CALCULATIONS OF TENSOR PRODUCTS OVER MODULAR REPRESENTATIONS IN CHARACTERISTIC P

by

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In papers by J.A. Green and R.-C. Renaud indecomposable modules of cyclic groups of prime power order are studied through the use of tensor products, as well as the characters on these modules. In this paper I will exhibit two decompositions, one for the product of any two indecomposable modules as a sum, and another for any single module as a product of basis elements. This will allow for easy computation of the character known as the Frobenius Perron dimension.

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ABSTRACT. In papers by J.A. Green and R.-C. Renaud indecomposable modules of cyclic groups of prime power order are studied through the use of tensor products, as well as the characters on these modules. In this paper I will exhibit two decompositions, one for the product of any two indecomposable modules as a sum, and another for any single module as a product of basis elements. This will allow for easy computation of the character known as the Frobenius Perron dimension.

1. BUILDING UP MOTIVATION

I want to start by discussing the setting in which tensor products are at their most intuitive and concrete: matrices. When referring to the tensor product of matrices I will be referring to how it acts by block matrices, namely:

For matrices 2 by 2 matrices A and B $A \otimes B = \left(\begin{array}{c|c} a_{11}B & a_{12}B \\ \hline a_{21}B & a_{22}B \end{array}\right)$

which is equivalent to $\left(\begin{array}{cc|cc} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{21} \\ \hline a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{21} \end{array}\right)$

Now I can start searching for interesting examples of matrices to study under the tensor product, which will lead me to group representations. In a field of prime characteristic p, we can represent the cyclic group with p elements by a Jordan block $J_{1,p-1}$. To be a little bit more concrete, we can let $p = 3$ and let $a = J_{1,2}$.

$$a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

This is a nice choice for our matrix because we can see that this matrix generates the cyclic group of order three:

$$a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, a^2 = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, a^3 = \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

.

We can now ask what would happen if we applied the tensor product to these matrices:

$$a \otimes a = \left(\begin{array}{ccc|ccc|ccc}
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)$$

$$a^2 \otimes a^2 = \left(\begin{array}{ccc|ccc|ccc}
1 & 2 & 1 & 2 & 4 & 2 & 1 & 2 & 1 \\
0 & 1 & 2 & 0 & 2 & 4 & 0 & 1 & 2 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 1 & 2 & 1 & 2 & 4 & 2 \\
0 & 0 & 0 & 0 & 1 & 2 & 0 & 2 & 4 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)$$

We could take into account the characteristic, noting that we can change every appearance of a 4 into a 1 in the last matrix. However, if we were to square $a \otimes a$ we would see that $(a \otimes a)^2 = a^2 \otimes a^2$, and furthermore, $(a \otimes a)^3 = I$. It might look at first glance as if the tensor product didn't change much about the underlying structure, and yet I will note that $a \otimes a^2 \neq a^2 \otimes a$.

While this was an interesting foray into matrices and motivations, the papers [G] and [R] together specify the multiplication operations with respect to the tensor product, allowing us to consider these objects without having to resort to dealing with their matrices. However, this is by no means the end of matrices in this work; they are of fundamental importance to most of what will follow.

2. Math introduction/Preliminaries

For a finite group G and a field K, any (K,G)-module may be regarded as isomorphic to a direct sum of indecomposable modules. This is particularly interesting for the decomposition into indecomposables of a tensor product of two modules.

This paper examines, as does [R], the case where G is a cyclic p-group and K is a field of characteristic p. We will exhibit a decomposition of any element into a product of basis elements that allows us to calculate the Frobenius Perron dimension of the element. We will then show a different decomposition that preserves a subcategory structure. Along the way, there will be numerous computations and examples of our results (I've tried to keep all of the interesting tangents to the end of the paper).

The choice of a decomposition is interesting because we want to know any interesting properties of elements V_r with $r > p$. For $0 \leq (r, s) \leq p$ the decomposition of the tensor product is given by the truncated Clebsch-Gordon rule.

$$V_i \otimes V_j = \sum_{\ell = max(i+j-k,0)}^{\min(i,j)} V_{i+j-2\ell}$$

This is listed as theorem one in [R] using the results from [G].

We have a couple of important notions to get out of the way before proceeding much further. The elements of our space we are dealing with are indecomposable modules and they will be denoted V_r for $0 < r \leq p^n$ where p^n is the order of the group we are examining. Taking notation from [G] and [R2], the algebra from which we are selecting out modules will be denoted $A_{p,\alpha}$. However, while we might have an element V_r in $A_{p,\alpha}$, [G] shows that there is a natural inclusion of $A_{p,\alpha} \subset A_{p,\alpha+1}$, allowing us to often disregard the specific algebra.

It is also worth stopping for a second to give a discussion of characters, particularly the Frobenius-Perron Dimension. A character is often meant to denote a group homomorphism $\chi : G \to F$ from the initial group to the multiplicative group of a field (most often the complex numbers). However, in this paper and in the references, a character is a ring homomorphism from our algebra to the real numbers $\chi : A_{p,\alpha} \to \mathbb{R}$. In particular, we are concerned mostly with the character $FPdim : A_{p,\alpha} \to \mathbb{R}$.

As [G] notes, for any non-trivial character ϕ we have $\phi(V_p) = 0$. From this is follows that $\phi(V_{np}) = 0$. There are also other implications for more general characters that I came across in [R2] after preparing most of this document.

In linear algebra, the Frobenius-Perron Dimension of a matrix is the largest eigenvalue. When we consider $p = 3, a = 2$, we have a matrix for V_5 as follows (how this was derived will be explained later)

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

meaning that FPdim(V_5) is the largest eigenvalue of the above matrix, which happens to be two. However, using the multiplication results can give us fruitful results: $V_5^2 = V_1 + V_5 + V_7 \implies V_5^3 = (V_1 \otimes V_5) + V_5^2 + (V_7 \otimes V_5) = V_5^2 + 2V_5$. This is great because we can move the terms to one side and then use the fact that FPdim is a character to calculate the dimension as the root of a polynomial:

$$V_5^3 - V_5^2 - 2V_5 = 0$$

$$FP(V_5^3 - V_5^2 - 2V_5) = FP(0)$$

$$FP(V_5^3) - FP(V_5^2) - FP(2V_5) = 0$$

$$FP(V_5)^3 - FP(V_5)^2 - 2FP(V_5) = 0$$

This implies that $FP(V_5)$ is either 2, 0 or -1. It can't be either of the last two because it must be positive, but it also must the largest value, so either way we have two, agreeing with the above result.

This also illustrates how useful it can be to deal directly with the character. It also shows us some nice symmetry because $V_4 = V_5 \otimes V_2 \implies$

$FP(V_4) = FP(V_2)FP(V_5) = FP(V_5)$. We know that $FP(V_2) = 1$ because $FP(V_1) = 1$: $FP(V_2)^2 = FP(V_2^2) = FP(V_1) = 1$. These kinds of computations and patterns were the impetus for the first part of the paper where I'll be specifying the FPdim of elements. These are also intimately connected to the matrices interlude near the end of the paper

Many results needed for computations in this paper come either directly from [G] or as lemmas from [R]. The versions we will need – with the summands of the form V_{np} removed, simplifying them drastically – are summarized below; references in brackets are to their papers.

3. Summary of Product Formulas

Let $q = p^\beta$ for $1 \leq \beta < \alpha$.

G-1 (cf. 2.5a). If $1 \leq (r, s) \leq q$ and $V_r \otimes V_s = V_{t_1} + ... + V_{t_b}$, then

$$V_{q-r} \otimes V_s = V_{q-t_1} + ... + V_{q-t_b}$$

Remark 3.1. *There's an instance of sloppy notation here, as indicated in the introduction. In this paper and in the references G-1 will almost always be used in A_α with V_{q-1} corresponding to the element of $A_{\alpha+1}$. In other words, having been well-established in [G] that we can move back and forth between A_α and $A_{\alpha+1}$, in A_n we can consider element $V_{(p-1)q+(p-1)}$ to be element V_{q-1} of A_{n+1} and apply the result*

G-2 (cf. 2.5b). If $1 \leq r \leq q$,

$$V_r \otimes V_{q-1} = V_{q-r}$$

G-3 (cf. 2.9c). If $q \leq r \leq pq$, with $r = r_0 q + r_1, 0 \leq r_1 < q$,

$$V_r \otimes V_{q-1} = V_{(r_0+1)q-r_1}$$

G-4 (cf. 2.8c). If $1 \leq r \leq q$,

$$V_r \otimes V_{q+1} = V_{q+r}$$

G-5 (cf. 2.8d). If $q \leq r \leq (p-1)q$, with $r = r_0 q + r_1, 0 \leq r_1 < q$,

$$V_r \otimes V_{q+1} = V_{(r_0+1)q+r_1}$$

G-6 (cf. 2.8e). If $(p-1)q \leq r \leq pq$,

$$V_r \otimes V_{q+1} = V_{r-q}$$

R-1 (cf. Lemma 2.3) For $0 \leq a < p$, with $0 \leq r < q$,

$$V_r \otimes V_{aq+1} = V_{aq+r}$$

R-2 (cf. Lemma 2.4) For $1 \leq a \leq p$, with $0 \leq r \leq q$,

$$V_r \otimes V_{aq-1} = V_{aq-r}$$

R-3 (cf. Lemma 2.6) For $0 \leq a \leq b < p$, with $a + b < p$,

$$V_{aq+1} \otimes V_{bq+1} = \sum_{i=1}^{a} (V_{(b-a+2i)q+1} + V_{(b-a+2i)q-1}) + V_{(b-a)q+1}$$

R-4 (cf. Lemma 2.9) For $0 \leq a \leq b < p$, with $a + b \leq p$,

$$V_{aq-1} \otimes V_{bq+1} = \sum_{i=1}^{a-1} (V_{(b-a+2i)q+1} + V_{(b-a+2i)q-1}) + V_{(b-a)q-1}$$

R-5 (cf. Lemma 2.10) For $0 \leq a \leq b < p$, with $a + b \geq p$,

$$V_{aq+1} \otimes V_{bq+1} = \sum_{i=1}^{p-b-1} (V_{(b-a+2i-1)q+1} + V_{(b-a+2i-1)q-1}) + V_{(b-a)q+1}$$

4. Comment on the product formulas, and the first decomposition

It may seem redundant to list G-2 and G-4 because their results are subsumed under R-1 and R-2 in the case that $a = 0$. However, the progression of formulas G-2 through G-6 shows how different methods needed to be employed to deal with the case $r > q$, resulting in considering $r = r_0 q + r_1$. This looks an awful lot like $aq + r$ in R-1, and this is the intuition behind this first decomposition:

Proposition 4.1. *For V_r in $A_{p,\alpha}$, $0 < r < p^{\alpha+1}$ and p does not divide r; there exists a decomposition of V_r into a direct sum of elements of the form $V_{r'}$ such that $r' < p$ and elements of the form V_{ap^γ} for $1 \leq a < p$ and $1 \leq \gamma \leq \alpha$*

Proof. Set $r = r_0 p^\beta + r_1$ where β is such that $p^\beta < r < p^{\beta+1}$. Apply R-4 to get

$$V_r = V_{r_0 q + r_1} = V_{a_1 q+1} \otimes V_{r_1}$$

Repeat this step on r_1 and continue iterating until $r_n < p$. This will give us

$$V_r = V_{a_1 q+1} \otimes ... \otimes V_{a_\beta q+1} \otimes V_{r_\beta}$$

Commenting further on this decomposition necessitates defining the Frobenius Perron dimension.

5. FPdim and some consequences

We now resort to one of the tricks for operations on small sets: multiplication tables. We see that in the case of $p = 5$ and $\alpha = 2$ we can construct a multiplication table giving us all of the information about products in $A_{p,\alpha}$:

$$\left(\begin{array}{c|cccccc}
\otimes & V_1 & V_2 & V_4 & V_5 & V_7 & V_8 \\
\hline
V_1 & V_1 & V_2 & V_4 & V_5 & v_7 & V_8 \\
V_2 & V_2 & V_1 & V_5 & V_4 & v_8 & V_7 \\
V_4 & V_4 & V_5 & V_1 + V_7 + V_5 & V_2 + V_8 + V_4 & V_4 & V_5 \\
V_5 & V_5 & V_4 & V_2 + V_8 + V_4 & V_1 + V_7 + V_5 & V_5 & V_4 \\
V_7 & V_7 & V_8 & V_4 & V_5 & V_1 & V_2 \\
V_8 & V_8 & V_7 & V_5 & V_4 & V_2 & V_1
\end{array}\right)$$

We can construct a new table using the relations we see above. Let each row denote the number of each element appearing in the product of V_5 with the elements on the far left column.

$$\begin{pmatrix} \otimes V_5 & \#(V_1) & \#(V_2) & \#(V_4) & \#(V_5) & \#(V_7) & \#(V_8) \\ \hline V_1 & 0 & 0 & 0 & 1 & 0 & 0 \\ V_2 & 0 & 0 & 1 & 0 & 0 & 0 \\ V_4 & 0 & 1 & 1 & 0 & 0 & 1 \\ V_5 & 1 & 0 & 0 & 1 & 1 & 0 \\ V_7 & 0 & 0 & 0 & 1 & 0 & 0 \\ V_8 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

We can then take the numerical data given in this table and store it in a much more useful way by simply considering it as a matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Since elements have matrices associated with them (different ones than were used to originally define the algebra), we can pick out distinguishing properties of the matrices. This is the goal of having characters.

Definition 5.1. *The mapping FPdim:* $A_{p,\alpha} \to \mathbb{R}$*, referred to as the Frobenius Perron dimension, takes elements of* V_r *in* $A_{p,\alpha}$ *to the maximal non-negative eigenvalue of the matrix formed by multiplication by* V_r*.*

Remark 5.2. *The matrix referenced in the definition of FPdim is usually referred to as the matrix of left multiplication by an element. Since our algebra is commutative this is justifiably dropped from our definition. However, it is important to be aware of in general.*

Because of the Frobenius-Perron theorem of linear algebra, this matrix has a non-negative eigenvalue. In fact, FPdim is the only character such that $FPdim(V_j \geq 0)$ for all j. More information on FPdim and its relation to tensor categories can be found in [EGNO]. And more information about characters on this algebra can be found in [R2].

Remark 5.3. *Regarding the matrix, we can also look at the two tables and note that the second can be obtained from the first by identifying V_5 with 1 and all other elements with 0.*

It is a known result (but also fun to discover on one's own) that for $r < p$, $FPdim(V_r) = \frac{k^j - k^{-j}}{k - k^{-j}}$, where $k = e^{\frac{\pi i}{p+1}}$. This follows directly from the truncated Clebsch-Gordan rule; see the matrix interlude for further discussion of this. Now this is great to know, but it is missing something: it does not yet tell us how to find the FPdim of all elements. However, given the decomposition derived in the last section, we can find FPdim of all elements if we can find the dimensions of all of the elements of the form V_{aq+1}. The results R-3, R-4, R-5 and G-2 are enough to specify the FPdim of V_{aq+1}. As was shown in the introduction with V_5, G-2 implies that $FPdim(V_r) = FPdim(V_{q-r})$, which is where this lemma will be applied.

Proposition 5.4. *The set of elements of the form V_{aq+1} constitute a closed set under the operation $\otimes$*

Proof. This breaks into a couple cases: For $0 \leq a \leq b < p$, with $a + b < p$, R-3 gives us

$$FP(V_{aq+1} \otimes V_{bq+1}) = FP\left(\sum_{i=1}^{a}(V_{(b-a+2i)q+1} + V_{(b-a+2i)q-1}) + V_{(b-a)q+1}\right)$$

Applying the linearity and G-2 give me further

$$= \sum_{i=1}^{a} FP(V_{(b-a+2i)q+1}) + \sum_{i=1}^{a} FP(V_{(p-b+a-2i)q+1}) + FP(V_{(b-a)q+1})$$

For $0 \leq a \leq b < p$, with $a + b \geq p$, R-5 gives us

$$FP(V_{aq+1} \otimes V_{bq+1}) = FP\left(\sum_{i=1}^{p-b-1}(V_{(b-a+2i-1)q+1} + V_{(b-a+2i-1)q-1}) + V_{(b-a)q+1}\right)$$

$$= \sum_{i=1}^{p-b-1} FP(V_{(b-a+2i-1)q+1}) + \sum_{i=1}^{p-b-1} FP(V_{(p-b+a-2i+1)q+1}) + FP(V_{(b-a)q+1})$$

These results, taken together, show that any product of elements of the for specified is a sum of eements of the same form, showing closure under the operation was was intended. □

This has come great computational implications which will be explored in the next section.

6. Examples

For some cases our matrices are prohibitively large to attempt to calculate their eigenvalues as the number of necessary computations grows proportional to $n^{2.376}$ for an efficient computer algorithm, and it grows as n^3 for how we would compute it by hand. Even exploring the case $p = 5, \alpha = 2$ is prohibitively difficult by hand, so we need to come up with some different ways around this. Let's get back to the polynomials and put a bit of our theory from the preceding sections into action by trying to find $FP(V_6)$ with $p = 5$. We can list all of the products relevant to this endeavor:

$$FP(V_6^2) = FP(V_{11} + V_{16} + V_1) = FP(V_{16}^2)$$

$$FP(V_6 \otimes V_{11}) = FP(V_{16} + V_{11} + V_6)$$

$$FP(V_6 \otimes V_{16}) = FP(V_{21} + V_6 + V_{11})$$

$$FP(V_6 \otimes V_{21}) = FP(V_{16})$$

$$FP(V_{11}^2) = FP(V_1 + V_6 + V_{11} + V_{16} + V_{21})$$

$$FP(V_{11} \otimes V_{16}) = FP(V_{11} + V_{16} + V_6)$$

$$FP(V_{11} \otimes V_{21}) = FP(V_{11})$$

$$FP(V_{16} \otimes V_{21}) = FP(V_6)$$

$$FP(V_{21} \otimes V_{21}) = FP(V_1) = 1$$

And from here we can continue to plug away at computation:

$$FP(V_6^3) = FP(V_6 \otimes V_6^2) = FP(V_6 \otimes [V_{11} + V_{16} + V_1])$$

$$= FP(V_{16} + V_{11} + V_6 + V_{21} + V_6 + V_{11} + V_6)$$

Next I can group some of these together according to the relation I found for V_6^2

$$= 2FP(V_{11}) + FP(V_{16}) + 3FP(V_6) + FP(V_{21}) + FP(V_1)$$

$$= FP(V_6^2) + 3FP(V_6) + FP(V_{11}) + FP(V_{21})$$

Now for the fourth degree case

$$FP(V_6^4) = FP(V_6 \otimes V_6^3)$$

$$= FP(V_6^3) + 3FP(V_6^2) + FP(V_1) + 2FP(V_6) + 2FP(V_{11}) + FP(V_{16}) + 2FP(V_{21})$$

I can simplify again according to the relation I found for V_6^2.

$$= FP(V_6^3) + 4FP(V_6^2) + 2FP(V_6) + FP(V_{11}) + 2FP(V_{21})$$

Proceeding in this way, we would eventually be able to find a polynomial for V_6 that would give us $FP(V_6)$. However, there clearly there must be a better way. Indeed, we could have noted from the beginning that $V_{21}^2 = V_1$ which implies that $FP(V_{21}) = 1 = FP(V_1)$. This would have given us an answer in the form of a fourth degree polynomial:

We know $FP(V_6^3) = FP(V_6^2) + 3FP(V_6) + FP(V_{11}) + FP(V_{21})$ which implies $FP(V_6^3) - FP(V_6^2) - 3FP(V_6) - FP(V_{11}) - FP(V_{21}) = 0$. Since this is equal to zero, we can add it to the right side of our fourth degree polynomial, which simplifies nicely:

$$FP(V_6^4) = 2FP(V_6^3) + 3FP(V_6^2) - FP(V_6) + FP(V_{21})$$

$$\implies FP(V_6)^4 - 2FP(V_6)^3 - 3FP(V_6)^2 + FP(V_6) - 1 = 0$$

This means that $FP(V_6)$ is the largest root of the polynomial $x^4 - 2x^3 - 3x^2 + x - 1 = 0$.

Noticing that $FP(V_{21}) = 1$ saved us quite a bit of hassle by not having to continue expanding polynomials, but there are more relations which further simplify these computations and lead to alternative methods. This is wonderful, because dealing with the polynomials by hand was time consuming and the sheer number of computations was so high it was likely to make a mistake (as I did, a couple times, while preparing these examples).

These examples have led back into theory, so let us be a little more rigorous with some of the results:

Lemma 6.1. $V_{aq+1} \otimes V_{(p-1)q+1} = V_{(p-1-a)q+1}$

I like to think of this as an analog for G-1 in this basis

Proof. By R-5, we have

$$V_{aq+1} \otimes V_{(p-1)q+1} = V_{(p-1-a)q+1} + \sum_{i=1}^{(p-1)-(p-1)} (V_{(b-a+2i-1)q\pm 1})$$

We simply have to observe that our given choice of products renders the sum empty. □

Corollary 6.2. $FP(V_{(p-1)q+1}) = 1$

Proof. If we let $a = p-1$ then we see $FP(V_{(p-1)q+1})^2 = 1$. This implies $FP(V_{(p-1)q+1}) = 1$ or $FP(V_{(p-1)q+1}) = -1$, but the latter cannot be the case by the definition of FPdim as non-negative □

Corollary 6.3. *If* $b = p-1-a$ *then* $FP(V_{aq+1}) = FP(V_{bq+1})$

Proof. By 6.1 and 6.2,

$$FP(V_{bq+1}) = FP(V_{aq+1} \otimes V_{(p-1)q+1}) = FP(V_{aq+1})$$

Getting back to examples, we can use these bits of information to find $FP(V_{11})$ with relatively minimal computation, without having to first know $FP(V_6)$:

$$FP(V_{11}^2) = FP(V_1 + V_6 + V_{11} + V_{16} + V_{21}) = 2 + 2FP(V_6) + FP(V_{11})$$

$$FP(V_{11}^3) = FP(V_{11})^2 + 2FP(V_{11}) + 2FP(V_6 + V_{11} + V_{16})$$

This last part can be simplified by subtracting out the original V_{11}^2 equation twice and the combining like terms:

$$FP(V_{11}^3) = 3FP(V_{11})^2 + 2FP(V_{11}) - 4$$

And there we have that $FP(V_{11})$ is the largest root of the polynomial $x^3 - 3x^2 + x - 4 = 0$

While this was indeed quicker and easier on the eyes, we can make it even easier. For both V_6 and V_{11} I was grouping things together in terms of powers as I went, but that is not necessary. If I do not do this V_{11} still remains relatively tame:

$$FP(V_{11}^2) = FP(V_1 + V_6 + V_{11} + V_{16} + V_{21})$$

$$FP(V_{11}^3) = FP(V_1 + 3V_6 + 5V_{11} + 3V_{16} + V_{21})$$

$$FP(V_{11}^4) = FP(5V_1 + 11V_6 + 13V_{11} + 11V_{16} + 5V_{21})$$

$$FP(V_{11}^5) = FP(13V_1 + 35V_6 + 45V_{11} + 35V_{16} + 13V_{21})$$

We have what can be thought of as a system of linear equations in five variables, so these are sufficient to specify $FP(V_{11})$ and it is the largest root of the polynomial $x^5 - 3x^4 - 2x^3 + 4x^2 = 0$. The fact that we have x^2 as our lowest power tells us that we could divide this by x^2 (corresponding to a root at $x = 0$) without changing $FP(V_{11})$. This would give us back exactly the result from above.

This is great because this matches exactly what we found in the previous part, but it's also nice because it makes explicit that dealing with these polynomials is the same as solving a system of linear equations in five variables. The important part here is that this can be done in five variables, meaning we can solve this system with a five by five matrix instead of a twenty by twenty.

Since we have the intuition that we can find these FPdim values by only considering elements of our basis in products, let us restrict ourselves to the components of the matrix involving basis elements. For example, if we have the multiplication table for $p = 5, \alpha = 2$ restricted to the elements in our basis, we have the following:

$$\begin{pmatrix} V_1 & V_6 & V_{11} & V_{16} & V_{21} \\ V_6 & V_{11} + V_{16} + V_1 & V_{11} + V_{16} + V_6 & V_{21} + V_6 + V_{11} & V_{16} \\ V_{11} & V_{11} + V_{16} + V_6 & V_1 + V_6 + V_{11} + V_{16} + V_{21} & V_{11} + V_{16} + V_6 & V_{11} \\ V_{16} & V_{21} + V_6 + V_{11} & V_{11} + V_{16} + V_6 & V_{11} + V_{16} + V_1 & V_6 \\ V_{21} & V_{16} & V_{11} & V_6 & V_1 \end{pmatrix}$$

This conveys all of the information I initially set out as product formulas. In addition, the matrix allows us to isolate matrices corresponding to each element, and from there we can find FPdim of all of the elements, but we can also see directly how this relates to the material with the polynomials:

$$Mat_{5,2}(V_{11}) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

As I noted in remark 5.3, this matrix is also the set of product relations for V_11,, which is part of the definition of FPdim. To see this is the case, we can look at it the same way we did originally with V_5 let each row be a basis vector $\{V_1, V_6, V_{11}, V_{16}, V_{21}\}$ and think of the columns (or rows if you decided to go with columns originally) as the coefficients of the basis vectors after multiplication by V_{11}. For instance, the second row is $0\ 1\ 1\ 1\ 0$, and we see that $V_6 \otimes V_{11} = 0(V_1) + 1(V_6) + 1V_{11} + 1(V_{16}) + 0(V_{21})$.

Now that we have gotten this matrix we could just calculate the eigenvalues to find FPdim, but we could also use the matrix to make the search for polynomials easier. By how we have defined the matrix, finding v_{11}^2 just requires examining the third row. Furthermore, we can find V_{11}^n by examining the third row of $(n-1)$ power of the matrix. This happens for the same reason that $FP(V_{11}^n) = (V_{11} \otimes V_{11}^{n-1})$: The nth matrix corresponds to V_{11}^n, so examining the third row gives the coefficients in the product $(V_{11} \otimes V_{11}^n)$, which can be shown by mathematical induction on n.

We could do this again with the matrix for V_6:

$$Mat_{5,2}(V_6) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

In this case the polynomials correspond to the second row because we are interested in the product of the second basis vector V_6 and a power of the matrix V_6^n.

The fact that I have been able to isolate just the elements of our basis, those of the form V_{ap^n+1}, allows me to characterize all of the elements in terms of FPdim values of its basis elements (and the remainder term V_r with $r < p$), which is why I have been referring to it as a basis. There's one further observation that makes the computation significantly easier for a general V_r:

Proposition 6.4. *For any natural numbers n and* $0 \leq a < p$, $FP(V_{ap^n+1}) = FP(V_{ap+1})$

Proof. By proposition 5.4, every V_{aq+1} is characterized in our basis by R-3 and R-5 independently of q. Thus, in their respective basis determined by q, they have the same matrix characterizing their FPdim. □

If I want to find $FP(v_{14})$ then it is simply $FP(v_{11}) \times FP(v_4)$. If I want to find $FP(v_{683})$ then we have $FP(v_{683}) = FP(v_{626}) \times FP(v_{51}) \times FP(v_6) \times FP(v_3) = FP(v_3) \times FP(v_6)^2 \times FP(v_{11})$

For any prime number p, we have reduced finding the FPdim of all elements, to a task of finding FPdim of $(2p-2)$ elements. And this gets even simpler when we note that by 6.2 at least (p-2) of these are not unique.

7. NON-(FP) PART OF THE PAPER

[G] originally justifies going from $\alpha = n$ to $\alpha = n+1$ by defining the multiplication relations for the element he denotes $w = V_{q+1} - V_{q-1}$ using arguments involving exact sequences and partitions. We also see V_{q+1} and V_{q-1} appear again in [R]'s product formulas, since we have elements of the form V_{aq+1} and V_{aq-1} in R-3 and R-4. My original intuition when attacking this problem is that I would be able to give a decomposition of elements into a sum of basis elements, like what is mentioned in 4.1, except I wanted the set to be closed under multiplication. For a while it looked like the set of elements of the form either V_{2aq+1} or V_{2aq-1} would provide that. However this does not hold true.

We began our discussion of polynomials and FPdim with the element V_5 with $p = 3, \alpha = 2$, and we found that we got a nice, terminating polynomial when we took higher powers that allowed us to calculate its FPdim. We then turned our attention to elements of the form V_{aq+1} in light of proposition 4.1 and [theorem 2], meaning that we would spend more time looking at V_4 than V_5. However, it is impossible to find a polynomial in V_4 for calculating its FPdim without invoking [theorem 2]. It was possible for V_5 because it is of the form V_{2aq-1} for $a = 1$. I unintentionally realized the application of this during the construction of this document when I originally tried to find a polynomial for V_6 with $p = 5, \alpha = 2$ before trying again with V_{11}.

Before we proceed further, there are a couple difficulties that need to be addressed. I had initialy conceived of using just elements of the form V_{2aq-1} and V_{2aq+1} to exhibit the basis for a subcategory consisting of p elements; however, this is not possible for a fixed q. In what follows I will show that I can exhibit a subcategory for a fixed q if I allow myself the 2p elements V_{aq-1} and V_{aq+1}. I will then also show that we can have a basis of the form V_{2ap^n-1} and V_{2ap^n+1} consisting of $(\alpha p - 1)$ elements as n ranges from 0 to $\alpha - 1$.

Theorem 7.1. *decomposition: The set of elements of that are either of the form V_{aq+1} or V_{aq-1} is closed under the operation $\otimes$*

Proof. We have eight different cases to consider here and it is instructive to have them listed out for explicit reference

Case 1: $V_{aq+1} \otimes V_{bq+1}$ with $0 \leq a \leq b < p$ and $a + b < p$

Case 2: $V_{aq+1} \otimes V_{bq+1}$ with $0 \leq a \leq b < p$ and $a + b \geq p$

Case 3: $V_{aq-1} \otimes V_{bq+1}$ with $0 \leq a \leq b < p$ and $a + b \leq p$

Case 4: $V_{aq+1} \otimes V_{bq-1}$ with $0 \leq a \leq b < p$ and $a + b \leq p$

Case 5: $V_{aq-1} \otimes V_{bq+1}$ with $0 \leq a \leq b < p$ and $a + b > p$

Case 6: $V_{aq+1} \otimes V_{bq-1}$ with $0 \leq a \leq b < p$ and $a + b > p$

Case 7: $V_{aq-1} \otimes V_{bq-1}$ with $0 \leq a \leq b < p$ and $a + b < p$

Case 8: $V_{aq-1} \otimes V_{bq-1}$ with $0 \leq a \leq b < p$ and $a + b \geq p$

Most of these can be solved by examining R-3, R-4, and R-5. In fact by examining the right sides of the equalities R-3, R-4, and R-5, we can see that they show the desired result for cases 1,3, and 2, respectively. Furthermore, we can apply G-1 twice to the left sides of R-3, R-4, and R-5 and to show cases 8, 7 and 6, respectively. This follows because [note the usage of q with respect to the remark after G-1] we have

$$V_r \otimes V_r = V_r \otimes V_r \otimes V_1 = V_r \otimes V_{q-1} \otimes V_r \otimes V_{q-1} = V_{q-r} \otimes V_{q-r}$$

meaning that we have directly reduced the later three cases to the former three by showing they have the same products given by R-3, R-4, R-5.

We only have left to consider cases 4 and 5, and by the same logic invoking G-1, these are equivalent for our purposes.

By R-2, for $0 < b \leq p$,

$$V_{q-1} \otimes V_{(b-1)q+1} = V_{bq-1}$$

Let $a < p + 1 - b$, then apply R-4 for $0 \leq a \leq b < p$ with $a - 1 + b < p$,

$$V_{aq+1} \otimes V_{bq-1} = V_{q-1} \otimes \left[\sum_{i=1}^{a} (V_{(b-a+2i)q\pm1}) + V_{(b-a)q+1} \right]$$

A bit of cleaning up with G-2 and G-3 yields our final result:

$$= \sum_{i=1}^{a} (V_{(b-a+2i+1)q-1} + V_{(b-a+2i-1)q+1}) + V_{(b-a)q+1}$$

□

Final matrix interlude

I have mentioned multiplication tables throughout this paper and how they can be useful as matrices, but I never explained how it was that I went about calculating them. As I hope this paper reflects, the essential points are R-1 and the truncated Clebsch-Gordon rule. When $\alpha = 1$ we can calculate multiplication tables just from the truncated Clebsch-Gordon rule, meaning that it isn't too difficult to get construct the tables and see their patterns. For instance V_3 in $A_{7,1}$ is given by

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

We could construct the full table by finding the matrices for all of the elements or by taking each $V_i \otimes V_j$. This essentially amounts to the difference between the elementary school multiplication table trick of counting in up by a certain number instead of doing each individual product. Putting them all together gives us the following:

$$\begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ v_2 & v_1 + v_3 & v_4 + v_2 & v_3 + v_5 & v_4 + v_6 & v_5 \\ v_3 & v_4 + v_2 & v_1 + v_3 + v_5 & v_2 + v_4 + v_6 & v_3 + v_5 & v_4 \\ v_4 & v_3 + v_5 & v_2 + v_4 + v_6 & v_1 + v_3 + v_5 & v_2 + v_4 & v_3 \\ v_5 & v_4 + v_6 & v_3 + v_5 & v_2 + v_4 & v_1 + v_3 & v_2 \\ v_6 & v_5 & v_4 & v_3 & v_2 & v_1 \end{pmatrix}$$

However, when $\alpha \neq 1$ we have some weirder multiplication tables that require us to employ R-1. First, note that the table for $A_{3,1}$ is as simple as can be:

$$\begin{pmatrix} v_1 & v_2 \\ v_2 & v_1 \end{pmatrix}$$

For $A_{3,2}$ we have the following, slightly more complicated table:

$$\begin{pmatrix} v_1 & v_2 & v_4 & v_5 & v_7 & v_8 \\ v_2 & v_1 & v_5 & v_4 & v_8 & v_7 \\ v_4 & v_5 & v_1+v_7+v_5 & v_2+v_8+v_4 & v_4 & v_5 \\ v_5 & v_4 & v_2+v_8+v_4 & v_1+v_7+v_5 & v_5 & v_4 \\ v_7 & v_8 & v_4 & v_5 & v_1 & v_2 \\ v_8 & v_7 & v_5 & v_4 & v_2 & v_1 \end{pmatrix}$$

The top two rows were obtained by effectively translating all of the elements in the block given by $A_{3,1}$ by multiplying by elements of the form V_{aq+1}. Then, once we have the top two rows, we also have the first two columns because our algebra is commutative, i.e. $V_i \otimes V_j = V_j \otimes V_i$. From the first two columns we can find the other columns by employing R-1 just as we did in the top row.

In [R], all of his subsequent lemmas came from the truncated Clebsch-Gordon rule and his first lemma, our formula R-1. In the same way, we constructed the multiplication table without needing any of the other product formulas because there results were implicit in the way we used R-1. This relationship, matrices informing our formulas and formulas informing our matrices, can be interesting and useful to investigate.

I would like to draw special attention to elements of the form V_{aq+1} again because we can use matrices to shed light on how we were able to find the smaller matrices we utilized in our computations of FPdim.

It becomes fruitful to employ a bit more notation here. Let $A_{p,\alpha}(V_r)$ denote the multiplication matrix associated with V_r in $A_{p,\alpha}$. Then we can see the following:

$$A_{3,2}(V_4) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

If we let $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ In general, let I_n and M_n denote the identity matrix of size n and the matrix with ones on the opposite diagonal of size n, respectively. In absence of an index I and M will denote the appropriate sizes of I_n and M_n for the space being discussed. Then we can rewrite $A_{3,2}(V_4)$ as a block matrix:

$$A_{3,2}(V_4) = \begin{pmatrix} 0 & I & 0 \\ I & M & I \\ 0 & I & 0 \end{pmatrix}$$

Remark 7.2. *We can always find a $FPdim(V_i)$ using the smallest α such that $p^\alpha > i$. In other words, the notion $FP(V_i)$ is well defined*

This becomes easy to see when we look at $A_{3,3}(V_4)$ as a block matrix:

$$A_{3,3}(V_4) = \begin{pmatrix} A_{3,2}(V_4) & 0 & 0 \\ 0 & A_{3,2}(V_4) & 0 \\ 0 & 0 & A_{3,2}(V_4) \end{pmatrix} = I_3 \otimes A_{3,2}(V_4)$$

And so we see $FP(A_{3,3}(V_4)) = FP(I_3 \otimes A_{3,2}(V_4)) = FP(A_{3,2}(V_4))$

Now, back to $A_{3,2}(V_4)$. We can see using 6.3 that $FP(V_4) = FP(V_5)$, but it is instructive to also note the similarities in their matrices. Using the same blocks that we did for $A_{3,2}(V_4)$, $A_{3,2}(V_5)$ is the following:

$$A_{3,2}(V_5) = \begin{pmatrix} 0 & M & 0 \\ M & I & M \\ 0 & M & 0 \end{pmatrix}$$

We can use this to show proposition 6.4 and to illustrate that FPdim respects the decomposition given in 4.1. As an example, consider V_6 in $A_{5,2}$. Restricted to the basis, it's given as

$$A_{5,2}(V_6) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

However, given that all of the elements in the product are of the V_{aq+1} we can find $V_6 \otimes V_{aq+r}$ by R-3, which gives us the following block matrix:

$$A_{5,2}(V_6) = \begin{pmatrix} 0 & I & 0 & 0 & 0 \\ I & 0 & I & I & 0 \\ 0 & I & I & I & 0 \\ 0 & I & I & 0 & I \\ 0 & 0 & 0 & I & 0 \end{pmatrix}$$

This is the tensor product of I_5 and the former matrix. We said we were dealing with V_6 but this also holds if we take V_{26} and find that it is the product of I_25 with its basis relations.

Also, note that we can apply $R-1$ twice to give us the matrix for V_9 in terms of the matrix for V_6. This gives us the following matrix

$$A_{5,2}(V_6) = \begin{pmatrix} 0 & M & 0 & 0 & 0 \\ M & 0 & M & M & 0 \\ 0 & M & M & M & 0 \\ 0 & M & M & 0 & M \\ 0 & 0 & 0 & M & 0 \end{pmatrix}$$

We see that we just ended up replacing I by M, but this is because V_6 is of the form V_{aq+1} and the matrix for V_1 is I, and V_9 is of the form V_{aq+4} and the matrix for V_4 is M. If we let V_r be as it is defined in proposition 4.1, we see that this says $FP(V_r) = FP(V_{a_1q+1} \otimes ... \otimes V_{a_\beta q+1} \otimes V_{r_\beta}) = FP(V_{a_1q+1}) \times ... \times FP(V_{a_\beta q+1}) \times FP(V_{r_\beta})$

The matrices V_6 and V_9 correspond to elements with the same FPdim, but if we were to use them to generate polynomials we would have different polynomials. The polynomials that we generated in earlier examples are divisible by the minimal polynomials of the matrix, which can be determined from the characteristic polynomial. The characteristic polynomial is given by the determinant: The characteristic polynomial of V_6 is given as a polynomial in λ by $det(V_6 - \lambda * I_5)$. We can show that the characteristic polynomials of these two matrices are different:

$$det \begin{pmatrix} -\lambda & 1 & 0 & 0 & 0 \\ 1 & -\lambda & 1 & 1 & 0 \\ 0 & 1 & 1-\lambda & 1 & 0 \\ 0 & 1 & 1 & -\lambda & 1 \\ 0 & 0 & 0 & 1 & -\lambda \end{pmatrix} = \lambda^5 - \lambda^4 - 5\lambda^3 + \lambda^2 + 3\lambda - 1$$

If we considered the larger matrix for V_6 instead of the one restricted to the basis, we would have the polynomial given by $(\lambda^5-\lambda^4-5\lambda^3+\lambda^2+3\lambda-1)^5$. The case for V_9 is harder because in the absence of a basis we have to consider the larger matrix (changing to the larger matrix means that what we will now use λ to denote what would previously have been referred to as λI_4)

$$det \begin{pmatrix} -\lambda & M & 0 & 0 & 0 \\ M & -\lambda & M & M & 0 \\ 0 & M & M-\lambda & M & 0 \\ 0 & M & M & -\lambda & M \\ 0 & 0 & 0 & M & -\lambda \end{pmatrix}$$

If we were to look at the minimal polynomials of V_6 and V_9, which is the polynomial formed by removing excess multiplicities of roots, we would see that the minimal polynomial of V_6 divides that of V_9 because the set of roots of V_9 includes all of the roots of V_6 and some of their negative. This is apparent if we remember that $V_9 = V_4 \otimes V_6$ and one is an eigenvalue of V_4.

So, we showed that V_9 and V_6 have different minimal polynomials despite having the same FPdim, but in the process we also showed that $A_{5,3}(V_6)$

doesn't have the same characteristic polynomial as $A_{5,2}(V_6)$, meaning that it does not make much sense to speak of the characteristic polynomial, just of the minimal polynomial which would divide any polynomial we could construct from the matrix to give us the FPdim.

As one last result, since our matrices are all symmetric (due to the commutativity of $\otimes$) and have entries of either 1 or 0, we can represent them as graphs. This was fruitful for me as a means for acquiring intuition on these problems. When investigating the FPdim of these elements, thinking of them as graphs allows us access to lemmas in graph theory that in some cases might have bounds on the possible reasonable values for our eigenvalues. Also, being able to move from a large matrix to a smaller on a closed basis can be more readily visualized with graphs because it corresponds to a component of the graph that is disconnected from the rest of the graph, while containing the essential information. There is also a notion of a tensor product on a graph (although it often goes by a variety of names other than tensor product), but it is a much less intuitive notion than on a matrix.

FURTHER DIRECTIONS

One of the first really interesting and satisfying results to come from this inquiry was noticing that when $p = 3$ all elements V_r have $FP(V_r) = 2^n$ for some n, and if we were to graph this with the indices (r) on the x-axis and the FPdim on the y-axis, we would get an interesting fractal shape reminiscent of Cantor's set. Now, this seems appealing to me from a purely aesthetic curiosity; however, it might have some implications for the original group C_3 or it might have a yet-undiscovered relation to some other area of math or application. There are so many good questions that could be asked and investigated!

[R2] is also interesting to peruse as a logical continuation about talking of characters. It elucidates some remarks made in [G] about the possible characters on A_{pa}. More specifically, it investigates the algebra of characters $A^*{}_{p,a}$ and proves some interesting properties. Among the things proven is that there is a unique non-trivial character with $\chi(V_r) \geq 0$. Thus, we could have defined FPdim by this property.

There are a lot of interesting connections here with other branches of math: [S]'s algorithm was referenced by [R], but it proceeded in a much more group-theoretic way than did [G]; there were no partitions or exact sequences, but there was reference to projective linear groups.

This study takes on a slightly more topological flavor in [BJ] because they investigate Adams operations, which (according to wikipedia) are a set of functorial ring homomorphisms from on a vector bundle V over a topological space X in K-theory. Where we have been discussing the tensor algebra of these groups, they are more interested in the exterior algebra, which is obtained by taking the quotient of the tensor algebra by elements of the form $k = v \otimes w + w \otimes v$. The exterior algebra is prominent in differential geometry, but having just taken quantum mechanics, the quotient operation defined there is akin to asking how entangled two particles are, in terms of quantum entanglement. This is a nice connection for this purpose because K-theory, referenced above as the context for Adams operations, stands for Knot theory.

Additionally, there is the possibility that we might find connections to the theory of subfactors. Subfactors are interesting because they involve, in a sense, factoring a Von Neumann algebra over a subset of itself. [J] is a great reference for interesting material. Particularly, why we care here: There are some results in the classification of subfactors in small degrees which seem very encouraging when looking at some of the FPdims we've_calculated because the smallest degress for subfactors are 1,2, $\frac{3+\sqrt{5}}{2}$, and_these all show up readily, as 2 is important to $p = 3$ and $\frac{3+\sqrt{5}}{2} = FP(V_6)$

with $p = 5$. However, it does not appear that this trend continues because the connectivity that allowed these trends to occur is violated with $_2\ P \geq 7$ because of R-3: When there are elements of the form V_{aq+1} and V_{bq+1} with $0 \leq a < p - 1$ and $0 \leq b < p1$ such that $a + b > p$ we have too much connectivity in the resulting graph to continue to resemble the relevant algebra for subfactors. All of this theory, some beautiful graphs, and a lot more can be found in [JMS]

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