A PLEA FOR PROBLEM SOLVING

A an excellent idea — hope you will pursue this with problems or a problem book say.

The professor whose name I have forgotten wrote his comment lightly in pencil I went over it in pen so it would xerox better

Marion Walter
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C-8

This paper is not a paper offering solutions to any of the many problems besieging mathematics teachers and mathematics teaching. Rather it will raise the issue of Problem Solving and merely point in a direction for investigating some of the ideas.

I have not taught in high school since 1951. Most of my judgments about students are based on the college students whom I have taught since then in New York, Ithaca and Boston.

Let me begin by stating what I mean by exercises, problem solving and Problem Solving (Capital P). All texts, the 'new' and the 'old' are full of exercises. If a student is taught quadratic equations, and is then asked to solve $x^2 + 45 = 14x$ and other equations of this type, I would call these exercises. They are an important and necessary part of the course and I am not arguing against them. If a student is asked a more difficult question on this topic, more difficult in the sense that routine application of the material just covered in the class will not enable the student to solve it--then I will call this type of question a problem (small p). It can be called a problem because the student has not been shown exactly how to solve it. An example of such a problem is:

Problem

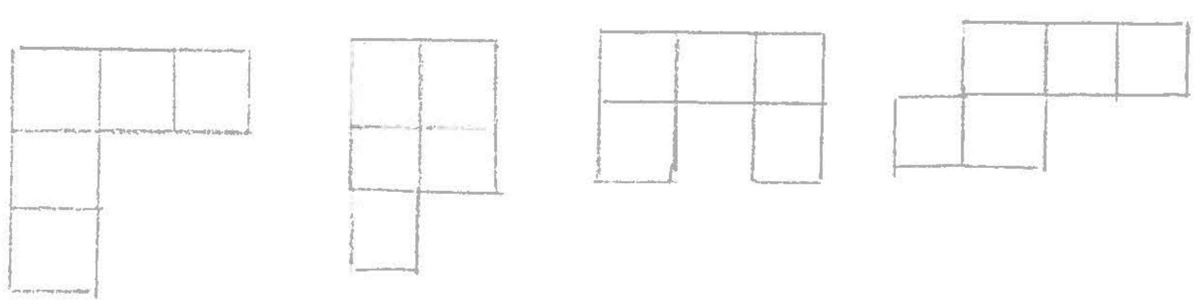
> Two students each set out to solve the same quadratic equation. The first student made a mistake in copying the constant term and got 3 and 2 as his result. The second student made a mistake in copying the coefficient of the first degree term and got 3 and -2 as his result. What is the solution of the correct equation?

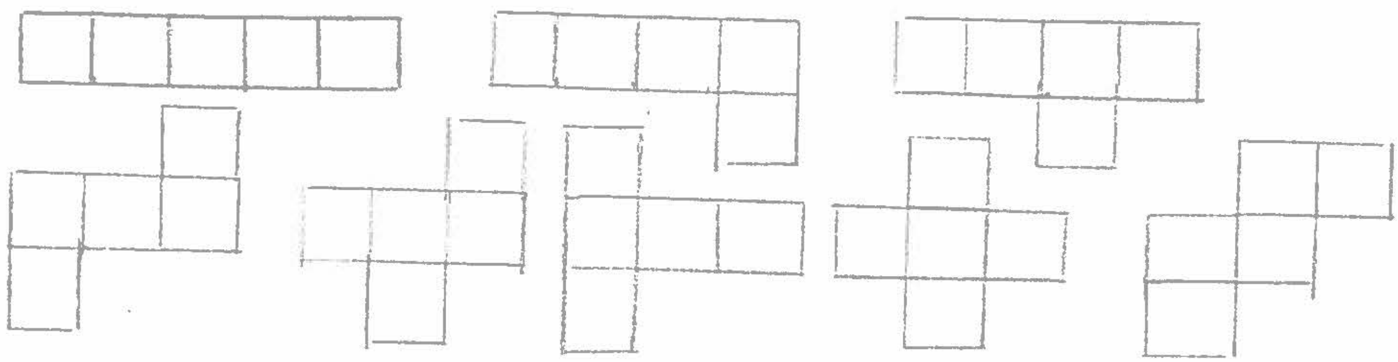
A comparable exercise would be:

Exercise Find the sum and product of the roots of $x^2 + 5x - 50 = 0$.

The above type of problem is desirable because not only does it call for some original thought (which exercises don't do), not only is the student asked to show initiative, encouraged to use his imagination; but it also serves to make the student think more deeply than the exercises do. Great progress has been made in this direction as can be seen, for example, in the S.M.S.G. and Illinois texts. Some of the very best problems of this type are given in Fisher and Ziebur, Integrated Algebra and Trigonometry. Problems of this type are usually on the material just or recently covered (except for miscellaneous review--which is a step in the right direction). The Problems (P) I am talking about are not necessarily connected with the material just covered that semester, problems that do not necessarily follow into a strict category, problems that do not necessarily have a complete solution. I am talking about Problems that do not come neatly packaged and labelled "Subject: Geometry", "Method: Locus". These Problems are not as easy to find. However here is an example of such a Problem.

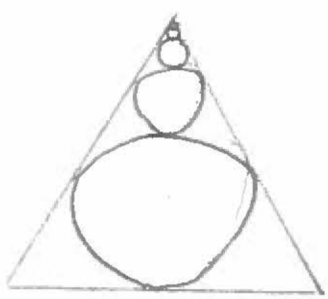
Given all possible arrangements of 5 squares





work out any theory you can about how and under what conditions they can be put into a box which measures 10 by 6 units.

Or a more well-known one:



Consider an equilateral △ and circles inscribed as shown. What is the sum of the lengths of the radii? Is this a problem in geometry? Infinite Series? Induction? [Consider it given not at the end of any of the appropriate topics].

Why do I think Problem Solving is important? There are several reasons. First of all when problems are always connected with work recently covered, the students quickly learn to only look toward these areas for solutions. Therefore the problems do not really give free rein to the students' imagination, creativeness and ingenuity. For example, if a student is asked to show $n! < (\frac{n+1}{2})^n$ at the end of the chapter on induction there isn't much joy in doing the problem. Suppose he is given the problem out of this context. He may come up with several solutions using properties of inequalities which he may have to prove, speculate and play with. Secondly, problems in real life which are mathematical in nature usually do not come labelled "Algebra", "Arithmetic" etc. More often the main and first problem is to decide what the problem is, what is wanted--and only then to look for a solution. Then one can ask, does it belong to a definite branch of mathematics? To more than one? Also how well do problems and exercises prepare students to evaluate and understand the many figures given in the Newspaper today? For the college-bound student, Problems are even more important. I have found that <u>on the whole</u>, college students are poor Problem Solvers though they may be good at problem solving. It is not that they are not bright enough--they are afraid to try a Problem if they do not right away have an idea at least of what branch of mathematics it belongs to--or in a given subject what method to use.[1]

As an example

> Given $ay + bx = c$.
>
> Find the minimum value of $\sqrt{x^2 + y^2}$.
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> Is this a problem in Calculus? Analytic Geometry? Or can it be reasoned out using plane geometry?

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[1] A rather narrow example of this is the case of Differential Equations. A student may know all the methods for solving a D.E. but will very quickly decide that she can't do the problem if she doesn't recognise the type of equation it is.

It is interesting to note that J.J. Gordon in perhaps a somewhat controversial book states "The tendency in both art and science, as well as in everyday experience, is to define the relevant in the narrowest almost legalistic sense, because superficially at least, the narrow limits offer a more comprehensive working position. Some finite part of the universe must be bitten off in order for it to be considered and examined. However, Synectics research into the creative process itself reveals those individuals (and groups) who are willing to defer the narrowing action are more imaginative and productive than individuals (or groups) who rapidly narrow the field."[2] In Problem solving, I feel we would not narrow the field for the student.

This narrowing of the point of view, can, I feel, be at times so hindering that sometimes a non-mathematician will solve a problem or puzzle before a mathematically trained person does, because while the trained person is looking somewhat blindly for the pigeon hole the other person is LOOKING at the problem.[3]

Therefore, whether students are going on with mathematics or not, this is one of the reasons I feel we should not only be teaching them subject matter but also teaching or guiding them in Problem solving. All of this applies of course also on the college level. My plea is for Problem solving at the earliest age. It is a strange

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[2] Gordon, J.J., Synectics, New York, Harper Brothers, 1961.

[3] "Look at the problem" was one of Professor Polya's favorite (and I might add most useful) exclamations in a course on "Methods of Advanced Calculus."

(ironical?) situation that at New York University, a Ph.D. candidate takes a Problem solving seminar as his <u>last</u> course (and usually puts it off as long as possible) because they find it the hardest--no wonder--this is the first time for many that they have contact with such varied problems at one time.

Let me next discuss some other advantages of Problem solving. At Simmons College over the last 6 years, I have often given Problem sheets to the students, especially during their first two years. These were sets of problems usually (not always) unrelated to the subject matter of the course. They were assigned over and above the regular work of the course and assignments. Unfortunately, I did not do this with any study on Problem solving in mind, but the following facts stand out:

1. The students tended to do the problems with great enthusiasm.
2. The students put greater effort into solving these problems than into their regular assignments.
3. There tended to be more discussion of these problems among the students and more involvement with them than with their regular work.
4. The students became less afraid to tackle problems--more daring in their approach to them.
5. They were disappointed if I gave none to a particular class.

Now, "Probably no subject in American College Curriculum is widely disliked as Mathematics"[4] and all the college students were

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[4] "Mathematics at Columbia", <u>Columbia College Today</u>, Winter, 1962-63.

after all, once elementary and high school students. The new math programs and the improved mathematics teaching has gone a long way to help erase this dislike of mathematics. I feel that Problem solving certainly, as far as my experience at Simmons is concerned, is certainly effective in stimulating interest. If someone should ask why should people be enthusiastic about mathematics, I will not mention all the reasons including aesthetic ones, but only point out the serious shortage of creative mathematicians.

Problem solving has other benefits. Under this type of Problem solving one can encourage intelligent guessing of answers or bounds to answers (not done enough at present) and encourage intuitive responses (the problem is not at the end of the chapter on trig.) "It is the intuitive mode that yields hypotheses quickly, that produces interesting combinations of ideas before their worth is known. It precedes proof, indeed it is what the techniques of analysis and proof are designed to test and check. It is founded on a kind of combinatorial playfulness that is only possible when the consequences of error are not overpowering or sinful".[5] In how many courses today are ingenious (but incorrect) answers counted as much as correct but routine solutions? Lobachevsky made a speculative assumption that one axiom did not hold. The geometry so obtained seemed at first "mere wilful play."[6] I feel much more

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[5] Bruner, Jerome, On Knowing, Cambridge, Harvard University Press, 1962.

[6] Gorden, op.cit. p. 126

encouragement should be given to original attempts--even if they are at first unsuccessful.

Also, if problems are given at the end of a chapter--or even mixed in review (where the student all too often merely recalls by rote what type the problem is) the problem is often viewed only from one direction (often recalled not viewed). "This can be a hindrance for a problem solving."[7] Duncker discovered "that fixation often interferes when the solution of a test problem requires the use of a familiar object in a novel way--fixation can be overcome and insight attained by a sudden shift in the way the problem or the objects involved in it are viewed."[8] Obviously this can be applied directly to my statements above.

From the point of view of Mathematics learned and insight gained, it is illuminating to solve a Problem by algebra, geometry or by algebra and common sense, or Calculus and Geometry and not enough of this is done at present.

Also we should make more of an attempt to give problems with insufficient or extraneous or incompatible data. (The newer texts make some effort in this direction.) Again I feel the emphasis is not sufficient.

To quote from an excellent book by W.W. Sawyer: "We ask rather is there any reason to suppose that this problem can be solved

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[7] Scheerer, Martin, "Problem-Solving," Scientific American, April, 1963, p. 121 (nice example)

[8] Scheerer, Ibid., p. 128

with the means we have at hand? Can it be broken into simpler problems? What is it that makes a problem solvable and how can we discover the nature of the problem we are dealing with?"[9] More emphasis is needed for these aspects of Problem solving than is given today.

Many suggestions on curriculum changes have been made by various responsible groups. For example, the O.E.E.C. made a resolution which read in part, "This group should work out a detailed synopsis of the entire subject matter of secondary school mathematics emphasizing the spirit in which it should be taught."[10]

The O.E.E.C. prepared such a report in 1960. Although it stated again on page 4, "the programs are offered also as suggestions to stimulate further thinking about the kinds of mathematics which should be taught in European secondary schools and the way in which this mathematics should be presented to the students", very little is said to support the belief that Problem solving (capital P) rather than exercises is of paramount importance. True, they state on p. 5 "it (the report) has attained a means to a unified approach to the entire curriculum, in which algebra, geometry, and the elements of analysis are no longer placed in separate compartments but are exhibited as having intimate and fruitful relations with one another." But I see little evidence of problems that can be so solved by such algebraic and geometric means, and comparisons of such methods. They do advocate the avoidance of "purposeless long numerical

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9 Sawyer, W.W., Prelude to Mathematics, Penguin Books, 1955, p. 2. As quoted in "Report of Commission on Math", coll. 214.

10 "New Thinking in School Mathematics", Organisation for European Economic Cooperation, Synopses for Modern Secondary School Mathematics, p.1

calculations and algebraic acrobatics...which have doubtful value as instruments for training in Mathematical thinking."[11] They even state "problems and exercises (they do not define what they mean by the different terms) must not be simple applications of the concepts taught." (I think exercises should be to a great extent).."it is necessary that they make an appeal to the student's interest, liking, and his desire to investigate, and that they develop in the students faculties of analysis and invention."[12] However, I see very little indication in the 300 page book as to how to do this except for the problems that are given and directly concerned with the subject matter. On the other hand, on page 85 they state, for some reason, only under the geometry section--and yet I feel it applies to all sections, "It is essential that the pupil learn to think creatively and intuitively. To this end, he must be given opportunities to find his own problems, to state his own solutions." This is not stated under the general aims! And again no indication is gigen as to how this is to be done. There were a few steps in the right direction. For example, p. 122, "at all times applications to all branches of knowledge and experience should be introduced.." but none are given. On p. 99 some interesting problems for discussion are stated. Even the report of the Commission on Mathematics does not give the ability to solve problems as one of their aims of mathematics in general education.[13]

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[11] Ibid. p. 9.

[12] Ibid. p. 9.

[13] Report of the Commission on Mathematics, "Program for College Preparatory Math", p. 11.

True, much of this is close to what everyone is talking about--the discovery method with which I agree but which I feel does not go far enough. There are also various things that go by this name. Polya uses it in the sense close to the one I mean. In some, the discovery is such the student couldn't possibly draw any other conclusion. (If you drop a brick on someone's foot, is he discovering it?) My plea is for all the best of discovery methods plus Problems from different fields with no routine solutions--preferably no unique way of solving the problem. I am therefore disappointed with the statement on p. 125, "To aid the student in making the abstractions that are characteristic of the algebra of the cycle, it is necessary to preserve not only a great number of examples (and counter examples)"[14]--I agree--but also exercises of the "discovery" type which develops in the student a disposition toward research. My feeling is that the exercises of this type as found in the S.M.S.G. and Illinois math while good as far as they go, are not sufficient.

What can be done about the situation? Clearly if we are to teach Problem solving, the teacher must have some of this ability. No mention is made of this in the Program for College Preparatory Math (p. 48) when discussing teacher training. I feel that the type of course that Professor Polya gives to high school teachers[15] is most valuable because it not only teaches subject matter--it teaches attitude, approach, etc. In other words, present teachers

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14 Ibid., p. 125

15 Polya, G., <u>Mathematical Discovery</u>, Vol. 1, New York, John Wiley and Sons, Inc., 1962.

must be trained. Hopefully, future teachers will be trained if this type of Problem solving is incorporated into Elementary and High Schools, + Colleges.

How can it be incorporated into the school program? There seem to be 3 possible ways.

1) Actual course taught exclusively with this emphasis.
2) Insure that part of each year or course is devoted to this.
3) Devote Math Club time and extra out of school hour time to it.

Which of these are most effective? I feel there must be experimentation to find out the optimum method. Method 3 is being used by a few schools in the U.S. Although students take PiMu Epsilon contest problems and others of this nature, this is usually not a regular activity in most high schools. An interesting comparison is drawn between the activities of this sort in Russia and in America in the April, 1963 issue of <u>The Mathematics Teacher</u>. I have not done research on which schools <u>are</u> active in the United States--but I am encouraged by the experiment described by Edwin Hirschi in the February, 1963 issue of <u>The Math Teacher</u>. Obviously, one of the first things to be done is to make a survey of activities that are going on and of their success. This brings one to the crucial question of how to measure success in this area and all the other psychological aspects which must be continued to be studied. Another large area for research is the finding of suitable Problems.

Perhaps we should also experiment with group problem solving-- where discussions could be carried on at once about relative interest of solutions. Problems that one person can begin but can't complete can be helped on by others. One would obtain many angles on a problem at once--thus teaching students to look for these on their own.

To end, let me say then that Problem solving deserves and requires more study, more experimentation and investigation so that it does not become the step child of subject matter lest we breed knowledgeable, informed, but unimaginative non-daring mathematicians.

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