

ABELIAN ARRANGEMENTS

by

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DISSERTATION ABSTRACT

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An abelian arrangement is a finite set of codimension one abelian subvarieties (possibly translated) in a complex abelian variety. We are interested in the topology of the complement of an arrangement. If the arrangement is unimodular, we provide a combinatorial presentation for a differential graded algebra (DGA) that is a model for the complement, in the sense of rational homotopy theory. Moreover, this DGA has a bi-grading that allows us to compute the mixed Hodge numbers. If the arrangement is chordal, then this model is a Koszul algebra. In this case, studying its quadratic dual gives a combinatorial description of the \mathbb{Q} -nilpotent completion of the fundamental group and the minimal model of the complement of the arrangement.

This dissertation includes previously unpublished co-authored material.

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For Piper and Persy.

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CHAPTER I

INTRODUCTION

This dissertation explores the topology of an elliptic analogue of hyperplane arrangements. In doing so, we highlight the analogy between three cases of arrangements: rational, trigonometric, and elliptic. The rational case consists of linear arrangements, which are finite sets of codimension one linear subspaces of a complex vector space. The trigonometric case consists of toric arrangements, which are finite sets of codimension one subtori in a complex torus. The elliptic case consists of abelian arrangements, which are finite sets of codimension one abelian subvarieties in a complex abelian variety. Moreover, we consider a fourth case of graphic arrangements of higher genus projective curves. In each case, we study a differential graded algebra (DGA) that is a model (in the sense of rational homotopy theory) of the complement of the arrangement.

For linear arrangements, the complement is formal, which means that the cohomology algebra with trivial differential is itself a model. A combinatorial presentation for this algebra is given by Orlik and Solomon [OS]. For toric arrangements, the complement is also formal, and in the unimodular case a combinatorial presentation of the cohomology ring is given by De Concini and Procesi [DCP1]. For abelian arrangements, the complement is *not* necessarily formal, but in the unimodular case a combinatorially presented model (with nontrivial differential) is given in Chapter III.

Totaro [Tot] and Kriz [Kri] each independently studied the cohomology of configuration spaces of smooth complex projective varieties. In particular, their work gives a presentation of a model for the cohomology in the special case of

an ordered configuration space on an elliptic curve (the complement of the braid arrangement), which we state as an example (Example 3.12). In Section 3.1, we generalize Totaro’s method to compute the rational cohomology of the complement of any abelian arrangement \mathcal{A} in a complex abelian variety X . We denote this complement by $M(\mathcal{A})$, and arrive at our results by studying the Leray spectral sequence of the inclusion $f : M(\mathcal{A}) \hookrightarrow X$. Specifically, we use Hodge theory to show degeneration of this spectral sequence at the E_3 term. Most results stated in the section are valid when considering a complex torus rather than an abelian variety; we need the algebraic structure when discussing the Hodge theory.

As seen in Section 3.2, our results are particularly nice in the case where \mathcal{A} is unimodular, which means that all multiple intersections of subvarieties in \mathcal{A} are connected. In this case, we give a presentation of a differential graded algebra $A(\mathcal{A})$ in terms of the combinatorics of \mathcal{A} (the partially ordered set consisting of all intersections of subvarieties in \mathcal{A}). The cohomology of $A(\mathcal{A})$ is isomorphic as a graded algebra to the cohomology of $M(\mathcal{A})$, by Theorem 3.8. Moreover, $A(\mathcal{A})$ admits a second grading, and its cohomology is canonically isomorphic as a bi-graded algebra to $\text{gr } H^*(M(\mathcal{A}); \mathbb{Q})$, the associated graded with respect to Deligne’s weight filtration. Thus it allows us to compute the mixed Hodge numbers of $M(\mathcal{A})$.

Remark 1.1. While the weight filtration on the cohomology of the complement of a linear or toric arrangement is trivial (by [Loo]), for an abelian arrangement it is always interesting. For a basic example, consider a punctured elliptic curve $M(\mathcal{A}) = E \setminus \{p_1, \dots, p_\ell\}$. Here, the first filtered piece of $H^1(M(\mathcal{A}); \mathbb{Q})$ consists of the image of the restriction map from $H^1(E; \mathbb{Q})$, which is neither trivial nor surjective.

This can be seen in the short exact sequence

$$0 \rightarrow H^1(E; \mathbb{Q}) \rightarrow H^1(E \setminus \{p_1, \dots, p_\ell\}; \mathbb{Q}) \rightarrow \mathbb{Q}(-1)^{\oplus(\ell-1)} \rightarrow 0.$$

Levin and Varchenko [LV] computed cohomology of elliptic arrangements with coefficients in a nontrivial rank one local system. Dupont [Dup1] also studied the more general case of the complement to a union of smooth hypersurfaces which intersect like hyperplanes in a smooth projective variety. Dupont used a similar but alternative method to that presented here to find the same model as described in Section 3.1, but he does not give the combinatorial presentation in Section 3.2.

In [Dup2], Dupont uses our decomposition on the Leray spectral sequence in Lemma 3.1 to show that all toric arrangements are formal. In [Suc], Suciu uses the model given in Theorem 3.8 to study resonance varieties and formality of elliptic arrangements.

In Chapter IV, we focus our attention on unimodular and supersolvable arrangements, which are classified by chordal graphs and are therefore called chordal arrangements. In this case, the above models are Koszul; this is due to Shelton and Yuzvinsky for linear arrangements [SY], and we prove it in the toric, abelian, and higher genus cases (Section 4.2). By studying the quadratic dual of the model, one can obtain a combinatorial presentation for a Lie algebra and use it to compute the \mathbb{Q} -nilpotent completion of the fundamental group and the minimal model. This is done by Papadima and Yuzvinsky in the linear case [PY], and the toric case is completely analogous. In the abelian and higher genus cases, the lack of formality makes this computation more subtle: we need to use nonhomogeneous quadratic duality, where the dual to a Koszul differential graded algebra is a quadratic-linear algebra. With

this tool, we extend Papadima and Yuzvinsky's results to the abelian and higher genus settings (Section 4.3).

We also prove that complements of chordal arrangements are rational $K(\pi, 1)$ spaces. In the rational and toric cases, this follows from formality and Koszulity [PY]. In the abelian case (where we lack formality) it is not automatic, but we obtain it from our concrete description of the minimal model (Corollary 4.13).

Bezrukavnikov [Bez] studied the Kriz-Totaro model of the configuration space of an arbitrary smooth, projective, complex curve; he showed that this model was Koszul, gave a presentation for the dual Lie algebra, and described the minimal model. The results in Chapter IV for chordal arrangements are generalizations of those given by Bezrukavnikov. This chapter also includes material from a joint project with Justin Hilburn.

CHAPTER II

PRELIMINARIES

In this chapter, we provide necessary definitions and collect known results from arrangement theory that will be used in the later chapters.

2.1. Definitions

We consider an arrangement $\mathcal{A} = \{H_1, \dots, H_\ell\}$ of smooth connected divisors in a smooth complex variety X , which intersect like hyperplanes. When we say that they **intersect like hyperplanes**, we mean that for every $p \in X$, there is a neighborhood $U \subseteq X$ of p , a neighborhood $V \subseteq T_p X$ of 0, and a homeomorphism $\phi : U \rightarrow V$ that induces $H_i \cap U \cong T_p H_i \cap V$ for all $H_i \in \mathcal{A}$.

There are three particularly interesting cases of arrangements. A **linear arrangement** is a finite set of codimension one linear subspaces in a complex vector space. A **toric arrangement** is a finite set of codimension one subtori in a complex torus. An **abelian arrangement** is a finite set of codimension one abelian subvarieties in a complex abelian variety. In each case, denote the complement of an arrangement \mathcal{A} in X by $M(\mathcal{A}) := X \setminus \cup_{H \in \mathcal{A}} H$. We will also consider the affine analogues of these arrangements, where we allow translations of such subvarieties. We say that an arrangement is **central** if the intersection of all of the subvarieties in \mathcal{A} is nonempty.

A **component** of an arrangement \mathcal{A} is a connected component of an intersection $H_S := \bigcap_{H \in S} H$ for some subset $S \subseteq \mathcal{A}$. Note that, except for linear arrangements, the intersections themselves need not be connected. We say that the arrangement is **unimodular** if the intersection H_S is connected for all subsets $S \subseteq \mathcal{A}$.

The **rank** of a component is defined as its complex codimension in X . Note that for a subset $S \subseteq \mathcal{A}$ with nonempty intersection H_S , the rank of its components is constant. Hence, we define the rank of a subset $S \subseteq \mathcal{A}$ as the rank of a connected component of H_S , which does not depend on the choice of component. We say that S is **independent** if $\text{rk}(S) = |S|$. Otherwise, $\text{rk}(S) < |S|$ and we say that S is **dependent**. If \mathcal{A} is central, then these definitions determine a matroid on \mathcal{A} , describing the combinatorics of the arrangement.

For a given order on the subvarieties in \mathcal{A} , a tuple of subvarieties in \mathcal{A} is called standard if the hyperplanes are written in increasing order. A standard tuple S is a **broken circuit** if there is some $H \in \mathcal{A}$ larger than all those in S such that $S \cup \{H\}$ is a circuit (that is, a minimally dependent set). We say that a standard tuple S is **nbc** (non-broken circuit) associated to F if F is a connected component of H_S and it does not contain any broken circuits.

Example 2.1. The terminology in arrangement theory is motivated by arrangements that arise from a matrix. We describe this with a special case of elliptic arrangements, where we have a complex elliptic curve E , $X = E^n$, and an $n \times \ell$ integer matrix. Each column corresponds to a map $\alpha_i : E^n \rightarrow E$. Assume that each column is primitive, so that each $H_i = \ker \alpha_i$ is a connected abelian subvariety of X .

In this case, an intersection H_S is the kernel of an $n \times |S|$ submatrix, taking the corresponding columns α_i for $H_i \in S$. The codimension of H_S is the rank of the corresponding matrix. In this way, the dependencies of the hyperplanes in \mathcal{A} correspond to the dependencies of the corresponding α_i 's in \mathbb{Z}^n .

Further suppose that \mathcal{A} is a unimodular arrangement and that the rank of the $n \times \ell$ matrix is equal to n . Then all $n \times n$ submatrices will have determinant ± 1 or 0. Otherwise, an intersection of subvarieties (that is, the kernel of the corresponding

submatrix) would be disconnected. This agrees with the usual notion of a unimodular matrix.

Let F be a component of the arrangement \mathcal{A} . For any point $x \in F$, define an arrangement $\mathcal{A}_F^{(x)}$ in the tangent space $T_x X$ consisting of hyperplanes $H_F^{(x)} := T_x H$ for all $H \supseteq F$. If X has complex dimension n , then $\mathcal{A}_F^{(x)}$ is a central hyperplane arrangement in $T_x X \cong \mathbb{C}^n$, and we denote its complement by

$$M(\mathcal{A}_F^{(x)}) = T_x X \setminus \cup_{H \supseteq F} H_F^{(x)}.$$

This arrangement may be referred to as the localization of \mathcal{A} at F , with respect to the point $x \in F$.

Remark 2.2. We say that a point $x \in F$ is a generic point of F if x is not contained in any smaller component of \mathcal{A} . By our assumption that the divisors intersect like hyperplanes, for a generic point $x \in F$, there is a neighborhood $U \subseteq X$ of x such that $U \cap M(\mathcal{A}) \cong M(\mathcal{A}_F^{(x)})$.

Remark 2.3. Also by our assumption that the divisors intersect like hyperplanes, the intersection lattice of the arrangement $\mathcal{A}_F^{(x)}$ does not depend on the choice of $x \in F$. Since the cohomology of $M(\mathcal{A}_F^{(x)})$ only depends on the combinatorics of $\mathcal{A}_F^{(x)}$ (see Theorem 2.7), we may write $H^*(M(\mathcal{A}_F); \mathbb{Q})$ to mean the cohomology of $M(\mathcal{A}_F^{(x)})$ for some (any) $x \in F$.

If \mathcal{A} is an abelian arrangement, then even more can be said. Not only does the cohomology not depend on the choice of $x \in F$, but for any two points x and y of F , we have a canonical homeomorphism (via translation) $M(\mathcal{A}_F^{(x)}) \cong M(\mathcal{A}_F^{(y)})$.

2.2. Graphic Arrangements

An **ordered graph** is a graph $\Gamma = (\mathcal{V}, \mathcal{E})$ with an ordering on the vertices \mathcal{V} . We will assume throughout that our graphs are simple (that is, they have no loops or multiple edges). An ordered graph can be considered as a directed graph in the following way: For each edge $e \in \mathcal{E}$, label its larger vertex by $h(e)$ (for “head” of an arrow) and its smaller vertex by $t(e)$ (for “tail” of an arrow). An order on the vertices of Γ induces an order on the edges by setting $e < e'$ if $h(e) < h(e')$ or if $h(e) = h(e')$ and $t(e) < t(e')$.

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be an ordered graph. Let C be \mathbb{C} , \mathbb{C}^\times , or a complex projective curve, and let $C^\mathcal{V}$ be the complex vector space (respectively torus or projective variety) whose coordinates are indexed by the vertices \mathcal{V} . For each edge $e \in \mathcal{E}$, let

$$H_e = \{x_\mathcal{V} \in C^\mathcal{V} \mid x_{h(e)} = x_{t(e)}\}.$$

The collection $\mathcal{A}(\Gamma, C) = \{H_e \mid e \in \mathcal{E}\}$ is a **graphic** arrangement in $C^\mathcal{V}$. In the case that C is \mathbb{C} , \mathbb{C}^\times , or a complex elliptic curve, $\mathcal{A}(\Gamma, C)$ is a linear, toric, or abelian arrangement, respectively.

Example 2.4. Let $C = \mathbb{C}$, \mathbb{C}^\times , or a complex projective curve. If Γ is the complete graph on n vertices, then $\mathcal{A} = \mathcal{A}(\Gamma, C)$ is the braid arrangement, and its complement $M(\mathcal{A})$ is the ordered configuration space of n points on C .

Let C be \mathbb{C} , \mathbb{C}^\times , or a complex projective curve, and let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a simple graph. If Γ is chordal (that is, every cycle with more than three vertices has a chord), then the graphic arrangement $\mathcal{A}(\Gamma, C)$ is said to be **chordal**.

A **perfect elimination ordering** is an order on the vertices so that for all $v \in \mathcal{V}$, v is a simplicial vertex (a vertex whose neighbors form a clique) in the graph

$\Gamma_v := \Gamma - \{v' \in \mathcal{V} \mid v' > v\}$. Such an ordering exists if and only if Γ is chordal [FG, p. 851]. From now on, we will use such an order when discussing chordal graphs.

Note that our definition of circuits (and hence broken circuit or nbc) coincides with that in graph theory. Let $F \subseteq \mathcal{E}$ be a flat of the matroid of Γ , and consider the subgraph $\Gamma[F]$ of Γ , which has edges F and vertices adjacent to edges in F . A set $S \subseteq \mathcal{E}$ is an nbc set associated to F if $S \subseteq F$ and S spans $\Gamma[F]$.

Remark 2.5. In the case of linear, toric, or abelian arrangements, the essential property that we need for our results in Chapter IV is that the arrangement is unimodular (for Theorems 2.10 and 3.8) and supersolvable (for Theorems 4.3 and 4.7). We could state all of our results in the language of unimodular and supersolvable arrangements; however, this isn't any more general than the language of chordal graphs. This is because Ziegler showed that a matroid is unimodular and supersolvable if and only if it is chordal graphic (Proposition 2.6 and Theorem 2.7 of [Zie]). In fact, since the edge set of $\Gamma - v$ is a modular hyperplane when v is a simplicial vertex [Zie, Proposition 4.4], the maximal chain of modular flats in the matroid corresponds exactly to our ordering on the vertices.

2.3. Rational Homotopy Theory and Quadratic Duality

In this section, we briefly state results that will be used to study the rational homotopy theory of chordal arrangements in Section 4.3. The fundamental problem of rational homotopy theory is to understand the topology of the \mathbb{Q} -completion of a space X , which is determined by the quasi-isomorphism type of a particular DGA $A_{PL}(X)$ with $H^*(A_{PL}(X), d) \cong H^*(X, \mathbb{Q})$ (see [Qui, Sul, BG]). A **model** for a space X is a differential graded algebra A which is quasi-isomorphic to $A_{PL}(X)$.

In the case that a model is Koszul, the following theorem tells us how to use nonhomogeneous quadratic duality (constructing a quadratic-linear algebra from a DGA, see [Pri, Pos, Bez]) to obtain more information about the rational homotopy theory of the space.

Theorem 2.6. [BH, Bez] Let X be a space with a quadratic model $A(X)$. Let $U(L)$ be the quadratic dual to $A(X)$, which is the universal enveloping algebra of a Lie algebra L .

1. $\overline{U(L)} \cong \overline{\mathbb{Q}[\pi_1(X)]}$, where the completions are each with respect to the augmentation ideal. This isomorphism respects the Hopf algebra structures.
2. The completion of L with respect to the filtration by bracket length is isomorphic to the Malcev Lie algebra of $\pi_1(X)$, which determines the Malcev (or \mathbb{Q} -nilpotent) completion of $\pi_1(X)$.
3. If $A(X)$ is Koszul, and L is graded by bracket length and finite dimensional in each bracket length, then the graded standard (or Chevalley-Eilenberg) complex of L , $\Omega(L^\bullet)$, is the minimal model of X .
4. Under the hypotheses of (3), the \mathbb{Q} -completion of X is a $K(\pi, 1)$ space.

Bezrukavnikov [Bez] laid the framework for this theorem, while the author and J. Hilburn [BH] stated this theorem more generally. Papadima and Yuzvinsky [PY] stated the analogous results in the case that X is formal.

2.4. Hyperplane Arrangements

For linear arrangements, a combinatorial presentation for the cohomology ring was first given by Orlik and Solomon [OS]. The fact that the complement of a

linear arrangement is formal (that is, its cohomology ring is a model for the space) is originally due to Brieskorn [Bri].

Theorem 2.7. [OT, Theorems 3.126&5.89] Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a hyperplane arrangement in a complex vector space, with complement $M(\mathcal{A})$. Then $M(\mathcal{A})$ is formal and $H^*(M(\mathcal{A}), \mathbb{Q})$ is isomorphic to the exterior algebra on the \mathbb{Q} -vector space spanned by $\{g_1, \dots, g_n\}$ modulo the ideal generated by

1. $g_{i_1} \cdots g_{i_k}$ whenever $S = \{H_{i_1}, \dots, H_{i_k}\}$ is such that $H_S = \emptyset$.
2. $\sum_{j=1}^k (-1)^j g_{e_1} \cdots \hat{g}_{e_j} \cdots g_{e_k}$ whenever $\{H_{i_1} < \dots < H_{i_k}\}$ is dependent.

The Orlik-Solomon algebra presented in Theorem 2.7 has a particularly nice basis indexed by the nbc sets (see [OT, Yuz]). That is, the set of $g_{i_1} \cdots g_{i_k}$ for all nbc-sets $\{H_{i_1} < \dots < H_{i_k}\}$ is a basis for $H^*(M(\mathcal{A}); \mathbb{Q})$. Moreover, this basis is compatible with the decomposition given by Brieskorn's Lemma:

Lemma 2.8. [Bri, p. 27] Let \mathcal{A} be a hyperplane arrangement with complement $M(\mathcal{A})$. Then

$$H^*(M(\mathcal{A})) = \bigoplus_F H^{\text{rk}(F)}(M(\mathcal{A}_F))$$

where the sum is taken over all components (flats) F of the arrangement.

A basis of $H^{\text{rk}(F)}(M(\mathcal{A}_F))$ is indexed by the nbc sets S associated to F .

Since we will restrict ourselves to graphic arrangements for the remainder of this section, we will state this presentation in the specific case of a graphic linear arrangement.

Example 2.9. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be an ordered graph, and let $\mathcal{A} = \mathcal{A}(\Gamma, \mathbb{C})$. $H^*(M(\mathcal{A}), \mathbb{Q})$ is isomorphic to the exterior algebra on the \mathbb{Q} -vector space spanned by $\{g_e \mid e \in \mathcal{E}\}$ modulo the ideal generated by

$\sum_j (-1)^j g_{e_1} \cdots \hat{g}_{e_j} \cdots g_{e_k}$ whenever $\{e_1 < \cdots < e_k\}$ is a cycle.

Yuzvinsky proved that the Orlik-Solomon ideal has a quadratic Gröbner basis when \mathcal{A} is supersolvable (eg. chordal), which implies that $H^*(M(\mathcal{A}))$ is Koszul [Yuz, Corollary 6.21]. Though Koszulity was first proven by Shelton and Yuzvinsky [SY], we outline Yuzvinsky's technique in [Yuz] applied to chordal arrangements as we will use similar techniques in the toric, abelian, and higher genus cases.

For ease of notation, whenever $S = \{e_1 < \cdots < e_k\}$ we will use $g_S := g_{e_1} \cdots g_{e_k}$ and $\partial g_S := \sum_j (-1)^j g_{e_1} \cdots \hat{g}_{e_j} \cdots g_{e_k}$. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a chordal graph with a perfect elimination ordering on the vertices (and edges).

First, Yuzvinsky showed that the set $G = \{\partial g_S \mid S \text{ is a circuit}\}$ is a Gröbner basis for the ideal $I = \langle G \rangle$ in the exterior algebra $\Lambda(g_e \mid e \in \mathcal{E})$, with the degree-lexicographic order such that $g_e < g_{e'}$ whenever $e < e'$. The leading (or initial) term of ∂g_S is $\text{In}(\partial g_S) = g_{S'}$ where $S' \subseteq S$ is the broken circuit associated to S . Recall that a subset G of an ideal I is a Gröbner basis if $\text{In}(I) = \langle \text{In}(G) \rangle$. To prove that this is a Gröbner basis, Yuzvinsky used the fact that the set of monomials not in $\text{In}(I)$ form a basis for $H^*(M(\mathcal{A})) = \Lambda(g_e)/I$. The set of monomials not in $\text{In}(I)$ is the basis $\{g_S \mid S \text{ is nbc}\}$.

Moreover, since Γ is chordal, this Gröbner basis can be reduced to a quadratic Gröbner basis. This is because we have the following property (which follows immediately from Proposition 6.19 of [Yuz]): $S \subseteq \mathcal{E}$ is an nbc set if and only if for all distinct $e, e' \in S$ we have $h(e) \neq h(e')$. A circuit S is not nbc, hence there exist distinct edges $e, e' \in S$ such that $h(e) = h(e')$. But then $\{e, e'\}$ contains (and hence is) a broken circuit, and so it is contained in some circuit T with $|T| = 3$. Thus $\text{In}(\partial g_T) = g_e g_{e'}$ divides $\text{In}(\partial g_S)$, and we can reduce our Gröbner basis to a quadratic one.

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a chordal graph, and let $\mathcal{A} = \mathcal{A}(\Gamma, \mathbb{C})$. Papadima and Yuzvinsky [PY] describe the **holonomy Lie algebra**, L , of $M(\mathcal{A})$ and show that it is the Lie algebra dual to the cohomology ring $H^*(M(\mathcal{A}))$. They also show that the standard complex of L is the minimal model of $M(\mathcal{A})$ [PY, Propositions 3.1&4.4]. Moreover, Kohno [Koh] shows that the holonomy Lie algebra is isomorphic to the Malcev Lie algebra $L(\pi_1(M(\mathcal{A})))$.

This Lie algebra L can be described as the free Lie algebra generated by c_e for $e \in \mathcal{E}$ modulo the relations

- (i) $[c_e, c_{e'}] = 0$ if e and e' are not part of a cycle of size 3, and
- (ii) $[c_{e_1}, c_{e_2} + c_{e_3}] = 0$ if $\{e_1, e_2, e_3\}$ is a cycle.

If X is a formal space, then $H^*(X)$ is Koszul if and only if X is rationally $K(\pi, 1)$ [PY, Theorem 5.1]. In particular, $M(\mathcal{A})$ is a rational $K(\pi, 1)$ space. Falk first showed that $M(\mathcal{A})$ is a rational $K(\pi, 1)$ space when studying the minimal model [Fal, Proposition 4.6], but the generality of Papadima and Yuzvinsky's arguments allows us to directly apply them to toric arrangements in Section 4.1.

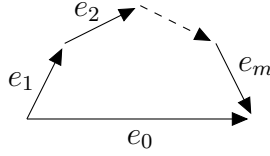
2.5. Toric Arrangements

De Concini and Procesi studied the cohomology of the complement of a toric arrangement. If \mathcal{A} is a unimodular toric arrangement, they show that the complement $M(\mathcal{A})$ is formal and give a presentation for the cohomology ring. Here, we state the result for graphic arrangements (which are always unimodular).

Theorem 2.10. [DCP1, Theorem 5.2] Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be an ordered graph, and let $\mathcal{A} = \mathcal{A}(\Gamma, \mathbb{C}^\times)$. Then $M(\mathcal{A})$ is formal and $H^*(M(\mathcal{A}), \mathbb{Q})$ is isomorphic to the exterior

algebra on the \mathbb{Q} -vector space spanned by $\{x_v, g_e \mid v \in \mathcal{V}, e \in \mathcal{E}\}$ modulo the ideal generated by:

- (ia) whenever e_0, e_1, \dots, e_m is a cycle with $t(e_0) = t(e_1)$, $h(e_0) = h(e_m)$, and $h(e_i) = t(e_{i+1})$ for $i = 1, \dots, m-1$ (as pictured below)



we have

$$g_{e_1} g_{e_2} \cdots g_{e_m} - \sum (-1)^{|I|+m+s_I} g_{e_{i_1}} \cdots g_{e_{i_k}} \psi_{e_{j_1}} \cdots \psi_{e_{j_{m-k-1}}} g_{e_0}$$

where the sum is taken over all $I = \{i_1 < \cdots < i_k\} \subsetneq \{1, \dots, m\}$ with complement $\{j_1 < \cdots < j_{m-k}\}$, $\psi_{e_\ell} = x_{h(e_\ell)} - x_{t(e_\ell)}$, and s_I is the parity of the permutation $(i_1, \dots, i_k, j_1, \dots, j_{m-k})$.

- (ib) if we again have a cycle, but have some arrows reversed, relabel the arrows so that $e_1 < \cdots < e_m < e_0$, then take the relation from (ia) and replace each ψ_{e_i} with $-\psi_{e_i}$ and each g_{e_i} with $-g_{e_i} - \psi_{e_i}$ whenever e_i points in the opposite direction of e_0 .

- (ii) $(x_{h(e)} - x_{t(e)})g_e$ for $e \in \mathcal{E}$.

The presentation encodes both the combinatorics of the arrangement and the geometry of the ambient space. The generators x_v come from the cohomology of \mathbb{C}^\times , while the generators g_e are similar to that of the Orlik-Solomon algebra for its rational counterpart. However, the toric analogue of the Orlik-Solomon relation is much more complicated.

2.6. Arrangements of Higher Genus Curves

We are particularly interested in the case of abelian arrangements, and the special case of elliptic arrangements. However, some results will hold when we consider projective curves of higher genus. The following result overlaps with Theorem 3.8 in the case of graphic elliptic arrangements.

By the work of Dupont and Bloch, we have the following presentation for graphic arrangements in the case that C is a complex projective curve of higher genus [Dup1].

Theorem 2.11. [Dup1] Let C be a complex projective curve with genus $g \geq 1$. Define the differential graded algebra $A(\mathcal{A})$ as the exterior algebra on the \mathbb{Q} -vector space spanned by

$$\{x_v^i, y_v^i, g_e \mid v \in \mathcal{V}, e \in \mathcal{E}, i = 1, \dots, g\}$$

modulo the ideal generated by the following relations:

- (i) $\sum_j (-1)^j g_{e_1} \cdots \hat{g}_{e_j} \cdots g_{e_k}$ whenever $\{e_1 < \cdots < e_k\}$ is a cycle,
- (ii) $(x_{h(e)}^i - x_{t(e)}^i)g_e, (y_{h(e)}^i - y_{t(e)}^i)g_e$, for each $e \in \mathcal{E}$,
- (iiia) $x_v^i y_v^j, x_v^i y_v^i, y_v^i y_v^j$ for $i \neq j$, and
- (iiib) $x_v^i y_v^i - x_v^j y_v^j$.

The differential is defined by putting $dx_v^i = dy_v^i = 0$ and

$$dg_e = x_{h(e)}^1 y_{h(e)}^1 + x_{t(e)}^1 y_{t(e)}^1 - \sum_{i=1}^g (x_{h(e)}^i y_{t(e)}^i + x_{t(e)}^i y_{h(e)}^i).$$

The DGA $A(\mathcal{A})$ is a model for $M(\mathcal{A})$.

Just as before, this algebra encodes both the combinatorics of the arrangement and the geometry of the ambient space. The generators x_v^i, y_v^i come from the

cohomology of C^ν , and we write these generators and relations here explicitly since we will use this presentation to show that the algebra is Koszul in Section 4.2. A more elegant way of writing the differential is to say that the generator g_e maps to $[\Delta_e] \in H^2(C^\nu)$, where Δ_e is the diagonal corresponding to the coordinates indexed by $h(e)$ and $t(e)$ in C^ν .

CHAPTER III

RATIONAL COHOMOLOGY

In this chapter, we study the rational cohomology of an abelian arrangement. We describe a method to compute it and give a combinatorial presentation in the case of unimodularity. We then discuss some more combinatorics and finish the chapter with some examples.

3.1. Leray Spectral Sequence

Let $\mathcal{A} = \{H_1, \dots, H_\ell\}$ be a set of smooth connected divisors that intersect like hyperplanes in a smooth complex variety X , and denote the complement of their union in X by $M(\mathcal{A})$. The inclusion $f : M(\mathcal{A}) \hookrightarrow X$ gives a Leray spectral sequence of the form

$$E_2^{p,q} = H^p(X; R^q f_* \mathbb{Q}) \implies H^{p+q}(M(\mathcal{A}); \mathbb{Q}).$$

Recall that $R^q f_* \mathbb{Q}$ is the sheafification of the presheaf on X taking an open set U to $H^q(U \cap M(\mathcal{A}); \mathbb{Q})$. To make use of this spectral sequence, we need to examine the sheaves $R^q f_* \mathbb{Q}$.

Let $x \in X$. In the following discussion, the localization of \mathcal{A} at a flat will always be with respect to the point x , and we will drop the superscript from our notation for localizations. Take the unique smallest component F_x of \mathcal{A} containing x . Note that x is a generic point of F_x , and so for a small neighborhood U around x , we have $U \cap M(\mathcal{A}) \cong M(\mathcal{A}_{F_x})$. This means that the stalk of our sheaf $R^q f_* \mathbb{Q}$ at x is given by

$$H^q(U \cap M(\mathcal{A}); \mathbb{Q}) \cong H^q(M(\mathcal{A}_{F_x}); \mathbb{Q}).$$

Note that the rank- q components (flats) of \mathcal{A}_{F_x} correspond exactly to the rank- q components of \mathcal{A} that contain F_x . For such an F , we can consider the usual localization of \mathcal{A}_{F_x} (a central hyperplane arrangement) at the component (flat) corresponding to F in \mathcal{A}_{F_x} , denoted by $(\mathcal{A}_{F_x})_F$. This is the same arrangement as \mathcal{A}_F . Then Brieskorn's Lemma (Lemma 2.8) implies that

$$H^q(M(\mathcal{A}_{F_x}); \mathbb{Q}) \cong \bigoplus_{\substack{F \supseteq F_x \\ \text{rk}(F)=q}} H^q(M((\mathcal{A}_{F_x})_F); \mathbb{Q}) \cong \bigoplus_{\substack{F \supseteq F_x \\ \text{rk}(F)=q}} H^q(M(\mathcal{A}_F); \mathbb{Q}).$$

Since x was a generic point of F_x , the rank- q components containing F_x are exactly the rank- q components containing x . Thus, the stalk at $x \in X$ can be decomposed as

$$(R^q f_* \mathbb{Q})_x \cong \bigoplus_{\substack{F \ni x \\ \text{rk}(F)=q}} H^q(M(\mathcal{A}_F); \mathbb{Q}).$$

Now, it is clear that the sheaf $R^q f_* \mathbb{Q}$ is supported on the union of the rank- q components of \mathcal{A} . We will define a sheaf ϵ_F , for each component F , that is supported on F . Then we will show that $R^q f_* \mathbb{Q}$ is isomorphic to the direct sum of these constant sheaves ϵ_F , taken over all rank- q components. This will help us prove the following lemma:

Lemma 3.1. Let $\mathcal{A} = \{H_1, \dots, H_\ell\}$ be a set of smooth connected divisors that intersect like hyperplanes in a smooth complex variety X . Then

$$H^p(X; R^q f_* \mathbb{Q}) \cong \bigoplus_{\text{rk}(F)=q} H^p(F; \mathbb{Q}) \otimes H^q(M(\mathcal{A}_F); \mathbb{Q}).$$

Proof. Let F be a component of rank q , and consider the divisors in \mathcal{A} that contain F . These form an arrangement in X , which we denote by $\mathcal{A}|_F$. Also denote its

complement in X by $M(\mathcal{A}|_F)$. The inclusion $g^F : M(\mathcal{A}|_F) \hookrightarrow X$ defines a sheaf $\epsilon_F := R^q g_*^F \mathbb{Q}$ on X . First, observe that the support of ϵ_F is equal to F . For any $x \in F$, there is a small neighborhood U of x in X such that $U \cap M(\mathcal{A}|_F) \cong M(\mathcal{A}_F)$. This means that the stalk at any point $x \in F$ is $(\epsilon_F)_x \cong H^q(M(\mathcal{A}_F); \mathbb{Q})$.

Moreover, the sheaf ϵ_F is constant on F . This is because, as we have previously discussed, $H^*(M(\mathcal{A}_F^{(x)}); \mathbb{Q})$, and hence the stalks of ϵ_F , can be canonically identified for any two points in F .

Let $\epsilon := \bigoplus_{\text{rk}(F)=q} \epsilon_F$, a sheaf on X . The stalk at $x \in X$ is

$$\epsilon_x = \bigoplus_{\text{rk}(F)=q} (\epsilon_F)_x \cong \bigoplus_{\substack{\text{rk}(F)=q, \\ x \in F}} H^q(M(\mathcal{A}_F); \mathbb{Q}).$$

For every open $U \subset X$, there is an inclusion $U \cap M(\mathcal{A}) \hookrightarrow U \cap M(\mathcal{A}|_F)$, which then induces a map $H^q(U \cap M(\mathcal{A}|_F); \mathbb{Q}) \rightarrow H^q(U \cap M(\mathcal{A}); \mathbb{Q})$. This gives a (presheaf) map $\epsilon \rightarrow R^q f_* \mathbb{Q}$. It is also an isomorphism on stalks, hence a sheaf isomorphism $\epsilon \cong R^q f_* \mathbb{Q}$.

Returning to the E_2 term of our Leray spectral sequence for the inclusion $f : M(\mathcal{A}) \hookrightarrow X$, we now have that

$$\begin{aligned} H^p(X; R^q f_* \mathbb{Q}) &\cong H^p(X; \epsilon) \\ &\cong \bigoplus_{\text{rk}(F)=q} H^p(X; \epsilon_F) \\ &\cong \bigoplus_{\text{rk}(F)=q} H^p(F; \mathbb{Q}) \otimes H^q(M(\mathcal{A}_F); \mathbb{Q}). \end{aligned}$$

□

If we further take X to be a projective variety, then the E_2 term of the spectral sequence is all that is needed to calculate the cohomology of $M(\mathcal{A})$.

Lemma 3.2. Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a set of smooth connected divisors that intersect like hyperplanes in a smooth complex projective variety X , and denote its complement by $M(\mathcal{A})$. Then all differentials d_j in the Leray spectral sequence for the inclusion $f : M(\mathcal{A}) \hookrightarrow X$ are trivial for $j > 2$.

Proof. To show that higher differentials are trivial, we consider the weight filtration on

$$H^p(X; R^q f_* \mathbb{Q}) \cong \bigoplus_{\text{rk}(F)=q} H^p(F; \mathbb{Q}) \otimes H^q(M(\mathcal{A}_F); \mathbb{Q}).$$

Note that since F is a smooth complex projective variety, $H^p(F; \mathbb{Q})$ is pure of weight p . Since $M(\mathcal{A}_F)$ is the complement of a rational hyperplane arrangement, $H^q(M(\mathcal{A}_F); \mathbb{Q})$ is pure of weight $2q$ (by [Sha]). This implies that $H^p(X; R^q f_* \mathbb{Q})$ is pure of weight $p + 2q$.

Now, the differentials must be strictly compatible with the weight filtration, as explained in the proof of Theorem 3 of [Tot]. Since the (p, q) position on the E_j term will also have weight $p + 2q$, the differential d_j will map something of weight $p + 2q$ to something of weight $(p + j) + 2(q - j + 1) = p + 2q - j + 2$. Being strictly compatible with weights implies that the only nontrivial differential must be when $j = 2$. \square

Moreover, if we consider only the cohomological grading (by $p + q$) on the E_2 term, we have the following theorem.

Theorem 3.3. Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a set of smooth connected divisors that intersect like hyperplanes in a smooth complex projective variety X . The rational cohomology of $M(\mathcal{A})$ is isomorphic as a graded algebra to the cohomology of $E_2(\mathcal{A})$ with respect to its differential.

Proof. By Lemma 3.2, the Leray spectral sequence degenerates at the E_3 term. This implies that the associated graded of $H^*(M(\mathcal{A}); \mathbb{Q})$ with respect to the Leray filtration is isomorphic to the cohomology of $E_2(\mathcal{A})$.

The groups $E_3^{p,q} = E_\infty^{p,q}$ that contribute to the k -th rational cohomology (when $p + q = k$) each have distinct weight (as described in the proof of Lemma 3.2), and so the Leray filtration is exactly the weight filtration. By the work of Deligne [Del, p. 81], the associated graded of $H^*(M(\mathcal{A}); \mathbb{Q})$ with respect to its weight filtration is isomorphic to $H^*(M(\mathcal{A}); \mathbb{Q})$ as an algebra. \square

Remark 3.4. The E_2 term of the spectral sequence forms a differential bi-graded algebra, denoted by $E_2(\mathcal{A})$. The main result of this section was that we have an isomorphism of algebras

$$H^*(E_2(\mathcal{A})) \cong \text{gr } H^*(M(\mathcal{A}); \mathbb{Q}),$$

where the right hand side is the associated graded with respect to the weight filtration. In particular, if we consider the bi-grading (and not just the cohomological grading) of $E_2(\mathcal{A})$, we have

$$H^{p,q}(E_2(\mathcal{A})) \cong \text{gr}_{p+2q} H^{p+q}(M(\mathcal{A}); \mathbb{Q}),$$

and we can compute the mixed Hodge numbers of $M(\mathcal{A})$.

Remark 3.5. The same method could be used to study the cohomology of an affine hyperplane arrangement in \mathbb{C}^n or of a toric arrangement in $(\mathbb{C}^\times)^n$. In fact, this is originally due to Looijenga [Loo]. In these cases, Lemma 3.1 applies, but Lemma 3.2 and Theorem 3.3 do not.

1. Let $\mathcal{A} = \{H_1, \dots, H_\ell\}$ be an affine arrangement of hyperplanes in a complex affine space X of dimension n , and denote the complement $M(\mathcal{A}) = X \setminus \cup_i H_i$.

For the Leray Spectral Sequence of the inclusion $f : M(\mathcal{A}) \hookrightarrow X$, the E_2 -term decomposes into $E_2^{0,q} = \bigoplus_{\text{rk}(F)=q} H^q(M(\mathcal{A}_F); \mathbb{Q})$ and $E_2^{p,q} = 0$ for $p \neq 0$. This forces the differentials to all be trivial, and we see that $E_2(\mathcal{A})$ is the Orlik-Solomon algebra $H^*(M(\mathcal{A}); \mathbb{Q})$.

2. Let $\mathcal{A} = \{H_1, \dots, H_\ell\}$ be an arrangement of codimension-one subtori in a complex torus $X = (\mathbb{C}^\times)^n$, and denote the complement by $M(\mathcal{A}) = X \setminus \bigcup_i H_i$. The E_2 -term of the Leray Spectral Sequence for the inclusion $f : M(\mathcal{A}) \hookrightarrow X$ decomposes into components, so that

$$E_2^{p,q} = \bigoplus_{\text{rk}(F)=q} H^p(F; \mathbb{Q}) \otimes H^q(M(\mathcal{A}_F); \mathbb{Q}).$$

Here, F is a complex torus and so $E_2^{p,q}$ is pure of weight $2(p+q)$. Since the differentials d_j respect the weights, d_j must be trivial for all j . Thus, $E_2(\mathcal{A}) \cong \text{gr}_L H^*(M(\mathcal{A}); \mathbb{Q})$, the associated graded with respect to the Leray filtration. This decomposition of the cohomology is the decomposition given by De Concini and Procesi in [DCP1, Remark 4.3]. However, $E_2(\mathcal{A})$ and $H^*(M(\mathcal{A}); \mathbb{Q})$ are not isomorphic as algebras in this case.

Remark 3.6. Another interesting result for an abelian arrangement \mathcal{A} in X comes from considering the deletion and restriction arrangements, with respect to some fixed $H_0 \in \mathcal{A}$. Here, we mean the analogous notion to the theory of hyperplane arrangements, where the deletion of H_0 is the arrangement $\mathcal{A}' = \mathcal{A} \setminus \{H_0\}$ in X and the restriction to H_0 is the arrangement \mathcal{A}'' of nonempty $\{H \cap H_0 : H \in \mathcal{A}'\}$ in H_0 . In the theory of hyperplane arrangements, the long exact sequence of the pair

$(M(\mathcal{A}'), M(\mathcal{A}))$ relates the cohomologies of $M(\mathcal{A})$, $M(\mathcal{A}')$, and $M(\mathcal{A}'')$, as follows:

$$\cdots \rightarrow H^i(M(\mathcal{A}')) \rightarrow H^i(M(\mathcal{A})) \rightarrow H^{i-1}(M(\mathcal{A}'')) \rightarrow H^{i+1}(M(\mathcal{A})) \rightarrow \cdots$$

Moreover, this long exact sequence splits into short exact sequences relating these cohomologies. In the abelian arrangement case, we can get the same kind of long exact sequence. However, it does not split into short exact sequences. To study the (nontrivial) boundary map, we can derive this long exact sequence in another way, by taking the long exact sequence induced by a short exact sequence of complexes

$$0 \rightarrow E_2(\mathcal{A}')^* \rightarrow E_2(\mathcal{A})^* \rightarrow E_2(\mathcal{A}'')^{*-1} \rightarrow 0.$$

The boundary map is then seen to be

$$\pi_* : H^{i-1}(M(\mathcal{A}''); \mathbb{Q})(-1) \rightarrow H^{i+1}(M(\mathcal{A}'); \mathbb{Q})$$

where $\pi : M(\mathcal{A}'') \hookrightarrow M(\mathcal{A}')$ is the closed immersion.

Remark 3.7. Dupont [Dup1] independently found the same differential graded algebra as described here. He considers the cohomology of the complement of a union $Y = Y_1 \cup \cdots \cup Y_\ell$ of smooth hypersurfaces which intersect like hyperplanes in a smooth complex projective variety X , and for simplicity he assumes that the arrangement is unimodular. Dupont's method uses the Gysin spectral sequence, which degenerates at the E_2 term and has a differential graded algebra $M^*(X, Y)$ as the E_1 term. Setting $\mathcal{A} = \{Y_1, \dots, Y_\ell\}$, the differential graded algebras $E_2(\mathcal{A})$ and $M^*(X, Y)$ are isomorphic. Moreover, Dupont constructs a wonderful compactification of these arrangements, so that the space $X \setminus Y$ can be realized as the complement of

a normal crossings divisor \tilde{Y} in a smooth projective variety \tilde{X} . Dupont also shows functoriality of M^* so that $M^*(X, Y)$ is quasi-isomorphic to $M^*(\tilde{X}, \tilde{Y})$.

By the work of Morgan [Mor], the differential graded algebra $M^*(\tilde{X}, \tilde{Y})$ is a model for the space $X \setminus Y = \tilde{X} \setminus \tilde{Y}$, in the sense of rational homotopy theory. Since our differential graded algebra $E_2(\mathcal{A})$ is isomorphic to $M^*(X, Y)$ and hence quasi-isomorphic to $M^*(\tilde{X}, \tilde{Y})$, $E_2(\mathcal{A})$ is a model for the space $M(\mathcal{A}) = X \setminus Y$.

3.2. Unimodular Abelian Arrangements

To explicitly describe the \mathbb{Q} -algebra structure of the E_2 term of the spectral sequence, we assume further that \mathcal{A} is a unimodular abelian arrangement. Recall that we allow the $H_i \in \mathcal{A}$ to be a translation of an abelian subvariety of X ; denote this subvariety by \bar{H}_i . For each $H_i \in \mathcal{A}$, let $E_i = X/\bar{H}_i$, an elliptic curve, so that \bar{H}_i is the kernel of the projection $\alpha_i : X \rightarrow E_i$. The E_2 term is a bi-graded algebra with a differential, which we denote by $E_2(\mathcal{A})$. The (p, q) -th graded term is isomorphic to

$$\bigoplus_{\text{rk}(F)=q} H^p(F; \mathbb{Q}) \otimes H^q(M(\mathcal{A}_F); \mathbb{Q})$$

by Lemma 3.1.

The multiplication of $E_2(\mathcal{A})$ can be described as follows: Let $x_1 \otimes y_1$ be in $H^{p_1}(F_1; \mathbb{Q}) \otimes H^{q_1}(M(\mathcal{A}_{F_1}); \mathbb{Q})$, and let $x_2 \otimes y_2$ be in $H^{p_2}(F_2; \mathbb{Q}) \otimes H^{q_2}(M(\mathcal{A}_{F_2}); \mathbb{Q})$. If $F_1 \cap F_2 = \emptyset$, then $(x_1 \otimes y_1) \cdot (x_2 \otimes y_2) = 0$. Otherwise, let $F = F_1 \cap F_2$ (which by unimodularity is a component of \mathcal{A}), $p = p_1 + p_2$, and $q = q_1 + q_2$. Also let, for $j = 1, 2$, $\gamma_j : F \hookrightarrow F_j$ and $\eta_j : M(\mathcal{A}_F) \hookrightarrow M(\mathcal{A}_{F_j})$ be the natural inclusions. Then

$$(x_1 \otimes y_1) \cdot (x_2 \otimes y_2) = (-1)^{q_1 p_2} (\gamma_1^*(x_1) \cup \gamma_2^*(x_2)) \otimes (\eta_1^*(y_1) \cup \eta_2^*(y_2)),$$

an element of $H^p(F; \mathbb{Q}) \otimes H^q(M(\mathcal{A}_F); \mathbb{Q})$.

In particular, consider the case that $F_1 = H_i$ and $F_2 = X$. For $1 \otimes g$ in $H^0(H_i; \mathbb{Q}) \otimes H^1(M(\mathcal{A}_{H_i}); \mathbb{Q})$ and $x \otimes 1$ in $H^p(X; \mathbb{Q}) \otimes H^0(M(\mathcal{A}_X); \mathbb{Q})$, we have

$$(1 \otimes g) \cdot (x \otimes 1) = (-1)^p \gamma_i^*(x) \otimes g \in H^p(H_i) \otimes H^q(M(\mathcal{A}_{H_i}); \mathbb{Q}).$$

Since H_i is (a possible translation of) the kernel of some map $\alpha_i : X \rightarrow E_i$, the kernel of γ_i^* contains the image of α_i^* in positive degree. This means that for $p > 0$, and any element $x \in H^p(X; \mathbb{Q})$ that is in the image of α_i^* , $(1 \otimes g) \cdot (x \otimes 1) = 0$.

We further observe that the row $q = 0$ inherits an algebra structure from $H^*(X; \mathbb{Q})$, and the column $p = 0$ inherits an algebra structure from the Orlik-Solomon algebra. In particular, if \mathcal{A} is central, then the column $p = 0$ inherits an algebra structure from $H^*(M(\mathcal{A}_0); \mathbb{Q})$ where \mathcal{A}_0 is the localization at the intersection of all hyperplanes in \mathcal{A} . These algebras are generated in degree one; moreover, they will generate the entire $E_2(\mathcal{A})$ algebra. This is because the map $\gamma^* : H^*(X; \mathbb{Q}) \rightarrow H^*(F; \mathbb{Q})$, where F is a component and $\gamma : F \hookrightarrow X$ is the natural inclusion, is surjective.

Since the algebra is generated by $E_2^{1,0}$ and $E_2^{0,1}$, it suffices to describe the differential on $H^0(H_i; \mathbb{Q}) \otimes H^1(M(\mathcal{A}_{H_i}); \mathbb{Q})$ for each $H_i \in \mathcal{A}$. This has a canonical generator, since the Orlik-Solomon algebra $H^*(M(\mathcal{A}_{H_i}); \mathbb{Q})$ has a canonical generator in degree one. The differential here is determined by the differential of the Leray spectral sequence for the inclusions $X \setminus H_i \hookrightarrow X$, which takes the generator to $[H_i] \in H^2(X; \mathbb{Q})$.

Now we will describe an algebra $A(\mathcal{A})$ determined by the arrangement \mathcal{A} and then prove in Theorem 3.8 that this algebra is isomorphic to $E_2(\mathcal{A})$. Let $B(\mathcal{A}) = H^*(X; \mathbb{Q})[g_1, \dots, g_\ell]$, a graded-commutative, bigraded algebra over \mathbb{Q} , where

$H^i(X; \mathbb{Q})$ has degree $(i, 0)$ and each g_j has degree $(0, 1)$. Let $I(\mathcal{A})$ be the ideal in $B(\mathcal{A})$ generated by the following relations:

- (1) $g_{i_1} \cdots g_{i_k}$ whenever $\bigcap_{j=1}^k H_{i_j} = \emptyset$.
- (2) $\sum_{j=1}^k (-1)^{j-1} g_{i_1} \cdots \hat{g}_{i_j} \cdots g_{i_k}$ whenever H_{i_1}, \dots, H_{i_k} are dependent.
- (3) $\alpha_i^*(x)g_i$, where α_i defines H_i and $x \in H^1(E_i; \mathbb{Q})$.

For notational purposes, we denote $g_S = g_{i_1} \cdots g_{i_k}$ for $S = \{H_{i_1}, \dots, H_{i_k}\}$ and $\partial g_S = \sum_{j=1}^k (-1)^{j-1} g_{i_1} \cdots \hat{g}_{i_j} \cdots g_{i_k}$.

Let $A(\mathcal{A}) = B(\mathcal{A})/I(\mathcal{A})$. Since $I(\mathcal{A})$ is homogeneous with respect to the grading on $B(\mathcal{A})$, $A(\mathcal{A})$ is a bi-graded algebra over \mathbb{Q} . Moreover, there is a differential on $A(\mathcal{A})$ defined by $dg_i = [H_i] \in H^2(X; \mathbb{Q})$ and $dx = 0$ for $x \in H^*(X; \mathbb{Q})$.

Theorem 3.8. Assume that $\mathcal{A} = \{H_1, \dots, H_\ell\}$ is a unimodular abelian arrangement. Then there is an isomorphism of bi-graded differential algebras

$$\phi : A(\mathcal{A}) \rightarrow E_2(\mathcal{A}).$$

Proof. First, we show that there is a surjective homomorphism ϕ , by defining a map $\theta : B(\mathcal{A}) \rightarrow E_2(\mathcal{A})$ which induces ϕ as follows: Let

$$\theta(g_i) := 1 \otimes e_i \in H^0(H_i; \mathbb{Q}) \otimes H^1(M(\mathcal{A}_{H_i}); \mathbb{Q})$$

where e_i is the canonical generator of $H^1(M(\mathcal{A}_{H_i}); \mathbb{Q})$, and for $x \in H^i(X; \mathbb{Q})$, let

$$\theta(x) := x \otimes 1 \in H^i(X; \mathbb{Q}) \otimes H^0(M(\mathcal{A}_X); \mathbb{Q}).$$

We have already observed that $E_2(\mathcal{A})$ is generated by $E_2^{1,0}$ and $E_2^{0,1}$. Even more explicitly, the elements $1 \otimes e_i$ and $x \otimes 1$ as above generate the algebra. Thus, θ is surjective.

By our observations above, it is easy to see that $\theta(g_S) = 0$ whenever $H_S = \emptyset$. For relation (2), suppose S is a dependent subset of \mathcal{A} . Then

$$\theta(\partial g_S) = 1 \otimes (\partial e_S) \in H^0(H_S; \mathbb{Q}) \otimes H^{\text{rk}(S)}(M(\mathcal{A}_{H_S}); \mathbb{Q})$$

which is zero since $\partial e_S = 0$ in the Orlik-Solomon algebra $H^*(M(\mathcal{A}_{H_S}); \mathbb{Q})$. Also, by our observations above, $\theta(\alpha_i^*(x)g_i)$ is equal to zero. Therefore, $\theta(I(\mathcal{A})) = 0$ and hence θ induces the desired surjection $\phi : A(\mathcal{A}) \rightarrow E_2(\mathcal{A})$.

We can decompose $B(\mathcal{A})$ with respect to the components of the arrangement, $B(\mathcal{A}) = \bigoplus_F B_F$, where B_F is the \mathbb{Q} -vector space spanned by xg_S for all standard tuples S of hyperplanes in \mathcal{A} whose intersection is exactly F , and all $x \in H^*(X; \mathbb{Q})$. The ideal $I(\mathcal{A})$ is homogeneous with respect to this grading. Thus, $A(\mathcal{A})$ can be decomposed by $A(\mathcal{A}) = \bigoplus_F A_F$, where $A_F = B_F/I_F$ with $I_F := I(\mathcal{A}) \cap B_F$.

The E_2 term of the Leray spectral sequence can also be graded by the components. Here, we have $E_2(\mathcal{A}) = \bigoplus_F E_2(F)$, where for each component F , $E_2(F) = H^*(F; \mathbb{Q}) \otimes H^{\text{rk}(F)}(M(\mathcal{A}_F); \mathbb{Q})$.

It suffices to show that, as \mathbb{Q} -vector spaces, $A_F \cong E_2(F)$. We do this by examining A_F . We have $B_F \cong \bigoplus_S H^*(X; \mathbb{Q}) \cdot g_S$, where the direct sum is taken over all standard tuples S of hyperplanes in \mathcal{A} with $H_S = F$. If we consider just the ideal I_1 generated by relations (1) and (2), then

$$B_F/(I_1 \cap B_F) \cong \bigoplus_S H^*(X; \mathbb{Q}) \cdot g_S,$$

where the sum is taken over all non-broken circuits S with $H_S = F$. This is because relations (1) and (2) are just the Orlik-Solomon relations on the g_i 's associated to F .

Next, we claim that relation (3) implies that for all $H_i \supseteq F$, all $S \subseteq \mathcal{A}$ with $H_S = F$, and all $x \in H^1(E_i; \mathbb{Q})$, we have $\alpha_i^*(x)g_S \in I$. This implies that relation (3) depends only on the component F , and not on the choice of subset S . This claim is clearly true when $H_i \in S$. If $H_i \notin S$, then take a maximal independent subset of S , denoted by T . Then $C := T \cup \{H_i\}$ is a dependent set, and $H_C = H_T = F$. We may assume, for ease of notation, that our hyperplanes are ordered so that $g_S = g_{(S-T)}g_T$ and $g_C = g_i g_T$. Then we have $g_T - g_i \partial g_T = \partial g_C \in I$, since C is dependent. This implies that

$$\begin{aligned} \alpha_i^*(x)g_S &= \alpha_i^*(x)g_{(S-T)}g_T \\ &= \alpha_i^*(x)g_{(S-T)}(g_T - g_i \partial g_T) + \alpha_i^*(x)g_{(S-T)}g_i \partial g_T \\ &\in I. \end{aligned}$$

Let J_F be the ideal in $H^*(X; \mathbb{Q})$ generated by $\alpha_i^*(x)$ for all $H_i \supseteq F$ and $x \in H^1(E_i; \mathbb{Q})$. Now, since $H^*(F; \mathbb{Q}) \cong H^*(X; \mathbb{Q})/J_F$, we must have that

$$A_F \cong \bigoplus H^*(F; \mathbb{Q}) \cdot g_S$$

where the sum is taken over all non-broken circuits S with $H_S = F$. This is then isomorphic to $H^*(F; \mathbb{Q}) \otimes H^q(M(\mathcal{A}_F); \mathbb{Q}) \cong E_2(F)$, since the non-broken circuits form a basis for $H^q(M(\mathcal{A}_F); \mathbb{Q})$. \square

Remark 3.9. If \mathcal{A} is not unimodular, we can still define the bi-graded differential algebra $A(\mathcal{A})$ and the homomorphism $\phi : A(\mathcal{A}) \rightarrow E_2(\mathcal{A})$, but it will no longer be

surjective. The problem is that if an intersection H_S of subvarieties has multiple components, the image of ϕ will include the element $1 \in H^0(H_S; \mathbb{Q})$, but it will not include the corresponding classes for the individual components.

3.3. Combinatorics

For both linear and toric central arrangements, the Poincaré polynomial of the complement is a specialization of the Tutte polynomial, which is determined by the associated matroid (see [DCP2, Moc]). While the combinatorial model in Theorem 3.8 gives a way of computing cohomology of a unimodular abelian arrangement, it is still unknown if there is a combinatorial formula for the Poincaré polynomial (or Betti numbers). However, this model does give us a combinatorial formula for the Euler characteristic and E-polynomial as a specialization of the Tutte polynomial. In fact, using the arithmetic Tutte polynomial (see [Moc]), these results will work for non-unimodular arrangements, but for simplicity we assume unimodularity.

The E-polynomial is a specialization of the Hodge polynomial, which can be computed using the bi-grading on $A(\mathcal{A})$ by

$$E(t) = \sum_{p,q} \dim(A^{p,q}) t^{p+2q} (-1)^{p+q}.$$

The Tutte polynomial for the matroid on \mathcal{A} is defined as

$$T(x, y) = \sum_{B \subseteq \mathcal{A}} (x-1)^{\text{rk}(\mathcal{A}) - \text{rk}(B)} (y-1)^{|B| - \text{rk}(B)}.$$

Theorem 3.10. Let \mathcal{A} be a unimodular central abelian arrangement with $\text{rk}(\mathcal{A}) = n$. Then the E-polynomial is $E(t) = (-t^2)^n T\left(\frac{2t-1}{t^2}, 0\right)$.

Proof. The proof is very similar to Moci's proof in the toric case (that the Poincaré polynomial of a toric arrangement is a specialization of the Tutte polynomial) [Moc, Theorem 5.10]. In fact, we will use Lemmas 5.5 and 5.9 from [Moc], whose proofs are valid in our case as well. These lemmas give us that for each component F , the number of non-broken circuits associated to F is

$$\text{nbc}(F) = (-1)^{\text{rk}(F)} \sum_{\substack{B \subseteq \mathcal{A} \\ H_B = F}} (-1)^{|B|}.$$

Then we have (where \mathcal{P} is the set of components of \mathcal{A}):

$$\begin{aligned} (-t^2)^n T\left(\frac{2t-1}{t^2}, 0\right) &= (-t^2)^n \sum_{B \subseteq \mathcal{A}} \left(\frac{(1-t)^2}{-t^2}\right)^{n-\text{rk}(B)} (-1)^{|B|-\text{rk}(B)} \\ &= \sum_{F \in \mathcal{P}} \sum_{\substack{B \subseteq \mathcal{A} \\ H_B = F}} (-t^2)^{\text{rk}(B)} (1-t)^{2(n-\text{rk}(B))} (-1)^{|B|-\text{rk}(B)} \\ &= \sum_{F \in \mathcal{P}} \left((-t^2)^{\text{rk}(F)} (1-t)^{2(n-\text{rk}(F))} (-1)^{\text{rk}(F)} \sum_{\substack{B \subseteq \mathcal{A} \\ H_B = F}} (-1)^{|B|} \right) \\ &= \sum_{F \in \mathcal{P}} \text{nbc}(F) (-t^2)^{\text{rk}(F)} (1-t)^{2(n-\text{rk}(F))} \\ &= \sum_{F \in \mathcal{P}} \left(\text{nbc}(F) (-t^2)^{\text{rk}(F)} \sum_p \binom{2(n-\text{rk}(F))}{p} (-t)^p \right) \\ &= \sum_q \sum_{\substack{F \\ \text{rk}(F)=q}} \left((-t^2)^q \text{nbc}(F) \sum_p \dim H^p(F) (-t)^p \right) \\ &= \sum_{p,q} \sum_{\substack{F \\ \text{rk}(F)=q}} \dim(H^p(F)) \dim(H^q(M(\mathcal{A}_F))) t^{p+2q} (-1)^{p+q} \\ &= E(t). \end{aligned}$$

□

Now, since the Euler characteristic is a specialization of the E-polynomial, we get the following corollary:

Corollary 3.11. If \mathcal{A} is a unimodular central abelian arrangement with $\text{rk}(\mathcal{A}) = n$, then the Euler characteristic of $M(\mathcal{A})$ is equal to

$$E(1) = (-1)^n T(1, 0) = (-1)^n \sum \text{NBC}(F),$$

where the sum on the right is over all dimension-zero components F .

3.4. Examples

We will state the case of an ordered configuration space (the complement of the braid arrangement) on an elliptic curve, which is the model give by Kriz [Kri] and Totaro [Tot].

Example 3.12. Let Γ be the complete graph on n vertices, and $\mathcal{A} = \mathcal{A}(\Gamma, E)$ for a complex elliptic curve E . Then $A(\mathcal{A})$ is a differential graded algebra which is the quotient of the graded-commutative \mathbb{Q} -algebra generated by

$$\{x_1, \dots, x_n, y_1, \dots, y_n, g_{ab} \mid 1 \leq a, b \leq n; a \neq b\},$$

with each x_i and y_i in degree $(1, 0)$ and g_{ab} in degree $(0, 1)$, by the following relations:

- (1) $g_{ab} = g_{ba}$,
- (2) $g_{ab}g_{ac} + g_{bc}g_{ba} + g_{ca}g_{cb} = 0$ for a, b, c distinct, and
- (3) $(x_a - x_b)g_{ab}, (y_a - y_b)g_{ab}$.

The differential is given by $g_{ab} = (x_a - x_b)(y_a - y_b)$.

Next, we will show an example in which the presentation from Theorem 3.8 is used to compute the cohomology of $M(\mathcal{A})$. Moreover, if we consider the bi-grading on $A(\mathcal{A}) \cong E_2(\mathcal{A})$, then we can compute the dimension of $\text{gr}_j H^i(M(\mathcal{A}); \mathbb{Q})$, the associated graded with respect to the weight filtration. By Remark 3.4, the (p, q) -th graded piece of $H^*(A(\mathcal{A}))$ will be isomorphic to $\text{gr}_{p+2q} H^{p+q}(M(\mathcal{A}); \mathbb{Q})$. We encode the information about dimension in a two-variable polynomial $H(t, u)$, where the coefficient of $t^i u^j$ is the dimension of $\text{gr}_j H^i(M(\mathcal{A}); \mathbb{Q})$.

Example 3.13. Let $X = E^2$ for an elliptic curve E , and let $\alpha_i : E^2 \rightarrow E$ be projection onto the i -th coordinate. Consider the arrangement $\mathcal{A} = \{H_1, H_2, H_3\}$ in E^2 with $H_1 = \ker \alpha_1$, $H_2 = \ker \alpha_2$, and $H_3 = \ker(\alpha_1 - \alpha_2)$. Pick generators x and y for $H^*(E; \mathbb{Q})$, where xy is the class of the identity of E . Then $H^*(E^2; \mathbb{Q})$ is generated by $x_i = \alpha_i^*(x)$ and $y_i = \alpha_i^*(y)$ for $i = 1, 2$. This then implies that the algebra $B(\mathcal{A})$ is the exterior algebra with generators $\{x_1, y_1, x_2, y_2, g_1, g_2, g_3\}$.

The relations in $I(\mathcal{A})$ can be written as

- (1) no relations of the type g_S (since all intersections are nonempty)
- (2) $g_2 g_3 - g_1 g_3 + g_1 g_2$ (since $\{H_1, H_2, H_3\}$ is minimally dependent)
- (3) $x_1 g_1, y_1 g_1, x_2 g_2, y_2 g_2, (x_1 - x_2) g_3$, and $(y_1 - y_2) g_3$.

The differential of $A(\mathcal{A}) = B(\mathcal{A})/I(\mathcal{A})$ is defined by $dx_i = 0$, $dy_i = 0$, $dg_1 = [H_1] = x_1 y_1$, $dg_2 = [H_2] = x_2 y_2$, and $dg_3 = [H_3] = x_1 y_1 - x_1 y_2 - x_2 y_1 + x_2 y_2$.

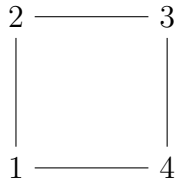
Computing cohomology, the polynomial described above becomes

$$H(t, u) = 1 + 4tu + 3t^2u^2 + 2t^2u^3.$$

Setting $u = 1$, we obtain the Poincaré polynomial $P(t) = 1 + 4t + 5t^2$. Setting $t = -1$, we obtain the E-polynomial $E(u) = 2u^3 + 3u^2 - 4u + 1$. Finally, the Euler characteristic is $P(-1) = E(1) = 2$.

To compute the next examples, and many more, we could use Macaulay2 [GS]. In particular, I have written a program that computes the two-variable Hodge polynomial for any elliptic arrangement defined from a unimodular matrix. The following is a list of some computations for graphic arrangements on four vertices.

Example 3.14. Consider the following graph, which defines an abelian arrangement by taking the columns of the given matrix.

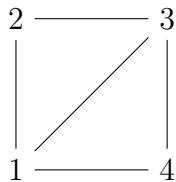


$$\begin{pmatrix} -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Using Macaulay2 to compute cohomology of our algebra, we find the two-variable Hodge polynomial

$$\begin{aligned} H(t, u) = & 2t^5u^7 + 3t^5u^6 + 4t^5u^5 + 4t^4u^6 + 6t^4u^5 + 19t^4u^4 \\ & + 2t^3u^5 + 3t^3u^4 + 32t^3u^3 + 24t^2u^2 + 8tu + 1. \end{aligned}$$

Example 3.15. Consider the following graph, which defines an abelian arrangement by taking the columns of the given matrix.



$$\begin{pmatrix} -1 & 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Using Macaulay2 to compute cohomology of our algebra, we find the two-variable Hodge polynomial

$$H(t, u) = 2t^5u^7 + 7t^5u^6 + 4t^5u^5 + 4t^4u^6 + 18t^4u^5 + 18t^4u^4 \\ + 2t^3u^5 + 15t^3u^4 + 30t^3u^3 + 4t^2u^3 + 23t^2u^2 + 8tu + 1.$$

Example 3.16. Consider the following graph, which defines an abelian arrangement by taking the columns of the given matrix.



$$\begin{pmatrix} -1 & 0 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

Using Macaulay2 to compute cohomology of our algebra, we find the two-variable Hodge polynomial

$$H(t, u) = 4t^5u^7 + 10t^5u^6 + 4t^5u^5 + 8t^4u^6 + 28t^4u^5 + 17t^4u^4 \\ + 4t^3u^5 + 26t^3u^4 + 28t^3u^3 + 8t^2u^3 + 22t^2u^2 + 8tu + 1.$$

Also note that the E-polynomial is

$$E(u) = H(-1, u) = -(u - 1)^2(2u - 1)(u^2 + 2u - 1)(2u^2 + 2u - 1)$$

and the Euler characteristic is equal to 0.

CHAPTER IV

RATIONAL HOMOTOPY THEORY

The results stated in this chapter are from a joint project with Justin Hilburn. This material was written entirely by me, with editorial assistance from my co-author. Moreover, I was the primary contributor to the proofs on Koszulity (Sections 4.1 and 4.2) and made a significant contribution to the application of quadratic duality to the rational homotopy theory of abelian arrangements (Section 4.3). Justin's contribution to the method of using quadratic duality to extract information about rational homotopy theory was invaluable.

4.1. Toric Arrangements

Since the complement to a chordal toric arrangement is formal (as in the linear case), we want to show that its cohomology ring is Koszul. Our argument will be similar to (but slightly more complicated than) the linear case. We will provide a \mathbb{Q} -basis for the cohomology ring, use it to show that our generating set of the ideal is a Gröbner basis, and then reduce the Gröbner basis to a quadratic one.

Lemma 4.1. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a chordal graph. Let F be a flat of the arrangement $\mathcal{A} = \mathcal{A}(\Gamma, \mathbb{C}^\times)$, and let S be a non-broken circuit associated to F . Define I_F to be the ideal generated by

$$\{x_{h(e)} - x_{t(e)} \mid e \in F\}$$

in $\Lambda(x_v \mid v \in \mathcal{V}) \cong H^*((\mathbb{C}^\times)^\mathcal{V})$, and let $H_F = \bigcap_{e \in F} H_e \subseteq (\mathbb{C}^\times)^\mathcal{V}$.

1. With the degree-lexicographic order and $x_v < x_{v'}$ whenever $v < v'$, the set

$$G_S := \{x_{h(e)} - x_{t(e)} \mid e \in S\}$$

is a Gröbner basis for I_F .

2. The set $\{x_{i_1} \cdots x_{i_r} \mid h(e) \notin \{i_1, \dots, i_r\} \text{ for each } e \in S\}$ is a basis for

$$H^*(H_F) \cong \Lambda(x_v \mid v \in \mathcal{V})/I_F$$

and this basis does not depend on the choice of nbc set S .

Proof. For linear relations, finding a Gröbner basis is equivalent to Gaussian elimination, and so consider the matrix M_F whose rows are indexed by edges $e \in F$, whose columns are indexed by the vertices in decreasing order, and whose entries are zero except $(M_F)_{e,h(e)} = 1$ and $(M_F)_{e,t(e)} = -1$ (so that row e corresponds to the element $x_{h(e)} - x_{t(e)}$). Note that since $|S| = \text{rk}(S) = \text{rk}(M_F)$, we may use row operations so that the rows corresponding to elements of S remain unchanged while all other rows are zero. Moreover, since $|\{h(e) \mid H_e \in S\}| = |S|$, the matrix is in row echelon form. Thus, G_S is a Gröbner basis for I_F .

For part (2), by Gröbner basis theory, the set of monomials not in $\text{In}(I_F)$ form a basis for $H^*(H_F)$. Since the ideal $\text{In}(I_F)$ is generated by $\text{In}(G_S) = \{x_{h(e)} \mid e \in S\}$, the monomials not in $\text{In}(I_F)$ are precisely those stated. Since S is an nbc set associated to F , it spans the subgraph $\Gamma[F]$. Thus $\{h(e) \mid e \in F\} = \{h(e) \mid e \in S\}$ and the basis given does not depend on S . \square

For ease of notation, we will use

$$x_{AGS} := x_{a_1} \cdots x_{a_r} g_{c_1} \cdots g_{c_k}$$

where $A = \{a_1 < \cdots < a_r\}$ and $S = \{c_1 < \cdots < c_k\}$. We will also denote relations (ia) and (ib) from Theorem 2.10 by r_S for a cycle S .

Lemma 4.2. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a chordal graph, and let $\mathcal{A} = \mathcal{A}(\Gamma, \mathbb{C}^\times)$. Define P to be the set of all monomials x_{AGS} such that S is a non-broken circuit and $h(e) \notin A$ for all $e \in S$. Then P is a basis for $H^*(M(\mathcal{A}))$.

Proof. There is a decomposition into the flats of \mathcal{A} [DCP1, Remark 4.3(2)] (see also Lemma 3.1), which is given by the following: For a flat F , let $H_F = \bigcap_{e \in F} H_e \subseteq (\mathbb{C}^\times)^\mathcal{V}$, and let V_F be the vector space spanned by g_S for all nbc sets S associated to F . Then

$$H^*(M(\mathcal{A})) = \bigoplus_F H^*(H_F) \otimes V_F.$$

Denote $H^*(H_F) \otimes V_F$ by A_F . To show that P is a basis for $H^*(M(\mathcal{A}))$, it suffices to show that

$$P \cap A_F = \{x_{AGC} \mid h(c) \notin A \text{ for } c \in C, C \text{ is an nbc set associated to } F\}$$

is a basis for A_F . But this follows from Lemma 4.1. □

Theorem 4.3. Let \mathcal{A} be a chordal toric arrangement. Then $H^*(M(\mathcal{A}))$ is Koszul.

Proof. Fix a degree-lexicographic order on $H^*(M(\mathcal{A}))$ that is induced by our order on \mathcal{V} . That is, $g_e < g_{e'}$ if $e < e'$, and $x_{h(e)} < g_e < x_{h(e)+1}$. We will show that

$$G = \{(x_{h(e)} - x_{t(e)})g_e, r_S \mid e \in \mathcal{E}, S \text{ a circuit}\}$$

is a Gröbner basis with this order which can be reduced to a quadratic Gröbner basis.

We have

$$\text{In}(G) = \{x_{h(e)}g_e, g_S \mid e \in \mathcal{E}, S \text{ is a broken circuit}\}.$$

Then P is the set of monomials that are not in $\langle \text{In}(G) \rangle$. Since $\langle \text{In}(G) \rangle \subseteq \text{In}(I)$, P contains the monomials that are not in $\text{In}(I)$. Since the set of monomials not in $\text{In}(I)$ is a basis for $H^*(M(\mathcal{A}))$ contained in the basis P , and $H^*(M(\mathcal{A}))$ is finite dimensional, we must have equality throughout. That means that the monomials in $\langle \text{In}(G) \rangle$ are exactly the monomials in $\text{In}(I)$. Since these ideals are generated by monomials, they must be equal. Note that the relations of type (ii) are already quadratic. In a similar way as in the linear case, we can reduce our relations r_S to quadratic ones as well. \square

Recall that if X is a formal space, then $H^*(X)$ is Koszul if and only if X is rationally $K(\pi, 1)$ [PY, Theorem 5.1]. We saw this result used in the linear case. Since formality holds in the toric case, we can apply their work to obtain the following theorem:

Theorem 4.4. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a chordal graph and $\mathcal{A} = \mathcal{A}(\Gamma, \mathbb{C}^\times)$.

1. The holonomy Lie algebra of $M(\mathcal{A})$ is the Lie algebra dual to $H^*(M(\mathcal{A}))$.
2. The minimal model of $M(\mathcal{A})$ is the standard (or Chevalley-Eilenberg) complex of L , $\Omega(L^\bullet)$.
3. $M(\mathcal{A})$ is a rational $K(\pi, 1)$ space.

4.2. Koszulity of $A(\mathcal{A})$

We first show that $A(\mathcal{A})$ is Koszul in the case of abelian arrangements, and then we extend to curves of higher genus.

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a chordal graph, and let E be a complex elliptic curve. For the chordal abelian arrangement $\mathcal{A} = \mathcal{A}(\Gamma, E)$, consider the algebra $A(\mathcal{A})$ from Theorem 3.8 (ignoring the differential). In this section, we will prove that $A(\mathcal{A})$ is Koszul. The proof is very similar to (but slightly more complicated than) the toric case.

Lemma 4.5. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a chordal graph. Let $F \subseteq \mathcal{E}$ be a flat of the arrangement $\mathcal{A} = \mathcal{A}(\Gamma, E)$, and let S be a non-broken circuit associated to F . Define I_F to be the ideal generated by

$$\{x_{h(e)} - x_{t(e)}, y_{h(e)} - y_{t(e)} \mid e \in F\}$$

in $\Lambda(x_v, y_v \mid v \in \mathcal{V}) \cong H^*(E^{\mathcal{V}})$, and let $H_F = \bigcap_{e \in F} H_e \subseteq E^{\mathcal{V}}$.

1. With the degree-lexicographic order and $x_v < y_v < x_{v'} < y_{v'}$ whenever $v < v'$, the set $G_S := \{x_{h(e)} - x_{t(e)}, y_{h(e)} - y_{t(e)} \mid e \in S\}$ is a Gröbner basis for I_F .
2. The set $\{x_{i_1} \cdots x_{i_r} y_{j_1} \cdots y_{j_t} \mid h(e) \notin \{i_1, \dots, i_r, j_1, \dots, j_t\}$ for each $e \in S\}$ is a basis for

$$H^*(H_F) \cong \Lambda(x_v, y_v \mid v \in \mathcal{V})/I_F$$

and this basis does not depend on the choice of nbc set S .

Proof. Consider the matrix M_F from the proof of Lemma 4.1. Build a 2×2 block matrix, where the upper left and lower right blocks are copies of M_F and the other blocks are zero. In the upper half of the matrix, row e corresponds to $x_{h(e)} - x_{t(e)}$, and in the lower half of the matrix, row e corresponds to $y_{h(e)} - y_{t(e)}$. By a similar

argument as before, we can eliminate rows that don't correspond to elements of S and we're left with a matrix in row echelon form. Thus, we have a Gröbner basis.

The proof of the second statement mimics the proof in the toric case, with $\text{In}(G_S) = \{x_{h(e)}, y_{h(e)} \mid e \in S\}$. \square

Lemma 4.6. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a chordal graph, and let $\mathcal{A} = \mathcal{A}(\Gamma, E)$. Define P to be the set of all monomials $x_A y_B g_S$ such that S is a non-broken circuit and $h(e) \notin (A \cup B)$ for all $e \in S$. Then P is a basis for $A(\mathcal{A})$.

Proof. By Lemma 3.1, there is a decomposition into the flats of \mathcal{A} , given by the following: For a flat F , let $H_F = \bigcap_{e \in F} H_e \subseteq E^\mathcal{V}$, and let V_F be the vector space spanned by g_S for all nbc sets S associated to F . Then

$$A(\mathcal{A}) = \bigoplus_F H^*(H_F) \otimes V_F.$$

Denote $H^*(H_F) \otimes V_F$ by A_F . To show that P is a basis for $A(\mathcal{A})$, it suffices to show that

$$P \cap A_F = \{x_A y_B g_S \mid h(c) \notin (A \cup B) \text{ for } c \in S, S \text{ is an nbc set associated to } F\}$$

is a basis for A_F . But this follows from Lemma 4.5. \square

Theorem 4.7. Let \mathcal{A} be a chordal abelian arrangement. Then $A(\mathcal{A})$ is Koszul.

Proof. Fix a degree-lexicographic order on $A(\mathcal{A})$ that is induced by our order on \mathcal{V} . That is, $g_e < g_{e'}$ if $e < e'$, and $x_{h(e)} < y_{h(e)} < g_e < x_{h(e)+1} < y_{h(e)+1}$. We claim that

$$G = \{(x_{h(e)} - x_{t(e)})g_e, (y_{h(e)} - y_{t(e)})g_e, \partial g_S \mid e \in \mathcal{E}, S \text{ a circuit}\}$$

is a Gröbner basis with this order. Here,

$$\text{In}(G) = \{x_{h(e)}g_e, y_{h(e)}g_e, g_S \mid e \in \mathcal{E}, S \text{ is a broken circuit}\}$$

and P from Lemma 4.6 is the set of monomials not in $\langle \text{In}(G) \rangle$. By an argument similar to that in the toric case, we can conclude that G is a Gröbner basis. Moreover, using the fact that we have a chordal graph, we can again reduce this (in the same way) to a quadratic Gröbner basis, thus proving Koszulity. \square

Now we extend our results to higher genus curves. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a chordal graph, and let C be a complex projective curve of genus $g > 1$. For the chordal arrangement $\mathcal{A} = \mathcal{A}(\Gamma, C)$, consider the algebra $A(\mathcal{A})$ from Theorem 2.11 (ignoring the differential). We will prove that $A(\mathcal{A})$ is Koszul. The proof is very similar to that of the abelian case just discussed.

Lemma 4.8. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a chordal graph. Let $F \subseteq \mathcal{E}$ be a flat of the arrangement $\mathcal{A} = \mathcal{A}(\Gamma, C)$, and let S be a non-broken circuit associated to F . Denote $H_F = \cap_{e \in F} H_e \subseteq C^{\mathcal{V}}$. Then $H^*(H_F) \cong \Lambda(x_v^i, y_v^i \mid v \in \mathcal{V}, i = 1, \dots, g) / I_F$ where I_F is the ideal generated by the relations

$$(i) \quad x_{h(e)}^i - x_{t(e)}^i, y_{h(e)}^i - y_{t(e)}^i \text{ for } e \in F,$$

$$(ii) \quad x_v^i x_v^j, y_v^i y_v^j, x_v^i y_v^j \text{ for } i \neq j, \text{ and}$$

$$(iii) \quad x_v^i y_v^i - x_v^j y_v^j.$$

This algebra has basis

$$\{x_{A_1}^1 \cdots x_{A_g}^g y_{B_1}^1 \cdots y_{B_g}^g \mid A_i \cap B_i = \emptyset \text{ for } i > 1; \{h(e) \mid e \in S\} \cap (A_i \cup B_i) = \emptyset \forall i\}$$

and this basis does not depend on the choice of S .

Proof. The relations generating the ideal form a Gröbner basis because the relations (ii) and (iii) form a Gröbner basis in the exterior algebra

$$\Lambda(x_v^i, y_v^i \mid v \notin \{h(e) \mid e \in F\}, i = 1, \dots, g)$$

Just as in the proof of Lemmas 4.1 and 4.5, we can reduce the relations to only needing $e \in S$, and the choice of S does not matter since $\{h(e) \mid e \in F\} = \{h(e) \mid e \in S\}$. The basis given is a basis because it is the set of monomials not in the initial ideal. \square

Lemma 4.9. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a chordal graph, and let $\mathcal{A} = \mathcal{A}(\Gamma, C)$. Define P to be the set of all monomials $x_{A_1}^1 \cdots x_{A_g}^g y_{A_1}^1 \cdots y_{A_g}^g g_S$ such that S is a non-broken circuit, $h(e) \notin (A_i \cup B_i)$ for all $e \in S$ and all i , and $A_i \cap B_i = \emptyset$ for $i > 1$. Then P is a basis for $A(\mathcal{A})$.

Proof. There is again a decomposition into the flats of \mathcal{A} , given by the following: For a flat F , let $H_F = \bigcap_{e \in F} H_e \subseteq C^\mathcal{V}$, and let V_F be the vector space spanned by g_S for all nbc sets S associated to F . Then

$$A(\mathcal{A}) = \bigoplus_F H^*(H_F) \otimes V_F.$$

Denote $H^*(H_F) \otimes V_F$ by A_F . To show that P is a basis for $A(\mathcal{A})$, it suffices to show that $P \cap A_F$ is a basis of A_F . But this follows from Lemma 4.8. \square

Theorem 4.10. Let C be a complex projective curve of genus $g > 1$, and let \mathcal{A} be a chordal arrangement in $C^\mathcal{V}$. Then $A(\mathcal{A})$ is Koszul.

Proof. Fix a degree-lexicographic order on $A(\mathcal{A})$ that is induced by our order on \mathcal{V} .

We claim that

$$G = \{(x_{h(e)}^i - x_{t(e)}^i)g_e, (y_{h(e)}^i - y_{t(e)}^i)g_e, \partial g_S, R \mid e \in \mathcal{E}, S \text{ a circuit}\}$$

is a Gröbner basis with this order, where R denotes the set of relations (iiia) and (iiib) in $A(\mathcal{A})$. Here,

$$\text{In}(G) = \{x_v^1 y_v^1, x_{h(e)}^i g_e, y_{h(e)}^i g_e, g_B \mid B \text{ a broken circuit}\}$$

and P from Lemma 4.9 is the set of monomials not in $\langle \text{In}(G) \rangle$. By an argument similar to the previous cases, we can conclude that G is a Gröbner basis. Moreover, using the fact that we have a chordal graph, we can again reduce this (in the same way) to a quadratic Gröbner basis, thus proving Koszulity. \square

4.3. Abelian Arrangements and Higher Genus Curves

Given a chordal abelian arrangement \mathcal{A} , we compute the quadratic dual to the quadratic DGA $A(\mathcal{A})$ and give a combinatorial presentation for the Lie algebra dual to $A(\mathcal{A})$. This then gives us a combinatorial description of $\overline{\mathbb{Q}[\pi_1(M(\mathcal{A}))]}$, the Malcev Lie algebra $L(\pi_1(M(\mathcal{A})))$, and the minimal model of $M(\mathcal{A})$. Finally, we will show that $M(\mathcal{A})$ is a rational $K(\pi, 1)$ space. These results are also stated here for projective curves of higher genus.

Fix a complex projective curve C of genus $g > 0$ and a chordal graph $\Gamma = (\mathcal{V}, \mathcal{E})$, and consider the chordal abelian arrangement $\mathcal{A} = \mathcal{A}(\Gamma, C)$. We will use quadratic-linear duality to study the rational homotopy theory of $M(\mathcal{A})$.

Let L be the free Lie algebra generated by a_v^i, b_v^i, c_e for $v \in \mathcal{V}$, $e \in \mathcal{E}$, and $i = 1, \dots, g$ subject to the following relations:

- (i) $[a_v^i, a_w^j] = [b_v^i, b_w^j] = 0$ for $v, w \in \mathcal{V}$,
- (ii) $[b_{h(e)}^i, a_{t(e)}^i] = [b_{t(e)}^i, a_{h(e)}^i] = c_e$ for $e \in \mathcal{E}$,
- (iib) $[a_v^i, b_w^j] = 0$ if $v \neq w$ and there is no edge connecting v and w , or if $v \neq w$ and $i \neq j$,
- (iic) $\sum_{i=1}^g [a_v^i, b_v^i] = \sum_{\substack{h(e)=v \\ \text{or } t(e)=v}} c_e$ for $v \in \mathcal{V}$,
- (iiia) $[a_v^i, c_e] = [b_v^i, c_e] = 0$ for $e \in \mathcal{E}$ and $h(e) \neq v \neq t(e)$,
- (iiib) $[a_{h(e)}^i + a_{t(e)}^i, c_e] = [b_{h(e)}^i + b_{t(e)}^i, c_e] = 0$ for $e \in \mathcal{E}$,
- (iva) $[c_e, c_{e'}] = 0$ whenever e and e' are not part of a 3-cycle, and
- (ivb) $[c_{e_1}, c_{e_2} + c_{e_3}] = 0$ whenever $\{e_1, e_2, e_3\}$ is a cycle.

It is clear that $\{a_v^i, b_v^i \mid v \in \mathcal{V}, i = 1, \dots, p\}$ is enough to generate the Lie algebra. However, when we consider the universal enveloping algebra $U(L)$ as a quadratic-linear algebra in Theorem 4.11, we want the classes c_e to lie in the first filtered piece. One may also note that relations (iiib) and (ivb) follow from the other relations.

The following theorem generalizes the elliptic curve case of the main theorem of [Bez]. Using the Lie algebra dual to $A(\mathcal{A})$, this theorem gives a description of the Malcev Lie algebra of $M(\mathcal{A})$ when \mathcal{A} is chordal.

Theorem 4.11. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a chordal graph, $\mathcal{A} = \mathcal{A}(\Gamma, C)$, and L be the Lie algebra described above. Then we have the following:

1. Consider the universal enveloping algebra $U(L)$ as a QLA whose first filtered piece is spanned by a_v^i, b_v^i, c_e for $v \in \mathcal{V}$ and $e \in \mathcal{E}$. Then $U(L)$ is a Koszul quadratic-linear algebra which is the nonhomogeneous quadratic dual to the Koszul DGA $A(\mathcal{A})$.
2. $\overline{U(L)} \cong \overline{\mathbb{Q}[\pi_1(M(\mathcal{A}))]}$, where the completions are each with respect to the augmentation ideal. This isomorphism respects the Hopf algebra structures.
3. $\overline{L} \cong L(\pi_1(M(\mathcal{A})))$, where the completion of L is with respect to the filtration by bracket length.
4. The completion of L with respect to the filtration by bracket length is isomorphic to the Malcev (or \mathbb{Q} -nilpotent) completion of $\pi_1(M(\mathcal{A}))$.

Proof. By computing the quadratic-linear algebra dual to $A(\mathcal{A})$, we see that it is indeed $U(L)$. Since $A(\mathcal{A})$ is Koszul by Theorem 4.7 or 4.10, $U(L)$ is also Koszul.

Statements (2) and (3) follow from Theorem 2.6. □

Since $A(\mathcal{A})$ is a Koszul model for $M(\mathcal{A})$, Theorem 2.6 gives us the following proposition and corollary, which describes the minimal model of $M(\mathcal{A})$ and shows that $M(\mathcal{A})$ is rationally $K(\pi, 1)$.

Proposition 4.12. Let C be a complex projective curve of genus $g \geq 1$, $\Gamma = (\mathcal{V}, \mathcal{E})$ a chordal graph, $\mathcal{A} = \mathcal{A}(\Gamma, C)$, and L be the Lie algebra described above. Consider L^\bullet with the grading by bracket length. Then the standard (or Chevalley-Eilenberg) complex $\Omega(L^\bullet)$ is the minimal model for $M(\mathcal{A})$.

Corollary 4.13. Let C be a complex projective curve of genus $g \geq 1$ and Γ a chordal graph, so that $\mathcal{A} = \mathcal{A}(\Gamma, C)$ is a chordal arrangement. Then the complement $M(\mathcal{A})$ is a rational $K(\pi, 1)$ space.

Remark 4.14. Not only is $M(\mathcal{A})$ rationally $K(\pi, 1)$, but it is not hard to show that $M(\mathcal{A})$ is also $K(\pi, 1)$. As an easy case, the punctured elliptic curve is homotopic to a wedge of circles and hence is $K(\pi, 1)$. Then by induction on $|\mathcal{V}|$ and using the long exact sequence in homotopy of a fibration, one can show if that if $\Gamma = (\mathcal{V}, \mathcal{E})$ is chordal, then the complement to $\mathcal{A}(\Gamma - v, E)$ is $K(\pi, 1)$. The fibration arises as the restriction of the projection $E^{\mathcal{V}} \rightarrow E^{\mathcal{V}-v}$ to $X_{\mathcal{A}(\Gamma, E)} \rightarrow X_{\mathcal{A}(\Gamma-v, E)}$, where $v \in V$ is the maximum vertex in our perfect elimination ordering. The fiber of this fiber bundle is homeomorphic to $E \setminus \{k \text{ points}\}$ where $k = |\mathcal{E} \setminus (\mathcal{E} - v)|$.

Remark 4.15. Since we have a description of the minimal model via this Lie algebra, this provides an alternative method of computing cohomology. We saw one method in Chapter III (particularly Section 3.4), but we could instead compute cohomology using Lie algebra homology.

Theorem 4.16. Let E be a complex elliptic curve of genus $g \geq 1$ and Γ a chordal graph, so that $\mathcal{A} = \mathcal{A}(\Gamma, E)$ is a chordal arrangement. If Γ has at least one cycle, then $M(\mathcal{A})$ is not formal.

Proof. By Corollary 4.13, $M(\mathcal{A})$ is rationally $K(\pi, 1)$. If it was also formal, then by [PY, Theorem 5.1], the cohomology of $A(\mathcal{A})$ would have to be a Koszul algebra. However, we argue that $H^*(A(\mathcal{A}))$ is not even generated in degree one, which implies that it is not quadratic (hence not Koszul). We do this by explicitly producing an element of $H^2(A(\mathcal{A}))$ that cannot be generated by anything of lower degree.

Label the vertices in the cycle by $e_1 < e_2 < e_3$, and label the vertices of the cycle by $v_1 = t(e_1) = t(e_2)$, $v_2 = h(e_1) = t(e_3)$, and $v_3 = h(e_2) = h(e_3)$. Denote $x_i = x_{v_i}$ and $g_i = g_{e_i}$. Consider the element $\gamma = (x_2 - x_3)g_2 + (x_1 - x_3)g_3 + (x_3 - x_2)g_1 \in A^{1,1}(\mathcal{A})$. It is easy to check that γ defines a nontrivial element of $H^2(A(\mathcal{A}))$ (that is, $d\gamma = 0$ and γ is not in the image of d).

Next, we claim that $H^{0,1} = 0$ so that $H^1(A(\mathcal{A})) = H^{1,0} \cong H^1(C^\mathcal{V})$. We need only show that $d : A^{0,1} \rightarrow A^{2,0}$ has a trivial kernel. Assume that some linear combination $\sum_{e \in \mathcal{E}} c_e g_e \in \ker(d)$. Then

$$\begin{aligned} 0 &= \sum_e c_e d g_e \\ &= \sum_e \left(c_e x_{h(e)}^1 y_{h(e)}^1 + c_e x_{t(e)}^1 y_{t(e)}^1 + \sum_i (c_e x_{h(e)}^i y_{t(e)}^i + c_e x_{t(e)}^i y_{h(e)}^i) \right) \\ &= \sum_{v,w \in \mathcal{V}} \sum_{i=1}^g c(v,w,i) x_v^i y_w^i + \sum_{v \in \mathcal{V}} c(v) x_v^1 y_v^1 \end{aligned}$$

where the coefficients are $c(v,w,i) = c_e$ if there is an edge e connecting v and w and 0 otherwise, and $c(v) = \sum_{\substack{h(e)=v \\ t(e)=v}} c_e$. Note that the set $\{x_v^1 y_v^1, x_v^i y_w^i \mid v \neq w\}$ is linearly independent, which implies our coefficients must be zero. In particular, each c_e must be zero.

Now, since $H^1(A(\mathcal{A})) \cong H^1(C^\mathcal{V})$, the subring of $H^*(A(\mathcal{A}))$ must only contain elements of pure weight. Our element γ is not of pure weight, and so it is not generated by $H^1(A(\mathcal{A}))$. \square

We end with an example, demonstrating non-formality.

Example 4.17. Consider the case of an elliptic curve. The braid arrangement of type A_2 corresponds to the complete graph Γ on three vertices $\mathcal{V} = \{1 < 2 < 3\}$ with edges labeled $\{12, 13, 23\}$.

The DGA $A(\mathcal{A})$ is the quotient of the exterior algebra $\Lambda(x_v, y_v, g_e)$ by the ideal generated by

$$(i) \quad (x_i - x_j)g_{ij}, (y_i - y_j)g_{ij}$$

$$(ii) \quad g_{12}g_{13} - g_{12}g_{23} + g_{13}g_{23}$$

with differential $dg_{ij} = (x_i - x_j)(y_i - y_j)$.

By a computation similar to that in Example 3.13, we get the two-variable Hodge polynomial

$$H(t, u) = 1 + 6tu + 12t^2u^2 + 2t^2u^3 + 10t^3u^3 + 4t^3u^4 + et^4u^4 + 2t^4u^5.$$

We can see explicitly that the degree one part of the cohomology is pure, but we have higher cohomology that is not pure, demonstrating that the algebra cannot be generated in degree one. In fact, our element γ from the proof of Theorem 4.16 and its y -counterpart are the only two extra generators that we need.

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