THE TRACIAL ROKHLIN PROPERTY FOR COUNTABLE DISCRETE AMENABLE GROUP ACTIONS ON NUCLEAR TRACIALLY APPROXIMATELY DIVISIBLE C^*-ALGEBRAS

by

MICHAEL YUAN SUN

A DISSERTATION

Presented to the Department of Mathematics
and the Graduate School of the University of Oregon
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

June 2014

DISSERTATION APPROVAL PAGE

Student: Michael Yuan Sun

Title: The Tracial Rokhlin Property for Countable Discrete Amenable Group Actions on Nuclear Tracially Approximately Divisible C^*-Algebras

This dissertation has been accepted and approved in partial fulfillment of the requirements for the Doctor of Philosophy degree in the Department of Mathematics by:

Huaxin Lin	Chair
N. Christopher Phillips	Core Member
Victor Ostrik	Core Member
Alexander Kleshchev	Core Member
Hailin Wang	Institutional Representative

and

Kimberly Andrews Espy	Vice President for Research & Innovation/ Dean of the Graduate School

Original approval signatures are on file with the University of Oregon Graduate School.

Degree awarded June 2014

DISSERTATION ABSTRACT

Michael Yuan Sun

Doctor of Philosophy

Department of Mathematics

June 2014

Title: The Tracial Rokhlin Property for Countable Discrete Amenable Group Actions on Nuclear Tracially Approximately Divisible C^*-Algebras

In this dissertation we explore the question of existence of a property of group actions on C^*-algebras known as the tracial Rokhlin property. We prove existence of the property in a very general setting as well as specialise the question to specific situations of interest.

For every countable discrete elementary amenable group G, we show that there always exists a G-action ω with the tracial Rokhlin property on any unital simple nuclear tracially approximately divisible C^*-algebra A. For the ω we construct, we show that if A is unital simple and $\mathcal{Z}$-stable with rational tracial rank at most one and G belongs to the class of countable discrete groups generated by finite and abelian groups under increasing unions and subgroups, then the crossed product $A \rtimes_\omega G$ is also unital simple and $\mathcal{Z}$-stable with rational tracial rank at most one.

We also specialise the question to UHF algebras. We show that for any countable discrete maximally almost periodic group G and any UHF algebra A, there exists a strongly outer product type action α of G on A. We also show the existence of countable discrete almost abelian group actions with the "pointwise" Rokhlin property on the universal UHF algebra. Consequently we get many examples of unital separable simple nuclear C^*-algebras with tracial rank zero and a unique tracial state appearing as crossed products.

CURRICULUM VITAE

NAME OF AUTHOR: Michael Yuan Sun

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:
University of Oregon, Eugene, United States
University of Sydney, Sydney, Australia

DEGREES AWARDED:
Doctor of Philosophy, Mathematics, 2014, University of Oregon
Master of Science, Mathematics research, 2009, University of Sydney
Bachelor of Science (Honors), Mathematics and Physics, 2008, University of Sydney

AREAS OF SPECIAL INTEREST:
Pure mathematics

PROFESSIONAL EXPERIENCE:

Created SUN Mathematics Limited Liability Company

GRANTS, AWARDS AND HONORS:

Borsting Prize for strongest graduating student, University of Oregon, 2014

Johnson Award, University of Oregon, 2013

Graduate Student Research Award, University of Oregon, 2012

Australian Postgraduate Award, 2009

Vice-Chancellors Research Award, University of Sydney, 2009

University of Sydney Medal, 2008

University of Sydney Scholarship for Outstanding Achievement, 2005-2008

PUBLICATIONS:

Ostrik, V., & Sun, M. Y. (2014). Level-rank duality via tensor categories. *Communications in Mathematical Physics*, 326, 49-61.

Sun, M. Y. (2011). Point stabilizers for the enhanced and exotic nilpotent cones. *Journal of Group Theory*, 14, 825-839.

ACKNOWLEDGEMENTS

I thank Professor N. Christopher Phillips for his careful proofreading of this dissertation, his generosity with his time and his willingness to help in general. I also want to thank my teacher Victor Ostrrik for his part in my mathematical development, it is both an honor and a privilege to be considered his student.

Dedicated to 林华新 and 韦郦珊, with love and admiration.

With fainting soul athirst for grace,

I wandered in a desert place.

And at the crossing of the ways...

you found me and I was saved.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

At San Francisco International Airport en route to a conference in Shanghai, Professor Lin asked me: “Is there an action of $\mathbb{Z}$ on the universal UHF algebra with the Rokhlin property?” The question was unexpected but seemed spontaneous and fascinating. I had never heard of the *Rokhlin property*, but I had learned what *UHF-algebras* were in class. Another question quickly interrupted the silence: “What if I gave you a $\mathbb{Z}$-action could you find me a $\mathbb{Z}^2$-action?” I was excited to hear about the $\mathbb{Z}$-action because this gave me a chance to get a clue into what the definition of the Rokhlin property actually was! This line of questioning continued and became increasingly difficult. We reached a point where it seemed that some groups just did not act with the Rokhlin property. Then came the question, which unbeknownst to me at that time, inspired this thesis:

“What about the *tracial Rokhlin property*?”

What was the Rokhlin property and why did people want to find examples of it in group actions on C^* algebras? How does its tracial analogue help with matters? These questions will mark the starting point of our investigations.

One fundamental way to investigate the structure of a C^*-algebra is through the study of its group actions. Not only does it reveal the inherent symmetries of the C^*-algebra, but actually allows one to exploit them to construct more C^*-algebras. An indispensable part of this theme is the crossed product construction. Given a discrete group G, a unital C^*-algebra A and a group action α of G on A, we can construct a C^*-algebra called the crossed product and denoted by the less common notation $A \rtimes_\alpha G$ to emphasize our assumptions on A and G . It has the following presentation:

$$A \rtimes_\alpha G = \langle a, u_g \,|\, a \in A, g \in G, \alpha_g(a) = u_g a u_g^* \rangle.$$

Asking about what sort of C^*-algebras one gets brings us to another aspect of the study of C^*-algebras: the classification of simple nuclear C^*-algebras using the Elliott invariant. It is remarkable that so basic an invariant can determine so much about the structure of a simple C^*-algebra. At the forefront of this success are the large classes of unital simple separable nuclear

($\mathcal{Z}$-stable) C^*-algebras which have *tracial rank* zero, tracial rank at most 1, *rational tracial rank* zero or rational tracial rank at most 1 (satisfying the Universal Coefficient Theorem (UCT)), which were discovered to be classifiable by Lin (26), (27), (29) and by Lin, Niu and Winter (49), (23). Classifiable C^*-algebras necessarily possess a property called $\mathcal{Z}$-stability which is attracting a lot of attention in work extending Elliott's classification program, while those C^*-algebras of tracial rank at most one are also known to be *tracially approximately divisible* (Lin (29, Theorem 5.4)).

From the point of view of classification, the crossed product construction gives a way to explicitly construct algebras which belong to a large class of classifiable C^*-algebras that were otherwise only identifiable by their Elliott invariant. Conversely, the classification of C^*-algebras allows one to distinguish between group actions through examining the crossed products, which also brings clarity to the crossed product construction itself by giving an identity to algebras otherwise only defined by generators and relations.

Of major interest with respect to these two themes would be a property of group actions that could, when acting on an algebra belonging to a classifiable class, produce a crossed product algebra that belonged to the same class. A long standing candidate for this property is one known as the *Rokhlin property.* For example, it can be shown that finite group actions with the Rokhlin property on approximately finite-dimensional C^*-algebras will have approximately finite dimensional crossed products (Phillips (42)). One major drawback to this property is that it seems too scarce to be able to fullfil the required role in many common situations. For example, in the case for finite groups, the size of the group will usually need to divide some key parameter of the C^*-algebra. Hence another property is sought that will command a similar influence but exist in greater abundance. In line with the theme of Lin's celebrated breakthrough in the classification program, a natural such weakening would manifest itself in the form of the *tracial Rokhlin property.* In this case, Phillips (42) gave the definition for finite groups having the tracial Rokhlin property on simple C^*-algebras and showed that having tracial rank zero is preserved by taking crossed products. Now that we see it shows some promise in achieving our purposes, what about the question of existence? Are actions with this property abundant enough to be of practical importance?

The question of how abundant this property is and the ways that it can manifest will be the main focus of this thesis. This is presented in the first part from Chapter III through to Chapter VII. The second part of this thesis, consisting of Chapter VII and Chapter IX, will verify that the actions constructed actually produce crossed products with the desired properties and hence allow us to construct certain algebras in a classifiable class as crossed products.

We have the privilege of this investigation because we have been recently blessed by Matui-Sato's (37) definition of the tracial Rokhlin property for amenable groups with a demonstration that the property can be used to preserve the property of $\mathcal{Z}$-stability from algebra to crossed product. We will henceforth refer to their definition as the *Matui-Sato tracial Rokhlin property.* Before this, a definition of the tracial Rokhlin property was only stated for $\mathbb{Z}$ (Osaka-Phillips (41)) and finite groups (Phillips (42)). Phillips (43) also showed the abundance of cyclic group actions with the tracial Rokhlin property, while for example Lin (25) gives an existence (and uniqueness) result for $\mathbb{Z}$ building on the work of Kishimoto and others.

In this thesis, we will construct for every countable discrete group G and $\mathcal{Z}$-stable C^*-algebra A, an action ω of G on A to prove the following theorem in Chapter VII:

Theorem (Corollary VII.14)**.** *Given a countable discrete elementary amenable group G and a unital simple separable $\mathcal{Z}$-stable tracially approximately divisible C^*-algebra A, then there exists a group action ω of G on A such that ω has the tracial Rokhlin property.*

Formally, we show that ω has the Matui-Sato tracial Rokhlin property as an action of G on $A \otimes \mathcal{Z}$. Built into the construction of ω is a family of G-actions γ on the Jiang-Su algebra $\mathcal{Z}$, which we also introduce in Chapter VI. An investigation into the classifiability of $\mathcal{Z} \rtimes_\gamma G$ and $A \rtimes_\omega G$ is undertaken in Chapter IX. There we obtain as part of Theorem IX.17 the following result:

Theorem. *Suppose A is a unital simple $\mathcal{Z}$-stable C^*-algebra with rational tracial rank at most one and G belongs to the class of groups generated by finite and abelian groups under increasing unions and taking subgroups. Then $A \rtimes_\omega G$ is a unital simple $\mathcal{Z}$-stable C^*-algebra with rational tracial rank at most one.*

One can also show for all G and all A in the above theorem satisfying certain UCT requirements that $A \rtimes_\omega G$ has tracial rank zero when A has tracial rank zero. So there is essentially always an action present on every unital simple separable nuclear tracial rank zero algebra that has the tracial Rokhlin property and a tracial rank zero crossed product, despite the problem of proving this in general being open. As part of our investigation we specialise to a very familiar class of C^*-algebras where many of the earlier investigations into the Rokhlin property began, the so-called *uniformly hyperfinite* algebras. We will see that part of our construction for group actions on $\mathcal{Z}$ will adapt particularly well to this new situation.

Uniformly hyperfinite algebras, or UHF algebras for short, represent some of the earliest and most fundamental examples of unital simple C^*-algebras. Their study and classification is attributed to Dixmier (5) and Glimm (9). Despite the classical status of UHF algebras in the theory of C^*-algebras, many new insights can still be gained from studying them and their group actions. One way to define a UHF algebra is to start with a sequence $(n_l)_{l\in\mathbb{N}}$ of strictly positive integers and then associate to it the C^*-algebra $M_{(n_l)_{l\in\mathbb{N}}}$ using an infinite tensor product (see Definition III.1). That is

$$M_{(n_l)_{l\in\mathbb{N}}} = M_{n_1} \otimes M_{n_2} \otimes \cdots \otimes M_{n_l} \otimes \ldots .$$

Looking at these UHF algebras together with their tensor product decomposition can lead us to an interesting point of view. For example, if $(n_l)_{l\in\mathbb{N}}$ is a constant sequence, the algebra $M_{(n_l)_{l\in\mathbb{N}}}$ is *strongly self-absorbing* in the sense of Winter (48). This perspective has had considerable success and has allowed one to *localise the Elliott conjecture* at a UHF algebra (see Winter (49)). Lin, Niu (23) and Winter (49) made use of this to extend Lin's celebrated classification of unital simple separable nuclear C^*-algebras A of tracial rank at most one and which satisfy the UCT to those A that only had to have tracial rank at most one *after* tensoring by a strongly self-absorbing UHF algebra. This is the class of unital separable simple nuclear $\mathcal{Z}$-stable C^*-algebras of rational tracial rank at most one and satisfies the UCT. It is also interesting to note that part of the argument used to prove that ω has the tracial Rokhlin property on $A \otimes \mathcal{Z}$ for all A was to first simulate it on $A \otimes \mathcal{Z}$ when A is a UHF algebra.

With these developments in mind we study group actions on UHF algebras. Given our definition, it is natural to look at those group actions that preserve some tensor product decomposition. Furthermore we will for convenience look at those actions that are *inner* on each factor, so that for each $l \in \mathbb{N}$, we have a group homomorphism

$$G \to U(M_{n_l}) \to \operatorname{Aut} M_{n_l}.$$

Putting this together we have (cf. Definition IV.3)

$$G \to \prod_{l=1}^{\infty} U(M_{n_l}) \to \operatorname{Aut}\left(\bigotimes_{l=1}^{\infty} M_{n_l}\right) = \operatorname{Aut} M_{(n_l)_{l\in\mathbb{N}}}.$$

These represent the class of actions that are most accessible to study. Among these actions we look for examples of actions satisfying some sort of *Rokhlin property*, *tracial Rokhlin property* or *strong outerness*, listed from strongest to weakest. The actions with these properties represent the unknown that we are trying to investigate and hope to gain insight from. Finding these properties together in the same action will serve to provide model group actions for the latter property. Again, the novelty of our investigation as compared to the many that have preceded us is that we now have at our disposal all of the groups included in Matui-Sato's tracial Rokhlin property. These at least include the countable discrete elementary amenable groups. Previous investigations have examined and exhausted actions of finite groups on UHF algebra with the Rokhlin property ((7), (8), (13), (11)), as well as single automorphisms of infinite order ((12), (1), (20)), which can be considered as actions by the group of integers. Higher rank free abelian groups followed not long after ((40), (39)). In all of these cases, product type actions were used as the model actions. The Klein bottle group was the first example of an infinite non-abelian group and its actions were classified by Matui-Sato (37). We will give a model product type action for this group in Chapter V.

The existence theorem (Corollary VII.14) mentioned above already shows that group actions on UHF algebras with the tracial Rokhlin property always exist. Now we see if they can always exist as a product type action.

If the action is outer, then the group homomorphism must be injective. In particular, we have an embedding

$$G \hookrightarrow \prod_{l=1}^{\infty} U(M_{n_l}).$$

These groups are known as *maximally almost periodic* groups. So our question, in its weakest version, becomes

Question. *Which maximally almost periodic groups can have strongly outer product type actions on a UHF algebra?*

We show in Chapter IV that the answer is "all of them". There we use the same method of obtaining strongly outer actions from faithful ones employed on the Jiang-Su algebra. This is particularly well-suited to the situation for UHF-algebras because it preserves the property of being a product type action. With a few adjustments we arrive at Theorem IV.12, which says

Theorem. *If G is a countable discrete maximally almost periodic group and A is any UHF algebra, then there is a strongly outer product type action of G on A. If G is also elementary amenable then α has the tracial Rokhlin property.*

We then give some examples of strongly outer abelian group actions in the following section of Chapter IV. In Chapter V we compare and contrast the abundance of strongly outer product type actions with the scarcity of actions with the Rokhlin property. We avoid a definition of the Rokhlin property for G by using a substitute called the *pointwise Rokhlin property* that serves to illustrate our point. In this case we quickly realise that the universal UHF algebra $\mathcal{Q}$ maximises our chances of finding group actions and we arrive at Theorem V.9.

Theorem. *If G is a countable discrete almost abelian group, then there is a product type action of G on $\mathcal{Q}$ with the (pointwise) Rokhlin property.*

In Chapter VI we focus purely on almost abelian group actions on $\mathcal{Q}$ and look for model actions within a particularly nice class of residually finite group actions.

In Chapter XIII we emphasise the convenience of the properties we have imposed when investigating the crossed products formed. For example, we know that being an inner action on each tensor factor will guarantee us that the crossed product is amenable quasidiagonal and satisfies the UCT. Strong outerness will tell us that the crossed product will have a unique tracial state, among other things.

Theorem. *Suppose G is a countable discrete maximally almost periodic amenable group, A is any UHF algebra and α is a product type action of G on A with the tracial Rokhlin property. Then $A \rtimes_\alpha G$ is unital simple separable nuclear with tracial rank zero, satisfies the Universal Coefficient Theorem and has a unique tracial state. Furthermore, if G is almost abelian, then $A \rtimes_\alpha G$ is also locally type I. Moreover, if G is finite, then $A \rtimes_\alpha G$ is approximately finite dimensional.*

As it might already have been hinted at, the organisation of this thesis will take on a somewhat reversed order to that in which the topics were just introduced. After some preliminary definitions and background material, we will develop our arguments from the more specific situation of single automorphisms on UHF algebras in Chapter III to the more general situation of group actions on tracially approximately divisible C^*-algebras in Chapter VII, finishing with an independent investigation into the crossed product algebras in Chapters VIII and IX. Indeed, Chapters III through VI concern themselve solely with UHF algebras, while Chapter VII moves on to the Jiang-Su algebra and exhibiting the general existence theorem. In principle, one could start with Chapter VII without compromising the logical integrity of the thesis. We have chosen this order of presentation so that the reader may first be exposed to a concrete elementary example of the Rokhlin property in action, tracial or otherwise, which contains much of the relevant insight into later developments and will serve as a platform for elevating these ideas to those situations.

Within the discussion of UHF-algebras we move from the very specific situation of single automorphisms in Chapter III to the least specific situation of strongly outer group actions in Chapter IV. From here we look to strengthen the condition of strongly outer to that of the pointwise Rokhlin property in Chapter V, which both limit the groups and algebras that we examine. We further specialise the situation in Chapter VI to the universal UHF algebra and have specific requirements on the way the groups act so as to ensure the nicest model actions for some of the more common groups obtained in Chapter V.

CHAPTER II

PRELIMINARIES

We give here some background material and basic definitions for C^*-algebras (see for example, Lin (21)). As the objects in this investigation will usually be loaded with adjectives we take the time here to list what many of the more common ones mean but will assume some basic knowledge of the more elementary jargon so as to avoid being overwhelmed by detail.

C^*-algebras: Separable, Simple, Nuclear, $\mathcal{Z}$-Stable, UCT

A C^*-algebra A is an algebra over $\mathbb{C}$ equipped with a norm $\|\cdot\|$ and a conjugate linear isometric involution $*$ such that

- $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$,
- $(ab)^* = b^*a^*$ for all $a, b \in A$,
- $\|a^*a\| = \|a\|^2$ for all $a \in A$,
- A is complete with respect to $\|\cdot\|$.

To have a homomorphism of C^*-algebras it suffices to have a homomorphism of their underlying algebras that preserves the $*$-operation. These are called $*$-homomorphisms, or $*$-isomorphisms if the underlying algebra homomorphisms are isomorphisms. For C^*-algebras A and B we write $A \cong B$ if A and B are isomorphic. We list some common assumptions on C^*-algebras in our investigation. We will reserve the upper case letter A for C^*-algebras. Also

$$a \approx_\epsilon b \text{ stands for } \|a - b\| < \epsilon.$$

Unital

A C^*-algebra A is called unital if it contains an element 1_A such that

- $\|1_A\| = 1$.
- $1_A b = b 1_A = b$ for all $b \in A$.

All of the C^*-algebras we consider will be unital.

Separable

A C^*-algebra is separable if there is a countable dense subset.

Simple

A C^*-algebra is simple if it has no proper closed two-sided ideals.

Nuclear (or Amenable)

We choose the more practical characterisation of amenability to reflect its use in this thesis. A C^*-algebra A is said to be nuclear (or amenable) if for any C^*-algebra B, the algebraic tensor product of A and B has only one possible norm so that the completion is a C^*-algebra. That is, there is only one possible way to define the C^*-algebra $A \otimes B$.

Let A and B be unital nuclear C^*-algebras. We have

$$A \otimes B = \langle a, b \,|\, a \in A, b \in B, ab = ba \rangle.$$

If $a \in A$ and $b \in B$ we denote their product in $A \otimes B$ by

$$a \otimes b = ab = ba$$

and note that

$$\|a \otimes b\| = \|a\|\|b\|.$$

Implicitly there are unital embeddings

$$A \to A \otimes B \colon a \mapsto a \otimes 1 \quad \text{and} \quad B \to A \otimes B \colon b \mapsto 1 \otimes b.$$

Also, for any unital C^*-algebra C, whenever there are pairs of unital embeddings

$$A \to C, \qquad B \to C$$

such that the images commute, there is a unital embedding

$$A \otimes B \to C.$$

$\mathcal{Z}$-stability

Let $\mathcal{Z}$ denote the Jiang-Su algebra (introduced further in Chapter 5). We say a C^*-algebra A is *$\mathcal{Z}$-stable* or *$\mathcal{Z}$-absorbing* if $A \otimes \mathcal{Z} \cong A$.

Universal Coefficient Theorem (UCT)

This is a technical condition that would take us too far astray to define. Suffice it to say that there are no nuclear C^*-algebras known to not satisfy the UCT and that the condition is needed to ensure the classification results in the section after the next section.

C^*-algebras: Projections, Unitaries and Traces

Let A be a unital C^*-algebra. Let M_n denote the algebra of $n \times n$ matrices, let 1_n denote its identity and let τ_n denote the trace normalised so that $\tau_n(1_n) = 1$.

Projections

An element $a \in A$ is called *self-adjoint* if $a^* = a$. Say a is positive and write $a \in A_+$ if a is self-adjoint and has real spectrum. If also $a^2 = a$ we call it a projection. We will usually reserve the lower case letters p and q to denote projections. We say projections p and q in A are *Murray von-Neumann equivalent* if there is a $v \in A$ such that $vv^* = p$ and $v^*v = q$. Let $p \in A$ be a projection and let $a \in A_+$. We write

$$p \preceq a$$

if p is Murray-von Neumann equivalent to a projection in $\overline{aAa}$.

Lemma II.1. *Suppose $\delta \in (0, 1/4)$ and $a \in A$ is self-adjoint and satisfies*

$$a^2 \approx_\delta a.$$

Then there is a projection p such that

$$p \approx_{2\delta} a.$$

Proof. See for example Lin (21, Lemma 2.5.5). □

Lemma II.2. *Let $\epsilon > 0$ and let $n \in \mathbb{N}$. There is a $\delta > 0$ such that for any projections $q_1, \ldots, q_n$ such that*

– $q_i q_j \approx_\delta 0$,

there exist mutually orthogonal projections $p_1, \ldots, p_n$ such that for $1 \leq i \leq n$

$$p_i \approx_\epsilon q_i.$$

Proof. See for example Lin (21, Lemma 2.5.6). □

Identifying $M_n(A)$ as a subalgebra of $M_{n+1}(A)$ via the embedding $a \mapsto \mathrm{diag}(a, 0)$ we define

$$M_\infty(A) = \bigcup_{n=1}^{\infty} M_n(A).$$

If $p_n \in M_n(A)$ and $q_{n'} \in M_{n'}(A)$ are projections, we say that they are *stably equivalent* if there is an l and an L such that $\mathrm{diag}(p_n, 1_l, 0_{L-l-n})$ is Murray-von Neumann equivalent to $\mathrm{diag}(q_{n'}, 1_l, 0_{L-l-n'})$ in $M_L(A)$. Writing $[p]$ for the equivalence class of p with respect to stable equivalence, we set

$$K_0(A)_+ = \{[p] \mid p = p^2 = p^* \in M_\infty(A)\}.$$

This is a cancellative semigroup under the addition defined by $[p] + [q] = [\mathrm{diag}(p, q)]$ and so it embeds into its Grothendieck group

$$K_0(A) = \{[p] - [q] \,|\, [p], [q] \in K_0(A)_+\}$$

of formal differences $[p] - [q]$ subject to the relation $[p_1] - [q_1] = [p_2] - [q_2]$ if $[p_1] + [q_2] = [p_2] + [q_1]$ in $K_0(A)_+$.

Unitaries

An element $u \in A$ is called a unitary if $u^*u = 1$ and $uu^* = 1$. We will write $U(A)$ for the group of unitaries in A and $U(A)_0$ for its path component of the identity. We have maps

$$U(M_n(A)) \to U(M_{n+1}(A)) \colon u \mapsto \mathrm{diag}(u, 1).$$

Define

$$K_1(A) = \varinjlim_n U(M_n(A))/U(M_n(A))_0,$$

which we note without proof is an abelian group.

Traces

A tracial state τ on A is a positive linear function $A \to \mathbb{C}$ such that

- $\tau(ab) = \tau(ba)$ for all $a, b \in A$,
- $\tau(1_A) = 1$.

It is convenient to note that for any $a, b \in A$ if $a \approx_\epsilon b$ then $\tau(a) \approx_\epsilon \tau(b)$ since $|\tau(x)| \leq \|x\|$ for all $x \in A$.

Lemma II.3. *If $p, q \in A$ are projections such that $p \approx_1 q$, then $\tau(p) = \tau(q)$.*

Proof. See for example (21, Lemma 2.5.1). □

We write $T(A)$ for the tracial state space of A, which is convex set. Denote by $\mathrm{Aff}(T(A))$ the set of real weak* continuous affine functions on $T(A)$, that is,

$$\mathrm{Aff}(T(A)) = \{f : T(A) \to \mathbb{R} \,|\, f \text{ is affine}\}.$$

For a projection $p \in M_n(A)$, define an associated evaluation function $\widehat{p} \in \mathrm{Aff}(T(A))$ by

$$\widehat{p}(\tau) = (\tau_n \otimes \tau)(p)$$

for all $\tau \in T(A)$. Then there is a group homomorphism

$$\rho_A : K_0(A) \to \mathrm{Aff}(T(A))$$

defined on $K_0(A)_+$ by

$$[p] \mapsto \widehat{p}.$$

For a unital simple C^*-algebra A with at least one tracial state, define a norm $\|\cdot\|_2$ for $a \in A$ by

$$\|a\|_2 = \sup_{\tau \in T(A)} \tau(a^*a)^{1/2}.$$

Suppose A is unital and simple with unique tracial state τ. Then we define the Gelfand-Naimark-Segal representation with respect to τ in this case. Define a positive definite sesquilinear form by

$$\langle a, b \rangle = \tau(b^*a)$$

for all $a, b \in A$.

Take the completion and call the Hilbert space obtained H_τ with $\widetilde{a}$ the image of $a \in A$. Then define $\pi_\tau : A \to B(H_\tau)$ by

$$\pi_\tau(a)\widetilde{b} = \widetilde{ab}.$$

We also have for all $a \in A$

$$\tau(a) = \langle \pi_\tau(a)\widetilde{1_A}, \widetilde{1_A} \rangle.$$

C^*-algebras: Tracial Rank and Elliott's Program

Tracial rank

A simple unital C^*-algebra is said to have *tracial rank zero* or be *tracially approximately finite dimensional* if for any $\epsilon > 0$, any $a \in A_+$ and every finite subset $\{a_1, a_2, \ldots, a_n\}$ in A, there is a finite dimensional subalgebra B with 1_B denoted by p, such that there exist $b_1, \ldots b_n \in B$ such that

- $[p, a_i] \approx_\epsilon 0$ for $1 \le i \le n$.

- $pa_ip \approx_\epsilon b_i$ for $1 \le i \le n$.

– $p \preceq a$.

If instead of a finite dimensional algebra, B is allowed to be of the form

$$\bigoplus_{i=1}^{m} p_i M_{n_i}(C(X)) p_i$$

where X a CW-complex of dimension at most one and p_i is a projection in $M_{n_i}(C(X))$, then we say that A has *tracial rank at most one*. Let $\mathcal{Q}$ be the universal UHF algebra (to be defined in chapter 2). Then A is said to have *rational tracial rank zero* or *rational tracial rank at most one* if $\mathcal{Q} \otimes A$ has tracial rank zero or tracial rank at most one respectively. We see that the weakest condition is to have rational tracial rank at most one and includes the algebras with the other conditions just defined.

Elliott invariant

Define the Elliott invariant for a unital C^*-algebra A to be

$$\mathrm{Ell}(A) = (K_0(A), K_0(A)_+, [1_A], K_1(A), T(A), \rho_A).$$

If A and B are unital C^*-algebras we say that $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$ if there are isomorphisms of groups and spaces

– $\varphi_0 : K_0(A) \to K_0(B)$,

– $\varphi_1 : K_1(A) \to K_1(B)$,

– $\lambda : T(B) \to T(A)$,

such that the following conditions are satisfied:

– $\varphi_0([1_A]) = [1_B]$,

– $\varphi_0(K_0(A)_+) = K_0^+(B)$,

and that the following diagram commutes:

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\rho_A} & \mathrm{Aff}(T(A)) \\ \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \mathrm{Aff}(\lambda)} \\ K_0(B) & \xrightarrow{\rho_B} & \mathrm{Aff}(T(B)). \end{array}$$

Elliott's classification program

The Elliott classification program is an attempt to find classes $\mathcal{A}$ (as large as possible) of unital separable simple nuclear C^*-algebras that can be classified using the Elliott invariant. That is to say, if A and B are in $\mathcal{A}$ and $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$, then $A \cong B$. Here are some of the recent breakthroughs in the program.

Theorem II.4 (Lin (26) (27)). *The class of unital separable simple nuclear C^*-algebras with tracial rank at most one and satisfying the UCT can be classified by the Elliott invariant.*

Theorem II.5 (Lin-Niu (23), Winter (49)). *The class of unital separable simple nuclear $\mathcal{Z}$-stable C^*-algebras with rational tracial rank at most one and satisfying the UCT can be classified by the Elliott invariant.*

Groups: Countable, Amenable, Elementary Amenable

We will reserve the upper case letter G for groups.

Discrete

A group is called discrete if its underlying set is equipped with the discrete topology. Consequently the multiplcation and inversion maps are automatically continuous and any map from G is automatically continuous. So any group is a discrete topological group and we will often omit the word "topological".

Countable

A group is countable if its underlying set is countable. This assumption will ensure the crossed product algebras we form are separable.

Amenable

We will see that the definition of the tracial Rokhlin property used here will only make sense for amenable groups.

Definition II.6 ((F,ϵ)-invariance)**.** Let G be a countable discrete group, let $F \subset G$ be a finite subset and let $\epsilon > 0$. We say a finite subset K of G is (F,ϵ)-invariant if $K \neq \emptyset$ and

$$\left| K \cap \bigcap_{g \in F} g^{-1}K \right| \geq (1-\epsilon)|K|.$$

Definition II.7 (Amenability)**.** A countable discrete group G is said to be *amenable* if an (F,ϵ)-invariant subset exists from any finite set F and $\epsilon > 0$. The group G is said to be *elementary amenable* if it is contained in the smallest class of groups that contains all abelian groups, all finite groups and is closed under taking subgroups, quotients, direct limits and extensions.

Crossed Products: Automorphisms and Actions

Let A be a unital C^*-algebra.

Automorphisms

An automorphism of A is a $*$-isomorphism from A to A. Let $\operatorname{Aut} A$ be the group of all automorphisms of A. For a unitary $u \in A$ define $\operatorname{Ad} u \in \operatorname{Aut} A$ for all $x \in A$ by

$$(\operatorname{Ad} u)(x) = uxu^*.$$

These are called inner automorphisms. Write (π_τ, H_τ) for the GNS-representation with respect to $\tau \in T(A)$. An automorphism is called *weakly inner* if it is inner when extended to an automorphism of $\pi_\tau(A)''$, the weak operator closure of $\pi_\tau(A)$, for all α-invariant $\tau \in T(A)$.

Group actions

Let G be a discrete group. We will usually denote a group action of G on A as a homomorphism

$$\alpha : G \to \operatorname{Aut} A$$

and the automorphism by which $g \in G$ acts as α_g. We ignore any continuity conditions as we will only look at discrete groups.

Crossed products

For any unital C^*-algebra A, any discrete group G and any action α of G on A, define the crossed product $A \rtimes_\alpha G$ to be the C^*-algebra with the presentation

$$A \rtimes_\alpha G = \langle a, u_g \,|\, a \in A, g \in G, \alpha_g(a) = u_g a u_g^* \rangle.$$

Implicitly the elements $a \in A$ satisfy the relations in A and the elements u_g for $g \in G$ satisfy the relations in G. That is, there is a unital embedding

$$A \to A \rtimes_\alpha G, \qquad a \mapsto a$$

and a group homomorphism

$$G \to U(A \rtimes_\alpha G), \qquad g \mapsto u_g.$$

For $g \in G$ we will refer to u_g as the canonical unitary implementing α_g. It is clear by definition that $A \rtimes_\alpha G$ is unital. Since G is discrete, finite sums of the form

$$\sum_g a_g u_g,$$

for $a_g \in A$ and $g \in G$, are dense in $A \rtimes_\alpha G$. We see that when G is countable $A \rtimes_\alpha G$ is separable. We see that for any unital C^*-algebra B, whenever there is a pair of maps

$$\psi : A \to B, \qquad \varphi : G \to U(B),$$

with ψ a unital embedding and φ a group homomorphism such that for all $a \in A$ and all $g \in G$,

$$\varphi(g)\psi(a)\varphi(g)^* = \psi(\alpha_g(a)),$$

then there is a canonical untial $*$-homomorphism $\Psi : A \rtimes_\alpha G \to B$ such that $\Psi|_A = \psi$ and $\Psi|_G = \varphi$.

Since G is discrete, there is also a *conditional expectation* $E : A \rtimes_\alpha G \to A$ defined so $au_g \mapsto 0$ for $g \neq 1$ and $au_1 \mapsto a$. If τ is a G invariant trace on A, then $\tau \circ E$ is a trace on $A \rtimes_\alpha G \to A$.

The Tracial Rokhlin Property

Definition II.8 (Tracial Rokhlin property)**.** A group action α of G on a C^*-algebra A has the *tracial Rokhlin property* if for every finite subset $F \subset G$, any $\epsilon_G > 0$, there is a finite (F, ϵ_G)-invariant subset K in G such that for every $\epsilon_A > 0$, every finite subset $\{x_1, \ldots, x_n\} \subset A$ and all non-zero $a \in A_+$, and mutually orthogonal projections $(p_k)_{k \in K}$ such that for all $h \in K$ and $g \in K \cap h^{-1}K$, and writing $p = \sum_{k \in K} p_k$, we have

- $[p_h, x_i] \approx_{\epsilon_A} 0$ for $1 \leq i \leq n$,
- $\alpha_h(p_g) \approx_{\epsilon_A} p_{hg}$,
- $1 - p \preceq a$.

The tracial Rokhlin property for amenable group actions is essentially new and was introduced by Matui-Sato. We will look in detail at aspects of this definition in Chapter III where the situation has been simplified as to be digestible.

There is also the weaker notion of this for algebras without projections called the *weak Rokhlin property* (Matui-Sato (37, Definition 2.5)).

Infinite Tensor Products and Strongly Outer Actions

Infinite tensor products

Definition II.9. For each $n \in \mathbb{N}$, let A_n be a unital nuclear C^*-algebra and let $\alpha_n \in \operatorname{Aut} A_n$. Then define the infinite tensor product as the C^*-algebra direct limit

$$\bigotimes_{n=1}^{\infty} A_n = \varinjlim \left(\bigotimes_{n=1}^{m} A_n, \operatorname{id} \otimes 1_{A_{m+1}} \right).$$

Also write $\bigotimes_{n=1}^{\infty} \alpha_n$ for the unique automorphism such that for $a_n \in A_n$

$$\left(\bigotimes_{n=1}^{\infty} \alpha_n \right) \left(\bigotimes_{n=1}^{m} a_n \right) = \bigotimes_{n=1}^{m} \alpha_n(a_n)$$

If A is a unital nuclear C^*-algebra and $A_n = A$ for all $n \in \mathbb{N}$, we sometimes write

$$A^{\otimes \mathbb{N}} = \bigotimes_{n=1}^{\infty} A.$$

If also $\alpha \in \operatorname{Aut} A$ and $\alpha_n = \alpha$ for all $n \in \mathbb{N}$, we sometimes write

$$\alpha^{\otimes \mathbb{N}} = \bigotimes_{n=1}^{\infty} \alpha.$$

If α is an action of G on A, we define $\alpha^{\otimes \mathbb{N}}$ by

$$(\alpha^{\otimes \mathbb{N}})_g = \alpha_g^{\otimes \mathbb{N}}$$

for all $g \in G$.

Strongly outer actions

Definition II.10. An action α of G on A is called *strongly outer* if α_g is not weakly inner for all $g \in G \setminus \{1\}$.

In the case an automorphism α is weakly inner we have for any invariant τ there is a unitary $u \in B(H_\tau)$ such that

$$\pi_\tau(\alpha(a)) = u \pi_\tau(a) u^*$$

for all $a \in A$. This means that there is a representation of the crossed product $\pi : A \rtimes \mathbb{Z} \to B(H_\tau)$ such that $\pi|_A = \pi_\tau$ and $\pi(u_1) = u$.

Definition II.11. Let $\widetilde{1_A} \in H_\tau$ be the vector obtained from 1_A in the GNS construction. Then we have the following positive state on $A \rtimes \mathbb{Z}$ extending τ:

$$\tau_{A \rtimes \mathbb{Z}}(x) = \langle \pi(x) \widetilde{1_A}, \widetilde{1_A} \rangle$$

Lemma II.12. *$\tau_{A \rtimes \mathbb{Z}}$ is a trace on $A \rtimes \mathbb{Z}$.*

Proof. By linearity and continuity of $\tau_{A \rtimes \mathbb{Z}}$ it suffices to show

$$\tau_{A \rtimes \mathbb{Z}}(a u^m b u^n) = \tau_{A \rtimes \mathbb{Z}}(b u^n a u^m)$$

for all $m, n \in \mathbb{Z}$, $a, b \in A$. We first show this for $m, n \geq 0$ by induction on $l = m + n$. This is clear for $l = 0$ since $\tau_{A \rtimes \mathbb{Z}}|_A = \tau_A$ is a trace.

Let $(v_k)_{k \in \mathbb{N}}$ be a sequence in A such that $v_k \to u$ in the weak operator topology. In the diagram below, the top equality is the induction hypothesis, the vertical arrows represent convergence, while the equality in the bottom row is the conclusion drawn from these facts, completing the induction step.

$$\begin{array}{ccc} \tau(a v_k u^{m-1} b u^n) & = & \tau(b u^k a v_k u^{m-1}) \\ \downarrow & & \downarrow \\ \tau(a u^m b u^n) & = & \tau(b u^n a u^m) \end{array}$$

Now we complete the proof in the case $m < 0$ or $n < 0$, by replacing the corresponding u by u^* in the argument above since v_k^* converges weakly to u^*. □

The following is based on Matui-Sato (37, Lemma 6.13).

Lemma II.13. *Let A_n be a sequence of unital simple nuclear C^*-algebras with unique tracial states τ_n and let $\alpha_n \in \mathrm{Aut}(A_n)$. For $A = \bigotimes_{n=1}^{\infty} A_n$ define $\alpha \in \mathrm{Aut}\, A$ by*

$$\alpha = \bigotimes_{n=1}^{\infty} \alpha_n.$$

Let τ be the unique tracial state on A. If there is a sequence of unitaries $v_n \in U(A_n)$ such that $\|\alpha_n(v_n) - v_n\|_2$ does not converge to 0, then α is not weakly inner.

Proof. Define a (central) sequence in A by

$$v(n) = 1 \otimes \cdots \otimes 1 \otimes v_n \otimes 1 \otimes \cdots,$$

where 1 appears in every factor except A_n. We will show that if α is weakly inner then

$$\|\alpha(v(n)) - v(n)\|_2 \to 0.$$

Assume there is a unitary $u \in \pi_\tau(A)''$ such that $\alpha_g^{\otimes \mathbb{N}} = \mathrm{Ad}\, u$ on $\pi_\tau(A)''$. This gives a representation of $A \rtimes_\alpha \mathbb{Z}$ on H_τ as described in the previous section and a sequence $(x_k)_{k \in \mathbb{N}}$ in

$\pi_\tau(A)$ such that $x_k \to u$ in the weak operator topology. Let $\epsilon > 0$, fix k so that

$$\|u - x_k\|_{2,A\rtimes\mathbb{Z}} \approx_{\epsilon/2} 0$$

by way of x_k strongly converging to u, and let n be large enough so that

$$[x_k, v(n)] \approx_\epsilon 0,$$

which is possible because $v(n)$ is a central sequence. Using $\|ab\|_2 \leq \|a\|_2\|b\|$ in the third step we now calculate:

$$\begin{aligned}
\|\alpha_g^{\otimes\mathbb{N}}(v(n)) - v(n)\|_{2,A} &= \|uv(n)u^* - v(n)\|_{2,A\rtimes\mathbb{Z}} \\
&= \|uv(n) - v(n)u\|_{2,A\rtimes\mathbb{Z}} \\
&\leq 2\|u - x_k\|_{2,A\rtimes\mathbb{Z}} + \|x_k v(n) - v(n)x_k\| \\
&\approx_\epsilon \|x_k v(n) - v(n)x_k\| \\
&\approx_\epsilon 0.
\end{aligned}$$

□

The following is essentially Matui-Sato (37, Lemma 6.13). We thank Y. Sato for communicating a proof of their lemma to us.

Corollary II.14. *Suppose A is a unital simple nuclear C^* algebra with a unique tracial state and $\alpha : G \to \mathrm{Aut}(A)$ corresponds to an action of G on A and $\ker\alpha = \{1_G\}$. Then the action $\alpha^{\otimes\mathbb{N}}$ of G on $A^{\otimes\mathbb{N}} = \bigotimes_{n=1}^\infty A$ is strongly outer.*

Proof. We use Lemma II.13 to show that for each $g \in G$ not 1, the automorphism that g acts by is not weakly inner. Since α is not trivial on A there is some $u \in A$ such that $\alpha(v) \neq v$. We can take v to be unitary since unitaries span A and α is linear. In particular, the sequence $\|\alpha_n(v_n) - v_n\|_2$ in Lemma II.13 with $\alpha_n = \alpha$ and $v_n = v$ for all $n \in \mathbb{N}$ is constant and non-zero. □

CHAPTER III

AUTOMORPHISMS OF UNIFORMLY HYPERFINITE C^*-ALGEBRAS

We look at the very specific situation of a single automorphism acting on a particularly nice class of C^*-algebras so that we may get a concrete understanding of the ideas involved. After introducing the definition and notation for UHF algebras in Section 1, we introduce the Rokhlin property for automorphisms of UHF algebras in Section 2 where we will embark on an informal discussion of the definition and present two fundamental examples of automorphisms with the Rokhlin property. Many of the observations discussed in Section 2 will be formalized into Lemmas and Propositions in Section 3, where the similar looking definition of the tracial Rokhlin property is introduced. Two particularly nice observations are presented in Section 3 as the "bump-up" and "cut-down" principles, which will be called upon in Chapter 2 and Chapter 3 respectively. An elementary lemma concerning the tensor product of automorphisms keeping the Rokhlin properties of the factors is stated and proved in Section 5.

Uniformly Hyperfinite C^*-algebras (UHF)

We will introduce the universal UHF algebra $\mathcal{Q}$ and other UHF algebras as infinite tensor products of full matrix algebras.

Definition III.1. Recalling the definition of the infinite tensor product (Definition II.9) we define for any sequence of strictly positive integers $(n_l)_{l\in\mathbb{N}}$ such that $n_l \geq 2$ for all $l \in \mathbb{N}$, its associated UHF algebra $M_{(n_l)_{l\in\mathbb{N}}}$ given by

$$\bigotimes_{l=1}^{\infty} M_{n_l} = M_{n_1} \otimes M_{n_2} \otimes \cdots \otimes M_{n_l} \otimes \ldots .$$

So we have in this notation a definition for the universal UHF algebra $\mathcal{Q}$ as

$$\mathcal{Q} = M_{(n)_{n\in\mathbb{N}}} = \bigotimes_{n=1}^{\infty} M_n .$$

Let $(n_l)_{l\in\mathbb{N}}$ and $(m_l)_{l\in\mathbb{N}}$ be sequences of strictly positive integers. We say that $(n_l)_{l\in\mathbb{N}}$ and $(m_l)_{l\in\mathbb{N}}$ are of the same *type* if

$$M_{(n_l)_{l\in\mathbb{N}}} \cong M_{(m_l)_{l\in\mathbb{N}}} .$$

We see that one way to get algebras of the same type is to regroup the factors. That is, a strictly positive sequence $(N_l)_{l\in\mathbb{N}}$ is called a *regrouping* of (n_l) if there is a strictly increasing sequence $(l_i)_{i=1}^{\infty}$ with $l_1 = 0$, such that

$$N_i = \prod_{l_i+1}^{l_{i+1}} n_j.$$

We say $(n_l)_{l\in\mathbb{N}}$ is of *infinite type* if

$$M_{(n_l)_{l\in\mathbb{N}}} \otimes M_{(n_l)_{l\in\mathbb{N}}} \cong M_{(n_l)_{l\in\mathbb{N}}},$$

which we note to be equivalent to saying that every prime divides infinitely many terms in the sequence. A special case of sequences of infinite type is the constant sequence $(n)_{l\in\mathbb{N}}$ for some $n \in \mathbb{N} \setminus \{0\}$, for which we will sometimes adopt the notation

$$M_{n^\infty} = M_{(n)_{l\in\mathbb{N}}}.$$

We will also use the notation $M_{(n_l)_{l\leq m}}$ to denote the corresponding finite dimensional subalgebra.

The Rokhlin Property for Automorphisms

Here we use the definition of the Rokhlin property for automorphisms as the focus of an informal discussion to introduce some basic ideas. We will eventually see how it is related to the definition in the preliminaries and later to the Matui-Sato definition of the tracial Rokhlin property.

Definition III.2. Let A be a unital C^*-algebra and $\alpha \in \operatorname{Aut} A$. We say α has the *Rokhlin property*, if for every $n' \in \mathbb{N}$, there exists $N' > n'$ such that for every $\epsilon > 0$ and every finite subset $\{a_1, \ldots, a_n\}$ in A, there exist mutually orthogonal projections $p_1, p_2, \ldots, p_{N'}$ such that

(i) $[p_i, a_j] \approx_\epsilon 0$ for $1 \leq i \leq N'$ and $1 \leq j \leq n$,

(ii) $\alpha(p_i) \approx_\epsilon p_{i+1}$ for $1 \leq i \leq N' - 1$,

(iii) $\sum_{i=1}^{N'} p_i = 1$.

So the Rokhlin property is about finding enough projections that satisfy certain conditions with respect to α. Notice that finding $N' > n'$ projections ensure that the order of α exceeds

n'. So as n' varies, this definition forces α to be of infinite order. The general definition involves two "towers" of projections but we will not deal with situations where it is not equivalent to the one tower version defined above and hence will stick with the one tower version for simplicity. We labelled these conditions (i), (ii) and (iii). Let us examine these conditions closely after seeing an example.

Consider for example the UHF algebra $M_{(2^l)_{l\in\mathbb{N}}}$. For each $l \in \mathbb{N}$ let $u_l \in U(M_{2^l})$ be the cyclic permutation of the standard basis vectors of order 2^l and let $\alpha \in \operatorname{Aut} M_{2^\infty}$ be defined by

$$\alpha = \bigotimes_{l=1}^{\infty} \operatorname{Ad} u_l.$$

We now show that α has the Rokhlin property. For any $n' \in \mathbb{N}$ we fix $N' = 2^{l'}$ for the minimum l' such that $2^{l'} > n'$. Let $\epsilon > 0$ and let $\{a_1, \dots, a_n\}$ be a finite subset of A. By the direct limit definition of M_{2^∞} there exists $L \geq l'$ and $a_1(L), \dots, a_n(L) \in \bigotimes_{l=1}^{L} M_{2^l}$ such that for $1 \leq i \leq n$, we have

$$a_i \approx_\epsilon a_i(L).$$

For $1 \leq j \leq 2^{L+1}$ let $e_{j,j} \in M_{2^{L+1}}$ denote the diagonal matrix units. Let $r(j)$ be the remainder when j is divided by $2^{l'}$ and for $1 \leq i \leq 2^{l'}$ set

$$p_i = \sum_{r(j)=i-1} e_{j,j}.$$

We check that these mutually orthogonal projections satisfy the requirements of the Rokhlin property. Let $1 \leq i \leq N'$ and $1 \leq j \leq n$. Since $a_j(L)$ and p_i lie in different tensor factors we have

$$\begin{aligned}[p_i, a_j] &\approx_{2\epsilon} [p_i, a_j(L)] \\ &= 0.\end{aligned}$$

Next let $1 \leq i \leq 2^{l'} - 1$. Then

$$\begin{aligned}
\alpha(p_i) &= \sum_{r(j)=i-1} \alpha(e_{j,j}) \\
&= \sum_{r(j)=i-1} \operatorname{Ad} u_{2^{l'}}(e_{j,j}) \\
&= \sum_{r(j)=i-1} \operatorname{Ad} u_{2^{l'}}(e_{j+1,j+1}) \\
&= \sum_{r(j)=i} \operatorname{Ad} u_{2^{l'}}(e_{j,j}) \\
&= p_{i+1}.
\end{aligned}$$

Finally we see that

$$\begin{aligned}
\sum_{i=1}^{2^{l'}} p_i &= \sum_{i=1}^{2^{l'}} \sum_{r(j)=i-1} e_{j,j} \\
&= \sum_{j=1}^{2^{L+1}} e_{j,j} \\
&= 1.
\end{aligned}$$

With this example in mind, we now return to examining the properties $(i) - (iii)$ and our informal discussion.

Property (i)

We saw for UHF algebras how tensor products work particularly well with condition (i), which after all is about commuting. Suppose the automorphism α preserves the tensor product decomposition of $M_{(n_l)_{l\in\mathbb{N}}}$ like in the above example. Then we would be able to always satisfy (i), (ii) and (iii) as long as we could satisfy just (ii) and (iii) outside any finite number of tensor factors. That is α has the Rokhlin property if it satisfies the Rokhlin property with condition (i) omitted on $M_{(n_l)_{l\geq m}}$ for all $m > 1$. The converse is also true because we can always include all of the matrix units of a finite number factors into the finite set which our projections will approximately commute with in (i) and hence the projections can be forced to be outside of those factors. (See Proposition III.10).

In a sense condition (i) is the only one that detects the whole algebra. Without it, the projections could potentially be found in any subalgebra left invariant by α. In this way to exhibit the Rokhlin property "globally" or for all n', one only needs to exhibit it "locally" for each n'. The problem with trying to take advantage of this for different algebras is that condition (iii) forces our invariant subalgebras to be unital subalgebras which are not always easy to find.

Recall that a central sequence $(a_n)_{n\in\mathbb{N}}$ in A is one such that for all $a \in A$, $aa_n - a_n a \to 0$ as $n \to \infty$. In a separable C^*-algebra to test the centrality condition for all $a \in A$, it suffices to test it for a countable dense subset. So if we take our finite subset $\{a_1, \ldots, a_n\}$ to be a finite subset of a countable dense subset and ϵ to decrease to 0 with n, then the projections $p_1, \ldots, p_{n'}$ we find (with n' fixed) form a central sequence. Conversely, a central sequence of projections will eventually satisfy (i) for any finite subset of A. The problem again is because of (iii), we may not be able to control the variance of N' and hence not be able to get well-defined sequences.

Property (ii)

Combined with property (i) we see that α must be an outer automorphism. This is because if there is a unitary u to implement α, we can include it in the finite subset for condition (i). In this case a projection commuting with u is fixed by α, which contradicts (ii). (See Lemma III.6).

Property (ii) also tells us that the projections we get are all Murray-von Neumann equivalent. In the case of a matrix algebra, this means that the projections all have the same rank or equivalently, all have the same number of 1' and 0's when diagonalized. We can essentially always assume our projections lie in some matrix algebra because these algebras are dense inside of a UHF algebra. Lemma II.2 will ensure that they remain mutually orthogonal and the fact that a projection that is close to 1 is actually equal to 1 will save condition (iii).

The actual orbit of p_1 under α is also a set of projections not fewer than N'. Since they are close to the projections given, they are "almost" mutually orthogonal. So if we were willing to allow almost mutually orthogonal projections in the definition, then we can enjoy exect equalities in (ii). Conversely, if we did have this, we can return to the original situation using Lemma II.2 to orthogonalise the projections.

For an automorphism with the Rokhlin property that does not necessarily preserve a tensor product decomposition, we can try to simulate a tensor product automorphism as follows. Fix an increasing sequence of integers $(n_l)_{l\in\mathbb{N}}$ to take on the roles of a varying n'. Find at least n_1 projections satisfying (ii), localise them to a matrix M_{N_1} where α might not act on the matrix but will look as if it does in its approximate action on the projections inside. Then find at least n_2 projections which approximately commute with the matrix units in M_{N_1} satisfying (ii). Then they approximately belong to $1_{N_1} \otimes M_{N_2}$ with an approximate action of α. Repeating this we get a tensor product decomposition $M_{(N_l)_{l\in\mathbb{N}}}$ for which the action of α approximately looks like that of Example III.2.

Property (iii)

This innocuous looking condition is responsible for the Rokhlin property being so hard to find in general. For example, if we were to find N' projections in a single matrix algebra M_l then condition (ii) implies that all of the projections have the same rank and hence combined with (iii) implies that l is divisible by N'. So as N' varies the matrices in our decomposition need to be divisible by infinite many different integers. This is actually not a problem for UHF algebras for infinite order automorphisms but will be for finite order automorphisms, while the problem for infinite order automorphisms will compound to more complicated algebras. Without (iii) we look at what the definition might look like to save effort later convincing the reader that the Matui-Sato tracial Rokhlin property is equivalent to the general one.

Definition. An automorphism α has the Rokhlin property without (iii) if for $n' \in \mathbb{N}$ there is $N' > n'$ and a central sequence of non-zero projections $(p_n)_{n\in\mathbb{N}}$ such that for $1 \leq i \neq j \leq N'$, we have

$$\lim_{n\to\infty} \alpha^i(p_n)\alpha^j(p_n) = 0.$$

Actions of $\mathbb{Z}$ with the Rokhlin Property

We notice that when α has the Rokhlin property that any power of α also satisfies the Rokhlin property, so that the Rokhlin property for a single infinite order automorphism is really a property for actions of the discrete group $\mathbb{Z}$. We see Example III.2 above determines a map $\mathbb{Z} \to \operatorname{Aut} M_{(2^l)_{l\in\mathbb{N}}}$ defined by $j \mapsto \alpha^j$ for all $j \in \mathbb{Z}$. We even see that there is a factorisation

$$\mathbb{Z} \to \prod_{l=1}^{\infty} U(M_{2^l}) \to \operatorname{Aut} M_{(2^l)_{l\in\mathbb{N}}}$$

defined by the lifting

$$j \mapsto (u_l)_{l=1}^{\infty}.$$

Actions of $\mathbb{Z}/k\mathbb{Z}$ with the Rokhlin Property

If α has finite order k, then we ignore n', require $N' = k$ and $\alpha(p_k) \approx_\epsilon p_1$ in the previous definition. That is:

Definition III.3. Let A be a unital C^*-algebra and $\alpha \in \operatorname{Aut} A$. We say α has the *order k Rokhlin property*, if for every $\epsilon > 0$ and every finite subset $\{a_1, \ldots, a_n\}$ in A, there exist k mutually orthogonal projections $p_1, p_2, \ldots, p_k$ such that

(i) $[p_i, a_j] \approx_\epsilon 0$ for all $i \leq k$ and all $j \leq n$,

(ii) $\alpha(p_i) \approx_\epsilon p_{i+1}$ for all $i \leq k$, with $p_{k+1} = p_1$,

(iii) $\sum_{i=1}^{N'} p_i = 1$.

If we review the construction of Example III.2, we see if we replace $(l)_{l\in\mathbb{N}}$ by $(k)_{l\in\mathbb{N}}$ and similar definitions for u_l, we would get an example of the Rokhlin property for an order k automorphism. As mentioned in the previous section, (iii) will now be a serious inhibition to finding such examples. Indeed with the above discussion we see that k must divide the matrix sizes of the matrix algebras appearing in the tensor product decomposition infinitely many times.

We also make the direct leap that this definition is really one for actions of the finite group $\mathbb{Z}/k\mathbb{Z}$, which can be readily extended to all finite groups. Below is the canonical example of a finite group acting with the Rokhlin property.

Proposition III.4. *Let G be a finite group acting on $\mathbb{C}^{|G|}$ via its left regular representation. This gives a group homomorphism*

$$G \to U(M_{|G|}),$$

which we duplicate infinitely many times to get

$$\alpha : G \hookrightarrow \prod_{n=1} U(M_{|G|}) \to \operatorname{Aut} M_{|G|^\infty}.$$

Suppose for every $g \in G$ that α_g has order $k(g)$. Then α_g has the order $k(g)$ Rokhlin property for all $g \in G$.

Proof. We will show that α has the pointwise Rokhlin property. Let $g \in G$ with $g \neq 1$ and let $\langle g \rangle$ denote the subgroup generated by g. If we restrict the left regular representation of G to $\langle g \rangle$, we get a decomposition into a direct sum of copies of the left regular representation for $\langle g \rangle$. Hence it suffices to assume that $G = \langle g \rangle$ and that $|G| = k$, the order of g. If we write $\mathbb{C}^k$ with respect to the basis $\{1, g, g^2, \ldots, g^{k-1}\}$ we get

$$\mathbb{C}^k = \mathbb{C}1 \oplus \mathbb{C}g \oplus \cdots \oplus \mathbb{C}g^{k-1},$$

we see that g acts as a cyclic permutation of the basis vectors of order k. Then Example III.2 combined with the comments directly above tells us this is an example of the Rokhlin property.

□

The Tracial Rokhlin Property for Automorphisms

Here we give the tracial Rokhlin property for automorphisms of UHF algebras. The only difference is in *(iii)*, where a weaker condition now resides.

Definition III.5. Let A be a UHF algebra and $\alpha \in \operatorname{Aut} A$. We say α has the *tracial Rokhlin property*, if for every $n' \in \mathbb{N}$, there exists $N' > n'$ such that for every $\epsilon > 0$ and every finite subset $\{a_1, \ldots, a_n\}$ in A, there exist mutually orthogonal projections $p_1, p_2, \ldots, p_{N'}$ such that with $p = \sum_{i=1}^{N'} p_i$ we have

(i) $[p_i, a_j] \approx_\epsilon 0$ for $1 \leq i \leq N'$ and for $1 \leq j \leq n$,

(ii) $\alpha(p_i) \approx_\epsilon p_{i+1}$ for $1 \leq i \leq N' - 1$,

(iii) $\tau(p) \approx_\epsilon 1$ for the unique tracial state τ.

We say α has the *order k tracial Rokhlin property* if in the above we ignore n' and require $N' = k$ as well as $\alpha(p_k) \approx_\epsilon p_1$.

Clearly if α has the (order k) Rokhlin property, it has the (order k) tracial Rokhlin property. This definition though slightly different than what appears in the literature is equivalent to such definitions.

Lemma III.6. *If α has the tracial Rokhlin property, then α is outer.*

Proof. Suppose $\alpha = \mathrm{Ad}\, u$ for some unitary $u \in A$. We can take $a_1 = u$ to be part of the finite subset and $\epsilon = 1/2$. Now property (i) gives us

$$[p_1, u] \approx_{\frac{1}{2}} 0.$$

But then property (ii) would imply

$$\begin{aligned} p_2 &\approx_{\frac{1}{2}} \alpha(p_1) \\ &= u p_1 u^* \\ &\approx_{\frac{1}{2}} u u^* p_1 \\ &= p_1. \end{aligned}$$

Hence we have the following contradiction:

$$\begin{aligned} p_1 &= p_1^2 \\ &\approx_1 p_1 p_2 \\ &= 0. \end{aligned}$$

□

Lemma III.7. *Suppose $\alpha \in \mathrm{Aut}\, M_{(n_l)_{l\in\mathbb{N}}}$ preserves the natural decomposition and has the tracial Rokhlin property. Then for any $n' \in \mathbb{N}$ there is $N' > n'$ such that for any finite subset $\{a_1, \ldots a_n\}$ and any $\epsilon > 0$ there exists $j \geq 1$ with $N = n_1 n_2 \cdots n_j$ and mutually orthogonal projections $p_1, \ldots, p_{N'}$ in M_N such that*

- $[p_i, a_j] \approx_\epsilon 0$ *for* $1 \leq i \leq N'$ *and* $1 \leq j \leq n$,
- $\alpha(p_i) \approx_\epsilon p_{i+1}$ *for* $1 \leq i \leq N' - 1$,
- $\tau(p_1 + \cdots + p_{N'}) \approx_\epsilon 1$.

If α has the order k tracial Rokhlin property, ignore n' and require $N' = k$ and $\alpha(p_k) \approx_\epsilon p_1$.

Proof. Let $n' \in \mathbb{N}$ and fix $N' > n'$ from α having the tracial Rokhlin property. Let $\epsilon > 0$ and let $\{a_1, \ldots a_n\}$ be a finite subset of $M_{(n_l)_{l\in\mathbb{N}}}$. Without loss of generality assume $\|a_j\| \leq 1$ for $1 \leq j \leq n$ and let $\delta = \delta(\epsilon/100, N')$ as in Lemma II.1. Without loss of generality we can assume

$\delta \leq \epsilon/100$ and $\delta \leq 1$. Since α has the tracial Rokhlin property for $N' > n'$, there exist $q_1, \ldots, q_{N'}$ mutually orthogonal projections such that

- $[q_i, a_j] \approx_\delta 0$ for $1 \leq i \leq N'$ and $1 \leq j \leq n$,
- $\alpha(q_i) \approx_\delta q_{i+1}$ for $1 \leq i \leq N' - 1$,
- $\tau(q_1 + \cdots + q_{N'}) \approx_\delta 1$.

By the direct limit definition, there exists $j \in \mathbb{N}$ such that with $N = n_1 n_2 \cdots n_j$, there are self-adjoint $q'_1, \ldots, q'_{N'} \in M_N \subset M_{(n_l)_{l\in\mathbb{N}}}$ such that for $1 \leq i \leq N'$, we have

$$q_i \approx_{\delta/32} q'_i.$$

We check

$$\begin{aligned} q'_i &\approx_{\delta/32} q_i \\ &= q_i^2 \\ &\approx_{\delta/32} q_i q'_i \\ &\approx_{\frac{\delta}{32}(1+\frac{\delta}{32})} (q'_i)^2. \end{aligned}$$

Apply Lemma II.1 to get projections p'_i such that for $1 \leq i \leq N'$, we have

$$q'_i \approx_{\delta/4} p'_i.$$

We check that

$$\begin{aligned} p'_i p'_j &\approx_{\delta/4} p'_i q'_j \\ &\approx_{\delta/4} p'_i q_j \\ &\approx_{\delta/2} q_i q_j \\ &= 0. \end{aligned}$$

Now apply Lemma II.2 to mutually orthogonalise the projections p'_i in M_N to obtain $p_1, \ldots, p_{N'} \in M_N$ so that

$$p_i \approx_{\epsilon/100} p'_i.$$

Now we check that $p_1, \ldots, p_{n'}$ satisfy our claim. Suppose $1 \leq i \leq N'$ and $1 \leq j \leq n$. We have

$$\begin{aligned}
[p_i, a_j] &\approx_{2\epsilon/100} [p_i', a_j] \\
&\approx_{\delta/2} [q_i', a_j] \\
&\approx_{2\delta} [q_i, a_j] \\
&\approx_{\delta} 0.
\end{aligned}$$

Now let $1 \leq i \leq N' - 1$. We have

$$\begin{aligned}
\alpha(p_i) &\approx_{\epsilon/100} \alpha(p_i') \\
&\approx_{\delta/2} \alpha(q_i') \\
&\approx_{\delta} \alpha(q_i) \\
&\approx_{\delta} q_{i+1} \\
&\approx_{\delta} q_{i+1}' \\
&\approx_{\delta/4} p_{i+1}' \\
&\approx_{\epsilon/100} p_{i+1}.
\end{aligned}$$

We also have

$$\begin{aligned}
\sum_{i=1}^{N'} \tau(p_i) &= \sum_{i=1}^{N'} \tau(q_i) \\
&\approx_{\delta} 1.
\end{aligned}$$

□

Lemma III.8. *Suppose $\alpha \in \operatorname{Aut} M_{(n_l)_{l \in \mathbb{N}}}$ preserves the natural decomposition and has the tracial Rokhlin property. Then for any $n' \in \mathbb{N}$, any finite subset $\{a_1, \ldots a_n\}$ and any $\epsilon > 0$ there exists $j \geq 1$ such that with $N = n_1 n_2 \cdots n_j$, there exist mutually orthogonal projections $p_1, \ldots, p_{n'}$ in M_N such that*

- $[p_i, a_j] \approx_\epsilon 0$ *for* $1 \leq i \leq n'$ *and* $1 \leq j \leq n$,
- $\alpha(p_i) \approx_\epsilon p_{i+1}$ *for* $1 \leq i \leq n' - 1$,
- $\tau(p_1 + \cdots + p_{n'}) \approx_\epsilon 1$.

If α has the order k tracial Rokhlin property, ignore n' and require $N' = k$ and $\alpha(p_k) \approx_\epsilon p_1$.

Proof. Let $n' \in \mathbb{N}$, let $\{a_1, \ldots, a_n\}$ be a finite subset and let $\epsilon > 0$. Use Lemma III.7 with some N' so large that $\frac{n'}{N'} < \frac{\epsilon}{2}$ and writing $N' = n'Q' + r'$ with $0 \leq r' < n'$ and $Q' \in \mathbb{N}$ gives r' satisifying $r'/N' < \epsilon/2$ and get projections $q_1, \ldots, q_{N'}$ relative to $\{a_1, \ldots, a_n\}$ and ϵ/N'. Then group the first $n'Q'$ projections into n' groups consisting of Q' projections in each group to get the projections in the conclusion. Let $r(j)$ be the remainder when j is divided by Q' and for $1 \leq i \leq n' - 1$ set

$$p_i = \sum_{\substack{r(j)=i \\ 1 \leq j \leq n'Q'}} q_j$$

and

$$p_{n'} = \sum_{\substack{r(j)=0 \\ 1 \leq j \leq n'Q'}} q_j.$$

□

Lemma III.9. *Let $\epsilon > 0$, let $N, N' \in \mathbb{N}$, let $\{e_{i,j} \,|\, 1 \leq i, j \leq N\}$ be a set of matrix units for M_N and let $x \in M_N \otimes M_{N'}$ satisfy*

$$[x, e_{i,j} \otimes 1_{N'}] \approx_\epsilon 0$$

for $1 \leq i, j \leq N$. Then there exists $b \in M_{N'}$ such that

$$x \approx_{10N^3\epsilon} 1 \otimes b.$$

Proof. Since the matrix units for a basis for M_N, there are unique $b_{i,j} \in M_{N'}$ for $i, j \leq N$ such that we can write

$$x = \sum_{i,j=1}^{N} e_{i,j} \otimes b_{i,j}.$$

If $N = 1$ there is nothing to prove, so let $N \geq 2$. Choose distinct $i_0, j_0 \in \{1, 2, \ldots, N\}$. We have from assumption

$$\begin{aligned} 0 &\approx_\epsilon [e_{i_0,i_0} \otimes 1, x] \\ &= \sum_{j=1}^{N} e_{i_0,j} \otimes b_{i_0,j} - \sum_{i=1}^{N} e_{i,i_0} \otimes b_{i,i_0}. \end{aligned}$$

Another interation gives

$$\begin{aligned} 0 &\approx_{2\epsilon} [e_{j_0,j_0}, [e_{i_0,i_0}, x]] \\ &= -e_{i_0,j_0} \otimes b_{i_0,j_0} - e_{j_0,i_0} \otimes b_{j_0,i_0}. \end{aligned}$$

One last iteration gives

$$\begin{aligned}0 &\approx_{4\epsilon} [e_{i_0,j_0},[e_{j_0,j_0},[e_{i_0,i_0},x]]] \\ &= (e_{j_0,j_0} - e_{i_0,i_0}) \otimes b_{j_0,i_0}.\end{aligned}$$

Hence

$$\begin{aligned}\|b_{j_0,i_0}\| &= \|(e_{j_0,j_0} - e_{i_0,i_0}) \otimes b_{j_0,i_0}\| \\ &\approx_{4\epsilon} 0.\end{aligned}$$

Therefore

$$x \approx_{4(N^2-N)\epsilon} \sum_{i=1}^{N} e_{i,i} \otimes b_{i,i}.$$

Let $j \neq 1$. We have

$$[e_{1,j}, x] \approx_{8(N^2-N)\epsilon} e_{1,j} \otimes b_{j,j} - e_{1,j} \otimes b_{11}$$

Therefore

$$b_{j,j} \approx_{(8(N^2-N)+1)\epsilon} b_{1,1}$$

Hence we have

$$\begin{aligned}\|x - 1 \otimes b_{1,1}\| &< 4(N^2-N)\epsilon + (N-1)(8(N^2-N)+1)\epsilon \\ &\leq 10N^3\epsilon.\end{aligned}$$

□

Proposition III.10. *Suppose* $\alpha \in \operatorname{Aut} M_{(n_l)_{l\in\mathbb{N}}}$ *preserves the natural decomposition. Then* α *has the tracial Rokhlin property if and only if there is a regrouping* $(N_l)_{l\in\mathbb{N}}$ *of* $(n_l)_{l\in\mathbb{N}}$ *such that* $M_{N_l} \subset M_{(N_l)_{l\in\mathbb{N}}}$ *contains mutually orthogonal projections* $p_1, \ldots, p_l$ *satisfying*

- $\alpha(p_i) \approx_{\frac{1}{2^l}} p_{i+1}$ *for* $1 \leq i \leq l-1$,
- $\tau(p_1 + \cdots + p_l) \approx_{\frac{1}{2^l}} 1$.

We can also replace $\frac{1}{2^l}$ *with any* ϵ_l *such that* $l\epsilon_l \to 0$.

Proof. Assume α has the tracial Rokhlin property. We define the integers N_l inductively on l. To get N_1, we apply Lemma III.8 with $n = 0$, $\epsilon = 1/2$ and $n' = 1$.

Now assume for an induction that we have found $N_1, \ldots, N_l$ with the properties required. Let $N = N_1 \cdots N_l$, let $\epsilon = 1/2^{l+5}$ and let $\{e_{i,j} \,|\, i, j \leq N\}$ be matrix units for M_N. By Lemma

III.7 there exists $s > 0$ such that with $N_{l+1} = n_{j+1} \dots n_{j+s}$, there are $l+1$ mutually orthogonal projections $q_1, \dots, q_{l+1}$ in $M_N \otimes M_{N_{l+1}}$ such that

- $[q_i, e_{i,j}] \approx_{\frac{\epsilon}{10N^3}} 0$ for all $i, j \leq N$,
- $\alpha(q_i) \approx_\epsilon q_{i+1}$ for $i \leq l$,
- $\tau(q_1 + \cdots + q_{l+1}) \approx_\epsilon 1$.

Now by Lemma III.9 we have $q_i \approx_\epsilon 1 \otimes b_i$ for some self-adjoint $b_i \in M_{N_{l+1}}$. Then $b_i^2 \approx_{4\epsilon} b_i$, so Lemma II.1 gives a projection $p_i \in M_{N_{l+1}}$ such that $p_i \approx_{8\epsilon} b_i$. We check that these projections satisfy our requirements.

Let $1 \leq i \leq l+1$. We have

$$\begin{aligned} \alpha(p_i) &\approx_\epsilon \alpha(b_i) \\ &\approx_\epsilon \alpha(q_i) \\ &\approx_\epsilon q_{i+1} \\ &\approx_\epsilon 1 \otimes b_{i+1} \\ &\approx_\epsilon p_{i+1}. \end{aligned}$$

We also have $\tau(p_i) = \tau(q_i)$ so the trace condition is satisfied. This completes the induction.

Conversely, let $n' \in \mathbb{N}$ and fix $N' > n'$. Let $\epsilon > 0$ and let $\{a_1, \dots, a_n\}$ be a finite subset of $M_{(N_l)_{l \in \mathbb{N}}}$. By the direct limit decomposition there exists $L > 0$ such that with $N = N_1 \cdots N_L$, there exist $a_1(N), \dots, a_n(N) \in M_N$ such that $a_j \approx_{\epsilon/2} a_j(N)$ for $1 \leq j \leq n$. Now choose $L' \in \mathbb{N}$ such that

- $L' > L$,
- $N'/L' < \epsilon/2$,
- $L'/2^{L'} < \epsilon/2$.

Then write L' in quotient remainder form as $L' = N'Q' + r'$ with $Q' \in \mathbb{N}$ and $0 \leq r' \leq N' - 1$. If $q_1, \dots, q_{L'}$ are the promised L' mutually orthogonal projections in $M_{L'}$ from the assumption, then define the projections $p_1, \dots, p_{N'}$ as follows: let $r(j)$ be the remainder when j is divided by N'. Then for $1 \leq i \leq N' - 1$ set

$$p_i = \sum_{\substack{r(j)=i \\ 1 \leq j \leq N'Q'}} q_j$$

and

$$p_{N'} = \sum_{\substack{r(j)=0 \\ 1 \leq j \leq N'Q'}} q_j.$$

We check these the projections satisfy the requirements of the tracial Rokhlin property. Let $1 \leq i \leq N'$. First we have

$$\begin{aligned}[p_i, a_j] &\approx_\epsilon [p_i, a_j(N)] \\ &= 0.\end{aligned}$$

Next for $1 \leq i \leq N' - 1$, we have

$$\begin{aligned}\alpha(p_i) &= \sum_{\substack{r(j)=i \\ 1 \leq j \leq N'Q'}} \alpha(q_j) \\ &\approx_{Q'/2^{L'}} \sum_{\substack{r(j)=i \\ 1 \leq j \leq N'Q'}} \alpha(q_{j+1}) \\ &= \sum_{\substack{r(j)=i+1 \\ 1 \leq j \leq N'Q'}} \alpha(q_j) \\ &= p_{i+1}.\end{aligned}$$

Since $Q' < L'$, we get $\alpha(p_i) \approx_\epsilon p_{i+1}$. Lastly, writing $p = p_1 + \cdots + p_{N'}$ and $q = q_1 + \cdots + q_{L'}$, we have

$$\begin{aligned}\tau(p) &= \frac{L' - r'}{L'} \tau(q) \\ &\approx_\epsilon \tau(q) \\ &\approx_\epsilon 1.\end{aligned}$$

□

Proposition III.11. *Suppose $\alpha \in \operatorname{Aut} M_{(n_l)_{l \in \mathbb{N}}}$ preserves the natural decomposition. Then α has the order k tracial Rokhlin property if and only if there is a regrouping $(N_l)_{l \in \mathbb{N}}$ of $(n_l)_{l \in \mathbb{N}}$ such that M_{N_l} contains mutually orthogonal projections $p_1, \ldots, p_k$ satisfying*

- $\alpha(p_i) \approx_{\frac{1}{2^l}} p_{i+1}$ *for* $i \leq k - 1$ *and* $\alpha(p_k) = p_1$,
- $\tau(p_1 + \cdots + p_k) \approx_{\frac{1}{2^l}} 1$.

We can also replace $\frac{1}{2^l}$ with any ϵ_l such that $\epsilon_l \to 0$.

"Bump-ups" and "Cut-downs"

We explore some relationships between the Rokhlin and tracial Rokhlin properties. The following can be thought of as ways of adding and removing the word "tracial" from the property at the cost of possibly changing the matrix sizes in the tensor product.

Lemma III.12 (Bump-up)**.** *Let $(n_l)_{l\in\mathbb{N}}$ be a sequence of positive integers and suppose that*

$$\alpha = \bigotimes_{l=1}^{\infty} \operatorname{Ad} u_l$$

for $u_l \in U(M_{n_l})$ has the tracial Rokhlin property. Then there is a regrouping $(N_l)_{l\in\mathbb{N}}$ of $(n_l)_{l\in\mathbb{N}}$ and unitaries $U_l \in U(M_{N_l})$ for $l \in \mathbb{N}$ such that

$$\alpha = \bigotimes_{l=1}^{\infty} \operatorname{Ad} U_l$$

and such that for any sequence $(s_l)_{l\in\mathbb{N}}$ there is a regrouping $(S_l)_{l\in\mathbb{N}}$ of $(s_l)_{l\in\mathbb{N}}$ and integers Q_l and r_l for each $l \in \mathbb{N}$ such that

$$\beta = \bigotimes_{n=1}^{\infty} \operatorname{Ad}(\operatorname{diag}(U_l \otimes 1_{Q_l}, 1_{r_l})) \in \operatorname{Aut} M_{(S_l)_{l\in\mathbb{N}}}$$

has the tracial Rokhlin property.

Proof. Suppose α has the tracial Rokhlin property. By Proposition III.10 there is a regrouping $(N_l)_{l\in\mathbb{N}}$ of $(n_l)_{l\in\mathbb{N}}$ such that M_{N_l} contains l mutually orthogonal projections $p_1, \ldots, p_l$ satisfying

- $\alpha(p_i) \approx_{1/2^l} p_{i+1}$ for $1 \leq i \leq l-1$,
- $\tau(p_1 + \cdots + p_l) \approx_{\frac{1}{2^l}} 1$.

Now regroup $(s_l)_{l\in\mathbb{N}}$ into S_l so that S_l is large enough relative to N_l to satisfy

$$\frac{N_l}{S_l} < \frac{1}{2^l}.$$

Then upon writing S_l in quotient remainder form

$$S_l = Q_l N_l + r_l,$$

for unique $Q_l \in \mathbb{N}$ and $0 \le r_l < N_l$, we have

$$r_l/S_l < 1/2^l.$$

We now check that β has the tracial Rokhlin property. By Proposition III.10 it suffices to check for each $l \in \mathbb{N}$ that M_{S_l} contains l mutually orthogonal projections $p_1, \ldots, p_l$ such that

- $\beta(p_i) \approx_{\frac{1}{2^l}} p_{i+1}$ for $1 \le i \le l-1$,
- $\tau(p_1 + \cdots + p_l) \approx_{\frac{1}{2^l}} 1$.

Now by construction, we can find l mutually orthogonal projections $p_1, p_2, \ldots, p_l$ in M_{N_l} such that

- $\alpha(p_i) \approx_{1/2^l} p_{i+1}$ for $1 \le i \le l-1$,
- $\tau(p_1 + \cdots + p_l) \approx_{1/2^l} 1$.

If we regard M_{N_l} as a subalgebra of M_{S_l} via the block diagonal embedding with Q_l copies of x and one zero block of size r_l, that is

$$x \mapsto \mathrm{diag}(x \otimes 1_{Q_l}, 0_{r_l}).$$

and identify the projections with their image under this embedding, then β restricts to α on M_{N_l}. So we only need to check the trace condition. Let τ_{S_l} denote the tracial state on M_{S_l} with identity 1_{S_l} and let τ_{N_l} denote the tracial state on M_{N_l}. Then

$$\begin{aligned}
\tau_A(p) &= \tau_{S_l}(p) \\
&= \tau_{S_l}(1_{N_l})\tau_{N_l}(p) \\
&= \frac{Q_j N_l}{S_l}\tau_{N_l}(p) \\
&= \left(1 - \frac{r_l}{S_l}\right)\tau_{N_l}(p) \\
&\approx_{\frac{1}{2^l}} 1 - \frac{r_l}{S_l} \\
&\approx_{\frac{1}{2^l}} 1.
\end{aligned}$$

Since $l/2^{l-1} \to 0$, this is enough to guarantee the tracial Rokhlin property by Proposition III.10.

□

Lemma III.13. *Let $(n_l)_{l\in\mathbb{N}}$ be a sequence of positive integers and suppose that*

$$\alpha = \bigotimes_{l=1}^{\infty} \operatorname{Ad} u_l$$

for $u_l \in U(M_{n_l})$ has the order k tracial Rokhlin property. Then there is a regrouping $(N_l)_{l\in\mathbb{N}}$ of $(n_l)_{l\in\mathbb{N}}$ and unitaries $U_l \in U(M_{N_l})$ for $l \in \mathbb{N}$ such that

$$\alpha = \bigotimes_{l=1}^{\infty} \operatorname{Ad} U_l$$

and such that for any sequence $(s_l)_{l\in\mathbb{N}}$ there is a regrouping $(S_l)_{l\in\mathbb{N}}$ of $(s_l)_{l\in\mathbb{N}}$ and integers Q_l and r_l for each $l \in \mathbb{N}$ such that

$$\beta = \bigotimes_{n=1}^{\infty} \operatorname{Ad}(\operatorname{diag}(U_l \otimes 1_{Q_l}, 1_{r_l})) \in \operatorname{Aut} M_{(S_l)_{l\in\mathbb{N}}}$$

has the order k tracial Rokhlin property.

We now exhibit the inverse principle for "residually finite" automorphisms.

Lemma III.14 (Cut-down)**.** *Let $(n_l)_{l\in\mathbb{N}}$ be a sequence of positive integers and suppose that $\alpha \in \operatorname{Aut} M_{(n_l)_{l\in\mathbb{N}}}$ is of order k and has the order k tracial Rokhlin property. Then there is a regrouping $(N_l)_{l\in\mathbb{N}}$ of $(n_l)_{l\in\mathbb{N}}$ with*

$$\alpha = \bigotimes_{l=1}^{\infty} \operatorname{Ad}(\operatorname{diag}(\lambda_1^{(l)}, \dots, \lambda_{N_l}^{(l)}))$$

for k-th roots of unity $\lambda_j \in \mathbb{C}$ with $1 \le j_1 < j_2 < \cdots < j_k \le N_l$ such that

$$\beta = \bigotimes_{l=1}^{\infty} \operatorname{Ad}(\operatorname{diag}(\lambda_{j_1}^{(l)}, \dots, \lambda_{j_k}^{(l)}))$$

has the order k Rokhlin property on M_{k^∞}.

Proof. Since α has the order k tracial Rokhlin property we can use Proposition III.10 to get a regrouping $(N_l)_{l\in\mathbb{N}}$ of $(n_l)_{l\in\mathbb{N}}$ so that for each $l \in \mathbb{N}$ there are k non-zero mutually orthogonal projections $p_1, \dots, p_k \in M_{N_l}$ such that, setting $p_{k+1} = p_1$, we have

– $\alpha(p_i) \approx_{\frac{1}{2^l}} p_{i+1}$ for $1 \le i \le k$

Let $\epsilon > 0$. We take $\delta = \delta(\epsilon, k)$ in Lemma II.2 and l large enough so that $\frac{1}{2^l} < \delta/10$. Let $\alpha = \operatorname{Ad} u$ on M_{N_l} for $u \in U(M_{N_l})$. We can decompose $\mathbb{C}^{N_l}$ into $e^{2\pi i j/k}$-eigenspaces V_j for u. That is

$$\mathbb{C}^{N_l} = \bigoplus_{j=0}^{k-1} V_j.$$

By taking a subprojection if necessary we can assume that p_1 has rank one. Let v be a unit vector that spans the range of p_1. Then we write for unique $v_j \in V_j$:

$$v = \sum_{j=0}^{k-1} v_j.$$

Let V be the subspace spanned by $\{v_j \,|\, 0 \leq j \leq k-1\}$. We see that both p_1 and u preserve V. Hence also $\alpha^j(p_1) = u^j p_1 u^{-j}$ preserves V for $0 \leq j \leq k-1$. Therefore we have k non-zero δ-approximately mutually orthogonal projections in $\operatorname{End}(V)$. By Lemma II.2 there exist k non-zero exactly mutually orthogonal projections $\operatorname{End}(V)$, making V a k-dimensional space. Hence we take $m_l = k$ and $j_1, \ldots, j_k$ so that $\lambda_{j_1}, \ldots, \lambda_{j_k}$ are k distinct roots of unity. □

Tensor Product of Automorphisms

We see here that tensor products of automorphisms with the tracial Rokhlin property will have the tracial Rokhlin property on the tensor product algebra. Most of the time we will only need one of the automorphisms to have the tracial Rokhlin property. Sometimes one of the automorphisms will only partially have the tracial Rokhlin property, in which case the other may be able to help. While these facts are kind of obvious, when combined with the observation that a tensor product of tensor product preserving automorphisms will be a tensor product preserving automorphism, we will certainly have something to savor.

Lemma III.15. *Let A and B be unital nuclear C^*-algebras, let $\alpha \in \operatorname{Aut} A$, let $\beta \in \operatorname{Aut} B$ and consider $\alpha \otimes \beta \in \operatorname{Aut}(A \otimes B)$. (For the claims about the tracial Rokhlin property we require that every trace on $A \otimes B$ restrict to an extremal trace on either A or B). We have:*

- *If α satisfies the Rokhlin property (resp. tracial Rokhlin property), then $\alpha \otimes \beta$ satisfies the Rokhlin property (resp. tracial Rokhlin property).*

– *If α satisfies the order k Rokhlin property (resp. order k tracial Rokhlin property), then $\alpha \otimes \beta$ satisfies the order k Rokhlin property (resp. order k tracial Rokhlin property).*

– *If α^k has the Rokhlin property (resp. tracial Rokhlin property) for some $k > 1$ and β has the order k Rokhlin property (resp. order k tracial Rokhlin property), then $\alpha \otimes \beta$ has the Rokhlin property (resp. tracial Rokhlin property).*

Proof. Assume α^k has the tracial Rokhlin property for some $k \geq 1$ and β has the order k tracial Rokhlin property. The case of the Rokhlin property is simpler. We allow $k = 1$ to include the first statement in the lemma. First for any $n' \in \mathbb{N}$ fix $N' > n'$ as in the definition of the tracial Rokhlin property for α^k. Let $\epsilon > 0$ and let $\{x_1, ..., x_n\}$ be a finite subset of $A \otimes B$. Since the algebraic tensor product is dense in $A \otimes B$, there exist $N \in \mathbb{N}$, $a_{i,j} \in A$ and $b_{i,j} \in B$ for $1 \leq i \leq N$ and $1 \leq j \leq n$ such that

$$x_j \approx_\epsilon \sum_{i=1}^{N} a_{i,j} \otimes b_{i,j} \text{ for } 1 \leq j \leq n.$$

Let

$$\delta = \frac{\epsilon}{\max_{j \leq n}(\sum_{i=1}^{N} \|b_{i,j}\|, \sum_{i=1}^{N} \|a_{i,j}\|)}.$$

By the choice of N', there exist mutually orthogonal projections $e_1, e_2, \ldots, e_{N'}$ such that if we write $e = e_1 + e_2 + \cdots + e_{N'}$, then we have

– $[e_l, a_{i,j}] \approx_\delta 0$ for $1 \leq l \leq N'$, $1 \leq i \leq N$ and $1 \leq j \leq n$,

– $[e_l, \alpha^{1-m}(a_{i,j})] \approx_\delta 0$ for $1 \leq l \leq N'$, $1 \leq m \leq k$, $1 \leq i \leq N$ and $1 \leq j \leq n$,

– $\alpha^k(e_l) \approx_\epsilon e_{l+1}$ for $1 \leq l \leq N' - 1$,

– $\tau_A(e) \approx_\epsilon 1$ for all $\tau_A \in T(A)$.

Since β has the order k tracial Rokhlin property, there exist mutually orthogonal projections $f_1, \ldots, f_k \in B$ for such that if we set $f = f_1 + f_2 + \cdots + f_k$ and $f_{k+1} = f_1$, we have

– $[f_m, b_{i,j}] \approx_\delta 0$ for $1 \leq m \leq k$, $1 \leq i \leq N$ and $1 \leq j \leq n$,

– $\beta(f_m) \approx_\epsilon f_{m+1}$ for $1 \leq m \leq k$,

– $\tau_B(f) \approx_\epsilon 1$ for all $\tau_B \in T(B)$.

Note if $k = 1$ we can take $f_1 = 1$. Set $p_{l,m} = \alpha^{m-1}(e_l) \otimes f_m$ for $1 \leq l \leq N'$ and $1 \leq m \leq k$ ordered lexicographically according first to l then m. Set $p = \sum_{l,m} p_{l,m}$. We now check that these witness the tracial Rokhlin property for $\alpha \otimes \beta$. Note that we can think of the first case as $k = 1$, $f_1 = 1_B$ and $p_{l,1} = p_l$ in the third case if we disregard β so that we can verify both cases simultaneously.

We first check that our projections approximately commute with our given finite subset. Let $1 \leq l \leq N'$, $1 \leq m \leq k$ and $1 \leq j \leq n$. We have

$$\begin{aligned}
p_{l,m}x_j &= (\alpha^{m-1}(e_l) \otimes f_m)x_j \\
&\approx_\epsilon (\alpha^{m-1}(e_l) \otimes f_m) \left(\sum_{i=1}^{N} a_{i,j} \otimes b_{i,j} \right) \\
&= \sum_{i=1}^{N} \alpha^{m-1}(e_l)a_{i,j} \otimes f_m b_{i,j} \\
&= \sum_{i=1}^{N} a_{i,j}\alpha^{m-1}(e_l) \otimes f_m b_{i,j} + \sum_{i=1}^{N} [\alpha^{m-1}(e_l), a_{i,j}] \otimes f_m b_{i,j} \\
&\approx_\epsilon \sum_{i=1}^{N} a_{i,j}\alpha^{m-1}(e_l) \otimes f_m b_{i,j} \\
&= \sum_{i=1}^{N} a_{i,j}\alpha^{m-1}(e_l) \otimes b_{i,j} f_m + \sum_{i=1}^{N} a_{i,j}\alpha^{m-1}(e_l) \otimes [f_m, b_{i,j}] \\
&\approx_\epsilon \sum_{i=1}^{N} a_{i,j}\alpha^{m-1}(e_l) \otimes b_{i,j} f_m \\
&= \left(\sum_{i=1}^{N} a_{i,j} \otimes b_{i,j} \right) (\alpha^{m-1}(e_l) \otimes f_m) \\
&\approx_\epsilon x_j(\alpha^{m-1}(e_l) \otimes f_m) \\
&= x_j p_{l,m}.
\end{aligned}$$

So $\|p_{l,m}x_j - x_j p_{l,m}\| < 4\epsilon$. We now check that $\alpha \otimes \beta$ cycles the projections. Let $1 \leq l \leq N' - 1$ and $1 \leq m \leq k$. We have

$$\begin{aligned}
(\alpha \otimes \beta)(p_{l,m}) &= (\alpha \otimes \beta)(\alpha^{m-1}(e_l) \otimes f_m) \\
&= \alpha^{m+1}(e_l) \otimes f_{m+1} \\
&\approx_\epsilon p_{l,m+1} \quad \text{if } m \neq k.
\end{aligned}$$

If $m = k$, we see from the same calculation as before

$$\begin{aligned}(\alpha \otimes \beta)(p_{l,m}) &\approx_\epsilon \alpha^k(e_l) \otimes f_{k+1}) \\ &\approx_\epsilon e_{l+1} \otimes f_1 \\ &= p_{l+1,1}.\end{aligned}$$

The remaining condition on the trace is seen to be satisfied as follows using Lin-Niu (22, Lemma 5.15) on the third line:

$$\begin{aligned}\tau(p) &= \tau\Big(\sum_{l,m} p_{l,m}\Big) \\ &= \tau(e \otimes f) \\ &= \tau_A(e) \otimes \tau_B(f) \\ &\approx_{2\epsilon} 1.\end{aligned}$$

($p = e \otimes f = 1$ if we are talking about the Rokhlin property.) Letting $k = 1$, $f_1 = 1_B$ and $p_{l,1} = p_l$ in the above calculations will give us the required calculations for p_l.

□

CHAPTER IV

GROUP ACTIONS ON UNIFORMLY HYPERFINITE C^*-ALGEBRAS

Maximally Almost Periodic Groups

Definition IV.1. A discrete group G is said to be *maximally almost periodic* if for every $g \in G \setminus \{1\}$ there exists $n \in \mathbb{N}$ and a group homomorphism $\varphi_n : G \to U(M_n)$ such that $\varphi_n(g) \neq 1$. Equivalently (for countable groups), G is maximally almost periodic if there is an embedding

$$G \hookrightarrow \prod_{n=1}^{\infty} U(M_n).$$

Lemma IV.2. *If G is a countable discrete maximally almost periodic group, then there is an embedding*

$$G \hookrightarrow \prod_{n=1}^{\infty} U(M_n)$$

such that the image of G intersects $\prod_{n=1}^{\infty} \mathbb{C}1_n$ trivially.

Proof. Starting with any embedding

$$\varphi : G \hookrightarrow \prod_{n=1}^{\infty} U(M_n),$$

we write φ_n for the map $G \to U(M_n)$. Define for all $n \in \mathbb{N}$ the map $\psi_n : G \to U(M_{n+1})$ by

$$\psi_n(g) = \operatorname{diag}(1, \varphi_n(g))$$

for all $g \in G$. Now define $\psi(g) = (1, \psi_1(g), \psi_2(g), \dots)$ for all $g \in G$ to give the required embedding. □

Product Type Actions

Since we have defined $\mathcal{Q}$ and other UHF algebras as tensor products, it will be natural to look at the actions that preserve this infinite tensor product structure. These can be represented by group homomorphisms that have the following factorisation:

$$G \to \prod_{l=1}^{\infty} \operatorname{Aut} M_{n_l} \to \operatorname{Aut}\left(\bigotimes_{l=1}^{\infty} M_{n_l}\right) = \operatorname{Aut} M_{(n_l)_{l\in\mathbb{N}}}.$$

Upon imposing that the action on each factor M_{n_l} be *inner* as well we get the next definition.

Definition IV.3 (product type action)**.** Define for any sequence of strictly positive integers $(n_l)_{l\in\mathbb{N}}$ the group homomorphism

$$\operatorname{Ad}_{(n_l)_{l\in\mathbb{N}}} : \prod_{l=1}^{\infty} U(M_{n_l}) \to \operatorname{Aut} M_{(n_l)_{l\in\mathbb{N}}},$$

written Ad when there is no confusion, for $u_{n_l} \in U(M_{n_l})$ by

$$(u_{n_l})_{l=1}^{\infty} \mapsto \bigotimes_{l=1}^{\infty} \operatorname{Ad} u_{n_l}$$

on the algebraic direct limit and then take the extension to the C^*-direct limit. We define an action of G to be a *product type action* if it is represented by a group homomorphism that has the following factorisation

$$G \to \prod_{l=1}^{\infty} U(M_{n_l}) \overset{\operatorname{Ad}}{\to} \operatorname{Aut} M_{(n_l)_{l\in\mathbb{N}}}$$

for some strictly positive sequence $(n_l)_{l\in\mathbb{N}}$.

We will now try to make explicit some ways to manipulate our definitions. For example, we may regroup and permute the factors in a tensor product decomposition and the algebras will remain of the same type via a canonical isomorphism and any product type actions will remain product type actions after conjugating by this canonical isomorphism.

Lemma IV.4 (Regrouping)**.** *Let $(l_i)_{i=1}^{\infty}$ be a strictly increasing sequence with $l_1 = 0$. Define m_i for $i \geq 1$ by*

$$m_i = \prod_{l_i+1}^{l_{i+1}} n_j.$$

In this case there is a canonical isomorphism

$$\Psi : M_{(n_l)_{l\in\mathbb{N}}} \to M_{(m_i)_{i\in\mathbb{N}}}$$

with

$$\Psi(M_{n_l+1} \otimes \cdots \otimes M_{n_{l+1}}) = M_{m_l}$$

and a canonical embedding

$$\varphi : \prod_{l=1}^{\infty} U(M_{n_l}) \to \prod_{i=1}^{\infty} U(M_{m_i})$$

such that the following diagram commutes:

$$\begin{array}{ccc} \prod_{l=1}^{\infty} U(M_{n_l}) & \xrightarrow{\text{Ad}} & \operatorname{Aut} M_{(n_l)_{l\in\mathbb{N}}}. \\ \varphi \big\downarrow & & \big\downarrow \Psi(\,)\Psi^{-1} \\ \prod_{i=1}^{\infty} U(M_{m_i}) & \xrightarrow{\text{Ad}} & \operatorname{Aut} M_{(m_i)_{i\in\mathbb{N}}}. \end{array}$$

Proof. In terms of the direct limit defintion of the infinite tensor product this corresponds to taking a subsequence of the connecting maps. Hence this will give the same limit with the new groupings preserved by the action. □

Lemma IV.5 (Reordering)**.** *Let* $d : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \setminus \{0\}$ *be any function and let* $\sigma : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ *be any bijection. Then there is a* $*$*-isomorphism*

$$\Psi_\sigma : \bigotimes_{m=1}^{\infty} \bigotimes_{n=1}^{\infty} M_{d(m,n)} \to \bigotimes_{l=1}^{\infty} M_{d(\sigma(l))}$$

such that

$$\Psi_\sigma(M_{d(m,n)}) = M_{d(\sigma(\sigma^{-1}(m,n)))}$$

and a canonical isomorphism

$$\varphi_\sigma : \prod_{m=1}^{\infty} \prod_{n=1}^{\infty} U(M_{d(m,n)}) \to \prod_{l=1}^{\infty} U(M_{d(\sigma(l))})$$

given by

$$((u_{d(m,n)})_{n=1}^{\infty})_{m=1}^{\infty} \mapsto (u_{\sigma(\sigma^{-1}(m,n))})_{\sigma^{-1}(m,n)=1}^{\infty}$$

such that the following diagram commutes

$$\begin{array}{ccc} \prod_{m=1}^{\infty}\prod_{n=1}^{\infty}U(M_{d(m,n)}) & \xrightarrow{\bigotimes \mathrm{Ad}} & \mathrm{Aut}\left(\bigotimes_{m=1}^{\infty}\bigotimes_{n=1}^{\infty}M_{d(m,n)}\right). \\ \varphi_\sigma \downarrow & & \downarrow \Psi_\sigma(\,)\Psi_\sigma^{-1} \\ \prod_{l=1}^{\infty}U(M_{d(\sigma(l))}) & \xrightarrow{\mathrm{Ad}} & \mathrm{Aut}\left(\bigotimes_{l=1}^{\infty}M_{d(\sigma(l))}\right). \end{array}$$

Proof. We first note that the domain algebra is simple so any unital $*$-homomorphism we define will be injective. Now we proceed to define the map. Let $m,n \in \mathbb{N}$ and let $\Psi_{m,n} : M_{d(m,n)} \to \bigotimes_{l=1}^{\infty} M_{d(\sigma(l))}$ be defined on matrix units $e_{i,j}^{(d(m,n))} \in M_{d(m,n)}$ by

$$e_{i,j}^{(d(m,n))} \mapsto \left(\bigotimes_{l=1}^{\sigma^{-1}(m,n)-1} 1_{d(\sigma(l))}\right) \otimes e_{i,j}^{(d(m,n))} \otimes \left(\bigotimes_{l=\sigma^{-1}(m,n)+1}^{\infty} 1_{d(\sigma(l))}\right).$$

for $1 \le i,j \le d(m,n)$. We note for later that for $m,n \in \mathbb{N}$ we have

$$\Psi_{m,n}(M_{d(m,n)}) = M_{d(\sigma(\sigma^{-1}(m,n))}.$$

Let $N > 1$. We see that since σ is injective, the images of $\Psi_{n,m}$ are in different tensor factors and in particular commute for $n = 1, \ldots, N$. Therefore we get a map

$$\bigotimes_{n=1}^{N} \Psi_{m,n} : \bigotimes_{n=1}^{N} M_{d(m,n)} \to \bigotimes_{l=1}^{\infty} M_{d(\sigma(l))}$$

satisfying for $e^{(d(m,n))} \in M_{d(m,n)}$ that

$$\bigotimes_{n=1}^{N} e^{(d(m,n))} \mapsto \prod_{n=1}^{N} \Psi_{m,n}(e^{(d(m,n))}).$$

Restricting this to $(\bigotimes_{n=1}^{N-1} M_{d(m,n)}) \otimes 1_{d(m,N)}$ we get

$$\left(\bigotimes_{n=1}^{N-1} e^{(d(m,n))}\right) \otimes 1_{d(m,N-1)} \mapsto \left(\prod_{n=1}^{N} \Psi_{m,n}(e^{(d(m,n))})\right) \Psi_{m,N}(1_{d(m,N)}).$$

which agrees with $\bigotimes_{n=1}^{N-1} \Psi_{m,n}$. Hence there is a map

$$\Psi_m : \varinjlim_{N} \left(\bigotimes_{n=1}^{N} M_{d(m,n)}, \mathrm{id} \otimes 1_{d(m,n)}\right) \to \bigotimes_{l=1}^{\infty} M_{d(\sigma(l))}.$$

We see that the images commute for $m = 1, \ldots, M$ since σ is injective so we have a map

$$\bigotimes_{m=1}^{M} \Psi_m : \bigotimes_{m=1}^{M} \bigotimes_{n=1}^{\infty} M_{d(m,n)} \to \bigotimes_{l=1}^{\infty} M_{d(\sigma(l))},$$

defined for $x_m \in \bigotimes_{n=1}^{\infty} M_{d(m,n)}$ by

$$\bigotimes_{m=1}^{M} x_m \mapsto \prod_{m=1}^{M} \Psi_m(x_m).$$

Restricting this to $(\bigotimes_{m=1}^{M-1} \bigotimes_{n=1}^{\infty} M_{d(m,n)}) \otimes 1$ we get

$$\bigotimes_{m=1}^{M-1} x_m \otimes 1 \mapsto \left(\prod_{m=1}^{M-1} \Psi_m(x_m) \right) \Psi_M(1)$$

which agrees with the map $\bigotimes_{m=1}^{M-1} \Psi_m$. Hence we have a map

$$\Psi : \varinjlim_{M} \bigotimes_{m=1}^{M} \bigotimes_{n=1}^{\infty} M_{d(m,n)} \to \bigotimes_{l=1}^{\infty} M_{d(\sigma(l))},$$

that is,

$$\Psi_\sigma : \bigotimes_{m=1}^{\infty} \bigotimes_{n=1}^{\infty} M_{d(m,n)} \to \bigotimes_{l=1}^{\infty} M_{d(\sigma(l))}.$$

Since Ψ_σ restricts to $\Psi_{m,n}$ on $M_{d(m,n)}$ we have

$$\Psi_\sigma(M_{d(m,n)}) = M_{d(\sigma(\sigma^{-1}(m,n))},$$

which also implies surjectivity, since these generate the target algebra as m and n run through $\mathbb{N}$ by surjectivity of σ. Hence Ψ_σ is a $*$-isomorphism. We also see that the diagram commutes. $\square$

Tracial Rokhlin Property and Strong Outerness

Definition IV.6. Suppose each $g \in G$ has order $1 \leq k(g) \leq \infty$ and α is a G action on some UHF algebra A. We say that an G-action α has the pointwise (tracial) Rokhlin property, if α_g has the order $k(g)$ (tracial) Rokhlin property for all $g \in G$. We understand the order ∞ (tracial) Rokhlin property to mean the (tracial) Rokhlin property.

Theorem IV.7. *Let A be a UHF algebra and let G be a countable discrete elementary amenable group. Then the following are equivalent:*

- *α has the tracial Rokhlin property,*
- *α has the pointwise tracial Rokhlin property,*
- *α is strongly outer.*

Proof. This is Matui-Sato (37, Theorem 3.7) specialized to actions of elementary amenable groups on UHF algebras. For automorphisms, let $G = \mathbb{Z}$ or $G = \mathbb{Z}/n\mathbb{Z}$. □

The equivalence of the last two conditions was already known to Kishimoto and others.

Existence of Strongly Outer Product Type Actions

We will present here our results concerning strongly outer product type actions on UHF algebras. We essentially show that every group that can act via a product type action can also act via a strongly outer product type action. We will show this for arbitrary UHF algebras by reducing to the case of the universal UHF algebra $\mathcal{Q}$ (or any UHF algebra of infinite type). The assumption that G is an elementary amenable group will be added later to upgrade the status of strongly outer actions to those with the tracial Rokhlin property. Recalling Definition IV.3, we consider actions corresponding to group homomorphisms of the form

$$G \hookrightarrow \prod_{l=1}^{\infty} U(M_{n_l}) \overset{\mathrm{Ad}}{\to} \mathrm{Aut}\, M_{(n_l)_{l\in\mathbb{N}}}.$$

We first look at the case of the universal UHF algebra $\mathcal{Q}$. Since we have chosen a standard tensor product decomposition for $\mathcal{Q}$ to be $M_{(n)_{n\in\mathbb{N}}}$, we will look for product type action in the following standard form:

$$G \hookrightarrow \prod_{n=1}^{\infty} U(M_n) \overset{\mathrm{Ad}}{\to} \mathrm{Aut}\, \mathcal{Q}.$$

There are other possible tensor product decompositions of $\mathcal{Q}$ and we now show that no generality is lost by trying to find actions in standard form.

Lemma IV.8. *Let G be a countable discrete group and suppose that for some sequence $(n_l)_{l\in\mathbb{N}}$ in $\mathbb{N}$ there exists a strongly outer action*

$$\alpha : G \hookrightarrow \prod_{l=1}^{\infty} U(M_{n_l}) \overset{\mathrm{Ad}}{\to} \operatorname{Aut} M_{(n_l)_{l\in\mathbb{N}}}.$$

Then there exists a strongly outer action

$$\beta : G \hookrightarrow \prod_{n=1}^{\infty} U(M_n) \overset{\mathrm{Ad}}{\to} \operatorname{Aut} \mathcal{Q}.$$

Proof. First we can if necessary regroup the tensor factors so that we can assume the sequence $(n_l)_{l\in\mathbb{N}}$ is strictly increasing. Let id denote the identity automorphism on $\bigotimes_{l=1}^{\infty} \bigotimes_{n_{l-1}<n<n_l} M_n$ and consider the action $\alpha \otimes \mathrm{id}$

$$\alpha \otimes \mathrm{id} : G \to \operatorname{Aut} \left(\bigotimes_{l=1}^{\infty} M_{n_l} \right) \otimes \left(\bigotimes_{l=1}^{\infty} \bigotimes_{n_{l-1}<n<n_l} M_n \right).$$

It is strongly outer by Theorem IV.7 combined with Lemma III.15 and factors through

$$\left(\prod_{l=1}^{\infty} U(M_{n_l}) \right) \times \left(\prod_{l=1}^{\infty} \prod_{n_{l-1}<n<n_l} U(M_n) \right).$$

Reordering the factors using Lemma IV.5, we get a conjugate action

$$\alpha \otimes \mathrm{id} : G \to \operatorname{Aut} \left(\bigotimes_{l=1}^{\infty} \bigotimes_{n_{l-1}<n\leq n_l} M_n \right).$$

factoring through

$$\prod_{l=1}^{\infty} \prod_{n_{l-1}<n\leq n_l} U(M_n) \overset{\mathrm{Ad}}{\to} \operatorname{Aut} \left(\bigotimes_{l=1}^{\infty} \bigotimes_{n_{l-1}<n\leq n_l} M_n \right).$$

Regrouping the factors now gives us our action in the required form. □

So any groups that can act via a product type action on some UHF algebra can also be found to act in standard form on $\mathcal{Q}$. We now show that any discrete group can act on $\mathcal{Q}$ via a strongly outer product type action in standard form.

Proposition IV.9. *Suppose G is a discrete group with a product type action*

$$\alpha : G \hookrightarrow \prod_{n=1}^{\infty} U(M_n) \stackrel{\mathrm{Ad}}{\to} \operatorname{Aut} \mathcal{Q},$$

such that $\ker \alpha = \{\mathrm{id}\}$. *Then there exists a strongly outer product type action*

$$\beta : G \hookrightarrow \prod_{n=1}^{\infty} U(M_n) \stackrel{\mathrm{Ad}}{\to} \operatorname{Aut} \mathcal{Q}.$$

Proof. We see that the action $\alpha^{\otimes \mathbb{N}}$ of G on $\mathcal{Q}^{\otimes \mathbb{N}}$ is strongly outer by Corollary II.14 with $A = \mathcal{Q}$. We also see that $\alpha^{\otimes \mathbb{N}}$ factorises as

$$\alpha^{\otimes \mathbb{N}} : G \to \left(\prod_{n=1}^{\infty} U(M_n) \right)^{\mathbb{N}} \to \operatorname{Aut}(\mathcal{Q}^{\otimes \mathbb{N}}),$$

which can be written in the more expanded form

$$\alpha^{\otimes \mathbb{N}} : G \to \prod_{m=1}^{\infty} \prod_{n=1}^{\infty} U(M_n) \to \operatorname{Aut} \left(\bigotimes_{m=1}^{\infty} \bigotimes_{n=1}^{\infty} M_n \right).$$

Rearrange the factors as in Lemma IV.5 to get a conjugate map (also call it $\alpha^{\otimes \mathbb{N}}$)

$$\alpha^{\otimes \mathbb{N}} : G \hookrightarrow \prod_{n=1}^{\infty} U(M_{n!}) \to \operatorname{Aut} \left(\bigotimes_{n=1}^{\infty} M_{n!} \right).$$

(If one wants to be more explicit with the application of Lemma IV.5, we take $d(m,n) = n$ and define σ using the sequence $(1,1)$, $(2,1)$, $(1,2)$, $(3,1)$, $(2,2)$, $(1,3)$ and so on where the pairs (m,n) such that $m + n = 2$ are exhausted first and then $m + n = 3$ and so on. We also have $\bigotimes_{m+n=l} M_{d(m,n)} = M_{(l-1)!}$.) Now we can apply Lemma IV.8 to get a strongly outer map β in the standard form required. ∎

Corollary IV.10. *Let G be a discrete maximally almost periodic group. There exists a strongly outer product type action*

$$\alpha : G \hookrightarrow \prod_{n=1}^{\infty} U(M_n) \stackrel{\mathrm{Ad}}{\to} \operatorname{Aut} \mathcal{Q}.$$

Proof. Lemma IV.2 gives us an embedding that we can compose with Ad to get a product type action α of the form

$$\alpha : G \hookrightarrow \prod_{n=1}^{\infty} U(M_n) \to \operatorname{Aut} \mathcal{Q}$$

such that $\ker \alpha = \{1\}$. Now Proposition IV.9 applies. □

We now show that no generality is lost when only considering $\mathcal{Q}$ as opposed to the other UHF algebras. The existence of strongly outer product type actions of G on $\mathcal{Q}$ will imply the existence of such G-actions on any UHF algebra (not necessarily of infinite type). We will eventually reduce to the following case, which is an example of the bump-up principle at work.

Lemma IV.11. *Suppose G is a discrete maximally almost periodic group and let $g \in G$ with order k for some $1 \leq k \leq \infty$. Then for any $(n_l)_{l\in\mathbb{N}}$ there is a regrouping $(N_l)_{l\in\mathbb{N}}$ and a map*

$$\beta[g] : G \hookrightarrow \prod_{l=1}^{\infty} U(M_{N_l}) \overset{\text{Ad}}{\to} \operatorname{Aut} M_{(N_l)_{l\in\mathbb{N}}},$$

such that $\beta[g]_g$ is an automorphism with the order k tracial Rokhlin property.

Proof. By Corollary IV.10 we have a strongly outer product type action of the form

$$\alpha : G \hookrightarrow \prod_{n=1}^{\infty} U(M_n) \overset{\text{Ad}}{\to} \operatorname{Aut} \mathcal{Q}.$$

In particular, α_g has the order k tracial Rokhlin property by Theorem IV.7 and we can apply Lemmas III.12 and III.13 for $(n_l)_{l\in\mathbb{N}}$ to get $(N_l)_{l\in\mathbb{N}}$ and sequences of integers $(Q_l)_{l\in\mathbb{N}}$, $(r_l)_{l\in\mathbb{N}}$ and an automorphism β_g of $M_{(N_l)_{l\in\mathbb{N}}}$ with the order k tracial Rokhlin property. Now define an embedding $\beta_l[g] : G \to U(M_{N_l})$ by $h \mapsto \operatorname{diag}(h \otimes 1_{Q_l}, 1_{r_l})$ for all $h \in G$. Combining these for all $l \in \mathbb{N}$, we get a map

$$\beta[g] : G \hookrightarrow \prod_{l=1}^{\infty} U(M_{N_l}) \overset{\text{Ad}}{\to} \operatorname{Aut} M_{(N_l)_{l\in\mathbb{N}}},$$

such that $\beta[g]_g$ is equal to β_g and hence has the order k tracial Rokhlin property. □

We come to our conclusion, which says that any group which can act on a UHF algebra with an injective product type action, acts on every UHF algebra with a strongly outer product type action.

Theorem IV.12. *Suppose G is any countable discrete maximally almost periodic group and A is any UHF algebra. Then there exists a strongly outer product type action α of G on A. Furthermore, if G is also elementary amenable, then α has the tracial Rokhlin property.*

Proof. Suppose that $A \cong M_{(n_l)_{l\in\mathbb{N}}}$ for some sequence $(n_l)_{l\in\mathbb{N}}$. We can partition this sequence into disjoint subsequences $(n_l(g))_{l\in\mathbb{N}}$ indexed by $g \in G$ whose union is $(n_l)_{l\in\mathbb{N}}$. Since G is a discrete maximally almost periodic group, by Lemma IV.11, for each $g \in G$ (of order $k(g)$), there is a sequence $(N_l(g))_{l\in\mathbb{N}}$ of the same type as $(n_l(g))_{l\in\mathbb{N}}$ (and therefore the union is also the same type) and a map

$$\beta[g] : G \hookrightarrow \prod_{l=1}^{\infty} U(M_{N_l(g)}) \xrightarrow{\mathrm{Ad}} \operatorname{Aut} M_{(N_l(g))_{l\in\mathbb{N}}},$$

such that $\beta[g]_g$ is an automorphism with the order $k(g)$ tracial Rokhlin property. Combining these maps into a map to the product, we get

$$\beta : G \hookrightarrow \prod_{g\in G}\prod_{l=1}^{\infty} U(M_n) \hookrightarrow \prod_{g\in G}\prod_{l=1}^{\infty} U(M_{N_l(g)}) \xrightarrow{\mathrm{Ad}} \operatorname{Aut}\Big(\bigotimes_{g\in G}\bigotimes_{l=1} M_{N_l(g)}\Big)$$

with the pointwise tracial Rokhlin property by Lemma III.15 and hence the action is strongly outer. Reordering the factors by Lemma IV.5 we get a conjugate product type action (which we will also call β)

$$\beta : G \hookrightarrow \prod_{l=1}^{\infty} U(M_{N_l}) \xrightarrow{\mathrm{Ad}} \operatorname{Aut}\Big(\bigotimes_{l=1}^{\infty} M_{N_l}\Big),$$

which therefore is also strongly outer. If G is elementary amenable, (37, Theorem 3.7) says that strongly outer actions are equivalent to actions with the tracial Rokhlin property, which completes the proof. $\square$

It would seem that most of our effort was exerted upon transfering the actions between different UHF algebras and that exhibiting a strongly outer product type action on any given UHF algebra of infinite type (as in Proposition IV.9) is quite direct.

This result highlights the robustness of the tracial Rokhlin property. We know this cannot be true for the Rokhlin property because we know that even finite group actions require the UHF algebra to be compatible with the group and essentially of infinite type. We will get a clearer picture of this comparison in the following sections.

Examples of Strongly Outer Product Type Actions

We give here a sufficient condition for a tensor product automorphism to be strongly outer in terms of only its trace in $U(M_n)$ for all $n \in \mathbb{N}$. We then use it to exhibit some examples of abelian group actions on $\mathcal{Q}$.

Abelian Groups

Proposition IV.13. *Every countable discrete abelian group is isomorphic to a subgroup of*

$$\bigoplus_{n=1}^{\infty}(\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}).$$

Proof. We sketch a proof relying on the basic theory of abelian groups. First every abelian group is a subgroup of a divisible group. Every divisible group is a direct sum of copies of $\mathbb{Q}$ and $\mathbb{Z}[1/p]/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$. So then every abelian group is a subgroup of a direct sum of copies of $\mathbb{Q}$ and copies of $\mathbb{Q}/\mathbb{Z}$. When our groups are countable we will need at most countably many summands. □

Strongly Outer Product Type Actions on $\mathcal{Q}$

Lemma IV.14. *Let A be a unital C^*-algebra and let τ be a tracial state on A. For every $\epsilon > 0$ there is $\delta > 0$ such that whenever $u \in A$ is a unitary satisfying* $\mathrm{Re}(1 - \tau(u)) < \delta$, *we have* $|1 - \tau(u)| < \epsilon$.

Proof. This proof was provided by N. C. Phillips. Without loss of generality let $\epsilon < 1$. Set $\delta = \epsilon^4/64$. Let τ be a tracial state on A and let $u \in A$ be a unitary such that $\mathrm{Re}(1 - \tau(u)) < \delta$. The Riesz Representation Theorem gives a Borel probability measure μ on S^1 such that $\tau(f(u)) = \int_{S^1} f\, d\mu$ for all $f \in C(S^1)$.

Set

$$E = \left\{\zeta \in S^1 \colon \mathrm{Re}(\zeta) < 1 - \epsilon^2/8\right\}.$$

We have

$$\frac{\epsilon^4}{64} = \delta > \mathrm{Re}(1 - \tau(u)) = \int_{S^1} \mathrm{Re}(1 - \zeta)\, d\mu(\zeta) \geq \int_E \mathrm{Re}(1 - \zeta)\, d\mu(\zeta) \geq \frac{\epsilon^2 \mu(E)}{8}.$$

So $\mu(E) < \epsilon^2/8$.

We claim that $\zeta \in S^1 \setminus E$ implies $|1-\zeta| \leq \epsilon/2$. To prove the claim, let $\zeta \in S^1 \setminus E$. Write $\zeta = \alpha + \beta i$ with $\alpha, \beta \in \mathbb{R}$. The definition of E implies that $\alpha \geq 1 - \epsilon^2/8$. Therefore

$$\beta^2 = 1 - \alpha^2 \leq 1 - \left(1 - \frac{\epsilon^2}{8}\right)^2 = \frac{\epsilon^2}{4} - \frac{\epsilon^4}{64}.$$

Also $1 - \alpha \leq \epsilon^2/8$. Therefore

$$|1-\zeta|^2 = (1-\alpha)^2 + \beta^2 \leq \frac{\epsilon^4}{64} + \left(\frac{\epsilon^2}{4} - \frac{\epsilon^4}{64}\right) = \frac{\epsilon^2}{4}.$$

The claim follows.

Now

$$\begin{aligned} |1-\tau(u)| &= \left|\int_{S^1} (1-\zeta)\, d\mu(\zeta)\right| \leq \int_{S^1} |1-\zeta|\, d\mu(\zeta) \\ &= \int_E |1-\zeta|\, d\mu(\zeta) + \int_{S^1\setminus E} |1-\zeta|\, d\mu(\zeta) \\ &\leq 2\mu(E) + \left(\frac{\epsilon}{2}\right)\mu(S^1 \setminus E) \leq \frac{\epsilon^2}{4} + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This completes the proof. □

Let τ be the unique tracial state on $\mathcal{Q}$. For unitaries u and v denote their commutator by $[u,v]_U = uvu^*v^*$.

Proposition IV.15. *Suppose there are sequences of unitaries $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ with $u_n, v_n \in U(M_n)$ such that $\tau([u_n, v_n]_{U(M_n)})$ does not converge to 1. Then $\alpha = \bigotimes \operatorname{Ad} u_n$ and $\beta = \bigotimes \operatorname{Ad} v_n$ are both strongly outer as automorphisms of $\mathcal{Q}$.*

Proof. Applying Lemma II.13 with $A_n = M_n$ and $[u_n, v_n]$ in place of u, we check

$$\begin{aligned} \|\beta(u(n)) - u(n)\|_2^2 &= \tau((v_n u_n v_n^* - u_n)^*(v_n u_n v_n^* - u_n)) \\ &= \tau(1 - u_n^* v_n u_n v_n^* - v_n u_n^* v_n^* u_n + 1) \\ &= 2 - \tau(u_n^* v_n u_n v_n^*) - \tau(v_n u_n^* v_n^* u_n) \\ &= 2 - \tau(v_n u_n v_n^* u_n^*) - \tau(u_n v_n u_n^* v_n^*) \\ &= 2 - \tau([u_n, v_n]^*) - \tau([u_n, v_n]) \\ &= 2[1 - \operatorname{Re}\tau([u_n, v_n])]. \end{aligned}$$

Note now that if $(w_n)_{n\in\mathbb{N}}$ is a sequence of unitaries and $\tau(w_n) \not\to 1$, then Re $\tau(w_n) \not\to 1$ by Lemma IV.14. Hence we see that β is strongly outer. Now $\tau([u_n, v_n]) = \tau([v_n, u_n^*])$ implies that α^{-1} is strongly outer and hence α is strongly outer. □

Lemma IV.16. *There is a strongly outer action*

$$\mathbb{R}/\mathbb{Z} \hookrightarrow \prod_{n=1}^{\infty} U(M_n) \xrightarrow{\mathrm{Ad}} \mathrm{Aut}\,\mathcal{Q}.$$

Proof. Define a homomorphism $\mathbb{R} \to U(M_n)$ by

$$v_n(r) = \mathrm{diag}(e^{2\pi i l r})_{l=1}^{n},$$

whose kernel is $\mathbb{Z}$, so we get an injective map

$$\mathbb{R}/\mathbb{Z} \hookrightarrow \prod_{n=1}^{\infty} U(M_n) \xrightarrow{\mathrm{Ad}} \mathrm{Aut}\,\mathcal{Q}.$$

We use the condition in Proposition IV.15 to check that it is strongly outer. Let u_n be the permutation matrix corresponding to the cycle $(123\ldots n)$. Then

$$\begin{aligned}
\tau([u_n, v_n(r)]) &= \tau(\mathrm{Ad}\,u_n(v_n(r))v_n(r)^*) \\
&= \tau(\mathrm{diag}(e^{2\pi i n r}, e^{-2\pi i r}, \ldots, e^{2\pi i (n-1) r})\,\mathrm{diag}(e^{-2\pi i l r})_{l=1}^{n}) \\
&= \tau(\mathrm{diag}(e^{2(n-1)\pi i r}, e^{-2\pi i r}, e^{-2\pi i r}, \ldots, e^{-2\pi i r})) \\
&= \frac{1}{n}(e^{2(n-1)\pi i r} + (n-1)e^{-2\pi i r}) \\
&= e^{-2\pi i r}(n^{-1}e^{-2n\pi i r} + n^{-1}(n-1)) \\
&\to e^{-2\pi i r}.
\end{aligned}$$

□

Lemma IV.17. *There is a strongly outer action*

$$\mathbb{R} \hookrightarrow \prod_{n=1}^{\infty} U(M_n) \xrightarrow{\mathrm{Ad}} \mathrm{Aut}\,\mathcal{Q}.$$

Proof. Let θ be an irrational number and let

$$\theta_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ \theta & \text{if } n \text{ is even.} \end{cases}$$

For each $r \in \mathbb{R}$ and $n \in \mathbb{N}$, define a homomorphism $\mathbb{R} \to U(M_n)$ by

$$v_n(r) = \operatorname{diag}(e^{2\pi\theta_n ilr})_{l=1}^{n},$$

We again use the condition in Proposition IV.15 to check that it is strongly outer. Let u_n be the permutation matrix corresponding to the cycle $(123\ldots n)$. Then

$$\begin{aligned}
\tau([u_n, v_n(r)]) &= \tau(\operatorname{Ad} u_n(v_n(r))v_n(r)^*) \\
&= \tau(\operatorname{diag}(e^{2\pi\theta_n inr}, e^{2\pi\theta_n ir}, \ldots, e^{2\pi\theta_n i(n-1)r}) \operatorname{diag}(e^{-2\pi\theta_n ilr})_{l=1}^{n}) \\
&= \tau(\operatorname{diag}(e^{2(n-1)\pi\theta_n ir}, e^{-2\pi\theta_n ir}, e^{-2\pi\theta_n ir}, \ldots, e^{-2\pi\theta_n ir})) \\
&= \frac{1}{n}(e^{2(n-1)\pi\theta_n ir} + (n-1)e^{-2\pi\theta_n ir}) \\
&= e^{-2\pi\theta_n ir}(n^{-1}e^{-2n\pi ir} + n^{-1}(n-1)).
\end{aligned}$$

Now two of the limit points of this, corresponding to odd and even n, are $e^{-2\pi\theta ir}$ and $e^{-2\pi ir}$. If these are both equal to 1 then r must be both irrational in the first case and rational in the second, which is impossible unless $r = 0$. □

Corollary IV.18. *There exist strongly outer product type actions of $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ on $\mathcal{Q}$.*

Proof. For $\mathbb{Q}$, restrict the homomorphism in Lemma IV.17 from $\mathbb{R}$ to $\mathbb{Q}$ and for $\mathbb{Q}/\mathbb{Z}$ restrict the map from Lemma IV.16. □

The above two lemmas represent the hardest case of abelian groups in two senses. The first is that these groups are not residually finite and the second is that showing they can act with strongly outer product type actions actually implies that every countable abelian group can also using the next lemma.

Lemma IV.19. *If for each $j \in \mathbb{N}$, G_j has a strongly outer product type action of the form*

$$\alpha_j : G_j \hookrightarrow \prod_{n=1}^{\infty} U(M_n) \stackrel{\mathrm{Ad}}{\to} \mathrm{Aut}\,\mathcal{Q},$$

then there exists a strongly outer product type action α of the infinite direct sum $G = \bigoplus_{j \in \mathbb{N}} G_j$ on $\mathcal{Q}$.

Proof. We will try to fit each of the direct summands into mutually disjoint subsequences of factors. For this it suffices to show that there is an appropriate embedding

$$\left(\prod_{n=1}^{\infty} U(M_n)\right) \oplus \left(\prod_{n=1}^{\infty} U(M_n)\right) \hookrightarrow \prod_{n=1}^{\infty} U(M_n)$$

First we see that there is an obvious embedding

$$\prod_{n=1}^{\infty} U(M_{n^2}) \hookrightarrow \prod_{n=1}^{\infty} U(M_n),$$

which when combined with the obvious embedding

$$U(M_n) \times U(M_n) \hookrightarrow U(M_{n^2})$$

gives the required embedding. This embedding is appropriate because when we restrict to G_j, the action is of the form $\alpha_j \otimes \mathrm{id}$, which is strongly outer by Theorem IV.7 combined with Lemma III.15. Map G_1 to the first factor and repeat the duplication process for the second factor. In this systematic way the embedding n-th summand can be defined for all n, which gives us an embedding of the direct sum over all n.

□

We have enough now to get a strongly outer product type action of any countable discrete abelian group.

Corollary IV.20. *Every countable discrete abelian group G has a strongly outer product type action of the form*

$$G \hookrightarrow \prod_{n=1}^{\infty} U(M_n) \stackrel{\mathrm{Ad}}{\to} \mathrm{Aut}\,\mathcal{Q},$$

Proof. Combining Lemmas IV.17 and IV.19 we see that there exists a strongly outer product type action of

$$\bigoplus_{n=1}^{\infty}(\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z})$$

on $\mathcal{Q}$. We know from Proposition IV.13 that every countable abelian group appears as a subgroup of this group. □

CHAPTER V

GROUP ACTIONS ON THE UNIVERSAL UHF ALGEBRA

Almost Abelian Groups

Definition V.1. A discrete group is said to be *almost abelian* if it has an abelian subgroup of finite index.

We will also observe throughout the course of this dissertation that countable discrete almost abelian groups are maximally almost periodic.

Lemma V.2. *Suppose that G is a discrete group with an abelian subgroup H of finite index. Then the subgroup*

$$N = \bigcap_{g \in G} gHg^{-1}$$

is normal and abelian with finite index. In particular, every almost abelian group is elementary amenable.

Proof. The subgroup N is abelian because it is a subgroup of H. It is a normal subgroup of G by construction. If suffices to show that G/N is finite in which case G is an extension of an abelian group N by a finite group G/N and we are done. Now N is exactly the kernel of the action of G on G/H, which is a finite set and hence G/N is isomorphic to a subgroup of the symmetric group $S_{[G:H]}$ making it finite. □

Lemma V.3. *Let G be a discrete group, let H be a subgroup of G with finite index k and let*

$$N = \bigcap_{g \in G} gHg^{-1}.$$

If there is a unitary representation of H on $\mathbb{C}^n$ with corresponding homomorphism

$$\rho_H : H \to U(M_n),$$

then there exists an induced representation with corresponding homomorphism given by

$$\rho_H^G : G \to U(M_n \otimes M_k)$$

such that the restriction of ρ_H^G to N is unitarily equivalent to k copies of the restriction of ρ_H to N. That is to say $\rho_H^G|_N = \rho_H|_N \otimes \mathrm{id}_{M_k}$.

Proof. Let $\{g_1, \dots, g_k\}$ be a set of coset representatives for H and define the induced representation on $\mathbb{C}^{nk}$ by first writing

$$\mathbb{C}^{nk} = g_1\mathbb{C}^n \oplus g_2\mathbb{C}^n \oplus \cdots \oplus g_k\mathbb{C}^n,$$

which now suggests how the G action is defined. Let $v_1, \dots, v_k \in \mathbb{C}^n$ and let σ be the permutation of $\{1, \dots, k\}$ induced by G acting on G/H. For $1 \leq j \leq k$ there exists $h_j \in H$ such that $gg_j = g_{\sigma(j)}h_j$. Then define

$$g(g_jv_j)_{1\leq j\leq k} = (g_{\sigma(j)}(h_jv_j))_{1\leq j\leq k}.$$

We can check that this defines a unitary representation of G with corresponding homomorphism

$$\rho_H^G : G \to U(M_n \otimes M_k).$$

Now let $h \in N$. We see that for $1 \leq j \leq k$ there is a unique $h_j \in N$ such that $h = g_jh_jg_j^{-1}$. This means that N preserves the decomposition

$$\mathbb{C}^{nk} = g_1\mathbb{C}^n \oplus g_2\mathbb{C}^n \oplus \cdots \oplus g_k\mathbb{C}^n.$$

We also see that with the obvious basis, h has the following diagonal matrix representation:

$$h \mapsto \mathrm{diag}(\rho_H|_N(h_1), \rho_H|_N(h_2), \dots, \rho_H|_N(h_k)).$$

This map is unitarily equivalent to the embedding

$$\rho_H|_N \otimes 1_{M_k} : N \to U(M_n) \otimes 1_{M_k} \hookrightarrow U(M_n) \otimes U(M_k),$$

with matrix representation

$$h \mapsto \rho_H|_N(h) \otimes 1_{M_k},$$

via the unitary matrix $\bigoplus_{1\leq j\leq k}(\rho_H^G(g_j)^*|_{g_j\mathbb{C}^n})$. $\square$

Lemma V.4. *Suppose G is a discrete almost abelian group. Then every irreducible representation of G is finite dimensional. Furthermore, there exists an $M \in \mathbb{N}$ such that the dimension of every irreducible representation is at most M.*

Proof. There being a uniform bound on the irreducible representations is due to Kaplansky (17, Theorem 1 and Theorem 3). □

Product Type Actions with the Pointwise Rokhlin Property

We begin this section by recalling Proposition III.4, which says that actions of a finite group G can be found on $A \otimes M_{|G|^\infty}$. We first generalise this to include groups that are certain extensions by finite groups and record it as Theorem V.7. We then use this along with our examples from Section 3 to show that almost abelian groups can act on $\mathcal{Q}$ with the pointwise Rokhlin property. In particular this result will be independent of Section 2. We then conclude with some remarks about why this is about as good as one can hope for.

Proposition V.5. *Let G be a discrete group with a normal subgroup N and let $q : G \to G/N$ be the quotient map. Suppose A and B are unital nuclear C^*-algebras, α is an action of G on A such that $\alpha|_N$ has the pointwise Rokhlin property, and β is an action of G/N on B that has the pointwise Rokhlin property. Then the action γ of G on $A \otimes B$, defined by $\gamma = \alpha \otimes (\beta \circ q)$, has the pointwise Rokhlin property.*

Proof. Note that if $g \in N$, then α_g is an automorphism with the Rokhlin property and if $g \notin N$, then $(\beta \circ q)_g$ is an automorphism with the Rokhlin property. We will now try to apply Lemma III.15 which involves considering the order of γ_g.

- Suppose γ_g has infinite order and $(\beta \circ q)_g$ has infinite order. Then $g \notin N$ meaning that $(\beta \circ q)_g$ has the Rokhlin property and we are done by applying Lemma III.15 with the roles of A and B reversed.

- If $(\beta \circ q)_g$ has finite order k and α_g has infinite order, then $g^k \in N$ and so $\alpha_g^k = \alpha_{g^k}$ has the Rokhlin property and $(\beta \circ q)_g$ has the Rokhlin property. This case is the content of Lemma III.15.

– If γ_g has finite order, then the orders of α_g and $(\beta \circ q)_g$ are also finite. Let k be the order of $(\beta \circ q)_g$. Then once again, $g^k \in N$ and α_g^k have the Rokhlin property, so Lemma III.15 applies.

□

Lemma V.6. *Suppose G is a countable discrete group, H is a subgroup of G with finite index k and let*

$$N = \bigcap_{g \in G} gHg^{-1}.$$

Also suppose that A is a UHF algebra and there is a product type action

$$\alpha_H : H \hookrightarrow \prod_{l=1}^{\infty} U(M_{n_l}) \to \operatorname{Aut} A$$

of H on A with the pointwise Rokhlin property. Then there is an extension of α to a product type action

$$\alpha_H^G : G \to \prod_{l=1}^{\infty} U(M_{n_l} \otimes M_{k^\infty}) \to \operatorname{Aut}\Big(M_{(n_l)_{l \in \mathbb{N}}} \otimes M_{k^\infty}\Big),$$

of G on $A \otimes M_{k^\infty}$ whose restriction to N has the pointwise Rokhlin property.

Proof. For each homomorphism $H \to U(M_{n_l})$ coming from α_H apply Lemma V.3 to get an induced homomorphism $G \to U(M_{n_l} \otimes M_k)$ such that when we put all of the maps together we get a map

$$\alpha_H^G : G \to \prod_{n=1}^{\infty} U(M_n \otimes M_k) \to \operatorname{Aut}\Big(\bigotimes_{n=1}^{\infty} (M_{n_l} \otimes M_k)\Big)$$

such that $\alpha_H^G|_N$ is conjugate to the action $\alpha_H|_N \otimes \mathrm{id}_{M_{k^\infty}}$, which has the pointwise Rokhlin property by Lemma III.15. □

Theorem V.7. *Suppose G is a countable discrete group with a subgroup H of finite index and normal subgroup of index k given by*

$$N = \bigcap_{g \in G} gHg^{-1}.$$

Suppose also that A is any UHF algebra. If H has a product type action on A with the pointwise Rokhlin property, then there is a product type action of G on $A \otimes M_{k^\infty}$ with the pointwise Rokhlin property.

Proof. Let q be the quotient map by N. By assumption, there exists a product type action

$$\alpha : H \hookrightarrow \prod_{l=1}^{\infty} U(M_{n_l}) \to \operatorname{Aut} A$$

of H on A which has the pointwise Rokhlin property. Let k' be the index of H which we note divides k. We then have by Proposition V.6 an action

$$\alpha_H^G : G \to \prod_{l=1}^{\infty} U(M_{n_l} \otimes M_{k'}) \to \operatorname{Aut} M_{(n_l k')_{l\in\mathbb{N}}}$$

of G on $A \otimes M_{k^\infty}$ whose restriction to N has the pointwise Rokhlin property. Now since G/N is a finite group of size k then by Proposition III.4 there is a product type action

$$\beta : G/H \to \prod_{l=1}^{\infty} U(M_k) \to \operatorname{Aut} M_{k^\infty}$$

of G/N on M_{k^∞} with the pointwise Rokhlin property. Combine these in the way specified by Proposition V.5 to get an action $\alpha_H^G \otimes (\beta \circ q)$ of G on $M_{(n_l k')_{l\in\mathbb{N}}} \otimes M_{k^\infty}$ with the pointwise Rokhlin property and that we easily see is of the form

$$G \hookrightarrow \left(\prod_{l=1}^{\infty} U(M_{n_l k})\right) \times \left(\prod_{l=1}^{\infty} U(M_k)\right) \to \operatorname{Aut}(M_{(n_l k')_{l\in\mathbb{N}}} \otimes M_{k^\infty}).$$

Reordering the factors we get a conjugate action (keeping the same name) of the form

$$\alpha_H^G \otimes (\beta \circ q) : G \hookrightarrow \prod_{n=1}^{\infty} U(M_{n_l k' k}) \to \operatorname{Aut}(M_{(n_l k' k)_{l\in\mathbb{N}}}).$$

Since k' divides k we have $M_{(n_l k' k)_{l\in\mathbb{N}}} \cong A \otimes M_{k^\infty}$ and we are done. □

We notice that when H is trivial, G is a finite group. We now look at the UHF algebra $\mathcal{Q}$, the only algebra that can possibly have an existence theorem for the pointwise Rokhlin property like that of Theorem IV.12.

Here we see the cut-down principle in action.

Lemma V.8. *Suppose G is a countable discrete abelian group and let $g \in G$ with finite order k. Then there is a map*

$$\alpha[g] : G \hookrightarrow \prod_{l=1}^{\infty} U(M_k) \hookrightarrow \operatorname{Aut} M_{k^\infty}$$

such that $\alpha[g]_g$ is an automorphism with the Rokhlin property.

Proof. By Corollary IV.20 there is a strongly outer action α of G on $\mathcal{Q}$. Hence α_g has the order k tracial Rokhlin property. Assume α_g has order k and without loss of generality the image of g is diagonal in $U(M_n)$. Let $(N_l)_{l\in\mathbb{N}}$ define a regrouping with respect to α_g as in Lemma III.14. Now regroup our action accordingly and by Lemma IV.4 we have a conjugate action

$$G \to \prod_{l=1}^{\infty} U(M_{N_l}) \to \operatorname{Aut} M_{(N_l)_{l\in\mathbb{N}}}.$$

Now since G is abelian, we can simultaneously diagonalise the image of G in $U(M_{N_l})$ and assume the first k entries are those for which α_g satisfies Lemma III.14. Then restricting to those entries we get a map

$$G \to U(M_{N_l}) \to U(M_k).$$

Combining these into a product gives

$$\alpha[g] : G \hookrightarrow \prod_{l=1}^{\infty} U(M_k) \hookrightarrow \operatorname{Aut} M_{k^\infty}$$

where the image of g has the Rokhlin property by Lemma III.14. □

Theorem V.9. *Let $\mathcal{Q}$ be the universal UHF algebra and let G be any countable discrete almost abelian group. Then there exists a product type action*

$$G \hookrightarrow \prod_{n=1}^{\infty} U(M_n) \to \operatorname{Aut} \mathcal{Q}$$

of G on $\mathcal{Q}$ with the pointwise Rokhlin property.

Proof. By Theorem V.7 it suffices to assume that G is abelian. In this case combining Corollary IV.20 with Lemma V.8 we have for each $g \in G$ of finite order $k(g)$ a map

$$\alpha[g] : G \hookrightarrow \prod_{l=1}^{\infty} U(M_{k(g)}) \hookrightarrow \operatorname{Aut} M_{k(g)^\infty}$$

such that $\alpha[g]_g$ is an automorphism with the strict Rokhlin property. For $g \in G$ with infinite order, let $\alpha[g]$ be any strongly outer product type action given by Corollary IV.20. The automorphism $\alpha[g]_g$ will have the Rokhlin property in this case because we have from Kishimoto (20, Theorem 1.4) every strongly outer $\mathcal{Z}$-action on $\mathcal{Q}$ is outer conjugate. In particular they are outer conjugate to the one from Proposition III.2 which has the Rokhlin property on $M_{2^\infty} \otimes \mathcal{Q}$. We can assume for all $g \in G$ that our maps are of the form

$$\alpha[g] : G \hookrightarrow \prod_{n=1}^{\infty} U(M_n) \hookrightarrow \operatorname{Aut} \mathcal{Q}.$$

Then combining these maps we get a map to the product

$$\alpha : G \hookrightarrow \prod_{m=1}^{\infty} \prod_{n=1}^{\infty} U(M_n) \hookrightarrow \operatorname{Aut} \left(\bigotimes_{m=1}^{\infty} \bigotimes_{n=1}^{\infty} M_n \right)$$

where every $g \in G$ acts via an automorphism with the Rokhlin property. Hence α has the pointwise Rokhlin property. We now apply Lemma IV.5 to get a product type action of the form required. □

Notice that since we made use of the example in Section 3 to provide the strongly outer actions of abelian groups, the above proof is logically independent of the results in Section 2.

We look at an example which was studied in Matui-Sato (37) as the first classification result for a group that was neither finite nor abelian. It is however almost abelian.

Thr Klein Bottle Group

The Klein bottle group has presentation

$$\mathbb{Z} \rtimes \mathbb{Z} = \langle a, b \,|\, bab^{-1} = a^{-1} \rangle.$$

We will omit the action in the notation since there is only one non-trivial action of $\mathbb{Z}$ on $\mathbb{Z}$. It was shown in Matui-Sato (37, Theorem 7.9) that every strongly outer action of $\mathbb{Z} \rtimes \mathbb{Z}$ on a UHF algebra A that absorbs M_{2^∞} is equivalent at least in the sense that the crossed products $A \rtimes (\mathbb{Z} \rtimes \mathbb{Z})$ are all the same (their equivalence is stronger) and the same is true on a UHF algebra with no tensor factor isomorphic to M_2. We will present here a model action of $\mathbb{Z} \rtimes \mathbb{Z}$ on M_{2^∞} with the pointwise Rokhlin property using the principles of Theorem V.9. Notice that $\mathbb{Z} \rtimes \mathbb{Z}$ is an almost abelian group with subgroup N generated by a and b^2 isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. This subgroup clearly has index 2 and is therefore normal. We have a $\mathbb{Z}$ action on M_{2^∞} given by Proposition III.2. Duplicating the finite stages of this map we get a map $\mathbb{Z} \oplus \mathbb{Z} \to U(M_{2^l}) \times U(M_{2^l})$ for each $l \in \mathbb{N}$. Embedding this into $U(M_{2^{2l}})$ we get a product type action of $\mathbb{Z} \oplus \mathbb{Z}$ on M_{2^∞} with the pointwise Rokhlin property. Now extend to $\mathbb{Z} \rtimes \mathbb{Z} \to U(M_{2^{2l}} \otimes M_2)$ by inducing as in Lemma V.3. Since N is normal we see that this will be enough to get the pointwise Rokhlin property upon taking a product. Let $g \notin N$. We see that there is a basis for $\mathbb{C}^{2^{2l}} \otimes \mathbb{C}^2$ for which g acts for some $h \in N$ as

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes h.$$

Putting this together as an infinite tensor product we have by Lemma III.15 that g acts with the order 2 Rokhlin property. Hence we have a model action of $\mathbb{Z} \rtimes \mathbb{Z}$ on M_{2^∞} with the pointwise Rokhlin property.

CHAPTER VI

MODEL GROUP ACTIONS ON THE UNIVERSAL UHF ALGEBRA

We look further into actions with the pointwise Rokhlin property and see if simpler model actions can be found for almost abelian groups. Whereas the previous chapter was themed around finite dimensional representations of groups, here we will be concerned with group actions on finite sets. We will not be able to get all of the almost abelian groups as before but many of the common ones do appear.

When restricting the adjoint action of $U(M_n)$ on M_n to the group of permutation matrices we get actions of subgroups of a product of symmetric groups on $\mathcal{Q}$. That is, subgroups of

$$\prod_{n=1}^{\infty} S_n,$$

where S_n is the symmetric group. These are known as residually finite groups and we will show that they behave well with respect to taking "almost". So to find almost abelian groups it suffices to look for abelian groups. Residually finite abelian groups are necessarily residually cyclic, in the sense that they will appear as a subgroup of a product of cyclic groups

$$\prod_{n=1}^{\infty} \mathbb{Z}/n\mathbb{Z}.$$

Actions of Abelian Groups

Abelian Groups

Here we give a list of abelian groups and their notation that are fundamental to the study of abelian groups. Let p be a prime number and let $r \in \mathbb{N}$ be a natural number.

Symbol	Description
$\mathbb{Z}$	Ring of integers
$\mathbb{Z}/p^r\mathbb{Z}$	Ring of integers modulo p^r
$\mathbb{Z}[p^{-1}]$	Ring of integers adjoin p^{-1}
$\mathbb{Z}_p$	Ring of p-adic integers
$\mathbb{Z}_{(p)}$	Ring of integers localised at the prime ideal (p)
$\mathbb{Z}[p^{-1}]/\mathbb{Z}$	Prüfer p-group
$\mathbb{Q}$	Ring of rational numbers

All these groups are countable except for the p-adic integers.

Definition VI.1 (p-adic integers)**.** The most convenient construction of the p-adic integers will be as the inverse limit of the system of quotient maps $\mathbb{Z}/p\mathbb{Z} \leftarrow \mathbb{Z}/p^2\mathbb{Z} \leftarrow \dots$, that is

$$\mathbb{Z}_p = \left\{ (a_r) \in \prod_{n=1}^{\infty} \mathbb{Z}/p^r\mathbb{Z} \,|\; a_k = a_l \bmod p^k \text{ for all } k \leq l \right\}.$$

Definition VI.2. An abelian group G is said to be p-divisible if for each $g \in G$ there is a $g' \in G$ such that $g = pg'$. It is divisible if it is p-divisible for all primes p.

The Prüfer p-groups and $\mathbb{Z}[\frac{1}{p}]$ are p-divisible, while $\mathbb{Q}$ is divisible.

Lemma VI.3. *If G is a p-divisible abelian group then G does not have subgroup of index a power of p. In particular, if G is divisible, then G does not have a proper subgroup of finite index.*

Proof. Suppose H is a subgroup of G with $[G : H] = p^r$ for some $r \geq 0$. Let $g \in G$. By divisibility there exists $g' \in G$ such that $g = p^r g'$. But this means that $g \in H$. Hence $r = 0$ and $H = G$. If G has a non-trivial finite index subgroup, then it must also have a index p subgroup for some prime p and therefore cannot be divisible. □

An element $g \in G$ is said to be a torsion element if it has finite order. The torsion elements form a subgroup. If this subgroup is zero, then G is said to be torsion-free. The groups $\mathbb{Z}/p^r\mathbb{Z}$ and the Prüfer p-groups are torsion groups while the other groups in Table 1 are torsion-free.

Theorem VI.4 (Prüfer)**.** *Suppose G is a countable abelian p-group. Then G is a direct sum of finite cyclic groups if and only if*

$$\bigcap_{n=1}^{\infty} p^n G = 0.$$

Proof. See for example Kaplansky (16, Theorem 11). □

A Class of Abelian Groups

Let $\mathbb{Z}/n\mathbb{Z}$ be the cyclic group of order n. Let $\mathcal{F}_0$ be the set of all abelian groups G for which there is an embedding $i : G \hookrightarrow \prod_{n=1}^{\infty} \mathbb{Z}/n\mathbb{Z}$ satisfying $i(G) \cap \bigoplus_{n=1}^{\infty} \mathbb{Z}/n\mathbb{Z} = \{0\}$.

We now investigate the kinds of groups that are in $\mathcal{F}_0$. As a part of this investigation we will see which of the basic abelian groups in the table of the previous section can and cannot be represented this way. By Lemma VI.3, we are fundamentally limited to non-divisible groups. That is

- $\mathbb{Z}[p^{-1}]/\mathbb{Z} \notin \mathcal{F}_0$ for all primes p,
- $\mathbb{Q} \notin \mathcal{F}_0$.

Lemma VI.5. *There is an isomorphism* $\prod_{n=1}^{\infty} \mathbb{Z}/n\mathbb{Z} \cong (\prod_{n=1}^{\infty} \mathbb{Z}/n\mathbb{Z})^{\mathbb{N}}$ *under which the subgroup* $\bigoplus_{n=1}^{\infty} \mathbb{Z}/n\mathbb{Z}$ *corresponds to* $\bigoplus_{n\in\mathbb{N}} \bigoplus_{n=1}^{\infty} \mathbb{Z}/n\mathbb{Z}$.

Proof. We only outline the proof to avoid being overwhelmed by notation. Decompose each $\mathbb{Z}/n\mathbb{Z}$ into a product of $\mathbb{Z}/p^r\mathbb{Z}$ according to the prime factorisation of n. Then reorder the factors accordingly so the same sequence of prime powers appears on the left and right hand side of the isomorphism. Since finite products are the same as finite direct sums, each summand and hence the infinite direct sum will be preserved by these oprations. □

Proposition VI.6. *The class* $\mathcal{F}_0$ *contains* $\prod_{n=1}^{\infty} \mathbb{Z}/n\mathbb{Z}$ *and is closed under countable direct sums and subgroups.*

Proof. For the first claim, take the diagonal embedding of $\prod_{n=1}^{\infty} \mathbb{Z}/n\mathbb{Z}$ into $(\prod_{n=1}^{\infty} \mathbb{Z}/n\mathbb{Z})^{\mathbb{N}}$, which clearly intersects the direct sum trivially. Hence the claim follows by Lemma VI.5. The closure under taking subgroups is obvious. Countable direct sums also follows from Lemma VI.5 since each summand can map into a different factor in the product. □

The following computation along with Prüfer's Theorem shows that the only countable torsion groups in $\mathcal{F}_0$ are direct sums of finite cyclic groups.

Lemma VI.7. *Suppose G is a residually finite abelian p-group. Then*

$$\bigcap_{n=1}^{\infty} p^n G = \{0\}.$$

Proof. We see for fixed m that $p^n(\mathbb{Z}/m\mathbb{Z})$ will not have p-torsion for sufficiently large n. Hence an element of the intersection cannot have p-torsion in any of its cyclic quotients and must therefore be zero. $\square$

Lemma VI.8. *Let p be a prime number and let $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$. There is a natural inclusion*

$$\mathbb{Z}_p \overset{i}{\hookrightarrow} \prod_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z},$$

which satisfies $i(\mathbb{Z}_p) \cap \bigoplus_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z} = \{0\}$. Consequently, $\mathbb{Z}_p \in \mathcal{F}_0$.

Proof. The construction of $\mathbb{Z}_p$ as a inverse limit given in Definition VI.1 gives a natural inclusion of $\mathbb{Z}_p$ into $\prod_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z}$ whose image must intersect $\bigoplus_{n=1}^{\infty} \mathbb{Z}/n\mathbb{Z}$ trivially since $\mathbb{Z}_p$ is torsion free. $\square$

Theorem VI.9. *Let P be the set of all primes. The class $\mathcal{F}_0$ contains all of the subgroups of the infinite direct sum*

$$\bigoplus_{r=1}^{\infty} \bigoplus_{p \in P} (\mathbb{Z}_{(p)} \oplus \mathbb{Z}/p^r\mathbb{Z}).$$

Proof. By Proposition VI.6 it suffices to show that $\mathbb{Z}_{(p)} \in \mathcal{F}_0$ and $\mathbb{Z}/p^r\mathbb{Z} \in \mathcal{F}_0$ for all primes $p \in P$ and $r \in \mathbb{N}$. Since $\mathbb{Z}_{(p)} \subset \mathbb{Z}_p$ for each $p \in P$, we get $\mathbb{Z}_{(p)} \in \mathcal{F}_0$ by Lemma VI.8. We also have $\mathbb{Z}/p^r\mathbb{Z} \in \mathcal{F}_0$ as an obvious consequence of Proposition VI.6. $\square$

The groups in our table which are not divisible are all in $\mathcal{F}_0$ and the divisible groups are definitely not in $\mathcal{F}_0$. Therefore in terms of describing a class of groups as subgroups of a fixed group, this is the largest fixed group possible. One might think that we could still include the inverse of some power of p to $\mathbb{Z}_{(p)} \subset \mathbb{Q}$ but this group will actually be isomorphic to $\mathbb{Z}_{(p)}$.

Lemma VI.10. *Let G be the subgroup of $\mathbb{Q}$ generated by $\mathbb{Z}_{(p)}$ and p^{-r} for some $r \geq 1$. Then G is isomorphic to $\mathbb{Z}_{(p)}$.*

Proof. Define a map $\varphi : G \to \mathbb{Z}_{(p)}$ by multiplication by p^r with inverse defined by composing the inclusion $\mathbb{Z}_{(p)} \hookrightarrow G$ with multiplication by p^{-r}. The main part of checking this is to show that if

$x \in \mathbb{Z}_{(p)}$ then $x/p^r \in G$. Since x can be written as a fraction m/n, it suffices to show that $\frac{1}{np^r} \in G$ for all n coprime to p.

Let n be coprime to p. Then there exists $a, b \in \mathbb{Z}$ so that $ap^r + bn = 1$ and we have

$$\frac{1}{p^r n} = \frac{ap^r + bn}{np^r} = \frac{a}{n} + \frac{b}{p^r} \in G.$$

□

Product Type Actions of Abelian Groups

Here we reveal the significance of the groups in $\mathcal{F}_0$.

Definition VI.11. An automorphism α is said to be uniformly outer if for every $a \in \mathcal{Q}$, every non-zero projection $p \in \mathcal{Q}$ and every $\epsilon > 0$, there exists $k > 0$ mutually orthogonal projections $p_1, \ldots, p_k$ such that $p = p_1 + \cdots + p_k$ and $p_i a \alpha(p_i) \approx_\epsilon 0$ for $1 \leq i \leq k$.

Let $\rho : \prod_{n=1}^\infty \mathbb{Z}/n\mathbb{Z} \to \prod_{n=1}^\infty U(M_n)$ be the product of the natural representations.

Proposition VI.12. *Suppose G is an abelian group for which there is an action α on $\mathcal{Q}$ of the form*

$$\alpha : G \overset{i}{\hookrightarrow} \prod_{n=1}^\infty \mathbb{Z}/n\mathbb{Z} \overset{\rho}{\hookrightarrow} \prod_{n=1}^\infty U(M_n) \overset{\mathrm{Ad}}{\to} \mathrm{Aut}\, \mathcal{Q}.$$

Then α has the pointwise Rokhlin property if and only if α satisfies $i(G) \cap \bigoplus_{n=1}^\infty \mathbb{Z}/n\mathbb{Z} = \{0\}$.

Proof. Since $\bigoplus_{n=1}^\infty \mathbb{Z}/n\mathbb{Z}$ acts by inner automorphisms, it will be necessary for $i(G)$ to intersect that subgroup trivially. We now prove that this condition is sufficient. We break the proof into two parts. In the first part we show that $g \notin \bigoplus_{n=1}^\infty \mathbb{Z}/n\mathbb{Z}$ implies $\mathrm{Ad}(\rho(i(g)))$ is uniformly outer and deduce that if g has infinite order, then it acts with the Rokhlin property. In the second part we verify directly that a group element of finite order k has the order k Rokhlin property.

Let $g \in G$ such that $i(g)$ is not in the direct sum. Let $a \in A$, let $p \in A$ be a non-zero projection and let $\epsilon > 0$. Without loss of generality we can assume $\|a\| = 1$. By the direct limit construction, there exists $N \in \mathbb{N}$, $b \in M_{N!}$ and a projection $q \in M_{N!}$ such that $a \approx_\epsilon b$ and $p \approx_\epsilon q$.

By assumption we can choose $n > N$ such that $i(g)_n$ has order $k \neq 1$. By the action of $\mathbb{Z}/n\mathbb{Z}$ on M_n we can find mutually orthogonal projections $q_1'', \ldots, q_k''$ that sum to 1_n and form an orbit in the obvious way under the action of the group generated by $i(g)_n$. Define

- $q'_1 = \sum_{i \neq k \text{ odd}} q''_i$,

- $q'_2 = \sum_{i \text{ even}} q''_i$ and

- $q'_3 = q''_k$ if k is odd and 0 otherwise.

Then define $p'_1 = pq'_1p$, $p'_2 = pq'_2p$ and $p'_3 = pq'_3p$. Note that $q_1 + q_2 + q_3 = p$ and q_1, q_2 and q_3 are close to some orthogonal projections p_1, p_2 and p_3 in pAp. Hence $p_1 + p_2 + p_3 = p$ since the sum is a projection close to the unit in pAp. We check the remaining condition. Let $1 \leq i \leq 3$. We have (noting α is isometric)

$$\begin{aligned} p_i a\alpha(p_i) &\approx p'_i b\alpha(p'_i) \\ &\approx qq'_i b\alpha(q'_i q) \\ &= qbq'_i\alpha(q'_i)\alpha(q) \\ &= 0. \end{aligned}$$

Hence g acts via a uniformly outer automorphism. By Kishimoto (20), any two automorphisms whose powers are all uniformly outer are outer conjugate. In particular g acts with the Rokhlin property since there exist automorphisms that act with the Rokhlin property.

Now assume g has finite order k and $k = p_1^{r_1} \cdots p_s^{r_s}$ is a unique factorisation into distint primes. The assumption that no power of $i(g)$ is in the direct sum guarantees for $1 \leq i \leq s$ that $p_i^{r_i}$ dividies infinitely many $i(g)_n$.

Let $\epsilon > 0$ and let $a_1, \ldots, a_n \in \mathcal{Q}$. We proceed to find k projections to witness the order k Rokhlin property.

Since $\mathcal{Q}$ is a direct limit, there exists $N \in \mathbb{N}$ and $b_1, \ldots b_n \in M_{N!}$ such that $a_i \approx_{\epsilon/2} b_i$. Also there exists for $1 \leq j \leq s$, $N_j > N$ such that $i(g)_{N_j}$ has order divisible by $P_j^{r_j}$. So $(g_{N_j})_{1 \leq j \leq s} \in \prod_{1 \leq j \leq s} \mathbb{Z}/N_j\mathbb{Z}$ has order divisible by k. We will now be able to find k projections in $M_{N_1} \otimes \cdots \otimes M_{N_s}$ to witness the order k Rokhlin property.

□

So $\mathcal{F}_0$ can also be defined as the set of all abelian groups G for which there is an action α on $\mathcal{Q}$ of the form

$$\alpha : G \overset{i}{\hookrightarrow} \prod_{n=1}^{\infty} \mathbb{Z}/n\mathbb{Z} \overset{\rho}{\hookrightarrow} \prod_{n=1}^{\infty} U(M_n) \overset{\text{Ad}}{\to} \operatorname{Aut} \mathcal{Q}$$

satisfying pointwise Rokhlin property.

Definition VI.13. A group G is said to be residually finite if for every $g \in G$ there is a a finite group F and a group homomorphism $\varphi : G \to F$ such that $\varphi(g) \neq 0$. Equivalently, there is an embedding

$$G \hookrightarrow \prod_{n=1}^{\infty} S_n,$$

where S_n is the symmetric group.

Definition VI.14. Suppose G is a group and H is a subgroup. Denote by G/H the set of cosets of H in G. Given an H-set Y, let $\sim$ be equivalence relation on $G \times Y$ generated by $(gh, y) \sim (g, hy)$. Then the induced G-set is defined to be

$$G \times_H Y = (G \times Y)/\sim .$$

Lemma VI.15. *Let G be a group and let H be a subgroup of finite index k. Let $\{g_1, \ldots, g_k\}$ be a set of coset representatives of H in G. Let Y be a H-set and write formally $gY = \{gy \,|\, y \in Y\}$. There is an isomorphism of G-sets*

$$G \times_H Y \cong \bigcup_{i=1}^{k} g_i Y,$$

where the action on the right hand side is the natural one.

Proof. Define $f : \bigcup_{i=1}^{k} g_i Y \to G \times_H Y$ by $g_i y \mapsto (g_i, y)$ and define $f^- : G \times_H Y \to \bigcup_{i=1}^{k} g_i Y$ as follows. For $g \in G$ choose i and $h \in H$ such that $g = g_i h$. Then set $f^-(g, y) = g_i(hy)$. The second map is well-defined because for $g = g_i h_i$, we have $f^-(gh, y) = g_i(h_i h y) = f^-(g, hy)$. We check that they are inverse to each other. First we have

$$f^- f(g_i y) = f^-(g_i, y) = g_i y$$

and in the reverse direction we have

$$f f^-(g, y) = f(g_i(h_i y)) = (g_i, h_i y) \sim (g_i h_i, y) = (g, y).$$

□

Let $\rho_n : S_n \to U(M_n)$ be the natural representation obtained from permuting the standard basis vectors of $\mathbb{C}^n$ and let $\rho = \prod_{n=1}^{\infty} \rho_n : \prod_{n=1}^{\infty} S_n \to \prod_{n=1}^{\infty} U(M_n)$. Define

$$\prod_{n=1}^{\infty} S_n \xrightarrow{\rho} \prod_{n=1}^{\infty} U(M_n) \xrightarrow{\mathrm{Ad}} \mathrm{Aut}\, \mathcal{Q}.$$

Let $\mathcal{F}$ be the set of all discrete groups G such that there is an action α of G on $\mathcal{Q}$ represented by

$$\alpha : G \hookrightarrow \prod_{n=1}^{\infty} S_n \xrightarrow{\rho} \prod_{n=1}^{\infty} U(M_n) \xrightarrow{\mathrm{Ad}} \mathrm{Aut}\, \mathcal{Q}$$

that has the pointwise Rokhlin property.

We now see how groups in $\mathcal{F}_0$ can be promoted to groups in $\mathcal{F}$.

Theorem VI.16. *Let G be a countable discrete group with an abelian subgroup H of finite index. If $H \in \mathcal{F}_0$, then $G \in \mathcal{F}$.*

Proof. Since $\mathcal{F}_0$ is closed under taking subgroups, we can replace H by $\bigcap_{g \in G} gHg^{-1}$ if necessary and assume H is normal. Let k be the index of the normal subgroup H. We know from Lemma V.2 that k is finite. Since $H \in \mathcal{F}_0$ there is an embedding

$$\iota : H \hookrightarrow \prod_{n=1}^{\infty} \mathbb{Z}/n\mathbb{Z}$$

such that for each $h \in H$ there are infinite many n for which h is able to act on the diagonal projections of M_n with no fixed points.

For each $n \in \mathbb{N}$ we have the coordinate map $\iota_n : H \to \mathbb{Z}/n\mathbb{Z}$ which corresponds to the action of H on a set $Y_n = \{e_{1,1}, \ldots, e_{n,n}\}$ of mutually orthogonal rank one projections in M_n. Take the fibred product to get a G-set

$$X_n = G \times_H Y_n,$$

where the elements of X_n correspond to rank one orthogonal projections in M_{nk}. So we have maps $\kappa_n : G \to S_{nk}$, which combine to form a map

$$\kappa : G \hookrightarrow \prod_{n=1}^{\infty} S_{nk}.$$

By construction, there will for each $g \in G$ be infinite many n for which g acts on X_n with no fixed points and hence g will act by a uniformly outer automorphism since this is the only property we needed in the proof of Proposition VI.12. Hence if g has infinite order we argue as above to conclude that g acts with the Rokhlin property.

So to show that this embedding will give an action of G with the pointwise Rokhlin property it suffices to consider elements in G of finite order.

Let $g \in G$ and suppose the image of g in G/H has order $m = P_1^{r_1} \cdots P_s^{r_s}$. Let $h = g^m$ with finite order $m' = P_1^{r'_1} \cdots P_s^{r'_s}$ (without loss of generality). Let $\epsilon > 0$ and let $\{a_1, \ldots, a_t\}$ be a finite subset. Choose N large enough so that elements in M_n commute with $\{a_1, \ldots, a_t\}$ up to ϵ for all $n > N$. Now find $N_1, N_2, \ldots, N_s > N$ so that we can find $P_l^{r'_l}$ mutually orthogonal projections cycled by h in M_{N_l}. We can write these projections as sums of the $e_{l,l}$ and call these $y_1^{(l)}, \ldots, y_{P_l^{r'_l}}^{(l)}$.

Now pick coset representatives for H in G of the following form

$$\begin{array}{ccccc} g_1 & gg_1 & g^2g_1 & \cdots & g^{m-1}g_1 \\ g_2 & gg_2 & g^2g_2 & \cdots & g^{m-1}g_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ g_{k/m} & gg_{k/m} & g^2g_{k/m} & \cdots & g^{m-1}g_{k/m}, \end{array}$$

We can write $X_n = \bigcup_{j,k} g^j g_k Y_n$, where $g^j g_k Y_n = \{g^j g_k e_{1,1}, \ldots, g^j g_k e_{n,n}\}$ corresponds to a set of rank one orthogonal projections. Define the rank $\frac{kN_l}{mP_l^{r'_l}}$ projections

$$x_{i,j}^{(l)} = g^j g_1 y_i^{(l)} + g^j g_2 y_i^{(l)} \cdots + g^j g_{k/m} y_i^{(l)}.$$

These form a single orbit of size $P_l^{r'_l} m$ from the action of the group generated by g. By adding projections we can define $z_{i,j}^{(l)}$ with orbit size $P_l^{r'_l + r_l}$. Define the rank $\prod_{l=1}^{s} \frac{kN_l}{P_l^{r'_l + r_l}}$ projection

$$p_1 = z_{1,1}^{(1)} \otimes z_{1,1}^{(2)} \otimes \cdots \otimes z_{1,1}^{(s)}$$

to generate our orbit of mm' projections inside $M_{kN_1} \otimes \cdots \otimes M_{kN_s}$ whose ranks sum to $(kN_1) \cdots (kN_s)$. □

CHAPTER VII

GROUP ACTIONS ON SIMPLE NUCLEAR C^*-ALGEBRAS

A Family of Group Actions on the Jiang-Su Algebra

We will introduce the Jiang-Su algebra $\mathcal{Z}$ as a very important example of a *strongly self-absorbing C^*-algebra*. Most of what we prove only uses this property of $\mathcal{Z}$ (along with its simplicity and having a unique tracial state, which are implied by the property). Another important example is the universal UHF-algebra $\mathcal{Q}$.

Definition VII.1 ((48))**.** A C^*-algebra A is called *strongly self-absorbing* if there is an isomorphism $\Psi : A \to A \otimes A$ and a sequence of unitaries $(v_n)_{n\in\mathbb{N}}$ in A such that for any $a \in A$, we have

$$\lim_{n\to\infty} \operatorname{Ad} v_n \circ \Psi(a) = a \otimes 1.$$

Definition VII.2. A C^*-algebra A is called $\mathcal{Z}$-stable if $A \otimes \mathcal{Z} \cong A$.

Lemma VII.3. *There is an isomorphism*

$$\mathcal{Z} \cong \varinjlim \left(\mathcal{Z}^{\otimes n}, \mathrm{id}_{\mathcal{Z}^{\otimes n}} \otimes 1\right).$$

Proof. This is well-known and first appeared as (15, Corollary 8.8). □

Definition VII.4. Let G be a countable discrete group and identify $\mathcal{Z}$ with $\mathcal{Z}^{\otimes G} = \bigotimes_{g\in G} \mathcal{Z}$ using Lemma VII.3 and countability of G. Now define an action β of G on $\mathcal{Z}$ via this identification by

$$\beta_g : \bigotimes_{h\in G} z_h \mapsto \bigotimes_{h\in G} z_{g^{-1}h}.$$

Again using Lemma VII.3 to identify $\mathcal{Z}$ with $\mathcal{Z}^{\otimes\mathbb{N}}$, we define the action γ of G on $\mathcal{Z}$ by

$$\gamma_g : \bigotimes_{n\in\mathbb{N}} z_n \mapsto \bigotimes_{n\in\mathbb{N}} \beta_g(z_n).$$

For any group automorphism φ, we define β^φ to be the action of G on $\mathcal{Z}$ given by $g \mapsto \beta_{\varphi(g)}$. So we have $\beta^{\mathrm{id}_G} = \beta$. We can define γ^φ analogously using γ instead of β and also get $\gamma^{\mathrm{id}_G} = \gamma$.

We first show that all of the β^φ are conjugate for different φ so we only need to consider $\beta^{\mathrm{id}_G} = \beta$ from here without loss of generality. A similar thing happens when a different ordering of G is taken to define the infinite tensor product.

Lemma VII.5. *Let φ be a group automorphism of G and let $\widehat{\varphi}$ denote the induced automorphism on $\mathcal{Z}$ given by*

$$\widehat{\varphi} : z = \bigotimes_{h \in G} z_h \mapsto \bigotimes_{h \in G} z_{\varphi(h)}.$$

Then for all $g \in G$, we have

$$\beta_g^\varphi = \widehat{\varphi} \circ \beta_g \circ \widehat{\varphi}^{-1}.$$

Proof. Let $g \in G$ and let $z = \bigotimes_{h \in G} z_h \in \mathcal{Z}^{\otimes G}$. We have

$$\begin{aligned}
(\widehat{\varphi}^{-1} \circ \beta_g^\varphi \circ \widehat{\varphi})(z) &= (\widehat{\varphi}^{-1} \circ \beta_g^\varphi \circ \widehat{\varphi}) \left(\bigotimes_{h \in G} z_h \right) \\
&= (\widehat{\varphi}^{-1} \circ \beta_g^\varphi) \left(\bigotimes_{h \in G} z_{\varphi(h)} \right) \\
&= \widehat{\varphi}^{-1} \left(\bigotimes_{h \in G} z_{\varphi(g^{-1})\varphi(h)} \right) \\
&= \bigotimes_{h \in G} z_{g^{-1}h} \\
&= \beta_g(z).
\end{aligned}$$

□

We now move from "one dimensional thinking" to "two dimensional thinking".

Theorem VII.6. *Suppose G is a countable discrete group. Then γ is conjugate to β. In particular, both γ and β are strongly outer.*

Proof. Since $\beta_g \neq \mathrm{id}$ for all $g \in G \setminus \{1\}$ and $\mathcal{Z}$ has unique trace, we apply Lemma **??** with $\alpha = \beta$ and $A = \mathcal{Z}$ to get that γ is strongly outer. Now we annotate with subscripts our copies of $\mathcal{Z}$ in the tensor product decomposition of $\mathcal{Z}^{\otimes G}$ to emphasise their position. That is,

$$\mathcal{Z}^{\otimes G} = \bigotimes_{h \in G} \mathcal{Z}_h.$$

The action β acts by permuting these factors. We note that for each h, the factor $\mathcal{Z}_h$ can be further decomposed using Lemma VII.3 into an infinite tensor product of copies of $\mathcal{Z}$, where we denote each copy by $\mathcal{Z}_h^{(l)}$ to emphasise its placement in the decomposition of $\mathcal{Z}_h$. That is,

$$\mathcal{Z}_h \cong \bigotimes_{l \in \mathbb{N}} \mathcal{Z}_h^{(l)}.$$

We see that for each $l \in \mathbb{N}$, β leaves the subalgebra $\mathcal{Z}^{(l)} = \bigotimes_{h \in G} \mathcal{Z}_h^{(l)}$ invariant and we recover the action β when we identify $\mathcal{Z}^{(l)}$ with $\mathcal{Z}$. Hence we have that β is conjugate to $\beta^{\otimes \mathbb{N}}$ acting on $\bigotimes_{l=1}^{\infty} \mathcal{Z}^{(l)}$, which is conjugate to γ acting on $\mathcal{Z}$. $\square$

Let G be a countable discrete group. Let A be a unital C^*-algebra. We use γ of the previous section to define an action $\omega(G, A)$ of G on $A \otimes \mathcal{Z}$. We show here that when G is elementary amenable and A is tracially approximately divisible then $\omega(G, A)$ has the tracial Rokhlin property in the sense of Matui-Sato, whose definition we record here.

The Action $\omega(G, A)$

Definition VII.7. Let γ be as in Definition VII.4. For any C^*-algebra A, define the action ω^A on $A \otimes \mathcal{Z}$ by

$$\omega^A = \mathrm{id}_A \otimes \gamma.$$

Remark VII.8. If A is unital, the action ω^A is pointwise approximately inner because all automorphisms on $\mathcal{Z}$ are approximately inner.

The Matui-Sato Tracial Rokhlin Property

Definition VII.9. A group action α of G on a C^*-algebra A has the *Matui-Sato tracial Rokhlin property* if for every finite subset $F \subset G$ and $\epsilon > 0$, there is a finite (F, ϵ)-invariant subset K in G and a central sequence $(p_n)_{n \in \mathbb{N}}$ in A consisting of projections such that for $g, h \in K$ with $g \neq h$

- $\lim_{n \to \infty} \alpha_g(p_n) \alpha_h(p_n) = 0$
- $\lim_{n \to \infty} \max_{\tau \in T(A)} |\tau(p_n) - |K|^{-1}| = 0.$

Let M_k denote the full $k \times k$ matrix algebra with identity written 1_k.

Lemma VII.10. *For any finite subset $F \subset G$ and $\epsilon > 0$, there is a finite (F, ϵ)-invariant subset K of G such that for each $n \in \mathbb{N}$ there is $m(n) \in \mathbb{N}$ and a projection $q_n \in M_{m(n)} \otimes \mathcal{Z}$ satisfying the following*

$$\omega_g^{M_{m(n)}}(q_n)\omega_h^{M_{m(n)}}(q_n) \approx_{1/n} 0 \quad \textit{for all } g, h \in K \textit{ with } g \neq h,$$

$$\tau(q_n) \approx_{1/n} |K|^{-1}.$$

Proof. Let F be a finite subset of G and let $\epsilon > 0$. By Lemma **??**, $\mathrm{id} \otimes \gamma$ is a strongly outer action on $\mathcal{Q} \otimes \mathcal{Z}$. Therefore by Theorem IV.7 it also has the tracial Rokhlin property. So we have from Definition VII.9 a finite subset K in G and a central sequence (q_n') consisting of projections in $\mathcal{Q} \otimes \mathcal{Z}$ such that for all $g, h \in K$ with $g \neq h$, we have

– $\lim_{n\to\infty}(\mathrm{id}_{\mathcal{Q}} \otimes \gamma_g)(q_n')(\mathrm{id}_{\mathcal{Q}} \otimes \gamma_h)(q_n') = 0$,

– $\lim_{n\to\infty} \max_{\tau \in T(\mathcal{Q}\otimes\mathcal{Z})} |\tau(q_n') - |K|^{-1}| = 0$.

By passing to a subsequence if necessary and noting that $\mathcal{Q} \otimes \mathcal{Z}$ has unique trace, we have

$$(\mathrm{id}_{\mathcal{Q}} \otimes \gamma_g)(q_n')(\mathrm{id}_{\mathcal{Q}} \otimes \gamma_h)(q_n') \approx_{1/3n} 0 \quad \text{for all } g, h \in K \text{ with } g \neq h,$$

$$\tau(q_n') \approx_{1/n} |K|^{-1}.$$

Since $\mathcal{Q}$ is a UHF-algebra, there are $m(n)$ and $q_n'' \in M_{m(n)}$ self-adjoint such that $q_n' \approx_{1/15n} q_n''$. Hence by functional calculus there is a projection $q_n \in M_{m(n)}$ such that $q_n \approx_{5/15n} q_n'$ (see for example (21, Lemma 2.5.5)). We see now, since automorphisms are isometric and $\mathrm{id}_{\mathcal{Q}} \otimes \gamma$ restricts to $\mathrm{id}_{m(n)} \otimes \gamma$ on $M_{m(n)} \otimes \mathcal{Z}$, that

$$\begin{aligned}(\mathrm{id}_{m(n)} \otimes \gamma_g)(q_n)(\mathrm{id}_{m(n)} \otimes \gamma_h)(q_n) &\approx_{1/3n} (\mathrm{id}_{\mathcal{Q}} \otimes \gamma_g)(q_n')(\mathrm{id}_{m(n)} \otimes \gamma_h)(q_n) \\ &\approx_{1/3n} (\mathrm{id}_{\mathcal{Q}} \otimes \gamma_g)(q_n')(\mathrm{id}_{\mathcal{Q}} \otimes \gamma_h)(q_n') \\ &\approx_{1/3n} 0 \quad \text{for all } g, h \in K \text{ with } g \neq h.\end{aligned}$$

We also have

$$\tau(q_n) = \tau(q_n') \approx_{1/n} |K|^{-1}.$$

□

Group Actions on Tracially Approximately Divisible C^*-algebras

Definition VII.11 (Tracially approximately divisible)**.** Let A be a unital simple separable C^*-algebra. We say that A is *tracially approximately divisible* if for every $\epsilon > 0$, every $l \in \mathbb{N}$, every finite subset $\{a_1, a_2, \ldots a_k\} \subset A$ and any non-zero $y \in A_+$, there exists a finite dimensional algebra B with each simple summand's rank exceeding l, and a $*$-homomorphism $\varphi : B \to A$, such that for all $i \leq k$ and $e \in B$ with $\|e\| \leq 1$,

- $[a_i, \varphi(e)] \approx_\epsilon 0$,
- $1 - \varphi(1_B) \precsim y$.

Simple tracially approximately divisible algebras automatically satisfy strict comparison because they are tracially $\mathcal{Z}$-stable (see (14, Definition 2.1, Theorem 3.3)). If we assume strict comparison, then the definition above is equivalent to the following definition, which will serve as our working definition.

Definition VII.12 (Tracially approximately divisible with strict comparison)**.** Let A be a unital simple separable C^*-algebra with strict comparison. We say that A is *tracially approximately divisible* if for every $\epsilon > 0$, every $n \in \mathbb{N}$, every finite subset $\{a_1, a_2, \ldots a_k\} \subset A$, there exists $N > n$ and a $*$-homomorphism $\varphi : M_N \to A$ such that for all $i \leq k$, $e \in M_N$ with $\|e\| \leq 1$ and $\tau \in T(A)$,

- $[a_i, \varphi(e)] \approx_\epsilon 0$,
- $\sup_{\tau \in T(A)} |1 - \tau(\varphi(1_N))| \approx_\epsilon 0$.

Theorem VII.13. *If A is a unital simple separable tracially approximately divisible C^*-algebra, G is a countable discrete elementary amenable group and ω^A is an action of G on $A \otimes \mathcal{Z}$ as in Definition VII.7, then ω^A has the tracial Rokhlin property.*

Proof. Let F be a finite subset of G and let $\epsilon > 0$. We aim to show that there is a finite (F, ϵ)-invariant subset K in G and a central sequence $(p_n)_{n \in \mathbb{N}}$ consisting of projections in $A \otimes \mathcal{Z}$ such that

- $\lim_{n \to \infty} \omega^A_g(p_n)\omega^A_h(p_n) = 0$ for all $g, h \in K$,
- $\lim_{n \to \infty} \max_{\tau \in T(A \otimes \mathcal{Z})} |\tau(p_n) - |K|^{-1}| = 0$.

We begin by introducing some notation for this proof. Define

$$\mathcal{Z}^{\otimes n} = \bigotimes_{j \leq n} \mathcal{Z} \quad \text{with action } \beta^{\otimes n} = \otimes_{j \leq n} \beta,$$

and

$$\mathcal{Z}^{\otimes (\mathbb{N} \setminus n)} = \bigotimes_{j > n} \mathcal{Z} \quad \text{with action } \beta^{\otimes (\mathbb{N} \setminus n)} = \otimes_{j > n} \beta.$$

There are obvious action preserving isomorphisms

$$\rho_n : (\mathcal{Z}, \gamma) \to (\mathcal{Z}^{\otimes n} \otimes \mathcal{Z}^{\otimes (\mathbb{N} \setminus n)}, \beta^{\otimes n} \otimes \beta^{\otimes (\mathbb{N} \setminus n)})$$

and

$$\sigma_n : (\mathcal{Z}, \gamma) \to (\mathcal{Z}^{\otimes (\mathbb{N} \setminus n)}, \beta^{\otimes (\mathbb{N} \setminus n)}).$$

Fix a dense sequence $x_1, x_2, \ldots$ in $A \otimes \mathcal{Z}$. We will proceed to define for each $n \in \mathbb{N}$ a projection p_n to satisfy our initial requirements. To do this, it will be helpful to also establish for $j \leq n$,

$$[p_n, x_j] \approx_{5/n} 0.$$

Let $n \in \mathbb{N}$. Find $a_{i,j} \in A$ and $z_{i,j} \in \mathcal{Z}$ such that for $j \leq n$, we have

$$x_j \approx_{1/n} \sum_{i=1}^{l(j)} a_{i,j} \otimes z_{i,j}.$$

Write

$$L_n = \sum_{j \leq n} \sum_{i \leq l(j)} \|a_{i,j}\|.$$

There exists $n' \in \mathbb{N}$ such that for all $j \leq n$ and $i \leq l(j)$, there are $z'_{i,j} \in \mathbb{Z}^{\otimes n'}$ satisfying

$$\rho_{n'}(z_{i,j}) \approx_{\frac{1}{nL_n}} z'_{i,j} \otimes 1_{\mathcal{Z}^{\otimes (\mathbb{N} \setminus n')}}.$$

Define an action preserving isomorphism

$$\chi_n : A \otimes \mathcal{Z} \to A \otimes \mathcal{Z}^{\otimes n'} \otimes \mathcal{Z}$$

by

$$\chi_n = (\mathrm{id}_{A\otimes\mathcal{Z}^{\otimes n'}} \otimes \sigma_{n'}^{-1}) \circ (\mathrm{id}_A \otimes \rho_{n'}).$$

For $j \leq n$ and $i \leq l(j)$, we get

$$\chi_n(x_j) \approx_{\frac{2}{n}} \sum_{i=1}^{l(j)} a_{i,j} \otimes z'_{i,j} \otimes 1_{\mathcal{Z}}. \tag{VII.1}$$

The calculation for (VII.1) is included here for convenience:

$$\begin{aligned}
\left\| \chi_n(x_j) - \sum_{i=1}^{l(j)} a_{i,j} \otimes z'_{i,j} \otimes 1_{\mathcal{Z}} \right\| &\approx_{1/n} \left\| \sum_{i=1}^{l(j)} a_{i,j} \otimes ((\mathrm{id} \otimes \sigma_{n'}^{-1})\rho_{n'}(z_{i,j}) - z'_{i,j} \otimes 1_{\mathcal{Z}}) \right\| \\
&= \left\| \sum_{i=1}^{l(j)} a_{i,j} \otimes ((\mathrm{id} \otimes \sigma_{n'}^{-1})(\rho_{n'}(z_{i,j}) - z'_{i,j} \otimes 1_{\mathcal{Z}^{\otimes(\mathbb{N}\setminus n')}})) \right\| \\
&\leq \sum_{i=1}^{l(j)} \|a_{i,j}\| \|((\mathrm{id} \otimes \sigma_{n'}^{-1})(\rho_{n'}(z_{i,j}) - z'_{i,j} \otimes 1))\| \\
&= \sum_{i=1}^{l(j)} \|a_{i,j}\| \|\rho_{n'}(z_{i,j}) - z'_{i,j} \otimes 1))\| \\
&\leq \sum_{i=1}^{l(j)} \|a_{i,j}\| \frac{1}{nL_n} \\
&= \frac{1}{nL_n} \sum_{i=1}^{l(j)} \|a_{i,j}\| \\
&\leq \frac{L_n}{nL_n} \\
&= \frac{1}{n}.
\end{aligned}$$

We now apply Lemma VII.10 to get an $m \in \mathbb{N}$, a finite (F, ϵ)-invariant subset K of G and a projection $q_n \in M_m \otimes \mathcal{Z}$ satisfying:

$$((\mathrm{id}_m \otimes \gamma_g)(q_n))\,((\mathrm{id}_m \otimes \gamma_h)(q_n)) \approx_{1/n} 0 \quad \text{for all } g, h \in K \text{ with } g \neq h,$$

$$(\tau_m \otimes \tau_{\mathcal{Z}})(q_n) \approx_{1/n} |K|^{-1}.$$

Thinking of q_n as a matrix with entries $y_{k,l} \in \mathcal{Z}$, we have

$$q_n = \sum_{k,l=1}^{m} e_{k,l} \otimes y_{k,l},$$

where the $e_{k,l}$ are standard matrix units. Also define for convenience,

$$L'_n = \sum_{j=1}^{n} \sum_{i=1}^{l(j)} \sum_{k,l=1}^{m} \|y_{k,l}\| \|z'_{i,j}\|.$$

Since A is tracially approximately divisible, there exists by Definition VII.12, an $m' \in \mathbb{N}$, a $*$-homomorphism $\varphi : M_{m'} \to A$, satisfying for all $j \leq n$, $i \leq l(j)$, and $e \in M_{m'}$ with $\|e\| \leq 1$,

$$[a_{i,j}, \varphi(e)] \approx_{\frac{1}{nL'_n}} 0 \tag{VII.2}$$

$$\tau(\varphi(1_{m'})) \approx_{\frac{1}{n}} 1 \tag{VII.3}$$

$$m' > mn. \tag{VII.4}$$

Now write $m' = Nm + r$ with $0 \leq r < m$, and $N \in \mathbb{N}$ and define an embedding

$$\psi_n : M_m \otimes \mathcal{Z} \hookrightarrow A \otimes \mathcal{Z}^{\otimes n'} \otimes \mathcal{Z}$$

on generators for $e \in M_m$ and $z \in \mathcal{Z}$ by

$$e \otimes z \mapsto \varphi(\mathrm{diag}(e \otimes 1_N, 0_r)) \otimes 1 \otimes z,$$

where $\mathrm{diag}(e \otimes 1_N, 0_r)$ denotes a block diagonal matrix with the first N blocks e and zeros for the remaining $r \times r$ block. We see that this embedding respects the group action and the image of q_n is

$$\psi_n(q_n) = \sum_{k,l=1}^{m} \varphi(\mathrm{diag}(e_{k,l} \otimes 1_N, 0_r)) \otimes 1 \otimes y_{k,l}. \tag{VII.5}$$

We now define the promised projections $p_n \in A \otimes \mathcal{Z}$ by

$$p_n = (\chi_n^{-1} \circ \psi_n)(q_n).$$

We first check for all $j \leq n$ that

$$[p_n, x_j] \approx_{5/n} 0.$$

Let $j \leq n$. Then

$$\begin{aligned}
\chi_n([p_n, x_j]) &= [\chi_n(p_n), \chi_n(x_j)] \\
&= [\psi_n(q_n), \chi_n(x_j)] \\
\text{(use (VII.1))} \quad &\approx_{\frac{4}{n}} \left[\psi_n(q_n), \sum_{i=1}^{l(j)} a_{i,j} \otimes z'_{i,j} \otimes 1_{\mathcal{Z}}\right] \\
\text{(use (VII.5))} \quad &= \sum_{i=1}^{l(j)} \sum_{k,l=1}^{m} [\varphi(\mathrm{diag}(e_{k,l} \otimes 1_N, 0_r)) \otimes 1 \otimes y_{k,l}, a_{i,j} \otimes z'_{i,j} \otimes 1_{\mathcal{Z}}] \\
&= \sum_{i=1}^{l(j)} \sum_{k,l=1}^{m} [\varphi(\mathrm{diag}(e_{k,l} \otimes 1_N, 0_r)), a_{i,j}] \otimes z'_{i,j} \otimes y_{k,l} \\
\text{(use (VII.2))} \quad &\approx_{1/n} 0.
\end{aligned}$$

We now show that $(p_n)_{n\in\mathbb{N}}$ satisfies the conditions in the definition of the tracial Rokhlin property.

- It is clear that K is a finite (F, ϵ)-invariant subset of G.
- The sequence $(p_n)_{n\in\mathbb{N}}$ is central: Let $x \in A \otimes \mathcal{Z}$ and $\epsilon > 0$. We have from density that for some $j \in \mathbb{N}$,

  $$x \approx_\epsilon x_j.$$

  Now let $n \geq j$ be such that $1/n < \epsilon$, then

  $$\begin{aligned}
  \chi_n([p_n, x]) &= [\chi_n(p_n), \chi_n(x)] \\
  &\approx_{2\epsilon} [\chi_n(p_n), \chi_n(x_j)] \\
  &\approx_{5/n} 0.
  \end{aligned}$$

  Hence for our choice of n we have

  $$[p_n, x] \approx_{7\epsilon} 0.$$

– Orthogonality: Let $g, h \in K$ with $g \neq h$. Then

$$\begin{aligned}\chi_n(\omega_g^A(p_n)) &= \omega_g^A(\chi_n(p_n)) \\ &= (\mathrm{id} \otimes \beta_g^{\otimes n'} \otimes \gamma_g)(\psi_n(q_n)) \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes \gamma_g)(\psi_n(q_n)) \\ &= \psi_n((\mathrm{id} \otimes \gamma_g)(q_n)).\end{aligned}$$

So

$$\begin{aligned}\chi_n(\omega_g^A(p_n)\omega_h^A(p_n)) &= \chi_n(\omega_g^A(p_n))\chi_n(\omega_h^A(p_n)) \\ &= \psi_n((\mathrm{id} \otimes \gamma_g)(q_n))\psi_n((\mathrm{id} \otimes \gamma_h)(q_n)) \\ &= \psi_n((\mathrm{id} \otimes \gamma_g)(q_n)(\mathrm{id} \otimes \gamma_h)(q_n)) \\ &\approx_{1/n} 0.\end{aligned}$$

– Trace condition: let $\tau \in T(A)$, and let $\tau_{\mathcal{Z}}$ and τ_k be the unique tracial states on $\mathcal{Z}$ and M_k respectively. Then

$$\begin{aligned}(\tau \otimes \tau_{\mathcal{Z}})(p_n) &= (\tau \otimes \tau_{\mathcal{Z}^{\otimes n'}} \otimes \tau_{\mathcal{Z}})(\chi_n(p_n)) \\ &= (\tau \otimes \tau_{\mathcal{Z}^{\otimes n'}} \otimes \tau_{\mathcal{Z}})(\psi_n(q_n)) \\ &= (\tau \otimes \tau_{\mathcal{Z}})(q_n) \\ &= (\tau \otimes \tau_{\mathcal{Z}})(\varphi(\mathrm{diag}(1_m \otimes 1_N, 0_r)) \otimes 1_{\mathcal{Z}})(\tau_m \otimes \tau_{\mathcal{Z}})(q_n) \\ &= \tau(\varphi(\mathrm{diag}(1_m \otimes 1_N, 0_r)))(\tau_m \otimes \tau_{\mathcal{Z}})(q_n) \\ &= \tau(\varphi(1_{m'}))\tau_{m'}(\mathrm{diag}(1_m \otimes 1_N, 0_r))(\tau_m \otimes \tau_{\mathcal{Z}})(q_n) \\ &= \frac{m' - r}{m'}\tau(\varphi(1_{m'}))(\tau_m \otimes \tau_{\mathcal{Z}})(q_n) \\ \text{(use (VII.3))} \quad &\approx_{1/n} \tau(\varphi(1_{m'}))(\tau_m \otimes \tau_{\mathcal{Z}})(q_n) \\ \text{(use (VII.4))} \quad &\approx_{1/n} (\tau_m \otimes \tau_{\mathcal{Z}})(q_n) \\ &\approx_{1/n} |K|^{-1}.\end{aligned}$$

Therefore we have

$$\max_{\tau \in T(A \otimes \mathcal{Z})} |\tau(p_n) - |K|^{-1}| \leq \frac{3}{n},$$

from which it follows the limit is 0. □

Corollary VII.14. *For any unital simple separable $\mathcal{Z}$-stable tracially approximately divisible C^*-algebra A and any discrete countable elementary amenable group G, there exists a pointwise approximately inner action ω of G on A with the tracial Rokhlin property. Furthermore, ω can be taken to be isomorphic to ω^A from Theorem VII.13.*

Proof. If A is $\mathcal{Z}$-stable, ω^A in Theorem VII.13 is such an action on A. □

Corollary VII.15. *For any unital simple separable nuclear tracially approximately divisible C^*-algebra A and any discrete countable elementary amenable group G, there exists a pointwise approximately inner action ω of G on A with the tracial Rokhlin property. Furthermore, ω can be taken to be isomorphic to ω^A from Theorem VII.13.*

Proof. Since simple tracially approximately divisible algebras are tracially $\mathcal{Z}$-stable ((14, Definition 2.1)), then A being nuclear implies it is in fact $\mathcal{Z}$-stable (see (14, Theorem 4.1)). Now use Corollary VII.14. □

Corollary VII.16. *If A is a unital simple separable nuclear infinite-dimensional C^*-algebra of tracial rank at most one and G is any discrete countable elementary amenable group, then there exists a pointwise approximately inner action ω of G on A with the tracial Rokhlin property. Furthermore, ω can be taken to be isomorphic to ω^A from Theorem VII.13.*

Proof. Lin (29, Theorem 5.4) shows that A is tracially approximately divisible. Since A is also nuclear, we can apply Corollary VII.15. □

CHAPTER VIII

CROSSED PRODUCTS FROM UNIFORMLY HYPERFINITE C^*-ALGEBRAS

We examine the crossed products from UHF algebras by product type actions with the tracial Rokhlin property and show that they belong to a familiar class of classifiable C^*-algebras.

Crossed Products by Product Type Actions

Recall we write $M_{(n_l)_{l\le m}}$ to stand for $M_{n_1}\otimes\cdots\otimes M_{n_m}$. We give a direct limit decomposition of $M_{(n_l)_{l\in\mathbb{N}}} \rtimes_\alpha G$ when α is a product type action with corresponding homomorphism of the form

$$\alpha : G \to \prod_{l=1}^{\infty} U(M_{n_l}) \overset{\mathrm{Ad}}{\to} \operatorname{Aut} M_{(n_l)_{l\in\mathbb{N}}}.$$

Let α_m denote the restriction of α to $M_{(n_l)_{l\le m}}$ and let $\alpha_{g,m}$ denote the restriction of α_g. Let 1_m be the identity in M_{n_m} and let

$$\phi_m : M_{(n_l)_{l\le m}} \to M_{(n_l)_{l\le m+1}}$$

be defined by $x \mapsto x \otimes 1_m$. Let

$$\phi_{n,\infty} : M_{(n_l)_{l\le m}} \to M_{(n_l)_{l\in\mathbb{N}}}$$

be the canonical embedding into the direct limit. For all $g \in G$, write $u_g^{(m)}$ for the canonical unitary implementing $\alpha_{g,m}$ in $M_{(n_l)_{l\le m}} \rtimes_{\alpha_m} G$. For the group homomorphism defined by $g \mapsto u_g^{(m+1)}$, we have

$$\begin{aligned} u_g^{(m+1)}\phi_m(x)(u_g^{(m+1)})^* &= (\alpha_g)_{m+1}(x\otimes 1_{m+1}) \\ &= \alpha_{g,m}(x)\otimes 1_{m+1} \\ &= \phi_m(\alpha_{g,m}(x)). \end{aligned}$$

Hence there is a $*$-homomorphism

$$\psi_m : M_{(n_l)_{l\le m}} \rtimes_{\alpha_m} G \to M_{(n_l)_{l\le m+1}} \rtimes_{\alpha_{m+1}} G$$

given on generators $x \in M_{(n_l)_{l\leq m}}$ and $u_g^{(m)}$ for $g \in G$ by

$$xu_g^{(m)} \mapsto \phi_m(x)u_g^{(m+1)}.$$

Denoting by $\psi_{m,\infty}$ the canonical map to $\varinjlim \left(M_{(n_l)_{l\leq m}} \rtimes_{\alpha_m} G, \psi_m\right)$ given by the completion of

$$\frac{\{(x_m) \in \prod_m M_{(n_l)_{l\leq m}} \mid \exists n : \psi_k(x_k) = x_{k+1} \forall k > n\}}{\{(x_m) \mid \lim_m x_m = 0\}}$$

we see for $x \in M_{(n_l)_{l\leq m}}$ and $g \in G$ that

$$\psi_{m,\infty}(xu_g^{(m)}) = (0, \dots, 0, xu_g^{(m)}, \phi_m(x)u_g^{(m+1)}, (\phi_{m+1} \circ \phi_m)(x)u_g^{(m+2)}, \dots).$$

Lemma VIII.1. *Let $\alpha : G \to \operatorname{Aut} M_{(n_l)_{l\in\mathbb{N}}}$ be a product type action preserving the natural decomposition of $M_{(n_l)_{l\in\mathbb{N}}}$. Then*

$$M_{(n_l)_{l\in\mathbb{N}}} \rtimes_\alpha G \cong \varinjlim \left(M_{(n_l)_{l\leq m}} \rtimes_{\alpha_m} G, \psi_m\right).$$

Proof. We will define $*$-homormorphisms in both directions and check they are inverse to each other. Let u_g implement α_g in $M_{(n_l)_{l\in\mathbb{N}}} \rtimes_\alpha G$. Similarly to the case for ψ_m we can define a map

$$\psi'_{m,\infty} : M_{(n_l)_{l\leq m}} \rtimes_{\alpha_m} G \to M_{(n_l)_{l\in\mathbb{N}}} \rtimes_\alpha G$$

on generators $x \in M_{(n_l)_{l\leq m}}$ and $u_g^{(n)}$ for $g \in G$ by

$$xu_g^{(m)} \mapsto \phi_{m,\infty}(x)u_g.$$

We check that

$$\begin{aligned}(\psi'_{m+1,\infty} \circ \psi_m)(xu_g^{(m)}) &= \psi'_{m+1,\infty}(\phi_m(x)u_g^{(m+1)}) \\ &= \phi_{m+1,\infty}(\phi_m(x))u_g \\ &= \phi_{m,\infty}(x)u_g \\ &= \psi'_{m,\infty}(xu_g^{(m)}).\end{aligned}$$

So we have a map

$$\varinjlim(\psi'_{m,\infty}) : \varinjlim \left(M_{(n_l)_{l\leq m}} \rtimes_{\alpha_m} G, \psi_m\right) \to M_{(n_l)_{l\in\mathbb{N}}} \rtimes_{\alpha} G.$$

Now we define the inverse map. We have a map

$$\psi_{m,\infty}|_{M_{(n_l)_{l\leq m}}} : M_{(n_l)_{l\leq m}} \to \varinjlim \left(M_{(n_l)_{l\leq m}} \rtimes_{\alpha_m} G, \psi_m\right)$$

and we check for $x \in M_{(n_l)_{l\leq m}}$

$$\begin{aligned}
(\psi_{m+1,\infty} \circ \phi_m)(x) &= (0, \dots, 0, \phi_m(x), (\phi_{m+1} \circ \phi_m)(x), \dots) \\
&= (0, \dots, 0, x, \phi_m(x), (\phi_{m+1} \circ \phi_m)(x), \dots) \\
&= \psi_{m,\infty}(x).
\end{aligned}$$

So we have a map from the universal property of the limit

$$\phi : M_{(n_l)_{l\in\mathbb{N}}} \to \varinjlim \left(M_{(n_l)_{l\leq m}} \rtimes_{\alpha_m} G, \psi_m\right).$$

now define a group homomorphism

$$G \to U\left(\varinjlim \left(M_{(n_l)_{l\leq m}} \rtimes_{\alpha_m} G, \psi_m\right)\right)$$

for $g \in G$ by

$$g \mapsto (u_g^{(m)})_{m\in\mathbb{N}}$$

and check the covariance condition on the generators. Let $x \in M_{(n_l)_{l\leq m}}$ and $g \in G$. We have

$$\begin{aligned}
(u_g^{(m)})_{n\in\mathbb{N}}\phi(x)(u_g^{(m)})^*_{m\in\mathbb{N}} &= (0, \dots, 0, u_g^{(m)} x (u_g^{(m)})^*, u_g^{(m+1)}\phi_m(x)(u_g^{(m+1)})^*, \dots) \\
&= (0, \dots, 0, \alpha_m(x), \alpha_{m+1}(\phi_m(x)), \dots) \\
&= (0, \dots, 0, \alpha_m(x), \phi_m(\alpha_m(x)), \dots) \\
&= \phi(\alpha_m(x)) \\
&= \phi(\alpha(x)).
\end{aligned}$$

Hence there is a map

$$\psi : M_{(n_l)_{l\in\mathbb{N}}} \rtimes_\alpha G \to \varinjlim \left(M_{(n_l)_{l\leq m}} \rtimes_{\alpha_m} G, \psi_m\right),$$

which we now check is inverse to $\varinjlim(\psi'_{m,\infty})$ on generators. Let $x \in M_{(n_l)_{l\leq m}}$ and let $g \in G$. We have

$$\begin{aligned}(\psi \circ \varinjlim(\psi'_{m,\infty}))(\psi_{m,\infty}(xu_g^{(m)})) &= \psi(\psi'_{m,\infty}(xu_g^{(m)})) \\ &= \psi(\phi_{m,\infty}(x)u_g) \\ &= \psi_{m,\infty}(xu_g^{(m)}).\end{aligned}$$

Let $x \in M_{(n_l)_{l\leq m}}$ and let $g \in G$. We have

$$\begin{aligned}(\varinjlim(\psi'_{m,\infty}) \circ \psi)(\phi_{m,\infty}(x)u_g) &= \varinjlim(\psi'_{m,\infty})(\phi(\phi_{m,\infty}(x))(u_g^{(m)})_{m\in\mathbb{N}}) \\ &= \varinjlim(\psi'_{m,\infty})(\psi_{m,\infty}(x)(u_g^{(m)})_{m\in\mathbb{N}}) \\ &= \psi'_{m,\infty}(xu_g^{(m)}) \\ &= \phi_{m,\infty}(x)u_g.\end{aligned}$$

□

Define the (full) group C^*-algebra $C^*(G)$ of a group G to be the C^*-algebra with the presentation

$$C^*(G) = \langle g \in G \,|\, g^* = g^{-1}\rangle.$$

Lemma VIII.2. *If G is a countable discrete amenable maximally almost periodic group, then $C^*(G)$ is amenable quasidiagonal and satisfies the UCT. Furthermore, if G is almost abelian then $C^*(G)$ is type I.*

Proof. That $C^*(G)$ is amenable when G is amenable is due to Guichardet (10). That $C^*(G)$ is quasidiagonal when G is amenable and maximally almost periodic is discussed in the introduction of (4) among other places. Also, $C^*(G)$ satisfies the UCT by (6, Proposition 6.1). That $C^*(G)$ is type I when G is almost abelian follows from Lemma V.4. □

Lemma VIII.3. *The map $\theta_m : M_{(n_l)_{l\leq m}} \otimes C^*(G) \to M_{(n_l)_{l\leq m}} \rtimes_{\alpha_m} G$ defined by*

$$\theta_m : x \otimes g \mapsto x(g_1^{-1} \otimes \cdots \otimes g_n^{-1})u_g$$

is a $*$-isomorphism with inverse defined by

$$\theta_m^{-1} : xu_g \mapsto x(g_1 \otimes \cdots \otimes g_n) \otimes g.$$

Proof. Note that the action of $\mathrm{Ad}((g_1^{-1} \otimes \cdots \otimes g_n^{-1})u_g)$ on $M_{(n_l)_{l \leq m}}$ is by design trivial and hence $(g_1^{-1} \otimes \cdots \otimes g_n^{-1})u_g$, which is the image of $1 \otimes g$, commutes with the image of $x \otimes 1$, which lies in $M_{(n_l)_{l \leq m}}$. Therefore θ_m is well-defined. For θ_m^{-1}, we have

$$\begin{aligned}
\theta_m^{-1}(u_g)x &= ((g_1 \otimes \cdots \otimes g_n) \otimes g)(x \otimes 1) \\
&= ((g_1 \otimes \cdots \otimes g_n)x) \otimes g \\
&= \alpha_g(x)\theta_m^{-1}(u_g),
\end{aligned}$$

and hence θ_m^{-1} is well-defined. Therefore both maps are well-defined by the universal properties of the presentations for these algebras. The fact that they are inverses only needs to be checked on the generators where it is obvious, hence θ_m and θ_m^{-1} are $*$-isomorphisms. □

We arrive at the following theorem.

Theorem VIII.4. *Let A be a UHF algebra, let G be a discrete group and let α be a product type action of G on A preserving the decomposition $A \cong M_{(n_l)_{l \in \mathbb{N}}}$ for some sequence $(n_l)_{l \in \mathbb{N}}$. Let g_m be the image of g in $U(M_{n_m})$ and suppose*

$$\Phi_m : M_{(n_l)_{l \leq m}} \otimes C^*(G) \to M_{(n_l)_{l \leq m}} \otimes C^*(G)$$

is the $*$-homomorphism defined on generators by

$$x \otimes g \mapsto ((x \otimes 1_{n_{m+1}})g_{n+1}) \otimes g.$$

Then

$$A \rtimes_\alpha G \cong \varinjlim \left(\Phi_m, M_{(n_l)_{l \leq m}} \otimes C^*(G)\right).$$

Proof. From Lemmas VIII.1 and VIII.3, it suffices to define the maps Φ_m so that the following diagrams commute

$$\begin{array}{ccc} M_{(n_l)_{l\leq m}} \rtimes_{\alpha_m} G & \xrightarrow{\psi_m} & M_{(n_l)_{l\leq m+1}} \rtimes_{\alpha_m} G. \\ {\scriptstyle\theta_m}\big\uparrow & & \big\downarrow{\scriptstyle\theta_{m+1}^{-1}} \\ M_{(n_l)_{l\leq m}} \otimes C^*(G) & \xrightarrow{\Phi_m} & M_{(n_l)_{l\leq m+1}} \otimes C^*(G). \end{array}$$

We have for $x \in M_{(n_l)_{l\leq m}}$ and $g \in G$ that

$$\begin{aligned} \Phi_m(x \otimes g) &= (\theta_{m+1}^{-1} \circ \psi_m \circ \theta_m)(x \otimes g) \\ &= (\theta_{m+1}^{-1} \circ \psi_m)(x(g_1^{-1} \otimes \cdots \otimes g_m^{-1})u_g^{(m)}) \\ &= \theta_{m+1}^{-1}(\phi_m(x(g_1^{-1} \otimes \cdots \otimes g_m^{-1}))u_g^{(m+1)}) \\ &= \phi_m(x(g_1^{-1} \otimes \cdots \otimes g_m^{-1}))((g_1 \otimes \cdots \otimes g_{m+1}) \otimes g) \\ &= (x(g_1^{-1} \otimes \cdots \otimes g_m^{-1}) \otimes 1_{m+1})((g_1 \otimes \cdots \otimes g_{m+1}) \otimes g) \\ &= ((x \otimes 1_{m+1})g_{m+1}) \otimes g \\ &= (\phi_m(x)g_{m+1}) \otimes g. \end{aligned}$$

□

Corollary VIII.5. *Suppose G is a countable discrete amenable maximally almost periodic group and suppose A is any UHF algebra. If α is a product type action of G on A, then $A \rtimes_\alpha G$ is nuclear quasidiagonal and satisfies the UCT. Furthermore, if G is almost abelian then $A \rtimes_\alpha G$ is locally type I.*

Crossed Products by Outer Actions

We note here the advantages of the tracial Rokhlin property for determining the crossed product. The following is based on Kishimoto (19, Lemma 4.3).

Lemma VIII.6. *Suppose α is an action of G on A with the pointwise tracial Rokhlin property and τ is any tracial state on $A \rtimes_\alpha G$. Then $\tau(au_g) = 0$ for all $a \in A$ and all $g \in G \setminus \{1\}$.*

Proof. Let $g \in G \setminus \{1\}$ with order k, let $a \in A$ and $\epsilon > 0$. Without loss of generality also assume $\|a\| \leq 1$. By definition of the order k tracial Rokhlin property, there exists $k' \geq k$ mutually orthogonal projections $p_1, \ldots, p_{k'}$ such that with $p = p_1 + \cdots + p_{k'}$, we have

- $[p_i, a] \approx_{\epsilon/2} 0$
- $\alpha_g(p_i) \approx_{\epsilon/2} p_{i+1}$ for $1 \leq i \leq k' - 1$
- $\tau(p) \approx_{\epsilon^2} 1$.

First we show that for $1 \leq i \leq k' - 1$

$$\|p_i a \alpha(p_i)\| < \epsilon. \tag{VIII.1}$$

Let $1 \leq i \leq k' - 1$. We have

$$\begin{aligned} p_i a \alpha(p_i) &\approx_{\epsilon/2} a p_i \alpha(p_i) \\ &\approx_{\epsilon/2} a p_i p_{i+1} \\ &= 0. \end{aligned}$$

Now we have

$$\begin{aligned} |\tau(au_g)| &= |\tau(pau_g) + \tau((1-p)au_g)| \\ &\leq |\tau(pau_g)| + \tau(1-p)^{1/2}\tau(a^*a)^{1/2} \\ &\approx_\epsilon |\tau(pau_g)| \\ &= \sum_{i=1}^{k'} \tau(p_i a u_g) \\ &= \sum_{i=1}^{k'} \tau(p_i a u_g p_i) \\ &= \sum_{i=1}^{k'} \tau(p_i a \alpha_g(p_i) u_g p_i) \\ &\leq \sum_{i=1}^{k'} \tau(p_i \|p_i a \alpha_g(p_i)\|^2 p_i)^{1/2} \tau(p_i)^{1/2} \\ &< \sum_{i=1}^{k'} \epsilon \tau(p_i) \\ &= \epsilon \tau(p) \\ &\approx_{\epsilon^2} \epsilon \end{aligned}$$

Hence $\tau(au_g) \approx_{2\epsilon+\epsilon^2} 0$. □

Proposition VIII.7. *Let G be a countable discrete group, let A be a UHF algebra and let α be an action of G on A. Then*

(i) if G is amenable and α is outer, then $A \rtimes_\alpha G$ is nuclear and simple,

(ii) if α has the pointwise tracial Rokhlin property, then $A \rtimes_\alpha G$ has a unique tracial state,

(iii) if G is amenable and α has the tracial Rokhlin property, then $A \rtimes_\alpha G$ is $\mathcal{Z}$-stable and has real rank zero.

Proof. We find general results from the literature and specialise them to our case.

(i) Rosenberg (45) shows that crossed products are amenable when G is amenable. Simplicity follows from Kishimoto (18, Theorem 3.1) since the action is outer.

(ii) Lemma VIII.6 shows that there is at most one trace τ on $A \rtimes_\alpha G$, namely the one given by composing the conditional expectation with the unique tracial state of A.

(iii) Matui-Sato (37, Theorem 4.9) tells us the crossed product is $\mathcal{Z}$-stable. In the presence of $\mathcal{Z}$-stability we use a characterisation of real rank zero from Rordam(44, Corollary 7.3(ii)). Since the image $K_0(A)$ under the trace is already dense in $\mathbb{R}$ and the image of $K_0(A \rtimes_\alpha G)$ contains the image of $K_0(A)$, then $A \rtimes_\alpha G$ has real rank zero.

□

The Crossed Products $M_{(n_l)_{l\in\mathbb{N}}} \rtimes_\alpha G$

Let A be a unital C^*-algebra, let G be a countable discrete group and let α be an action of G on A. It is clear that $A \rtimes_\alpha G$ is unital and separable. We will also assume that A is a UHF algebra and note the advantages for determining $A \rtimes_\alpha G$ when α is a product type action. Nonetheless this is not enough to get everything that one would desire. For example, we know that with only this we cannot guarantee that $A \rtimes_\alpha G$ has a unique tracial state or has real rank zero (see for example (20, Theorem 1.3)). However, these missing properties are exactly why we looked for the tracial Rokhlin property.

We combine the advantages of the tracial Rokhlin property with being a product type action in the following theorem.

Theorem VIII.8. *Suppose G is a countable discrete maximally almost periodic amenable group, A is any UHF algebra and α is a product type action of G on A with the tracial Rokhlin property. Then $A \rtimes_\alpha G$ is unital simple separable nuclear with tracial rank zero and satisfies the Universal Coefficient Theorem. Moreover, $A \rtimes_\alpha G$ has a unique tracial state.*

Proof. By a result of W. Winter (50, Theorem 2.1), to prove $A \rtimes_\alpha G$ has tracial rank zero, it suffices to show $A \rtimes_\alpha G$ is $\mathcal{Z}$-stable with real rank zero and finite decomposition rank. We have real rank zero and $\mathcal{Z}$-stability from Proposition IX.16 (iii). So it suffices to show that $A \rtimes_\alpha G$ has finite decomposition rank. Now using Matui-Sato (38, Corollary 1.2), it suffices to show that $A \rtimes_\alpha G$ is simple nuclear quasidiagonal with a unique tracial state. Simplicity and nuclearity come from Proposition IX.16 (i), while quasidiagonality follows from Corollary VIII.5. Having a unique tracial state comes from Proposition IX.16 (ii). The Universal Coefficient Theorem is from Corollary VIII.5. □

Corollary VIII.9. *Suppose G is a countable discrete maximally almost periodic elementary amenable group, A is any UHF algebra and α is a strongly outer product type action of G on A. Then $A \rtimes_\alpha G$ is unital simple separable nuclear with tracial rank zero and satisfies the Universal Coefficient Theorem. Moreover, $A \rtimes_\alpha G$ has a unique tracial state. Furthermore, if G is almost abelian then $A \rtimes_\alpha G$ is locally type I.*

Proof. When G is elementary amenable, the tracial Rokhlin property is equivalent to strong outerness by Matui-Sato's theorem, reproduced here as Theorem IV.7. Hence Theorem VIII.8 applies. □

CHAPTER IX

CROSSED PRODUCTS FROM JIANG-SU-STABLE ALGEBRAS

The Crossed Products $\mathcal{Z} \rtimes G$

$\mathcal{Z} \rtimes_\beta G$ and $\mathcal{Z} \rtimes_\gamma G$

Let β and γ be as in Definition VII.4. Simple C^*-algebras with rational tracial rank zero are important because they help define a large class of C^*-algebras which can be classified by their Elliott invariants. We investigate the classifiability of $\mathcal{Z} \rtimes_\gamma G$ by examining its rational tracial rank, that is, the tracial rank of $\mathcal{Q} \otimes (\mathcal{Z} \rtimes_\gamma G)$. We investigate $\mathcal{Z} \rtimes_\beta G$ and $\mathcal{Z} \rtimes_\gamma G$ simultaneously because they are isomorphic.

We first summarize what can be deduced in a straightforward manner from the literature about $\mathcal{Z} \rtimes_\gamma G$ in the following proposition.

Proposition IX.1. *If G is a countable discrete elementary amenable group, then $\mathcal{Z} \rtimes_\gamma G$*

- *is unital and separable,*
- *is simple,*
- *has a unique tracial state,*
- *is nuclear,*
- *and is $\mathcal{Z}$-stable.*

Proof. It is clear that $\mathcal{Z} \rtimes_\gamma G$ is unital and separable. For unique trace it suffices to show that $\mathcal{Q} \otimes (\mathcal{Z} \rtimes_\gamma G)$ has unique trace. In this case we have $\mathcal{Q} \otimes (\mathcal{Z} \rtimes_\gamma G) \cong (\mathcal{Q} \otimes \mathcal{Z}) \rtimes_{\mathrm{id} \otimes \gamma} G$ and $\mathrm{id} \otimes \gamma = \omega^{\mathcal{Q}}$ is a strongly outer action (by Lemma II.13) on $\mathcal{Q} \otimes \mathcal{Z}$, which is isomorphic to $\mathcal{Q}$. So Proposition VIII.7(ii) tells us $\mathcal{Q} \rtimes G$ has unique trace. For nuclearity use Rosenberg (45, Theorem 1). For $\mathcal{Z}$-stablity use Matui-Sato (37, Corollary 4.11). □

We could also let $A = \mathcal{Z}$ in Proposition IX.16 and apply Proposition IX.13 to obtain the above proposition minus the claim about unique trace.

Definition IX.2. If $A \subset B(H)$ is separable then A is a quasidiagonal set of operators if there exists an increasing sequence of finite rank projections, $q_1 \leq q_2 \leq q_3 \cdots$, such that for all $a \in A$, $[a, q_n] \to 0$ and $q_n \to 1_H$ strongly. A separable C^*-algebra is quasidiagonal if it has a faithful representation whose image is a quasidiagonal set of operators.

Remark IX.3. It is clear from this definition that a subalgebra of a quasidiagonal C^*-algebra is quasidiagonal.

Corollary IX.4. *The following are equivalent:*

- $\mathcal{Z} \rtimes_\gamma G$ *is quasidiagonal*
- $\mathcal{Z} \rtimes_\gamma G$ *has rational tracial rank zero*
- $\mathcal{Z} \rtimes_\gamma G$ *has rational tracial rank at most one.*

Proof. If $\mathcal{Z} \rtimes_\gamma G$ is quasidiagonal, then Proposition IX.1 combined with Matui-Sato (38, Theorem 6.1) allows us to conclude that $\mathcal{Z} \rtimes_\gamma G$ has rational tracial rank zero. The next implication is obvious. Finally, if $\mathcal{Z} \rtimes_\gamma G$ has rational tracial rank at most one, then it is isomorphic to a subalgebra of $\mathcal{Q} \otimes (\mathcal{Z} \rtimes_\gamma G)$. Since $\mathcal{Q} \otimes (\mathcal{Z} \rtimes_\gamma G)$ has tracial rank at most one, it is quasidiagonal by Lin (32, Corollary 6.7). Hence Remark IX.3 tells us that $\mathcal{Z} \rtimes_\gamma G$ is quasidiagonal. □

Lemma IX.5. *Let G_i be a sequence of groups acting of $\mathcal{Z}$ with the β action in each case denoted β_i. If $G = \varinjlim(G_i, \varphi_i)$ with φ_i injective and $\mathcal{Z} \rtimes_{\beta^i} G_i$ is quasidiagonal for all i, then $\mathcal{Z} \rtimes_\beta G$ is quasidiagonal.*

Proof. Use the injectivity of φ_i to define a $*$-homomorphism

$$\Psi_i : \mathcal{Z}^{\otimes G_i} \to \mathcal{Z}^{\otimes G_{i+1}}$$

on generators by

$$z = \bigotimes_{g \in G_i} z_g \mapsto 1_{\mathcal{Z}^{\otimes (G_{i+1} \setminus \varphi_i(G_i))}} \otimes \bigotimes_{h \in \varphi_i(G_i)} z_{\varphi_i^{-1}(h)}.$$

We check this is covariant with respect to β^i. For $g \in G_i$, we have

$$\begin{aligned}
\varphi_i(g)\Psi_i(z)\varphi_i(g)^* &= \beta^{i+1}_{\varphi_i(g)}\left(1_{\mathcal{Z}^{\otimes(G_{i+1}\setminus\varphi_i(G_i))}} \otimes \bigotimes_{h\in\varphi_i(G_i)} z_{\varphi_i^{-1}(h)}\right) \\
&= 1_{\mathcal{Z}^{\otimes(G_{i+1}\setminus\varphi_i(G_i))}} \otimes \bigotimes_{h\in\varphi_i(G_i)} z_{\varphi_i^{-1}(\varphi_i(g)^{-1}h)} \\
&= 1_{\mathcal{Z}^{\otimes(G_{i+1}\setminus\varphi_i(G_i))}} \otimes \bigotimes_{h\in\varphi_i(G_i)} z_{g^{-1}\varphi_i^{-1}(h)} \\
&= \Psi_i\left(\bigotimes_{h\in G_i} z_{g^{-1}h}\right) \\
&= \Psi_i(\beta^i_g(z)).
\end{aligned}$$

Hence we have a sequence of injective maps $\Psi_i \rtimes \varphi_i : \mathcal{Z} \rtimes_{\beta^i} G_i \to \mathcal{Z} \rtimes_{\beta^{i+1}} G_{i+1}$ of quasidiagonal algebras, hence the limit is quasidiagonal (see (2, Section 9)). Now we show that this limit is isomorphic to $\mathcal{Z}^{\otimes G} \rtimes_\beta G$. First notice that $\varinjlim(\mathcal{Z}^{\otimes G_i}, \Psi_i) \cong \mathcal{Z}^{\otimes G}$ and get the obvious maps

$$\mathcal{Z}^{\otimes G_i} \hookrightarrow \mathcal{Z}^{\otimes G} \hookrightarrow \mathcal{Z}^{\otimes G} \rtimes_\beta G$$

and

$$G_i \hookrightarrow G \hookrightarrow \mathcal{Z}^{\otimes G} \rtimes_\beta G.$$

We can show covariance in much the same way as before to get

$$\mathcal{Z}^{\otimes G_i} \rtimes_{\beta^i} G_i \hookrightarrow \mathcal{Z}^{\otimes G} \rtimes_\beta G.$$

We check that these maps are compatible with the increasing i and conclude there is an injective map

$$\varinjlim(\mathcal{Z}^{\otimes G_i} \rtimes_{\beta^i} G_i, \Psi_i \rtimes \varphi_i) \hookrightarrow \mathcal{Z}^{\otimes G} \rtimes_\beta G.$$

This map is also surjective because its image contains $\mathcal{Z}^{\otimes G}$ and G, which generate the target algebra. □

Lemma IX.6. *If H is a subgroup of G and $\mathcal{Z} \rtimes_\gamma G$ is quasidiagonal, then $\mathcal{Z} \rtimes_\gamma H$ is quasidiagonal.*

Proof. It suffices to show that the obvious map $\mathcal{Z} \rtimes_\gamma H \to \mathcal{Z} \rtimes_\gamma G$ is injective. One way to see this is to recall that $\mathcal{Z} \rtimes_\gamma H$ is simple. □

Proposition IX.7. *If G is a finite group, then $\mathcal{Z} \rtimes_\gamma G$ is quasidiagonal.*

Proof. Let $n = |G|$. We define an embedding of $\mathcal{Z} \rtimes_\gamma G$ into $M_n(\mathcal{Z})$ as follows. First define $A \to M_n(\mathcal{Z})$ for $a \in A$ by

$$a \mapsto \operatorname{diag}(\gamma_g(a))_{g \in G}.$$

Then define $G \to U(M_n) \otimes 1_{\mathcal{Z}} \to U(M_n(\mathcal{Z}))$ via its left regular representation. One check that these satisfy the covariance relations and hence we get our embedding after noting $\mathcal{Z} \rtimes_\gamma G$ is simple. □

Proposition IX.8. *If G is a countable discrete abelian group, then $\mathcal{Z} \rtimes_\gamma G$ is quasidiagonal.*

Proof. Lin (33, Theorem 9.11) shows that if the crossed product of any AH-algebra and any finitely generated abelian group has an invariant tracial state, then the crossed product is quasidiagonal. We apply this to $(\mathcal{Q} \otimes \mathcal{Z}) \rtimes_{\mathrm{id} \otimes \gamma} G \cong \mathcal{Q} \otimes (\mathcal{Z} \rtimes_\gamma G)$ when G is a finitely generated abelian group to show the crossed product is quasidiagonal. Now $\mathcal{Z} \rtimes_\gamma G$ is a subalgebra of $\mathcal{Q} \otimes (\mathcal{Z} \rtimes_\gamma G)$ and hence quasidiagonal. By Lemma **??**, $\mathcal{Z} \rtimes_\beta G$ is quasidiagonal for finitely generated groups G. The condition on G being finitely generated can be removed by Lemma IX.5. Finally, one last application of Lemma **??** gives the result. □

Remark. The quasidiagonality of $\mathcal{Z} \rtimes_\gamma \mathbb{Z}$ can be shown directly using earlier results of Brown (3) and even earlier results of Voiculescu (47), applied again to $\mathcal{Q} \rtimes_\omega \mathbb{Z}$.

We summarize the findings of this section in the next theorem.

Theorem IX.9. *Let γ be as in Definition VII.4 and let $\mathcal{C}$ be the class of countable discrete groups generated by abelian groups and finite groups under increasing unions and taking subgroups. Then $\mathcal{Z} \rtimes_\gamma G$ is a unital separable simple nuclear $\mathcal{Z}$-stable C^*-algebra with rational tracial rank zero for any $G \in \mathcal{C}$.*

Proof. We combine Lemmas IX.6 and IX.5 with Propositions IX.7 and IX.8 to get $\mathcal{Z} \rtimes_\gamma G$ is quasidiagonal for all $G \in \mathcal{C}$ and hence has rational tracial rank zero by Corollary IX.4. The remaining properties are those listed in Proposition IX.1. □

Corollary IX.10. *Let γ be as in Theorem IX.9. Then $\mathcal{Z} \rtimes_\gamma \mathbb{Z}$ is unital separable simple nuclear $\mathcal{Z}$-stable with rational tracial rank zero and has a unique tracial state as well as and satisfying the UCT. We also have for $i = 0$ or 1 that*

$$K_i(\mathcal{Z} \rtimes_\gamma \mathbb{Z}) = \mathbb{Z}.$$

Moreover, if α is any other strongly outer $\mathbb{Z}$-action on $\mathcal{Z}$, then there exists an automorphism σ of $\mathcal{Z}$ and a unitary $u \in \mathcal{Z}$ such that

$$\alpha = \operatorname{Ad} u \circ \sigma \circ \gamma \circ \sigma^{-1}.$$

In particular, $\mathcal{Z} \rtimes_\alpha \mathbb{Z} \cong \mathcal{Z} \rtimes_\gamma \mathbb{Z}$.

Proof. Since $\mathbb{Z} \in \mathcal{C}$, putting $G = \mathbb{Z}$ in Theorem IX.9 shows that $\mathcal{Z} \rtimes_\gamma \mathbb{Z}$ is unital separable simple nuclear $\mathcal{Z}$-stable with unique tracial state and rational tracial rank zero. Crossed products by $\mathbb{Z}$ always satisfy the UCT. The K-groups are obtained using the Pimsner-Voiculescu six-term exact sequence. The uniqueness statement is due to Sato (46, Theorem 1.3). □

The Crossed Products $\mathcal{Z} \rtimes_\omega G$

We make use of Matui-Sato (37) to show that γ and $\omega^{\mathcal{Z}}$ are equivalent in some sense as actions on $\mathcal{Z}$. The following lemma unpackages the definition of cocycle conjugacy as applied to actions on $\mathcal{Z}$.

Lemma IX.11. *There exists $\theta \in \operatorname{Aut}(\mathcal{Z})$, and collections of unitaries $(v'_g)_{g\in G}$ and $(v_n)_{n\in\mathbb{N}}$ such that for each $g \in G$,*

$$\theta\gamma_g\theta^{-1} = \operatorname{Ad} v'_g\omega_g^{\mathcal{Z}}$$

$$\lim_{n\to\infty} v_n\omega_g^{\mathcal{Z}}(v_n^*) = v'_g.$$

Proof. We know that γ has the weak Rokhlin property by Theorem VII.6. Therefore (37, Theorem 4.9) applies (specialized to actions on $\mathcal{Z}$). □

Definition IX.12. We say that two actions are stably outer conjugate with respect to θ, v and v' if they satisfy the conclusion of Lemma IX.11.

Proposition IX.13. *The actions γ and $\omega^{\mathcal{Z}}$ are stably outer conjugate. If they are stably outer conjugate with respect to θ, v and v', then there is an isomorphism*

$$\Psi : \mathcal{Z} \rtimes_{\gamma} G \to \mathcal{Z} \rtimes_{\omega^{\mathcal{Z}}} G$$

$$zu_g \mapsto \theta(z)v'_g u'_g,$$

where u_g and u'_g are the standard unitaries implementing γ and $\omega^{\mathcal{Z}}$ respectively in the crossed product.

Proof. The first part is a restatement of Lemma IX.11, from which showing the crossed products are isomorphic is standard. □

The Crossed Products $A \rtimes_{\omega} G$

Let A be a unital C^*-algebra and let G be a discrete group. Let ω be as in Theorem VII.13 and let γ be as in Definition VII.4. If A is $\mathcal{Z}$-stable, then ω^A is conjugate to an action of G on A that we will also call ω.

Proposition IX.14. *For $g \in G$ let u_g and u'_g be the implementing unitaries for ω_g and γ_g respectively. There is an isomorphism*

$$i : (A \otimes \mathcal{Z}) \rtimes_{\omega} G \to A \otimes (\mathcal{Z} \rtimes_{\gamma} G),$$

such that

$$i : (a \otimes z)u_g \mapsto a \otimes (zu'_g).$$

Proof. Define

$$i_A : A \to A \otimes (\mathcal{Z} \rtimes_{\gamma} G)$$

by

$$a \mapsto a \otimes 1_{\mathcal{Z} \rtimes G},$$

which is obviously a $*$-homomorphisms and define

$$i_{\mathcal{Z}} : \mathcal{Z} \to A \otimes (\mathcal{Z} \rtimes_{\gamma} G)$$

by

$$z \mapsto 1_A \otimes (zu_1),$$

which again is obviously a $*$-homomorphism. Now since the image of i_A and $i_{\mathcal{Z}}$ commute, there is a map

$$i_{A\otimes\mathcal{Z}} : A \otimes \mathcal{Z} \to A \otimes (\mathcal{Z} \rtimes_\gamma G)$$

given on generators by

$$a \otimes z \mapsto a \otimes zu_1.$$

We also have a group homomorphism

$$u : G \to U(A \otimes (\mathcal{Z} \rtimes_\gamma G))$$

given by

$$g \mapsto 1 \otimes u_g.$$

We check that these two maps are covariant:

$$\begin{aligned} u(g)i_{A\otimes\mathcal{Z}}(a \otimes z)u(g)^* &= (1 \otimes u_g)(a \otimes zu_1)(1 \otimes u_g^*) \\ &= a \otimes (u_g z u_g^* u_1) \\ &= a \otimes (\gamma_g(z)u_1) \\ &= i_{A\otimes\mathcal{Z}}(a \otimes \gamma_g(z)) \\ &= i_{A\otimes\mathcal{Z}}(\omega_g(a \otimes z)). \end{aligned}$$

Hence by the universal property of crossed products, we get the desired map i. We now construct its inverse map j by similar considerations.

Let v_g be the unitary implementing the action of ω. Define

$$j_A : A \to (A \otimes \mathcal{Z}) \rtimes_\omega G$$

by

$$a \mapsto (a \otimes 1)v_1,$$

which is clearly a $*$-homomorphisms along with

$$j_{\mathcal{Z}} : \mathcal{Z} \to (A \otimes \mathcal{Z}) \rtimes_{\omega} G$$

$$z \mapsto (1 \otimes z)v_1.$$

Finally, define

$$v : G \to U((A \otimes \mathcal{Z}) \rtimes_{\omega} G)$$

$$g \mapsto v_g.$$

We see that $j_{\mathcal{Z}}$ and v are covariant:

$$\begin{aligned} v(g)j_{\mathcal{Z}}(z)v(g) &= v_g(1 \otimes z)v_1v_g^* \\ &= \omega_g(1 \otimes z)v_1 \\ &= (1 \otimes \gamma_g(z))v_1 \\ &= j_{\mathcal{Z}}(\gamma_g(z)). \end{aligned}$$

So by the universal property, there is a $*$-homomorphism

$$j_{\mathcal{Z} \rtimes G} : \mathcal{Z} \rtimes_{\gamma} G \to (A \otimes \mathcal{Z}) \rtimes_{\omega} G$$

$$zu_g \mapsto (1 \otimes z)v_g.$$

We check that the image of j_A commutes with the image of $j_{\mathcal{Z} \rtimes G}$:

$$\begin{aligned} j_A(a)j_{\mathcal{Z} \rtimes G}(zu_g) &= (a \otimes 1)v_1(1 \otimes z)v_g \\ &= (1 \otimes z)(a \otimes 1)v_g \\ &= (1 \otimes z)v_g\omega_g(a \otimes 1)v_1 \\ &= (1 \otimes z)v_g(a \otimes \gamma_g(1))v_1 \\ &= (1 \otimes z)v_g(a \otimes 1)v_1 \\ &= j_{\mathcal{Z} \rtimes G}(zu_g)j_A(a). \end{aligned}$$

Hence we get a map by the universal property of tensor products

$$j : A \otimes (\mathcal{Z} \rtimes_\gamma) G \to (A \otimes \mathcal{Z}) \rtimes_\omega G$$

$$a \otimes (zu_g) \mapsto (a \otimes z) v_g.$$

We check that the maps i and j are inverse to each other.

- $j \circ i = \mathrm{id}$

$$\begin{aligned} j(i((a \otimes z)v_g)) &= j(a \otimes (zu_g)) \\ &= (a \otimes z)v_g. \end{aligned}$$

- $i \circ j = \mathrm{id}$

$$\begin{aligned} i(j(a \otimes (zu_g))) &= i((a \otimes z)v_g) \\ &= a \otimes (zu_g). \end{aligned}$$

□

Lemma IX.15. *If A is $\mathcal{Z}$-stable, then there is a $*$-isomorphism*

$$\mathcal{Q} \otimes (A \rtimes_\omega G) \cong (\mathcal{Q} \otimes A) \otimes ((\mathcal{Z} \rtimes_\gamma G) \otimes \mathcal{Q}).$$

Proof. We use Proposition IX.14 and $\mathcal{Q} \cong \mathcal{Q} \otimes \mathcal{Q}$ to write

$$\begin{aligned} \mathcal{Q} \otimes ((A \otimes \mathcal{Z}) \rtimes_\omega G) &\cong \mathcal{Q} \otimes (A \otimes (\mathcal{Z} \rtimes_\gamma G)) \\ &\cong (\mathcal{Q} \otimes A) \otimes ((\mathcal{Z} \rtimes_\gamma G) \otimes \mathcal{Q}). \end{aligned}$$

Now since A is $\mathcal{Z}$-stable, we are done. □

Proposition IX.16. *Suppose A is a unital separable $\mathcal{Z}$-stable C^*-algebra and G is any countable discrete amenable group. Then $A \rtimes_\omega G$ is a unital separable $\mathcal{Z}$-stable C^*-algebra and we also have:*

- *If A is simple, then $A \rtimes_\omega G$ is simple.*
- *If A is nuclear, then $A \rtimes_\omega G$ is nuclear.*
- $T(A \rtimes_\omega G) = T(A)$.

– *If A has real rank zero, then $A \rtimes_\omega G$ has real rank zero.*

Proof. It is clear that the crossed product is unital and separable since A is unital and separable, and G is countable. If A is simple, then Lemma II.13 shows that ω is pointwise outer. Hence Kishimoto (18, Theorem 3.1) shows the (reduced) crossed product is simple. Nuclearity follows from Rosenberg (45, Theorem 1). For $\mathcal{Z}$-stablity we use Proposition IX.14 to get

$$\begin{aligned}\mathcal{Z} \otimes (A \rtimes_\omega G) &\cong \mathcal{Z} \otimes (A \otimes (\mathcal{Z} \rtimes_\gamma G)) \\ &\cong (\mathcal{Z} \otimes A) \otimes (\mathcal{Z} \rtimes_\gamma G). \\ &\cong A \otimes (\mathcal{Z} \rtimes_\gamma G) \\ &\cong A \rtimes_\omega G.\end{aligned}$$

For the claim about the tracial state spaces, let τ_γ be the unique tracial state on $\mathcal{Z} \rtimes_\gamma G$ (Proposition IX.1) and define a map $T(A) \to T(A \otimes (\mathcal{Z} \rtimes_\gamma G))$ given by $\tau \mapsto \tau \otimes \tau_\gamma$. This map is obviously affine and injective, while for surjectivity we make use of a brief argument which can be found as (22, Lemma 5.15). Now by Proposition IX.14 we have $T(A \otimes (\mathcal{Z} \rtimes_\gamma G)) \cong T(A \rtimes_\omega G)$. Since our algebras are $\mathcal{Z}$-stable, we use the characterization of real rank zero by Rørdam (44, Theorem 7.2) that $K_0(A)$ is uniformly dense in the space of affine functions on $T(A)$ under the standard mapping ρ_A gotten by evaluation. Since $K_0(A) \subset K_0(A \rtimes_\omega G)$ via $p \mapsto p \otimes 1$, under the identification $T(A) = T(A \rtimes_\omega G)$, $\rho_{A \rtimes_\omega G}(K_0(A)) = \rho_A(K_0(A))$, which is already uniformly dense. Hence the image of $\rho_{A \rtimes_\omega G}$ is uniformly dense and we are done. □

Theorem IX.17. *Suppose A is a unital separable simple nuclear $\mathcal{Z}$-stable C^*-algebra. Let $\mathcal{C}$ be as in Theorem IX.9, let $G \in \mathcal{C}$, let ω be as in Theorem VII.13 and let γ be as in Definition VII.4. Then ω is isomorphic to an action of G on A and $A \rtimes_\omega G$ is a unital separable simple nuclear $\mathcal{Z}$-stable C^*-algebra. Furthermore:*

– *If A has rational tracial rank at most one, then $A \rtimes_\omega G$ has rational tracial rank at most one.*

– *If A has rational tracial rank zero, then $A \rtimes_\omega G$ has rational tracial rank zero.*

– *If A has tracial rank at most one, satisfies the UCT and $\mathcal{Z} \rtimes_\gamma G$ satisfies the UCT, then $A \rtimes_\omega G$ has tracial rank at most one and satisfies the UCT.*

– *If A has tracial rank zero, satisfies the UCT and $\mathcal{Z} \rtimes_\gamma G$ satisfies the UCT, then $A \rtimes_\omega G$ has tracial rank zero and satisfies the UCT.*

Proof. Since A is $\mathcal{Z}$-stable, ω is isomorphic to an action on A. Since $G \in \mathcal{C}$ and A satisfy the hypotheses of Proposition IX.16, the conditions of being unital, separable, simple, nuclear and $\mathcal{Z}$-stable, are retained by the crossed product. To determine the rational tracial rank of $A \rtimes_\omega G$ we use Lemma IX.15 and apply Hu-Lin-Xue (31, Theorem 4.8), which says that the tracial rank of a tensor product is bounded by the sum of the tracial ranks of the factors, to the algebra on the right hand side of the lemma. Since $G \in \mathcal{C}$, the tracial rank of $\mathcal{Q} \otimes (\mathcal{Z} \rtimes_\gamma G)$ is zero by Theorem IX.9, which means the tracial rank is bounded by the rational tracial rank of A. This gives us both claims about rational tracial rank. Now we address the claim for A being of tracial rank at most one. We will use (30, Theorem 4.7) with our A as their B and $\mathcal{Z} \rtimes_\gamma G$ as their A to show that $A \otimes (\mathcal{Z} \rtimes_\gamma G)$ has tracial rank at most one and satisfies the UCT. Hence by Proposition IX.14 $A \rtimes_\omega G$ has tracial rank at most one and satisfies the UCT. Now if A was also tracial rank zero, then it is real rank zero and Proposition IX.16 tells us that $A \rtimes_\omega G$ is real rank zero. But we know that if an algebra is unital simple of tracial rank at most one and real rank zero, then it has tracial rank zero (see for example (28, Lemma 3.2)). □

Here is a curious result in the converse direction.

Theorem IX.18. *Suppose A is a unital separable simple nuclear $\mathcal{Z}$-stable C^*-algebra satisfying the UCT. Let $\mathcal{C}$ be as in Theorem IX.9, let ω be as in Theorem VII.13 and let γ be as in Definition VII.4. If there exists $G \in \mathcal{C}$ such that $\mathcal{Z} \rtimes_\gamma G$ satisfies the UCT and $A \rtimes_\omega G$ has rational tracial rank at most one, then A has rational tracial rank at most one.*

Proof. Let $B = \mathcal{Z} \rtimes_\gamma G$ and note $A \otimes B = A \rtimes_{\omega^A} G$ by Proposition IX.14. We now apply Lin-Sun (30, Theorem 4.8 (1,2,13)) with reference to Propositions IX.1 and IX.16, and Theorem IX.9 to verify the hypotheses there. Hence we get a conclusion that is equivalent (by (22, Theorem 3.6)) to our claim. □

We specialise Theorems IX.17 and IX.18 to the case of the integers.

Corollary IX.19. *Suppose A is a unital separable simple nuclear $\mathcal{Z}$-stable C^*-algebra satisfying the UCT and ω is as in Theorem VII.13. Then $A \rtimes_\omega \mathbb{Z}$ is a unital separable simple nuclear $\mathcal{Z}$-stable C^*-algebra satisfying the UCT. We also have:*

- *A has rational tracial rank at most one if and only if* $A \rtimes_\omega \mathbb{Z}$ *has rational tracial rank at most one.*
- *If A has rational tracial rank zero, then* $A \rtimes_\omega \mathbb{Z}$ *has rational tracial rank zero.*
- *If A has tracial rank at most one, then* $A \rtimes_\omega \mathbb{Z}$ *has tracial rank at most one.*
- *If A has tracial rank zero, then* $A \rtimes_\omega \mathbb{Z}$ *has tracial rank zero.*
- $K_i(A \rtimes_\omega \mathbb{Z}) = K_0(A) \oplus K_1(A)$ *for* $i = 0$ *or* 1.

Proof. The group $\mathbb{Z}$ is amenable and so Proposition IX.16 tells us that $A \rtimes_\omega \mathbb{Z}$ is unital separable simple nuclear and $\mathcal{Z}$-stable. The UCT will always be preserved by crossed products by $\mathbb{Z}$. For the claim about rational tracial rank, we note that $\mathbb{Z} \in \mathcal{C}$ and that the forward implication is given by Theorem IX.17 with $G = \mathbb{Z}$ and the converse by Theorem IX.18 with $G = \mathbb{Z}$. Since $\mathcal{Z} \rtimes_\gamma \mathbb{Z}$ always satisfies the UCT, the next three claims all follow from Theorem IX.17 with $G = \mathbb{Z}$. For the calculation of K-groups, we use the Künneth Theorem for tensor products combined with knowing the K-groups of $\mathcal{Z} \rtimes_\gamma \mathbb{Z}$ from Corollary IX.10. □

REFERENCES CITED

[1] O. Bratteli, D. Evans, A. Kishimoto, *The Rohlin property for quasi free automorphisms of the Fermion algebra*, Proc. London Math. Soc. (3) 71 (1995) 675–694.

[2] N. P. Brown, *On quasidiagonal C^*-algebras*, arxiv:0008181v1.

[3] N. P. Brown, *AF embeddability of crossed products of AF algebras by the integers*, J. Funct. Anal. 160, 150-175 (1998).

[4] J. Carrion, M. Dadarlat, C. Eckhardt, *On groups with quasidiagonal C*-algebras* J. Funct. Anal. 265 (2013), no. 1, 135–152.

[5] J. Dixmier, *On some C^*-algebras considered by Glimm*, J. Funct. Anal. (1964)

[6] S. Echterhoff, W. Lück, N. C. Phillips, and S. Walters, *The structure of crossed products of irrational rotation algebras by finite subgroups of* $\mathrm{SL}_2(\mathbb{Z})$, J. Reine Angew. Math. **639** (2010), 173–221.

[7] T. Fack, O. Marechal, *Sur la classification des symetries des C^*-algebres UHF* Canad. J. Math. vol xxxi no. 3 1979 pp 496–523.

[8] T. Fack, O. Marechal, *Sur la classification des automorphismes periodiques des C^*-algebres UHF*, J. Funct. Anal. 40 (1981), no. 3, 267–301.

[9] J. Glimm, *On a certain class of operator algebras*, Trans. Amer. Math. Soc. **95** 318-340 (1960).

[10] A. Guichardet, *Tensor products of C^*-algebras*, Arhus University Lecture Notes Series 12 (1969).

[11] R. Herman, V. Jones, *Models of finite group actions*, Math. Scand. 52 (1983), 312–320

[12] R. Herman, A. Ocneanu, *Stability for integer actions on UHF algebras*, J. Funct. Anal. 59 (1984), no. 1, 132–144..

[13] R. Herman, V. Jones, *Period two automorphisms of UHF C^*-algebras*, J. Funct. Anal. 45 (1982), no.2, 169–176.

[14] I. Hirshberg, J. Orovitz, *Tracially $\mathcal{Z}$-absorbing C^*-algebras*, J. Funct. Anal. 265 (2013), no. 5, 765–785.

[15] X. Jiang, H. Su, *On a simple unital projectionless C^*-algebra*, Amer. J. Math. 121, no. 2, (1999), 359–413.

[16] I. Kaplansky *Infinite abelian groups* Revised edition, Michigan University Press, 1968.

[17] I. Kaplansky *Groups with representations of bounded degree*, Canad. J. Math. 105-112. MR 10, 428 (1949).

[18] A. Kishimoto, *Outer automorphisms and reduced crossed products of simple C^*-algebras*, Commun. Math. Phys. 81, 429–435 (1981).

[19] A. Kishimoto, *The Rohlin property for shifts on UHF algebras and automorphisms of Cuntz algebras*, J. Funct. Anal. 140 (1996), no. 1, 100–123.

[20] A. Kishimoto, *The Rohlin property for automorphisms of UHF algebras*, J. Reine Angew. Math. 465, 183–196 (1995).

[21] H. Lin, *An introduction to the classification of amenable C^*-algebras*, World Scientific 2001.

[22] H. Lin, N. Zhuang, *The range of a class of classifiable separable simple amenable C^*-algebras*, J. Funct. Anal. 260 (2011), no. 1, 1–29.

[23] H. Lin, N. Zhuang, *Lifting KK-elements, asymptotic unitary equivalence and classification of simple C^*-algebras*, Adv. Math. 219 (2008), no. 5, 1729–1769.

[24] H. Lin, *On local AH-algebras*, arxiv:1104.0445.

[25] H. Lin, *Kishimoto's Conjugacy Theorems in simple C^*-algebras of tracial rank one*, arxiv:1305.4994

[26] H. Lin, *Classification of simple C^*-algebras with tracial toplogical rank zero*, Duke Math. J. **125**, No. 1, (2004).

[27] H. Lin, *Classification of simple tracially AF C^*-algebras*, Canad. J. Math. Vol. 53, No. 1, pp 161–194 (2001).

[28] H. Lin, *The Rokhlin property for automorphisms on simple C^*-algebras*, Operator theory, operator algebras, and applications, 189–215, Contemp. Math., 414, Amer. Math. Soc., Providence, RI, 2006.

[29] H. Lin, *Simple nuclear C^*-algebras of tracial topological rank one*, J. Funct. Anal. 251 (2007), no. 2, 601–679.

[30] H. Lin, W. Sun, *Tensor products of classifiable C^*-algebras*, 2012, arxiv:1203.3737v1.

[31] S. Hu, H. Lin, Y. Xue, *The tracial topological rank of C^*-algebras (II)*, Indiana Univ. Math. J. **53**, No. 6 (2004), 1579–1606.

[32] H. Lin, *The tracial topological rank of C^*-algebras*, Proc. London Math. Soc. (3) 83 (2001) 199–234.

[33] H. Lin, *AF-embeddings of the crossed products of AH-algebras by finitely generated abelian groups.*, Int. Math. Res. Pap. (2008), no. 3.

[34] H. Lin *The Rokhlin property for automorphisms on simple C^*-algebras*, arxiv:0602513v2.

[35] H. Matui, Y. Sato, *Strict comparison and Z-absorption of nuclear C^*-algebras*, Acta Math. 209 (2012), no. 1, 179–196.

[36] H. Matui, Y. Sato, *Z-stability of crossed products by strongly outer actions I*, Commun. Math. Phys. 314 (2012), 193–228.

[37] H. Matui, Y. Sato, *Z-stability of crossed products by strongly outer actions II*, arxiv:1205.1590v2.

[38] H. Matui, Y. Sato, *Decomposition rank of UHF-absorbing C^*-algebras*, arxiv:1303.4371v2.

[39] H. Matui, *$\mathbb{Z}^N$-actions on UHF algebras of infinite type*, J. Reine Angew. Math. 657, 225-244 (2011).

[40] H. Nakamura, *The Rohlin property for $\mathbb{Z}^2$-actions on UHF algebras*, Surikaisekikenkyusho Kokyuroku (1997), no. 977, 92–101.

[41] H. Osaka, N. C. Phillips, *Stable and real rank for crossed products by automorphisms with the tracial Rokhlin property*, Ergodic Theory Dynam. Systems 26 (2006), no.5, 1579–1621.

[42] N. C. Phillips, *The tracial Rokhlin property for actions of finite groups on C^*-algebras*, Amer. J. Math. (2011), no. 3, 581–636.

[43] N. C. Phillips, *The tracial Rokhlin property is generic*, arxiv:1209.3859v1.

[44] M. Rørdam, *The stable and real rank of $\mathcal{Z}$-absorbing C^*-algebras*, Int. J. Math. 15, no. 10 (2004) 1065–1084.

[45] J. Rosenberg, *Amenability of crossed products of C^*-algebras*, Comm. Math. Phys. 57, (1977) no. 2, 187–191.

[46] Y. Sato, *The Rohlin property for automorphisms of the Jiang-Su algebra*, J. Funct. Anal. 259 (2010) no. 2, 453–476.

[47] D. Voiculescu, *Almost inductive limit automorphisms and embeddings into AF-algebras*, Ergodic Theory Dynam. Systems 6 (1986), no. 3, 475–484.

[48] A. Toms, W. Winter, *Strongly self-absorbing C^*-algebras*, Trans. Amer. Math. Soc. 359 (2007), no. 8, 3999–4029.

[49] W. Winter *Localizing the Elliott conjecture at strongly self-absorbing C*-algebras*, arxiv:0708.0283v3.

[50] W. Winter, *C^*-algebras with local finite decomposition rank*, J. Funct. Anal. 243 (2007), no. 2, 394–425.