THE HOMOTOPY CALCULUS OF CATEGORIES AND GRAPHS

by

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A DISSERTATION

Presented to the Department of Mathematics
and the Graduate School of the University of Oregon
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

June 2015

DISSERTATION APPROVAL PAGE

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Title: The Homotopy Calculus of Categories and Graphs

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Degree awarded June 2015

DISSERTATION ABSTRACT

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We construct categories of spectra for two model categories. The first is the category of small categories with the canonical model structure, and the second is the category of directed graphs with the Bisson-Tsemo model structure. In both cases, the category of spectra is homotopically trivial. This implies that the Goodwillie derivatives of the identity functor in each category, if they exist, are weakly equivalent to the zero spectrum. Finally, we give an infinite family of model structures on the category of small categories.

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ACKNOWLEDGEMENTS

First and foremost, I would like to thank my advisor, Hal Sadofsky, for always being patient and encouraging. I could not have done this without your guidance. Thank you to my friends and family for all of your support over the years. I would also like to thank my husband, John Jasper, for motivating me to persevere in this endeavor.

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CHAPTER I

INTRODUCTION

A model category is essentially a category in which there is a notion of homotopy. The standard example is $\mathcal{T}op$, the category of topological spaces. Another example is the category $\mathcal{C}h(R)$ of bounded-below chain complexes of modules over an associative unital ring R, where homotopy means chain homotopy. More formally, a model category is a category that has all small limits and colimits together with a model structure, which consists of three classes of morphisms called weak equivalences, fibrations, and cofibrations satisfying certain axioms [Hov99].

In this dissertation, we consider model categories of graphs and categories. Bisson and Tsemo [BT+09] defined a model structure on a category of directed graphs, denoted $\mathcal{G}ph_*$, in which a graph map is a weak equivalence if it is bijective on n-cycles for all $n \geq 1$. A similar but more complex category is $\mathcal{C}at_*$, in which the objects are pointed small categories and the morphisms are functors [Rez96]. In $\mathcal{C}at_*$, the weak equivalences are equivalences of categories.

In the early 1990's, Goodwillie invented a method for analyzing certain functors $F : \mathcal{T}op \to \mathcal{T}op$ in a way that mimics the method of Taylor series from calculus [Goo03]. The Taylor series of a function $f : \mathbb{R} \to \mathbb{R}$ gives us a way to approximate the value of f using polynomial functions. Furthermore, under the right circumstances, we can make our approximation arbitrarily close to the correct answer, and the series converges to the value of the function. The goal of Goodwillie calculus, also known as homotopy calculus, is to approximate F with other functors that are homotopically simpler so that the values of the approximations approach the value of the functor, at least for some inputs. These approximating functors are called polynomial functors and are denoted P_nF. There are natural maps between these polynomial functors, which results in a "Taylor tower"

$$\dots P_nF \to P_{n-1}F \to \dots \to P_0F = *.$$

Ideally, the inverse limit of this tower is F. We define D_nF to be the fiber of the map $P_nF \to P_{n-1}F$. Each D_nF corresponds to a spectrum, which is called the n^{th} derivative of F.

Goodwillie's methods have been generalized to other model categories. Although the domain and codomain of a functor need not be the same, the derivatives of the identity functor $\mathbb{1} : \mathcal{C} \to \mathcal{C}$ on a model category $\mathcal{C}$ are of particular interest. Here the analogy to regular calculus becomes more tenuous as $\mathbb{1}$ is rarely polynomial itself. The complexity of $\mathbb{1}$ in terms of homotopy calculus is related to the complexity of the homotopy theory of $\mathcal{C}$.

The derivatives of $\mathbb{1}$ often have the structure of an operad, which is an object that models a certain algebraic property. For example, there is an operad that encodes commutativity and another that encodes associativity. There is one operad, called the Lie operad, that determines if a vector space has the structure of a Lie algebra. Walter showed that the derivatives of the identity functor on $\mathcal{DGL}_r$, the category of differential graded Lie algebras over $\mathbb{Q}$ that are zero below level r, have the structure of the Lie operad [Wal06].

In all known examples, the derivatives of a functor $F : \mathcal{C} \to \mathcal{C}$ are objects in the category of spectra on $\mathcal{C}$ [Hov01], which is denoted $\mathcal{S}p^{\mathbb{N}}(\mathcal{C}, \Sigma)$ and is called the stabilization of $\mathcal{C}$. The objects of $\mathcal{S}p^{\mathbb{N}}(\mathcal{C}, \Sigma)$ are sequences $\{X_i\}_{i \geq 0}$ of objects of $\mathcal{C}$ together with structure maps $\Sigma X_i \to X_{i+1}$, where $\Sigma : \mathcal{C} \to \mathcal{C}$ is the suspension functor. Morphisms in $\mathcal{S}p^{\mathbb{N}}(\mathcal{C},\Sigma)$ act level-wise and respect the structure maps. We show that $\Sigma^2 X$ is weakly equivalent to the zero object in $\mathcal{C}at_*$ for all $X \in \mathrm{Ob}(\mathcal{C}at_*)$ and, moreover, that any object in $\mathcal{S}p^{\mathbb{N}}(\mathcal{C}at_*,\Sigma)$ is weakly equivalent to the zero spectrum $\{0,0,\ldots\}$. Therefore we expect the derivatives of the identity functor $\mathbb{1} : \mathcal{C}at_* \to \mathcal{C}at_*$ to be homotopically trivial. In other words, the derivatives would have the structure of the trivial operad.

We apply similar methods to the category $\mathcal{G}ph_*$. We construct the suspension functor $\Sigma : \mathcal{G}ph_* \to \mathcal{G}ph_*$ and show that ΣX is weakly equivalent to the zero object of the category for all $X \in \mathrm{Ob}(\mathcal{G}ph_*)$. Although this is not enough to conclude that there is a category of spectra $Sp^{\mathbb{N}}(\mathcal{G}ph_*,\Sigma)$ having the properties necessary to apply Goodwillie's work, we show that if the derivatives of the identity functor in this category exist, then they are homotopically trivial.

Overview

In chapter 2, we explain Goodwillie's construction and give many of the definitions related to model categories that we will use throughout this work. In chapter 3, we review Walter's result in detail. Chapter 4 contains a description of the model category structure on $\mathcal{C}at_*$, and we show that $\mathcal{C}at_*$ has the properties necessary to apply Goodwillie's method. Then we construct the stabilization of $\mathcal{C}at_*$ and prove that the stabilization is homotopically trivial. We discuss a version of the Bisson-Tsemo category of directed graphs in chapter 5 and show that the stabilization of this category, if it exists, must be homotopically trivial as well. Finally, in chapter 6 we present a model structure on $\mathcal{C}at_*$ for each positive integer n and make a conjecture about how the stabilization of $\mathcal{C}at_*$ depends on the choice of model structure.

CHAPTER II

HOMOTOPY CALCULUS

In calculus, one studies functions $f : \mathbb{R} \to \mathbb{R}$ by studying the derivatives of f. In particular, given a function f that is infinitely differentiable at a point $x = a$, one may approximate f using its n^{th} Taylor polynomial $T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$. Then there is some interval on which the Taylor series converges to the value of the function, that is,

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i.$$

In the 1990's, Goodwillie [Goo90], [Goo91], [Goo03] invented an analogous method for analyzing functors on $\mathcal{T}op$, the category of topological spaces, or from $\mathcal{T}op$ to $\mathcal{S}p$, the category of spectra. This method, which is now known as Goodwillie calculus or homotopy calculus, has since been applied to functors on other appropriately nice categories. Essentially, one needs a category in which there is some notion of homotopy. Such categories are called model categories.

2.1. Model Categories

Before we can define a model category, we will need some preliminary definitions.

Definition 2.1. [Hov99, 1.1.1] Let $\mathcal{C}$ be a category.

1. A map f in $\mathcal{C}$ is a *retract* of a map g in $\mathcal{C}$ if and only if there is a commutative diagram of the form

$$\begin{array}{ccccc} A & \longrightarrow & C & \longrightarrow & A \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & D & \longrightarrow & B \end{array}$$

where the horizontal composites are identities.

2. A *functorial factorization* is an ordered pair (α, β) of functors $\mathrm{Mor}\, \mathcal{C} \to \mathrm{Mor}\, \mathcal{C}$ such that $f = \beta(f) \circ \alpha(f)$ for all $f \in \mathrm{Mor}\, \mathcal{C}$.

Definition 2.2. [Hov99, 1.1.2] Suppose $i : A \to B$ and $p : X \to Y$ are maps in a category $\mathcal{C}$. Then i has the *left lifting property with respect to* p and p has the *right lifting property with respect to* i if, for every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y \end{array}$$

there is a lift $h : B \to X$ such that $hi = f$ and $ph = g$.

Definition 2.3. [Hov99, 1.1.3] A *model structure* on a category $\mathcal{C}$ consists of three subcategories of $\mathcal{C}$ called weak equivalences, cofibrations, and fibrations, and two functorial factorizations (α, β) and (γ, δ) satisfying the following properties:

1. (2-out-of-3) If f and g are morphisms of $\mathcal{C}$ such that gf is defined and two of f, g and gf are weak equivalences, then so is the third.

2. (Retracts) If f and g are morphisms of $\mathcal{C}$ such that f is a retract of g and g is a weak equivalence, cofibration, or fibration, then so is f.

3. (Lifting) Define a map to be a *trivial cofibration* if it is both a cofibration and a weak equivalence. Similarly, define a map to be a *trivial fibration* if it is both a fibration and a weak equivalence. Then trivial cofibrations have the left lifting property with respect to fibrations, and cofibrations have the left lifting property with respect to trivial fibrations.

4. (Factorization) For any morphism f, $\alpha(f)$ is a cofibration, $\beta(f)$ is a trivial fibration, $\gamma(f)$ is a trivial cofibration, and $\delta(f)$ is a fibration.

Definition 2.4. [Hov99, 1.1.4] A *model category* is a category $\mathcal{C}$ with all small limits and colimits together with a model structure on $\mathcal{C}$.

We will denote weak equivalences by $\xrightarrow{\simeq}$, cofibrations by $\hookrightarrow$, and fibrations by $\twoheadrightarrow$.

Since a model category $\mathcal{C}$ has all small limits, $\mathcal{C}$ contains the limit of the empty diagram, which is a terminal object $\mathbf{T}$ of $\mathcal{C}$. Similarly, $\mathcal{C}$ having small colimits means that $\mathcal{C}$ contains the colimit of the empty diagram, which is an initial object $\mathbf{I}$ of $\mathcal{C}$. Therefore any model category contains an initial object and a terminal object, though these are not necessarily the same.

An object X is called *cofibrant* if the unique map $\mathbf{I} \to X$ is a cofibration and *fibrant* if the unique map $X \to \mathbf{T}$ is a fibration. It is always possible to replace an object X by a cofibrant or fibrant object by part (4) of Definition 2.3. That is, we may factor the map $\mathbf{I} \to X$ as $\mathbf{I} \hookrightarrow Q(X) \xrightarrow{\simeq} X$, and we may factor the map $X \to \mathbf{T}$ as $X \xrightarrow{\simeq} R(X) \twoheadrightarrow \mathbf{T}$. We call $Q(X)$ the cofibrant replacement of X and $R(X)$ the fibrant replacement of X.

For any model category, we can construct an associated model category which is *pointed*, that is, which has a zero object $\mathbf{0}$.

Definition 2.5. [Hir03, 7.6.1] If $\mathcal{C}$ is a category and A is an object of $\mathcal{C}$, then the *category of objects of* $\mathcal{C}$ *under* A, denoted $A \downarrow \mathcal{C}$, is the category in which

- an object is a map $A \to X$ in $\mathcal{C}$,
- a map from $g : A \to X$ to $h : A \to Y$ is a map $f : X \to Y$ in $\mathcal{C}$ such that $f \circ g = h$, and
- composition of maps is defined by composition of maps in $\mathcal{C}$.

This is also called the *coslice category of* $\mathcal{C}$ *with respect to* A.

Theorem 2.6. [Hir03, 7.6.4] *If* $\mathcal{M}$ *is a model category, then the category* $A \downarrow \mathcal{M}$ *is a model category in which a map is a weak equivalence, fibration, or cofibration if it is one in* $\mathcal{M}$.

Given a model category $\mathcal{M}$ with terminal object $\mathbf{T}$, the coslice category $\mathbf{T} \downarrow \mathcal{M}$ is isomorphic to the category of pointed objects of $\mathcal{M}$ and basepoint-preserving morphisms. That is, we may force $\mathbf{T}$ to be an initial object of $\mathcal{M}$ as well as a terminal object. The weak equivalences, fibrations, and cofibrations are then precisely the largest possible subsets of those classes in the unpointed case.

The archetypal example of a model category (and the motivation for its definition) is the category of topological spaces.

Example 2.7. [DS95, 8.3] Call a map in $\mathcal{T}op$

- a weak equivalence if it is a weak homotopy equivalence,
- a fibration if it is a Serre fibration, and
- a cofibration if it has the left lifting property with respect to trivial fibrations.

With these choices, $\mathcal{T}op$ is a model category.

Example 2.8. [DS95, 7.2] Another example of a model category is $\mathcal{C}h(R)$, the category of non-negatively graded chain complexes of R-modules, where R is an associative ring with unit. There is a model structure on $\mathcal{C}h(R)$ in which a map $f : M \to N$ is

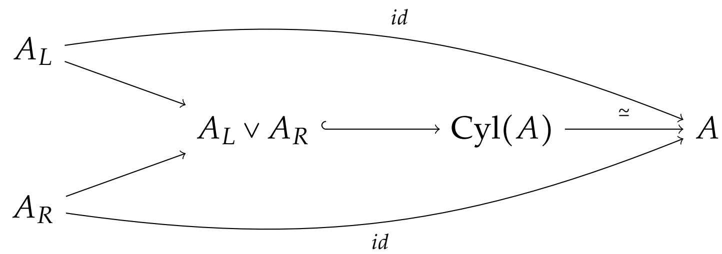
- a weak equivalence if f induces isomorphisms on homology $H_k M \to H_k N$ for all $k \geq 0$,
- a cofibration if for each $k \geq 0$ the map $f_k : M_k \to N_k$ is injective with a projective R-module as its cokernel, and
- a fibration if for each $k > 0$ the map $f_k : M_k \to N_k$ is surjective.

Many constructions in $\mathcal{T}op$ have analogs in other model categories. For example, one may construct suspension objects and loop objects in any pointed model category.

Definition 2.9. [Qui67, I.2.9] Given a cofibrant object A in a pointed model category $\mathcal{M}$, the *suspension* ΣA is the pushout of the diagram

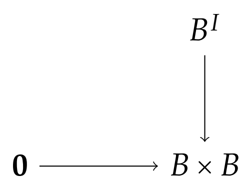
$$\begin{array}{ccc} A \vee A & \longrightarrow & \mathrm{Cyl}(A) \\ \downarrow & & \\ \mathbf{0} & & \end{array}$$

where a cylinder object $\mathrm{Cyl}(A)$ is an object of $\mathcal{M}$ that satisfies the following diagram:

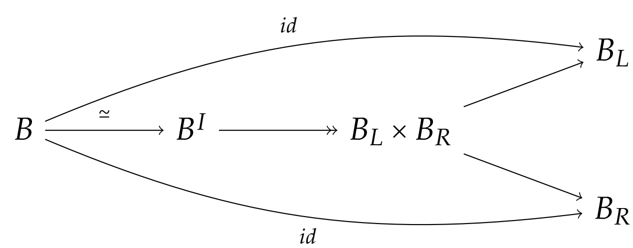


Here the maps $A_L \to A_L \vee A_R$ and $A_R \to A_L \vee A_R$ are inclusions, and the composition $A_L \vee A_R \hookrightarrow \mathrm{Cyl}(A) \to A$ is the coproduct map.

Definition 2.10. [Qui67, I.2.9] Given a fibrant object B in a pointed model category $\mathcal{M}$, ΩB ("loops on B") is the pullback of the diagram



where a path object B^I is an object of $\mathcal{M}$ that satisfies the following diagram:



Here the composition $B \to B^I \twoheadrightarrow B_L \times B_R$ is the diagonal map, and the maps $B_L \times B_R \to B_L$, $B_L \times B_R \to B_R$ are projections.

Note that cylinder objects and path objects exist in any model category $\mathcal{M}$ because $\mathcal{M}$ has functorial factorizations.

Proposition 2.11. *$\Omega B \simeq X$, where X is the homotopy pullback of the diagram*

$$\begin{array}{ccc} & & \mathbf{0} \\ & & \big\downarrow \\ \mathbf{0} & \longrightarrow & B \end{array}$$

Proof. Let $i : B \to B^I$ and $p : B^I \to B \times B$ be the maps from Definition 2.10. Let $\pi_L : B \times B \to B$ and $\pi_R : B \times B \to B$ be the projections onto the left and right factors of the product, respectively. Note that by the two-out-of-three property, $\pi_R \circ p$ is a weak equivalence. Also, $\pi_R \circ p$ is a fibration since both π_R and p are fibrations. Similarly, $\pi_L \circ p$ is a trivial fibration.

Let Y be the pullback of the diagram

$$\begin{array}{ccc} & & B^I \\ & & \big\downarrow{\scriptstyle \pi_R \circ p} \\ \mathbf{0} & \longrightarrow & B \end{array}$$

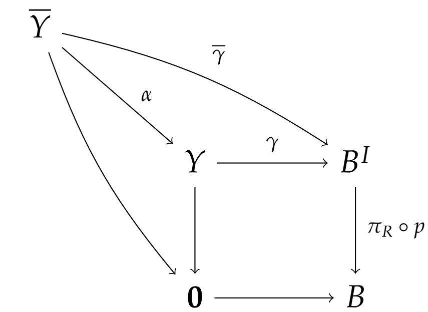
Let $\gamma : Y \to B^I$ and $0 : Y \to \mathbf{0}$ be the maps defined by this pullback. The class of trivial fibrations in $\mathcal{M}$ is closed under pullbacks [Hir03, 7.2.12], so $0 : Y \to \mathbf{0}$ is a trivial fibration. Note $\pi_R \circ p \circ \gamma = 0$.

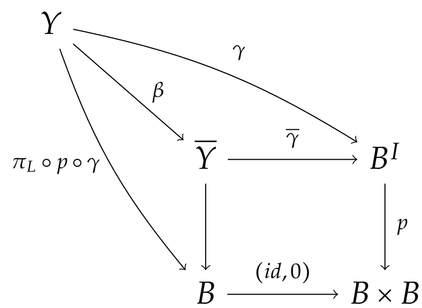
Let $\overline{Y}$ be the pullback of the diagram

$$\begin{array}{ccc} & & B^I \\ & & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow[(id,0)]{} & B \times B \end{array}$$

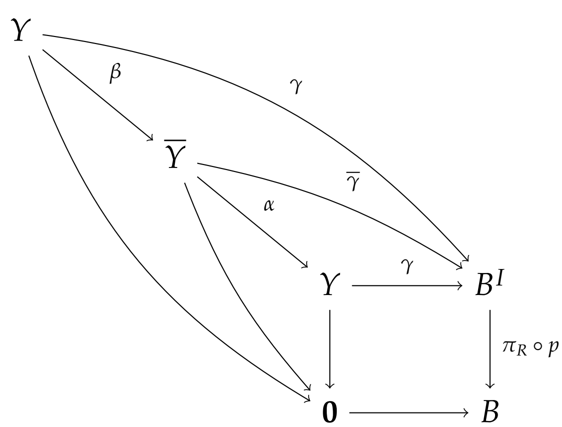
Let $\overline{\gamma} : \overline{Y} \to B^I$ and $\delta : \overline{Y} \to B$ be the maps defined by this pullback. Note that $p \circ \overline{\gamma} = (id, 0) \circ \delta$, so $\pi_L \circ p \circ \overline{\gamma} = \delta$ and $\pi_R \circ p \circ \overline{\gamma} = 0$.

We claim that $Y \cong \overline{Y}$. To see this, consider the following diagrams:





By the above relations, both diagrams commute, and so α and β are unique. Consider the commutative diagram



Since the square is a pullback, the map $\alpha \circ \beta : Y \to Y$ is unique. However, id_Y also makes the diagram commute, and so $\alpha \circ \beta = id_Y$. A similar argument shows that $\beta \circ \alpha = id_{\overline{Y}}$. Therefore Y and Y' are isomorphic.

Since $Y \simeq \mathbf{0}$ and $\pi_L \circ p \circ \gamma : Y \to B$ is a fibration, X is weakly equivalent to the pullback $\overline{X}$ of the diagram

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle \pi_L \circ p \circ \gamma} \\ \mathbf{0} & \longrightarrow & B \end{array}$$

Thus we have two pullback squares

$$\begin{array}{ccccc} \overline{X} & \longrightarrow & Y & \xrightarrow{\gamma} & B^I \\ \downarrow & & \downarrow{\scriptstyle \pi_L \circ p \circ \gamma} & & \downarrow{\scriptstyle p} \\ \mathbf{0} & \longrightarrow & B & \xrightarrow[(id,0)]{} & B \times B \end{array}$$

It follows that the outer square is also a pullback square. Therefore X is weakly equivalent to the pullback ΩB of the diagram

$$\begin{array}{ccc} & & B^I \\ & & \downarrow{\scriptstyle p} \\ \mathbf{0} & \longrightarrow & B \times B \end{array}$$

□

An important class of model categories are those which are cofibrantly generated. We describe this below.

Definition 2.12. [Hov99, 2.1.9] Let $\mathcal{C}$ be a category containing all small colimits. Let I be a set of maps in $\mathcal{C}$. A *relative I-cell complex* is a transfinite composition of pushouts of elements of I. The collection of relative I-cell complexes is denoted by I-cell.

Definition 2.13. [Hov99, 2.1.3] Let $\mathcal{C}$ be a category containing all small colimits, $\mathcal{D}$ a collection of morphisms in $\mathcal{C}$, A an object of $\mathcal{C}$, and κ a cardinal. We say that A is *κ-small relative to $\mathcal{D}$* if, for all κ-filtered ordinals λ and all λ-sequences

$$X_0 \to X_1 \to \ldots \to X_\beta \to \ldots$$

such that each map $X_\beta \to X_{\beta+1}$ is in $\mathcal{D}$ for $\beta + 1 < \lambda$, the map of sets $\operatorname{colim}_{\beta<\lambda} \mathcal{C}(A, X_\beta) \to \mathcal{C}(A, \operatorname{colim}_{\beta<\lambda} X_\beta)$ is an isomorphism. We say that A is *small relative to $\mathcal{D}$* if it is κ-small relative to $\mathcal{D}$ for some κ. We say that A is *small* if A is small relative to $\mathcal{C}$.

Definition 2.14. [Hov99, 2.1.17] A model category $\mathcal{M}$ is *cofibrantly generated* if there are sets of maps I and J in $\mathcal{M}$ such that

- the domains of the maps of I are small relative to I-cell;
- the domains of the maps of J are small relative to J-cell;
- the fibrations are exactly those maps that have the right lifting property with respect to J;
- the trivial fibrations are exactly those maps that have the right lifting property with respect to I.

We call I the *set of generating cofibrations* and J the *set of generating trivial cofibrations*.

It turns out to be very important that a model category have enough small objects. A related notion is that of a compact object. We say that an object X is *compact* if mapping out of it commutes with filtered colimits. That is, given a functor $F : \mathcal{D} \to \mathcal{C}$, X is compact if $\operatorname{colim} \mathcal{C}(X, F(D)) \to \mathcal{C}(X, \operatorname{colim} F(D))$ whenever $\mathcal{D}$ is a filtered category.

Definition 2.15. [Wal06, 2.4.3] A nonempty category $\mathcal{D}$ is *filtered* if

- for every pair of objects A, B in $\mathcal{D}$ there exists an object D in $\mathcal{D}$ and maps $A \to D$, $B \to D$, and
- for every pair of maps $f, g : A \to B$, there exists an object D in $\mathcal{D}$ and a morphism $h : B \to D$ such that $hf = hg$.

The following definitions will help us understand especially nice model categories, i.e. those that are cellular (2.17), combinatorial (2.19), accessible (2.20), proper (2.22), or simplicial (2.24).

Definition 2.16. [Hir03, 10.9.1] Let $\mathcal{C}$ be a category that is closed under pushouts. The map $f : A \to B$ in $\mathcal{C}$ is an *effective monomorphism* if f is the equalizer of the pair of natural inclusions $B \rightrightarrows B \sqcup_A B$, i.e. if

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle f}\downarrow & & \downarrow \\ B & \longrightarrow & B \sqcup_A B \end{array}$$

is a pullback diagram.

Definition 2.17. [Hir03, 12.1.1] A model category $\mathcal{M}$ is *cellular* if there is a set of cofibrations I and a set of trivial cofibrations J making $\mathcal{M}$ into a cofibrantly generated model category and also satisfying the following conditions:

- The domains and codomains of I are compact relative to I.

- The domains of J are small relative to I.
- Cofibrations are effective monomorphisms.

Definition 2.18. [Dug01, 2.2] A category $\mathcal{C}$ is *locally presentable* if it has all small colimits and if there is a regular cardinal λ and a set of objects $\mathcal{A}$ in $\mathcal{C}$ such that

- every object in $\mathcal{A}$ is small with respect to λ-filtered colimits, and
- every object of $\mathcal{C}$ can be expressed as a λ-filtered colimit of elements of $\mathcal{A}$.

Definition 2.19. [Dug01, 2.1] A model category $\mathcal{M}$ is *combinatorial* if it is cofibrantly generated and the underlying category is locally presentable.

Definition 2.20. [AR94, 2.1] A category $\mathcal{C}$ is *accessible* if there is a regular cardinal λ such that

- $\mathcal{C}$ has λ-directed colimits, and
- $\mathcal{C}$ has a set $\mathcal{A}$ of λ-presentable objects such that every object is a λ-directed colimit of objects from $\mathcal{A}$.

Proposition 2.21. [Cen04] The coslice category of an accessible category is accessible.

Definition 2.22. [Hir03, 13.1.1] A model category $\mathcal{M}$ is *left proper* if every pushout of a weak equivalence along a cofibration is a weak equivalence. $\mathcal{M}$ is *right proper* if every pullback of a weak equivalence along a fibration is a weak equivalence. If $\mathcal{M}$ is both left proper and right proper, we say that $\mathcal{M}$ is *proper*.

Proposition 2.23. [Hir03, 13.1.3] If every object of $\mathcal{M}$ is cofibrant, then $\mathcal{M}$ is left proper. If every object of $\mathcal{M}$ is fibrant, then $\mathcal{M}$ is right proper.

Definition 2.24. [Hir03, 9.1.6] A *simplicial model category* is a model category $\mathcal{M}$ that is enriched over simplicial sets. That is, for every two objects X and Y of $\mathcal{M}$, $\mathrm{Map}(X,Y)$ is a simplicial set, and these sets satisfy composition, associativity, and unital conditions. Also:

- For every two objects X and Y of $\mathcal{M}$ and every simplicial set K, there are objects $X \otimes K$ and Y^K of $\mathcal{M}$ such that there are isomorphisms of simplicial sets
$$\mathrm{Map}(X \otimes K, Y) \cong \mathrm{Map}(K, \mathrm{Map}(X,Y)) \cong \mathrm{Map}(X, Y^K)$$
that are natural in X, Y, and K.

- If $i : A \to B$ is a cofibration in $\mathcal{M}$ and $p : X \to Y$ is a fibration in $\mathcal{M}$, then the map of simplicial sets
$$\mathrm{Map}(B,X) \xrightarrow{i^* \times p_*} \mathrm{Map}(A,X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B,Y)$$
is a fibration and is a trivial fibration if either i or p is a weak equivalence.

  This condition is known as the *pushout-product axiom*.

The "right" notion of equivalence of two model categories is that of a Quillen equivalence.

Definition 2.25. [Hov99, 1.3.1] Let $\mathcal{C}$ and $\mathcal{D}$ be model categories. A pair of adjoint functors $F : \mathcal{C} \leftrightarrows \mathcal{D} : U$ is a *Quillen adjunction* if the following equivalent definitions are satisfied:

- F preserves cofibrations and trivial cofibrations.

- U preserves fibrations and trivial fibrations.

We call F a *left Quillen functor* and U a *right Quillen functor*.

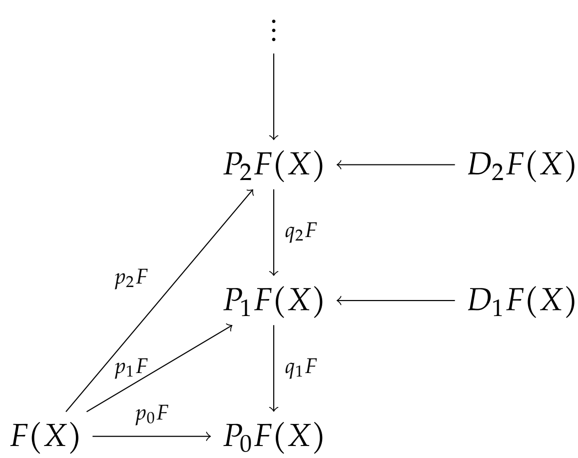
Definition 2.26. [Hov99, 1.3.12] A Quillen adjunction is a *Quillen equivalence* if whenever X is a cofibrant object of $\mathcal{C}$ and Y is a fibrant object of $\mathcal{D}$, a morphism $X \to U(Y)$ is a weak equivalence in $\mathcal{C}$ if and only if the adjunct morphism $F(X) \to Y$ is a weak equivalence in $\mathcal{D}$.

A Quillen equivalence preserves the model structure on a category. In other words, if two model categories are Quillen equivalent, then they have the same homotopy theory. For example, the geometric realization and singular functors $|-| : sSet \leftrightarrows \mathcal{T}op : Sing$ provide a Quillen equivalence between the category $sSet$ of simplicial sets and $\mathcal{T}op$. The advantage is that one can transport combinatorial results from $sSet$ to $\mathcal{T}op$, where calculations tend to be significantly more difficult.

2.2. The Taylor Tower

The goal of this section is to explain Goodwillie's main theorem [Goo03, 1.13], which is stated as follows.

Theorem 2.27. *A homotopy functor between two appropriate model categories $F : \mathcal{C} \to \mathcal{D}$ determines a tower of functors $P_n F : \mathcal{C} \to \mathcal{D}$ with maps from F*



where the functors P_nF are n-excisive, p_nF is universal among maps from F to n-excisive functors, the maps q_nF are fibrations, the functors $D_nF = hofib(q_nF)$ are n-homogeneous, and all the maps are natural.

The sequence

$$\ldots \to P_2F \to P_1F \to P_0F$$

is called the *Taylor tower of F*. Ideally, but not necessarily, the inverse limit of the tower is F. In this section, we explain when and how we construct this tower.

Hypothesis 2.28. The tools of homotopy calculus are applicable in model categories that are pointed, left proper, and simplicial [Kuh07]. Let $\mathcal{C}$ and $\mathcal{D}$ be categories that satisfy these conditions. Additionally, we assume that in $\mathcal{D}$, the sequential homotopy colimit of homotopy cartesian cubes is homotopy cartesian. Let $F : \mathcal{C} \to \mathcal{D}$ be a homotopy functor, i.e. a functor that preserves weak equivalences. We also require that F commutes with filtered homotopy colimits.

Let S be a finite set. Denote the power set of S by $\mathcal{P}(S) = \{T \subseteq S\}$, which is partially ordered by inclusion. Let $\mathcal{P}_0 = \mathcal{P}(S) \setminus \varnothing$ and $\mathcal{P}_1 = \mathcal{P}(S) \setminus S$. An n-cube in $\mathcal{C}$ is a functor $\chi : \mathcal{P}(S) \to \mathcal{C}$ with $|S| = n$. We say χ is *(homotopy) cartesian* if $\chi(\varnothing) \to \text{holim}_{T \in P_0(S)} \chi(T)$ is a weak equivalence, and χ is *(homotopy) cocartesian* if $\text{hocolim}_{T \in P_1(S)} \to \chi(S)$ is a weak equivalence.

Example 2.29. A 0-cube $\chi(\varnothing)$ is cartesian if and only if it is cocartesian if and only if $\chi(\varnothing) \simeq \mathbf{0}$.

A 1-cube $f : \chi(\varnothing) \to \chi(\{1\})$ is cartesian if and only if it is cocartesian if and only if it is a weak equivalence.

A 2-cube

$$\begin{array}{ccc} \chi(\varnothing) & \longrightarrow & \chi(\{1\}) \\ \downarrow & & \downarrow \\ \chi(\{2\}) & \longrightarrow & \chi(\{1,2\}) \end{array}$$

is cartesian if it is a homotopy pullback square and cocartesian if it is a homotopy pushout square.

We say that χ is *strongly cocartesian* if every two-dimensional face of χ is cocartesian.

Definition 2.30. [Kuh07, 4.10] A functor $F : \mathcal{C} \to \mathcal{D}$ is *n-excisive* or *polynomial of degree at most n* if, whenever χ is a strongly cocartesian $(n+1)$-cube in $\mathcal{C}$, then $F(\chi)$ is a cartesian cube in $\mathcal{D}$.

For example, the identity functor in $\mathcal{S}p$ is 1-excisive because homotopy pushout squares and homotopy pullback squares coincide in this category. Another example of a 1-excisive functor is the spectrification functor $\Sigma^\infty : \mathcal{T}op \to \mathcal{S}p$, which preserves homotopy pushouts. In fact, both of these functors are n-excisive for all $n \geq 1$ by the following lemma. For this reason, we simply refer to 1-excisive functors as *excisive* or *linear*.

Lemma 2.31. [Kuh07, 4.16] If F is n-excisive, then F is k-excisive for all $k \geq n$.

We wish to construct an n-excisive functor P_nF for each $n \geq 0$. The idea will be to construct particular strongly cocartesian $(n+1)$-cubes for which this condition is satisfied and then to show that the condition must be satisfied for all strongly cocartesian $(n+1)$-cubes.

If T is a finite set and X is an object in $\mathcal{C}$, let $X * T$ ("X join T") be the homotopy cofiber of the folding map $\sqcup_T X \to X$. That is, $X * T$ is the homotopy pushout of the diagram

$$\begin{array}{ccc} \sqcup_T X & \longrightarrow & X \\ \downarrow & & \\ \mathbf{0} & & \end{array}$$

For example, in $\mathcal{T}op_*$:

- $X * \varnothing \simeq X$
- $X * \{1\} \simeq CX$
- $X * \{1,2\} \simeq \Sigma X$.

In general, if T is a k-element set, then $X * T$ is equivalent to k cones identified at the open ends.

Let $X \in \mathrm{Ob}(\mathcal{C})$, let $|S| = n+1$, and let $T \subseteq \mathcal{P}(S)$. It turns out that the map $T \to X * T$ defines a strongly cocartesian $(n+1)$-cube. The following functor is an intermediary step in defining the n-excisive functors.

Definition 2.32. [Kuh07, 5.1] Let $T_n F : \mathcal{C} \to \mathcal{D}$ be defined by

$$T_n F(X) = \mathrm{holim}_{T \in P_0(S)} F(X * T).$$

Note that there is a natural transformation $t_n(F) : F \to T_n F$ since the initial object of the cube is $F(X * \varnothing) \simeq F(X)$. Furthermore, $t_n(F)$ is a weak equivalence if F is n-excisive.

For example, $T_1F(X)$ is the homotopy pullback of

$$\begin{array}{ccc} & & F(X*\{1\}) \\ & & \downarrow \\ F(X*\{2\}) & \longrightarrow & F(X*\{1,2\}) \end{array}$$

or, equivalently,

$$\begin{array}{ccc} & & F(CX) \\ & & \downarrow \\ F(CX) & \longrightarrow & F(\Sigma X) \end{array}$$

If $F(\mathbf{0}) \simeq \mathbf{0}$, we say that F is *reduced*. In this case, $F(CX) \simeq \mathbf{0}$, so $T_1F(X)$ is the homotopy pullback of

$$\begin{array}{ccc} & & \mathbf{0} \\ & & \downarrow \\ \mathbf{0} & \longrightarrow & F(\Sigma X) \end{array}$$

which is weakly equivalent to $\Omega F(\Sigma X)$ by Proposition 2.11.

This process can be repeated. That is, $T_1^2F(X) \simeq T_1(T_1F)(X)$ is the homotopy pullback of the diagram

$$\begin{array}{ccc} & & T_1F(X*\{1\}) \\ & & \downarrow \\ T_1F(X*\{2\}) & \longrightarrow & T_1F(X*\{1,2\}) \end{array}$$

But $T_1F(X * \{1\}) \simeq T_1F(\mathbf{0}) \simeq \Omega F(\Sigma(\mathbf{0})) \simeq \Omega F(\mathbf{0}) \simeq \Omega(\mathbf{0}) \simeq \mathbf{0}$, and $T_1F(\Sigma X) \simeq \Omega F(\Sigma\Sigma X) \simeq \Omega F(\Sigma^2 F)$. Finally, the homotopy pullback of

$$\begin{array}{ccc} & & \mathbf{0} \\ & & \downarrow \\ \mathbf{0} & \longrightarrow & T_1\Omega F(\Sigma^2 X) \end{array}$$

is $T_1^2 F(X) \simeq \Omega^2 F(\Sigma^2 X)$.

Definition 2.33. [Kuh07, 5.2] Let $P_n F : \mathcal{C} \to \mathcal{D}$ be defined by

$$P_n F(X) = \text{hocolim}\left\{F(X) \xrightarrow{t_n(F)} T_n F(X) \xrightarrow{t_n(T_n F)} T_n^2 F(X) \to \ldots\right\}.$$

So in our example,

$$\begin{aligned} P_1 F(X) &= \text{hocolim}\left\{F(X) \to \Omega F(\Sigma X) \to \Omega^2 F(\Sigma^2 X) \to \ldots\right\} \\ &\simeq \text{hocolim}_{n\to\infty} \Omega^n F(\Sigma^n X). \end{aligned}$$

The composition of the t_n's gives a natural transformation $p_n F : F \to P_n F$. If F is n-excisive, then $p_n F$ is the homotopy colimit of weak equivalences, so $p_n F$ is a weak equivalence.

The following results are proven in [Goo03] when $\mathcal{C} = \mathcal{T}op_*$ and $\mathcal{D}$ is $\mathcal{T}op_*$ or $\mathcal{S}p$, but the proofs apply equally well for any functor F and categories $\mathcal{C}, \mathcal{D}$ satisfying Hypothesis 2.28.

Proposition 2.34.

1. *$P_n F$ is a homotopy functor.*

2. *$P_n F$ is n-excisive.*

3. *$P_n F$ is universal among n-excisive functors G such that $F \to G$.*

Next, we show that there are natural maps $q_nF : P_nF \to P_{n-1}F$ for all $n \geq 1$. Note that there is a natural map $q_{n,1} : T_nF \to T_{n-1}F$ because T_nF comes from the homotopy limit of an $n+1$-cube, and $T_{n-1}F$ comes from the homotopy limit of one of the n-dimensional faces of this cube. Similarly, the inclusion of n-cubes into $(n+1)$-cubes gives natural maps $q_{n,i} : T_n^iF \to T_{n-1}^iF$ for all $i \geq 1$. Moreover, there is a commutative diagram

$$\begin{array}{ccccccc}
F & \xrightarrow{t_nF} & T_nF & \xrightarrow{t_nT_nF} & T_n^2F & \xrightarrow{t_nT_n^2F} & \dots \\
\downarrow = & & \downarrow q_{n,1} & & \downarrow q_{n,2} & & \\
F & \xrightarrow{t_{n-1}F} & T_{n-1}F & \xrightarrow{t_{n-1}T_{n-1}F} & T_{n-1}^2F & \xrightarrow{t_{n-1}T_{n-1}^2F} & \dots
\end{array}$$

We define q_n as the induced map of the horizontal homotopy colimits.

The homotopy fiber of the map $q_nF : P_nF \to P_{n-1}F$ is denoted D_nF. Equivalently, for each $X \in \mathrm{Ob}(\mathcal{C})$, $D_nF(X) \simeq \mathrm{holim}(\mathbf{0} \to P_{n-1}F(X) \xleftarrow{q_n} P_nF(X))$. In the analogy with calculus, P_nF is like the n^{th} Taylor polynomial of a function. The functor D_nF behaves like the n^{th} term in the Taylor series. In particular, one can recover P_nF if one knows $P_{n-1}F$ and D_nF. Pictured as in Theorem 2.27, we call P_nF the n^{th} stage of the Taylor tower of F and D_nF the n^{th} layer. Each functor D_nF is *homogeneous* of degree n, which means $P_nD_nF \simeq D_nF$ (D_nF is n-excisive) and $P_{n-1}D_nF \simeq *$ (D_nF is n-reduced) [Goo03, 1.17]. That is, homogeneous functors are n-excisive with trivial $(n-1)$-excisive part. This terminology is inspired by homogeneous polynomials, which are concentrated in a single degree.

The following two results are often helpful in proving that D_nF is n-excisive and n-reduced. Again, although Goodwillie assumed that $\mathcal{C} = \mathcal{T}op_*$ and $\mathcal{D}$ is $\mathcal{T}op_*$ or $\mathcal{S}p$, the proofs hold for any functor F and categories $\mathcal{C}, \mathcal{D}$ satisfying Hypothesis 2.28.

Proposition 2.35. [Goo91, 3.4] Let $\Delta : \mathcal{C} \to \mathcal{C}^n$ be the diagonal inclusion. If $F : \mathcal{C}^n \to \mathcal{D}$ is k_i-excisive in the i^{th} variable for all $1 \leq i \leq n$, then the composition $F \circ \Delta$ is k-excisive, where $k = k_1 + \ldots + k_n$.

Proposition 2.36. [Goo03, 3.1] Let $\Delta : \mathcal{C} \to \mathcal{C}^n$ be the diagonal inclusion. If $F : \mathcal{C}^n \to \mathcal{D}$ is 1-reduced in the i^{th} variable for all $1 \leq i \leq n$, then the composition $F \circ \Delta$ is n-reduced.

If $F : \mathcal{T}op_* \to \mathcal{T}op_*$, then

$$D_n F(X) \simeq \Omega^\infty (A \wedge (\Sigma^\infty X)^{\wedge n})_{h\Sigma_n}$$

where A is a spectrum with an action of the n^{th} symmetric group Σ_n [Goo03]. This spectrum A is called the n^{th} *derivative of* F and is denoted $\partial_n F$. In cases where $\mathcal{C}$ is a sufficiently nice simplicial model category, given a homotopy functor $F : \mathcal{C} \to \mathcal{C}$, the layers of the Taylor tower have the form

$$D_n F(X) \simeq \Omega^\infty_{\mathcal{C}} (\partial_n F \otimes (\Sigma^\infty_{\mathcal{C}} X)^{\otimes n})_{h\Sigma_n}$$

where $\partial_n F$ is a $\mathcal{C}$-spectrum with Σ_n-action, $\otimes$ is the symmetric monoidal product in the category of spectra on $\mathcal{C}$ induced by the smash product of simplicial sets, and $\Sigma^\infty_{\mathcal{C}}$ and $\Omega^\infty_{\mathcal{C}}$ are the canonical Quillen adjoint pair between $\mathcal{C}$ and the category of spectra on $\mathcal{C}$ [Wal06, p. 5]. We will explain what we mean by the category of spectra on $\mathcal{C}$ in section 4.2.

The identity functor is especially interesting to study because the derivatives of the identity functor provide a measure of how complicated the homotopy theory of a category is. There are several categories in which the derivatives of the identity functor are known to have extra structure, specifically, the structure of an operad.

2.3. Operads

Definition 2.37. [MSS02, I.1.4] Let $\mathcal{V}$ be a symmetric monoidal category. An operad $\mathcal{O}$ in $\mathcal{V}$ consists of a set of objects $O(n)$, $n \geq 1$ of $\mathcal{V}$ equipped with

- an action of the symmetric group Σ_n on $O(n)$,
- an element $e \in O(1)$ called the identity, and
- structure maps, also known as composition operations,

$$\gamma_{k;n_1,\ldots,n_k} : O(k) \otimes O(n_1) \otimes O(n_2) \otimes \ldots \otimes O(n_k) \to O(n_1 + n_2 + \ldots + n_k).$$

These data must also satisfy compatibility and associativity conditions, but we do not list those here. Operads are useful for bookkeeping because they are an abstraction of a family of composable functions of n variables for various n. Thus operads encode algebraic properties.

Definition 2.38. We say that an object A in $\mathcal{V}$ is an *algebra over the operad* $\mathcal{O}$ if there is a family of Σ_n-equivariant morphisms $\mu_n : O(n) \otimes A^{\otimes n} \to A$ which are compatible with the structure morphisms of $\mathcal{O}$.

That is, we interpret $O(n)$ as objects of actual n-ary operations on an object A. We discuss a few common examples.

Example 2.39. Let $\mathcal{A}ss$ denote the associative operad, so-named because a vector space V over a field k is an algebra over the associative operad if and only if V

is an associative algebra. Here $A(n) = k\langle\Sigma_n\rangle$, the vector space over k with basis $\sigma_1, \dots, \sigma_{n!} \in \Sigma_n$. The action of Σ_n on $A(n)$ is the composition of permutations, so

$$\left(\sum_{i=1}^{n!} k_i\sigma_i\right) g = \sum_{i=1}^{n!} k_i(\sigma_i g).$$

The identity is the unit in $A(1)$. Finally, the maps

$$\gamma_{n;m_1,\dots,m_n} : A(n) \otimes A(m_1) \otimes \dots A(m_n) \to A(m_1 + \dots + m_n)$$

simply multiply elements.

Suppose that a vector space V is an algebra over $\mathcal{A}ss$. The map $\mu_n : A(n) \otimes V^{\otimes n} \to V$ is given by $\mu_n(\sigma, v_1, \dots v_n) = v_{\sigma^{-1}(1)} \cdot \dots \cdot v_{\sigma^{-1}(n)}$. For example, (123) is a basis element of $A(3)$, and $(123) \otimes v \otimes w \otimes z = zvw$. The existence of the map $A(2) \otimes V \otimes V \to V$ shows that V has multiplication, so V is an algebra, and the associativity condition from the definition of an operad implies that V is associative.

Example 2.40. Let $\mathcal{C}om$ denote the commutative operad, so-named because a vector space V over k is a commutative algebra if and only if V is an algebra over the commutative operad. Here $C(n) = k$ for all $n \geq 1$. The action of Σ_n on $C(n)$ is the trivial action, i.e. $g \cdot x = x$ for all $g \in \Sigma_n$ and $x \in C(n) = k$. The identity is the unit in $C(1)$. Again, the structure maps multiply elements.

Suppose that a vector space V is an algebra over the commutative operad. The map $\mu_n : C(n) \otimes V^{\otimes n} \to V$ is given by $k_1 \cdot (v_1, \dots, v_n) \mapsto k_1 v_1 \dots v_n$. As before, the algebra structure comes from μ_2. By the equivariance condition, given $\sigma \in \Sigma_n$,

$$\mu_n(\sigma(k_1), \sigma(v_1, \dots, v_n)) = \mu_n(k_1, v_1, \dots, v_n),$$

or

$$k_1 v_{\sigma(1)} \dots v_{\sigma(n)} = k_1 v_1 \dots v_n,$$

so the algebra must be commutative.

Example 2.41. Let $\mathcal{L}ie$ denote the Lie operad. A vector space over k is a Lie algebra if and only if it is an algebra over the Lie operad. Here $\mathcal{L}ie(n)$ is the differential graded vector space concentrated in degree 0 generated by all abstract bracket expressions of n elements (e.g. $[x_1,[x_2,\dots,[x_{n-1},x_n]\cdots]]$) modulo anti-symmetry and the Jacobi identity. Note that the dimension of $\mathcal{L}ie(n)$ as a k-vector space is $(n-1)!$ [GK+94, 1.3.9]. The action of Σ_n on $\mathcal{L}ie(n)$ is generated by the permutation of the elements in a bracket expression with no negative signs. The identity is the unit in $\mathcal{L}ie(1) \cong k$. The maps μ_n "plug in" elements. For example, given a Lie algebra $\mathfrak{g}$, $\mu_3 : \mathcal{L}ie(3) \otimes \mathfrak{g}^{\otimes 3} \to \mathfrak{g}$ behaves as

$$\begin{aligned}
[x_1,[x_2,x_3]] \otimes x \otimes y \otimes z &\mapsto [x,[y,z]] \\
[x_2,[x_3,x_1]] \otimes x \otimes y \otimes z &\mapsto [y,[z,x]] \\
[x_3,[x_1,x_2]] \otimes x \otimes y \otimes z &\mapsto [z,[x,y]]
\end{aligned}$$

The link between Goodwillie calculus and operads is that there are several categories in which the derivatives of the identity functor are known to be the objects of an operad. Such categories include

- $\mathcal{T}op_*$ [Chi05],
- $\mathcal{S}p$ [AC11],
- the category of algebras over an operad in the category of symmetric spectra [HH13], and

- $\mathcal{DGL}_r$, the category of differential graded Lie algebras over $\mathbb{Q}$ that are **0** below grading r [Wal06].

In fact, the derivatives of the identity functor in $\mathcal{DGL}_r$ are the objects of the $\mathcal{L}ie$ operad, though the grading is shifted. We prove this result in the next chapter.

CHAPTER III

DIFFERENTIAL GRADED LIE ALGEBRAS

A *differential graded vector space* (DG) is $V = (V_\bullet, d_V)$, where $V_\bullet$ is a graded vector space and $d_V = \{d_i \mid d_i : V_i \to V_{i-1}\}$ is a differential. If $v \in V_n$, we say that the *degree of* v is n and write $|v| = n$. Given differential graded vector spaces $V = (V_\bullet, d_V)$ and $W = (W_\bullet, d_W)$, a *DG map* $f : V \to W$ is a family of vector space maps $f_k : V_n \to W_{n+k}$ for all n. The integer k is called the *degree of* f. Let $\mathcal{DG}$ denote the category of differential graded $\mathbb{Q}$-vector spaces with degree 0 maps. A *quasi-isomorphism* of DGs is a map that induces an isomorphism on homology.

A DG V is called *r-reduced* if $V_k = 0$ for all $k < r$. We write $\mathcal{DG}_r$ for the full subcategory of $\mathcal{DG}$ consisting of all r-reduced DGs. The r-reduction functor $\mathrm{red}_r : \mathcal{DG} \to \mathcal{DG}_r$ acts by $(\mathrm{red}_r V)_k = V_k$ for $k > r$, $(\mathrm{red}_r V)_k = 0$ for $k < r$, and $(\mathrm{red}_r V)_r = \ker(d_r : V_r \to V_{r-1})$. Note that $H_k(\mathrm{red}_r(V)) = H_k(V)$ for all $k \geq r$, and $H_k(\mathrm{red}_r(V)) = 0$ for all $k < r$. There are also shift functors $s, s^{-1} : \mathcal{DG} \to \mathcal{DG}$ given by $(s(V))_i = V_{i-1}$ and $(s^{-1}(V))_i = V_{i+1}$. The category of graded vector spaces is denoted $\mathcal{G}$, and the category of r-reduced graded vector spaces is denoted $\mathcal{G}_r$.

We are interested in differential graded Lie algebras, which are DGs with extra structure.

Definition 3.1. [Wal06, 4.1] A *differential graded Lie algebra* (DGL) $L = (L_\bullet, d_L, [-,-]_L)$ consists of a differential graded vector space $(L_\bullet, d_L)$ equipped with a linear, degree zero, graded DG map

$$[-,-] : (L_\bullet, d_L) \otimes (L_\bullet, d_L) \to (L_\bullet, d_L)$$

satisfying

- $[x,y] = -(-1)^{|x|\cdot|y|}[y,x]$ (graded anti-symmetry) and
- $[x,[y,z]] + (-1)^{|x|(|y|+|z|)}[y,[z,x]] + (-1)^{|z|(|x|+|y|)}[z,[x,y]] = 0$ (graded Jacobi identity).

We call $[-,-]$ the *Lie bracket* map. Because the Lie bracket is a DG map, $d_L[x,y] = [d_Lx,y] + (-1)^{|x|}[x,d_Ly]$. Maps of DGLs are degree 0 graded vector space maps $f : V_\bullet \to W_\bullet$ such that $f(d_Vx) = d_Wf(x)$ and $f([x,y]) = [f(x),f(y)]$. We denote the category of DGLs with these maps by $\mathcal{DGL}$.

A DGL L is r-reduced if $L_k = 0$ for all $k < r$. The category of all r – reduced DGLs and DGL maps is denoted $\mathcal{DGL}_r$. The reduction functor $\mathrm{red}_r : \mathcal{DGL} \to \mathcal{DGL}_r$ is the restriction of the reduction functor on $\mathcal{DG}$. The category of graded Lie algebras and degree 0 Lie algebra maps is denoted $\mathcal{GL}$. Given $V \in \mathrm{Ob}(\mathcal{DG})$, we write $[V]_{DGL}$ to represent the DGL that has V as the underlying DG and a trivial Lie bracket, i.e. $[v,w] = 0$ for all $v,w \in V$. Similarly, we write $[L]_{DG}$ when we want to consider an object L of $\mathcal{DGL}$ as an object of $\mathcal{DG}$ and $[L]_{GL}$ when we want to forget the differential of L.

We define $(-)^{\mathrm{ab}} : \mathcal{DGL}_r \to \mathcal{DG}_r$ by $(L)^{\mathrm{ab}} = L/[-,-]$, the graded abelianization of L. By $(L)^{\mathrm{ab}}$, we really mean $[(L)^{\mathrm{ab}}]_{DG}$ since $(L)^{\mathrm{ab}}$ is actually an object of $\mathcal{DGL}$ with a trivial Lie bracket. Note that $(-)^{\mathrm{ab}} : \mathcal{DGL}_r \rightleftarrows \mathcal{DG} : [-]_{DGL} \circ \mathrm{red}_r$ are adjoint functors.

Quillen showed that $\mathcal{DGL}_r$ is a model category for each $r \geq 1$.

Theorem 3.2. [Qui69, 5.1] There is a model structure on $\mathcal{DGL}_r$, $r \geq 1$ in which the weak equivalences are quasi-isomorphisms, the fibrations are degree-wise surjections in degrees $k > r$, and the cofibrations are maps which have the left lifting property with respect to trivial fibrations.

We need to describe the cofibrant objects in this category. Given $V \in \mathrm{Ob}(\mathcal{G}_r)$, we construct $\mathbb{L}_V \in \mathrm{Ob}(\mathcal{DGL}_r)$ by taking the tensor algebra $TV = \bigoplus_k V^{\otimes k}$, defining the Lie bracket as $[x, y] = x \otimes y - (-1)^{|x|\cdot|y|} y \otimes x$, and taking $\mathbb{L}_V$ to be the sub-Lie algebra of TV generated by V. We say that $L \in \mathrm{Ob}(\mathcal{DGL}_r)$ is *free* if $[L]_{GL} \cong \mathbb{L}_V$ for some graded vector space V. The cofibrant objects in $\mathcal{DGL}_r$ are these free DGLs [Wal06, 4.2.3]. We write $f\mathcal{DGL}_r$ to denote the full subcategory of free DGLs in $\mathcal{DGL}_r$.

Note that the functor $\mathbb{L}_{(-)} : \mathcal{G}_r \to \mathcal{GL}_r$ defined by $V \mapsto \mathbb{L}_V$ is the left adjoint of the forgetful functor $[-]_{G_r} : \mathcal{GL}_r \to \mathcal{G}_r$. We point out that $\mathbb{L}_{(-)}$ is not left adjoint to the forgetful functor $[-]_{DG_r} : \mathcal{DGL}_r \to \mathcal{DG}_r$, as one might expect from the name "free DGL." That is, $\mathbb{L}_V$ is free in $\mathcal{GL}_r$ but not in $\mathcal{DGL}_r$. The forgetful functor $[-]_{DG_r}$ does have a left adjoint $\mathbb{L}_{(-)} : \mathcal{DG}_r \to \mathcal{DGL}$, which is defined as above but on differential graded vector spaces rather than graded vector spaces, thus with consequences for the differential on the resulting DGL. DGLs which are free in this sense are known in the literature as "truly free" DGLs.

Homotopy theorists are interested in differential graded Lie algebras because $\mathcal{DGL}_1$ and the category of rational simply connected topological spaces, $\mathcal{T}op_{\mathbb{Q}}$, have equivalent homotopy categories. In particular, Quillen [Q69] constructed a chain of Quillen equivalences from $\mathcal{T}op_{\mathbb{Q}}$ to $\mathcal{DGL}_1$ in which the homology of the DGL corresponding to a rational simply-connected space $X_{\mathbb{Q}}$ is equal to the shifted homotopy of $X_{\mathbb{Q}}$.

Because of this relationship, it is reasonable to ask what the category of spectra on $\mathcal{DGL}_1$ is, and more generally, what the category of spectra on $\mathcal{DGL}_{r-1}$

is for $r \geq 2$. One may construct rational spectra by considering the stabilization of $\mathcal{DGL}_{r-1}$. In particular, we have the following definition.

Definition 3.3. [Wal06, 6.2.1] A *$\mathcal{DGL}$-spectrum E* is a sequence of DGLs $\{L_i\}_{i\geq 0}$ equipped with maps $L_i \to \Omega L_{i+1} = [\mathrm{red}_r(s^{-1}[L_{i+1}]_{DG})]_{DGL}$.

In fact, the category of $\mathcal{DGL}$-spectra is isomorphic to $\mathcal{DG}$ [Wal06, 6.2.3]. Therefore $\mathcal{DG}$ is to $\mathcal{DGL}$ as $\mathcal{S}p$ is to $\mathcal{T}op$. Differential graded rational vector spaces behave like rational spectra, and differential graded rational vector spaces that are r-reduced behave like $(r-1)$ – connected rational spectra. Moreover, there is a spectrification functor analogous to $\Sigma^\infty : \mathcal{T}op \to \mathcal{S}p$.

Definition 3.4. [Wal06, 6.2.5] Define $\Sigma^\infty_{\mathcal{DGL}} : \mathcal{DGL}_{r-1} \to \mathcal{DG}_r$ by $\Sigma^\infty_{\mathcal{DGL}}(L) = s(L)^{\mathrm{ab}}$ and $\Omega^\infty_{\mathcal{DGL}} : \mathcal{DG}_r \to \mathcal{DGL}_{r-1}$ by $\Omega^\infty_{\mathcal{DGL}}(V) = [s^{-1}\mathrm{red}_r V]_{DGL}$.

Lemma 3.5. [Wal06, 6.2.6] $\Sigma^\infty_{\mathcal{DGL}}$ and $\Omega^\infty_{\mathcal{DGL}}$ are adjoint functors.

Walter's main theorem is stated as follows.

Theorem 3.6. [Wal06, 8.1.1] The n^{th} derivative of the identity functor $\mathbb{1}_{\mathcal{DGL}} : \mathcal{DGL}_{r-1} \to \mathcal{DGL}_{r-1}$, $r \geq 2$ is $\mathcal{L}ie(n)$ graded in degree $(1-n)$ with Σ_n-action twisted by the sign of permutations.

The goal of this chapter is to present Walter's proof. First, Walter shows that Theorem 2.27 applies in $\mathcal{DGL}_{r-1}$. In particular, since $\mathbb{1}_{\mathcal{DGL}}$ is a homotopy functor, there is a universal approximating tower of fibrations

$$\ldots \to P_n\mathbb{1}_{\mathcal{DGL}} \to P_{n-1}\mathbb{1}_{\mathcal{DGL}} \to \ldots \to P_1\mathbb{1}_{\mathcal{DGL}} \to P_0\mathbb{1}_{\mathcal{DGL}}$$

with homotopy fibers $D_n\mathbb{1}_{\mathcal{DGL}} \to P_n\mathbb{1}_{\mathcal{DGL}} \to P_{n-1}\mathbb{1}_{\mathcal{DGL}}$.

Let $L \in \mathrm{Ob}(f\mathcal{DGL}_{r-1})$. It suffices to consider free DGLs because the cofibrant replacement of any DGL is free, and $D_n\mathbb{1}_{\mathcal{DGL}}$ is a homotopy functor. Because of our analogy with $\mathcal{T}op$, we expect these homotopy fibers to have the form

$$\begin{aligned} D_n\mathbb{1}_{\mathcal{DGL}}(L) &\simeq \Omega^{\infty}_{\mathcal{DGL}}(\partial_n\mathbb{1}_{\mathcal{DGL}} \otimes (\Sigma^{\infty}_{\mathcal{DGL}}L)^{\otimes n})_{\Sigma_n} \\ &\simeq \left[s^{-1}(\partial_n\mathbb{1}_{\mathcal{DGL}} \otimes (s(L)^{\mathrm{ab}})^{\otimes n})_{\Sigma_n}\right]_{DGL} \\ &\cong \left[(s^{n-1}\partial_n\mathbb{1}_{\mathcal{DGL}} \otimes ((L)^{\mathrm{ab}})^{\otimes n})_{\Sigma_n}\right]_{DGL} \end{aligned}$$

where $\partial_n\mathbb{1}_{\mathcal{DGL}}$ is a DG with Σ_n-action. Here Σ_n acts on $[s(L)^{\mathrm{ab}}]^{\otimes n}_{DG}$ and $[(L)^{\mathrm{ab}}]^{\otimes n}_{DG}$ by permutation of elements with signs according to the Koszul convention and on s^{n-1} by multiplication by $(-1)^{sgn(\sigma)}$. That is, the Σ_n-equivariant isomorphism on the last line is given by applying s^{-1} to the Σ_n-equivariant map $(sx_1 \otimes \cdots \otimes sx_n) \mapsto (-1)^k s^n \otimes x_1 \otimes \cdots \otimes x_n$, where $k = \sum_{j=1}^{n-1}\sum_{i=1}^{j}|x_i|$ is the sign incurred by moving all of the s's to the beginning of the expression.

We will show that the Taylor tower of $\mathbb{1}_{\mathcal{DGL}}$ is the same as a tower of quotients of the lower central series of L. Recall that the lower central series of L is

$$L \supset [L,L] \supset [L,[L,L]] \supset \dots$$

Let $L = \Gamma^1(L)$, $[L,L] = \Gamma^2(L)$, $[L[L,L]] = \Gamma^3(L)$, etc. Then this chain of inclusions induces the following tower by taking quotients:

$$\begin{array}{c}\vdots \\ \downarrow \\ L/\Gamma^3(L) =: B_2(L) \\ \downarrow \\ L/\Gamma^2(L) =: B_1(L) \\ \downarrow \\ L/L = \mathbf{0}\end{array}$$

Each map is degree-wise surjective, so each map is a fibration in DGL. The limit of this tower is L. This is because L being $(r-1)$-reduced means that $\Gamma^n(L)$ is $n(r-1)$ reduced, so $L_k = (L/\Gamma^n(L))_k$ for n sufficiently large. When L is free, this tower is called the bracket-length filtration of L since, for $L = (\mathbb{L}_V, d)$, we have $B_n(L) = ((T^{\leq n}V) \cap \mathbb{L}_V, d = d_0 + \ldots + d_n)$. That is, $B_n(L)$ consists of the elements of $\mathbb{L}_V$ with bracket-length at most n. Here d_i is a differential on $\mathbb{L}_V$ that increases bracket length by i.

Let $H_n = \mathrm{hofib}(B_n \xrightarrow{f} B_{n-1})$. By [Wal06, 4.2.13], $H_n = \mathrm{red}_r((s^{-1}B_{n-1} \times B_n), d, [-,-])$, where

- $d(s^{-1}a, b) = (s^{-1}(f(b) - d(a)), db)$
- $[s^{-1}a, b] = \frac{1}{2}s^{-1}[a, f(b)]_{B_{n-1}}$
- $[b_1, b_2] = [b_1, b_2]_{B_n}$
- $[s^{-1}a_1, s^{-1}a_2] = 0$

Note $H_n(L)/(T^n \cap \mathbb{L}_V) \simeq \mathrm{hofib}(B_{n-1} \xrightarrow{id} B_{n-1})$, and this homotopy fiber has zero homology. Hence $H_n(L)/(T^n \cap \mathbb{L}_V)$ has zero homology, meaning the inclusion $(T^n \cap \mathbb{L}_V) \hookrightarrow H_n(L) = (s^{-1}B_{n-1} \times B_n)$ given by $x \mapsto (0,[x])$ is a quasi-isomorphism. Therefore $H_n(L) \simeq T^n \cap \mathbb{L}_V$, i.e. $H_n(L)$ is all elements of L of bracket length n.

However, $[(\mathcal{L}ie(n) \otimes ((L)^{\text{ab}})^{\otimes n})_{\Sigma_n}]_{DGL}$ also represents the elements of L with bracket length n. This is because $\mathcal{L}ie(n)$ is the n^{th} space of the Lie operad, and $[(\mathcal{L}ie(n) \otimes ((L)^{\text{ab}})^{\otimes n})_{\Sigma_n}]_{DGL}$ is a free Lie algebra concentrated in bracket length n. Since $L = \mathbb{L}_V$ and the free object of the Lie operad on V both give a free Lie algebra on $(L)^{\text{ab}} \cong V$, we must have $H_n(L) \simeq [(\mathcal{L}ie(n) \otimes ((L)^{\text{ab}})^{\otimes n})_{\Sigma_n}]_{DGL}$.

Next, we show that $H_n : f\mathcal{DGL}_{r-1} \to \mathcal{DGL}_{n(r-1)}$ is an n-homogeneous functor. Consider the case when $n = 1$. We observe that $H_1(L) \simeq T^1 \cap \mathbb{L}_V \simeq (L)^{\text{ab}}$. We must show that $P_1H_1 = H_1$ and $P_0H_1 \simeq *$. First, consider the homotopy pushout square of free DGLs

$$\begin{array}{ccc} \mathbb{L}_V & \longrightarrow & \mathbb{L}_W \\ \downarrow & & \downarrow \\ \mathbb{L}_X & \longrightarrow & \mathbb{L}_{X \oplus sV \oplus W} \end{array}$$

Applying H_1 to this square, we get

$$\begin{array}{ccc} V & \longrightarrow & W \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \oplus sV \oplus W \end{array}$$

Since $V \simeq X \oplus s^{-1}(X \oplus sV \oplus W) \oplus W$, this is a homotopy pullback square, and hence H_1 is 1-excisive. Certainly if $V \simeq 0$, then $\mathbb{L}_V \simeq 0$. Also, if $\mathbb{L}_V \simeq 0$, then $H_1(\mathbb{L}_V) = (\mathbb{L}_V)^{\text{ab}} \simeq V \simeq 0$. Therefore H_1 is 1-reduced. By Proposition 2.35, since the functor $(-)^{\text{ab}}$ is 1-excisive, the functor $((-)^{\text{ab}})^{\otimes n}$ is n-excisive. Similarly, by Proposition 2.36, $(-)^{\mathrm{ab}}$ being 1-reduced implies that $((-)^{\mathrm{ab}})^{\otimes n}$ is n-reduced. Cartesian cubes of dimension $n+1$ are preserved by tensoring with $\mathcal{L}ie(n)$. It follows that H_n is n-excisive and n-reduced.

Now we show that B_n is n-excisive for all n by induction. Note that $B_0 = L/L = \mathbf{0}$, so the fiber sequence $H_1 \to B_1 \to B_0$ implies $H_1 \simeq B_1$. Since H_1 is 1-homogenous, so is B_1.

Assume that for all $1 \le i \le n-1$, B_i is i-excisive. Note this implies that $B_0, \dots, B_{n-1}$ are n-excisive. Consider the following diagram.

$$\begin{array}{ccccc} H_n(L) & \longrightarrow & B_n(L) & \longrightarrow & B_{n-1}(L) \\ \downarrow & & \downarrow & & \downarrow \\ P_nH_n(L) & \longrightarrow & P_nB_n(L) & \longrightarrow & P_nB_{n-1}(L) \end{array}$$

Since H_n and B_{n-1} are n-excisive, the outside vertical maps are equivalences. Each of the horizontal sequences induces a long exact sequence in homology. By the Five Lemma, $H_*(B_n(L)) \to H_*(P_nB_n(L))$ is an isomorphism. Thus $B_n(L) \to P_nB_n(L)$ is a weak equivalence, so B_n is n-excisive.

Therefore this tower is an approximating tower of fibrations of n-excisive functors converging to $\mathbb{1}_{\mathcal{DGL}}$ in the sense that the maps $L \to B_n(L)$ are vector space isomorphisms up to degree $n(r-1)$. By the universality of $P_n\mathbb{1}_{\mathcal{DGL}}$, we know $P_n\mathbb{1}_{\mathcal{DGL}} \simeq P_nB_n$, and $P_nB_n \simeq B_n$ since B_n is n-excisive. Therefore the bracket-length filtration is the rational Taylor tower of $\mathbb{1}_{\mathcal{DGL}}$ evaluated on L. By defining $\partial_n\mathbb{1}_{\mathcal{DGL}}$ to be $\mathcal{L}ie(n)$ graded in degree $(1-n)$ with Σ_n-action twisted by the sign of permutations, we find that $[(\mathcal{L}ie(n) \otimes ((L)^{\mathrm{ab}})^{\otimes n})_{\Sigma_n}]_{DGL}$ matches the expected $D_n\mathbb{1}_{\mathcal{DGL}}(L) \simeq \left[(s^{n-1}\partial_n\mathbb{1}_{\mathcal{DGL}} \otimes ((L)^{\mathrm{ab}})^{\otimes n})_{\Sigma_n}\right]_{DGL}$.

CHAPTER IV

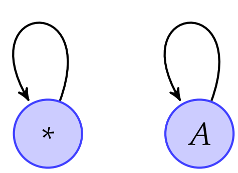
THE CATEGORY OF SMALL CATEGORIES

In this chapter, we consider $\mathcal{C}at_*$, the category of pointed small categories and functors between them. Our goal is to calculate the derivatives of the identity functor on $\mathcal{C}at_*$. As we have previously noted, the derivatives should be objects in the category of spectra on $\mathcal{C}at_*$. In order to construct this category of spectra, we must first determine how the suspension functor behaves in $\mathcal{C}at_*$.

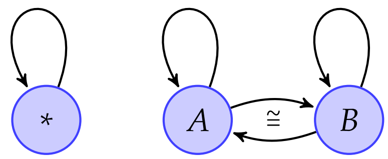
The zero object $\mathbf{0}$ in $\mathcal{C}at_*$ is the category with one object and only the identity morphism. Then any category $\mathcal{C}$ has a chosen morphism $\mathbf{0} \to \mathcal{C}$. We call the image of this morphism the basepoint of $\mathcal{C}$ and denote it by *. We use the canonical model structure on $\mathcal{C}at$, which was first explicitly described by Rezk [Rez96, 3.1]. That is, we define a morphism to be

- a weak equivalence if it is an equivalence of categories (equivalently, if it is fully faithful and essentially surjective),
- a cofibration if it is injective on objects, and
- a fibration if it is an isofibration. An *isofibration* is a functor $F : \mathcal{C} \to \mathcal{D}$ such that for any object $C \in \mathrm{Ob}(\mathcal{C})$ and any isomorphism $\phi : F(C) \xrightarrow{\cong} D$, there is an isomorphism $\psi : C \xrightarrow{\cong} C'$ such that $F(\psi) = \phi$.

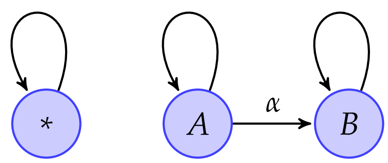
Let $\mathbf{T}$ be the category with two objects and only identity morphisms:



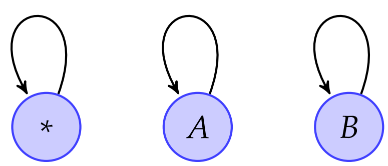
Let $\mathbf{I}$ be the category with a disjoint basement, two other objects, and unique isomorphisms between them:



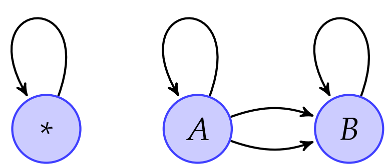
Let $\mathbf{1}$ be the category with a disjoint basepoint, two other objects, and a map α between them:



Let $\dot{\mathbf{1}}$ be the maximal subcategory of $\mathbf{1}$ not containing α:



Finally, let $\mathbf{P} = \mathbf{1} \sqcup_{\dot{\mathbf{1}}} \mathbf{1}$ be the category consisting of a disjoint basepoint, two other objects, and a pair of parallel arrows:



That is, $\mathbf{P}$ is the pushout of the diagram $(\mathbf{1} \leftarrow \dot{\mathbf{1}} \rightarrow \mathbf{1})$, where both of the maps are inclusions.

Proposition 4.1. *The canonical model structure on $\mathcal{C}at_*$ is cofibrantly generated with generating trivial cofibration $J = \{j\}$, where $j = \mathbf{T} \hookrightarrow \mathbf{I}$, and generating cofibrations $I = \{u, v, w\}$, where $u : \mathbf{0} \to \mathbf{T}$ and $v : \dot{\mathbf{1}} \to \mathbf{1}$ are inclusions and $w : \mathbf{P} \to \mathbf{1}$ identifies the parallel arrows.*

Proof. This is the pointed version of the structure given by Rezk, and the proof is analogous. First, consider a commutative square

$$\begin{array}{ccc} \mathbf{T} & \longrightarrow & \mathcal{C} \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle F} \\ \mathbf{I} & \longrightarrow & \mathcal{D} \end{array}$$

Note that a lift exists precisely when F is an isofibration, so J determines the fibrations.

Next, we show that a functor F is a trivial fibration if and only if F has the right lifting property with respect to every map in I. Each map in I is a cofibration, and hence every trivial fibration has the right lifting property with respect to each. Conversely, suppose $F : \mathcal{C} \to \mathcal{D}$ has the right lifting property with respect to each map in I. Since F has the right lifting property with respect to u, F is surjective on objects. Since F has the right lifting property with respect to v, F is surjective on Hom sets. Finally, since F has the right lifting property with respect to w, F is injective on Hom sets. Therefore F is a weak equivalence. Note that F also has the right lifting property with respect to j, and so F is a trivial fibration.

□

Proposition 4.2. *$\mathcal{C}at_*$ is a proper model category.*

Proof. Clearly, all objects in $\mathcal{C}at_*$ are cofibrant. It is also true that all objects in $\mathcal{C}at_*$ are fibrant. It suffices to show that every map $\mathcal{C} \to \mathbf{0}$ has the right lifting property with respect to $j : \mathbf{T} \hookrightarrow \mathbf{I}$. Indeed, given a commutative square

$$\begin{array}{ccc} \mathbf{T} & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ \mathbf{I} & \longrightarrow & \mathbf{0} \end{array}$$

we define a lift by sending both non-basepoint objects of $\mathbf{I}$ to the image of the non-basepoint object of $\mathbf{T}$ in $\mathcal{C}$ and the isomorphism of $\mathbf{I}$ to the identity morphism on that object. By Proposition 2.23, $\mathcal{C}at_*$ is proper.

□

Proposition 4.3. *$\mathcal{C}at_*$ is a simplicial model category.*

Proof. Rezk shows that $\mathcal{C}at$ satisfies the pushout-product axiom [Rez96, 5.1], and the proof for the pointed case is analogous. We define a pair of adjoint functors

$$\pi : sSet_* \rightleftarrows \mathcal{C}at_* : \mu$$

as follows. Let μ take a category $\mathcal{C}$ to the simplicial nerve of the subcategory $\mathcal{C}' \subseteq \mathcal{C}$ having $\mathrm{Ob}\,\mathcal{C}' = \mathrm{Ob}\,\mathcal{C}$ and having as morphisms the isomorphisms of $\mathcal{C}$. Let π take a simplicial set K to the category πK, where $\mathrm{Ob}(\pi K) = K_0$, and where there is a generating isomorphism $k : d_1 k \to d_0 k$ for each $k \in K_1$ subject to the relation $d_0 l \cdot d_2 l = d_1 l$ for each $l \in K_2$. The category πK is called the *fundamental groupoid of* K.

We show that (π, μ) is a Quillen adjunction by showing that π preserves cofibrations and trivial cofibrations. Since the cofibrations in $sSet_*$ are injective on

n-simplices for all n, it is immediate that π preserves cofibrations. Consider the generating trivial cofibrations in $sSet_*$

$$\iota_{n,k} : \Lambda^k[n] \to \Delta[n],\ n \geq 1,\ 0 \leq k \leq n.$$

For $n > 1$, the map $\pi\iota_{n,k}$ is an isomorphism because both the source and target are mapped to the category with $n+1$ objects and trivial automorphism groups. Also, $\pi\Delta[1]$ is the category with two objects and unique isomorphisms between them, so $\pi\iota_{1,k} : \mathbf{0} \to \pi\Delta[1]$ is an equivalence of categories for $k = 0, 1$. Therefore each $\pi\iota_{n,k}$ is a trivial cofibration in $\mathcal{C}at_*$. Since π preserves the generating trivial cofibrations, π preserves all trivial cofibrations.

Let $\mathcal{D}^{\mathcal{C}} = \mathrm{Hom}(\mathcal{C}, \mathcal{D})$ denote the category whose objects are functors $F : \mathcal{C} \to \mathcal{D}$ and whose morphisms are natural transformations. Define the tensoring $\otimes : \mathcal{C}at_* \times sSet_* \to \mathcal{C}at_*$ by $\mathcal{C} \otimes K = (\mathcal{C} \times \pi K)/(* \times \pi K)$. By $* \times \pi K$, we mean the category where $\mathrm{Ob}(* \times \pi K) \cong \mathrm{Ob}(\pi K)$ and $\mathrm{Mor}(* \times \pi K) \cong \mathrm{Mor}(\pi K)$. Define the powering $\mathcal{C}at_* \times sSet_*^{op} \to \mathcal{C}at_*$ by $\mathcal{C}^K = \mathcal{C}^{\pi K}$ and the enrichment $\mathcal{C}at_*^{op} \times \mathcal{C}at_* \to sSet_*$ by $\mathrm{Map}_*(\mathcal{C}, \mathcal{D}) = \mu(\mathcal{D}^{\mathcal{C}})$.

We must show that these definitions satisfy the isomorphisms of Definition 2.24. Let $\mathcal{C}, \mathcal{D} \in \mathrm{Ob}(\mathcal{C}at_*)$ and $K \in \mathrm{Ob}(sSet_*)$. Then

$$\begin{aligned}\mathrm{Map}_*(\mathcal{C}\otimes K,\mathcal{D}) &= \mu(\mathrm{Hom}(\mathcal{C}\otimes K,\mathcal{D})) \text{ by definition of the enrichment}\\ &\cong \mu(\mathrm{Hom}(\pi K,\mathrm{Hom}(\mathcal{C},\mathcal{D})))\\ &\cong \mu(\mathrm{Hom}(\pi K,\mathcal{D}^{\mathcal{C}}))\\ &\cong \mathrm{Map}_*(K,\mu(\mathcal{D}^{\mathcal{C}}))\\ &\cong \mathrm{Map}_*(K,\mathrm{Map}_*(\mathcal{C},\mathcal{D})) \text{ by definition of the powering}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Map}_*(\mathcal{C}\otimes K,\mathcal{D}) &= \mu(\mathrm{Hom}(\mathcal{C}\otimes K,\mathcal{D})) \text{ by definition of the enrichment}\\ &\cong \mu(\mathrm{Hom}(\mathcal{C},\mathrm{Hom}(\pi K,\mathcal{D})))\\ &= \mu(\mathrm{Hom}(\mathcal{C},\mathcal{D}^{K})) \text{ by definition of the powering}\\ &\cong \mathrm{Map}_*(\mathcal{C},\mathcal{D}^{K}) \text{ by definition of the enrichment.}\end{aligned}$$

□

4.1. Suspensions of Categories

We will identify the cylinder objects in $\mathcal{C}at_*$ and use this to construct suspensions (see Definition 2.9). Fortunately, there is a natural choice of cylinder object in any simplicial model category [Pel11, 1.6.6].

Proposition 4.4. *Given $\mathcal{C} \in Ob(\mathcal{C}at_*)$, a cylinder object for $\mathcal{C}$ is $Cyl(\mathcal{C}) = \mathcal{C} \otimes \Delta^1$.*

Proof. Label the objects in $\pi\Delta^1$ by $*$ and C, and let f denote the unique isomorphism $* \to C$. The objects of $\mathrm{Cyl}(\mathcal{C})$ are pairs of the form $(A,*)$ or (A,C), where $A \in \mathrm{Ob}(\mathcal{C})$. Note that $(*,*) = (*,C)$, so $\mathrm{Cyl}(\mathcal{C})$ has the same objects as $\mathcal{C} \vee \mathcal{C}$. In particular, $\mathcal{C} \vee \mathcal{C} \hookrightarrow \mathrm{Cyl}(\mathcal{C})$. Furthermore, there is a unique isomorphism $f_A = (id_A, f)$ between $(A, *)$ and (A, C) for all $A \in \mathrm{Ob}(\mathcal{C})$, so $\mathrm{Mor}_{\mathrm{Cyl}(\mathcal{C})}((A,C),(A',C')) \cong \mathrm{Mor}_{\mathcal{C}}(A,A')$. Therefore the map given by projection onto the first factor $\mathrm{Cyl}(\mathcal{C}) \to \mathcal{C}$ is a weak equivalence.

□

Definition 4.5. Given two objects a and b in a category $\mathcal{C}$, if there is a zig-zag of morphisms from a to b, then we say that a and b are *in the same component* of $\mathcal{C}$.

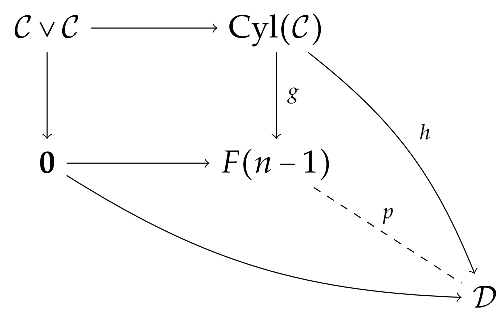
Proposition 4.6. *Given a category $\mathcal{C}$, $\Sigma\mathcal{C}$ has one object A, and $End(A) = F(n-1)$, the free group on $n-1$ generators, where n is the number of components of $\mathcal{C}$.*

Proof. In taking the pushout of the diagram

$$\begin{array}{ccc} \mathcal{C} \vee \mathcal{C} & \longrightarrow & \mathrm{Cyl}(\mathcal{C}) \\ \downarrow & & \\ \mathbf{0} & & \end{array}$$

we identify all the objects of $\mathrm{Cyl}(\mathcal{C})$ with the unique object of $\mathbf{0}$ and all the morphisms of $\mathcal{C} \vee \mathcal{C}$ with the unique morphism of $\mathbf{0}$. If there is a morphism $\alpha : * \to a$ in $\mathcal{C}$, then $f_a \circ \alpha = \alpha$ implies that $im(f_a) \circ id_{\mathbf{0}} \simeq id_{\mathbf{0}}$, or equivalently, $im(f_a) \simeq id_{\mathbf{0}}$ in the pushout. In turn, the uniqueness of the isomorphisms f_a and f_b implies that all morphisms in the same component as the basepoint of $\mathrm{Cyl}(\mathcal{C})$ get identified to $id_{\mathbf{0}}$ in the pushout. If a and b are in a component of $\mathcal{C}$ that does not contain the basepoint and there is a morphism $\alpha : a \to b$, then $f_b \circ \alpha = \alpha \circ f_a$ implies that $im(f_b) \circ id_{\mathbf{0}} \simeq id_{\mathbf{0}} \circ f_a$, or equivalently, $im(f_a) \simeq im(f_b)$ in the pushout. Therefore we are left with one isomorphism for each component of $\mathcal{C}$ that does not contain the basepoint.

We have shown that the category $F(n-1)$ completes the diagram. In order to prove that $F(n-1)$ is the pushout, we must show that for any other category $\mathcal{D}$ that makes the diagram commute, there is a unique map $p : F(n-1) \to \mathcal{D}$ that makes both triangles commute.



Since the outer square commutes, h sends all objects of Cyl($\mathcal{C}$) to the basepoint of $\mathcal{D}$, which we denote by $*_{\mathcal{D}}$, and all morphisms of Cyl($\mathcal{C}$) to isomorphisms of $*_{\mathcal{D}}$. In particular, h maps all morphisms of $\mathcal{C}$ to the identity morphism of $*_{\mathcal{D}}$, and the same argument as above shows that if a and b are in the same component of $\mathcal{C}$, then $h(f_a) = h(f_b)$.

Define $p : F(n-1) \to \mathcal{D}$ by $p(*_{F_{n-1}}) = *_{\mathcal{D}}$ and $p(g(f_a)) = h(f_a)$ for all $a \in \mathrm{Ob}(\mathcal{C})$. Since p is determined by g and h, p is unique. We remark that p defines an isomorphism between $F(n-1)$ and some subcategory of $\mathcal{D}$.

□

Corollary 4.7. *For any category $\mathcal{C}$, $\Sigma^2\mathcal{C} \simeq \mathbf{0}$.*

Proof. For any category $\mathcal{C}$, $\Sigma\mathcal{C}$ has one component. Thus $\Sigma^2\mathcal{C}$ has one object A and $\mathrm{End}(A) \simeq F(0) \simeq id_A$.

□

Similarly, there is a standard path object in any simplicial model category [Pel11, 1.6.7].

Proposition 4.8. *Given $\mathcal{C} \in Ob(\mathcal{C}at_*)$, a path object for $\mathcal{C}$ is $\mathcal{C}^I = \mathrm{Hom}(\pi\Delta^1, \mathcal{C})$.*

Proof. The objects in $\mathcal{C}^I$ are isomorphisms $g : A_0 \to A_1$ in $\mathcal{C}$. Morphisms $u : g \to h$ between g and $h : B_0 \to B_1$ are pairs $u = (u_0, u_1)$ so that $u_1 \circ g = h \circ u_0$. The basepoint of $\mathcal{C}^I$ is the morphism $\pi\Delta^1 \to *$.

Note that there is a functor $f : \mathcal{C} \to \mathcal{C}^I$ given by $A \mapsto id_A$ and that f is a weak equivalence. Let $(s,t) : \mathcal{C}^I \to \mathcal{C} \times \mathcal{C}$, where s is the source map and t is the target map. Then the composition $\mathcal{C} \to \mathcal{C}^I \to \mathcal{C} \times \mathcal{C}$ is the diagonal map. Also, (s,t) is a fibration. Let $a : A_0 \to A_1$ be an object of $\mathcal{C}^I$, and let $(u_0, u_1) : (A_0, A_1) \to (B_0, B_1)$ be an isomorphism in $\mathcal{C} \times \mathcal{C}$. Then there is a unique isomorphism $b : B_0 \to B_1$ making the square commute. Thus $u = (u_0, u_1)$ defines an isomorphism $a \to b$ in $\mathcal{C}^I$ such that $(s,t)(u) = (u_0, u_1)$.

□

Proposition 4.9. *For any category $\mathcal{C}$, $\Omega^2\mathcal{C} \simeq \mathbf{0}$.*

Proof. The pullback $\Omega\mathcal{C}$ of the diagram

$$\begin{array}{ccc} & & \mathcal{C}^I \\ & & \downarrow{\scriptstyle (s,t)} \\ \mathbf{0} & \longrightarrow & \mathcal{C} \times \mathcal{C} \end{array}$$

has the set of objects $\{a : A_0 \to A_1 \mid s(a) = t(a) = *\}$. That is, $\Omega\mathcal{C}$ has one object for each isomorphism of the basepoint of $\mathcal{C}$. Let $f, g \in \mathrm{Ob}(\Omega\mathcal{C})$, so $f, g : * \to *$. A morphism $u : f \to g$ in $\Omega\mathcal{C}$ is a pair of morphisms $u = (u_0, u_1)$ satisfying the commutative diagram

$$\begin{array}{ccc} * & \xrightarrow{f} & * \\ {\scriptstyle u_0}\downarrow & & \downarrow{\scriptstyle u_1} \\ * & \xrightarrow[g]{} & * \end{array}$$

and such that $(s,t)(u) = (u_0, u_1) = (id, id)$. Thus $\mathrm{Mor}(f,g) = (id, id)$ if $f = g$. If $f \neq g$, then there are no such commutative diagrams, and so $\mathrm{Mor}(f,g) = \varnothing$. Finally, $\Omega^2\mathcal{C}$ is the pullback of the diagram

$$\begin{array}{ccc} & & \Omega\mathcal{C}^I \\ & & \downarrow \\ \mathbf{0} & \longrightarrow & \Omega\mathcal{C} \times \Omega\mathcal{C} \end{array}$$

which has one object for each isomorphism of the basepoint of $\Omega\mathcal{C}$. The basepoint of $\Omega\mathcal{C}$ corresponds to the identity morphism of the basepoint of $\mathcal{C}$, and the only morphism of the basepoint of $\Omega\mathcal{C}$ is (id, id). Therefore $\Omega^2\mathcal{C} \simeq \mathbf{0}$.

□

4.2. Categories of Spectra

Now we are ready to define the category of spectra on $\mathcal{C}at_*$. Hovey [Hov01] gives a method for constructing a category of spectra $Sp^{\mathbb{N}}(\mathcal{C}, G)$ for certain model categories $\mathcal{C}$ and functors $G : \mathcal{C} \to \mathcal{C}$.

Definition 4.10. [Hov01, 1.1] Let G be a left Quillen endofunctor of a left proper cellular model category $\mathcal{C}$. Define $Sp^{\mathbb{N}}(\mathcal{C}, G)$, the *category of spectra on* $\mathcal{C}$, as follows. A *spectrum* X is a sequence $\{X_n\}_{n\geq 0}$ of objects of $\mathcal{C}$ together with structure maps $\sigma : GX_n \to X_{n+1}$ for all n. A *map of spectra* from X to Y is a collection of maps $f_n : X_n \to Y_n$ so that the following diagram commutes for all n:

$$\begin{array}{ccc} GX_n & \xrightarrow{\sigma_X} & X_{n+1} \\ {\scriptstyle Gf_n}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ GY_n & \xrightarrow{\sigma_Y} & Y_{n+1} \end{array}$$

We are interested in the category $Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$. By Proposition 2.23, since all objects of $\mathcal{C}at_*$ are cofibrant, $\mathcal{C}at_*$ is left proper. However, we still cannot use Hovey's construction as it is given.

Proposition 4.11. *$\mathcal{C}at_*$ is not cellular.*

Proof. Consider the generating cofibration $w : \mathbf{P} \to \mathbf{1}$. Note $\mathbf{1} \sqcup_{\mathbf{P}} \mathbf{1} = \mathbf{1}$, so $\mathrm{eq}(\mathbf{1} \rightrightarrows \mathbf{1} \sqcup_{\mathbf{P}} \mathbf{1}) = \mathbf{1}$, not $\mathbf{P}$. Therefore w is not an effective monomorphism. □

Fortunately, Hovey only requires cellularity in order to guarantee that a particular Bousfield localization exists. We will take advantage of the fact that localizations also exist in left proper combinatorial model categories. We will say more about this later, but for now, we proceed with the construction.

Given $n \geq 0$, the evaluation functor $\mathrm{Ev}_n : Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma) \to \mathcal{C}at_*$ takes X to X_n. The evaluation functor has a left adjoint $F_n : \mathcal{C}at_* \to Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$ defined by $(F_n X)_m = \Sigma^{m-n} X$ if $m \geq n$ and $(F_n X)_m = \mathbf{0}$ for $m < n$ with the obvious structure maps. By results in [Hov01], $Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$ is bicomplete and can be given what is called the projective model structure. This is a preliminary step toward making the model structure that we actually wish to use, the stable model structure.

Definition 4.12. In the *projective model structure* on $Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$, a map of spectra $f : X \to Y$ is

- a weak equivalence if $f_n : X_n \to Y_n$ is a weak equivalence in $\mathcal{C}at_*$ for all $n \geq 0$.

- a fibration if $f_n : X_n \to Y_n$ is a fibration in $\mathcal{C}at_*$ for all $n \geq 0$.

- a cofibration if f has the left lifting property with respect to all trivial fibrations in $Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$.

Furthermore, this model structure is cofibrantly generated with generating cofibrations $I_\Sigma = \bigcup_n F_n I$ and generating trivial cofibrations $J_\Sigma = \bigcup_n F_n J$, where I and J are the generating cofibrations and generating trivial cofibrations of $\mathcal{C}at_*$. Note that the zero object in $Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$ is the spectrum $\mathbf{0}$, where $\mathbf{0}_n = \mathbf{0}$ for all $n \geq 0$.

The functor Σ extends to a functor on $Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$ called the *prolongation* of Σ. The prolongation of Σ is given by $\Sigma : Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma) \to Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$, where $(\Sigma X)_n = \Sigma X_n$ with the obvious structure maps. The functor Σ is a left Quillen functor with respect to the projective model structure. However, Σ is not a Quillen equivalence. For example, $\Sigma F(1) \simeq \mathbf{0}$, but $F(1)$ is not weakly equivalent to $\Omega\mathbf{0} \simeq \mathbf{0}$. Our goal is to define a model structure on $Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$, called the stable model structure, with respect to which Σ is a Quillen equivalence. In order to do so, we need $Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$ to be a combinatorial model category. We begin by showing that $\mathcal{C}at_*$ is combinatorial.

Proposition 4.13. *$\mathcal{C}at_*$ is a proper combinatorial simplicial model category.*

Proof. We have already seen that $\mathcal{C}at_*$ is cofibrantly generated (Proposition 4.1), proper (Proposition 4.2), and simplicial (Proposition 4.3). Note that $\mathcal{C}at$ is locally (finitely) presentable because it is equivalent to the category of models of a finite limit sketch [BW08]. In particular, this category is accessible. By Proposition 2.21, $\mathcal{C}at_*$ is accessible. Therefore $\mathcal{C}at_*$ is also locally presentable and hence combinatorial.

□

Proposition 4.14. [Hov01, 1.15] A map $i : A \to B$ in $Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$ is a (trivial) cofibration in the projective model structure if and only if the maps $A_0 \to B_0$ and $A_n \sqcup_{\Sigma A_{n-1}} \Sigma B_{n-1} \to B_n$ for $n \geq 1$ are (trivial) cofibrations in $\mathcal{C}at_*$.

Proposition 4.15. *Every object in $Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$ is cofibrant with respect to the projective model structure.*

Proof. Let $B \in \mathrm{Ob}(Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma))$. Certainly $\mathbf{0} \to B_0$ is a cofibration in $\mathcal{C}at_*$. Also, $\mathbf{0} \sqcup_{\mathbf{0}} \Sigma B_{n-1} \simeq \Sigma B_{n-1} \to B_n$ is a cofibration in $\mathcal{C}at_*$ for all $n \geq 1$ since ΣB_{n-1} has one object.

□

The above proposition together with Proposition 2.23 implies that $Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$ is left proper. We also could have used the following lemma.

Lemma 4.16. [Hov01, 1.14] With the projective model structure, $Sp^{\mathbb{N}}(\mathcal{C}, \Sigma)$ is left proper (resp. right proper, proper) if $\mathcal{C}$ is left proper (resp. right proper, proper).

Lemma 4.17. [Sch97, 2.1.5] Let $\mathcal{C}$ be a pointed proper simplicial model category which admits the small object argument. Then $Sp^{\mathbb{N}}(\mathcal{C}, \Sigma)$ is a simplicial model category.

In particular, if X is a spectrum, define $X \otimes K$ by $(X \otimes K)_n = X_n \otimes K$. The powering and enrichment over simplicial sets is also defined level-wise.

Proposition 4.18. *$Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$ is a left proper combinatorial simplicial model category.*

Proof. By Lemmas 4.16 and 4.17, since $\mathcal{C}at_*$ is proper and simplicial, so is $Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$. Also, we showed in the proof of Proposition 4.13 that $\mathcal{C}at_*$ is locally presentable. Thus there is a set of morphisms K that generate all morphisms in $\mathcal{C}at_*$ over colimits. Let $K_\Sigma = \bigcup_n F_n K$. Then K_Σ is a set, and K_Σ generates the morphisms of $Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$ over colimits because colimits in $Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$ are computed level-wise. Since $Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$ is cofibrantly generated, we conclude that $Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$ is combinatorial.

□

Now we describe Bousfield localization. Bousfield localization is a procedure that takes a model structure and produces a new one with the same cofibrations and more weak equivalences. It allows us to turn the maps of a set $\mathcal{S}$ into weak equivalences while keeping the model category axioms satisfied.

Let A be an object in a simplicial model category $\mathcal{M}$. By [Hir03, 16.1.3]), there is a functorial cosimplicial resolution of A induced by the functorial factorizations of $\mathcal{M}$. This cosimplicial resolution $A^\bullet$ is given by $A^n = QA \otimes \Delta[n]$. In general, $A^\bullet$ is a cofibrant replacement for the constant cosimplicial object $cc_* A$ in the Reedy model structure on $\mathcal{M}^\Delta$. By mapping out of this cosimplicial resolution, we get a simplicial set $\mathrm{Map}_l(A^\bullet, RX)$. The face and degeneracy maps of this simplicial set are induced by the coface and codegeneracy maps of $A^\bullet$.

Dually, there is a functorial simplicial resolution $X_\bullet$ of X, where $X_\bullet$ is a fibrant replacement of the constant simplicial object $cs_* X$ in the Reedy model category structure on $\mathcal{M}^{\Delta^{op}}$. By mapping into it, we get a simplicial set $\mathrm{Map}_r(QA, X_\bullet)$. We define the homotopy function complex

$$\mathrm{map}(A, X) := \mathrm{Map}_r(QA, RX) \cong \mathrm{Map}_l(QA, RX),$$

where these two sets are naturally isomorphic in the homotopy category of simplicial sets. In fact, $\mathrm{map}(A, X) \cong \mathrm{Map}(QA, RX)$.

Definition 4.19. [Hov01, 2.1] Let $\mathcal{S}$ be a set of maps in a simplicial model category $\mathcal{M}$.

1. An $\mathcal{S}$*-local object* of $\mathcal{M}$ is a fibrant object W such that, for every $f : A \to B$ in $\mathcal{S}$, the induced map $\mathrm{map}(B, W) \to \mathrm{map}(A, W)$ is a weak equivalence of simplicial sets.

2. An $\mathcal{S}$*-local equivalence* is a map $g : A \to B$ in $\mathcal{M}$ such that the induced map $\mathrm{map}(B, W) \to \mathrm{map}(A, W)$ is a weak equivalence of simplicial sets for all $\mathcal{S}$-local objects W.

The stable model structure on $Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$ is a Bousfield localization of the projective model structure. This process does not necessarily produce a model structure given any model category $\mathcal{M}$ and any set of maps $\mathcal{S}$, but it does when $\mathcal{M}$ is combinatorial. For the following theorem, the reader should take the universe $\mathbf{X}$ to be $\mathcal{S}et$, the category of sets. Then, for example, "$\mathbf{X}$-combinatorial" matches our definition of combinatorial.

Theorem 4.20. [Bar10, 4.7] If $\mathcal{M}$ is left proper and $\mathbf{X}$-combinatorial ($\mathbf{X}$ some universe), and $\mathcal{S}$ is an $\mathbf{X}$-small set of homotopy classes of morphisms of $\mathcal{M}$, the left Bousfield localization $L_{\mathcal{S}}\mathcal{M}$ of $\mathcal{M}$ along any set representing $\mathcal{S}$ exists and satisfies the following conditions.

- The model category $L_{\mathcal{S}}\mathcal{M}$ is left proper and $\mathbf{X}$-combinatorial.

- As a category, $L_{\mathcal{S}}\mathcal{M}$ is simply $\mathcal{M}$.

- The cofibrations of $L_{\mathcal{S}}\mathcal{M}$ are exactly those of $\mathcal{M}$.

- The fibrant objects of $L_{\mathcal{S}}\mathcal{M}$ are the fibrant $\mathcal{S}$-local objects of $\mathcal{M}$.

- The weak equivalences of $L_{\mathcal{S}}\mathcal{M}$ are the $\mathcal{S}$-local equivalences.

Definition 4.21. Define a set of maps in $Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$ by $\mathcal{S} = \{F_{n+1}\Sigma\mathcal{C} \xrightarrow{s_n^{\mathcal{C}}} F_n\mathcal{C}\}$, as $\mathcal{C}$ runs through the set of domains and codomains of the maps of I and $n \geq 0$. Here $s_n^{\mathcal{C}}$ is adjoint to the identity map of $\Sigma\mathcal{C}$. The *stable model structure* on $Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$ is the localization of the projective model structure with respect to $\mathcal{S}$. We call the $\mathcal{S}$-local weak equivalences *stable equivalences* and the $\mathcal{S}$-local fibrations *stable fibrations*.

Theorem 4.22. [Hov01, 3.8] The functor $\Sigma : Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma) \to Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$ is a Quillen equivalence with respect to the stable model structure.

We now have a model structure on our category of spectra that makes $Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$ analogous to $\mathcal{S}p$. However, it turns out that $Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$ is actually very simple homotopically.

Proposition 4.23. *Every object in $Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$ is cofibrant with respect to the stable model structure.*

Proof. By Proposition 4.15, every object in $Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$ is cofibrant with respect to the projective model structure, and cofibrations in the stable model structure are the same as in the projective model structure.

□

Proposition 4.24. *Every object in $Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$ is stably equivalent to* $\mathbf{0}$.

Proof. Certainly $\mathbf{0}$ is fibrant. Also, $\Omega^2(\mathbf{0}) \simeq \mathbf{0}$ for all $n \geq 0$. This is because Ω is a right adjoint, $\mathbf{0}$ is the limit of the empty diagram, and right adjoints preserve limits. Finally, for any (cofibrant) object $X \in \mathrm{Ob}(Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma))$, $\Sigma^2 X \simeq \mathbf{0}$ in the projective model structure, and all level-wise weak equivalences are still weak equivalences after Bousfield localization. Since Σ is a Quillen equivalence, having a weak equivalence $\Sigma^2 X \xrightarrow{\simeq} \mathbf{0}$ implies that the adjunct morphism $X \to \Omega^2(\mathbf{0}) \simeq \mathbf{0}$ is a weak equivalence.

□

Note that we could repeat this argument in any similar category in which there is some number of suspensions after which any object becomes equivalent to the zero object. This leads to the following theorem.

Theorem 4.25. *If $\mathcal{M}$ is a pointed, left proper, simplicial, cellular or combinatorial model category, and if there is $n \in \mathbb{Z}_{\geq 0}$ such that $\Sigma^n X \simeq \mathbf{0}$ for all $X \in Ob(\mathcal{M})$, then every object in $Sp^{\mathbb{N}}(\mathcal{M}, \Sigma)$ is stably equivalent to* $\mathbf{0}$.

Since $\mathcal{C}at_*$ satisfies Hypothesis 2.28 and the identity functor $\mathbb{1}$ is a homotopy functor, Goodwillie's construction applies. In particular, we expect the n^{th} layer of the Taylor tower of $\mathbb{1}_{\mathcal{C}at_*}$ to have the form

$$D_n \mathbb{1}_{\mathcal{C}at_*}(\mathcal{C}) \simeq \Omega^\infty(\partial_n \mathbb{1}_{\mathcal{C}at_*} \otimes (\Sigma^\infty \mathcal{C})^{\otimes n})_{h\Sigma_n},$$

where $\Omega^\infty = \mathrm{Ev}_0$, $\Sigma^\infty = F_0$, and the n^{th} derivative $\partial_n \mathbb{1}_{\mathcal{C}at_*}$ is an object of $Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$. Hence $\partial_n \mathbb{1}_{\mathcal{C}at_*} \simeq \mathbf{0}$ for all n, which implies that $D_n \mathbb{1}_{\mathcal{C}at_*}(\mathcal{C}) \simeq \mathbf{0}$ for all n and for all $\mathcal{C} \in \mathrm{Ob}(\mathcal{C}at_*)$. Again, we may generalize this result to any similar category.

Corollary 4.26. *In any category $\mathcal{M}$ satisfying the conditions of Theorem 4.25, the derivatives of the identity functor $\partial_* \mathbb{1}_{\mathcal{M}}$ exist and $\partial_* \mathbb{1}_{\mathcal{M}} \simeq \mathbf{0}$.*

Remark 4.27. On the surface, this example seems analogous to the example in calculus of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$. For all i, the i^{th} derivative at $a = 0$ is $f^{(i)}(0) = 0$. Hence the Taylor series of f about $a = 0$ is $T(x) = \sum_{i=0}^{\infty} \frac{f^i(0)}{i!} x^i = 0$. Thus the series converges for all x, but $T(x) = f(x)$ if and only if $x = 0$. Here $D_n \mathbb{1}_{\mathcal{C}at_*}(\mathcal{C}) \simeq \mathbf{0}$ for all $\mathcal{C} \in \text{Ob}(\mathcal{C}at_*)$, so $P_n \mathbb{1}_{\mathcal{C}at_*}(\mathcal{C}) \simeq \mathbf{0}$ for all $\mathcal{C} \in \text{Ob}(\mathcal{C}at_*)$. Therefore the inverse limit of the Taylor tower of $\mathbb{1}_{\mathcal{C}at_*}$ exists and is equivalent to $\mathbf{0}$ for all $\mathcal{C}$, and so the value of the limit of the Taylor tower is equivalent to the value of the identity functor if and only if $\mathcal{C} \simeq \mathbf{0}$. However, the analogy would be better if the only linear function $\psi : \mathbb{R} \to \mathbb{R}$ were $\psi(x) = 0$. In this case, the only possible derivatives of a function f would be $f^{(i)} = 0$, just as we determined that $\partial_* \mathbb{1}_{\mathcal{C}at_*} \simeq \mathbf{0}$ by showing that this was the only option. Furthermore, we would expect the derivatives of any functor $F : \mathcal{C}at_* \to \mathcal{C}at_*$ to be objects in $Sp^{\mathbb{N}}(\mathcal{C}at_*, \Sigma)$, and so $\partial_* F$ would also be weakly equivalent to $\mathbf{0}$. This means that no endofunctors of $\mathcal{C}at_*$ are analytic.

In short, the world of categories is very different from the usual setting of calculus.

CHAPTER V

THE CATEGORY OF DIRECTED GRAPHS

Definition 5.1. A directed graph $X = (V(X), E(X), s, t)$, where $V(X)$ is a set of vertices, $E(X)$ is a set of edges, and $s, t : E(X) \to V(X)$ are functions that specify the source and target vertices of each edge. If e is an edge with $s(e) = v$ and $t(e) = w$, we write $e : v \to w$. Denote the set of edges in X with source vertex x by $X(x, -)$.

A morphism of directed graphs $f : X \to Y$ is a pair of functions $f_V : V(X) \to V(Y)$ and $f_E : E(X) \to E(Y)$ such that $s \circ f_E = f_V \circ s$ and $t \circ f_E = f_V \circ t$.

Let $\mathcal{G}ph$ be the category of directed graphs and graph morphisms. Note that the terminal object in $\mathcal{G}ph$ is the graph with one vertex and one edge. We force this graph to be the initial object in our category as well. Thus every graph X comes with a chosen graph morphism $\mathbf{0} \to X$, which means that X has a designated looped basepoint, denoted $*$. Let $\mathcal{G}ph_*$ be the category of pointed graphs and basepoint-preserving graph morphisms.

The categorical product of two graphs $A \times B$ has vertex set $V(A) \times V(B)$ and edge set $E(A) \times E(B)$, where there is an edge $e = (e_1, e_2) : (v_1, w_1) \to (v_2, w_2)$ in $A \times B$ if $e_1 : v_1 \to v_2$ is an edge in A and $e_2 : w_1 \to w_2$ is an edge in B. The basepoint of $A \times B$ is the vertex $(*, *)$ together with the loop at this vertex. The categorical coproduct is the wedge, so $V(A \vee B) = (V(A) \sqcup V(B))/(*_A \sim *_B)$ and $E(A \vee B) = E(A) \sqcup E(B)$, where the loops at the basepoints of A and B have also been identified.

Let C_n be the unpointed graph with n vertices labeled $0, 1, \ldots, n-1$ and n edges labeled e_i for $i = 0, \ldots, n-1$, where $s(e_i) = i$ and $t(e_i) = i + 1 \bmod n$. The image of C_n in a graph is called an *n-cycle*. We denote the cycle graph with a disjoint basepoint by $C_{n,*}$.

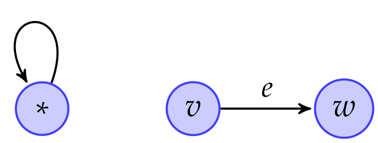
A *tree* T is an unpointed graph with a unique vertex r called the *root* of T such that there are no edges with target r and such that, for every other vertex x in T, there is a unique path in T from r to x. Let $T_{i_1}, \ldots, T_{i_k}$ be trees. Let $T_{n,k}$ be a graph obtained by taking $C_{n,*} \sqcup T_{i_1} \sqcup \ldots \sqcup T_{i_k}$ and identifying the root of T_{i_j} with the vertex v_{i_j} of $C_{n,*}$. Note that $C_{n,*}$ can also be considered as a graph of the type $T_{n,k}$, where each T_{i_j} is a tree with one vertex.

Bisson and Tsemo [BT+09] showed that there is a model structure on $\mathcal{G}ph$ in which the

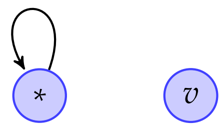
- weak equivalences are acyclic graph morphisms. A graph morphism $f : X \to Y$ is *acyclic* when f is a bijection on n-cycles for all $n \geq 1$.
- fibrations are surjectings. A graph morphism $f : X \to Y$ is a *surjecting* when the induced function $f : X(x, -) \to Y(f(x), -)$ is surjective for all $x \in V(X)$.
- cofibrations are those maps that have the left lifting property with respect to trivial fibrations.

We use the Bisson-Tsemo model structure and Theorem 2.6 to put a model structure on $\mathcal{G}ph_*$.

Let G be the following graph:



Let G' be the subgraph



Let $s : G' \hookrightarrow G$ and $i_n : \mathbf{0} \hookrightarrow C_{n,*}$ be inclusions. Let $j_n : C_{n,*} \vee C_{n,*} \to C_{n,*}$ be the coproduct graph morphism. Finally, set $K = \{i_n, j_n : n \geq 1\}$.

The following statements are based on results in [BT11]. The proofs for $\mathcal{G}ph_*$ are analogous to the proofs for $\mathcal{G}ph$.

Proposition 5.2. *1. The above model structure on $\mathcal{G}ph_*$ is cofibrantly generated with generating trivial cofibration $J = \{s\}$ and generating cofibrations $I = J \cup K$.*

2. A graph X is cofibrant if and only if X is a wedge of graphs of the form $T_{n,k}$.

3. A graph X is fibrant if and only if every vertex of X is the source of some edge.

As before, we wish to construct the suspension functor, and so we must first construct cylinder objects.

Proposition 5.3. *Given any graph A, $Cyl(A) = (A_L \vee A_R)/\sim$, where $v_L \sim v_R$ for all vertices $v \in V(A)$ and $e_L \sim e_R$ if $e \in E(A)$ is part of a cycle.*

Proof. Note that $\mathrm{Cyl}(A)$ has the same cycles as A because we have identified the two copies of each cycle in $A_L \vee A_R$. Thus $\mathrm{Cyl}(A) \xrightarrow{\simeq} A$, where the map from $\mathrm{Cyl}(A) \to A$ collapses any remaining duplicate edges. Clearly the composition $A \vee A \to \mathrm{Cyl}(A) \to A$ is the identity on each copy of A. It remains to show that the gluing map $q : A_L \vee A_R \to \mathrm{Cyl}(A)$ is a cofibration. Let $f : X \to Y$ be a trivial fibration, and consider a commutative square

$$\begin{array}{ccc} A \vee A & \xrightarrow{j} & X \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle f} \\ \mathrm{Cyl}(A) & \xrightarrow{g} & Y \end{array}$$

Suppose $e \in E(\mathrm{Cyl}(A))$ is part of a cycle C. Then $g(e)$ is part of a cycle $g(C)$ in Y, which lifts to a unique cycle C' in X since f is a trivial fibration. Define a lift h on e by $h(e) = e'$, where e' is the edge of C' satisfying $f(e') = g(e)$. Since e is part of a cycle, there are two edges (which are also parts of cycles) which q maps to e. Denote these by e_L and e_R. By the commutativity of the square, since $gq(e_L) = gq(e_R)$, we have $fj(e_L) = fj(e_R)$. Since f is a weak equivalence and $j(e_L)$ and $j(e_R)$ are each part of a cycle in X, we conclude $j(e_L) = j(e_R)$. Therefore h makes both triangles commute.

Now suppose $e \in E(\mathrm{Cyl}(A))$ is not part of a cycle. Then there is a unique edge e'' in $A \vee A$ that is also not part of a cycle satisfying $q(e'') = e$. Define a lift by $h(e) = j(e'')$. By the commutativity of the diagram, $gq(e'') = fj(e'')$. Thus $gq(e'') = fhq(e'')$, or $g(e) = fh(e)$, so $g = fh$.

□

Theorem 5.4. *For any cofibrant graph A, $\Sigma A \simeq \mathbf{0}$.*

Proof. In taking the pushout of the diagram below, we identify all the vertices of $\mathrm{Cyl}(A)$ with the unique vertex of $\mathbf{0}$ and all the edges of $\mathrm{Cyl}(A)$ with the unique edge of $\mathbf{0}$ because both of these maps are surjective.

$$\begin{array}{ccc} A \vee A & \longrightarrow & \mathrm{Cyl}(A) \\ \downarrow & & \\ \mathbf{0} & & \end{array}$$

□

Theorem 5.5. *For any fibrant graph B, $\Omega B \simeq \mathbf{0}$.*

Proof. Let B have vertex set V and edge set E. We define B^I as well as the map $B^I \to B \times B$ using induction. Let $B_0 = B$, and let $f_0 : B_0 \to B \times B$ be the diagonal map. For each vertex $v \in V$, and for each edge $e \in E \times E$ such that $s(e) = (v, v)$ and $e \notin im(f_0)$, add an edge e' and vertex v' to B_0 so that $s(e') = (v, v)$ and $t(e') = v'$. Call this new graph B_1. There is a map $f_1 : B_1 \to B \times B$ such that $f_1|_{B_0} = f_0$ and $f_1(e') = e$. Thus f_1 surjects onto the vertices of $im(f_0)$.

Suppose we have constructed $B_1, \ldots, B_n$ together with maps $f_i : B_i \to B \times B$ so that $f_i|_{B_{i-1}} = f_{i-1}$ and f_i surjects onto the vertices of $im(f_{i-1})$ for each $1 \leq i \leq n$. For each vertex $v \in V(B_n)$ and for each edge $e \in E \times E$ such that $s(e) = f_n(v)$ and $e \notin im(f_n)$, add an edge e' and vertex v' to B_n so that $s(e') = v$ and $t(e') = v'$. Call this new graph B_{n+1}. Define f_{n+1} by $f_{n+1}|_{B_n} = f_n$ and $f_{n+1}(e') = e$. Note f_{n+1} surjects onto the vertices of $im(f_n)$.

Define $B^I = \varinjlim B_i$, and define $f : B^I \to B \times B$ by $f = \varinjlim f_i$. By construction, f is a surjecting. Also, B^I is weakly equivalent to B because B^I was formed by attaching trees to B.

Recall that ΩB is the pullback of the diagram

$$\begin{array}{ccc} & & B^I \\ & & \big\downarrow f \\ \mathbf{0} & \longrightarrow & B \times B \end{array}$$

Thus the graph ΩB has vertices $V = \{(v, *) \in V(B^I) \times V(\mathbf{0}) \mid f(v) = *_{B \times B}\}$ and edges $E = \{(e, *) \in E(B^I) \times E(\mathbf{0}) \mid f(e)$ is the 1-cycle at $*_{B \times B}\}$. By construction, the subgraph of B^I that maps to the basepoint of $B \times B$ contains the basepoint of

B^I and a (possibly infinite) disjoint union of edges that are not part of a cycle. Therefore $\Omega B \simeq \mathbf{0}$.

□

One would hope that $\mathcal{G}ph_*$ is left proper and either cellular or combinatorial so that we may repeat the argument we used to construct a category of spectra for $\mathcal{C}at_*$. However, only one of these properties holds.

Proposition 5.6. *$\mathcal{G}ph_*$ is combinatorial.*

Proof. An unpointed directed graph is equivalent to a presheaf over the unpointed category with two objects and two parallel non-identity morphisms. Here the objects correspond to sets of vertices and edges, and the two morphisms specify the source and target vertex for each edge. A category of presheaves is locally presentable; indeed, another definition of a locally presentable category is as a localization of a category of presheaves [AR94]. Therefore $\mathcal{G}ph$ is locally presentable, and hence $\mathcal{G}ph_*$ is locally presentable.

□

Proposition 5.7. *$\mathcal{G}ph_*$ is not cellular.*

Proof. Consider the generating cofibration $X = C_{n,*} \vee C_{n,*} \xrightarrow{j_n} C_{n,*} = Y$. Note $Y \sqcup_X Y = Y$, so $\mathrm{eq}(Y \rightrightarrows Y \sqcup_X Y) = Y$, not X. Therefore j_n is not an effective monomorphism.

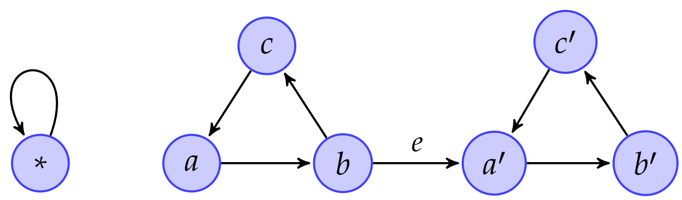
□

Proposition 5.8. *$\mathcal{G}ph_*$ is not left proper.*

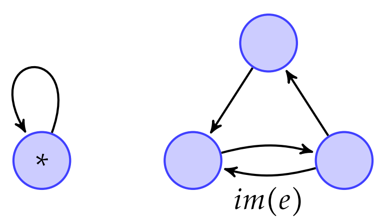
Proof. We provide an example of a weak equivalence f that is not preserved by pushout along a cofibration. Consider the generating cofibration $j_3 : C_{3,*} \vee C_{3,*} \to C_{3,*}$ and the pushout diagram

$$\begin{array}{ccc} C_{3,*} \vee C_{3,*} & \xrightarrow{f} & X \\ \downarrow{\scriptstyle j_3} & & \\ C_{3,*} & & \end{array}$$

in which X is the graph



and f is the inclusion. In the pushout, we glue together the vertices a and a', b and b', and c and c' along with the corresponding pairs of edges. However, the edge e is not in the image of f. Hence the pushout is



This graph is not weakly equivalent to $C_{3,*}$ because it contains a 2-cycle.

□

This means that the construction we used to define a category of spectra on $\mathcal{C}at_*$ does not apply to $\mathcal{G}ph_*$. It may still be the case that $\mathcal{S}p^{\mathbb{N}}(\mathcal{G}ph_*, \Sigma)$ exists. However, since $\Sigma X \simeq \mathbf{0}$ for all $X \in \mathrm{Ob}(\mathcal{G}ph_*)$, we would expect any category of spectra on $\mathcal{G}ph_*$ to be homotopically trivial.

CHAPTER VI

OTHER MODEL STRUCTURES ON CAT

The canonical model structure is not the only model structure on $\mathcal{C}at_*$ that we could have used. In this chapter, we describe an infinite family of model structures on $\mathcal{C}at$, the category of unpointed small categories, which we believe induce homotopically trivial categories of spectra on $\mathcal{C}at_*$. We obtain these model structures by transporting them from other model categories across Quillen adjunctions. Recall that by the remarks following Theorem 2.6, each model structure on $\mathcal{C}at$ determines a model structure on $\mathcal{C}at_*$.

Definition 6.1. [Hir03, 10.5.15] Let I be a set of morphisms in a model category $\mathcal{M}$. We say that *I permits the small object argument* if the domains of the maps in I are small with respect to I-cell.

Theorem 6.2. [Hir03, 11.3.2] Let $\mathcal{M}$ be a cofibrantly generated model category with generating cofibrations I and generating trivial cofibrations J. Let $\mathcal{N}$ be a category that is closed under small limits and colimits, and let $F : \mathcal{M} \leftrightarrows \mathcal{N} : U$ be a pair of adjoint functors. If we let $FI = \{Fu \mid u \in I\}$ and $FJ = \{Fv \mid v \in J\}$, and if

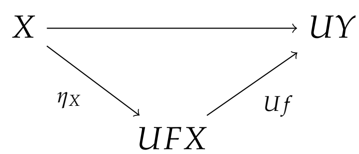
- both of the sets FI and FJ permit the small object argument, and
- U takes relative FJ-cell complexes to weak equivalences,

then there is a cofibrantly generated model category structure on $\mathcal{N}$ in which FI is a set of generating cofibrations, FJ is a set of generating trivial cofibrations, and the weak equivalences are the maps that U takes to weak equivalences in $\mathcal{M}$. Furthermore, with respect to this model category structure, (F, U) is a Quillen adjunction.

This theorem is due to Kan and is often called the transfer theorem. We note that the new model structure is not necessarily Quillen equivalent to the original model structure.

Proposition 6.3. [MP12, 16.2.3] If the pair (F, U) satisfies the hypotheses of the transfer theorem, then (F, U) is a Quillen equivalence if and only if the unit of the adjunction $\eta_X : X \to UFX$ is a weak equivalence in $\mathcal{M}$ for all cofibrant $X \in \text{Ob}(\mathcal{M})$.

Proof. $(\Leftarrow)$ Suppose that $\eta_X : X \to UFX$ is a weak equivalence for all cofibrant $X \in \text{Ob}(\mathcal{M})$. Fix a particular X, and suppose that $f : FX \to Y$ is a weak equivalence in $\mathcal{N}$. Consider the commutative diagram



Since f is a weak equivalence, Uf is a weak equivalence. Therefore the composition $Uf \circ \eta_X : X \to UY$ is a weak equivalence.

Now suppose that the adjunct morphism $X \to UY$ is a weak equivalence. By the two-out-of-three property, Uf is also a weak equivalence. By the transfer theorem, f is a weak equivalence.

$(\Rightarrow)$ Let $X \in \text{Ob}(\mathcal{M})$ be cofibrant, and suppose (F, U) is a Quillen equivalence. Let $FX \xrightarrow{R} Y$ be a fibrant replacement of FX in $\mathcal{N}$. Then

$$FX \xrightarrow{id} FX \xrightarrow{R} Y$$

is a weak equivalence, and thus the adjunct composition

is a weak equivalence. Also, UR is a weak equivalence because U preserves weak equivalences. By the two-out-of-three property, $X \xrightarrow{\simeq} UFX$.

□

Now we consider the "n-equivalence" model structures on $\mathcal{T}op$, which we define in the next paragraph. We will use these and Theorem 6.2 to put a model structure on $\mathcal{C}at_*$ for each $n \geq 1$.

Definition 6.4. [EDHP95, 2.1] A map of topological spaces $f : X \to Y$ is a *weak n-equivalence* if for all $k = 0, \ldots, n$ and for each $x \in X$, the induced map $\pi_k(f) : \pi_k(X, x) \to \pi_k(Y, f(x))$ is an isomorphism. The identity of the empty space is also a weak n-equivalence.

A map of simplicial sets $f : X \to Y$ is a *weak n-equivalence* if $|f| : |X| \to |Y|$ is a weak n-equivalence in $\mathcal{T}op$.

For all $n \geq 1$, there is a model structure on $sSet$ [EDHP95] in which

- the weak equivalences are weak n-equivalences.
- the fibrations are n-fibrations. We say $f : X \to Y$ is an *n-fibration* if f has the right lifting property with respect to $\Lambda^p_k \to \Delta[p]$ for $0 < p \leq n+1$, $0 \leq k \leq p$ and with respect to $\Lambda^{n+2}_k \to \dot{\Delta}[n+2]$, $0 \leq k \leq n+2$.
- the cofibrations are *n-cofibrations*, i.e. maps that have the left lifting property with respect to all trivial n-fibrations.

This model structure is cofibrantly generated with generating cofibrations

$$I = \{\dot{\Delta}[q] \to \Delta[q] \mid 0 \leq q \leq n+1\}$$

and generating trivial cofibrations

$$J = \{\Lambda_k^p \to \Delta[p] \mid 0 < p \leq n+1,\ 0 \leq k \leq p\} \cup \{\Lambda_k^{n+2} \to \dot{\Delta}[n+2] \mid 0 \leq k \leq n+2\}.$$

We call $sSet$ with this model structure $sSet_n$. If $n \neq m$, then $sSet_n$ and $sSet_m$ are not equivalent. In addition, each of these categories is different from $sSet$ with the Quillen model structure, in which the weak equivalences are those morphisms whose geometric realization is a weak homotopy equivalence of topological spaces, the cofibrations are level-wise injections, and the fibrations are Kan fibrations [Hir03, 7.10.8].

In [Tan13], Tanaka puts a cofibrantly generated model structure on $\mathcal{C}at$ using the pair of adjoint functors $c : sSet_1 \leftrightarrows \mathcal{C}at : N$. Here c is the categorization functor, which is defined as follows. The set of objects in cX is X_0, and morphisms in cX are freely generated by the set X_1 subject to relations given by elements of X_2, namely, $x_1 = x_2 x_0$ in cX if there exists a 2-simplex x such that $d_2 x = x_2$, $d_0 x = x_0$, and $d_1 x = x_1$. The right adjoint N is the nerve functor, which is defined by $N_n\mathcal{C} = \mathcal{C}at([n], \mathcal{C})$ and maps

$$d_i(f_1, \dots, f_n) = (f_1, \dots, f_{i-1}, f_{i+1} \circ f_i, f_{i+1}, \dots, f_n)$$

$$s_j(f_1, \dots, f_n) = (f_1, \dots, f_j, 1, f_{j+1}, \dots, f_n).$$

We remark that (c, N) are the same functors as (π, μ) from Chapter IV. We adopt different notation because we are now considering unpointed categories.

Tanaka shows that (c, N) is a Quillen equivalence. He calls $\mathcal{C}at$ with the model structure obtained using the transfer theorem and this adjunction $\mathcal{C}at_1$. This model structure is also related to the canonical model structure on $\mathcal{C}at$.

Proposition 6.5. [Tan13, 3.20] Let I be the unpointed category with two objects and one morphism between them. Let $S^\infty = c\Delta_1$ be the unpointed category with two objects and a unique isomorphism between them. The Tanaka model structure on $\mathcal{C}at$ is the left Bousfield localization of the canonical model structure on $\mathcal{C}at$ with respect to the inclusion $\phi : I \to S^\infty$.

More generally, we consider the adjunction $c : sSet_n \leftrightarrows \mathcal{C}at : N$.

Proposition 6.6. *For $n \geq 2$, $\mathcal{C}at$ with the model structure obtained by using the transfer theorem with (c, N) is not Quillen equivalent to $sSet_n$.*

Proof. Since the fibrations in $sSet_n$ have the right lifting property with respect to maps of the form $\Lambda^p_k \to \Delta[p]$ for $0 < p \leq n+1$, $0 \leq k \leq p$, in particular, the trivial fibrations of $sSet_n$ are surjective on k-simplices for all $0 \leq k \leq n+1$. Since the inclusion $\varnothing \to \dot{\Delta}[3]$ has the left lifting property with respect to all such maps, $\dot{\Delta}[3]$ is cofibrant. However, $Nc\dot{\Delta}[3] \simeq \Delta[3] \not\simeq \dot{\Delta}[3]$. Applying Proposition 6.3, we conclude that the adjunction is not a Quillen equivalence.

□

Upon reflection, it is unsurprising that $c : sSet \leftrightarrows \mathcal{C}at : N$ is only a Quillen equivalence when $n = 1$ because c only uses the information of a simplicial set up to the 2-simplices. Now we consider another pair of adjoint functors between $sSet$ and $\mathcal{C}at$.

Theorem 6.7. [Tho80] Let Sd be the subdivision endofunctor of $sSet$, and let Ex be its right adjoint. Then $\mathcal{C}at$ with the model structure obtained by using the transfer theorem and the adjunction $cSd^2 : sSet \rightleftarrows Cat : Ex^2N$ is Quillen equivalent to $sSet$ with the Quillen model structure.

Proposition 6.8. *There is a model structure on Cat where*

- *the weak equivalences are those maps f such that Ex^2Nf is a weak n-equivalence.*
- *the fibrations are those maps f such that Ex^2Nf is an n-fibration.*
- *the cofibrations have the left lifting property with respect to all trivial fibrations.*

We call Cat with this model structure Cat_n.

Proof. We must show that Cat_n satisfies the hypothesis of the transfer theorem. First, Cat_n is closed under small limits and colimits because Cat is. Next, cSd^2I and cSd^2J permit the small object argument because the domains of these sets contain finitely many objects and morphisms. Finally, we must show that Ex^2N takes relative cSd^2J-cell complexes to weak equivalences. Note that every cSd^2J-cell complex is cSd^2 applied to a relative J-cell complex. Let $f \in J$-cell, so f is a weak equivalence [Hov99, 2.1.19]. By [FL81], $Id \to Ex^2NcSd^2$ is a weak homotopy equivalence, and so $f \simeq Ex^2NcSd^2f$. Thus Ex^2NcSd^2f is a weak equivalence, i.e. Ex^2N takes relative cSd^2J-cell complexes to weak equivalences.

□

Proposition 6.9. *Cat_n is Quillen equivalent to $sSet_n$.*

Proof. By the transfer theorem, (cSd^2, Ex^2N) is a Quillen adjunction. Since $X \xrightarrow{\simeq} Ex^2NcSd^2(X)$ for all $X \in \mathrm{Ob}(sSet_n)$, in particular, this holds for all cofibrant objects of $sSet_n$. By Proposition 6.3, (cSd^2, Ex^2N) is a Quillen equivalence.

□

Let $\varnothing \neq X \in \mathrm{Ob}(\mathcal{T}op)$. Note that $\pi_k(\Sigma^n X) = 0$ for all $k < n$. Therefore in the $(n-1)$-equivalence model structure on $\mathcal{T}op$, $\Sigma^n X \simeq *$. If $\mathcal{T}op_n$ satisfies the hypotheses of Theorem 4.25, then this theorem implies that every object in $Sp^{\mathbb{N}}((\mathcal{T}op_n)_*, \Sigma)$ is stably equivalent to **0**. Hovey [Hov01, 5.5] shows that under certain conditions, a Quillen equivalence $\Phi : \mathcal{C} \to \mathcal{D}$ induces a Quillen equivalence $Sp^{\mathbb{N}(\Phi)} : Sp^{\mathbb{N}}(\mathcal{C}, \Sigma) \to Sp^{\mathbb{N}}(\mathcal{D}, \Sigma)$. In light of these results, we make the following conjecture.

Conjecture 6.10. The category of spectra $Sp^{\mathbb{N}}((\mathcal{C}at_n)_*, \Sigma)$ exists, and every object in $Sp^{\mathbb{N}}((\mathcal{C}at_n)_*, \Sigma)$ is stably equivalent to **0**.

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