REPRESENTATIONS OF THE ORIENTED BRAUER CATEGORY

by

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A DISSERTATION

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DISSERTATION ABSTRACT

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We study the representations of a certain specialization $\mathcal{OB}(\delta)$ of the oriented Brauer category in arbitrary characteristic p. We exhibit a triangular decomposition of $\mathcal{OB}(\delta)$, which we use to show its irreducible representations are labelled by the set of all p-regular bipartitions. We then explain how its locally finite dimensional representations can be used to categorify the tensor product $V(-\varpi_{m'}) \otimes V(\varpi_m)$ of an integrable lowest weight and highest weight representation of the Lie algebra $\mathfrak{sl}_{\Bbbk}$. This is an example of a slight generalization of the notion of tensor product categorification in the sense of Losev and Webster and is the main result of this paper. We combine this result with the work of Davidson to describe the crystal structure on the set of irreducible representations. We use the crystal to compute the decomposition numbers of standard modules as well as the characters of simple modules assuming $p = 0$. We give another proof of the classification of irreducible modules over the walled Brauer algebra. We use this classification to prove that the irreducible $\mathcal{OB}(\delta)$-modules are infinite dimensional unless $\delta = 0$, in which case they are all infinite dimensional except for the irreducible module labelled by the empty bipartition, which is one dimensional.

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DEDICATION

To Ron Clayborn (1955 - 2014). Rock on!

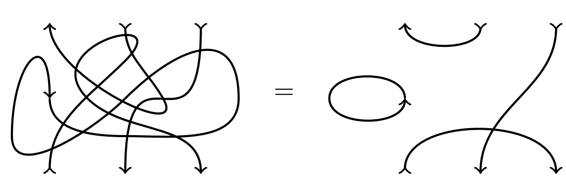
TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

INTRODUCTION

Let $\Bbbk$ be an algebraically closed field of characteristic $p \geq 0$. The *oriented Brauer category* $\mathcal{OB}$ is the free $\Bbbk$-linear symmetric monoidal category generated by a single object $\uparrow$ and its dual $\downarrow$. Its objects are finite sequences of the symbols $\uparrow, \downarrow$, including the empty sequence $\varnothing$, which is the unit object. The set of such sequences is denoted $\langle \downarrow, \uparrow \rangle$. If $\mathsf{a}, \mathsf{b} \in \langle \downarrow, \uparrow \rangle$, then the space $\mathrm{Hom}_{\mathcal{OB}}(\mathsf{a}, \mathsf{b})$ has a basis consisting of *oriented Brauer diagrams with bubbles* of type $\mathsf{a} \to \mathsf{b}$. Such a diagram is obtained by drawing the sequence a below b and pairing the vertices $\uparrow, \downarrow$ by drawing strings in the space between a, b. The strings are allowed to cross, and may connect any two vertices as long as they induce an orientation on the string. For example, a pair $\uparrow, \uparrow$ may be connected by a string if and only if one belongs to a and the other to b; and a pair $\uparrow, \downarrow$ may be connected if and only if they both belong to a or both belong to b. We shall refer to strings which pair two vertices from the bottom row *caps* and those which pair two vertices from the top row *cups*. All other strings will be called *vertical strings*. Additionally, there may be some number of closed (oriented) curves (called *bubbles*) in the space between a, b. Two such diagrams D_1, D_2 are equivalent if they have the same number of bubbles and the remaining strings partition the vertices in a and b in the same way. That is, the apparent topological structure of oriented Brauer diagrams is irrelevant; they are a convenient way of visually representing combinatorial information (see section III.1 for their combinatorial definition). For example, the following two oriented Brauer diagrams with bubbles of type $\uparrow\downarrow\downarrow \to \uparrow\downarrow\downarrow$ are equivalent:



Given two such diagrams D_1, D_2, their composition $D_1 \circ D_2$ is given by vertically stacking D_1 on top of D_2 to get another such diagram with bubbles. The identity morphism of an object a

is simply a row of parallel vertical strings, orientations determined by a.

The $\Bbbk$-linear category $\mathcal{OB}$ is actually linear over the polynomial algebra over $\Bbbk$ by letting the polynomial generator act by adding a bubble. By fixing $\delta \in \Bbbk$ and specializing the polynomial generator at δ we obtain a $\Bbbk$-linear monoidal category $\mathcal{OB}(\delta)$. This amounts to introducing a relation to the presentation of $\mathcal{OB}$ identifying a bubble with $\delta \cdot \varnothing$, where $\varnothing$ denotes the empty diagram, which is the identity morphism of the identity object. So $\mathrm{Hom}_{\mathcal{OB}(\delta)}(\mathsf{a}, \mathsf{b})$ has basis consisting of oriented Brauer diagrams of type $\mathsf{a} \to \mathsf{b}$ *without* bubbles, and composition is performed by vertically stacking diagrams and removing any bubbles formed, multiplying by δ for each bubble removed.

The algebra $\mathrm{End}_{\mathcal{OB}(\delta)}(\downarrow^r\uparrow^s)$ is isomorphic to the walled Brauer algebra $B_{r,s}(\delta)$, which was introduced independently by Turaev [T] and Koike [Ko] in the late 1980s, motivated in part by a Schur-Weyl duality between $B_{r,s}(\delta)$ and $GL_m(\mathbb{C})$ arising from mutually commuting actions on the "mixed" tensor space $V^{\otimes r} \otimes W^{\otimes s}$, where V is the natural representation of $GL_m(\mathbb{C})$ and $W = V^*$. The walled Brauer algebra is spanned by *walled Brauer diagrams* which are obtained by drawing two rows of $r + s$ vertices, one above the other, and drawing strings between pairs of vertices. We imagine a wall separating the first r vertices and the last s vertices in both rows. We require that the endpoints of each string are either on the same row of vertices and opposite sides of the wall, or else they are on opposite rows and the same side of the wall. So forgetting orientations defines a linear isomorphism $\mathrm{End}_{\mathcal{OB}(\delta)}(\downarrow^r\uparrow^s) \to B_{r,s}(\delta)$. The multiplication in $B_{r,s}(\delta)$ is such that this map is an algebra isomorphism. It is worth noting that the Karoubi envelope of $\mathcal{OB}(\delta)$ is Deligne's tensor category $\underline{\mathrm{Rep}}(GL_\delta)$ (see [CW]).

The main goal of this paper is to show how $\mathcal{OB}(\delta)$ can be used to categorify a tensor product of representations of the Lie algebra $\mathfrak{sl}_\Bbbk$. The Lie algebra $\mathfrak{sl}_\Bbbk$ is defined as the Kac Moody Lie algebra associated to the graph with vertices $\Bbbk$ and an edge between i and $i+1$ for each $i \in \Bbbk$. Writing $\delta = m - m'$ for $m, m' \in \Bbbk$ we will ultimately be able to categorify $V(-\varpi_{m'}) \otimes V(\varpi_m)$, the tensor product of an integrable lowest weight representation and an integrable highest weight representation of lowest (resp. highest) weight $-\varpi_{m'}$ (resp. ϖ_m), where ϖ_i is the i^{th} fundamental dominant weight of $\mathfrak{sl}_\Bbbk$ (see section II.3).

For convenience, we often replace $\mathcal{OB}(\delta)$ with a locally unital algebra $OB(\delta)$ whose representations are equivalent to those of $\mathcal{OB}(\delta)$, and we suppress δ in the notation, writing OB for $OB(\delta)$. Briefly, a *locally unital algebra* A is a nonunital algebra with a distinguished collection of mutually orthogonal idempotents $\{1_i : i \in I\}$ satisfying $A = \bigoplus_{i,j \in I} 1_i A 1_j$. It is *locally finite dimensional* if each $1_i A 1_j$ is finite dimensional. In our situation $I = \langle \downarrow, \uparrow \rangle$, 1_{a} is the identity morphism of the object a, and

$$1_{\mathsf{a}} OB 1_{\mathsf{b}} = \mathrm{Hom}_{\mathcal{OB}(\delta)}(\mathsf{b}, \mathsf{a}),$$

which is finite dimensional. The multiplication making $OB = \bigoplus_{\mathsf{a},\mathsf{b} \in \langle \downarrow, \uparrow \rangle} 1_{\mathsf{a}} OB 1_{\mathsf{b}}$ into a locally finite dimensional locally unital algebra is induced by composition in $\mathcal{OB}(\delta)$. By a locally finite dimensional module over a locally unital algebra A, we mean an A-module V satisfying $V = \bigoplus_{i \in I} 1_i V$ and $\dim 1_i V < \infty$ for $i \in I$. The category of such modules is denoted A-mod.

Let $\mathbb{K} = \bigoplus_{\mathsf{a} \in \langle \downarrow, \uparrow \rangle} \Bbbk \cdot 1_{\mathsf{a}}$. The first important observation made in this thesis is that OB has a triangular decomposition $OB = OB^- \otimes_{\mathbb{K}} OB^0 \otimes_{\mathbb{K}} OB^+$ (see section III.2), where OB^+ (resp. OB^-) is the subalgebra spanned by all diagrams with no cups (resp. caps) and no crossings among vertical strings. The role of the Cartan subalgebra is played by OB^0, which is the subalgebra spanned by diagrams with no cups or caps. We observe in section III.4 that OB^0 is Morita equivalent to

$$\bigoplus_{r,s \geq 0} \Bbbk S_r \otimes \Bbbk S_s. \tag{I.0.0.1}$$

Thus the simple modules of OB^0 are parametrized by the set Λ of *p-regular bipartitions*, that is, pairs of partitions which have no p rows of the same length (or simply all pairs of partitions if $p = 0$). We use this triangular decomposition to define the *standard modules* by analogy with the Verma modules. Explicitly, we use a natural projection $OB^0 \otimes_{\mathbb{K}} OB^+ \to OB^0$ to inflate the action of OB^0 on its projective indecomposable modules $Y(\lambda)$, $\lambda \in \Lambda$, to an action of $OB^0 \otimes_{\mathbb{K}} OB^+$, and then induce to an action of OB to obtain the standard module $\Delta(\lambda)$. This construction (inflation followed by induction) defines the *standardization functor* $\Delta : OB^0\text{-mod} \to OB\text{-mod}$.

Mimicking standard arguments from Lie theory, we show in section III.6 that the standard modules have unique irreducible quotients, giving a complete set of inequivalent irreducible OB-modules $\{L(\lambda) : \lambda \in \Lambda\}$. In section III.3 we define a preorder on the set Λ. This preorder is essentially a version of the "inverse dominance order" of Losev and Webster (see Definition 3.2 of [LW]). It should be viewed as the appropriate analog of the Bruhat order from Lie theory. We prove the following theorems in sections IV.3 and V.2, respectively.

Theorem. *If $L(\lambda)$ arises as a subquotient of $\Delta(\mu)$ then $\lambda \leq \mu$.*

Theorem. *The projective cover $P(\lambda)$ of $L(\lambda)$ surjects onto $\Delta(\lambda)$ with kernel having a finite filtration with sections $\Delta(\mu)$ with $\mu > \lambda$.*

This essentially means that the category OB-mod is *locally standardly stratified*, which is a mild weakening of the notion of a standardly stratified category introduced by Losev and Webster in

[LW] as part of the structure required for a tensor product categorification. The remaining data of a tensor product categorification is a categorical action in the sense of Rouquier (see [R]). Roughly, this means that for each $i \in \Bbbk$ there are biadjoint endofunctors E_i, F_i of OB-mod which induce an action of $\mathfrak{sl}_{\Bbbk}$ on the split Grothendieck group of the category of finitely generated projective OB-modules. Additionally, these functors need to be equipped with some endomorphisms (natural transformations) satisfying certain relations as prescribed by Rouquier.

The notion of a categorical action of a Kac-Moody algebra $\mathfrak{g}$ was first defined in a paper of Chuang and Rouquier, [CR] in the case $\mathfrak{g} = \mathfrak{sl}_2$. The general case was defined in [R] and also in the work of Khovanov and Lauda, [KL1]-[KL3]. This new direction was first motivated by the observation that many categories occurring in representation theory, such as the representations of symmetric groups, of Hecke algebras, of the general linear groups or of Lie algebras of type A have endofunctors that on the level of the Grothendieck group give actions of Kac-Moody Lie algebras of type A. These ideas led Chuang and Rouquier first to the definition of categorical actions of type A algebras, which they then generalized to Kac-Moody algebras of arbitrary type.

To construct the categorical $\mathfrak{sl}_{\Bbbk}$-action on OB-mod we begin by defining endofunctors E, F of OB-mod in section IV.1 as tensoring with certain bimodules. The functors E, F are analogous to the induction and restriction functors for the symmetric group. In fact, using obvious notation for induction and restriction between products of symmetric groups (see I.0.0.1), we define endofuctors of OB^0-mod

$$E^{\uparrow} = \bigoplus_{r,s\geq 0} \mathrm{res}^{r,s+1}_{r,s} \qquad E^{\downarrow} = \bigoplus_{r,s\geq 0} \mathrm{ind}^{r+1,s}_{r,s}$$

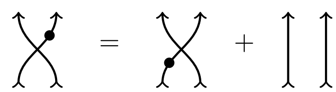
$$F^{\uparrow} = \bigoplus_{r,s\geq 0} \mathrm{ind}^{r,s+1}_{r,s} \qquad F^{\downarrow} = \bigoplus_{r,s\geq 0} \mathrm{res}^{r+1,s}_{r,s}$$

and we show in Theorem IV.2.1 that we have short exact sequences of functors

$$0 \to \Delta \circ E^{\uparrow} \to E \circ \Delta \to \Delta \circ E^{\downarrow} \to 0$$

$$0 \to \Delta \circ F^{\downarrow} \to F \circ \Delta \to \Delta \circ F^{\uparrow} \to 0.$$

To split the functors E, F into direct sums of E_i, F_i, we next construct certain endomorphisms of the bimodules defining E, F, and define E_i, F_i to be the generalized i-eigenspace of E, F, respectively. These endomorphisms come naturally from the *affine oriented Brauer category* $\mathcal{AOB}$, which is obtained from $\mathcal{OB}$ by adjoining an additional monoidal generator, an endomorphism of $\uparrow$, along with an extra relation. If we depict this morphism diagrammatically as a dot on a string oriented upward, then morphisms in $\mathcal{AOB}$ can be unambiguously be represented as *dotted* oriented Brauer diagrams, i.e. oriented Brauer diagrams with some nonnegative number of dots on each segment of each string (including bubbles). In particular, we view $\mathcal{OB}$ as a subcategory of $\mathcal{AOB}$. The extra relation imposed in $\mathcal{AOB}$ is depicted diagrammatically as



This relation comes from the degenerate affine Hecke algebra $\mathcal{H}_n$. If S_n is represented by permutation diagrams and the polynomial generator x_i is represented as a dot on string i, then it is easy to see that $\mathrm{End}_{\mathcal{AOB}}(\uparrow^n)$ is isomorphic to the degenerate affine Hecke algebra $\mathcal{H}_n$. For any choice of $m \in \Bbbk$, there is a full functor $\mathcal{AOB} \to \mathcal{OB}$ which restricts to the identity functor on the subcategory $\mathcal{OB}$ and sends a dot on an $\uparrow$-string on the leftmost boundary to m (Theorem III.1.4). Note that with the identifications $\mathrm{End}_{\mathcal{AOB}}(\uparrow^n) = \mathcal{H}_n$ and $\mathrm{End}_{\mathcal{OB}}(\uparrow^n) = \Bbbk S_n$, a dot on the i^{th} string is sent to $m + L_i$, where L_i is the i^{th} Jucys-Murphy element. We can use this functor to interpret dotted diagrams as morphisms in $\mathcal{OB}$.

The Jucys-Murphy elements split the induction and restriction functors $F^{\downarrow}, E^{\downarrow}, E^{\uparrow}, F^{\uparrow}$ into generalized eigenspaces:

$$E^{\uparrow} = \bigoplus_{i \in \Bbbk} E_i^{\uparrow} \qquad E^{\downarrow} = \bigoplus_{i \in \Bbbk} E_i^{\downarrow}$$
$$F^{\uparrow} = \bigoplus_{i \in \Bbbk} F_i^{\uparrow} \qquad F^{\downarrow} = \bigoplus_{i \in \Bbbk} F_i^{\downarrow}.$$

It follows from [Groj] that the functors $F_i^{\downarrow}, E_i^{\downarrow}, E_i^{\uparrow}, F_i^{\uparrow}$ give a categorical $\mathfrak{sl}_{\Bbbk}^{\downarrow} \oplus \mathfrak{sl}_{\Bbbk}^{\uparrow}$-action on OB^0 -mod under the assignments

$$e_i^{\downarrow} \mapsto E_i^{\downarrow} \qquad f_i^{\downarrow} \mapsto F_i^{\downarrow}$$
$$e_i^{\uparrow} \mapsto E_i^{\uparrow} \qquad f_i^{\uparrow} \mapsto F_i^{\uparrow}$$

(see Theorem V.3.1).

As a vector space, the bimodule defining the functor F is $\bigoplus_{\mathsf{a} \in \langle \downarrow, \uparrow \rangle} OB1_{\mathsf{a}\uparrow}$. The endomorphism of this bimodule which we use to split F is given on $OB1_{\mathsf{a}\uparrow}$ by multiplying by a dot on the bottom right $\uparrow$. The endomorphism for E is defined similarly, using a dot on a $\downarrow$ string.

Theorem. *The above short exact sequences split:*

$$0 \to \Delta \circ E_i^{\uparrow} \to E_i \circ \Delta \to \Delta \circ E_i^{\downarrow} \to 0$$

$$0 \to \Delta \circ F_i^{\downarrow} \to F_i \circ \Delta \to \Delta \circ F_i^{\uparrow} \to 0.$$

We use these split sequences to compute the formal characters of the standardized Specht modules $\widetilde{\Delta}(\lambda)$, that is, the modules obtained by applying the standardization functor Δ to the Specht modules for OB^0. The coefficients are the numbers of paths of various types in the *branching graph*. The vertices of the branching graph are all bipartitions and there is an edge between bipartitions λ and μ whenever $\mu = (\mu^{\downarrow}, \mu^{\uparrow})$ is obtained by adding a box to $\lambda = (\lambda^{\downarrow}, \lambda^{\uparrow})$. If the box is added to row i and column j of $\lambda^{\uparrow}$, then the edge is colored $m + j - i$, read mod p. If the box is added to row i and column j of $\lambda^{\downarrow}$, then the edge is colored $m' + i - j$, mod p (see section IV.3).

We now state our main theorem (Theorem V.3.2).

Theorem. *The endofunctors E_i, F_i of OB-mod together with certain natural transformations defined in detail in section IV.2 define a categorical $\mathfrak{sl}_{\Bbbk}$-action. This action is compatible with the locally stratified structure on OB-mod and makes OB-mod into a (generalized) tensor product categorification of $V(-\varpi_{m'}) \otimes V(\varpi_m)$ in the sense of [LW].*

This theorem allows us to apply the main result of Davidson [D] which implies that $E_i L(\lambda)$ is either zero or else its head and socle are both isomorphic to some simple module parameterized by some bipartition $\widetilde{e}_i\lambda$. This result of Davidson is a generalization of a result of Chuang and Rouquier [CR], which in turn extended ideas of Grojnowski, Vazirani and Kleshchev (see [K]). Letting $L(\widetilde{e}_i\lambda)$ denote the zero module in the event that $E_i L(\lambda) = 0$, we then have $\mathrm{head}(E_i L(\lambda)) = L(\widetilde{e}_i\lambda)$ for all $i \in \Bbbk$ and $\lambda \in \Lambda$. A similar statement holds for $F_i L(\lambda)$: $\mathrm{head}(F_i L(\lambda)) = L(\widetilde{f}_i\lambda)$. The main results of this paper and [D] also enable us to compute the bipartitions $\widetilde{e}_i\lambda, \widetilde{f}_i\lambda$ explicitly, which we do in section V.4. The graph whose vertices are Λ with an edge colored i between λ and $\widetilde{f}_i\lambda$ whenever $\widetilde{f}_i\lambda \neq 0$ is called the *crystal graph*. It is the Kashiwara tensor product of the crystals associated to the $\mathfrak{sl}_{\Bbbk}$-modules $V(-\varpi_{m'}), V(\varpi_m)$, which are known (see section 11.1 of [K]). In characteristic zero we are able to use the crystal to describe the actions of E_i, F_i on the projective covers of the simple modules, which we require for our proof of the following theorem (see section VI.1).

Theorem. *If $p = 0$ the composition multiplicities of standard modules and the characters of the simple modules can be computed by explicit combinatorics as described in chapter VI of the present thesis.*

As a final application we recover the classification of irreducible $B_{r,s}(\delta)$-modules, which was first proved by A. Cox, M. De Visscher, S. Doty and P. Martin, (see [CDDM]). We then use this classification to show that $L(\lambda)$ is (globally) finite dimensional if and only if $\delta = 0$ and $\lambda = (\varnothing, \varnothing)$, in which case it is one dimensional.

Organization of thesis

In Chapter II we recall some classical theory and introduce some slight modifications of standard notions. We recite some facts about the representation theory of the symmetric group, which is central to our approach. Next we define locally unital algebras and develop some theory including versions of Schur's Lemma and the Krull-Schmidt theorem. We also define the Lie algebra $\mathfrak{sl}_{\Bbbk}$, which is the Kac-Moody algebra underlying the categorical action we present here. We then recall some of the classical theory of quotient functors and then discuss a weakened version of standard stratification (see [LW]).

Chapter III introduces our main object of study, the oriented Brauer category. We describe its triangular decomposition, which enables us to use some techniques from Lie theory. We show in section III.4 that its Cartan subalgebra is Morita equivalent to the direct sum of all group algebras of products of two symmetric groups. After defining a duality functor on a certain category of representations of the oriented Brauer category, we prove the classification of its simple modules.

The endofunctors E, F of OB-mod leading to the categorical $\mathfrak{sl}_{\Bbbk}$-action are defined in chapter IV. We also define the endofunctors $E^{\uparrow}, E^{\downarrow}, F^{\uparrow}, F^{\downarrow}$ of OB^0-mod and explain their relation to E, F, namely the short exact sequences mentioned above. We use this relation to compute the formal characters of the standardized Specht modules in terms of the branching graph, which is defined in section IV.3.

In chapter V we prove our main theorem. First we show that OB-mod is standardly stratified in sections V.1 and V.2. We then explain the compatibility of this stratified structure with the categorical $\mathfrak{sl}_{\Bbbk}$-action in section V.3. The description of the crystal graph is given in section V.4.

In the final chapter of this thesis we use the above crystal to determine the composition multiplicities of the standard modules and the characters of the simple modules, assuming $p = 0$. Finally, we deduce the known classification of simple $B_{r,s}(\delta)$-modules due to Cox et al. (see [CDDM]) and use it to prove $L(\lambda)$ is (globally) finite dimensional if and only if $\delta = 0$ and $\lambda = (\varnothing, \varnothing)$, in which case it is one dimensional.

CHAPTER II

PRELIMINARIES

In this chapter we gather some basic facts which we shall need throughout this paper. In section II.1 we recall some classical results about the representation theory of the symmetric group. In section II.2 we define the notion of a locally unital algebra and develop some basic theory, similar to that of unital algebras. Next we introduce the Lie algebra $\mathfrak{sl}_{\Bbbk}$ and its weights in section II.3. We recall some basic facts about quotient functors and idempotent truncations in section II.4. Finally, we define locally stratified categories in section II.5.

II.1. The Symmetric Group

Let S_n denote the symmetric group on n letters and $\Bbbk S_n$ -mod the category of finite dimensional $\Bbbk S_n$-modules. A *central character* is an algebra homomorphism $\chi : Z(\Bbbk S_n) \to \Bbbk$. The central characters split $\Bbbk S_n$ -mod into blocks:

$$\Bbbk S_n\text{-mod} = \bigoplus_{\chi} \Bbbk S_n\text{-mod}[\chi],$$

where $\Bbbk S_n$ -mod$[\chi]$ consists of those S_n-modules V such that $(z - \chi(z))^N V = 0$ for $N \gg 0$ (see section 1.1 in [K]).

We identify a partition with its Young diagram as usual. We use the English convention so that the top row represents the first part of the partition. For $p > 0$, a partition λ is called *p-regular* if for any $k > 0$ we have $\#\{j : \lambda_j = k\} < p$. In terms of the Young diagram, this means that no p rows have the same length. By definition, all partitions are 0-regular. Let $\mathcal{P}_p(n)$ be the set of p-regular partitions of n.

Given a partition λ of n, we define the *content* of a box in row i and column j to be $j - i \pmod p$. The *content* of λ is the tuple $\mathrm{cont}(\lambda) = (\gamma_i)_{i \in \mathbb{Z}\cdot 1_{\Bbbk}}$, where γ_i is the number of boxes of content i.

For any partition λ of n there is an explicit construction (see [J]) of a finite dimensional

$\Bbbk S_n$-module $S(\lambda)$, the corresponding *Specht module*. Assuming λ is p-regular, it is known that $S(\lambda)$ has irreducible head, which we denote by $D(\lambda)$.

Theorem II.1.1. *The modules $\{D(\lambda) : \lambda \in \mathcal{P}_p(n)\}$ form a complete set of inequivalent irreducible $\Bbbk S_n$-modules. Moreover, for $\lambda, \mu \in \mathcal{P}_p(n)$ we have:*

(i) $D(\lambda)$ is self-dual;

(ii) modules $D(\lambda)$ and $D(\mu)$ belong to the same block of $\Bbbk S_n$ if and only if $\operatorname{cont}(\lambda) = \operatorname{cont}(\mu)$.

Proof. This is part of Theorem 11.2.1 in [K]. □

Since $\Bbbk S_n$ is a symmetric algebra, the projective cover of $D(\lambda)$ is isomorphic to its injective envelope, the corresponding *Young module* $Y(\lambda)$. Actually Young modules are defined for any partition, but shall only need those associated to p-regular partitions which happen to be projective covers and injective envelopes as previously mentioned.

Define the k^{th} *Jucys-Murphy element* $L_k \in \Bbbk S_n$ by

$$L_k := \sum_{1 \leq m < k} (m,\, k).$$

Embedding $\Bbbk S_{n-1} \subset \Bbbk S_n$ with respect to the first $n-1$ letters, it is easy to see that L_n commutes with $\Bbbk S_{n-1}$. In particular, the Jucys-Murphy elements commute with each other. Now let $\underline{i} = (i_1, \ldots, i_n) \in \Bbbk^n$ and let V be a finite dimensional $\Bbbk S_n$-module. We let $V_{\underline{i}}$ denote its simultaneous generalized eigenspace of $L_1, \ldots, L_n$ corresponding to the tuple of eigenvalues $\underline{i}$. That is,

$$V_{\underline{i}} = \{v \in V : (L_k - i_k)^N v = 0 \text{ for } k = 1, \ldots, n, \text{ and } N \gg 0\}.$$

Now define the *formal character* of a $\Bbbk S_n$-module to be the following element of the free $\mathbb{Z}$-module on basis $\{e^{\underline{i}} : \underline{i} \in \Bbbk\}$.

$$\operatorname{ch} V := \sum_{\underline{i} \in \Bbbk^n} \left(\dim V_{\underline{i}}\right) e^{\underline{i}}.$$

Lemma II.1.2. *The characters of the irreducible $\Bbbk S_n$-modules are $\mathbb{Z}$-linearly independent.*

Proof. This is Lemma 11.2.5 in [K]. □

II.2. Locally Unital Algebras

By a *locally unital $\Bbbk$-algebra* we mean a non-unital associative $\Bbbk$-algebra A containing a distinguished collection of mutually orthogonal idempotents $\{1_i : i \in I\}$, for some index set I, such that

$$A = \bigoplus_{i,j \in I} 1_i A 1_j. \tag{II.2.0.1}$$

We can build a locally unital $\Bbbk$-algebra A out of a small $\Bbbk$-linear category $\mathcal{A}$ as follows. Set

$$A = \bigoplus_{\mathsf{a},\mathsf{b} \in \mathrm{ob}(\mathcal{A})} \mathrm{Hom}_{\mathcal{A}}(\mathsf{a}, \mathsf{b}) \tag{II.2.0.2}$$

and define multiplication of two composable morphisms to be their composition, and let the product of two non-composable morphisms be zero. It is easy to see that this makes A into a locally unital $\Bbbk$-algebra, with the identity morphisms in $\mathcal{A}$ serving as the system of idempotents. For example, we have $1_{\mathsf{a}} A 1_{\mathsf{b}} = \mathrm{Hom}_{\mathcal{A}}(\mathsf{b}, \mathsf{a})$ so that (II.2.0.1) is satisfied.

A homomorphism of locally unital algebras is an algebra homomorphism which sends distinguished idempotents to distinguished idempotents. This corresponds to the notion of a $\Bbbk$-linear functor $\mathcal{A} \to \mathcal{B}$ if A, B are built from $\mathcal{A}, \mathcal{B}$ as above.

By a *module* over a locally unital $\Bbbk$-algebra $A = \bigoplus_{i,j \in I} 1_i A 1_j$, we will always mean a *locally unital module*, that is, a module V such that

$$V = \bigoplus_{i \in I} 1_i V. \tag{II.2.0.3}$$

We denote by A-Mod the category of all such modules. The nonzero spaces $1_i V$ are called *weight spaces*, and the nonzero elements of $1_i V$ are called *weight vectors*. We note that if A is built out of a small $\Bbbk$-linear category $\mathcal{A}$ as above then A-Mod is equivalent to the category of representations of $\mathcal{A}$, ie. $\Bbbk$-linear functors $\mathcal{A} \to \Bbbk$-Vec to the category of $\Bbbk$-vector spaces. A module V is said to be *locally finite dimensional* if each $1_i V$ is finite dimensional. The category of all locally finite dimensional A-modules will be denoted A-mod. A locally unital algebra A is called *locally finite dimensional* if each $1_i A 1_j$ is finite dimensional.

Suppose A is a locally unital algebra, and V is a finitely generated A-module with generators $v_1, \ldots, v_n$. Since A is locally unital, $1_i v_j$ is nonzero for only finitely many choices of i, j, and v_j is the sum of the nonzero $1_i v_j$. So we may assume $v_1, \ldots, v_n$ are weight vectors. Let V_0 denote their $\Bbbk$-linear span in V. Then V_0 is a *homogeneous generating subspace*, that is, a subspace V_0 satisfying $V_0 = \bigoplus_{i \in I} 1_i V_0$ which generates V. We have shown that V has a finite dimensional homogeneous generating subspace.

Proposition II.2.1. *If A is locally finite dimensional, then every finitely generated A-module is* *locally finite dimensional.*

Proof. Choose a finite dimensional homogeneous generating subspace V_0 of V. Then V is a quotient of the finitely generated projective module

$$P = \bigoplus_{i \in I} A1_i^{\oplus \dim 1_i V_0}. \tag{II.2.0.4}$$

In particular, $1_j V$ is the image of $1_j P = \bigoplus_{i\in I} 1_j A 1_i^{\oplus \dim 1_i V_0}$. Now $1_j A 1_i^{\oplus \dim 1_i V_0}$ is nonzero for only finitely many i, and these summands are all finite dimensional as A is locally finite dimensional and V_0 is finite dimensional. So $1_j V$ is the image of the finite dimensional vector space $1_j P$, which proves the proposition. $\square$

We have the following version of Schur's lemma.

Lemma II.2.2. *If A is locally finite dimensional, and L a simple A-module, then* $\mathrm{End}_A(L) = \Bbbk$.

Proof. Let $f \in \mathrm{End}_A(L)$. For each $i \in I$, f restricts to a linear endomorphism f_i of $1_i L$, which is finite dimensional by Proposition II.2.1. Fix any $i \in I$. As $\Bbbk$ is algebraically closed, f_i has some eigenvalue λ. Then $\ker(f - \lambda \cdot \mathrm{Id}) \neq 0$ implies that $f = \lambda \cdot \mathrm{Id}$. $\square$

Proposition II.2.3. *Assume A is any locally unital algebra. Let V be a finitely generated A-module and W a locally finite dimensional A-module. Then* $\mathrm{Hom}_A(V, W)$ *is finite dimensional. In particular, if A is locally finite dimensional, then* $\mathrm{End}_A(V)$ *is finite dimensional.*

Proof. Choose a finite dimensional homogeneous generating subspace V_0 of V. There is a finite subset $J \subset I$ such that $V_0 = \bigoplus_{i\in J} 1_i V_0$. Any $f \in \mathrm{Hom}_A(V, W)$ is determined by its restriction to V_0. So we have

$$\dim \mathrm{Hom}_A(V, W) \leq \sum_{i \in J} \dim \mathrm{Hom}_{\Bbbk}(1_i V_0, 1_i W) = \sum_{i\in J} \dim 1_i V_0 \cdot \dim 1_i W < \infty.$$

$\square$

Corollary II.2.4. *If A is locally finite dimensional, then every finitely generated A-module V has a direct sum decomposition into finitely many indecomposable modules. This decomposition is essentially unique in the sense of the Krull-Schmidt theorem.*

Proof. If V is indecomposable, then $\mathrm{End}_A(V)$ is a finite dimensional algebra in which 1 is a primitive idempotent, hence a local ring. This observation proves uniqueness in the usual way (see Theorem 7.5 of [La]).

To prove existence, suppose there is some finitely generated module which is not a finite direct sum of indecomposables. Choose such a module V, which has a finite dimensional homogeneous generating subspace V_0 satisfying $V_0 = \bigoplus_{i \in J} 1_i V$ for some finite subset $J \subset I$. Choose V with the dimension of such V_0 minimal. Then as V is not indecomposable, we have $V = V_1 \oplus V_2$ for some nonzero submodules V_1, V_2. It is clear that each V_k is generated by $\bigoplus_{i \in J} 1_i V_k$, which has smaller dimension than V_0. By minimality, we must have that V_1, V_2 are each a finite direct sum of indecomposables. Therefore so is V. $\square$

Proposition II.2.5. *If A is locally finite dimensional, then every simple A-module L has a projective cover.*

Proof. We know L is a quotient of some finitely generated projective module (see (II.2.0.4)). Take an indecomposable summand P mapping onto L. Call this map π. Then to see that P is a projective cover of L, suppose $Q \subset P$ is a submodule which maps onto L. Then projectivity gives a map $g : P \to Q$ such that $\pi \circ f \circ g = \pi$, where $f : Q \to P$ is inclusion. Then $1 - f \circ g$ is a non-unit in $\mathrm{End}_A(P)$, which is a local ring as P is indecomposable. Therefore $f \circ g$ is an isomorphism. In particular f is surjective, so $Q = P$. So P is a projective cover of L. $\square$

Let L be a simple A-module. For $V \in A\text{-Mod}$ we define the *composition multiplicity* of L in V as usual by

$$[V : L] = \sup \#\{i : V_{i+1}/V_i \cong L\} \tag{II.2.0.5}$$

the supremum being taken over all filtrations by submodules $0 = V_0 \subset \cdots \subset V_n = V$. If $[V : L] \neq 0$, we call L a *composition factor* of V, even if V does not have a composition series.

Proposition II.2.6. *If V is locally finite dimensional, then $[V : L]$ is finite for any simple L. Moreover, if P is a projective cover of L, then we have*

$$[V : L] = \dim \mathrm{Hom}_A(P, V)$$

Proof. Choose $i \in I$ so that $1_i L \neq 0$. Suppose n sections of a given filtration of V are isomorphic to L. Then $n \leq \dim 1_i V$. Hence $[V : L] \leq \dim 1_i V < \infty$. In particular, the supremum $\sup \#\{i : V_{i+1}/V_i \cong L\}$ is attained by some finite filtration of V. It remains to show that $\dim \mathrm{Hom}_A(P, V)$ is equal to the number of sections in this filtration which are isomorphic to L. This follows from exactness of $\mathrm{Hom}_A(P, ?)$. $\square$

Right modules and bimodules are defined for locally unital algebras in the obvious way. If $(A, I), (B, J)$ are locally unital algebras and V is a (B, A)-bimodule, then $V \otimes_A ?$ is a well-defined functor A-Mod $\to$ B-Mod. Moreover, if V is locally finite dimensional as a B-module, then $V \otimes_A ?$ is a well-defined functor A-mod $\to$ B-mod. However, if I is infinite then $\mathrm{Hom}_B(V, ?)$ is not a well-defined functor B-Mod $\to$ A-Mod. This is because $\mathrm{Hom}_B(V, W)$ need not be a locally unital module if I is infinite:

$$\mathrm{Hom}_B(V, W) = \mathrm{Hom}_B\left(\bigoplus_{i \in I} V1_i, W\right) = \prod_{i \in I} \mathrm{Hom}_B(V1_i, W) \tag{II.2.0.6}$$

Instead, the right adjoint of $V \otimes_A ?$ is $\bigoplus_{i \in I} \mathrm{Hom}_B(V1_i, ?)$. In particular, if $B \subset A$ is a subalgebra with $I = J$, then induction $A \otimes_B ?$ is left adjoint to restriction $\bigoplus_{j \in J} \mathrm{Hom}_A(A1_j, ?) \cong A \otimes_A ?$, which is left adjoint to coinduction $\bigoplus_{i \in I} \mathrm{Hom}_B(A1_i, ?)$.

II.3. The Lie Algebra $\mathfrak{sl}_{\Bbbk}$

Make $\Bbbk$ into a graph by connecting i and $i+1$ for each $i \in \Bbbk$. The result is a disjoint union of graphs of type A_∞ if $p = 0$ or $\widetilde{A}_{p-1}$ if $p > 0$. We let $\mathfrak{sl}_{\Bbbk}$ denote the Kac-Moody Lie algebra associated to this graph (see chapter 1 of [Kac]). So

$$\mathfrak{sl}_{\Bbbk} = \bigoplus \widehat{\mathfrak{sl}}_p$$

where $\widehat{\mathfrak{sl}}_0 = \mathfrak{sl}_\infty$ and the sum is over the cosets of $\mathbb{Z} \cdot 1_{\Bbbk}$ in $\Bbbk$.

Alternatively, $\mathfrak{sl}_{\Bbbk}$ can be described as follows. First assume $p > 0$ and choose a set $\widetilde{\Bbbk}$ of $\mathbb{Z} \cdot 1_{\Bbbk}$-coset representatives in $\Bbbk$. Let $\mathfrak{h}$ be the vector space with basis $\{\alpha_i^\vee : i \in \Bbbk\} \cup \{d_i : i \in \widetilde{\Bbbk}\}$. Define the *weight lattice*

$$P = \{\lambda \in \mathfrak{h}^* : \lambda(\alpha_i^\vee) \in \mathbb{Z}, \text{ for all } i \in \Bbbk\}. \tag{II.3.0.7}$$

Inside P there are elements $\{\varpi_i : i \in \Bbbk\} \cup \{\delta_i : i \in \widetilde{\Bbbk}\}$ which are dual to the chosen basis of $\mathfrak{h}$. That is,

$$\begin{aligned} \varpi_i(\alpha_j^\vee) &= \delta_{i,j} & \qquad \varpi_i(d_j) &= 0 \\ \delta_i(\alpha_j^\vee) &= 0 & \qquad \delta_i(d_j) &= \delta_{i,j}, \end{aligned}$$

where $\delta_{i,j}$ is the Kronecker delta. We define the *simple roots* $\alpha_i = 2\varpi_i - \varpi_{i-1} - \varpi_{i+1}$ and set $c_{ij} = \alpha_i(\alpha_j^\vee)$.

Then $\mathfrak{sl}_{\Bbbk}$ is generated by $\{e_i, f_i\}_{i\in\Bbbk}$ and $\mathfrak{h}$ subject only to the following relations:

$$[h,h'] = 0 \qquad [e_i, f_i] = \delta_{i,j}\alpha_i^\vee$$

$$[h,e_i] = \alpha_i(h)e_i \qquad [h,f_i] = -\alpha_i(h)f_i$$

$$(\text{ad } e_i)^{1-c_{ij}}(e_j) = 0 \qquad (\text{ad } f_i)^{1-c_{ij}}(f_j) = 0$$

for all $h, h' \in \mathfrak{h}$ and $i \in \Bbbk$.

In characteristic zero $\mathfrak{sl}_{\Bbbk}$ can be described as the set of finitely supported, traceless matrices $(a_{ij})_{i,j\in\Bbbk}$ with $a_{ij} = 0$ unless i, j lie in the same coset of $\mathbb{Z}\cdot 1_{\Bbbk}$. Explicitly, $a_{ij} \neq 0$ for only finitely many choices of i, j, each such pair necessarily lying in the same coset of $\mathbb{Z}\cdot 1_{\Bbbk}$, and $\sum_i a_{ii} = 0$. The Lie bracket is given by the commutator of matrices. Let $\mathfrak{h}$ be the *Cartan subalgebra* of diagonal matrices in $\mathfrak{sl}_{\Bbbk}$. Let e_i denote that matrix with a 1 in row i and column $i+1$ and zeros elsewhere. Let f_i be the transpose of e_i and set $\alpha_i^\vee = [e_i, f_i]$. Define the weight lattice P as in (II.3.0.7). Inside P we have the diagonal coordinate function ε_i defined by $\varepsilon_i(\text{diag}(a_j)) = a_i$. Set $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. Note that the infinite sum $\varpi_i := \sum\limits_{i-j\in\mathbb{Z}_{\geq 0}\cdot 1_{\Bbbk}} \varepsilon_j$ can be interpreted as an element of $\mathfrak{h}^*$. That is, ϖ_i is the function $\mathfrak{h} \to \Bbbk$ sending $\text{diag}(a_i)$ to $\sum\limits_{i-j\in\mathbb{Z}_{\geq 0}\cdot 1_{\Bbbk}} a_j$, which is a finite sum as matrices in $\mathfrak{h}$ have finite support.

In any characteristic, the weight lattice P is partially ordered by *dominance*. Explicitly, given $x, y \in P$, we define $x \leq y$ if $y - x \in \bigoplus_{i\in\Bbbk} \mathbb{Z}_{\geq 0}\cdot \alpha_i$.

II.4. Quotient Functors

Let $\mathcal{A}$ be an abelian category. A full subcategory $\mathcal{C}$ is called a *Serre subcategory* (sometimes called a *thick subcategory*) if for every short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

in $\mathcal{A}$, M is an object in $\mathcal{C}$ if and only if M', M'' are objects in $\mathcal{C}$ (see chapter 3 of [G]). Given an abelian category $\mathcal{A}$ and a Serre subcategory $\mathcal{C}$, we can form the *quotient category* $\mathcal{A}/\mathcal{C}$ as follows.

- The objects of $\mathcal{A}/\mathcal{C}$ are the same as those of $\mathcal{A}$.
- $\text{Hom}_{\mathcal{A}/\mathcal{C}}(M,N) = \varinjlim\limits_{M',N'} \text{Hom}_{\mathcal{A}}(M', N/N')$,

where the limit runs through all subobjects $M' \subset M$, $N' \subset N$ such that M/M' and N' are objects of $\mathcal{C}$. We define a functor $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ by $\pi(M) = M$ for any object M, and given a morphism

$f : M \to N$, $\pi(f)$ is the image of f in the inductive limit $\varinjlim_{M',N'} \mathrm{Hom}_{\mathcal{A}}(M', N/N')$. We call π a *quotient functor*, or the *canonical functor* $\mathcal{A} \to \mathcal{A}/\mathcal{C}$.

Proposition II.4.1. *If $\mathcal{C}$ is a Serre subcategory of $\mathcal{A}$, then the category $\mathcal{A}/\mathcal{C}$ is abelian and the canonical functor $\mathcal{A} \to \mathcal{A}/\mathcal{C}$ is an exact additive functor.*

Proof. This is Lemme 1 and Proposition 1 of [G] combined. See section 1 of chapter 3 of [G] for their proofs. □

The above quotient has the following universal property.

Corollary II.4.2. *Let $\mathcal{A}, \mathcal{D}$ be abelian categories. If $\mathcal{C}$ is a Serre subcategory of $\mathcal{A}$ and $G : \mathcal{A} \to \mathcal{D}$ is an exact functor with $G(M) = 0$ for any object M of $\mathcal{C}$, then there is a unique functor $H : \mathcal{A}/\mathcal{C} \to \mathcal{D}$ such that $G = H \circ \pi$. Moreover, the functor H is exact.*

Proof. The first statement is Corollaire 2 of [G]. Exactness of H follows from Corollaire 3 of [G]. See section 1 of chapter 3 of [G] for their proofs. □

We conclude this section by giving an important example of a quotient functor, namely *idempotent truncation* (see section V.1).

Let A be a locally finite dimensional locally unital algebra, and let $e \in A$ be a nonzero idempotent. Then e lies in the span of finitely many of the $1_i A 1_j$, which implies that the unital algebra eAe lies in the same (finite dimensional) subspace. We recall the functor $f : A\text{-mod} \to eAe\text{-mod}$ from section 6.2 of [Gr]. If $V \in A$-mod, then the subspace eV is an eAe-module, and we define $f(V) = eV$. If $\theta : V \to V'$ is a morphism in A-mod, then $f(\theta)$ is the restriction of f to eV. It is easy to see that f is exact.

Theorem II.4.3. *The functor f is the quotient functor by the Serre subcategory consisting of modules V with $eV = 0$.*

Proof. First note that the functor $h = Ae \otimes_{eAe} ?$ is isomorphic to a right inverse to f. Let $\mathcal{D}$ be an abelian category and $G : A\text{-mod} \to \mathcal{D}$ an exact functor satisfying $G(V) = 0$ whenever $eV = 0$. If $H : eAe\text{-mod} \to \mathcal{D}$ satisfies $G = H \circ f$, then composing on the right with h gives $H = G \circ h$. On the other hand, if we define $H = G \circ h$, then we have $G = H \circ f$ as follows. The product map $m : Ae \otimes_{eAe} eV \to V$ fits into an exact sequence

$$0 \to K \to Ae \otimes_{eAe} eV \xrightarrow{m} V \to C \to 0$$

where $K = \ker f$ and $C = \operatorname{coker} f$. Since the restriction of m to $eAe \otimes_{eAe} eV$ is an isomorphism, we have $f(K) = f(C) = 0$ and therefore $G(K) = G(C) = 0$. Applying the exact functor G to the above exact sequence now shows that G is isomorphic to $G \circ h \circ f = H \circ f$. □

Theorem II.4.4. *Suppose $\{V_\lambda : \lambda \in \Lambda\}$ is a full set of irreducible modules in A-*mod*, indexed by a set Λ, and let $\Lambda' = \{\lambda \in \Lambda : eV_\lambda \neq 0\}$. Then $\{eV_\lambda : \lambda \in \Lambda'\}$ is a full set of irreducible modules in eAe-*mod*.*

Proof. This is Theorem 6.2g of [Gr]. The proof given in section 6.2 of [Gr] is valid for any locally unital algebra, providing we interpret $(1-e)V$ as the kernel of the linear endomorphism of V given by multiplication by e. □

II.5. Standard Stratification

We need to relax the finiteness conditions slightly in the definition of a standardly stratified category given in [LW]. Specifically, let $\mathcal{C}$ denote an abelian category which is equivalent to A-mod for some locally finite dimensional locally unital $\Bbbk$-algebra A. Let Λ be a preordered set with a fixed bijection $\lambda \mapsto L(\lambda)$ to the isomorphism classes of simple objects in $\mathcal{C}$. Fix a projective cover $P(\lambda)$ for $L(\lambda)$ in $\mathcal{C}$, which exists by Proposition II.2.5.

Let Ξ be the poset induced by Λ. That is, Ξ is the quotient of Λ by the equivalence relation which identifies λ, μ whenever $\lambda \leq \mu$ and $\lambda \geq \mu$. Given $\xi \in \Xi$, let $\mathcal{C}_{\leq \xi}$ (resp. $\mathcal{C}_{<\xi}$) be the full subcategory of $\mathcal{C}$ consisting of all modules whose composition factors $L(\lambda)$ satisfy $[\lambda] \leq \xi$ (resp. $[\lambda] < \xi$). Note that $\mathcal{C}_{<\xi}$ is a Serre subcategory of $\mathcal{C}_{\leq\xi}$ (see section II.4). Set $\mathcal{C}_\xi := \mathcal{C}_{\leq\xi}/\mathcal{C}_{<\xi}$. For $\lambda \in \xi$, let $L_\xi(\lambda)$ denote the image of $L(\lambda)$ in $\mathcal{C}_\xi$. Let $P_\xi(\lambda)$ denote its projective cover in $\mathcal{C}_\xi$. Let π_ξ denote the (exact) quotient functor $\mathcal{C}_{\leq\xi} \to \mathcal{C}_\xi$ (see section II.4). By the general theory pi_ξ has a left adjoint. We fix one and denote it henceforth by Δ_ξ.

We call the category $\mathcal{C}$ as above a *locally stratified* category if Δ_ξ is exact and, setting $\Delta(\lambda) = \Delta_{[\lambda]}(P_{[\lambda]}(\lambda))$, there is an epimorphism $P(\lambda) \to \Delta(\lambda)$ whose kernel admits a finite filtration by objects $\Delta(\mu)$ with $\mu > \lambda$. Said differently, we define a *standard filtration* or Δ-*flag* of a module to be a finite filtration with with sections isomorphic to standard modules. Then $P(\lambda)$ is required to have a standard filtration with $\Delta(\lambda)$ at the top, and other sections isomorphic to $\Delta(\mu)$ for $\mu > \lambda$.

Given a locally stratified category $\mathcal{C}$, we define the *associated graded category* to be $\operatorname{gr}\mathcal{C} = \bigoplus_{\xi\in\Xi} \mathcal{C}_\xi$. We call $\Delta = \bigoplus \Delta_\xi : \operatorname{gr}\mathcal{C} \to \mathcal{C}$ the *standardization functor.* The objects $\Delta(\lambda)$ are called the *standard objects* and the objects $\overline{\Delta}(\lambda) = \Delta_{[\lambda]}(L_{[\lambda]}(\lambda))$ are called the *proper standard objects.*

We remark that more classical notion of a highest weight category of [CPS] is a special case of this notion of local stratification. If each $\mathcal{C}_\xi$ is equivalent to the category of finite dimensional $\Bbbk$-vector spaces, then $\mathcal{C}$ is called a *locally highest weight category*. This is the case if the preorder on Λ is actually a partial order. If in addition, the set Λ is finite, then the subcategory of $\mathcal{C}$ consisting of objects of finite length is a highest weight category in the sense of [CPS].

CHAPTER III

THE ALGEBRA OB

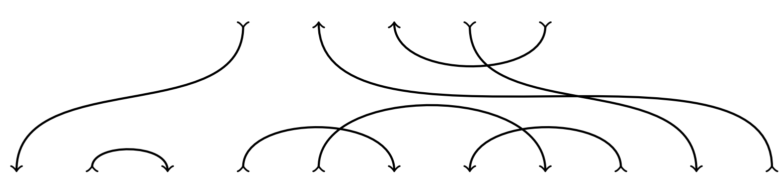
In this chapter we meet the oriented Brauer categories and their locally unital algebra counterparts. Their definitions are given in section III.1. We describe its triangular decomposition in section III.2. The poset Λ parameterizing the simple OB-modules is defined in section III.3. We then show in section III.4 that OB^0 and $\bigoplus_{r,s\geq 0} \Bbbk S_r \otimes \Bbbk S_s$ are Morita equivalent. In section III.5 we define a duality functor on OB-mod. We then give the classification of simple OB-modules in section III.6.

III.1. Oriented Brauer Categories

Let $\langle \downarrow, \uparrow \rangle$ denote the set of all words in the alphabet $\{\uparrow, \downarrow\}$, including the empty word $\varnothing$. Given two words $\mathsf{a} = \mathsf{a}_1 \dots \mathsf{a}_n, \mathsf{b} = \mathsf{b}_1 \dots \mathsf{b}_m \in \langle \downarrow, \uparrow \rangle$, an *oriented Brauer diagram* of type $\mathsf{a} \to \mathsf{b}$ is a diagrammatic representation of a bijection

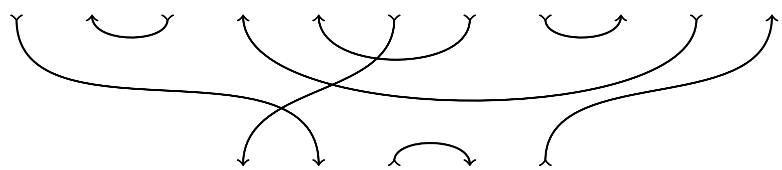
$$\{i : \mathsf{a}_i = \uparrow\} \cup \{i : \mathsf{b}_i = \downarrow\} \to \{i : \mathsf{a}_i = \downarrow\} \cup \{i : \mathsf{b}_i = \uparrow\}.$$

We draw such a diagram by aligning the sequences a, b in two rows, b above a, and drawing consistently oriented strands between a and b connecting pairs of letters prescribed by the bijection. For example,

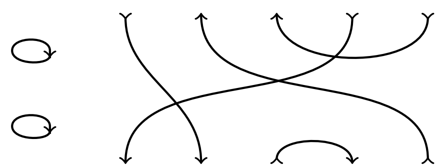


is a diagram of type $\downarrow\uparrow\downarrow\uparrow\uparrow\downarrow\downarrow\downarrow\uparrow\downarrow\uparrow \to \downarrow\uparrow\uparrow\downarrow\downarrow$. Two diagrams are *equivalent* if they are of the same type and represent the same bijection. The strands connecting a vertex in a to one in b are called *vertical strands* and all other strands are called *horizontal*. The horizontal strands which connect two vertices in a are called *caps*, while those which connect two vertices in b are called *cups*. Given a diagram D of type $\mathsf{a} \to \mathsf{b}$ we define the source of D to be $s(D) = \mathsf{a}$ and the target of D to be $t(D) = \mathsf{b}$. We also define D' to be the diagram obtained from D by switching the orientation of every strand.

Given $\mathsf{a}, \mathsf{b}, \mathsf{c} \in \langle \downarrow, \uparrow \rangle$, we may stack an oriented Brauer diagram of type $\mathsf{b} \to \mathsf{c}$ on top of one of type $\mathsf{a} \to \mathsf{b}$ to obtain an oriented Brauer diagram of type $\mathsf{a} \to \mathsf{c}$ along with some number of loops made up of strands which were connected only to vertices in b, which we call *bubbles*. Two oriented Brauer diagrams with bubbles are *equivalent* if they have the same number (possibly zero) of bubbles (regardless of orientation), and the underlying oriented Brauer diagrams obtained by ignoring the bubbles are equivalent in the earlier sense. For example, if we stack the above diagram on top of



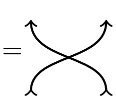
we get



The oriented Brauer category $\mathcal{OB}$

Now we can define the *oriented Brauer category* $\mathcal{OB}$ to be the $\Bbbk$-linear category with objects $\langle \downarrow, \uparrow \rangle$ and morphisms $\mathrm{Hom}_{\mathcal{OB}}(\mathsf{a}, \mathsf{b})$ consisting of all formal $\Bbbk$-linear combinations of equivalence classes of oriented Brauer diagrams with bubbles of type $\mathsf{a} \to \mathsf{b}$. The composition $D_1 \circ D_2$ of diagrams with bubbles is given by stacking D_1 on top of D_2. This is clearly associative. There is also a tensor product making $\mathcal{OB}$ into a strict monoidal category. If D_1, D_2 are diagrams, then $D_1 \otimes D_2$ is obtained by horizontally stacking D_1 to the left of D_2. We will often omit $\otimes$ from the notation.

As a $\Bbbk$-linear monoidal category, $\mathcal{OB}$ is generated by objects $\uparrow, \downarrow$ and the morphisms $c : \varnothing \to \uparrow\downarrow$, $d : \downarrow\uparrow \to \varnothing$ and $s : \uparrow\uparrow \to \uparrow\uparrow$ given by

$c =$  $d =$  $s =$ 

which satisfy the following relations.

$$(\uparrow d) \circ (c \uparrow) = \uparrow \tag{III.1.0.1}$$

$$(d \downarrow) \circ (\downarrow c) = \downarrow \tag{III.1.0.2}$$

$$(\uparrow s) \circ (s \uparrow) \circ (\uparrow s) = (s \uparrow) \circ (\uparrow s) \circ (s \uparrow) \tag{III.1.0.3}$$

$$s^2 = \uparrow\uparrow \tag{III.1.0.4}$$

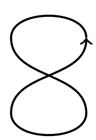
$$(d \uparrow\downarrow) \circ (\downarrow s \downarrow) \circ (\downarrow\uparrow c) \text{ is invertible.} \tag{III.1.0.5}$$

We denote the inverse of the diagram in III.1.0.5 by t.

Theorem III.1.1. *As a $\Bbbk$-linear monoidal category, $\mathcal{OB}$ is generated by the objects $\uparrow, \downarrow$ and morphisms c, d, s, t subject only to the relations III.1.0.1-III.1.0.5, where t is the inverse referred to in relation III.1.0.5.*

Proof. See section 3.1 of [BCNR] for the proof of this theorem. Ultimately it is a consequence of a more general result of Turaev for ribbon categories. □

Let $\delta \in \Bbbk$. We shall be studying the representation theory of the category $\mathcal{OB}(\delta)$ obtained from $\mathcal{OB}$ by imposing the additional relation $d \circ t \circ c = \delta$, which is represented diagrammatically as



$= \delta.$ (III.1.0.6)

Since diagrams in $\mathcal{OB}$ are viewed up to equivalence, this relation says that a bubble is equal to $\delta \cdot \varnothing$ in $\mathcal{OB}(\delta)$. So in $\mathcal{OB}(\delta)$, composition $D_1 \circ D_2$ of two diagrams is defined by stacking D_1 on top of D_2, then removing all bubbles and multiplying by the scalar δ^n, where n is the number of bubbles removed. Then $\mathrm{Hom}_{\mathcal{OB}(\delta)}(\mathsf{a}, \mathsf{b})$ has a basis consisting of equivalence classes of oriented Brauer diagrams of type $\mathsf{a} \to \mathsf{b}$ with no bubbles.

The affine oriented Brauer category $\mathcal{AOB}$

The *affine oriented Brauer category $\mathcal{AOB}$* is the monoidal category generated by objects $\uparrow, \downarrow$ and morphisms $c : \varnothing \to \uparrow\downarrow$, $d : \downarrow\uparrow \to \varnothing$, $s : \uparrow\uparrow \to \uparrow\uparrow$, and $x : \uparrow \to \uparrow$ subject to (III.1.0.1)-(III.1.0.5) plus one extra relation

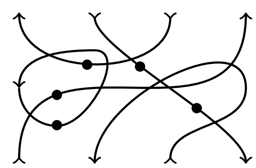
$$(\uparrow x) \circ s = s \circ (x \uparrow) + \uparrow\uparrow . \tag{III.1.0.7}$$

Note that by Theorem III.1.1 there is a functor $\mathcal{OB} \to \mathcal{AOB}$ sending the generators of $\mathcal{OB}$

to those of $\mathcal{AOB}$ with the same name. Hence we can interpret any oriented Brauer diagram with bubbles as a morphism in $\mathcal{AOB}$. Let us now represent the new generator x by the diagram

$x =$ 

We define a *dotted oriented Brauer diagram with bubbles* to be an oriented Brauer diagram with bubbles, such that each segment is decorated in addition with some non-negative number of dots, where a *segment* means a connected component of the diagram obtained when all crossings are deleted. The following is a typical example of a dotted oriented Brauer diagram with bubbles.

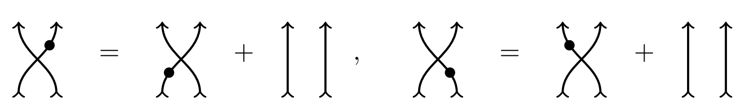


Two dotted oriented Brauer diagrams with bubbles are *equivalent* if one can be obtained from the other by continuously deforming strands through other strands and crossings, and also by sliding dots along strands *without* pulling them past any crossings.

Any dotted oriented Brauer diagram with bubbles is equivalent to one that is a vertical composition of diagrams of the form $\mathsf{a}c\mathsf{b}$, $\mathsf{a}d\mathsf{b}$, $\mathsf{a}s\mathsf{b}$, $\mathsf{a}x\mathsf{b}$ and $\mathsf{a}t\mathsf{b}$ for various $\mathsf{a}, \mathsf{b} \in \langle\downarrow, \uparrow\rangle$. Hence it can be interpreted as a morphism in $\mathcal{AOB}$. Moreover, the resulting morphism is well defined independent of the choices made, and it depends only on the equivalence class of the original diagram. For example, the following diagram x' represents the morphism $(d \downarrow) \circ (\downarrow x \downarrow) \circ (\downarrow c) \in \mathrm{End}_{\mathcal{AOB}}(\downarrow)$:

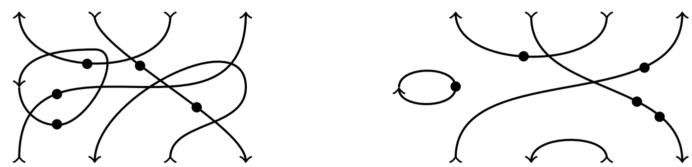
$x' =$ 

Also, we can represent the relation (III.1.0.7) as the first of the following two diagrammatic relations; the second follows from the first by composing with s on top and bottom:



These local relations explain how to move dots past crossings in any diagram, introducing an "error term" with fewer dots.

A dotted oriented Brauer diagram with bubbles is *normally ordered* if it is equivalent to a tensor (horizontal) product of diagrams $b_1 \dots b_n D$, where $b_1, \dots, b_n$ are each a clockwise, crossing free bubble with some nonnegative number of dots and D is a dotted oriented Brauer diagram without bubbles, with all dots on outward-pointing boundary segments, i.e. segments which intersect the boundary at a point that is directed out of the picture. For example, of the two diagrams below, the one on the left is not normally ordered, but the diagram on the right is.



(III.1.0.8)

Theorem III.1.2. *For* $\mathsf{a}, \mathsf{b} \in \langle \downarrow, \uparrow \rangle$, *the space* $\mathrm{Hom}_{\mathcal{AOB}}(\mathsf{a}, \mathsf{b})$ *has basis given by equivalence classes of normally ordered dotted oriented Brauer diagrams with bubbles of type* $\mathsf{a} \to \mathsf{b}$.

Proof. This is Theorem 1.2 of [BCNR] and is proved in section 5.4 of that paper. $\square$

The cyclotomic oriented Brauer category $\mathcal{COB}$

Suppose $\mathcal{M}$ is a monoidal category. A *right tensor ideal* $\mathcal{I}$ of $\mathcal{M}$ is the data of a submodule $\mathcal{I}(\mathsf{a}, \mathsf{b}) \subseteq \mathrm{Hom}_{\mathcal{M}}(\mathsf{a}, \mathsf{b})$ for each pair of objects a, b in $\mathcal{M}$, such that for all objects $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}$ we have $h \circ g \circ f \in \mathcal{I}(\mathsf{a}, \mathsf{d})$ whenever $f \in \mathrm{Hom}_{\mathcal{M}}(\mathsf{a}, \mathsf{b})$, $g \in \mathcal{I}(\mathsf{b}, \mathsf{c})$, $h \in \mathrm{Hom}_{\mathcal{M}}(\mathsf{c}, \mathsf{d})$, and $g \otimes 1_{\mathsf{c}} \in \mathcal{I}(\mathsf{a} \otimes \mathsf{c}, \mathsf{b} \otimes \mathsf{c})$ whenever $g \in \mathcal{I}(\mathsf{a}, \mathsf{b})$. The *quotient* $\mathcal{M}/\mathcal{I}$ of $\mathcal{M}$ by right tensor ideal $\mathcal{I}$ is the category with the same objects as $\mathcal{M}$ and morphisms given by $\mathrm{Hom}_{\mathcal{M}/\mathcal{I}}(\mathsf{a}, \mathsf{b}) := \mathrm{Hom}_{\mathcal{M}}(\mathsf{a}, \mathsf{b})/\mathcal{I}(\mathsf{a}, \mathsf{b})$.

Let $\ell \geq 1$ be a fixed *level* and $f(u) \in \Bbbk[u]$ be a monic polynomial of degree ℓ in the auxiliary variable u. The *cyclotomic oriented Brauer category* $\mathcal{COB}$ is the quotient of $\mathcal{AOB}$ by the right tensor ideal generated by $f(x) \in \mathrm{End}_{\mathcal{AOB}}(\uparrow)$. Since $\mathcal{COB}$ has the same objects as $\mathcal{AOB}$ while its morphism spaces are quotients of those in $\mathcal{AOB}$, any morphism in $\mathcal{AOB}$ can be viewed as one in $\mathcal{COB}$.

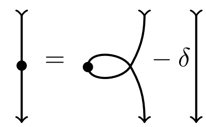
Theorem III.1.3. *For* $\mathsf{a}, \mathsf{b} \in \langle \downarrow, \uparrow \rangle$, *the space* $\mathrm{Hom}_{\mathcal{COB}}(\mathsf{a}, \mathsf{b})$ *has basis given by equivalence classes of normally ordered dotted oriented Brauer diagrams with bubbles of type* $\mathsf{a} \to \mathsf{b}$, *subject to the additional constraint that each segment is decorated by at most* $(\ell - 1)$ *dots.*

Proof. This is Theorem 1.5 of [BCNR]. It is proved is sections 5.1-3 of that paper. $\square$

Theorem III.1.4. *Suppose that* $f(u) = u - m \in \Bbbk[u]$ *is monic of degree one. Then the functor* $\mathcal{OB} \to \mathcal{COB}$ *defined as the composite first of the functor* $\mathcal{OB} \to \mathcal{AOB}$ *then the quotient functor* $\mathcal{AOB} \to \mathcal{COB}$ *is an isomorphism.*

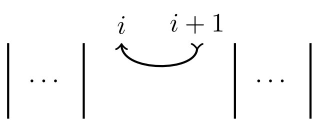
Proof. This is Theorem 3.3 of [BCNR]. $\square$

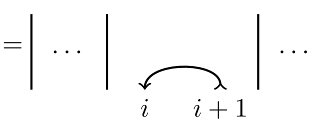
The above theorem implies in particular that the functor $\mathcal{OB} \to \mathcal{AOB}$ is faithful. It also shows that there is a functor $\mathcal{AOB} \to \mathcal{OB}$ which restricts to identity on $\mathcal{OB}$ and sends $x\mathsf{a}$ to $m1_{\uparrow\mathsf{a}}$ for any $\mathsf{a} \in \langle\downarrow,\uparrow\rangle$. Using this functor we can interpret dotted diagrams as morphisms in $\mathcal{OB}$, which we shall do from now on. For example the relation

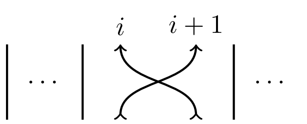


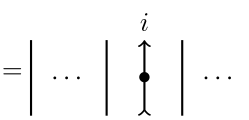
shows that the functor $\mathcal{AOB} \to \mathcal{OB}$ sends $x'\mathsf{a}$ to $m'1_{\downarrow\mathsf{a}}$, which is how we interpret a dot on a downward-oriented string on the left edge of a diagram. Lemma III.1.5 below explains how to interpret a dot found anywhere in a diagram. We stress that from now on all diagrams refer to morphisms in $\mathcal{OB}$; this will not be ambiguous as we shall not need the category $\mathcal{AOB}$ again.

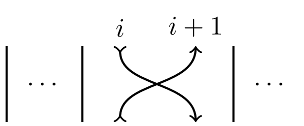
Define $c_i1_\mathsf{a}, d_i1_\mathsf{a}, s_i, x_i1_\mathsf{a}, t_i1_\mathsf{a}, s_i'1_\mathsf{a}, x_i'1_\mathsf{a}$ (also written $1_\mathsf{b}c_i, 1_\mathsf{b}d_i, 1_\mathsf{b}s_i, 1_\mathsf{b}x_i, 1_\mathsf{b}t_i, 1_\mathsf{b}s_i', 1_\mathsf{b}x_i'$, respectively) to be the following morphisms in $\mathcal{OB}$, for all $i, \mathsf{a}, \mathsf{b}$ for which they make sense. Here, i labels the ith string from the left. The object on the bottom of each diagram is a, and the one on top is b (for different choices of a, b).

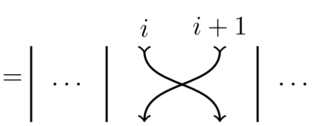
$c_i1_\mathsf{a} = 1_\mathsf{b}c_i =$ 

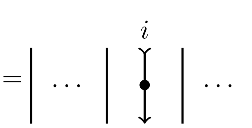
$d_i1_\mathsf{a} = 1_\mathsf{b}d_i =$ 

$s_i1_\mathsf{a} = 1_\mathsf{b}s_i =$ 

$x_i1_\mathsf{a} = 1_\mathsf{b}x_i =$ 

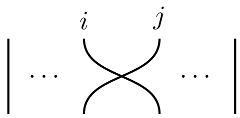
$t_i1_\mathsf{a} = 1_\mathsf{b}t_i =$ 

$s_i'1_\mathsf{a} = 1_\mathsf{b}s_i' =$ 

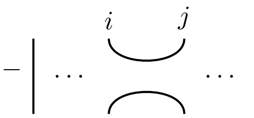
$x_i'1_\mathsf{a} = 1_\mathsf{b}x_i' =$ 

If D is a diagram and $D = 1_\mathsf{a}D1_\mathsf{b}$, then we allow ourselves to write compositions of D with the above morphisms without writing the $1_\mathsf{a}, 1_\mathsf{b}$. For example, if $\mathsf{a}_i =\downarrow$ and $\mathsf{a}_{i+1} =\uparrow$, then we can define $t_i^{-1}1_\mathsf{a} = d_is_{i+1}c_{i+2}1_\mathsf{a}$ and if $\mathsf{b}_i =\uparrow$ and $\mathsf{b} =\downarrow$, then $1_\mathsf{b}t_i^{-1} = 1_\mathsf{b}d_is_{i+1}c_{i+2}$. Note that the symbols c_i, d_i, s_i, x_i, etc. are ambiguous until an object is specified by multiplying by some well-defined morphism in $\mathcal{OB}$.

To express $x_i 1_{\mathsf{a}\uparrow}, x'_i 1_{\mathsf{a}\downarrow}$ in terms of undotted oriented Brauer diagrams we have some more notation to introduce. For $\mathsf{a} \in \langle\downarrow,\uparrow\rangle$, $1 \leq i < j \leq \ell(\mathsf{a})$, define $(i\,j)1_{\mathsf{a}}$ to be whichever one of the following two morphisms $\mathsf{a} \to \mathsf{a}$ matches the orientations of the vertices $\mathsf{a}_i, \mathsf{a}_j$.



if $\mathsf{a}_i = \mathsf{a}_j$

$-$ 

if $\mathsf{a}_i \neq a_j$

Lemma III.1.5.

$$\textit{If } \mathsf{a}_i = \uparrow \textit{ then } x_i 1_{\mathsf{a}} = m1_{\mathsf{a}} + \sum_{1\leq j<i} (j\,i)1_{\mathsf{a}} \textit{ and}$$

$$\textit{if } \mathsf{a}_i = \downarrow \textit{ then } x'_i 1_{\mathsf{a}} = m'1_{\mathsf{a}} - \sum_{1\leq j<i} (j\,i)1_{\mathsf{a}}.$$

Proof. These two statements are proved by induction. The induction step follows from the relations

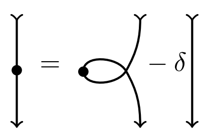
$$x_{i+1}1_{\mathsf{a}} = s_i x_i s_i 1_{\mathsf{a}} + (i \quad i+1)1_{\mathsf{a}} \qquad \text{if } \mathsf{a}_i = \mathsf{a}_{i+1} = \uparrow$$

$$x'_{i+1}1_{\mathsf{a}} = s'_i x'_i s'_i 1_{\mathsf{a}} + (i \quad i+1)1_{\mathsf{a}} \qquad \text{if } \mathsf{a}_i = \mathsf{a}_{i+1} = \downarrow$$

$$x_{i+1}1_{\mathsf{a}} = t_i x_i t_i^{-1} 1_{\mathsf{a}} + (i \quad i+1)1_{\mathsf{a}} \qquad \text{if } \mathsf{a}_i = \downarrow, \mathsf{a}_{i+1} = \uparrow$$

$$x'_{i+1}1_{\mathsf{a}} = t_i^{-1} x'_i t_i 1_{\mathsf{a}} + (i \quad i+1)1_{\mathsf{a}} \qquad \text{if } \mathsf{a}_i = \uparrow, \mathsf{a}_{i+1} = \downarrow$$

The base case is easy. We already know $x_1 1_{\mathsf{a}} = m1_{\mathsf{a}}$. Now $x'_1 1_{\mathsf{a}} = m'1_{\mathsf{a}}$ follows from the relation



□

III.2. Triangular Decomposition

We apply the construction of a locally unital algebra starting from the small $\mathbb{k}$-linear category $\mathcal{OB}(\delta)$ to obtain the algebra $OB(\delta)$ (see section II.2). Since we shall never consider the algebra built out of $\mathcal{OB}$, we can unambiguously abbreviate $OB(\delta)$ to OB. We note that then OB is locally finite dimensional.

We now describe a triangular decomposition for OB analogous to that of the universal enveloping algebra of a semisimple Lie algebra (see sections 0.5 and 1.1 of [H]). Let $\mathbb{K}$ denote the span of the distinguished idempotents: $\mathbb{K} = \bigoplus_{\mathsf{a}\in\langle\downarrow,\uparrow\rangle} \mathbb{k}\cdot 1_{\mathsf{a}}$. Let OB^+ (resp. OB^-) denote the span of all diagrams with no cups (resp. caps) and no crossings among vertical strands. Also let OB^0 denote the span of all diagrams with no cups or caps. Then OB^+, OB^-, OB^0 are subalgebras containing $\mathbb{K}$. Observe that if V is a right $\mathbb{K}$-module and W a left $\mathbb{K}$-module, then $V \otimes_{\mathbb{K}} W \cong \bigoplus_{\mathsf{a} \in \langle \downarrow, \uparrow \rangle} V 1_{\mathsf{a}} \otimes_{\Bbbk} 1_{\mathsf{a}} W$.

Proposition III.2.1. *The product map $OB^- \otimes_{\mathbb{K}} OB^0 \otimes_{\mathbb{K}} OB^+ \to OB$ is an isomorphism of vector spaces.*

Proof. Surjectivity is clear as $OB^- \otimes_{\mathbb{K}} OB^0 \otimes_{\mathbb{K}} OB^+ = \bigoplus_{\mathsf{a},\mathsf{b} \in \langle \downarrow, \uparrow \rangle} OB^- 1_{\mathsf{a}} \otimes_{\Bbbk} 1_{\mathsf{a}} OB^0 1_{\mathsf{b}} \otimes_{\Bbbk} 1_{\mathsf{b}} OB^+$. Any element of the kernel of the product map can be written as $\sum_{\underline{D}} c_{\underline{D}} \underline{D}$, summing over all $\underline{D} = D^- \otimes D^0 \otimes D^+$ with D^-, D^0, D^+ in the respective standard bases for OB^-, OB^0, OB^+. We then have $\sum_{\underline{D}} c_{\underline{D}} D^+ D^0 D^- = 0$. Now the nonzero $D^+ D^0 D^-$ are all distinct diagrams, so linear independence of diagrams shows $c_{\underline{D}} = 0$ whenever $\underline{D} \neq 0$, so injectivity is proved. □

Define $OB^\sharp$ (resp. $OB^\flat$) to be the subalgebra spanned by all diagrams with no cups (resp. caps). So $OB^\sharp, OB^\flat$ are the images of $OB^0 \otimes_{\mathbb{K}} OB^+, OB^- \otimes_{\mathbb{K}} OB^0$ under the product map. The algebras $OB^\sharp, OB^\flat$ are graded by the number of caps or cups appearing in a diagram, respectively. So $OB^\sharp = \bigoplus_{d \geq 0} OB^\sharp[d]$, where $OB^\sharp[d]$ is the span of all diagrams in $OB^\sharp$ which have exactly d caps. Similarly $OB^+ = \bigoplus_{d \geq 0} OB^+[d]$. Also $OB^\flat = \bigoplus_{d \leq 0} OB^\flat[d]$, where $OB^\flat[d]$ is the span of all diagrams in $OB^\flat$ which have exactly $-d$ cups, and similarly define $OB^-[d]$. Observe that $OB^0 = OB^\sharp[0]$ is a quotient of $OB^\sharp$ by the two-sided ideal $\bigoplus_{d \geq 1} OB^\sharp[d]$. Pulling back through this quotient gives an exact functor $OB^0\text{-mod} \to OB^\sharp\text{-mod}$. To simplify the notation, we shall simply view OB^0-modules as $OB^\sharp$-modules in this way without further comment. Composing this with $OB \otimes_{OB^\sharp} ?$ defines a functor $\Delta : OB^0\text{-mod} \to OB\text{-mod}$. Later in this thesis, we will show that OB-mod is locally standardly stratified in the sense of section II.5 with gr $OB\text{-mod} = OB^0\text{-mod}$ and standardization functor Δ; see Proposition V.1.2 below. We therefore call Δ the *standardization functor*. Since $OB = OB^- \otimes_{\mathbb{K}} OB^\sharp$, it is easy to see that as a right $OB^\sharp$-module, OB is isomorphic to a direct sum of copies of $1_{\mathsf{a}} OB^\sharp$, for various a's. Hence OB is a projective right $OB^\sharp$ module, so that Δ is an exact functor.

If V is an OB^0-module, then

$$\Delta V = OB^- \otimes_{\mathbb{K}} OB^\sharp \otimes_{OB^\sharp} V = OB^- \otimes_{\mathbb{K}} V = \bigoplus_{\mathsf{a}} OB^- 1_{\mathsf{a}} \otimes_{\Bbbk} 1_{\mathsf{a}} V.$$

Let B_{a} be a basis for $1_{\mathsf{a}} V$. Then ΔV has a basis consisting of $D \otimes v$ with D a diagram with no caps and no crossings among vertical strands, and $v \in B_{s(d)}$.

III.3. Weights

A *bipartition* is a pair of partitions. Recall the definition of a p-regular partition from section II.1. A bipartition is *p-regular* if both its constituent partitions are p-regular. Let $\overline{\Lambda}$ denote the set of all bipartitions and Λ the set of all p-regular bipartitions. Let $\overline{\Lambda}_{r,s}$ be the set of all bipartitions $\lambda = (\lambda^{\downarrow}, \lambda^{\uparrow})$ such that $\lambda^{\downarrow} \vdash r$ and $\lambda^{\uparrow} \vdash s$, and let $\Lambda_{r,s} = \Lambda \cap \overline{\Lambda}_{r,s}$. For such a bipartition we let $|\lambda| = r + s$. We define a preorder on $\overline{\Lambda}$ (hence on Λ) as follows. Recall that $m, m' \in \Bbbk$ are fixed and $\delta = m - m'$. Given a box in row i and column j, the *$\uparrow$-content* of this box is $\mathrm{cont}_{\uparrow}(\square) = m + j - i$, and its *$\downarrow$-content* is $\mathrm{cont}_{\downarrow}(\square) = m' + i - j$. The *content* of a box in either constituent partition of a bipartition $\lambda = (\lambda^{\downarrow}, \lambda^{\uparrow})$ refers to its $\downarrow$-content if it belongs to $\lambda^{\downarrow}$ and its $\uparrow$-content if it belongs to $\lambda^{\uparrow}$.

Recall the weight lattice P, which is partially ordered by dominance (see section II.3). Now define the *$\uparrow$-content* of a bipartition $\lambda = (\lambda^{\downarrow}, \lambda^{\uparrow})$ to be

$$\mathrm{cont}_{\uparrow}(\lambda) = \sum_{\square \in \lambda^{\uparrow}} \alpha_{\mathrm{cont}_{\uparrow}(\square)} \in P.$$

We also define the *$\downarrow$-content* of a bipartition to be

$$\mathrm{cont}_{\downarrow}(\lambda) = \sum_{\square \in \lambda^{\downarrow}} \alpha_{\mathrm{cont}_{\downarrow}(\square)}.$$

Next we define the *content* of $\lambda = (\lambda^{\downarrow}, \lambda^{\uparrow})$ to be

$$\mathrm{cont}(\lambda) = \mathrm{cont}_{\uparrow}(\lambda) - \mathrm{cont}_{\downarrow}(\lambda).$$

Now given two bipartitions, λ, μ, we define $\lambda \leq \mu$ if $\mathrm{cont}(\lambda) = \mathrm{cont}(\mu)$ and $\mathrm{cont}_{\uparrow}(\lambda) \geq \mathrm{cont}_{\uparrow}(\mu)$, or equivalently, if $\mathrm{cont}(\lambda) = \mathrm{cont}(\mu)$ and $\mathrm{cont}_{\downarrow}(\lambda) \geq \mathrm{cont}_{\downarrow}(\mu)$.

Note that this preorder depends on δ, but not on m, m' individually. This preorder induces a partial order on the set Ξ of equivalence classes of Λ under the equivalence relation given by $\lambda \sim \mu$ if $\lambda \leq \mu$ and $\mu \leq \lambda$. So $\lambda \sim \mu$ if and only if $(\mathrm{cont}_{\downarrow}(\lambda), \mathrm{cont}_{\uparrow}(\lambda)) = (\mathrm{cont}_{\downarrow}(\mu), \mathrm{cont}_{\uparrow}(\mu))$. Note that in characteristic zero, the preorder on $\overline{\Lambda} = \Lambda$ is already a partial order, ie. $\Xi = \Lambda$. Also note that if $\delta \notin \mathbb{Z} \cdot 1_{\Bbbk}$ and $\lambda \leq \mu$, then $\mathrm{cont}_{\uparrow}(\lambda) = \mathrm{cont}_{\uparrow}(\mu)$ and $\mathrm{cont}_{\downarrow}(\lambda) = \mathrm{cont}_{\downarrow}(\mu)$. That is, the partial order on Ξ is trivial if $\delta \notin \mathbb{Z} \cdot 1_{\Bbbk}$. In any case, we see that $\lambda \leq \mu$ implies $|\lambda| \geq |\mu|$, and therefore $\overline{\Lambda}$ is upper finite. We remark that the above partial order is a special case of the "inverse dominance order" mentioned in Definition 3.2 of [LW].

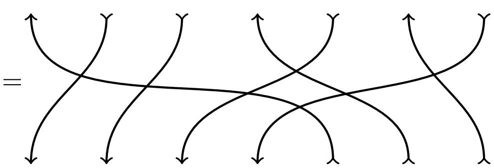
Example III.3.1. *The bipartitions* $(\varnothing,\varnothing)$ *and* $(\varnothing,\square)$ *are not comparable because the first has content equal to 0 while the second has content equal to* α_m*. However, we do have* $(\varnothing,\varnothing) > (\square,\square)$ *assuming* $\delta = 0$*.*

Example III.3.2. *If* $p = 2$*, then* $(\ydiagram{4},\square) \sim \left(\ydiagram{3,1},\square\right)$ *because they both have* $\downarrow$*-content equal to* $2\alpha_{m'} + 2\alpha_{m'+1}$ *and* $\uparrow$*-content equal to* α_m*. So these two 2-regular bipartitions are identified in* Ξ*. In a similar fashion one can see that* $\Lambda \neq \Xi$ *whenever* $p > 0$*.*

III.4. OB^0 and the Symmetric Groups

For $\mathsf{a} \in \langle\downarrow,\uparrow\rangle$, define $\ell_\downarrow(\mathsf{a}) = \#\{i : \mathsf{a}_i = \downarrow\}$, $\ell_\uparrow(\mathsf{a}) = \#\{i : \mathsf{a}_i = \uparrow\}$, and $\ell(\mathsf{a}) = \ell_\uparrow(\mathsf{a}) + \ell_\downarrow(\mathsf{a})$. Let $\langle\downarrow,\uparrow\rangle_{r,s} = \{\mathsf{a} \in \langle\downarrow,\uparrow\rangle : \ell_\downarrow(\mathsf{a}) = r \text{ and } \ell_\uparrow(\mathsf{a}) = s\}$. For $\mathsf{a},\mathsf{b} \in \langle\downarrow,\uparrow\rangle_{r,s}$ let ${}_{\mathsf{a}}\sigma_{\mathsf{b}}$ denote the unique OB-diagram $\mathsf{b} \to \mathsf{a}$ in OB^0 such that no strings of the same orientation (up or down) cross. For convenience, we shall write ${}_{\mathsf{a}}\sigma_{r,s}$ for ${}_{\mathsf{a}}\sigma_{(\downarrow^r\uparrow^s)}$ and ${}_{r,s}\sigma_{\mathsf{a}}$ for ${}_{(\downarrow^r\uparrow^s)}\sigma_{\mathsf{a}}$. For example,

$${}_{(\uparrow\downarrow\downarrow\uparrow\downarrow\uparrow\downarrow)}\sigma_{4,3} =$$



Note that we evidently have ${}_{\mathsf{c}}\sigma_{\mathsf{b}\mathsf{b}}\sigma_{\mathsf{a}} = {}_{\mathsf{c}}\sigma_{\mathsf{a}}$ and ${}_{\mathsf{a}}\sigma_{\mathsf{a}} = 1_{\mathsf{a}}$. Let $r,s \geq 0$. We have an algebra isomorphism

$$\mathrm{Mat}_{\binom{r+s}{r}}(\Bbbk S_r \otimes \Bbbk S_s) \to \bigoplus_{\mathsf{a},\mathsf{b}\in\langle\downarrow,\uparrow\rangle_{r,s}} 1_{\mathsf{a}} OB^0 1_{\mathsf{b}}, \qquad \sum_{\mathsf{a},\mathsf{b}\in\langle\downarrow,\uparrow\rangle_{r,s}} \tau_{\mathsf{a},\mathsf{b}} e_{\mathsf{a},\mathsf{b}} \mapsto \sum_{\mathsf{a},\mathsf{b}\in\langle\downarrow,\uparrow\rangle_{r,s}} {}_{\mathsf{a}}\sigma_{r,s}\, \tau_{\mathsf{a},\mathsf{b}}\, {}_{r,s}\sigma_{\mathsf{b}},$$

where the rows and columns of matrices are indexed by $\langle\downarrow,\uparrow\rangle_{r,s}$, and $e_{\mathsf{a},\mathsf{b}}$ is the corresponding matrix unit. This implies that

$$OB^0 = \bigoplus_{r,s\geq 0}\left(\bigoplus_{\mathsf{a},\mathsf{b}\in\langle\downarrow,\uparrow\rangle_{r,s}} 1_{\mathsf{a}} OB^0 1_{\mathsf{b}}\right) \text{ is Morita equivalent to } S := \bigoplus_{r,s\geq 0} \Bbbk S_r \otimes \Bbbk S_s.$$

For $\lambda = (\lambda^\downarrow, \lambda^\uparrow) \in \Lambda$, the outer tensor product of Specht modules $S(\lambda^\downarrow) \boxtimes S(\lambda^\uparrow)$ is an S-module with simple head whose projective cover coincides with its injective hull (see section II.1). Transporting $S(\lambda^\downarrow) \boxtimes S(\lambda^\uparrow)$ through the Morita equivalence gives us the *Specht module* $S(\lambda)$. We denote its irreducible head by $D(\lambda)$. The projective cover (and injective hull) of $D(\lambda)$ shall be denoted $Y(\lambda)$. Define the *standard modules* $\Delta(\lambda) = \Delta Y(\lambda)$, and the *proper standard modules* $\overline{\Delta}(\lambda) = \Delta D(\lambda)$. Write $\widetilde{\Delta}(\lambda)$ for the standardized Specht modules $\Delta S(\lambda)$.

III.5. Duality

Given a diagram $D \in 1_{\mathsf{a}} OB 1_{\mathsf{b}}$, let $\tau(D)$ be the diagram in $1_{\mathsf{b}} OB 1_{\mathsf{a}}$ which represents the inverse of the bijection represented by D (see section III.1). That is, $\tau(D)$ is obtained by flipping the diagram D vertically, then reversing all orientations. For example, $\tau(d) = c'$ (recall the definition of the diagram D' from section III.1). Then τ extends to an anti-involution of OB which fixes $\mathbb{K}$, preserves OB^0, and swaps OB^+ with OB^-. Given an OB-module V, we give an OB-module structure to the *restricted dual*

$$V^* = \bigoplus_{\mathsf{a} \in \langle \downarrow, \uparrow \rangle} \mathrm{Hom}_{\Bbbk}(1_{\mathsf{a}} V, \Bbbk) \tag{III.5.0.9}$$

via the formula

$$(D \cdot f)(v) = f(\tau(D)v), \qquad D \in OB, f \in V^*, v \in V. \tag{III.5.0.10}$$

We have an exact contravariant involution $V \mapsto V^*$ of OB-mod. Since τ preserves OB^0 we similarly get a duality functor on OB^0-mod. Passing through the Morita equivalence described in section III.4, we obtain the usual duality functor for the symmetric groups.

III.6. Classification of Simple Modules

Theorem III.6.1. *Let $\lambda = (\lambda^{\downarrow}, \lambda^{\uparrow}) \in \Lambda$. Then $\overline{\Delta}(\lambda)$ is an indecomposable module which has a unique maximal submodule. Let $L(\lambda)$ denote its unique irreducible quotient. Then $\{L(\lambda) : \lambda \in \Lambda\}$ is a complete set of inequivalent irreducible OB-modules.*

Proof. Let $r = |\lambda^{\downarrow}|, s = |\lambda^{\uparrow}|$, and let $\mathsf{a} \in \langle \downarrow, \uparrow \rangle_{r,s}$. Then

$$1_{\mathsf{a}} \overline{\Delta}(\lambda) = 1_{\mathsf{a}} OB^- \otimes_{\mathbb{K}} D(\lambda) = 1_{\mathsf{a}} \otimes_{\Bbbk} D(\lambda),$$

which generates $\overline{\Delta}(\lambda)$. This shows that any proper submodule of $\overline{\Delta}(\lambda)$ must lie in the subspace

$$\bigoplus_{\substack{\mathsf{a} \in \langle \downarrow, \uparrow \rangle_{r+t,s+t} \\ t \geq 1}} 1_{\mathsf{a}} \overline{\Delta}(\lambda) \subsetneq \overline{\Delta}(\lambda),$$

and hence so does the sum of all proper submodules, proving the first statement.

Let L be an irreducible OB-module. Choose $\mathsf{a} \in \langle \downarrow, \uparrow \rangle$ of minimal length such that $1_{\mathsf{a}} L \neq 0$. Let V be the OB^0-submodule generated by $1_{\mathsf{a}} L$. Note that $1_{\mathsf{b}} V = 1_{\mathsf{b}} OB^0 1_{\mathsf{a}} L = 0$ unless $\mathsf{a}, \mathsf{b} \in \langle \downarrow, \uparrow \rangle_{r,s}$ for some $r, s \geq 0$. Let $v, w \in V$ be nonzero. There is some $f \in 1_{t(w)} OB 1_{t(v)}$ with $fv = w$. Since caps act as zero on V, f can be chosen in the span of diagrams having no caps. But then we must have $f \in OB^0$ as $1_{t(w)} f v = w \neq 0$.

Hence we have $V = OB^0 1_{\mathsf{a}} L \cong D(\lambda)$ for some $\lambda \in \Lambda_{r,s}$ and therefore $\overline{\Delta}(\lambda) \cong OB \otimes_{OB^\sharp} V$, where V is viewed as an $OB^\sharp$-module as usual. Then

$$\mathrm{Hom}_{OB}(\overline{\Delta}(\lambda), L) = \mathrm{Hom}_{OB^\sharp}(V, L) = \mathrm{Hom}_{OB^0}(V, L) \neq 0, \tag{III.6.0.11}$$

where the last equality follows from the fact that the image of any OB^0-homomorphism $V \to L$ lies in V, and caps act as zero on V. Then there is a surjective module homomorphism $\overline{\Delta}(\lambda) \to L$, so that $L \cong L(\lambda)$. □

It is worth emphasizing a fact which was deduced in the proof of this theorem. If V is an OB-module, let n be the minimal length of $\mathsf{a} \in \langle \downarrow, \uparrow \rangle$ such that $1_{\mathsf{a}} V \neq 0$. Then we call $\bigoplus_{\ell(\mathsf{a})=n} 1_{\mathsf{a}} V$ the *shortest word space* of V. It is a submodule of the restriction of V to OB^0. The fact we wish to emphasize is that the shortest word space of $L(\lambda)$ is isomorphic to $D(\lambda)$, as an OB^0-module.

We deduce that

$$L(\lambda)^* \cong L(\lambda) \tag{III.6.0.12}$$

by comparing shortest word spaces as OB^0-modules, using the fact that simple OB^0-modules are self dual.

CHAPTER IV

BRANCHING RULES

In this chapter we construct the advertised categorical action. We begin by defining functors E, F in section IV.1. Then in section IV.2 we construct certain short exact sequences of functors which are central to many of our arguments. For example, in section IV.3 we use our short exact sequences to compute the formal characters of the modules $\widetilde{\Delta}(\lambda)$. These characters are described in terms of the branching graph, which is defined in the same section.

IV.1. Functors E, F

Recall that our main theorem will be that locally finite dimenstional OB-modules categorify a tensor product of representations of $\mathfrak{sl}_{\Bbbk}$. In this section we begin gathering the data of this categorification.

Define OB-bimodules $OB_{\uparrow}$, $OB_{\downarrow}$, ${}_{\uparrow}OB$, ${}_{\downarrow}OB$ as follows.

$$1_{\mathsf{a}} OB_{\uparrow} 1_{\mathsf{b}} = 1_{\mathsf{a}} OB 1_{\mathsf{b}\uparrow} \qquad 1_{\mathsf{a}} OB_{\downarrow} 1_{\mathsf{b}} = 1_{\mathsf{a}} OB 1_{\mathsf{b}\downarrow}$$

$$1_{\mathsf{a}} ({}_{\uparrow}OB) 1_{\mathsf{b}} = 1_{\mathsf{a}\uparrow} OB 1_{\mathsf{b}} \qquad 1_{\mathsf{a}} ({}_{\downarrow}OB) 1_{\mathsf{b}} = 1_{\mathsf{a}\downarrow} OB 1_{\mathsf{b}}$$

The left (resp. right) actions on $OB_{\uparrow}, OB_{\downarrow}$ (resp. ${}_{\uparrow}OB$, ${}_{\downarrow}OB$) are just the usual multiplication. The remaining actions are defined as follows. For $f \in OB$ and

$$x \in OB_{\uparrow}, \qquad x \cdot f = x \circ (f \uparrow)$$

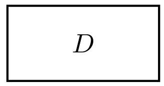
$$x \in OB_{\downarrow}, \qquad x \cdot f = x \circ (f \downarrow)$$

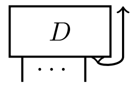
$$x \in {}_{\uparrow}OB, \qquad f \cdot x = (f \uparrow) \circ x$$

$$x \in {}_{\downarrow}OB, \qquad f \cdot x = (f \downarrow) \circ x$$

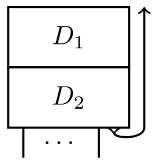
Proposition IV.1.1. *$OB_{\downarrow} \cong {}_{\uparrow}OB$ and $OB_{\uparrow} \cong {}_{\downarrow}OB$.*

Proof. We have a linear isomorphism $\varphi : OB_{\downarrow} \to {}_{\uparrow}OB$,

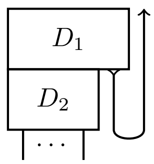


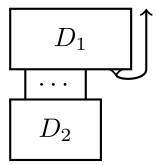
$\mapsto$


which is a bimodule homomorphism as follows:

$\varphi(D_1 \circ D_2) =$


$= D_1 \cdot \varphi(D_2),$ and

$\varphi(D_1 \cdot D_2) =$


$=$


$= \varphi(D_1) \circ D_2.$

The other isomorphism is obtained from φ by reversing the orientation of the added string. $\square$

We now have two functors OB-mod $\to OB$-mod:

$$E = OB_{\downarrow} \otimes_{OB} ? \cong {}_{\uparrow}OB \otimes_{OB} ?$$

$$F = OB_{\uparrow} \otimes_{OB} ? \cong {}_{\downarrow}OB \otimes_{OB} ?$$

The functors E, F are exact because they are biadjoint. For example, the counit of the adjunction (E, F) is induced by the bimodule homorphism $OB_{\downarrow} \otimes_{OB} OB_{\uparrow} \to OB$ given on $OB_{\downarrow} \otimes_{OB} OB_{\uparrow} 1_{\mathsf{a}} = OB_{\mathsf{a}\uparrow\downarrow}$ by right multiplication by $\mathsf{a}c$. The unit of this adjunction is induced by the bimodule homomorphism $OB \to OB_{\uparrow} \otimes_{OB} OB_{\downarrow}$ given on $OB_{1\mathsf{a}}$ by right multiplication by $\mathsf{a}d$, identifying $OB_{\uparrow} \otimes_{OB} OB_{\downarrow} 1_{\mathsf{a}} = OB1_{\mathsf{a}\downarrow\uparrow}$. The unit and counit of the adjunction (F, E) are defined similarly, replacing c, d with c', d'.

IV.2. Analogues for OB^0

We similarly define some endofunctors of OB^0-mod. Define OB^0-bimodules $OB^0_{\uparrow}$, $OB^0_{\downarrow}$, ${}_{\uparrow}OB^0$, ${}_{\downarrow}OB^0$ by replacing "OB" by "OB^0" everywhere in the definitions of $OB_{\uparrow}$, $OB_{\downarrow}$, ${}_{\uparrow}OB$, ${}_{\downarrow}OB$, respectively. Now we define functors by tensoring with these bimodules:

$$E^{\uparrow} = {}_{\uparrow}OB^0 \otimes_{OB^0} ? \qquad\qquad E^{\downarrow} = OB^0_{\downarrow} \otimes_{OB^0} ?$$

$$F^{\uparrow} = OB^0_{\uparrow} \otimes_{OB^0} ? \qquad\qquad F^{\downarrow} = {}_{\downarrow}OB^0 \otimes_{OB^0} ?$$

These functors correspond, under the Morita equivalence $OB^0 \cong S$ discussed in section III.4, to induction and restriction in the two tensor factors. Specifically, recall that $\Bbbk S_r$ is embedded in $\Bbbk S_{r+1}$ with respect to the first r letters. This induces embeddings $\Bbbk S_{r,s} \subset \Bbbk S_{r+1,s}$ and $\Bbbk S_{r,s} \subset \Bbbk S_{r,s+1}$, where $\Bbbk S_{r,s}$ is shorthand for $\Bbbk S_r \otimes \Bbbk S_s$. These last two embeddings give induction and restriction functors $\mathrm{ind}_{r,s}^{r+1,s}$, $\mathrm{res}_{r,s}^{r+1,s}$ and $\mathrm{ind}_{r,s}^{r,s+1}$, $\mathrm{res}_{r,s}^{r,s+1}$. Then the Morita equivalence identifies

$$E^{\uparrow} = \bigoplus_{r,s\geq 0} \mathrm{res}_{r,s}^{r,s+1} \qquad\qquad E^{\downarrow} = \bigoplus_{r,s\geq 0} \mathrm{ind}_{r,s}^{r+1,s}$$

$$F^{\uparrow} = \bigoplus_{r,s\geq 0} \mathrm{ind}_{r,s}^{r,s+1} \qquad\qquad F^{\downarrow} = \bigoplus_{r,s\geq 0} \mathrm{res}_{r,s}^{r+1,s}$$

Theorem IV.2.1. *There exist short exact sequences of functors* OB^0-mod $\to OB$-mod

$$0 \to \Delta \circ E^{\uparrow} \to E \circ \Delta \to \Delta \circ E^{\downarrow} \to 0$$

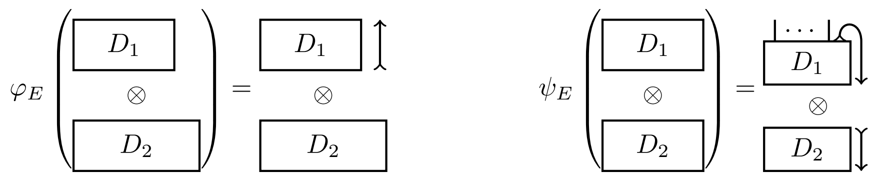
$$0 \to \Delta \circ F^{\downarrow} \to F \circ \Delta \to \Delta \circ F^{\uparrow} \to 0$$

Proof. The functors appearing in the short exact sequences are given by tensoring by certain bimodules. So to get these short exact sequences, it suffices to find short exact sequences of (OB, OB^0)-bimodules of the form

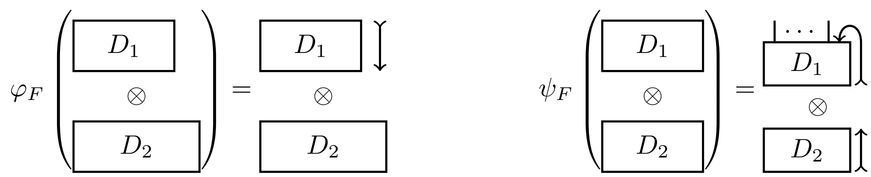
$$E: \qquad 0 \to OB \otimes_{OB^\sharp} {}_{\uparrow}OB^0 \xrightarrow{\varphi_E} {}_{\uparrow}OB \otimes_{OB^\sharp} OB^0 \xrightarrow{\psi_E} OB \otimes_{OB^\sharp} OB^0_{\downarrow} \to 0 \tag{IV.2.0.1}$$

$$F: \qquad 0 \to OB \otimes_{OB^\sharp} {}_{\downarrow}OB^0 \xrightarrow{\varphi_F} {}_{\downarrow}OB \otimes_{OB^\sharp} OB^0 \xrightarrow{\psi_F} OB \otimes_{OB^\sharp} OB^0_{\uparrow} \to 0. \tag{IV.2.0.2}$$

To define the above homomorphisms, we write pure tensors $D_1 \otimes D_2$ in the above bimodules by drawing D_1 over D_2, with the tensor sign separating them. Then the above maps can be depicted as follows:

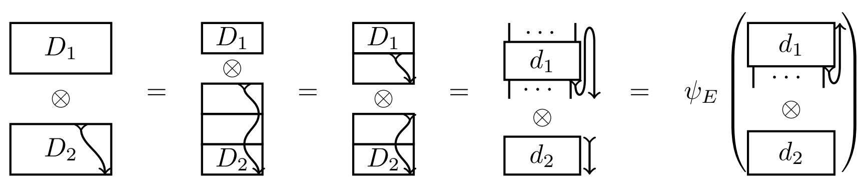


(IV.2.0.3)



(IV.2.0.4)

It is easy to see that these maps are well defined and make IV.2.0.1 and IV.2.0.2 into chain complexes of bimodules. We check that IV.2.0.1 is exact. Exactness of IV.2.0.2 is proved similarly. To see that ψ_E is surjective, let $D_1 \in OB$, $D_2 \in OB^0_\downarrow$ be diagrams. Then



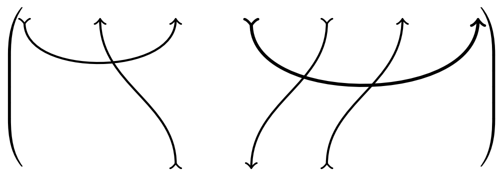
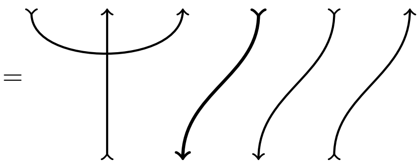
where the string shown in diagram D_2 is the one connected to its bottom-right vertex, d_2 is obtained from D_2 by deleting this string, and d_1 is the diagram above the tensor sign in the third frame.

To see that $\ker \psi_E \subset \operatorname{im} \phi_E$ we first define I to be the standard basis of OB^- consisting of diagrams with no caps and no crossings among vertical strings. Now let I_1 to be the set of all diagrams in I whose target object ends in $\uparrow$ and this vertex is on a cup. We define I_2 to be the set of all diagrams in I whose target object ends in $\uparrow$ and this vertex is on a vertical string. Now observe that

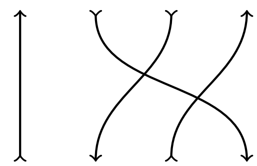
$$_{\uparrow}OB \otimes_{OB^\sharp} OB^0 = \bigoplus_{D \in I_1 \cup I_2} D \otimes_{\Bbbk} 1_{s(D)} OB^0.$$

So any $x \in \ker \psi_E$ can be written as $x = \sum_{D \in I_1 \cup I_2} D \otimes f_D$, with $f_D \in OB^0$. Now if $D \in I_1$, and has a cup connected to its top-right vertex and some other vertex i, then let $\hat{D}$ denote the diagram in I obtained from D by replacing the cup with a new vertical string connected to i on top (its vertex on bottom is determined by the fact that vertical strings are not allowed to cross). Let $\tau_D = 1_{s(\hat{D})} \tau_D 1_{s(D)\downarrow}$ be the diagram in OB^0 which, when placed beneath $\hat{D}$, connects the "new" vertex in $s(\hat{D})$ to the bottom-right vertex, and none of the other vertical strings cross each other.

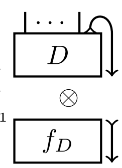
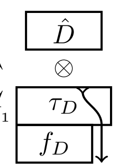
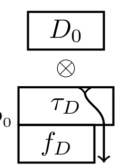
For example, if

$D =$  then $\hat{D} =$ 

and the corresponding τ is



Now

$0 = \psi_E(x) = \sum_{D \in I_1}$  $= \sum_{D \in I_1}$  $= \sum_{D_0 \in I,\, D \in I_1,\, \hat{D} = D_0}$ 

Therefore $\sum_{D \in I_1,\, \hat{D} = D_0} \tau_D \circ (f_D \downarrow) = 0$ for all $D_0 \in I$. Now since each diagram appearing in $\tau_D \circ (f_D \downarrow)$ has its bottom right vertex connected to some vertex on top, determined by $D \in I_1$, we have by linear independence of diagrams that $\tau_D \circ (f_D \downarrow) = 0$ for all $D_0 \in I$ and all $D \in I_1$ with $\hat{D} = D_0$, ie. for all $D \in I_1$. Then $f_D = 0$ for all $D \in I_1$, whence $x = \sum_{D \in I_2} D \otimes f_D \in \operatorname{im} \varphi_E$.

It remains to show that φ_E is injective. Note that $\varphi_E = \bigoplus_{D \in I} \varphi_E^{(D)}$, where $\varphi_E^{(D)}$ is the restriction of φ_E to $D \otimes_{\Bbbk} {}_{\uparrow}OB^0$, which is a linear isomorphism onto $(D \uparrow) \otimes_{\Bbbk} OB^0$. Then since $D \mapsto D \uparrow$ is an injective map $I \to I_1$, we have that φ_E is injective. $\square$

Recall from theorem III.1.4 that we can interpret dotted diagrams as elements of OB. It is immediate from the definition of the bimodule $OB_{\uparrow}$ that the linear endomorphism of $OB_{\uparrow}$ given on $OB1_{\mathsf{a}\uparrow}$ by right multiplication by $\mathsf{a}x$ is a bimodule homomorphism, which we denote x. This induces an endomorphism of the functor F, denoted X. Since $\Bbbk$ is algebraically closed and X preserves the individual spaces $1_{\mathsf{a}}V$ which are finite dimensional, we see that we have a decomposition $F = \bigoplus_{i \in \Bbbk} F_i$, where F_i is the generalized i-eigenspace of X acting on F. By adjointness, the endomorphism X induces an endomorphism X of E which is given on $OB1_{\mathsf{a}\downarrow}$ by right multiplication by $\mathsf{a}x'$. The functor E splits into the direct sum of E_i, defined as the generalized i-eigenspace of x' acting on E. It should be noted that using the isomorphism from proposition IV.1.1 we can alternatively describe $X \in \operatorname{End} F$ as left multplication on ${}_{\downarrow}OB$ by a dot on a "down" strand, and

$X \in \operatorname{End} E$ as left multiplication on ${}_{\uparrow}OB$ by a dot on an "up" strand. Note that $E^2 = (E\,{}_{\uparrow}OB) \otimes_{OB} ?$ and $1_{\mathsf{a}} E\,{}_{\uparrow}OB = 1_{\mathsf{a}\uparrow\uparrow} OB$. So we also have an endomorphism T of E^2 induced by the bimodule endomorphism given on $1_{\mathsf{a}\uparrow\uparrow}OB$ by left multiplication by the diagram $\mathsf{a}s$.

To similarly refine the functors $E^{\uparrow}, F^{\uparrow}, E_{\downarrow}, F_{\downarrow}$, we introduce the *Jucys-Murphy elements*:

$$z_{\uparrow} 1_{\mathsf{a}\uparrow} = \sum_{\substack{1 \le j < \ell(\mathsf{a}) \\ \mathsf{a}_j = \uparrow}} (j\, i) 1_{\mathsf{a}\uparrow} \qquad z_{\downarrow} 1_{\mathsf{a}\downarrow} = \sum_{\substack{1 \le j < \ell(\mathsf{a}) \\ \mathsf{a}_j = \downarrow}} (j\, i) 1_{\mathsf{a}\downarrow}$$

It is easy to show that $(D \uparrow) z_{\uparrow} 1_{\mathsf{a}\uparrow} = z_{\uparrow} 1_{\mathsf{b}\uparrow} (D \uparrow)$ and $(D \uparrow) z_{\uparrow} 1_{\mathsf{a}\uparrow} = z_{\uparrow} 1_{\mathsf{b}\uparrow} (D \uparrow)$ for $D \in 1_{\mathsf{b}} OB^0 1_{\mathsf{a}}$. Therefore the linear endomorphism of ${}_{\uparrow}OB^0$ (resp. $OB^0_{\uparrow}$) given on $1_{\mathsf{a}\uparrow}OB^0$ (resp. $OB^0 1_{\mathsf{a}\uparrow}$) by left (resp. right) multiplication by $m1_{\mathsf{a}\uparrow} + z_{\uparrow}1_{\mathsf{a}\uparrow}$ is an OB^0-bimodule homomorphism. Call this endomorphism $x_{\uparrow}$. Similarly, we have an OB^0-bimodule endomorphism of ${}_{\downarrow}OB^0$ (resp. $OB^0_{\downarrow}$) given on $1_{\mathsf{a}\downarrow}OB^0$ (resp. $OB^0 1_{\mathsf{a}\downarrow}$) by left (resp. right) multiplication by $m' 1_{\mathsf{a}\downarrow} - z_{\downarrow} 1_{\mathsf{a}\downarrow}$. Call this endomorphism $x_{\downarrow}$.

Since $x^{\downarrow}, x^{\uparrow}$ preserve the spaces $1_{\mathsf{a}}V$, we have decompositions

$$E^{\uparrow} = \bigoplus_{i \in \Bbbk} E_i^{\uparrow} \qquad E^{\downarrow} = \bigoplus_{i \in \Bbbk} E_i^{\downarrow}$$

$$F^{\uparrow} = \bigoplus_{i \in \Bbbk} F_i^{\uparrow} \qquad F^{\downarrow} = \bigoplus_{i \in \Bbbk} F_i^{\downarrow}$$

where the subscript i means to take the generalized i-eigenspace of $x^{\uparrow}$ or $x^{\downarrow}$ as appropriate. For example, since $E^{\uparrow} = {}_{\uparrow}OB^0 \otimes_{OB^0} ?$, $x^{\uparrow}$ induces an endomorphism of $E^{\uparrow}$, and $E_i^{\uparrow}$ is the corresponding generalized i-eigenspace.

Corollary IV.2.2. *There exist short exact sequences of functors OB^0-mod $\to OB$-mod*

$$0 \to \Delta \circ E_i^{\uparrow} \to E_i \circ \Delta \to \Delta \circ E_i^{\downarrow} \to 0$$

$$0 \to \Delta \circ F_i^{\downarrow} \to F_i \circ \Delta \to \Delta \circ F_i^{\uparrow} \to 0$$

Proof. First note that each of the functors $E_i, F_i, E_i^{\uparrow}, F_i^{\uparrow}, E_i^{\downarrow}, F_i^{\downarrow}$ can be alternately described as tensoring with its value on the appropriate regular module. For example, if ${}_{\uparrow}OB_i^0$ denotes the generalized i-eigenspace of $x^{\uparrow}$ acting on ${}_{\uparrow}OB^0$ (that is, ${}_{\uparrow}OB_i^0 = E_i^{\uparrow} OB^0$), then $E_i^{\uparrow} = {}_{\uparrow}OB_i^0 \otimes_{OB^0} ?$. Indeed, by the definition of $x^{\uparrow}$ it is clear that $E_i^{\uparrow} \supset {}_{\uparrow}OB_i^0 \otimes_{OB^0} ?$. Then taking the direct sum over

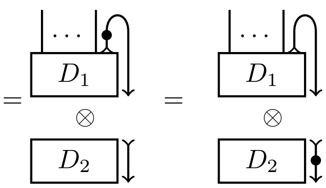
$i \in \Bbbk$ produces an equality. Therefore none of the inclusions $E_i^{\uparrow} \supset {}_{\uparrow}OB_i^0 \otimes_{OB^0} ?$ can be proper.

Now it suffices to verify that the short exact sequences of bimodules in the proof of theorem IV.2.1 intertwine the homomorphism x with the analogues for OB^0. The homomorphisms x, x' obviously preserve the images of $OB \otimes_{OB^\sharp} {}_{\downarrow}OB^0$ and $OB \otimes_{OB^\sharp} {}_{\uparrow}OB^0$.

Let us show that the short exact sequence for E intertwines x' with the analogues for OB^0. The proof for the short exact sequence for F is similar. Let $D_1 \in 1_{\mathsf{a}}OB, D_2 \in {}_{\uparrow}OB^0$ be diagrams with $(D_1 \uparrow)D_2 \neq 0$. Then

$$x\varphi_E(D_1 \otimes D_2) = x_{\ell(\mathsf{a})+1}(D_1 \uparrow) \otimes D_2 = (D_1 \uparrow)x_{\ell(\mathsf{a})+1} \otimes D_2 = \varphi_E(D_1 \otimes x_{\ell(\mathsf{a})+1} D_2) = \varphi_E x_{\uparrow}(D_1 \otimes D_2).$$

Now let $D_1 \in 1_{\mathsf{a}\uparrow}OB1_{\mathsf{b}}, D_2 \in 1_{\mathsf{b}}OB^0 1_{\mathsf{c}}$ be diagrams. Then

$\psi_E x(D_1 \otimes D_2) =$  $= x_{\downarrow}\psi_E(D_1 \otimes D_2)$ (IV.2.0.5)

Note that by Lemma III.1.5 the diagram beneath the tensor symbol in the third frame above represents

$$(m' 1_{\mathsf{b}\downarrow} - z_{\downarrow} 1_{\mathsf{b}\downarrow}) \circ (D_2 \downarrow) = (D_2 \downarrow) \circ (m' 1_{\mathsf{c}\downarrow} - z_{\downarrow} 1_{\mathsf{c}\downarrow}).$$

□

IV.3. Characters

Let $V \in OB$-mod and $\mathsf{a} \in \langle \downarrow, \uparrow \rangle$, $n = \ell(\mathsf{a})$. For each $i = 1, \ldots, n$ let x_i be the endomorphism of $1_{\mathsf{a}}V$ given by left multiplication by $x_i 1_{\mathsf{a}}$ if $\mathsf{a}_i = \uparrow$ or $x_i' 1_{\mathsf{a}}$ if $\mathsf{a}_i = \downarrow$. Suppose $n \geq 1$. Then in the finite dimensional commutative subalgebra of $1_{\mathsf{a}}OB1_{\mathsf{a}}$ generated by $x_1, \ldots, x_n$, there exist idempotents $\{1_{\mathsf{a};\underline{i}} : \underline{i} \in \Bbbk^n\}$ such that $1_{\mathsf{a};\underline{i}}$ acts on any module as projection onto the simultaneous generalized $\underline{i}$-eigenspace of $x_1, \ldots, x_n$. Let $1_{\varnothing;\varnothing} = 1_{\varnothing}$. We define the *formal character* of an OB-module V to be the formal sum

$$\operatorname{ch} V = \sum_{\mathsf{a},\underline{i}} (\dim 1_{\mathsf{a};\underline{i}} V) e^{\mathsf{a}_{\underline{i}}} \qquad \text{(IV.3.0.6)}$$

where $\mathsf{a}_{\underline{i}}$ is the sequence of subscripted symbols obtained from a by labelling a_j with the subscript i_j. Note that since our interpretation of $x_i 1_{\mathsf{a}}$ in OB depends on our choice of m, m', so does $\operatorname{ch} V$. Let $\operatorname{Supp}(V)$ be the set of all $(\mathsf{a}, \underline{i})$ such that $1_{\mathsf{a};\underline{i}} V \neq 0$.

Proposition IV.3.1. *The characters of irreducible OB-modules are linearly independent.*

Proof. Suppose $\sum_{\lambda \in \Lambda} a_\lambda \mathrm{ch}\ L(\lambda) = 0$ is a nontrivial relation. Choose λ_0, with $|\lambda_0|$ minimal, such that $a_{\lambda_0} \neq 0$. If $\lambda_0 \vdash (r, s)$ define a linear map T by $Te^{\mathsf{a}_{\underline{i}}} = 0$ for $\mathsf{a} \neq \downarrow^r\uparrow^s$, and $Te^{\downarrow^r_{\underline{i}}\uparrow^s_{\underline{j}}} = e^{(i_1-m,\ldots,i_r-m,m'-j_1,\ldots,m'-j_s)}$. Suppose $a_\lambda T\mathrm{ch}\ L(\lambda) \neq 0$. Then $\sum_{\underline{i}}(\dim 1_{\downarrow^r\uparrow^s;\underline{i}}L(\lambda))e^{\underline{i}} \neq 0$ so $\lambda \vdash (r-t, s-t)$ for some $t \geq 0$. But $a_\lambda \neq 0$ implies $|\lambda| \geq |\lambda_0|$, which yields $t = 0$, so that $\lambda \vdash (r, s)$. Then if $D(\lambda)$ is the the irreducible $kS_r \otimes kS_s$-module labeled by λ, then we have $T\mathrm{ch}\ L(\lambda) = \mathrm{ch}\ D(\lambda)$, where ch $D(\lambda)$ denotes the usual character of modules for the symmetric group (see section II.1), because $1_{\downarrow^r\uparrow^s;\underline{i}}L(\lambda) = 1_{\downarrow^r\uparrow^s;\underline{i}}D(\lambda)$. We then have $\sum_{\lambda \vdash (r,s)} a_\lambda \mathrm{ch}\ D(\lambda) = 0$ which implies $a_\lambda = 0$ whenever $\lambda \vdash (r, s)$ (see Lemma II.1.2). In particular, $a_{\lambda_0} = 0$. □

We can describe the character of $\widetilde{\Delta}(\lambda)$ combinatorially. We define the *branching graph* $\mathbb{B}$ by taking $\overline{\Lambda}$ for the set of vertices. There is an edge between λ and μ whenever μ is obtained from λ by adding a single box to either of the constituent partitions. We color the edge with the content of the box added. For example, the part of the branching graph involving only bipartitions of size 3 or less is shown in Figure 1, in the case $p = m = m' = 0$.

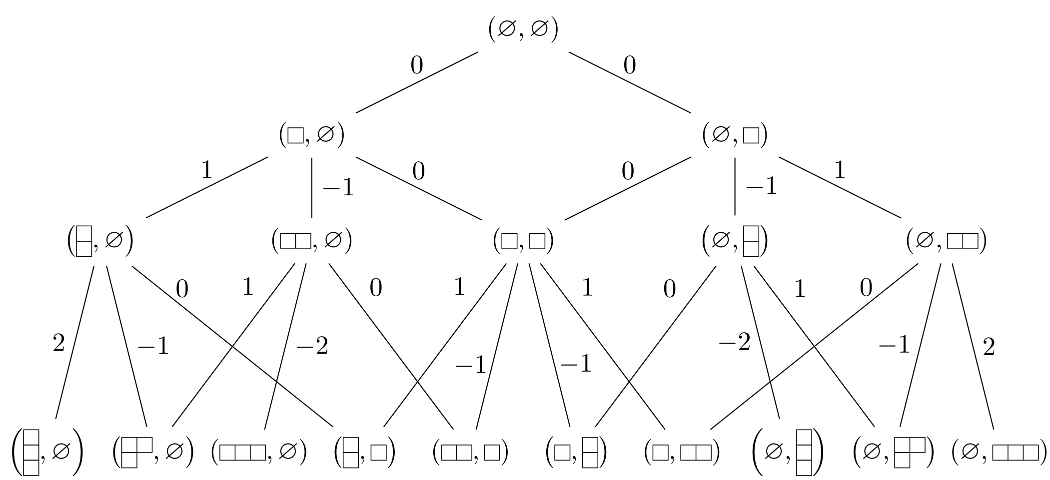


Figure 1: Branching graph when $p = m = m' = 0$.

By a *path* in $\mathbb{B}$ we mean a finite sequence of vertices $\lambda_0\lambda_1 \ldots \lambda_n$ with each λ_i, λ_{i+1} connected by an edge. We require $\lambda_0 = (\varnothing, \varnothing)$. The *type* of a path is $\mathsf{a}_{\underline{i}} = (\mathsf{a}_1)_{i_1} \ldots (\mathsf{a}_n)_{i_n}$ where i_j is the color of the j^{th} edge traversed in the path and $\mathsf{a}_j \in \{\downarrow, \uparrow\}$ is determined as follows. If λ_{j+1} is obtained from λ_j by adding a box to $\lambda_j^\uparrow$ or removing a box from $\lambda_j^\downarrow$ then $\mathsf{a}_j =\uparrow$. Otherwise $\mathsf{a}_j =\downarrow$.

Proposition IV.3.2.
$$\operatorname{ch} \widetilde{\Delta}(\lambda) = \sum_p e^{\operatorname{type}(p)}$$
where the sum is over all paths to λ in $\mathbb{B}$.

Proof. We show $1_{\mathsf{a};\underline{i}}\widetilde{\Delta}(\lambda)$ has dimension equal to the number of paths in $\mathbb{B}$ to λ of type $\mathsf{a}_{\underline{i}}$ by induction on $\ell(\mathsf{a})$. We have $\dim 1_{\varnothing;\varnothing}\widetilde{\Delta}(\lambda) = \dim 1_{\varnothing}\widetilde{\Delta}(\lambda)$, which is 0 if $\lambda \neq (\varnothing, \varnothing)$ and 1 if $\lambda = (\varnothing, \varnothing)$. In both cases this equals the number of paths to λ of type $\varnothing$.

The induction step follows from the identities $1_{\mathsf{a}\uparrow;\underline{i}i}V = 1_{\mathsf{a};\underline{i}}E_iV$ and $1_{\mathsf{a}\downarrow,\underline{i}i}V = 1_{\mathsf{a},\underline{i}}F_iV$. For example, since $E_i\widetilde{\Delta}(\lambda)$ has a filtration with sections isomorphic to $\widetilde{\Delta}(\mu)$, where μ is obtained from λ either by adding a box of $\downarrow$-content i to $\lambda^{\downarrow}$ or removing a box of $\uparrow$-content i from $\lambda^{\uparrow}$, each possible such μ appearing exactly once, it follows that $\dim 1_{\mathsf{a}\uparrow;\underline{i}i}\widetilde{\Delta}(\lambda) = \sum_\mu \dim 1_{\mathsf{a};\underline{i}}\widetilde{\Delta}(\mu)$ summing over all μ obtained from λ by removing a box of $\downarrow$-content i from $\lambda^{\downarrow}$ or adding a box of $\uparrow$-content i to $\lambda^{\uparrow}$. The case involving $F_i\widetilde{\Delta}(\lambda)$ is similar. $\square$

Example IV.3.3. *We compute the terms $e^{\mathsf{a}_{\underline{i}}}$ in $\operatorname{ch}\widetilde{\Delta}(\square, \varnothing)$ with $\ell(\mathsf{a}) \leq 3$. There is only one path to $(\square, \varnothing)$ of length ≤ 2, which contributes a term $e^{\downarrow_0}$. There are six paths to $(\square, \varnothing)$ of length 3. Two of these begin by departing from $(\varnothing, \varnothing)$, and then immediately returning to $(\varnothing, \varnothing)$ before proceeding to $(\square, \varnothing)$. These two paths contribute $e^{\downarrow_0\uparrow_0\downarrow_0} + e^{\uparrow_0\downarrow_0\downarrow_0}$. Another path to $(\square, \varnothing)$ of length 3 is $(\varnothing, \varnothing) \to (\varnothing, \square) \to (\square, \square) \to (\square, \varnothing)$, which contributes the term $e^{\uparrow_0\downarrow_0\downarrow_0}$. The remaining three paths to $(\square, \varnothing)$ of length 3 all begin by following the edge $(\varnothing, \varnothing) \to (\varnothing, \square)$, then visiting a bipartition of size 3 before returning to $(\square, \varnothing)$. These paths contribute $e^{\downarrow_0\downarrow_1\uparrow_1} + e^{\downarrow_0\downarrow_{-1}\uparrow_{-1}} + e^{\downarrow_0\uparrow_0\downarrow_0}$. So the terms $e^{\mathsf{a}_{\underline{i}}}$ in $\operatorname{ch}\widetilde{\Delta}(\square, \varnothing)$ with $\ell(\mathsf{a}) \leq 3$ are: $e^{\downarrow_0} + 2e^{\downarrow_0\uparrow_0\downarrow_0} + 2e^{\uparrow_0\downarrow_0\downarrow_0} + e^{\downarrow_0\downarrow_1\uparrow_1} + e^{\downarrow_0\downarrow_{-1}\uparrow_{-1}}$.*

Corollary IV.3.4. *If $[\widetilde{\Delta}(\lambda) : L(\mu)] \neq 0$, then there is a path in $\mathbb{B}$ to μ of length $|\mu|$, and a path to λ of the same type.*

Proof. Choose $\mathsf{a} \in \langle \downarrow, \uparrow \rangle_{|\mu^{\downarrow}|,|\mu^{\uparrow}|}$ so $1_{\mathsf{a}}L(\mu) \neq 0$. Then we can choose $\underline{i}$ so that $(\mathsf{a}, \underline{i}) \in \operatorname{Supp} L(\mu) \subset \widetilde{\Delta}(\mu)$. Then proposition IV.3.2 guarantees a path to μ of type $\mathsf{a}_{\underline{i}}$. But also $(\mathsf{a}, \underline{i}) \in \operatorname{Supp} L(\mu) \subset \operatorname{Supp}\widetilde{\Delta}(\lambda)$, so there is a path to λ of type $\mathsf{a}_{\underline{i}}$. $\square$

Theorem IV.3.5. *If $[\widetilde{\Delta}(\lambda) : L(\mu)] \neq 0$, then $\mu \leq \lambda$.*

Proof. From the corollary we have a path to μ of length $|\mu|$ and a path to λ of the same type. We will prove that the existence of such a pair of paths implies $\mu \leq \lambda$ by induction on $|\mu|$. If $|\mu| = 0$, then $\lambda = \mu = (\varnothing, \varnothing)$ so $\mu \leq \lambda$ is trivial. Now assume $|\mu| \geq 1$. Delete the last edge in each path to obtain paths to μ' and λ', each of length $|\mu'|$. By induction, $\mu' \leq \lambda'$, ie. $\operatorname{cont}(\lambda') = \operatorname{cont}(\mu')$

and $\mathrm{cont}_{\uparrow}(\mu') \geq \mathrm{cont}_{\uparrow}(\lambda')$. Since the type of the deleted edge in each path is the same, we have $\mathrm{cont}(\mu) = \mathrm{cont}(\lambda)$.

Suppose the type of deleted edge is $\uparrow_i$. Then $\mathrm{cont}_{\uparrow}(\mu) = \mathrm{cont}_{\uparrow}(\mu') + \alpha_i$. Now if λ is obtained from λ' by adding a box to $\lambda'^{\uparrow}$ then $\mathrm{cont}_{\uparrow}(\lambda) = \mathrm{cont}_{\uparrow}(\lambda') + \alpha_i$ so that $\mathrm{cont}_{\uparrow}(\mu) - \mathrm{cont}_{\uparrow}(\lambda) = \mathrm{cont}_{\uparrow}(\mu') - \mathrm{cont}_{\uparrow}(\lambda') \geq 0$. On the other hand, if λ is obtained from λ' by removing a box from $\lambda'^{\downarrow}$, then $\mathrm{cont}_{\uparrow}(\lambda) = \mathrm{cont}_{\uparrow}(\lambda')$ so that $\mathrm{cont}_{\uparrow}(\mu) - \mathrm{cont}_{\uparrow}(\lambda) = \mathrm{cont}_{\uparrow}(\mu') + \alpha_i - \mathrm{cont}_{\uparrow}(\lambda') \geq \alpha_i > 0$.

The case when the type of the last edge is $\downarrow_i$ is treated similarly. $\square$

CHAPTER V

TENSOR PRODUCT CATEGORIFICATION

In this chapter we assemble the pieces of our main theorem (Theorem V.3.2). In section V.1 we show that we have the data of a locally stratified structure on OB-mod, namely that our standardization functor Δ is the left adjoint of a quotient functor (see section II.5). We go on to show in section V.2 that the axioms of a locally stratified structure are met. In section V.3 we show that the functors E_i, F_i define a categorical $\mathfrak{sl}_{\Bbbk}$-action on OB-mod and state our main theorem (Theorem V.3.2). Then in section V.4 we describe the crystal of OB, which is a consequence of Theorem V.3.2 and the main result of [D].

V.1. Standardization

The poset Ξ parameterizes the blocks of the category of finite dimensional OB^0-modules. For $\xi \in \Xi$, the block OB^0-mod$[\xi]$ consists of finite dimensional OB^0-modules whose composition factors are $D(\lambda)$, with $\lambda \in \xi$. Denote by Δ_ξ the restriction of Δ to OB^0-mod$[\xi]$. Then we have seen that the image of Δ_ξ lies in OB-mod$_{\leq \xi}$, which is the full subcategory of OB-mod consisting of modules V with $[V : L(\lambda)] = 0$ unless $\lambda \in \xi'$ for some $\xi' \in \Xi$ satisfying $\xi' \leq \xi$.

We've defined Δ_ξ as inclusion OB^0-mod$[\xi] \hookrightarrow OB^0$-mod, followed by the inflation OB^0-mod $\to OB^\sharp$-mod, followed by induction to OB. Therefore, Δ_ξ is left adjoint to the functor π_ξ defined as the composite $\mathrm{pr}_\xi \circ R \circ \mathrm{res}^{OB}_{OB^\sharp}$, where pr_ξ is projection to the block OB^0-mod$[\xi]$, and R is the right adjoint to the inflation functor. We have $RV = \{v \in V : OB^\sharp[1]v = 0\}$.

For $V \in OB$-mod$_{\leq \xi}$, let $1_\xi = \sum_{\mathsf{a} \in \langle \downarrow, \uparrow \rangle_{r,s}} 1_{\mathsf{a}}$, where the representatives of ξ are bipartitions of (r, s). Then clearly $\pi_\xi V \subset 1_\xi V$ and $1_\xi V \subset R \circ \mathrm{res}^{OB}_{OB^\sharp} V$. So $\pi_\xi V = 1_\xi V$ once we can see that $1_\xi V \in OB^0$-mod$[\xi]$. This is so because the composition factors of V are $L(\lambda)$ for $[\lambda] \leq \xi$, and for such λ we have $1_\xi L(\lambda) = 0$ unless $\lambda \in \xi$, in which case $1_\xi L(\lambda) = D(\lambda)$ as we've noted before. In fact, we now see that $\pi_\xi V$ is the shortest word space of V whenever $V \in OB$-mod$_{\leq \xi}$ and $\pi_\xi V \neq 0$.

Now it is clear that π_ξ is exact and commutes with the duality. Therefore it has a right adjoint ∇_ξ, which is obtained from Δ_ξ by composing with duality on the left and right. The result is that ∇_ξ is given by inclusion $OB^0\text{-mod}[\xi] \hookrightarrow OB^0\text{-mod}$, followed by the inflation functor $OB^0\text{-mod} \to OB^\flat\text{-mod}$ (defined by letting cups act as zero), followed by the coinduction to OB. Explicitly, $\nabla_\xi = \bigoplus_{\mathsf{a}\in\langle\downarrow,\uparrow\rangle} \mathrm{Hom}_{OB^\flat}(OB1_{\mathsf{a}}, ?)$. Here, and forever, we view any OB^0-module as an $OB^\flat$-module via the inflation functor as we do with inflation to $OB^\sharp$. We set $\nabla = \bigoplus_{\xi\in\Xi} \nabla_\xi$ and call this the *costandardization* functor. We define the *costandard modules* $\nabla(\lambda) = \nabla Y(\lambda)$ and the *proper costandard modules* $\overline{\nabla}(\lambda) = \nabla D(\lambda)$. Write $\widetilde{\nabla}(\lambda)$ for $\nabla S(\lambda)$.

Proposition V.1.1. *We have $\Delta(\lambda)^* \cong \nabla(\lambda)$ and $\overline{\Delta}(\lambda)^* \cong \overline{\nabla}(\lambda)$. In particular,*

$$\left[\overline{\nabla}(\mu) : L(\lambda)\right] = \left[\overline{\Delta}(\mu) : L(\lambda)\right].$$

Proof. The stated isomorphisms follow from the fact that $D(\lambda)$ and $Y(\lambda)$ are self-dual. Now since $L(\lambda)$ is self-dual, the statement about composition multiplicities follows. □

Proposition V.1.2. *$OB\text{-mod}_\xi = OB^0\text{-mod}[\xi]$ and π_ξ is the quotient functor from section II.5.*

Proof. This is a special case of Theorem II.4.3 in light of the above description of π_ξ.

□

V.2. Projective Modules

Given $V \in OB\text{-mod}$ a *standard filtration* or Δ-*flag* of V is a finite filtration

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

with $V_i/V_{i-1} \cong \Delta(\lambda_i)$ for $\lambda_i \in \Lambda$. Let $(V : \Delta(\lambda))$ denote the multiplicity of $\Delta(\lambda)$ as a section of such a filtration of V. Theorem V.2.2 below proves that this is independent of the standard filtration chosen. We begin with a lemma.

Lemma V.2.1. *For any $\lambda, \mu \in \Lambda$, we have*

$$\dim \mathrm{Ext}^i_{OB}(\Delta(\mu), \overline{\nabla}(\lambda)) = \delta_{i,0}\delta_{\lambda,\mu}.$$

Proof. Using the triangular decomposition of OB, we can see that $\mathrm{res}^{OB}_{OB^\flat} \circ \Delta = OB^\flat \otimes_{OB^0} ?$

$$\begin{aligned}
\mathrm{Ext}^i_{OB}(\Delta(\mu), \overline{\nabla}(\lambda)) &= \mathrm{Ext}^i_{OB}\left(\Delta(\mu), \bigoplus_{\mathsf{a}} \mathrm{Hom}_{OB^\flat}(OB1_{\mathsf{a}}, D(\lambda))\right) \\
&= \mathrm{Ext}^i_{OB^\flat}(OB^\flat \otimes_{OB^0} Y(\mu), D(\lambda)) \\
&= \mathrm{Ext}^i_{OB^0}(Y(\mu), D(\lambda)).
\end{aligned}$$

□

Theorem V.2.2. *Let $V \in OB$-mod have a standard filtration. Then for any $\lambda \in \Lambda$, we have*

$$(V : \Delta(\lambda)) = \dim \mathrm{Hom}_{OB}(V, \overline{\nabla}(\lambda)). \tag{V.2.0.1}$$

Proof. Induct on the length of the filtration. If $V = \Delta(\mu)$, then both sides of V.2.0.1 are equal to $\delta_{\lambda,\mu}$ by the lemma. For the induction step, if $\Delta(\mu)$ is at the top of a standard filtration of V, apply $\mathrm{Hom}_{OB}(?, \overline{\nabla}(\lambda))$ to $0 \to W \to V \to \Delta(\mu) \to 0$. By Lemma V.2.1 the first Ext^1 term is zero. Therefore by induction, we have

$$\begin{aligned}
\dim \mathrm{Hom}_{OB}(V, \overline{\nabla}(\lambda)) &= \dim \mathrm{Hom}_{OB}(\Delta(\mu), \overline{\nabla}(\lambda)) + \dim \mathrm{Hom}_{OB}(W, \overline{\nabla}(\lambda)) \\
&= \delta_{\lambda,\mu} + (W : \Delta(\lambda)) \\
&= (V : \Delta(\lambda)).
\end{aligned}$$

□

Proposition V.2.3. *Let $V \in OB$-mod have a Δ-flag: $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ with $V_i/V_{i-1} \cong \Delta(\lambda_i)$ for some $\lambda_i \in \Lambda$, $i = 1, \ldots, n$. Suppose $V = V' \oplus V''$ with $V', V'' \in OB$-mod. Then V', V'' both have Δ-flags.*

Proof. If $n = 1$, then $V \cong \Delta(\lambda_1)$ is indecomposable, so there is nothing to prove. Suppose $n > 1$. Choose $\lambda \in \Lambda$ maximal such that $L(\lambda)$ appears as a section of some filtration of V. If there are multiple candidates for λ, choose $|\lambda|$ minimal.

Now $L(\lambda)$ is a quotient of some submodule $\tilde{V} \subset V$. Then since $D(\lambda)$ is a summand of $L(\lambda)$ as an OB^0-module, we have a nonzero OB^0-module homomorphism $\tilde{V} \to D(\lambda)$. By projectivity of $Y(\lambda)$, we get a nonzero OB^0-module homomorphism $Y(\lambda) \to \tilde{V} \hookrightarrow V$. In fact, this is an $OB^\sharp$-module homomorphism, because its image lies in the direct sum of all $1_{\mathsf{a}}V$ with $a \in \langle \downarrow, \uparrow \rangle_{r,s}$, where

$\lambda \vdash (r,s)$. Caps act as zero on this space because $1_{\mathsf{a}}V = \sum_{i=1}^{n} \dim 1_{\mathsf{a}}\Delta(\lambda_i) = 0$ for any a with $\ell(\mathsf{a}) < |\lambda|$. We therefore have a nonzero induced homomorphism $\Delta(\lambda) \to V$. Then relabeling V' and V'' if necessary, we have a nonzero homomorphism $\varphi \in \mathrm{Hom}_{OB}(\Delta(\lambda), V')$.

Choose i minimal so that $\operatorname{im}\varphi \subset V_i$. Then the composite

$$\Delta(\lambda) \xrightarrow{\varphi} V_i \to V_i/V_{i-1} \cong \Delta(\lambda_i)$$

is nonzero, which implies $\lambda \leq \lambda_i$. By maximality of λ, we then have $\lambda = \lambda_i$, and this composite must be an isomorphism. Therefore φ is injective and $\Delta(\lambda) \cap V_{i-1} = 0$. We have $V/\Delta(\lambda) = (V'/\Delta(\lambda)) \oplus V''$. If $V/\Delta(\lambda)$ has a shorter Δ-flag than V, then induction gives Δ-flags for $V'/\Delta(\lambda)$ and V'', hence for V' also.

To see that $V/\Delta(\lambda)$ has a shorter Δ-flag than V, observe that the short exact sequence $0 \to V_{i-1} \to V_i \to \Delta(\lambda) \to 0$ splits because $\Delta(\lambda) \cap V_{i-1} = 0$. Thefore $V_i/\Delta(\lambda) \cong V_{i-1}$ so that the short exact sequence $0 \to V_i/\Delta(\lambda) \to V/\Delta(\lambda) \to V/V_i \to 0$ becomes $0 \to V_{i-1} \to V/\Delta(\lambda) \to V/V_i \to 0$. Since V_{i-1} has a Δ-flag of length $i-1$ and V/V_i has one of length $n-i$, we see that $V/\Delta(\lambda)$ has one of length $n-1$, as needed. □

Let $\lambda \in \Lambda_{r,s}$. We construct the projective cover $P(\lambda)$ of $L(\lambda)$ as follows. Recall that the shortest word space of $L(\lambda)$ is isomorphic as an OB^0-module to $D(\lambda)$, ie.

$$\bigoplus_{\mathsf{a} \in \langle \downarrow, \uparrow \rangle_{r,s}} 1_{\mathsf{a}} L(\lambda) \cong D(\lambda).$$

Therefore $\dim \mathrm{Hom}_{OB^0}(Y(\lambda), L(\lambda)) = \dim \mathrm{Hom}_{OB^0}(Y(\lambda), D(\lambda)) = 1$. Consider the projective module $\hat{P}(\lambda) = OB \otimes_{OB^0} Y(\lambda)$. We have $\dim \mathrm{Hom}_{OB}(\hat{P}(\lambda), L(\lambda)) = 1$, and so $L(\lambda)$ is covered by some unique indecomposable projective summand $P(\lambda)$.

Proposition V.2.4. *$P(\lambda)$ has a Δ-flag, with $\Delta(\lambda)$ at the top, and all other sections isomorphic to $\Delta(\mu)$ with $\lambda < \mu$.*

Proof. We shall show that $\hat{P}(\lambda)$ has a Δ-flag, and conclude from Proposition V.2.3 that $P(\lambda)$ does also. In particular, the following version of BGG reciprocity follows from V.2.2, II.2.6, and V.1.1:

$$(P(\lambda) : \Delta(\mu)) = \left[\overline{\Delta}(\mu) : L(\lambda)\right] \qquad \text{(V.2.0.2)}$$

Now Theorem IV.3.5 proves the description of sections of the Δ-flag.

To show $\hat{P}(\lambda) = OB \otimes_{OB^\sharp} OB^\sharp \otimes_{OB^0} Y(\lambda)$ has a Δ-flag, we define an infinite descending filtration on $OB^\sharp$ by setting $OB^\sharp_i = \bigoplus_{k \geq i} OB^\sharp[k]$. It follows that $OB^\sharp \otimes_{OB^0} Y(\lambda)$ has a *finite* filtration with sections $OB^\sharp[k] \otimes_{OB^0} Y(\lambda)$, $0 \leq k \leq \min\{|\lambda^\downarrow|, |\lambda^\uparrow|\}$, and so $\hat{P}(\lambda)$ has a finite filtration with sections $\Delta\left(OB^\sharp[k] \otimes_{OB^0} Y(\lambda)\right)$.

Note that $OB^\sharp[k] \otimes_{OB^0} Y(\lambda)$ is a summand of the OB^0-module $OB^\sharp[k]$, which is isomorphic to

$$\bigoplus_{r,s \geq 0} OB^0 1_{r,s} \otimes_{\Bbbk} 1_{r,s} OB^+[k] \cong \bigoplus_{r,s \geq 0} (\dim 1_{r,s} OB^+[k]) OB^0 1_{r,s},$$

which is projective. Hence $OB^\sharp[k] \otimes_{OB^0} Y(\lambda)$ is a finite dimensional projective OB^0-module, and therefore has a filtration with sections $Y(\mu)$ for various bipartitions μ of $(|\lambda^\downarrow| - k, |\lambda^\uparrow| - k)$. This finishes the proof. $\square$

We have now proved the following theorem.

Theorem V.2.5. *The preorder on Λ defined in section III.3 makes OB-mod into a locally stratified category with standard objects $\Delta(\lambda)$, $\lambda \in \Lambda$. If $p = 0$ then OB-mod is a locally highest weight category.*

V.3. Categorical Action

Let $\mathcal{C} = OB$-mod. We have shown that $\mathcal{C}$ is a locally stratified category with gr $\mathcal{C}$ isomorphic to the category of finite dimensional OB^0-modules. We also have categorical actions on $\mathcal{C}$ and gr $\mathcal{C}$. The categorical action on gr $\mathcal{C}$ is well-known, and the categorical action on $\mathcal{C}$ was described in section IV.1. We now review these categorical actions, and then discuss their compatibility with the local stratification.

We denote by $K_0(\mathcal{C})$ (resp. $K_0(\text{gr } \mathcal{C})$) the split Grothendieck group of the category of finitely generated projective OB-modules (resp. OB^0-modules). That is, $K_0(\mathcal{C})$ is the free abelian group on the isomorphism classes of finitely generated projective OB-modules modulo $[B] - [A] - [C]$ whenever $B \cong A \oplus C$, and similarly with OB^0. We let $\mathcal{C}^\Delta$ denote the subcategory of $\mathcal{C}$ consisting of all modules with a Δ-flag. We denote by $G_0(\mathcal{C}^\Delta)$ its Grothendieck group. That is, $G_0(\mathcal{C}^\Delta)$ is the free abelian group on the isomorphism classes of modules in $\mathcal{C}^\Delta$ modulo $[B] - [A] - [C]$ for any $A, B, C \in \mathcal{C}^\Delta$ exhibiting a short exact sequence $0 \to A \to B \to C \to 0$. We define $[\mathcal{C}] = \mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{C})$, $[\text{gr } \mathcal{C}] = \mathbb{C} \otimes_{\mathbb{Z}} K_0(\text{gr } \mathcal{C})$, and $[\mathcal{C}^\Delta] = \mathbb{C} \otimes_{\mathbb{Z}} G_0(\mathcal{C}^\Delta)$.

Theorem V.3.1. *The operators $E_i^\downarrow, F_i^\downarrow, E_i^\uparrow, F_i^\uparrow$, $i \in \Bbbk$ on $[gr\ \mathcal{C}]$ satisfy the relations of the Chevalley* *generators of* $\mathfrak{sl}_{\Bbbk}^{\downarrow} \oplus \mathfrak{sl}_{\Bbbk}^{\uparrow}$ *under the assignments*

$$e_i^{\downarrow} \mapsto E_i^{\downarrow} \qquad f_i^{\downarrow} \mapsto F_i^{\downarrow}$$

$$e_i^{\uparrow} \mapsto E_i^{\uparrow} \qquad f_i^{\uparrow} \mapsto F_i^{\uparrow}.$$

Moreover, this module is isomorphic to the tensor product of basic modules $V(-\varpi_{m'}) \otimes V(\varpi_m)$*, and the weight space decomposition of this module coincides with the decomposition of finite dimensional* OB^0*-modules into blocks.*

Proof. This follows from [Groj] in light of the identifications

$$E_i^{\uparrow} = \bigoplus_{r,s\geq 0} (i-m) - \mathrm{res}_{r,s}^{r,s+1} \qquad E_i^{\downarrow} = \bigoplus_{r,s\geq 0} (m'-i) - \mathrm{ind}_{r,s}^{r+1,s}$$

$$F_i^{\uparrow} = \bigoplus_{r,s\geq 0} (i-m) - \mathrm{ind}_{r,s}^{r,s+1} \qquad F_i^{\downarrow} = \bigoplus_{r,s\geq 0} (m'-i) - \mathrm{res}_{r,s}^{r+1,s}$$

□

The categorical $\mathfrak{sl}_{\Bbbk}$-action on $\mathcal{C}$ is a slightly modified version of Rouquier's definition in [R]. That is, we have an adjoint pair (E, F) of exact functors $\mathcal{C} \to \mathcal{C}$ and endomorphisms $X \in \mathrm{End}\, E$, $T \in \mathrm{End}\, E^2$ (see IV.1). We have $E = \bigoplus_{i\in\Bbbk} E_i$ and $F = \bigoplus_{i\in\Bbbk} F_i$, where E_i (resp. F_i) is the generalized i-eigenspace of X acting on E (resp. F). The functor F is isomorphic to a left adjoint of E. The action on E^n of $X_i = E^{n-i} X E^{i-1}$ for $1 \leq i \leq n$ and of $T_i = E^{n-i-1} T E^{i-1}$ for $1 \leq i \leq n-1$ induce an action of the degenerate affine Hecke algebra.

Theorem V.3.1 says that $[\mathrm{gr}\ \mathcal{C}] = V(-\varpi_{m'}) \otimes V(\varpi_m)$ as $\mathfrak{sl}_{\Bbbk}^{\downarrow} \oplus \mathfrak{sl}_{\Bbbk}^{\uparrow}$-modules. Pulling this action back through the diagonal map $\mathfrak{sl}_{\Bbbk} \to \mathfrak{sl}_{\Bbbk}^{\downarrow} \oplus \mathfrak{sl}_{\Bbbk}^{\uparrow}$ gives an integrable $\mathfrak{sl}_{\Bbbk}$-module. This module is isomorphic to $[\mathcal{C}^{\Delta}]$ via the standardization functor (see corollary IV.2.2). Then since $[\mathcal{C}]$ embeds into $[\mathcal{C}^{\Delta}]$, we see that $[\mathcal{C}]$ is an integrable $\mathfrak{sl}_{\Bbbk}$-module. Moreover, this embedding is an isomorphism $[\mathcal{C}] \cong [\mathcal{C}^{\Delta}]$. To see this, choose a total order of Λ refining its preorder. Then note that with respect to the ordered bases $\{[P(\lambda)] : \lambda \in \Lambda\}$ for $[\mathcal{C}]$ and $\{[\Delta(\lambda)] : \lambda \in \Lambda\}$ for $[\mathcal{C}^{\Delta}]$, the embedding is given by an upper unitriangular matrix (see Proposition V.2.4). Therefore the embedding is an isomorphism. Note that although we don't have a decomposition of $\mathcal{C}$ into blocks, we do have $[\mathcal{C}] = \bigoplus_{\xi\in\Xi} [\mathcal{C}_{\xi}]$.

The local stratification and the categorical action on $\mathcal{C}$ are compatible in the following sense. First, the poset Ξ can be identified with pairs of weights of $V(-\varpi_{m'}), V(\varpi_m)$ via the map $[\lambda] \mapsto (-\varpi_{m'} + \mathrm{cont}_{\downarrow} \lambda, \varpi_m - \mathrm{cont}_{\uparrow} \lambda)$. This map is clearly surjective, and the remarks in section III.3

show that it is a well-defined injection. Transporting the partial order on Ξ through this bijection induces the *inverse dominance order* (see Definition 3.2 of [LW]) on the set of pairs of weights of $V(-\varpi_{m'}), V(\varpi_m)$. That is, for two such pairs $\underline{\mu} = (\mu_1, \mu_2), \underline{\nu} = (\nu_1, \nu_2)$, we have $\underline{\mu} \leq \underline{\nu}$ if and only if $\mu_1 \geq \mu_2$ and $\mu_1 + \mu_2 = \nu_1 + \nu_2$. Second, gr $\mathcal{C}$ carries a categorical $\mathfrak{sl}_{\mathbb{k}}^{\downarrow} \oplus \mathfrak{sl}_{\mathbb{k}}^{\uparrow}$ action with $[\text{gr } \mathcal{C}] \cong V(-\varpi_{m'}) \otimes V(\varpi_m)$ and the weight ξ subcategory of gr $\mathcal{C}$ is the quotient $\mathcal{C}_\xi$. Lastly, for each $M \in \mathcal{C}_\xi$ the object $E_i\Delta(M)$ admits a filtration with successive quotients being $\Delta(E_i^{\downarrow}M), \Delta(E_i^{\uparrow}M)$, and similarly with F_i.

We have proved our main theorem, which we now state.

Theorem V.3.2. *The endofunctors E_i, F_i of OB-mod define a categorical $\mathfrak{sl}_{\mathbb{k}}$-action. This action is compatible with the locally stratified structure on OB-mod and categorifies $V(-\varpi_{m'}) \otimes V(\varpi_m)$.*

V.4. Crystal Graph Structure

We modify the description of the crystal associated to $V(\varpi_0)$ from ([K]) to get the crystals associated to $V(-\varpi_{m'})$ and $V(\varpi_m)$. Fix a partition λ. Label all addable nodes of $\uparrow$-content i by $+$ and all removable nodes of $\uparrow$-content i by $-$. The *$\uparrow_i$-signature* of λ is the sequence of pluses and minuses obtained by going along the rim of the Young diagram of λ from bottom left to top right and reading off all the signs. The *reduced $\uparrow_i$-signature* of λ is obtained from its $\uparrow_i$-signature by successively deleting all neighboring pairs of the form $-+$. The reduced $\uparrow_i$-signature is a sequence of $+$'s followed by a sequence of $-$'s.

To define (*reduced*) *$\downarrow_i$-signature*, label all addable nodes of $\downarrow$-content i by $-$ and all removable nodes of $\downarrow$-content i by $+$. The *$\downarrow_i$-signature* of λ is the sequence of pluses and minuses obtained by going along the rim of the Young diagram of λ from top right to bottom left and reading off all the signs. The *reduced $\downarrow_i$-signature* of λ is obtained from its $\downarrow_i$-signature by successively deleting all neighboring pairs of the form $-+$. The reduced $\downarrow_i$-signature is a sequence of $+$'s followed by a sequence of $-$'s.

We make the set of p-regular partitions into the crystal associated to $V(\varpi_m)$ as follows. We define

$$\varepsilon_i^{\uparrow}(\lambda) = \#\{-\text{'s in the reduced } \uparrow_i \text{-signature of } \lambda\}$$

$$\varphi_i^{\uparrow}(\lambda) = \#\{+\text{'s in the reduced } \uparrow_i \text{-signature of } \lambda\}.$$

If $\varepsilon_i^{\uparrow}(\lambda) = 0$, set $\tilde{e}_i^{\uparrow}\lambda = 0$. Otherwise define $\tilde{e}_i^{\uparrow}\lambda$ to be the partition obtained by removing the node in λ corresponding to the leftmost $-$ in its reduced $\uparrow_i$-signature. If $\varphi_i^{\uparrow}(\lambda) = 0$, set $\tilde{f}_i^{\uparrow}\lambda = 0$. Otherwise define $\tilde{f}_i^{\uparrow}\lambda$ to be the partition obtained by adding the node in λ corresponding to the rightmost $+$ in its reduced $\uparrow_i$-signature. Finally, set $\mathrm{wt}^{\uparrow}(\lambda) = \varpi_m - \mathrm{cont}_{\uparrow}\lambda$.

We make the set of p-regular partitions into the crystal associated to $V(-\varpi_{m'})$ as follows. We define

$$\varepsilon_i^{\downarrow}(\lambda) = \#\{-\text{'s in the reduced } \downarrow_i \text{-signature of } \lambda\}$$

$$\varphi_i^{\downarrow}(\lambda) = \#\{+\text{'s in the reduced } \downarrow_i \text{-signature of } \lambda\}.$$

If $\varepsilon_i^{\downarrow}(\lambda) = 0$, set $\tilde{e}_i^{\downarrow}\lambda = 0$. Otherwise define $\tilde{e}_i^{\downarrow}\lambda$ to be the partition obtained by adding the node in λ corresponding to the leftmost $-$ in its reduced $\downarrow_i$-signature. If $\varphi_i^{\downarrow}(\lambda) = 0$, set $\tilde{f}_i^{\downarrow}\lambda = 0$. Otherwise define $\tilde{f}_i^{\downarrow}\lambda$ to be the partition obtained by removing the node in λ corresponding to the rightmost $+$ in its reduced $\downarrow_i$-signature. Finally, set $\mathrm{wt}^{\downarrow}(\lambda) = -\varpi_{m'} + \mathrm{cont}_{\downarrow}\lambda$.

The crystal associated to $V(-\varpi_{m'}) \otimes V(\varpi_m)$ is the Kashiwara tensor product of the above crystals. Given a bipartition $\lambda = (\lambda^{\downarrow}, \lambda^{\uparrow})$, its *i-signature* is obtained by concatenating the $\downarrow_i$-signature of $\lambda^{\downarrow}$ followed by the $\uparrow_i$-signature of $\lambda^{\uparrow}$. The *reduced i-signature* of λ is obtained from its i-signature by successively deleting all neighboring pairs of the form $-+$. The reduced i-signature is a sequence of $+$'s followed by a sequence of $-$'s.

We make Λ into the crystal associated to $V(-\varpi_{m'}) \otimes V(\varpi_m)$ by defining

$$\varepsilon_i(\lambda) = \#\{-\text{'s in the reduced } i\text{-signature of } \lambda\}$$

$$\varphi_i(\lambda) = \#\{+\text{'s in the reduced } i\text{-signature of } \lambda\}.$$

If $\varepsilon_i(\lambda) = 0$, set $\tilde{e}_i\lambda = 0$. Otherwise define $\tilde{e}_i\lambda$ to be the bipartition obtained by adding or removing the node in λ corresponding to the leftmost $-$ in its reduced $\downarrow_i$-signature (add the node if it belongs to $\lambda^{\downarrow}$ and remove it if it belongs to $\lambda^{\uparrow}$). If $\varphi_i(\lambda) = 0$, set $\tilde{f}_i\lambda = 0$. Otherwise define $\tilde{f}_i\lambda$ to be the partition obtained by adding or removing the node in λ corresponding to the rightmost $+$ in its reduced $\downarrow_i$-signature (add the node if it belongs to $\lambda^{\uparrow}$ and remove it if it belongs to $\lambda^{\downarrow}$). Finally, set $\mathrm{wt}(\lambda) = \mathrm{wt}^{\downarrow}(\lambda^{\downarrow}) + \mathrm{wt}^{\uparrow}(\lambda^{\uparrow})$.

Example V.4.1. *Let $m = m' = 0$, $p = 2$. Then the 0-signature of $\left(\ydiagram{2,1}, \varnothing\right)$ is $---+$ (the three $-$'s correspond to the three* addable *nodes in $\ydiagram{2,1}$ of content 0, and the $+$ corresponds to the addable node in $\varnothing$), and its reduced 0-signature is $--$. The node corresponding to the rightmost $-$ in the reduced signature is the addable node in the first row of $\ydiagram{2,1}$. Thus $\tilde{e}_0\left(\ydiagram{2,1}, \varnothing\right) = \left(\ydiagram{3,1}, \varnothing\right)$.*

Theorem V.4.2. *Suppose $E_iL(\lambda) \neq 0$. Then the head and socle of $E_iL(\lambda)$ are both isomorphic to* *the simple module $L(\tilde{e}_i\lambda)$. Moreover, if $\varepsilon_i(\lambda) = 1$, then $E_iL(\lambda) = L(\tilde{e}_i\lambda)$. The same result holds with E_i replaced by F_i, $\tilde{e}_i$ replaced by $\tilde{f}_i$, and ε_i replaced by φ_i.*

Proof. This follows from the main result of [D], combined with our main theorem above, which verifies that the hypotheses of [D] are all satisfied. □

We now define the *crystal graph*, which is a subgraph of the branching graph from section IV.3. The set of vertices is Λ, and whenever $\lambda \in \Lambda$ and $\tilde{f}_i\lambda \neq 0$ we connect λ and $\tilde{f}_i\lambda$ with an edge colored i.

Example V.4.3. *Let $m = m' = 0$, $p = 2$. The the part of the crystal graph involving bipartitions of size up to 4 is shown in Figure 2.*

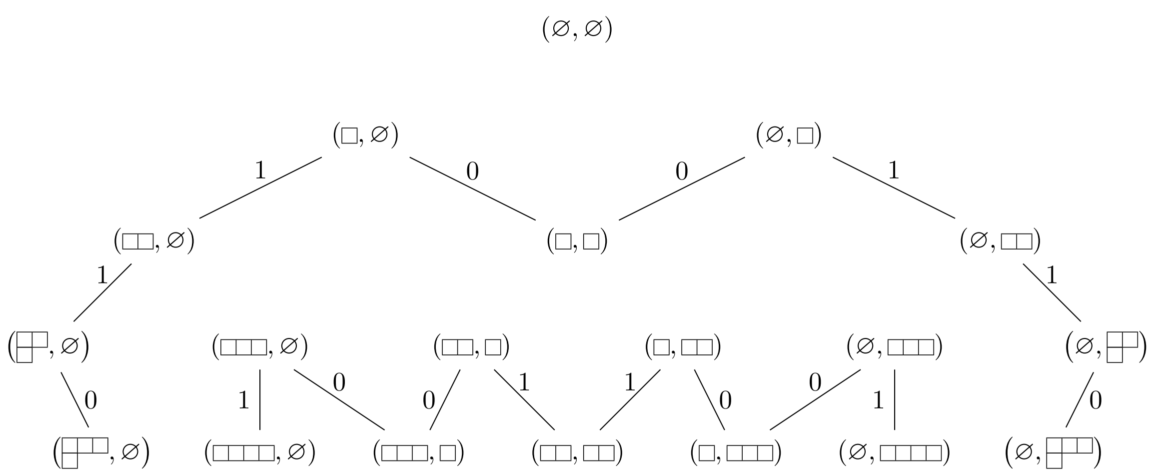
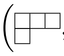
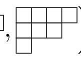


Figure 2: Crystal graph when $m = m' = 0$, $p = 2$.

As the above example shows, the crystal graph is not connected in general. For instance, $(\varnothing, \varnothing)$ is isolated if $m = m' = 0$. Also, it is easy to describe the connected component containing $(\square, \square)$ in the case $m = m' = 0$, $p = 2$. It is a graph of type A_∞, with the colors of edges forming the pattern $\ldots, 0, 0, 1, 1, 0, 0, 1, 1, \ldots$. Part of this component is visible in the above example. Starting at $(\square, \square)$, there are two possible routes. One route passes through $(\varnothing, \square)$ and the other through $(\square, \varnothing)$. With each additional edge, a box is added to the nonempty constituent partition, alternating between the first and second rows.

In fact the crystal graph has infinitely many connected components if $p = 0$ and $m, m' \in \mathbb{Z} \cdot 1_{\Bbbk}$. Presumably this assertion is also true when $p > 0$, but we will not attempt to prove that here. For the $p = 0$ case, first define the *k-value* of a bipartition $(\lambda^{\downarrow}, \lambda^{\uparrow})$ to be the smallest integer $k \geq -\min(m, m')$ such that the union of a horizontal strip of height $k + m$ and a vertical strip of width $k + m'$ can cover the diagram composed of the Young diagram of $\lambda^{\downarrow}$ rotated through 180° and adjoined to Young diagram of $\lambda^{\uparrow}$ with corner vertices touching. Equivalently it is the smallest $k \geq -\min(m, m')$ such that $\lambda_i^{\uparrow} + \lambda_{m+k+2-i}^{\downarrow} \leq k + m'$ for some $1 \leq i \leq k + m + 1$. In other words, λ is a $(k + m, k + m')$-cross bipartition in the sense of Comes and Wilson.

Example V.4.4. *If $m = 0$ and $m' = 2$, then the k-value of the bipartition* ( , ) *is 2 (See Figure 3).*

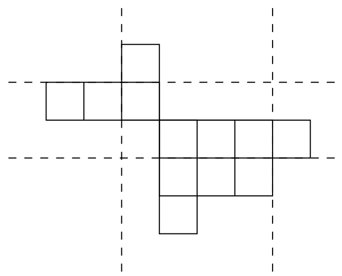


Figure 3: Illustration of k-value.

Proposition V.4.5. *Let $p = 0$ and $m, m' \in \mathbb{Z} \cdot 1_{\Bbbk}$. Any two bipartitions in the same connected component of the crystal graph have the same k-value. In particular, the crystal graph has infinitely many connected components.*

Proof. Suppose $\lambda_i^{\uparrow} + \lambda_{m+k+2-i}^{\downarrow} \leq k + m'$ for some $1 \leq i \leq k + m + 1$ and $\mu = \tilde{f}_j \lambda$. If $\mu_i^{\uparrow} + \mu_{m+k+2-i}^{\downarrow} > k + m'$ then we must have $\lambda_i^{\uparrow} + \lambda_{m+k+2-i}^{\downarrow} = k + m'$ and that $\tilde{f}_j$ adds a box either to row i of $\lambda^{\uparrow}$ or to row $m + k + 2 - i$ of $\lambda_{\downarrow}$. In this case, we find that both $\lambda^{\downarrow}, \lambda^{\uparrow}$ have an addable node of content $m - i + 1 + \lambda_i^{\uparrow}$. Hence the j-signature of λ is $-+$, which reduces to $\varnothing$. It follows that $\tilde{f}_j \lambda = 0$, which is a contradiction. This shows that the k-value of μ is less than or equal to that of λ. But the above argument applies to the situation $\mu = \tilde{e}_j \lambda$, implying that the k-value of μ is greater than or equal to that of λ. Hence they have the same k-value. The last statement follows from the fact that a bipartition of any k-value exists. $\square$

CHAPTER VI

APPLICATIONS

This final chapter contains some applications of our results in the case $p = 0$. We recall that $\Delta(\lambda) = \overline{\Delta}(\lambda)$ under this assumption. In section VI.1 we compute the composition multiplicities of the standard objects explicitly using the combinatorics of arc diagrams. Then in section VI.2 we explain how to compute the characters of the simple modules using the results of section VI.1 together with our computation of the characters of the standard modules in terms of the branching graph (see section IV.3). Finally, in section VI.3 we give a proof of the known classification of simple $B_{r,s}(\delta)$-modules due to Cox et al. (see [CDDM]) and use it to prove $L(\lambda)$ is (globally) finite dimensional if and only if $\delta = 0$ and $\lambda = (\varnothing, \varnothing)$, in which case it is one dimensional.

VI.1. Decomposition Numbers in Characteristic 0

We assume for the remainder of this paper that $p = 0$. In this case we are able to use the crystal to compute $[\Delta(\lambda) : L(\mu)] = (P(\mu) : \Delta(\lambda))$. For this it will be convenient to depict a bipartition by its *marker* rather than its Young diagram. The notion of a marker is a variation on the weight diagrams introduced by Brundan and Stroppel. The marker of a bipartition $\lambda = (\lambda^{\downarrow}, \lambda^{\uparrow})$ is obtained by decorating the integer points on the y-axis with the symbols $>, <, \times, \circ$ as follows. First define the sets

$$I_{\uparrow}(\lambda) = \{\lambda_1^{\uparrow} + m, \lambda_2^{\uparrow} + m - 1, \lambda_3^{\uparrow} + m - 2, \dots\}$$

$$I_{\downarrow}(\lambda) = \{m' + 1 - \lambda_1^{\downarrow}, m' + 2 - \lambda_2^{\downarrow}, m' + 3 - \lambda_3^{\downarrow}, \dots\}$$

The ith vertex is labelled

$$\begin{cases} < & \text{if } i \text{ lies in } I_{\uparrow} \text{ but not } I_{\downarrow} \\ > & \text{if } i \text{ lies in } I_{\downarrow} \text{ but not } I_{\uparrow} \\ \times & \text{if } i \text{ lies in both } I_{\uparrow} \text{ and } I_{\downarrow} \\ \circ & \text{if } i \text{ lies in neither of } I_{\uparrow} \text{ and } I_{\downarrow} \end{cases}$$

Observe that

$$\mathrm{wt}^{\uparrow}(\lambda^{\uparrow}) = \sum_{i \in I_{\uparrow}(\lambda)} \varepsilon_i \text{ and } \mathrm{wt}^{\downarrow}(\lambda^{\downarrow}) = \sum_{i \in I_{\downarrow}(\lambda)} \varepsilon_i$$

so that

$$\mathrm{wt}(\lambda) = \sum_{i \in I_{\uparrow}(\lambda) \cap I_{\downarrow}(\lambda)} \varepsilon_i - \sum_{i \in I_{\uparrow}(\lambda)^c \cap I_{\downarrow}(\lambda)^c} \varepsilon_i.$$

That is, $\mathrm{wt}(\lambda)$ specifies the vertices which are labelled $\circ$ and $\times$.

Suppose $\lambda < \mu$. Then $\mathrm{wt}(\lambda) = \mathrm{wt}(\mu)$ and $\mathrm{wt}^{\uparrow}(\lambda^{\uparrow}) > \mathrm{wt}^{\uparrow}(\mu^{\uparrow})$. That is, the markers for λ and μ have $\circ$ and $\times$ at all the same vertices and

$$\sum_{i \in I_{\uparrow}(\lambda)} \varepsilon_i - \sum_{i \in I_{\uparrow}(\mu)} \varepsilon_i > 0$$

is a finite sum of $\varepsilon_i - \varepsilon_j$, $i < j$. In terms of markers, this says that μ is obtained from λ by successively switching $\binom{<}{>}$ to $\binom{>}{<}$ (not just adjacent vertices).

Conversely, if there is a $>$ at vertex i and a $<$ at vertex j $(i < j)$ in the marker for λ, then switching the labels of these vertices doesn't affect the positions of vertices labelled $\circ$ and $\times$, so $\mathrm{wt}(\lambda)$ is unchanged. But switching the labels adds $\varepsilon_i - \varepsilon_j$ to $\mathrm{wt}^{\uparrow}(\lambda^{\uparrow})$, which means that switching the labels produced a bipartition $\mu > \lambda$: $\binom{<}{>} < \binom{>}{<}$. We conclude that λ is maximal if and only if in its marker every $<$ appears above every $>$.

The *left arc diagram* associated to λ (denoted by λ) is obtained by drawing non-crossing rays and arcs in the left half plane incident to some subset of the vertices in the marker in such a way that

- vertices at the bottom ends of arcs are labelled $>$;
- vertices at the top ends of arcs are labelled $<$;
- vertices at the right ends of rays are labelled either by $<$ or by $>$ in such a way that all rays labelled $>$ appear above all rays labelled $<$;
- all remaining vertices not at the ends of arcs or rays are labelled either by $\circ$ or by $\times$.

This diagram can be realized by successively drawing arcs connecting pairs of vertices $i < j$ with vertex i labelled $<$ and vertex j labelled $>$ and having no unpaired vertices labelled $>$ or $<$ in between vertices i and j. Once no more such pairs can be found, draw a ray at all unpaired vertices. We define the *defect* of λ to be the number of arcs in its left arc diagram. A bipartition is maximal if and only if its defect is zero.

Given any markers λ, μ with $\mathrm{wt}(\lambda) = \mathrm{wt}(\mu)$, it makes sense to glue the left arc diagram for λ onto the marker μ to obtain a composite diagram $\lambda\mu$ in which the endpoints of each arc and ray of the left arc diagram of λ are labelled either $<$ or $>$ by the marker μ. We say $\lambda\mu$ is *well-oriented* if

- each arc has exactly one label $<$ and one label $>$ making it into either a *counterclockwise* or *clockwise* arc;
- all rays labelled $>$ are above all rays labelled $<$.

Now we can state the result promised at the beginning of the section.

Theorem VI.1.1. *If $p = 0$ then*

$$[\Delta(\lambda) : L(\mu)] = (P(\mu) : \Delta(\lambda)) = \begin{cases} 1 & \text{if } \mu\lambda \text{ is well-oriented} \\ 0 & \text{otherwise} \end{cases}$$

To prove this theorem we will need to know the action of E_i, F_i on $\Delta(\lambda)$ and $P(\lambda)$ in terms of markers. Note that $I_\uparrow(\lambda)$ consists exactly of those numbers which appear as the $\uparrow$-content of some node in row i and column $\lambda_i + 1$, $i = 1, 2, \ldots$, and $I_\downarrow(\lambda)$ consists of the $\downarrow$-contents of nodes in row i and column λ_i, $i = 1, 2, \ldots$. From this observation, we see that if $\lambda^\downarrow$ has an addable (resp. removable) node of $\downarrow$-content i then $I_\downarrow(\lambda)$ contains $i+1$ and not i (resp. contains i and not $i+1$). Similarly, if $\lambda^\uparrow$ has an addable (resp. removable) node of $\uparrow$-content i then $I_\uparrow(\lambda)$ contains i and not $i+1$ (resp. contains $i+1$ and not i).

The following lemma is a consequence of Corollary IV.2.2.

Lemma VI.1.2. *For $\lambda \in \Lambda$, $i \in \mathbb{Z}$ and symbols $x, y \in \{\circ, >, <, \times\}$, let $\lambda\binom{y}{x}$ be the marker obtained from λ by relabelling its ith and $(i+1)$th vertices by x and y, respectively. Then if $v_\lambda = [\Delta(\lambda)]$ we have:*

(i) if $\lambda = \lambda\binom{>}{\circ}$, $\lambda\binom{<}{\circ}$, $\lambda\binom{\times}{>}$ or $\lambda\binom{\times}{<}$ then $e_i v_\lambda = v_\mu$ where μ is obtained from λ by switching the labels on its ith and $(i+1)$th vertices;

(ii) if $\lambda = \lambda\binom{>}{<}$ then $e_i v_\lambda = v_\mu$ where $\mu = \lambda\binom{\circ}{\times}$;

(iii) if $\lambda = \lambda\binom{<}{>}$ then $e_i v_\lambda = v_\mu$ where $\mu = \lambda\binom{\circ}{\times}$;

(iv) if $\lambda = \lambda\binom{\times}{\circ}$ then $e_i v_\lambda = v_\mu + v_\nu$ where $\mu = \lambda\binom{<}{>}$ and $\nu = \lambda\binom{>}{<}$;

(v) in all other situations $e_i v_\lambda = 0$.

There are also analogous formulae for $f_i v_\lambda$, which may be obtained from the above by interchanging the roles of $\circ$ *and* $\times$.

Observe that since $p = 0$, the i-signature of any λ consists of at most two symbols. So $\varepsilon_i(\lambda) \leq 2$ and $\varphi_i(\lambda) \leq 2$ for all i. Interpreting $L(\tilde{e}_i\lambda)$ (resp. $L(\tilde{f}_i\lambda)$) as zero whenever $\tilde{e}_i\lambda = 0$ (resp. $\tilde{f}_i\lambda = 0$), Theorem V.4.2 implies

$$E_i L(\lambda) = L(\tilde{e}_i\lambda) \text{ whenever } \varepsilon_i(\lambda) \neq 2$$

$$F_i L(\lambda) = L(\tilde{f}_i\lambda) \text{ whenever } \varphi_i(\lambda) \neq 2.$$

Lemma VI.1.3. *Let the notation be as in Lemma VI.1.2. Then if $p_\lambda = [P(\lambda)]$ we have:*

(i) if $\lambda = \lambda\left(\begin{smallmatrix} > \\ \circ \end{smallmatrix}\right)$, $\lambda\left(\begin{smallmatrix} < \\ \circ \end{smallmatrix}\right)$, $\lambda\left(\begin{smallmatrix} \times \\ > \end{smallmatrix}\right)$ or $\lambda\left(\begin{smallmatrix} \times \\ < \end{smallmatrix}\right)$ then $e_i p_\lambda = p_\mu$ where μ is obtained from λ by switching the labels on its ith and $(i+1)$th vertices;

(ii) if $\lambda = \lambda\left(\begin{smallmatrix} > \\ < \end{smallmatrix}\right)$ then $e_i p_\lambda = p_\mu$ where $\mu = \lambda\left(\begin{smallmatrix} \circ \\ \times \end{smallmatrix}\right)$;

(iii) if $\lambda = \lambda\left(\begin{smallmatrix} < \\ > \end{smallmatrix}\right)$ then $e_i p_\lambda = 2p_\mu$ where $\mu = \lambda\left(\begin{smallmatrix} \circ \\ \times \end{smallmatrix}\right)$;

(iv) if $\lambda = \lambda\left(\begin{smallmatrix} \times \\ \circ \end{smallmatrix}\right)$ then $e_i p_\lambda = p_\mu$ where $\mu = \lambda\left(\begin{smallmatrix} < \\ > \end{smallmatrix}\right)$;

(v) if $\lambda = \lambda\left(\begin{smallmatrix} > \\ > \end{smallmatrix}\right)$ and vertex $i+1$ is connected to vertex $j > i+1$ in $\underline{\lambda}$ then $e_i p_\lambda = p_\mu$, where μ is obtained from λ by relabelling vertices $i, i+1$ and j by the symbols $\times$, $\circ$ and $>$, respectively;

(vi) if $\lambda = \lambda\left(\begin{smallmatrix} < \\ < \end{smallmatrix}\right)$ and vertex i is connected to vertex $j < i$ in $\underline{\lambda}$ then $e_i p_\lambda = p_\mu$, where μ is obtained from λ by relabelling vertices j, i and $i+1$ by the symbols $<$, $\times$ and $\circ$, respectively;

(vii) in all other situations $e_i p_\lambda = 0$.

There are also analogous formulae for $f_i p_\lambda$, which may be obtained from the above by interchanging the roles of $\circ$ *and* $\times$.

Proof. In fact we shall only need parts (i), (iv) to prove theorem VI.1.1, so we prove these needed parts only. The remaining part may be proved in a similar fashion to Theorem 3.9 of [BS]. Since $E_i P(\lambda)$ is a finitely generated projective module, it is a finite direct sum: $E_i P(\lambda) = \bigoplus_\mu m_\mu P(\mu)$ where

$$m_\mu = \dim \operatorname{Hom}(E_i P(\lambda), L(\mu)) = \dim \operatorname{Hom}(P(\lambda), F_i L(\mu)).$$

If $\varphi_i(\mu) \le 1$, then $F_i L(\mu) = L(\tilde{f}_i \mu)$ so $m_\mu = 1$ if $\lambda = \tilde{f}_i \mu$ (i.e. if $\mu = \tilde{e}_i \lambda$) and $m_\mu = 0$ otherwise. If $\varphi_i(\mu) = 2$ then $\mu = \mu(\begin{smallmatrix}\circ\\ \times\end{smallmatrix})$ so

$$
\begin{aligned}
m_\mu &\le \dim \operatorname{Hom}(P(\lambda), F_i \Delta(\mu)) \\
&= \dim \operatorname{Hom}(P(\lambda), \Delta(\mu(\begin{smallmatrix}>\\ <\end{smallmatrix}))) + \dim \operatorname{Hom}(P(\lambda), \Delta(\mu(\begin{smallmatrix}<\\ >\end{smallmatrix}))).
\end{aligned}
$$

So if $m_\mu \ne 0$ then $\lambda \le \mu(\begin{smallmatrix}>\\ <\end{smallmatrix})$ and $\lambda \le \mu(\begin{smallmatrix}>\\ <\end{smallmatrix})$, which shows that λ does not have a $\circ$ or $\times$ at vertices i or $i+1$. We have shown that $e_i p_\lambda = p_{\tilde{e}_i \lambda}$ for any λ which does not have a $\circ$ or $\times$ at vertices i or $i+1$. This proves (i), (iv), and (vii). $\square$

Proof of Theorem VI.1.1. We prove by induction on the defect of μ that $p_\mu = [P(\mu)]$ is the sum of all $v_\lambda = [\Delta(\lambda)]$ such that λ is obtained from μ by switching the orientations of some subset of the arcs in μ, i.e. switching the labels of the endpoints of the arcs.

If the defect of μ is zero, then μ is maximal, and we have $P(\mu) = \Delta(\mu)$ so $p_\mu = v_\mu$ as needed. Suppose μ has positive defect. Find a pair of vertices $i < j$ connected by an arc in μ with no vertices labelled $<$ or $>$ in between i and j. If vertex $j-1$ is labelled $\times$, then $P(\mu) = E_{j-1} P(\nu)$, where ν is obtained by switching the $j-1, j$ vertices of μ. Or if vertex $j-1$ is labelled $\circ$, then $P(\mu) = F_{j-1} P(\nu)$, where ν is obtained by switching the $j-1, j$ vertices of μ. We can thus write $P(\mu)$ as a composition of various E_k, F_k applied to $P(\nu)$, where ν is obtained from μ by moving the $<$ at vertex j past any $\circ$'s and $\times$'s onto vertex $i+1$. Now $P(\nu) = F_i P(\nu(\begin{smallmatrix}\circ\\ \times\end{smallmatrix}))$. Since $P(\nu(\begin{smallmatrix}\circ\\ \times\end{smallmatrix}))$ has smaller defect than $P(\mu)$, we have that $p_{\nu(\begin{smallmatrix}\circ\\ \times\end{smallmatrix})}$ is the sum of all v_λ such that λ is obtained from $\nu(\begin{smallmatrix}\circ\\ \times\end{smallmatrix})$ by switching the orientations of some subset of the arcs in its left arc diagram.

Now p_μ is obtained by applying the above composition of the various e_k, f_k to p_ν. Each v_λ appearing in the expression for $p_{\nu(\begin{smallmatrix}\circ\\ \times\end{smallmatrix})}$ satisfies $\lambda = \lambda(\begin{smallmatrix}\circ\\ \times\end{smallmatrix})$. For such λ we have $f_i v_\lambda = v_{\lambda(\begin{smallmatrix}>\\ <\end{smallmatrix})} + v_{\lambda(\begin{smallmatrix}<\\ >\end{smallmatrix})}$ which means p_ν is the sum of all v_λ such that λ is obtained from ν by switching the orientations of some subset of the arcs in its left arc diagram. Now the effect of applying the composition of the e_k, f_k to each of the v_λ is to move the label at vertex $i+1$ back up to vertex j past the $\circ$'s and $\times$'s from before. This is so because the rules used in selecting the various E_k, F_k apply to the standard modules as well. That is, if vertex $j-1$ in the marker of μ is labelled $\times$, then $\Delta(\mu) = E_{j-1} \Delta(\nu)$, where ν is obtained by switching the $j-1, j$ vertices of μ, and so on. $\square$

Example VI.1.4. *Let $\delta = 0$. We determine all μ such that $L(\mu)$ is a composition factor of $\Delta(\lambda)$, where $\lambda = (\square, \square)$. The marker for λ is $\begin{smallmatrix}<\\ >\end{smallmatrix}$ (where all vertices above these are $>$ and all vertices below these are $<$). The possible well-oriented composite diagrams $\mu\lambda$ are shown in Figure 4.*

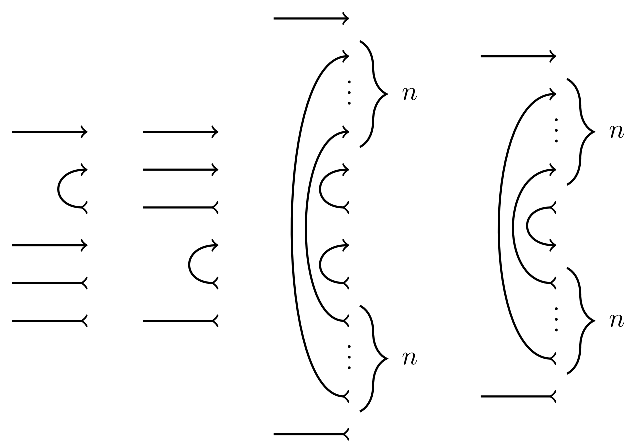


Figure 4: Well-oriented composite diagrams.

where $n \geq 0$ and all vertices not shown are at the ends of rays. The corresponding μ's are shown in Figure 5.

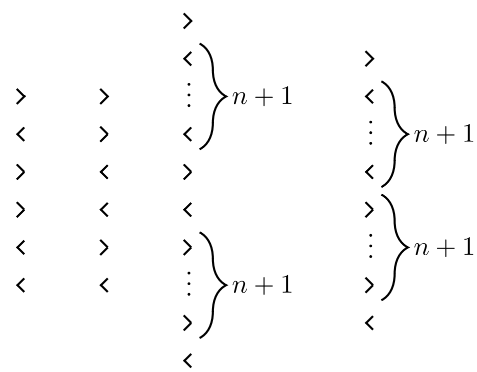


Figure 5: Markers of the composition factors of a standard module.

The first two of these correspond to the partitions $\left(\begin{smallmatrix}\square\\\square\end{smallmatrix},\square\square\right),\left(\square\square,\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)$*, respectively. The fourth corresponds to the bipartition with both constituent partitions being the partition of $(n+1)^2$ with $n+1$ equal parts. The third corresponds to the bipartition obtained from the fourth by removing the two removable nodes.*

VI.2. Characters of Simple Modules in Characterstic 0

We have a method for computing the first several terms of the characters of simples modules in characteristic zero. Suppose we want to know the terms $e^{\mathsf{a}_{\underline{i}}}$ in $\operatorname{ch} L(\lambda)$ with $\ell(\mathsf{a}) \leq n$ (call this $\operatorname{ch}_n L(\lambda)$).

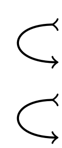
Theorem VI.1.1 tells us how to write $\operatorname{ch}_n \Delta(\lambda)$ in terms of $\operatorname{ch}_n L(\mu)$. Fix a total order refining the partial order on the set of bipartitions $\mu \leq \lambda$ such that $|\mu| \leq n$. Then the matrix whose entry in row μ and column ν is $[\Delta(\mu) : L(\nu)]$ is upper unitriangular, hence invertible. Row λ (the first row) of the the inverse matrix gives $\operatorname{ch}_n L(\lambda)$ in terms of $\operatorname{ch}_n \Delta(\mu)$, which we can compute using the branching graph.

Example VI.2.1. *Let $m = m' = 0$. We compute $\operatorname{ch}_5 L(\varnothing, \varnothing)$. Figure 6 shows the upper unitriangular matrix described above.*

	$(\varnothing,\varnothing)$	$(\square,\square)$	$(\ydiagram{1,1},\ydiagram{2})$	$(\ydiagram{2},\ydiagram{1,1})$	$(\ydiagram{1,1,1},\ydiagram{3})$	$(\ydiagram{2,1},\ydiagram{2,1})$	$(\ydiagram{3},\ydiagram{1,1,1})$	$\dots$
$(\varnothing,\varnothing)$	1	1	0	0	0	0	0	
$(\square,\square)$	0	1	1	1	0	1	0	
$(\ydiagram{1,1},\ydiagram{2})$	0	0	1	0	1	1	0	
$(\ydiagram{2},\ydiagram{1,1})$	0	0	0	1	0	1	1	
$(\ydiagram{1,1,1},\ydiagram{3})$	0	0	0	0	1	0	0	
$(\ydiagram{2,1},\ydiagram{2,1})$	0	0	0	0	0	1	0	
$(\ydiagram{3},\ydiagram{1,1,1})$	0	0	0	0	0	0	1	
$\vdots$								

Figure 6: Decomposition numbers of a standard module.

One can compute the entries in column ν by writing p_ν in terms of v_μ. For example column $(\ydiagram{2,1},\ydiagram{2,1})$ is computed as follows. Vertices $-1, 0, 1, 2$ of the marker for $(\ydiagram{2,1},\ydiagram{2,1})$ are shown below with its left arc diagram (all vertices not shown are at the ends of rays).



So the markers for those μ such that $\Delta(\mu)$ appears in the Δ-flag of $P(\nu)$ are obtained by replacing the above four vertices with each of

$$\begin{array}{cccc} < & > & < & > \\ > & < & > & < \\ < & < & > & > \\ > & > & < & < \end{array}$$

These correspond to $\mu = (\ydiagram{2,1},\ydiagram{2,1}), (\ydiagram{2},\ydiagram{1,1}), (\ydiagram{1,1},\ydiagram{2}), (\square,\square)$, respectively.

We invert the upper left 4×4 submatrix and find the first row is $1, -1, 1, 1$. So

$$
\begin{aligned}
\mathrm{ch}_5\, L(\varnothing, \varnothing) &= \mathrm{ch}_5\, \Delta(\varnothing, \varnothing) - \mathrm{ch}_5\, \Delta\left(\square, \square\right) + \mathrm{ch}_5\, \Delta\left(\begin{smallmatrix}\square\\ \square\end{smallmatrix}, \square\square\right) + \mathrm{ch}_5\, \Delta\left(\square\square, \begin{smallmatrix}\square\\ \square\end{smallmatrix}\right) \\
&= (e^{\varnothing} + e^{\downarrow_0\uparrow_0} + e^{\uparrow_0\downarrow_0} + e^{\downarrow_0\uparrow_0\downarrow_0\uparrow_0} + e^{\downarrow_0\uparrow_0\uparrow_0\downarrow_0} + e^{\uparrow_0\downarrow_0\uparrow_0\downarrow_0} + e^{\uparrow_0\downarrow_0\downarrow_0\uparrow_0} + e^{\downarrow_0\downarrow_1\uparrow_1\uparrow_0} + e^{\downarrow_0\downarrow_{-1}\uparrow_{-1}\uparrow_0} \\
&\qquad + e^{\downarrow_0\uparrow_0\downarrow_0\uparrow_0} + e^{\downarrow_0\uparrow_0\uparrow_0\downarrow_0} + e^{\uparrow_0\uparrow_1\downarrow_1\downarrow_0} + e^{\uparrow_0\uparrow_{-1}\downarrow_{-1}\downarrow_0} + e^{\uparrow_0\downarrow_0\uparrow_0\downarrow_0} + e^{\uparrow_0\downarrow_0\downarrow_0\uparrow_0}) \\
&\quad - (e^{\downarrow_0\uparrow_0} + e^{\uparrow_0\downarrow_0} + e^{\downarrow_0\uparrow_0\downarrow_0\uparrow_0} + e^{\downarrow_0\uparrow_0\uparrow_0\downarrow_0} + e^{\uparrow_0\downarrow_0\downarrow_0\uparrow_0} + e^{\uparrow_0\downarrow_0\uparrow_0\downarrow_0} + e^{\downarrow_0\downarrow_1\uparrow_1\uparrow_0} + e^{\downarrow_0\downarrow_{-1}\uparrow_{-1}\uparrow_0} \\
&\qquad + e^{\downarrow_0\uparrow_0\downarrow_0\uparrow_0} + e^{\downarrow_0\uparrow_0\uparrow_0\downarrow_0} + e^{\uparrow_0\uparrow_1\downarrow_1\downarrow_0} + e^{\uparrow_0\uparrow_{-1}\downarrow_{-1}\downarrow_0} + e^{\uparrow_0\downarrow_0\uparrow_0\downarrow_0} + e^{\uparrow_0\downarrow_0\downarrow_0\uparrow_0} \\
&\qquad + e^{\downarrow_0\downarrow_1\uparrow_0\uparrow_1} + e^{\downarrow_0\downarrow_{-1}\uparrow_0\uparrow_{-1}} + e^{\downarrow_0\uparrow_0\downarrow_1\uparrow_1} + e^{\downarrow_0\uparrow_0\downarrow_{-1}\uparrow_{-1}} + e^{\downarrow_0\uparrow_0\uparrow_{-1}\downarrow_{-1}} + e^{\downarrow_0\uparrow_0\uparrow_1\downarrow 1} \\
&\qquad + e^{\uparrow_0\uparrow_1\downarrow_0\downarrow_1} + e^{\uparrow_0\uparrow_{-1}\downarrow_0\downarrow_{-1}} + e^{\uparrow_0\downarrow_0\uparrow_1\downarrow_1} + e^{\uparrow_0\downarrow_0\uparrow_{-1}\downarrow_{-1}} + e^{\uparrow_0\downarrow_0\downarrow_{-1}\uparrow_{-1}} + e^{\uparrow_0\downarrow_0\downarrow_1\uparrow_1}) \\
&\quad + (e^{\downarrow_0\downarrow_1\uparrow_0\uparrow_1} + e^{\downarrow_0\uparrow_0\downarrow_1\uparrow_1} + e^{\downarrow_0\uparrow_0\uparrow_1\downarrow_1} + e^{\uparrow_0\uparrow_1\downarrow_0\downarrow_1} + e^{\uparrow_0\downarrow_0\uparrow_1\downarrow_1} + e^{\uparrow_0\downarrow_0\downarrow_1\uparrow_1}) \\
&\quad + (e^{\downarrow_0\downarrow_{-1}\uparrow_0\uparrow_{-1}} + e^{\downarrow_0\uparrow_0\downarrow_{-1}\uparrow_{-1}} + e^{\downarrow_0\uparrow_0\uparrow_{-1}\downarrow_{-1}} + e^{\uparrow_0\uparrow_{-1}\downarrow_0\downarrow_{-1}} + e^{\uparrow_0\downarrow_0\uparrow_{-1}\downarrow_{-1}} + e^{\uparrow_0\downarrow_0\downarrow_{-1}\uparrow_{-1}}) \\
&= e^{\varnothing}
\end{aligned}
$$

The relevant edges of the branching graph are shown in Figure 7.

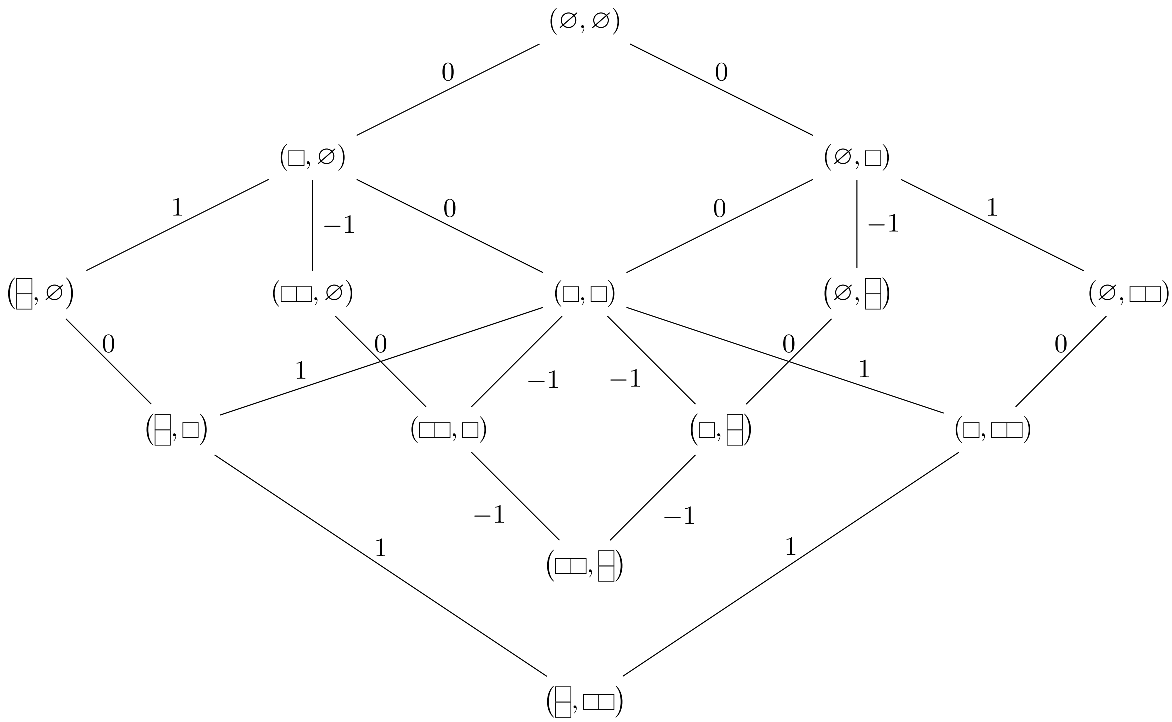


Figure 7: Branching graph (computation of character of standard module).

In this case we can actually see that $\operatorname{ch} L(\varnothing,\varnothing) = e^{\varnothing}$ *because* $\bigoplus_{\mathsf{a}\neq\varnothing} 1_{\mathsf{a}}\Delta(\varnothing,\varnothing)$ *is a submodule of* $\Delta(\varnothing,\varnothing)$ *(*$\delta = 0$*) and* $\dim 1_{\varnothing}\Delta(\varnothing,\varnothing) = 1$*, so that* $L(\varnothing,\varnothing) = 1_{\varnothing}L(\varnothing,\varnothing)$ *is one dimensional.*

VI.3. Representations of the Walled Brauer Algebra

Recall the walled Brauer algebra $B_{r,s}(\delta) = \operatorname{End}_{\mathcal{OB}(\delta)}(\downarrow^r\uparrow^s) = 1_{\downarrow^r\uparrow^s} OB 1_{\downarrow^r\uparrow^s}$ from the introduction of this thesis. Since $B_{r,s}(\delta)$ is an idempotent truncation of OB, Theorem II.4.4 describes the simple $B_{r,s}(\delta)$-modules. First let $L_{r,s}(\lambda) = 1_{\downarrow^r\uparrow^s}L(\lambda)$ for $\lambda \in \Lambda$, and set $\Lambda_{r,s}(\delta) = \{\lambda \in \Lambda : L_{r,s}(\lambda) \neq 0\}$. Then the simple $B_{r,s}(\delta)$-modules are $\{L_{r,s}(\lambda) : \lambda \in \Lambda_{r,s}(\delta)\}$. Using a result of Cox et al. (see [CDDM]) we are able to describe the set $\Lambda_{r,s}(\delta)$.

Theorem VI.3.1. *The set* $\{L_{r,s}(\lambda) : \lambda \in \Lambda_{r,s}(\delta)\}$ *is a complete set of inequivalent irreducible*

$B_{r,s}(\delta)$-modules, where

$$\Lambda_{r,s}(\delta) = \begin{cases} \bigcup_{t=0}^{\min(r,s)} \Lambda_{r-t,s-t} & \delta \neq 0 \text{ or } r = s = 0 \\ \bigcup_{t=0}^{\min(r,s)} \Lambda_{r-t,s-t} \setminus \{(\varnothing,\varnothing)\} & \delta = 0 \text{ and } r + s > 0. \end{cases} \tag{VI.3.0.1}$$

Proof. It is clear that $L_{r,s}(\lambda) = 0$ unless $\lambda \in \bigcup_{t=0}^{\min(r,s)} \Lambda_{r-t,s-t}$. We have also observed that $L(\varnothing,\varnothing) = 1_\varnothing L(\varnothing,\varnothing)$ is one dimensional if δ, so $L_{r,s}(\varnothing,\varnothing) = 0$ if $r + s > 0$. This shows that $\Lambda_{r,s}(\delta)$ contained in the set on the right hand side of (VI.3.0.1). It remains to show these two sets have the same size. This follows from [CDDM], where the authors show that the right hand set labels the simple $B_{r,s}(\delta)$-modules by another method. □

We can now show that most simple OB-modules are (globally) infinite dimensional.

Corollary VI.3.2. *The simple module $L(\lambda)$ is (globally) finite dimensional if and only if $\delta = 0$ and $\lambda = (\varnothing,\varnothing)$.*

Proof. We have already observed the "if" part of the statement. Suppose $\delta \neq 0$ or $\lambda \neq (\varnothing,\varnothing)$. Then by Theorem VI.3.1 we have $\lambda \in \Lambda_{r+t,s+t}(\delta)$ for all $t \geq 0$, where $\lambda \vdash (r,s)$. Hence $1_{\downarrow^{r+t}\uparrow^{s+t}} L(\lambda) \neq 0$ for all $t \geq 0$, showing that $L(\lambda)$ has infinitely many nonzero weight spaces. So $L(\lambda)$ is infinite dimensional. □

Remark VI.3.3. *It would have been nice to find a proof of Theorem VI.3.1 without appealing to [CDDM], but we have been unable to do this.*

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