt{1-(2x-1)^2}}dx \quad \text{for} \quad x \in [0,1].$$

Limiting Kernels at the Origin

As mentioned in the Introduction, for general X, the orthogonal polynomials were studied in [25] where the authors Moreno, Martinez-Finkelshtein, and Sousa derived an asymptotic form for the orthogonal polynomials and for the weighted Christoffel-Darboux kernel K_N using Riemann-Hilbert problem techniques. Here we present the results on K_N. With n a nonnegative integer, denote by $(x)_n$ to be the Pochhammer symbol. Let

$$\begin{aligned} G(a;x) &= {}_1F_1(a;1;x)e^{-\frac{x}{2}} \\ &= e^{-\frac{x}{2}}\sum_{k=0}^{\infty}\frac{(a)_k}{(k!)^2}x^k. \end{aligned}$$

Let $X > 0$ and

$$\lambda = \frac{i\log X}{2\pi}.$$

With K_N as defined in (IV.10) and the polynomials, weight, and normalization corresponding to the polynomials $\{P_n^{(\alpha,\alpha)}\}$, the limiting Christoffel-Darboux kernel scaled at the origin is

$$\begin{aligned} &\lim_{N\to\infty}\frac{\pi}{N}K_N\left(\frac{\pi x}{N},\frac{\pi y}{N}\right) \\ &= \begin{cases} -\dfrac{1}{2\pi i}\dfrac{\log X}{X-1}\dfrac{G(1+\lambda;-2\pi ix)G(\lambda;-2\pi iy)-G(1+\lambda;-2\pi iy)G(\lambda;-2\pi ix)}{x-y} & \text{if } x\neq y \\ \dfrac{\log X}{X-1}\big(G'(1+\lambda;-2\pi ix)G(\lambda;-2\pi ix)-G(1+\lambda;-2\pi ix)G'(\lambda;-2\pi ix)\big) & \text{if } x=y. \end{cases} \end{aligned}$$

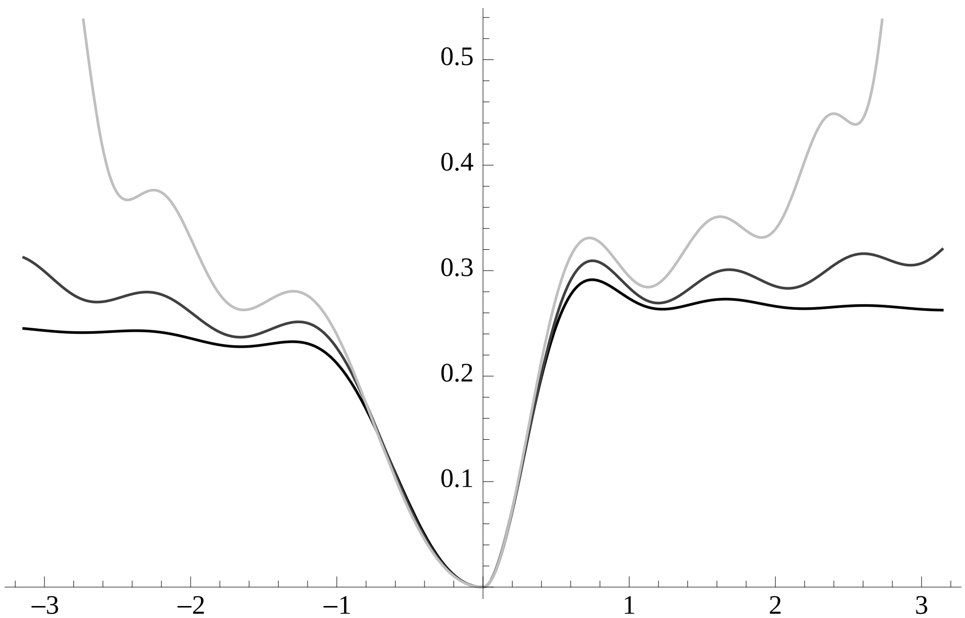

Figure 4.2.: Plots of correlation functions for $\alpha = 0$ (Legendre weight) as a function of $x - y$.

Figure 4.2. is the graphs of the second correlation functions for $X = \dfrac{1}{10}$ with $N = 10$ (lightest) and $N = 18$ (darker). The darkest graph is the limiting correlation function ($N \to \infty$) for $X = \dfrac{1}{10}$. The graphs show that in the scaling $x \mapsto \dfrac{\pi x}{N}$, the local behaviors of the particles detect the jump of the weight at the origin.

The limiting kernel anywhere else in the bulk is the sine kernel (I.8). This follows from Doron Lubinsky's result on universality in the bulk stated in Theorem 1.1 of [22], which states that on an interval inside the support of a regular weight which is positive and continuous, the weighted and scaled Christoffel-Darboux kernel converges to the sine kernel.

The kernel behaves as we would expect when $X \to 1$ in that

$$\lim_{X\to 1}\left(-\frac{1}{2\pi i}\right)\frac{\log X}{X-1}\frac{G(1+\lambda;-2\pi ix)G(\lambda;-2\pi iy)-G(1+\lambda;-2\pi iy)G(\lambda;-2\pi ix)}{x-y}$$
$$=\frac{\sin\pi(x-y)}{\pi(x-y)}$$

since

$$G(1+\lambda;-2\pi ix)\to e^{-\pi ix}$$

$$G(\lambda;-2\pi ix)\to e^{\pi ix}$$

as $X \to 1$. On the other hand, the kernel diverges as $X \to 0^+$ (hence does not converge to the Bessel kernel (I.10)) since the proper scaling at the hard edge of the Jacobi ensemble is $x \mapsto \dfrac{x}{N^2}$.

Edge Kernels at the Origin When $X = 0$ and $X = 1$

We derive new edge kernels for the current ensemble which are closely related to some of the classical kernels. For $X = 1$ in the Jacobi orthogonal polynomial ensemble, by scaling $x \mapsto \dfrac{\pi x}{N}$, the kernel converges to the sine kernel (I.8). So Theorem IV.1 gives us the scaling and limiting kernel of our current ensemble:

$$\frac{\sqrt{\pi}}{\sqrt{N}}S_N\left(\frac{\sqrt{\pi}x}{\sqrt{N}},\frac{\sqrt{\pi}y}{\sqrt{N}}\right)\to\frac{\sin\pi(x^2-y^2)}{\pi(x-y)} \tag{IV.16}$$

where S_N is the kernel defined in (IV.7). When $X = 0$ in the (shifted) Jacobi polynomial ensemble, the particles are supported on $[0,1]$, so 0 is now a hard edge.

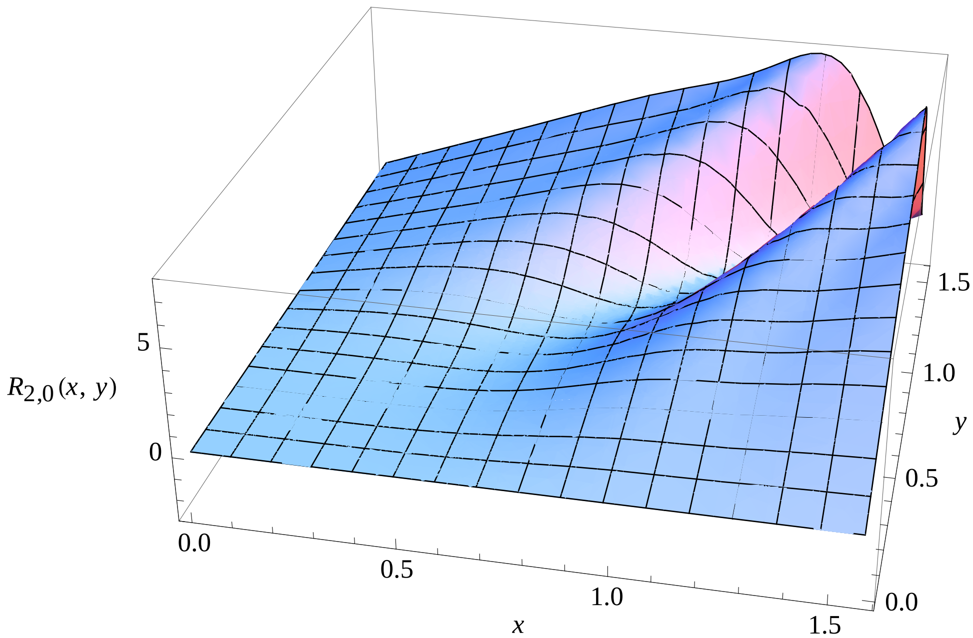

Figure 4.3.: Plot of $R_{2,0}$ with kernel (IV.16).

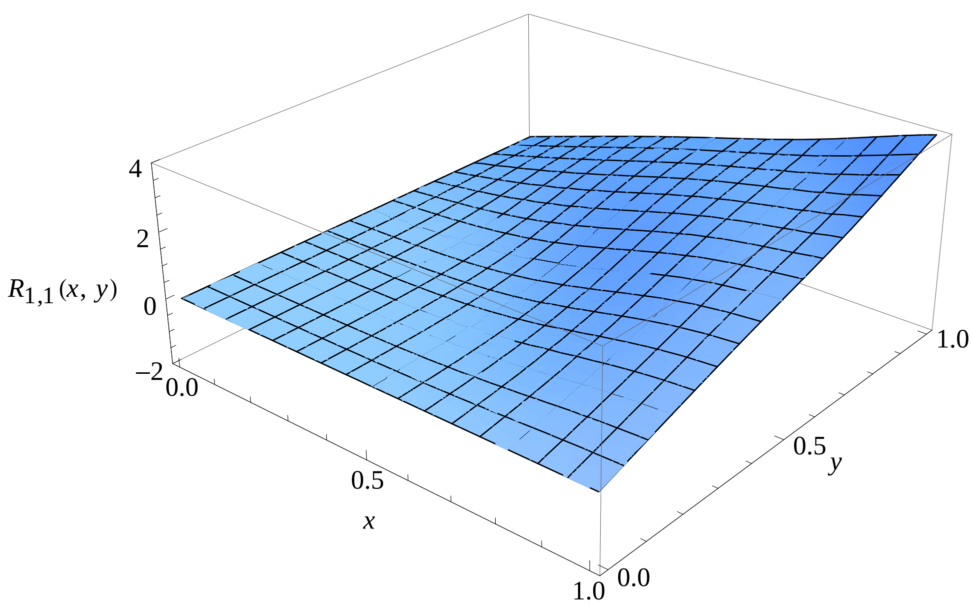

Figure 4.4.: Plot of $R_{1,1}$ with kernel (IV.16).

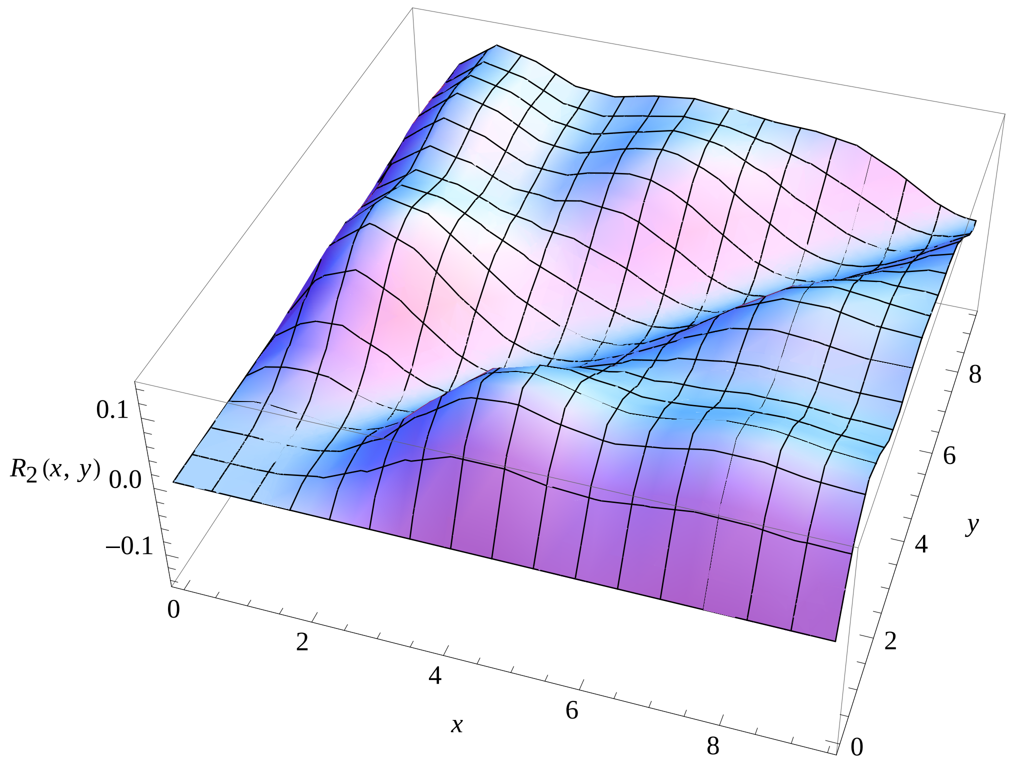

Figure 4.5.: Plot of R_2 with kernel (IV.17).

By scaling $x \mapsto \dfrac{x}{4N^2}$, the kernel converges to the Bessel kernel (I.10). By Theorem IV.1,

$$\frac{1}{2N} S_N \left(\frac{x}{2N}, \frac{y}{2N} \right) \to \frac{y J_\alpha(x) J'_\alpha(y) - x J_\alpha(y) J'_\alpha(x)}{2(x-y)}. \tag{IV.17}$$

Example: Hermite

For

$$w_X(x) = \begin{cases} e^{-x^2} & \text{if } 0 < x < \infty \\ X \cdot e^{-x^2} & \text{if } -\infty < x \leq 0, \end{cases}$$

the orthogonal polynomials were studied in [19] where the authors derived the asymptotics of the Hankel determinant. When $X = 1$, the orthogonal polynomials are the Hermite polynomials and the orthogonal polynomial ensemble is the GUE. When $X = 0$, the orthogonal polynomials are the *half-range Hermite polynomials* [4], but the author could not find this ensemble in the literature. We omit the derivations of the density and kernel in this situation, and we refer the reader to the many references on the GUE [41]. Instead, we examine the orthogonal polynomials with respect to the weight w_X.

Orthogonal Polynomials

We derive the three-term recurrence relation for the orthogonal polynomials for all values of the fugacity parameter. The method for deriving this relation is generalized from the calculation for the three-term relation of the half-range Hermite polynomials in [4]. The resulting polynomials give an interpolation between the half-range Hermite and Hermite polynomials.

Proposition IV.1. *For a fixed X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of numbers which satisfy*

$$\begin{aligned}\alpha_0 &= \frac{-X+1}{\sqrt{\pi}(X+1)}\\ \beta_0 &= 0\\ \beta_n + \beta_{n-1} + \alpha_{n-1}^2 &= \frac{2n-1}{2}\\ \alpha_n\alpha_{n-1}\beta_n &= \left(\frac{n}{2} - \beta_n\right)^2.\end{aligned}$$

Then the polynomials $\{p_n\}$ satisfy the three-term recurrence relation

$$p_0(x) = 1$$

and

$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x).$$

Remark IV.2. By letting

$$\gamma_n = \frac{n}{2} - \beta_n,$$

so that

$$\alpha_n^2 = \gamma_{n+1} + \gamma_n,$$

we can reduce to a single sequence $\{\gamma_n\}$ with three-term recurrence relation

$$0 = (\gamma_{n+1} + \gamma_n)(\gamma_n + \gamma_{n-1})\left(\frac{n}{2} - \gamma_n\right)^2 - \gamma_n^4.$$

These polynomials were studied in [19] by Alexander Its and Igor Krasovsky. In particular, they derive an asymptotic formula for the Hankel determinant using Riemann Hilbert techniques. This allows them to compute certain asymptotics of the polynomials.

Proofs

Proof of Theorem IV.1

We follow the calculation in Theorem 3.1 of [30] and Theorem 3 of [7]. The first step is to express $Z_{L,M}$ as defined in (IV.2) in terms of Pfaffians. Let $\{\rho_n\}$ be any family of monic polynomials where $\deg(\rho_n) = n$. It is a well-known identity that the Vandermonde product in (IV.1)

$$\Delta(\boldsymbol{x}_L \vee -\boldsymbol{x}_L \vee i\boldsymbol{y}_M \vee -i\boldsymbol{y}_M)$$

can be expressed as the determinant of the matrix

$$\begin{aligned}
&\mathbf{V}_{L,M}(\boldsymbol{x}, \boldsymbol{y}) \\
&= \left[\begin{matrix}
\rho_0(x_1) & \rho_0(-x_1) & \cdots & \rho_0(x_L) & \rho_0(-x_L) \\
\rho_1(x_1) & \rho_1(-x_1) & \cdots & \rho_1(x_L) & \rho_1(-x_L) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\rho_{2N-1}(x_1) & \rho_{2N-1}(-x_1) & \cdots & \rho_{2N-1}(x_L) & \rho_{2N-1}(-x_L)
\end{matrix}\right. \\
&\qquad\qquad \left.\begin{matrix}
\rho_0(y_1) & \rho_0(-y_1) & \cdots & \rho_0(y_M) & \rho_0(-y_M) \\
\rho_1(y_1) & \rho_1(-y_1) & \cdots & \rho_1(y_M) & \rho_1(-y_M) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\rho_{2N-1}(y_1) & \rho_{2N-1}(-y_1) & \cdots & \rho_{2N-1}(y_M) & \rho_{2N-1}(-y_M)
\end{matrix}\right].
\end{aligned}$$

We introduce the following notation: $\mathbf{V}_L^{\mathfrak{t}}(\boldsymbol{x}_L)$ is the minor of $\mathbf{V}_{L,M}(\boldsymbol{x}_L)$ consisting of the columns $1, \ldots, 2L$ and rows $\mathfrak{t}(1), \ldots, \mathfrak{t}(2L)$, and $\mathbf{V}_M^{\mathfrak{t}'}(\boldsymbol{y}_M)$ is the minor with columns $2L+1, \ldots, 2N$ and rows $\mathfrak{t}'(1), \ldots, \mathfrak{t}'(2M)$. We expand the determinant using the Laplace expansion to write

$$\det \mathbf{V}(\boldsymbol{x}_L, \boldsymbol{y}_M) = \sum_{\mathfrak{t}:\underline{2L} \nearrow \underline{2N}} \operatorname{sgn} \mathfrak{t} \cdot \det \mathbf{V}_L^{\mathfrak{t}}(\boldsymbol{x}_L) \cdot \det \mathbf{V}_M^{\mathfrak{t}'}(\boldsymbol{y}_M).$$

Then

$$
\begin{aligned}
Z_{L,M} &= \frac{1}{L!M!}\int_{[0,\infty)^L}\int_{[0,\infty)^M} e^{-E(\boldsymbol{x}_L,\boldsymbol{y}_M)}\, d\boldsymbol{x}_L d\boldsymbol{y}_M \\
&= \frac{1}{L!M!}\int_{[0,\infty)^L}\int_{[0,\infty)^M} (-i)^M \det \mathbf{V}_{L,M}(\boldsymbol{x}_L,\boldsymbol{y}_M) \\
&\quad \times \left(\prod_{l=1}^{L} w(x_l)\right)\left(\prod_{m=1}^{M} w(y_m)\right) d\boldsymbol{x}_L d\boldsymbol{y}_M \\
&= \frac{1}{L!M!}\sum_{\mathfrak{t}:\underline{2L}\nearrow\underline{2N}} \operatorname{sgn}\mathfrak{t}\int_{[0,\infty)^L}\int_{[0,\infty)^M} (-i)^M \det \mathbf{V}_L^{\mathfrak{t}}(\boldsymbol{x}_L)\det \mathbf{V}_M^{\mathfrak{t}'}(\boldsymbol{y}_M) \\
&\quad \times \left(\prod_{l=1}^{L} w(x_l)\right)\left(\prod_{m=1}^{M} w(y_m)\right) d\boldsymbol{x}_L d\boldsymbol{y}_M \\
&= \sum_{\mathfrak{t}:\underline{2L}\nearrow\underline{2N}} \operatorname{sgn}\mathfrak{t}\cdot\frac{1}{L!}\int_{[0,\infty)^L} \det \mathbf{V}_L^{\mathfrak{t}}(\boldsymbol{x}_L)\left(\prod_{l=1}^{L} w(x_l)\right) d\boldsymbol{x}_L \\
&\quad \times \frac{1}{M!}\int_{[0,\infty)^M} (-i)^M \det \mathbf{V}_M^{\mathfrak{t}'}(\boldsymbol{y}_M)\left(\prod_{m=1}^{M} w(y_m)\right) d\boldsymbol{y}_M.
\end{aligned}
$$

We again use the Laplace expansion for the two determinants:

$$Z_{L,M} = \sum_{\mathfrak{t}:\underline{2L}\nearrow\underline{2N}} \operatorname{sgn}\mathfrak{t}\cdot\frac{1}{L!}\int_{[0,\infty)^L} \sum_{\substack{\sigma\in S_{2L}\\ \sigma(2l-1)<\sigma(2l)\ \forall l}} \operatorname{sgn}\sigma$$

$$\times\prod_{l=1}^{L}\big(\rho_{\mathfrak{t}(\sigma(2l-1))-1}(x_l)\rho_{\mathfrak{t}(\sigma(2l))-1}(-x_l)-\rho_{\mathfrak{t}(\sigma(2l-1))-1}(-x_l)\rho_{\mathfrak{t}(\sigma(2l))-1}(x_l)\big)\,w(x_l)\,d\boldsymbol{x}_L$$

$$\times\frac{1}{M!}\int_{[0,\infty)^M}\sum_{\substack{\tau\in S_{2M}\\ \tau(2m-1)<\tau(2m)\ \forall m}}\operatorname{sgn}\tau$$

$$\times\prod_{m=1}^{M}(-i)\big(\rho_{\mathfrak{t}'(\tau(2m-1))-1}(y_m)\rho_{\mathfrak{t}'(\tau(2m))-1}(-y_m)$$

$$-\rho_{\mathfrak{t}'(\tau(2m-1))-1}(-y_m)\rho_{\mathfrak{t}'(\tau(2m))-1}(y_m)\big)w(y_m)\,d\boldsymbol{y}_M$$

$$=\sum_{\mathfrak{t}:\underline{2L}\nearrow\underline{2N}} \operatorname{sgn}\mathfrak{t}\cdot\frac{1}{L!}\sum_{\substack{\sigma\in S_{2L}\\ \sigma(2l-1)<\sigma(2l)\ \forall l}} \operatorname{sgn}\sigma$$

$$\times\prod_{l=1}^{L}\int_0^\infty\big(\rho_{\mathfrak{t}(\sigma(2l-1))-1}(x)\rho_{\mathfrak{t}(\sigma(2l))-1}(-x)-\rho_{\mathfrak{t}(\sigma(2l-1))-1}(-x)\rho_{\mathfrak{t}(\sigma(2l))-1}(x)\big)\,w(x)\,dx$$

$$\times\frac{1}{M!}\sum_{\substack{\tau\in S_{2M}\\ \tau(2m-1)<\tau(2m)\ \forall l}}\operatorname{sgn}\tau$$

$$\times\prod_{m=1}^{M}(-i)\int_0^\infty\big(\rho_{\mathfrak{t}'(\tau(2m-1))-1}(y)\rho_{\mathfrak{t}'(\tau(2m))-1}(-y)$$

$$-\rho_{\mathfrak{t}'(\tau(2m-1))-1}(-y)\rho_{\mathfrak{t}'(\tau(2m))-1}(y)\big)w(y)\,d\boldsymbol{y}_M$$

$$=\sum_{\mathfrak{t}:\underline{2L}\nearrow\underline{2N}} \operatorname{sgn}\mathfrak{t}\cdot\frac{1}{2^L L!}\sum_{\sigma\in S_{2L}}\operatorname{sgn}\sigma$$

$$\times\prod_{l=1}^{L}\int_0^\infty\big(\rho_{\mathfrak{t}(\sigma(2l-1))-1}(x)\rho_{\mathfrak{t}(\sigma(2l))-1}(-x)-\rho_{\mathfrak{t}(\sigma(2l-1))-1}(-x)\rho_{\mathfrak{t}(\sigma(2l))-1}(x)\big)\,w(x)\,dx$$

$$\times\frac{1}{2^M M!}\sum_{\tau\in S_{2M}}\operatorname{sgn}\tau$$

$$\times\prod_{m=1}^{M}(-i)\int_0^\infty\big(\rho_{\mathfrak{t}'(\tau(2m-1))-1}(y)\rho_{\mathfrak{t}'(\tau(2m))-1}(-y)$$

$$-\rho_{\mathfrak{t}'(\tau(2m-1))-1}(-y)\rho_{\mathfrak{t}'(\tau(2m))-1}(y)\big)w(y)\,d\boldsymbol{y}_M$$

which is equal to

$$\sum_{\mathfrak{t}:\underline{2L}\nearrow\underline{2N}} \operatorname{sgn}\mathfrak{t}\cdot \operatorname{Pf}\left[\langle \rho_{\mathfrak{t}(n)-1}|\rho_{\mathfrak{t}(m)-1}\rangle_1\right]_{n,m=1}^{2L} \operatorname{Pf}\left[\langle \rho_{\mathfrak{t}'(n)-1}|\rho_{\mathfrak{t}'(m)-1}\rangle_2\right]_{n,m=1}^{2M}.$$

Using the formula for the Pfaffian of the sum of two $2N\times 2N$ anti-symmetric matrices

$$\operatorname{Pf}(A+B)=\sum_{n=0}^{N}\sum_{\mathfrak{t}:\underline{2n}\nearrow\underline{2N}} \operatorname{sgn}\mathfrak{t}\cdot \operatorname{Pf}A_{\mathfrak{t}}\cdot \operatorname{Pf}B_{\mathfrak{t}'}, \tag{IV.18}$$

the weighted sum is

$$\begin{aligned}
Z_N(X) &= \sum_{(L,M)} X^M Z_{L,M}\\
&= \sum_{M=0}^{N} X^M \sum_{\mathfrak{t}:\underline{2L}\nearrow\underline{2N}} \operatorname{sgn}\mathfrak{t}\cdot \operatorname{Pf}\left[\langle \rho_{\mathfrak{t}(n)-1}|\rho_{\mathfrak{t}(m)-1}\rangle_1\right]_{n,m=1}^{2L}\\
&\quad \times \operatorname{Pf}\left[\langle \rho_{\mathfrak{t}'(n)-1}|\rho_{\mathfrak{t}'(m)-1}\rangle_2\right]_{n,m=1}^{2M}\\
&= \operatorname{Pf}\left[\langle \rho_{n-1}|\rho_{m-1}\rangle_1 + X\langle \rho_{n-1}|\rho_{m-1}\rangle_2\right]_{n,m=1}^{2N}.
\end{aligned}$$

Proof of Proposition IV.1

A direct calculation shows that for $m\le 2n+1$,

$$\begin{aligned}
&\langle x^m|p_n(x^2)\rangle_1 + X\langle x^m|p_n(x^2)\rangle_2\\
&= \int_0^\infty (p_n(x^2)x^m - p_n(x^2)(-x)^m)w(x)\,dx\\
&\quad - iX\int_0^\infty (p_n(-x^2)(ix)^m - p_n(-x^2)(-ix)^m)w(x)\,dx\\
&= (1-(-1)^m)\int_0^\infty (p_n(x^2)x^m - iXp_n(-x^2)(ix)^m)w(x)\,dx.
\end{aligned}$$

This is zero unless m is odd. So let $m=2k+1$. Then the above expression is equal to

$$2\int_0^\infty \left(p_n(x^2)x^{2k+1} - (-1)^{k+1}Xp_n(-x^2)x^{2k+1}\right) w(x)\,dx.$$

A change of variables gives

$$\int_0^\infty \left(p_n(x)x^k + Xp_n(-x)(-x)^k\right) w(\sqrt{x})\,dx = \langle p_n|x^k\rangle.$$

This is zero unless $k = n$ in which case $m = 2n+1$. Finally, for the odd degree polynomials,

$$\begin{aligned}
&\langle xp_n(x^2)|p_m(x^2)\rangle_1 + X\langle xp_n(x^2)|p_m(x^2)\rangle_2 \\
&\quad = 2\int_0^\infty \left(xp_n(x^2)p_m(x^2) + Xxp_n(-x^2)p_m(-x^2)\right) w(\sqrt{x})\,dx \\
&\quad = \int_0^\infty (p_n(x)p_m(x) + Xp_n(-x)p_m(-x))\, w(\sqrt{x})\,dx \\
&\quad = \langle p_n|p_m\rangle \\
&\quad = r_n(X)\delta_{n,m}.
\end{aligned}$$

This proves the proposition.

Proof of Corollary IV.3

Let

$$X_k = \text{Bernoulli}(\lambda_k) - \lambda_k$$

and define σ_k and s_n to be the nonnegative numbers such that

$$\sigma_k^2 = \mathbb{E}[X_k^2] = \lambda_k(1-\lambda_k) \quad \text{and} \quad s_n^2 = \sum_{k=1}^n \sigma_k^2.$$

Let $\mathbb{P}$ be the probability measure on the measure space of the random variable X. The Lindeberg condition (Theorem 27.2 in [5]) states that for all $\varepsilon > 0$, if

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{k=1}^{n} \int_{|X_k| \geq \varepsilon s_n} X_k^2 \, d\mathbb{P} = 0,$$

then

$$\frac{1}{s_n} \sum_{k=1}^{n} X_k$$

converges weakly to the standard normal distribution. If

$$0 < \liminf_{k\to\infty} \lambda_k \leq \limsup_{k\to\infty} \lambda_k < 1,$$

then there exists $\delta > 0$ and K_1 sufficiently large such that $\delta < \lambda_k < 1 - \delta$ when $k \geq K_1$. For such k, $\lambda_k(1-\lambda_k) > \delta^2$ which implies that

$$\lim_{n\to\infty} s_n^2 = \infty.$$

Therefore, there exists K_2 depending on ε such that $n \geq K_2$ implies $\varepsilon s_n > 1$. Then because $X_k < 1$ for all k,

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{k=1}^{n} \int_{|X_k| \geq \varepsilon s_n} X_k^2 \, d\mathbb{P} = \sum_{k=1}^{K_2} \int_{|X_k| \geq \varepsilon s_n} X_k^2 \, d\mathbb{P} \cdot \lim_{n\to\infty} \frac{1}{s_n^2} = 0,$$

and the Lindeberg condition is satisfied.

Proof of Theorem IV.1

Formula (IV.9) was proved in Theorem 4 of [7] and Theorem 3.1 of [30]. Here we present the main ideas and leave out most of the details. Let δ

be the Dirac delta function, let $\{a_1, \dots, a_N, b_1, \dots, b_N\}$ be indeterminants, let $\{x_1, \dots, x_N, y_1, \dots, y_N\} \subset [0, \infty)$, and let

$$d\nu_1(x) = dx + \sum_{n=1}^{N} a_n d\delta_{x_n}$$
$$d\nu_2(x) = dx + \sum_{n=1}^{N} b_n d\delta_{y_n}.$$

$$Z_N^{\nu_1,\nu_2}(X) = \sum_{(L,M)} \frac{X^M}{L!M!} \int_{[0,\infty)^L} \int_{[0,\infty)^M} e^{-E(\boldsymbol{x}_L, \boldsymbol{y}_M)} \left(\prod_{l=1}^{L} d\nu_1(x_l) \right) \left(\prod_{m=1}^{M} d\nu_2(y_m) \right)$$

Then

$$\begin{aligned}
\frac{Z_N^{\nu_1,\nu_2}(X)}{Z_N(X)} &= \frac{1}{Z_N(X)} \sum_{(L,M)} \frac{X^M}{L!M!} \int_{[0,\infty)^L} \int_{[0,\infty)^M} e^{-E(\boldsymbol{x}_L, \boldsymbol{y}_M)} \\
&\quad \times \left(\prod_{l=1}^{L} d\nu_1(x_l) \right) \left(\prod_{m=1}^{M} d\nu_2(y_m) \right) \\
&= \frac{1}{Z_N(X)} \sum_{(L,M)} \sum_{l=0}^{L} \sum_{m=0}^{M} \sum_{\mathfrak{u}:\underline{l} \nearrow \underline{L}} \sum_{\mathfrak{v}:\underline{m} \nearrow \underline{M}} \frac{X^M}{L!M!} \left(\prod_{j=1}^{l} a_{\mathfrak{u}(j)} \right) \left(\prod_{k=1}^{m} b_{\mathfrak{v}(k)} \right) \\
&\quad \times \int_{[0,\infty)^{L-l}} \int_{[0,\infty)^{M-m}} e^{-E(\boldsymbol{x}_{\mathfrak{u}} \vee \boldsymbol{u}_{L-l}, \boldsymbol{y}_{\mathfrak{v}} \vee \boldsymbol{v}_{M-m})} \, d\boldsymbol{u}_{L-l} d\boldsymbol{v}_{M-m}.
\end{aligned}$$

Expanding the product of the measures and using the symmetry of the integrand, the above is equal to

$$\frac{1}{Z_N(X)} \sum_{(L,M)} \sum_{l=0}^{L} \sum_{m=0}^{M} \sum_{\mathfrak{u}:\underline{l} \nearrow \underline{L}} \sum_{\mathfrak{v}:\underline{m} \nearrow \underline{M}} \left(\prod_{j=1}^{l} a_{\mathfrak{u}(j)} \right) \left(\prod_{k=1}^{m} b_{\mathfrak{v}(k)} \right) R_{l,m}^N(\boldsymbol{x}_{\mathfrak{u}} \vee \boldsymbol{y}_{\mathfrak{v}}).$$

Therefore $\dfrac{Z_N^{\nu_1,\nu_2}(X)}{Z_N(X)}$ is a generating function for the correlation functions. On the other hand,

$$\begin{aligned}
\langle f|g \rangle_1^{\nu_1} + X \langle f|g \rangle_2^{\nu_2} &= \langle f|g \rangle_1 + X \langle f|g \rangle_2 + \sum_{n=1}^{N} a_n (f(x_n)g(-x_n) - g(x_n)f(-x_n)) w(x_n) \\
&\quad - iX \sum_{n=1}^{N} b_n (f(ix_n)g(-ix_n) - g(ix_n)f(-ix_n)) w(x_n).
\end{aligned}$$

Let $\mathbf{A}$ be the $2N \times 4N$ matrix

$$\Big[\sqrt{a_m}\sqrt{w(x_m)}q_n(x_m) \quad \sqrt{a_m}\sqrt{w(x_m)}q_n(-x_m) \quad \cdots$$
$$\sqrt{b_k}e^{\frac{3\pi i}{4}}X^{\frac{1}{2}}\sqrt{w(x_k)}q_n(ix_k) \quad \sqrt{b_k}e^{\frac{3\pi i}{4}}X^{\frac{1}{2}}\sqrt{w(x_k)}q_n(-ix_k)\Big]$$

for $n = 0, \ldots, 2N-1$ and $m, k = 1, \ldots, N$, let $\mathbf{Z}$ be the $2N \times 2N$ Gram matrix

$$\mathbf{Z} = \begin{bmatrix} 0 & r_0 & & & \\ -r_0 & 0 & & & \\ & & \ddots & & \\ & & & 0 & r_{N-1} \\ & & & -r_{N-1} & 0 \end{bmatrix},$$

and define $\mathbf{J}$ to be the $4N \times 4N$ matrix

$$\mathbf{J} = \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{bmatrix}.$$

For any square matrix $\mathbf{M}$, we denote $\mathbf{M}^T$ as the transpose and $\mathbf{M}^{-T}$ as the inverse transpose of $\mathbf{M}$. Then

$$\frac{Z_N^{\nu_1,\nu_2}(X)}{Z_N(X)} = \frac{\mathrm{Pf}(\mathbf{Z} + \mathbf{A}\mathbf{J}\mathbf{A}^T)}{\mathrm{Pf}(\mathbf{Z})}.$$

Using the the Pfaffian Cauchy-Binet formula (Theorem B.1 in [7])

$$\begin{aligned} &= \frac{\mathrm{Pf}(-\mathbf{J}^{-T} - \mathbf{A}^T\mathbf{Z}^{-T}\mathbf{A})}{\mathrm{Pf}(-\mathbf{J}^{-T})} \\ &= \mathrm{Pf}(\mathbf{J} + \mathbf{A}^T\mathbf{Z}^{-T}\mathbf{A}). \end{aligned} \tag{IV.19}$$

Recalling the four 2×2 matrix kernels as defined in (IV.8), matrix multiplication results in a $2(l+m) \times 2(l+m)$ matrix of 2×2 blocks

$$\mathbf{A}^T\mathbf{Z}^{-T}\mathbf{A} = \begin{bmatrix} \sqrt{a_j a_{j'}}\mathbf{K}_N^{1,1}(x_j, x_{j'}) & \sqrt{a_j b_{k'}}\mathbf{K}_N^{1,2}(x_j, y_{k'}) \\ \sqrt{a_j b_k}\mathbf{K}_N^{2,1}(y_k, x_{j'}) & \sqrt{b_k b_{k'}}\mathbf{K}_N^{2,2}(y_k, y_{k'}) \end{bmatrix}$$

for $j, j' = 1, \ldots, l$ and $k, k' = 1, \ldots, m$, and each of the scalar kernel entries are expressed in terms of

$$S_N(x, y) = \sqrt{w(x)w(y)} \sum_{n=0}^{2N-1} \frac{q_{2n+1}(x)q_{2n}(-y) - q_{2n+1}(-y)q_{2n}(x)}{r_n}.$$

by the relationship (IV.8). Substituting in (IV.6),

$$S_N(x, y) = (x + y)\sqrt{w(x)w(y)} \sum_{n=0}^{N-1} \frac{p_n(x^2)p_n(y^2)}{r_n}.$$

Finally, the expression (IV.7) comes from the Christoffel-Darboux formula. After expanding (IV.19) using (IV.18), we get

$$\begin{aligned}
\frac{Z_N^{\nu_1,\nu_2}(X)}{Z_N(X)} &= \frac{1}{Z_N(X)} \sum_{n=0}^{N} \sum_{\mathfrak{t}:\underline{2n}\nearrow\underline{2N}} \operatorname{sgn} \mathfrak{t} \cdot \operatorname{Pf}(\mathbf{A}^T \mathbf{Z}^{-T} \mathbf{A})_{\mathfrak{t}} \\
&= \frac{1}{Z_N(X)} \sum_{(L,M)} \sum_{l=0}^{L} \sum_{m=0}^{M} \sum_{\mathfrak{u}:\underline{2l}\nearrow\underline{2L}} \sum_{\mathfrak{v}:\underline{2m}\nearrow\underline{2M}} \left(\prod_{j=1}^{2l} a_{\mathfrak{u}(j)} \right) \left(\prod_{k=1}^{2m} b_{\mathfrak{v}(k)} \right) \\
&\quad \times \operatorname{Pf} \mathbf{K}_{l,m}(\boldsymbol{x}_{\mathfrak{u}}, \boldsymbol{y}_{\mathfrak{v}})
\end{aligned}$$

where the matrix $\mathbf{K}_{l,m}$ is given by (IV.8). Finally, matching coefficients gives the result.

Proof of Proposition IV.1

Let

$$\begin{aligned}
f_N(x) &= \frac{a_1 N^{\alpha_1}}{N} K_N \left(a_1 N^{\alpha_1} x, a_1 N^{\alpha_1} x\right) \quad \text{for} \quad x \in \mathbb{R} \\
g_N(x) &= \frac{\sqrt{a_1 N^{\alpha_1}}}{N} S_N \left(\sqrt{a_1 N^{\alpha_1}} x, \sqrt{a_1 N^{\alpha_1}} x\right) \quad \text{for} \quad x \geq 0.
\end{aligned}$$

Note that

$$g_N(x) = 2x f_N(x^2) \quad \text{for} \quad x \geq 0$$

$$-iX \cdot g_N(ix) = 2x f_N(-x^2) \quad \text{for} \quad x \geq 0.$$

Let V_N a random variable with density f_N and define

$$V_N^+ = \max\{V_N, 0\} \quad \text{and} \quad V_N^- = \min\{V_N, 0\}.$$

with be the portion of the random variable such that the range of V_N is the nonnegative reals. We claim that $\sqrt{V_N^+}$ has density $2xf_N(x^2)$ for $x \geq 0$ (if V_N^+ has range $[0, s]$, then $\sqrt{V_N^+}$ has range $[0, \sqrt{s}]$). To see this, for $a > 0$,

$$\begin{aligned} \mathbb{P}\left\{\sqrt{V_N^+} < a\right\} &= \mathbb{P}\{V_N^+ < a^2\} \\ &= \int_0^{a^2} f_N(x)\,dx \\ &= 2\int_0^a x f_N(x^2)\,dx. \end{aligned}$$

Now, an equivalent statement for weak convergence is that for any continuous function h,

$$\int h(V_N^+)\,d\mathbb{P} \to \int h(V^+)\,d\mathbb{P}.$$

By setting $h(x) = \sqrt{x}$ and letting g be any continuous function, we calculate

$$\begin{aligned} \int g(\sqrt{V_N^+})\,d\mathbb{P} &= \int (g \circ h)(V_N^+)\,d\mathbb{P} \\ &\to \int (g \circ h)(V^+)\,d\mathbb{P} \\ &= \int g(\sqrt{V^+})\,d\mathbb{P} \end{aligned}$$

and conclude that $\sqrt{V_N^+}$ converges weakly to $\sqrt{V^+}$. Finally, $\sqrt{V^+}$ has density $2xf(x^2)\big|_{x\geq 0}$, which is (IV.13). Likewise, the same calculations show that $\sqrt{-V_N^-}$ converges weakly to the random variable whose density is $2xf(-x^2)$ for $x \geq 0$ to prove (IV.14).

Proof of Theorem IV.1

Note that

$$\sqrt{a_1 N^{\alpha_1} + \frac{x}{a_2 N^{\alpha_2}}} + O\left(\frac{1}{N^{\frac{3}{2}\alpha_1 + 2\alpha_2}}\right) = \sqrt{a_1 N^{\alpha_1}} + \frac{x}{2\sqrt{a_1} a_2 N^{\frac{\alpha_1}{2}+\alpha_2}}.$$

Let

$$\begin{aligned} x_N &= \sqrt{a_1 N^{\alpha_1}} + \frac{x}{2\sqrt{a_1} a_2 N^{\frac{\alpha_1}{2}+\alpha_2}} \\ y_N &= \sqrt{a_1 N^{\alpha_1}} + \frac{y}{2\sqrt{a_1} a_2 N^{\frac{\alpha_1}{2}+\alpha_2}}. \end{aligned}$$

Then

$$\begin{aligned} &\frac{1}{2\sqrt{a_1} a_2 N^{\frac{\alpha_1}{2}+\alpha_2}} S_N\left(x_N, y_N\right) \\ &\quad = \frac{1}{2\sqrt{a_1} a_2 N^{\frac{\alpha_1}{2}+\alpha_2}} \left(2\sqrt{a_1 N^{\alpha_1}} + \frac{x+y}{2\sqrt{a_1} a_2 N^{\frac{\alpha_1}{2}+\alpha_2}}\right) \\ &\quad\quad \times K_N\left(\left(\sqrt{a_1 N^{\alpha_1}} + \frac{x}{2\sqrt{a_1} a_2 N^{\frac{\alpha_1}{2}+\alpha_2}}\right)^2, \left(\sqrt{a_1 N^{\alpha_1}} + \frac{y}{2\sqrt{a_1} a_2 N^{\frac{\alpha_1}{2}+\alpha_2}}\right)^2\right) \\ &\quad = \frac{1}{a_2 N^{\alpha_2}} K_N\left(\left(\sqrt{a_1 N^{\alpha_1} + \frac{x}{a_2 N^{\alpha_2}}} + O(N^{-\frac{3}{2}\alpha_1 - 2\alpha_2})\right)^2,\right. \\ &\quad\quad\quad\quad \left.\left(\sqrt{a_1 N^{\alpha_1} + \frac{y}{a_2 N^{\alpha_2}}} + O(N^{-\frac{3}{2}\alpha_1 - 2\alpha_2})\right)^2\right) \\ &\quad\quad + O(N^{-\alpha_1 - 2\alpha_2}) \\ &\quad = \frac{1}{a_2 N^{\alpha_2}} K_N\left(a_1 N^{\alpha_1} + \frac{x}{a_2 N^{\alpha_2}} + O(N^{-\alpha_1 - 2\alpha_2}), a_1 N^{\alpha_1} + \frac{y}{a_2 N^{\alpha_2}} + O(N^{-\alpha_1 - 2\alpha_2})\right) \\ &\quad\quad + O(N^{-\alpha_1 - 2\alpha_2}). \end{aligned}$$

Let

$$h_N(x,y) = \frac{1}{a_2 N^{\alpha_2}} K_N \left(a_1 N^{\alpha_1} + \frac{x}{a_2 N^{\alpha_2}}, a_1 N^{\alpha_1} + \frac{y}{a_2 N^{\alpha_2}} \right).$$

Then h_N converges uniformly to A, and because A is continuous,

$$\begin{aligned}
&\frac{1}{a_2 N^{\alpha_2}} K_N \left(a_1 N^{\alpha_1} + \frac{x}{a_2 N^{\alpha_2}} + O(N^{-\alpha_1 - 2\alpha_2}), a_1 N^{\alpha_1} + \frac{y}{a_2 N^{\alpha_2}} + O(N^{-\alpha_1 - 2\alpha_2}) \right) \\
&\qquad + O(N^{-\alpha_1 - 2\alpha_2}) \\
&= h_N \left(x + O(N^{-\alpha_1 - \alpha_2}), y + O(N^{-\alpha_1 - \alpha_2}) \right) + O(N^{-\alpha_1 - 2\alpha_2}) \\
&\to A(x,y).
\end{aligned}$$

Next, we show the scalar kernels which correspond to particle interactions from different rays converge to 0. We prove this for the kernel $S_N(x, -y)$ only, since the proofs of the other kernels are the same. Now

$$\begin{aligned}
&\frac{1}{2\sqrt{a_1} a_2 N^{\frac{\alpha_1}{2} + \alpha_2}} S_N(x_N, -y_N) \\
&\quad = \frac{1}{2\sqrt{a_1} a_2 N^{\frac{\alpha_1}{2} + \alpha_2}} \sqrt{w(x_N) w(y_N)} \frac{p_N(x_N^2) p_{N-1}(y_N^2) - p_N(y_N^2) p_{N-1}(x_N^2)}{r_{N-1}(x_N + y_N)}
\end{aligned}$$

The denominator is simplified as

$$x_N + y_N = 2\sqrt{a_1 N^{\alpha_1}} + \frac{x+y}{2\sqrt{a_1} a_2 N^{\frac{\alpha_1}{2} + \alpha_2}}.$$

In contrast, the higher order term $\sqrt{a_1 N^{\alpha_1}}$ in the denominator of $S_N(x_N, y_N)$ cancels and simplifies to

$$x_N - y_N = \frac{x-y}{2\sqrt{a_1} a_2 N^{\frac{\alpha_1}{2} + \alpha_2}}.$$

Other than the denominators, the two scaled kernels are identical. Therefore, because $S_N(x, y)$ converges and $S_N(x, -y)$ has a higher order of N in the denominator, $S_N(x, -y)$ will converge to 0 in the limit.

All of the other scalar kernels of $\mathbf{K}_N^{1,2}$ and $\mathbf{K}_N^{2,1}$ also fail to eliminate the $\sqrt{a_1 N^{\alpha_1}}$ factor, so they converge to 0 in the scaled limit.

Proof of Theorem IV.2

Recall the radial external field Q in the complex plane introduced in the Setup. Suppose the particle support is $\{z \in \mathbb{C} : |z| \leq c\}$ for $c \in (0, \infty]$. Now, in the orthogonal polynomial ensemble, the external field Q_X (given by the relation $w_X = e^{-Q_X}$) can be expressed in terms of Q and is dependent on X. This relation is

$$Q_X(x) = \begin{cases} Q(\sqrt{x}) & \text{if } x \geq 0 \\ Q(\sqrt{-x}) - \log X & \text{if } x < 0. \end{cases}$$

We can decompose $Q_X(x)$ as

$$Q_X(x) = Q_1(x) + \begin{cases} -\log X & \text{if } -c \leq x \leq 0 \\ 0 & \text{if } 0 < x \leq c \\ \infty & \text{otherwise.} \end{cases}$$

Let

$$d\mu_{\boldsymbol{x}_N} = \frac{1}{N} \sum_{n=1}^{N} \delta_{x_n}.$$

The probability density of points is

$$
\begin{aligned}
P_N(\boldsymbol{x}_N) &= \frac{1}{Z_N} \cdot \prod_{n<m} (x_n - x_m)^2 \cdot \exp\left(-\sum_{n=1}^{N} Q_X(x_n) \right) \\
&= \frac{1}{Z_N} \exp\left(-\sum_{n=1}^{N} Q_X(x_n) - \sum_{n \neq m} \log |x_n - x_m|^{-1} \right) \\
&= \frac{1}{Z_N} \exp\Big(-N^2 \Big(\frac{1}{N} \int_{-c}^{c} Q_1(s)\, d\mu_{\boldsymbol{x}_N}(s) - \frac{1}{N} \int_{-c}^{0} \log X \, d\mu_{\boldsymbol{x}_N}(s) \\
&\quad + \int_{-c}^{c} \int_{-c}^{c} \log |s-t|^{-1}\, d\mu_{\boldsymbol{x}_N}(s) d\mu_{\boldsymbol{x}_N}(t) \Big) \Big).
\end{aligned}
$$

In the limit, the external potential will disappear. We must scale the location of the particles by $x \mapsto a_1 N^{\alpha_1} x$ (this is the same scaling as from (IV.13)) so that

$$
Q_1(a_1 N^{\alpha_1} s) \sim N Q_1(s) \tag{IV.20}
$$

Then the energies between the external field and the particle interaction will be on the same order. With this scaling we have

$$
\begin{aligned}
P_N(\boldsymbol{x}_N) = \frac{1}{Z_N} \exp\Big(-N^2 \Big(\int_{-c}^{c} Q_1(s)\, d\mu_{\boldsymbol{x}_N}(s) - \frac{1}{N} \int_{-c}^{0} \log X \, d\mu_{\boldsymbol{x}_N}(s) \\
+ \int_{-c}^{c} \int_{-c}^{c} \log |s-t|^{-1}\, d\mu_{\boldsymbol{x}_N}(s) d\mu_{\boldsymbol{x}_N}(t) \Big) \Big).
\end{aligned}
$$

Now for $0 < X < \infty$, the "jump" portion of the external potential is at a lower order than the portion from the remaining energy, so this part will disappear in the limit. A physical system generally tends to a state which minimizes energy. This leads to the energy minimizing problem, which is the unique solution (called the *equilibrium measure*) to

$$
\inf \left\{ \int_{-c}^{c} Q_1(s)\, d\mu(s) + \int_{-c}^{c} \int_{-c}^{c} \log |t-s|^{-1}\, d\mu(t) d\mu(s) \right\}
$$

where the infimum is taken over all Borel probability measures μ.

Proof of Proposition IV.1

We follow the proof in [4]. Let

$$r_n = \langle p_n | p_n \rangle$$

$$s_n = \langle x p_n | p_n \rangle.$$

Then the orthogonality relations combined with the three-term relation give

$$\alpha_n = \frac{s_n}{r_n}$$
$$\beta_n = \frac{r_n}{r_{n-1}}.$$

Using

$$r_n = \langle x p_{n-1} | p_n \rangle,$$

we use the three-term relation to obtain

$$\beta_n + \beta_{n-1} + \alpha_{n-1}^2 = \frac{1}{r_{n-1}} \langle x^2 p_{n-1} | p_{n-1} \rangle$$
$$\alpha_n + \alpha_{n-1} = \frac{1}{r_n} \langle x^2 p_{n-1} | p_n \rangle.$$

Now using integration by parts,

$$\begin{aligned}
\langle x^2 p_{n-1} | p_{n-1} \rangle &= \int_0^\infty x^2 \left((p_{n-1}(x))^2 + X(p_{n-1}(-x))^2 \right) e^{-x^2} dx \\
&= -\frac{1}{2} \int_0^\infty x \left((p_{n-1}(x))^2 + X(p_{n-1}(-x))^2 \right) \frac{d}{dx} \left(e^{-x^2} \right) dx \\
&= \frac{1}{2} \int_0^\infty \frac{d}{dx} \left(x \left((p_{n-1}(x))^2 + X(p_{n-1}(-x))^2 \right) \right) e^{-x^2} dx \\
&= \frac{1}{2} r_{n-1} + \int_0^\infty \left(x p_{n-1}(x) p'_{n-1}(x) + X(-x) p_{n-1}(-x) p'_{n-1}(-x) \right) e^{-x^2} \, dx \\
&= \frac{1}{2} r_{n-1} + \langle p_{n-1} | x p'_{n-1} \rangle \\
&= \frac{2n-1}{2} r_{n-1}
\end{aligned}$$

which yields

$$\beta_n + \beta_{n-1} + \alpha_{n-1}^2 = \frac{2n-1}{2}.$$

Again, integration by parts gives

$$\begin{aligned}
\langle x^2 p_{n-1} | p_n \rangle &= \int_0^\infty x^2 \left(p_n(x) p_{n-1}(x) + X p_n(-x) p_{n-1}(-x) \right) e^{-x^2} dx \\
&= -\frac{1}{2} \int_0^\infty x \left((p_n(x) p_{n-1}(x) + X p_n(-x) p_{n-1}(-x) \right) \frac{d}{dx} \left(e^{-x^2} \right) dx \\
&= \frac{1}{2} \int_0^\infty \frac{d}{dx} \left(x \left((p_n(x) p_{n-1}(x) + X p_n(-x) p_{n-1}(-x) \right) \right) e^{-x^2} dx \\
&= \frac{1}{2} \int_0^\infty \left(x p'_n(x) p_{n-1}(x) - X x p'_n(-x) p_{n-1}(x) \right) e^{-x^2} dx \\
&= \frac{1}{2} \langle x p_{n-1} | p'_n \rangle.
\end{aligned}$$

Now, using

$$x p_{n-1}(x) = p_n(x) + \alpha_{n-1} p_{n-1}(x) + \beta_{n-1} p_{n-2}(x)$$

$$p'_n(x) = (x - \alpha_{n-1}) p'_{n-1}(x) + p_{n-1}(x) - \beta_{n-1} p'_{n-2}(x)$$

and

$$\frac{1}{2}\langle xp_{n-2}|p'_{n-1}\rangle = r_{n-1}(\alpha_{n-1}+\alpha_{n-2}),$$

we substitute and simply:

$$\frac{1}{2}\langle xp_{n-1}|p'_{n}\rangle = \frac{1}{2}\alpha_{n-1}r_{n-1} + \beta_{n-1}r_{n-1}(\alpha_{n-1}+\alpha_{n-2}).$$

This calculation gives us the second relation

$$\beta_n(\alpha_n+\alpha_{n-1}) = \frac{1}{2}\alpha_{n-1} + \beta_{n-1}(\alpha_{n-1}+\alpha_{n-2}).$$

Multiplying both sides by α_{n-1} and substituting the first recursion relation for α_{n-1}^2 gives

$$\alpha_n\alpha_{n-1}\beta_n = \left(\frac{n}{2}-\beta_n\right)^2.$$

CHAPTER V

REFLECTED PARTICLES ON THE UNIT CIRCLE

Fix $\alpha, \gamma > -1$ and let N be a nonnegative integer. Consider the ensemble of N conjugate pairs of charge-one particles interacting logarithmically on the unit circle with Jacobi potential

$$Q(\theta) = -\alpha \log\left(2\sin^2\frac{\theta}{2}\right) - \gamma \log\left(2\cos^2\frac{\theta}{2}\right) \quad \text{for} \quad 0 \leq \theta \leq 2\pi.$$

We study the point process of the N points $\{e^{i\theta_1}, \ldots, e^{i\theta_N}\}$ on the upper half of the unit circle ($0 \leq \theta_j \leq \pi$). We get a Pfaffian point process, and the skew-orthogonal polynomials can be expressed in terms of the Jacobi polynomials.

This ensemble is similar to the reflection ensemble studied in Chapter IV in the situation where all the particles lie on one of the axes (when the fugacity $X = 0$ or $X = \infty$). In that case, N particles live on the half line with N corresponding particles reflected about the origin. Compare this model magnified at the origin to our current ensemble magnified at $\theta = 0$ or $\theta = \pi$ so that the circle locally looks like its tangent lines. Because these two locations are also points of reflection, the two particle models are locally identical. Therefore, we should expect the local statistics in these regions to be the same, hence it would not be surprising if the limiting kernels are akin.

The total potential energy of the system is

$$E(\boldsymbol{\theta}_N) = -\sum_{n>m}^{N} \left(\log\left|e^{i\theta_n} - e^{i\theta_m}\right| + \log\left|e^{-i\theta_n} - e^{-i\theta_m}\right|\right) - \sum_{n,m=1}^{N} \log\left|e^{i\theta_n} - e^{-i\theta_m}\right|$$
$$+ \sum_{n=1}^{N} Q(\theta_n).$$

The external field induces the weight

$$w^{\alpha,\gamma}(\theta) = \left(2\sin^2\frac{\theta}{2}\right)^{\alpha}\left(2\cos^2\frac{\theta}{2}\right)^{\gamma} \quad \text{for} \quad 0 \le \theta \le 2\pi,$$

so the Boltzman factor with inverse temperature equal to one is

$$\begin{aligned} e^{-E(\boldsymbol{\theta}_N)} &= \left(\prod_{n>m}^{N} \left|e^{i\theta_n} - e^{i\theta_m}\right|\left|e^{-i\theta_n} - e^{-i\theta_m}\right|\right)\left(\prod_{n,m=1}^{N} \left|e^{i\theta_n} - e^{-i\theta_m}\right|\right)\left(\prod_{n=1}^{N} w^{\alpha,\gamma}(\theta_n)\right) \\ &= \prod_{n>m}^{N} \left(e^{i\theta_n} - e^{i\theta_m}\right)\left(e^{-i\theta_n} - e^{-i\theta_m}\right)\left(e^{i\theta_n} - e^{-i\theta_m}\right)\left(e^{-i\theta_n} - e^{i\theta_m}\right) \\ &\quad \times \left(\prod_{n=1}^{N} i\left(e^{-i\theta_n} - e^{i\theta_n}\right)\right)\left(\prod_{n=1}^{N} w^{\alpha,\gamma}(\theta_n)\right). \end{aligned}$$

The Partition Function

We follow the standard steps in studying a Pfaffian point process. We first define $\langle\cdot|\cdot\rangle^{\alpha,\gamma}$ to be the skew-inner product given by

$$\langle\rho_n|\rho_m\rangle^{\alpha,\gamma} = i\int_0^{\pi} \left(\rho_n(e^{i\theta})\rho_m(e^{-i\theta}) - \rho_n(e^{-i\theta})\rho_m(e^{i\theta})\right) w^{\alpha,\gamma}(\theta)\, d\theta \tag{V.1}$$

and $\{\rho_n\}$ is any family of monic polynomials such that $\deg(\rho_n) = n$.

Theorem V.1. *The partition function can be expressed as a Pfaffian of a* $2N \times 2N$ *antisymmetric matrix:*

$$Z_{N,\alpha,\gamma} = \mathrm{Pf}\left[\langle \rho_n | \rho_m \rangle^{\alpha,\gamma}\right]_{n,m=0}^{2N-1}.$$

Skew-Orthogonal Polynomials

For

$$r_n^{\alpha,\gamma} = 2 \cdot \frac{2^{\alpha+\gamma+1}}{2n+\alpha+\gamma+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\gamma+1)}{\Gamma(n+\alpha+\gamma+1)n!}, \tag{V.2}$$

let $P_n^{(\alpha,\gamma)}$ be the nth degree Jacobi polynomial with parameters α and γ, i.e. the orthogonal polynomials which satisfy the relations

$$\int_0^\pi P_n^{(\alpha,\gamma)}(\cos\theta) P_m^{(\alpha,\gamma)}(\cos\theta) \sin\theta \cdot w^{\alpha,\gamma}(\theta)\, d\theta = \delta_{n,m} \frac{r_n^{\alpha,\gamma}}{2}.$$

Theorem V.1. *The monic skew-orthogonal polynomials with respect to the skew-inner product (V.1) are defined by*

$$q_{2n}^{\alpha,\gamma}(\theta) = e^{in\theta} P_n^{(\alpha,\gamma)}(\cos\theta)$$

$$q_{2n+1}^{\alpha,\gamma}(\theta) = e^{-i\theta}(q_{2n+2}^{\alpha,\gamma}(e^{i\theta}) - q_{2n}^{\alpha,\gamma}(e^{i\theta})),$$

with normalization $r_n^{\alpha,\gamma}$ *(V.2), i.e.*

$$r_n^{\alpha,\gamma} = \langle q_{2n}^{\alpha,\gamma} | q_{2n+1}^{\alpha,\gamma} \rangle^{\alpha,\gamma}.$$

From here on out, we fix α and γ and omit this dependence in the notation.

Correlation Functions and Kernels

Corollary V.1. *The correlation function R_n^N can be expressed as the Pfaffian of a $2n \times 2n$ antisymmetric matrix consisting of 2×2 matrix block kernels: For*

$$S_N(\theta, \phi) = i\sqrt{w(\theta)w(\phi)} \sum_{n=0}^{N-1} \frac{q_{2n}(e^{i\theta})q_{2n+1}(e^{-i\phi}) - q_{2n+1}(e^{i\theta})q_{2n}(e^{-i\phi})}{r_n}$$

$$DS_N(\theta, \phi) = S_N(\theta, -\phi)$$

$$IS_N(\theta, \phi) = S_N(-\theta, \phi)$$

and

$$\boldsymbol{K}_N(\theta, \phi) = \begin{bmatrix} DS_N(\theta, \phi) & S_N(\theta, \phi) \\ -S_N(\phi, \theta) & IS_N(\theta, \phi) \end{bmatrix},$$

we have

$$R_n^N(\boldsymbol{\theta}_n) = \mathrm{Pf}\left[\boldsymbol{K}_N(\theta_k, \theta_l)\right]_{k,l=1}^{n} \tag{V.3}$$

Proof of Corollary V.1. The proof follows from the proof of Theorem IV.1 with $X = 0$. In essence, by manipulating the weight, the partition function becomes a multivariate polynomial whose coefficients are the correlation functions. □

Limiting Density and Kernels

Unlike the ensemble studied in Chapter IV, the kernels can not be reduced to a Christoffel-Darboux kernel. While in the previous section we used results from orthogonal polynomial ensemble theory to help us with the asymptotic calculations, for this model we must directly calculate the asymptotic density and kernels.

Theorem V.1. *The density of particles converges to the uniform distribution.*

Remark V.2. Despite the repulsion at the edges $\theta = 0$ and $\theta = \pi$, the push from the bulk is greater and evenly distributes the particles on the circle.

Theorem V.3. *Rescaling the locations of the particles by* $\theta \mapsto \dfrac{\theta}{N} + \dfrac{\pi}{2}$*, the bulk kernels at* $\dfrac{\pi}{2}$ *are*

$$\begin{aligned}
\lim_{N\to\infty} \frac{i}{N} S_N\left(\frac{\theta}{N} + \frac{\pi}{2}, \frac{\phi}{N} + \frac{\pi}{2}\right) &= e^{i(\theta-\phi)}\frac{\sin(\theta-\phi)}{\pi(\theta-\phi)} \\
\lim_{N\to\infty} \frac{i}{N} DS_N\left(\frac{\theta}{N} + \frac{\pi}{2}, \frac{\phi}{N} + \frac{\pi}{2}\right) &= 0 \\
\lim_{N\to\infty} \frac{i}{N} IS_N\left(\frac{\theta}{N} + \frac{\pi}{2}, \frac{\phi}{N} + \frac{\pi}{2}\right) &= 0.
\end{aligned}$$

Theorem V.4. *Rescaling the locations of the particles by* $\theta \mapsto \dfrac{\theta}{N}$*, the edge kernels at* 0 *are*

$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{N} S_N\left(\frac{\theta}{N}, \frac{\phi}{N}\right) &= \frac{\theta+\phi}{2}\int_0^1 te^{it(\theta-\phi)} J_\alpha(t\theta)J_\alpha(t\phi)dt \\
&\quad + \frac{i}{2}\int_0^1 te^{it(\theta-\phi)}(\phi J_\alpha(t\theta)J_\alpha'(t\phi) - \theta J_\alpha(t\phi)J_\alpha'(t\theta))dt \\
\lim_{N\to\infty} \frac{1}{N} DS_N\left(\frac{\theta}{N}, \frac{\phi}{N}\right) &= \frac{\theta-\phi}{2}\int_0^1 te^{it(\theta+\phi)} J_\alpha(t\theta)J_\alpha(t\phi)dt \\
&\quad + \frac{i}{2}\int_0^1 te^{it(\theta+\phi)}(\phi J_\alpha(t\theta)J_\alpha'(t\phi) - \theta J_\alpha(t\phi)J_\alpha'(t\theta))dt \\
\lim_{N\to\infty} \frac{1}{N} IS_N\left(\frac{\theta}{N}, \frac{\phi}{N}\right) &= \frac{\phi-\theta}{2}\int_0^1 te^{-it(\theta+\phi)} J_\alpha(t\theta)J_\alpha(t\phi)dt \\
&\quad + \frac{i}{2}\int_0^1 te^{-it(\theta+\phi)}(\phi J_\alpha(t\theta)J_\alpha'(t\phi) - \theta J_\alpha(t\phi)J_\alpha'(t\theta))dt.
\end{aligned}$$

Corollary V.5. *The kernels at the edge can be rewritten as*

$$\begin{aligned}
S(\theta,\phi) &= e^{i(\theta-\phi)}\frac{\phi J_\alpha(\theta)J'_\alpha(\phi)-\theta J_\alpha(\phi)J'_\alpha(\theta)}{2(\theta-\phi)}\\
DS(\theta,\phi) &= e^{i(\theta+\phi)}\frac{\phi J_\alpha(\theta)J'_\alpha(\phi)-\theta J_\alpha(\phi)J'_\alpha(\theta)}{2(\theta+\phi)}\\
IS(\theta,\phi) &= -e^{-i(\theta+\phi)}\frac{\phi J_\alpha(\theta)J'_\alpha(\phi)-\theta J_\alpha(\phi)J'_\alpha(\theta)}{2(\theta+\phi)}.
\end{aligned}$$

Compare these kernels to (IV.17).

Proofs

Recall that we fix α and γ and omit them from the notation. Let $P_n^{(\alpha,\gamma)} = P_n$.

Proof of Theorem V.1

This proof is essentially the same as the proof of Theorem IV.1 with the simplification $X = 0$. In summary, we first write the Boltzman factor as

$$\begin{aligned}
e^{-E(\boldsymbol{\theta}_N)} &= \left(\prod_{n=1}^{N} i\cdot w(\theta_n)\right)\\
&\quad\times\det\begin{bmatrix}
\rho_0(e^{i\theta_1}) & \rho_0(e^{-i\theta_1}) & \cdots & \rho_0(e^{i\theta_N}) & \rho_0(e^{-i\theta_N})\\
\rho_1(e^{i\theta_1}) & \rho_1(e^{-i\theta_1}) & \cdots & \rho_1(e^{i\theta_N}) & \rho_1(e^{-i\theta_N})\\
\vdots & \vdots & & \vdots & \vdots\\
\rho_{2N-1}(e^{i\theta_1}) & \rho_{2N-1}(e^{-i\theta_1}) & \cdots & \rho_{2N-1}(e^{i\theta_N}) & \rho_{2N-1}(e^{-i\theta_N})
\end{bmatrix}.
\end{aligned}$$

Recall $\langle\cdot|\cdot\rangle$ is the skew inner product given by

$$\langle\rho_n|\rho_m\rangle = i\int_0^\pi \big(\rho_n(e^{i\theta})\rho_m(e^{-i\theta}) - \rho_n(e^{-i\theta})\rho_m(e^{i\theta})\big)\, w(\theta)\, d\theta.$$

Then, using Laplace's expansion and Fubini's theorem,

$$\begin{aligned}
Z_N &= \frac{1}{N!} \sum_{\substack{\sigma \in S_{2N} \\ \sigma(2n-1)<\sigma(2n)\,\forall n}} \operatorname{sgn}\sigma \\
&\quad \times \prod_{n=1}^{N} i \int_0^\pi \left(\rho_{\sigma(2n-1)-1}(e^{i\theta})\rho_{\sigma(2n)-1}(e^{-i\theta}) - \rho_{\sigma(2n-1)-1}(e^{-i\theta})\rho_{\sigma(2n)-1}(e^{i\theta})\right) w(\theta)\, d\theta \\
&= \frac{1}{2^N N!} \sum_{\sigma \in S_{2N}} \operatorname{sgn}\sigma \\
&\quad \times \prod_{n=1}^{N} i \int_0^\pi \left(\rho_{\sigma(2n-1)-1}(e^{i\theta})\rho_{\sigma(2n)-1}(e^{-i\theta}) - \rho_{\sigma(2n-1)-1}(e^{-i\theta})\rho_{\sigma(2n)-1}(e^{i\theta})\right) w(\theta)\, d\theta \\
&= \operatorname{Pf}\left[\langle \rho_n | \rho_m \rangle\right]_{n,m=0}^{2N-1}.
\end{aligned}$$

Proof of Theorem V.1

Let $2n+1 \geq m$. Then

$$\begin{aligned}
\langle e^{in\theta} P_n(\cos\theta) | e^{im\theta} \rangle &= i \int_0^\pi P_n(\cos\theta)(e^{i(m-n)\theta} - e^{-i(m-n)\theta}) w(\theta)\, d\theta \\
&= 2 \int_0^\pi P_n(\cos\theta) \frac{\sin(m-n)\theta}{\sin\theta} \sin\theta\, w(\theta)\, d\theta.
\end{aligned}$$

Let $U_n(x)$ be the Chebyshev polynomial of the second kind of degree n. Now

$$\frac{\sin(m-n)\theta}{\sin\theta} = \begin{cases} U_{m-n-1}(\cos\theta) & \text{if } m \geq n+1 \\ 0 & \text{if } m = n \\ -U_{n-m-1}(\cos\theta) & \text{if } m \leq n-1. \end{cases}$$

Thus,

$$\langle e^{in\theta}P_n(\cos\theta)|e^{im\theta}\rangle = \begin{cases} 2\displaystyle\int_{-1}^{1} P_n(\cos\theta)U_{m-n-1}(\cos\theta)\sin\theta\cdot w(\theta)\,d\theta & \text{if } m\geq n+1 \\ 0 & \text{if } n=m \\ -2\displaystyle\int_{-1}^{1} P_n(\cos\theta)U_{n-m-1}(\cos\theta)\sin\theta\cdot w(\theta)\,d\theta & \text{if } m\leq n-1. \end{cases}$$

Now all the cases are equal to zero unless $m = 2n+1$. This proves the construction of the even skew-polynomials. For the odd polynomials, let $m \leq 2n-1$. Then

$$\begin{aligned}\langle e^{-i\theta}(q_{2n+2}(e^{i\theta}) - q_{2n}(e^{i\theta}))|e^{im\theta}\rangle &= \langle (q_{2n+2}(e^{i\theta}) - q_{2n}(e^{i\theta}))|e^{i(m+1)\theta}\rangle \\ &= \langle q_{2n+2}(e^{i\theta})|e^{i(m+1)\theta}\rangle^{\alpha,\gamma} - \langle q_{2n}(e^{i\theta})|e^{i(m+1)\theta}\rangle \\ &= 0.\end{aligned}$$

The normalization of the polynomials are

$$\begin{aligned} r_n &= \langle q_{2n+1}(e^{i\theta})|q_{2n}(e^{i\theta})\rangle \\ &= \langle e^{in\theta}P_n(\cos\theta)|e^{-i\theta}(e^{i(n+1)\theta}P_{n+1}(\cos\theta) - e^{in\theta}P_n(\cos\theta))\rangle \\ &= \langle P_n(\cos\theta)|P_{n+1}(\cos\theta)\rangle^{\alpha,\gamma} - \langle e^{i\theta}P_n(\cos\theta)|P_n(\cos\theta)\rangle \\ &= -i\int_0^{\pi}(P_n(\cos\theta))^2(e^{i\theta} - e^{-i\theta})w(\theta)\,d\theta \\ &= 2\int_0^{\pi}(P_n(\cos\theta))^2\sin\theta\, w(\theta)\,d\theta. \end{aligned}$$

Proof of Theorem V.1

Let

$$a_{N-1} = \frac{(2N+\alpha+\gamma)(2N+\alpha+\gamma-1)}{2N(N+\alpha+\gamma)}$$

be the ratio of the Nth and $(N-1)$th leading coefficient of the Jacobi polynomials.

We have

$$\begin{aligned}\frac{1}{N}S_N(\theta,\theta) &= \frac{iw(\theta)}{N}\sum_{n=0}^{N-1}\frac{q_{2n}(e^{i\theta})q_{2n+1}(e^{-i\theta})-q_{2n+1}(e^{i\theta})q_{2n}(e^{-i\theta})}{r_n}\\ &= \frac{2w(\theta)\sin\theta}{N}\sum_{n=0}^{N-1}\frac{(P_n(\cos\theta))^2}{r_n}.\end{aligned}$$

Using the Christoffel-Darboux formula, the above expression is equal to

$$\frac{2w(\theta)\sin\theta}{N}\cdot\frac{P_N'(\cos\theta)P_{N-1}(\cos\theta)-P_{N-1}'(\cos\theta)P_N(\cos\theta)}{a_{N-1}r_{N-1}}. \tag{V.4}$$

Now (Section 4.5 of [36])

$$P_n'(x) = \frac{-n((2n+\alpha+\gamma)x+\gamma-\alpha)P_n(x)+2(n+\alpha)(n+\gamma)P_{n-1}(x))}{(2n+\alpha+\gamma)(1-x^2)}.$$

Substituting this back into (V.4) and some simplification gives

$$\begin{aligned}\frac{1}{N}S_N(\theta,\theta) = {}& \frac{2w(\theta)}{Na_{N-1}r_{N-1}\sin\theta}\\ &\times\Bigg\{\left(-\cos\theta-\frac{N(\gamma-\alpha)}{2N+\alpha+\gamma}+\frac{(N-1)(\gamma-\alpha)}{2(N-1)+\alpha+\gamma}\right)\\ &\times P_N(\cos\theta)P_{N-1}(\cos\theta)\\ &+\frac{2(N+\alpha)(N+\gamma)}{2N+\alpha+\gamma}(P_{N-1}(\cos\theta))^2\\ &-\frac{2(N+\alpha-1)(N+\gamma-1)}{2(N-1)+\alpha+\gamma}P_N(\cos\theta)P_{N-2}(\cos\theta)\Bigg\}.\end{aligned} \tag{V.5}$$

We break (V.5) down into the two factors. First,

$$\frac{2w(\theta)}{Na_{N-1}r_{N-1}\sin\theta} = \frac{(N+\alpha+\gamma)\Gamma(N)\Gamma(N+\alpha+\gamma)(\sin\frac{\theta}{2})^{2\alpha}(\cos\frac{\theta}{2})^{2\gamma}}{(2N+\alpha+\gamma)\Gamma(N+\alpha)\Gamma(N+\gamma)\sin\theta}.$$

Now for $\alpha\in\mathbb{R}$,

$$\frac{\Gamma(n+\alpha)}{\Gamma(n)n^\alpha} = 1+O(n^{-1}). \tag{V.6}$$

Thus,

$$\frac{2w(\theta)}{Na_{N-1}r_{N-1}\sin\theta} \to \frac{(\sin\frac{\theta}{2})^{2\alpha}(\cos\frac{\theta}{2})^{2\gamma}}{2\sin\theta}. \tag{V.7}$$

For the second factor of (V.5), we use the asymptotics of the Jacobi polynomial which can be found in Theorem 8.21.8 of [36]: For $0 < \theta < \pi$,

$$P_n(\cos\theta) = \frac{\cos[(n+\frac{1}{2}(\alpha+\gamma+1))\theta - (\alpha+\frac{1}{2})\frac{\pi}{2}]}{\sqrt{\pi n}(\sin\frac{\theta}{2})^{\alpha+\frac{1}{2}}(\cos\frac{\theta}{2})^{\gamma+\frac{1}{2}}} + O(n^{-\frac{3}{2}}). \tag{V.8}$$

The second factor of (V.5) consists of three summands. For the first summand, note that

$$-\cos\theta - \frac{N(\gamma-\alpha)}{2N+\alpha+\gamma} + \frac{(N-1)(\gamma-\alpha)}{2(N-1)+\alpha+\gamma} = O(1). \tag{V.9}$$

Therefore, using (V.8) and (V.9),

$$\left(-\cos\theta - \frac{N(\gamma-\alpha)}{2N+\alpha+\gamma} + \frac{(N-1)(\gamma-\alpha)}{2(N-1)+\alpha+\gamma}\right) P_N(\cos\theta)P_{N-1}(\cos\theta) \to 0. \tag{V.10}$$

For the asymptotics of the next two summands, note that

$$\begin{aligned} \frac{2(N+\alpha)(N+\gamma)}{2N+\alpha+\gamma} &= N + O(1) \\ \frac{2(N+\alpha-1)(N+\gamma-1)}{2(N-1)+\alpha+\gamma} &= N + O(1). \end{aligned}$$

Then

$$\begin{aligned} &\frac{2(N+\alpha)(N+\gamma)}{2N+\alpha+\gamma}(P_{N-1}(\cos\theta))^2 - \frac{2(N+\alpha-1)(N+\gamma-1)}{2(N-1)+\alpha+\gamma}P_N(\cos\theta)P_{N-2}(\cos\theta) \\ &\quad \sim N(P_{N-1}(\cos\theta))^2 - P_N(\cos\theta)P_{N-2}(\cos\theta) \end{aligned}$$

Now with

$$c_n = n + \frac{1}{2}(\alpha + \gamma + 1)$$
$$d = \left(\alpha + \frac{1}{2}\right)\frac{\pi}{2}, \tag{V.11}$$

and using (V.8),

$$
\begin{aligned}
&(P_{N-1}(\cos\theta))^2 - P_N(\cos\theta)P_{N-2}(\cos\theta) \\
&\quad = \frac{1}{\pi(\sin\frac{\theta}{2})^{2\alpha+1}(\cos\frac{\theta}{2})^{2\gamma+1}} \\
&\qquad \times \left(\frac{\cos^2[c_{N-1}\cdot\theta - d]}{N-1} - \frac{\cos[c_N\cdot\theta - d]\cos[c_{N-2}\cdot\theta - d]}{\sqrt{N(N-2)}} + O(N^{-\frac{3}{2}})\right).
\end{aligned}
$$

Now

$$\frac{\cos^2[c_{N-1}\cdot\theta - d]}{N-1} = \frac{\cos^2[c_{N-1}\cdot\theta - d]}{N} + \frac{\cos^2[c_{N-1}\cdot\theta - d]}{N^2 - N}$$

and with

$$\sqrt{x(x-2)} = x + O(1),$$

then

$$\frac{\cos[c_N\cdot\theta - d]\cos[c_{N-2}\cdot\theta - d]}{\sqrt{N(N-2)}} = \frac{\cos[c_N\cdot\theta - d]\cos[c_{N-2}\cdot\theta - d]}{N} + O(N^{-2})$$

so that

$$
\begin{aligned}
&(P_{N-1}(\cos\theta))^2 - P_N(\cos\theta)P_{N-2}(\cos\theta) \\
&\quad \sim \frac{\cos^2[c_{N-1}\cdot\theta - d] - \cos[c_N\cdot\theta - d]\cos[c_{N-2}\cdot\theta - d]}{\pi N(\sin\frac{\theta}{2})^{2\alpha+1}(\cos\frac{\theta}{2})^{2\gamma+1}}.
\end{aligned}
$$

Using trigonometric identities,

$$\cos^2[c_{N-1}\cdot\theta - d] - \cos[c_N\cdot\theta - d]\cos[c_{N-2}\cdot\theta - d] = \sin^2\theta.$$

Therefore,

$$N(P_{N-1}(\cos\theta))^2 - P_N(\cos\theta)P_{N-2}(\cos\theta) \to \frac{\sin^2\theta}{\pi(\sin\frac{\theta}{2})^{2\alpha+1}(\cos\frac{\theta}{2})^{2\gamma+1}}. \tag{V.12}$$

Plugging (V.7), (V.10), and (V.12) back into (V.5),

$$\frac{1}{N}S_N(\theta,\theta) \to \frac{1}{\pi}.$$

Proof of Theorem V.3

Let

$$\begin{aligned} W_N(\theta,\phi) &= \sqrt{w\left(\frac{\theta}{N}+\frac{\pi}{2}\right)w\left(\frac{\phi}{N}+\frac{\pi}{2}\right)} \\ &= 2^{\alpha+\gamma}\left(\sin\left[\frac{1}{2}\left(\frac{\theta}{N}+\frac{\pi}{2}\right)\right]\sin\left[\frac{1}{2}\left(\frac{\theta}{N}+\frac{\pi}{2}\right)\right]\right)^{\alpha} \\ &\quad\times\left(\cos\left[\frac{1}{2}\left(\frac{\theta}{N}+\frac{\pi}{2}\right)\right]\cos\left[\frac{1}{2}\left(\frac{\phi}{N}+\frac{\pi}{2}\right)\right]\right)^{\gamma} \end{aligned} \tag{V.13}$$

and let

$$\begin{aligned} A_N(\theta,\phi) &= \frac{W_N(\theta,\phi)}{N}\left(e^{-i\frac{\theta}{N}}+e^{i\frac{\phi}{N}}\right)\sum_{n=0}^{N-1}e^{i\frac{n}{N}(\theta-\phi)}\frac{P_n\left(\cos\left[\frac{\theta}{N}+\frac{\pi}{2}\right]\right)P_n\left(\cos\left[\frac{\phi}{N}+\frac{\pi}{2}\right]\right)}{r_n} \\ B_N(\theta,\phi) &= \frac{W_N(\theta,\phi)}{N}\sum_{n=0}^{N-1}e^{i\frac{n}{N}(\theta-\phi)} \\ &\quad\times\frac{P_n\left(\cos\left[\frac{\theta}{N}+\frac{\pi}{2}\right]\right)P_{n+1}\left(\cos\left[\frac{\phi}{N}+\frac{\pi}{2}\right]\right)-P_{n+1}\left(\cos\left[\frac{\theta}{N}+\frac{\pi}{2}\right]\right)P_n\left(\cos\left[\frac{\phi}{N}+\frac{\pi}{2}\right]\right)}{r_n}. \end{aligned} \tag{V.14}$$

Then

$$\begin{aligned}
&\frac{i}{N} S_N\left(\frac{\theta}{N}+\frac{\pi}{2}, \frac{\phi}{N}+\frac{\pi}{2}\right) \\
&\quad = \frac{iW_N(\theta,\phi)}{N} \\
&\quad\quad \times \sum_{n=0}^{N-1} \frac{q_{2n}\left(e^{i\left(\frac{\theta}{N}+\frac{\pi}{2}\right)}\right) q_{2n+1}\left(e^{-i\left(\frac{\phi}{N}+\frac{\pi}{2}\right)}\right) - q_{2n+1}\left(e^{i\left(\frac{\theta}{N}+\frac{\pi}{2}\right)}\right) q_{2n}\left(e^{-i\left(\frac{\phi}{N}+\frac{\pi}{2}\right)}\right)}{r_n} \\
&\quad = A_N(\theta,\phi) + iB_N(\theta,\phi).
\end{aligned}$$

First we simplify B_N. We now use the asympotics of the Jacobi polynomials (V.8):

$$\begin{aligned}
&W_N(\theta,\phi)\cdot \frac{P_n\left(\cos\left[\frac{\theta}{N}+\frac{\pi}{2}\right]\right) P_{n+1}\left(\cos\left[\frac{\phi}{N}+\frac{\pi}{2}\right]\right) - P_{n+1}\left(\cos\left[\frac{\theta}{N}+\frac{\pi}{2}\right]\right) P_n\left(\cos\left[\frac{\phi}{N}+\frac{\pi}{2}\right]\right)}{r_n} \\
&\quad = \frac{W_N(\theta,\phi)}{\pi\sqrt{n(n+1)}\left(\sin\left[\frac{1}{2}\left(\frac{\theta}{N}+\frac{\pi}{2}\right)\right]\sin\left[\frac{1}{2}\left(\frac{\theta}{N}+\frac{\pi}{2}\right)\right]\right)^{\alpha+\frac{1}{2}}\left(\cos\left[\frac{1}{2}\left(\frac{\theta}{N}+\frac{\pi}{2}\right)\right]\cos\left[\frac{1}{2}\left(\frac{\phi}{N}+\frac{\pi}{2}\right)\right]\right)^{\gamma+\frac{1}{2}} r_n} \\
&\quad\quad \times \Bigg\{ \cos\left[c_n\left(\frac{\theta}{N}+\frac{\pi}{2}\right)-d\right]\cos\left[c_{n+1}\left(\frac{\phi}{N}+\frac{\pi}{2}\right)-d\right] \\
&\quad\quad - \cos\left[c_{n+1}\left(\frac{\theta}{N}+\frac{\pi}{2}\right)-d\right]\cos\left[c_n\left(\frac{\phi}{N}+\frac{\pi}{2}\right)-d\right]\Bigg\} + \frac{1}{r_n}O(n^{-2})
\end{aligned}$$

where c_n and d are defined in (V.11). Since $r_n = O(n^{-1})$,

$$\frac{1}{r_n}O(n^{-2}) = O(n^{-1}).$$

Now

$$\begin{aligned}
\cos\left[c_n\left(\frac{\theta}{N}+\frac{\pi}{2}\right)-d\right] &= \cos\left[\frac{n\theta}{N}+\frac{\pi n}{2}+\frac{\pi}{4}(\gamma-\alpha)+\frac{\theta}{2N}(\alpha+\gamma+1)\right] \\
&= \cos\left[\frac{n\theta}{N}+\frac{\pi n}{2}+\frac{\pi}{4}(\gamma-\alpha)\right]\cos\left[\frac{\theta}{2N}(\alpha+\gamma+1)\right] \\
&\quad - \sin\left[\frac{n\theta}{N}+\frac{\pi n}{2}+\frac{\pi}{4}(\gamma-\alpha)\right]\sin\left[\frac{\theta}{2N}(\alpha+\gamma+1)\right] \\
&= \cos\left[\frac{n\theta}{N}+\frac{\pi n}{2}+\frac{\pi}{4}(\gamma-\alpha)\right]\left(1+O(N^{-2})\right)+O(N^{-1}) \\
&= \cos\left[\frac{n\theta}{N}+\frac{\pi n}{2}+\frac{\pi}{4}(\gamma-\alpha)\right]+O(N^{-1}) \qquad \text{(V.15)}
\end{aligned}$$

and similarly

$$
\begin{aligned}
&\cos\left[c_{n+1}\left(\frac{\theta}{N}+\frac{\pi}{2}\right)-d\right] \\
&\quad=\cos\left[\frac{n\theta}{N}+\frac{\pi}{2}(n+1)+\frac{\pi}{4}(\gamma-\alpha)+\frac{\theta}{2N}(\alpha+\gamma+1)+\frac{\theta}{N}\right] \\
&\quad=\cos\left[\frac{n\theta}{N}+\frac{\pi}{2}(n+1)+\frac{\pi}{4}(\gamma-\alpha)\right]\cos\left[\frac{\theta}{2N}(\alpha+\gamma+1)+\frac{\theta}{N}\right] \\
&\qquad-\ \sin\left[\frac{n\theta}{N}+\frac{\pi}{2}(n+1)+\frac{\pi}{4}(\gamma-\alpha)\right]\sin\left[\frac{\theta}{2N}(\alpha+\gamma+1)+\frac{\theta}{N}\right] \\
&\quad=\cos\left[\frac{n\theta}{N}+\frac{\pi}{2}(n+1)+\frac{\pi}{4}(\gamma-\alpha)\right]\left(1+O(N^{-2})\right)+O(N^{-1}) \\
&\quad=\cos\left[\frac{n\theta}{N}+\frac{\pi}{2}(n+1)+\frac{\pi}{4}(\gamma-\alpha)\right]+O(N^{-1}) \\
&\quad=-\sin\left[\frac{n\theta}{N}+\frac{\pi n}{2}+\frac{\pi}{4}(\gamma-\alpha)\right]+O(N^{-1}).
\end{aligned}
$$

So using trigonometric identities,

$$
\begin{aligned}
&\cos\left[c_n\left(\frac{\theta}{N}+\frac{\pi}{2}\right)-d\right]\cos\left[c_{n+1}\left(\frac{\phi}{N}+\frac{\pi}{2}\right)-d\right] \\
&\qquad-\ \cos\left[c_{n+1}\left(\frac{\theta}{N}+\frac{\pi}{2}\right)-d\right]\cos\left[c_n\left(\frac{\phi}{N}+\frac{\pi}{2}\right)-d\right] \\
&\quad=\sin\left[\frac{n\theta}{N}+\frac{\pi n}{2}+\frac{\pi}{4}(\gamma-\alpha)\right]\cos\left[\frac{n\phi}{N}+\frac{\pi n}{2}+\frac{\pi}{4}(\gamma-\alpha)\right] \\
&\qquad-\ \sin\left[\frac{n\phi}{N}+\frac{\pi n}{2}+\frac{\pi}{4}(\gamma-\alpha)\right]\cos\left[\frac{n\theta}{N}+\frac{\pi n}{2}+\frac{\pi}{4}(\gamma-\alpha)\right]+O(N^{-1}) \\
&\quad=\sin\left[\frac{n}{N}(\theta-\phi)\right]+O(N^{-1}).
\end{aligned}
\tag{V.16}
$$

Recalling the value of r_n from (V.2), we can simplify

$$
\begin{aligned}
&\frac{W_N(\theta,\phi)}{\pi\sqrt{n(n+1)}\left(\sin\left[\frac{1}{2}\left(\frac{\theta}{N}+\frac{\pi}{2}\right)\right]\sin\left[\frac{1}{2}\left(\frac{\theta}{N}+\frac{\pi}{2}\right)\right]\right)^{\alpha+\frac{1}{2}}\left(\cos\left[\frac{1}{2}\left(\frac{\theta}{N}+\frac{\pi}{2}\right)\right]\cos\left[\frac{1}{2}\left(\frac{\phi}{N}+\frac{\pi}{2}\right)\right]\right)^{\gamma+\frac{1}{2}} r_n} \\
&\quad=\frac{1}{\left(\sin\left[\frac{1}{2}\left(\frac{\theta}{N}+\frac{\pi}{2}\right)\right]\sin\left[\frac{1}{2}\left(\frac{\theta}{N}+\frac{\pi}{2}\right)\right]\cos\left[\frac{1}{2}\left(\frac{\theta}{N}+\frac{\pi}{2}\right)\right]\cos\left[\frac{1}{2}\left(\frac{\phi}{N}+\frac{\pi}{2}\right)\right]\right)^{\frac{1}{2}}} \\
&\qquad\times\frac{(2n+\alpha+\gamma+1)\Gamma(n+\alpha+\gamma+1)\Gamma(n+1)}{4\pi\sqrt{n(n+1)}\Gamma(n+\alpha+1)\Gamma(n+\gamma+1)}.
\end{aligned}
$$

We break this above expression into its two factors. For the first factor, we use

$$\sqrt{x(x+1)} = x + O(1)$$

and (V.6) to get

$$\begin{aligned}
\frac{(2n+\alpha+\gamma+1)\Gamma(n+\alpha+\gamma+1)\Gamma(n+1)}{4\pi\sqrt{n(n+1)}\Gamma(n+\alpha+1)\Gamma(n+\gamma+1)} &= \frac{(2n+\alpha+\gamma+1)(1+O(n^{-1}))}{4\pi(n+O(n^{-1}))} \\
&= \frac{1}{2\pi} + O(n^{-1}). \qquad \text{(V.17)}
\end{aligned}$$

For the second factor, we use

$$\begin{aligned}
\sin\left[\frac{1}{2}\left(\frac{\theta}{N}+\frac{\pi}{2}\right)\right] &= \frac{\sqrt{2}}{2}\left(\sin\frac{\theta}{2N} + \cos\frac{\theta}{2N}\right) \\
&= \frac{\sqrt{2}}{2} + O(N^{-1})
\end{aligned}$$

and

$$\cos\left[\frac{1}{2}\left(\frac{\theta}{N}+\frac{\pi}{2}\right)\right] = \frac{\sqrt{2}}{2} + O(N^{-1}).$$

This leads to

$$\begin{aligned}
&\frac{1}{\left(\sin\left[\frac{1}{2}\left(\frac{\theta}{N}+\frac{\pi}{2}\right)\right]\sin\left[\frac{1}{2}\left(\frac{\theta}{N}+\frac{\pi}{2}\right)\right]\cos\left[\frac{1}{2}\left(\frac{\theta}{N}+\frac{\pi}{2}\right)\right]\cos\left[\frac{1}{2}\left(\frac{\phi}{N}+\frac{\pi}{2}\right)\right]\right)^{\frac{1}{2}}} \\
&\quad = \frac{1}{\frac{1}{2}+O(N^{-1})} \\
&\quad = 2 + O(N^{-1}). \qquad \text{(V.18)}
\end{aligned}$$

So (V.16), (V.17), and (V.18) together gives

$$\begin{aligned}
B_N(\theta,\phi) &\sim \frac{1}{\pi N}\sum_{n=0}^{N-1} e^{i\frac{n}{N}(\theta-\phi)}\sin\left[\frac{n}{N}(\theta-\phi)\right] \\
&\to \frac{1}{\pi}\int_0^1 e^{it(\theta-\phi)}\sin\left[t(\theta-\phi)\right]dt. \qquad \text{(V.19)}
\end{aligned}$$

Next we simplify A_N. We use the asymptotic formula for the Jacobi polynomials (V.8), substitute in the values of W_N (V.13) and r_n (V.2) and simplify:

$$
\begin{aligned}
&W_N(\theta,\phi)\frac{P_n\left(\cos\left[\frac{\theta}{N}+\frac{\pi}{2}\right]\right)P_n\left(\cos\left[\frac{\phi}{N}+\frac{\pi}{2}\right]\right)}{r_n} \\
&\quad= \frac{(2n+\alpha+\gamma+1)\Gamma(n+\alpha+\gamma+1)\Gamma(n+1)}{4\pi n\Gamma(n+\alpha+1)\Gamma(n+\gamma+1)} \\
&\qquad\times \frac{\cos\left[c_n\left(\frac{\theta}{N}+\frac{\pi}{2}\right)-d\right]\cos\left[c_n\left(\frac{\phi}{N}+\frac{\pi}{2}\right)-d\right]}{\left(\sin\left[\frac{1}{2}\left(\frac{\theta}{N}+\frac{\pi}{2}\right)\right]\sin\left[\frac{1}{2}\left(\frac{\theta}{N}+\frac{\pi}{2}\right)\right]\cos\left[\frac{1}{2}\left(\frac{\theta}{N}+\frac{\pi}{2}\right)\right]\cos\left[\frac{1}{2}\left(\frac{\phi}{N}+\frac{\pi}{2}\right)\right]\right)^{\frac{1}{2}}}+O(n^{-1}).
\end{aligned}
$$

Using (V.15),

$$
\begin{aligned}
&\cos\left[c_n\left(\frac{\theta}{N}+\frac{\pi}{2}\right)-d\right]\cos\left[c_n\left(\frac{\phi}{N}+\frac{\pi}{2}\right)-d\right] \\
&\quad= \cos\left[\frac{n\theta}{N}+\frac{\pi n}{2}+\frac{\pi}{4}(\gamma-\alpha)\right]\cos\left[\frac{n\phi}{N}+\frac{\pi n}{2}+\frac{\pi}{4}(\gamma-\alpha)\right]+O\left(N^{-1}\right) \\
&\quad= \left(\cos\left[\frac{n\theta}{N}+\frac{\pi}{4}(\gamma-\alpha)\right]\cos\left[\frac{\pi n}{2}\right]-\sin\left[\frac{n\theta}{N}+\frac{\pi}{4}(\gamma-\alpha)\right]\sin\left[\frac{\pi n}{2}\right]\right) \\
&\qquad\times\left(\cos\left[\frac{n\phi}{N}+\frac{\pi}{4}(\gamma-\alpha)\right]\cos\left[\frac{\pi n}{2}\right]-\sin\left[\frac{n\phi}{N}+\frac{\pi}{4}(\gamma-\alpha)\right]\sin\left[\frac{\pi n}{2}\right]\right) \\
&\qquad+O\left(N^{-1}\right) \\
&\quad= \begin{cases}\cos\left[\frac{n\theta}{N}+\frac{\pi}{4}(\gamma-\alpha)\right]\cos\left[\frac{n\phi}{N}+\frac{\pi}{4}(\gamma-\alpha)\right] & \text{if } n \text{ even}\\ \sin\left[\frac{n\theta}{N}+\frac{\pi}{4}(\gamma-\alpha)\right]\sin\left[\frac{n\phi}{N}+\frac{\pi}{4}(\gamma-\alpha)\right] & \text{if } n \text{ odd}\end{cases} \\
&\qquad+O(N^{-1}) \\
&\quad:= h_{N,n}(\theta,\phi)+O(N^{-1}).
\end{aligned}
$$

Now (V.17), (V.18), and

$$e^{-i\frac{\theta}{N}}+e^{i\frac{\phi}{N}}=2+O(N^{-1})$$

give

$$A_N(\theta,\phi) \sim \frac{2}{\pi N}\sum_{n=0}^{N-1} e^{i\frac{n}{N}(\theta-\phi)} h_{N,n}(\theta,\phi).$$

Since we will be taking $N \to \infty$, we may assume $N = 2K$. Then

$$\begin{aligned}A_N(\theta,\phi) \sim \frac{2}{\pi N}\sum_{k=0}^{K-1}\Bigg\{ & e^{i\frac{2k}{N}(\theta-\phi)} \cos\left[\frac{2k\theta}{N}+\frac{\pi}{4}(\gamma-\alpha)\right]\cos\left[\frac{2k\phi}{N}+\frac{\pi}{4}(\gamma-\alpha)\right] \\ &+ e^{i\frac{2k+1}{N}(\theta-\phi)}\sin\left[\frac{(2k+1)\theta}{N}+\frac{\pi}{4}(\gamma-\alpha)\right]\sin\left[\frac{(2k+1)\phi}{N}+\frac{\pi}{4}(\gamma-\alpha)\right]\Bigg\}\end{aligned}$$

We next use

$$\begin{aligned} e^{i\frac{2k+1}{N}(\theta-\phi)} &= e^{i\frac{2k}{N}(\theta-\phi)}(1+O(N^{-1})) \\ \sin\left[\frac{(2k+1)\theta}{N}+\frac{\pi}{4}(\gamma-\alpha)\right] &= \sin\left[\frac{2k\theta}{N}+\frac{\pi}{4}(\gamma-\alpha)\right] + O(N^{-1}).\end{aligned}$$

Then

$$\begin{aligned}A_N(\theta,\phi) &\sim \frac{2}{\pi N}\sum_{k=0}^{K-1}\Bigg\{ e^{i\frac{2k}{N}(\theta-\phi)} \cos\left[\frac{2k\theta}{N}+\frac{\pi}{4}(\gamma-\alpha)\right]\cos\left[\frac{2k\phi}{N}+\frac{\pi}{4}(\gamma-\alpha)\right] \\ &\qquad + e^{i\frac{2k}{N}(\theta-\phi)}\sin\left[\frac{2k\theta}{N}+\frac{\pi}{4}(\gamma-\alpha)\right]\sin\left[\frac{2k\phi}{N}+\frac{\pi}{4}(\gamma-\alpha)\right] + O(N^{-1})\Bigg\} \\ &= \frac{2}{\pi N}\sum_{k=0}^{K-1}\left(e^{i\frac{2k}{N}(\theta-\phi)}\cos\left[\frac{2k}{N}(\theta-\phi)\right]+O(N^{-1})\right) \\ &\sim \frac{2}{\pi N}\sum_{k=0}^{K-1}\left(e^{i\frac{2k}{N}(\theta-\phi)}\cos\left[\frac{2k}{N}(\theta-\phi)\right]\right) \\ &\to \frac{1}{\pi}\int_0^1 e^{it(\theta-\phi)}\cos\left[t(\theta-\phi)\right]dt. \end{aligned} \tag{V.20}$$

Putting (V.19) and (V.20) together gives

$$\begin{aligned}\frac{i}{N}S_N\left(\frac{\theta}{N}+\frac{\pi}{2},\frac{\phi}{N}+\frac{\pi}{2}\right) \to \frac{1}{\pi}\int_0^1 e^{it(\theta-\phi)}\cos\left[t(\theta-\phi)\right]dt \\ + \frac{i}{\pi}\int_0^1 e^{it(\theta-\phi)}\sin\left[t(\theta-\phi)\right]dt \\ = \frac{1}{\pi}\int_0^1 e^{2it(\theta-\phi)}dt \\ = e^{i(\theta-\phi)}\frac{\sin(\theta-\phi)}{\pi(\theta-\phi)}.\end{aligned}$$

For DS_N, the A_N term differs from $(V.14)$ in that $e^{-i\frac{\theta}{N}}+e^{i\frac{\phi}{N}}$ is instead

$$e^{-i\frac{\phi}{N}}-e^{i\frac{\theta}{N}}=O(N^{-1}).$$

The B_N term for DS_N is

$$\begin{aligned}B_N(\theta,\phi)=\frac{W_N(\theta,\phi)}{N}\sum_{n=0}^{N-1}(-1)^n e^{i\frac{n}{N}(\theta+\phi)} \\ \times\frac{P_n\left(\cos\left[\frac{\theta}{N}+\frac{\pi}{2}\right]\right)P_{n+1}\left(\cos\left[\frac{\phi}{N}+\frac{\pi}{2}\right]\right)-P_{n+1}\left(\cos\left[\frac{\theta}{N}+\frac{\pi}{2}\right]\right)P_n\left(\cos\left[\frac{\phi}{N}+\frac{\pi}{2}\right]\right)}{r_n}.\end{aligned}$$

Now

$$\left|\frac{P_n\left(\cos\left[\frac{\theta}{N}+\frac{\pi}{2}\right]\right)P_{n+1}\left(\cos\left[\frac{\phi}{N}+\frac{\pi}{2}\right]\right)-P_{n+1}\left(\cos\left[\frac{\theta}{N}+\frac{\pi}{2}\right]\right)P_n\left(\cos\left[\frac{\phi}{N}+\frac{\pi}{2}\right]\right)}{r_n}\right|\to 0$$

as $n\to\infty$ for all N. This shows that $DS_N\to 0$. The proof for IS_N is same as the proof for DS_N.

Proof of Theorem V.4

Let

$$C_N(\theta,\phi) = \frac{i\sqrt{w\left(\frac{\theta}{N}\right) w\left(\frac{\phi}{N}\right)}}{N}\left(e^{-i\frac{\theta}{N}} - e^{i\frac{\phi}{N}}\right) \times \sum_{n=0}^{N-1} e^{i\frac{n}{N}(\theta-\phi)} \frac{P_n(\cos\frac{\theta}{N})P_n(\cos\frac{\phi}{N})}{r_n}$$

$$D_N(\theta,\phi) = \frac{i\sqrt{w\left(\frac{\theta}{N}\right) w\left(\frac{\phi}{N}\right)}}{N}\sum_{n=0}^{N-1} e^{i\frac{n}{N}(\theta-\phi)} \times \frac{P_n(\cos\frac{\theta}{N})P_{n+1}(\cos\frac{\phi}{N}) - P_n(\cos\frac{\phi}{N})P_{n+1}(\cos\frac{\theta}{N})}{r_n}.$$

Then it follows that

$$\frac{1}{N}S_N\left(\frac{\theta}{N},\frac{\phi}{N}\right) = C_N(\theta,\phi) + D_N(\theta,\phi). \tag{V.21}$$

We first simplify C_N. Note that

$$e^{-i\frac{\theta}{N}} - e^{i\frac{\phi}{N}} = -\frac{i}{N}(\theta+\phi) + O(N^{-2})$$

We use the asymptotics for Jacobi polynomials

$$P_n(\cos\theta) = \frac{2^{\frac{\alpha+\gamma}{2}}\Gamma(n+\alpha+1)}{\sqrt{w(\theta)}\left(n+\frac{\alpha+\gamma+1}{2}\right)^{\alpha} n!}\cdot\sqrt{\frac{\theta}{\sin\theta}}J_\alpha\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\theta\right] + O(n^{-\frac{3}{2}}). \tag{V.22}$$

Then

$$C_N(\theta,\phi) \sim \frac{2^{\alpha+\gamma}(\theta+\phi)+O(N^{-1})}{N^2}\cdot\sqrt{\frac{\theta\phi}{N^2\sin\frac{\theta}{N}\sin\frac{\phi}{N}}} \times \sum_{n=0}^{N-1} e^{i\frac{n}{N}(\theta-\phi)}\left(\frac{\Gamma(n+\alpha+1)}{\left(n+\frac{\alpha+\gamma+1}{2}\right)^{\alpha} n!}\right)^2 \times \frac{J_\alpha\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\theta}{N}\right] J_\alpha\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\phi}{N}\right]}{r_n}.$$

Now using the identity [42]

$$J_\alpha(x+y) = \sum_{k=-\infty}^{\infty} J_{\alpha-k}(x) J_k(y), \tag{V.23}$$

we have

$$\begin{aligned}
J_\alpha & \left[\left(n + \frac{\alpha+\gamma+1}{2}\right)\frac{\theta}{N}\right] \\
&= \sum_{k=-\infty}^{\infty} J_{\alpha-k}\left[\frac{n\theta}{N}\right] J_k\left[\frac{(\alpha+\gamma+1)\theta}{2N}\right] \\
&= J_\alpha\left[\frac{n\theta}{N}\right] J_0\left[\frac{(\alpha+\gamma+1)\theta}{2N}\right] \\
&\quad + \sum_{k=1}^{\infty}\left(J_{\alpha-k}\left[\frac{n\theta}{N}\right] J_k\left[\frac{(\alpha+\gamma+1)\theta}{2N}\right] + J_{\alpha+k}\left[\frac{n\theta}{N}\right] J_{-k}\left[\frac{(\alpha+\gamma+1)\theta}{2N}\right]\right).
\end{aligned}$$

Using

$$J_{-n}(x) = (-1)^n J_n(x),$$

the above expression is equal to

$$\begin{aligned}
J_\alpha & \left[\frac{n\theta}{N}\right] J_0\left[\frac{(\alpha+\gamma+1)\theta}{2N}\right] \\
&+ \sum_{k=1}^{\infty}\left(J_{\alpha-k}\left[\frac{n\theta}{N}\right] J_k\left[\frac{(\alpha+\gamma+1)\theta}{2N}\right] + (-1)^k J_{\alpha+k}\left[\frac{n\theta}{N}\right] J_k\left[\frac{(\alpha+\gamma+1)\theta}{2N}\right]\right).
\end{aligned}$$

The Bessel function can be expanded as

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)}\left(\frac{x}{2}\right)^{2m+\alpha}. \tag{V.24}$$

In particular,

$$\begin{aligned}
J_0\left[\frac{(\alpha+\gamma+1)\theta}{2N}\right] &= 1 + O(N^{-2}) \\
J_k\left[\frac{(\alpha+\gamma+1)\theta}{2N}\right] &= O(N^{-k})
\end{aligned}$$

This leads to

$$J_\alpha\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\theta}{N}\right]=J_\alpha\left[\frac{n\theta}{N}\right]+O(N^{-1})$$

and consequently

$$\begin{aligned} J_\alpha\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\theta}{N}\right] J_\alpha\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\phi}{N}\right] \\ = J_\alpha\left[\frac{n\theta}{N}\right] J_\alpha\left[\frac{n\phi}{N}\right]+O(N^{-1}). \end{aligned} \tag{V.25}$$

Using (V.6) we have

$$\left(\frac{\Gamma(n+\alpha+1)}{\left(n+\frac{\alpha+\gamma+1}{2}\right)^\alpha n!}\right)^2 = 1+O(n^{-1}) \tag{V.26}$$

and

$$\frac{1}{r_n}=\frac{n}{2^{\alpha+\gamma+1}}+O(1). \tag{V.27}$$

So (V.25), (V.26), and (V.27) imply

$$\begin{aligned} C_N(\theta,\phi) &\sim \frac{\theta+\phi}{2N}\cdot\sqrt{\frac{\theta\phi}{N^2\sin\frac{\theta}{N}\sin\frac{\phi}{N}}}\sum_{n=0}^{N-1}\frac{n}{N}\cdot e^{i\frac{n}{N}(\theta-\phi)}J_\alpha\left[\frac{n\theta}{N}\right]J_\alpha\left[\frac{n\phi}{N}\right] \\ &\to \frac{\theta+\phi}{2}\int_0^1 t\cdot e^{it(\theta-\phi)}J_\alpha[t\theta]J_\alpha[t\phi]\,dt. \end{aligned}$$

Next we simplify D_N. After applying the asymptotics (V.22) to D_N, we get

$$\begin{aligned} D_N(\theta,\phi) \sim \frac{2^{\alpha+\gamma}i}{N}\cdot\sqrt{\frac{\theta\phi}{N^2\sin\frac{\theta}{N}\sin\frac{\phi}{N}}}\sum_{n=0}^{N-1}e^{i\frac{n}{N}(\theta-\phi)} \\ \times\frac{\Gamma(n+\alpha+1)\Gamma(n+\alpha+2)g_{N,n}(\theta,\phi)}{\left(n+\frac{\alpha+\gamma+1}{2}\right)^\alpha\left(n+1+\frac{\alpha+\gamma+1}{2}\right)^\alpha n!(n+1)!r_n} \end{aligned}$$

where

$$\begin{aligned} g_{N,n}(\theta,\phi) = J_\alpha\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\theta}{N}\right] J_\alpha\left[\left(n+1+\frac{\alpha+\gamma+1}{2}\right)\frac{\phi}{N}\right] \\ - J_\alpha\left[\left(n+1+\frac{\alpha+\gamma+1}{2}\right)\frac{\theta}{N}\right] J_\alpha\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\phi}{N}\right]. \end{aligned}$$

Now using the identities (V.23) and (V.24) we have

$$
\begin{aligned}
&J_\alpha\left[\left(n+1+\frac{\alpha+\gamma+1}{2}\right)\frac{\theta}{N}\right]\\
&\quad=\sum_{k=-\infty}^{\infty} J_{\alpha-k}\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\theta}{N}\right] J_k\left[\frac{\theta}{N}\right]\\
&\quad=J_\alpha\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\theta}{N}\right] J_0\left[\frac{\theta}{N}\right]+\sum_{k=1}^{\infty} J_{\alpha-k}\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\theta}{N}\right] J_k\left[\frac{\theta}{N}\right]\\
&\qquad+\sum_{k=1}^{\infty}(-1)^k J_{\alpha+k}\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\theta}{N}\right] J_k\left[\frac{\theta}{N}\right]\\
&\quad=J_\alpha\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\theta}{N}\right](1+O(N^{-2}))+\frac{\theta}{2N}J_{\alpha-1}\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\theta}{N}\right]\\
&\qquad+O(N^{-2})\sum_{k=2}^{\infty} J_{\alpha-k}\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\theta}{N}\right] J_k\left[\frac{\theta}{N}\right]\\
&\qquad-\frac{\theta}{2N}J_{\alpha+1}\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\theta}{N}\right]\\
&\qquad+O(N^{-2})\sum_{k=2}^{\infty}(-1)^k J_{\alpha+k}\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\theta}{N}\right] J_k\left[\frac{\theta}{N}\right].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
g_{N,n}(\theta,\phi) = {} & J_\alpha\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\theta}{N}\right]\left\{J_\alpha\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\phi}{N}\right]\right. \\
& \left. + \frac{\phi}{2N}\left(J_{\alpha-1}\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\phi}{N}\right] - J_{\alpha+1}\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\phi}{N}\right]\right)\right\} \\
& - J_\alpha\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\phi}{N}\right]\left\{J_\alpha\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\theta}{N}\right]\right. \\
& \left. + \frac{\theta}{2N}\left(J_{\alpha-1}\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\theta}{N}\right] - J_{\alpha+1}\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\theta}{N}\right]\right)\right\} \\
& + O(N^{-2}) \\
= {} & \frac{\phi}{2N}J_\alpha\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\theta}{N}\right] \\
& \times\left(J_{\alpha-1}\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\phi}{N}\right] - J_{\alpha+1}\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\phi}{N}\right]\right) \\
& - \frac{\theta}{2N}J_\alpha\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\phi}{N}\right] \\
& \times\left(J_{\alpha-1}\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\theta}{N}\right] - J_{\alpha+1}\left[\left(n+\frac{\alpha+\gamma+1}{2}\right)\frac{\theta}{N}\right]\right) \\
& + O(N^{-2}).
\end{aligned}
$$

We again use (V.23) and (V.24) to get

$$
\begin{aligned}
& \frac{\phi}{2N}J_\alpha\left[\frac{n\theta}{N}\right]\left(J_{\alpha-1}\left[\frac{n\phi}{N}\right] - J_{\alpha+1}\left[\frac{n\phi}{N}\right]\right) \\
& \qquad - \frac{\theta}{2N}J_\alpha\left[\frac{n\phi}{N}\right]\left(J_{\alpha-1}\left[\frac{n\theta}{N}\right] - J_{\alpha+1}\left[\frac{n\theta}{N}\right]\right) + O(N^{-2}).
\end{aligned}
$$

Now

$$
J'_\alpha(z) = \frac{1}{2}\left(J_{\alpha-1}(z) - J_{\alpha+1}(z)\right).
$$

Thus,

$$
g_{N,n}(\theta,\phi) = \frac{\phi}{N}J_\alpha\left[\frac{n\theta}{N}\right]J'_\alpha\left[\frac{n\phi}{N}\right] - \frac{\theta}{N}J_\alpha\left[\frac{n\phi}{N}\right]J'_\alpha\left[\frac{n\theta}{N}\right] + O(N^{-2}). \quad \text{(V.28)}
$$

Next, using (V.27),

$$\frac{\Gamma(n+\alpha+1)\Gamma(n+\alpha+2)}{\left(n+\frac{\alpha+\gamma+1}{2}\right)^{\alpha}\left(n+1+\frac{\alpha+\gamma+1}{2}\right)^{\alpha} n!(n+1)!r_n} = \frac{n}{2^{\alpha+\gamma+1}} + O(1). \qquad \text{(V.29)}$$

Thus, using (V.28) and (V.29),

$$\begin{aligned} D_N(\theta,\phi) \sim\ & \frac{i2^{\alpha+\gamma}}{N} \cdot \sqrt{\frac{\theta\phi}{N^2 \sin\frac{\theta}{N}\sin\frac{\phi}{N}}} \sum_{n=0}^{N-1} \frac{1}{N}\left(\frac{n}{2^{\alpha+\gamma+1}} + O(1)\right) e^{i\frac{n}{N}(\theta-\phi)} \\ & \times \left(\phi J_\alpha\left[\frac{n\theta}{N}\right] J'_\alpha\left[\frac{n\phi}{N}\right] - \theta J_\alpha\left[\frac{n\phi}{N}\right] J'_\alpha\left[\frac{n\theta}{N}\right] + O(N^{-1})\right) \\ & \to \frac{i}{2}\int_0^1 te^{it(\theta-\phi)}\left(\phi J_\alpha(t\theta)J'_\alpha(t\phi) - \theta J_\alpha(t\phi)J'_\alpha(t\theta)\right)dt. \end{aligned}$$

The proofs for the edge asymptotics of DS_N and IS_N follow directly from the proof for S_N.

Proof of Corollary V.5

We use the identity from Equation 6 of [2]

$$\int xJ_\alpha(ax)J_\alpha(bx)dx = \frac{x(bJ_\alpha(ax)J'_\alpha(bx) - aJ_\alpha(bx)J'_\alpha(ax))}{a^2-b^2}.$$

Then

$$\begin{aligned} & \int_0^1 te^{it(\theta-\phi)}J_\alpha(t\theta)J_\alpha(t\phi)dt \\ & = \int_0^1 e^{it(\theta-\phi)}\frac{d}{dt}\left(\frac{t(\phi J_\alpha(t\theta)J'_\alpha(t\phi) - \theta J_\alpha(t\phi)J'_\alpha(t\theta))}{\theta^2-\phi^2}\right)dt \\ & = e^{i(\theta-\phi)}\frac{\phi J_\alpha(\theta)J'_\alpha(\phi) - \theta J_\alpha(\phi)J'_\alpha(\theta)}{\theta^2-\phi^2} \\ & \quad - \int_0^1 i(\theta-\phi)e^{it(\theta-\phi)}\frac{t(\phi J_\alpha(t\theta)J'_\alpha(t\phi) - \theta J_\alpha(t\phi)J'_\alpha(t\theta))}{\theta^2-\phi^2}dt. \end{aligned}$$

Thus

$$\begin{aligned}
&\frac{\theta+\phi}{2}\int_0^1 te^{it(\theta-\phi)}J_\alpha(t\theta)J_\alpha(t\phi)dt \\
&\quad = e^{i(\theta-\phi)}\frac{\phi J_\alpha(\theta)J'_\alpha(\phi)-\theta J_\alpha(\phi)J'_\alpha(\theta)}{2(\theta-\phi)} \\
&\qquad - \frac{i}{2}\int_0^1 te^{it(\theta-\phi)}(\phi J_\alpha(t\theta)J'_\alpha(t\phi)-\theta J_\alpha(t\phi)J'_\alpha(t\theta))dt
\end{aligned}$$

and the result follows. The proofs for the remaining kernels are exactly the same.

We record the kernel entries for the $\beta = 2$ (CUE), $\beta = 1$ (COE), and $\beta = 4$ (CSE) circular ensemble. We will interpret N as the total number of charge-$\sqrt{\beta}$ particles on the unit circle interacting logarithmically with no external field. The partition function is

$$Z_{N;\beta} = \int_{[-\pi,\pi)^N} \prod_{n>m} \left| e^{i\theta_n} - e^{i\theta_m} \right|^{\beta} d\boldsymbol{\theta}_N,$$

and the correlation functions are

$$R^N_{n;\beta}(\boldsymbol{\theta}_n) = \mathrm{Pf} \begin{bmatrix} DS_{N;\beta}(\theta_j, \theta_k) & S_{N;\beta}(\theta_j, \theta_k) \\ -S_{N;\beta}(\theta_k, \theta_j) & IS_{N;\beta}(\theta_j, \theta_k) \end{bmatrix}_{j,k=1}^{n}.$$

APPENDIX A

$\beta = 2$ (CUE)

The CUE is a determinantal point process, and the correlation function is

$$R^N_{n;2}(\boldsymbol{\theta}_n) = \det \Big[S_{N;2}(\theta_j, \theta_k)\Big]_{j,k=1}^n$$

where

$$S_{N;2}(\theta, \phi) = \frac{1}{2\pi} \sum_{n=1}^{N} e^{i(n-1)(\theta-\phi)}. \tag{A.1}$$

By scaling $\theta \mapsto \dfrac{2\pi}{N}$, the limiting kernel is

$$\begin{aligned} S_2(\theta, \phi) &= \lim_{N\to\infty} \frac{2\pi}{N} S_{N;2}\left(\frac{2\pi\theta}{N}, \frac{2\pi\phi}{N}\right) \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i \left(\frac{n-1}{N}\right)(\theta-\phi)} \\ &= \int_0^1 e^{2\pi i(\theta-\phi)t}\, dt \\ &= e^{\pi i(\theta-\phi)} \frac{\sin \pi(\theta-\phi)}{\pi(\theta-\phi)}. \end{aligned} \tag{A.2}$$

The determinantal point process can also be considered as a Pfaffian point process, for

$$R^N_{n;2}(\boldsymbol{\theta}_n) = \det \Big[S_{N;2}(\theta_j, \theta_k)\Big]_{j,k=1}^n = \mathrm{Pf} \begin{bmatrix} 0 & S_{N;2}(\theta_j, \theta_k) \\ -S_{N;2}(\theta_k, \theta_j) & 0 \end{bmatrix}_{j,k=1}^n . \tag{A.3}$$

In other words,

$$DS_{N;2}(\theta, \phi) = 0$$

$$IS_{N;2}(\theta, \phi) = 0.$$

APPENDIX B

$\beta = 1$ (COE)

Let N be even and denote

$$(2n-1)!! = \prod_{k=1}^{n}(2k-1).$$

The partition function of this ensemble is given by

$$\begin{aligned} Z_{N;1} &= \int_{[-\pi,\pi)^N} \prod_{n>m} \left|e^{i\theta_n} - e^{i\theta_m}\right| d\boldsymbol{\theta}_N \\ &= \operatorname{Pf} \mathbf{A}_N \\ &= \frac{(8\pi)^{\frac{N}{2}}}{(N-1)!!} \end{aligned}$$

where $\mathbf{A}_N$ is defined in (III.3). The kernel entries are

$$\begin{aligned} S_{N;1}(\theta,\phi) &= \frac{1}{\pi}\sum_{n=1}^{\frac{N}{2}} \cos\left[\left(\frac{2n-1}{2}\right)(\theta-\phi)\right] \\ DS_{N;1}(\theta,\phi) &= \frac{i}{\pi N}\sum_{n=1}^{\frac{N}{2}} \left(\frac{2n-1}{2}\right)\sin\left[\left(\frac{2n-1}{2}\right)(\theta-\phi)\right] \\ IS_{N;1}(\theta,\phi) &= -\frac{2iN}{\pi}\sum_{n=1}^{\frac{N}{2}} \left(\frac{2}{2n-1}\right)\sin\left[\left(\frac{2n-1}{2}\right)(\theta-\phi)\right] + \operatorname{sgn}(\theta-\phi) \end{aligned} \quad (B.1)$$

By scaling $\theta \mapsto \dfrac{2\pi\theta}{N}$, the limiting kernels are

$$
\begin{aligned}
\frac{2\pi}{N} S_{N;1}\left(\frac{2\pi\theta}{N}, \frac{2\pi\phi}{N}\right) &= \frac{2}{N}\sum_{n=1}^{\frac{N}{2}} \cos\left[\pi\left(\frac{2n-1}{N}\right)(\theta-\phi)\right] \\
&\to \int_0^1 \cos\left[\pi(\theta-\phi)t\right]\, dt \\
&= \frac{\sin\left[\pi(\theta-\phi)\right]}{\pi(\theta-\phi)}, \\
\frac{2\pi}{N} DS_{N;1}\left(\frac{2\pi\theta}{N}, \frac{2\pi\phi}{N}\right) &= \frac{i}{4}\cdot\frac{2}{N}\sum_{n=1}^{\frac{N}{2}} \left(\frac{2n-1}{N}\right)\sin\left[\pi\left(\frac{2n-1}{N}\right)(\theta-\phi)\right] \\
&\to \frac{i}{4}\int_0^1 t\cdot\sin\left[\pi(\theta-\phi)t\right]\, dt \\
&= \frac{i}{4}\left(\frac{\sin\left[\pi(\theta-\phi)\right]}{\pi^2(\theta-\phi)^2} - \frac{\cos\left[\pi(\theta-\phi)\right]}{\pi(\theta-\phi)}\right), \\
\frac{2\pi}{N} IS_{N;1}\left(\frac{2\pi\theta}{N}, \frac{2\pi\phi}{N}\right) &= -4i\cdot\frac{2}{N}\sum_{n=1}^{\frac{N}{2}} \left(\frac{1}{\frac{2n-1}{N}}\right)\sin\left[\pi\left(\frac{2n-1}{N}\right)(\theta-\phi)\right] + 2\pi\cdot\operatorname{sgn}(\theta-\phi) \\
&\to -4i\int_0^1 \frac{\sin\left[\pi(\theta-\phi)t\right]}{t}\, dt + 2\pi\cdot\operatorname{sgn}(\theta-\phi). \qquad \text{(B.2)}
\end{aligned}
$$

Remark B.1. In Chapter III, N was the total charge of the system. In the $\beta = 1$ case, the total charge is equal to the total number of particles.

APPENDIX C

$\beta = 4$ (CSE)

The partition function of this ensemble is given by

$$\begin{aligned} Z_{N;4} &= \int_{[-\pi,\pi)^N} \prod_{n>m} \left| e^{i\theta_n} - e^{i\theta_m} \right|^4 d\boldsymbol{\theta}_N \\ &= \operatorname{Pf} \mathbf{B}_{2N} \\ &= (2\pi)^N (2N-1)!! \end{aligned}$$

where $\mathbf{B}_N$ is defined in (IV.15). The kernel entries are

$$\begin{aligned} S_{N;4}(\theta,\phi) &= \frac{1}{2\pi} \sum_{n=1}^{N} \cos\left[\left(\frac{2n-1}{2}\right)(\theta-\phi)\right] \\ DS_{N;4}(\theta,\phi) &= \frac{iN}{2\pi} \sum_{n=1}^{N} \left(\frac{2}{2n-1}\right) \sin\left[\left(\frac{2n-1}{2}\right)(\theta-\phi)\right] \\ IS_{N;4}(\theta,\phi) &= -\frac{i}{2\pi N} \sum_{n=1}^{N} \left(\frac{2n-1}{2}\right) \sin\left[\left(\frac{2n-1}{2}\right)(\theta-\phi)\right]. \end{aligned} \tag{C.1}$$

By scaling $\theta \mapsto \dfrac{2\pi\theta}{N}$, the limiting kernel entries are

$$\begin{aligned} S_4(\theta,\phi) &= \lim_{N\to\infty} \frac{2\pi}{N} S_{N;4}\left(\frac{2\pi\theta}{N}, \frac{2\pi\phi}{N}\right) \\ &= \lim_{N\to\infty} \frac{1}{2} \cdot \frac{2}{N} \sum_{n=1}^{N} \cos\left[\pi\left(\frac{2n-1}{N}\right)(\theta-\phi)\right] \\ &= \frac{1}{2} \int_0^2 \cos\left[\pi(\theta-\phi)t\right] dt \\ &= \frac{\sin\left[2\pi(\theta-\phi)\right]}{2\pi(\theta-\phi)}, \end{aligned}$$

$$
\begin{aligned}
DS_4(\theta,\phi) &= \lim_{N\to\infty} 2\pi D_{N;4}\left(\frac{2\pi\theta}{N},\frac{2\pi\phi}{N}\right)\\
&= \lim_{N\to\infty} \frac{i}{2}\cdot\frac{2}{N}\sum_{n=1}^{N}\frac{1}{\left(\frac{2n-1}{2N}\right)}\sin\left[\pi\left(\frac{2n-1}{N}\right)(\theta-\phi)\right]\\
&= i\int_0^2 \frac{\sin\left[\pi(\theta-\phi)t\right]}{t}\,dt,
\end{aligned}
$$

$$
\begin{aligned}
IS_4(\theta,\phi) &= \lim_{N\to\infty} \frac{2\pi}{N^2} IS_{N;4}\left(\frac{2\pi\theta}{N},\frac{2\pi\phi}{N}\right)\\
&= \lim_{N\to\infty} -\frac{i}{2}\cdot\frac{2}{N}\sum_{n=1}^{N}\left(\frac{2n-1}{2N}\right)\sin\left[\pi\left(\frac{2n-1}{N}\right)(\theta-\phi)\right]\\
&= -\frac{i}{4}\int_0^2 t\cdot\sin\left[\pi(\theta-\phi)t\right]dt\\
&= i\left(\frac{\cos\left[2\pi(\theta-\phi)\right]}{2\pi(\theta-\phi)}-\frac{\sin\left[2\pi(\theta-\phi)\right]}{4\pi^2(\theta-\phi)^2}\right). \qquad \text{(C.2)}
\end{aligned}
$$

Remark C.1. We make the same remark from the previous section. In Chapter III, N represented the total charge of the system. This interpretation changes the kernels. In this case, N must be even and the total number of particles is $\frac{N}{2}$. Making this change,

$$
\begin{aligned}
S_{N;4}(\theta,\phi) &= \frac{1}{2\pi}\sum_{n=1}^{\frac{N}{2}}\cos\left[\left(\frac{2n-1}{2}\right)(\theta-\phi)\right]\\
DS_{N;4}(\theta,\phi) &= \frac{iN}{2\pi}\sum_{n=1}^{\frac{N}{2}}\left(\frac{2}{2n-1}\right)\sin\left[\left(\frac{2n-1}{2}\right)(\theta-\phi)\right]\\
IS_{N;4}(\theta,\phi) &= -\frac{i}{2\pi N}\sum_{n=1}^{\frac{N}{2}}\left(\frac{2n-1}{2}\right)\sin\left[\left(\frac{2n-1}{2}\right)(\theta-\phi)\right], \qquad \text{(C.3)}
\end{aligned}
$$

and scaling by $\theta \mapsto \dfrac{2\pi\theta}{N}$, the limiting kernels are

$$\begin{aligned} S_4(\theta,\phi) &= \frac{\sin\left[\pi(\theta-\phi)\right]}{2\pi(\theta-\phi)} \\ DS_4(\theta,\phi) &= i\int_0^1 \frac{\sin\left[\pi(\theta-\phi)t\right]}{t}\,dt \\ IS_4(\theta,\phi) &= -\frac{i}{4}\left(\frac{\sin\left[\pi(\theta-\phi)\right]}{\pi^2(\theta-\phi)^2} - \frac{\cos\left[\pi(\theta-\phi)\right]}{\pi(\theta-\phi)}\right). \end{aligned} \tag{C.4}$$

REFERENCES CITED

[1] M. Adler, P. J. Forrester, and T. Nagao, *Classical skew orthogonal polynomials and random matrices*, J. Stat. Phys. **99** (2000), 141–170.

[2] S. K. H. Auluck, *Some integral identities involving products of general solutions of Bessel's equation of integral order*, preprint (2010), http://arxiv.org/abs/1006.4417.

[3] J. Baik, P. Deift, and K. Johansson, *On the distribution of the length of the longest increasing subsequence of random permutations*, J. Amer. Math. Soc. **12** (1999), 1119–1178.

[4] J. S. Ball, *Half-range generalized Hermite polynomials and the related Gaussian quadratures*, SIAM J. Numer. Anal. **40** (2003), 2311–2317.

[5] P. Billingsley, *Probability and Measure*, John Wiley & Sons, Inc., New York, NY, 1995.

[6] A. Borodin and E. Kanzieper, *A note on the Pfaffian integration theorem*, J. Phys. A: Math. Theor. **40** (2007), 849–855.

[7] A. Borodin and C. D. Sinclair, *The Ginibre ensemble of real random matrices and its scaling limits*, Comm. Math. Phys. **291** (2009), 177–224.

[8] T. Cover and J. Thomas, *Elements of Information Theory*, John Wiley & Sons, Inc., Hoboken, NJ, 2006.

[9] I. Dumitriu and A. Edelman, *Matrix models for beta ensembles*, J. Math. Phys. **43** (2002), 5830–5847.

[10] F. J. Dyson, *Statistical theory of the energy levels of complex systems I*, J. Math. Phys. **3** (1962), 140–156.

[11] F. J. Dyson and M. L. Mehta, *Statistical theory of the energy levels of complex systems. V*, J. Math. Phys. **4** (1963), 713–719.

[12] P. J. Forrester, *An exactly solvable two-component solvable classical coulomb system*, J. Aust. Math. Soc. Series B **26** (1984), 119–128.

[13] P. J. Forrester, *Log-Gases and Random Matrices*, Princeton University Press, Princeton, NJ, 2010.

[14] P. J. Forrester and A. Mays, *Pfaffian point processes for the Gaussian real generalized eigenvalue problem*, Prob. Theory and Related Fields **10** (2010), 1–47.

[15] P. J. Forrester and C. D. Sinclair, *A generalized plasma and interpolation between classical random matrix ensembles*, J. Stat. Phys. **143** (2011), 326–345.

[16] J. Ginibre, *Statistical ensemble of complex, quaternion, and real matrices*, J. Math. Phys. **6** (1965), 440–449.

[17] I. Gohberg, S. Goldberg, and N. Krupnik. *Traces and Determinants of Linear Operators*, Birkhauser Verlag, Basel, 2000.

[18] I. J. Good, *Short proof of a conjecture by Dyson*, J. Math. Phys. **11** (1970), 1884.

[19] A. Its and I. Krasovsky, *Hankel determinant and orthogonal polynomials for the Gaussian weight with a jump*, Contemp. Math. **458** (2008), 215–247.

[20] M. Jimbo, T. Miwa, Y. Mori, and M. Sato, *Density matrix of an impenetrable Bose gas and the fifth painleve transcendent*, Physica D **1** (1980), 80–158.

[21] R. Killip and I. Nenciu, *Matrix models for circular ensemble*, Int. Math. Res. Not. (2008), 2665–2701.

[22] D. S. Lubinsky, *A new approach to universality involving orthogonal polynomials*, Ann. Math. **170** (2009), 915–939.

[23] M. L. Mehta, *Random Matrices*, Elsevier/Academic Press, Amsterdam, 2004.

[24] Y. A. Melnikov, *Green's Functions and Infinite Products: Bridging the Divide*, Birkhauser, Boston, MA, 2011.

[25] A. F. Moreno, A. Martinez-Finkelshtein, and V.L Sousa, *Asymptotics of orthogonal polynomials for a weight with a jump on* $[-1, 1]$, Constr. Approx. **33** (2011), 219–263.

[26] T. Nagao, *Universal correlations near a singularity of random matrix spectrum*, J. Phys. Soc. Jpn. **64** (1995), 3675 – 3681.

[27] T. Nagao and M. Wadati, *Correlation functions of random matrix ensembles related to classical orthogonal polyomials*, J. Phys. Soc. Jpn. **60** (1991), 3298–3322.

[28] T. Nagao and M. Wadati, *Eigenvalue distribution of random matrices at the spectrum edge*, J. Phys. Soc. Jpn. **62** (1993), 3845–3856.

[29] J. Ramirez, B. Rider, and B. Virag, *Beta ensembles, stochastic Airy spectrum, and a diffusion*, J. Amer. Math. Soc. **24** (2011), 919–944.

[30] B. Rider, C. D. Sinclair, and Y. Xu, *A solvable mixed charge ensemble on the line: global results*, Probab. Theory Related Fields **155** (2013), 127–164.

[31] W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, New York, 1964.

[32] B. Simon, *Weak convergence of CD kernels and applications*, Duke Math. J. **146** (2009), 305–330.

[33] C. D. Sinclair, *The range of multiplicative functions on* $\mathbb{C}[x], \mathbb{R}[x]$, *and* $\mathbb{Z}[x]$, Proc. London Math. Soc. **96** (2008), 697–737.

[34] C. D. Sinclair, *Correlation functions for* $\beta = 1$ *ensembles of matrices of odd size*, J. Stat. Phys. **136** (2009), 17–33.

[35] C. D. Sinclair, *Ensemble averages when* β *is a square integer*, Monatsh. Math. **166** (2012), 121–144.

[36] G. Szego, *Orthogonal Polynomials*, American Mathematical Society, Providence, RI, 1975.

[37] T. Tao and V. Vu, *Random matrices: The four moment theorem for Wigner ensembles*, Arxiv preprint arXiv:1112.1976, 2011

[38] C. A. Tracy and H. Widom, *Level spacing distributions and the Airy kernel*, Comm. Math. Phys. **159** (1994), 151–174.

[39] C. A. Tracy and H. Widom, *Level spacing distributions and the Bessel kernel*, Comm. Math. Phys. **161** (1994), 289–309.

[40] C. A. Tracy and H. Widom, *On orthogonal and sympletic matrix ensembles*, Comm. Math. Phys. **177** (1996), 727–754.

[41] C. A. Tracy and H. Widom, *Correlation functions, cluster functions and spacing distributions for fandom matrices*, J. Stat. Phys. **83** (1998), 809–835.

[42] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, University Press, Cambridge, 1944.

[43] D. Wells, *The Penguin Dictionary of Curious and Interesting Numbers*, Penguin Books Ltd, Harmondswork, Middlesex, England, 1986.

[44] E. P. Wigner, *Characteristic vectors of bordered matrices with infinite dimensions*, Annals Math. **62** (1955), 548–564.