THE EINSTEIN CONSTRAINT EQUATIONS ON ASYMPTOTICALLY EUCLIDEAN MANIFOLDS

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JAMES DILTS

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Student: James Dilts

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This dissertation has been accepted and approved in partial fulfillment of the requirements for the Doctor of Philosophy degree in the Department of Mathematics by:

James Isenberg	Chair
Dr. Weiyong He	Core Member
Dr. Peng Lu	Core Member
Dr. Marcin Bownik	Core Member
Dr. Graham Kribs	Institutional Representative

and

Scott L. Pratt	Dean of the Graduate School

Original approval signatures are on file with the University of Oregon Graduate School.

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DISSERTATION ABSTRACT

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In this dissertation, we prove a number of results regarding the conformal method of finding solutions to the Einstein constraint equations. These results include necessary and sufficient conditions for the Lichnerowicz equation to have solutions, global supersolutions which guarantee solutions to the conformal constraint equations for near-constant-mean-curvature (near-CMC) data as well as for far-from-CMC data, a proof of the limit equation criterion in the near-CMC case, as well as a model problem on the relationship between the asymptotic constants of solutions and the ADM mass. We also prove a characterization of the Yamabe classes on asymptotically Euclidean manifolds and resolve the (conformally) prescribed scalar curvature problem on asymptotically Euclidean manifolds for the case of nonpositive scalar curvatures.

Many, though not all, of the results in this dissertation have been previously published in [Dil14], [DIMM14], [DL14], [DM15], and [DGI15]. This dissertation includes previously published coauthored material.

CURRICULUM VITAE

NAME OF AUTHOR: James Dilts

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:
University of Oregon, Eugene, OR
Brigham Young University, Provo, UT

DEGREES AWARDED:
Doctor of Philosophy, Mathematics, 2015, University of Oregon
Bachelor of Science, Mathematics, 2010, Brigham Young University

AREAS OF SPECIAL INTEREST:
Mathematical Relativity
Differential Geometry
Partial Differential Equations

PROFESSIONAL EXPERIENCE:

Graduate Teaching Fellow & Graduate Research Fellow, University of Oregon, 2010-2015

PUBLICATIONS:

J. Dilts, D. Maxwell, Yamabe Classification and Prescribed Scalar Curvature in the Asymptotically Euclidean Setting, arXiv:1503.04172. (Submitted)

J. Dilts, J. Leach, A limit equation criterion for solving the Einstein constraint equations on manifolds with ends of cylindrical type, Annales Henri Poincaré. (In press)

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For my wife and kids, who put up with my strange obsession with odd little symbols.

TABLE OF CONTENTS

Chapter Page

LIST OF TABLES

CHAPTER I

INTRODUCTION

General relativity, Albert Einstein's theory of gravity, has proven remarkably successful in describing the universe from planetary to intergalactic scales. In this theory, Einstein made the surprising claim that gravity is equivalent to the curvature of space [Ein15b]. In other words, mass and energy bend and stretch space itself.

In our solar system, the stretching is very slight; even passing over the surface of the sun, the error is much less than one percent. However, this slight stretching has been confirmed by numerous tests. The first physical confirmation was the orbit of Mercury. The oval orbit of mercury precesses (rotates) by a small amount each year. However, the observed precession is about eight percent greater than Newtonian gravity predicts. By taking into account the stretching of space, Einstein [Ein15a] correctly explained the observed precession.

Some other confirmations of the accuracy of general relativity include the bending of light in gravitational fields, the gravitational red-shift effect, and the Shapiro time delay. General relativity has become the most accurate theory of gravity known. It has led to remarkable technologies, such as GPS, and remarkable physical predictions, such as the Big Bang and black holes, cf. [Wal84].

In general relativity, the universe is described by a Lorentzian manifold, called a spacetime. Vectors in the spacetime with positive inner product represent space-like directions, while those with negative inner product represent time-like directions. Those with zero inner product can be interpreted as the directions that light can travel. As mentioned earlier, mass and energy stretch spacetime itself.

This is represented by the equations of general relativity:

$$R_{\mu\nu} - \frac{1}{2}R\gamma_{\mu\nu} + \Lambda\gamma_{\mu\nu} = \kappa T_{\mu\nu}. \tag{1.1}$$

Here, $\gamma_{\mu\nu}$ is the metric, $R_{\mu\nu}$ and R are the Ricci and scalar curvatures respectively, Λ is the cosmological constant and κ is a constant depending on the units chosen. In what follows, we choose units such that $\kappa = 1$. The tensor $T_{\mu\nu}$ is the stress-energy tensor, which combines both mass and energy into one object.

While easy to write down, the Einstein equations (1.1) are not easy to solve. Minkowski space, $\mathbb{R}^n$ equipped with a flat Lorentzian metric, solves them trivially. The first non-trivial exact example is the Schwarzschild solution. This solution is spherically symmetric in spatial (spacelike) directions, and describes the space surrounding a star. For many years, general relativity was a business of finding special, symmetric solutions to the Einstein equations. Many of these are quite important, such as the Kerr metric, which generalizes the Schwarzschild metric to allow rotation and represents, it is thought, the end state of black holes, and the FLRW metric, a family of spatially homogeneous and isotropic solutions that are the basis of the Big Bang and the standard model of cosmology. See [Wal84] for more information on these solutions.

In most scientific theories, one wants to be able to specify initial conditions, such as the location of particles and their momenta, and then evolve the system to predict where the particles will be in the future. This is called the initial value problem. It took about forty years before the initial value problem for general relativity was put on a firm theoretical footing.

In Newtonian physics, the initial data of particles and their momenta is freely specifiable. This is not the case in general relativity. In general relativity, initial data is (usually) given on a Riemannian spatial submanifold of the spacetime. Using the Gauss and Codazzi equations, one can reduce the Einstein equations (1.1) on the submanifold to the Einstein constraint equations,

$$R_g + (\mathrm{tr}_g K)^2 - |K|_g^2 = T_{nn}, \tag{1.2a}$$

$$\nabla_g^i K_{ij} - \nabla_j(\mathrm{tr}_g K) = 2T_{in}, \tag{1.2b}$$

where g is the induced Riemannian metric, K is the second fundamental form, n is the unit normal direction, and latin indices indicate spatial directions. Equation (1.2a) is known as the Hamiltonian constraint while equation (1.2b) is known as the momentum constraint. A standard reference on these equations is [BI04].

The constraint equations (1.2) must hold on any spatial submanifold of a spacetime satisfying the Einstein equations (1.1). In 1952, Yvonne Choquet-Bruhat [FB52] proved the converse: given a Riemannian manifold (M, g) and a symmetric 2-tensor K, there is a spacetime satisfying the Einstein equations (1.1) where (M, g) is a submanifold and K is the second fundamental form of this submanifold. Later, Choquet-Bruhat and Robert Geroch [CBG69] proved the existence of an appropriate "maximal" spacetime containing (M, g), called the maximal globally hyperbolic development.

Due to these results, instead of trying to find and classify all solutions of the Einstein equations, one may instead find and classify the solutions of the constraint equations. In addition, the initial value problem is vital in finding solutions in complicated situations, such as inspiraling binary black holes; exact solutions are difficult, if not impossible, to find, but computers can approximate these solutions using the initial value formulation.

Thus one would like to understand the full set of solutions to the constraint equations (1.2), and, in particular, to parameterize this set. The constraint equations are an underdetermined system of elliptic PDEs. Roughly speaking, for an n dimensional spacetime, there are n functions determined by the constraint equations, while the rest of the data is freely specifiable. However, it is not immediately obvious which of the quantities or components of tensors we should attempt to specify and which we should attempt to solve for. A useful decomposition of the data is needed. One of the most useful decompositions is known as the conformal method.

1.1. The Conformal Method

The conformal method was developed by Lichnerowicz, Choquet-Bruhat, and York in order to parameterize all the solutions of the constraint equations (1.2). To date it has been the most successful method in doing so.

The main idea behind the conformal method, as described in the previous section, is to decompose the initial data into freely specifiable and determined data. Over the years, several variations of the conformal method have been introduced, which did not appear to be equivalent. Fortunately, David Maxwell [Max14a] recently showed that all of the conformal methods lead to the same set of solutions, and so are effectively equivalent. Indeed, there is a straightforward transformation of the specifiable data from any of the methods to data from any of the others. Because of this, we present and use the method that appears to have the most advantages, which Maxwell refers to as the "conformal thin sandwich-Hamiltonian" formulation, or CTS-H for short.

In this method, the initial data consists not only of a Riemannian manifold $(M, \overline{g})$ and a symmetric 2-tensor $\overline{K}$, but also of a function $\overline{N}$, called the lapse function. When solving for the complete spacetime, the lapse function controls the relative length of the unit normal to the submanifold and the coordinate vector ∂_t. However, the $\overline{N}$ found via the conformal method need not be used in finding the spacetime; it is called the lapse function due to the derivation of the CTS formulations.

In the CTS-H formulation of Einstein's theory with matter sources, one first specifies a manifold M and a background metric g. One then chooses functions τ, r, a function $N > 0$ going to 1 at infinity, a vector field J, and a transverse-traceless (i.e., divergence-free and trace-free) symmetric 2-tensor σ. We call $(g, \tau, N, \sigma, r, J)$ the "seed data." One then seeks a function $\phi > 0$ and a vector field W solving the conformal constraint equations:

$$-a\Delta\phi + R\phi + \kappa\tau^2\phi^{q-1} - \left|\sigma + \frac{1}{2N}LW\right|^2 \phi^{-q-1} - r\phi^{-q/2} = 0 \tag{1.3a}$$

$$\mathrm{div}\frac{1}{2N}LW = \kappa\phi^q d\tau + J. \tag{1.3b}$$

Here, all quantities and operators are calculated relative to g, $q = \frac{2n}{n-2}$, $\kappa = \frac{n-1}{n}$, $a = \frac{4(n-1)}{n-2}$, and L is the conformal Killing operator, defined by

$$LW_{ab} = \nabla_a W_b + \nabla_b W_a - \frac{2}{n}\nabla_c W^c g_{ab}. \tag{1.4}$$

We refer to Equation (1.3a) as the Lichnerowicz equation, while equation (1.3b) is called the vector equation. The system is also called the LCBY (Lichnerowicz-Choquet-Bruhat-York) equations.

Once (ϕ, W) is found, the initial data solving (1.2) is reconstructed as follows:

$$\overline{g}_{ab} = \phi^{q-2} g_{ab}, \tag{1.5}$$

$$\overline{K}_{ab} = \phi^{-2}\left(\sigma_{ab} + \frac{1}{2N} LW_{ab}\right) + \frac{1}{n}\tau \overline{g}_{ab}, \tag{1.6}$$

$$T_{nn} = \phi^{-\frac{3}{2}q+1} r, \tag{1.7}$$

$$T_{in} = \phi^{-q} J. \tag{1.8}$$

We make several notes on this. First, given ϕ, York [Yor73] proved that such a decomposition of $\overline{K}$ exists and is unique. Indeed the decomposition is $L^2(M)$ orthogonal, i.e.,

$$\int_M \langle \sigma, LW \rangle = 0, \tag{1.9}$$

though they are not in general orthogonal pointwise. Next, note that $\tau = \mathrm{tr}_{\overline{g}} \overline{K}$. Thus τ represents the mean curvature of the initial data. Also, though we could allow $r < 0$, this represents a negative energy density. For the rest of the thesis we assume the weak energy condition, which in this case is equivalent to saying that $r \geq 0$.

Finally, note that the metric g was, in the end, only specified up to a conformal factor. Perhaps the greatest strength of the CTS-H formulation over the other formulations of the conformal method is that it is conformally covariant. Specifically, we have the following proposition, as proven in [Max14a, Prop 6.4].

Proposition 1.1.1. *Let $(g, \tau, N, \sigma, r, J)$ be CTS-H seed data, and let ψ be a smooth positive function. Then (ϕ, W) solve the conformal constraint equations* (1.3) *for the data $(g, \tau, N, \sigma, r, J)$ if and only if $(\psi^{-1}\phi, W)$ solve the conformal constraint* *equations for the data*

$$(\psi^{q-2}g, \tau, \psi^{q}N, \psi^{-2}\sigma, \psi^{-\frac{3}{2}q+1}r, \psi^{-q}J). \tag{1.10}$$

Both yield the same solution $(\overline{g}, \overline{K})$ *of the constraint equations.*

As a consequence of this, when we attempt to find solutions to the conformal constraint equations, we can, without loss of generality, do all calculations with respect to any convenient representative $\widetilde{g} \in [g]$. Usually we use this freedom to choose a representative with convenient scalar curvature. In the more traditional conformal method, the corresponding conformal constraint equations (essentially (1.3) with $N \equiv \frac{1}{2}$) are not conformally covariant.

The Lichnerowicz equation (1.3a) is a semilinear elliptic PDE. If $\sigma + \frac{1}{2N}LW, r \equiv 0$, the Lichnerowicz equation becomes the (conformally) prescribed scalar curvature equation. In particular, if ϕ solves (1.3a) with $\sigma + \frac{1}{2N}LW, r \equiv 0$, then the scalar curvature of $\phi^{q-2}g$ is $-\kappa\tau^2$. The prescribed scalar curvature problem is closely related to the solvability of the Lichnerowicz equation (and the conformal constraint equations overall) as we see below.

The vector equation (1.3b) is a linear elliptic PDE. In the absence of conformal Killing fields, the vector equation is completely understood. However, in the presence of conformal Killing fields, it is not well understood. A conformal Killing field V is one such that $LV \equiv 0$, and represents a symmetry of some conformally related metric. On a compact manifold, in the vacuum case (i.e.,

$(r, J) \equiv 0)$,

$$\begin{aligned} 0 &= \int_M \frac{-1}{4N} \langle LW, LV \rangle \\ &= \int_M \left\langle V, \operatorname{div} \frac{1}{2N} LW \right\rangle \\ &= \int \kappa \phi^q \langle V, d\tau \rangle \end{aligned} \tag{1.11}$$

by integration by parts. (The adjoint of L is $-2\mathrm{div}$.) Thus $\phi^q d\tau$ must be L^2 orthogonal to all conformal Killing fields for there to be a solution W to the vector equation (1.3b). In the case $d\tau \equiv 0$, i.e., the constant mean curvature (CMC) case, this is not a problem. However, in general, since ϕ is also unknown, this is a serious complication. The "drift formulation" by Maxwell [Max15b], described below in Section 1.3, is an extension of the CTS-H formulation that, among other things, attempts to resolve this problem. In this thesis, we assume that the metric does not allow any conformal Killing fields. Fortunately, it is well known [BCS05] that generic metrics do not admit any conformal Killing fields.

Earlier, we described the CTS-H conformal method as splitting the initial data into freely specifiable and determined data. This is not precisely the case. For instance, on a compact manifold, if τ is a constant, then the solution to the vector equation is $W \equiv 0$. If, in addition, $R > 0$ and $\sigma, r \equiv 0$, the maximum principle implies that the Lichnerowicz equation (1.3a) has no positive solution. Thus the seed data is not freely specifiable. The goal, then, becomes to determine which seed data sets lead to (a hopefully unique) solution of the conformal constraint equations. If fully understood, this leads to a parameterization of the solutions to the constraint equations (1.2). The case that is most fully understood is the case where M is a compact manifold without boundary.

1.2. The Compact Case

The simplest case is the constant mean curvature case, i.e., when $d\tau \equiv 0$. In this case, the conformal constraint equations (1.3) decouple. If J is L^2 orthogonal to any conformal Killing fields, the vector equation (1.3b) has a solution, and W is not dependent on ϕ. Thus we can reduce the conformal constraint equations to a single equation:

$$-a\Delta\phi + R\phi + \kappa\tau^2\phi^{q-1} - \beta^2\phi^{-q-1} - r\phi^{-q/2} = 0 \tag{1.12}$$

where $\beta = \left|\sigma + \frac{1}{2N}LW\right|$.

As before, the sign of the scalar curvature R affects whether or not equation (1.12) has any solutions. This leads us to the Yamabe problem. The Yamabe problem asks whether a metric can be conformally transformed to one with constant scalar curvature. The answer is yes (cf. [LP87]), with the sign of the target scalar curvature being prescribed by a conformal invariant called the Yamabe invariant. The Yamabe invariant of a metric, $Y(g)$, is defined by

$$Y(g) := \inf_{u \in C^\infty(M), u \not\equiv 0} \frac{\int_M a|\nabla u|^2 + Ru^2}{\|u\|_q^2}. \tag{1.13}$$

If $Y(g) > 0$, we say g is Yamabe positive, and similar for Yamabe null and negative. The resolution of the Yamabe problem says that g is Yamabe positive if and only if g can be conformally transformed to a metric with constant positive scalar curvature, and similar statements hold for Yamabe null and negative metrics. Since the CTS-H method is conformally covariant (cf. Proposition 1.1.1), we can assume the scalar curvature R is constant of the appropriate sign.

For compact manifolds, the CMC case (with $r \equiv 0$) was completed by Jim Isenberg in [Ise95]. The case $r \geq 0$ is proven essentially the same way, and so we include it here also. The solvability of equation (1.12) is detailed in Table 1.

TABLE 1. Solvability of the CMC Conformal Constraint Equations

	$\tau = 0, \beta, r \equiv 0$	$\tau = 0, \beta, r \not\equiv 0$	$\tau \neq 0, \beta, r \equiv 0$	$\tau \neq 0, \beta, r \not\equiv 0$
$Y(g) > 0$	No	Yes	No	Yes
$Y(g) = 0$	Yes*	No	No	Yes
$Y(g) < 0$	No	No	Yes	Yes

In all cases the solution ϕ to equation (1.12) is unique, except in the case $Y(g) = 0$, $\tau = 0$ and $\beta, r \equiv 0$ (marked with a $*$), in which case there is a one parameter homothety family of solutions, namely $\phi \in \mathbb{R}^+$. This gives a complete parameterization of the CMC solutions of the constraint equations (1.2), cf. [Ise87]. Similar results have been found for other topologies and asymptotic conditions.

If every spacetime could be evolved from CMC initial data, Isenberg's work would be enough to parameterize the solutions of the Einstein equations. Unfortunately, not all spacetimes can be obtained this way, as proved in [CIP05]. It is not known whether or not generic spacetimes can be obtained from CMC initial data. Thus for a complete parameterization of solutions to the Einstein equations, we must consider the conformal constraint equations with generic mean curvature τ.

We first consider the Lichnerowicz equation (1.3a). Unsurprisingly, the solvability of the Lichnerowicz equation mirrors the solvability of the CMC conformal constraint problem as tabulated in Table 1, but with one caveat. On a compact manifold, if $Y(g) < 0$, the Lichnerowicz equation has a solution if and only if g can be conformally transformed to a metric with scalar curvature $-\kappa\tau^2$. For

$\tau^2 > 0$, this is always true. For τ with zeroes, the solvability is discussed in [Rau95], [DM15] and Chapter IV below. This problem is completely understood.

Excepting that caveat, one might expect the solvability of the generic conformal constraint equations to mirror that of the CMC case, as shown in Table 2.

TABLE 2. Hypothesized Solvability of the Conformal Constraint Equations

	$\tau \not\equiv 0, \beta, r \equiv 0$	$\tau \not\equiv 0, \beta, r \not\equiv 0$
$Y(g) > 0$	No	Yes
$Y(g) = 0$	No	Yes
$Y(g) < 0$	Yes	Yes

For nearly CMC data, this solvability is realized, at least in the case where there are no conformal Killing fields. The near-CMC conditions typically come in two flavors. If τ is a constant for which the CMC conformal constraint equations have a solution, then the inverse function theorem can be used to show that any nearby τ (in $W^{1,p}_{\delta-1}$) also leads to a solution. In the second case, the condition is that $\|d\tau\|_p$ is sufficiently small compared to $\inf \tau$. Using these types of conditions, the Yamabe negative near-CMC case was settled in 1996 [IM96], the nonexistence cases for Yamabe nonnegative metrics in 2004 [IÓM04], and the remaining cases in 2008 [ACI08]. (These results prove results for manifolds with scalar curvature of a strict sign; Maxwell's conformal covariance of the CTS-H formulation [Max14a] is needed to make them apply to the entire Yamabe classes.) All of these results rely on there being no conformal Killing fields, for the reasons discussed above.

The only generic result known for the arbitrary mean curvature case was proven by Holst, Nagy, and Tsogtgerel [HNT09], then improved by Maxwell [Max09]. This result essentially says that on a Yamabe positive compact manifold, given an arbitrary τ, if σ, r, and J are small enough, then the conformal constraint equations (1.3) have a (not necessarily unique) solution. More recently, however, Nguyen [Ngu14] showed that all such solutions are merely rescalings of perturbations off of the maximal ($\tau \equiv 0$) case. We discuss this in Corollary 5.2.7. Thus the only generic far-from-CMC result known is, essentially, a near-CMC result.

Another attempted method to find solutions to the conformal constraint equations is the "limit equation" criterion. First explored by Dahl, Gicquaud, and Humbert [DGH12], this method says that either the conformal constraint equations or the limit equation

$$\operatorname{div}\frac{1}{2N}LW = \alpha_0\sqrt{\kappa}\,|LW|\,\frac{d\tau}{2N\tau} \tag{1.14}$$

(for some $\alpha_0 \in (0,1]$) have a (nontrivial) solution. As was suspected, Nguyen recently showed [Ngu14] that both can in fact have solutions.

The limit equation was originally found via a subcriticality argument. If the exponent of ϕ in the vector equation (1.3b) is reduced by epsilon, the coupling of the conformal constraint equations is weak enough so that solutions are relatively simple to find. As $\epsilon \to 0$, if these subcritical solutions are bounded, they must converge to a solution to the conformal constraint equations. If they are instead unbounded, it can be shown that they converge to a solution of the limit equation (1.14).

The limit equation criterion is that if the limit equation has no solutions, then the conformal constraint equations must have a solution. While this method may be used for the far-from-CMC case, so far, it has only been used to find solutions in the near-CMC case, as in [DGH12].

In order to better explore the far-from-CMC regime, Maxwell studied a model problem with high symmetry [Max11]. He studied seed data on T^n with the flat metric, where the data depended on only one coordinate (i.e., with U^{n-1} symmetry). For some particular data, he showed that if τ was sufficiently far-from-CMC in some sense, then there were no solutions to the conformal constraint equations (in the symmetry class of the data). Given such a τ, however, if the transverse traceless part of the data were sufficiently small, then there were at least two solutions.

This is in contrast to the CMC and near-CMC case, where solutions are unique. Also, the nonexistence for far-from-CMC data is in contradiction with the hypothesized solvability described in Table 2. However, this model problem may be a special case for several reasons. First, the background metric is flat, which is known to be a very special case, even in the CMC theory. Second, the background metric has conformal Killing fields, which is known to be non-generic. Third, the mean curvature function τ has jump discontinuities. Finally, the non-existence/non-uniqueness only occurs when τ changes signs. However, this could also represent new phenomena, or, perhaps, limitations of the conformal method.

Maxwell [Max14b] later studied a related problem. On flat T^n, and arbitrary τ, again with U^{n-1} symmetry, he found seed data that led to either flat Kasner or static-toroidal solutions of the constraint equations. It was shown that there was in fact a one parameter family of solutions to the conformal constraint equations if and only if

$$\tau^* := \frac{\int_{S^1} N\tau dx}{\int_{S^1} N dx} = 0, \tag{1.15}$$

where the integrals are respect to the flat metric. While τ^* appears to be determined by N and τ, the reality is more complicated.

Recall that the CTS-H method is conformally covariant, as described in Proposition 1.1.1. However, equation (1.15) is not conformally covariant. Thus, if we started with arbitrary seed data on T^n that happened to lead to one of these solutions, we would need to calculate τ^* with respect to the solution metric and not the background metric. In general, then, there is no way of determining whether or not a set of seed data leads to a one parameter family until after the solution has already been found. This presents serious problems for the goal of parameterizing all solutions to the constraint equations (1.2), since these one parameter families are essentially impossible to detect.

1.3. The Drift Formulation

In an attempt to avoid the pitfalls for parameterizing solutions to the constraint equations described in the last section, Maxwell introduced the drift formulation of the conformal method, originally in [Max14b], and expanded in [Max15b]. In the standard CTS-H method, the mean curvature τ is specified in the seed data, and is unchanged by the conformal factor found by solving the Lichnerowicz equation (1.3a). This, however, makes calculating τ^* impossible without first finding the solution to the conformal constraint equations (1.3).

Maxwell's idea was to specify the constant τ^* directly, and then define τ by adding τ^* to a conformally varying term, given by something he calls a drift, for reasons explained in [Max15b]. Since τ^* is specified directly, the one parameter families of solutions described in the previous section occur only when τ^* is specified to be zero. The drift formulation also has the advantage of making it possible to find solutions even in the presence of conformal Killing fields.

Though Maxwell introduces several possible ways to construct such τ, we discuss only one. In this formulation, which he calls CTS-H with volumetric drift, the drift is given by a vector field V which is specified up to a conformal Killing field Q. Given seed data $(g, \tau^*, V, N, \sigma, r, J)$, one tries to find a solution (ϕ, W, Q) to

$$-a\Delta\phi + R\phi + \kappa \left(\tau^* + \frac{\phi^{-2q}}{N}\operatorname{div}(\phi^q(V+Q))\right)^2 \phi^{q-1} - \left|\sigma + \frac{1}{2N}LW\right|^2 \phi^{-q-1} - r\phi^{-q/2} = 0 \tag{1.16a}$$

$$\operatorname{div}\frac{1}{2N}LW = \kappa\phi^q d\left(\tau^* + \frac{\phi^{-2q}}{N}\operatorname{div}(\phi^q(V+Q))\right) + J., \tag{1.16b}$$

which is the same as (1.3), except that we replaced τ with

$$\tau := \tau^* + \frac{\phi^{-2q}}{N}\operatorname{div}(\phi^q(V+Q)). \tag{1.17}$$

The data is then reassembled as before, except

$$\overline{K}_{ab} = \phi^{-2}\left(\sigma_{ab} + \frac{1}{2N}LW_{ab}\right) + \frac{1}{n}\left(\tau^* + \frac{\phi^{-q}}{N}\overline{\operatorname{div}}V\right)\overline{g}_{ab}, \tag{1.18}$$

where $\overline{\operatorname{div}}$ is the divergence with respect to $\overline{g}$. Note that for $V \equiv 0$, this method reduces to the CMC CTS-H method.

The drift method has several advantages over the CTS-H formulation. First, the one parameter families found in [Max14b] occur if and only if $\tau^* = 0$. Thus at least that obstruction to parameterization is overcome. Also, Maxwell proved [Max15b, Thm 10.1] that the vector equation (1.16b) has a solution for some Q, even if g has conformal Killing fields. Thus it becomes possible to solve the constraint equations in the presence of conformal Killing fields.

However, the drift formulation equations (1.16) are much more complicated analytically. For example, in the original vector equation (1.3b), one can find an upper bound on LW based on an upper bound for ϕ. In the drift vector equation, a similar upper bound naively requires bounds on $\|\phi\|_{C^2}$. Perturbation methods, such as those used to produce solutions in the CTS-H formulation, are expected to extend to the drift setting [Max15a]. Since all known generic results for the CTS-H method are near-CMC results, this would show that the drift formulation is at least as useful as the CTS-H method. The drift formulation is a promising approach to finding the parameterization of the constraint equations.

1.4. This Dissertation

In this dissertation, we discuss the conformal constraint equations, in particular focusing on the asymptotically Euclidean (AE) case. In Chapter II, we introduce AE manifolds and the appropriate Banach spaces for analysis, and then discuss elliptic operator theory on AE manifolds. Because the vector equation (1.3b) is relatively simple, we discuss its solvability in this chapter. This chapter serves as a common introduction to all the subsequent chapters.

In Chapter III, we discuss the solvability of the Lichnerowicz equation (1.3a). In particular, we show that the Lichnerowicz equation is solvable if an only if the metric can be conformally transformed to one with scalar curvature $-\kappa\tau^2$. We then leverage this result to obtain a circumstance where the conformal constraint equations do not admit a solution, and also show an example of the blowup of solutions. The results in this chapter, though written solely by the author of this dissertation, will appear in [DGI15].

In Chapter IV we discuss when the prescribed scalar curvature problem from the previous chapter has a solution. We give a necessary and sufficient condition for the problem to have a solution; namely, that the zero set of the prescribed scalar curvature has positive Yamabe invariant, as defined in this chapter. Because of this problem's close relation to AE Yamabe classes, we also give a characterization of the AE Yamabe classes. This chapter is taken from [DM15].

In Chapter V, we prove the existence of solutions for the conformal constraint equations for arbitrary mean curvature, assuming the tensor σ and the matter terms r and J are sufficiently small. We also show existence in the near-CMC case. This part is taken from [DIMM14]. We also discuss a new solvability criterion, related to Nguyen's "local supersolution" from [Ngu14].

In Chapter VI, we discuss the limit equation criterion for AE manifolds. Unfortunately, we only show that the solution of the limit equation is nontrivial in the near-CMC case. We show that arbitrarily near-CMC data, in the sense required, does not ever occur. This is unpublished joint work with Romain Gicquaud and Jim Isenberg.

In Chapter VII, we discuss the relation of the ADM mass to the asymptotics of the solution of the conformal constraint equations (1.3). We then present a model problem for this relation, which shows that the ADM mass is not monotonically dependent on the asymptotics of the solution.

CHAPTER II

ASYMPTOTICALLY EUCLIDEAN MANIFOLDS

Perhaps the simplest solution to the Einstein constraint equations (1.1) is Euclidean space, with the second fundamental form K and stress-energy tensor T vanishing. Physically this represents space with no matter and no tidal forces. Heuristically, far from any mass and energy, space should become more and more like Euclidean space. Far from any star, gravity becomes very weak. Mathematically, this kind of initial data is represented by asymptotically Euclidean (AE) manifolds.

A manifold (M^n, g) is called asymptotically Euclidean (AE) if there exists a compact set $K \subset M$ such that $M \setminus K$ is a (finite) collection of components E_i, each diffeomorphic to the exterior of a ball in Euclidean space, $\mathbb{R}^n \setminus B_R(0)$, and on each end, g is asymptotic to the Euclidean metric g_{Euc}. The E_i are called the ends of M.

In order to be precise, we must first define appropriate weighted Sobolev and Hölder norms. First, fix a Euclidean coordinate system on each end, i.e., a distinguished diffeomorphism from E_i to $\mathbb{R}^n \setminus B_R(0)$. Let $\rho \geq 1$ be a smooth function which agrees with the radial coordinate on each end. We say a function $f \in W^{s,p}_\delta(M)$ if

$$\sum_{|j| \leq s} \|\rho^{-\delta - \frac{n}{p} + |j|} \nabla^j f\|_{L^p} < \infty, \tag{2.1}$$

where j is a multi-index, and ∇^j is calculated with respect to a frame agreeing with the Euclidean frame on each end. We denote this quantity by $\|f\|_{W^{s,p}_\delta(M)}$, without the (M) if the domain is understood. If $s = 0$, we denote the space as $L^p_\delta(M)$ and the norm as $\|f\|_{p,\delta}$. To extend this space and norm to tensors, we require the same regularity and decay for each component of the tensor with respect to the Euclidean frame in the background Euclidean metric g_{Euc}. Note that our convention on δ is chosen such that $f \in W^{s,p}_\delta(M)$ implies that, using the little-o notation, f is $o(\rho^\delta)$; other conventions exist in the literature.

For $\alpha \in [0,1]$, we say a function $f \in C^{s,\alpha}_\delta(M)$ if

$$\sup_{B, |j| \leq s} \left\{ |\nabla^j f| \rho^{|j|-\delta}, [\nabla^s f]_{B;\alpha} \rho^{s-\delta} \right\} < \infty, \tag{2.2}$$

where the supremum is over all balls $B \subset M$ of unit radius, and $[\cdot]_{B;\alpha}$ is the Hölder seminorm on that ball. We denote $C^{s,0}_\delta(M)$ by $C^s_\delta(M)$ for simplicity. We extend this space to tensors similarly.

We then say that g is a $W^{s,p}_\delta$ AE manifold if $\delta < 0$ and

$$g|_{E_i} - g_{Euc} \in W^{s,p}_\delta \tag{2.3}$$

on each end E_i. We similarly define $C^{s,\alpha}_\delta$ AE manifolds.

We state some basic properties of these spaces in the following two propositions, the first of which is taken from [Max05, Lemma 1]:

Proposition 2.0.1 (Properties of Weighted Sobolev Spaces)**.** *The following properties hold for the weighted Sobolev spaces defined by* (2.1)*:*

1. *If $p \geq q$ and $\delta' < \delta$ then $L^p_{\delta'} \subset L^q_\delta$ and the inclusion is continuous.*

2. *For $s \geq 1$ and $\delta' < \delta$ the inclusion $W^{s,p}_{\delta'} \subset W^{s-1,p}_\delta$ is compact.*

3. *If $s < n/p$ then $W^{s,p}_\delta \subset L^r_\delta$ where $r = np/(n - sp)$. If $s = n/p$ then $W^{s,p}_\delta \subset L^r_\delta$ for all $r \geq p$. If $s > n/p$ then $W^{s,p}_\delta \subset C^0_\delta$. These inclusions are continuous, and the last is compact.*

4. If $m \leq \min(j,s)$, $p \leq q$, $\epsilon > 0$, and $m < j + s - n/q$, then multiplication is a continuous bilinear map from $W^{j,q}_{\delta_1} \times W^{s,p}_{\delta_2}$ to $W^{m,p}_{\delta_1+\delta_2+\epsilon}$ for any $\epsilon > 0$. In particular, if $s > n/p$ and $\delta < 0$, then $W^{s,p}_{\delta}$ is an algebra.

Proposition 2.0.2 (Properties of Weighted Hölder Spaces)**.** *The following properties hold for the weighted Holder spaces defined by* (2.2)*:*

1. *If $s + \alpha \geq s' + \alpha'$, $\alpha \neq 1$, and $\delta \leq \delta'$ then the inclusion $C^{s,\alpha}_{\delta} \subset C^{s',\alpha'}_{\delta'}$ is continuous.*

2. *If $s + \alpha > s' + \alpha'$ and $\delta < \delta'$ then the inclusion $C^{s,\alpha}_{\delta} \subset C^{s',\alpha'}_{\delta'}$ is compact.*

3. *Assume $s + \alpha \leq s' + \alpha'$. Then multiplication is a continuous bilinear map from $C^{s,\alpha}_{\delta} \times C^{s',\alpha'}_{\delta'}$ to $C^{s,\alpha}_{\delta+\delta'}$. In particular, if $\delta \leq 0$, then $C^{s,\alpha}_{\delta}$ is an algebra.*

The standard Poincaré and Sobolev inequalities on $\mathbb{R}^n$, with appropriately chosen weights, also hold on AE manifolds, as shown in [DM15, Lem 2.1].

Lemma 2.0.3. *There exist constants c_1, c_2 such that*

$$\|\nabla u\|_{p,-n/p} \geq c_1 \|u\|_{p,1-n/p} \tag{2.4}$$

$$\|\nabla u\|_2 \geq c_2 \|u\|_q \tag{2.5}$$

for all $u \in u \in W^{1,2}_{\delta^*}(M)$ and $p \in [1,n)$.

We refer the reader to [Bar86] for further properties of weighted Sobolev spaces.

2.1. Elliptic Operators

Elliptic operator theory on AE manifolds is well established, going back at least to [McO79]. For more references, see also [Max05], [CBIY00], and appendix C in [CMP12]. While more general results are available, we focus our attention on the Laplacian and vector Laplacian. The following result is adapted from [DIMM14].

Proposition 2.1.1. *Suppose* (M^n, g) *is a* $W^{s,p}_{\gamma}$ *AE manifold, with* $s \geq 2$*,* $s > n/p$*, and* $\gamma < 0$*. Suppose* $V \in W^{s-2,p}_{\gamma-2}$*. Let* $\mathcal{P}$ *be either the operator* $-a\Delta + V$ *or the operator* $\operatorname{div}\frac{1}{2N}L$*, where* L *is the conformal Killing operator* (1.4)*. Then for* $\delta \in (2-n, 0)$ *the operator*

$$\mathcal{P} : W^{s,p}_{\delta} \to W^{s-2,p}_{\delta-2} \tag{2.6}$$

is Fredholm of index zero, and

$$\|u\|_{W^{s,p}_{\delta}} \leq C \left(\|Pu\|_{W^{s-2,p}_{\delta-2}} + \|u\|_{L^p_{\delta'}} \right) \tag{2.7}$$

holds for some $C > 0$*, any* δ' *and all* $u \in W^{s,p}_{\delta}$*. The map* (2.6) *is an isomorphism if and only if* $\mathcal{P}$ *has trivial null space in* $W^{s,p}_{\delta}$*. If* (2.6) *is an isomorphism, then the estimate* (2.7) *can be strengthened to*

$$\|u\|_{W^{s,p}_{\delta}} \leq C\|\mathcal{P}u\|_{W^{s-2,p}_{\delta-2}}. \tag{2.8}$$

Similarly, if (M, g) *is a* $C^{s,\alpha}_{\gamma}$ *AE manifold, then*

$$\mathcal{P} : C^{s,\alpha}_{\delta} \longrightarrow C^{s-2,\alpha}_{\delta-2} \tag{2.9}$$

is Fredholm of index zero, with a corresponding a priori estimate. If $\mathcal{P}$ *has trivial nullspace in* $C^{s,\alpha}_{\delta}$*, then there exists a constant* $C > 0$ *such that*

$$\|u\|_{C^{s,\alpha}_{\delta}} \leq C\|\mathcal{P}u\|_{C^{s-2,\alpha}_{\delta-2}}. \tag{2.10}$$

Because Proposition 2.1.1 requires $s \geq 2$, and for simplicity, we will assume $s = 2$ for the rest of this paper, unless mentioned otherwise. In other words, g is either a $W^{2,p}_\gamma$ or $C^{2,\alpha}_\gamma$ AE manifold. In any case, if $g \in W^{s,p}$ for $s > 2$ and $s > n/p$, Sobolev embedding 2.0.1, implies that $g \in W^{2,p'}$ with $p' > n/2$.

We now prove two maximum principles, taken from [Max05].

Proposition 2.1.2 (A Maximum Principle for AE Manifolds)**.** *Suppose* (M, g) *and* V *are as in Proposition 2.1.1, and suppose* $V \geq 0$*. Suppose* $u \in W^{2,p}_\delta$ *for some* $\delta < 0$*. If*

$$-a\Delta u + Vu \geq 0, \tag{2.11}$$

then $u \geq 0$*.*

Proof. Let

$$v = (u + \epsilon)^- := \min\{0, u + \epsilon\} \tag{2.12}$$

for some $\epsilon > 0$. Since $u \to 0$ on each end, we see that v is compactly supported. By Sobolev embedding, $v \in W^{1,2}$ as well. Since

$$\int_M a|\nabla v|^2 = \int_M -av\Delta u \leq \int_M -Vuv \leq 0, \tag{2.13}$$

we know $u \geq -\epsilon$. Letting $\epsilon \to 0$, we find $u \geq 0$. □

Proposition 2.1.3 (A Strong Maximum Principle for AE Manifolds)**.** *Suppose* (M, g) *and* V *are as in Proposition 2.1.1. Suppose* $u \in W^{2,p}_{loc}$ *is nonnegative and satisfies*

$$-a\Delta u + Vu \geq 0. \tag{2.14}$$

If $u(x) = 0$ *somewhere, then* $u \equiv 0$*.*

Proof. Suppose $u(x) = 0$. The weak Harnack inequality from [Tru73a] applies to u; i.e., for some radius R sufficiently small and some exponent q sufficiently large, there exists $C > 0$ such that

$$\|u\|_{L^q(B_{2R}(x))} \leq C \inf_{B_R(x)} u = 0. \tag{2.15}$$

Thus u vanishes on a neighborhood of x, and a connectivity argument shows that $u \equiv 0$. $\square$

In order to discuss when the operator $\mathcal{P} = -a\Delta + V$ is an isomorphism, we need to first discuss the Yamabe invariant. The Yamabe invariant on AE manifolds is defined similarly to how it is defined on compact manifolds (cf. Equation (1.13)), except that the test functions must have compact support. Precisely,

$$Y(g) := \inf_{u \in C_0^\infty(M), u \not\equiv 0} \frac{\int_M a|\nabla u|^2 + Ru^2}{\|u\|_q^2}, \tag{2.16}$$

where $C_0^\infty(M)$ represents smooth functions with compact support. As before, we say g is Yamabe positive if $Y(g) > 0$. The Yamabe classes on AE manifolds appear to behave very differently than the Yamabe classes on compact manifolds, but they are in fact equivalent to each other in some sense. This idea is discussed further in Chapter IV. We now can prove the following isomorphism theorem.

Proposition 2.1.4. *The operator $-a\Delta + V$ from Equation* (2.6) *is an isomorphism either if $V \geq 0$ or if $V = R_g$ and g is Yamabe positive.*

No $C^{2,\alpha}_\gamma$ AE manifold allows a conformal Killing field in $C^{2,\alpha}_\delta$. Also, if $p > n$, no $W^{2,p}_\gamma$ AE manifold allows a conformal Killing field in $W^{2,p}_\delta$. Thus the operator

$\operatorname{div}\frac{1}{2N}L$ *is always an isomorphism on* $C^{2,\alpha}_\delta$*, and is an isomorphism on* $W^{2,p}_\delta$ *if* $p > n$ *or if the metric admits no conformal Killing fields.*

Proof. Suppose $V \geq 0$ and $(-a\Delta + V)u = 0$. By the maximum principle 2.1.2, $u \geq 0$ and $-u \geq 0$. Thus $u \equiv 0$, and so $-a\Delta + V$ is an isomorphism.

Suppose $V = R_g$ and g is Yamabe positive, but that $-a\Delta + R$ is not an isomorphism. Then there exists a nontrivial solution $u \in W^{2,p}_\delta \subset C^{0,\alpha}_\delta$ solving $(-a\Delta + R)u = 0$. By Sobolev embedding 2.0.1, $u \in L^q$. Integration by parts implies that $\int a|\nabla u|^2 + Ru^2 = 0$. Estimating u by smooth functions with compact support, we find that $Y(g) = 0$ (see the definition (2.16)), which is a contradiction.

The facts about conformal Killing fields are found in [Max05]. That the kernel of $\operatorname{div}\frac{1}{2N}L$ is the set of conformal Killing fields follows from the calculation (1.11). □

On compact manifolds, the constants are harmonic functions; i.e., they satisfy $\Delta u = 0$. On AE manifolds, the constants are harmonic functions, but in addition, if for each end E_i, we specify a constant u_i, there is a unique harmonic function u such that $u - u_i \in W^{2,p}_\delta(E_i)$ (cf. [DIMM14, Lem 4.1]). We introduce the following notation.

Definition 2.1.5. *For any set of constants* u_i*, the "asymptotic function"* $\mathring{u}$ *is the unique harmonic function such that* $\mathring{u} \to u_i$ *on* E_i*. Such a function has the same regularity as the metric; i.e., if* (M, g) *is a* $W^{2,p}_\gamma$ *AE manifold,* $\mathring{u} \in W^{2,p}$ *and* $\mathring{u} - u_i \in W^{2,p}_\delta(E_i)$ *for any* $\delta \in (2-n, 0)$*. The existence of such a function is guaranteed by [DIMM14, Lem 4.1]. When we refer to* $\mathring{u}$ *we do not mention the constants* u_i*.*

Corollary 2.1.6. *The function* $\mathring{u}$ *satisfies* $\min_i u_i \leq \mathring{u} \leq \max_i u_i$ *with equality if and only if* $\min_i u_i = \max_i u_i$*.*

Proof. The maximum principle 2.1.2 implies that $\min_i u_i \le \mathring{u} \le \max_i u_i$. If $\min_i u_i = \mathring{u}$ somewhere, the strong maximum principle 2.1.3 implies that $\min_i u_i = \sup u = \max_i u_i$. If $\min_i u_i = \max_i u_i$, all the u_i are the same, and so $\mathring{u} \equiv u_i$ is the desired harmonic function. □

When searching for solutions to the Lichnerowicz equation (1.3a), there is no reason to restrict ourselves to ϕ such that $\phi \to 1$ on each end, since ϕ approaching any other constant simply scales the Euclidean coordinates on that end. Indeed, as we see in Chapter V, assuming ϕ approaches some other constant can assist in finding solutions. Thus, we generally assume $\phi \to \mathring{u}$ on each end.

While the operator $\operatorname{div}\frac{1}{2N}L$ appears in a linear equation (1.3b), the Lichnerowicz equation (1.3a) is semilinear, and so Proposition 2.1.4 is not sufficient to find solutions of this equation. A useful tool for finding solutions to semilinear equations is the method of sub and supersolutions.

Consider the nonlinear problem

$$-a\Delta u = f(x, u) \tag{2.17}$$

for a function $f(x, y) : M \times \mathbb{R} \to \mathbb{R}$ which takes the form $f(x, y) = \sum_{i=1}^{j} a_i(x) y^{b_i}$ for specified functions a_i and constants b_i, where we use the convention that $y^{b_i} \equiv 1$ if $b_i = 0$. We also assume that $a_i(x) \in L^p_{\delta-2}$ for some $\delta < 0$ (or, similarly, that $a_i(x) \in C^{0,\alpha}_{\delta-2}$). Note that, depending on the value(s) of b_i, y^{b_i} is smooth on $(0, \infty)$, $[0, \infty)$, or $(-\infty, \infty)$. We say a function f is " regular" if it satisfies these properties, and the largest interval for which all the y^{b_i} are smooth is f's "interval of regularity" I. Note that the Lichnerowicz equation (1.3a) takes this form, as long as we require sufficient regularity of the seed data. Recall that u_- is called a subsolution of (2.17) if $-a\Delta u_- \le f(x, u_-)$, and similarly (with $\ge$ replacing $\le$) for a supersolution u_+.

Theorem 2.1.7 (Sub and Supersolution Theorem for AE Manifolds)**.** *Let (M, g) be a $W^{2,p}_\gamma$ AE manifold with $p > n/2$ and $\gamma < 0$. Suppose $f(x, y)$ is regular (as defined above) for some $\delta \in (2-n, 0)$. Suppose that there are sub and supersolutions $u_\pm \in L^\infty$ such that $u_- \le u_+$ and $\inf u_- \in I$. Suppose $\mathring{u}$ is such that, sufficiently far out on each end, $u_- \le \mathring{u} \le u_+$. Then Equation (2.17) admits a solution u such that $u_- \le u \le u_+$ and $u - \mathring{u} \in W^{2,p}_\delta$.*

A similar theorem holds for $C^{2,\alpha}_\gamma$ AE manifolds if f is $C^{0,\alpha}_\delta$ regular and $u_\pm \in C^{0,\alpha}$. The solution then satisfies $u - \mathring{u} \in C^{2,\alpha}_\delta$.

Remark 2.1.8. *This is essentially Theorem 1 in Appendix B.2. in [CBIY00], but with lower regularity requirements, and generalized asymptotics. We mirror their proof. Note that if some $b_i < 0$, the theorem requires $u_- > \epsilon > 0$ for some $\epsilon > 0$.*

Proof. We only prove the Sobolev case. The Hölder case is proven similarly.

We construct a solution by induction, starting from ϕ_-. Let k be a positive function on M such that $k \in L^p_\delta$ and

$$k(x) + \sup_{y \in \mathrm{Range}(u_\pm)} f_y(x, y) \ge 0. \tag{2.18}$$

Such a k exists by our assumptions on $u_\pm$ and f.

Let $v_1 \in W^{2,p}_\delta$ be the unique solution to

$$-a\Delta v_1 + k v_1 = f(x, u_-) + k(u_- - \mathring{u}) \tag{2.19}$$

and let $u_1 = v_1 + \mathring{u}$. The solution v_1 exists by Proposition 2.1.4.

Using the equality and inequality satisfied by v_1 and u_- respectively, we find that

$$-a\Delta(u_1 - u_-) + k(u_1 - u_-) \geq 0. \tag{2.20}$$

By the maximum principle 2.1.2, $u_1 \geq u_-$. Similarly,

$$-a\Delta(u_+ - u_1) + k(u_+ - u_1) \geq f(x, u_+) - f(x, u_-) + k(u_+ - u_-) \tag{2.21}$$

$$= (u_+ - u_-)\left(k + \int_0^1 f_y(x, u_- + t(u_+ - u_-))dt\right) \tag{2.22}$$

$$\geq 0, \tag{2.23}$$

where the last line holds by our assumption on k, Equation (2.18). Again by the maximum principle 2.1.2, $u_1 \leq u_+$.

We then let $u_i = v_i + \mathring{u}$, where $v_i \in W^{2,p}_\delta$ solves

$$-a\Delta v_i + kv_i = f(x, u_{i-1}) - kv_{i-1}. \tag{2.24}$$

Again using the maximum principle, we can show that u_i is an increasing sequence; i.e.,

$$u_- \leq u_1 \leq u_2 \leq \cdots \leq u_{i-1} \leq u_i \leq \cdots \leq u_+. \tag{2.25}$$

Since the u_i constitute a bounded increasing sequence, the u_i converge to some function u with $u_- \leq u \leq u_+$. We claim that u is a solution of Equation (2.17).

From Proposition 2.1.1, we have

$$\|v_{i+1}\|_{W^{2,p}_\delta} \leq C\|f(x, u_i) - kv_i\|_{L^p_\delta}. \tag{2.26}$$

The right hand side is uniformly bounded by our assumptions on k and f, and since v_i and u_i are bounded. Thus v_i is uniformly bounded in $W^{2,p}_\delta$.

The compact embedding of $W^{2,p}_\delta$ into $C^{0,\alpha}_{\delta'}$ for any $\delta' > \delta$ and some $\alpha > 0$ from Proposition 2.0.1 implies that $u_i \to u$ in $C^{0,\alpha}_{\delta'}$, and that $u - \mathring{u} \in W^{2,p}_\delta$. This convergence implies that $f(x, u_{i-1}) - kv_{i-1}$ converges in L^p_δ, and so, since $-a\Delta + k$ is an isomorphism, u_i must converge to u in $W^{2,p}_\delta$. Thus $-a\Delta u = f(x, u)$, as desired. $\square$

CHAPTER III

SOLVABILITY OF THE LICHNEROWICZ EQUATION

The results in this chapter, though written solely by the author of this dissertation, will appear in [DGI15], with coauthors Romain Gicquaud and James Isenberg.

The Lichnerowicz equation (1.3a) is a semilinear elliptic equation. Because of the mixed sign of the exponents, it is of a type not generally studied. However, with appropriate sign restrictions on the coefficients, we can fully understand this equation. Recall that in the compact case, the solvability of the Lichnerowicz equation is given by Table 2, with one caveat. Namely, if g is Yamabe negative, the Lichnerowicz equation is solvable if and only if g can be conformally transformed to a metric with scalar curvature $-\kappa\tau^2$. The main result of this chapter is that, regardless of Yamabe class, the Lichnerowicz equation on AE manifolds is solvable if and only if g can be conformally transformed to a metric with scalar curvature $-\kappa\tau^2$.

First we must discuss what kind of data we are looking for when we discuss asymptotically Euclidean initial data. Clearly we want an AE manifold, $(M, \overline{g})$. The usual regularity we desire for $\overline{g}$ is $W^{2,p}_\delta$. However, we also need $\overline{K}$ to decay at infinity. Heuristically, we want this so that our spacelike slice $(M, \overline{g})$ is not curled up inside the spacetime near infinity. We thus require $\overline{K} \in W^{1,p}_{\delta-1}$. In order to guarantee this regularity, we require that our seed data satisfies

$$(g - g_{Euc}, \tau, N-1, \sigma, r, J) \in W^{2,p}_\delta \times W^{1,p}_{\delta-1} \times W^{2,p}_\delta \times L^{2p}_{\delta-1} \times L^p_{\delta-2} \times L^p_{\delta-2}, \tag{3.1}$$

with $p > n/2$ and $\delta \in (2-n, 0)$, and similarly for $C^{2,\alpha}_\delta$ seed data. We then seek a solution (ϕ, W) with $\phi - \mathring{u} \in W^{2,p}_\delta$ and $W \in W^{2,p}_\delta$. The reconstructed initial data $(\overline{g}, \overline{K}, T_{nn}, T_{in})$ from (1.5) then has the desired regularity.

We must make one additional restriction on the sign of the seed data. In particular, we must require $r \geq 0$. This is known as the weak energy condition, and is simply requiring that the matter density can never be negative. Equivalently, it says that gravity is always an attractive force. This is a physically reasonable assumption, though not strictly necessary for general relativity. For instance, solutions of the Einstein equations with stable, traversable wormholes require matter with negative energy density.

We can now prove the main result of this chapter.

Theorem 3.0.9 (Curvature Criterion for AE Solutions to the Lichnerowicz Equation)**.** *Suppose that (M, g) is a $W^{2,p}_\delta$ AE manifold with $p > n/2$ and $\delta \in (2-n, 0)$. Assume that r, $\left|\sigma + \frac{1}{2N} LW\right|^2$ and τ^2 are all contained in $L^p_{\delta-2}$, and that $r \geq 0$. Then the Lichnerowicz equation* (1.3a) *has a positive solution ϕ with $\phi - \mathring{u} \in W^{2,p}_\delta$ if and only if there exists a positive conformal factor ψ with $\psi - \mathring{u}' \in W^{2,p}_\delta$ such that $\overline{g} = \psi^{q-2} g$ has scalar curvature $-\kappa\tau^2$. The $\mathring{u}$ and $\mathring{u}'$ are two positive asymptotic functions, as defined in Definition 2.1.5. A similar result holds for $C^{2,\alpha}_\delta$ regularity.*

Proof. $(\Rightarrow)$ Suppose there is such a solution ϕ to the Lichnerowicz equation. It is well known that the desired ψ is a solution to

$$-a\Delta\psi + R\psi + \kappa\tau^2\psi^{q-1} = 0. \tag{3.2}$$

Equation (3.2) clearly satisfies the conditions of the sub and supersolution theorem 2.1.7 as a consequence of the regularity we have presumed. Note that the scalar curvature R must be in $L^p_{\delta-2}$. For $\beta \geq 1$, $\beta\phi$ is a supersolution for (3.2). If $\beta > \sup \mathring{u}'/\mathring{u}$, $\beta\phi$ satisfies the conditions of Theorem 2.1.7. For the subsolution, we take $\psi_- \equiv 0$. This is certainly regular. Also, we note that since the exponents in (3.2) are positive, 0 lies in the interval of regularity for $f(x, y)$. Together, these conditions and Theorem 2.1.7 guarantee the existence of a solution $\psi \geq 0$ of (3.2) with the properties we desire, except that it may be zero somewhere.

However, we can easily argue that ψ cannot be 0 anywhere. Suppose it were zero at some point. Since $\psi \in W^{2,p}_{loc}$, the strong maximum principle 2.1.3 implies that $\psi \equiv 0$. But $\psi \to \mathring{u}$ at infinity, a contradiction. Thus $\psi > 0$, proving the implication.

($\Leftarrow$) Suppose there is such a conformal factor ψ. Note that ψ must then satisfy Equation (3.2). For $\beta \leq 1$, $\beta\psi$ is a subsolution for the Lichnerowicz equation. If $\beta < \inf \mathring{u}/\mathring{u}'$, then $\beta\psi$ satisfies the conditions of Theorem 2.1.7.

To help find the supersolution, we use the conformal covariance of the Lichnerowicz equation 1.1.1 and the conformal factor ψ to assume that the scalar curvature is $-\kappa\tau^2$.

Proposition (2.1.4) shows that there exist solutions v_ϵ to the linear problem

$$-a\Delta v_\epsilon + \epsilon\kappa\tau^2 v_\epsilon = \epsilon\left(\left|\sigma + \frac{1}{2N}LW\right|^2 + r\right) \tag{3.3}$$

such that $v_\epsilon - 1 \in W^{2,p}_\delta$ for each $\epsilon \in [0, 1]$. Note that $v_0 \equiv 1$, and that the solution map is continuous in ϵ. We claim that $v_\epsilon > 0$ for all $\epsilon \in [0, 1]$. By continuity, the set of ϵ for which $v_\epsilon > 0$ is open. Suppose some ϵ were on the boundary of the set for which $v_\epsilon > 0$. By continuity, $v_\epsilon \geq 0$, and $v_\epsilon = 0$ somewhere. By the strong maximum principle 2.1.3, $v_\epsilon \equiv 0$. However, this contradicts that $v_\epsilon \to 1$ at infinity. Thus the set of ϵ for which $v_\epsilon > 0$ is open. Since this set is also nonempty, it is all of $[0, 1]$.

We claim $\beta v := \beta v_1$ is a supersolution to the Lichnerowicz equation (with $R = -\kappa\tau^2$) for large β. Indeed, if we plug βv into the Lichnerowicz equation, we get

$$\begin{aligned} &-\kappa\tau^2\beta v - \kappa\tau^2\beta v + \left|\sigma + \frac{1}{2N}LW\right|^2 \beta + r\beta \\ &\qquad + \kappa\tau^2(\beta v)^{q-1} - \left|\sigma + \frac{1}{2N}LW\right|^2 (\beta v)^{-q-1} - r(\beta v)^{-q/2} \\ &= \kappa\tau^2\left[(\beta v)^{q-1} - 2\beta v\right] + \left|\sigma + \frac{1}{2N}LW\right|^2 \left[\beta - (\beta v)^{-q-1}\right] + r\left[\beta - (\beta v)^{-q/2}\right] \geq 0 \end{aligned} \tag{3.4}$$

for sufficiently large β. If $\beta > \mathring{u}/\mathring{u}'$, the sub and supersolution theorem 2.1.7 provides the desired solution to the Lichnerowicz equation. □

In light of this theorem, we make the following definition.

Definition 3.0.10. *The seed data $(g, \tau, N, \sigma, r, J)$ is said to be "admissible" if there is a conformal factor transforming g to a metric with scalar curvature $-\kappa\tau^2$.*

It is important to understand for which AE metrics we can make this conformal transformation. This question is resolved in Chapter IV. We note here, however, that it is well known (cf. [Max05]) that g is Yamabe positive if and only if g can be conformally transformed to a metric with identically vanishing scalar curvature. We use this fact later in this chapter.

If the seed data is admissible, the Lichnerowicz equation has a solution asymptotic to any desired asymptotic function. This solution is unique.

Theorem 3.0.11 (Uniqueness of Solutions to the Lichnerowicz Equation)**.** *Suppose that (M, g) is a $W^{2,p}_\delta$ AE manifold with $p > n/2$ and $\delta \in (2-n, 0)$. Assume that r, $\left|\sigma + \frac{1}{2N}LW\right|^2$ and τ^2 are all contained in $L^p_{\delta-2}$, and that $r \geq 0$. If ϕ_1, ϕ_2 both solve the Lichnerowicz equation, and are such that $\phi_1 - \phi_2 \in W^{2,p}_\delta$, then $\phi_1 = \phi_2$.*

Proof. The following proof is taken from [CBIP06, Thm 8.3].

We use ϕ_i as a conformal factor and use the conformal covariance of the Lichnerowicz equation (cf. Proposition 1.1.1) to get

$$R_{\phi_i^{q-2}g} = -\kappa\tau^2 + |\sigma + LW|^2\phi_i^{-2q} + r\phi_i^{-\frac{3}{2}q+1}. \tag{3.5}$$

Therefore, using $u := \phi_2/\phi_1$, we obtain

$$\begin{aligned} - a\Delta_{\phi_1^{q-2}g}u + \left(\kappa\tau^2 - \left|\sigma + \frac{1}{2N}LW\right|^2 \phi_1^{-2q} + r\phi_1^{-\frac{3}{2}q+1}\right) u \\ = \left(\kappa\tau^2 - \left|\sigma + \frac{1}{2N}LW\right|^2 \phi_2^{-2q} + r\phi_2^{-\frac{3}{2}q+1}\right) u^{q-1}. \end{aligned} \tag{3.6}$$

This equation may be written as

$$-a\Delta_{\phi_1^{q-2}g}(u-1) + \xi(\phi_1, \phi_2)(u-1) = 0 \tag{3.7}$$

where $\xi(\phi_1, \phi_2)$ is a positive expression in terms of ϕ_i and the seed data. Also, as long as the ϕ_i are continuous and positive, $\xi \in L^p_{\delta-2}$. The operator $-a\Delta + \xi$ is thus an isomorphism (cf. Proposition 2.1.4). Thus $u - 1 \equiv 0$ and so $\phi_1 \equiv \phi_2$. □

3.1. Non-existence of Solutions to the Constraints

Theorem 3.0.9 allows us to find seed data which cannot lead to solutions of the conformal constraint equations (1.3).

Theorem 3.1.1. *Suppose g is Yamabe non-positive. If $\tau \equiv 0$, then the conformal constraint equations allow no solutions. The class of such metrics is non-empty.*

Proof. By Theorem 3.0.9, the Lichnerowicz equation is solvable if and only if g can be conformally transformed to a metric with scalar curvature $-\kappa\tau^2 \equiv 0$. However, since g is not Yamabe positive, there is no such conformally related metric (cf. [Max05]). Thus there can be no solution to the conformal constraint equations.

Friedrich in [Fri11] showed the existence of an AE Yamabe null manifold. In Proposition 4.4.3, we show that if a Yamabe null or negative compact manifold is decompactified in a particular way, the related AE metric is also Yamabe null or negative. □

In general, if one cannot conformally transform the scalar curvature to $-\kappa\tau^2$, there cannot be a solution to the conformal constraint equations. In Chapter IV, we give a characterization of when this is possible.

In addition, we can show that if τ_i approaches some τ that does not allow solutions, then any solutions ϕ_i, W_i of the constraint equations corresponding to seed data with τ_i must blow up as $i \to \infty$. We focus on the case $\tau \equiv 0$, but the techniques work in more general cases. We first prove a lemma.

Lemma 3.1.2. *Suppose $\tau_1^2 \geq \tau_2^2$, with $\tau_i \in W^{1,p}_{\delta-1}$. Suppose ϕ solves the Lichnerowicz equation with τ_2 and any σ and LW. Suppose a conformal factor ψ transforming the scalar curvature to $-\kappa\tau_1^2$ exists, and that $\psi - \psi \in W^{2,p}_{\delta}$. Then $\phi \geq \psi$.*

Proof. Let $\phi/\psi = \widetilde{\phi}$. By conformal covariance 1.1.1, $\widetilde{\phi}$ solves

$$-a\widetilde{\Delta}\widetilde{\phi} - \kappa\tau_1^2\widetilde{\phi} + \kappa\tau_2^2\widetilde{\phi}^{q-1} - \psi^{-2}\left|\sigma + \frac{1}{2N}\widetilde{L}W\right|^2_{\psi^{q-2}g}\widetilde{\phi}^{-q-1} - \psi^{-\frac{3}{2}q+1}r\widetilde{\phi}^{-q/2} = 0, \quad (3.8)$$

where $\widetilde{\Delta}$ and $\widetilde{L}$ are operators with respect to $\psi^{q-2}g$.

Suppose, by way of contradiction, that $\phi < \psi$ somewhere. Thus $\widetilde{\phi} < 1$ somewhere. Since $\widetilde{\phi} \to 1$ at infinity, it must have a global minimum at some point $p \in M$. On some small ball $B(p)$ around p, $\widetilde{\phi} < 1$, and so

$$-\kappa\tau_1^2\widetilde{\phi} + \kappa\tau_2^2\widetilde{\phi}^{q-1} \leq 0 \quad (3.9)$$

on $B(p)$. Clearly, then, $-\widetilde{\Delta}\widetilde{\phi} \geq 0$ on $B(p)$.

Let

$$v = (\widetilde{\phi} - \inf_{\partial B(p)} \widetilde{\phi})^- := \min\{0, \widetilde{\phi} - \inf_{\partial B(p)} \widetilde{\phi}\} \leq 0. \quad (3.10)$$

Since $\widetilde{\phi}(p)$ is a global minimum, $v = 0$ on $\partial B(p)$. Thus

$$\int_{B(p)} a|\nabla v|^2 = \int_{B(p)} -a\widetilde{\Delta}\widetilde{\phi}v \leq 0, \quad (3.11)$$

and so $v \equiv 0$ on $B(p)$. Thus $\inf \widetilde{\phi} \geq \inf_{\partial B(p)} \widetilde{\phi}$, and so $\widetilde{\phi}$ is constant on $B(p)$. By continuity, we can similarly argue that $\widetilde{\phi} < 1$ everywhere. This contradicts that $\widetilde{\phi} \to 1$ at infinity. Thus $\widetilde{\phi} \geq 1$, and so $\phi \geq \psi$. $\square$

Using a prescribed scalar curvature result from Chapter IV, we can show that sequences of solutions with $\tau_i \to 0$ blow up.

Theorem 3.1.3. *Suppose we have seed data $(g, \tau_i, N, \sigma, r, J)$ as in (3.1). Suppose g is Yamabe non-positive, and has no conformal Killing fields. Suppose (ϕ_i, W_i) are* *solutions to the conformal constraint equations* (1.3) *for the data with* τ_i*. If* $\tau_i \to 0$ *in* $C^0_{\delta-1} \cap W^{1,p}_{\delta-1}$*, then* $\sup \phi_i \to \infty$*.*

Proof. Since $\tau_i \to 0$ in $C^0_{\delta-1}$, $\kappa\tau_i^2 \leq C\rho^{2\delta-2}$ for some $C > 0$. Since the τ_i are admissible, Lemma 4.3.3 shows that there exists a conformal factor ψ transforming g to a metric with scalar curvature $-C\rho^{2\delta-2}$. Then, by Lemma 3.1.2, $\phi_i \geq \psi > 0$ for all i. Thus the ϕ_i are uniformly bounded below.

Suppose the ϕ_i are bounded above. Then $\phi_i^q d\tau_i \to 0$ in $L^p_{\delta-2}$. By the continuity of the vector equation, and since there are no conformal Killing fields on g (cf. Proposition 2.1.4), $W_i \to 0$ in $W^{2,p}_\delta$.

Since the ϕ_i are bounded above and below,

$$F(\phi_i) := R\phi_i + \tau_i^2\phi_i^{q-1} - \left|\sigma + \frac{1}{2N}LW_i\right|^2 \phi_i^{-q-1} - r\phi_i^{-q/2} \tag{3.12}$$

is bounded in $L^p_{\delta-2}$. Since ϕ_i solves the Lichnerowicz equation (1.3a), estimate (2.7) shows that the ϕ_i are uniformly bounded in $W^{2,p}_\delta$. By compact embedding, ϕ_i converge strongly in L^∞ to some ϕ_∞. Thus $F(\phi_i)$ converges strongly in $L^p_{\delta-2}$ to $F(\phi_\infty)$. This in turn shows that ϕ_i converges in $W^{2,p}_\delta$ to ϕ_∞.

However, since $\tau_i \to 0$, $\tau_i^2\phi_i^{q-1} \to 0$. Also, $LW_i \to LW_\infty$, where LW_∞ is the solution of

$$\operatorname{div}\frac{1}{2N}LW_\infty = J. \tag{3.13}$$

Thus

$$\left|\sigma + \frac{1}{2N}LW_i\right|^2 \phi_i^{-q-1} \to \left|\sigma + \frac{1}{2N}LW_\infty\right|^2 \phi_\infty^{-q-1}. \tag{3.14}$$

Since ϕ_i converge in $W^{2,p}_\delta$, ϕ_∞ solves

$$-a\Delta\phi_\infty + R\phi_\infty - \left|\sigma + \frac{1}{2N}LW_\infty\right|^2 \phi_\infty^{-q-1} - r\phi_\infty^{-q/2} = 0. \tag{3.15}$$

which is impossible because, in this equation, $\tau \equiv 0$, which is not admissible. Thus ϕ_i cannot be bounded above. $\square$

Theorem 3.1.4. *Suppose we have seed data* $(g, \tau_i, N, \sigma, r, J)$ *as in* (3.1)*. Suppose g is Yamabe non-positive, and has no conformal Killing fields. Suppose* (ϕ_i, W_i) *are solutions to the conformal constraint equations* (1.3) *for the data with τ_i. If $\tau_i \to 0$ in $C^0_{\delta-1} \cap W^{1,p}_{\delta-1}$, and $\tau_i \geq \tau_{i+1}$, then for any choice of radial function $\rho \geq 1$ and for any $p > n$, one of the following occurs:*

- *for all $\eta \in (2-n, 0)$, $\|\tau_i^2 \phi_i^{q-1}\|_{L^p_{\eta-2}}$ is unbounded.*
- *for all $\eta \in \mathbb{R}$, $\|\phi_i\|_{L^p_\eta}$ is unbounded.*

Proof. Since the τ_i are admissible, let ψ_i be the conformal factors transforming g to a metric with scalar curvature $-\kappa\tau_i^2$. Suppose, by way of contradiction, that both $\|\tau_i^2\psi_i^{q-1}\|_{L^p_{\eta-2}}$ and $\|\psi_i\|_{L^p_{\eta'}}$ are bounded, for some choices of p, η, η' and radial function ρ. By the estimate (2.7),

$$\|\psi_i\|_{W^{2,p}_\eta} \leq C\|\tau_i^2\psi_i^{q-1}\|_{L^p_{\eta-2}} + C\|\psi_i\|_{L^p_{\eta'}}, \tag{3.16}$$

which is bounded by assumption. Since ψ_i is uniformly bounded in $W^{2,p}_\eta$, by Sobolev embedding 2.0.1 a subsequence, which we also denote by ψ_i, converges in C^0. Mirroring the proof of Theorem 3.1.3, we can find a limit ψ_∞ which again contradicts that g is Yamabe nonpositive. Thus either $\|\tau_i^2\psi_i^{q-1}\|_{L^p_{\eta-2}}$ or $\|\psi_i\|_{L^p_{\eta'}}$ is unbounded.

By Lemma 3.1.2, $\phi_i \geq \psi_i$, and so ϕ_i must be unbounded in the same way as ψ_i. $\square$

CHAPTER IV

PRESCRIBED SCALAR CURVATURE AND YAMABE CLASSES

This work is (lightly) adapted from a paper posted to arXiv.com [DM15] in March, 2015. David Maxwell and I collaborated on the original research, and wrote the paper together. This paper is quoted with his permission.

One formulation of the prescribed scalar curvature problem asks: given Riemannian manifold (M^n, g) and some function R', is there a conformally related metric g' with scalar curvature R'? If we define $g' = \phi^{N-2} g$ for $N := \frac{2n}{n-2}$, [1] this is equivalent to finding a positive solution of

$$-a\Delta\phi + R\phi = R'\phi^{N-1}. \tag{4.1}$$

On a compact manifold the Yamabe invariant of the conformal class of g poses an obstacle to the solution of (4.1). For example, in the case where M is connected and R' is constant, problem (4.1) is known as the Yamabe problem, and it admits a solution if and only if the sign of the Yamabe invariant agrees with the sign of R' [Yam60][Tru68][Aub76][Sch84]. More generally, if R' has constant sign, we can conformally transform to a metric with scalar curvature R' only if the sign of the Yamabe invariant agrees with the sign of the scalar curvature. Hence it is natural to divide conformal classes into three types, Yamabe positive, negative, and null, depending on the sign of the Yamabe invariant.

We are interested in solving equation (4.1) on a class of complete Riemannian manifolds that, loosely speaking, have a geometry approximating Euclidean space

[1] In this chapter, and this chapter alone, we use the notation $N := \frac{2n}{n-2}$ instead of q. Since the lapse N does not appear in the chapter, there should be no confusion about the two N's.

at infinity. These asymptotically Euclidean (AE) manifolds also possess a Yamabe invariant, but the relationship between the Yamabe invariant and problem (4.1) was not, up until this work, well understood in the AE setting, except for some results concerning Yamabe positive metrics. We have the following consequences of [Max05] Proposition 3.

1. An AE metric can be conformally transformed to an AE metric with zero scalar curvature if and only if it is Yamabe positive. As a consequence, since the scalar curvature of an AE metric decays to zero at infinity, only Yamabe positive AE metrics can be conformally transformed to have constant scalar curvature.

2. Yamabe positive AE metrics have conformally related AE metrics with everywhere positive scalar curvature, and conformally related AE metrics with everywhere negative scalar curvature.

3. If an AE metric admits a conformally related metric with non-negative scalar curvature, then it is Yamabe positive.

Note that it was at one time believed that transformation to zero scalar curvature is possible if and only if the manifold is Yamabe non-negative [CB81]. The proof of this contention in [CB81] contains an error, and the statement and proof were corrected in [Max05]. See also [Fri11], which shows that there exist Yamabe-null AE manifolds and hence the hypotheses of [CB81] and [Max05] are genuinely different.

As a consequence of the above three facts, the situation on an AE manifold is somewhat different from the compact setting. In particular, although positive scalar curvature is a hallmark of Yamabe positive metrics, negative scalar curvature does not characterize Yamabe-negative metrics. Indeed, reporting joint work with David Maxwell, we show in this chapter that given an AE metric g, and a strictly negative function R' that decays to zero suitably at infinity, the conformal class of g includes a metric with scalar curvature equal to R' regardless of the sign of the Yamabe invariant. So every strictly negative scalar curvature is attainable for every conformal class, but zero scalar curvature is attainable only for Yamabe positive metrics. Thus we are lead to investigate the role of the Yamabe class in the boundary case of prescribed non-positive scalar curvature.

Rauzy treated the analogous problem on smooth compact Riemannian manifolds in [Rau95], which contains the following statement. Suppose $R' \le 0$ and $R' \not\equiv 0$. Observe that if R' is the scalar curvature of a metric conformally related to some g, then g must be Yamabe-negative, and without loss of generality we assume that g has constant negative scalar curvature R. Then there is a metric in the conformal class of g with scalar curvature R' if and only if

$$a\lambda_{R'} > -R \tag{4.2}$$

where a is the constant from equation (4.1) and where

$$\lambda_{R'} = \inf\left\{ \frac{\int |\nabla u|^2}{\int u^2} : u \in W^{1,2}, u \ge 0, u \not\equiv 0, \int R'u = 0 \right\}. \tag{4.3}$$

Rauzy's condition (4.2) is not immediately applicable on asymptotically Euclidean manifolds, in part because of the initial transformation to constant negative scalar curvature. However, recalling that R is constant we can write $a\lambda_{R'} + R$ as the infimum of

$$\frac{\int a|\nabla u|^2 + Ru^2}{\int u^2} \tag{4.4}$$

over functions u supported in the region where $R' = 0$. So, in effect, inequality (4.2) expresses the positivity of the first eigenvalue of the conformal Laplacian of the constant scalar curvature metric g on the region $\{R' = 0\}$. The connection between the first eigenvalue of the conformal Laplacian and prescribed scalar curvature problems is well known, but its use is more technical on non-compact manifolds where true eigenfunctions need not exist. For example, [FCS80] shows that a metric on a noncompact manifold can be conformally transformed to a scalar flat one if and only if the first eigenvalue of the conformal Laplacian is positive on every bounded domain.

In this chapter, following [DM15], we extend these ideas in a number of ways to solve the prescribed non-positive scalar curvature problem on asymptotically Euclidean manifolds, and we obtain a related characterization of the Yamabe class of an AE metric. In particular, we show the following.

- Every measurable subset $V \subseteq M$ can be assigned a number $y(V)$ that generalizes the Yamabe invariant of a manifold. The invariant depends on the conformal class of the AE metric, but is independent of the conformal representative.

- We can assign every measurable subset $V \subseteq M$ a number $\lambda_\delta(V)$ that generalizes the first eigenvalue of the conformal Laplacian. These numbers are not conformal invariants, and are not even canonically defined as they depend on a choice of parameters (a number δ and a choice of weight function at infinity). Nevertheless the sign of $\lambda_\delta(V)$ agrees with the sign of $y(V)$, regardless of the choice of these parameters.

- Given an AE metric g and a candidate scalar curvature $R' \leq 0$, there is a metric in the conformal class of g with scalar curvature equal to R' if and only if $\{R' = 0\}$ is Yamabe positive; i.e., $y(\{R' = 0\}) > 0$.

- A metric is Yamabe positive if and only if for every function $R' \leq 0$ there is a conformally related metric with scalar curvature equal to R'.

- A metric is Yamabe null if and only if for every function $R' \leq 0$, except for $R' \equiv 0$, there is a conformally related metric with scalar curvature equal to R'.

- A metric is Yamabe negative if and only if there is a function $R' \leq 0$, $R' \not\equiv 0$, such that no conformally related metric has scalar curvature equal to R'. We also present some results concerning which scalar curvatures have Yamabe positive zero sets.

- Additionally, a metric is Yamabe positive/negative/null if and only if it admits a conformal compactification to a metric with the same Yamabe type.

These results carry over to compact manifolds, where we obtain some technical improvements. First, Rauzy's condition (4.2) is equivalent to our condition $y(\{R' = 0\}) > 0$ (or equivalently $\lambda_\delta(\{R' = 0\}) > 0$). But the condition $y(\{R' < 0\}) > 0$ can be checked without reference to a particular background metric. Moreover, we work with fairly general metrics ($W^{2,p}_{\mathrm{loc}}$ with $p > n/2$), and candidate scalar curvatures in $L^p(M)$. Finally, there is an error in Rauzy's proof, closely related to the gap in Yamabe's original attempt at the Yamabe problem, that we correct in our presentation. [2]

[2] We thank Rafe Mazzeo for useful conversations concerning this correction.

The prescribed scalar curvature problem on AE manifolds for $R' \geq 0$, or for functions R' which change sign, remains open. Of course if $R' \geq 0$ the problem can only be solved if the manifold is Yamabe positive, but it is not known the extent to which this condition is sufficient. For prescribed scalar curvatures that change sign, little is known for any Yamabe class. Nevertheless, the case $R' \leq 0$ that we treat here has an interesting application to general relativity; see below. For comparison, we note that the prescribed scalar curvature problem on a compact manifold is also not yet fully solved. On a Yamabe-positive manifold it is necessary that $R' > 0$ somewhere, and on a Yamabe-null manifold it is necessary that either $R' \equiv 0$, or $R' > 0$ somewhere and $\int R' < 0$ when computed with respect to the scalar flat conformal representative. See [ES86] which shows that these conditions are sufficient in some cases. See also [BE87] for obstructions posed by conformal Killing fields.

4.1. Asymptotically Euclidean Manifolds

We mention a few extensions of what has been discussed above in Chapter II which will be applicable in this chapter. We will work exclusively with $W^{2,p}_{\alpha}$ AE metrics with $p > n/2$, and we henceforth assume

$$p > n/2 \quad \text{and} \quad \alpha < 0. \tag{4.5}$$

A $W^{2,p}_{\alpha}$ metric is Hölder continuous and has curvatures in $L^p_{\alpha-2}$.

The Laplacian and conformal Laplacian of a $W^{2,p}_{\alpha}$ metric are well-defined as maps from $W^{2,q}_{\delta}$ to $L^q_{\delta-2}$ for $q \in (1,p]$, they are Fredholm with index 0 if $\delta \in (2-n,0)$, and indeed the Laplacian is an isomorphism in this range; see, e.g., [Bar86]

Proposition 2.2. Note that [Bar86] works on a manifold diffeomorphic to R^n, but the results we cite from [Bar86] extend to manifolds with general topology and any finite number of ends.

Many of the results in this chapter hold for both asymptotically Euclidean and compact manifolds, and indeed we can often treat a $W^{2,p}$ metric on a compact manifold as a $W^{2,p}_\alpha$ metric on an asymptotically Euclidean manifold with zero ends, in which case the weight function ρ is irrelevant and could be set to 1 if desired. For the sake of brevity, throughout Section 4.2 we interpret a compact manifold as an AE manifold with zero ends. In the remaining sections there are differences between the two cases and we assume that AE manifolds have at least one end.

The weight parameter

$$\delta^* = \frac{2-n}{2} \tag{4.6}$$

plays a prominent role in this chapter, and it reflects the minimum decay needed to ensure $\int |\nabla u|^2$ is finite. At this decay rate, $L^N_{\delta^*} = L^N$ and we have the inequalities that generalize the Poincaré and Sobolev inequalities, Lemma 2.0.3.

Lemma 2.0.3 evidently fails on compact manifolds, as can be seen by taking u to be a constant. For our proofs that treat the compact and non-compact case simultaneously it will be helpful to have a suitable inequality that works in both settings. Observe that for any $\delta > 0$ there exists c_2 such that

$$\|u\|_{2,\delta} + \|\nabla u\|_2 \geq c_2 \|u\|_N. \tag{4.7}$$

This follows from the standard Sobolev inequality on compact manifolds and follows trivially from inequality (2.5) on non-compact manifolds. Recall, again, that in this chapter, $N := \frac{2n}{n-2}$.

4.2. The Yamabe Invariant of a Measurable Set

Throughout this section, let (M, g) be a $W^{2,p}_\alpha$ AE manifold with $p > n/2$ and $\alpha < 0$, with the convention that a compact manifold is an AE manifold with zero ends. For $u \in C_c^\infty(M)$ (i.e., smooth functions of compact support), $u \not\equiv 0$, the Yamabe quotient of u is

$$Q^y_g(u) = \frac{\int a|\nabla u|^2 + Ru^2}{\|u\|_N^2} \tag{4.8}$$

and the Yamabe invariant of g is the infimum of Q^y_g taken over $C_c^\infty(M)$. Here and in other notations we drop the decoration g when the metric is understood. Our principal goal in this section is to define a similar conformal invariant for arbitrary measurable subsets of M and to analyze its properties.

It will be convenient to work with a complete function space, and we claim that the domain of Q^y can be extended to $W^{1,2}_{\delta^*} \setminus \{0\}$ where δ^* is defined in equation (4.6). To see this, first note from the embedding properties of weighted Sobolev spaces that $W^{1,2}_{\delta^*}$ embeds continuously in $L^N = L^N_{\delta^*}$ and that $u \mapsto \nabla u$ is continuous from $W^{1,2}_{\delta^*}$ to L^2; indeed δ^* is the minimum decay needed to ensure these conditions. To treat the scalar curvature term in Q^y, we have the following.

Lemma 4.2.1. *The map*

$$u \mapsto \int Ru^2 \tag{4.9}$$

is weakly continuous on $W^{1,2}_{\delta^*}$. Moreover, for any $\delta > \delta^*$ and $\epsilon > 0$, there is constant $C > 0$ such that

$$\left| \int Ru^2 \right| \le \epsilon \|\nabla u\|_2^2 + C\|u\|^2_{2,\delta}. \tag{4.10}$$

Proof. Recall that $R \in L^p_{\alpha-2}$ where $p > n/2$ and $\alpha < 0$. So there is an $s \in (0,1)$ such that

$$\frac{1}{p} = s\frac{2}{n}. \tag{4.11}$$

Set $\sigma = \delta^* - \alpha/2$. Since $s < 1$ and $\sigma > \delta^*$, $W^{1,2}_{\delta^*}$ embeds compactly in $W^{s,2}_{\sigma}$, where the interpolation (Sobolev) space $W^{s,2}_{\sigma}$ is described in [Tri76a][Tri76b]. Moreover, $W^{s,2}_{\sigma}$ embeds continuously in L^q_{σ} where

$$\frac{1}{q} = \frac{1}{2} - \frac{s}{n} = \frac{1}{2}\left(1 - \frac{1}{p}\right). \tag{4.12}$$

Since

$$\frac{1}{p} + \frac{2}{q} = 1 \tag{4.13}$$

and since

$$\alpha - 2 + 2\sigma = 2\delta^* - 2 = -n, \tag{4.14}$$

Hölder's inequality implies the map (4.9) is continuous on L^q_{σ}, and from the previously mentioned compact embedding the map (4.9) is therefore weakly continuous on $W^{1,2}_{\delta^*}$. Moreover, Hölder's inequality implies there is a constant C such that

$$\left|\int Ru^2\right| \leq C\|u\|^2_{W^{s,2}_{\sigma}}. \tag{4.15}$$

From interpolation [Tri76b] we have

$$\|u\|_{W^{s,2}_{\sigma}} \leq C\|u\|^s_{W^{1,2}_{\delta^*}}\|u\|^{1-s}_{2,\delta} \tag{4.16}$$

where δ satisfies

$$s\delta^* + (1-s)\delta = \sigma. \tag{4.17}$$

Since $\sigma = \delta^* - \alpha/2$, we find

$$\delta = \delta^* - \frac{\alpha/2}{1-s}, \tag{4.18}$$

and since $\alpha < 0$ and $s \in (0,1)$, $\delta > \delta^*$. Indeed, by raising α close to zero, or lowering p close to $n/2$ (which raises s up to 1), we can obtain any particular $\delta > \delta^*$. We conclude from inequalities (4.15), (4.16) and the arithmetic-geometric mean inequality that

$$\left| \int Ru^2 \right| \le \epsilon \|\nabla u\|^2_{W^{1,2}_{\delta^*}} + C\|u\|^2_{2,\delta}. \tag{4.19}$$

This establishes inequality (4.10) on a compact manifold, and we obtain (4.10) in the non-compact case by applying the Poincaré inequality (2.4). □

Corollary 4.2.2. *The map*

$$u \mapsto \int a|\nabla u|^2 + Ru^2 \tag{4.20}$$

is weakly upper semicontinuous on $W^{1,2}_{\delta^*}$.

Proof. This follows from the weak upper semicontinuity of $u \mapsto \int |\nabla u|^2$ along with Lemma 4.2.1. □

Definition 4.2.3. *Let $V \subseteq M$ be a measurable set. The test functions supported in V are*

$$A(V) := \left\{ u \in W^{1,2}_{\delta^*}(M) : u \not\equiv 0, u|_{V^c} = 0 \right\}, \tag{4.21}$$

where V^c is the complement of V.

Definition 4.2.4. *Let $V \subseteq M$ be measurable. The Yamabe invariant of V is*

$$y_g(V) = \inf_{u \in A(V)} Q^y(u). \tag{4.22}$$

If V has measure zero, and hence $A(V)$ is empty, we use the convention $y_g(V) = \infty$.

Since $C_c^\infty(M)$ is dense in $W^{1,2}_{\delta^*}(M)$, for $V = M$, this agrees with the usual definition of the Yamabe invariant.

In principle, the infimum in the definition of the Yamabe invariant could be $-\infty$. The following estimate, which will be useful later in the paper as well, shows that this is not possible.

Lemma 4.2.5. *Let $\delta \in \mathbb{R}$. There exist positive constants C_1 and C_2 such that for all $u \in W^{1,2}_{\delta^*}$,*

$$\|u\|_{W^{1,2}_{\delta^*}} \leq C_1 \left[\int a|\nabla u|^2 + Ru^2 \right] + C_2 \|u\|^2_{2,\delta}. \tag{4.23}$$

Proof. It is enough to establish inequality (4.23) assuming $\delta > \delta^*$. From Lemma 4.2.1, there is a constant C such that

$$\left| \int Ru^2 \right| \leq \frac{a}{2} \int |\nabla u|^2 + C\|u\|^2_{2,\delta} \tag{4.24}$$

and hence

$$\int a|\nabla u|^2 + Ru^2 \geq \frac{a}{2} \int |\nabla u|^2 - C\|u\|^2_{2,\delta}. \tag{4.25}$$

Consequently

$$\int |\nabla u|^2 \leq \frac{2}{a} \left[\int a|\nabla u|^2 + Ru^2 \right] + \frac{2C}{a} \|u\|^2_{2,\delta}. \tag{4.26}$$

Inequality (4.23) now follows trivially in the compact case, and follows from the Poincaré inequality (2.4) in the non-compact case. □

Lemma 4.2.6. *For every measurable set V, $y(V) > -\infty$.*

Proof. Let u_k be some minimizing sequence for Q^y normalized so that $\|u_k\|_N = 1$. Lemma 4.2.5 and the continuous embedding $L^N \hookrightarrow L^2_\delta$ implies that u_k is uniformly bounded in $W^{1,2}_{\delta^*}$. Estimate (4.10) then implies that $Q(u_k)$ is uniformly bounded below. $\square$

As one might expect, $y(V)$ is a conformal invariant.

Lemma 4.2.7. *Suppose $g' = \phi^{N-2} g$ is a conformally related metric with $\phi - 1 \in W^{2,p}_{\alpha}$. Then*

$$y_{g'}(V) = y_g(V). \tag{4.27}$$

Proof. The conformal transformation laws

$$\begin{aligned} dV_{g'} &= \phi^N dV_g \\ R_{g'} &= \phi^{1-N}(-a\Delta_g \phi + R_g \phi) \end{aligned} \tag{4.28}$$

together with an integration by parts imply

$$\int_M |\nabla u|^2_{g'} + R_{g'} u^2 \; dV_{g'} = \int_M |\nabla(\phi u)|^2_g + R_g (\phi u)^2 \; dV_g \tag{4.29}$$

for all $u \in W^{1,2}_{\delta^*}(M)$. Since $\| \cdot \|_{g',N} = \|\phi \cdot \|_{g,N}$, it follows that

$$Q^y_{g'}(u) = Q^y_g(\phi u) \tag{4.30}$$

for all $u \in W^{1,2}_{\delta^*}(M)$ as well. Since $A(V)$ is invariant under multiplication by ϕ, $y_{g'}(V) = y_g(V)$. $\square$

We will primarily be interested in the sign of the Yamabe invariant.

Definition 4.2.8. *A measurable set $V \subseteq M$ is called Yamabe positive, negative, or null depending on the sign of $y_g(V)$.*

The Yamabe invariant involves the critical Sobolev exponent N and hence can be technically difficult to work with. On a compact manifold, however, the sign of the Yamabe invariant can be determined from the sign of the first eigenvalue of the conformal Laplacian. These eigenvalues enjoy superior analytical properties (for instance, it is simpler to show that the related eigenfunctions exist), and we now describe how to extend this approach to measurable subsets of compact or asymptotically Euclidean manifolds.

For $\delta > \delta^*$ we define the Rayleigh quotients

$$Q_{g,\delta}(u) = \frac{\int a|\nabla u|^2 + Ru^2}{\|u\|_{2,\delta}^2}. \tag{4.31}$$

Our previous arguments for the Yamabe quotient imply that $Q_{g,\delta}$ is well-defined for any $u \in W^{1,2}_{\delta^*} \setminus \{0\}$, and indeed $Q_{g,\delta}$ is continuous on this set.

Definition 4.2.9. *The first δ-weighted eigenvalue of the conformal Laplacian is*

$$\lambda_{g,\delta}(V) = \inf_{u \in A(V)} Q_{g,\delta}(u). \tag{4.32}$$

By convention, if V has measure zero then $\lambda_{g,\delta}(V) = \infty$. We will write Q_δ and λ_δ when the metric is understood.

The value of $\lambda_\delta(V)$ is not particularly meaningful; it depends on the choice of weight function ρ and it is not a conformal invariant. Nevertheless, its sign is a conformal invariant independent of the choice of ρ.

Proposition 4.2.10. *For any measurable set $V \subseteq M$, the following are equivalent:*

1. $y(V) > 0$.

2. $\lambda_\delta(V) > 0$ *for all* $\delta > \delta^*$.

3. $\lambda_\delta(V) > 0$ *for some* $\delta > \delta^*$.

Proof. We assume that V has positive measure since the equivalence is trivial otherwise. The implication $1 \Rightarrow 2$ follows from the inequality $\|u\|_{2,\delta} \le C\|u\|_N$ applied to Q^y. The implication $2 \Rightarrow 3$ is trivial. So it remains to show that $3 \Rightarrow 1$.

Let V be a measurable set with $\lambda_\delta(V) > 0$ for some $\delta > \delta^*$. Suppose to produce a contradiction that $y(V) \le 0$. Then there is a sequence $u_k \in A(V)$, normalized so that $\int a|\nabla u_k|^2 + \|u_k\|_{2,\delta}^2 = 1$, such that $Q^y(u_k) \le 1/k$. Then

$$\lambda_\delta(V)\|u_k\|_{2,\delta}^2 \le \int a|\nabla u_k|^2 + Ru_k^2 \le \frac{1}{k}\|u_k\|_N^2 \le \frac{c}{k}\left[\int a|\nabla u_k|^2 + \|u_k\|_{2,\delta}^2\right] \le \frac{c}{k} \tag{4.33}$$

by the Sobolev inequality (4.7). In particular, $\|u_k\|_{2,\delta}^2 \to 0$. Using inequality (4.33), we also find that

$$\int Ru_k^2 \le \frac{c}{k} - \int a|\nabla u|^2 \to -1. \tag{4.34}$$

However, by Lemma 4.2.1, there exists $C > 0$ such that

$$\left|\int Ru_k^2\right| \le \frac{a}{2}\|\nabla u_k\|_2^2 + C\|u_k\|_{2,\delta}^2 \to \frac{1}{2}, \tag{4.35}$$

which is a contradiction. $\square$

Corollary 4.2.11. *For a measurable set* $V \subseteq M$*, the signs of* $y(V)$ *and* $\lambda_\delta(V)$ *are the same for any* $\delta > \delta^*$.

Proof. Proposition 4.2.10 shows that $y(V)$ is positive if and only if $\lambda_\delta(V)$ is also. Choosing an appropriate test function shows that $y(V)$ is negative if and only if $\lambda_\delta(V)$ is also. Together, these imply that $y(V)$ is zero if and only if $\lambda_\delta(V)$ is. $\square$

The decay rate δ^* is critical for Corollary 4.2.11. For $\delta < \delta^*$, $W^{1,2}_{\delta^*}$ is not contained in L^2_δ and hence our definition of λ_δ does not extend to this range. One could minimize Q_δ over smooth functions instead to define λ_δ, but using rescaled bump functions on large balls as test functions, it can be shown that $\lambda_\delta(\mathbb{R}^n) = 0$ for $\delta < \delta^*$, despite the fact that Lemma 2.0.3 implies $y(\mathbb{R}^n) > 0$. Note that we have not addressed equality in the threshold case $\delta = \delta^*$.

We now turn to continuity properties of λ_δ. Monotonicity is obvious from the definition.

Lemma 4.2.12. *Let $\delta > \delta^*$. If V_1 and V_2 are measurable sets with $V_1 \subseteq V_2$, then $\lambda_\delta(V_1) \geq \lambda_\delta(V_2)$.*

Note that Lemma 4.2.12 holds even for $V_1 = \emptyset$, and that this relies on our definition $\lambda_\delta(\emptyset) = y(\emptyset) = \infty$. To obtain more refined properties of λ_δ, we start by showing that minimizers of the Rayleigh quotients exist and are generalized eigenfunctions.

Proposition 4.2.13. *Let V be a measurable set with positive measure and let $\delta > \delta^*$. There exists a non-negative $u \in A(V)$ that minimizes Q_δ over $A(V)$. Moreover, on any open set contained in V,*

$$-a\,\Delta\,u + Ru = \lambda_\delta(V)\rho^{2(\delta^*-\delta)}u. \tag{4.36}$$

Proof. Let u_k be a minimizing sequence in $A(V)$; this uses the hypothesis that V has positive measure. Without loss of generality we may assume that each $\|u_k\|_{2,\delta} = 1$. Since

$$a\int_M |\nabla u_k|^2 + Ru_k^2 = Q_\delta(u_k), \tag{4.37}$$

and since u_k is a minimizing sequence, Lemma 4.2.5 implies $\{u_k\}$ is bounded in $W^{1,2}_{\delta^*}(M)$ and hence converges weakly in $W^{1,2}_{\delta^*}(M)$ and strongly in $L^2_\delta(M)$ to a limit $u \in W^{1,2}_{\delta^*}(M)$ with $\|u\|_{2,\delta} = 1$. Since each $u_k = 0$ on V^c, from the strong L^2_δ convergence we see $u = 0$ on V^c, and since $u \not\equiv 0$ we conclude that $u \in A(V)$. Weak upper semicontinuity (Corollary 4.2.2) implies that u minimizes Q_δ over the test functions $A(V)$. Noting that $|u|$ is also a minimizer, we may assume $u \geq 0$.

Suppose V contains an open set Ω. Then any $\phi \in C^\infty_c(\Omega)$ with $\phi \not\equiv 0$ belongs to $A(V)$, and we can differentiate $Q_\delta(u + t\phi)$ at $t = 0$ to find that u is a weak solution in Ω of equation (4.36). $\square$

Lemma 4.2.14 (Continuity from above)**.** *Let $V \subseteq M$ be a measurable set. If $\{V_k\}$ is a decreasing sequence of measurable sets with $\cap V_k = V$, then*

$$\lim_{k\to\infty} \lambda_\delta(V_k) = \lambda_\delta(V). \tag{4.38}$$

Proof. From the elementary monotonicity of λ_δ, $\Lambda = \lim_{k\to\infty} \lambda_\delta(V_k)$ exists and

$$\lambda_\delta(V_k) \leq \Lambda \leq \lambda_\delta(V) \tag{4.39}$$

for each k. So it is enough to show that

$$\Lambda \geq \lambda_\delta(V). \tag{4.40}$$

We may assume that Λ is finite, for inequality (4.40) is trivial otherwise. As a consequence, each V_k is nonempty and Proposition 4.2.13 provides minimizers u_k

of Q_δ over $A(V_k)$ satisfying $\|u_k\|_{2,\delta} = 1$. For each k, since $\|u_k\|_{2,\delta} = 1$,

$$\int a|\nabla u_k|^2 + Ru_k^2 \leq \Lambda. \tag{4.41}$$

From inequality (4.41) and the boundedness of the sequence in $L^2_\delta(M)$, Lemma 4.2.5 implies that the sequence is bounded in $W^{1,2}_{\delta^*}(M)$. A subsequence converges weakly in $W^{1,2}_{\delta^*}(M)$ and strongly in $L^2_\delta(M)$ to a limit v with $\|v\|_{2,\delta} = 1$. From weak upper semicontinuity (Corollary 4.2.2) we conclude that $Q_\delta(v) \leq \Lambda$ as well. Moreover, $v \in A(V)$ since $v = 0$ on V_k^c. So $\lambda_\delta(v) \leq \Lambda$. □

Note that Lemma 4.2.14 is false for the Yamabe invariant. For example, one can take a sequence of balls in $\mathbb{R}^n$ that shrink down to the empty set. It is easy to see that the Yamabe invariant is scale invariant and hence is a finite constant along the sequence. Yet the Yamabe invariant of the empty set is infinite. In contrast, if $V_n \searrow \emptyset$, Lemma 4.2.14 implies $\lambda_\delta(V_n) \to \infty$, and in particular at some point along the sequence $\lambda_\delta(V_n) > 0$. The following result, which is an extension of [Rau95] Lemma 2 to the AE setting, shows that in fact $\lambda_\delta(V)$ is positive so long as a certain weighted volume is sufficiently small.

Lemma 4.2.15 (Small sets are Yamabe positive)**.** *For any $\mu > n$, there exists $C > 0$ such that if $Vol_\mu(V) := \int_V \rho^{-\mu} < C$, V is Yamabe positive.*

Proof. Suppose that $u \in A(V)$. Define δ by $(-2\delta - n)\frac{n}{2} = -\mu$. Note that $\mu > n$ implies that $\delta > \delta^*$. Then, by Hölder's inequality,

$$\|u\|_{2,\delta}^2 = \int u^2 \rho^{-2\delta - n} \leq \left(\int u^N\right)^{2/N} \left(\int_V \rho^{(-2\delta-n)\frac{n}{2}}\right)^{2/n} = \|u\|_N^2 \mathrm{Vol}_\mu(V)^{2/n}. \tag{4.42}$$

By the Sobolev inequality (4.7), there exists C_1 such that

$$\|u\|_N^2 \leq C_1 \left[\int a|\nabla u|^2 + \|u\|_{2,\delta}^2 \right]. \tag{4.43}$$

We also note that Lemma 4.2.1 implies there exists C_2 such that

$$-C_2\|u\|_{2,\delta}^2 \leq \frac{1}{2}\int a|\nabla u|^2 + \int Ru^2. \tag{4.44}$$

Let η be defined by $\eta \mathrm{Vol}_\mu(V)^{2/n} C_1 = \frac{1}{2}$. Using inequalities (4.42)-(4.44), we calculate

$$\begin{aligned}
(\eta - C_2)\|u\|_{2,\delta}^2 &\leq \eta\|u\|_N^2 \mathrm{Vol}_\mu(V)^{2/n} + \int Ru^2 + \frac{1}{2}\int a|\nabla u|^2 \\
&\leq \eta \mathrm{Vol}_\mu(V)^{2/n} C_1 \left[\int a|\nabla u|^2 + \|u\|_{2,\delta}^2 \right] + \int Ru^2 + \frac{1}{2}\int a|\nabla u|^2 \\
&= \int \left(a|\nabla u|^2 + Ru^2\right) + \frac{1}{2}\|u\|_{2,\delta}^2.
\end{aligned} \tag{4.45}$$

Dividing through by $\|u\|_{2,\delta}^2$, inequality (4.45) reduces to

$$\eta - C_2 - \frac{1}{2} \leq Q_\delta(u). \tag{4.46}$$

As $\mathrm{Vol}_\mu(V) \to 0$, $\eta \to \infty$. Thus there is a $C > 0$ such that if $\mathrm{Vol}_\mu(V) < C$, then $Q_\delta(u)$ has a uniform positive lower bound for all $u \in A(V)$. Thus $\lambda_\delta(V) > 0$, and so V is Yamabe positive by Corollary 4.2.11. $\square$

In Section 4.4 below we discuss the relationship between the Yamabe invariant of an AE manifold and its compactification. After compactification, for $\mu = 2n$, the condition $\mathrm{Vol}_\mu(V) < C$ corresponds to the condition that the usual volume of the compactified set is sufficiently small. This is exactly Rauzy's condition, and the other choices of μ provide a mild generalization of his result.

Lemma 4.2.16 (Strict monotonicity at connected, open sets). *Let $\delta > \delta^*$ and let Ω be a connected open set. For any measurable set E in Ω with positive measure,*

$$\lambda_\delta(\Omega \setminus E) > \lambda_\delta(\Omega). \tag{4.47}$$

Proof. Let $V = \Omega \setminus E$. We may assume V has positive measure, for inequality (4.47) is trivial otherwise.

Suppose to the contrary that $\lambda_\delta(V) = \lambda_\delta(\Omega)$. Since V has positive measure, Proposition 4.2.13 provides a function $u \in A(V)$ with $Q_\delta(u) = \lambda_\delta(V)$. Hence u also is a minimizer of Q_δ over $A(\Omega)$, and Proposition 4.2.13 implies that u weakly solves

$$-a\Delta u + \left[R - \lambda_\delta \rho^{2(\delta^*-\delta)}\right] u = 0 \tag{4.48}$$

on Ω. Local regularity implies that $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$, and we may assume after adjusting u on a set of zero measure that u is continuous. Since E has positive measure, we can still conclude that u vanishes at some point in Ω. Following the argument of Lemma 4 from [Max05], we may apply the weak Harnack inequality of [Tru73b] to conclude that u vanishes everywhere on the connected set Ω, and hence on all of M. Since $u \in A(\Omega)$, this is a contradiction. □

The connectivity hypothesis in Lemma 4.2.16 is necessary to obtain strict monotonicity. For example, two disjoint unit balls in R^n have the same first eigenvalue as a single unit ball. On the other hand, the assumption that Ω is open is not optimal, and relaxing this condition would require a suitable replacement for the weak Harnack inequality.

Although we have not established continuity from below for λ_δ, it holds in certain cases. The following is a prototypical result that suffices for our purposes.

Lemma 4.2.17 (Continuity from below; prototype)**.** *Suppose V is measurable. Let $x_0 \in M$ and let $B_r(x_0)$ be the ball of radius r about x_0. Then for any $\delta > \delta^*$*

$$\lim_{r\to 0} \lambda_\delta(V \setminus B_r) = \lambda_\delta(V). \tag{4.49}$$

Proof. Let u be a function in $A(V)$ that minimizes Q_δ. Let χ_r be a radial bump function that equals 0 on $B_r(x_0)$, equals 1 outside $B_{2r}(x_0)$, and has its gradient bounded by $2/r$. Defining $u_r = \chi_r u$ we claim that $u_r \to u$ in $W^{1,2}_{\delta^*}(M)$. Assuming this for the moment, we conclude from the continuity of Q_δ that

$$\lambda_\delta(V) \le \lambda_\delta(V \setminus B_r) \le Q_\delta(u_r) \to Q_\delta(u) = \lambda_\delta(V) \tag{4.50}$$

and hence we obtain equality (4.49).

To show that $u_r \to u$ in $W^{1,2}_{\delta^*}$, since $u_r \to u$ in $L^2_{\delta^*}$, it is enough to show that $\int |\nabla(u-u_r)|^2 \to 0$. However,

$$\int |\nabla(u-u_r)|^2 \le 2\int (1-\chi_r)^2|\nabla u|^2 + u^2|\nabla(1-\chi_r)|^2. \tag{4.51}$$

The first term on the right-hand side of inequality (4.51) evidently converges to zero. For the second, we note from Hölder's inequality that

$$\int_{B_{2r}} u^2 \le \left[\int_{B_{2r}} u^N\right]^{\frac{2}{N}} \left[\int_{B_{2r}} 1\right]^{\frac{2}{n}} \le Cr^2 \left[\int_{B_{2r}} u^N\right]^{\frac{2}{N}}. \tag{4.52}$$

Since $u \in L^N_{\text{loc}}$, $\int_{B_{2r}} u^N \to 0$ as $r \to 0$. Since $\nabla(1 - \chi_r)$ is bounded by c/r, we conclude that the second term of the right-hand side of inequality (4.51) also converges to zero.

$\square$

4.3. Prescribed Non-Positive Scalar Curvature

In this section, we prove the following necessary and sufficient condition for an AE Riemannian manifold with at least one end to be conformally related to one which has scalar curvature equal to a specified nonpositive function.

Theorem 4.3.1. *Let (M^n, g) be a $W^{2,p}_\alpha$ AE manifold with $p > n/2$ and $\alpha \in (2-n, 0)$. Suppose $R' \in L^p_{\alpha-2}$ is non-positive. Then the following are equivalent:*

1. *There exists a positive function ϕ with $\phi - 1 \in W^{2,p}_\alpha$ such that the scalar curvature of $g' = \phi^{N-2} g$ is R'.*
2. *$\{R' = 0\}$ is Yamabe positive.*

For compact Yamabe negative manifolds we have the following analogous result. Since Rauzy's condition (4.2) is equivalent to the set $\{R' = 0\}$ being Yamabe positive, this theorem is a generalization to lower regularity and a correction of the proof of part of Theorem 1 in Rauzy's work [Rau95].

Theorem 4.3.2. *Let (M^n, g) be a $W^{2,p}$ compact Yamabe negative manifold with $p > n/2$. Suppose $R' \in L^p$ is non-positive. Then the following are equivalent:*

1. *There exists a positive function ϕ with $\phi \in W^{2,p}$ such that the scalar curvature of $g' = \phi^{N-2} g$ is R'.*
2. *$\{R' = 0\}$ is Yamabe positive.*

For the most part, the proof of Theorem 4.3.2 can be obtained from the proof of Theorem 4.3.1 by treating a compact manifold as an asymptotically Euclidean manifold with zero ends. So we focus on Theorem 4.3.1 and then present the few additional arguments needed to prove Theorem 4.3.2 at the end of the section.

Turning to Theorem 4.3.1, the proof that 1) implies 2) is short, so we delay it and concentrate on the direction 2) implies 1). Suppose that $\{R' = 0\}$ is Yamabe positive. We show that we can make the desired conformal change using a sequence of results proved over the remainder of this section. It suffices to work under the following simplifying hypotheses.

1. We may assume that the prescribed scalar curvature R' is bounded since Lemma 4.3.3, which we prove next, shows that we can the lower scalar curvature after first solving the problem for a scalar curvature that is truncated below.

2. We may assume $\{R' = 0\}$ contains a neighborhood of infinity, since continuity from above (Lemma 4.2.14) shows that we can truncate R' in a "small" neighborhood of infinity such that its zero set remains Yamabe positive, and we can subsequently lower the scalar curvature after solving the modified problem.

3. We may assume that the initial scalar curvature satisfies $R = 0$ in a neighborhood of infinity, since Lemma 4.3.4, which we prove below, shows that we can initially conformally transform to such a scalar curvature, and since the hypotheses of Theorem 4.3.1 are conformally invariant.

Lemma 4.3.3. *Suppose (M, g) is a $W^{2,p}_{\alpha}$ AE manifold with $p > n/2$ and $\alpha \in (2-n, 0)$. Suppose $R' \in L^p_{\alpha-2}$. If $R_g \geq R'$, then there exists a positive ϕ with $\phi - 1 \in W^{2,p}_{\alpha}$ such that $g' = \phi^{N-2} g$ has scalar curvature R'.*

Proof. We seek a solution to $-a\Delta\phi + R_g\phi = R'\phi^{q-1}$. Note that 0 is a subsolution and, since $R_g \geq R'$, 1 is a supersolution. By [Max05] Proposition 2, there exists a solution ϕ with $0 \leq \phi \leq 1$ and $\phi - 1 \in W^{2,p}_{\alpha}$. Since $\phi \geq 0$ solves $-a\Delta\phi + (R - R'\phi^{q-2})\phi = 0$, and since $\phi \to 1$ at infinity, the weak Harnack inequality [Tru73b] implies that ϕ is positive. $\square$

Lemma 4.3.4. *Suppose (M, g) is a $W^{2,p}_{\alpha}$ AE manifold with $p > n/2$ and $\alpha \in (2-n, 0)$. There exists $\phi > 0$ with $\phi - 1 \in W^{2,p}_{\alpha}$ such that the metric $g' = \phi^{N-2} g$ has zero scalar curvature on some neighborhood of infinity.*

Proof. We prove this result for a manifold with one end; the extension to several ends can be done by repeated application of our argument. Let E_r be the region outside the coordinate ball of radius r in end coordinates. By Lemma 4.2.15, $y(E_r) > 0$ for r large enough. Following Proposition 3 in [Max05] we claim that

$$-a\Delta + \eta R : \{u \in W^{2,p}_{\alpha}(E_R) : u|_{\partial E_r} = 0\} \to L^p_{\alpha-2}(E_R) \tag{4.53}$$

is an isomorphism for all $\eta \in [0, 1]$. Because we assume homogenous boundary conditions, the argument in Propositions 1.6 through 1.14 in [Bar86] showing that $-a\Delta + \eta R$ is Fredholm of index zero requires no changes except imposing the boundary condition. Suppose, then, to produce a contradiction, that there exists a nontrivial u in the kernel. An argument parallel to Lemma 3 in [Max05] implies that $u \in W^{2,p}_{\alpha'}$ for any $\alpha' \in (2-n, 0)$. In particular, the extension of u by zero to M belongs to $W^{1,2}_{\delta^*}(M)$ and hence also to $A(E_r)$. Integration by parts implies

$Q^y(u) = 0$, which contradicts the fact that E_r is Yamabe positive. Thus $-a\Delta + \eta R$ is an isomorphism.

Let u_η be the nontrivial solution in $\{u \in W^{2,p}_\alpha(E_r) : u|_{\partial E_r} = 0\}$ of

$$-a\Delta u_\eta + \eta R u_\eta = -\eta R. \tag{4.54}$$

Then $\phi_\eta := u_\eta + 1$ solves

$$-a\Delta \phi_\eta + \eta R \phi_\eta = 0 \tag{4.55}$$

on E_r. Let $I = \{\eta \in [0,1] : \phi_\eta > 0\}$. Since $\phi_0 \equiv 1$, I is nonempty. The set of solutions u_η such that $u_\eta > -1$ is open in $W^{2,p}_\alpha \subset C^0_\alpha$. Thus, by the continuity of the map $\eta \mapsto u_\eta$, I is open. Suppose $\eta_0 \in \overline{I}$. If $\phi_{\eta_0} = 0$ somewhere, the weak Harnack inequality [Tru73b] implies that $\phi_{\eta_0} \equiv 0$, which contradicts the fact that $\phi_{\eta_0} \to 1$ at infinity. Thus $\phi_{\eta_0} > 0$ on E_r, and so I is closed. Thus $I = [0,1]$, and $\phi_1 > 0$. We set ϕ to be an arbitrary positive $W^{2,p}_\alpha$ extension of $\phi_1|_{E_r}$; ϕ satisfies the properties claimed in this lemma. $\square$

Consider the family of functionals

$$F_q(u) = \int a|\nabla u|^2 + \int R(u+1)^2 - \frac{2}{q}\int R' \, |u+1|^q \tag{4.56}$$

for $q \in [2, N)$.

Broadly, the strategy of the proof of Theorem 4.3.1 is to construct minimizers u_q of the subcritical functionals, and then establish sufficient control to show that $(1 + u_q)$ converges in the limit $q \to N$ to the desired conformal factor. The following uniform coercivity estimate, which we prove following a variation of techniques found in [Rau95], is the key step in showing the existence of subcritical minimizers.

Proposition 4.3.5 (Coercivity of F_q)**.** *Suppose that* $\{R' = 0\}$ *is Yamabe positive, that* $\delta > \delta^*$*, and that* $q_0 \in (2, N)$*. For every* $B \in \mathbb{R}$ *there is a* $K > 0$ *such that for all* $q \in [q_0, N)$ *and all* $u \in W^{1,2}_{\delta^*}$ *with* $u \geq -1$*, if* $\|u\|_{2,\delta} > K$ *then* $F_q(u) > B$*.*

Proof. For $\eta > 0$ let

$$A_\eta = \left\{ u \in W^{1,2}_{\delta^*}, u \geq -1 : \int |R'||u|^2 \leq \eta \|u\|^2_{2,\delta} \int |R'| \right\}. \tag{4.57}$$

Morally, $u \in A_\eta$ if it is concentrated on the zero set

$$Z = \{R' = 0\}, \tag{4.58}$$

with greater concentration as $\eta \to 0$.

Fix a constant $\mathcal{L} \in (0, \lambda_\delta(Z))$. We first claim that there is an $\eta_0 < 1$ such that if $u \in A_{\eta_0}$, then

$$\int a|\nabla u|^2 + Ru^2 \geq \mathcal{L}\|u\|^2_{2,\delta}. \tag{4.59}$$

Suppose to the contrary that this is false, and let η_k be a sequence converging to 0. We can then construct a sequence v_k with each $v_k \in A_{\eta_k}$ such that $\|v_k\|_{2,\delta} = 1$ and

$$\int a|\nabla v_k|^2 + Rv_k^2 < \mathcal{L}. \tag{4.60}$$

Note that $\mathcal{L}$ is finite even if $\lambda_\delta(Z) = \infty$. So from the boundedness of the sequence v_k in L^2_δ and Lemma 4.2.5, the sequence v_k is bounded in $W^{1,2}_{\delta^*}$, and a subsequence (which we reduce to) converges weakly in $W^{1,2}_{\delta^*}$ and strongly in L^2_δ to a limit v with

$\|v\|_{2,\delta} = 1$. Now

$$0 \leq \int |R'|v_k^2 \leq \eta_k \int |R'| \to 0. \tag{4.61}$$

Since $|R'|v_k^2 \to |R'|v^2$ in L^1 we conclude that $v = 0$ outside of Z. From weak upper semicontinuity (Corollary 4.2.2) we conclude that

$$\int a|\nabla v|^2 + Rv^2 \leq \mathcal{L} \tag{4.62}$$

as well. However, since v is supported in Z

$$\int a|\nabla v|^2 + Rv^2 \geq \lambda_\delta(Z)\|v\|_{2,\delta}^2 = \lambda_\delta(Z) > \mathcal{L}, \tag{4.63}$$

which is a contradiction, and establishes inequality (4.59).

Let $B \in \mathbb{R}$ and suppose $q \in (q_0, N)$, $u \in W_{\delta^*}^{1,2}$ and $u \geq -1$. We wish to show that there is a K independent of q so that if $\|u\|_{2,\delta} > K$ then $F_q(u) > B$. It is enough to find a choice of K under two cases depending on whether $u \in A_{\eta_0}$ or not. If u is concentrated on Z, the coercivity follows from the fact that Z is Yamabe positive (as used to obtain inequality (4.59)), and if u is not concentrated on Z then the coercivity follows from the fact that $R' < 0$ away from Z.

Suppose that $u \notin A_{\eta_0}$, so

$$\int |R'||u|^2 > \eta_0 \|u\|_{2,\delta}^2 \int |R'|. \tag{4.64}$$

We calculate

$$
\begin{aligned}
F_q(u) &= \int a|\nabla u|^2 + \int R(u+1)^2 + \frac{2}{q}\int |R'||u+1|^q \\
&\geq \int a|\nabla u|^2 - 2\int |R|(u^2+1) + \frac{2}{q}\int |R'|(|u|^q - 1) \\
&\geq \int \frac{a}{2}|\nabla u|^2 - C\|u\|_{2,\delta}^2 - 2\int |R| + \frac{2}{q}\int |R'|(|u|^q - 1) \\
&\geq \int \frac{a}{2}|\nabla u|^2 - C\|u\|_{2,\delta}^2 - 2\int \left(|R| + \frac{1}{q}|R'|\right) + \frac{2}{q}\int |R'||u|^q.
\end{aligned}
\tag{4.65}
$$

Here we have applied Lemma 4.2.1, and have used the fact that $(u+1)^q \geq |u|^q - 1$ for $u \geq -1$. Inequality (4.64) and Hölder's inequality imply

$$
\eta_0\|u\|_{2,\delta}^2 \int |R'| < \int |R'||u|^2 \leq \left(\int |R'||u|^q\right)^{\frac{2}{q}} \left(\int |R'|\right)^{1-\frac{2}{q}} \tag{4.66}
$$

and hence

$$
(\eta_0)^{\frac{q}{2}}\|u\|_{2,\delta}^q \int |R'| \leq \int |R'||u|^q. \tag{4.67}
$$

Using the fact that $\eta_0 < 1$ and $q < N$, inequalities (4.65) and (4.67) imply at last that

$$
F_q(u) \geq \int \frac{a}{2}|\nabla u|^2 - C\|u\|_{2,\delta}^2 - 2\int \left(|R| + \frac{1}{q}|R'|\right) + \frac{2}{q}(\eta_0)^{\frac{N}{2}}\|u\|_{2,\delta}^q \int |R'|. \tag{4.68}
$$

We note that $\int |R'| > 0$, for otherwise condition (4.64) is impossible, and hence the coefficient on $\|u\|_{2,\delta}^q$ is positive. Since $q > 2$, there is a K such that if $\|u\|_{2,\delta} > K$, then $F_q(u) \geq B$. Note that since C is independent of $q \geq q_0$, so is the choice of K.

Now suppose $u \in A_{\eta_0}$, so inequality (4.59) holds. Then for any $\epsilon > 0$,

$$\begin{aligned}
F_q(u) &\geq \int a|\nabla u|^2 + \int R(u+1)^2 \\
&= \int a|\nabla u|^2 + Ru^2 + \int R\left[(u+1)^2 - u^2\right] \\
&\geq \int a|\nabla u|^2 + Ru^2 - \int |R| \left[\epsilon u^2 + 1 + \frac{1}{\epsilon}\right] \\
&\geq (1-\epsilon)\left[\int a|\nabla u|^2 + Ru^2\right] + \epsilon \int (a|\nabla u|^2 - 2|R|u^2) - \left(1 + \frac{1}{\epsilon}\right)\int |R| \\
&\geq (1-\epsilon)\mathcal{L}\|u\|_{2,\delta}^2 + \epsilon\left(\int \frac{a}{2}|\nabla u|^2 - C\|u\|_{2,\delta}^2\right) - \left(1+\frac{1}{\epsilon}\right)\int |R| \\
&\geq \left[(1-\epsilon)\mathcal{L} - \epsilon C\right]\|u\|_{2,\delta}^2 + \epsilon \int \frac{a}{2}|\nabla u|^2 - \left(1+\frac{1}{\epsilon}\right)\int |R|.
\end{aligned} \tag{4.69}$$

Here we have applied Lemma 4.2.1, inequality (4.59), and the fact that $(u+1)^2 - u^2 \leq \epsilon u^2 + 1 + (1/\epsilon)$ for all $u \geq -1$ and all $\epsilon > 0$. We can pick ϵ sufficiently small so that the coefficient of $\|u\|_{2,\delta}$ in the final expression of inequality (4.69) is at least $\mathcal{L}/2$. Hence there is a K such that if $\|u\|_{2,\delta} \geq K$, then $F_q(u) \geq B$. Since C is independent of $q \geq q_0$, so is ϵ and the choice of K. □

Lemma 4.3.6. *For $q < N$ the operator F_q is weakly upper semicontinuous on $W^{1,2}_{\delta^*}$.*

Proof. Lemma 4.2.1 together with the weak continuity of continuous linear maps implies that

$$u \mapsto \int a|\nabla u|^2 + R(u+1)^2 \tag{4.70}$$

is weakly upper semicontinuous on $W^{1,2}_{\delta^*}$. Hence it suffices to show that

$$u \mapsto \int R'|u+1|^{q-1} \tag{4.71}$$

is weakly continuous on $W^{1,2}_{\delta^*}$. But fixing $\delta > \delta^*$ we know that the embedding $W^{1,2}_{\delta^*} \hookrightarrow L^q_\delta$ is compact and that the map (4.71) is continuous on L^q_δ. □

We now obtain existence of subcritical minimizers from the coercivity of F_q, along with uniform estimates in $W^{1,2}_{\delta^*}$ for the minimizers.

Lemma 4.3.7. *For any $q_0 \in (2, N)$, for each $q \in [q_0, N)$, there exists $u_q > -1$, bounded in $W^{1,2}_{\delta^*}$ and independent of q, which minimizes F_q and is a weak solution of*

$$-a\Delta(u_q + 1) + R(u_q + 1) = R'(u_q + 1)^{q-1}. \tag{4.72}$$

Moreover, $u_q \in W^{2,p}_{\sigma}$ for every $\sigma \in (2 - n, 0)$.

Proof. Let $B = \int R + \int |R'|$, let $\delta > \delta^*$, and let $q_0 \in (2, N)$. Observe that

$$F_q(0) \leq B \tag{4.73}$$

for all $q \in (q_0, N)$. Let K be the constant associated with B, δ and q_0 obtained from Proposition 4.3.5. Fix $q \in (q_0, N)$ and let u_k be a minimizing sequence in $W^{1,2}_{\delta^*}$ for F_q. Without loss of generality, we can assume each $u_k \geq -1$ since $F_q(u_k) = F_q(\max(u_k, -2 - u_k))$. We can assume that each $F_q(u_k) \leq F_q(0) \leq B$ and hence Proposition 4.3.5 implies that each $\|u_k\|_{2,\delta} \leq K$. Since

$$\int a|\nabla u_k|^2 + R(1 + u_k)^2 \leq F_q(u_k) < B \tag{4.74}$$

as well, Lemma 4.2.5 implies that there is a $C > 0$ such that each $\|u_k\|_{W^{1,2}_{\delta^*}} \leq C$. Note that C depends on K and B, which are independent of $q \geq q_0$. A subsequence (which we reduce to) converges weakly in $W^{1,2}_{\delta^*}$ and strongly in L^q_δ to a limit $u_q \geq -1$. Lemma 4.3.6 shows that F_q is weakly upper semicontinuous, so u_q is a minimizer. Moreover, $\|u_q\|_{W^{1,2}_{\delta^*}} \leq C$ as well.

Since u_q is a minimizer, we find that $(1+u_q)$ is a weak solution of

$$\left[-a\Delta + R - R'(1+u_q)^{q-2}\right](1+u_q) = 0. \tag{4.75}$$

Since $R' \in L^\infty_{\rm loc}$ and since $u_q \in L^N_{\rm loc}$, an easy computation shows that $R'(1+u_q)^{q-2} \in L^r_{\rm loc}$ for some $r > n/2$. Since $R \in L^p_{\rm loc}$ and $g \in W^{2,p}_{\rm loc}$ with $p > n/2$, we find that the coefficients of the differential operator in brackets in equation (4.75) satisfy the hypotheses of the weak Harnack inequality of [Tru73b]. Hence, since $1 + u_q \geq 0$ and since the manifold is connected, either $1 + u_q > 0$ everywhere or $u_q \equiv -1$. But u_q decays at infinity, and so we conclude that $1 + u_q$ is everywhere positive.

We now bootstrap the regularity of u_q, which we know initially belongs to $L^N_{\delta^*}$. Fix $\sigma \in (2-n, 0)$. Suppose it is known that for some $r \geq N$ that $u_q \in L^r_{\rm loc}$. From equation (4.75), u_q solves

$$-a\Delta u_q = R'(1+u_q)^{q-1} - R(1+u_q). \tag{4.76}$$

Recall that $R' \in L^\infty_{\rm loc}$ and $R \in L^p_{\rm loc}$ and both have compact support. Then $R'(1+u_q)^{q-1}$ belongs to $L^{t_1}_\sigma$ with

$$\frac{1}{t_1} = \frac{q-1}{r} \leq \frac{1}{r} + \frac{q-2}{N} < \frac{N-1}{N}, \tag{4.77}$$

and $R(1+u_q)$ belongs to $L^{t_2}_\sigma$ with

$$\frac{1}{t_2} = \frac{1}{r} + \frac{1}{p}. \tag{4.78}$$

Let $t = \min(t_1, t_2)$ and note that $t < p$ since $t_2 < p$. From [Bar86] Proposition 1.6 we see that u_q is a strong solution of (4.76) and from [Bar86] Proposition 2.2, which implies $\Delta : W^{2,t}_\sigma \to L^t_\sigma$ is an isomorphism for $1 < t \leq p$, we conclude that $u_q \in W^{2,t}_\sigma$. From Sobolev embedding we obtain $u_q \in L^{r'}_\sigma$ where

$$\frac{1}{r'} = \frac{1}{t} - \frac{2}{n}, \tag{4.79}$$

so long as $1/t > n/2$, at which point the bootstrap changes as discussed below.

Now

$$\begin{aligned} \frac{1}{t_1} - \frac{2}{n} &\leq \frac{1}{r} + \frac{q-2}{N} - \frac{2}{n} \\ &= \frac{1}{r} + \frac{q}{N} - \left[\frac{2}{N} + \frac{2}{n}\right] \\ &= \frac{1}{r} + \left[\frac{q}{N} - 1\right]. \end{aligned} \tag{4.80}$$

Also,

$$\frac{1}{t_2} - \frac{2}{n} = \frac{1}{r} + \left[\frac{1}{p} - \frac{2}{n}\right]. \tag{4.81}$$

Let $\epsilon = \min(1 - q/N, 2/n - 1/p)$ and note that ϵ is positive and independent of r. Inequalities (4.80) and (4.81) imply

$$\frac{1}{r'} \leq \frac{1}{r} - \epsilon \tag{4.82}$$

Hence, after a finite number of iterations (depending on the size of ϵ, and hence on how close q is to N) we can reduce $1/r$ by multiples of ϵ until $1/r \leq \epsilon$. At this point the bootstrap changes, and in at most two more iterations we can conclude that $u_q \in L^\infty_\sigma$ and also $u_q \in W^{2,p}_\sigma$.

□

The uniform $W^{1,2}_{\delta^*}$ bounds of Lemma 4.3.7 are enough to obtain the existence of a solution u in $W^{2,N/(N-1)}_\sigma$ of equation (4.72) with $q = N$. At the end of Section IV.6 of [Rau95] it is claimed that on a compact manifold in the smooth setting that elliptic regularity now implies u is smooth. But in fact this is not quite enough regularity to start a bootstrap: $W^{2,N/(N-1)}_\sigma$ embeds continuously in L^N_σ, which implies no more regularity than was known initially. To start a bootstrap and ensure the continuity of u we need the following improved estimate, which follows a modification of the strategy of [LP87] Proposition 4.4.

Lemma 4.3.8. *For each compact set K, the minimizers u_q are uniformly bounded in $L^M(K)$ for some $M > N$.*

Proof. Let χ be a smooth positive function with compact support that equals 1 in a neighborhood of K. Let $v = \chi^2(1+u_q)^{1+2\sigma}$ where u_q is a subcritical minimizer and where σ is a small constant to be chosen later. Note that since $u_q \in L^\infty_{\text{loc}} \cap W^{1,2}_{\text{loc}}$, $v \in W^{1,2}_{\delta^*}$. Setting $w = (1+u_q)^{1+\sigma}$, a short computation shows that

$$\int \chi^2 |\nabla w|^2 = -2\frac{1+\sigma}{1+2\sigma}\int \langle \chi\nabla w, w\nabla\chi\rangle + \frac{(1+\sigma)^2}{1+2\sigma}\int \langle \nabla u_q, \nabla v\rangle . \tag{4.83}$$

Applying Young's inequality to the first term on the right-hand side of equation (4.83) and merging a resulting piece into the left-hand side we conclude there is a constant C_1 such that

$$\|\chi\nabla w\|_2^2 \le C_1 \|w\nabla\chi\|_2^2 + 2\frac{(1+\sigma)^2}{1+2\sigma}\int \langle \nabla u_q, \nabla v\rangle . \tag{4.84}$$

Since u_q is a subcritical minimizer,

$$\begin{aligned} a\int \langle \nabla u_q, \nabla v\rangle &= \int R'(1+u_q)^{q-2}\chi^2 w^2 - \int R\chi^2 w^2 \\ &\le \left| \int R\chi^2 w^2 \right| \\ &\le \epsilon \|\nabla(\chi w)\|_2^2 + C_\epsilon \|\chi w\|_2^2. \end{aligned} \tag{4.85}$$

We applied Lemma 4.2.1 in the last line and used the fact that for functions with support contained in a fixed compact set, weighted and unweighted norms are equivalent. Note also that the fact that $R' \leq 0$ everywhere is used in going from line 1 to line 2 in (4.85). Noting that there is a constant C_2 such that

$$\|\nabla(\chi w)\|_2^2 \leq C_2(\|\chi \nabla w\|_2^2 + \|w \nabla \chi\|_2^2), \tag{4.86}$$

we can combine inequalities (4.84), (4.85), and (4.86) to conclude that, if ϵ is sufficiently small to absorb the term from inequality (4.85) into the left-hand side, then there is a constant C_3 such that

$$\|\nabla(\chi w)\|_2^2 \leq C_3 \left[\|w \nabla \chi\|_2^2 + \|w\chi\|_2^2\right]. \tag{4.87}$$

Finally, from the Sobolev inequality (4.7), there is a constant C_4 such that

$$\|\chi w\|_N^2 \leq C_4 \left[\|w \nabla \chi\|_2^2 + \|w\chi\|_2^2\right] \tag{4.88}$$

as well. Now u_q is bounded uniformly in L^N on the support K' of χ, and hence we can take σ sufficiently small so that w is bounded independent of q in $L^2(K')$ as well. Thus $(1 + u_q)$ is bounded uniformly in $L^M(K)$ for $M = N(1 + \sigma)$. □

Corollary 4.3.9. *Let p be the exponent such that g is a $W^{2,p}_\alpha$ AE manifold and let $\sigma \in (2 - n, 0)$. The subcritical minimizers u_q are bounded in $W^{2,p}_\sigma$ as $q \to N$.*

Proof. Consider a subcritical minimizer u_q, which is a weak solution of

$$-a\,\Delta\, u_q = -R(1 + u_q) + R'(1 + u_q)^{q-1}. \tag{4.89}$$

Let K be a compact set containing the support of R and R', and let $M > N$ be an exponent such that we have uniform bounds on u_q in $L^M(K)$. We wish to bootstrap this to better regularity for u_q.

Since the bootstrap for the two terms is different, we concentrate first on the interesting term, $R'(1+u_q)^{q-1}$, and suppose for the moment that the other term is absent. Let us write

$$\frac{1}{M} = \frac{1}{N} - \epsilon \tag{4.90}$$

for some $\epsilon > 0$. Now

$$|R'(1+u_q)^{q-1}| \leq |R'|(1+|1+u_q|^{N-1}). \tag{4.91}$$

Since R' is bounded, the term $R'|1+u_q|^{N-1}$ belongs to $L^s(K)$ with

$$\begin{aligned} \frac{1}{s} &= \frac{1}{M}(N-1) \\ &= \left(\frac{1}{N} - \epsilon\right)(N-1) \\ &= \frac{2}{n} + \frac{1}{N} - \epsilon(N-1). \end{aligned} \tag{4.92}$$

Since R' is zero outside of K we conclude $R'(1+u_q)^{q-1} \in L^s_\sigma$. Note that the norm of $R'(1+u_q)^{q-1}$ in L^s_σ depends on the norm of u_q in $L^M(K)$ but is otherwise independent of q. Since the functions u_q are uniformly bounded in $L^M(K)$, we obtain control of $R'(1+u_q)^{q-1}$ in L^s_σ independent of q.

If $s \leq p$ then $s \in (1,p]$ and we cite [Bar86] Proposition 2.2 to conclude $u_q \in W^{2,s}_\sigma$ and therefore $u_q \in L^{M'}(K)$ with

$$\frac{1}{M'} = \frac{1}{s} - \frac{2}{n} = \frac{1}{N} - \epsilon(N-1). \tag{4.93}$$

Similarly, after k iterations of this process we would find u_q belongs to $W^{2,s}_\sigma$ with

$$\frac{1}{s} = \frac{2}{n} + \frac{1}{N} - \epsilon(N-1)^k \tag{4.94}$$

unless $s > p$, at which point the bootstrap terminates at $u_q \in W^{2,p}_\sigma$ with norm depending on $\|u_q\|_{L^M(K)}$ (which is independent of q) and on the number of iterations needed to reach $s \leq p$. Note that since $N > 2$, we will reach the condition $s \geq p$ in a finite number of steps independent of q.

Now consider the bootstrap for the term $-R(1+u_q)$ alone. Write

$$\frac{1}{p} = \frac{2}{n} - \epsilon' \tag{4.95}$$

for some $\epsilon' > 0$. The term $-R(1+u_q)$ then belongs to $L^t(K)$ with

$$\frac{1}{t} = \frac{1}{p} + \frac{1}{M} = \frac{2}{n} - \epsilon' + \frac{1}{M}. \tag{4.96}$$

Note that $1 < t < p$ and hence [Bar86] Proposition 2.2 implies $u_q \in W^{2,t}_\sigma$. Note that the norm of u_q in $W^{2,t}_\sigma$ depends on the norm of u_q in $L^M(K)$ but is otherwise independent of q. Consequently u_q is controlled in $L^{M'}(K)$ independent of q where

$$\frac{1}{M'} = \frac{1}{t} - \frac{2}{n} = \frac{1}{M} - \epsilon'. \tag{4.97}$$

After k iterations we would find instead

$$\frac{1}{M'} = \frac{1}{M} - k\epsilon' \tag{4.98}$$

and the bootstrap stops in finitely many steps independent of q if $k\epsilon' > 1/M$, at which point we find that $u_q \in W^{2,p}_\sigma$, with norm independent of q. There is an exceptional case if $k\epsilon' = 1/M$, but it can be avoided by an initial perturbation of M.

The bootstrap in the full case follows from combining these arguments. $\square$

Proof of Theorem 4.3.1. **(2. implies 1.)** The u_q are uniformly bounded in $W^{2,p}_\sigma$ by Corollary 4.3.9 for any $\sigma \in (2-n, 0)$. Thus they converge to some u strongly in $W^{1,2}_{\delta^*}$ and uniformly on compact sets. In particular, since the u_q weakly solve (4.72), $\phi := u + 1$ weakly solves

$$-a\Delta\phi + R\phi = R'\phi^{N-1}. \tag{4.99}$$

Since each $u_q \geq -1$, $\phi \geq 0$, and since $\phi \to 1$ at infinity, $\phi \not\equiv 0$. Hence the weak Harnack inequality [Tru73b] implies $\phi > 0$.

Since $\sigma \in (2-n, 0)$ is arbitrary, $\phi - 1 \in W^{2,p}_\alpha$ in particular. Note that the rapid decay $\sigma \approx 2-n$ uses the fact that $R = 0$ near infinity. The lesser decay rate α in the statement of the theorem stems from the fact that we may have used a conformal factor in $W^{2,p}_\alpha$ to initially set $R = 0$ near infinity or to lower the scalar curvature after changing it to R'.

(1. implies 2.) Let $Z = \{R' = 0\}$. The case where Z has zero measure is trivial, for then $y(Z) = \infty > 0$. Hence we assume Z has positive measure and suppose there exists a conformally related metric g' with scalar curvature R'. Let $\delta > \delta^*$ be fixed and let u be a minimizer of $Q_{g',\delta}$ over $A(Z)$ as provided by Proposition 4.2.13. Note that

$$\int R'u^2 dV_{g'} = 0 \tag{4.100}$$

since $R' = 0$ on Z and $u = 0$ on Z^c. Hence

$$\lambda_{g',\delta}(Z) = Q_{g',\delta}(u) = a\frac{\int |\nabla u|^2_{g'} dV_{g'}}{\|u\|_{g',2,\delta}}. \tag{4.101}$$

In particular, $\lambda_{g',\delta}(Z) \geq 0$, and $\lambda_{g',\delta}(Z) = 0$ only if u is constant. But Z has positive measure, and therefore $A(Z)$ does not contain any constants. Hence $\lambda_{g',\delta}(Z) > 0$, and Proposition 4.2.10 implies that Z is Yamabe positive. $\square$

This completes the proof of Theorem 4.3.1. Turning to the compact case (Theorem 4.3.2) recall that we started the AE argument with the following simplifying hypotheses:

1. The prescribed scalar curvature R' is bounded.
2. The prescribed scalar curvature R' has compact support.
3. The initial scalar curvature R has compact support.

The last two of these are trivial if M is compact, and the first is justified by Lemma 4.3.10 below, which shows that we can lower scalar curvature after first solving the problem for a scalar curvature that is truncated below. In the compact case we require an additional condition which will be used in Lemma 4.3.11.

4. We may assume that the initial scalar curvature R is continuous and negative. Indeed, from Proposition 4.2.13 there is a positive function ϕ solving $-a\Delta\phi + R\phi = \lambda_\delta(M)\phi$ on M. Note that $\lambda_\delta(M) < 0$ since g is Yamabe negative. Using ϕ as the conformal factor we obtain a scalar curvature $\lambda_\delta(M)\phi^{2-N}$. The hypotheses of Theorem 4.2 are conformally invariant and hence unaffected by this change.

Lemma 4.3.10. *Suppose (M, g) is a $W^{2,p}$ compact Yamabe negative manifold. Suppose $R' \in L^p$. If $0 \geq R \geq R'$, then there exists a positive ϕ with $\phi \in W^{2,p}$ such that $g' = \phi^{N-2} g$ has scalar curvature R'.*

Proof. We wish to solve

$$-a\Delta\phi + R\phi = R'\phi^{N-1}. \tag{4.102}$$

Note that $\phi_+ = 1$ is a supersolution of equation (4.102). To find a subsolution first observe that $R \not\equiv 0$ since the manifold is Yamabe negative. So, since $-R \geq 0$ and $-R \not\equiv 0$, for each $\epsilon > 0$ there exists a unique $\phi_\epsilon \in W^{2,p}$ solving

$$-a\Delta\phi_\epsilon - R\phi_\epsilon = -R + \epsilon R'. \tag{4.103}$$

If $\epsilon = 0$ the solution is 1, and since $W^{2,p}$ embeds continuously in C^0 we can fix $\epsilon > 0$ such that $\phi_\epsilon > 1/2$ everywhere. We claim that $\phi_- := \eta\phi_\epsilon$ is a subsolution if $\eta > 0$ is sufficiently small. Indeed,

$$\begin{aligned} -a\Delta\phi_- + R\phi_- &= \eta\left[R(2\phi_\epsilon - 1)\right] + \eta\epsilon R' \\ &\leq \eta\epsilon R'. \end{aligned} \tag{4.104}$$

So ϕ_- is a subsolution so long as

$$\eta\epsilon R' \leq R'\phi_-^{N-1}. \tag{4.105}$$

A quick computation shows that inequality (4.105) holds if η is small enough so that $\eta^{2-N} \geq \phi_\epsilon^{N-1}/\epsilon$ everywhere. We can also take η small enough so that $\phi_- \leq \phi_+ = 1$, and hence there exists a solution $\phi \in W^{2,p}$ with $\phi \geq \phi_- > 0$ of equation (4.102) ([Max05] Proposition 2). □

The remainder of the proof of Theorem 4.3.2 closely follows the proof of Theorem 4.3.1 by treating a compact manifold as an asymptotically Euclidean manifold with zero ends. In particular, the cited results of Section 4.2 apply equally in both cases, and differences arise only when the following facts are cited.

- A constant function in $W^{1,2}_{\delta^*}$ is identically zero.

- The Laplacian is an isomorphism from $W^{2,p}_{\sigma}$ to L^p_{σ} for $\sigma \in (2-n, 0)$.

Twice, we use the property that constants in $W^{1,2}_{\delta^*}$ vanish: once in Lemma 4.3.7 in showing $1 + u_q \not\equiv 0$, and once in the final proof of Theorem 4.3.1 showing that in the limit $1 + u \not\equiv 0$ as well. The following lemma provides the alternative argument needed to ensure that these functions do not vanish identically in the compact case.

Lemma 4.3.11. *Suppose (M, g) is compact and that R_g is continuous and negative. Fix $q_0 \in (2, N)$. Then $\|1 + u_q\|_2 \geq C$ for some C independent of $q \in (q_0, N)$. Moreover, the limit $1 + u$ is not identically zero.*

Proof. Note that for any constant k,

$$F_q(k) = (1+k)^2 \int R - \frac{2}{q}(1+k)^q \int R'. \tag{4.106}$$

Since $\int R < 0$, for any $k \neq -1$ close enough to -1, $F_q(k) < 0$. Indeed, there are constants $k_0 > -1$ and $c > 0$ such that $F_q(k_0) < -c$ for all $q \in (q_0, N)$. But then

$$\int R(1+u_q)^2 \leq F_q(u_q) \leq F_q(k_0) \leq -c \tag{4.107}$$

since u_q minimizes F_q. Since R is continuous, and thus bounded below, $\|1 + u_q\|_2 \geq C$ for some C independent of $q \in (q_0, N)$. Since $u_q \to u$ in L^2, we also have $\|1 + u\|_2 \geq C$, and so $1 + u$ is not identically zero. $\square$

We use the fact that $\|\Delta u\|_{p,\sigma}$ controls $\|u\|_{W^{2,p}_\sigma}$ twice as well, once in the bootstrap of Lemma 4.3.7 and once in the bootstrap of Lemma 4.3.9. However, on a compact manifold, $\|u\|_{W^{2,p}}$ is controlled by the sum of $\|\Delta u\|_p$ and $\|u\|_2$, and the coercivity estimate from Proposition (4.3.5) ensures that $\|u_q\|_2$ is uniformly bounded as $q \to N$. This provides the needed extra control for the bootstraps and completes the proof of Theorem 4.3.2.

4.4. Yamabe Classification

In this section we provide two characterizations of the Yamabe class of an asymptotically Euclidean manifold, one in terms of the prescribed scalar curvature problem and one in terms of the Yamabe type of the manifold's compactification. Note that throughout this section AE manifolds have at least one end.

Theorem 4.4.1. *Suppose (M, g) is a $W^{2,p}_\alpha$ AE manifold with $p > n/2$ and $\alpha \in (2-n, 0)$. Let $\mathcal{R}_{\le 0}$ be the set of non-positive elements of $L^p_{\alpha-2}$.*

1. *M is Yamabe positive if and only if the set of non-positive scalar curvatures of metrics conformally equivalent to g is $\mathcal{R}_{\le 0}$.*
2. *M is Yamabe null if and only if the set of non-positive scalar curvatures of metrics conformally equivalent to g is $\mathcal{R}_{\le 0} \setminus \{0\}$.*
3. *M is Yamabe negative if and only if the set of non-positive scalar curvatures of metrics conformally equivalent to g is a strict subset of $\mathcal{R}_{\le 0} \setminus \{0\}$.*

Proof. It suffices to prove the forward implications.

1) Suppose M is Yamabe positive, and hence so is every subset. If $R' \in \mathcal{R}_{\le 0}$, then $\{R' = 0\}$ is Yamabe positive and Theorem 4.3.1 implies $[g]$ includes a metric with scalar curvature R'.

2) Suppose M is Yamabe null. Since M is open and connected, Lemma 4.2.16 implies that if $E \subseteq M$ has positive measure, then $M \setminus E$ is Yamabe positive. Hence for any $R' \in \mathcal{R}_{\leq 0}$ with $R' < 0$ on a set of positive measure, $\{R' = 0\}$ is Yamabe positive, and Theorem 4.3.1 implies we can conformally transform to a metric with scalar curvature R'. But $R' \equiv 0$ is impossible, for otherwise Theorem 4.3.1 would imply M is Yamabe positive.

3) Suppose M is Yamabe negative. Since M is open, Lemma 4.2.17 shows that there is a nonempty open set $W \subseteq M$ such that $M \setminus W$ is also Yamabe negative. Suppose $R' \in L^p_{\alpha-2}$ is non-positive and supported in W. Then $\{R' = 0\}$ contains $M \setminus W$ and is hence Yamabe negative. But then Theorem 4.3.1 shows that we cannot conformally transform to a metric with scalar curvature R'. In particular, $R' \equiv 0$ is one of the unattainable scalar curvatures. $\square$

While Theorem 4.4.1 completely describes the set of allowable scalar curvatures in cases 1) and 2), it does not in case 3). Of course, we already have demonstrated a necessary and sufficient criterion for being able to make the conformal change: the zero set of R' must be Yamabe positive. Nevertheless, it would be desirable to describe this situation more explicitly, and there are a few things that can be said. First, by Lemma 4.2.15, if $R' \in \mathcal{R}_{\leq 0}$ and the weighted volume of $\{R' = 0\}$ is sufficiently small, then $\{R' = 0\}$ is Yamabe positive, and thus g is conformally equivalent to a metric with scalar curvature R'. In particular, if $R' < 0$ everywhere, then it is attainable. Conversely, by Lemma 4.2.17, for any sequence $\{R'_k\} \subset \mathcal{R}_{\leq 0}$ such that $\{R'_k < 0\} \subset B_{1/k}(x_0)$ for some fixed $x_0 \in M$, then for k large enough, $\{R'_k = 0\}$ is Yamabe negative, and thus g is not conformally equivalent to a metric with scalar curvature R'_k. That is, the strictly negative part of R' cannot be constrained to a small ball. Similarly, an argument analogous to the proof of Lemma 4.2.17 shows that the complement of a sufficiently "small" neighborhood of infinity is Yamabe negative, and hence the strictly negative part of R' cannot be constrained to a small neighborhood of infinity.

Our second characterization of the Yamabe class of an AE manifold involves its compactification. An AE manifold can be compactified using a conformal factor that decays suitably at infinity, and a compact manifold can be transformed into an AE manifold using a conformal factor with a suitably singularity. We would like to show that the sign of the Yamabe invariant is preserved under these operations, and we begin by laying out the details of the compactification/decompactification procedure. In particular, there is a precise relationship between the decay of the metric at infinity and its smoothness at the point of compactification.

Lemma 4.4.2. *Let $p > n/2$ and let $\alpha = \frac{n}{p} - 2$, so $-2 < \alpha < 0$. Suppose (M, g) is a $W^{2,p}_\alpha$ AE manifold. There is a smooth conformal factor ϕ that decays to infinity at the rate ρ^{2-n} such that $\bar{g} = \phi^{N-2} g$ extends to a $W^{2,p}$ metric on the compactification $\overline{M}$.*

Conversely, suppose $(\overline{M}, \overline{g})$ is a compact $W^{2,p}$ manifold, with $p > n/2$ and $p \neq n$. Given a finite set $\mathcal{P}$ of points in $\overline{M}$ there is conformal factor $\overline{\phi}$ that is smooth on $M = \overline{M} \setminus \mathcal{P}$, has a singularity of order $|x|^{2-n}$ at each point of $\mathcal{P}$, and such that $g = \overline{\phi}^{N-2} \overline{g}$ is a $W^{2,p}_\alpha$ AE manifold with $\alpha = \frac{n}{p} - 2$.

Proof. For simplicity we treat the case of only one end.

Let (M, g) be a $W^{2,p}_\alpha$ AE manifold and let z^i be the Euclidean end coordinates on M, so

$$g_{ij} = e_{ij} + k_{ij}, \tag{4.108}$$

with $k \in W^{2,p}_{\alpha}$. Let x^i be coordinates given by the Kelvin transform $x^i = z^i/|z|^2$, so $z^i = x^i/|x|^2$ as well.

We define a conformal factor $\phi = |z|^{2-n}$ near infinity, and extend it to be smooth on the rest of M. Let $\overline{g} = \phi^{N-2} g$ and let $\overline{M}$ be the one-point compactification of M, with P being the point at infinity. We wish to show that $\overline{g}$ extends to a $W^{2,p}(\overline{M})$ metric.

Near P, $\phi^{N-2} = |z|^{-4}$ and

$$\overline{g}_{ij} = e_{ij} + \overline{k}_{ij} \tag{4.109}$$

where

$$\overline{k}_{ij} := k_{ij} - \frac{4}{|x|^2} x^a k_{a(i} x_{j)} + \frac{4}{|x|^4} x^a x^b k_{ab} x_i x_j = O(k). \tag{4.110}$$

and $x_a = e_{ab} x^b$. Since $\overline{k}_{ij} \to 0$ at P, we set $\overline{g}_{ij}(P) = e_{ij}$ to obtain a continuous metric, and we need to show that $\overline{k} \in W^{2,p}(\overline{M})$. Since $\overline{k} \in W^{2,p}_{\text{loc}}(M)$, and since a point is a removable set, we need only show that the second derivatives of $\overline{k}$ belong to $L^p(B)$ for some coordinate ball B containing P.

Let $\bar{\partial}$ represent the derivatives in x^i coordinates. Since $\frac{\partial z}{\partial x} = O(|x|^{-2})$, we calculate

$$\begin{aligned} \bar{\partial}\overline{k} &= O(\partial k) O(|z|^2) + O(k) O(|z|) \\ \bar{\partial}^2 \overline{k} &= O(\partial^2 k) O(|z|^4) + O(\partial k) O(|z|^3) + O(k) O(|z|^2). \end{aligned} \tag{4.111}$$

In order to show $\bar{\partial}^2 \overline{k} \in L^p(B)$, it is sufficient to show that each of the three terms in equation (4.111) is in $L^p(B)$.

Note that near infinity

$$d\overline{V} = \phi^N dV = |z|^{-2n} dV. \tag{4.112}$$

Hence the L^p norm of the $O(k)O(|z|^2)$ term of equation (4.111) is controlled by

$$\begin{aligned} \int \left(O(k)O(|z|^2)\right)^p |z|^{-2n} dV &= \int O\left(|k|^p\right) O\left(|z|^{2p-2n}\right) dV \\ &\leq C\|k\|^p_{W^{2,p}_\alpha}, \end{aligned} \tag{4.113}$$

where we have used the equality

$$2p - 2n = -n - \alpha p \tag{4.114}$$

and expression (2.1) defining the weighted norm. Hence the $O(k)O(|z|^2)$ term of equation (4.111) belongs to $L^p(B)$. The two remaining terms have the same asymptotics and similar calculations show that they belong to $L^p(B)$ as well.

For the converse, consider a $W^{2,p}$ compact manifold $(\overline{M}, \overline{g})$ with $p > n/2$ and $p \neq n$. Let P be a point to remove to obtain $M = \overline{M} \setminus \{P\}$. Since $\overline{g}$ is continuous we can find smooth coordinates x^i near P such that $\overline{g} = e + \overline{k}$ for some $\overline{k} \in W^{2,p}$ which vanishes at P. Moreover, if $p > n$ then $\overline{g}$ has Hölder continuous derivatives and the proof of Proposition 1.25 in [Aub98] shows we can additionally assume these are normal coordinates (i.e., the first derivatives of $\overline{k}$ vanish at P). Finally, since the result we seek only involves properties of $\overline{k}$ local to P, we can assume that $\overline{k} = 0$ except in a small coordinate ball B near P.

We claim there is a constant C such that

$$\int_B \frac{|\overline{k}|^p}{|x|^{2p}} \leq C \int_B |\bar{\partial}^2 \overline{k}|^p d\overline{V} \qquad \text{and} \tag{4.115}$$

$$\int_B \frac{|\bar{\partial}\overline{k}|^p}{|x|^p} d\overline{V} \leq C \int_B |\bar{\partial}^2 \overline{k}|^p d\overline{V}. \tag{4.116}$$

Assuming for the moment that this claim is true, let $z_i = x_i/|x|^2$. Let $\overline{\phi} = |x|^{2-n}$ near P and extend $\overline{\phi}$ as a positive smooth function on the remainder of M. Let $g = \overline{\phi}^{N-2}\overline{g}$. Near P, $\overline{\phi}^{N-2} = |x|^{-4}$ and so $g = e + k$ near infinity, where

$$k_{ij} := \overline{k}_{ij} - \frac{4}{|z|^2} z^a \overline{k}_{a(i} z_{j)} + \frac{4}{|z|^4} z^a z^b \overline{k}_{ab} z_i z_j = O(\overline{k}). \tag{4.117}$$

Since $k \in W^{2,p}_{\text{loc}}$, we need only establish the desired asymptotics at infinity.

A computation similar to the one leading to equation (4.111) shows

$$\begin{aligned} \partial k &= O(\bar{\partial}\overline{k})O(|x|^2) + O(\overline{k})O(|x|) \\ \partial^2 k &= O(\bar{\partial}^2\overline{k})O(|x|^4) + O(\bar{\partial}\overline{k})O(|x|^3) + O(\overline{k})O(|x|^2). \end{aligned} \tag{4.118}$$

Also, $d\overline{V} = |z|^{-2n} dV$ near P. Hence

$$\begin{aligned} \int |\partial^2 k|^p |z|^{4p-2n} dV &= \int |\partial^2 k|^p |x|^{-4p} |x|^{2n} dV \qquad (4.119) \\ &= \int \left(O(\bar{\partial}^2\overline{k})\right)^p + \left(O(\bar{\partial}\overline{k})O(|x|^{-1})\right)^p + \left(O(\overline{k})O(|x|^{-2})\right)^p d\overline{V}. \end{aligned}$$

$$(4.120)$$

From inequalities (4.115) and (4.116), quantity (4.120) is finite. Noting

$$4p - 2n = -n - \alpha p + 2p \tag{4.121}$$

we conclude $|\partial^2 k| \in L^p_{\alpha-2}$, as desired. A similar calculation shows that $|\partial k| \in L^p_{\alpha-1}$ and $|k| \in L^p_{\alpha}$. This concludes the proof, up to establishing inequalities (4.115) and (4.116).

Theorem 1.3 of [Bar86] implies that

$$\int_B \frac{|f|^p}{|x|^{2p}} d\overline{V} \le c \int_B \frac{|\bar{\partial} f|^p}{|x|^p} d\overline{V} \le C \int_B |\bar{\partial}^2 f|^p d\overline{V} < \infty \tag{4.122}$$

for smooth functions f that are compactly supported in B and vanish in a neighborhood of P. This inequality relies on the fact that $p \neq n$, which corresponds to the condition $\delta = 0$ in [Bar86] Theorem 1.3.

Let f_n be a sequence of smooth functions vanishing near P that converges to $\overline{k}$ in $W^{2,p}$; such a sequence exists since $\overline{k} = 0$ at P, since $\partial \overline{k} = 0$ at P if $p > n$, and since we have assumed that $\overline{k}$ vanishes outside of B. By reduction to a subsequence we may assume that the values and first derivatives of sequence converge pointwise a.e., and using Fatou's Lemma we find

$$\begin{aligned} \int_B \frac{|\overline{k}|^p}{|x|^{2p}} &\le \liminf_{n\to\infty} \int_B \frac{|f_n|^p}{|x|^{2p}} \\ &\le C \lim_{n\to\infty} \int_B |\bar{\partial}^2 f_n|^p d\overline{V} \\ &= C \int_B |\bar{\partial}^2 \overline{k}|^p d\overline{V} < \infty. \end{aligned} \tag{4.123}$$

This is inequality (4.115), and a similar argument shows that inequality (4.116) holds as well. □

The threshold $\alpha = -2$ in Lemma 4.4.2 arises because there is a connection between the rate of decay of the AE metric and the rate of convergence of the metric at the point of compactification in a chosen coordinate system: roughly speaking, decay of order ρ^α corresponds to convergence at a rate of $r^{-\alpha}$. For a generic smooth metric we can use normal coordinates to obtain convergence at a rate of r^2, but we cannot expect to do better generally. Hence the decompactification of a smooth metric typically does not decay faster than ρ^{-2}. Looking at the proof of Lemma 4.4.2, we note that it can be readily extended to $s > 2$ to show that a $W^{s,p}_{\alpha}$ AE metric with $s \geq 2$, $p > n/s$ and $\alpha = (n/p) - s$ can be compactified to a $W^{s,p}$ metric. But the decay condition $\alpha = (n/p) - s$ is quite restrictive for $s > 2$: smooth metrics decompactify generally to metrics with decay $O(\rho^{-2})$, but compactification of a $W^{s,p}_{-2}$ metric would not be known to be C^3, regardless of how high s and p are. A more refined analysis for $s > 2$ would need to take into account asymptotics of the Weyl or Cotton-York tensor, and we point to Herzlich [Her97] for related results in the C^s setting.

Proposition 4.4.3. *Let (M, g) and $(\overline{M}, \overline{g})$ be a pair of manifolds as in Lemma 4.4.2, related by $g = \overline{\phi}^{N-2}\overline{g}$. Then $y_g(M) = y_{\overline{g}}(\overline{M})$.*

Proof. For simplicity we assume that M has one end. Let $P \in \overline{M}$ be the singular point of $\overline{\phi}$. Note that $W^{1,2}_c(M)$ is dense in $W^{1,2}_{\delta^*}(M)$ and that

$$S_P := W^{1,2}(\overline{M}) \cap \{u : u|_{B_r(P)} = 0 \text{ for some } r > 0\} \tag{4.124}$$

is dense in $W^{1,2}(\overline{M})$ since $2 < n$. From upper semicontinuity of the Yamabe quotient, the Yamabe invariants of g and $\overline{g}$ can be computed by minimizing the Yamabe quotient over $W^{1,2}_c$ and S_P respectively. Note that $u \mapsto \overline{\phi}u$ is a bijection between $W^{1,2}_c(M)$ and S_p. The proof of Lemma 4.2.7 shows that for $u \in W^{1,2}_c$,

$$Q^y_g(u) = Q^y_{\overline{g}}(\overline{\phi}u) \tag{4.125}$$

and hence $y_g(M) = y_{\overline{g}}(\overline{M})$. $\square$

Combining Lemma 4.4.2 and Proposition 4.4.3 we obtain our second classification.

Proposition 4.4.4. *Let (M, g) be a $W^{2,p}_{\alpha}$ AE manifold with $\alpha \leq \frac{n}{p} - 2$. Then (M, g) is Yamabe positive/negative/null if and only if some conformal compactification, as described in Lemma 4.4.2, has the same Yamabe type.*

Consequently, Yamabe classification on AE manifolds has the same topological flavor as in the compact setting. For instance, since the torus does not allow a Yamabe positive metric, the decompactified torus, which is diffeomorphic to $\mathbb{R}^n$ with a handle, does not allow a metric with nonnegative scalar curvature.

We mention an application of Proposition 4.4.4 to general relativity. Recall the Einstein constraint equations, (1.2), which initial data $(M, g, K, T_{nn}, T_{ni})$ in general relativity must satisfy. It is natural to suppose that the energy density T_{nn} is everywhere nonnegative, which is known as the weak energy condition. If the initial data is maximal, i.e., if the mean curvature $\operatorname{tr} K$ is zero, then the weak energy condition implies $R \geq 0$. Thus, if the compactification of an AE manifold has a topology that does not admit a Yamabe positive metric, then the original AE manifold does not allow maximal initial data satisfying the weak energy condition. We mention that the results in [IMP02] show that every AE manifold does admit *some* solution of the constraints.

CHAPTER V

SOLUTIONS OF THE CONFORMAL CONSTRAINT EQUATIONS

In this chapter, we find solutions of the conformal constraint equations (1.3) following the setup introduced in [HNT09] and [Max09]. This method, based on a fixed point theorem, is the first to allow solutions with arbitrary mean curvatures. In Sections 5.1 and 5.2, we prove the appropriate analogues on asymptotically Euclidean manifolds. This is mostly work with Isenberg, Mazzeo, and Meier from [DIMM14].

Nguyen [Ngu14] later showed via a scaling argument that these solutions with arbitrary mean curvatures could alternatively be interpreted as rescalings of perturbations of CMC results. We discuss this argument. Also, Nguyen presented a new method for finding solutions to the conformal constraint equations, using half-continuity and a fixed point theorem. In Section 5.3, we present a simpler proof of his result, and marginally strengthen it.

5.1. The Fixed Point Approach

A standard method of solving differential (and other) equations is the fixed point method. In this method, one first finds a functional whose fixed points are solutions of the desired equation. One then uses a priori estimates and other properties of the functional to fulfill the conditions of one of the many fixed point theorems, such as the Schauder fixed point theorem. These theorems guarantee fixed points of functionals under very general circumstances. A good introduction to these techniques is found in [Bro04].

One of the first and most important fixed point theorems is the Brouwer fixed point theorem. In its basic form, it says that any continuous function from a ball in $\mathbb{R}^n$ to the same ball has a fixed point. Many other fixed point theorems, such as the ones we use in this chapter, the Schauder fixed point theorem and the Leray-Schauder alternative, are based on the Brouwer fixed point theorem. We now state the Schauder fixed point theorem.

Theorem 5.1.1 (Schauder Fixed Point Theorem)**.** *Let S be a closed convex subset of a normed linear space X and let $F : S \to S$ be a compact map. Then F has a fixed point.*

We introduce the map F that we use in this chapter. In essence, the map F is an iteration map, taking a function ϕ, solving the vector equation for some W using that ϕ, and then solving the Lichnerowicz equation using that W. In this way, we iterate the coupling, which allows us to find solutions at each step. This is similar to the proof of the sub and supersolution theorem 2.1.7, where we set up an iteration scheme, using the solution of the previous step in order to fix the nonlinearity of the equation $-a\Delta u = f(x, u)$.

To be more precise, for any positive function $v \in L^\infty$, let $W(v)$ be the solution in $W^{2,p}_\delta$ of the vector equation (1.3b) with v replacing ϕ. Let $G(W)$ to be the solution of the Lichnerowicz equation using W such that $G(W) - \mathring{u} \in W^{2,p}_\delta$, where $\mathring{u}$ is the desired asymptotic function. Let $E : W^{2,p}_\delta \to L^\infty$ be the compact Sobolev embedding map give by Proposition 2.0.1. We then define $F(v) := (E \circ G \circ W)(v)$. Clearly, if $F(\phi) = \phi$, then $(\phi, W(\phi))$ is a solution to the conformal constraint equations (1.3).

In order for F to be well defined, there are two requirements. First, for the vector equation to have a solution, we must require that g has no conformal Killing fields. Recall that if $p > n$ and g is a $W^{2,p}_\delta$ or $C^{2,\alpha}_\delta$ AE manifold, then this is known to be true (cf. Theorem 2.1.4). Next, for the Lichnerowicz equation to have a solution, we must assume that the seed data is admissible (cf. Definition 3.0.10); i.e., that there is a conformal factor ψ that transforms the metric to one with scalar curvature $-\kappa\tau^2$.

In order to define the domain set S, recall the definition of global sub and supersolutions.

Definition 5.1.2. *Functions $\phi_\pm$ are "global sub and supersolutions" of the Lichnerowicz equation* (1.3a) *if for any $\phi \leq \phi_+$, ϕ_+ is a supersolution and ϕ_- is a subsolution of the Lichnerowicz equation with $W = W(\phi)$.*

Suppose we could find global sub and supersolutions $\phi_\pm$ with properties as in the sub and supersolution theorem 2.1.7. Then $S = \{\phi \in L^\infty : \phi_- \leq \phi \leq \phi_+\}$ is clearly closed and convex, and by the sub and supersolution theorem, $F : S \to S$.

In order to show that F is compact, first recall that the composition of compact maps and continuous maps is compact. The solution map W is continuous by the continuity of ϕ^q and of $\left(\mathrm{div}\frac{1}{2N}L\right)^{-1}$. The map E is compact. Thus we only need to show that G is continuous. We show that it is in fact C^1 in the following lemma.

Lemma 5.1.3. *Given appropriately regular and admissible data, the solution map $G : W^{2,p}_\delta \to W^{2,p}_\delta$ is continuously Gâteaux differentiable. A similar statement holds for $C^{2,\alpha}_\delta$ data.*

Proof. This lemma follows from the implicit function theorem. The proof we give here is essentially the same as is used in proving [Max09, Prop 13], which is the equivalent result for compact manifolds. Since the proof is identical for $C^{2,\alpha}_\delta$ data, we only prove the first case. For brevity, we equivalently prove that the solution map G is continuous on the quantity $\beta = \sigma + \frac{1}{2N} LW$.

Let $\beta_0 \in W^{1,p}_{\delta-1}$ and set $\psi_0 = G(\beta_0)$. We then define $\widetilde{G}(\beta) := \psi_0^{-1} G(\psi_0^2 \beta)$. Thus, for $\widetilde{g} = \psi_0^{q-2} g$ and $\widetilde{r} = \psi_0^{-\frac{3}{2}q+1}$, conformal covariance 1.1.1 implies that $\phi = \widetilde{G}(\beta)$ is the solution of

$$-a\Delta_{\widetilde{g}}\phi + R_{\widetilde{g}}\phi + \kappa\tau^2\phi^{q-1} - |\beta|^2_{\widetilde{g}}\phi^{-q-1} - \widetilde{r}\phi^{-\frac{q}{2}} = 0 \tag{5.1}$$

such that $\phi - 1 \in W^{2,p}_{\delta}$.

Thus to show that G is continuous in a neighborhood of β_0, we only need to show that $\widetilde{G}$ is continuous near $\psi_0^{-2}\beta_0$. We remark that $\widetilde{G}(\psi_0^{-2}\beta_0) \equiv 1$.

We define the map $\Phi : W^{2,p}_{\delta} \times W^{1,p}_{\delta-1} \to L^p_{\delta-2}$ by

$$\Phi(\phi, \beta) = -a\Delta_{\widetilde{g}}\phi + R_{\widetilde{g}}\phi + \kappa\tau^2\phi^{q-1} - |\beta|^2_{\widetilde{g}}\phi^{-q-1} - \widetilde{r}\phi^{-\frac{q}{2}}. \tag{5.2}$$

Note that $\Phi(\widetilde{G}(\beta), \beta) = 0$. The Gâteaux derivative of Φ is given by

$$\begin{aligned} D\Phi_{\phi,\beta}(h,k) = -a\Delta_{\widetilde{g}}h + R_{\widetilde{g}}h + \kappa(q-1)\tau^2\phi^{q-2}h \\ + (q+1)|\beta|^2\phi^{-q-2}h - 2\phi^{-q-1}\langle\beta, k\rangle + \frac{q}{2}r\phi^{-\frac{q}{2}-1}h. \end{aligned} \tag{5.3}$$

Thus

$$D\Phi_{1,\beta_0}(h,0) = -a\Delta_{\widetilde{g}}h + R_{\widetilde{g}}h + \kappa(q-1)\tau^2 h - (q+1)|\beta|^2 h - \frac{q}{2}rh. \tag{5.4}$$

However, since $G(\psi_0^{-2}\beta_0) \equiv 1$, we have that

$$R_{\tilde{g}} = -\kappa\tau^2 + |\beta_0|^2 + r, \tag{5.5}$$

and so

$$D\Phi_{1,\beta_0}(h,0) = -a\Delta_{\tilde{g}}h + \left[\kappa(q-2)\tau^2 + (q+2)|\beta|^2 + \frac{q+2}{2}r\right] h. \tag{5.6}$$

Since the coefficient of h is positive and is contained in $L^p_{\delta-2}$ by assumption, we see from Proposition 2.1.4 that $D\Phi_{1,\beta_0} : W^{2,p}_\delta \to L^p_{\delta-2}$ is an isomorphism. The implicit function theorem on Banach spaces then implies that G is C^1 in a neighborhood of β_0. □

On compact manifolds, Maxwell [Max09] showed that, in fact, no global subsolution is needed to construct a solution of the conformal constraint equations. For Yamabe nonnegative metrics, he proves this using a Green's function argument to find a uniform lower bound on solutions. In the Yamabe negative case, he uses a conformal factor transforming the metric to one with scalar curvature $-\kappa\tau^2$ as a global subsolution. In the asymptotically Euclidean case, this conformal factor argument works for all Yamabe classes, which makes the argument simpler. We obtain the following existence theorem.

Theorem 5.1.4. *Assume that the admissible seed data* $(g, \tau, N, \sigma, r, J)$ *has the regularity specified in* (3.1)*. Assume that* $r \geq 0$ *and that* g *admits no conformal Killing fields. Suppose there exists a positive global supersolution* ϕ_+*, satisfying the hypotheses of the sub and supersolution theorem 2.1.7. Then for any asymptotic function* $\mathring{u}$ *with asymptotics less than that of* ϕ_+*, there exist* (ϕ, W) *solving the* *conformal constraint equations* (1.3) *such that* ϕ *is positive and* $\phi - \mathring{u}$ *and* W *are in* $W^{2,p}_\delta$.

A similar statement holds for $C^{2,\alpha}_\delta$ *data.*

Proof. We first find a global subsolution. Since the data is admissible, let ψ be the positive conformal factor transforming the metric to one with scalar curvature $-\kappa\tau^2$. Let $\phi_- = \alpha\psi$. As in the proof of Theorem 3.0.9, ϕ_- is a subsolution of the Lichnerowicz equation (1.3a) for any $\alpha \leq 1$, regardless of what W is. Thus ϕ_- is a global subsolution. We then choose α small enough such that $\phi_- \leq \phi_+$ and such that the asymptotics of $\mathring{u}$ are greater than those of ϕ_-.

The sub and supersolution theorem 2.1.7 and the uniqueness theorem 3.0.11 now guarantee that the solution map G is well defined. Since $\phi_\pm$ are global sub and supersolutions, F maps $S := \{\phi \in L^\infty : \phi_- \leq \phi \leq \phi_+\}$ into itself. As discussed above, F is also a compact map. Thus the Schauder fixed point theorem 5.1.1 shows there is a fixed point ϕ of F. Thus $(\phi, W(\phi))$ is a solution to the conformal constraint equations. $\square$

Theorem 5.1.4 shows that, for any fixed choice of the seed data $(g, \tau, N, \sigma, r, J)$ and supersolution ϕ_+ satisfying the hypotheses of the theorem, there is at least a k-dimensional family of solutions (where k is the number of ends), parameterized by the product of the intervals $(0, \phi_{+,i}]$, where $\phi_+ \to \phi_{+,i}$ on the end E_i. This nonuniqueness leads one to enquire about the full extent of these families of solutions: for what asymptotic functions $\mathring{u}$ are there solutions of the conformal constraint equations? Unfortunately, neither the necessary analysis of the linearizations of the operators in (1.3), nor the a priori estimates for the solutions, is clear at this time, so we do not yet have more definitive results on the full family of solutions.

5.2. Global Supersolutions

We have reduced the problem finding solutions of the conformal constraint equations (1.3) to that of finding an appropriate global supersolution. We now present two lemmas which prove useful in finding such supersolutions.

Lemma 5.2.1. *Assume that g is a $W^{2,p}_{\delta}$ AE metric with vanishing scalar curvature. There is a unique solution w to $-a\Delta w = \rho^{\gamma-2}$ with $w = c_\gamma\,\rho^\gamma + \hat{w}$, $c_\gamma = (\gamma^2 + (n-2)\gamma)^{-1}$, and $\hat{w} \in W^{2,p}_{\gamma'}$ where $\gamma' = 2\gamma$ if this number is greater than $2-n$ (or else $\gamma' \in (2-n, \gamma)$).*

Similarly, if g is a $C^{2,\alpha}_{\delta}$ AE metric then this unique solution w decomposes as $c_\gamma \rho^\gamma + \hat{w}$, with $\hat{w} \in C^{2,\alpha}_{\gamma'}$.

Proof. Write $w = c_\gamma\,\rho^\gamma + \hat{w}$ and let $\overline{g}$ be a $W^{2,p}$ metric which agrees with g away from the ends but is exactly Euclidean on each E_j. Then we must solve

$$(-a\Delta + R)\hat{w} = \left(\rho^{\gamma-2} - c_\gamma(-a\Delta_{\overline{g}} + R_{\overline{g}})\rho^\gamma\right) - c_\gamma\left((-a\Delta + R) - (-a\Delta_{\overline{g}} + R_{\overline{g}})\right)\rho^\gamma. \tag{5.7}$$

The first term on the right is L^p with compact support, while the second term lies in $L^p_{2\gamma-2}$, so the entire right hand side lies in $L^p_{2\gamma-2} \subset L^p_{\gamma'-2}$. The result follows from Theorem 2.1.4.

The proof in the Hölder setting is the same. □

The second lemma is a slight weakening of the elliptic estimate (2.8), which is adequate for our purposes.

Lemma 5.2.2. *If (M, g) is AE and has no conformal Killing fields, and if $f \in L^p_{\delta-2}$ with $p > n$, then the unique solution $W \in W^{2,p}_\delta$ to* $\operatorname{div}\frac{1}{2N}LW = f$ *satisfies*

$$\|LW\|_\infty \leq C_1 \rho^{\delta-1} \|f\|_{p,\delta-2}. \tag{5.8}$$

Proof. Combining (2.8) and Sobolev embedding 2.0.1, we get

$$\rho^{1-\delta}|LW| \leq \|LW\|_{C^0_{\delta-1}} \leq C_1' \|LW\|_{W^{1,p}_{\delta-1}} \leq C_1' \|W\|_{W^{2,p}_\delta} \leq C_1 \|f\|_{p,\delta-2}, \tag{5.9}$$

which implies (5.8). □

In the first main result of this section we construct global supersolutions, allowing the mean curvature to be arbitrary but requiring that the other data (except the metric) be quite small.

Theorem 5.2.3 (Far-from-CMC Global Supersolution)**.** *Suppose that (M, g) is a $W^{2,p}_\gamma$ Yamabe positive AE manifold, with $p > n$ and $\gamma \in (2-n, 0)$, and set $\delta = \gamma/2$. Fix $\tau \in W^{1,p}_{\delta-1}$ and $N \in W^{2,p}_\delta$. Suppose $\sigma \in L^\infty_{\delta-1}$, nonnegative $r \in L^\infty_{2\delta-2}$ and $J \in L^p_{\delta-2}$ are sufficiently small (depending on τ, g and n). Then, for any $\mathring{u}$, there exists a global supersolution $\phi_+ > 0$ with $\phi_+ - \eta\mathring{u} \in W^{2,p}_{\gamma'}$ for some constant $\eta > 0$ and any $\gamma' > \gamma$.*

Similarly, if (M, g) is a $C^{2,\alpha}_\gamma$ Yamabe positive AE manifold, and if the corresponding Hölder norms of σ, J and ρ are sufficiently small, then there exists a global supersolution ϕ_+ with $\phi_+ - \eta\mathring{u} \in C^{2,\alpha}_\gamma$ for some $\eta > 0$.

The main ideas used in this proof are similar to those used in the compact case, but there are new issues arising in the construction of the supersolution on each end. Because the proofs in the Sobolev and Hölder settings are identical, we present only the former.

Proof. Since g is Yamabe positive, by conformal covariance 1.1.1, we may assume without loss of generality that $R \equiv 0$. By Lemma 5.2.1, there exists a (unique) $\Psi = \mathring{u} + c_\gamma \rho^\gamma + \hat{\Psi}$, with $\hat{\Psi} \in W^{2,p}_{2\gamma}$, such that

$$-a\Delta\Psi = \rho^{\gamma-2}, \tag{5.10}$$

or equivalently

$$-a\Delta(\Psi - \mathring{u}) = \rho^{\gamma-2}. \tag{5.11}$$

Note that, by the maximum principles 2.1.2 and 2.1.3, $\Psi > 0$.

Now set $\phi_+ = \eta\Psi$, where the constant $\eta > 0$ is to be chosen below. We claim that, for appropriate η, ϕ_+ is a global supersolution. To verify this, we first note that from (5.8), with $f = \kappa\phi^q d\tau + J$, we have

$$\|LW\|_\infty \leq C\rho^{\delta-1}\left(\|d\tau\|_{p,\delta-2}\|\phi\|_\infty^q + \|J\|_{p,\delta-2}\right), \tag{5.12}$$

and hence

$$\left|\sigma + \frac{1}{2N}LW\right|^2 \leq C\rho^{2\delta-2}(\|d\tau\|^2_{p,\delta-2}\|\phi\|^{2q}_\infty + \|\sigma\|^2_{\infty,\delta-1} + \|J\|^2_{p,\delta-2}). \tag{5.13}$$

Since Ψ decays at the precise rate ρ^{γ} (and is strictly positive), then deleting subscripts denoting the norms for simplicity, we calculate

$$
\begin{aligned}
- a\Delta\phi_+ + \kappa\tau^2\phi_+^{q-1} - \left|\sigma + \frac{1}{2N}LW\right|^2 \phi_+^{-q-1} - r\phi_+^{-q/2} \geq \\
\eta\,\rho^{\gamma-2} - \rho^{2\delta-2}\left(C_1\eta^{q-1} + C_2\eta^{-q-1}(\|\sigma\|^2 + \|J\|^2) + C_3\eta^{-q/2}\|r\|\right).
\end{aligned}
\tag{5.14}
$$

The constants C_1, C_2 and C_3 depend only on ρ, N, and the dimension n. Since $2\delta - 2 = \gamma - 2 < 0$ and $q - 1 > 1$, we first choose η sufficiently small so that

$$
\frac{1}{2}\eta\,\rho^{\gamma-2} - C_1\eta^{q-1}\rho^{2\delta-2} > 0, \tag{5.15}
$$

and then choose $\|\sigma\|$, $\|J\|$ and $\|\rho\|$ sufficiently small (depending on C_1, F, n and η), so that

$$
\frac{1}{2}\eta\,\rho^{\gamma-2} - \rho^{2\delta-2}\left(C_2\eta^{-q-1}(\|\sigma\|^2 + \|J\|^2) + C_3\eta^{-q/2}\|\rho\|\right) > 0 \tag{5.16}
$$

as well. This proves that ϕ_+ is a global supersolution. $\square$

The second main result is the existence of a global supersolution for near-CMC data, i.e., where $d\tau$ is sufficiently small as compared to τ.

Theorem 5.2.4 (Near-CMC Global Supersolution)**.** *Suppose that (M, g) is a $C^{2,\alpha}_{\gamma}$, Yamabe positive AE manifold, where $\gamma \in (2-n, 0)$, and set $\delta = \gamma/2$. Fix data $\tau \in C^{1,\alpha}_{\delta-1}$, $N \in C^{2,\alpha}_{\delta}$, $\sigma \in C^{0,\alpha}_{\delta-1}$, nonnegative $r \in C^{0,\alpha}_{\delta-2}$ and $J \in C^{0,\alpha}_{\delta-2}$. Then, there exists a constant $B > 0$, depending on the seed data, but not on τ, such that if τ satisfies $\tau^2 - B\|d\tau\|^2_{C^{0,\alpha}_{\delta-2}}\rho^{2\delta-2} \geq 0$, then there exists a global supersolution $\phi_+ > 0$ with $\phi_+ - \eta \in W^{2,p}_{\gamma}$ for any constant $\eta > 0$ sufficiently large.*

Remark 5.2.5. *The hypothesis* $\tau^2 - B\|d\tau\|^2_{C^{0,\alpha}_{\delta-2}}\rho^{2\delta-2} \geq 0$ *is precisely where the use of Hölder rather than Sobolev data is important for asymptotically Euclidean data. Indeed, if* τ *satisfies this inequality, then in particular,* $\tau \geq C\rho^{\delta-1}$*, so the norm of* τ *in* $L^p_{\delta-1}$ *is necessarily infinite.*

The condition $\tau^2 - B\|d\tau\|^2_{C^{0,\alpha}_{\delta-2}}\rho^{2\delta-2} \geq 0$*, which in particular imposes a lower bound on the decay of* τ *and requires that* τ *never vanishes, may not be fulfilled by any functions* τ*. Indeed, as we show in Chapter VI, similar near-CMC conditions are not always fulfilled.*

Proof. Since τ never vanishes, the data is admissible (cf. Definition 3.0.10), and so by conformal covariance 1.1.1, we may assume without loss of generality that $R = -\kappa\tau^2$. By Theorem 2.1.4, there exists a solution u to

$$-a\Delta u = r + |\sigma|^2 \tag{5.17}$$

with $u - 1 \in C^{2,\alpha}_{\delta}$. Indeed, this is equivalent to

$$-a\Delta(u-1) = r + |\sigma|^2 \geq 0, \tag{5.18}$$

and so the maximum principle 2.1.2 shows that $u \geq 1$. By the estimate (2.10), $\sup u$ is bounded and depends only on $r, |\sigma|$ and g.

Now set $\phi_+ = \eta u$, where η is chosen below, and using estimate (5.12) and the inequality $\phi \leq \phi_+$, we have

$$
\begin{aligned}
|LW|^2 \leq C^2 \rho^{2\delta-2} ((\sup \phi)^q \|d\tau\|_{C^{0,\alpha}_{\delta-2}} + \|J\|_{C^{0,\alpha}_{\delta-2}})^2 \\
\leq 2C^2 \rho^{2\delta-2} ((\sup \eta u)^{2q} \|d\tau\|^2_{C^{0,\alpha}_{\delta-2}} + \|J\|^2_{C^{0,\alpha}_{\delta-2}}) \\
\leq 2C^2 \rho^{2\delta-2} \left((\sup u)^{2q} \|d\tau\|^2_{C^{0,\alpha}_{\delta-2}} (\eta u)^{2q} + \|J\|^2_{C^{0,\alpha}_{\delta-2}} \right); \quad (5.19)
\end{aligned}
$$

the constant C is the same one appearing in (5.12).

Dropping the subscripts on the norms, and using the fact that $\tau^2 \geq C\rho^{2\delta-2}$ for some $C > 0$, we calculate

$$
\begin{aligned}
- a\Delta\phi_+ - \kappa\tau^2\phi_+ + \kappa\tau^2\phi_+^{q-1} - \left|\sigma + \frac{1}{2N} LW\right|^2 \phi_+^{-q-1} - r\phi_+^{-q/2} \\
\geq \kappa\tau^2 \left[(\eta u)^{q-1} - \eta u\right] + r\left[\eta - (\eta u)^{-q/2}\right] - \left|\sigma + \frac{1}{2N} LW\right|^2 \phi_+^{-q-1} + |\sigma|\eta \\
\geq \frac{1}{2}\kappa\tau^2 \left[(\eta u)^{q-1} - \eta u - C\|J\|\|\tau\|^{-2}(\eta u)^{-q-1}\right] + (\eta u)^{q-1} \left[\frac{1}{2}\kappa\tau^2 - C\rho^{2\delta-2}(\sup u)^{2q}\|d\tau\|^2\right] \\
\geq 0, \quad (5.20)
\end{aligned}
$$

for all η large enough, as long as

$$
C\rho^{2\delta-2}(\sup u)^{2q}\|d\tau\|^2 \leq \tau^2, \tag{5.21}
$$

which has been assumed. This shows that ϕ_+ is a global supersolution. $\square$

We now present a scaling argument of Nguyen [Ngu14], which shows that the far-from-CMC result 5.2.3 is simply a rescaling of a near-CMC result.

Proposition 5.2.6. (ϕ, W) *is a solution of the conformal constraint equation* (1.3) *for the seed data* $(g, \tau, N, \sigma, r, J)$ *if and only if* $(C^{-1}\phi, C^{-q/2-1}W)$ *is a solution of the conformal constraint equations for the data* $(g, C^{q/2-1}\tau, N, C^{-q/2-1}\sigma, C^{-q/2-1}r, C^{-q/2-1}J)$.

Proof. Plugging the scaled data and solutions into conformal constraint equations gives this immediately. □

Corollary 5.2.7. *Given a far-from-CMC solution to the conformal constraint equations, as given by Theorem 5.2.3, there is a solution of the conformal constraint equations equivalent to it, in the sense of Proposition 5.2.6, which is a perturbation of the CMC case,* $\tau \equiv 0$.

It is well known (see [CBIY00]) that perturbations of the CMC case always lead to solutions of the conformal constraint equations. This means that the far-from-CMC result is simply the rescaling of a previously known near-CMC result. This means, unfortunately, that virtually nothing is known about the far-from-CMC case in the general sense, i.e., with σ, r and J arbitrary. We do note that most existing near-CMC results require that $d\tau$ be sufficiently small as compared to $\inf \tau^2$, or similar, while the near-CMC condition for the perturbation of $\tau \equiv 0$ is that the $W^{1,p}_{\delta-1}$ norm is sufficiently small. In particular, τ can have zeroes.

It is also interesting to consider the range of allowed asymptotic functions. The far-from-CMC result allows only very small asymptotic functions, while this rescaled near-CMC result allows arbitrarily large asymptotic functions, as long as τ is sufficiently small. Since all asymptotic functions are allowed if the seed data (τ, σ, r, J) vanishes (since we are then just solving $\Delta\phi = 0$), this leads one to wonder whether large asymptotic functions are only ever attainable if τ is small.

5.3. Local Supersolutions

The search for global supersolutions for more general cases than those considered in the previous section has been very difficult, partly since any such construction seems to require an estimate on LW like (5.12), which in turn seems to allow only either τ to be small or the rest of the data to be small. However, requiring a global supersolution in order for solutions of the conformal constraints to exist is likely stricter than necessary. In order to give new tools for finding solutions to the conformal constraint equations, Nguyen introduced the idea of "local supersolutions."

Definition 5.3.1. *A function $\phi_+ \in L^\infty$ is a "local supersolution" of the conformal constraint equations if for every positive function $\phi \leq \phi_+$ with $\phi = \phi_+$ somewhere, there exists $p \in M$ such that $F(\phi)(p) \leq \phi(p)$. (For the definition of F, see Section 5.1.)*

Local supersolutions are more general than global supersolutions.

Proposition 5.3.2. *A global supersolution is a local supersolution.*

Proof. Let ϕ_+ be a global supersolution, and suppose ϕ is a positive function such that $\phi \leq \phi_+$ and $\phi(p) = \phi_+(p)$ at $p \in M$. Then, by the definition of global supersolution, $F(\phi)(p) \leq \phi_+(p) = \phi(p)$, and thus ϕ_+ is a local supersolution. □

Nguyen originally used the idea of half-continuity along with a fixed point theorem allowing for half-continuous maps to show that the existence of a local supersolution leads to a solution of the conformal constraint equations. The following result is Theorem 4.12 in [Ngu14].

Theorem 5.3.3. *Suppose the seed data* $(g, \sigma, 1/2, \tau, 0, 0)$*, M compact, $p > n$, are such that the zero set of τ has zero measure and g allows no conformal Killing fields. If g is Yamabe nonnegative, assume $\sigma \not\equiv 0$. If g is Yamabe negative, assume there is a conformal factor changing the metric to one with scalar curvature $-\kappa\tau^2$. Let $F : L^\infty \to L^\infty$ be the iteration map for the constraint equations, as described in Section 5.1. If there exists a local supersolution ψ, then there exists a fixed point ϕ for F with $\phi \leq \psi$. In particular, the conformal constraint equations have a solution.*

The seeming advantage of this result over using a global supersolution is that one needs only to check that the solution is smaller at a single point, rather than on the entire manifold. We present a simpler proof of a slightly stronger result. The new proof is based on the Leray-Schauder alternative.

Theorem 5.3.4 (Leray-Schauder Alternative)**.** *Let $F : X \to X$ be a compact, continuous map of a normed linear space. Let Ω be a bounded star-shaped open subset of X containing 0, and suppose that $x \in \partial\Omega$ implies that $F(x) \neq \lambda x$ for any $\lambda > 1$. Then F has a fixed point on $\overline{\Omega}$.*

Theorem 5.3.5. *Let $(g, \tau, N, \sigma, r, J)$ be seed data. Suppose g has no conformal Killing fields. If M is AE, suppose the data is admissible. If M compact and g is Yamabe nonnegative, assume $\sigma \not\equiv 0$. If M is compact and g is Yamabe negative, assume there is a conformal factor changing the metric to one with scalar curvature $-\kappa\tau^2$. Suppose there exists $\psi \in L^\infty$ such that for any $0 < \phi \leq \psi$, with* $\inf |\phi - \psi| = 0$*, that $F(\phi) \neq \lambda\phi$ for all $\lambda > 1$. Then there exists a fixed point ϕ for F with $\phi \leq \psi$. In particular, the conformal constraint equations have a solution.*

Proof. Define $F'(\phi) = F(|\phi|)$. The map $F : L^\infty \to L^\infty$ is a compact continuous map by the proof of Theorem 5.1.4, and so F' is as well. Thus we only need to find an appropriate set Ω, as in Theorem 5.3.4.

Let $\Omega = \{\phi \in L^\infty : -\psi < \phi < \psi\}$. Clearly Ω is bounded, open, star-shaped, and contains zero. Suppose there were some $\phi \in \partial\Omega$ and $\lambda > 1$ such that $F'(\phi) = \lambda\phi$. In particular, this means that $\inf|\phi - \psi| = 0$. Since F' outputs only positive solutions, $\phi > 0$. By assumption, there are no such ϕ. Thus Ω fulfills the conditions of Theorem 5.3.4, and so F' has a fixed point ϕ with $\phi > 0$. Since $\phi > 0$, $F'(\phi) = F(\phi)$, and so F has a fixed point as well. $\square$

Note that for M compact, $\inf|\phi - \psi| = 0$ means $\phi = \psi$ somewhere. For AE manifolds, though, ϕ may not equal ψ anywhere.

Other than applying to AE manifolds, Theorem 5.3.5 has a few advantages over Theorem 5.3.3. The main advantage is that one only needs to show that $F(\phi) \neq \lambda\phi$ instead of showing that $F(\phi)(p) \leq \phi(p)$ for appropriate ϕ, a slightly more general condition. This allows one to assume that the solution for the Lichnerowicz equation is a multiple of the function one began with, rather than an arbitrary solution. Unfortunately, no local supersolution that is not also a global supersolution has yet been found.

CHAPTER VI

THE LIMIT EQUATION CRITERION

Another method of finding solutions to the conformal constraint equations (1.3) is the limit equation criterion, originally introduced in [DGH12]. This result says that if a particular equation, called the limit equation, has *no* solutions, then the conformal constraint equations have a solution. To be precise, the main result of [DGH12] says the following:

Theorem 6.0.6. *Suppose the seed data* $(g, \sigma, 1/2, \tau, 0, 0)$*,* M *compact,* $p > n$*, are such that* $\tau > 0$ *and* g *allows no conformal Killing fields. If* g *is Yamabe nonnegative, assume* $\sigma \not\equiv 0$*. If* g *is Yamabe negative, assume there is a conformal factor changing the metric to one with scalar curvature* $-\kappa\tau^2$*. Then at least one of he following holds:*

- *The conformal constraint equations* (1.3) *admit a solution* (ϕ, W) *with* $\phi > 0$*. Furthermore, the set of solutions* $(\phi, W) \in W^{2,p} \times W^{2,p}$ *is compact.*
- *There exists a nontrivial solution* $W \in W^{2,p}$ *of the limit equation*

$$\mathrm{div} LW = \alpha_0 \sqrt{1/\kappa} |LW| \frac{d\tau}{\tau} \tag{6.1}$$

for some $\alpha_0 \in (0, 1]$*.*

The name "limit equation" comes from the original method of proof. In [DGH12], they first show that a "subcritical" version of the conformal constraint equations always have a solution. They make the equations subcritical by changing the exponent of ϕ in the vector equation (1.3b) from q to $q - \epsilon$. This allows global supersolutions to be easily found. Dahl, Gicquaud, and Humbert then show that if a sequence of these solutions with $\epsilon \to 0$ are bounded, then they must converge to a solution of the original equation. If they are unbounded, then they must converge to a solution of the limit equation (6.1). Since then, another, simpler method has been found for setting up the sequence (cf. [Ngu14]), though the argument for convergence is essentially the same.

The two conclusions of Theorem 6.0.6 are not a dichotomy. Nguyen in [Ngu14] showed that there is seed data on the sphere that allows solutions to both the conformal constraint equations and the limit equation. Thus, unfortunately, the use of the limit equation is limited to the case mentioned earlier. If one can show, for particular seed data, that the limit equation has no solutions, then the conformal constraint equations must have a solution.

Though the limit equation criterion itself is not limited to this case, the criterion has only been successfully applied for near-CMC seed data. In particular, in [DGH12], they find that the limit equation (6.1) has no nontrivial solutions if either the C^0 or L^n norm of $d\tau/\tau$ is sufficiently small. The reason it is difficult to prove stronger results is that the usual method of proving nonexistence is to find an estimate for the right hand side of (6.1) that provides a contradiction. In order to do this, seemingly the only tool that one has is to make $d\tau/\tau$ small, since LW appears on both sides of the limit equation.

In this chapter, we prove most of the limit equation criterion in the AE setting, except for the vital condition that the solution W of the limit equation must be nontrivial. The proof breaks down only at this point. It can be repaired if the data is near-CMC. Since the compact case has only been applied to near-CMC data, this seems like a reasonable assumption. However, since we must also require that $\tau \to 0$ at infinity and that $\tau > 0$, arbitrarily near-CMC data does not exist, and thus the near-CMC condition is very difficult to check. This chapter is based on an unpublished collaboration with Jim Isenberg and Romain Gicquaud.

6.1. Setup on Asymptotically Euclidean Manifolds

The main difficulty in translating the limit equation criterion to asymptotically Euclidean manifolds is the fall off rate of τ. In the original proof, the assumption that $\tau > \epsilon > 0$ is vital; this is most easily seen by noticing that we divide by τ in the limit equation (6.1). However, for the seed data to lead to asymptotically Euclidean initial data, τ must converge to zero. These competing conditions lead to most of the additional difficulty in the AE case.

The proof is simpler when using Hölder norms for essentially the same reasons as discussed in Remark 5.2.5. Put together, our assumptions are as follows.

Assumption 6.1.1 (Seed Data Assumption)**.** *The seed data $(g, \tau, N, \sigma, r, J)$ satisfy the conditions*

- (M, g) *is a* $C^{2,\alpha}_\delta$ *AE manifold, with* $\delta \in (2-n, 0)$*, allowing no conformal Killing fields.*
- $\tau \in C^{1,\alpha}_{\delta-1}$*, and* $|\tau| \geq C\rho^{\delta-1} > 0$*. Without loss of generality, assume that* $\tau > 0$*.*
- $N - 1 \in C^{2,\alpha}_\delta$*.*
- $\sigma \in C^{0,\alpha}_{\delta-1}$*. Thus* $|\sigma| \leq C\tau$*.*
- $0 \leq r$*, and* $r \in C^{0,\alpha}_{\delta-2}$*. Thus* $r \leq C\tau$*.*
- $J \in C^{0,\alpha}_{\delta-2}$*.*

We can now state the main result of this chapter.

Theorem 6.1.2. *Suppose the seed data satisfies Assumption 6.1.1. Then at least one of the following holds:*

- *For any asymptotic function* $\mathring{u}$*, the conformal constraint equations* (1.3) *admit a solution* (ϕ, W) *with* $\phi > 0$ *and* $\phi - \mathring{u}$ *and* W *in* $C^{2,\alpha}_{\delta}$*.*

- *There exists a (perhaps trivial) solution* $W \in W^{2,p}_{\delta}$ *of the limit equation*

$$\mathrm{div}\frac{1}{2N}LW = \alpha_0\sqrt{1/\kappa}|LW|\frac{d\tau}{2N\tau} \tag{6.2}$$

  for some $\alpha_0 \in [0,1]$*. Furthermore,* $|W| \leq C\rho^{\delta}$ *and* $|LW| \leq C\rho^{\delta-1}$ *for some* C *dependent only on* g *and* $\|d\tau\|_{C^0_{\delta-2}}$*. If*

$$\kappa\tau^2 - \frac{1}{4N^2}|LW_0|^2 \geq c\tau^2 \tag{6.3}$$

  for some $c > 0$ *and for all solutions* W_0 *of the vector equation* (1.3b) *with* $J \equiv 0$ *and* $\phi \leq 1$*, then the solution* W *of the limit equation is nontrivial, and* $\alpha_0 \neq 0$*.*

The near-CMC condition (6.3) is phrased oddly because it is easier to work with later. Similar to estimate (5.8),

$$\begin{aligned} \|LW_0\|_{C^0_{\delta-1}} &\leq C\|\phi^q|d\tau| + J\|_{C^0_{\delta-2}} \\ &\leq C\|d\tau\|_{C^0_{\delta-2}}. \end{aligned} \tag{6.4}$$

Note that the constant C only depends on g. Thus a sufficient condition for (6.3) to hold is that there exists a $C > 0$ depending only on g and N, such that $\tau^2 - C\|d\tau\|^2_{C^0_{\delta-2}}\rho^{2\delta-2} \geq c\tau^2$. This is more clearly a near-CMC condition.

The simpler proof of the limit equation criterion presented in [Ngu14] is based on the Schaefer fixed point theorem.

Theorem 6.1.3 (Schaefer Fixed Point Theorem)**.** *Assume that $F : X \to X$ is a compact map on a Banach space, and assume that the set*

$$K = \{x \in X : \exists t \in [0,1] \text{ such that } x = tF(x)\} \tag{6.5}$$

is bounded. Then F has a fixed point.

Let F be the iteration map, as in Chapter V, giving a solution of the Lichnerowicz equation with asymptotic function $\mathring{u}$. Recall that F is a compact map on L^∞. We use the Schaefer fixed point theorem by setting

$$K := \{\phi \in L^\infty : \exists t \in [0,1] \text{ such that } \phi = tF(\phi)\}. \tag{6.6}$$

Note that the definition of K does not directly mention the asymptotic function of ϕ. However, since $\phi = tF(\phi)$, we know that $\phi - t\mathring{u} \in W^{2,p}_\delta$, for some $t \in (0,1]$.

6.2. Convergence of Solutions

In this section, we show that if K is unbounded, then the limit equation has a solution. By conformal covariance, and since $\tau^2 > 0$, it follows from the prescribed scalar curvature theorem 4.3.1 that we can assume without loss of generality that $R = -\kappa\tau^2$. The definition of K shows that K being unbounded is equivalent to the existence of an unbounded sequence (ϕ_i, t_i) with $t_i \in (0, 1]$ solving

$$-a\Delta\phi_i - \kappa\tau^2\phi_i + \kappa\tau^2\phi_i^{q-1} - \left|\sigma + \frac{1}{2N}LW_i\right|^2 \phi_i^{-q-1} - r\phi_i^{-q/2} = 0 \tag{6.7a}$$

$$\operatorname{div}\frac{1}{2N}LW = \kappa(t_i\phi_i)^q d\tau + J. \tag{6.7b}$$

Lemma 6.2.1. *Suppose such a sequence (ϕ_i, t_i) exists. Then there exists a (perhaps trivial) solution $W \in W^{2,p}_\delta$ of the limit equation*

$$\operatorname{div}\frac{1}{2N}LW = \alpha_0\sqrt{1/\kappa}|LW|\frac{d\tau}{2N\tau} \tag{6.8}$$

for some $\alpha_0 \in [0, 1]$. Furthermore, $|W| \leq C\rho^\delta$ and $|LW| \leq C\rho^{\delta-1}$ for some C dependent only on g and $\|d\tau\|_{C^0_{\delta-2}}$. If

$$\kappa\tau^2 - \frac{1}{4N^2}|LW_0|^2 \geq c\tau^2 \tag{6.9}$$

for some $c > 0$ and for all solutions W_0 of the vector equation (1.3b) *with $J \equiv 0$ and $\phi \leq 1$, then the solution W of the limit equation is nontrivial, and $\alpha_0 \neq 0$.*

Proof. We claim the sequence $W_i := W(\phi_i)$ (sub)converges to a solution of the limit equation, up to rescaling. In the original proof of the limit equation criterion, the sequence of subcritical solutions was renormalized by an energy based on $|LW_i| := |LW(t_i\phi_i)|$. While this energy has some advantages which we discuss below, its use requires proving an estimate of the form $\sup\phi_i^q \leq \sup\{1, \int|LW|^2\}$. Despite some effort, we were unable to prove an analogous estimate for AE data. The problem is that $|LW_i|$ falls off as τ, but the convergence of a renormalized $|LW_i|$ is only in a weaker space. It is essentially for this same reason that we are only able to prove nontriviality of the solution of the limit equation for near-CMC data.

Instead of an energy based on $|LW_i|$, we bound $\sup \phi_i$ directly. By assumption $\sup \phi_i \to \infty$. Let $\Gamma_i = \sup \phi_i$. We start by rescaling the seed data and solutions by certain powers of the energy. In particular, we rescale σ, r, J, ϕ_i, and W_i (we do not rescale the metric, N, or τ) as follows

$$\widetilde{\phi}_i := \Gamma_i^{-1}\phi_i, \quad \widetilde{W}_i := \Gamma_i^{-q}W_i, \quad \widetilde{\sigma}_i := \Gamma_i^{-q}\sigma, \quad \widetilde{r}_i := \Gamma_i^{\frac{-3q}{2}+1}r, \quad \widetilde{J}_i := \Gamma_i^{-q}J. \tag{6.10}$$

Then, dividing the conformal constraint equations (6.7) by certain powers of the energy, and substituting in these rescaled quantities, we obtain

$$\frac{1}{\Gamma_i^{q-2}}\left(-a\Delta\widetilde{\phi}_i - \kappa\tau^2\widetilde{\phi}_i\right) + \kappa\tau^2\widetilde{\phi}_i^{q-1} - \left|\widetilde{\sigma}_i + \frac{1}{2N}L\widetilde{W}_i\right|^2 \widetilde{\phi}_i^{-q-1} - \widetilde{r}_i\widetilde{\phi}_i^{-q/2} = 0, \tag{6.11a}$$

$$\operatorname{div}\frac{1}{2N}L\widetilde{W}_i = \kappa(t_i\widetilde{\phi}_i)^q d\tau + \widetilde{J}_i. \tag{6.11b}$$

Proceeding, we substantially follow the original proof from [DGH12]. Similar to the estimate (6.4),

$$\begin{aligned}\|W_i\|_{C^{1,\beta}_\delta} &\le C\|t_i^q\phi_i^q|d\tau| + J\|_{C^0_{\delta-2}}\\ &\le C\Gamma_i^q\left(\|d\tau\|_{C^0_{\delta-2}} + \|J/\Gamma_i^q\|_{C^0_{\delta-2}}\right).\end{aligned} \tag{6.12}$$

Note that the constant C only depends on g, N, and our choice of $\beta > \alpha$. Consequently, $\widetilde{W}_i$ is uniformly bounded in $C^{1,\beta}_\delta$ and $L\widetilde{W}_i$ is uniformly bounded in $C^{0,\beta}_{\delta-1}$. Using the compact Sobolev embedding 2.0.2, a subsequence of $\widetilde{W}_i$ converges in $C^{1,\alpha}_{\delta'}$ for any $\delta' > \delta$ to some $\widetilde{W}_\infty \in C^{1,\beta}_\delta$. Thus, in particular,

$$|\widetilde{W}_\infty| \le C\rho^\delta \quad \text{and} \quad |L\widetilde{W}_\infty| \le C\rho^{\delta-1}, \tag{6.13}$$

for C dependent only on g, N, and β. We cannot, however, be certain that $\widetilde{W}_\infty \not\equiv 0$.

Heuristically, as $i \to \infty$ we would expect all the terms in the rescaled Lichnerowicz equation (6.11a) except the τ^2 and $L\widetilde{W}_i$ terms to go to zero. Thus, we define the function $\widetilde{\phi}_\infty$ by

$$\kappa\tau^2\widetilde{\phi}_\infty^{q-1} := \frac{1}{4N^2}|L\widetilde{W}_\infty|^2\widetilde{\phi}_\infty^{-q-1}, \tag{6.14}$$

which reduces to

$$\widetilde{\phi}_\infty^q = \frac{|L\widetilde{W}_\infty|}{2N\sqrt{\kappa}\tau}. \tag{6.15}$$

Comparing expression (6.15) with the rescaled vector constraint equation (6.11b), we see that if $\widetilde{\phi}_i \to \widetilde{\phi}_\infty$ in an appropriate space, and if $\widetilde{W}_\infty$ is a solution (in an appropriate sense) to the limit of equation (6.11b) as $i \to \infty$, then it follows that $\widetilde{W}_\infty$ is a solution to the limit equation (6.8). So long as $\widetilde{\phi}_\infty$ is not identically zero, this solution is non-trivial. Therefore, we focus on verifying these limits.

For any $\epsilon > 0$, we claim there is an i_0 such that if $i \geq i_0$ that

$$|\widetilde{\phi}_i - \widetilde{\phi}_\infty| < \epsilon\rho^\epsilon. \tag{6.16}$$

If (6.16) holds (for small ϵ), it then follows that $\widetilde{\phi}_i \to \widetilde{\phi}_\infty$ in $C^0_{\epsilon'}$ for any small $\epsilon' > 0$. Recalling the definition of $\widetilde{\phi}_\infty$ from (6.15), we let $\widetilde{\phi}_+$ be any C^2 function for which the inequality

$$\widetilde{\phi}_\infty + \frac{\epsilon}{2}\rho^\epsilon \leq \widetilde{\phi}_+ \leq \widetilde{\phi}_\infty + \epsilon\rho^\epsilon. \tag{6.17}$$

holds. We claim that $\widetilde{\phi}_+ \geq \widetilde{\phi}_i$ everywhere for i large enough. Since $\widetilde{\phi}_i \leq 1$, it is immediately clear that this is true except on a compact set depending only on ϵ. On that compact set, we claim that $\widetilde{\phi}_+$ is a (pointwise) supersolution of the rescaled Lichnerowicz equation (6.11a) for all i is large enough.

Multiplying the rescaled Lichnerowicz equation (6.11a) by $\widetilde{\phi}_+^{q+1}$, we need to verify the inequality

$$\frac{\widetilde{\phi}_+^{q+1}}{\Gamma^{q-2}}\left(-a\Delta\widetilde{\phi}_+ - \kappa\tau^2\widetilde{\phi}_+\right) + \kappa\tau^2\widetilde{\phi}_+^{2q} \geq \left|\widetilde{\sigma} + \frac{1}{2N}L\widetilde{W}_i\right|^2 + \widetilde{r}\widetilde{\phi}^{q/2+1}. \tag{6.18}$$

Since, by definition,

$$\widetilde{\phi}_+^{2q} \geq \left(\widetilde{\phi}_\infty + \frac{\epsilon}{2}\rho^\epsilon\right)^{2q} \geq \widetilde{\phi}_\infty^{2q} + \left(\frac{\epsilon}{2}\rho^\epsilon\right)^{2q}, \tag{6.19}$$

inequality (6.18) is satisfied provided that

$$\begin{aligned}\frac{\widetilde{\phi}_+^{q+1}}{\Gamma_i^{q-2}}\left(-a\Delta\widetilde{\phi}_+ - \kappa\tau^2\widetilde{\phi}_+\right) + \kappa\tau^2\left(\frac{\epsilon}{2}\rho^\epsilon\right)^{2q}& \\ \geq \left|\widetilde{\sigma}_i + \frac{1}{2N}L\widetilde{W}_i\right|^2 - \frac{1}{2N}|L\widetilde{W}_\infty|^2 + \widetilde{r}_i\widetilde{\phi}^{q/2+1}.&\end{aligned} \tag{6.20}$$

Recalling that $L\widetilde{W}_i \to L\widetilde{W}_\infty$, we readily verify that all of the terms in equation (6.20) go to zero pointwise as $i \to \infty$ except for the ϵ term. Thus for any fixed compact set, there exists an i_0 such that for all $i \geq i_0$, $\widetilde{\phi}_+$ is a pointwise supersolution on that compact set.

We want to use the sub and supersolution theorem to prove that $\widetilde{\phi}_+ \geq \widetilde{\phi}_i$ for large i. Since $\widetilde{\phi}_+$ is only a supersolution on a compact set, we cannot use the sub and supersolution theorem on AE manifolds 2.1.7. However, in the complement of the compact set $\{\frac{\epsilon}{2}\rho^\epsilon \leq 1\}$, we know $\widetilde{\phi}_+ \geq \widetilde{\phi}_i$. Thus we can use the sub and supersolution theorem on compact manifolds with boundary (cf. [Dil14]) to find a solution of (6.11a) less than $\widetilde{\phi}_+$ on that compact set. (For convenience, we can use $\alpha\Gamma_i^{-1}$ for the subsolution. It is a global subsolution for any $\alpha \leq 1$, independent of ϵ.) Since such solutions are unique, it must be $\widetilde{\phi}_i$. Thus we obtain the pointwise inequality

$$\widetilde{\phi}_i \leq \widetilde{\phi}_+ \leq \widetilde{\phi}_\infty + \epsilon\rho^\epsilon. \tag{6.21}$$

everywhere on M.

We prove a similar (subsolution type) result for any C^2 function $\widetilde{\phi}_-$ satisfying

$$\widetilde{\phi}_\infty - \epsilon\rho^\epsilon \leq \widetilde{\phi}_- \leq \widetilde{\phi}_\infty - \frac{\epsilon}{2}\rho^\epsilon. \tag{6.22}$$

Since $\widetilde{\phi}_i > 0$, we only need to show that $\widetilde{\phi}_-$ is a subsolution of the Lichnerowicz equation where $\widetilde{\phi}_-$ is positive. The proof follows similarly, using $\widetilde{\phi}_+$ as our supersolution. We thus have

$$\widetilde{\phi}_i \geq \widetilde{\phi}_\infty - \epsilon\rho^\epsilon \tag{6.23}$$

for large enough i.

Using the two inequalities (6.21) and (6.23), we conclude that $\widetilde{\phi}_i \to \widetilde{\phi}_\infty$ in $C^0_{\epsilon'}$ for any $\epsilon' > 0$. This implies that $\widetilde{\phi}_i^q d\tau \to \widetilde{\phi}_\infty^q d\tau$ in $L^p_{\delta-2+\epsilon' q}$. Also, a subsequence of t_i^q converges to a number $\alpha_0 \in [0,1]$. Thus $\widetilde{W}_i \to \widetilde{W}_\infty$ in $W^{2,p}_{\delta+\epsilon' q}$ for $\epsilon' < -\delta/q$. The $\widetilde{W}_\infty$ weakly satisfies the limit equation (6.8). Since the right hand side of the limit equation is in $C^{0,\alpha}_{\delta-2}$ (since $L\widetilde{W}_\infty \in C^{0,\beta}_{\delta-1}$), we can conclude by Proposition 2.1.1 that $\widetilde{W}_\infty \in C^{2,\alpha}_\delta$.

Our only remaining task is to verify that $W_\infty \not\equiv 0$ in the near-CMC case. In this case, if $\alpha_0 = 0$, any solution of the limit equation is a conformal Killing field, which implies that $W_\infty \equiv 0$, a contradiction (cf. Proposition 2.1.4). To show that W_∞ is not trivial, we repeat the previous argument with a few changes. This implies that $\widetilde{W}_\infty \not\equiv 0$.

Assume that $L\widetilde{W}_\infty \equiv 0$, which by definition implies that $\widetilde{\phi}_\infty \equiv 0$. Let $\widetilde{\phi}_+$ be a constant less than 1. We can derive the equivalent of equation (6.20) for this $\widetilde{\phi}_+$, namely

$$-\frac{\widetilde{\phi}_+^{q+2}}{\Gamma_i^{q-2}}\kappa\tau^2\widetilde{\phi}_+ + \kappa\tau^2\widetilde{\phi}_+^{2q} \geq \left|\widetilde{\sigma}_i + \frac{1}{2N}L\widetilde{W}_i\right|^2 + \widetilde{r}_i\widetilde{\phi}^{q/2+1}. \tag{6.24}$$

We note that as $i \to \infty$, the J term in equation (6.12) goes to zero. Thus, by our near-CMC assumption,

$$\kappa\tau^2\widetilde{\phi}_+^{2q} - \left|\widetilde{\sigma}_i + \frac{1}{2N}L\widetilde{W}_i\right|^2 \geq c\tau^2 \tag{6.25}$$

for large enough i, $\widetilde{\phi}_+$ sufficiently close to 1, and some small $c > 0$. The other terms in (6.24) go to zero in $C^0_{2\delta-2}$, and so for i large enough, (6.24) holds on all of M.

Using the sub and supersolution theorem on AE manifolds, we can deduce

$$\widetilde{\phi}_i \leq \widetilde{\phi}_+ < 1. \tag{6.26}$$

Since $\widetilde{\phi}_i = 1$ somewhere, this is a contradiction. □

The main result is now easily proved.

Proof of Theorem 6.1.2. Let F be the iteration map, as in Chapter V, giving a solution of the Lichnerowicz equation with asymptotic function $\mathring{u}$. Let

$$K := \{\phi \in L^\infty : \exists t \in [0,1] \text{ such that } \phi = tF(\phi)\} \tag{6.27}$$

If K is bounded, then the Schaefer fixed point theorem 6.1.3 gives a solution to the conformal constraint equations. If K is unbounded, Lemma 6.2.1 shows the limit equation has a solution. $\square$

We now make a number of remarks on the proof of Lemma 6.2.1, which is the heart of the limit equation criterion. On compact manifolds, it is easy to show that $\widetilde{\phi}_\infty$ is nontrivial. By definition, $\widetilde{\phi}_i = 1$ at some point p_i, and since M is compact, the p_i converge to some point p_∞. Since the $\widetilde{\phi}_i$ converge in L^∞, we have $\widetilde{\phi}_\infty(p_\infty) = 1$, and so $\widetilde{\phi}_\infty$ and $\widetilde{W}_\infty$ are nontrivial. On AE manifolds, this argument breaks down at two points. First, since M is noncompact, the points p_i may wander off to infinity. Indeed, if they were contained on a compact set, then the $\widetilde{\phi}_i$ would converge to a nontrivial $\widetilde{\phi}_\infty$ in a similar fashion. Second, we are only able to show that $\widetilde{\phi}_i$ converges to $\widetilde{\phi}_\infty$ in $C^0_{\epsilon'}$ for $\epsilon' > 0$, and thus $\widetilde{W}_i$ only converges to $\widetilde{W}_\infty$ in $C^{1,\beta}_{\delta+\epsilon' q}$. In particular, if $\widetilde{W}_\infty$ were trivial, $|L\widetilde{W}_i|^2$ converges to zero only in $C^0_{2\delta-2+2\epsilon' q}$, and so we can only show the inequality (6.24) on compact sets, unless we make the near-CMC assumption that we did. Fixing either of these two points would show that the solution to the limit equation is nontrivial.

Another idea, as discussed above, is to use some relative of $\|LW_i\|_2$ as the energy. If $L\widetilde{W}_i$ converges in that norm, then $\|L\widetilde{W}_\infty\|_2$ would be 1, and thus nontrivial. This type of argument works in the compact case, and is in fact what Dahl, Gicquaud, and Humbert originally did in [DGH12]. Unfortunately, proving convergence of the $\widetilde{\phi}_i$ requires an estimate related to $\sup \phi_i^q \leq C \max\{\|LW_i\|_2, 1\}$. We have tried to prove such an estimate for AE manifolds, but were unable to do so. The problem is, again, that $|LW_i|$ and τ have the same falloff at infinity.

One possible reason why the proof of the limit equation criterion for AE manifolds has proven difficult is its relationship with asymptotic functions. The limit equation criterion 6.1.2, as long as the limit equation has no solutions, gives a solution for *any* asymptotic constant. In chapter V, however, except in the near-CMC case, our proof strongly depends on the asymptotic function being sufficiently small. Using the rescaling of Proposition 5.2.6, the asymptotic function is allowed to be larger, as long as τ scaled towards zero, a kind of near-CMC condition. Thus, perhaps, the limit equation criterion for more arbitrary τ may be easier to prove as long as the asymptotic function is sufficiently small. Unfortunately, we were also unable to leverage this idea.

We can use a modification of the argument in Lemma 6.2.1 to show that if τ is sufficiently near-CMC, then the conformal constraint equations have a solution.

Proposition 6.2.2. *If*

$$\kappa\tau^2\left(\frac{1}{2}\right)^{2q} - \frac{1}{4N^2}|LW_0|^2 \geq c\tau^2 \tag{6.28}$$

for some $c > 0$ and all solutions W_0 of the vector equation (1.3b) *with $J \equiv 0$ and $\phi \leq 1$, then the conformal constraint equations have a solution as in Theorem 6.1.2.*

Proof. Suppose τ satisfies (6.28). Let K be as in (6.27). Suppose K is unbounded. Then we can proceed as in the proof of Lemma 6.2.1.

Let $\widetilde{W}_\infty$ and $\widetilde{\phi}_\infty$ be constructed as in the proof of Lemma 6.2.1. Then, using, $\widetilde{\phi}_\infty$ to define W_0, condition (6.28) reduces to

$$\widetilde{\phi}_\infty^{2q} := \frac{|L\widetilde{W}_\infty|^2}{4N^2\kappa\tau^2} \leq \left(\frac{1}{2}\right)^{2q} - \epsilon \tag{6.29}$$

for some small $\epsilon > 0$. In particular, $\widetilde{\phi}_\infty$ is bounded above by a number strictly less than 1/2.

Let $\widetilde{\phi}_+$ be a constant function such that $\widetilde{\phi}_\infty + \frac{1}{2} < \widetilde{\phi}_+ < 1 \leq \widetilde{\phi}_\infty + 1$.

Using the near-CMC condition (6.28), $\widetilde{\phi}_+ \geq \widetilde{\phi}_i$ everywhere, as is shown in the proof of Lemma 6.2.1. But $\widetilde{\phi}_i = 1$ somewhere and $\widetilde{\phi}_+ < 1$. This is a contradiction, so for such τ, K must be bounded. Thus the Schaefer fixed point theorem 6.1.3 gives a solution to the conformal constraint equations. $\square$

As discussed in the introduction of this chapter, since the limit equation criterion has only been applied in the near-CMC case, it might seem reasonable to assume that $\kappa\tau^2 - \frac{1}{4N^2}|LW_0|^2 \geq c\tau^2$. Unfortunately, we can show that τ cannot be arbitrarily near-CMC, in the sense that there are constants $C > 0$ sufficiently large such that no $\tau > \rho^{\delta-1}$ satisfies the related inequality $\tau^2 - C\|d\tau\|^2_{C^0_{\delta-2}}\rho^{2\delta-2} \geq c\tau^2$. This makes it very difficult to verify that the near-CMC condition in the limit equation criterion 6.1.2 or Proposition 6.2.2 is fulfilled.

Proposition 6.2.3. *There is a constant $C > 0$ sufficiently large such that for any $C' > 0$, no τ satisfying $\tau > C'\rho^{\delta-1}$ also satisfies*

$$\tau^2 - C\|d\tau\|^2_{C^0_{\delta-2}}\rho^{2\delta-2} \geq 0. \tag{6.30}$$

Proof. Suppose to the contrary that there are τ_i such that, after scaling, $\tau_i \geq \rho^{\delta-1}$ and $\|d\tau_i\|_{C^0_{\delta-2}} < 1/i$. Using the Poincaré inequality 2.0.3, for $p \in (n/2, n)$, we have

$$\|\tau_i\|_{p,1-n/p} \leq C\|d\tau_i\|_{p,-n/p} \leq C\|d\tau_i\|_{C^0_{\delta-2}} \leq C\frac{1}{i}. \tag{6.31}$$

This contradicts the fact that $\tau_i \geq \rho^{\delta-1}$. $\square$

CHAPTER VII

ADM MASS AND THE ASYMPTOTIC FUNCTION

Ó Murchadha [ÓM05] showed that, in the compact, CMC case, the volume of the solution to the conformal constraint equations (1.3) is monotonically related to the constant curvature τ. In particular, instead of specifying a constant τ, one could equivalently specify the volume of the solution manifold.

In the asymptotically Euclidean case, τ must vanish at infinity, and so there is only one choice for constant τ, $\tau \equiv 0$. However, there is a new "constant" that one may specify, the asymptotic function. An AE manifold does not have (finite) volume, but it seems logical to ask what, if anything, the choice of asymptotic function controls. In analog to volume for the compact case, the ADM mass is a natural candidate.

In general relativity, it is very difficult to define the mass of a non-isolated object, such as a star. However, there are good definitions for the mass of an entire system. One such definition is the ADM mass, a metric invariant for AE manifolds first described by Arnowitt, Deser, and Misner in [ADM61]. This mass describes the total mass of all matter in the AE manifold, as measured by the mass's effects on the asymptotics of the metric. The usual definition for the ADM mass is

$$M_{ADM}(g) := \frac{1}{16\pi} \lim_{t\to\infty} \sum_i \int_{S_t} (g_{ij,i} - g_{ii,j})\nu_e^j dS_e, \tag{7.1}$$

where S_t is the Euclidean coordinate sphere of radius t on an end, ν_e^j is the Euclidean unit normal to S_t and dS_e is the Euclidean spherical volume element. Bartnik [Bar86] showed that the ADM mass is independent of the choice of

Euclidean coordinates. As expected, for Euclidean space, the ADM mass is zero. For the Schwarzschild family of solutions, the mass is exactly the standard mass parameter M.

If g does not fall off to the Euclidean metric e fast enough, the mass may not exist. For this reason, we assume that $\delta < 1 - n/2$ for this chapter. If g is a $W^{2,p}_\delta$ AE metric, then $M_{ADM}(g)$ exists. Also, since the mass is dependent on only one end of the AE manifold, we ignore the other ends, and use an asymptotic constant c instead of the more general asymptotic function $\mathring{u}$. This makes no difference in our calculations below.

Suppose $\phi - c := f \in W^{2,p}_\delta$. Let $\widetilde{g} = \phi^{q-2} g$, as usual. In order to calculate the mass, we have to use coordinates such that $\widetilde{g} \to e$ in those coordinates. If $c \neq 1$, this means we must scale the coordinates. Let x^i be the Euclidean coordinates for g. Let $\overline{x}^i = c^{q/2-1} x^i$. Then, denoting $\widetilde{g}$ in $\overline{x}$ coordinates as $\overline{g}$, we have

$$\overline{g}_{ij} d\overline{x}^i d\overline{x}^j = \overline{g}_{ij} c^{2-q} dx^i dx^j = \widetilde{g}_{ij} dx^i dx^j = \phi^{q-2} g_{ij} dx^i dx^j. \tag{7.2}$$

Thus, as functions on M (not as tensors), $\overline{g}_{ij} = (\phi/c)^{q-2} g_{ij}$.

Let $\overline{e}$ be the Euclidean metric in the $\overline{x}$ coordinates. Then $dS_{\overline{e}} = c^{(q/2-1)(n-1)} dS_e$ and $\frac{\partial f}{\partial \overline{x}} = \frac{\partial f}{\partial x} c^{1-q/2}$. Finally, since we are integrating over spheres, tracing with $\nu^{\overline{j}}_{\overline{e}}$ picks the radial component. Since the radial direction is the same for both metrics, the scaling of the derivatives/metric takes care of this term.

Combining these facts, we have

$$\begin{aligned} M_{ADM}(\overline{g}) &= \frac{1}{16\pi} \lim_{t\to\infty} \sum_i \int_{S_t} (\overline{g}_{ij,\overline{i}} - \overline{g}_{ii,\overline{j}}) \nu^{\overline{j}}_{\overline{e}} dS_{\overline{e}} &(7.3)\\ &= \frac{c^{(q/2-1)(n-2)}}{16\pi} \lim_{t\to\infty} \sum_i \int_{S_t} \left[\left((\phi/c)^{q-2} g_{ij}\right)_{,i} - \left((\phi/c)^{q-2} g_{ii}\right)_{,j} \right] \nu^j_e dS_e. &(7.4) \end{aligned}$$

Using $(q/2-1)(n-2)=2$ and $(\phi/c)^{q-2}=1+(q-2)f/c+L.O.T.$, and ignoring the lower order terms, which vanish in the limit, we find

$$M_{ADM}(\overline{g}) = c^2 M_{ADM}(g) + \frac{c(q-2)}{16\pi}\lim_{t\to\infty}\sum_i \int_{S_t}\left[g_{ij}f_{,i} - g_{ii}f_{,j}\right]v_e^j dS_e \tag{7.5}$$

$$= c^2 M_{ADM}(g) + \frac{c(q-2)(1-n)}{16\pi}\lim_{t\to\infty}\int_{S_t}\partial_r f dS_e, \tag{7.6}$$

where ∂_r is the Euclidean radial derivative.

Starting with $W^{2,p}_\delta$ seed data, the Lichnerowicz equation (1.3a) implies that $\Delta\phi \in L^1$. Applying integration by parts,

$$\int_{B_t}\Delta\phi dV_g = \int_{S_t}\nabla_i\phi\nu_g^i dS_g. \tag{7.7}$$

If $t\to\infty$, we can drop lower order terms, and thus

$$\int_M \Delta\phi dV_g = \lim_{t\to\infty}\int_{S_t}\partial_r\phi dS_e. \tag{7.8}$$

Thus

$$M_{ADM}(\overline{g}) = c^2 M_{ADM}(g) + \frac{c(q-2)(n-1)}{16\pi a}\int_M -a\Delta\phi dV_g. \tag{7.9}$$

7.1. Model Problem

We now discuss a (relatively) simple model problem. Assume we have seed data with regularity as in Assumption 3.1, with $p > n$. Let $\tau \equiv 0$, i.e., the CMC case. In this case, the metric g must be Yamabe positive in order for the Lichnerowicz equation (1.3a) to have a solution, and so, without loss of generality, we may assume that $R \equiv 0$. For simplicity, we also assume that $r \equiv 0$ and $J \equiv 0$.

Also, since $d\tau \equiv 0$, we know $LW \equiv 0$. We ignore the degenerate case $\sigma \equiv 0$. Thus the conformal constraint equations reduce to a single equation,

$$-a\Delta\phi - |\sigma|^2\phi^{-q-1}. \tag{7.10}$$

Let ϕ_c be the solution to (7.10) such that $\phi_c - c \in W^{2,p}_\delta$. Such solutions exist for all c by Theorem 3.0.9. We first prove a lemma.

Lemma 7.1.1. *Let $\eta = 2\delta - 4 + n$.There exists a positive solution $u \in W^{2,p}_{\eta+2}$ of*

$$-a\Delta u - \rho^\eta = 0 \tag{7.11}$$

such that $C_0\rho^{2-n} \le u \le C_1\rho^{\eta+2}$.

Proof. Note that $\eta \in (-n, -2)$ since $\delta \in (2-n, 1-n/2)$. Thus Proposition 2.1.1 shows that $\Delta : W^{2,p}_{\eta+2} \to L^p_\eta$ is an isomorphism, and so there exists a solution $u \in W^{2,p}_{\eta+2}$ of

$$-a\Delta u - \rho^\eta = 0 \tag{7.12}$$

such that $u \le C\rho^{\eta+2}$. The maximum principle 2.1.2 shows that u is positive.

We claim that $u \ge C\rho^{2-n}$ for some $C > 0$. A straightforward extension of [Bar86] shows that $\Delta : u \mapsto (\Delta u, u|_{\partial E_R})$ is an isomorphism between the spaces $W^{2,p}_\delta(E_R) \to L^p_\delta(E_R) \times W^{2-1/p,p}(\partial E_R)$, where E_R is the exterior of the ball of radius R in the end. Using this result, we can also extend the sub and supersolution theorem 2.1.7 to allow (smooth) internal boundaries with Dirichlet boundary data.

Let $v \in W^{2,p}_\delta$ be the unique solution of

$$\Delta v = 0 \text{ on } E_R \quad \text{and} \quad v = 1 \text{ on } \partial E_R. \tag{7.13}$$

Clearly, for small enough $\alpha > 0$, αv is a subsolution to (7.12) on E_R. Since u is a solution, and thus a supersolution, the extension of the sub and supersolution theorem shows that $u \geq \alpha v$.

We claim that $v > C\rho^{2-n}$ for some $C > 0$, which completes the proof. Consider the functions $v_\pm = \alpha_\pm(C_\pm \rho^{2-n} \mp \rho^{2-n+\delta/2})$ for appropriately chosen constants α_+,, α_-, C_+, and C_-. We claim that the functions $v_\pm$ are respectively super and subsolutions of the boundary value problem (7.13). Indeed, using the decomposition

$$\Delta = \Delta_e + k^{ij}\partial_i\partial_j + g^{ij}\Gamma^k_{ij}\partial_k, \tag{7.14}$$

and the fact that $W^{2,p}_\delta \subset C^1_\delta$, it is clear that $\Delta\rho^{2-n} = O(\rho^{-n+\delta})$ since $\Delta_e\rho^{2-n} = 0$. (Recall that ρ is the radial coordinate sufficiently far out on each end.) We similarly get that $\Delta\rho^{2-n+\delta/2} = O(\rho^{-n+\delta/2})$. This is the highest order term that remains. Then, because of this term's sign, it eventually dominates, making $v_\pm$ a super or subsolution. Finally, $\alpha_\pm$ can be made large or small to ensure the boundary condition falls between $v_\pm$. Again using the extension of the sub and supersolution theorem, along with the fact that Δ is an isomorphism, $v_- \leq v \leq v_\pm$, and so $v > C\rho^{2-n}$, as claimed. □

We now list a number of properties of the solutions ϕ_c of the reduced Lichnerowicz equation (7.10) and their derivatives, $\delta\phi_c := \frac{\partial}{\partial c}\phi_c$.

Proposition 7.1.2 (Properties of ϕ_c)**.** *The function ϕ_c satisfies*

$$c < \phi_c \leq c^{-q/2}C_2 u + c, \tag{7.15}$$

for some $C_2 > 0$ independent of c. Also,

$$0 < \phi_1 - 1 \leq \phi_c \tag{7.16}$$

for all $c > 0$. Finally,

$$\sup\left\{0, \frac{q+1}{c}(c - \phi_c) + 1\right\} \leq \delta\phi_c \leq 1. \tag{7.17}$$

In particular, as $c \to \infty$, $\phi_c \to c$ and $\delta\phi_c \to 1$.

Proof. First, note that c is a subsolution for (7.10). We claim that $c^{-q/2}C_2u + c$ is a supersolution for (7.10) for some $C_2 > 0$ independent of c. Note that $c < c^{-q/2}Cu+c$ since $u > 0$, and so the sub and supersolution theorem 2.1.7 combined with the uniqueness of solutions of the Lichnerowicz equation from Theorem 3.0.11 shows that $c \leq \phi_c \leq c^{-q/2}C_2u + c$.

For $c^{-q/2}C_2u + c$ to be a supersolution to the reduced Lichnerowicz equation (7.10), we must show

$$-a\Delta(c^{-q/2}Cu + c) - |\sigma|^2(c^{-q/2}Cu + c)^{-q-1} \geq 0. \tag{7.18}$$

Using $|\sigma|^2 \leq S\rho^{2\delta-2}$ and $C_0\rho^{2-n} \leq u \leq C_1\rho^{\eta+2}$, where $\eta = 2\delta - 4 + n$, this reduces to

$$c{-}q/2C_2\rho^\eta \geq |\sigma|^2(c^{-q/2}C_2u + c)^{-q-1} \tag{7.19}$$

$$C_2C_0c^{-q/2-1}\rho^{2-n} + 1 \geq S^{1/(q+1)}C_2^{-1/(q+1)}c^{\frac{q}{2(q+1)}-1}\rho^{\frac{2-n}{q+1}}. \tag{7.20}$$

For some C_2 large enough, independent of c, this is true, and so $c^{-q/2}C_2u + c$ is a supersolution. The calculation to show this is long and unenlightening, so we do not include it. Essentially, the 1 bounds the right hand side if ρ and/or c are large, while the other term bounds the right hand side if ρ and/or c are small.

Note that $\phi_c - c \in W^{2,p}_\delta$ and that $-a\Delta(\phi_c - c) \geq 0$. Since $\phi_c \geq c$ and $\phi_c \not\equiv c$, by the strong maximum principle 2.1.3, $\phi_c - c > 0$, and so $\phi_c > c$.

A quick calculation shows that $\phi_1 - 1 + \epsilon$ is a subsolution to (7.10) for any $0 < \epsilon < 1$. Using ϕ_c as a supersolution, the sub and supersolution theorem 2.1.7 combined with uniqueness from Theorem 3.0.11 show that $\phi_c \geq \phi_1 - 1 + \epsilon$. Letting $\epsilon \to 0$, we have $\phi_c \geq \phi_1 - 1 > 0$.

Taking the variation of (7.10), we find that $\delta\phi_c$ satisfies

$$(-a\Delta + (q+1)|\sigma|^2\phi_c^{-q-2})\delta\phi_c = 0. \tag{7.21}$$

Since ϕ_c changes at a rate of one near infinity, we require $\delta\phi_c \to 1$ at infinity. By Proposition 2.1.4, $\delta\phi_c - 1 \in W^{2,p}_\delta$. Then, since $(-a\Delta + (q+1)|\sigma|^2\phi_c^{-q-2})(\delta\phi_c - 1) \leq 0$, the maximum principle 2.1.2 shows that $\delta\phi_c - 1 \leq 0$, and so $\delta\phi_c \leq 1$.

To show that $\delta\phi_c \geq 0$, note that for $c' > c$, $\phi_{c'}$ is a supersolution for ϕ_c, i.e., for the reduced Lichnerowicz equation (7.10). Thus ϕ_c is nondecreasing, and so $\delta\phi_c \geq 0$.

We claim that $\frac{q+1}{c}(c - \phi_c) + 1$ is a subsolution of (7.21), which then implies that it is a lower bound for $\delta\phi_c$. Indeed,

$$\begin{aligned}\left(-a\Delta + (q+1)|\sigma|^2\phi_c^{-q-2}\right)\left[\frac{q+1}{c}(c-\phi_c)+1\right] &= |\sigma|^2\phi_c^{-q-1}\left[-\frac{q+1}{c}(q+2) + \frac{(q+1)(q+2)}{\phi_c}\right] \\ &\leq 0\end{aligned} \tag{7.22}$$

since $\phi_c \geq c$. $\square$

With those properties of ϕ_c, we can now understand how the integral term in (7.9) behaves as $c \to 0$ or $c \to \infty$. In this model problem, the integral term, modulo a constant, becomes $c \int_M |\sigma|^2 \phi_c^{-q-1}$.

Proposition 7.1.3. *For all c large enough, the integral term in* (7.9) *strictly decreases and approaches* 0 *as $c \to \infty$.*

If σ has compact support, the integral term goes to zero as $c \to 0$.

If $|\sigma|^2 \geq C\rho^\alpha$ for some $\alpha > \frac{2n}{n-1}\delta - \frac{n}{n-1}$ and $C > 0$, the integral term becomes unbounded as $c \to 0$.

Remark 7.1.4. *The lower bound on σ need not hold on all of M. Indeed, it is sufficient for σ to be bounded below only on some wedge of positive angle, perhaps far out on the end. Also, note that, for $\alpha = 2\delta - 2$, the inequality for α reduces to $\delta < 1 - n/2$, which was already assumed. Finally, note that the δ from the lower bound on α is the $\delta \in (2-n, 0)$ from the inequality $|\sigma|^2 \leq C\rho^{2\delta-2}$. In particular, the result can hold even if σ falls off faster than the metric.*

Proof. For all c,

$$0 < c \int_M |\sigma|^2 \phi_c^{-q-1} \leq c^{-q} \int_M |\sigma|^2, \tag{7.23}$$

and so the integral term approaches zero as $c \to \infty$.

The derivative of the mass as a function of c is

$$\frac{\partial}{\partial c} M_{ADM}(\overline{g}) = 2cM_{ADM}(g) + C_0 \int |\sigma|^2 \phi_c^{-q-1} \left[1 - (q+1)\frac{c}{\phi_c}\delta\phi_c\right]. \tag{7.24}$$

Since $c/\phi_c \to 1$ and $\delta\phi_c \to 1$ uniformly in c, the integrand in (7.24) is negative for large c. Thus the integral term decreases monotonically for all c large enough.

Suppose σ has compact support. Then

$$c\int_M |\sigma|^2\phi_c^{-q-1} \le c\int_M |\sigma|^2(\phi_1 - 1)^{-q-1}. \tag{7.25}$$

Since $\phi_1 - 1 > 0$ and σ has compact support, the integral term is finite. Thus, as $c \to 0$, the integral term of the mass (7.9) vanishes.

Suppose that $|\sigma|^2 \ge C\rho^\alpha$ for some $\alpha > \frac{2n}{n-1}\delta - \frac{n}{n-1}$ and $C > 0$. Recall that by Equation (7.15) and Lemma 7.1.1, $\phi_c \le c^{-q/2}C\rho^{\eta+2} + c$, where $\eta := 2\delta - 4 + n$.

Dropping all constants not depending on c, one has on an end E,

$$\int_E |\sigma|^2\phi_c^{-q-1}dV \ge \int_E \frac{\rho^{\alpha+n-1}}{(c^{-q/2}\rho^{\eta+2} + c)^{q+1}}d\rho d\sigma \tag{7.26}$$

$$= c^{-q-1+\frac{(1+q/2)(\alpha+n)}{\eta+2}} \int_{Cc^{\frac{1+q/2}{-\eta-2}}}^{\infty} \frac{r^{\alpha+n-1}}{(1+r^{\eta+2})^{q+1}}dr \tag{7.27}$$

The first line is true for some integration form $d\sigma$. For the second line, we pulled out c's and used the substitution $\rho = rc^{\frac{1+q/2}{\eta+2}}$, and integrated out the $d\sigma$. Since the final integral converges as $c \to 0$, we may bound it by a constant. Thus,

$$\int_E |\sigma|^2\phi_c^{-q-1}dV \ge C_0 c^{-q-1+\frac{(1+q/2)(\alpha+n)}{\eta+2}}. \tag{7.28}$$

If $-q - 1 + \frac{(1+q/2)(\alpha+n)}{\eta+2} < -1$, then the integral term of the mass (7.9) blows up as $c \to 0$. Using $\eta = 2\delta - 4 + n$, this condition reduces to $\alpha > \frac{2n}{n-1}\delta - \frac{n}{n-1}$. This establishes the final claim of the proposition. $\square$

While the original hope was that mass and asymptotic constants were in one to one correspondence, Proposition 7.1.3 unfortunately shows that this is not the case. If $M_{ADM}(g) = 0$, and there are σ which have compact support, then for those σ, the mass is not a monotonic function of c. For any positive mass, since

$c^2 M_{ADM}(g)$ is zero as $c \to 0$ and unbounded as $c \to \infty$, for σ which do not fall off very quickly, the mass is again not monotonic as a function of c.

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