

MOTIVIC INTEGRAL OF K3 SURFACES OVER A NON-ARCHIMEDEAN  
FIELD

by

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## DISSERTATION ABSTRACT

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We prove a formula expressing the motivic integral [LS] of a K3 surface over  $\mathbb{C}((t))$  with semi-stable reduction in terms of the associated limit mixed Hodge structure. Secondly, for every smooth variety over a complete discrete valuation field we define an analogue of the monodromy pairing, constructed by Grothendieck in the case of Abelian varieties, and prove that our monodromy pairing is a birational invariant of the variety. Finally, we propose a conjectural formula for the motivic integral of maximally degenerate K3 surfaces over an arbitrary complete discrete valuation field and prove this conjecture for Kummer K3 surfaces.

This dissertation includes previously published co-authored material.

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## CHAPTER I

### INTRODUCTION

The majority of this dissertation was published in *Advances in Mathematics* 228, no. 5. The only sections not included in that publication are 1.1, 1.2, and 3.2. Vadim Vologodsky was a co-author for that publication and contributed to sections 1.6, 1.7, 2.2, 4.1, 4.2, and 4.3.

#### 1.1. The Weil Measure on Varieties

Let  $K$  be a non-archimedean local field,  $R \subset K$  the maximal compact subring,  $\mathfrak{m} \subset R$  the maximal ideal,  $\pi$  the uniformizer, and  $k = R/\mathfrak{m}$  the residue field with  $|k| = q$ .

For a smooth complete  $n$ -dimensional variety,  $X$ , over  $K$ , Weil observed that a top degree differential form  $\omega \in \Gamma(X, \omega_X)$  defines a finite real valued measure  $|\omega|$  on the set of  $K$ -points of  $X$  in the following way.

Let  $x \in X$  be a  $K$ -point,  $t_1, \dots, t_n$  local  $p$ -adic analytic parameters at  $x$ . Then  $t_1, \dots, t_n$  define a bi-analytic homeomorphism  $\theta: U \rightarrow \mathbb{A}^n(K)$  of an open subset  $U \subset X(K)$  containing  $x$  with an open subset  $\theta(U) \subset \mathbb{A}^n(K)$ . Let  $|dt|$  be the standard Haar measure on  $\mathbb{A}^n(K)$  normalized by the condition that the measure of the maximal ideal  $\mathfrak{m} \subset R$  equals 1 and let  $g dt$  be the pullback of  $\omega$  with respect to  $\theta^{-1}$ . The measure  $|\omega|$  on  $U$  is then defined to be the pullback with respect to  $\theta$  of the measure  $\|g(t)\| |dt|$  on  $\theta(U)$ .

Change of variables implies that this measure is independent of the choice of local coordinates and therefore agrees on intersections of open sets. Thus  $|\omega|$  is a global measure. However, this measure still depends on our choice of  $\omega$ .

## 1.2. Volume of Calabi-Yau Varieties over Local Fields

Now assume that  $X$  is a Calabi-Yau variety *i.e.*, the canonical bundle  $\omega_X$  is trivial. A weak Néron model of  $X$ ,  $\mathcal{V}$ , is a smooth  $R$ -scheme whose generic fiber is  $X$  and such that every point of  $X$  with values in an unramified  $K' \supset K$  extends to a  $R'$ -point of  $\mathcal{V}$ .<sup>1</sup>

**Example 1.2.1.** *If  $\overline{X}$  is a proper regular model of  $X$  over  $R$ , then  $\mathcal{V} := \overline{X} - X_{\text{sing}}$  is a weak Néron model.*

If  $X$  is a Calabi-Yau variety with non-zero form  $\omega$  and  $\mathcal{V}$  is a weak Néron model of  $X$  over the ring of integers  $R \subset K$ , we have

$$\int_{X(K)} |\omega| = \sum_i |V_i^\circ(k)| q^{-m_i}, \quad (1.1)$$

where  $V_i^\circ$  are the connected components of the special fiber of  $\mathcal{V}$ ,  $r_i \in \mathbb{Z}$  are defined from the equation  $\text{div } \omega = \sum_i m_i [V_i^\circ]$ , where  $\omega$  is regarded as a rational section of the relative canonical bundle,  $\Omega_{\mathcal{V}/\text{spec } R}^n$ . Let us explain formula 1.1 in more detail.

Let

$$\phi: \mathcal{V} \rightarrow \mathcal{V}(k)$$

---

<sup>1</sup>A weak Néron model always exists but it is almost never unique.

be the reduction map. If  $\bar{x} \in \mathcal{V}(k)$  is a closed  $k$ -point of  $\mathcal{V}$  and  $t_1, \dots, t_n$  are local parameters at  $\bar{x}$ , then the functions  $t_1, \dots, t_n$  define a bi-analytic homeomorphism

$$f: \phi^{-1}(\bar{x}) \rightarrow \mathbb{A}^n(\mathfrak{m}),$$

where  $\phi^{-1}(\bar{x})$  is the fiber of  $\phi$  over  $\bar{x}$ . So we have

$$\int_{X(K)} |\omega| = \sum_i |V_i^\circ(k)| \int_{\phi^{-1}(\bar{x}_i)} |\omega|$$

where  $\bar{x}_i$  is a closed  $k$ -point of  $V_i^\circ(k)$ .

On a Zariski neighborhood of  $\bar{x}_i$  we can write  $g dt_1 \wedge \dots \wedge dt_n = \omega$ , where  $g$  is  $\pi^{m_i}$  times an invertible function. Thus using the homeomorphism  $f$  we have

$$\int_{\phi^{-1}(\bar{x}_i)} |\omega| = \int_{\mathbb{A}^n(\mathfrak{m})} \|g(t)\| |dt| = \int_{\mathbb{A}^n(\mathfrak{m})} \frac{1}{q^{m_i}} |dt| = \frac{1}{q^{m_i}}$$

In particular, the quantity at the right-hand side of equation (1.1) does not depend on the choice of  $\mathcal{V}$  (but does depend on  $\omega$ ). The renormalized integral

$$\int_{X(K)} := \sum_i [V_i^\circ(k)] q^{-m_i + \min_i m_i}, \quad (1.2)$$

is an invariant of  $X$ .

If  $X$  has a smooth and proper model over  $R$  the Grothendieck-Lefschetz formula together with the Proper Base Change and Local Acyclicity theorems yield a cohomological interpretation for the normalized measure:

$$\int_{X(K)} = \sum_j (-1)^j \text{Tr}(F^{-1}, H^j(X_{\bar{K}}, \mathbb{Q}_l)), \quad (1.3)$$

where  $l$  is prime number different from the characteristic of  $k$  and  $F \in Gal(\overline{K}/K)$  is a lifting of the Frobenius automorphism  $Fr \in Gal(\overline{k}/k)$ .<sup>2</sup>

The main question we wished to consider is whether it is possible to find a generalization of the formula (1.3) to the case of bad reduction? The case of Abelian varieties is treated in [V]. In general this question is very difficult. However, we do have the following result.

*If  $X$  admits a proper strictly semi-stable model over  $R$  then the formula (1.3) holds modulo  $q - 1$ .*

*Proof.* Let  $\overline{X}$  be a strictly semi-stable model of  $X$ . We have  $\int_{X(K)} \equiv |Y_{sm}(k)| \pmod{(q-1)}$ . Grothendieck-Lefschetz lets us count smooth points via stalks of the local cohomology of the sheaf of nearby cycles. If the point isn't smooth, strictly semi-stability implies the alternating sum of traces is the alternating sum of exterior powers of some vector space which is 0. Then  $\int_{X(K)} \equiv |Y_{sm}(k)| \pmod{(q-1)}$ . On the other hand, by the Grothendieck-Lefschetz formula

$$\sum_j (-1)^j Tr(F^{-1}, H^j(X_{\overline{K}}, \mathbb{Q}_l)) = \sum_j (-1)^j Tr(F^{-1}, H^j(Y_{\overline{k}}, \Psi(\mathbb{Q}_l))) = \sum_{y \in Y(k)} \sum_i (-1)^i Tr(F^{-1}, \mathcal{H}^i(\Psi(\mathbb{Q}_l))_y).$$

If  $y \in Y_{sm}(k)$  the corresponding sum equals 1 reduces to fact that alternating sum of traces is the number of points. If  $y \in Y_{sing}(k)$  then  $\mathcal{H}^i(\Psi(\mathbb{Q}_l))_y \simeq \bigwedge^i T(-i)$ , where  $T$  is a  $\mathbb{Q}_l$ -vector space with the trivial action of  $Gal(\overline{K}/K)$ .

---

<sup>2</sup>In fact, the action of  $Gal(\overline{K}/K)$  on the  $l$ -adic cohomology factors through the Galois group of the residue field.

Thus,

$$\sum_i (-1)^i \text{Tr}(F^{-1}, \mathcal{H}^i(\Psi(\mathbb{Q}_l))_y) \equiv \sum_i (-1)^i \dim \bigwedge^i T \equiv 0 \pmod{(q-1)}.$$

□

### 1.3. Motivic Integral of a Calabi-Yau Variety

Motivic integration was introduced by M. Kontsevich in a 1995 lecture in Orsay, where he announced an affirmative solution to the following conjecture of Batyrev: *two birationally equivalent Calabi-Yau manifolds have the same Hodge numbers*. This conjecture was motivated by work in theoretical physics, namely string theory, which predicts that as a manifold the universe is locally a product of  $\mathbb{R}^4$  (space-time) and a compact component (a Calabi-Yau manifold). The conjecture means that two such special manifolds share the same important numerical invariants, provided that they contain large enough isomorphic open subsets.

Kontsevich gave a remarkably elegant and conceptual proof of this result, essentially as a corollary of his theory of motivic integration. He was inspired by the theory of p-adic integration, which Batyrev himself had used to prove a weaker form of his conjecture. Later in [LS] Loeser and Sebag generalized Kontsevich's construction providing, in particular, a motivic refinement of Weil's measure discussed in I.1.

Let  $R$  be a complete discrete valuation ring with fraction field  $K$  and perfect residue field  $k$ . By a Calabi-Yau variety  $X$  over  $K$  we mean a smooth projective scheme  $X$  over  $K$ , of pure dimension  $d$ , with trivial canonical bundle  $\omega_X := \Omega_{X/K}^d$ . In [LS], Loeser and Sebag associated with any Calabi-Yau variety  $X$  over  $K$  a canonical

element

$$\int_X \in K_0(\text{Var}_k)_{loc}$$

of the ring  $K_0(\text{Var}_k)_{loc}$ , where  $K_0(\text{Var}_k)_{loc}$  is obtained from the Grothendieck ring  $K_0(\text{Var}_k)$  of algebraic varieties over  $k$  by inverting the class  $[\mathbb{A}_k^1]$  of the affine line.

The motivic integral  $\int_X$  can be computed from a weak Néron model of  $X$ . Recall, that a weak Néron model of a smooth proper scheme  $X$  over  $K$  is a smooth scheme  $\mathcal{V}$  of finite type over  $R$  together with an isomorphism  $\mathcal{V} \otimes_R K \simeq X$  satisfying the following property: for every finite unramified extension  $R' \supset R$  with fraction field  $K'$ , the canonical map  $\mathcal{V}(R') \rightarrow X(K')$  is bijective. According to ([BLR], §3.5, Theorem 3), every smooth proper  $K$ -scheme  $X$  admits a weak Néron model. We note that a weak Néron model is almost never unique: for example, if  $\overline{X}$  is a proper regular model of  $X$  over  $R$ , then the smooth locus  $\overline{X}_{sm}$  of  $\overline{X}$  is a weak Néron model of  $X$  (see Lemma 2.3.2).

Given a Calabi-Yau variety  $X$  over  $K$ , a weak Néron model  $\mathcal{V}$  of  $X$ , and a nonzero top degree differential form  $\omega \in \Gamma(X, \omega_X)$ , we can view  $\omega$  as a rational section of the canonical bundle  $\omega_{\mathcal{V}/R}$  on  $\mathcal{V}$ . The divisor of  $\omega$  is supported on the special fiber  $V^\circ$  of  $\mathcal{V}$ . Thus, we can write

$$\text{div } \omega = \sum_i m_i V_i^\circ, \tag{1.4}$$

where  $V_1^\circ, \dots, V_s^\circ$  are the irreducible components of the special fiber  $V^\circ$ . The motivic integral of  $X$  is defined by the formula<sup>3</sup>

$$\int_X := \sum_i [V_i^\circ] (m_i - \min_i m_i). \tag{1.5}$$

---

<sup>3</sup>We note that our terminology and notation are different from those used by Loeser and Sebag. Notation for  $\int_X$  in [LS] is  $[\overline{X}]$ . The name ‘‘motivic integral’’ is reserved in *loc. cit.* for a more general construction that associates with any smooth proper  $K$ -scheme  $X$  and a top degree differential form  $\omega \in \Gamma(X, \omega_X)$  an element  $\int_X \omega$  of a certain completion of the motivic ring  $K_0(\text{Var}_k)_{loc}$ .

Here, given an element  $[Z] \in K_0(\text{Var}_k)_{loc}$  and an integer  $n$ , we write  $[Z](n)$  for its Tate twist:

$$[Z](n) := [Z] \cdot [\mathbb{A}^1]^{-n}.$$

A key result proven by Loeser and Sebag ([LS], Theorem 4.4.1) is that the right-hand side of equation (1.5) is independent of the choice of  $\mathcal{V}$  and  $\omega$ .

If  $k = \mathbb{F}_q$ , the image of the motivic integral under the homomorphism

$$K_0(\text{Var}_{\mathbb{F}_q})_{loc} \rightarrow \mathbb{Z}_{(q)} \quad [Z] \rightsquigarrow |Z(\mathbb{F}_q)| \tag{1.6}$$

is equal to the volume  $\int_{X(K)} |\omega|$ , for an appropriately normalized  $\omega \in \Gamma(X, \omega_X)$  ([LS], §4.6).

In this paper we express the motivic integral of K3 surfaces over  $\mathbb{C}((t))$  with strictly semi-stable reduction in terms of the associated limit mixed Hodge structures. We also compute the motivic integral of some K3 surfaces over an arbitrary complete discrete valuation field. To our knowledge the only class of varieties, for which similar formulas were previously known, is the class of abelian varieties (see, *e.g.* [SGA7], Exposé IX, [V], [HN1], [HN2]), where the computation is based on the theory of Néron models, and, in particular, for  $K = \mathbb{C}((t))$ , on the Hodge theoretic description of the special fiber of the Néron model. Unfortunately, K3 surfaces do not have a Néron model, in general, which makes our problem substantially more difficult.

Let us describe the organization of the paper in more detail.

#### 1.4. Limit Mixed Hodge Structure

In §II we explain some preliminary material, the most important of which is the notion of limit mixed Hodge structure associated with a variety over the field of formal

Laurent series  $\mathbb{C}((t))$ . Schmid and Steenbrink associated with every smooth projective variety over the field  $K_{mer}$  of meromorphic functions on an open neighborhood of zero in the complex plane a mixed Hodge structure, called the limit mixed Hodge structure. In §2.2, using Log Geometry, we extend the Steenbrink-Schmid construction to smooth projective varieties over  $\mathbb{C}((t))$ .

### 1.5. Motivic Integral of K3 Surfaces over $\mathbb{C}((t))$

In order to state our first main result we need to introduce a bit of notation. Let  $X$  be a smooth projective K3 surface over  $K = \mathbb{C}((t))$  and let  $H^2(\lim X) = (H^2(\lim X, \mathbb{Z}), W_i^{\mathbb{Q}}, F^i)$  be the corresponding limit mixed Hodge structure (see §2.2). Assume that the monodromy acts on  $H^2(\lim X, \mathbb{Z})$  by a unipotent operator. Then its logarithm  $N$  is known to be integral ([FS], Prop. 1.2):

$$N : H^2(\lim X, \mathbb{Z}) \rightarrow H^2(\lim X, \mathbb{Z}). \quad (1.7)$$

Set  $W_i^{\mathbb{Z}} = W_i^{\mathbb{Q}} \cap H^2(\lim X, \mathbb{Z})$ . The morphisms

$$Gr N^i : W_{i+2}^{\mathbb{Z}}/W_{i+1}^{\mathbb{Z}} \rightarrow W_{2-i}^{\mathbb{Z}}/W_{1-i}^{\mathbb{Z}}, \quad i = 1, 2 \quad (1.8)$$

are injective and have finite cokernels. Let  $r_i(X, K)$  be their orders. In §III we prove the following result.

**Theorem 1.** *Let  $X$  be a smooth projective K3 surface over  $K = \mathbb{C}((t))$ . Assume that  $X$  has a strictly semi-stable model over  $R = \mathbb{C}[[t]]$  and that the operator  $N$  is not equal to 0. Let  $s$  be the smallest integer such that  $N^s = 0$ . Then  $s$  is either 2 or 3 and for every finite extension  $K_e \supset K$  of degree  $e$  the motivic integral of the K3 surface  $X_e = X \otimes_K K_e$  over  $K_e$  is given by the following formulas.*



(a) If  $s = 2$  then

$$\begin{aligned} \int_{X_e} &= 2\mathbb{Z}(0) - (e\sqrt{r_1(X, K)} + 1)[E(X)] + 20\mathbb{Z}(-1) \\ &\quad + (e\sqrt{r_1(X, K)} - 1)[E(X)](-1) + 2\mathbb{Z}(-2), \end{aligned} \quad (1.9)$$

where  $E(X)$  is the elliptic curve defined by the weight 1 Hodge structure on  $W_1^{\mathbb{Z}} = W_1^{\mathbb{Q}} \cap H^2(\lim X, \mathbb{Z})$  and  $\mathbb{Z}(n) := [\mathbb{A}^1]^{-n}$ ,  $n \in \mathbb{Z}$ .

(b) If  $s = 3$  then

$$\begin{aligned} \int_{X_e} &= \left( \frac{e^2 r_2(X, K)}{2} + 2 \right) \mathbb{Z}(0) + (20 - e^2 r_2(X, K)) \mathbb{Z}(-1) \\ &\quad + \left( \frac{e^2 r_2(X, K)}{2} + 2 \right) \mathbb{Z}(-2). \end{aligned} \quad (1.10)$$

Note, that if  $N = 0$  the K3 surface  $X$  has a smooth proper model over  $R$  whose special fiber  $Y$  (and thus the motivic integral) is determined by the polarized pure Hodge structure  $H^2(\lim X, \mathbb{Z})$ .

Let us explain the idea of our proof assuming that  $e = 1$ . First, using the theory of Hilbert schemes and Artin's approximation theorem, we reduce the proof to the case when  $X$  is obtained by the restriction of a smooth family  $\mathcal{X}$  of K3 surfaces over a smooth punctured complex curve  $C = \overline{C} - a$  to the formal punctured neighborhood of the point  $a \in \overline{C}$ . The rest of the proof is based on a result of Kulikov, [Ku], asserting the existence of a (non-unique) strictly semi-stable model  $\overline{\mathcal{X}} \xrightarrow{\overline{\pi}} \overline{C}$  such that the log canonical bundle  $\omega_{\overline{\mathcal{X}}/\overline{C}}(\log)$  is trivial over an open neighborhood of the special fiber  $Y$ . For any such model, we have

$$\int_X = [Y_{sm}],$$

where  $Y_{sm} \subset Y$  is the smooth locus of  $Y$ . It is shown in [Ku] that the special fiber  $Y$  of a Kulikov model has a very special form. If  $s = 2$  the Clemens polytope  $Cl(Y)$  of  $Y$  (see §2.1) is a partition of an interval and all but two irreducible components of  $Y$  are ruled surfaces fibered over elliptic curves, all of which are isomorphic to a single elliptic curve  $E$ . The two components corresponding to the boundary points of  $Cl(Y)$  are rational surfaces. If  $s = 3$  then all the irreducible components of  $Y$  are rational surfaces and the Clemens polytope  $Cl(Y)$  is a triangulation of a sphere. Next, using results of Friedman and Scattone ([FS], [Fr]) we prove that the Steenbrink weight spectral sequence for Kulikov's model  $\overline{\mathcal{X}} \xrightarrow{\overline{\pi}} \overline{C}$  (and therefore by the Weak Factorization Theorem ([KonSo], Theorem 9), for every strictly semi-stable model of  $X$ ) degenerates integrally at the second term. Of course, the degeneration of the weight spectral sequence with rational coefficients is a corollary of Hodge Theory and holds in general, but the degeneration over  $\mathbb{Z}$  is a special non-trivial property of K3 surfaces. This, combined with the generalized Picard-Lefschetz formula, implies that, for  $s = 2$ , the Hodge structure on  $H^1(E)$  is isomorphic to that on  $W_1^{\mathbb{Z}}$  and that the number of irreducible components of  $Y$  equals  $\sqrt{r_1(X, K)} + 1$ . Similarly, as proven in [FS], for  $s = 3$ , the combinatorics of  $Y$  (*i.e.*, the number of irreducible, components, double curves and triple points) is completely determined by the monodromy action on the integral lattice  $H^2(\lim X, \mathbb{Z})$ . This, together with a variant of A'Campo's formula (Proposition 2.3.1), completes the proof.

## 1.6. Monodromy Pairing

*In this section, Vadim Vologodsky authored the statements and proofs of the lemmas which we developed jointly over many hours of meetings.*

In §IV we introduce a generalization of the invariant  $r_2(X, K)$ , that we defined in §1.5 for K3 surfaces over  $\mathbb{C}((t))$ , to the case of an arbitrary smooth variety over a complete discrete valuation field. Our construction is based on the theory of analytic spaces over non-archimedean fields developed by Berkovich [Ber1]. For a complete discrete valuation field  $K$  we denote by  $\widehat{K}$  the completion of an algebraic closure of  $K$ . One of the key features of Berkovich's theory is that the underlying topological space  $|X_{\widehat{K}}^{an}|$  of the analytification of a scheme  $X$  over  $K$  has interesting topological invariants (in contrast with the space  $X(\widehat{K})$  equipped with the usual topology, which is totally disconnected). In particular, if  $X$  is the generic fiber of a proper strictly semi-stable scheme  $\overline{X}$  over  $R$  the space  $|X_{\widehat{K}}^{an}|$  is homotopy equivalent to the Clemens polytope of the special fiber  $Y$ . We denote by  $\Gamma_C^m(X)$  the singular cohomology of the space  $|X_{\widehat{K}}^{an}|$  with coefficients in a ring  $C$ . In Theorem 4, we prove that, for every prime  $\ell$  different from the characteristic of the residue field of  $K$ , and for every smooth scheme  $X$ , there is a canonical isomorphism of  $\text{Gal}(\overline{K}/K)$ -modules

$$\gamma : \Gamma_{\mathbb{Q}_\ell}^m(X) \xrightarrow{\sim} \text{Im} \left( H^m(X_{\overline{K}}, \mathbb{Q}_\ell)(m) \xrightarrow{N^m} H^m(X_{\overline{K}}, \mathbb{Q}_\ell) \right), \quad (1.11)$$

where  $N$  is the logarithm of the monodromy operator. In particular, the dimension of the vector space on the right-hand side of (1.11) is independent of  $\ell$ . Let us note, that a different description of the space  $\Gamma_{\mathbb{Q}_\ell}^m(X)$  in the case of finite residue field was obtained earlier by Berkovich [Ber4].

If  $d$  is the dimension of  $X$ , we use (1.11) to define a non-degenerate pairing

$$\Gamma_{\mathbb{Q}}^d(X) \otimes \Gamma_{\mathbb{Q}}^d(X) \rightarrow \mathbb{Q}. \quad (1.12)$$

In the special case when  $X$  is proper, the pairing (1.11) is given by the formula

$$(x, y) = (-1)^{\frac{d(d-1)}{2}} \langle \gamma(x), y' \rangle, \quad (1.13)$$

where  $y' \in H^d(X_{\bar{K}}, \mathbb{Q}_\ell)$  is an element such that  $N^d y' = \gamma(y)$  and  $\langle, \rangle$  is the Poincaré pairing on  $H^d(X_{\bar{K}}, \mathbb{Q}_\ell)$ . We prove in Theorem 5 that (1.13) is independent of  $\ell$  and positive. Moreover, the groups  $\Gamma_{\mathcal{C}}^m(X)$  and the monodromy pairing (1.12) are birational invariants of  $X$ .

We define a numeric (birational) invariant  $r_d(X, K)$  of  $X$  to be the discriminant of the dual pairing

$$\Gamma_d(X) \otimes \Gamma_d(X) \rightarrow \mathbb{Q}, \quad (1.14)$$

where  $\Gamma_d(X)$  is  $\text{Hom}(\Gamma_{\mathbb{Z}}^d(X), \mathbb{Z})$ .

In remark 4.2.1, we define for a polarized projective variety  $X$  and any integer  $m$  a more general positive pairing  $\Gamma_{\mathbb{Q}}^m(X) \otimes \Gamma_{\mathbb{Q}}^m(X) \rightarrow \mathbb{Q}$  which in the case of semi-stable abelian variety  $A$  and its dual  $A'$  boils down, after some identifications, to the monodromy pairing  $\Gamma_1(A) \otimes \Gamma_1(A') \rightarrow \mathbb{Z}$  defined by Grothendieck ([SGA7], Exp. IX). In particular, the number  $r_d(A, K)$  is non-zero if and only if  $A$  is completely degenerate in which case  $r_d(A, K)$  is equal to  $d!|\pi_0(\mathcal{V}(A) \otimes \bar{k})|$ , where  $\mathcal{V}(A)$  is the Néron model of  $A$ .

### 1.7. Motivic Integral of Maximally Degenerate K3 Surfaces

*In this section, Vadim Vologodksy authored the statements and proofs of the lemmas which we developed jointly over many hours of meetings.*

We say that a  $d$ -dimensional Calabi-Yau variety over a complete discrete valuation field  $K$  is maximally degenerate if  $\Gamma_{\mathbb{Q}}^d(X) \neq 0$ . According to (1.11),  $X$

is maximally degenerate if and only if for some (and, hence, for any) prime  $\ell \neq \text{char } k$  the map

$$H^d(X_{\overline{K}}, \mathbb{Q}_\ell)(m) \xrightarrow{N^d} H^d(X_{\overline{K}}, \mathbb{Q}_\ell)$$

is not zero<sup>4</sup>. We conjecture that for every maximally degenerate K3 surface over  $K$  there exists a finite extension  $K' \supset K$  such that, for every finite extension  $L \supset K$  of ramification index  $e$  containing  $K'$ , we have

$$\int_{X_L} = \left( \frac{e^2 r_2(X, K)}{2} + 2 \right) \mathbb{Q}(0) + (20 - e^2 r_2(X, K)) \mathbb{Q}(-1) + \left( \frac{e^2 r_2(X, K)}{2} + 2 \right) \mathbb{Q}(-2).$$

If  $\text{char } k = 0$  our conjecture follows from part (b) of Theorem 1. In §?? we prove this conjecture in the case of Kummer K3 surfaces over an arbitrary complete discrete valuation field  $K$  with  $\text{char } k \neq 2$  by constructing explicitly a poly-stable formal model of the analytic space  $X^{an}$ .

The groups  $\Gamma_{\mathbb{Z}}^d(X)$  that we used to define the invariant  $r_d(X, K)$  can be interpreted as the weight 0 part of the limit motive of  $X$  (Remark 4.1.2). It would be interesting to define geometrically the limit 1-motive attached to  $X$  and use it to compute the motivic integral for K3 surfaces which are not maximally degenerate.

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<sup>4</sup>There is an extensive literature on maximally degenerate Calabi-Yau varieties over  $\mathbb{C}((t))$ . See *e.g.* [Mo1], [LTY].

## CHAPTER II

### PRELIMINARIES

#### 2.1. Clemens Polytope and Nerve of a Strictly Semi-Stable Scheme

Let  $R$  be a complete discrete valuation ring with residue field  $k$  and fraction field  $K$ . Recall that a scheme  $\overline{X}$  of finite type over  $\text{spec } R$  is strictly semi-stable if every point  $x \in \overline{X}$  has a Zariski neighborhood  $x \in U \subset \overline{X}$  such that the morphism  $U \rightarrow \text{spec } R$  factors through an étale morphism

$$U \rightarrow \text{spec } R[T_0, \dots, T_d]/(T_0 \cdots T_r - t), \quad 0 \leq r \leq d,$$

for a uniformizer  $t$  of  $K$ . If  $k$  is perfect,  $\overline{X}$  is a strictly semi-stable scheme if and only if it is regular and flat over  $R$ , the generic fiber  $X = \overline{X} \times_R K$  is smooth over  $K$  and the special fiber  $Y = \overline{X} \times_R k$  is a reduced strictly normal crossing divisor on  $\overline{X}$ .

Let  $\overline{X}$  be a strictly semi-stable scheme. Then the irreducible components  $V_1, \dots, V_m$  of  $Y$  as well as the schemes

$$Y^{(q)} = \coprod_{i_0 < \dots < i_q} V_{i_0} \cap \dots \cap V_{i_q} \tag{2.1}$$

are smooth. It is convenient to encode the combinatorial structure of  $Y$  by a certain topological space. To do this we need to introduce some terminology.

By an abstract triangulated set we mean a contravariant functor  $\tilde{\Delta} \rightarrow \text{Sets}$ , where  $\tilde{\Delta}$  is the category whose objects are finite totally ordered sets  $[q] := \{0, \dots, q\}$ ,  $q \in \mathbb{Z}_{\geq 0}$  and whose morphisms are strictly increasing maps. Thus, giving an abstract triangulated set  $S_\bullet$  amounts to giving a set  $S_q$  of “ $q$ -simplices” for each  $q \in \mathbb{Z}_{\geq 0}$

together with “boundary maps”  $\delta_j : S_q \rightarrow S_{q-1}$ ,  $j = 0, \dots, q$ , subject to certain conditions<sup>1</sup>. We shall write  $|S_\bullet|$  for the realization of  $S_\bullet$  ([GM], §1.1).

Given a strictly semi-stable scheme  $\overline{X}$  consider the abstract triangulated set whose  $q$ -dimensional simplices are indexed by the set  $\pi_0 \left( Y_{\overline{k}}^{(q)} \right)$ . The boundary maps  $\delta_j : \pi_0 \left( Y_{\overline{k}}^{(q)} \right) \rightarrow \pi_0 \left( Y_{\overline{k}}^{(q-1)} \right)$ ,  $j = 0, \dots, q$ , are given by the maps

$$\pi_0 \left( V_{i_0, \overline{k}} \cap \dots \cap V_{i_q, \overline{k}} \right) \rightarrow \pi_0 \left( V_{i_0, \overline{k}} \cap \dots \cap V_{i_{j-1}, \overline{k}} \cap V_{i_{j+1}, \overline{k}} \cap \dots \cap V_{i_q, \overline{k}} \right)$$

induced by the injections

$$V_{i_0} \cap \dots \cap V_{i_q} \hookrightarrow V_{i_0} \cap \dots \cap V_{i_{j-1}} \cap V_{i_{j+1}} \cap \dots \cap V_{i_q}.$$

The realization of this triangulated set is a topological space which we call (following [KonSo]) the Clemens polytope of  $Y$  and denote by  $Cl(Y)$ . Although the abstract triangulated set we constructed depends upon the choice of ordering on the set of irreducible components  $V_i$ , the homeomorphism type of the topological space  $Cl(Y)$  does not.

**Proposition 2.1.1.** *Let  $\overline{X}$  be a strictly semi-stable model of  $X$  over  $\text{spec } R$  with special fiber  $Y$  then for every abelian group  $C$ ,*

$$H_{sing}^*(Cl(Y), C) \cong H_{Zar}^*(Y_{\overline{k}}, \underline{C})$$

---

<sup>1</sup>The category of abstract triangulated sets can be viewed as a full subcategory of the category of simplicial sets: if  $S'_\bullet$  is a simplicial set such that the boundary of each nondegenerate simplex of  $S'_\bullet$  is nondegenerate then nondegenerate simplices of  $S'_\bullet$  together with the boundary maps form an abstract triangulated set. This yields an equivalence between the full subcategory of the category of simplicial sets whose objects satisfy the above property and the category of abstract triangulated sets ([GM], §1.6).

*Proof.* To simplify our notation we assume that  $k = \bar{k}$ . Consider the complex

$$i_{0*}\underline{C} \xrightarrow{\partial_0} i_{1*}\underline{C} \xrightarrow{\partial_1} \dots .$$

where  $i_q : Y^{(q)} \hookrightarrow Y$ . The differentials  $\partial_i$  are characterized by the property that the induced map on global sections  $\Gamma(i_{q*}\underline{C}) = C[\pi_0(Y^{(q)})] \rightarrow C[\pi_0(Y^{(q+1)})] = \Gamma(i_{q+1*}\underline{C})$  equals  $\sum_j (-1)^j \delta_j^*$ . This complex is a resolution of  $\underline{C}$ . Since each  $Y^{(q)}$  is a disjoint union of smooth irreducibles and constant sheaves on irreducibles are flabby it follows that the sheaves  $i_{q*}\underline{C}$  are flabby. Thus the complex of global sections

$$\Gamma(i_{0*}\underline{C}) \xrightarrow{\partial_0} \Gamma(i_{1*}\underline{C}) \xrightarrow{\partial_1} \dots .$$

computes the Zariski cohomology  $H_{Zar}^*(Y, \underline{C})$ . On the other hand, this complex is the simplicial complex of  $Cl(Y)$ .

□

Assume that  $\bar{X}$  is a proper semi-stable scheme over  $R$ . Then, by the Proper Base Change theorem (*e.g.*, [D]), for every torsion abelian group  $C$  we have canonical morphisms of  $\text{Gal}(\bar{K}/K)$ -modules

$$H_{Zar}^*(Y_{\bar{k}}, \underline{C}) \rightarrow H_{et}^*(Y_{\bar{k}}, \underline{C}) \cong H_{et}^*(\bar{X}_{R^{sh}}, \underline{C}) \rightarrow H_{et}^*(X_{\bar{K}}, \underline{C}), \quad (2.2)$$

where  $R^{sh}$  denotes a strict Heselization of  $R$ . Applying (2.2) to  $C = \mathbb{Z}/\ell^n\mathbb{Z}$  and passing to the limit, we obtain a canonical morphism

$$H_{sing}^*(Cl(Y), \mathbb{Z}_\ell) \rightarrow H^*(X_{\bar{K}}, \mathbb{Z}_\ell). \quad (2.3)$$



We will see in §IV that the groups  $H_{sing}^*(Cl(Y), C)$  and the morphism (2.3) depend only on the generic fiber  $X$  and not on the choice of proper strictly semi-stable model  $\bar{X}$ .

**Remark 2.1.2.** *Let us explain the relation of the notion of Clemens polytope to a more general notion of nerve of a scheme, introduced in ([Ber3]). For a reduced scheme  $Y$  over  $k$ , let  $Nor(Y) \subset Y$  be the normal locus of  $Y$ , which is an open subset of  $Y$ , and let  $Y^{[0]} = Y$ ,  $Y^{[i+1]} = Y^{[i]} \setminus Nor(Y^{[i]})$ ,  $i \geq 0$ . The irreducible components of  $Y^{[i]} \setminus Y^{[i+1]}$  are called strata of  $Y$ . The set,  $Str(Y)$ , of all strata has a natural partial order: for strata  $x, y \in Str(Y)$ , we say that  $x \leq y$  if  $y$  is contained in the closure of  $x$ . We denote by  $N(Y)$  the nerve of the partially ordered set  $Str(Y)$ . If  $\bar{X}$  is a strictly semi-stable scheme over  $R$ , the triangulated space  $|N(Y \otimes \bar{k})|$  is obtained from  $Cl(Y)$  by subdivision. In particular, the spaces  $|N(Y \otimes \bar{k})|$  are  $Cl(Y)$  homeomorphic.*

## 2.2. Limit Mixed Hodge Structure Associated with a Variety over $\mathbb{C}((t))$

*In this section, Vadim Vologodksy authored the statements and proofs of the lemma which we developed jointly over many hours of meetings.*

In [St1], Steenbrink associated with every smooth projective variety over the field  $K_{mer}$  of meromorphic functions on an open neighborhood of zero in the complex plane a mixed Hodge structure, called the limit mixed Hodge structure. Another construction of the same mixed Hodge structure had been given earlier by Schmid, [Sch]. In this section, we explain how to extend the Steenbrink-Schmid construction to smooth projective varieties over the field of formal Laurent series  $K = \mathbb{C}((t))$ . A rough idea: generalizing a construction by Steenbrink, [St3], we attach a mixed Hodge structure to every projective normal crossing (not necessarily reduced) log scheme over the log point. Applying this construction to the special fiber  $Y$  of a

normal crossing model  $\overline{X}$  of  $X$  over  $R = \mathbb{C}[[t]]$  we get our  $H^m(\lim X)$ . We then prove independence of the choice of a model and functoriality.

We shall summarize the properties of our construction in the following theorem.

**Theorem 2.** *For every non-negative integer  $m$ , there exists a contravariant functor*

$$SmPr_K \rightarrow \widetilde{MHS}, \quad (2.4)$$

$$X \rightsquigarrow H^m(\lim X) = (H^m(\lim X, \mathbb{Z}), W_i^{\mathbb{Q}}, F_i, T)$$

from the category of smooth projective varieties over  $K = \mathbb{C}((t))$  to the category of mixed Hodge structures equipped with an endomorphism  $T$  of the underlying abelian group with the following properties.

(a) *If we write  $T_{\mathbb{Q}} = SU$  for the factorization of the endomorphism  $T_{\mathbb{Q}} \in \text{End}(H^m(\lim X, \mathbb{Q}))$  into the product of semi-simple and unipotent endomorphisms,  $S$  and  $U$  respectively, such that  $ST_{\mathbb{Q}} = T_{\mathbb{Q}}S$  and  $UT_{\mathbb{Q}} = T_{\mathbb{Q}}U$ , then  $N = \log U$  is a morphism of rational mixed Hodge structures*

$$N : H^m(\lim X) \otimes \mathbb{Q} \rightarrow H^m(\lim X) \otimes \mathbb{Q}(-1)$$

*and  $S$  is a finite order automorphism of  $H^m(\lim X) \otimes \mathbb{Q}$ .*

(b) *The functor (2.4) is compatible with base change. That is, if  $K_e = \mathbb{C}((t^{\frac{1}{e}})) \supset K$  is a finite extension and  $X \rightsquigarrow X_{K_e}$  is the base change functor, we have a functorial isomorphism*

$$(H^m(\lim X, \mathbb{Z}), W_i^{\mathbb{Q}}, F^i, T^e) \simeq (H^m(\lim X_{K_e}, \mathbb{Z}), W_i^{\mathbb{Q}}, F^i, T).$$

(c) If  $\overline{X}$  is a strictly semi-stable scheme over  $R = \mathbb{C}[[t]]$ ,  $X$  and  $Y$  are the generic and special fibers of  $\overline{X}$  respectively, and  $Y^{(q)} \hookrightarrow Y$  is the closed subscheme defined in (2.1), one has the weight spectral sequence  $E_r^{pq}(\overline{X})$  which converges to  $H^*(\lim X)$  in the category of mixed  $\mathbb{Z}$ -Hodge structures with the first term given by the formula:

$$E_1^{pq}(\overline{X}) = \bigoplus_{i, i-p \geq 0} H^{q+2p-2i}(Y^{(2i-p)})(p-i).$$

The sequence  $E_r^{pq}(\overline{X}) \otimes \mathbb{Q}$  degenerates at  $E_2$  terms.

(d) If  $\mathcal{X}$  is a smooth projective variety over  $K_{mer}$  the limit mixed Hodge structure  $H^m(\lim(\mathcal{X} \otimes_{K_{mer}} K))$  is canonically isomorphic to the one constructed by Schmid and Steenbrink ([Sch], [St1], [St2]).

*Proof.* Let

$$(\overline{X}, M_{\overline{X}}) \rightarrow (\text{spec } R, M_R = R - 0)$$

be a proper smooth morphism of fine and saturated (fs for short) log schemes ([Il2], §1). Assume that the log structure on  $(\overline{X}, M_{\overline{X}})$  is vertical *i.e.*, the induced log structure on  $j : X = \overline{X} \otimes_R K \hookrightarrow \overline{X}$  is trivial. A basic example of this situation is a regular proper  $R$ -scheme  $\overline{X}$  such that its reduced special fiber  $Y_{red}$  is a normal crossing divisor on  $\overline{X}$  endowed with the log structure

$$M_{\overline{X}} = j_* \mathcal{O}_X^* \cap \mathcal{O}_{\overline{X}}. \quad (2.5)$$

The special fiber  $Y = \overline{X} \otimes_R \mathbb{C}$  with the induced log structure is a proper smooth log scheme over the log point

$$\pi : (Y, M_Y) \rightarrow (\text{spec } \mathbb{C})_{log}.$$

Following ([KN] §1) we consider the associated map of topological spaces

$$\pi : Y^{\log} \rightarrow (\text{spec } \mathbb{C})^{\log} = S^1,$$

where  $S^1 \subset \mathbb{C}$  is the unit circle. The map  $\pi$  is a locally trivial fibration over  $S^1$  ([NO], Theorem 5.1). Let  $\exp(2\pi i\tau) : \mathbb{R}^1 \rightarrow S^1$  be the universal cover, and let  $\tilde{Y}^{\log}$  be the fiber product  $Y^{\log} \times_{S^1} \mathbb{R}^1$ . The topological space  $\tilde{Y}^{\log}$  carries a canonical automorphism that takes a point  $(y, a) \in Y^{\log} \times_{S^1} \mathbb{R}^1$  to  $(y, a + 2\pi i)$ . We will write  $T_Y$  for the induced automorphism of the cohomology group  $H^m(\tilde{Y}^{\log}, \mathbb{Z})$ . The following lemma implies that the cohomology of  $\tilde{Y}^{\log}$  depends only on the generic fiber of  $\bar{X}$ .

**Lemma 2.2.1.** *Let  $f : (\bar{X}, M_{\bar{X}}) \rightarrow (\bar{X}', M_{\bar{X}'})$  be a log morphism of smooth proper vertical fs log schemes over  $(\text{spec } R, M_R)$ . Assume that the induced morphism of the generic fibers  $f_K : \bar{X} \otimes K \rightarrow \bar{X}' \otimes K$  is an isomorphism. Then, for every non-negative integer  $m$ , the morphism*

$$f^* : H^m(\tilde{Y}^{\log}, \mathbb{Z}) \rightarrow H^m(\tilde{Y}'^{\log}, \mathbb{Z})$$

*is an isomorphism.*

*Proof.* Let  $n$  be a positive integer. The comparison theorems of Kato and Nakayama (see, e.g. [Il2], Th. 5.9, Cor. 8.4) imply the existence of the commutative diagram below.

$$\begin{array}{ccc} H^m(\tilde{Y}'^{\log}, \mathbb{Z}/n\mathbb{Z}) & \xrightarrow{\sim} & H_{\text{et}}^m(\bar{X}' \otimes \bar{K}, \mathbb{Z}/n\mathbb{Z}) \\ \downarrow f^* & & \downarrow f_K^* \\ H^m(\tilde{Y}^{\log}, \mathbb{Z}/n\mathbb{Z}) & \xrightarrow{\sim} & H_{\text{et}}^m(\bar{X} \otimes \bar{K}, \mathbb{Z}/n\mathbb{Z}) \end{array} \quad (2.6)$$

Since the groups  $H^m(\tilde{Y}^{log}, \mathbb{Z})$ ,  $H^m(\tilde{Y}^{log}, \mathbb{Z})$  are finitely generated the lemma follows.  $\square$

Let us explain how the formation  $(T_Y, H^m(\tilde{Y}^{log}, \mathbb{Z}))$  is compatible with base change. For a positive integer  $e$ , the fs log scheme  $(\text{spec } R_e = \text{spec } \mathbb{C}[[t^{\frac{1}{e}}]], M_{R_e} = R_e - 0)$  is smooth over  $(\text{spec } R, M_R)$ . Let  $(\bar{X}_e, M_{\bar{X}_e})$  be the fiber product

$$(\bar{X}, M_{\bar{X}}) \otimes_{(\text{spec } R, M_R)} (\text{spec } R_e, M_{R_e})$$

in the category of fs log schemes<sup>2</sup>. As the functor  $(Y, M_Y) \rightsquigarrow Y^{log}$  commutes with fiber products we have a Cartesian diagram of topological spaces

$$\begin{array}{ccc} Y_e^{log} & \longrightarrow & Y^{log} \\ \downarrow & & \downarrow \\ S^1 & \longrightarrow & S^1, \end{array} \tag{2.7}$$

where the lower horizontal map is an  $e$ -fold cover. We get from (2.7) a canonical isomorphism

$$H^m(\tilde{Y}^{log}, \mathbb{Z}) \xrightarrow{\sim} H^m(\tilde{Y}_e^{log}, \mathbb{Z}) \tag{2.8}$$

that carries  $T_{Y_e}$  to  $T_{\tilde{Y}_e}$ .

Assume, in addition, that the log scheme  $\pi : (Y, M_Y) \rightarrow (\text{spec } \mathbb{C})_{log}$  satisfies the following condition:

---

<sup>2</sup>Warning: the functor that takes a fs log scheme to the underlying scheme does not commute with the fiber products.

(U): for every closed point  $y \in Y$ , the cokernel of the morphism  $\pi^* : \mathbb{Z} = K^*/R^* \rightarrow (M_Y^{gr}/\mathcal{O}_Y^*)_y$  is torsion free.

In ([IKN], Theorem 6.3 and Theorem 7.1) Illusie, Kato and Nakayama proved that under the above assumption the relative log de Rham cohomology  $H^m(\overline{X}, \Omega_{\overline{X}/R}^*(log))$  is a free  $R$ -module, the residue of the logarithmic Gauss-Manin connection on  $H^m(\overline{X}, \Omega_{\overline{X}/R}^*(log))$  is nilpotent, the Hodge spectral sequence, defined by the “stupid” filtration  $\sigma_{\geq \bullet}$  on  $\Omega_{\overline{X}/R}^*(log)$ , degenerates at the  $E_1$  term and the Hodge filtration

$$H^m(\overline{X}, \sigma_{\geq j} \Omega_{\overline{X}/R}^*(log)) \hookrightarrow H^m(\overline{X}, \Omega_{\overline{X}/R}^*(log)),$$

splits (*i.e.* the associated graded  $R$ -module is free). Moreover, there is a canonical isomorphism<sup>3</sup>

$$H^m(\tilde{Y}^{log}, \mathbb{C}) \simeq H^m(Y, \Omega_{Y/\mathbb{C}}^*(log)) \tag{2.9}$$

compatible with the base change  $Y \rightsquigarrow Y_e$ . Set

$$F^j H^m(Y, \Omega_{Y/\mathbb{C}}^*(log)) := H^m(Y, \sigma_{\geq j} \Omega_{Y/\mathbb{C}}^*(log)) \hookrightarrow H^m(Y, \Omega_{Y/\mathbb{C}}^*(log)).$$

As an immediate corollary of the Illusie-Kato-Nakayama results we get the following statement.

**Lemma 2.2.2.** (a) *Let  $f : (\overline{X}, M_{\overline{X}}) \rightarrow (\overline{X}', M_{\overline{X}'})$  be a log morphism of smooth proper vertical fs log schemes over  $(\text{spec } R, M_R)$  satisfying the condition (U).*

*Assume that the induced morphism of generic fibers  $f_K : \overline{X} \otimes K \rightarrow \overline{X}' \otimes K$*

---

<sup>3</sup>The isomorphism (2.9) depends on the choice of a uniformizer of  $R$ . Our choice is  $t$ .

is an isomorphism. Then, for every non-negative integer  $m$ , the morphism

$$f^* : F^\bullet H^m(Y', \Omega_{Y'/\mathbb{C}}^*(\log)) \rightarrow F^\bullet H^m(Y, \Omega_{Y/\mathbb{C}}^*(\log))$$

is a filtered isomorphism.

- (b) For a smooth proper vertical fs log scheme  $(\bar{X}, M_{\bar{X}})$  satisfying the condition (U) and a positive integer  $e$  the canonical morphism

$$F^\bullet H^m(Y, \Omega_{Y/\mathbb{C}}^*(\log)) \rightarrow F^\bullet H^m(Y_e, \Omega_{Y_e/\mathbb{C}}^*(\log))$$

is a filtered isomorphism.

Assume, in addition, that  $\bar{X}$  is projective. Let

$$W_\bullet = W_\bullet H^m(\tilde{Y}^{\log}, \mathbb{Q}) \subset H^m(\tilde{Y}^{\log}, \mathbb{Q})$$

be the monodromy filtration defined by the nilpotent endomorphism  $N_Y = \log T_Y$  of  $H^m(\tilde{Y}^{\log}, \mathbb{Q})$ :

$$N_Y W_i \subset W_{i-2},$$

$$\text{Gr } N_Y^i : W_{i+m}^{\mathbb{Q}}/W_{i+m-1}^{\mathbb{Q}} \xrightarrow{\sim} W_{m-i}^{\mathbb{Q}}/W_{m-i-1}^{\mathbb{Q}}.$$

**Lemma 2.2.3.** *For every smooth projective vertical fs log scheme  $(\bar{X}, M_{\bar{X}})$  satisfying the condition (U) the triple  $(H^m(\tilde{Y}^{\log}, \mathbb{Z}), W_i H^m(\tilde{Y}^{\log}, \mathbb{Q}), F^j H^m(Y_{an}, \Omega_{Y_{an}/\mathbb{C}}^*(\log)))$  together with the isomorphism (2.9) constitute a mixed Hodge structure.*

*Proof.* By the semi-stable reduction theorem ([KKMS], p. 198) we can find an integer  $e$ , a projective strictly semi-stable scheme  $\overline{X}'$  over  $R_e$  and log morphism

$$f : (\overline{X}', M_{\overline{X}'}) \rightarrow (\overline{X}_e, M_{\overline{X}_e}),$$

where  $M_{\overline{X}'}$  is given by (2.5) and  $f$  is an isomorphism over the generic point of  $R_e$ . The Lemmas 2.2.1 and 2.2.2 reduce the proof to the case when  $\overline{X} = \overline{X}'$ . In this case our assertion is proven in ([KawNam], p. 405-406 and [St3], §5.6).  $\square$

To construct the functor (2.4) we define an auxiliary subcategory  $SS_R$  of the category of schemes over  $R$  whose objects are regular projective  $R$ -schemes  $\overline{X}$  such that the reduced special  $Y_{red}$  is a strict normal crossing divisor on  $\overline{X}$ . Let  $\mathcal{S} \subset Mor(SS_R)$  be the subset that consists of morphisms  $f : \overline{X} \rightarrow \overline{X}'$  such that  $f_K : \overline{X} \otimes K \simeq \overline{X}' \otimes K$ .

**Lemma 2.2.4.** *The set  $\mathcal{S}$  is a left multiplicative system in  $Mor(SS_R)$  ([KS], §7). Moreover, the functor*

$$SS_R \rightarrow SmPr_K$$

*that takes  $\overline{X}$  to  $\overline{X} \otimes K$  exhibits the category  $SmPr_K$  as the localization of  $SS_R$  by  $\mathcal{S}$ .*

*Proof.* The Lemma follows from the Hironaka theorem on resolution of singularities immediately.  $\square$

Thus, by the universal property of the localization giving a functor from the category  $SmPr_K$  to another category is equivalent to giving a functor from  $SS_R$  that takes every morphism in  $\mathcal{S}$  to an isomorphism. We define a functor  $\Psi : SS_R \rightarrow \widetilde{MHS}$  as follows. Let  $\overline{X}$  be an object of  $SS_R$ , and let  $M_{\overline{X}}$  be the canonical log structure given by the formula (2.5). For sufficiently divisible integer  $e$  the log scheme  $(Y_e, M_{Y_e})$



satisfies the property (U). We set

$$\begin{aligned} \Psi(\overline{X}) &= (H^m(\tilde{Y}^{log}, \mathbb{Z}) \\ &\simeq H^m(\tilde{Y}_e^{log}, \mathbb{Z}), W_i H^m(\tilde{Y}_e^{log}, \mathbb{Q}), F^j H^m(Y_e, \Omega_{Y_e/\mathbb{C}}^*(log)), T_Y). \end{aligned} \tag{2.10}$$

The right-hand side of (2.10) is independent of  $e$  and is naturally promoted to a contravariant functor  $\Psi : SS_R \rightarrow \widetilde{MHS}$ . By lemma 2.2.1  $\Psi$  takes every morphism in  $\mathcal{S}$  to an isomorphism. The functor (2.4) is constructed. Let us check the required properties of (2.4).

a) The only non-trivial statement is that  $S$  preserves the Hodge filtration on  $H^m(Y_e, \Omega_{Y_e/\mathbb{C}}^*(log))$ . Consider the action of the group  $\mathbb{Z}/e\mathbb{Z}$  on the log scheme  $(\overline{X}_e, M_{\overline{X}_e})$  induced by the Galois action on  $R_e$ . The restriction of this action to  $(Y_e, M_{\overline{X}_e})$  yields an action on  $H^m(Y_e, \Omega_{Y_e/\mathbb{C}}^*(log))$ . One easily checks that the action of the generator  $1 \in \mathbb{Z}/e\mathbb{Z}$  on  $H^m(Y_e, \Omega_{Y_e/\mathbb{C}}^*(log))$  equals  $S$ . The compatibility with the Hodge filtration follows immediately.

b) This follows from (2.8) and Lemma 2.2.2.

c) Denote by  $Y_{an} = Y(\mathbb{C})$  the analytic space associated with  $Y$ . Let  $\tilde{p} : \tilde{Y}^{log} = Y^{log} \times_{S^1} \mathbb{R}^1 \rightarrow Y_{an}$  be the composition of the projection to the first factor and the canonical map  $p : Y^{log} \rightarrow Y_{an}$ . The complex  $R\tilde{p}_*\mathbb{Z} \in D^b(Sh(Y_{an}))$  has a canonical automorphism  $T_Y$  induced by the automorphism of the space  $\tilde{Y}^{log}$ . In ([KawNam], p. 405-406), Kawamata and Namikawa put a weight filtration on the complex  $R\tilde{p}_*\mathbb{Q}$  and proved that this filtration yields the required spectral sequence with rational coefficients. Thus, we just need to lift the Kawamata-Namikawa filtration to  $R\tilde{p}_*\mathbb{Z}$ . The required canonical lifting is provided by the following result.

**Lemma 2.2.5.** (cf. [Sa], Prop. 2.7) Assume that  $\overline{X}$  is a strictly semi-stable scheme over  $R$  of relative dimension  $d$ .

(a) The complex  $R\tilde{p}_*\mathbb{Z}$  is a  $(-d)$ -shifted perverse sheaf on  $Y$  (i.e.,  $R\tilde{p}_*\mathbb{Z}[d]$  is a perverse sheaf). Moreover, the canonical filtration  $\tau_{\leq i}R\tilde{p}_*\mathbb{Z}$  is a filtration by  $(-d)$ -shifted perverse subsheaves and it coincides with the filtration on  $R\tilde{p}_*\mathbb{Z}$  by kernels of  $(T-1)^{i+1}$  (computed in the abelian category of  $(-d)$ -shifted perverse sheaves):

$$\tau_{\leq i}R\tilde{p}_*\mathbb{Z} = \text{Ker} \left( (T-1)^{i+1} : R\tilde{p}_*\mathbb{Z} \rightarrow R\tilde{p}_*\mathbb{Z} \right).$$

In particular,  $(T-1)^{d+1}$  is 0 on  $R\tilde{p}_*\mathbb{Z}$ .

(b) Let  $0 \subset W_{-d}R\tilde{p}_*\mathbb{Z} \subset \cdots \subset W_iR\tilde{p}_*\mathbb{Z} \subset W_dR\tilde{p}_*\mathbb{Z} = R\tilde{p}_*\mathbb{Z}$  be the monodromy filtration on  $R\tilde{p}_*\mathbb{Z}$  viewed as an object of the abelian category of  $(-d)$ -shifted perverse sheaves equipped with the nilpotent endomorphism  $T-1$ . Then, for every integer  $r$ , we have an isomorphism

$$\text{Gr}_r^W R\tilde{p}_*\mathbb{Z} \simeq \bigoplus_{\substack{i-j=r \\ i, j \geq 0}} \wedge^{i+j+1} (M_{Y_{an}}^{gr} / \mathcal{O}_{Y_{an}}^*) \simeq \bigoplus_{i-j=r} a_{i+j*} \mathbb{Z}[-i-j],$$

where  $a_q$  denotes the embedding  $Y^{(q)} \hookrightarrow Y$ . The first isomorphism is canonical, the second one depends on the order of the set of irreducible components of  $Y$ .

(c) The Verdier dual complex  $\mathcal{D}_Y(R\tilde{p}_*\mathbb{Z})$  is quasi-isomorphic to  $R\tilde{p}_*\mathbb{Z}[2 \dim X]$ .

*Proof.* For the first statement it suffices to prove that, for every prime number  $\ell$ , the complex  $\mathbb{Z}_\ell \otimes R\tilde{p}_*\mathbb{Z} = R\tilde{p}_*\mathbb{Z}_\ell$  has the corresponding properties. According to the comparison results of Kato and Nakayama (see, e.g. [Il2], Th. 5.9, Cor. 8.4) the complex  $R\tilde{p}_*\mathbb{Z}_\ell$  is quasi-isomorphic to the complex of nearby cycles  $R\Psi\mathbb{Z}_\ell$  computed

using the étale topology. The results for  $R\Psi\mathbb{Z}_\ell$  are proven in ([Sa], Lemma 2.5 and Cor. 2.6). The proof of the second statement is parallel to the proof of the analogous result for  $R\Psi\mathbb{Z}_\ell$  ([Sa], Prop. 2.7). For the last statement of the Lemma observe that  $R\tilde{p}_*\mathbb{Z}$  is quasi-isomorphic to  $Rp_{1*}\mathbb{Z}$ , where  $p_1 : Y_1^{log} \rightarrow Y_{an}$  is the restriction of the map  $Y^{log} \rightarrow Y_{an} \times S^1$  to the fiber over  $Y_{an} \times \{1\}$ . As the map  $p_1$  is proper, we have

$$\mathcal{D}_Y(Rp_{1*}\mathbb{Z}) \simeq Rp_{1*}\mathcal{D}_{Y_1^{log}}\mathbb{Z}.$$

Finally, a simple local computation shows that  $\mathcal{D}_{Y_1^{log}}\mathbb{Z} \simeq \mathbb{Z}[2 \dim X]$ . □

d) The last assertion of Theorem 2 is proven in ([IKN] Theorem 8.3, [St2], Appendix). The proof of Theorem 2 is now completed.

□

**Remark 2.2.6.** *We expect that the functor (2.4) extends to the category of smooth quasi-compact rigid analytic varieties over  $\mathbb{C}((t))$  (cf. [A2]).*

We finish this subsection by recalling a variant of the Picard-Lefschetz formula for semi-stable degenerations. Let  $\bar{X}$  be a projective strictly semi-stable scheme over  $R$  of relative dimension  $d$ , and let  $Y$  be its special fiber. The simplicial complex that computes the homology of the Clemens polytope  $Cl(Y)$  coincides with the complex

$$E_1^{-d,2d}(\bar{X})(d) \rightarrow E_1^{-d+1,2d}(\bar{X})(d) \rightarrow \dots \rightarrow E_1^{0,2d}(\bar{X})(d),$$

where  $E_r^{pq}(\bar{X})$  is the weight spectral sequence from Theorem 2. From this we get a canonical morphism

$$H^m(\lim X, \mathbb{Z}) \rightarrow E_2^{-d+m,2d}(\bar{X})(d) \simeq H_m(Cl(Y)) \tag{2.11}$$

As the weight spectral sequence degenerates rationally in  $E_2$  terms (2.11) yields an isomorphism

$$H_m(Cl(Y)) \otimes \mathbb{Q} \xrightarrow{\simeq} Gr_{2d}^{W^{\mathbb{Q}}} H^m(\lim X)(d). \quad (2.12)$$

We apply this to  $m = d$ . If

$$\langle, \rangle: W_0^{\mathbb{Q}} H^d(\lim X) \otimes Gr_{2d}^{W^{\mathbb{Q}}} H^d(\lim X)(d) \rightarrow \mathbb{Q}$$

denotes the pairing induced by Poincaré duality (Lemma 2.2.5, c)) then, for every

$$x = \sum_{v \in \pi_0(Y^{(d)})} a_v v, \quad y = \sum_{v \in \pi_0(Y^{(d)})} b_v v \in H_d(Cl(Y)) \otimes \mathbb{Q},$$

we have

$$(-1)^{\frac{d(d-1)}{2}} \langle Gr N^d \gamma(x), \gamma(y) \rangle = \sum_v a_v b_v. \quad (2.13)$$

This follows from compatibility of the weight spectral sequence with Poincaré duality and the monodromy action ([Sa], Cor. 2.6 and Prop. 2.15).

### 2.3. Motivic Serre Invariant

Let  $R$  be a complete discrete valuation ring with perfect residue field  $k$  and fraction field  $K$ . The motivic Serre invariant of a smooth proper variety  $X$  over  $K$  is the class of the special fiber  $V^0$  of a weak Néron model  $\mathcal{V}$  of  $X$  in the quotient ring

$$K_0(Var_k)_{loc} \rightarrow K_0(Var_k)_{loc} / (\mathbb{Z}(1) - \mathbb{Z}).$$

It is shown in ([LS], Theorem 4.5.1) that the motivic Serre invariant  $S(X)$  is well defined *i.e.*, independent of the choice of  $\mathcal{V}$ . If  $X$  is a Calabi-Yau variety  $S(X)$  equals the image of the motivic integral  $\int_X$  in the quotient ring.

Let  $K = \mathbb{C}((t))$ . In the following Proposition, which is a refinement of A'Campo's formula for the Euler characteristic of the motivic integral<sup>4</sup>, we denote by  $S^H(X)$  the image of  $S(X)$  under the ring homomorphism

$$K_0(\text{Var}_{\mathbb{C}})_{\text{loc}}/(\mathbb{Z}(1) - \mathbb{Z}) \rightarrow K_0(\text{MHS})/(\mathbb{Z}(1) - \mathbb{Z}) \quad (2.14)$$

that takes the class of a variety  $Z$  to the virtual mixed Hodge structure  $\sum(-1)^i[H_c^i(Z, \mathbb{Z})]$ .

**Proposition 2.3.1.** *Let  $X$  be a smooth projective variety over  $\mathbb{C}((t))$ . Assume that  $X$  has a projective strictly semi-stable model  $\bar{X}$  over  $\mathbb{C}[[t]]$ . Then  $S^H(X)$  is equal to the class of  $\sum(-1)^i[H^i(\lim X)]$ .*

*Proof.* We start with the following general (and well known) observation.

**Lemma 2.3.2.** *Let  $R$  be a complete discrete valuation ring with perfect residue field  $k$  and fraction field  $K$ , and let  $\bar{X}$  be a proper flat scheme over  $R$ . Assume that  $\bar{X}$  is regular and that the generic fiber  $X = \bar{X} \otimes_R K$  is smooth over  $K$ . Then the smooth locus  $\bar{X}_{sm}$  of the morphism  $\bar{X} \rightarrow \text{spec } R$  is a weak Néron model of  $X$ .*

*Proof.* Since  $X$  is smooth we have that  $\bar{X}_{sm} \otimes_R K = X$ . Let  $R' \supset R$  be a finite unramified extension with fraction field  $K'$ . We need to show that every morphism  $x : \text{spec } K' \rightarrow X$  extends to an  $R'$ -morphism  $\bar{x} : \text{spec } R' \rightarrow \bar{X}_{sm}$ . As  $\bar{X}$  is proper over  $R$ ,  $x$  extends to an  $R'$ -morphism  $\bar{x} : \text{spec } R' \rightarrow \bar{X}$ . We claim that  $\bar{x}$  takes the

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<sup>4</sup>Related results were obtained by Nicaise [Ni].

closed point of  $\text{spec } R'$  to a smooth point,  $y$ , of the special fiber  $Y = \overline{X} \otimes_R k$ . Since  $k$  is perfect, it suffices to check that  $y$  is a regular point of  $Y$  ([SGA1], II, Cor. 5.3). Indeed, let  $\mathcal{O}_{\overline{X},y}$  (resp.  $\mathcal{O}_{Y,y}$ ) be the local ring of  $\overline{X}$  (resp.  $Y$ ) at  $y$  and let  $m_{\overline{X},y} \subset \mathcal{O}_{\overline{X},y}$  (resp.  $m_{Y,y} \subset \mathcal{O}_{Y,y}$ ) be the maximal ideal. We have a surjective morphism

$$m_{\overline{X},y}/m_{\overline{X},y}^2 \twoheadrightarrow m_{Y,y}/m_{Y,y}^2 \quad (2.15)$$

of finite-dimensional vector spaces over  $\mathcal{O}_{\overline{X},y}/m_{\overline{X},y}$ . Let us show that the image in  $m_{\overline{X},y}/m_{\overline{X},y}^2$  of a uniformizer  $t \in R$  is not equal to 0. Indeed, we have a morphism  $\mathcal{O}_{\overline{X},y} \xrightarrow{\overline{x}^*} R'$  induced by  $\overline{x}$  such that the composition  $R \rightarrow \mathcal{O}_{\overline{X},y} \xrightarrow{\overline{x}^*} R'$  is the identity morphism. Since  $K'$  is unramified over  $K$ ,  $t$  is also a uniformizer for  $R'$ . Therefore,  $t$  does not belong to  $m_{\overline{X},y}^2$ . We proved that the image of  $t$  in  $m_{\overline{X},y}/m_{\overline{X},y}^2$  is not 0. On the other hand, its image in  $m_{Y,y}/m_{Y,y}^2$  is 0. Hence, morphism (2.15) is not injective and, therefore,

$$\dim m_{\overline{X},y}/m_{\overline{X},y}^2 > \dim m_{Y,y}/m_{Y,y}^2.$$

On the other hand, since  $\overline{X}$  is regular, we have that  $\dim m_{\overline{X},y}/m_{\overline{X},y}^2$  equals the Krull dimension of  $\mathcal{O}_{\overline{X},y}$ . Thus,  $\dim m_{Y,y}/m_{Y,y}^2 \leq \dim \mathcal{O}_{\overline{X},y} - 1 = \dim \mathcal{O}_{Y,y}$ . Hence,  $Y$  is regular and, therefore, smooth at point  $y$ . It follows that the map  $\overline{x} : \text{spec } R' \rightarrow \overline{X}$  factors through  $\overline{X}_{sm} \subset \overline{X}$ .  $\square$

We now come back to the proof of Proposition 2.3.1. According to the above lemma the smooth locus  $\mathcal{V}$  of  $\overline{X}$  is a weak Néron model of  $X$ . Using notation of (2.1) and the inclusion-exclusion formula we find

$$[V^0] = \sum_{j=0}^{\dim X} ((-1)^j (j+1) [Y^{(j)}]).$$

On the other hand, by part c) of the Theorem 2 the class  $\sum(-1)^i[H^i(\lim X)]$  is equal to the image under (2.14) of the class

$$[\lim X] = \sum_{j=0}^{\dim X} \left( (-1)^j [Y^{(j)}] \sum_{a=0}^j \mathbb{Z}(-a) \right).$$

Comparing the two formulas we complete the proof of Proposition 2.3.1. □

Let  $\chi : K_0(\text{Var}_{\mathbb{C}}) \rightarrow \mathbb{Z}$  be the ring homomorphism defined by

$$\chi([Z]) = \sum (-1)^i \dim H_c^i(Z, \mathbb{C}).$$

Notice that since  $\chi(\mathbb{Z}(1) - \mathbb{Z}) = 0$ ,  $\chi$  factors uniquely through  $K_0(\text{Var}_{\mathbb{C}})_{loc}/(\mathbb{Z}(1) - \mathbb{Z})$ .

We have the following corollary of Proposition 2.3.1.

**Corollary 2.3.3** (cf. A'Campo ([AC])). *Let  $X$  be a smooth projective variety over  $K = \mathbb{C}((t))$ . Assume that  $X$  has a projective strictly semi-stable model  $\bar{X}$  over  $\mathbb{C}[[t]]$ .*

*Then*

$$\chi(S(X)) = \sum (-1)^i \dim H^i(\lim X, \mathbb{C}).$$

In the rest of this subsection, we explain an analogue of the above Proposition for the finite residue field case. Let  $K$  be a local field with residue field  $k = \mathbb{F}_q$ , and let

$$K_0(\text{Var}_{\mathbb{F}_q})_{loc}/(\mathbb{Z}(1) - \mathbb{Z}) \rightarrow \mathbb{Z}/(q-1),$$

be the homomorphism induced by (1.6). The image of  $S(X)$  in  $\mathbb{Z}/(q-1)$  is the classical Serre invariant which we denote by  $S^q(X)$ .

**Proposition 2.3.4.** *Let  $X$  be a smooth proper variety over  $K$ . Assume that  $X$  has a proper strictly semi-stable model over the ring of integers  $R$ . Then the Serre invariant*

of  $X$  is given by the formula

$$\sum_j \text{Tr}(F^{-1}, H^j(X_{\bar{K}}, \mathbb{Q}_\ell)) \quad (2.16)$$

where  $F \in \text{Gal}(\bar{K}/K)$  is a lifting of the Frobenius automorphism  $Fr \in \text{Gal}(\bar{k}/k)$  and  $\ell$  is a prime number different from the characteristic of  $k$ .

*Proof.* This can be proved as its Hodge analogue above using the  $\ell$ -adic weight spectral sequence. We give a different proof. Let  $\bar{X}$  be a strictly semi-stable model of  $X$ . Then the Serre invariant of  $X$  equals  $|Y_{sm}(k)|$  modulo  $(q-1)$ . On the other hand, if  $\Psi(\mathbb{Q}_\ell)$  is the complex of nearby cycles (viewed as a complex of  $\ell$ -adic sheaves on  $Y$ ), by the Grothendieck-Lefschetz formula we have

$$\begin{aligned} \sum_j (-1)^j \text{Tr}(F^{-1}, H^j(X_{\bar{K}}, \mathbb{Q}_\ell)) &= \sum_j (-1)^j \text{Tr}(F^{-1}, H^j(Y_{\bar{k}}, \Psi(\mathbb{Q}_\ell))) = \\ &= \sum_{y \in Y(k)} \sum_i (-1)^i \text{Tr}(F^{-1}, \mathcal{H}^i(\Psi(\mathbb{Q}_\ell))_y). \end{aligned} \quad (2.17)$$

If  $y \in Y_{sm}(k)$ , the corresponding internal sum equals 1. If  $y \in Y_{sing}(k)$  then  $\mathcal{H}^i(\Psi(\mathbb{Q}_\ell))_y \simeq \bigwedge^i T(-i)$ , where  $T$  is a vector space with the trivial action of  $\text{Gal}(\bar{K}/K)$  ([SGA7], Exposé I, Th. 3.3). Thus, for  $y \in Y_{sing}(k)$ , we have

$$\sum_i (-1)^i \text{Tr}(F^{-1}, \mathcal{H}^i(\Psi(\mathbb{Q}_\ell))_y) \equiv \sum_i (-1)^i \dim \bigwedge^i T \equiv 0 \pmod{(q-1)}.$$

It follows that the right-hand side of (2.17) is equal to  $|Y_{sm}(k)|$  modulo  $(q-1)$  which is the Serre invariant of  $X$ .  $\square$



## CHAPTER III

### MOTIVIC INTEGRAL OF K3 SURFACES OVER $\mathbb{C}((T))$

In this chapter we will prove Theorem 1 stated in the introduction. Without loss of generality we may assume that the ramification index  $e$  is equal to 1. Indeed, by Theorem 2 part (b), the formulas (1.9) and (1.10) for the pair  $(X/K, e)$  are equivalent to those for the pair  $(X_{K_e}/K_e, 1)$ . If  $X$  admits a strictly semi-stable model over  $R$  then  $X_{K_e}$  admits a strictly semi-stable model over  $R_e$  ([Sa], Lemma 1.11). We will write  $r_i$  for  $r_i(X, K)$ .

#### 3.1. Approximation of Varieties over the Formal Disk

We will need the following version of Artin's Approximation Theorem.

**Proposition 3.1.1.** *Let  $k$  be a field of characteristic 0, and let  $\bar{X}$  be a projective strictly semi-stable scheme over  $R = k[[t]]$ . For every positive integer  $n$  there exist*

- (1) *a smooth curve  $\bar{C}$  over  $k$  with a point  $a \in \bar{C}(k)$ ,*
- (2) *an étale morphism  $h : \bar{C} \rightarrow \mathbb{A}_k^1 = \text{spec } k[t]$  that carries  $a$  to 0,*
- (3) *a flat projective scheme  $\bar{\mathcal{X}}$  over  $\bar{C}$ ,*
- (4) *an isomorphism of schemes over  $R_n = \text{spec } k[t]/t^{n+1}$ :*

$$\bar{X} \times_{\text{spec } R} \text{spec } R_n \simeq \bar{\mathcal{X}} \times_{\bar{C}} \text{spec } R_n.$$

Here  $\text{spec } R_n$  is viewed as a scheme over  $\overline{C}$  via the unique morphism  $\tilde{i}_n : \text{spec } R_n \rightarrow \overline{C}$  that carries the point 0 to  $a$  and makes the following diagram commutative

$$\begin{array}{ccc} & & \overline{C} \\ & \nearrow \tilde{i}_n & \downarrow h \\ \text{spec } R_n & \xrightarrow{i_n} & \text{spec } k[t] \end{array}$$

If  $\overline{C}, a, h, \overline{\mathcal{X}}$  are as above, the scheme  $\overline{\mathcal{X}}$  is regular in an open neighborhood of its special fiber  $Y'$  and  $Y'$  is a reduced divisor on  $\overline{\mathcal{X}}$  with strict normal crossings. In addition, if  $X$  is a  $d$ -dimensional Calabi-Yau variety the collection  $\overline{C}, a, h, \overline{\mathcal{X}}$  can be chosen so that the line bundle  $\Omega_{\mathcal{X}/C}^d$  is trivial and

$$\int_X = \int_{\mathcal{X} \times_C \text{spec } K'} . \quad (3.1)$$

Here we set  $C = \overline{C} - a$ ,  $\mathcal{X} = \overline{\mathcal{X}} \times_{\overline{C}} C$ , and  $K'$  denotes the fraction field of the completed local ring  $R' = \hat{\mathcal{O}}_{\overline{C}, a}$ .

*Proof.* Choose an embedding  $\overline{\mathcal{X}} \hookrightarrow \mathbb{P}_R^n$  and let  $\nu : \text{spec } R \rightarrow \text{Hilb}(\mathbb{P}_R^n)$  be the corresponding morphism to the Hilbert scheme. Using Artin's Formal Approximation Theorem (see *e.g.*, [BLR], §3.6) on the morphism  $\nu$  we obtain (1)-(4). Next, we claim that the scheme  $\overline{\mathcal{X}}' = \overline{\mathcal{X}} \times_{\overline{C}} \text{spec } R'$  is regular. As  $\overline{\mathcal{X}}'$  is proper over  $R'$  and the set of its regular points is open ([EGA] IV, 6.12.5) it suffices to show that the local ring of any point of the special fiber  $Y'$  is regular which in turn follows from property (4) and the regularity of  $\overline{\mathcal{X}}$ . Moreover,  $Y'$  being isomorphic to the special fiber of a strictly semi-stable scheme  $\overline{\mathcal{X}}$  is a strict normal crossing divisor on  $\overline{\mathcal{X}}'$  and on  $\overline{\mathcal{X}}$ . Note that under our assumption that  $\text{char } k = 0$  this implies strict semi-stability of  $\overline{\mathcal{X}}'$ .

Suppose that  $X$  is a Calabi-Yau variety. Then the divisor of any nonzero relative log form  $\omega \in H^0(\overline{X}, \Omega_{\overline{X}/R}^d(\log))$  is supported on the special fiber  $Y$  of  $\overline{X}$ . Write  $\text{div}(\omega) = \sum_i n_i [V_i]$ , where  $V_i$  are the irreducible components of  $Y$ . Assume that the quadruple  $\overline{C}, a, h, \overline{X}$  satisfies properties (1)-(4) with  $n \geq \sum_i n_i$ . To prove the last assertion of the proposition, formula (3.1), we will show that there exists a section  $\omega' \in H^0(\overline{X}', \Omega_{\overline{X}'/R'}^d(\log))$  whose divisor is supported on the special fiber  $Y'$  of  $\overline{X}'$  and such that via the isomorphism  $Y \simeq Y'$  from (4)

$$\text{div}(\omega) = \text{div}(\omega'). \quad (3.2)$$

Indeed, by Lemma 4.1 from [KawNam], for every proper strictly semi-stable scheme  $\overline{X}$  over  $R$  the  $R$ -module  $H^0(\overline{X}, \Omega_{\overline{X}/R}^d(\log))$  is free and, in addition, we have

$$H^0(\overline{X}, \Omega_{\overline{X}/R}^d(\log)) \otimes_R R_n \xrightarrow{\sim} H^0(\overline{X} \otimes R_n, \Omega_{\overline{X} \otimes R_n / R_n}^d(\log)).$$

Applying this result to  $\overline{X}$  and  $\overline{X}'$  we find that  $H^0(\overline{X}, \Omega_{\overline{X}/R}^d(\log))$  and  $H^0(\overline{X}', \Omega_{\overline{X}'/R'}^d(\log))$  are free modules of rank 1 over  $R$  and  $R'$  respectively and that (4) induces an isomorphism

$$\theta : H^0(\overline{X}, \Omega_{\overline{X}/R}^d(\log)) \otimes_R R_n \xrightarrow{\sim} H^0(\overline{X}', \Omega_{\overline{X}'/R'}^d(\log)) \otimes_R R_n.$$

(The  $R$ -action on  $H^0(\overline{X}', \Omega_{\overline{X}'/R'}^d(\log))$  comes via the isomorphism  $R \xrightarrow{\sim} R'$  induced by  $h$ .) We claim that a section  $\omega' \in H^0(\overline{X}', \Omega_{\overline{X}'/R'}^d(\log))$  such that  $\theta(\omega \otimes 1) = \omega' \otimes 1$  does the job. Our claim is local: it suffices to show that, for a closed point  $b \in \overline{X}$  and local regular functions  $f, g \in \mathcal{O}_{\overline{X}, b}$  such that  $\text{div}(f)$  is supported on  $Y$ ,  $\sum_i \text{ord}_{V_i} f \leq n$ , and  $f - g \in (t^{n+1})$ , one has  $\text{div}(f) = \text{div}(g)$ . Let  $x_i$  be a system of local parameters

at  $b$  such that  $t = x_1 \cdots x_m$ . Then, locally around  $b$ , we have  $f = x_1^{n_1} \cdots x_m^{n_m} u$ , where  $u$  is invertible and  $\sum_i n_i \leq n$ . If  $n_1 > 0$ ,  $g \in f + (t^{n+1})$  is divisible by  $x_1$  and  $\frac{f}{x_1} - \frac{g}{x_1} \in (t^n)$ . Arguing by induction we see that  $g$  is divisible by  $x_1^{n_1} \cdots x_m^{n_m}$  and

$$\frac{f}{x_1^{n_1} \cdots x_m^{n_m}} - \frac{g}{x_1^{n_1} \cdots x_m^{n_m}} \in (t).$$

In particular,  $g = x_1^{n_1} \cdots x_m^{n_m} u'$  for some invertible  $u'$ .

To complete the proof of the proposition let us explain how (3.2) implies (3.1). Suppose that the pair  $\overline{\mathcal{X}}, \omega' \in H^0(\overline{\mathcal{X}}', \Omega_{\overline{\mathcal{X}}'/R'}^d(\log))$  is chosen such that that equation (3.2) holds. Then, in particular,  $\omega'$  restricts to a non-vanishing differential form on  $X'$ . Thus,  $X'$  is a Calabi-Yau variety. Secondly, by Lemma 2.3.2 the schemes  $\overline{X}_{sm}$  and  $\overline{X}'_{sm}$  are weak Néron models of  $X$  and  $X'$  respectively. Moreover, by property (4) and (3.2) there exists an isomorphism between the special fibers of  $\overline{X}_{sm}$  and  $\overline{X}'_{sm}$  that carries  $\text{div}(\omega)$  to  $\text{div}(\omega')$ . Using (1.5) formula (3.1) follows.  $\square$

### 3.2. Kulikov Model

It is enough to prove Theorem 1 in the case where  $X$  is the restriction of a strictly semi-stable family over a complex curve.

Indeed, apply Proposition 3.1.1 to a strictly semi-stable model  $\overline{X}$  of  $X$ . As the limit mixed Hodge structure of a strictly semi-stable scheme depends only on its special fiber together with its log structure which, in turn, is determined by its first infinitesimal neighborhood  $\overline{X} \otimes_R R/t^2$ , the formulas (1.9), (1.10) for  $X$  are equivalent to those for  $\mathcal{X} \times_C \text{spec } K'$ .

Let  $X$  be a K3 surface over  $K$ , which is the restriction of a strictly semi-stable family over a complex curve. In ([Ku], Theorem 2), Kulikov demonstrated the following,

**Theorem 3** (Kulikov). *Let  $\overline{\mathcal{X}} \rightarrow \overline{C}$  be a projective strictly semi-stable morphism whose restriction  $\mathcal{X}$  to  $C$  is a smooth family of K3 surfaces. Then there is a proper strictly semi-stable complex analytic space over  $\overline{C}$ ,  $\overline{\mathcal{X}}' \rightarrow \overline{C}$ , together with a bimeromorphic map  $\overline{\mathcal{X}}' \dashrightarrow \overline{\mathcal{X}}$ , which commutes with the projections to  $\overline{C}$  and induces an isomorphism  $\overline{\mathcal{X}}' \times_{\overline{C}} C \xrightarrow{\sim} \mathcal{X} \times_{\overline{C}} C$ , such that the log canonical bundle  $\Omega_{\overline{\mathcal{X}}'/\overline{C}}^2(\log)$  is trivial over a neighborhood of the special fiber of  $\overline{\mathcal{X}}'$ .*

One refers to  $\overline{\mathcal{X}}'$  as a Kulikov model for  $\mathcal{X} \rightarrow C$ . It is shown in [Ku] that the special fiber,  $Y'$ , of any Kulikov model has a very special form of one of the following types (depending on the number  $s$  defined in Theorem 1)

- (I) ( $s = 1$ )  $Y'$  is a smooth K3 surface
- (II) ( $s = 2$ )  $Y'$  is a chain of smooth surfaces  $V_0, \dots, V_m$  ruled by elliptic curves, with smooth rational surfaces on either end and each double curve  $V_i \cap V_{i+1}$  is a smooth elliptic curve.
- (III) ( $s = 3$ )  $Y'$  is a union of smooth rational surfaces whose pairwise intersections are smooth rational curves and the Clemens polytope of  $Y$  is a triangulation of  $S^2$ .

In addition, for  $s = 2$ , Friedman showed in ([Fr], Theorem 2.2) that a Kulikov model can be chosen so that all the ruled elliptic surfaces in  $Y'$  are minimal *i.e.*,  $\mathbb{P}^1$ -fibrations over an elliptic curve. We shall call such model *special*.

We do not know whether, for given  $\mathcal{X} \rightarrow C$ , there exists a Kulikov model which is a scheme. However, we will show that Theorem 1 and Proposition 4.8 can still be

proved using Kulikov's Theorem. In fact, the only part of the argument in the proof of Theorem 1 where we need the algebraicity of Kulikov's model  $\overline{\mathcal{X}}'$  is the proof of formula 3.3 stated below.

**Lemma 3.2.1.** *If  $\overline{\mathcal{X}}'$  is a Kulikov model and  $Y'$  is its special fiber, we have*

$$\int_X = [Y'_{sm}]. \quad (3.3)$$

Here  $X$  is the K3 surface over  $K$  obtained from  $\mathcal{X}$  by the base change.

Note that since the irreducible components of  $Y'$  are projective, the smooth locus of  $Y'_{sm}$  has the structure of an algebraic variety and, hence,  $[Y'_{sm}]$  makes sense as an element of the Grothendieck group of varieties.

*Proof of Lemma 3.2.1.* Let  $\overline{\mathcal{X}}' \rightarrow \overline{C}$  be any proper complex analytic model for  $\mathcal{X} \rightarrow C$ , which is bimeromorphically isomorphic to an algebraic model  $\overline{\mathcal{X}} \rightarrow \overline{C}$ . Then, according to a theorem of Artin ([Ar], Theorem 7.3),  $\overline{\mathcal{X}}'$  has a unique structure of an algebraic space such that the map  $\overline{\mathcal{X}}' \rightarrow \overline{C}$  is algebraic. Since any smooth algebraic surface is quasi-projective, the smooth locus  $Y'_{sm}$  of the special fiber of  $\overline{\mathcal{X}}' \rightarrow \overline{C}$  acquires the structure of an algebraic variety. Now, assume that  $\overline{\mathcal{X}}'$  is regular and let  $V_i^\circ$  be irreducible components of  $Y'_{sm}$ . Define integers  $m_i$  by the formula 1.4. The lemma will follow from a more general claim:

$$\int_X := \sum_i [V_i^\circ] (m_i - \min_i m_i). \quad (3.4)$$

To prove (3.4) observe, first, that the formula is true if  $\overline{\mathcal{X}}'$  is a scheme. Indeed, in this case  $\overline{\mathcal{X}}' \otimes R - Y'_{sing}$  is a weak Néron model ( Lemma 2.3.2), and (3.4) boils down to the definition of motivic integral. Hence, it suffices to check that the right-hand

side of (3.4) is independent of the choice of a regular model  $\overline{\mathcal{X}}'$ . Using the Weak Factorization Theorem for algebraic spaces ([AKMW], Theorem 0.3.1) it is enough to show this for two models, one of which is obtained from the other one by blowing up at a smooth subvariety of the special fiber. In this case the assertion follows by a direct inspection.  $\square$

Next, for the the proof of Proposition 4.3.1. It suffices to prove the following result.

**Lemma 3.2.2.** *With the assumption of Kulikov's Theorem,  $X := \mathcal{X} \times_C \text{spec } K$ , there is a homotopy equivalence between the topological space  $|X_{\overline{K}}^{an}|$  and the Clemens polytope  $Cl(Y')$  of the special fiber of  $\overline{\mathcal{X}}'$ , which identifies the canonical map*

$$\Gamma_{\mathbb{Z}}^*(X) \rightarrow H^*(\lim X, \mathbb{Z})$$

with the map

$$H^*(Cl(Y')) \rightarrow H^*(\lim X, \mathbb{Z})$$

coming from the weight spectral sequence.

*Proof.* Berkovich's result ([Ber3], §5) applied to the strictly semi-stable model  $\overline{\mathcal{X}}$  (which is a scheme) implies the assertion of the lemma with  $Y'$  replaced by  $Y$ . Next, let  $\overline{\mathcal{X}} \rightarrow \overline{C}$  and  $\overline{\mathcal{X}}' \rightarrow \overline{C}$  be any proper regular models for  $\mathcal{X} \rightarrow C$  in the category of algebraic spaces. Assume that the *reduced* special fibers  $Y_{red}$  and  $Y'_{red}$  are strict normal crossing divisors. We claim that there is a homotopy equivalence between the Clemens polytopes  $Cl(Y)$  and  $Cl(Y')$ , which identifies the maps  $H^*(Cl(Y)) \rightarrow H^*(\lim X, \mathbb{Z}), H^*(Cl(Y')) \rightarrow H^*(\lim X, \mathbb{Z})$ . Indeed, using the Weak Factorization Theorem ([AKMW], Theorem 0.3.1) we may assume that  $\overline{\mathcal{X}}$  is obtained from  $\overline{\mathcal{X}}'$  by

blowing up at an admissible subvariety of the special fiber. In this case the assertion can be checked directly (see [S], Lemma in §2).  $\square$

### 3.3. Type II Degeneration

Suppose that  $\bar{X}$  is a type II special Kulikov model. Let  $V_0, \dots, V_m$  be the irreducible components of  $Y$  such that  $V_0$  and  $V_m$  are rational surfaces, and let  $C_i = V_i \cap V_{i+1}$  be the double curves.

**Lemma 3.3.1.** (1) *Let  $E_1, \dots, E_{m-1}$  be ruling elliptic curves for  $V_1, \dots, V_{m-1}$ . Then*

$$C_i \cong E_i \cong E_j \cong C_j \text{ for all } i \text{ and } j.$$

(2) *At least one of the rational components,  $V_0$  or  $V_m$ , is not minimal.*

*Proof.* (1). We will first prove that  $E_1 \cong E_2$ . Let  $C_1$  and  $C_2$  be elliptic curves given by the intersection  $V_1 \cap V_2$  and  $V_2 \cap V_3$  respectively. We have the following diagram

$$\begin{array}{ccccc} C_1 & \hookrightarrow & V_2 & \hookleftarrow & C_2 \\ & \searrow & \downarrow h & \swarrow & \\ & f_1 & E_2 & f_2 & \end{array}$$

Notice that the maps  $f_1$  and  $f_2$  cannot be constant since this would imply the existence of injections of  $C_1$  and  $C_2$  into rational curves. Thus  $f_1$  and  $f_2$  must be finite. The triviality of the log canonical bundle  $\Omega_Y^2(\log)$  implies that for the canonical class  $K_{V_2}$  we have  $K_{V_2} = -[C_1] - [C_2]$ . On the other hand, the restriction of  $K_{V_2}$  to a smooth fiber,  $h^{-1}(a)$ , of the map  $h : V_2 \rightarrow E_2$  is isomorphic to  $K_{h^{-1}(a)}$ . As  $h^{-1}(a)$  is a smooth rational curve, we have that  $\deg(K_{V_2}|_{h^{-1}(a)}) = -2$  which implies the degree of the divisor  $-[C_1] - [C_2]$  intersected with the fiber  $h^{-1}(a)$  is  $-2$ . Hence the images of  $C_1$  and  $C_2$  in  $V_2$  have only one intersection point with a generic fiber which implies  $f_1$



and  $f_2$  are one-to-one and  $C_1 \cong E_2 \cong C_2$ . We then apply the same method of proof to show that  $C_2 \cong E_3 \cong C_3$  and so on.

(2). We claim that for a minimal ruled elliptic surface  $V_i$  and two disjoint sections  $C_{i-1}, C_i \subset V_i$ , we have

$$([C_{i-1}])_{V_i}^2 = -([C_i])_{V_i}^2.$$

Indeed, the Néron-Severi group of  $V_i$  is generated by the class  $[C_i]$  of  $C_i$  and the class  $[\mathbb{P}^1]$  of a smooth fiber of the map  $V_i \rightarrow E_i$ . If  $[C_{i-1}] = [C_i] + c[\mathbb{P}^1]$ , we have

$$0 = (c[\mathbb{P}^1])_{V_i}^2 = ([C_{i-1}] - [C_i])_{V_i}^2 = ([C_{i-1}])_{V_i}^2 + ([C_i])_{V_i}^2.$$

On the other hand, since  $Y$  is the special fiber of a semi-stable degeneration, we have for every  $i$

$$([C_i])_{V_i}^2 = -([C_i])_{V_{i+1}}^2.$$

Combining the two formulas we see that  $([C_0])_{V_0}^2 = -([C_{m-1}])_{V_m}^2$ . In particular, at least for one of the rational components, say  $V_0$ , the self-intersection of the double curve lying on it is non-positive. Thus,  $(K_{V_0})_{V_0}^2 = (-[C_0])_{V_0}^2 \leq 0$ . Using Noether's formula ([Bea] I.14) it follows that  $V_0$  is not minimal.  $\square$

Let  $E$  be an elliptic curve such that  $E \cong C_i$  for all  $i$ . Then we get from (3.3)

$$\int_X = \sum_{i=0}^m [V_i] - 2 \sum_{i=0}^{m-1} [C_i] = \sum_{i=0}^m [V_i] - 2m[E].$$

Since  $V_0$  and  $V_m$  are both rational surfaces we have  $[V_0] = \mathbb{Z} + a_0\mathbb{Z}(-1) + \mathbb{Z}(-2)$  and  $[V_m] = \mathbb{Z} + a_m\mathbb{Z}(-1) + \mathbb{Z}(-2)$ . Each  $V_i$  for  $1 \leq i \leq m-1$  is birationally equivalent to  $\mathbb{P}^1 \times E$ . Thus, by ([Bea] II.11),  $[V_i] = [E \times \mathbb{P}^1] + a_i\mathbb{Z}(-1)$  for  $1 \leq i \leq m-1$ . Letting

$a = \sum_{i=0}^m a_i$  we have

$$\begin{aligned}
\int_X &= 2\mathbb{Z} + a\mathbb{Z}(-1) + (m-1)[E] \cdot [\mathbb{P}^1] + 2\mathbb{Z}(-2) - 2m[E] \\
&= 2\mathbb{Z} + a\mathbb{Z}(-1) + (m-1)[E] + (m-1)[E](-1) + 2\mathbb{Z}(-2) - 2m[E] \\
&= 2\mathbb{Z} + a\mathbb{Z}(-1) - (m+1)[E] + (m-1)[E](-1) + 2\mathbb{Z}(-2).
\end{aligned}$$

Using Corollary 2.3.3 and the fact that the Euler characteristic of a K3 surface is 24 it follows that  $a = 20$ . Thus we have the formula

$$\int_X = 2\mathbb{Z} - (m+1)[E] + 20\mathbb{Z}(-1) + (m-1)[E](-1) + 2\mathbb{Z}(-2).$$

Now we want to express the number of double curves  $m$  and the class of the elliptic curve  $[E]$  in terms of the limit mixed Hodge structure  $H^2(\lim X)$ . First, we show that the *integral* weight spectral sequence  $E_r^{pq}$  from Theorem 2 degenerates at the second term. Since it degenerates rationally it will suffice to show that the  $E_2$  terms are torsion free. The nontrivial portion of the first term of the spectral sequence is

$$\begin{aligned}
\oplus_{i=0}^{m-1} H^2(C_i)(-1) &\xrightarrow{\delta_4} \oplus_{i=0}^m H^4(V_i) \\
\oplus_{i=0}^{m-1} H^1(C_i)(-1) &\xrightarrow{\delta_3} \oplus_{i=1}^{m-1} H^3(V_i) \\
\oplus_{i=0}^{m-1} H^0(C_i)(-1) &\xrightarrow{\delta_2} \oplus_{i=0}^m H^2(V_i) \xrightarrow{\delta'_2} \oplus_{i=0}^{m-1} H^2(C_i) \\
&\oplus_{i=1}^{m-1} H^1(V_i) \xrightarrow{\delta_1} \oplus_{i=0}^{m-1} H^1(C_i) \\
&\oplus_{i=0}^m H^0(V_i) \xrightarrow{\delta_0} \oplus_{i=0}^{m-1} H^0(C_i)
\end{aligned} \tag{3.5}$$

The first and the last complexes compute (co)homology of the Clemens polytope of  $Y$  and, hence, are quasi-isomorphic to  $\mathbb{Z}$ . Consider the middle complex. The map  $\delta_2$  is injective since  $\delta_2 \otimes \mathbb{Q}$  is. Let us prove that  $\delta'_2$  is surjective. For every  $(u_0, \dots, u_m) \in \bigoplus_{i=0}^m H^2(V_i)$ , we have

$$\delta'_2(u_0, \dots, u_m) = ((u_0)|_{C_0} - (u_1)|_{C_0}, \dots, (u_{m-1})|_{C_{m-1}} - (u_m)|_{C_{m-1}}).$$

For every  $1 \leq i \leq m-1$  the restriction morphisms  $H^2(V_i) \rightarrow H^2(C_i)$ ,  $H^2(V_i) \rightarrow H^2(C_{i-1})$  are surjective because  $V_i$  is ruled over  $C_i$  and over  $C_{i-1}$ . By part (2) of Lemma 3.3.1 one of the rational surfaces, say  $V_0$ , is not minimal. If  $D$  is a smooth rational  $-1$ -curve on  $V_0$ , we have

$$-1 = (K_{V_0} \cdot D)_{V_0} = (-C_0 \cdot D)_{V_0}.$$

In particular, the restriction morphism  $H^2(V_0) \rightarrow H^2(C_0)$  is surjective. Surjectivity of  $\delta'_2$  follows. Thus, the third complex in (3.5) has nontrivial cohomology only in the middle degree. As the complex is self-dual, the middle cohomology group must be torsion free. Consider the fourth complex. Identifying  $H^1(C_i)$  with  $H^1(E) =: H$ , we find that the fourth complex is isomorphic to

$$H^{\oplus m-1} \xrightarrow{\delta_1} H^{\oplus m}$$

with the differential given by the formula

$$\delta_1(u_1, \dots, u_{m-1}) = (u_1, u_2 - u_1, \dots, u_{m-1} - u_{m-2}, -u_{m-1}).$$

In particular, it has nontrivial cohomology group only in a single degree and this group is isomorphic to  $H$ . The second complex in (3.5) is dual to the fourth one. This completes the proof of degeneration.

Since the spectral sequence degenerates at  $E_2$  and the  $E_2$  terms are torsion free it follows that  $W_1^{\mathbb{Z}} = \text{Coker}(\delta_1) \cong H = H^1(E, \mathbb{Z})$ . Thus  $W_1^{\mathbb{Z}}$  determines the elliptic curve  $E$ .

It remains to prove that  $m^2 = r_1$ .<sup>1</sup> Indeed, we have the following commutative diagram of abelian groups

$$\begin{array}{ccccc}
 W_3^{\mathbb{Z}} = E_2^{-1,3} \simeq H & \xrightarrow{\Delta} & H^{\oplus m} & \xrightarrow{\delta_3} & H^{\oplus m-1} \\
 & \searrow & \downarrow \text{Id} & \searrow N & \\
 H^{\oplus m-1} & \xrightarrow{\delta_1} & H^{\oplus m} & \xrightarrow{\Sigma} & H \simeq E_2^{1,0} = W_1^{\mathbb{Z}}
 \end{array}$$

where  $\Delta$  is the diagonal map,  $\Sigma$  is the summation map, and  $\delta_3$  is given by the formula

$$\delta_3(u_0, \dots, u_{m-1}) = (u_1 - u_0, \dots, u_{m-1} - u_{m-2}).$$

It follows that

$$N = \Sigma \circ \Delta = m \text{Id},$$

and thus we have

$$r_1 := |\text{Coker}(W_3^{\mathbb{Z}} \xrightarrow{N} W_1^{\mathbb{Z}})| = |\text{Coker}(H \xrightarrow{m} H)| = m^2.$$

This completes this proof of the Theorem for type II degenerations.

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<sup>1</sup>This fact is stated without proof in [FS].

### 3.4. Type III Degeneration

Suppose that  $\overline{X}$  is a type III Kulikov degeneration. In ([FS], Prop. 7.1), Friedman and Scattone proved that the number of triple points of  $Y$  is equal to  $r_2$ . Then since the Clemens polytope of  $Y$  is a triangulation of  $S^2$  it follows that the number of double curves in  $Y$  is equal to  $\frac{3}{2}r_2$  and using Euler's formula for triangulations of a sphere we have that the number of irreducible components of  $Y$  equals  $\frac{r_2}{2} + 2$ . We know that each irreducible component  $V_i$  of  $Y$  is a smooth rational surface and each  $C_j$  is a smooth rational curve. Thus for each  $C_j$  we have  $[C_j] = \mathbb{Z} + \mathbb{Z}(-1)$  and since every non-singular rational surface can be obtained by blowing up either the projective plane or a Hirzebruch surface it follows that  $[V_i] = \mathbb{Z} + a_i\mathbb{Z}(-1) + \mathbb{Z}(-2)$  for some  $a_i \in \mathbb{Z}_{\geq 0}$ . Let  $a = \sum_i a_i$ . Then, we have

$$\begin{aligned} \int_X &= \sum_{i \in \pi_0(Y^{(0)})} [V_i] - 2 \sum_{j \in \pi_0(Y^{(1)})} [C_j] + 3r_2\mathbb{Z} \\ &= \left(\frac{r_2}{2} + 2\right)\mathbb{Z} + a\mathbb{Z}(-1) + \left(\frac{r_2}{2} + 2\right)\mathbb{Z}(-2) - 3r_2(\mathbb{Z} + \mathbb{Z}(-1)) + 3r_2\mathbb{Z} \\ &= \left(\frac{r_2}{2} + 2\right)\mathbb{Z}(-2) + (a - 3r_2)\mathbb{Z}(-1) + \left(\frac{r_2}{2} + 2\right)\mathbb{Z} \end{aligned}$$

Finally, using Proposition 2.3.1 it follows that

$$a - 3r_2 = 20 - r_2.$$

**Remark 3.4.1.** *We claim that in notation of §3.4 the canonical map (2.11)*

$$W_4^{\mathbb{Z}}/W_3^{\mathbb{Z}} \xrightarrow{\gamma} H_2(Cl(Y)) \quad (3.6)$$

is an isomorphism. Indeed, let  $x$  be a generator of  $W_4^{\mathbb{Z}}/W_3^{\mathbb{Z}}$ , and let

$$\gamma(x) = \sum_{i \in \pi_0(Y^{(2)})} b_i \delta_i,$$

where  $\delta_i$  are 2-simplices of  $Cl(Y)$ . Then, since  $\gamma(x) \in H_2(Cl(Y))$ , the boundary of the 2-dimensional chain  $\sum_{i \in \pi_0(Y^{(2)})} b_i \delta_i$  is 0. As the  $\delta_i$  form a triangulation of a compact connected manifold it follows that all the numbers  $|b_i|$  are equal one to the other<sup>2</sup>. If  $b$  denotes their common value, we have by the Picard-Lefschetz formula (2.13)

$$- \langle Gr N^2 \gamma(x), \gamma(x) \rangle = \sum_{i \in \pi_0(Y^{(2)})} b_i^2 = |\pi_0(Y^{(2)})| b^2.$$

The number at the left-hand side of the above formula equals  $r_2$ . Thus by Friedman-Scattone's result  $b = 1$  and therefore  $\gamma(x)$  is a generator of  $H_2(Cl(Y))$ .

It follows from a general result of Berkovich explained in the next section that the group  $H_2(Cl(Y))$  and morphism (3.6) are independent of the choice of a strictly semi-stable model  $\overline{X}$ . Thus, it is an isomorphism for every such model.

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<sup>2</sup>Indeed, every 1-simplex  $\epsilon$  of the triangulation has precisely two 2-simplices, say  $\delta_i$  and  $\delta_j$ , adjacent to it. Thus, in order to have the coefficient at  $\epsilon$  of the boundary of  $\gamma(x)$  vanish  $|b_i|$  must be equal to  $|b_j|$ .

## CHAPTER IV

### THE MONODROMY PAIRING

*In this section, Vadim Vologodsky authored some of the statements and proofs of the results presented herein, which we developed jointly over many hours of meetings. This chapter was revised many times before publication and discerning exactly which of us is the author of an exact piece would be somewhat difficult to ascertain.*

Let  $K$  be a complete discrete valuation field, and let  $\widehat{K}$  be the completion of an algebraic closure  $\overline{K}$  of  $K$ . In [Ber1], Berkovich developed a theory of analytic spaces over  $K$ . The underlying topological space  $|X_{\widehat{K}}^{an}|$  of the analytification of a scheme  $X$  over  $K$  has interesting topological invariants (in contrast with the space  $X(\widehat{K})$  equipped with the usual topology, which is totally disconnected). In particular, if  $X$  is the generic fiber of a proper strictly semi-stable scheme  $\overline{X}$  over  $R$  the space  $|X_{\widehat{K}}^{an}|$  is homotopy equivalent to the Clemens polytope of the special fiber  $Y$  ([Ber3], §5). In this section we construct a positive pairing on the singular cohomology group  $H^m(|X_{\widehat{K}}^{an}|, \mathbb{Q})$  that generalizes Grothendieck's monodromy pairing in the case of abelian varieties. Applications to motivic integrals are discussed in the last section.

#### 4.1. Cohomology of the Analytic Space Associated with a Smooth Scheme

Let  $R$  be a complete discrete valuation domain,  $K$  its fraction field,  $k$  the residue field, and let  $I \subset G = \text{Gal}(\overline{K}/K)$  be the inertia subgroup. We denote by  $s$  and  $\eta$  the closed and generic points of  $\text{spec } R$  respectively. For a prime number  $\ell$  different from

char  $k$ , we have a canonical surjection ([SGA7], I, §0.3)

$$\chi : I \rightarrow \mathbb{Z}_\ell(1)(\bar{k}).$$

If  $\rho : G \rightarrow \text{Aut}(V)$  is a finite rank  $\mathbb{Z}_\ell$ -representation of  $G$  there is a canonical  $G$ -homomorphism:

$$N : V \otimes \mathbb{Q}_\ell(1) \rightarrow V \otimes \mathbb{Q}_\ell,$$

defined as follows. The composition  $\text{Log} \circ \rho$  with the  $\ell$ -adic logarithm  $\text{Aut}(V) \rightarrow \text{End}(V \otimes \mathbb{Q}_\ell)$  restricted to the inertia subgroup  $I$  factors through  $\chi$ . The map  $\mathbb{Z}_\ell(1) \rightarrow \text{End}(V \otimes \mathbb{Q}_\ell)$  yields  $N$ .

Denote by  $\widehat{K}$  the completion of the algebraic closure  $\overline{K}$  with respect to the unique valuation  $\overline{K}^* \rightarrow \mathbb{Q}$  extending the valuation on  $K$ . For a smooth scheme  $X$  of finite type over  $K$ , let  $X_{\widehat{K}}^{an}$  be the  $\widehat{K}$ -analytic space associated with  $X \otimes_K \widehat{K}$  ([Ber1], §3.4), and let  $|X_{\widehat{K}}^{an}|$  be the underlying topological space. According to ([Ber3], Theorem 9.1; [HL], Theorem 13.1.7)  $|X_{\widehat{K}}^{an}|$  is a paracompact locally contractible topological space homotopy equivalent to a finite CW complex. In particular, the singular cohomology groups

$$\Gamma_C^m(X) = H^m(|X_{\widehat{K}}^{an}|, C)$$

with coefficients in a ring  $C$  are finitely generated  $C$ -modules. The action of the Galois group  $G$  on  $|X_{\widehat{K}}^{an}|$  induces one on  $\Gamma_C^m(X)$ . In ([HL], Theorem 13.1.8) Hrushovski and Loeser proved that there exists a finite normal extension  $K' \supset K$  such that the morphism

$$H^m(|X_{K'}^{an}|, C) \rightarrow H^m(|X_{\widehat{K}}^{an}|, C) = \Gamma_C^m(X)$$



is an isomorphism<sup>1</sup>. It follows, that the action of  $G$  on  $\Gamma_C^m(X)$  factors through a finite quotient  $G = \text{Gal}(\overline{K}/K) \twoheadrightarrow \text{Gal}(K'/K)$ .

**Theorem 4.** *For every smooth variety  $X$  and every prime number  $\ell \neq \text{char } k$ , the canonical morphism ([Ber2], Theorem 7.5.4; [Ber3], Theorem 3.2)*

$$\gamma : \Gamma_{\mathbb{Z}_\ell}^m(X) \rightarrow H^m(X_{\overline{K}}^{an}, \mathbb{Z}_\ell) \simeq H^m(X_{\overline{K}}, \mathbb{Z}_\ell) \quad (4.1)$$

*induces an isomorphism of  $G$ -modules*

$$\Gamma_{\mathbb{Q}_\ell}^m(X) \xrightarrow{\sim} \text{Im}(H^m(X_{\overline{K}}, \mathbb{Q}_\ell)(m) \xrightarrow{N^m} H^m(X_{\overline{K}}, \mathbb{Q}_\ell)). \quad (4.2)$$

We will write  $N^m H^m(X_{\overline{K}}, \mathbb{Q}_\ell)$  for the right-hand side of (4.2).

*Proof.* Without loss of generality we may assume that  $k$  is separably closed and that  $X$  is irreducible. We first prove the theorem assuming that  $X$  is projective and has a strictly semi-stable model  $\overline{X}$  over  $R$ . In this case, according to a key result of Berkovich ([Ber3], §5),  $\Gamma_C^m(X)$  is isomorphic to the singular cohomology of the Clemens polytope of the special fiber of  $\overline{X}$ . On the other hand, we consider the weight filtration  $W_i$  on  $H^m(X_{\overline{K}}, \mathbb{Q}_\ell)$  ([RZ], [Sa]). Interpreting the cohomology of the Clemens polytope as the weight zero part of  $H^m(X_{\overline{K}}, \mathbb{Q}_\ell)$  we find that (4.2) is equivalent to a special case of Deligne's monodromy conjecture which asserts that,

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<sup>1</sup>This result was announced in ([Ber3], Theorem 10.1), however the proof in *loc.cit.* is not correct: the assertion on p.82 that a proper hyper-covering of a scheme  $X$  induces a hyper-covering of the topological space  $|X^{an}|$  is false. Example: take the hyper-covering associated with the  $r$ -fold étale cover  $\mathbb{G}_m \rightarrow \mathbb{G}_m$ . If the associated simplicial topological space over  $|\mathbb{G}_m^{an}|$  were a hyper-covering one would get an isomorphism between the cohomology of the contractible space  $|\mathbb{G}_m^{an}|$  and the group cohomology  $H^*(\mathbb{Z}/r\mathbb{Z}, A)$ . In fact,  $\Gamma_A^*(X)$  is an interesting example of cohomology theory that does not have the étale descent property.

for every integer  $0 \leq i \leq m$ , the morphism

$$N^i : Gr_W^{m+i} H^m(X_{\bar{K}}, \mathbb{Q}_\ell)(i) \rightarrow Gr_W^{m-i} H^m(X_{\bar{K}}, \mathbb{Q}_\ell)$$

is an isomorphism. We prove Deligne's conjecture for  $i = m$  using the method of Steenbrink (who proved it for all  $i$  and  $k = \mathbb{C}$ ). To prove the theorem for arbitrary smooth  $X$  we show that the functors  $\Gamma_{\mathbb{Q}_\ell}^m$  and  $N^m H^m$ , first, admit transfers for finite morphisms and, second, take every dominant open embedding  $U \hookrightarrow X$  to an isomorphism. Finally, we use de Jong's alteration result to complete the proof.

*Step 1.* Assuming that  $X$  has a projective strictly semi-stable model  $\bar{X}$  over  $R$ ,  $X \simeq \bar{X} \otimes_R K$ . Denote by  $D_i$ ,  $i = 1, 2, \dots, s$  the irreducible components of the special fiber  $Y = \bar{X} \otimes k$ ;

$$Y^{(q)} = \bigsqcup_{I \subset \{1, \dots, s\}, |I|=q+1} \bigcap_{i \in I} D_i,$$

and by  $\pi_0(Y^{(q)})$  the set of connected components of  $Y^{(q)}$ . We have a commutative diagram

$$\begin{array}{ccc} \Gamma_{\mathbb{Q}_\ell}^m(X) & \xrightarrow{\gamma} & H^m(X_{\bar{K}}, \mathbb{Q}_\ell) \\ \simeq \downarrow & \nearrow \rho & \\ H_{sing}^m(Cl(Y), \mathbb{Q}_\ell) \simeq E_2^{m,0}(\bar{X}) & & \end{array} \quad (4.3)$$

where  $E_2^{m,0}(\bar{X})$  is the weight zero term of the weight spectral sequence converging to  $H^m(X_{\bar{K}}, \mathbb{Q}_\ell)$  [Sa]. According to [Na] the weight spectral sequence degenerates at  $E_2$ ; in particular the morphism  $\rho$  is injective. Since the range of the weight filtration on  $H^m(X_{\bar{K}}, \mathbb{Q}_\ell)$  is at most  $2m$  and  $N$  shifts the filtration by 2, we have

$$N^m H^m(X_{\bar{K}}, \mathbb{Q}_\ell) \subset Im(\rho). \quad (4.4)$$

Consider the commutative diagram

$$\begin{array}{ccc}
E_2^{-m,2m}(\overline{X})(m) & \leftarrow & H^m(X_{\overline{K}}, \mathbb{Q}_\ell)(m) \\
\downarrow \overline{N}^m & & \downarrow N^m \\
E_2^{m,0}(\overline{X}) & \hookrightarrow & H^m(X_{\overline{K}}, \mathbb{Q}_\ell),
\end{array} \tag{4.5}$$

The upper horizontal arrow in this diagram is the projection to the weight  $2m$  quotient. We will prove, following the method of ([St1], §5), that, for every  $m$ , one has

$$\overline{N}^m : E_2^{-m,2m}(\overline{X})(m) \xrightarrow{\sim} E_2^{m,0}(\overline{X}). \tag{4.6}$$

This trivially holds for  $d := \dim X < m$  because in this case both sides of (4.6) equal 0. Let us prove (4.6) for  $m = d$ . Consider the following commutative diagram [Sa]

$$\begin{array}{ccccccc}
E_1^{d-1,0} & \longrightarrow & H^0(Y^{(d)}, \mathbb{Q}_\ell) = E_1^{d,0} & \longrightarrow & E_2^{d,0} & \rightarrow & 0 \\
& & \uparrow & & \nearrow & & \\
0 \rightarrow E_2^{-d,2d}(d) & \longrightarrow & H^0(Y^{(d)}, \mathbb{Q}_\ell) = E_1^{-d,2d}(d) & \longrightarrow & E^{-d+1,2d}(d) & & 
\end{array}$$

where

$$E_1^{d-1,0} = H^0(Y^{(d-1)}, \mathbb{Q}_\ell), \quad E^{-d+1,2d}(d) = H^2(Y^{(d-1)}, \mathbb{Q}_\ell)(1),$$

the diagonal morphism is  $\overline{N}^d$ , and the vertical arrow is the identity morphism. The rows of the above diagram are exact and dual to one another. In particular, we have a non-degenerate pairing

$$\langle, \rangle : E_2^{d,0} \otimes E_2^{-d,2d}(d) \rightarrow \mathbb{Q}_\ell$$

that identifies  $E_2^{-d,2d}(d)$  with  $H_d(Cl(Y)) \otimes \mathbb{Q}_\ell$ . Next, consider the symmetric form

$$E_2^{-d,2d}(d) \otimes E_2^{-d,2d}(d) \rightarrow \mathbb{Q}_\ell, \quad x \otimes y \mapsto \langle \bar{N}^d x, y \rangle. \quad (4.7)$$

We claim that (4.7) is non-degenerate. In fact, if

$$x = \sum_{v \in \pi_0(Y^{(d)})} a_v v, \quad y = \sum_{v \in \pi_0(Y^{(d)})} b_v v \in E_2^{-d,2d}(\bar{X})(d) \subset \mathbb{Q}_\ell [\pi_0(Y^{(d)})],$$

we have

$$\langle N^d x, y \rangle = \sum a_v b_v.$$

Thus (4.7) comes by extension of scalars from a positive form

$$H_d(Cl(Y), \mathbb{Q}) \otimes H_d(Cl(Y), \mathbb{Q}) \rightarrow \mathbb{Q}. \quad (4.8)$$

This proves that the morphism (4.6) is injective; since  $\dim E_2^{-d,2d} = \dim E_2^{d,0}$ , it must be an isomorphism.

Assume that  $0 < m < d$ . Choose an embedding  $\bar{X} \hookrightarrow \mathbb{P}_R^N$  and a generic hyperplane section  $\bar{Z} = \bar{X} \cap \mathbb{P}_R^{N-d+m}$  of dimension  $m$ ;  $Z = X \cap \mathbb{P}_K^{N-d+m}$ . Then  $\bar{Z}$  is again strictly semi-stable and the embedding  $i : \bar{Z} \hookrightarrow \bar{X}$  induces a morphism of spectral sequences  $E_r^{p,q}(\bar{X}) \rightarrow E_r^{p,q}(\bar{Z})$ . By the Hard Lefschetz Theorem the composition of the restriction morphism and the Poincaré pairing

$$H^m(X_{\bar{K}}, \mathbb{Q}_\ell) \otimes H^m(X_{\bar{K}}, \mathbb{Q}_\ell) \rightarrow H^m(Z_{\bar{K}}, \mathbb{Q}_\ell) \otimes H^m(Z_{\bar{K}}, \mathbb{Q}_\ell) \rightarrow \mathbb{Q}_\ell(-m)$$

is non-degenerate. The induced isomorphism  $H^m(X_{\bar{K}}, \mathbb{Q}_\ell) \rightarrow (H^m(X_{\bar{K}}, \mathbb{Q}_\ell))^*(-m)$  takes  $E_2^{m,0}(\bar{X}) \subset H^m(X_{\bar{K}}, \mathbb{Q}_\ell)$  to  $(E_2^{-m,2m}(\bar{X}))^* \subset (H^m(X_{\bar{K}}, \mathbb{Q}_\ell))^*$ . Thus

$$\dim E_2^{m,0}(\bar{X}) \leq \dim E_2^{-m,2m}(\bar{X}).$$

Let us show that (4.6) is injective. It is enough to check that in the commutative diagram

$$\begin{array}{ccc} E_2^{-m,2m}(\bar{X})(m) & \xrightarrow{\bar{N}^m} & E_2^{m,0}(\bar{X}) \\ \downarrow i^* & & \downarrow i^* \\ E_2^{-m,2m}(\bar{Z})(m) & \xrightarrow{\sim} & E_2^{m,0}(\bar{Z}). \end{array} \quad (4.9)$$

the left downward arrow is an injection. We have

$$\begin{array}{ccc} E_2^{-m,2m}(\bar{X})(m) & \hookrightarrow & E_1^{-m,2m}(\bar{X})(m) = H^0(Y^{(m)}, \mathbb{Q}_\ell) \\ \downarrow i^* & & \downarrow i^* \\ E_2^{-m,2m}(\bar{Z})(m) & \longrightarrow & E_1^{-m,2m}(\bar{Z})(m) = H^0(Y^{(m)} \cap \mathbb{P}_k^{N-d+m}, \mathbb{Q}_\ell). \end{array} \quad (4.10)$$

In this commutative diagram the upper horizontal arrow is an injection because the incoming differential  $0 = E_1^{-m-1,2m}(\bar{X}) \xrightarrow{d_1} E_1^{-m,2m}(\bar{X})$  is trivial. The right downward arrow is an injection because  $\mathbb{P}_k^{N-n+m}$  intersects every connected component of  $Y^{(m)}$ . This completes the proof of (4.6) and that of (4.2).

*Step 2.* Hrushovski and Loeser proved in ([HL], Th. 13.1.8) that for every smooth variety  $X$  and an open dense subset  $U \subset X$  the restriction morphism

$$\Gamma_C^m(X) \rightarrow \Gamma_C^m(U) \quad (4.11)$$

is an isomorphism. Let us show that the functor at the right-hand side of (4.2) has the same property:

$$N^m H^m(X_{\bar{K}}, \mathbb{Q}_\ell) \xrightarrow{\sim} N^m H^m(U_{\bar{K}}, \mathbb{Q}_\ell). \quad (4.12)$$

We first prove (4.12) in the case when  $X$  is the generic fiber of a projective strictly semi-stable pair  $(\bar{X}, \bar{Z} = \bar{Z}_f \cup Y)$  over  $R$  ([deJ1], §6.3) and  $j : U \hookrightarrow X$  is the complement to  $Z = \bar{Z} \otimes K$  in  $X$ . Denote by  $T$  the special fiber  $\bar{Z}_f \otimes k$  of the flat part of  $\bar{Z}$  and by  $\bar{j} : Y - T \hookrightarrow Y$  the embedding. The idea of the following argument (that goes back to Nakayama [Na]) is the following. When the residue field  $k$  is finite (4.12) can be derived from the Weil conjectures, proven by Deligne, and the formula (4.6) proven in Step 1 (*cf.* [Ber4] p. 672). In general, the works of Fujiwara, Kato and Nakayama on logarithmic étale cohomology [Il2] imply that  $\ell$ -adic cohomology groups of  $X$  and  $U$  depend only on the special fibers,  $Y$  and  $Y - T$ , endowed with their natural log structures (that, in turn, are determined by the first infinitesimal neighborhood of  $Y$  (*resp.*  $Y - T$ ) in  $\bar{X}$  (*resp.*  $\bar{X} - \bar{Z}_f$ )). Then, a specialization argument enables one to reduce to the finite field case. Let us explain the details.

For a scheme  $S_{log}$  over the log point  $(\text{spec } k)_{log}$  we denote by  $R\tilde{c}_*$  the functor from the derived category of  $\ell$ -adic sheaves on the Kummer étale site,  $S_{log}^{ket}$ , to the derived category of  $\ell$ -adic sheaves on  $S$  equipped with an endomorphism of weight 2 *i.e.*, a morphism  $N : \mathcal{F} \rightarrow \mathcal{F}(-1)$  ([Il2], §8, p. 308). Consider the log structure on the scheme  $\bar{X}$  associated with the divisor  $Y$ , and let  $Y_{log} = (Y, M_Y)$  be the special fiber with the induced log structure. According to ([Il2], §8, Cor. 8.4.3) the action of the wild inertia  $P \subset \text{Gal}(\bar{K}/K)$  on the complexes of nearby cycles  $\Psi\mathbb{Q}_\ell$ ,  $\Psi Rj_*\mathbb{Q}_\ell$  is trivial. Therefore we can and we will view the nearby cycles as objects of the derived category of  $\ell$ -adic sheaves on  $Y$  endowed with an endomorphism  $N$  of weight 2. Then,

we have

$$\Psi \mathbb{Q}_\ell \simeq R\tilde{\epsilon}_*(\mathbb{Q}_\ell),$$

$$\Psi Rj_* \mathbb{Q}_\ell \simeq R\tilde{\epsilon}_*(R\bar{j}_* \mathbb{Q}_\ell).$$

We have to prove that the morphism

$$N^m H^m(Y_{\bar{k}}, R\tilde{\epsilon}_*(\mathbb{Q}_\ell)) \rightarrow N^m H^m(Y_{\bar{k}}, R\tilde{\epsilon}_*(R\bar{j}_* \mathbb{Q}_\ell))$$

is an isomorphism. This will follow from a more general fact about log schemes over  $(\text{spec } k)_{\log}$ .

Let  $Y_{\log} = (Y, M_Y)$  be a fs log scheme over  $(\text{spec } k)_{\log}$ , and let  $T \hookrightarrow Y$  be closed subscheme. We will say that  $(Y_{\log}, T)$  is a standard log strictly semi-stable pair if, for some integers  $0 \leq a \leq b \leq d$ , there is an isomorphism between  $Y_{\log}$  and the special fiber the log scheme  $\text{spec } R[x_0, \dots, x_d]/(x_0 \cdots x_a - \pi)$  (with the log structure defined by the divisor  $\pi = 0$ ) that takes  $T$  to the subscheme given by the equation  $x_{a+1} \cdots x_b = 0$ . We will say that  $(Y_{\log}, T)$  is a log strictly semi-stable pair if every point of  $Y$  has a Zariski neighborhood  $U$  such that  $(U_{\log}, T \cap U)$  admits a strict étale morphism to a standard log strictly semi-stable pair. If this is the case, every irreducible component  $T_i$  of  $T = T_1 \cup \dots \cup T_n$  with the log structure induced from  $Y$  and  $T_i \cap (T_1 \cup \dots \cup T_{i-1}) \subset T_i$  is again a log strictly semi-stable pair.

Let  $(Y_{\log}, T)$  be a proper log strictly semi-stable pair. In ([Na], §1), Nakayama constructed the weight spectral sequence  $E_r^{pq}$  converging to  $H^m(Y \otimes \bar{k}, R\tilde{\epsilon}_*(\mathbb{Q}_\ell))$  and proved that it degenerates in the  $E_2$ -terms. In particular, for every integer  $m$ , the canonical morphism

$$H_{\text{sing}}^m(Cl(Y)) \otimes \mathbb{Q}_\ell \simeq E_2^{m,0} \rightarrow H^m(Y \otimes \bar{k}, R\tilde{\epsilon}_*(\mathbb{Q}_\ell))$$

is an embedding.

**Lemma 4.1.1.** *For every proper log strictly semi-stable pair  $(Y_{\log}, T)$  the composition*

$$H_{\text{sing}}^m(Cl(Y_{\bar{k}}), \mathbb{Q}_\ell) \hookrightarrow H^m(Y_{\bar{k}}, R\tilde{c}_*(\mathbb{Q}_\ell)) \rightarrow H^m(Y_{\bar{k}}, R\tilde{c}_*(R\bar{j}_*\mathbb{Q}_\ell)) \quad (4.13)$$

is a monomorphism whose image contains  $N^m H^m(Y_{\bar{k}}, R\tilde{c}_*(R\bar{j}_*\mathbb{Q}_\ell))$ .

*Proof.* The specialization argument of Nakayama [Na] reduces the statement to the case when  $k$  is a finite field; in the rest of the proof we will be assuming that this is the case. The vector spaces appearing in (4.13) carry an action of the Galois group  $\text{Gal}(\bar{k}/k)$ . Let us look at the action of the Frobenius element  $Fr \in \text{Gal}(\bar{k}/k)$ . For a finite-dimensional  $\ell$ -adic representation  $V$  of  $\text{Gal}(\bar{k}/k)$  we denote by  $V_0$  the largest invariant subspace of  $V$  such that all the eigenvalues of  $Fr$  on  $V_0$  are roots of unity. Looking at the weight spectral sequence we see that

$$E_2^{m,0} = (H^m(Y_{\bar{k}}, R\tilde{c}_*(\mathbb{Q}_\ell))_0.$$

Thus, to prove the lemma it suffices to show the following:

(a)

$$(H^m(Y_{\bar{k}}, R\tilde{c}_*(\mathbb{Q}_\ell))_0 \xrightarrow{\sim} (H^m(Y_{\bar{k}}, R\tilde{c}_*(R\bar{j}_*\mathbb{Q}_\ell))_0. \quad (4.14)$$

(b) The eigenvalues of  $Fr$  acting on  $H^m(Y_{\bar{k}}, R\tilde{c}_*(R\bar{j}_*\mathbb{Q}_\ell))$  are Weil numbers of weights from 0 to  $2m$ .

Arguing by induction on  $d = \dim Y$  we assume that the above assertions hold for log strictly semi-stable pairs of dimension less than  $d$ . Let  $T_1, \dots, T_n$  be irreducible components of  $T$ , let  $Y_j$  be the complement to  $\bigcup_{i \leq j} T_i$  in  $Y$ . Consider the Gysin exact



sequence

$$\begin{aligned} \cdots \rightarrow H^{m-2}((T_{j+1} \cap Y_j) \otimes \bar{k}, R\tilde{\epsilon}_*(\mathbb{Q}_\ell))(-1) \rightarrow H^m(Y_j \otimes \bar{k}, R\tilde{\epsilon}_*(\mathbb{Q}_\ell)) \\ \rightarrow H^m(Y_{j+1} \otimes \bar{k}, R\tilde{\epsilon}_*(\mathbb{Q}_\ell)) \rightarrow H^{m-1}((T_{j+1} \cap Y_j) \otimes \bar{k}, R\tilde{\epsilon}_*(\mathbb{Q}_\ell))(-1) \rightarrow \cdots \end{aligned}$$

By our induction assumption the boundary terms of the sequence have weights between 2 and  $2m$ . Induction on  $j$  proves the first claim (4.14). The second claim also follows from the above and from the fact that  $H^m(Y \otimes \bar{k}, R\tilde{\epsilon}_*(\mathbb{Q}_\ell))$  has weights between 0 and  $2m$ .  $\square$

As we know from Step 1, for a projective strictly semi-stable scheme  $\bar{X}$  over  $R$ , we have

$$N^m H^m(X_{\bar{K}}, \mathbb{Q}_\ell) \xrightarrow{\sim} E_2^{m,0}.$$

This together with the Lemma 4.1.1 complete the proof of (4.12) for strictly semi-stable pairs.

Before going further, recall that, for every generically finite surjective morphism  $f : X' \rightarrow X$  of smooth connected varieties, the induced map

$$f^* : H^m(X_{\bar{K}}, \mathbb{Q}_\ell) \rightarrow H^m(X'_{\bar{K}}, \mathbb{Q}_\ell)$$

is injective. In fact, the canonical isomorphism  $\mathbb{Q}_\ell \xrightarrow{\sim} Rf^! \mathbb{Q}_\ell$  defines by adjunction a morphism

$$Rf_* \mathbb{Q}_\ell \xrightarrow{\sim} Rf_! \mathbb{Q}_\ell \rightarrow \mathbb{Q}_\ell.$$

In turn, the latter yields the transfer morphism

$$f_* : H^m(X'_{\overline{K}}, \mathbb{Q}_\ell) \rightarrow H^m(X_{\overline{K}}, \mathbb{Q}_\ell)$$

such that the composition  $f_* f^*$  equals multiplication by the degree of  $f$  over the generic point.

Let us return to the proof of (4.12). Without loss of generality we may assume that  $X$  is connected. Then, by de Jong's result ([deJ1], §6.3) we can find a proper generically finite surjective morphism  $f : X' \rightarrow X$  such that  $X'$  is an open subscheme of a connected projective strictly semi-stable scheme  $\overline{X}'$  over a finite extension  $R' \supset R$  and such that  $(\overline{X}', \overline{X}' - X')$  is a strictly semi-stable pair. Applying de Jong's result once again, we find a proper generically finite surjective morphism  $g : \overline{X}'' \rightarrow \overline{X}'$ , with connected  $\overline{X}''$ , such that  $(\overline{X}'', \overline{X}'' - (fg)^{-1}(U))$  is a projective strictly semi-stable pair over some  $R'' \supset R'$ . Diagram:

$$\begin{array}{ccccc}
(fg)^{-1}(U) & \hookrightarrow & X'' & \hookrightarrow & \overline{X}'' \\
\downarrow & & \downarrow & & \downarrow g \\
f^{-1}(U) & \hookrightarrow & X' & \hookrightarrow & \overline{X}' \\
\downarrow & & \downarrow f & & \\
U & \hookrightarrow & X & & 
\end{array} \tag{4.15}$$

We know that (4.12) is true for the embeddings  $X' \hookrightarrow \overline{X}' \otimes K$  and  $g^{-1}f^{-1}(U) \hookrightarrow \overline{X}'' \otimes K$ .<sup>2</sup> Define a morphism  $u : N^m H^m(U_{\overline{K}}, \mathbb{Q}_\ell) \rightarrow N^m H^m(X_{\overline{K}}, \mathbb{Q}_\ell)$  to be the

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<sup>2</sup>Indeed,  $(\overline{X}', \overline{X}' - X')$  is a strictly semi-stable pair over  $R'$ . Therefore, we have  $N^m H^m(\overline{X}' \times_{R'} \overline{K}, \mathbb{Q}_\ell) \xrightarrow{\sim} N^m H^m(X' \times_{R'} \overline{K}, \mathbb{Q}_\ell)$ . This implies that the morphism  $N^m H^m(\overline{X}' \times_R \overline{K}, \mathbb{Q}_\ell) \rightarrow N^m H^m(X' \times_R \overline{K}, \mathbb{Q}_\ell)$  is an isomorphism as well.

composition

$$\begin{aligned}
N^m H^m(U_{\bar{K}}, \mathbb{Q}_\ell) &\xrightarrow{(fg)^*} N^m H^m((fg)^{-1}(U)_{\bar{K}}, \mathbb{Q}_\ell) \simeq N^m H^m(\bar{X}'' \otimes \bar{K}, \mathbb{Q}_\ell) \\
&\xrightarrow{g^*} N^m H^m(\bar{X}' \otimes \bar{K}, \mathbb{Q}_\ell) \xrightarrow{Res} N^m H^m(X' \otimes \bar{K}, \mathbb{Q}_\ell) \xrightarrow{f^*} N^m H^m(X \otimes \bar{K}, \mathbb{Q}_\ell).
\end{aligned}$$

An easy diagram chase shows that  $u$  divided by the degree of the morphism  $fg$  over the generic point is the two-sided inverse to the restriction morphism (4.12).

*Step 3.* Let  $f : U' \rightarrow U$  be a finite surjective morphism of connected smooth varieties. Assume that the corresponding extension  $\text{Rat}(X) \subset \text{Rat}(X')$  of the field of rational functions is normal and let  $G$  be its Galois group. Then, the pullback morphism  $f^*$  induces an isomorphism

$$N^m H^m(U_{\bar{K}}, \mathbb{Q}_\ell) \xrightarrow{\sim} (N^m H^m(U'_{\bar{K}}, \mathbb{Q}_\ell))^G.$$

Let us show the functor  $\Gamma_{\mathbb{Q}}^m$  has the same property:

$$\Gamma_{\mathbb{Q}}^m(U) \xrightarrow{\sim} (\Gamma_{\mathbb{Q}}^m(U'))^G. \quad (4.16)$$

Indeed, by ([Ber2], Prop. 4.2.4), the cohomology of the topological space  $|U_{\hat{K}}^{an}|$  with rational coefficients coincides with the étale cohomology of the analytic space  $U_{\hat{K}}^{an}$  with coefficients in  $\mathbb{Q}$ . Next, since the functor of  $G$ -invariants is exact in any  $\mathbb{Q}$ -linear abelian category, we have

$$(H_{et}^m(U_{\hat{K}}^{an}, \mathbb{Q}))^G \simeq H_{et}^m(U_{\hat{K}}^{an}, (f_* \mathbb{Q})^G).$$

We complete the proof of (4.16) by showing that the canonical morphism  $\mathbb{Q} \rightarrow (f_*\mathbb{Q})^G$  is an isomorphism. In fact, the weak base change theorem ([Ber2] Th. 5.3.1) reduces the statement to the case when  $U_{\widehat{K}}$  is a single point. In this case  $G$  acts transitively on points of  $U'_{\widehat{K}}$  and our assertion follows.

*Step 4.* Now we can complete the proof of Theorem 4. We may assume that  $X$  is connected. Then, by ([deJ2], Th 5.9), there exists a proper generically finite surjective morphism  $f : X' \rightarrow X$  such that the field extension  $\text{Rat}(X) \subset \text{Rat}(X')$  is normal,  $X'$  is an open subscheme of a connected projective strictly semi-stable scheme  $\overline{X}'$  over a finite extension  $R' \supset R$ . Let  $U$  be an open dense subset of  $X$  over which  $f$  is finite. By the result of Step 1 the Theorem is true for  $\overline{X}'$ .<sup>3</sup> Then, by Step 2 it is true for  $f^{-1}(U)$  and thus, by Step 3, for  $U$ . Applying the result of Step 2 once again we complete the proof of Theorem 4.

□

**Remark 4.1.2.** *The groups  $\Gamma_{\mathbb{Z}}^*(X)$  are related to the weight zero part of motivic vanishing cycles  $\Psi(X) \in DM_{gm}^{eff}(k)$  of  $X$  ([A1], [A2]). Namely, if  $\text{char } k = 0$ , one has*

$$\Gamma_{\mathbb{Z}}^m(X) \simeq \text{Hom}_{DM_{gm}^{eff}(\overline{k})}(\Psi(X), \mathbb{Z}[m]).$$

**Remark 4.1.3.** *Assume that  $K = \mathbb{C}((t))$ . For every smooth projective  $X/K$  there is a canonical morphism (cf. [Ber5], Theorem 5.1)*

$$\Gamma_{\mathbb{Z}}^m(X) \rightarrow W_0^{\mathbb{Q}} \cap H^m(\lim X, \mathbb{Z}) \tag{4.17}$$

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<sup>3</sup>Indeed, the result of Step 1 implies that the morphism  $H^m(|X'^{an} \times_{K'} \widehat{K}|, \mathbb{Q}_{\ell}) \rightarrow N^m H^m(X' \times_{K'} \overline{K}, \mathbb{Q}_{\ell})$  is an isomorphism. This implies that  $H^m(|X'^{an} \times_K \widehat{K}|, \mathbb{Q}_{\ell}) \rightarrow N^m H^m(X' \times_K \overline{K}, \mathbb{Q}_{\ell})$  is also an isomorphism.

that induces an isomorphism modulo torsion

$$\Gamma_{\mathbb{Q}}^m(X) \simeq W_0^{\mathbb{Q}} H^m(\lim X). \quad (4.18)$$

Morphism (4.17) can be constructed as follows. Pick a finite extension  $K' \supset K$  and strictly semi-stable model  $\overline{X}_{R'}$  of  $X_{K'} = X \otimes_K K'$  over the integral closure  $R'$  of  $R$  in  $K'$ . Then (4.17) is defined to be the composition

$$\begin{aligned} \Gamma_{\mathbb{Z}}^m(X) &\simeq \Gamma_{\mathbb{Z}}^m(X_{K'}) \xrightarrow{\sim} H^m(Cl(Y)) \\ &\rightarrow W_0^{\mathbb{Q}} \cap H^m(\lim X_{K'}, \mathbb{Z}) \simeq W_0^{\mathbb{Q}} \cap H^m(\lim X, \mathbb{Z}), \end{aligned} \quad (4.19)$$

where  $Y$  is the special fiber of  $\overline{X}_{R'}$  and the map  $H^m(Cl(Y)) \rightarrow W_0^{\mathbb{Q}} \cap H^m(\lim X_{K'}, \mathbb{Z})$  comes from the weight spectral sequence (see §2.2). As the weight spectral sequence with rational coefficients degenerates at  $E_2$  terms the above composition is an isomorphism up to torsion. The composition of (4.19) with the embedding  $W_0^{\mathbb{Q}} \cap H^m(\lim X, \mathbb{Z}) \hookrightarrow H^m(X_{\overline{K}}, \mathbb{Z}_{\ell})$  equals the canonical morphism  $\Gamma_{\mathbb{Z}}^m(X) \rightarrow H^m(X_{\overline{K}}, \mathbb{Z}_{\ell})$  from Theorem 4. Thus, the morphism  $\Gamma_{\mathbb{Z}}^m(X) \rightarrow W_0^{\mathbb{Q}} \cap H^m(\lim X, \mathbb{Z})$  induced by (4.19) is independent of the choice of  $K'$  and  $\overline{X}_{R'}$ .

In general, morphism (4.17) is not bijective.

We conjecture that for every smooth proper variety  $X$  over  $K$ , one has

$$\dim_{\mathbb{Q}} \Gamma_{\mathbb{Q}}^m(X) \leq \dim_K H^m(X, \mathcal{O}_X). \quad (4.20)$$

Conjecture (4.20) is motivated by the following result.

**Proposition 4.1.4.** *The inequality (4.20) is true if either of the following conditions holds.*

(a)  $\text{char } k = 0$ .

(b)  $K$  is a finite extension of  $\mathbb{Q}_p$ .

*Proof.* When proving the first part of the Proposition, we may assume that  $R = \mathbb{C}[[t]]$  and  $X$  is the generic fiber of a strictly semi-stable scheme  $\overline{X}$  over  $R$  ([HL], Theorem 13.1.8). In this case, we have

$$\Gamma_{\mathbb{Q}}^m(X) \simeq H_{Zar}^m(Y, \mathbb{Q}) \simeq W_0^{\mathbb{Q}} H^m(\lim X).$$

where  $Y$  is the special fiber of  $\overline{X}$ . The first part of the Proposition now follows from the inequality  $\dim_{\mathbb{Q}} W_0^{\mathbb{Q}} \leq \dim_{\mathbb{C}} F^0/F^1 = \dim_K H^m(X, \mathcal{O}_X)$ . For the second part, recall from ([Ber2], Theorem 1.1) that  $\Gamma_K^m(X)$  is isomorphic to the subspace of the p-adic cohomology  $H^m(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} K$  that consists of smooth vectors *i.e.*, vectors whose stabilizer in  $G$  is open. Thus,

$$\dim_K \Gamma_K^m(X) \leq \dim_K (H^m(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p)^G = \dim_K H^m(X, \mathcal{O}_X).$$

The last equality follows from the Hodge-Tate decomposition proven by Faltings [Fa].

□

## 4.2. The Monodromy Pairing

Let  $X$  be a smooth variety over a complete discrete valuation field  $K$  and  $d = \dim X$ . In this subsection we define a canonical positive symmetric form (that we shall call the monodromy pairing)

$$(\cdot, \cdot) : \Gamma_{\mathbb{Q}}^d(X) \otimes \Gamma_{\mathbb{Q}}^d(X) \rightarrow \mathbb{Q}. \quad (4.21)$$

The group  $\Gamma_{\mathbb{Z}}^d(X)$  as well as the monodromy pairing depends only on the class of  $X$  modulo birational equivalence.

First, we define a pairing

$$(\cdot, \cdot)_\ell : N^d H^d(X_{\overline{K}}, \mathbb{Q}_\ell) \otimes N^d H^d(X_{\overline{K}}, \mathbb{Q}_\ell) \rightarrow \mathbb{Q}_\ell.$$

By ([deJ1], Th 4.1, Rem. 4.2), there exists a proper generically finite surjective morphism  $f : X' \rightarrow X$  such that  $X'$  is an open subscheme of a smooth projective variety  $\tilde{X}'$  over a finite extension  $K' \supset K$ . Let  $r$  be the degree of  $f$  over the generic point. Consider the morphism

$$\overline{f}^* : N^d H^d(X_{\overline{K}}, \mathbb{Q}_\ell) \xrightarrow{f^*} N^d H^d(X'_{\overline{K}}, \mathbb{Q}_\ell) \xleftarrow{\sim} N^d H^d(\tilde{X}'_{\overline{K}}, \mathbb{Q}_\ell).^4$$

The left arrow is an isomorphism by (4.12). Given  $x, y \in N^d H^d(X_{\overline{K}}, \mathbb{Q}_\ell)$  we set

$$(x, y)_\ell = \frac{(-1)^{\frac{d(d-1)}{2}}}{r} \langle \overline{f}^*(x), \overline{f}^*(y') \rangle, \quad (4.22)$$

where  $y' \in H^d(X_{\overline{K}}, \mathbb{Q}_\ell)(d)$  is an element such that  $N^d y' = y$  and

$$\langle, \rangle : H^d(\tilde{X}'_{\overline{K}}, \mathbb{Q}_\ell) \otimes H^d(\tilde{X}'_{\overline{K}}, \mathbb{Q}_\ell)(d) \rightarrow \mathbb{Q}_\ell$$

is the Poincaré pairing. Let us check that  $(\cdot, \cdot)_\ell$  is well defined. Indeed, if  $y''$  is another element such that  $N^d y'' = y$ , then

$$\langle \overline{f}^*(x), \overline{f}^*(y' - y'') \rangle = \langle N^d \overline{f}^*(x'), \overline{f}^*(y' - y'') \rangle$$

---

<sup>4</sup>We write  $X'_{\overline{K}}$  for the fiber product of  $X'$  and  $\text{spec } \overline{K}$  over  $\text{spec } K$ .

$$= (-1)^d \langle \bar{f}^*(x'), N^d \bar{f}^*(y' - y'') \rangle = 0.$$

The independence of the choice of  $X'$ ,  $f$  and  $\tilde{X}'$  follows from the fact that given another such triple  $X''$ ,  $g$  and  $\tilde{X}''$  we can find a smooth projective scheme over some finite extension of  $K$  that admits proper generically finite surjective morphisms to both  $\tilde{X}'$  and  $\tilde{X}''$ .

Let us also remark that the pairing  $(\cdot, \cdot)_\ell$  is symmetric.

**Theorem 5.** *For every smooth connected variety  $X$  of dimension  $d$ , the restriction (4.21) of the pairing  $(\cdot, \cdot)_\ell$  to the subspace*

$$\Gamma_{\mathbb{Q}}^d(X) \hookrightarrow N^m H^m(X_{\bar{K}}, \mathbb{Q}_\ell)$$

*takes values in  $\mathbb{Q}$  and is independent of  $\ell \neq \text{char } k$ . Moreover, the pairing (4.21) is positively defined (and, in particular, non-degenerate).*

*Proof.* Thanks to the birational invariance property of  $\Gamma^d(X)$  (4.11) and de Jong's semi-stable reduction theorem ([deJ1], §6.3) it is enough to prove the theorem in the case when  $X$  is the generic fiber of a strictly semi-stable projective scheme  $\bar{X}$  over  $R$ . In this case, using (4.3) we have a canonical isomorphism  $\Gamma_{\mathbb{Q}}^d(X) \simeq H^d(Cl(Y), \mathbb{Q})$  that identifies, by the Picard-Lefschetz formula (*cf.* (2.13)), the pairing  $(\cdot, \cdot)_\ell$  restricted to  $\Gamma_{\mathbb{Q}}^d(X)$  with the dual of the pairing (4.8).  $\square$

**Remark 4.2.1.** *The construction of the monodromy pairing can be generalized as follows. For a pair  $(X, \mu)$ , where  $X$  is a smooth projective variety over  $K$  and  $\mu \in H^2(X, \mathbb{Q}_\ell(1))$  is the class of an ample line bundle over  $X$ , and an integer  $m \leq d$ , we define a positive symmetric form*

$$(\cdot, \cdot)_\mu : \Gamma_{\mathbb{Q}}^m(X) \otimes \Gamma_{\mathbb{Q}}^m(X) \rightarrow \mathbb{Q} \tag{4.23}$$



to be the composition

$$\Gamma_{\mathbb{Q}}^m(X) \otimes \Gamma_{\mathbb{Q}}^m(X) \rightarrow N^m H^m(X_{\bar{K}}, \mathbb{Q}_{\ell}) \otimes N^m H^m(X_{\bar{K}}, \mathbb{Q}_{\ell}) \xrightarrow{(\cdot, \cdot)_{\ell, \mu}} \mathbb{Q}_{\ell},$$

where  $(x, N^m y')_{\ell, \mu} = (-1)^{\frac{d(d-1)}{2}} \langle x, y' \mu^{d-m} \rangle$ . Let us prove that (4.23) is independent of  $\ell$  and positive. Without loss of generality, we may assume that  $\mu$  is the class of very ample line bundle  $L$ . Let  $X \hookrightarrow \mathbb{P}_K^N$  be the corresponding embedding, and let  $Z = X \cap \mathbb{P}_K^{N-d+m} \xrightarrow{i} X$  be a generic hyperplane section of dimension  $m$ . Then,  $(\cdot, \cdot)_{\mu}$  equals the composition

$$\Gamma_{\mathbb{Q}}^m(X) \otimes \Gamma_{\mathbb{Q}}^m(X) \xrightarrow{i^* \otimes i^*} \Gamma_{\mathbb{Q}}^m(Z) \otimes \Gamma_{\mathbb{Q}}^m(Z) \xrightarrow{(\cdot, \cdot)} \mathbb{Q}.$$

By Theorem 4 and the Hard Lefschetz Theorem the restriction morphism  $i^* : \Gamma_{\mathbb{Q}}^m(X) \rightarrow \Gamma_{\mathbb{Q}}^m(Z)$  is injective. Our claim follows from Theorem 5.

**Remark 4.2.2.** Assume that  $K = \mathbb{C}((t))$ . For a smooth projective  $d$ -dimensional scheme  $X$  over  $K$  the isomorphism

$$\Gamma_{\mathbb{Q}}^d(X) \simeq W_0^{\mathbb{Q}} H^d(\lim X)$$

from Remark 4.1.3 carries the monodromy pairing on  $\Gamma_{\mathbb{Q}}^d(X)$  to the pairing

$$(\cdot, \cdot) : W_0^{\mathbb{Q}} H^d(\lim X) \otimes W_0^{\mathbb{Q}} H^d(\lim X) \rightarrow \mathbb{Q}$$

defined by the formula (cf. (4.22))

$$(x, y) = (-1)^{\frac{d(d-1)}{2}} \langle x, y' \rangle,$$

where  $x \in W_0^{\mathbb{Q}}$ ,  $y' \in W_{2d}^{\mathbb{Q}}/W_{2d-1}^{\mathbb{Q}}$  is such that  $\text{Gr}N^d(y') = y$ , and  $\langle \cdot, \cdot \rangle: W_0^{\mathbb{Q}} \otimes W_{2d}^{\mathbb{Q}}/W_{2d-1}^{\mathbb{Q}} \rightarrow \mathbb{Q}$  denotes the Poincaré pairing.

**Example 4.2.3.** Let  $A$  be a  $d$ -dimensional abelian variety over  $K$  with semi-stable reduction. According to ([Ber1], §6.5), after replacing  $K$  by a finite unramified extension, we can represent the analytic space  $A^{\text{an}}$  as the quotient of  $G^{\text{an}}$  by  $\Lambda$ , where  $G^{\text{an}}$  is the analytic group associated with a semi-abelian variety  $0 \rightarrow \bar{T} \rightarrow \bar{G} \rightarrow \bar{B} \rightarrow 0$  over  $R$  and  $\Lambda \xrightarrow{u} \bar{G}(K)$  a lattice. Moreover, the map  $|G_{\widehat{K}}^{\text{an}}| \rightarrow |A_{\widehat{K}}^{\text{an}}|$  exhibits  $|G_{\widehat{K}}^{\text{an}}|$  as a universal cover of  $|A_{\widehat{K}}^{\text{an}}|$ . In particular,  $\Gamma_m(A) := H_m(|A_{\widehat{K}}^{\text{an}}|) \simeq \bigwedge^m \Lambda$ . A polarization,  $\mu$ , of  $A$  defines an isogeny  $\mu_* : \Lambda \rightarrow \Xi$ , where  $\Xi$  is the group of characters of  $\bar{G}$ . Using ([C], Theorem 2.1), we see that the pairing

$$\Gamma_1(A) \otimes \Gamma_1(A) \rightarrow \mathbb{Q}$$

derived from (4.23) equals the pullback of Grothendieck's monodromy pairing

$$\Lambda \otimes \Xi \xrightarrow{u \otimes \text{Id}} \bar{G}(K)/\bar{G}(R) \otimes \Xi \rightarrow \Xi^* \otimes \Xi \rightarrow \mathbb{Z}$$

via  $\text{Id} \otimes \mu_* : \Lambda \otimes \Lambda \rightarrow \Lambda \otimes \Xi$ , divided by the degree  $\mu^d \in \mathbb{Z}$  of the polarization.

### 4.3. A Birational Invariant

Let  $X$  be a smooth connected variety over a complete discrete valuation field  $K$  and  $d = \dim X$ . Assume that  $\Gamma^d(X)_{\mathbb{Q}} \neq 0$ . Let  $\text{Disc}(\cdot, \cdot) \in \mathbb{Q}^*$  be the discriminant of the monodromy pairing (4.21) relative to the lattice  $\Gamma_{\mathbb{Z}}^d(X)/\Gamma_{\mathbb{Z}}^d(X)_{\text{tor}} \subset \Gamma_{\mathbb{Q}}^d(X)$ , and let

$$r_d(X, K) = \frac{1}{\text{Disc}(\cdot, \cdot)}. \quad (4.24)$$

Since the group  $\Gamma_{\mathbb{Z}}^d(X)$  and the monodromy pairing (4.21) are birational invariants of  $X$  so is the number  $r_d(X, K)$ . If  $K \subset K'$  is a finite extension of ramification index  $e$ , we have

$$r_d(X \otimes K', K') = e^{d \dim \Gamma_{\mathbb{Q}}^d(X)} r_d(X, K).$$

In the remaining part of this section we shall relate the invariant  $r_d(X, K)$  defined here to the one introduced in §1.5 for K3 surfaces over  $\mathbb{C}((t))$ .

**Proposition 4.3.1.** *Let  $X$  be a smooth projective K3 surface over  $K = \mathbb{C}((t))$  and let  $H^2(\lim X)$  be the corresponding limit mixed Hodge structure (see §2.2). Set  $W_i^{\mathbb{Z}} := W_i^{\mathbb{Q}} \cap H^2(\lim X, \mathbb{Z})$ . Assume that the monodromy acts on  $H^2(\lim X, \mathbb{Z})$  by a unipotent operator and let  $N : H^2(\lim X, \mathbb{Z}) \rightarrow H^2(\lim X, \mathbb{Z})$  be its logarithm (which is integral by ([FS], Prop. 1.2)). Then*

- (a) *The topological space  $|X_{\widehat{K}}^{an}|$  is contractible unless  $N^2 \neq 0$ . If  $N^2 \neq 0$  the space  $|X_{\widehat{K}}^{an}|$  is homotopy equivalent to a 2-dimensional sphere and the canonical map (see Remark 4.1.3)*

$$\Gamma_{\mathbb{Z}}^2(X) \rightarrow W_0^{\mathbb{Z}} \tag{4.25}$$

*is an isomorphism.*

- (b) *Assume that  $N^2 \neq 0$ . Then the number  $r_2(X, \mathbb{C}((t)))$  defined by (4.24) is equal to the order of the following group*

$$\text{Coker} (W_4^{\mathbb{Z}}/W_3^{\mathbb{Z}} \xrightarrow{Gr N^2} W_0^{\mathbb{Z}}). \tag{4.26}$$

*Proof.* It is enough to prove the proposition in the case where  $X$  is the restriction of a strictly semi-stable family over a smooth curve. Indeed, at the expense of a finite extension of  $K$  we may choose a strictly semi-stable model  $\overline{X}$  for  $X$ . The space

$|X_{\widehat{K}}^{an}|$  is homotopy equivalent to the Clemens polytope of the special fiber  $Y$  of  $\overline{X}$  ([Ber3], §5). Applying Proposition 3.1.1 to  $\overline{X}$  we find a proper strictly semi-stable family  $\overline{\mathcal{X}}$  over a smooth pointed curve  $a \in \overline{C}$ , whose fiber over the first infinitesimal neighborhood of point  $a$  is isomorphic to  $\overline{X} \otimes_R R/t^2$  and whose generic fiber is a K3 surface. As the limit mixed Hodge structure of a strictly semi-stable scheme depends only on the first infinitesimal neighborhood of special fiber the validity of the proposition does not change if we replace  $X$  by  $\mathcal{X} \times_C \text{spec } K'$ .

Thus, we may assume that  $X$  has a Kulikov model over  $R = \mathbb{C}[[t]]$  (see §3.2). If  $\overline{X}$  is a Kulikov model, then the Clemens polytope  $Cl(Y)$  of the special fiber of  $\overline{X}$  is homeomorphic to a point or to an interval for type I or II degenerations and it is homeomorphic to a 2-dimensional sphere for type III degenerations. This proves the first part of the proposition except for the claim that morphism (4.25) is an isomorphism. Using Berkovich's result ([Ber3], §5), the latter is equivalent the following assertion: the canonical morphism

$$H^2(Cl(Y)) \rightarrow W_0^{\mathbb{Z}}$$

coming from the weight spectral sequence (see Theorem 2) is an isomorphism. In fact, the (equivalent) dual statement,

$$W_4^{\mathbb{Z}}/W_3^{\mathbb{Z}} \xrightarrow{\sim} H_2(Cl(Y))$$

is proven (using a deep result of Friedman-Scattone [FS]) in Remark 3.4.1. This completes the proof of the first part of the proposition.

Part (b) of the proposition follows from the fact that (4.25) is an isomorphism and Remark 4.2.2. □

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