REPRESENTATIONS OF KHOVANOV-LAUDA-ROUQUIER ALGEBRAS OF AFFINE LIE TYPE

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DISSERTATION ABSTRACT

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We study representations of Khovanov-Lauda-Rouquier (KLR) algebras of affine Lie type. Associated to every convex preorder on the set of positive roots is a system of cuspidal modules for the KLR algebra. For a balanced order, we study imaginary semicuspidal modules by means of 'imaginary Schur-Weyl duality'. We then generalize this theory from balanced to arbitrary convex preorders for affine ADE types. Under the assumption that the characteristic of the ground field is greater than some explicit bound, we prove that KLR algebras are properly stratified. We introduce affine zigzag algebras and prove that these are Morita equivalent to arbitrary imaginary strata if the characteristic of the ground field is greater than the bound mentioned above. Finally, working in finite or affine affine type A, we show that skew Specht modules may be defined over the KLR algebra, and real cuspidal modules associated to a balanced convex preorder are skew Specht modules for certain explicit hook shapes.

This dissertation contains previously published (unpublished) co-authored material.

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Stratifying KLR algebras of affine Lie type (joint with A. Kleshchev), 25 pages (submitted)

Affine zigzag algebras and imaginary strata for KLR algebras (joint with A. Kleshchev), 33 pages (submitted)

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For Lindsey

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CHAPTER I

INTRODUCTION

This chapter contains coauthored material, both published and unpublished. In fact, this dissertation is a compilation of four previously existing articles, three of which are joint work with Alexander Kleshchev. Chapter III has appeared as [29], which is accepted for publication. Chapter IV has appeared as [30, 28], and chapter V has appeared as [40], all of which have been submitted for publication. This introduction and Chapter II contain portions of the introductions and preliminary sections of [29, 30, 28, 40]. Chapter VI contains calculations performed in [29, 28]. Interested readers are encouraged to read and refer to the original sources rather than this dissertation.

Let $\mathfrak{g}$ be a Kac-Moody Lie algebra, with associated symmetrizable Cartan matrix $\mathtt{C}$. Let k be an arbitrary field of characteristic p. There is a certain $\mathbb{Z}$-graded associative k-algebra R_α called a *Khovanov-Lauda-Rouquier (or KLR) algebra* associated to every α in the positive root lattice. These algebras were introduced independently by Khovanov and Lauda [20] and Rouquier [42] in 2008.

The main result about KLR algebras, and the motive for their introduction, is that they categorify the upper half of the quantum group $\mathcal{U}_q(\mathfrak{g})^+$. This means that one may recover the quantum group structure from categories of representations of the KLR algebra by *decategorifying*—taking the Grothendieck group. Writing ${}_{\mathcal{A}}\mathbf{f}$ for the $\mathbb{Z}[q, q^{-1}]$-subalgebra in $\mathcal{U}_q(\mathfrak{g})^+$ generated by divided powers, we have

$$\bigoplus_\alpha [\mathrm{Proj}(R_\alpha)] \cong {}_{\mathcal{A}}\mathbf{f}, \qquad \bigoplus_\alpha [\mathrm{Rep}(R_\alpha)] \cong {}_{\mathcal{A}}\mathbf{f}^*,$$

where $\mathrm{Proj}(R_\alpha)$ (resp. $\mathrm{Rep}(R_\alpha)$) is the full subcategory of projective (resp. finite dimensional) modules in the category R_α-mod of finitely generated graded R_α-modules. The multiplication and co-multiplication structures in the Grothendieck group come, respectively, from certain induction and restriction functors:

$$\mathrm{Ind}_{\alpha,\beta}^{\alpha+\beta} : (R_\alpha \otimes R_\beta)\text{-mod} \to R_{\alpha+\beta}\text{-mod}, \qquad \mathrm{Res}_{\alpha,\beta}^{\alpha+\beta} : R_{\alpha+\beta}\text{-mod} \to (R_\alpha \otimes R_\beta)\text{-mod}.$$

It will be convenient to write $M \circ N$ for $\mathrm{Ind}_{\alpha,\beta}^{\alpha+\beta} M \boxtimes N$.

The existence of interesting morphisms between objects in R_α-mod, invisible at the level of the quantum group, yield a rich structure and make the representation theory of KLR algebras a compelling and fruitful area for research.

In this dissertation we study the representation theory of KLR algebras of untwisted affine Lie type. Though there are different ways to study this subject, the focus in this paper is to investigate the presence of a stratified structure on R_α-mod, and the so-called cuspidal modules and standard modules associated with this structure.

In [32] Kleshchev and Ram gave a classification via Lyndon words of simple representations of KLR algebras of finite type, as irreducible heads of certain standard modules. McNamara [38] generalized the Lyndon word approach to arbitrary convex orders on the positive root system in finite type. For KLR algebras of *affine* Lie type two different approaches to the theory of standard modules were proposed by Tingley and Webster [44] and Kleshchev [24]. This dissertation builds on the approach of [24], which we describe in the next section.

1.1. Cuspidal systems

Let the Cartan matrix $\mathtt{C}$ be of arbitrary *untwisted affine type*. In particular, the simple roots α_i are labeled by $i \in I = \{0, 1, \ldots, l\}$, where 0 is the affine vertex of the corresponding Dynkin diagram. We have an (affine) root system Φ and the corresponding finite root subsystem $\Phi' = \Phi \cap \mathbb{Z}\text{-}\operatorname{span}(\alpha_1, \ldots, \alpha_l)$. Denote by Φ'_+ and Φ_+ the sets of *positive* roots in Φ' and Φ, respectively. Then $\Phi_+ = \Phi_+^{\mathrm{im}} \sqcup \Phi_+^{\mathrm{re}}$, where $\Phi_+^{\mathrm{im}} = \{n\delta \mid n \in \mathbb{Z}_{>0}\}$ for the null-root δ, and

$$\Phi_+^{\mathrm{re}} = \{\beta + n\delta \mid \beta \in \Phi'_+,\ n \in \mathbb{Z}_{\geq 0}\} \sqcup \{-\beta + n\delta \mid \beta \in \Phi'_+,\ n \in \mathbb{Z}_{>0}\}.$$

As in [1], a *convex preorder* on Φ_+ is a preorder $\preceq$ such that the following three conditions hold for all $\beta, \gamma \in \Phi_+$:

$$\beta \preceq \gamma \text{ or } \gamma \preceq \beta; \tag{1.1}$$

$$\text{if } \beta \preceq \gamma \text{ and } \beta + \gamma \in \Phi_+, \text{ then } \beta \preceq \beta + \gamma \preceq \gamma; \tag{1.2}$$

$$\beta \preceq \gamma \text{ and } \gamma \preceq \beta \text{ if and only if } \beta \text{ and } \gamma \text{ are proportional.} \tag{1.3}$$

Convex preorders are known to exist. Let us fix an arbitrary convex preorder $\preceq$ on Φ_+. From (1.3) we have that $\beta \preceq \gamma$ and $\gamma \preceq \beta$ happens for $\beta \neq \gamma$ if and only if both β and γ are imaginary. We write $\beta \prec \gamma$ if $\beta \preceq \gamma$ but $\gamma \not\preceq \beta$. The following set is *totally ordered* with respect to $\preceq$:

$$\Psi := \Phi_+^{\mathrm{re}} \cup \{\delta\}. \tag{1.4}$$

It is easy to see that the set of real roots splits into two disjoint infinite sets

$$\Phi^{\mathrm{re}}_{\succ} := \{\beta \in \Phi^{\mathrm{re}}_{+} \mid \beta \succ \delta\} \text{ and } \Phi^{\mathrm{re}}_{\prec} := \{\beta \in \Phi^{\mathrm{re}}_{+} \mid \beta \prec \delta\}.$$

If μ is a partition of n we write $\mu \vdash n$ and $n = |\mu|$. By an *l-multipartition* of n, we mean a tuple $\underline{\mu} = (\mu^{(1)}, \dots, \mu^{(l)})$ of partitions such that $|\mu^{(1)}| + \cdots + |\mu^{(l)}| = n$. The set of all l-multipartitions of n is denoted by $\mathscr{P}_n$, and $\mathscr{P} := \sqcup_{n \geq 0} \mathscr{P}_n$. A *root partition of* α is a pair $(M, \underline{\mu})$, where M is a tuple $(m_\rho)_{\rho \in \Psi}$ of non-negative integers such that $\sum_{\rho \in \Psi} m_\rho \rho = \alpha$, and $\underline{\mu}$ is an l-multipartition of m_δ. It is clear that all but finitely many integers m_ρ are zero, so we can always choose a finite subset

$$\rho_1 > \cdots > \rho_s > \delta > \rho_{-t} > \cdots > \rho_{-1}$$

of Ψ such that $m_\rho = 0$ for ρ outside of this subset. Then, denoting $m_u := m_{\rho_u}$, we can write any root partition of α in the form

$$(M, \underline{\mu}) = (\rho_1^{m_1}, \dots, \rho_s^{m_s}, \underline{\mu}, \rho_{-t}^{m_{-t}}, \dots, \rho_{-1}^{m_{-1}}), \tag{1.5}$$

where all $m_u \in \mathbb{Z}_{\geq 0}$, $\underline{\mu} \in \mathscr{P}$, and

$$\sum_{u=1}^{s} m_u \rho_u + |\underline{\mu}|\delta + \sum_{u=1}^{t} m_{-u}\rho_{-u} = \alpha.$$

We write $\Pi(\alpha)$ for the set of root partitions of α. The set $\Pi(\alpha)$ has a natural partial order '$\leq$', see §III.

Let $\rho \in \Phi_+$. For $M \in R_\rho$-mod, we say that M is *semicuspidal* if $\mathrm{Res}^{\rho}_{\beta,\gamma} M \neq 0$ implies that β is a sum of positive roots less than or equal to ρ, and γ is a sum of positive roots greater than or equal to ρ. We say that M is *cuspidal* if these inequalities are strict. If ρ is imaginary and M is semicuspidal, we say that M is *imaginary*.

A *cuspidal system* (for a fixed convex preorder) is the following data:

(Cus1) A cuspidal irreducible R_ρ-module L_ρ assigned to every $\rho \in \Phi_+^{\mathrm{re}}$.

(Cus2) An irreducible imaginary $R_{n\delta}$-module $L(\underline{\mu})$ assigned to every $\underline{\mu} \in \mathscr{P}_n$. It is required that $L(\underline{\lambda}) \not\cong L(\underline{\mu})$ unless $\underline{\lambda} = \underline{\mu}$.

It is proved in [24] that (for a fixed convex preorder) cuspidal modules exist and are determined uniquely up to an isomorphism.

Given a root partition $\pi = (\rho_1^{m_1}, \dots, \rho_s^{m_s}, \underline{\mu}, \rho_{-t}^{m_{-t}}, \dots, \rho_{-1}^{m_{-1}}) \in \Pi(\alpha)$ as above, the corresponding *standard module* is:

$$\bar{\Delta}(\pi) := q^{\mathtt{sh}(\pi)} L_{\rho_1}^{\circ m_1} \circ \cdots \circ L_{\rho_s}^{\circ m_s} \circ L(\underline{\mu}) \circ L_{\rho_{-t}}^{\circ m_{-t}} \circ \cdots \circ L_{\rho_{-1}}^{m_{-1}}, \tag{1.6}$$

where $q^{\mathtt{sh}(\pi)}$ means that grading is shifted by a certain integer $\mathtt{sh}(\pi)$.

Theorem. (Cuspidal Systems) [24, Main Theorem] *For any convex preorder there exists a cuspidal system*

$$\{L_\rho \mid \rho \in \Phi_+^{\mathrm{re}}\} \cup \{L(\underline{\mu}) \mid \underline{\mu} \in \mathscr{P}\}.$$

Moreover:

(i) *For every root partition π, the standard module $\bar{\Delta}(\pi)$ has irreducible head; denote this irreducible module $L(\pi)$.*

(ii) *$\{L(\pi) \mid \pi \in \Pi(\alpha)\}$ is a complete and irredundant system of irreducible R_α-modules up to isomorphism and degree shift.*

(iii) *For every root partition* π*, we have* $L(\pi)^{\circledast} \cong L(\pi)$*.*

(iv) *For all root partitions* $\pi, \sigma \in \Pi(\alpha)$*, we have that* $[\bar{\Delta}(\pi) : L(\pi)]_q = 1$*, and* $[\bar{\Delta}(\pi) : L(\sigma)]_q \neq 0$ *implies* $\sigma \leq \pi$*.*

(v) *The induced module* $L_\rho^{\circ n}$ *is irreducible for all* $\rho \in \Phi_+^{\mathrm{re}}$ *and* $n \in \mathbb{Z}_{>0}$*.*

1.2. Imaginary Schur-Weyl duality

The above theorem gives a 'rough classification' of irreducible R_α-modules. The main problem is that we did not give a canonical definition of individual irreducible imaginary modules $L(\underline{\mu})$. We just know that the amount of such modules for $R_{n\delta}$ is equal to the number of l-multipartitions of n, and so we have labeled them by such multipartitions in an arbitrary way.

In this work we address this issue. Our approach relies on the so-called *imaginary Schur-Weyl duality.* This theory in particular allows us to construct an equivalence between an appropriate category of imaginary representations of KLR algebras and the category of representations of the classical Schur algebra.

Let us make an additional assumption that the convex preorder is *balanced*, which means that

$$\Phi_{\succ}^{\mathrm{re}} = \{\beta + n\delta \mid \beta \in \Phi'_+,\ n \in \mathbb{Z}_{\geq 0}\}. \tag{1.7}$$

This is equivalent to

$$\alpha_i \succ n\delta \succ \alpha_0 \qquad (i \in I', n \in \mathbb{Z}_{>0}). \tag{1.8}$$

Of course, we then also have $\Phi_{\prec}^{\mathrm{re}} = \{-\beta + n\delta \mid \beta \in \Phi'_+,\ n \in \mathbb{Z}_{>0}\}$. Balanced convex preorders always exist, see for example [2].

1.21. Imaginary Schur-Weyl duality

The first steps towards imaginary Schur-Weyl duality have already been made in [24]. First of all there are the *minuscule* representations—the irreducible imaginary representations which correspond to l-multipartitions of 1. There are exactly l such multipartitions, namely $\underline{\mu}(1), \dots, \underline{\mu}(l)$, where

$$\underline{\mu}(i) := (\emptyset, \dots, \emptyset, (1), \emptyset, \dots, \emptyset)$$

with the partition (1) in the ith position. For each $i = 1, \dots, l$, we have defined an irreducible R_δ-module $L_{\delta,i}$, see [24, Section 5], and set

$$L(\underline{\mu}(i)) := L_{\delta,i} \qquad (1 \le i \le l).$$

The *imaginary tensor space of color* i is the $R_{n\delta}$-module

$$M_{n,i} := L_{\delta,i}^{\circ n} \qquad (1 \le i \le l).$$

In [24, Lemma 5.7] it is proved that any composition factor of a *mixed tensor space*

$$M_{n_1,1} \circ \cdots \circ M_{n_l,l}$$

is imaginary. We call composition factors of $M_{n,i}$ *irreducible imaginary modules of color* i. The following theorem reduces the study of irreducible imaginary modules to irreducible imaginary modules of a fixed color:

Theorem. (Reduction to One Color) [24, Theorem 5.10] *Suppose that for each $n \in \mathbb{Z}_{\ge 0}$ and $i \in I'$, we have an irredundant family $\{L_i(\lambda) \mid \lambda \vdash n\}$ of irreducible* *imaginary $R_{n\delta}$-modules of color i. For a multipartition $\underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(l)}) \in \mathscr{P}_n$, define*

$$L(\underline{\lambda}) := L_1(\lambda^{(1)}) \circ \cdots \circ L_l(\lambda^{(l)}).$$

Then $\{L(\underline{\lambda}) \mid \underline{\lambda} \in \mathscr{P}_n\}$ is a complete and irredundant system of irreducible imaginary $R_{n\delta}$-modules. In particular, the given modules $\{L_i(\lambda) \mid \lambda \vdash n\}$ give all the irreducible imaginary modules of color i up to isomorphism.

In view of this theorem, we need to construct irreducible imaginary $R_{n\delta}$-modules $L_i(\lambda)$ of color i. We will now fix i and drop the index i from our notation. We must describe the composition factors of the imaginary tensor space $M_n = M_{n,i}$ and show that they are naturally labeled by the partitions λ of n.

The $R_{n\delta}$-module structure on the imaginary tensor space M_n yields an algebra homomorphism $R_{n\delta} \to \mathrm{End}_F(M_n)$. Define the *imaginary Schur algebra* $\mathscr{S}_n$ as the image of $R_{n\delta}$ under this homomorphism. In other words,

$$\mathscr{S}_n = R_{n\delta}/\mathrm{Ann}_{R_{n\delta}}(M_n).$$

Modules over $R_{n\delta}$ which factor through to $\mathscr{S}_n$ will be called *imaginary modules* (of color i). Thus the *category of imaginary $R_{n\delta}$-modules* is the same as the category of $\mathscr{S}_n$-modules.

It is clear that M_n and its composition factors are imaginary modules. Conversely, any irreducible $\mathscr{S}_n$-module appears as a composition factor of M_n. So our new notion of an imaginary module fits with the old notion of an irreducible imaginary module in the sense of cuspidal systems.

The first major result of this dissertation is:

Theorem 1. (Imaginary Schur-Weyl Duality)

(i) *M_n is a projective $\mathscr{S}_n$-module.*

(ii) *The endomorphism algebra* $\mathrm{End}_{R_{n\delta}}(M_n) = \mathrm{End}_{\mathscr{S}_n}(M_n)$ *of the imaginary tensor space M_n is isomorphic to the group algebra $F\mathfrak{S}_n$ of the symmetric group $\mathfrak{S}_n$ (concentrated in degree zero). Thus M_n can be considered as a right $F\mathfrak{S}_n$-module.*

(iii) $\mathrm{End}_{F\mathfrak{S}_n}(M_n) = \mathscr{S}_n$.

Parts (i) and (ii) of Theorem 3 are Theorem 3.44, and part (iii) is Theorem 3.101(ii).

In view of Theorem 1, we have an exact functor

$$\gamma_n : \mathscr{S}_n\text{-mod} \to F\mathfrak{S}_n\text{-mod}, \quad V \mapsto \mathrm{Hom}_{\mathscr{S}_n}(M_n, V). \tag{1.9}$$

Unfortunately, γ_n is not an equivalence of categories, unless the characteristic of the ground field is zero or greater than n, since in general the $\mathscr{S}_n$-module M_n is not a projective generator. In order to resolve this problem, we need to upgrade from imaginary Schur-Weyl duality to imaginary Howe duality.

1.22. Imaginary Howe duality

Let $\mathtt{x}_n := \sum_{g \in \mathfrak{S}_n} g$. In view of Theorem 1, M_n is a right $F\mathfrak{S}_n$-module. Define the *imaginary divided* and *exterior* powers respectively as follows:

$$Z_n := \{m \in M_n \mid mg - \mathtt{sgn}(g)m = 0 \text{ for all } g \in \mathfrak{S}_n\},$$

$$\Lambda_n := M_n \mathtt{x}_n.$$

For $h \in \mathbb{Z}_{>0}$, denote by $X(h,n)$ the set of all compositions of n with h parts:

$$X(h,n) := \{(n_1, \dots, n_h) \in \mathbb{Z}^h_{\geq 0} \mid n_1 + \cdots + n_h = n\}.$$

The corresponding set of partitions is

$$X_+(h,n) := \{(n_1, \dots, n_h) \in X(h,n) \mid n_1 \geq \cdots \geq n_h\}.$$

For a composition $\nu = (n_1, \dots, n_h) \in X(h,n)$, we define the functors of *imaginary induction* and *imaginary restriction* as

$$I^n_\nu := \mathrm{Ind}_{n_1\delta,\dots,n_h\delta} : R_{n_1\delta,\dots,n_h\delta}\text{-mod} \to R_{n\delta}\text{-mod}$$

and

$${}^*I^n_\nu := \mathrm{Res}_{n_1\delta,\dots,n_h\delta} : R_{n\delta}\text{-mod} \to R_{n_1\delta,\dots,n_h\delta}\text{-mod}\,.$$

These functors 'respect' the categories of imaginary representations. For example, given imaginary $R_{n_b\delta}$-modules V_b for $b = 1, \dots, h$, the module $I^n_\nu(V_1 \boxtimes \cdots \boxtimes V_h)$ is also imaginary. Define

$$Z^\nu := I^n_\nu(Z_{n_1} \boxtimes \cdots \boxtimes Z_{n_h}),$$
$$\Lambda^\nu := I^n_\nu(\Lambda_{n_1} \boxtimes \cdots \boxtimes \Lambda_{n_h}).$$

Now, let $S_{h,n}$ be the classical Schur algebra, whose representations are the same as the degree n polynomial representations of the general linear group $GL_h(F)$, see [12]. In particular, it is a finite dimensional quasi-hereditary algebra with irreducible, standard, costandard, and indecomposable tilting modules

$$L_h(\lambda),\ \Delta_h(\lambda),\ \nabla_h(\lambda),\ T_h(\lambda) \qquad (\lambda \in X_+(h,n)).$$

Theorem 2. (Imaginary Howe Duality)

(i) *For each $\nu \in X(h,n)$ the $\mathscr{S}_n$-module Z^ν is projective. Moreover, for any $h \geq n$, we have that $Z := \bigoplus_{\nu \in X(h,n)} Z^\nu$ is a projective generator for $\mathscr{S}_n$.*

(ii) *The endomorphism algebra $\mathrm{End}_{\mathscr{S}_n}(Z)$ is isomorphic to the classical Schur algebra $S_{h,n}$. Thus Z can be considered as a right $S_{h,n}$-module.*

(iii) $\mathrm{End}_{S_{h,n}}(Z) = \mathscr{S}_n$.

Part (i) of Theorem 4 is Theorem 3.74(iii), part (ii) is Theorem 3.66, while part (iii) follows from (i) and (ii) and general Morita theory.

1.23. Morita equivalence

Theorem 2 allows us to plug in Morita theory to define mutually inverse equivalences of categories

$$\alpha_{h,n} : \mathscr{S}_n\text{-mod} \to S_{h,n}\text{-mod}, \quad V \mapsto \mathrm{Hom}_{\mathscr{S}_n}(Z, V) \tag{1.10}$$

$$\beta_{h,n} : S_{h,n}\text{-mod} \to \mathscr{S}_n\text{-mod}, \quad W \mapsto Z \otimes_{S_{h,n}} W. \tag{1.11}$$

Denoting by $f_{h,n}$ the usual Schur functor, as for example in [12, §6], by definitions we then have a commutative triangle (up to isomorphism of functors):

$$\begin{tikzcd}
 & S_{h,n}\text{-mod} \arrow[dl, "f_{h,n}"'] \arrow[dr, shift left, "\beta_{h,n}"] & \\
F\mathfrak{S}_n\text{-mod} & & \mathscr{S}_n\text{-mod} \arrow[ll, "\gamma_n"'] \arrow[ul, shift left, "\alpha_{h,n}"]
\end{tikzcd}$$

Let $\lambda \in X_+(n,n)$ and $h \geq n$. We can also consider λ as an element of $X_+(h,n)$. Define the graded $\mathscr{S}_n$-modules (hence, by inflation, also graded $R_{n\delta}$-modules):

$$L(\lambda) := \beta_{h,n}(L_h(\lambda)), \tag{1.12}$$

$$\Delta(\lambda) := \beta_{h,n}(\Delta_h(\lambda)), \tag{1.13}$$

$$\nabla(\lambda) := \beta_{h,n}(\nabla_h(\lambda)), \tag{1.14}$$

$$T(\lambda) := \beta_{h,n}(T_h(\lambda)). \tag{1.15}$$

Theorem 3. (Imaginary Schur Algebra) *The imaginary Schur algebra $\mathscr{S}_n$ is a finite dimensional quasi-hereditary algebra with irreducible, standard, costandard, and indecomposable tilting modules*

$$L(\lambda),\ \Delta(\lambda),\ \nabla(\lambda),\ T(\lambda) \qquad (\lambda \in X_+(h,n)).$$

We also study an imaginary analogue of Ringel duality, certain *Gelfand-Graev* fragments of the graded character of imaginary representations, and an imaginary analogue of the Jacobi-Trudi formula.

1.3. Stratifying KLR algebras of affine ADE types

Restricting attention to KLR algebras R_α of untwisted affine ADE types, we are able to generalize much of the theory of the previous section from balanced to arbitrary convex preorders. In particular, we obtain an analogue of the imaginary Howe duality theory in complete generality. To do this, we take advantage of two added ingredients: the recent work of McNamara [39], which gives the desired result in characteristic zero, and reduction modulo p.

1.31. Stratifying KLR algebras

Under the assumption $p = 0$, it is proved in [39] that R_α is properly stratified. Informally, this means that the category R_α-mod of finitely generated graded R_α-modules is stratified by the categories B_ξ-mod for much simpler algebras B_ξ. Our goal then is to apply reduction modulo p arguments to generalize this result to the case where p is greater than some explicit bound, related to the bound appearing in James' Conjecture.

We define the *semicuspidal algebra* $C_{n\alpha}$ so that the category of finitely generated semicuspidal $R_{n\alpha}$-modules is equivalent to $C_{n\alpha}$-mod. Projective indecomposable modules in $C_{n\alpha}$-mod are used to define standard modules for R_θ. We show that our definitions, which use parabolic induction of semicuspidal representations, agree with a general categorical definition of standard modules. We then verify the flatness condition in the definition of properly stratified algebras. To verify the standard filtration condition we need a certain Ext result, following McNamara's argument in [39]. With this theorem in hand, a standard argument gives:

Theorem 4. *Let $\alpha = \sum_{i\in I} n_i\alpha_i \in Q_+$ and assume that $p > \min\{n_i \mid i \in I\}$. For any convex preorder on Φ_+, the algebra R_α is properly stratified.*

1.32. Affine zigzag algebras and imaginary strata

Description of the algebras B_ξ in the previous section are easily reduced to the semicuspidal cases, which split into real and imaginary subcases. In the real case we have $B_{n\alpha} \cong k[z_1, \dots, z_n]^{\mathfrak{S}_n}$, the algebra of symmetric polynomials in n variables, but the imaginary case $B_{n\delta}$ is not so easy to understand.

Working with a balanced convex order, we prove that $B_\delta \cong k[z] \otimes \mathsf{A}$, where A is the zigzag algebra of [14] corresponding to the underlying finite Dynkin diagram Γ'

obtained by deleting the affine node from Γ, and $k[z]$ is the polynomial algebra. In order to describe the higher imaginary strata, we introduce the rank n *affine zigzag algebra* $\mathsf{A}_n^{\mathrm{aff}}$, which is defined for any connected graph without loops. We show that $B_{n\delta}$ is (graded) Morita equivalent to the affine zigzag algebra $\mathsf{A}_n^{\mathrm{aff}}$ corresponding to Γ' if $p > \min\{n_i \mid i \in I\}$ (or $p = 0$).

Denoting

$$\Delta_{n\delta} := \bigoplus_{\underline{\lambda} \in \mathscr{P}_n} \Delta(\underline{\lambda}) \quad \text{and} \quad B_{n\delta} := \mathrm{End}_{R_{n\delta}}(\Delta_{n\delta})^{\mathrm{op}},$$

we have that $B_{n\delta}$ is the basic algebra Morita equivalent to $C_{n\delta}$. It turns out that the parabolically induced module $\Delta_\delta^{\circ n}$, which can be considered as a $C_{n\delta}$-module, is always projective in the category $C_{n\delta}$-mod. However, it is a projective generator in $C_{n\delta}$-mod if and only if $p > n$ or $p = 0$. So under these assumptions, the endomorphism algebra of $\Delta_\delta^{\circ n}$ is Morita equivalent to $C_{n\delta}$ and $B_{n\delta}$. Otherwise, it is Morita equivalent to their idempotent truncations. The following result is proved under no restrictions on p. In fact, it holds over an arbitrary commutative unital ground ring k.

Theorem 5. *Assume that the convex preorder on Φ_+ is balanced. Then we have an isomorphism of graded algebras*

$$\mathrm{End}_{R_{n\delta}}(\Delta_\delta^{\circ n})^{\mathrm{op}} \cong \mathsf{A}_n^{\mathrm{aff}},$$

where $\mathsf{A}_n^{\mathrm{aff}}$ is the affine zigzag algebra of type Γ'. In particular, $B_\delta \cong k[z] \otimes \mathsf{A}$.

1.4. Skew Specht modules and real cuspidal modules

Restricting attention to affine type A, we examine a connection between classical representation theory of symmetric groups and the cuspidal system theory of KLR algebras.

1.41. Skew Specht modules

Let $\mathcal{O}$ be a commutative ring with identity, and let $\mathfrak{S}_d$ be the symmetric group on d letters. To every partition λ of d, or equivalently, every Young diagram with d nodes, there is an associated $\mathcal{O}\mathfrak{S}_d$-module $S^\lambda_{\mathcal{O}}$ called a *Specht module*, which has $\mathcal{O}$-basis in correspondence with standard λ-tableaux. Over the complex numbers, the group algebra of $\mathfrak{S}_d$ is semisimple, and it is well known that $\{S^\lambda_{\mathbb{Z}} \mid \lambda \vdash d\}$ is a complete set of *irreducible* representations. For $k \leq d$, we consider $\mathfrak{S}_k$ a subgroup of $\mathfrak{S}_d$ with respect to the first k letters, and denote the copy of $\mathfrak{S}_k$ embedded in $\mathfrak{S}_d$ with respect to the last k letters as $\mathfrak{S}'_k$. For $\lambda \vdash d$ and $\mu \vdash k$,

$$S^{\lambda/\mu}_{\mathbb{Z}} := \mathrm{Hom}_{\mathfrak{S}_k}(S^\mu_{\mathbb{Z}}, \mathrm{Res}_{\mathfrak{S}_k} S^\lambda_{\mathbb{Z}}) \tag{1.16}$$

is a $\mathbb{Z}\mathfrak{S}'_{d-k}$-module. In fact, $S^{\lambda/\mu}_{\mathbb{Z}} \neq 0$ if and only if the Young diagram for μ is contained in that of λ, so going forward we assume that is the case. The set of nodes λ/μ in the complement is called a *skew diagram*, and $S^{\lambda/\mu}_{\mathbb{Z}}$ is called a *skew Specht module*. As a $\mathbb{Z}$-vector space, $S^{\lambda/\mu}_{\mathbb{Z}}$ has basis in correspondence with standard λ/μ-tableaux, and there is an analogue of Young's orthogonal form for skew Specht modules. When $\mathcal{O} = \mathbb{F}$ is a field of positive characteristic, semisimplicity fails, but skew Specht modules still arise as subquotients of restrictions of Specht modules to Young subgroups.

More generally, to an l-multipartition $\boldsymbol{\lambda}$, one may associate a Specht module $S^{\boldsymbol{\lambda}}$ over a cyclotomic Hecke algebra of level l, of which the group algebra of $\mathfrak{S}_d$ is a special (level one) case. Brundan and Kleshchev [6] showed that over an arbitrary field such algebras are isomorphic to a certain cyclotomic quotient R_d^Λ of the Khovanov-Lauda-Rouquier (KLR) algebra $R_d = \bigoplus_{\mathrm{ht}(\alpha)=d} R_\alpha$. In [27], Kleshchev, Mathas and Ram gave a presentation for $S^{\boldsymbol{\lambda}}$ over R_α, in terms of a 'highest weight' generator $v^{\boldsymbol{\lambda}}$ and relations which include a homogeneous version of the classical Garnir relations for Specht modules.

In this work we define graded skew Specht modules over R_α by extending, in the most obvious way, the presentation of [27] to skew diagrams $\boldsymbol{\lambda}/\boldsymbol{\mu}$. We prove that this yields a graded R_α-module $S^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ with homogeneous basis in correspondence with standard $\boldsymbol{\lambda}/\boldsymbol{\mu}$-tableaux. We show that for $\boldsymbol{\lambda}$ of *content* $\beta+\alpha$, the $R_\beta \otimes R_\alpha$-module $\mathrm{Res}_{\beta,\alpha} S^{\boldsymbol{\lambda}}$ has an explicit graded filtration with subquotients of the form $S^{\boldsymbol{\mu}} \boxtimes S^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$.

1.42. Real cuspidal modules

Our motivation for constructing graded skew Specht modules arose from the study of real cuspidal modules over KLR algebras of affine type $\mathtt{A}$. We prove the following theorem for every balanced convex preorder $\succeq$.

Theorem 6. *For a real root $\alpha \in \Phi_+$, the irreducible cuspidal module L_α is isomorphic to a skew Specht module $S^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$, where $\boldsymbol{\lambda}/\boldsymbol{\mu}$ is an explicit skew hook diagram (dependent on $\succeq$).*

This gives a presentation for cuspidal modules, along with a description of the graded character which can be read off from the skew hook diagram. This result can be seen as an affine analogue of a result by Kleshchev and Ram [32, §8.4], which showed that in *finite* type $\mathtt{A}$, the cuspidal modules are Specht modules associated to certain hook partitions.

1.5. Overview

In Chapter II we present preliminary definitions and results which will be required throughout the dissertation. Preliminaries which are specific to a given chapter will appear in that section. In Chapter III we prove imaginary Schur-Weyl duality. In Chapter IV we generalize the results of III to arbitrary convex orders in affine $\mathtt{ADE}$ types, present proof of the stratification result, and describe the imaginary strata. In Chapter V we define skew Specht modules in type $\mathtt{A}$, and use these to describe real cuspidal modules. Technical calculations and results needed to prove results in Chapters III and IV are relegated to Chapter VI.

CHAPTER II

PRELIMINARIES

This chapter contains sections of the the articles [29, 30, 28, 40]. The papers [29, 30, 28] were co-authored with Alexander Kleshchev. We developed the results in the co-authored material jointly over many meetings, and, by the nature of collaborative mathematical work, it is difficult to attribute exact portions of the co-authored material to either Kleshchev or myself individually.

2.1. Ground rings

Throughout the paper, F is a field of arbitrary characteristic $p \geq 0$. We also often work over a ring $\mathcal{O}$, which is assumed to be either $\mathbb{Z}$ or F. Denote the ring of Laurent polynomials in the indeterminate q by $\mathscr{A} := \mathbb{Z}[q, q^{-1}]$. We use quantum integers

$$[n]_q := (q^n - q^{-n})/(q - q^{-1}) \in \mathscr{A} \qquad (n \in Z),$$

and the quantum factorials $[n]_q^! := [1]_q[2]_q \dots [n]_q$.

2.2. Symmetric groups and Schur algebras

In this section we review the standard facts and combinatorics related to symmetric groups and Schur algebras.

2.21. Partitions and compositions

We denote by $X(h, n)$ the set of all compositions of n with h parts (some of which could be zero), $X_+(h, n)$ the set of al partitions of n with at most h parts, and

$X_+(n) = X_+(n,n)$ the set of partitions of n. Sometimes we write $\lambda \vdash n$ to indicate that $\lambda \in X_+(n)$ and $\lambda \vDash n$ to indicate that $\lambda \in X(n,n)$. The standard *dominance order* on $X(h,n)$ is denoted by "$\leq$".

We will use the special elements $\varepsilon_1, \dots, \varepsilon_h \in X(h,1)$, where

$$\varepsilon_m = (0, \dots, 0, 1, 0, \dots, 0)$$

with 1 in the mth position. For a composition $\mu \vDash n$ we denote by $\mu^+ \vdash n$ the unique partition obtained from μ by a permutation of its parts. For $\lambda \vdash n$, we have its *transpose* partition $\lambda^{\mathrm{tr}} \vdash n$.

If $p > 0$, then $\lambda \in X_+(h,n)$ is *p-restricted* if $\lambda_r - \lambda_{r+1} < p$ for all $r = 1, 2, \dots, h-1$. A *p-adic expansion* of λ is some (non-unique) way of writing $\lambda = \lambda(0) + p\lambda(1) + p^2\lambda(2) + \dots$ such that each $\lambda(i) \in X_+(h, n(i))$ is p-restricted. This can be applied to a partition $\lambda \vdash n$ considered as an element of $X_+(n,n)$, in which case the nth part $\lambda_n \leq 1$, and so the p-adic expansion is unique.

2.22. Coset representatives

Let $\lambda = (\lambda_1, \dots, \lambda_a) \vDash n$, and let $\mathfrak{S}_\lambda$ be the corresponding standard parabolic subgroup of $\mathfrak{S}_n$, i.e. $\mathfrak{S}_\lambda$ is the row stabilizer in $\mathfrak{S}_n$ of the *row leading tableau* $\mathtt{T}^\lambda$ obtained by allocating the numbers $1, \dots, n$ into the boxes of λ from left to right in each row starting from the first row and going down. The *column leading tableau* $\mathtt{T}_\lambda$ obtained by allocating the numbers $1, \dots, n$ into the boxes of λ from top to bottom in each column starting from the first column and going to the right. Denote

$$\mathtt{x}_\lambda = \sum_{w \in \mathfrak{S}_\lambda} w, \quad \mathtt{y}_\lambda = \sum_{w \in \mathfrak{S}_\lambda} \mathtt{sgn}(w) w \qquad (\lambda \vDash n),$$

where $\mathtt{sgn}(w)$ is the sign of the permutation $w \in \mathfrak{S}_n$. Sometimes we also use the notation $\mathtt{sgn}_{\mathfrak{S}_n}$ and $\mathtt{sgn}_{\mathfrak{S}_\lambda}$ to denote the sign representations of the corresponding groups.

Let $x \in \mathfrak{S}_n$. If $x\mathfrak{S}_\lambda x^{-1}$ is a standard parabolic subgroup, say $\mathfrak{S}_\mu$ for some composition μ, we write $\mu =: x\lambda$ and say that *x permutes the parts of λ*, i.e. in that case we have

$$x\mathfrak{S}_\lambda x^{-1} = \mathfrak{S}_{x\lambda}.$$

We recall some standard facts on minimal length coset representatives in symmetric groups, see e.g. [8, Section 1]. For $\lambda \vDash n$, denote by $\mathcal{D}_n^\lambda$ (resp. ${}^\lambda\mathcal{D}_n$) the set of the minimal length left (resp. right) coset representatives of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$. Note that the *permutation module*

$$\mathrm{Per}^\lambda := \mathrm{ind}_{\mathcal{O}\mathfrak{S}_\lambda}^{\mathcal{O}\mathfrak{S}_n} \mathtt{triv}_{\mathfrak{S}_\lambda} \simeq \mathcal{O}\mathfrak{S}_n \mathtt{x}_\lambda$$

has an $\mathcal{O}$-basis $\{g \otimes 1 \mid g \in \mathcal{D}_n^\lambda\}$, and similarly for the *signed permutation module*

$$\mathrm{SPer}^\lambda := \mathrm{ind}_{\mathcal{O}\mathfrak{S}_\lambda}^{\mathcal{O}\mathfrak{S}_n} \mathtt{sgn}_{\mathfrak{S}_\lambda} \simeq \mathcal{O}\mathfrak{S}_n \mathtt{y}_\lambda \simeq \mathrm{Per}^\lambda \otimes \mathtt{sgn}_{\mathfrak{S}_n}.$$

More generally, if $\nu \vDash n$ and λ is a refinement of ν, denote $\mathcal{D}_\nu^\lambda := \mathcal{D}_n^\lambda \cap \mathfrak{S}_\nu$ and ${}^\lambda\mathcal{D}_\nu := {}^\lambda\mathcal{D}_n \cap \mathfrak{S}_\nu$. Then $\mathcal{D}_\nu^\lambda$ (resp. ${}^\lambda\mathcal{D}_\nu$) is set of the minimal length left (resp. right) coset representatives of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_\nu$. Moreover,

$$\mathcal{D}_n^\lambda = \{xy \mid x \in \mathcal{D}_n^\nu,\ y \in \mathcal{D}_\nu^\lambda\} \text{ and } {}^\lambda\mathcal{D}_n = \{yx \mid x \in {}^\nu\mathcal{D}_n,\ y \in {}^\lambda\mathcal{D}_\nu\}. \tag{2.1}$$

For two compositions $\lambda, \mu \vDash n$ set ${}^\lambda\mathcal{D}_n^\mu := \mathcal{D}_n^\mu \cap {}^\lambda\mathcal{D}_n$. Then ${}^\lambda\mathcal{D}_n^\mu$ is the set of the minimal length $(\mathfrak{S}_\lambda, \mathfrak{S}_\mu)$-double coset representatives in $\mathfrak{S}_n$. If $x \in {}^\lambda\mathcal{D}_n^\mu$, then

$\mathfrak{S}_\lambda \cap x\mathfrak{S}_\mu x^{-1}$ is a standard parabolic in $\mathfrak{S}_n$. This standard parabolic corresponds to certain composition of n, which we denote $\lambda \cap x\mu$. Similarly, $x^{-1}\mathfrak{S}_\lambda x \cap \mathfrak{S}_\mu$ is the standard parabolic corresponding to a composotion $x^{-1}\lambda \cap \mu$. Thus:

$$\mathfrak{S}_\lambda \cap x\mathfrak{S}_\mu x^{-1} = \mathfrak{S}_{\lambda\cap x\mu}, \quad x^{-1}\mathfrak{S}_\lambda x \cap \mathfrak{S}_\mu = \mathfrak{S}_{x^{-1}\lambda\cap\mu} \qquad (x \in {}^\lambda\mathcal{D}_n^\mu). \tag{2.2}$$

Moreover, x permutes the parts of $x^{-1}\lambda \cap \mu$, and $x(x^{-1}\lambda \cap \mu) = \lambda \cap x\mu$, so

$$x\mathfrak{S}_{x^{-1}\lambda\cap\mu}x^{-1} = \mathfrak{S}_{\lambda\cap x\mu}.$$

For $\lambda \vdash n$ define u_λ to be the unique element of ${}^{\lambda^{\mathrm{tr}}}\mathcal{D}^\lambda$ such that $\mathfrak{S}_{\lambda^{\mathrm{tr}}} \cap u_\lambda \mathfrak{S}_\lambda u_\lambda^{-1} = \{1\}$; in other words, u_λ is defined from $u_\lambda \mathtt{T}^\lambda = \mathtt{T}_\lambda$.

Lemma 2.3. [8, Lemma 4.1] If $\lambda \vdash n$, then $\mathtt{y}_{\lambda^{\mathrm{tr}}}\mathcal{O}\mathfrak{S}_n\mathtt{x}_\lambda$ is an $\mathcal{O}$-free $\mathcal{O}$-module of rank one, generated by the element $\mathtt{y}_{\lambda^{\mathrm{tr}}}u_\lambda\mathtt{x}_\lambda$.

2.23. Schur algebras

The necessary information on Schur algebras is conveniently gathered in [5, Section 1]. We recall only some most often needed facts for reader's convenience. The *Schur algebra* $S_{h,n} = S_{h,n,\mathcal{O}}$ is defined to the endomorphism algebra

$$S_{h,n} := \mathrm{End}_{\mathcal{O}\mathfrak{S}_n}\Big(\bigoplus_{\nu\in X(h,n)} \mathrm{Per}^\nu\Big),$$

writing endomorphisms commuting with the left action of $\mathcal{O}\mathfrak{S}_n$ on the right.

Let $\lambda, \mu \in X(h,n)$ and $u \in \mathfrak{S}_n$. The right multiplication in $\mathcal{O}\mathfrak{S}_n$ by

$$g^u_{\mu,\lambda} := \sum_{w\in\mathfrak{S}_\mu u\,\mathfrak{S}_\lambda \cap {}^\mu\mathscr{D}} w \tag{2.4}$$

induces a well-defined homomorphism of left $\mathcal{O}\mathfrak{S}_n$-modules

$$\varphi^u_{\mu,\lambda} : \mathrm{Per}^\mu \to \mathrm{Per}^\lambda .$$

Extending $\varphi^u_{\mu,\lambda}$ to all of $\bigoplus_{\nu \in X(h,n)} \mathrm{Per}^\nu$ by letting it act as zero on Per^ν for $\nu \neq \mu$, we obtain a well-defined element

$$\varphi^u_{\mu,\lambda} \in S_{h,n}. \tag{2.5}$$

Lemma 2.6. $S_{h,n}$ is $\mathcal{O}$-free with basis $\{\varphi^u_{\mu,\lambda} \mid \mu, \lambda \in X(h,n),\ u \in {}^\mu\mathcal{D}^\lambda\}$.

Lemma 2.7. For $h \geq n$, the $\mathcal{O}$-linear map $\kappa : F\mathfrak{S}_n \to S_{h,n}$, defined on a basis element $w \in \mathfrak{S}_n$ by $\kappa(w) := \varphi^w_{(1^n),(1^n)}$, is a (unital) ring embedding.

One can also define the Schur algebra using the signed permutation modules. So consider instead the algebra

$$\mathrm{End}_{\mathcal{O}\mathfrak{S}_n}\Big(\bigoplus_{\nu \in X(h,n)} \mathrm{SPer}^\nu\Big).$$

For $\lambda, \mu \in X(h,n)$ and $u \in \mathfrak{S}_n$ set

$$s^u_{\mu,\lambda} := \sum_{w \in \mathfrak{S}_\mu u\, \mathfrak{S}_\lambda \cap {}^\mu\mathscr{D}} \mathtt{sgn}(w) w. \tag{2.8}$$

Lemma 2.9. The algebras $S_{h,n}$ and $\mathrm{End}_{\mathcal{O}\mathfrak{S}_n}\Big(\bigoplus_{\nu \in X(h,n)} \mathrm{SPer}^\nu\Big)$ are isomorphic, the natural basis element $\varphi^u_{\mu,\lambda}$ of $S_{h,n}$ corresponding under the isomorphism to the endomorphism which is zero on SPer^ν for $\nu \neq \mu$ and sends SPer^μ into SPer^λ via the homomorphism induced by right multiplication in $\mathcal{O}\mathfrak{S}_n$ by $s^u_{\mu,\lambda}$.

2.24. Representation theory of Schur algebras

We recall some facts about the representation theory of $S_{h,n}$, assuming now that $\mathcal{O} = F$. All the results gathered here are explained in detail and properly referenced in [5, Section 1]. First of all, it is known that the elements

$$e(\mu) := \varphi^1_{\mu,\mu} \in S_{h,n} \qquad (\mu \in X(h,n)) \tag{2.10}$$

are idempotents. We have a *weight space decomposition* for $W \in S_{h,n}$-mod:

$$W = \bigoplus_{\mu \in X(h,n)} e(\mu)W.$$

The subspaces $e(\mu)W$ are the *weight spaces of* W.

The irreducible $S_{h,n}$-modules are parametrized by the elements of $X_+(h,n)$. We write $L_h(\lambda)$ for the irreducible $S_{h,n}$-module corresponding to $\lambda \in X_+(h,n)$. In particular, $L_h(\lambda)$ has highest weight λ, i.e. $e(\lambda)L_h(\lambda) \neq 0$ and $e(\mu)L_h(\lambda) = 0$ for all $\mu \in X(h,n)$ with $\mu \not\leq \lambda$. It is known that $S_{h,n}$ is a quasi-hereditary algebra with weight poset $(X_+(h,n), \leq)$. In particular, we have associated to $\lambda \in X_+(h,n)$ the standard and costandard modules $\Delta_h(\lambda)$ and $\nabla_h(\lambda)$ such that $\Delta_h(\lambda)$ (resp. $\nabla_h(\lambda)$) has simple head (resp. socle) isomorphic to $L_h(\lambda)$, and all other composition factors are of the form $L_h(\mu)$ with $\mu < \lambda$.

For $\lambda \in X_+(h,n)$ and $\nu \in X(h,n)$, denote by $k_{\lambda,\nu}$ the dimension of the ν-weight space of $L_h(\lambda)$:

$$k_{\lambda,\nu} := \dim e(\nu)L_h(\lambda). \tag{2.11}$$

In particular, if $\operatorname{char} F = 0$, then $L_h(\lambda) = \Delta_h(\lambda)$ and so it is well-known that $k_{\lambda,\nu} = K_{\lambda,\nu}$, where

$$K_{\lambda,\nu} := \sharp\{\text{semistandard } \lambda\text{-tableaux of type } \nu\}, \tag{2.12}$$

also known as the (λ, ν)-Kostka number. To give the necessary definitions, we consider λ as a partition, and so we can speak of the corresponding Young diagram. A λ-*tableau* is an allocation of numbers from the set $\{1, \dots, h\}$ (possibly with repetitions) into the boxes of the Young diagram λ. A λ-tableau is of type ν if each $1 \le k \le n$ appears in it exactly ν_k times. A λ-tableau is *column strict* if its entries increase down the columns. A λ-tableau is *row weak* if its entries weakly increase from left to right along the rows. A λ-tableau is *semistandard* if it is row weak and column strict.

The algebra $S_{h,n}$ possesses an anti-automorphism τ defined on the standard basis elements by $\tau(\varphi^u_{\mu,\lambda}) = \varphi^{u^{-1}}_{\lambda,\mu}$. Using this, we define the contravariant dual M^τ of an $S_{h,n}$-module M to be the dual vector space M^* with action defined by $(s \cdot f)(m) = f(\tau(s)m)$ for all $s \in S_{h,n}, m \in M, f \in M^*$. We have $L_h(\lambda)^\tau \simeq L_h(\lambda)$ and $\Delta_h(\lambda)^\tau \simeq \nabla_h(\lambda)$ for all $\lambda \in X_+(h,n)$.

Given a left $S_{h,n}$-module M, we write $\tilde{M}$ for the right $S_{h,n}$-module equal to M as a vector space with right action defined by $ms = \tau(s)m$ for $m \in M, s \in S_{h,n}$. This gives us modules $\tilde{L}_h(\lambda), \tilde{\Delta}_h(\lambda)$ and $\tilde{\nabla}_h(\lambda)$ for each $\lambda \in X_+(h,n)$.

Lemma 2.13. $S_{h,n}$ has a filtration as an $(S_{h,n}, S_{h,n})$-bimodule with factors isomorphic to $\Delta_h(\lambda) \otimes \tilde{\Delta}_h(\lambda)$, each appearing once for each $\lambda \in X_+(h,n)$ and ordered in any way refining the dominance order on partitions so that factors corresponding to more dominant λ appear lower in the filtration.

We have an algebra map $S_{h,n+l} \to S_{h,n} \otimes S_{h,l}$, which enables us to view the tensor product $M \otimes M'$ of an $S_{h,n}$-module M and an $S_{h,l}$-module M' as an $S_{h,n+l}$-module. Let $V_h = L_h((1))$ be the *natural module*. The nth tensor power $V_h^{\otimes n}$ can be regarded as an $S_{h,n}$-module. We also have the *symmetric*, *divided* and *exterior powers*: $S^n(V_h) = \nabla_h((n))$, $Z^n(V_h) = \Delta_h((n))$, $\Lambda^n(V_h) = L_h((1^n))$. More generally, given $\nu = (n_1, \ldots, n_a) \in X(h,n)$, define

$$S^\nu(V_h) \quad := S^{n_1}(V_h) \otimes \cdots \otimes S^{n_a}(V_h), \tag{2.14}$$

$$Z^\nu(V_h) \quad := Z^{n_1}(V_h) \otimes \cdots \otimes Z^{n_a}(V_h), \tag{2.15}$$

$$\Lambda^\nu(V_h) \quad := \Lambda^{n_1}(V_h) \otimes \cdots \otimes \Lambda^{n_a}(V_h). \tag{2.16}$$

all of which can be regarded as $S_{h,n}$-modules.

Lemma 2.17. For $\nu \in X(h,n)$ we have:

(i) the left ideal $S_{h,n}e(\nu)$ of $S_{h,n}$ is isomorphic to $Z^\nu(V_h)$ as an $S_{h,n}$-module;

(ii) providing $h \geq n$, the left ideal $S_{h,n}\kappa(\mathrm{y}_\nu)$ of $S_{h,n}$ is isomorphic to $\Lambda^\nu(V_h)$ as an $S_{h,n}$-module, where $\kappa : F\mathfrak{S}_n(V) \to S_{h,n}$ is the embedding of Lemma 2.7.

A finite dimensional $S_{h,n}$-module M has a *standard* (resp. *costandard*) filtration if M has a filtration $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_r = M$ such that each factor M_i/M_{i-1} is isomorphic to a direct sum of copies of $\Delta_h(\lambda)$ (resp. $\nabla_h(\lambda)$) for some fixed $\lambda \in X_+(h,n)$ (depending on i).

Lemma 2.18. If M, M' are modules with standard (resp. costandard) filtrations then so is $M \otimes M'$.

In particular, Lemma 2.18 implies that for any $\nu \in X(h,n)$, the modules $S^\nu(V_h)$, and $\Lambda^\nu(V_h)$ have costandard filtrations, while $Z^\nu(V_h)$ and $\Lambda^\nu(V_h)$ have standard filtrations.

Lemma 2.19. Let $h \geq n$ and $\lambda \in X_+(h,n)$, we have that the homomorphism space $\mathrm{Hom}_{S_{h,n}}(Z^\lambda(V_h), \Lambda^{\lambda^{\mathrm{tr}}}(V_h))$ is one-dimensional, and the image of any non-zero such homomorphism is isomorphic to $\Delta_h(\lambda)$.

If $p > 0$, then for and $h, n, r \in \mathbb{Z}_{\geq 0}$, there is a *Frobenius homomorphism*

$$F_r : S_{h,np^r} \to S_{h,n},$$

twisting with which one gets the *Frobenius twist* functor

$$S_{h,n}\text{-mod} \to S_{h,np^r}\text{-mod}, \ M \mapsto M^{[r]}.$$

For example $L(\lambda)^{[r]} \simeq L(p^r\lambda)$. The Steinberg tensor product theorem is:

Lemma 2.20. Suppose that $\lambda \in X_+(h,n)$ has p-adic expansion $\lambda = \lambda(0) + p\lambda(1) + p^2\lambda(2) + \dots$. Then, $L_h(\lambda) \simeq L_h(\lambda(0)) \otimes L_h(\lambda(1))^{[1]} \otimes L_h(\lambda(2))^{[2]} \otimes \dots$.

2.25. Induction and restriction for Schur algebras

For a composition $\chi = (h_1, \dots, h_a) \vDash h$ there is a natural *Levi Schur subalgebra*

$$S_{\chi,n} \simeq \bigoplus_{n_1+\dots+n_a=n} S_{h_1,n_1} \otimes \dots \otimes S_{h_a,n_a} \subseteq S_{h,n}, \tag{2.21}$$

and the usual restriction, and induction functors:

$$\mathrm{res}^{S_{h,n}}_{S_{\chi,n}} : S_{h,n}\text{-mod} \to S_{\chi,n}\text{-mod}, \quad \mathrm{ind}^{S_{h,n}}_{S_{\chi,n}} : S_{\chi,n}\text{-mod} \to S_{h,n}\text{-mod}\,.$$

Moreover, fix $l \le h$ and embed $X(l,k)$ into $X(h,k)$ in the natural way. Let e be the idempotent

$$e = e_{h,l} := \sum_{\mu \in X(l,k)} e(\mu) \in S_{h,n}. \tag{2.22}$$

Lemma 2.23. We have $S_{l,n} \simeq eS_{h,n}e$.

Then, we have the Schur functor

$$\mathrm{trun}^{S_{h,n}}_{S_{l,n}} : S_{h,n}\text{-mod} \to S_{l,n}\text{-mod},\ M \mapsto eM \tag{2.24}$$

and its left adjoint

$$\mathrm{infl}^{S_{h,n}}_{S_{l,n}} : S_{l,n}\text{-mod} \to S_{h,n}\text{-mod},\ N \mapsto S_{h,n}e \otimes_{eS_{h,n}e} N. \tag{2.25}$$

Lemma 2.26. If $n \le l \le h$, then the functors $\mathrm{trun}^{S_{h,n}}_{S_{l,n}}$ and $\mathrm{infl}^{S_{h,n}}_{S_{l,n}}$ are mutually quasi-inverse equivalences of categories.

Lemma 2.27. Let $l < h$ and $\mu = (\mu_1, \dots, \mu_h) \in X_+(h,n)$.

(i) If $\mu_{l+1} \neq 0$ then $\mathrm{trun}^{S_{h,n}}_{S_{l,n}} L_h(\mu) = \mathrm{trun}^{S_{h,n}}_{S_{l,n}} \Delta_h(\mu) = \mathrm{trun}^{S_{h,n}}_{S_{l,n}} \nabla_h(\mu) = 0$.

(ii) If $\mu_{l+1} = 0$, we may regard μ as an element of $X_+(l,n)$, and then we have $\mathrm{trun}^{S_{h,n}}_{S_{l,n}} L_h(\mu) \simeq L_l(\mu)$, $\mathrm{trun}^{S_{h,n}}_{S_{l,n}} \Delta_h(\mu) \simeq \Delta_l(\mu)$, and $\mathrm{trun}^{S_{h,n}}_{S_{l,n}} \nabla_h(\mu) \simeq \nabla_l(\mu)$.

Lemma 2.28. If $\chi = (h_1, \dots, h_a) \vDash h$ and $\nu = (n_1, \dots, n_a) \vDash n$, with $h_r \ge n_r$ for all $r = 1, \dots, a$, then

$$\mathrm{ind}^{S_{h,n}}_{S_{\chi,n}}(? \boxtimes \cdots \boxtimes ?) \qquad \text{and} \qquad (\mathrm{infl}^{S_{h,n_1}}_{S_{h_1,n_1}} ?) \otimes \cdots \otimes (\mathrm{infl}^{S_{h,n_a}}_{S_{h_a,n_a}} ?)$$

are isomorphic functors from S_{h_1,n_1}-mod $\times \cdots \times S_{h_a,n_a}$-mod to $S_{h,n}$-mod.

2.26. Schur functors

Here we review the material of [5, Section 3.1] for future references. Let $\mathscr{S}$ be a finite dimensional F-algebra, $P \in \mathscr{S}$-mod be a projective module, and $H = \mathrm{End}_{\mathscr{S}}(P)$, writing endomorphisms commuting with the left $\mathscr{S}$-action on the right. Define the functors:

$$\alpha := \mathrm{Hom}_{\mathscr{S}}(P, ?) : \mathscr{S}\text{-mod} \to H\text{-mod},$$
$$\beta := P \otimes_H ? : H\text{-mod} \to \mathscr{S}\text{-mod}\,.$$

The α is exact, and β is left adjoint to α.

Given an $\mathscr{S}$-module V, let $O_P(V)$ denote the largest submodule V' of V such that $\mathrm{Hom}_{\mathscr{S}}(P, V') = 0$. Let $O^P(V)$ denote the submodule of V generated by the images of all $\mathscr{S}$-homomorphisms from P to V. Any $\mathscr{S}$-module homomorphism $V \to W$ sends $O_P(V)$ into $O_P(W)$ and $O^P(V)$ into $O^P(W)$, so we can view O_P and O^P as functors $\mathscr{S}$-mod $\to \mathscr{S}$-mod. Finally, any homomorphism $V \to W$ induces a well-defined $\mathscr{S}$-module homomorphism $V/O_P(V) \to W/O_P(W)$. We thus obtain an exact functor $A_P : \mathscr{S}\text{-mod} \to \mathscr{S}\text{-mod}$ defined on objects by $V \mapsto V/O_P(V)$. The following two lemmas can be found for example in [5, 3.1a, 3.1c]:

Lemma 2.29. The functors $\alpha \circ \beta$ and $\alpha \circ A_P \circ \beta$ are both isomorphic to the identity.

Lemma 2.30. If $V, W \in \mathscr{S}$-mod satisfy $O^P(V) = V$ and $O_P(W) = 0$, then $\mathrm{Hom}_{\mathscr{S}}(V, W) \simeq \mathrm{Hom}_H(\alpha(V), \alpha(W))$.

The main result on the functors α, β, proved for example in [5, 3.1d], is:

Theorem 2.31. The functors α and $A_P \circ \beta$ induce mutually inverse equivalences of categories between H-mod and the full subcategory of $\mathscr{S}$-mod consisting of all $V \in \mathscr{S}$-mod such that $O_P(V) = 0$ and $O^P(V) = V$.

An easy consequence is the following relation between the irreducible modules, see [5, 3.1e]:

Lemma 2.32. Let $\{E_m \mid m \in M\}$ be a complete set of non-isomorphic irreducible $\mathscr{S}$-modules appearing in the head of P. For all $m \in M$, set $D_m := \alpha(E_m)$. Then, $\{D_m \mid m \in M\}$ is a complete irredundant set of irreducible H-modules, and $A_P \circ \beta(D_m) \simeq E_m$.

Finally, we will make use of the following more explicit description of the effect of the composite functor $A_P \circ \beta$ on left ideals of H, see [5, 3.1f]:

Lemma 2.33. Suppose that every composition factor of the socle of P also appears in its head. Then for any left ideal J of H, we have $A_P \circ \beta(J) \simeq PJ$.

2.3. Lie theoretic notation

Throughout the paper

$$\mathtt{C} = (\mathtt{c}_{ij})_{i,j \in I}$$

is a *Cartan matrix* of *untwisted affine type*, see [17, §4, Table Aff 1]. We have

$$I = \{0, 1, \dots, l\},$$

where 0 is the affine vertex. Following [17, §1.1], let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of the Cartan matrix $\mathtt{C}$, so we have simple roots $\{\alpha_i \mid i \in I\}$, simple coroots $\{\alpha_i^\vee \mid i \in I\}$, and a bilinear form $(\cdot, \cdot)$ on $\mathfrak{h}^*$ such that

$$\mathtt{c}_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$$

for all $i, j \in I$. We normalize $(\cdot, \cdot)$ so that $(\alpha_i, \alpha_i) = 2$ if α_i is a short simple root.

The *fundamental dominant weights* $\{\Lambda_i \mid i \in I\}$ have the property $\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{i,j}$, where $\langle \cdot, \cdot \rangle$ is the natural pairing between $\mathfrak{h}^*$ and $\mathfrak{h}$. We have the *integral weight lattice* $P = \oplus_{i \in I} \mathbb{Z} \cdot \Lambda_i$ and the set of *dominant weights* $P_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \cdot \Lambda_i$. For $i \in I$ we define

$$q_i := q^{(\alpha_i, \alpha_i)/2}, \quad [n]_i := [n]_{q_i}, \quad [n]_i^! := [n]_{q_i}^!. \tag{2.34}$$

Denote $Q_+ := \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \cdot \alpha_i$. For $\alpha \in Q_+$, we write $\mathrm{ht}(\alpha)$ for the sum of its coefficients when expanded in terms of the α_i's.

Let $\mathfrak{g}' = \mathfrak{g}(\mathtt{C}')$ be the finite dimensional simple Lie algebra whose Cartan matrix $\mathtt{C}'$ corresponds to the subset of vertices $I' := I \setminus \{0\}$. The affine Lie algebra $\mathfrak{g} = \mathfrak{g}(\mathtt{C})$ is then obtained from $\mathfrak{g}'$ by a procedure described in [17, Section 7]. We denote by W (resp. W') the corresponding *affine Weyl group* (resp. *finite Weyl group*). It is a Coxeter group with standard generators $\{r_i \mid i \in I\}$ (resp. $\{r_i \mid i \in I'\}$), see [17, Proposition 3.13].

Let Φ' and Φ be the root systems of $\mathfrak{g}'$ and $\mathfrak{g}$ respectively. Denote by Φ'_+ and Φ_+ the set of *positive* roots in Φ' and Φ, respectively, cf. [17, §1.3]. Let

$$\delta = a_0\alpha_0 + a_1\alpha_1 + \cdots + a_l\alpha_l. \tag{2.35}$$

By [17, Table Aff 1], we always have

$$a_0 = 1. \tag{2.36}$$

We have

$$\delta - \alpha_0 = \theta, \tag{2.37}$$

where θ is the highest root in the finite root system Φ'. Finally, $\Phi_+ = \Phi_+^{\mathrm{im}} \sqcup \Phi_+^{\mathrm{re}}$, where

$$\Phi_+^{\mathrm{im}} = \{n\delta \mid n \in \mathbb{Z}_{>0}\}$$

and

$$\Phi_+^{\mathrm{re}} = \{\beta + n\delta \mid \beta \in \Phi'_+,\ n \in \mathbb{Z}_{\geq 0}\} \sqcup \{-\beta + n\delta \mid \beta \in \Phi'_+,\ n \in \mathbb{Z}_{>0}\}.$$

2.4. KLR algebras

Define the polynomials in the variables u, v

$$\{Q_{ij}(u,v) \in F[u,v] \mid i,j \in I\}$$

as follows. For the case where the Cartan matrix $\mathtt{C} \neq \mathtt{A}_1^{(1)}$, choose signs ε_{ij} for all $i,j \in I$ with $\mathtt{c}_{ij} < 0$ so that $\varepsilon_{ij}\varepsilon_{ji} = -1$. Then set:

$$Q_{ij}(u,v) := \begin{cases} 0 & \text{if } i = j; \\ 1 & \text{if } \mathtt{c}_{ij} = 0; \\ \varepsilon_{ij}(u^{-\mathtt{c}_{ij}} - v^{-\mathtt{c}_{ji}}) & \text{if } \mathtt{c}_{ij} < 0. \end{cases} \tag{2.38}$$

For type $A_1^{(1)}$ we define

$$Q_{ij}(u,v) := \begin{cases} 0 & \text{if } i = j; \\ (u-v)(v-u) & \text{if } i \neq j. \end{cases} \tag{2.39}$$

Fix $\alpha \in Q_+$ of height d. Let

$$I_\alpha = \{\boldsymbol{i} = (i_1, \dots, i_d) \in I^d \mid \alpha_{i_1} + \dots + \alpha_{i_d} = \alpha\}. \tag{2.40}$$

The *KLR-algebra* $R_\alpha = R_\alpha(\mathcal{O})$ is an associative graded unital $\mathcal{O}$-algebra, given by the generators

$$\{1_{\boldsymbol{i}} \mid \boldsymbol{i} \in I_\alpha\} \cup \{y_1, \dots, y_d\} \cup \{\psi_1, \dots, \psi_{d-1}\} \tag{2.41}$$

and the following relations for all $\boldsymbol{i}, \boldsymbol{j} \in I_\alpha$ and all admissible r, t:

$$1_{\boldsymbol{i}} 1_{\boldsymbol{j}} = \delta_{\boldsymbol{i},\boldsymbol{j}} 1_{\boldsymbol{i}}, \quad \textstyle\sum_{\boldsymbol{i} \in I_\alpha} 1_{\boldsymbol{i}} = 1; \tag{2.42}$$

$$y_r 1_{\boldsymbol{i}} = 1_{\boldsymbol{i}} y_r; \quad y_r y_t = y_t y_r; \tag{2.43}$$

$$\psi_r 1_{\boldsymbol{i}} = 1_{s_r \boldsymbol{i}} \psi_r; \tag{2.44}$$

$$(y_t \psi_r - \psi_r y_{s_r(t)}) 1_{\boldsymbol{i}} = \delta_{i_r, i_{r+1}} (\delta_{t,r+1} - \delta_{t,r}) 1_{\boldsymbol{i}}; \tag{2.45}$$

$$\psi_r^2 1_{\boldsymbol{i}} = Q_{i_r, i_{r+1}}(y_r, y_{r+1}) 1_{\boldsymbol{i}} \tag{2.46}$$

$$\psi_r \psi_t = \psi_t \psi_r \qquad (|r - t| > 1); \tag{2.47}$$

$$\begin{aligned} &(\psi_{r+1}\psi_r\psi_{r+1} - \psi_r\psi_{r+1}\psi_r) 1_{\boldsymbol{i}} \\ =&\delta_{i_r, i_{r+2}} \frac{Q_{i_r, i_{r+1}}(y_{r+2}, y_{r+1}) - Q_{i_r, i_{r+1}}(y_r, y_{r+1})}{y_{r+2} - y_r} 1_{\boldsymbol{i}}. \end{aligned} \tag{2.48}$$

The *grading* on R_α is defined by setting:

$$\deg(1_{\boldsymbol{i}}) = 0, \quad \deg(y_r 1_{\boldsymbol{i}}) = (\alpha_{i_r}, \alpha_{i_r}), \quad \deg(\psi_r 1_{\boldsymbol{i}}) = -(\alpha_{i_r}, \alpha_{i_{r+1}}).$$

It is pointed out in [21] and [42, §3.2.4] that up to isomorphism R_α depends only on the Cartan matrix and α.

Fix in addition a dominant weight $\Lambda \in P_+$. The corresponding *cyclotomic KLR algebra* R_α^Λ is the quotient of R_α by the following ideal:

$$J_\alpha^\Lambda := (y_1^{\langle \Lambda, \alpha_{i_1}^\vee \rangle} 1_{\boldsymbol{i}} \mid \boldsymbol{i} = (i_1, \ldots, i_d) \in I_\alpha). \tag{2.49}$$

For each element $w \in S_d$ fix a reduced expression $w = s_{r_1} \ldots s_{r_m}$ and set

$$\psi_w := \psi_{r_1} \ldots \psi_{r_m}.$$

In general, ψ_w depends on the choice of the reduced expression of w.

Theorem 2.50. [20, Theorem 2.5], [42, Theorem 3.7] The elements

$$\{\psi_w y_1^{m_1} \ldots y_d^{m_d} 1_{\boldsymbol{i}} \mid w \in S_d,\ m_1, \ldots, m_d \in \mathbb{Z}_{\geq 0},\ \boldsymbol{i} \in I_\alpha\}$$

form an $\mathcal{O}$-basis of R_α.

There exists a homogeneous algebra anti-involution

$$\tau : R_\alpha \longrightarrow R_\alpha, \quad 1_{\boldsymbol{i}} \mapsto 1_{\boldsymbol{i}}, \quad y_r \mapsto y_r, \quad \psi_s \mapsto \psi_s \tag{2.51}$$

for all $\boldsymbol{i} \in I_\alpha,\ 1 \leq r \leq d$, and $1 \leq s < d$. If $M = \bigoplus_{d \in \mathbb{Z}} M_d$ is a finite dimensional graded R_α-module, then the *graded dual* $M^\circledast$ is the graded R_α-module such that $(M^\circledast)_n := \mathrm{Hom}_{\mathcal{O}}(M_{-n}, \mathcal{O})$, for all $n \in \mathbb{Z}$, and the R_α-action is given by $(xf)(m) = f(\tau(x)m)$, for all $f \in M^\circledast, m \in M, x \in R_\alpha$.

We remark that there is also a diagrammatic presentation of KLR algebras given in [20]. This presentation is particularly convenient for large calculations, and we will make use of it in Chapters V and VI.

2.41. Basic representation theory of R_α

Let H be any ($\mathbb{Z}$-)graded F-algebra. By a module V over H, we *always* mean a graded left H-module. We denote by H-Mod the abelian category of all graded left H-modules, with morphisms being *degree-preserving* module homomorphisms, which we denote by Hom. Let H-mod denote the abelian subcategory of all *finite dimensional* graded H-modules, and $[H\text{-mod}]$ be the corresponding Grothendieck group. Then $[H\text{-mod}]$ is an $\mathscr{A}$-module via $q^m[M] := [q^m M]$, where $q^m M$ denotes the module obtained by shifting the grading up by m, i.e. $(q^m M)_n := M_{n-m}$. We denote by $\hom_H(M,N)$ the space of morphism in H-Mod, i.e. degree zero homogeneous H-module homomorphisms. Similarly we have $\mathrm{ext}^m_H(M,N)$.

For $n \in \mathbb{Z}$, let $\mathrm{Hom}_H(M,N)_n := \hom_H(q^n M, N)$ denote the space of all homomorphisms that are homogeneous of degree n. Set

$$\mathrm{Hom}_H(M,N) := \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_H(M,N)_n.$$

For graded H-modules M and N we write $M \cong N$ to mean that M and N are isomorphic as graded modules and $M \simeq N$ to mean that they are isomorphic as H-modules after we forget the gradings.

For a finite dimensional graded vector space $V = \oplus_{n\in\mathbb{Z}} V_n$, its *graded dimension* is $\dim_q V := \sum_{n\in\mathbb{Z}} (\dim V_n) q^n \in \mathscr{A}$. Given $M, L \in H$-mod with L irreducible, we write $[M : L]_q$ for the corresponding *graded composition multiplicity*, i.e. $[M : L]_q := \sum_{n\in\mathbb{Z}} a_n q^n$, where a_n is the multiplicity of $q^n L$ in a graded composition series of M.

Going back to the algebras $R_\alpha = R_\alpha(F)$, every irreducible graded R_α-module is finite dimensional [20, Proposition 2.12], and there are finitely many irreducible modules in R_α-mod up to isomorphism and grading shift [20, §2.5]. A prime field is a splitting field for R_α, see [20, Corollary 3.19], so working with irreducible R_α-modules we do not need to assume that F is algebraically closed. Finally, for every irreducible module L, there is a unique choice of the grading shift so that we have $L^\circledast \xrightarrow{\sim} L$ [20, Section 3.2]. When speaking of irreducible R_α-modules we often assume by fiat that the shift has been chosen in this way.

For $\boldsymbol{i} \in I_\alpha$ and $M \in R_\alpha$-mod, the $\boldsymbol{i}$*-word space* of M is $M_{\boldsymbol{i}} := 1_{\boldsymbol{i}} M$. We have $M = \bigoplus_{\boldsymbol{i} \in I_\alpha} M_{\boldsymbol{i}}$. We say that $\boldsymbol{i}$ is a *word of* M if $M_{\boldsymbol{i}} \neq 0$. A non-zero vector $v \in M_{\boldsymbol{i}}$ is called a *vector of word* $\boldsymbol{i}$. Note from the relations that $\psi_r M_{\boldsymbol{i}} \subseteq M_{s_r \boldsymbol{i}}$.

Let M be a finite dimensional graded R_α-module. Define the *q-character* of M as follows:

$$\mathrm{ch}_q\, M := \sum_{\boldsymbol{i} \in I_\alpha} (\dim_q M_{\boldsymbol{i}}) \boldsymbol{i} \in \mathscr{A} I_\alpha.$$

The q-character map $\mathrm{ch}_q : R_\alpha\text{-mod} \to \mathscr{A} I_\alpha$ factors through to give an *injective* $\mathscr{A}$-linear map from the Grothendieck group $\mathrm{ch}_q : [R_\alpha\text{-mod}] \to \mathscr{A} I_\alpha$, see [20, Theorem 3.17].

2.42. Induction, coinduction, and duality for KLR algebras

Given $\alpha, \beta \in Q_+$, we set $R_{\alpha,\beta} := R_\alpha \otimes R_\beta$. Let $M \boxtimes N$ be the outer tensor product of the R_α-module M and the R_β-module N. There is an injective homogeneous (non-unital) algebra homomorphism $R_{\alpha,\beta} \hookrightarrow R_{\alpha+\beta}$ mapping $1_{\boldsymbol{i}} \otimes 1_{\boldsymbol{j}}$ to $1_{\boldsymbol{ij}}$, where $\boldsymbol{ij}$ is the concatenation of the two sequences. The image of the identity element of $R_{\alpha,\beta}$ under this map is

$$1_{\alpha,\beta} := \sum_{\boldsymbol{i} \in I_\alpha,\ \boldsymbol{j} \in I_\beta} 1_{\boldsymbol{ij}}.$$

Let $\mathrm{Ind}_{\alpha,\beta}^{\alpha+\beta}$ and $\mathrm{Res}_{\alpha,\beta}^{\alpha+\beta}$ be the corresponding induction and restriction functors:

$$\mathrm{Ind}_{\alpha,\beta}^{\alpha+\beta} := R_{\alpha+\beta}1_{\alpha,\beta}\otimes_{R_{\alpha,\beta}} ? : R_{\alpha,\beta}\text{-mod} \to R_{\alpha+\beta}\text{-mod},$$
$$\mathrm{Res}_{\alpha,\beta}^{\alpha+\beta} := 1_{\alpha,\beta}R_{\alpha+\beta}\otimes_{R_{\alpha+\beta}} ? : R_{\alpha+\beta}\text{-mod} \to R_{\alpha,\beta}\text{-mod}\,.$$

We often omit upper indices and write simply $\mathrm{Ind}_{\alpha,\beta}$ and $\mathrm{Res}_{\alpha,\beta}$.

Note that $\mathrm{Res}_{\alpha,\beta}$ is just left multiplication by the idempotent $1_{\alpha,\beta}$, so it is exact and sends finite dimensional modules to finite dimensional modules. By [20, Proposition 2.16], $1_{\alpha,\beta}R_{\alpha+\beta}$ is a free left $R_{\alpha,\beta}$-module of finite rank, so $\mathrm{Res}_{\alpha,\beta}$ also sends finitely generated projectives to finitely generated projectives. Similarly, $R_{\alpha+\beta}1_{\alpha,\beta}$ is a free right $R_{\alpha,\beta}$-module of finite rank, so $\mathrm{Ind}_{\alpha,\beta}$ is exact and sends finite dimensional modules to finite dimensional modules. The functor $\mathrm{Ind}_{\alpha,\beta}$ is left adjoint to $\mathrm{Res}_{\alpha,\beta}$, and it sends finitely generated projectives to finitely generated projectives.

These functors have obvious generalizations to $n \geq 2$ factors:

$$\mathrm{Ind}_{\gamma_1,\dots,\gamma_n} : R_{\gamma_1,\dots,\gamma_n}\text{-mod} \to R_{\gamma_1+\dots+\gamma_n}\text{-mod},$$
$$\mathrm{Res}_{\gamma_1,\dots,\gamma_n} : R_{\gamma_1+\dots+\gamma_n}\text{-mod} \to R_{\gamma_1,\dots,\gamma_n}\text{-mod}\,.$$

If $M_a \in R_{\gamma_a}$-Mod, for $a = 1, \dots, n$, we define

$$M_1 \circ \cdots \circ M_n := \mathrm{Ind}_{\gamma_1,\dots,\gamma_n} M_1 \boxtimes \cdots \boxtimes M_n. \tag{2.52}$$

In view of [20, Lemma 2.20], we have

$$\mathrm{ch}_q\,(M_1 \circ \cdots \circ M_n) = \mathrm{ch}_q\,(M_1) \circ \cdots \circ \mathrm{ch}_q\,(M_n). \tag{2.53}$$

Finally, the functors of induction and restriction have parabolic analogues. For example, given a family $(\alpha_b^a)_{1\le a\le n,\ 1\le b\le m}$ of elements of Q_+, set $\sum_{a=1}^n \alpha_b^a =: \beta_b$ for all $1\le b\le m$. Then we have obvious functors

$$\mathrm{Ind}^{\beta_1;\dots;\beta_m}_{(\alpha_1^1,\dots,\alpha_1^n);\dots;(\alpha_m^1,\dots,\alpha_m^n)} \qquad \text{and} \qquad \mathrm{Res}^{\beta_1;\dots;\beta_m}_{(\alpha_1^1,\dots,\alpha_1^n);\dots;(\alpha_m^1,\dots,\alpha_m^n)}.$$

While the induction functor $\mathrm{Ind}_{\gamma_1,\dots,\gamma_n}$ is left adjoint to the functor $\mathrm{Res}_{\gamma_1,\dots,\gamma_n}$, the *right* adjoint is given by the coinduction:

$$\mathrm{Coind}_{\gamma_1,\dots,\gamma_n} = \mathrm{Coind}^{\gamma_1+\dots+\gamma_n}_{\gamma_1,\dots,\gamma_n} := \mathrm{Hom}_{R_{\gamma_1,\dots,\gamma_n}}(1_{\gamma_1,\dots,\gamma_n} R_{\gamma_1+\dots+\gamma_n}, ?)$$

Induction and coinduction are related as follows:

For $\underline{\gamma} := (\gamma_1,\dots,\gamma_n)\in Q_+^n$, we denote

$$d(\underline{\gamma}) := \sum_{1\le m<k\le n} (\gamma_m,\gamma_k).$$

Lemma 2.54. [35, Theorem 2.2] Let $\underline{\gamma} := (\gamma_1,\dots,\gamma_n)\in Q_+^n$, and V_m be a finite-dimensional R_{γ_m}-module for $m=1,\dots,n$. Then

$$\mathrm{Ind}_{\gamma_1,\dots,\gamma_n} V_1\boxtimes\cdots\boxtimes V_n \cong q^{d(\underline{\gamma})}\,\mathrm{Coind}_{\gamma_n,\dots,\gamma_1} V_n\boxtimes\cdots\boxtimes V_1.$$

Lemma 2.55. Let $\underline{\gamma} := (\gamma_1,\dots,\gamma_n)\in Q_+^n$, and V_m be a finite dimensional R_{γ_m}-module for $m=1,\dots,n$. Then

$$(V_1\circ\cdots\circ V_n)^{\circledast} \cong q^{d(\underline{\gamma})}(V_n^{\circledast}\circ\cdots\circ V_1^{\circledast}).$$

Proof. Follows from Lemma 2.54 by uniqueness of adjoint functors as in [23, Theorem 3.7.5] $\square$

Lemma 2.56. Let $V \in R_\theta$-mod, $\boldsymbol{i} \in I^\theta$, and $v \in 1_{\boldsymbol{i}}V$ be a non-zero homogeneous vector with $R_\theta v = V$. Assume that there is only one irreducible R_θ-module L up to $\simeq$ with $1_{\boldsymbol{i}}L \neq 0$ and $[V : L]_q \neq 0$. Then $\operatorname{head} V \simeq L$.

Proof. If W is the radical of V then $V/W \cong \oplus_r m_r(q) L_r$ for simple modules L_r, with $L_r \not\cong L_s$ for $r \neq s$, and multiplicities $m_r(q) \in \mathbb{Z}[q, q^{-1}]$. By assumptions, there exists r such that $L \cong L_r$, $m_s(q) = 0$ for $s \neq r$, and $v + W \in m_r(q)L_r$. Finally, $v + W$ generates $m_r(q)L_r$, so $m_r(q)$ is of the form q^d. $\square$

2.43. Crystal operators and extremal words

The theory of crystal operators has been developed in [20], [35] and [18] following ideas of Grojnowski [13], see also [23]. We review necessary facts for the reader's convenience.

Let $\alpha \in Q_+$ and $i \in I$. It is known that $R_{n\alpha_i}$ is a nil-Hecke algebra with unique (up to a degree shift) irreducible module

$$L(i^n) = q_i^{n(n-1)/2} L(i)^{\circ n}.$$

Moreover, $\dim_q L(i^n) = [n]^!_i$ We have functors

$$e_i : R_\alpha\text{-mod} \to R_{\alpha-\alpha_i}\text{-mod},\ M \mapsto \operatorname{Res}^{R_{\alpha-\alpha_i,\alpha_i}}_{R_{\alpha-\alpha_i}} \circ \operatorname{Res}_{\alpha-\alpha_i,\alpha_i} M,$$
$$f_i : R_\alpha\text{-mod} \to R_{\alpha+\alpha_i}\text{-mod},\ M \mapsto \operatorname{Ind}_{\alpha,\alpha_i} M \boxtimes L(i).$$

If $L \in R_\alpha$-mod is irreducible, we define

$$\tilde{f}_i L := \mathrm{head}(f_i L), \quad \tilde{e}_i L := \mathrm{soc}\,(e_i L).$$

A fundamental fact is that $\tilde{f}_i L$ is again irreducible and $\tilde{e}_i L$ is irreducible or zero. We refer to $\tilde{e}_i$ and $\tilde{f}_i$ as the crystal operators. These are operators on $B \cup \{0\}$, where B is the set of isomorphism classes of the irreducible R_α-modules for all $\alpha \in Q_+$. Define $\mathrm{wt} : B \to P,\ [L] \mapsto -\alpha$ if $L \in R_\alpha$-mod.

Theorem 2.57. [35] B with the operators $\tilde{e}_i, \tilde{f}_i$ and the function wt is the crystal graph of the negative part $U_q(\mathfrak{n}_-)$ of the quantized enveloping algebra of $\mathfrak{g}$.

For $M \in R_\alpha$-mod, define

$$\varepsilon_i(M) = \max\{k \geq 0 \mid e_i^k(M) \neq 0\}.$$

Then $\varepsilon_i(M) = \max\{\varepsilon_i(\boldsymbol{j}) \mid \boldsymbol{j}$ is a word of $M\}$, where for $\boldsymbol{j} = (j_1, \dots, j_d) \in I$,

$$\varepsilon_i(\boldsymbol{j}) := \max\{k \geq 0 \mid j_{d-k+1} = \cdots = j_d = i\} \tag{2.58}$$

is the length of the longest i-tail of $\boldsymbol{j}$. Define also

$$\varepsilon_i^*(M) := \max\{k \geq 0 \mid j_1 = \cdots = j_k = i \text{ for a word } \boldsymbol{j} = (j_1, \dots, j_d) \text{ of } M\}$$

to be the length of the longest i-head of the words of M.

Proposition 2.59. [35, 20] Let L be an irreducible R_α-module, $i \in I$, and $\varepsilon = \varepsilon_i(L)$.

(i) $e_i L$ is either zero or it has a simple socle; denote this socle $\tilde{e}_i L$ interpreted as 0 if $e_i L = 0$;

(ii) f_iL has simple head denoted $\tilde{f}_iL$;

(iii) $\tilde{e}_i\tilde{f}_iL \simeq L$ and if $\tilde{e}_iL \neq 0$ then $\tilde{f}_i\tilde{e}_iL \simeq L$;

(iv) $\varepsilon = \max\{k \geq 0 \mid \tilde{e}_i^k(L) \neq 0\}$;

(v) $\mathrm{Res}_{\alpha-\varepsilon\alpha_i,\varepsilon\alpha_i} L \simeq \tilde{e}_i^{\varepsilon} L \boxtimes L(i^n)$.

Recall from (2.49) the cyclotomic ideal J_α^Λ. We have an obvious functor of inflation $\mathrm{infl}^\Lambda : R_\alpha^\Lambda\text{-mod} \to R_\alpha\text{-mod}$ and its left adjoint

$$\mathrm{pr}^\Lambda : R_\alpha\text{-mod} \to R_\alpha^\Lambda\text{-mod},\ M \mapsto M/J_\alpha^\Lambda M.$$

Lemma 2.60. [35, Proposition 2.4] Let L be an irreducible R_α-module. Then $\mathrm{pr}^\Lambda L \neq 0$ if and only if $\varepsilon_i^*(L) \leq \langle \Lambda, \alpha_i^\vee \rangle$ for all $i \in I$.

Let $M \in R_\alpha$-mod and $\boldsymbol{i} = i_1^{a_1} \dots i_b^{a_b}$, with $a_1, \dots, a_b \in \mathbb{Z}_{>0}$, be a word of M. Then $\boldsymbol{i}$ is *extremal* for M if

$$a_b = \varepsilon_{i_b}(M),\ a_{b-1} = \varepsilon_{i_{b-1}}(\tilde{e}_{i_b}^{a_b} M)\ ,\ \dots\ ,\ a_1 = \varepsilon_{i_1}(\tilde{e}_{i_2}^{a_2} \dots \tilde{e}_{i_b}^{a_b} M).$$

It follows that $i_k \neq i_{k+1}$ for all $k = 1, \dots, b-1$.

Lemma 2.61. [24, Lemma 2.10] Let L be an irreducible R_α-module, and $\boldsymbol{i} = i_1^{a_1} \dots i_b^{a_b} \in I_\alpha$ be an extremal word for L with $i_k \neq i_{k+1}$. Set $N := \sum_{m=1}^b a_m(a_m - 1)(\alpha_{i_m}, \alpha_{i_m})/4$. Then

$$\dim_q L_{\boldsymbol{i}} = \prod_{k=1}^b [a_k]_{i_k}^! \quad \text{and} \quad \dim 1_{\boldsymbol{i}} L_N = \dim 1_{\boldsymbol{i}} L_{-N} = 1.$$

2.44. Mackey Theorem

We state a slight generalization of Mackey Theorem of Khovanov and Lauda [20, Proposition 2.18]. First some notation. Given $\underline{\kappa} = (\kappa_1, \dots, \kappa_N) \in Q_+^N$, and a permutation $x \in \mathfrak{S}_N$, we denote

$$x\underline{\kappa} := \big(\kappa_{x^{-1}(1)}, \dots, \kappa_{x^{-1}(N)}\big).$$

Correspondingly, define the integer

$$s(x, \underline{\kappa}) := -\sum_{1 \le m < k \le N,\ x(m) > x(k)} (\kappa_m, \kappa_k).$$

Writing $R_{\underline{\kappa}}$ for $R_{\kappa_1,\dots,\kappa_N}$, there is an obvious natural algebra isomorphism

$$\varphi^x : R_{x\underline{\kappa}} \to R_{\underline{\kappa}}$$

permuting the components. Composing with this isomorphism, we get a functor

$$R_{\underline{\kappa}}\text{-mod} \to R_{x\underline{\kappa}}\text{-mod},\ M \mapsto {}^{\varphi^x}M.$$

Making an additional shift, we get a functor

$$R_{\underline{\kappa}}\text{-mod} \to R_{x\underline{\kappa}}\text{-mod},\ M \mapsto {}^{x}M := q^{s(x,\underline{\kappa})}({}^{\varphi^x}M).$$

For the purposes of the following theorem, let us fix

$$\underline{\gamma} = (\gamma_1, \dots, \gamma_n) \in Q_+^n \quad \text{and} \quad \underline{\beta} = (\beta_1, \dots, \beta_m) \in Q_+^m$$

with

$$\gamma_1 + \cdots + \gamma_n = \beta_1 + \cdots + \beta_m =: \alpha.$$

Denote $d_b^a := \mathrm{ht}(\alpha_b^a)$ and $d := \mathrm{ht}(\alpha)$.

Let $\mathcal{D}(\underline{\beta}, \underline{\gamma})$ be the set of all tuples $\underline{\alpha} = (\alpha_b^a)_{1 \le a \le n,\ 1 \le b \le m}$ of elements of Q_+ such that $\sum_{b=1}^m \alpha_b^a = \gamma^a$ for all $1 \le a \le n$ and $\sum_{a=1}^n \alpha_b^a = \beta_b$ for all $1 \le b \le m$.

For each $\underline{\alpha} \in \mathcal{D}(\underline{\beta}, \underline{\gamma})$, we define permutations $x(\underline{\alpha}) \in \mathfrak{S}_{mn}$ and $x(\underline{\alpha}) \in \mathfrak{S}_d$. The permutation $x(\underline{\alpha})$ maps

$$(\alpha_1^1, \ldots, \alpha_m^1, \alpha_1^2, \ldots, \alpha_m^2 \ldots, \alpha_1^n, \ldots, \alpha_m^n)$$

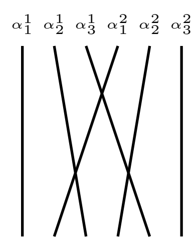
to

$$(\alpha_1^1, \ldots, \alpha_1^n, \alpha_2^1, \ldots, \alpha_2^n, \ldots, \alpha_m^1, \ldots, \alpha_m^n).$$

On the other hand, $w(\underline{\alpha})$ is the corresponding permutation of the blocks of sizes d_b^a.

Example 2.62. Assume that $n = 2$, $m = 3$, and all $d_b^a = 2$. Then $x(\underline{\alpha}) \in \mathfrak{S}_6$ is the permutation which maps $1 \mapsto 1, 2 \mapsto 3, 3 \mapsto 5, 4 \mapsto 2, 5 \mapsto 4, 6 \mapsto 6$. In diagrammatic form:

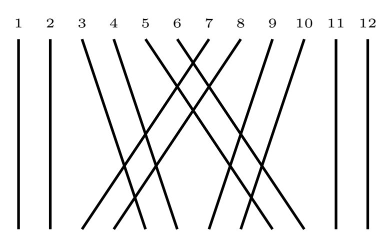
$x(\underline{\alpha}) =$



.

On the other hand, $w(\underline{\alpha}) \in \mathfrak{S}_{12}$ is the corresponding block permutation:

$w(\underline{\alpha}) =$



.

Let $M \in R_{\underline{\gamma}}$-mod. We can now consider the $R_{\alpha_1^1,\dots,\alpha_1^n;\dots;\alpha_m^1,\dots,\alpha_m^n}$-module

$$ {}^{x(\underline{\alpha})}\big(\mathrm{Res}^{\gamma_1;\dots;\gamma_n}_{\alpha_1^1,\dots,\alpha_m^1;\dots;\alpha_1^n,\dots,\alpha_m^n} M\big). $$

Finally, let $\leq$ be a total order refining the Bruhat order on $\mathfrak{S}_d$, and for $\underline{\alpha} \in \mathcal{D}(\underline{\beta}, \underline{\gamma})$, consider the submodules

$$ F_{\leq \underline{\alpha}}(M) := \sum_{w \in \mathcal{D}(\underline{\beta},\underline{\gamma}),\ w \leq w(\underline{\alpha})} R_{\underline{\beta}} \psi_w \otimes 1_{\underline{\alpha}} M \subseteq \mathrm{Res}^{\alpha}_{\underline{\beta}} \mathrm{Ind}^{\alpha}_{\underline{\gamma}} M, $$
$$ F_{< \underline{\alpha}}(M) := \sum_{w \in \mathcal{D}(\underline{\beta},\underline{\gamma}),\ w < w(\underline{\alpha})} R_{\underline{\beta}} \psi_w \otimes 1_{\underline{\alpha}} M \subseteq \mathrm{Res}^{\alpha}_{\underline{\beta}} \mathrm{Ind}^{\alpha}_{\underline{\gamma}} M. $$

Theorem 2.63. Let

$$ \underline{\gamma} = (\gamma_1, \dots, \gamma_n) \in Q_+^n \quad \text{and} \quad \underline{\beta} = (\beta_1, \dots, \beta_m) \in Q_+^m $$

with

$$ \gamma_1 + \cdots + \gamma_n = \beta_1 + \cdots + \beta_m =: \alpha, $$

and $M \in R_{\underline{\gamma}}$-mod. With the notation as above, the filtration $(F_{\leq \underline{\alpha}}(M))_{\underline{\alpha} \in \mathcal{D}(\underline{\beta},\underline{\gamma})}$ is a filtration of $\mathrm{Res}_{\underline{\beta}} \mathrm{Ind}_{\underline{\gamma}} M$ as an $R_{\underline{\beta}}$-module. Moreover, the subquotients of the filtration are:

$$ \begin{aligned} F_{\leq \underline{\alpha}}(M)/F_{< \underline{\alpha}}(M) &\cong \mathrm{Ind}^{\underline{\beta}}_{x(\underline{\alpha}) \cdot \underline{\alpha}} \big({}^{x(\underline{\alpha})} \big(\mathrm{Res}^{\underline{\gamma}}_{\underline{\alpha}} M\big)\big). \\ &= \mathrm{Ind}^{\beta_1;\dots;\beta_m}_{\alpha_1^1,\dots,\alpha_1^n;\dots;\alpha_m^1,\dots,\alpha_m^n} {}^{x(\underline{\alpha})}\big(\mathrm{Res}^{\gamma_1;\dots;\gamma_n}_{\alpha_1^1,\dots,\alpha_m^1;\dots;\alpha_1^n,\dots,\alpha_m^n} M\big). \end{aligned} $$

Proof. For $m = n = 2$ this follows from [20, Proposition 2.18]. The general case can be proved by the same argument or deduced from the case $m = n = 2$ by induction. □

2.5. Convex preorders and root partitions

We now describe the theory of cuspidal systems from [24]. Recall the notion of a convex preorder on Φ_+ from (1.1)–(1.3). General theory of cuspidal systems is valid for an arbitrary convex preorder, but for the theory of imaginary representations we will need an additional assumption that the preorder is balanced, see (1.7), (1.8).

Recall that $I' = \{1, \dots, l\}$. We will consider the set $\mathscr{P}$ of l-multipartitions

$$\underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(l)}),$$

where each $\lambda^{(i)} = (\lambda_1^{(i)}, \lambda_2^{(i)}, \dots)$ is a usual partition. We denote

$$|\underline{\lambda}| := \sum_{i \in I'} |\lambda^{(i)}|.$$

For $n \in \mathbb{Z}_{\geq 0}$, the set of all $\underline{\lambda} \in \mathscr{P}$ such that $|\underline{\lambda}| = n$ is denoted $\mathscr{P}_n$.

Recall the totally ordered set Ψ defined in (1.4). Denote by $\mathtt{Se}$ the set of all finitary tuples $M = (m_\rho)_{\rho \in \Psi} \in \mathbb{Z}_{\geq 0}^{\Psi}$ of non-negative integers. The left lexicographic order on $\mathtt{Se}$ is denoted $\leq_l$ and the right lexicographic order on $\mathtt{Se}$ is denoted $\leq_r$. We will use the following *bilexicographic* partial order on $\mathtt{Se}$:

$$M \leq N \qquad \text{if and only if} \qquad M \leq_l N \text{ and } M \geq_r N.$$

Let

$$\pi = (M, \underline{\mu}) = (\rho_1^{m_1}, \dots, \rho_s^{m_s}, \underline{\mu}, \rho_{-t}^{m_{-t}}, \dots, \rho_{-1}^{m_{-1}})$$

be a root partition as in (4.13), so that $M \in \mathtt{Se}$ and $\underline{\mu} \in \mathscr{P}_{m_\delta}$. For $\rho \in \Psi$, we define $M_\rho := m_\rho \rho$, and consider a tuple $|M| = (M_\rho)_{\rho \in \Psi} \in Q_+^{\Psi}$. Ignoring trivial terms, we can also write

$$|M| = (m_1\rho_1, \dots, m_s\rho_s, m_\delta\delta, m_{-t}\rho_{-t}, \dots, m_{-1}\rho_{-1}).$$

Then we have a parabolic subalgebra

$$R_{|M|} = R_{m_1\rho_1,\dots,m_s\rho_s,m_\delta\delta,m_{-t}\rho_{-t},\dots,m_{-1}\rho_{-1}} \subseteq R_\alpha.$$

We will use the following partial order on the set $\Pi(\alpha)$ of root partitions of α:

$$(M, \underline{\mu}) \leq (N, \underline{\nu}) \text{ if and only if } M \leq N \text{ and if } M = N \text{ then } \underline{\mu} = \underline{\nu}. \tag{2.64}$$

2.6. Cuspidal systems and standard modules

Let $\preceq$ be an arbitrary convex preorder on Φ_+. Recall the definition of a cuspidal system

$$\{L_\rho \mid \rho \in \Phi_+^{\mathrm{re}}\} \cup \{L(\underline{\mu}) \mid \underline{\mu} \in \mathscr{P}\}$$

from §1.1.

For every $\alpha \in Q_+$ and $\pi = (M, \underline{\mu}) \in \Pi(\alpha)$ as in (4.13), we define an integer

$$\mathtt{sh}(\pi) = \mathtt{sh}(M, \underline{\mu}) := \sum_{\rho \in \Phi_+^{\mathrm{re}}} (\rho, \rho) m_\rho (m_\rho - 1)/4, \tag{2.65}$$

the irreducible $R_{|M|}$-module

$$L_\pi = L_{M,\underline{\mu}} := q^{\mathtt{sh}(\pi)}\, L_{\rho_1}^{\circ m_1} \boxtimes \cdots \boxtimes L_{\rho_s}^{\circ m_s} \boxtimes L(\underline{\mu}) \boxtimes L_{\rho_{-t}}^{\circ m_{-t}} \boxtimes L_{\rho_{-1}}^{m_{-1}}, \tag{2.66}$$

and the *standard module*

$$\bar{\Delta}(\pi) = \bar{\Delta}(M, \underline{\mu}) := q^{\mathtt{sh}(\pi)}\, L_{\rho_1}^{\circ m_1} \circ \cdots \circ L_{\rho_s}^{\circ m_s} \circ L(\underline{\mu}) \circ L_{\rho_{-t}}^{\circ m_{-t}} \circ L_{\rho_{-1}}^{m_{-1}}. \tag{2.67}$$

Note that $\bar{\Delta}(M, \underline{\mu}) = \mathrm{Ind}_{|M|} L_{M,\underline{\mu}}$.

Theorem 2.68. [24] Given a convex preorder there exists a corresponding cuspidal system $\{L_\rho \mid \rho \in \Phi_+^{\mathrm{re}}\} \cup \{L(\underline{\lambda}) \mid \underline{\lambda} \in \mathscr{P}\}$. Moreover:

(i) For every root partition π, the standard module $\bar{\Delta}(\pi)$ has irreducible head; denote this irreducible module $L(\pi)$.

(ii) $\{L(\pi) \mid \pi \in \Pi(\alpha)\}$ is a complete and irredundant system of irreducible R_α-modules up to isomorphism and degree shift.

(iii) For every root partition π, we have $L(\pi)^{\circledast} \cong L(\pi)$.

(iv) For all $\pi, \sigma \in \Pi(\alpha)$, we have that $[\bar{\Delta}(\pi) : L(\pi)]_q = 1$, and $[\bar{\Delta}(\pi) : L(\sigma)]_q \neq 0$ implies $\pi \leq \sigma$.

(v) For all $(M, \underline{\mu}), (N, \underline{\nu}) \in \Pi(\alpha)$, we have that $\mathrm{Res}_{|M|} L(M, \underline{\mu}) \cong L_{M,\underline{\mu}}$ and $\mathrm{Res}_{|N|} L(M, \underline{\mu}) \neq 0$ implies $N \leq M$.

(vi) The induced module $L_\rho^{\circ n}$ is irreducible for all $\rho \in \Phi_+^{\mathrm{re}}$ and $n \in \mathbb{Z}_{>0}$.

CHAPTER III

IMAGINARY SCHUR-WEYL DUALITY

The work in this chapter has appeared in the article [29], which has been accepted for publication in the Memoirs of the American Mathematical Society. It is co-authored with Alexander Kleshchev. We developed the results in the co-authored material jointly over many meetings, and, by the nature of collaborative mathematical work, it is difficult to attribute exact portions of the co-authored material to either Kleshchev or myself individually.

3.1. Imaginary tensor space

In this chapter we assume that the fixed convex preorder we are working with is balanced, so that $\alpha_i \succ n\delta \succ \alpha_0$ for all $i \in I'$ and $n \in \mathbb{Z}_{>0}$. It turns out that the theory of imaginary representations is independent of the choice of a *balanced* convex preorder. Denote

$$e := \mathrm{ht}(\delta).$$

Recall the irreducible imaginary representations of $R_{n\delta}$ defined by the property (Cus2) in §1.1. The irreducible imaginary representations of R_δ are called *minuscule imaginary representations*. The minuscule imaginary representations can be *canonically* labeled by the elements of $\mathscr{P}_1$ as explained below.

Lemma 3.1. [24, Lemma 5.1] Let L be an irreducible R_δ-module. The following are equivalent:

(i) L is minuscule imaginary;

(ii) L factors through to the cyclotomic quotient $R_\delta^{\Lambda_0}$;

(iii) we have $i_1 = 0$ for any word $\boldsymbol{i} = (i_1, \ldots, i_e)$ of L.

We always consider $R^{\Lambda_0}_\alpha$-modules as R_α-modules via $\mathrm{infl}^{\Lambda_0}$.

Proposition 3.2. [24, Lemma 5.2, Corollary 5.3] Let $i \in I'$.

(i) The cuspidal module $L_{\delta-\alpha_i}$ factors through $R^{\Lambda_0}_{\delta-\alpha_i}$ and it is the only irreducible $R^{\Lambda_0}_{\delta-\alpha_i}$-module.

(ii) The minuscule imaginary modules are exactly

$$\{L_{\delta,i} := \tilde{f}_i L_{\delta-\alpha_i} \mid i \in I'\}.$$

(iii) $e_j L_{\delta,i} = 0$ for all $j \in I \setminus \{i\}$. Thus, for each $i \in I'$, the minuscule imaginary module $L_{\delta,i}$ can be characterized uniquely up to isomorphism as the irreducible $R^{\Lambda_0}_\delta$-module such that $i_e = i$ for all words $\boldsymbol{i} = (i_1, \ldots, i_e)$ of $L_{\delta,i}$.

For each $i \in I'$, we refer to the minuscule module $L_{\delta,i}$ described in Proposition 3.2 as the minuscule module of *color* i. Let

$$\underline{\mu}(i) := (\emptyset, \ldots, \emptyset, (1), \emptyset, \ldots, \emptyset) \in \mathscr{P}_1 \qquad (i \in I') \tag{3.3}$$

be the l-multipartition of 1 with (1) in the ith component. We associate to it the minuscule module $L_{\delta,i}$:

$$L(\underline{\mu}(i)) := L_{\delta,i} \qquad (i \in I'). \tag{3.4}$$

Lemma 3.5. [24, Lemma 5.4] Let $i \in I'$. Then $\varepsilon_i(L_{\delta,i}) = 1$.

The minuscule modules are defined over $\mathbb{Z}$, see [24, Remark 5.5]. To be more precise, for each $i \in I'$, there exists an $R_\delta(\mathbb{Z})$-module $L_{\delta,i,\mathbb{Z}}$ which is free finite rank over $\mathbb{Z}$ and such that $L_{\delta,i,\mathbb{Z}} \otimes F$ is the minuscule imaginary module $L_{\delta,i,F}$ over $R_\delta(F)$ for any ground field F. In particular,

$$\mathrm{End}_{R_\delta}(L_{\delta,i,\mathcal{O}}) \cong \mathcal{O}. \tag{3.6}$$

The *imaginary tensor space of color* i is the $R_{n\delta}$-module

$$M_{n,i} := L_{\delta,i}^{\circ n}. \tag{3.7}$$

In this definition we allow n to be zero, in which case $M_{0,i}$ is the trivial module over the trivial algebra R_0. A composition factor of $M_{n,i}$ is called an *irreducible imaginary module of* color i. Color is well-defined in the following sense: if $n > 0$ and L is an irreducible imaginary $R_{n\delta}$-module of color i, then L cannot be irreducible imaginary of color $j \in I'$. Indeed, every word appearing in the character of $M_{n,i}$, and hence in the character of L, ends on i.

Lemma 3.8. [24, Lemma 5.7] Any composition factor of $M_{n_1,1} \circ \cdots \circ M_{n_l,l}$ is imaginary.

The following theorem provides a 'reduction to one color':

Theorem 3.9. [24, Theorem 5.10] Suppose that for each $n \in \mathbb{Z}_{\geq 0}$ and $i \in I'$, we have an irredundant family $\{L_i(\lambda) \mid \lambda \vdash n\}$ of irreducible imaginary $R_{n\delta}$-modules of color i. For a multipartition $\underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(l)}) \in \mathscr{P}_n$, define

$$L(\underline{\lambda}) := L_1(\lambda^{(1)}) \circ \cdots \circ L_l(\lambda^{(l)}).$$

Then $\{L(\underline{\lambda}) \mid \underline{\lambda} \in \mathscr{P}_n\}$ is a complete and irredundant system of irreducible imaginary $R_{n\delta}$-modules. In particular, the given modules $\{L_i(\lambda) \mid \lambda \vdash n\}$ give all the irreducible imaginary modules of color i up to isomorphism.

Corollary 3.10. Suppose that for each $n \in \mathbb{Z}_{\geq 0}$ and for each $i = 1, \dots, l$, we have an irredundant family $\{L_i(\lambda) \mid \lambda \vdash n\}$ of irreducible imaginary $R_{n\delta}$-modules of color i. Then each irreducible imaginary $R_{n\delta}$-module of color i is isomorphic to one of the modules $L_i(\lambda)$ for some $\lambda \vdash n$.

Proof. Let L be an irreducible imaginary $R_{n\delta}$-module of color i. By Theorem 3.9, we must have $L \simeq L_1(\mu^{(1)}) \circ \cdots \circ L_l(\mu^{(l)})$ for some multipartition $(\mu^{(1)}, \dots, \mu^{(l)}) \in \mathscr{P}_n$. It remains to note that $\mu^{(j)} = \emptyset$ for all $j \neq i$, for otherwise j would arise as a last letter of some word arising in the character of L, giving a contradiction. $\square$

If the Cartan matrix $\mathtt{C}$ is symmetric, then the minuscule representations can be described very explicitly as certain special homogeneous representations, see [24, Sections 5.4,5.5].

Lemma 3.11. [24, Lemma 5.16] Let $i \in I'$. Then we can write $\Lambda_0 - \delta + \alpha_i = w(i)\Lambda_0$ for a unique $w(i) \in W$ which is Λ_0-minuscule.

By the theory of homogeneous representations [24, Sections 5.4,5.5], the minuscule element $w(i)$ constructed in Lemma 3.11 is of the form $w_{C(i)}$ for some strongly homogeneous component $C(i)$ of $G_{\delta - \alpha_i}$.

Lemma 3.12. [24, Lemma 5.17] Let $i \in I'$, $d := e-1 = \mathrm{ht}(\delta - \alpha_i)$ and $\boldsymbol{j} = (j_1, \dots, j_d) \in C(i)$. Then the cuspidal module $L_{\delta-\alpha_i}$ is the homogeneous module $L(C(i))$, and we have:

(i) $j_1 = 0$;

(ii) j_d is connected to i in the Dynkin diagram, i.e. $a_{j_d,i} = -1$

(iii) if $j_b = i$ for some b, then there are at least three indices b_1, b_2, b_3 such that $b < b_1 < b_2 < b_3 \leq d$ such that $a_{i,b_1} = a_{i,b_2} = a_{i,b_3} = -1$.

Now we can describe the minuscule modules as homogeneous modules:

Proposition 3.13. [24, Proposition 5.19] Let $i \in I'$. The set of concatenations

$$C_i := \{\boldsymbol{j} i \mid \boldsymbol{j} \in C(i)\}$$

is a homogeneous component of G_δ, and the corresponding homogeneous R_δ-module $L(C_i)$ is isomorphic to the minuscule imaginary module $L_{\delta,i}$.

Example 3.14. Let $\mathtt{C} = \mathtt{A}_l^{(1)}$ and $i \in I'$. Then $L_{\delta,i}$ is the homogeneous irreducible R_δ-module with character

$$\mathrm{ch}_q\, L_{\delta,i} = 0\big((1, 2, \dots, i-1) \circ (l, l-1, \dots, i+1)\big)i.$$

For example, $L_{\delta,1}$ and $L_{\delta,l}$ are 1-dimensional with characters

$$\mathrm{ch}_q\, L_{\delta,1} = (0, l, l-1, \dots, 1), \quad \mathrm{ch}_q\, L_{\delta,l} = (01 \dots l),$$

while for $l \geq 3$, the module $L_{\delta,l-1}$ is $(l-2)$-dimensional with character

$$\mathrm{ch}_q\, L_{\delta,l-1} = \sum_{r=0}^{l-3} (0, 1, \dots, r, l, r+1, \dots, l-1).$$

Example 3.15. Let $\mathtt{C} = \mathtt{D}_l^{(1)}$ and $i \in I'$. By Proposition 3.13, we have that $L_{\delta,i}$ is the homogeneous module $L(C_i)$, where C_i is the connected component in G_δ containing the following word:

$$\begin{cases} (0, 2, 3, \dots, l-2, l, l-1, l-2, \dots, i+1, 1, 2, \dots, i) & \text{if } i \leq l-1, \\ (0, 2, 3, \dots, l-2, l-1, 1, 2, \dots, l-2, l) & \text{if } i = l. \end{cases} \tag{3.16}$$

If the Cartan matrix $\mathtt{C}$ is non-symmetric, the explicit construction of the minuscule representations $L_{\delta,i}$ is more technical. It is explained in Chapter VI.

3.11. Imaginary tensor space and its parabolic analogue

Fix $i \in I'$, and recall from (3.7) the imaginary tensor space $M_{n,i} = L_{\delta,i}^{\circ n}$ of color i. We are going to study irreducible imaginary $R_{n\delta}$-modules of color i, i.e. composition factors of $M_{n,i}$. Since i is going to be fixed throughout, we usually simplify our notation and write M_n for $M_{n,i}$, L_δ for $L_{\delta,i}$, etc. Recall that we denote by e the height of null-root δ.

Throughout we fix an extremal word

$$\boldsymbol{i} := (i_1, \dots, i_e) \tag{3.17}$$

of L_δ so that the top degree component $(1_{\boldsymbol{i}} L_\delta)_N$ of the word space $1_{\boldsymbol{i}} L_\delta$ is 1-dimensional, see Lemma 2.61. To be more precise, for a symmetric Cartan matrix $\mathtt{C}$, the module L_δ is homogeneous by Proposition 3.13, i.e. all its word spaces are 1-dimensional, and we can take $\boldsymbol{i}$ to be an arbitrary word of L_δ. For non-symmetric $\mathtt{C}$, we make a specific choice of $\boldsymbol{i}$ as in (6.8), (6.10), (6.12), (6.14) in types $\mathtt{B}_1^{(1)}, \mathtt{C}_1^{(1)}, \mathtt{F}_4^{(1)}, \mathtt{G}_2^{(1)}$ respectively.

Pick a non-zero vector $v \in (1_{\boldsymbol{i}} L_\delta)_N$. Recall that L_δ is defined over $\mathbb{Z}$, so we may assume that $(1_{\boldsymbol{i}} L_\delta)_N = \mathcal{O} \cdot v$. Denote

$$v_n := v \otimes \cdots \otimes v \in L_\delta^{\boxtimes n}. \tag{3.18}$$

We identify $L_\delta^{\boxtimes n}$ with the submodule $1 \otimes L_\delta^{\boxtimes n} \subseteq M_n = \mathrm{Ind}_{\delta,\dots,\delta} L_\delta^{\boxtimes n}$, so v_n can will be considered as an element of M_n. Note that v_n generates M_n as an $R_{n\delta}$-module, and that

$$v_n \in (1_{\boldsymbol{i}^n} M_n)_{nN}.$$

By degrees,

$$y_r v_n = 0 \qquad (1 \leq r \leq n).$$

More generally, let $\nu = (n_1, \dots, n_a) \vDash n$. Consider the parabolic subalgebra

$$R_{\nu,\delta} := R_{n_1\delta,\dots,n_a\delta} \subseteq R_{n\delta}, \tag{3.19}$$

and consider the $R_{\nu,\delta}$-module

$$M_\nu := M_{n_1} \boxtimes \cdots \boxtimes M_{n_a},$$

with generator

$$v_\nu := v_{n_1} \otimes \cdots \otimes v_{n_a}.$$

By transitivity of induction this module embeds naturally into M_n as an $R_{\nu,\delta}$-submodule.

Lemma 3.20. $M_\nu^{\circledast} \cong M_\nu$. In particular, every composition factor of the socle of M_ν appears in its head.

Proof. This is [24, Lemma 5.6]. □

We denote by $e\nu$ the composition

$$e\nu := (en_1, \dots, en_a) \vDash en.$$

The following lemma immediately follows from the Basis Theorem 2.50:

Lemma 3.21. Let $\nu \vDash n$. Then

$$M_\nu = \bigoplus_{w \in \mathcal{D}_{e\nu}^{(e^n)}} \psi_w \otimes L_\delta^{\boxtimes n}$$

as $\mathcal{O}$-modules. In particular,

$$M_n = \bigoplus_{w \in \mathcal{D}_{en}^{(e^n)}} \psi_w \otimes L_\delta^{\boxtimes n}$$

as $\mathcal{O}$-modules.

Define

$$V_n := \mathrm{Res}_{\delta,\dots,\delta} M_n. \tag{3.22}$$

More generally, for a composition $\nu = (n_1, \dots, n_a) \vDash n$, set

$$V_\nu := \mathrm{Res}^{n_1\delta;\dots;n_a\delta}_{\delta,\dots,\delta} M_\nu \cong V_{n_1} \boxtimes \cdots \boxtimes V_{n_a}.$$

Clearly $v_\nu \in V_\nu$.

To describe V_n and V_ν, we introduce the *block permutation group* B_n as the subgroup of $\mathfrak{S}_{en}$ generated by the *block permutations* $w_1, \dots, w_{n-1}$, where w_r is the product of transpositions

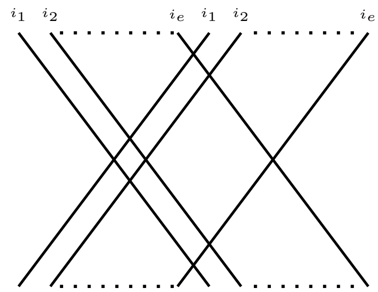
$$w_r := \prod_{b=re-e+1}^{re} (b, b+e) \qquad\qquad (1 \le r < n). \tag{3.23}$$

The group B_n is isomoprphic to the symmetric group $\mathfrak{S}_n$ via

$$\iota : \mathfrak{S}_n \xrightarrow{\sim} \mathscr{B}_n,\ s_r \mapsto w_r \qquad (1 \le r < n).$$

Note that each element $\iota(w) \in B_n$ belongs to $\mathcal{D}_{en}^{(e^n)}$. For example, if $n = 2$ then, in terms of Khovanov-Lauda diagrams [20] we have

$$\psi_{w_1} 1_{\boldsymbol{i}^2} =$$



.

Define

$$\sigma_r := \psi_{w_r} \qquad (1 \leq r < n),$$

and

$$\sigma_w := \sigma_{r_1} \dots \sigma_{r_m} \qquad (w \in \mathfrak{S}_n),$$

where we have picked a reduced decomposition $w = s_{r_1} \dots s_{r_m}$.

Let us write δ^n for $(\delta, \dots, \delta)$ with n terms. By definition, $V_n = \mathrm{Res}_{\delta^n} M_n$ is an R_{δ^n}-module.

Proposition 3.24. We have:

(i) As an R_{δ^n}-module, V_n has a filtration with $n!$ composition factors $\cong L_\delta^{\boxtimes n}$.

(ii) As an $\mathcal{O}$-module, $V_n = \bigoplus_{w \in \mathfrak{S}_n} V(w)$, where $V(w) := \sigma_w \otimes L_\delta^{\boxtimes n}$.

(iii) $1_{\boldsymbol{i}^n} M_n = \oplus_{w \in \mathfrak{S}_n} (\sigma_w \otimes (1_{\boldsymbol{i}} L_\delta)^{\boxtimes n})$

(iv) $(1_{\boldsymbol{i}^n} M_n)_{nN}$ is the top degree component of the weight space $1_{\boldsymbol{i}^n} M_n$, and $(1_{\boldsymbol{i}^n} M_n)_{nN} = \oplus_{w \in \mathfrak{S}_n} \mathcal{O} \cdot (\sigma_w \otimes v_n)$.

Proof. (i) follows by an application of the Mackey Theorem 2.63, using the property (Cus2) of L_δ and the fact that $(\delta, \delta) = 0$ to deduce that all grading shifts are trivial.

(ii) is proved by a word argument. Indeed, given words $\boldsymbol{i}^{(1)}, \dots, \boldsymbol{i}^{(n)}$ of L_δ, we have $i_1^{(1)} = \dots = i_1^{(n)} = 0$ by Lemma 3.1(iii). So, the only shuffles of $\boldsymbol{i}^{(1)}, \dots, \boldsymbol{i}^{(n)}$ which lie in V_n are permutations of these words. So the result follows from Lemma 3.21.

(iii) follows from (ii), and (iv) follows from (iii). $\square$

Corollary 3.25. All $R_{n\delta}$-endomorphisms of M_n are of degree zero, and $\dim \mathrm{End}_{R_{n\delta}}(M_n) \leq n!$.

Proof. This follows from the adjointness of the functors Ind and Res and Proposition 3.24(i). $\square$

3.2. Imaginary Schur-Weyl duality

In this section, we prove the key fact that $\mathrm{End}_{R_{n\delta}}(M_n)$ is isomorphic to the group algebra of the symmetric group $\mathfrak{S}_n$. We distinguish between the cases where the Cartan matrix $\mathtt{C}$ is symmetric and non-symmetric. The symmetric case can be handled nicely using the work [19]. For the non-symmetric case we have to appeal to the computations made in Chapter VI.

Assume in this paragraph that $\mathtt{C}$ is symmetric. We review the Kang-Kashiwara-Kim intertwiners [19] adapted to our needs. Definition 1.4.5 of [19] yields a *non-zero* $R_{2\delta}$-homomorphism

$$\tau : M_2 \to q^{(\delta,\delta)-2(\delta,\delta)_n+2s} M_2,$$

where $(\cdot,\cdot)_n$ and s are as in [19, §1.3,(1.4.8)]. (This homomorphism would be denoted r_{L_δ,L_δ} in [19].) Since $(\delta,\delta) = 0$, and all endomorphisms of M_2 are of degree zero by Corollary 3.25, it follows that $s = (\delta,\delta)_n$ and we actually have $\tau : M_2 \to M_2$. Now, it follows from [19, Proposition 1.4.4(iii)], the adjointness of Ind and Res, and

Proposition 3.24(ii) that

$$\tau(v^1 \otimes v^2) = \sigma_1 \cdot (v^2 \otimes v^1) + c(v^1, v^2)\, v^1 \otimes v^2 \qquad (v^1, v^2 \in L_\delta) \tag{3.26}$$

for some $c(v^1, v^2) \in \mathcal{O}$. In particular

$$\tau(v_2) = (\sigma_1 + c)v_2 \tag{3.27}$$

for some constant $c \in \mathcal{O}$.

Even if $\mathtt{C}$ is not symmetric, there is an endomorphism τ of M_2 with the property (3.27), see Chapter VI. So from now on we use it in all cases.

Inserting the endomorphism τ into the rth and $r+1$st positions in $M_n = L_\delta^{\circ n}$, yields endomorphisms

$$\tau_r : M_n \to M_n,\ v_n \mapsto (\sigma_r + c)v_n \qquad (1 \leq r < n). \tag{3.28}$$

We note that the elements τ_r go back to [27], where a special case of Theorem 3.29 below is checked, see [27, Theorem 4.13].

We always consider the group algebra $\mathcal{O}\mathfrak{S}_n$ as a graded algebra concentrated in degree zero.

Theorem 3.29. The endomorphisms τ_r satisfy the usual Coxeter relations of the standard generators of the symmetric group $\mathfrak{S}_n$, i.e. $\tau_r^2 = 1$, $\tau_r\tau_s = \tau_s\tau_r$ for $|r-s| > 1$, and $\tau_r\tau_{r+1}\tau_r = \tau_{r+1}\tau_r\tau_{r+1}$. This defines a (degree zero) homomorphism

$$F\mathfrak{S}_n \to \mathrm{End}_{R_{n\delta}}(M_n)^{\mathrm{op}}$$

which is an isomorphism.

Proof. If $\mathtt{C}$ is symmetric, we use the elements φ_w from [19, Lemma 1.3.1(iii)]. Then τ_r's satisfy braid relations, as noted in [19, p.16]. For the quadratic relations, by definition, τ_r^2 maps v_n to $((z'-z)^{-2s}\varphi_{w_r}^2 v_n)|_{z=z'=0}$, where the action is taking place in $(L_\delta)_z \circ (L_\delta)_{z'}$ and we consider v_n as a vector of $(L_\delta)_z \circ (L_\delta)_{z'}$ in the obvious way. Since $y_s v_n = 0$ in L_δ we have $y_s v_n = z v_n$ in $(L_\delta)_z$ for all s. So the product in the right hand side of [19, Lemma 1.3.1(iv)] is easily seen to act with the scalar $(z'-z)^{2(\delta,\delta)_n}$ on the vector $v_n \in (L_\delta)_z \circ (L_\delta)_{z'}$. Since we already know that $s = (\delta,\delta)_n$, it follows that $\tau_r^2 v_n = v_n$. Since v_n generates M_n as an $R_{n\delta}$-module, we deduce that $\tau_r^2 = 1$.

If $\mathtt{C}$ is not symmetric, then we check in Proposition 6.16 that the τ_r still satisfy the quadratic and braid relations.

For an arbitrary $\mathtt{C}$ let $w \in \mathfrak{S}_n$ with reduced decomposition $w = s_{r_1} \dots s_{r_m}$. Then in view of (3.28), for $\tau_w := \tau_{r_1} \dots \tau_{r_m}$ (the product in $\mathrm{End}_{R_{n\delta}}(M_n)^{\mathrm{op}}$), we have

$$\tau_w(v_n) = (\sigma_{r_1} + c) \dots (\sigma_{r_m} + c) v_n. \tag{3.30}$$

It follows that

$$\tau_w(v_n) = \sigma_w(v_n) + \sum_{u<w} c_u \sigma_u v_n \qquad (c_u \in \mathcal{O}), \tag{3.31}$$

where $<$ is the Bruhat order. In view of Proposition 3.24(ii), the elements $\{\tau_w \mid w \in \mathfrak{S}_n\}$ are linearly independent, and the result follows from Corollary 3.25. $\square$

In view of the theorem, we can now consider M_n as an $(R_{n\delta}, \mathcal{O}\mathfrak{S}_n)$-bimodule, with the right action $mw = \tau_w(m)$ for $m \in M_n$ and $w \in \mathfrak{S}_n$, where the linear transform action τ_w is defined by (3.30).

Corollary 3.32. We have:

(i) As R_{δ^n}-modules, $V_n \cong \bigoplus_{w \in \mathfrak{S}_n} L_\delta^{\boxtimes n} w \cong (L_\delta^{\boxtimes n})^{\oplus n!}$.

(ii) As $\mathcal{O}$-modules,

$$1_{\boldsymbol{i}^n} M_n = \oplus_{w \in \mathfrak{S}_n} (1_{\boldsymbol{i}} L_\delta)^{\boxtimes n} w \quad \text{and} \quad (1_{\boldsymbol{i}^n} M_n)_{nN} = \oplus_{w \in \mathfrak{S}_n} \mathcal{O} \cdot v_n w.$$

Proof. Since $L_\delta^{\boxtimes n}$ is irreducible as an R_{δ^n}-module, the result now follows from Theorem 3.29 and Proposition 3.24. □

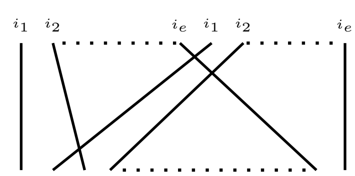
Let $u_0 \in \mathfrak{S}_{2e}$ be the minimal length element such that

$$u_0 \boldsymbol{i}^2 = \boldsymbol{i}^{\{2\}} := (i_1, i_1, i_2, i_2, \ldots, i_e, i_e).$$

Example 3.33. If C is symmetric, we know that L_δ is homogeneous, and so $i_m \neq i_{m+1}$ for all $1 \leq m < e$. So in that case, we have that

$$u_0 : n \mapsto \begin{cases} 2n-1 & \text{if } 1 \leq n \leq e, \\ 2(n-e) & \text{if } e < n \leq 2e. \end{cases} \tag{3.34}$$

In terms of Khovanov-Lauda diagrams,

$\psi_{u_0} 1_{\boldsymbol{i}} =$  .

Note that in all cases, we can write

$$\sigma_1 = \psi_{u'} \psi_{u_0} \tag{3.35}$$

for some $u' \in \mathfrak{S}_{2e}$.

Lemma 3.36. The constant c appearing in (3.27) is equal to ± 1. Moreover, $\psi_{u_0}\sigma_1 v_2 = -2c\psi_{u_0}v_2$.

Proof. By Theorem 3.29, we have $\tau_1^2 = 1$. It follows that $(\sigma_1 + c)^2 v_2 = v_2$ or $\sigma_1^2 v_2 = (-2c\sigma_1 + 1 - c^2)v_2$.

On the other hand, note that $\psi_{u_0}v_2 \neq 0$ spans the top degree component of the word space $(M_2)_{\boldsymbol{i}^{\{2\}}}$. It follows that $\psi_{u_0}\sigma_1 v_2 = d\psi_{u_0}v_2$ for some constant d. Multiplying on the left with $\psi_{u'}$ as in (3.35), this yields $\sigma_1^2 v_2 = d\sigma_1 v_2$. Comparing with the previous paragraph, we conclude that $1 - c^2 = 0$ and $d = -2c$. □

Lemma 3.37. Let $w \in \mathfrak{S}_n$, and $v^1, \dots, v^n \in L_\delta$. Then

$$(v^1 \otimes \cdots \otimes v^n)w \equiv \sigma_w(v^{w1} \otimes \cdots \otimes v^{wn}) \pmod{\sum_{u<w} \sigma_u \otimes L_\delta^{\boxtimes n}}$$

and

$$\sigma_w(v^1 \otimes \cdots \otimes v^n) \equiv (v^{w^{-1}1} \otimes \cdots \otimes v^{w^{-1}n})w \pmod{\sum_{u<w} L_\delta^{\boxtimes n} u}\,.$$

Proof. The second statement follows from the first. Further, note that the first statement in the special case where $v^1 = \cdots = v^n = v_1$ is contained in (3.31). For the general case, write $v^1 = x_1 v_1, \dots, v^n = x_n v_1$ for some $x_1, \dots, x_n \in R_\delta$. Then, using (3.31) and considering $x_1 \otimes \cdots \otimes x_n \in R_{\delta^n}$ as an element of $R_{n\delta} \supseteq R_{\delta^n}$, we get

$$\begin{aligned}
(v^1 \otimes \cdots \otimes v^n)w &= (x_1 v_1 \otimes \cdots \otimes x_n v_1)w \\
&= (x_1 \otimes \cdots \otimes x_n)((v_1 \otimes \cdots \otimes v_1)w) \\
&= (x_1 \otimes \cdots \otimes x_n)(\sigma_w(v_1 \otimes \cdots \otimes v_1) + \sum_{u<w} c_u \sigma_u(v_1 \otimes \cdots \otimes v_1)).
\end{aligned}$$

Using the fact that all our vectors belong to V_n, Proposition 3.24(ii), and the relations in $R_{n\delta}$, we have for any $u \leq w$ that

$$
\begin{aligned}
(x_1 \otimes \cdots \otimes x_n)\sigma_u(v_1 \otimes \cdots \otimes v_1) &\equiv \sigma_u(x_{u1}v_1 \otimes \cdots \otimes x_{un}v_1) \ (\mathrm{mod}\ \sum_{x<u} \sigma_x \otimes L_\delta^{\boxtimes n}) \\
&\equiv \sigma_u(v^{u1} \otimes \cdots \otimes v^{un}) \ (\mathrm{mod}\ \sum_{x<u} \sigma_x \otimes L_\delta^{\boxtimes n}),
\end{aligned}
$$

which proves the lemma. □

Remark 3.38. If $\mathtt{C}$ is symmetric, then an induction on the Bruhat order and (3.26) allow us to strengthen Lemma 3.37 as follows: for any $v^1, \dots, v^n \in L_\delta$, we have

$$
(v^1 \otimes \cdots \otimes v^n)w = \sigma_w(v^{w1} \otimes \cdots \otimes v^{wn}) + \sum_{u<w} c_u \sigma_u(v^{u1} \otimes \cdots \otimes v^{un}) \tag{3.39}
$$

for some scalars $c_u \in \mathcal{O}$ (depending on $v^1, \dots, v^n$).

If $c = -1$, it will be convenient to change the sign, so let us redefine τ_r so that

$$
\tau_r(v_n) := (c\sigma_r + 1)v_n \qquad (1 \leq r < n). \tag{3.40}
$$

Remark 3.41. The constant c in general depends on the choice of the signs $\varepsilon_{i,j}$ in the definition of the KLR algebra. For symmetric $\mathtt{C}$, it can be proved that $c = \prod_{1\leq r<s\leq e} \epsilon_{i_r,i_s}$. We are not going to need this result.

3.3. Imaginary Schur algebras

A key role in this paper is played by the *imaginary Schur algebra*

$$
\mathscr{S}_n = \mathscr{S}_{n,\mathcal{O}} := R_{n\delta}/\operatorname{Ann}_{R_{n\delta}}(M_n),
$$

and its parabolic analogue for $\nu = (n_1, \dots, n_a) \vDash n$:

$$\mathscr{S}_\nu = \mathscr{S}_{\nu,\mathcal{O}} := R_{\nu,\delta}/\operatorname{Ann}_{R_{\nu,\delta}}(M_\nu) \cong \mathscr{S}_{n_1} \otimes \cdots \otimes \mathscr{S}_{n_a}. \tag{3.42}$$

Modules over $R_{n\delta}$ which factor through $\mathscr{S}_n$ will be called *imaginary modules*. Thus the category of imaginary $R_{n\delta}$-modules is the same as the category of $\mathscr{S}_n$-modules.

We make use of the following useful criterion:

Lemma 3.43. **('Schubert's Criterion')** Let A be a (graded) algebra and $0 \to Z \to P \to M \to 0$ be a short exact sequence of (graded) A-modules with P (graded) projective. If every (degree zero) A-module homomorphism from P to M annihilates Z, then M is a (graded) projective $A/\operatorname{Ann}_A(M)$-module.

Proof. The proof given in [5, Lemma 3.2a] goes through for the graded setting. $\square$

Now we can prove our first key result.

Theorem 3.44. M_ν is a projective $\mathscr{S}_\nu$-module, and

$$\operatorname{End}_{\mathscr{S}_\nu}(M_\nu) \cong \mathcal{O}\mathfrak{S}_\nu. \tag{3.45}$$

Proof. The second statement comes from Theorem 3.29. It suffices to prove the first statement for the special case $\nu = (n)$. We will apply Schubert's Criterion to see that M_n is projective as a $\mathscr{S}_n$-module. Let $P := q^{nN} R_{n\delta} 1_{\boldsymbol{i}^n}$ Then we have a (homogeneous) surjection $\pi : P \twoheadrightarrow M_n,\ 1_{\boldsymbol{i}^n} \mapsto v_n$.

To verify the assumptions in Lemma 3.43, it suffices to show that every (homogeneous) homomorphism $\varphi : P \to M$ can be written as $\varphi = f \circ \pi$ for $f \in \operatorname{end}_{R_{n,\delta}}(M_n)$. Since P is generated by $1_{\boldsymbol{i}^n}$, it suffices to prove that $\varphi(1_{\boldsymbol{i}^n}) =$

$f(\pi(1_{\boldsymbol{i}^n})) = f(v_n)$. By Proposition 3.24(iv) and Corollary 3.32(ii), the vector $\varphi(1_{\boldsymbol{i}^n}) \in (1_{\boldsymbol{i}^n} M_n)_{nN}$ can be written as a linear combination

$$\varphi(1_{\boldsymbol{i}^n}) \in (1_{\boldsymbol{i}^n} M_n)_{nN} = \sum_{w \in \mathfrak{S}_n} c_w v_n w = \sum_{w \in \mathfrak{S}_n} c_w \tau_w(v_n) \qquad (c_w \in \mathcal{O}).$$

So we can take $f = \sum_{w \in \mathfrak{S}_n} c_w \tau_w$. Now apply Schubert's Criterion to see that M_n is projective. □

3.31. Characteristic zero theory

In this section, we assume that $\mathcal{O} = F$. If the characteristic of F is zero or greater than n, the imaginary Schur algebra is semisimple and Morita equivalent to $F\mathfrak{S}_n$:

Theorem 3.46. Assume that $\operatorname{char} F = 0$ or $\operatorname{char} F > n$. Then $\mathscr{S}_n$ is semisimple, M_n is a projective generator for $\mathscr{S}_n$, and $\mathscr{S}_n$ is Morita equivalent to $F\mathfrak{S}_n$.

Proof. Under the assumptions on the characteristic, the endomorphism algebra of the $\mathscr{S}_n$-module M_n, which we know is isomorphic to $F\mathfrak{S}_n$, is semisimple. In view of Lemma 2.33, we conclude that M_n is semisimple as an $\mathscr{S}_n$-module. By definition, the imaginary Schur algebra $\mathscr{S}_n$ is semisimple, and the theorem now follows from Morita theory. □

The theorem defines a Morita equivalence

$$\gamma_n : \mathscr{S}_n\text{-mod} \to F\mathfrak{S}_n\text{-mod}\,.$$

One can easily show that for $N \in \mathscr{S}_n$-mod and $M \in \mathscr{S}_m$-mod, there is a functorial isomorhism $\gamma_{n+m}(N \circ M) \cong \operatorname{ind}_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{n+m}} \gamma_n(N) \boxtimes \gamma_m(M)$. We will not do it now, since more general result will be obtained (for an arbitrary ground) field in Section 3.61 using Schur algebras.

3.4. Imaginary induction and restriction

Throughout the section $\nu = (n_1, \dots, n_a) \vDash n$. Recall the parabolic subalgebra $R_{\nu,\delta} \subseteq R_{n\delta}$ from (3.19).

Consider the functors of *imaginary induction* and *imaginary restriction*:

$$I_\nu^n := \mathrm{Ind}_{n_1\delta,\dots,n_a\delta}^{n\delta} : R_{\nu,\delta}\text{-mod} \to R_{n\delta}\text{-mod},$$
$${}^*I_\nu^n := \mathrm{Res}_{n_1\delta,\dots,n_a\delta}^{n\delta} : R_{n\delta}\text{-mod} \to R_{\nu,\delta}\text{-mod}\,.$$

Let $I_{\nu,\delta} \subseteq I_{n\delta}$ be the set of the concatenations $\boldsymbol{j} = \boldsymbol{j}(1)\dots\boldsymbol{j}(a)$ such that $\boldsymbol{j}(b) \in I_{n_b\delta}$ for all $b = 1, \dots, a$. Set

$$1_{\nu,\delta} := \sum_{\boldsymbol{j} \in I_{\nu,\delta}} 1_{\boldsymbol{j}}.$$

Then $1_{\nu,\delta}$ is the identity in $R_{\nu,\delta}$ and ${}^*I_\nu^n M = 1_{\nu,\delta} M$. The functor I_ν^n is left adjoint to the functor ${}^*I_\nu^n$. The following result (partially) describes the right adjoint:

Lemma 3.47. For a composition $\nu = (n_1, \dots, n_a) \vDash n$ consider the opposite composition $\nu^{\mathrm{op}} := (n_a, \dots, n_1) \vDash n$. Let V be an $R_{n\delta}$-module, and V_b be an $R_{\nu_b\delta}$-module for $b = 1, \dots, a$. Then there is a functorial isomorphism

$$\mathrm{Hom}_{R_{\nu,\delta}}({}^*I_\nu^n V, V_1 \boxtimes \cdots \boxtimes V_a) \cong \mathrm{Hom}_{R_{n\delta}}(V, I_{\nu^{\mathrm{op}}}^n V_a \boxtimes \cdots \boxtimes V_1).$$

Proof. This follows from Lemma 2.54. □

In view of Theorem 3.29, M_ν is an $(R_{\nu,\delta}, \mathcal{O}\mathfrak{S}_\nu)$-bimodule, so we can regard $I_\nu^n M_\nu$ as an $(R_{n\delta}, \mathcal{O}\mathfrak{S}_\nu)$-bimodule. Similarly, M_n is an $(R_{n\delta}, \mathcal{O}\mathfrak{S}_n)$-bimodule, so we can regard ${}^*I_\nu^n M_n$ as an $(R_{\nu,\delta}, \mathcal{O}\mathfrak{S}_n)$-bimodule. Recall that we identify $L_\delta^{\boxtimes n}$ with a natural R_{δ^n}-submodule of M_n.

Lemma 3.48. We have:

(i) $I_\nu^n M_\nu \cong M_n$ as $(R_{n\delta}, \mathcal{O}\mathfrak{S}_\nu)$-bimodules.

(ii) ${}^*I_\nu^n M_n \cong M_\nu \otimes_{\mathcal{O}\mathfrak{S}_\nu} \mathcal{O}\mathfrak{S}_n$ as $(R_{\nu,\delta}, \mathcal{O}\mathfrak{S}_n)$-bimodules.

(iii) We have the following decompositions of $\mathcal{O}$-modules:

$$
{}^*I_\nu^n M_n = \bigoplus_{x \in \mathcal{D}_{e\nu}^{(e^n)},\ y \in {}^\nu\mathcal{D}_n} \psi_x \sigma_y L_\delta^{\boxtimes n} = \bigoplus_{x \in \mathcal{D}_{e\nu}^{(e^n)},\ y \in {}^\nu\mathcal{D}_n} \psi_x L_\delta^{\boxtimes n} y
$$

Proof. (i) By transitivity of induction, $I_\nu^n M_\nu \cong M_n$ as $R_{n\delta}$-modules. By definition, the isomorphism is compatible with the right $\mathcal{O}\mathfrak{S}_\nu$-module structures.

(iii) Recall the decomposition $M_n = \bigoplus_{w \in \mathcal{D}_{en}^{(e^n)}} \psi_w L_\delta^{\boxtimes n}$ from Lemma 3.21. Note, using word argument, that $\psi_w L_\delta^{\boxtimes n} \subseteq {}^*I_\nu^n M_n = 1_{\nu,n} M_n$ for $w \in \mathcal{D}_{en}^{(e^n)}$ if and only if w can be written (uniquely) as $w = x\iota(y)$, where $x \in \mathcal{D}_{e\nu}^{(e^n)}$, $y \in {}^\nu\mathcal{D}_n$, and $\ell(w) = \ell(x) + \ell(\iota(y))$, and otherwise $\psi_w L_\delta^{\boxtimes n} \cap 1_{\nu,n} M_n = 0$. This gives the first decomposition.

To deduce the second decomposition from the first, observe by a word argument, that each $\psi_x L_\delta^{\boxtimes n} y \subseteq {}^*I_\nu^n M_n$. Next, note using Lemma 3.21 that each $\psi_x \sigma_y L_\delta^{\boxtimes n} \cong L_\delta^{\boxtimes n}$ as vector spaces. As $\psi_x L_\delta^{\boxtimes n} \to \psi_x L_\delta^{\boxtimes n} y$ is an invertble linear transformation, we also have that $\psi_x L_\delta^{\boxtimes n} y \cong L_\delta^{\boxtimes n}$ as vector spaces. Now, by dimensions, it suffices to prove that the sum $\sum_{x \in \mathcal{D}_{e\nu}^{(e^n)}, y \in {}^\nu\mathcal{D}_n} \psi_x L_\delta^{\boxtimes n} y$ is direct. Well, if

$$
\sum_{x \in \mathcal{D}_{e\nu}^{(e^n)}, y \in {}^\nu\mathcal{D}_n} \psi_x v_{x,y} y = 0 \tag{3.49}
$$

with $v_{x,y} \in L_\delta^{\boxtimes n}$, let x, y be chosen so that $xy \in \mathfrak{S}_{en}$ is Bruhat maximal with $v_{x,y} \neq 0$. Rewriting the left hand side of (3.49) using Lemma 3.37, gives $\psi_x \sigma_y v_{x,y} + (*) = 0$, where

$$(*) \in \sum_{x' \in \mathcal{D}_{e\nu}^{(e^n)},\ y' \in {}^{\nu}\mathcal{D}_n,\ x'y' \not\geq xy} \psi_{x'} \sigma_{y'} L_\delta^{\boxtimes n}.$$

We get a contradiction.

(ii) follows from (iii). □

In view of Lemma 3.48(ii), the $R_{\nu,\delta}$-action on ${}^*I_\nu^n M_n$ factors through the quotient $\mathscr{S}_\nu$, so ${}^*I_\nu^n M_n$ is a $\mathscr{S}_\nu$-module in a natural way. In Corollary 3.75 we will prove a stronger result that the functor I_ν^n sends $\mathscr{S}_\nu$-modules to $\mathscr{S}_n$-modules and the functor ${}^*I_\nu^n$ sends $\mathscr{S}_n$-modules to $\mathscr{S}_\nu$-modules.

Corollary 3.50. The following pairs of functors are isomorphic:

(i) $I_\nu^n \circ (M_\nu \otimes_{\mathcal{O}\mathfrak{S}_\nu} ?)$ and $(M_n \otimes_{\mathcal{O}\mathfrak{S}_n} ?) \circ \mathrm{ind}_{\mathfrak{S}_\nu}^{\mathfrak{S}_n} : \mathcal{O}\mathfrak{S}_\nu\text{-mod} \to R_{n,\delta}\text{-mod}$.

(ii) ${}^*I_\nu^n \circ (M_n \otimes_{\mathcal{O}\mathfrak{S}_n} ?)$ and $(M_\nu \otimes_{\mathcal{O}\mathfrak{S}_\nu} ?) \circ \mathrm{res}_{\mathfrak{S}_\nu}^{\mathfrak{S}_n} : \mathcal{O}\mathfrak{S}_n\text{-mod} \to R_{\nu,\delta}\text{-mod}$.

Proof. (i) Take $N \in \mathcal{O}\mathfrak{S}_\nu$-mod. Using Lemma 3.48(i), we have natural isomorphisms

$$\begin{aligned} M_n \otimes_{\mathcal{O}\mathfrak{S}_n} \mathrm{ind}_{\mathfrak{S}_\nu}^{\mathfrak{S}_n} N &= M_n \otimes_{\mathcal{O}\mathfrak{S}_n} \mathcal{O}\mathfrak{S}_n \otimes_{\mathcal{O}\mathfrak{S}_\nu} N \cong M_n \otimes_{\mathcal{O}\mathfrak{S}_\nu} N \\ &\cong (I_\nu^n M_\nu) \otimes_{\mathcal{O}\mathfrak{S}_\nu} N \cong I_\nu^n (M_\nu \otimes_{\mathcal{O}\mathfrak{S}_\nu} N), \end{aligned}$$

as required.

(ii) Using Lemma 3.48(ii), for an $\mathcal{O}\mathfrak{S}_n$-module N, we have natural isomorphisms

$$\begin{aligned} {}^*I_\nu^n (M_n \otimes_{\mathcal{O}\mathfrak{S}_n} N) &\cong ({}^*I_\nu^n M_n) \otimes_{\mathcal{O}\mathfrak{S}_n} N \cong (M_\nu \otimes_{\mathcal{O}\mathfrak{S}_\nu} \mathcal{O}\mathfrak{S}_n) \otimes_{\mathcal{O}\mathfrak{S}_n} N \\ &\cong M_\nu \otimes_{\mathcal{O}\mathfrak{S}_\nu} \mathrm{res}_{\mathfrak{S}_\nu}^{\mathfrak{S}_n} N, \end{aligned}$$

as required. $\square$

We need a version of the Mackey Theorem for imaginary induction and restriction. Recall the notation from Section 2.22. In particular, given two compositions $\lambda, \mu \vDash n$ and $x \in {}^{\lambda}\mathcal{D}_n^{\mu}$ we have compositions $\lambda \cap x\mu$ and $x^{-1}\lambda \cap \mu$. Moreover, the corresponding parabolic algebras $R_{\lambda\cap x\mu,\delta}$ and $R_{x^{-1}\lambda\cap\mu,\delta}$ are naturally isomorphic via an isomorphism

$$\Pi_x : R_{\lambda\cap x\mu,\delta} \xrightarrow{\sim} R_{x^{-1}\lambda\cap\mu,\delta}, \tag{3.51}$$

which permutes the components. Composing with this isomorphism we get a functor

$$R_{x^{-1}\lambda\cap\mu,\delta}\text{-mod} \to R_{\lambda\cap x\mu,\delta}\text{-mod},\ \ M \mapsto {}^{x}M.$$

Note that we do not need any grading shifts. With this notation, we have:

Theorem 3.52. **(Imaginary Mackey Theorem)** Let $\lambda, \mu \vDash n$, and M be an $\mathscr{S}_\mu$-module. Then there is a filtration of ${}^*I_\lambda^n I_\mu^n M$ with subfactors

$$I^{\lambda}_{\lambda\cap x\mu}\, {}^{x}({}^*I^{\mu}_{x^{-1}\lambda\cap\mu} M) \qquad (x \in {}^{\lambda}\mathcal{D}_n^{\mu}).$$

Proof. This follows from the usual Mackey Theorem 2.63 using the fact that all composition factors of M are imaginary in the sense of (Cusp2). $\square$

3.5. Imaginary Howe duality

The imaginary Schur-Weyl duality described previously is not quite sufficient to describe the composition factors of M_n at least when the characteristic of the ground field is positive. The problem is that even though M_n is a projective module over the imaginary Schur algebra, it is not in general a projective generator. We construct the desired projective generator as a direct sum $Z = \bigoplus_{\nu\in X(h,n)} Z^\nu$ of 'imaginary divided powers' modules, and the endomorphism algebra of Z turns out to be the classical Schur algebra $S_{h,n}$. This leads to an equivalence of module categories for the imaginary and the classical Schur algebras. First, we need to develop a theory of "Gelfand-Grave modules".

Throughout the section, $\nu = (n_1, \dots, n_a) \vDash n \in \mathbb{Z}_{>0}$.

3.51. Gelfand-Graev modules

Denote by w_0 the longest element of $\mathfrak{S}_n$, and for $i \in I$, consider the element

$$\gamma_{n,i} := \psi_{w_0} \prod_{m=1}^{n} y_m^{m-1} \in R_{n\alpha_i},$$

and the $R_{n\alpha_i}$-module

$$\Gamma_{n,i} := q_i^{-n(n-1)/2} R_{n\alpha_i}\gamma_{n,i}.$$

The following is well-known:

Proposition 3.53. The algebra $R_{n\alpha_i}$ is isomorphic to the affine nil-Hecke algebra and has unique (up to isomorphism and degree shift) irreducible module, denoted $L(i^n)$ with formal character $[n]^!_i(i^n)$. Moreover:

(i) $\gamma_{n,i}$ is a primitive idempotent in $R_{n\alpha_i}$. In particular, $\Gamma_{n,i}$ is a projective indecomposable $R_{n\alpha_i}$-module. In fact, $\Gamma_{n,i}$ is the projective cover of $L(i^n)$.

(ii) $\Gamma_{n,i}$ is isomorphic to the polynomial representation of the affine nil-Hecke algebra $R_{n\alpha_i}$ (with degree shifted down by $(\alpha_i, \alpha_i)n(n-1)/4$); in particulr, $\Gamma_{n,i}$ has an

$\mathcal{O}$-basis

$$\{y_1^{b_1} \dots y_n^{b_n} \gamma_{n,i} \mid b_1, \dots, b_n \in \mathbb{Z}_{\geq 0}\},$$

and the formal character $q_i^{-n(n-1)/2}(1-q_i^2)^{-n}(i^n)$.

(iii) Let $(m_1, \dots, m_s) \models n$. Then

$$\mathrm{Res}_{m_1\alpha_i,\dots,m_s\alpha_i} \Gamma_{n,i} \cong q_i^{-n(n-1)/2+\sum_{i=1}^{s} m_i(m_i-1)/2} \Gamma_{m_1,i} \boxtimes \cdots \boxtimes \Gamma_{m_s,i}.$$

Proof. For (i) and (ii), see for example [20, section 2.2] or [26, Theorem 4.12]. Part (iii) follows easily from (i) and (ii) by characters. □

Now, recall the word $\boldsymbol{i} = (i_1, \dots, i_e)$ from (3.17). We rewrite:

$$\boldsymbol{i} = j_1^{m_1} \dots j_r^{m_r} \tag{3.54}$$

with $j_k \neq j_{k+1}$ for all $k = 1, 2, \dots, r-1$. Define the *Gelfand-Graev idempotent*:

$$\gamma_{n,\delta} := \gamma_{nm_1,j_1} \otimes \gamma_{nm_2,j_2} \otimes \cdots \otimes \gamma_{nm_r,j_r} \in R_{nm_1\alpha_{j_1},nm_2\alpha_{j_2},\dots,nm_r\alpha_{j_r}} \subseteq R_{n\delta}.$$

By Proposition 3.53(i),

$$\Gamma_n := \prod_{k=1}^{r} q_{j_k}^{-nm_k(nm_k-1)/2} R_{n\delta}\gamma_{n,\delta} \cong \Gamma_{nm_1,j_1} \circ \Gamma_{nm_2,j_2} \circ \cdots \circ \Gamma_{nm_r,j_r}$$

is a projective $R_{n\delta}$-module which we refer to as the *Gelfand-Graev* module. By Proposition 3.53(ii),

$$\mathrm{ch}_q\, \Gamma_n = \prod_{k=1}^{r} q_{j_k}^{-nm_k(nm_k-1)/2}(1-q_{j_k}^2)^{-nm_k}\, j_1^{nm_1} \circ j_2^{nm_2} \circ \dots \circ j_r^{nm_r}.$$

More generally, we consider the *(parabolic) Gelfand-Graev idempotent*

$$\gamma_{\nu,\delta} := \gamma_{n_1,\delta} \otimes \cdots \otimes \gamma_{n_a,\delta} \in R_{\nu,\delta} \subseteq R_{n\delta}, \tag{3.55}$$

and the projective $R_{\nu,\delta}$-module

$$\Gamma_\nu := \prod_{b=1}^{a} \prod_{k=1}^{r} q_{j_k}^{-n_b m_k (n_b m_k - 1)/2} R_{\nu,\delta}\gamma_{\nu,\delta} \cong \Gamma_{n_1} \boxtimes \cdots \boxtimes \Gamma_{n_a}$$

with character

$$\begin{aligned}\operatorname{ch}_q \Gamma_\nu = &\prod_{b=1}^{a} \prod_{k=1}^{r} q_{j_k}^{-n_b m_k (n_b m_k - 1)/2} (1 - q_{j_k}^2)^{-n_b m_k} \\ &\times (j_1^{n_1 m_1} \circ \cdots \circ j_r^{n_1 m_r}) \dots (j_1^{n_a m_1} \circ \cdots \circ j_r^{n_a m_r}).\end{aligned}$$

Lemma 3.56. We have

$$\operatorname{Res}_{nm_1\alpha_{j_1},\dots,nm_r\alpha_{j_r}} M_n \simeq L(j_1^{nm_1}) \boxtimes \cdots \boxtimes L(j_r^{nm_r}).$$

Proof. The lemma is obtained by an application of the Mackey Theorem or a character computation. $\square$

Proposition 3.57. We have:

(i) ${}^*I_\nu^n \Gamma_n \cong \Gamma_\nu \oplus X$, where X is a projective module over $R_{\nu,\delta}$ such that $\operatorname{Hom}_{R_{\nu,\delta}}(X, M_\nu) = 0$.

(ii) $\operatorname{Hom}_{R_{n\delta}}(\Gamma_n, M_n) \cong \mathcal{O}$.

(iii) We have an isomorphism of right modules over $\mathcal{O}\mathfrak{S}_n = \mathrm{End}_{R_{n\delta}}(M_n)$:

$$\mathrm{Hom}_{R_{n\delta}}(\Gamma_n, M_n) \cong \mathtt{sgn}_{\mathfrak{S}_n}.$$

Proof. (ii) By Frobenius Reciprocity, $\mathrm{Hom}_{R_{n\delta}}(\Gamma_n, M_n)$ is isomorphic to

$$\mathrm{Hom}_{R_{nm_1\alpha_{j_1},\dots,nm_r\alpha_{j_r}}}(\Gamma_{nm_1,j_1} \boxtimes \cdots \boxtimes \Gamma_{nm_r,j_r}, \mathrm{Res}_{nm_1\alpha_{j_1},\dots,nm_r\alpha_{j_r}} M_n).$$

By Proposition 3.53, $\Gamma_{nm_1,j_1} \boxtimes \cdots \boxtimes \Gamma_{nm_r,j_r}$ is the projective cover of $L := L(j_1^{nm_1}) \boxtimes \cdots \boxtimes L(j_r^{nm_r})$. The result now follows from (3.6) and Lemma 3.56.

(i) By the Mackey Theorem and Proposition 3.53(iii), the module

$$^*I_\nu^n \Gamma_n = \mathrm{Res}_{n_1\delta,\dots,n_a\delta} \mathrm{Ind}_{nm_1\alpha_{j_1},\dots,nm_r\alpha_{j_r}} \Gamma_{nm_1,j_1} \boxtimes \cdots \boxtimes \Gamma_{nm_r,j_r}$$

has filtration with factors of the form

$$(\Gamma_{m_{1,1},j_1} \circ \Gamma_{m_{1,2},j_2} \circ \cdots \circ \Gamma_{m_{1,r},j_r}) \boxtimes \cdots \boxtimes (\Gamma_{m_{a,1},j_1} \circ \Gamma_{m_{a,2},j_2} \circ \cdots \circ \Gamma_{m_{a,r},j_r}),$$

where $\sum_{s=1}^r m_{t,s}\alpha_{j_s} = n_t\delta$ for all $t = 1,\dots,a$, $\sum_{t=1}^a m_{t,s} = nm_s$ for all $s = 1,\dots,r$, and we ignore grading shifts. All of this modules are projective, so we actually have a direct sum. One of the terms is Γ_ν—it corresponds to taking $m_{t,s} = n_t m_s$ for all $1 \le t \le a, 1 \le s \le r$.

Now, note, using Lemma 3.47 that

$$\begin{aligned}\mathrm{Hom}_{R_{\nu,\delta}}(^*I_\nu^n \Gamma_n, M_\nu) &= \mathrm{Hom}_{R_{\nu,\delta}}(^*I_\nu^n \Gamma_n, M_{n_1} \boxtimes \cdots \boxtimes M_{n_a})\\ &\cong \mathrm{Hom}_{n\delta}(\Gamma_n, I_{\nu^{\mathrm{op}}}^n(M_{n_a} \boxtimes \cdots \boxtimes M_{n_1})) \cong \mathrm{Hom}_{n\delta}(\Gamma_n, M_n).\end{aligned}$$

By part (ii), the latter Hom-space is isomorphic to $\mathcal{O}$. On the other hand, again by (ii), we have that $\mathrm{Hom}_{\nu,\delta}(\Gamma_\nu, M_\nu) \cong \mathcal{O}$. We conclude that Γ_ν appears in ${}^*I_\nu^n\Gamma_n$ with graded multiplicity 1, and other projective summands do not have non-trivial homomorphisms to M_ν, as required.

(iii) Note that Γ_n is generated by a vector of the word $\boldsymbol{j} := (j_1^{nm_1}, \dots, j_r^{nm_r})$. Under any homomorphism from Γ_n to M_n, this generating vector is mapped to a vector in the word space $1_{\boldsymbol{j}} M_n$. So, it suffices to show that an arbitrary $w \in \mathfrak{S}_n$ acts on the whole word space $1_{\boldsymbol{j}} M_n$ with the scalar $\mathtt{sgn}(w)$. Let u be the shortest element of $\mathfrak{S}_{ne}$ such that $u \cdot \boldsymbol{i}^n = \boldsymbol{j}$. Then any other vector in $1_{\boldsymbol{j}} M_n$ can be written in the form $\{\psi_x \psi_u v_n\}$ for some x. So it suffices to prove that $\psi_u v_n s_r = -\psi_u v_n$ for an arbitrary simple generator s_r of $\mathfrak{S}_n$ with $1 \le r < n$.

Recall the definition of $u_0 \in \mathfrak{S}_{2e}$ from Lemma 3.36. For $1 \le r < n$, let

$$\varphi_r : \mathfrak{S}_{2e} \simeq \mathfrak{S}_{1^{(r-1)e}} \times \mathfrak{S}_{2e} \times \mathfrak{S}_{1^{(n-r-1)e}} \hookrightarrow \mathfrak{S}_{ne}$$

be the natural embedding and $u_0(r) := \varphi_r(u_0)$.

There exists $u' \in \mathfrak{S}_d$ such that $\psi_u = \psi_{u'} \psi_{u_0(r)}$. So by Lemma 3.36, we have

$$\psi_u \sigma_r(v_n) = \psi_{u'} \psi_{u_0(r)} \sigma_r v_n = -2c \psi_{u'} \psi_{u_0(r)} v_n = -2c \psi_u v_n.$$

Therefore, using (3.40), we get

$$\begin{aligned} \psi_u v_n s_r &= \psi_u \tau_r(v_n) = \psi_u (c\sigma_r + 1)(v_n) \\ &= c\psi_u \sigma_r(v_n) + \psi_u v_n = -2\psi_u v_n + \psi_u v_n = -\psi_u v_n, \end{aligned}$$

completing the proof. □

Remark 3.58. In type $\mathtt{A}_l^{(1)}$, we can strengthen Proposition 3.57(i) to claim that ${}^*I_\nu^n \Gamma_n \cong \Gamma_\nu$. Indeed, in this case each simple root appears in δ with multiplicity one, from which one can easily deduce that $\mathrm{ch}_q {}^*I_\nu^n \Gamma_n = \mathrm{ch}_q \Gamma_\nu$.

3.52. Imaginary symmetric, divided, and exterior powers

Let

$$\mathtt{x}_n := \sum_{g \in \mathfrak{S}_n} g \qquad \text{and} \qquad \mathtt{y}_n := \sum_{g \in \mathfrak{S}_n} \mathtt{sgn}(g) g.$$

Define *imaginary symmetric, divided, and exterior powers* as the following $R_{n\delta}$-modules:

$$S_n := M_n / \operatorname{span}\{mg - \mathtt{sgn}(g)m \mid g \in \mathfrak{S}_n,\ m \in M_n\},$$

$$Z_n := \{m \in M_n \mid mg - \mathtt{sgn}(g)m = 0 \text{ for all } g \in \mathfrak{S}_n\},$$

$$\Lambda_n := M_n \mathtt{x}_n.$$

These $R_{n\delta}$-modules factor through the quotient $\mathscr{S}_n$ to induce well-defined $\mathscr{S}_n$-modules. It is perhaps unfortunate that our symmetric powers correspond to the sign representation and our exterior powers correspond to the trivial representation; curiously this is the same phenomenon as for finite GL_n, cf. [5, section 3.3].

Note that $\Lambda_n = M_n \mathtt{x}_n \neq 0$ and $M_n \mathtt{y}_n \neq 0$ for example by Theorem 3.29. Finally, by definition, $M_n \mathtt{y}_n$ is a submodule of Z_n. Recall the word $\boldsymbol{i}$ from (3.17).

Lemma 3.59. We have $1_{\boldsymbol{i}^n} \Lambda_n = (1_{\boldsymbol{i}} L_\delta)^{\boxtimes n} \mathtt{x}_n$ and $1_{\boldsymbol{i}^n} M_n \mathtt{y}_n = (1_{\boldsymbol{i}} L_\delta)^{\boxtimes n} \mathtt{y}_n$. Moreover, if $\mathcal{O} = F$ (i.e. $\mathcal{O}$ is a field) then Λ_n and $M_n \mathtt{y}_n$ are irreducible $R_{n\delta}$-modules.

Proof. We prove the lemma for Λ_n, the argument for $M_n \mathtt{y}_n$ being similar.

By Corollary 3.32, the word space $1_{\boldsymbol{i}^n} M_n$ is isomorphic to free right module over $\mathcal{O}\mathfrak{S}_n$ with basis $(1_{\boldsymbol{i}} L_\delta)^{\boxtimes n}$, and under this isomorphism the generator $v_n \in M_n$ corresponds to $1 \in \mathcal{O}\mathfrak{S}_n$. Therefore $1_{\boldsymbol{i}^n}\Lambda_n = 1_{\boldsymbol{i}^n} M_n \mathtt{x}_n = (1_{\boldsymbol{i}} L_\delta)^{\boxtimes n}\mathtt{x}_n$.

We know that M_n is a projective $\mathscr{S}_n$-module and every composition factor of its socle appears in its head, see Theorem 3.44 and Lemma 3.20. Also, left ideal $F\mathfrak{S}_n\mathtt{x}_n$ is an irreducible $F\mathfrak{S}_n$-module. Using these remarks, the irreducibility of $\Lambda_n = M_n\mathtt{x}_n$ follows from Lemmas 2.33 and 2.32. □

Lemma 3.60. We have $S_n \cong M_n \otimes_{\mathcal{O}\mathfrak{S}_n} \mathtt{sgn}_{\mathfrak{S}_n}$. Moreover, if $\mathcal{O} = F$ then:

(i) S_n has simple head isomorphic to $M_n\mathtt{y}_n$, and no other composition factors of S_n are isomorphic to quotients of M_n.

(ii) $Z_n \cong (S_n)^{\circledast}$.

(iii) Z_n has simple socle isomorphic to $M_n\mathtt{y}_n$, and no other composition factors of Z_n are isomorphic to submodules of M_n.

Proof. Write $M := M_n$ for short. By definition, $M \otimes_{\mathcal{O}\mathfrak{S}_n} \mathtt{sgn}_{\mathfrak{S}_n}$ is the quotient of $M \otimes_{\mathcal{O}} \mathtt{sgn}_{\mathfrak{S}_n}$ by $\mathrm{span}\{mg \otimes 1 - \mathtt{sgn}(g)m \otimes 1 \mid g \in \mathfrak{S}_n,\ m \in M\}$. If we identify $M \otimes_{\mathcal{O}} \mathtt{sgn}_{\mathfrak{S}_n}$ and M as $\mathcal{O}$-modules in the natural way, this immediately gives the first statement.

(i) Let α, β and A_P be the functors defined in Section 2.26, taking the projective module P to be the $\mathscr{S}_n$-module M of Theorem 3.44. Then, the previous paragraph shows that $S_n \cong \beta(\mathtt{sgn}_{\mathfrak{S}_n})$. As every composition factor of the socle of M appears in its head by Lemma 3.20, we conclude using Lemmas 2.33 and 3.59 that

$$A_P \circ \beta(\mathtt{sgn}_{\mathfrak{S}_n}) \cong A_P(S_n) \cong M\mathtt{y}_n$$

is an irreducible $\mathscr{S}_n$-module. We deduce that $M\mathtt{y}_n$ appears in the head of S_n and no other composition factors of S_n appear in the head of M. Since S_n is a quotient of M, this means that S_n has simple head.

(ii) In view of Lemma 3.20, we choose some isomorphism $\varphi : M \to M^{\circledast}$ of $R_{n\delta}$-modules. This choice induces an isomorphism $\kappa : \mathrm{End}_{R_{n\delta}}(M) \to \mathrm{End}_{R_{n\delta}}(M^{\circledast})$ with $f\kappa(\theta) = \varphi((\varphi^{-1}f)\theta)$ for all $f \in M^{\circledast}$ and $\theta \in \mathrm{End}_{R_{n\delta}}(M)$, writing endomorphisms on the right. On the other hand, there is a natural anti-isomorphism $\sharp : \mathrm{End}_{R_{n\delta}}(M) \to \mathrm{End}_{R_{n\delta}}(M^{\circledast})$ defined by letting $\theta^{\sharp}$ be the dual map to $\theta \in \mathrm{End}_{R_{n\delta}}(M)$. Now if we set $\sigma := \kappa^{-1} \circ \sharp$, we have defined an anti-automorphism of $F\mathfrak{S}_n = \mathrm{End}_{R_{n\delta}}(M)$. Define a non-degenerate bilinear form on M by $(v, w) := \varphi(v)(w)$ for $v, w \in M$. For any $h \in F\mathfrak{S}_n$ we have

$$(v\sigma(h), w) = (\varphi^{-1}(\varphi(v)h^{\sharp}), w) = (\varphi(v)h^{\sharp})(w) = \varphi(v)(wh) = (v, wh).$$

By definition, $S_n = M/\operatorname{span}\{vh - \mathtt{sgn}(h)v \mid h \in F\mathfrak{S}_n,\ v \in M\}$. So

$$\begin{aligned} S_n^{\circledast} &\cong \{w \in M \mid (w, vh - \mathtt{sgn}(h)v) = 0 \text{ for all } v \in M, h \in F\mathfrak{S}_n\} \\ &= \{w \in M \mid (w\sigma(h) - \mathtt{sgn}(h)w, v) = 0 \text{ for all } v \in M, h \in F\mathfrak{S}_n\} \\ &= \{w \in M \mid wh = \mathtt{sgn}(\sigma(h))w \text{ for all } h \in F\mathfrak{S}_n\}. \end{aligned}$$

To complete the proof, it remains to show that $\mathtt{sgn}(\sigma(h)) = \mathtt{sgn}(h)$ for all $h \in F\mathfrak{S}_n$. We can consider $\mathtt{sgn} \circ \sigma$ as a linear representation of $F\mathfrak{S}_n^{\mathrm{op}}$, so we either have $\mathtt{sgn} \circ \sigma = \mathtt{sgn}$ as required, or $\mathtt{sgn} \circ \sigma = \mathrm{id}$. In the latter case, $S_n^{\circledast}$ contains Λ_n as an irreducible submodule, see Lemma 3.59, whence S_n contains $M\mathtt{x}_n$ in its head. But this is not so by (i), unless $M\mathtt{x}_n \cong M\mathtt{y}_n$, in which case, applying Lemma 2.32, the sign representation of $F\mathfrak{S}_n$ is isomorphic to its trivial representation and we are done.

(iii) This follows from (ii) by dualizing, using (i). $\square$

3.53. Parabolic analogues

For $\nu = (n_1, \dots, n_a) \vDash n$, let

$$\mathtt{x}_\nu := \sum_{g \in \mathfrak{S}_\nu} g \qquad \text{and} \qquad \mathtt{y}_\nu := \sum_{g \in \mathfrak{S}_\nu} \mathtt{sgn}(g) g.$$

We have the parabolic analogues of symmetric, divided and exterior powers, namely the $\mathscr{S}_\nu$-modules

$$\begin{aligned} S_\nu &:= M_\nu / \operatorname{span}\{mg - \mathtt{sgn}(g)m \mid g \in \mathfrak{S}_\nu,\ m \in M_\nu\}, \\ Z_\nu &:= \{m \in M_\nu \mid mg - \mathtt{sgn}(g)m = 0 \text{ for all } g \in \mathfrak{S}_\nu\}, \\ \Lambda_\nu &:= M_\nu \mathtt{x}_\nu. \end{aligned}$$

In view of (3.42), if $\nu = (n_1, \dots, n_a)$ then $S_\nu \cong S_{n_1} \boxtimes \cdots \boxtimes S_{n_a}$, and similarly for Z, Λ. In view of this observation, the basic properties of S_ν, Z_ν and Λ_ν follow directly from Lemmas 3.59 and 3.60.

Lemma 3.61. For any $\nu \vDash n$, we have ${}^*I_\nu^n S_n \cong S_\nu$ and ${}^*I_\nu^n Z_n \cong Z_\nu$.

Proof. We prove the first statement, the second one then follows from Lemma 3.60(ii) since the restriction functor ${}^*I_\nu^n$ commutes with duality. By Lemma 3.60, we have $S_n \cong M_n \otimes_{F\mathfrak{S}_n} \mathtt{sgn}_{\mathfrak{S}_n}$ and $S_\nu \cong M_\nu \otimes_{F\mathfrak{S}_\nu} \mathtt{sgn}_\nu$. Now, using Corollary 3.50(ii), we get

$${}^*I_\nu^n S_n \cong {}^*I_\nu^n (M_n \otimes_{F\mathfrak{S}_n} \mathtt{sgn}_{\mathfrak{S}_n}) \cong M_\nu \otimes_{F\mathfrak{S}_\nu} (\operatorname{res}^{\mathfrak{S}_n}_{\mathfrak{S}_\nu} \mathtt{sgn}_{\mathfrak{S}_n}) \cong M_\nu \otimes_{F\mathfrak{S}_\nu} \mathtt{sgn}_{\mathfrak{S}_\nu},$$

which is isomorphic to S_ν, as required. $\square$

Define $R_{n\delta}$-modules

$$S^\nu := M_n/\operatorname{span}\{mg - \mathtt{sgn}(g)m \mid g \in \mathfrak{S}_\nu,\ m \in M_n\},$$
$$Z^\nu := \{m \in M_n \mid mg = \mathtt{sgn}(g)m \text{ for all } g \in \mathfrak{S}_\nu\},$$
$$\Lambda^\nu := M_n \mathtt{x}_\nu.$$

If we identify $M_n = I_\nu^n M_\nu$ as $(R_{n\delta}, \mathcal{O}\mathfrak{S}_\nu)$-bimodules as in Lemma 3.48(i), it is easy to check that the quotient S^ν of M_n is identified with the quotient $I_\nu^n S_\nu$ of $I_\nu^n M_\nu$. Similarly we get the analogous results for Z and Λ. Thus:

$$S^\nu \cong I_\nu^n S_\nu, \quad Z^\nu \cong I_\nu^n Z_\nu, \quad \Lambda^\nu \cong I_\nu^n \Lambda_\nu. \tag{3.62}$$

Note that Z^ν contains $M_n \mathtt{y}_\nu$ as a submodule.

Lemma 3.63. We have $S^\nu \cong M_n \otimes_{\mathcal{O}\mathfrak{S}_n} (\operatorname{ind}_{\mathfrak{S}_\nu}^{\mathfrak{S}_n} \mathtt{sgn}_{\mathfrak{S}_\nu})$. Moreover, if $\mathcal{O} = F$ then:

(i) $(S^\nu)^\circledast \cong Z^{\nu^{\mathrm{op}}}$, where $\nu^{\mathrm{op}} = (n_a, \dots, n_1)$ is the opposite composition;

(ii) No composition factors of $Z^\nu/M_n \mathtt{y}_\nu$ are isomorphic to submodules of M_n.

Proof. By Lemma 3.60, we have $S_\nu \cong M_\nu \otimes_{F\mathfrak{S}_\nu} \mathtt{sgn}_{\mathfrak{S}_\nu}$. Therefore, using Corollary 3.50(i) and (3.62) , we get that

$$M_n \otimes_{\mathcal{O}\mathfrak{S}_n} (\operatorname{ind}_{\mathfrak{S}_\nu}^{\mathfrak{S}_n} \mathtt{sgn}_{\mathfrak{S}_\nu}) \cong I_\nu^n S_\nu \cong S^\nu.$$

(i) Using (3.62), Lemma 2.55, and Lemma 3.60(ii), we have

$$\begin{aligned}(S^\nu)^{\circledast} &\cong I^n_\nu(S_{n_1}\boxtimes\cdots\boxtimes S_{n_a})^{\circledast}\\ &\cong I^n_{\nu^{\mathrm{op}}}(S^{\circledast}_{n_a}\boxtimes\cdots\boxtimes S^{\circledast}_{n_1})\\ &\cong I^n_{\nu^{\mathrm{op}}}(Z_{n_a}\boxtimes\cdots\boxtimes Z_{n_1})\cong Z^{\nu^{\mathrm{op}}}.\end{aligned}$$

(ii) Let α, β and A_P be the functors defined in Section 2.26, taking the projective module P to be the $\mathscr{S}_n$-module M_ν of Theorem 3.44. Now, $\mathrm{ind}_{\mathfrak{S}_\nu}^{\mathfrak{S}_n}\mathtt{sgn}_{\mathfrak{S}_\nu}$ is the left ideal $F\mathfrak{S}_n\mathtt{y}_\nu$ of $F\mathfrak{S}_n$. So, by Lemmas 3.20 and 2.33, we get $A_P\circ\beta(\mathrm{ind}_{\mathfrak{S}_\nu}^{\mathfrak{S}_n}\mathtt{sgn}_{\mathfrak{S}_\nu})\cong M_n\mathtt{y}_\nu$. Using the first statement of the lemma and the definition of the functor $A_P\circ\beta$, we see that $S^\nu\cong\beta(\mathrm{ind}_{\mathfrak{S}_\nu}^{\mathfrak{S}_n}\mathtt{sgn}_{\mathfrak{S}_\nu})$ is an extension of $M_n\mathtt{y}_\nu$ and a module having no composition factors in common with the head (or equivalently by Lemma 3.20 the socle) of M_n. Now (ii) follows on dualizing using (i) and Lemma 2.55. □

3.54. Schur algebras as endomorphism algebras

Recalling the theory of Schur algebras from Section 2.23, fix an integer $h\geq n$ and let $S_{h,n}=S_{h,n,\mathcal{O}}$ denote classical the Schur algebra, always considered as a graded algebra in a *trivial* way, i.e. concentrated in degree zero.

Recall the elements $\varphi^u_{\mu,\lambda}$ from (2.5) and $g^u_{\mu,\lambda}$ from (2.4). Our first connection between $R_{n\delta}$ and the Schur algebra arises as follows:

Theorem 3.64. Let $\mathcal{O}=F$. Then there is an algebra isomorphism

$$S_{h,n}\xrightarrow{\sim}\mathrm{End}_{\mathscr{S}_n}\left(\bigoplus\nolimits_{\nu\in X(h,n)}\Lambda^\nu\right),$$

under which the natural basis element $\varphi^u_{\mu,\lambda}$ of $S_{h,n}$ maps to the endomorphism which is zero on the summands Λ^ν for $\nu \neq \mu$ and sends Λ^μ into Λ^λ via the homomorphism induced by right multiplication in M_n by $g^u_{\mu,\lambda}$.

Proof. Let $A_P \circ \beta$ denote the equivalence of categories from Theorem 2.31, for the projective $\mathscr{S}_n$-module $P = M_n$, see Theorem 3.44. By Lemmas 3.20 and 2.33, we have

$$\bigoplus\nolimits_{\nu \in X(h,n)} \Lambda^\nu \cong A_P \circ \beta \left(\bigoplus\nolimits_{\nu \in X(h,n)} \mathrm{Per}^\nu \right).$$

So the endomorphism algebras of $\bigoplus_{\nu \in X(h,n)} \Lambda^\nu$ and $\bigoplus_{\nu \in X(h,n)} \mathrm{Per}^\nu$ are isomorphic. The latter is $S_{h,n}$ by definition. It remains to check that the image of $\varphi^u_{\mu,\lambda}$ under the functor $A_P \circ \beta$ is precisely the endomorphism described. This follows using Lemma 2.33 one more time. □

Note in the theorem above and in the similar results below that the algebra $S_{h,n}$ acts with degree zero homogeneous endomorphisms, so in particular we have

$$\mathrm{End}_{\mathscr{S}_n} \left(\bigoplus\nolimits_{\nu \in X(h,n)} \Lambda^\nu \right) = \mathrm{end}_{\mathscr{S}_n} \left(\bigoplus\nolimits_{\nu \in X(h,n)} \Lambda^\nu \right).$$

Recalling that $S_{h,n}$ can also be described as the endomorphism algebra

$$\mathrm{End}_{F\mathfrak{S}_n}(\bigoplus_{\nu \in X(h,n)} \mathrm{SPer}^\nu),$$

and the elements (2.8), the same argument as in the proof of Theorem 3.64 shows:

Proposition 3.65. Let $\mathcal{O} = F$. Then there is an algebra isomorphism

$$S_{h,n} \xrightarrow{\sim} \mathrm{End}_{\mathscr{S}_n} \left(\bigoplus\nolimits_{\nu \in X(h,n)} M_n \mathtt{y}_\nu \right),$$

under which the natural basis element $\varphi^u_{\mu,\lambda}$ of $S_{h,n}$ maps to the endomorphism which is zero on the summands $M_n \mathtt{y}_\nu$ for $\nu \neq \mu$ and sends $M_n \mathtt{y}_\mu$ into $M_n \mathtt{y}_\lambda$ via the homomorphism induced by right multiplication in M_n by $s^u_{\mu,\lambda}$.

Our final endomorphism algebra result is as follows:

Theorem 3.66. Let $\mathcal{O} = F$. Then there is an algebra isomorphism

$$S_{h,n} \stackrel{\sim}{\longrightarrow} \mathrm{End}_{\mathscr{S}_n}\left(\bigoplus_{\nu \in X(h,n)} Z^\nu\right),$$

under which the natural basis element $\varphi^u_{\mu,\lambda}$ of $S_{h,n}$ maps to the endomorphism which is zero on the summands Z^ν for $\nu \neq \mu$ and sends Z^μ into Z^λ via the homomorphism induced by right multiplication in M_n by $s^u_{\mu,\lambda}$.

Proof. First, we check that the endomorphisms in the statement of the theorem are well-defined. For this we need to see that, as submodules of M_n, $Z^\mu s^u_{\mu,\lambda} \subseteq Z^\lambda$. To prove this, it suffices by definition of Z^λ to prove that $Z^\mu s^u_{\mu,\lambda}(s_r - 1) = 0$ for all simple transpositions $s_r \in \mathfrak{S}_\lambda$. Right multiplication by $s^u_{\mu,\lambda}(s_r - 1)$ yields an $R_{n\delta}$-module homomorphism from Z^μ to M_n. Considering two cases: where $\mu = (1^n)$ and $\mu \neq (1^n)$, we see that the element $s^u_{\mu,\lambda}(s_r - 1)$ always annihilates the submodule $M_n \mathtt{y}_\mu$ of Z^μ. So in fact, $s^u_{\mu,\lambda}(s_r - 1)$ must annihilate all of Z^μ by Lemma 3.63(ii).

Let S be the subalgebra of $\mathrm{End}_{R_{n\delta}}\left(\bigoplus_{\nu \in X(h,n)} Z^\nu\right)$ consisting of all endomorphisms which preserve the subspace $\bigoplus_{\nu \in X(h,n)} M_n \mathtt{y}_\nu \subseteq \bigoplus_{\nu \in X(h,n)} Z^\nu$. Restriction gives an algebra homomorphism

$$S \to \mathrm{End}_{R_{n\delta}}\left(\bigoplus_{\nu \in X(h,n)} M_n \mathtt{y}_\nu\right),$$

which is injective by Lemma 3.63(ii) and surjective by the previous paragraph and Proposition 3.65. This shows in particular that the endomorphisms of the module $\bigoplus_{\nu \in X(h,n)} Z^\nu$ defined in the statement of the theorem are linearly independent and span S. It remains to check using dimensions that S equals all of $\mathrm{End}_{R_{n\delta}}\left(\bigoplus_{\nu \in X(h,n)} Z^\nu\right)$. On expanding the direct sums, this will follow if we can show that

$$\dim \mathrm{Hom}_{R_{n\delta}}(Z^\mu, Z^\lambda) = \dim \mathrm{Hom}_{F\mathfrak{S}_n}(\mathrm{SPer}^\lambda, \mathrm{SPer}^\mu)$$

for all $\lambda, \mu \in X(h,k)$. We calculate using Lemmas 3.63 and 2.29:

$$\begin{aligned}
\mathrm{Hom}_{R_{n\delta}}(Z^\mu, Z^\lambda) &\cong \mathrm{Hom}_{R_{n\delta}}(S^{\lambda^{\mathrm{op}}}, S^{\mu^{\mathrm{op}}}) \\
&\cong \mathrm{Hom}_{R_{n\delta}}(\beta(\mathrm{SPer}^{\lambda^{\mathrm{op}}}), \beta(\mathrm{SPer}^{\mu^{\mathrm{op}}})) \\
&\cong \mathrm{Hom}_{F\mathfrak{S}_n}(\mathrm{SPer}^{\lambda^{\mathrm{op}}}, \alpha \circ \beta(\mathrm{SPer}^{\mu^{\mathrm{op}}})) \\
&\cong \mathrm{Hom}_{F\mathfrak{S}_n}(\mathrm{SPer}^{\lambda^{\mathrm{op}}}, \mathrm{SPer}^{\mu^{\mathrm{op}}}) \\
&\cong \mathrm{Hom}_{F\mathfrak{S}_n}(\mathrm{SPer}^{\lambda}, \mathrm{SPer}^{\mu}),
\end{aligned}$$

as desired. □

3.55. Projective generator for imaginary Schur algebra

Recalling the idempotents $\gamma_{n,\delta}$ and $\gamma_{\nu,\delta}$ from Section 3.51, we introduce the following temporary notation:

$$Y_n := R_{n\delta}\gamma_n M_n, \quad Y_\nu := R_{\nu,\delta}\gamma_\nu M_\nu. \tag{3.67}$$

Later it will turn out that $Y_n = Z_n$ and $Y_\nu = Z_\nu$. It easy to see that

$$Y_\nu \cong Y_{n_1} \boxtimes \cdots \boxtimes Y_{n_a}.$$

Recall for the next lemma that by definition, Z_ν is a submodule of M_ν.

Lemma 3.68. If $\mathcal{O} = F$, then:

(i) Y_ν is the image of any non-zero element of the one dimensional space $\mathrm{Hom}_{R_{\nu,\delta}}(\Gamma_\nu, Z_\nu)$. Moreover, the latter Hom-space is concentrated in degree zero.

(ii) ${}^*I_\nu^n Y_n \cong Y_\nu$.

Proof. (i) By Proposition 3.57, we have $\mathrm{Hom}_{R_{\nu,\delta}}(\Gamma_\nu, M_\nu) \cong F$, and the image of any non-zero map in this homomorphism space is contained in Z_ν.

(ii) By (i), Y_n is a non-zero submodule of M_n, so $\mathrm{Hom}_{R_{n\delta}}(Y_n, M_n) \neq 0$. Now, using Lemmas 3.48(i) and 3.47, we get

$$0 \neq \mathrm{Hom}_{R_{n\delta}}(Y_n, M_n) \cong \mathrm{Hom}_{R_{n\delta}}(Y_n, I_{\nu^{\mathrm{op}}}^n M_{\nu^{\mathrm{op}}}) \cong \mathrm{Hom}_{R_{\nu,\delta}}({}^*I_\nu^n Y_n, M_\nu).$$

In particular, ${}^*I_\nu^n Y_n \neq 0$.

Now let $\theta : \Gamma_n \to Z_n$ be a non-zero homomorphism. By (i), we have $\mathrm{im}\,\theta = Y_n$. By Proposition 3.57, we have ${}^*I_\nu^n \Gamma_n \cong \Gamma_\nu \oplus X$ for X with $\mathrm{Hom}_{R_{\nu,\delta}}(X, M_\nu) = 0$, and by Lemma 3.61, we have ${}^*I_\nu^n Z_n \cong Z_\nu$. So, applying the exact functor ${}^*I_\nu^n$ to θ and restricting to Γ_ν, we obtain a homomorphism $\bar{\theta} : \Gamma_\nu \to Z_\nu$ with image ${}^*I_\nu^n Y_n$, which is non-zero by the previous paragraph. By (i), the image of $\bar{\theta}$ is Y_ν. $\square$

By Proposition 3.57, we have that $Y_\nu = \mathrm{im}\,\mathtt{p}_\nu$ for a map $\mathtt{p}_\nu : \Gamma_\nu \to M_\nu$ which spans the one-dimensional space $\mathrm{Hom}_{R_{\nu,\delta}}(\Gamma_\nu, M_\nu)$. By functoriality, this map induces a map

$$\mathtt{p}^{\nu} : I_{\nu}^{n}\Gamma_{\nu} \to I_{\nu}^{n}Y_{\nu}.$$

Lemma 3.69. Any map $f : I_{\nu}^{n}\Gamma_{\nu} \to M_n$ factors through $\mathtt{p}^{\nu}$, i.e. there exists a unique map $\bar{f} : I_{\nu}^{n}Y_{\nu} \to M_n$ such that $f = \bar{f} \circ \mathtt{p}^{\nu}$. In particular, for any submodule $N \subseteq M$, we have $\dim_q \operatorname{Hom}_{R_{n\delta}}(I_{\nu}^{n}\Gamma_{\nu}, N) = \dim_q \operatorname{Hom}_{R_{n\delta}}(I_{\nu}^{n}Y_{\nu}, N)$.

Proof. By adjointness of I_{ν}^{n} and ${}^*I_{\nu}^{n}$, the map f is functorially induced by a map $f_{\nu} : \Gamma_{\nu} \to {}^*I_{\nu}^{n}M_n$. By Lemma 3.48(ii), we have ${}^*I_{\nu}^{n}M_n \cong M_{\nu} \otimes_{\mathcal{O}\mathfrak{S}_{\nu}} \mathcal{O}\mathfrak{S}_n$. By Proposition 3.57(ii), the map f_{ν} factors through $\mathtt{p}_{\nu}$, i.e. there exists $\bar{f}_{\nu} : Y_{\nu} \to {}^*I_{\nu}^{n}M_n$ such that $f_{\nu} = \bar{f}_{\nu} \circ \mathtt{p}_{\nu}$. Now take $\bar{f}$ to be functorially induced by $\bar{f}_{\nu}$. □

Lemma 3.70. Let $\mathcal{O} = F$. For any $\lambda, \mu \vDash n$, all of the spaces

$$\operatorname{Hom}_{R_{n\delta}}(I_{\lambda}^{n}Y_{\lambda}, I_{\mu}^{n}Y_{\mu}), \tag{3.71}$$

$$\operatorname{Hom}_{R_{n\delta}}(I_{\lambda}^{n}\Gamma_{\lambda}, I_{\mu}^{n}Y_{\mu}), \tag{3.72}$$

$$\operatorname{Hom}_{R_{n\delta}}(I_{\lambda}^{n}\Gamma_{\lambda}, I_{\mu}^{n}Z_{\mu}) \tag{3.73}$$

have (graded) dimension equal to $|{}^{\lambda}\mathcal{D}_n^{\mu}|$.

Proof. We consider only (3.72), since by Lemma 3.69, the result for (3.71) follows from that for (3.72), and the proof for (3.73) is similar to that for (3.72), using that ${}^*I_{\lambda}^{n}Z_n \cong Z_{\lambda}$ according to Lemma 3.61.

By adjointness of I_{λ}^{n} and ${}^*I_{\lambda}^{n}$, we have

$$\operatorname{Hom}_{R_{n\delta}}(I_{\lambda}^{n}\Gamma_{\lambda}, I_{\mu}^{n}Y_{\mu}) \cong \operatorname{Hom}_{R_{\lambda,\delta}}(\Gamma_{\lambda}, {}^*I_{\lambda}^{n}I_{\mu}^{n}Y_{\mu}).$$

Since Γ_λ is projective, the Imaginary Mackey Theorem 3.52 show that

$$\dim_q \operatorname{Hom}_{R_{\lambda,\delta}}(\Gamma_\lambda, {}^*I^n_\lambda I^n_\mu Y_\mu) = \sum_{x \in {}^\lambda\mathcal{D}^\mu_n} \dim_q H_x,$$

where

$$H_x = \operatorname{Hom}_{R_{\lambda,\delta}}(\Gamma_\lambda, I^\lambda_{\lambda\cap x\mu}{}^x({}^*I^\mu_{x^{-1}\lambda\cap\mu} Y_\mu))$$

By Lemma 3.68, we have ${}^*I^\mu_{x^{-1}\lambda\cap\mu} Y_\mu \cong Y_{x^{-1}\lambda\cap\mu}$, so

$$H_x \cong \operatorname{Hom}_{R_{\lambda,\delta}}(\Gamma_\lambda, I^\lambda_{\lambda\cap x\mu} Y_{\lambda\cap x\mu}).$$

Note that the composition $\lambda \cap x\mu$ is a refinement of λ. Denote by ν the composition obtained by from $\lambda \cap x\mu$ by taking the parts of this refinement within each part λ_m of λ in the opposite order. By Lemmas 3.47 and 3.68 and Proposition 3.57(i)(ii), we now have

$$H_x \cong \operatorname{Hom}_{R_{\nu,\delta}}({}^*I^\lambda_\nu \Gamma_\lambda, Y_\nu) \cong \operatorname{Hom}_{R_{\nu,\delta}}(\Gamma_\nu, Y_\nu) \cong F.$$

This completes the proof. □

Recall that $X_+(n)$ can be identified with the set of the partitions of n. The following theorem is the main result of the chapter:

Theorem 3.74. If $\mathcal{O} = F$ then:

(i) The submodules Z_n and Y_n of M_n coincide. So Z_n can be characterized as the image of any non-zero homomorphism from Γ_n to M_n.

(ii) The number of non-isomorphic composition factors of the $R_{n\delta}$-module M_n is equal to $|X_+(n)|$.

(iii) Z^ν is a projective $\mathscr{S}_n$-module, for all $\nu \vDash n$. Moreover, for any $h \geq n$, we have that $\bigoplus_{\nu \in X(h,n)} Z^\nu$ is a projective generator for $\mathscr{S}_n$.

Proof. Fix some $h \geq n$ and set

$$
\begin{aligned}
Z &:= \textstyle\bigoplus_{\nu \in X(h,n)} I_\nu^n Z_\nu = \bigoplus_{\nu \in X(h,n)} Z^\nu, \\
Y &:= \textstyle\bigoplus_{\nu \in X(h,n)} I_\nu^n Y_\nu, \\
\Gamma &:= \textstyle\bigoplus_{\nu \in X(h,n)} I_\nu^n \Gamma_\nu.
\end{aligned}
$$

As Y_ν is a non-zero submodule of Z_ν, it contains the simple socle $M_\nu \mathtt{y}_\nu$ of Z_ν as a submodule, see Lemma 3.60(iii). Applying I_ν^n to the inclusions $M_\nu \mathtt{y}_\nu \subseteq Y_\nu \subseteq Z_\nu$, we see that $M_n \mathtt{y}_\nu \subseteq I_\nu^n Y_\nu \subseteq I_\nu^n Z_\nu$ as naturally embedded submodules of M_n. So

$$\textstyle\bigoplus_{\nu \in X(h,n)} M_n \mathtt{y}_\nu \subseteq Y \subseteq Z.$$

Also observe that Y is a quotient of Γ, since each Y_ν is a quotient of Γ_ν. By Lemmas 3.70 and 2.6, we have

$$
\begin{aligned}
\dim \operatorname{Hom}_{R_{n\delta}}(Y,Y) &= \dim \operatorname{Hom}_{R_{n\delta}}(\Gamma, Y) = \dim \operatorname{Hom}_{R_{n\delta}}(\Gamma, Z) \\
&= \sum_{\lambda,\mu \in X(h,n)} |{}^\lambda \mathcal{D}_n^\mu| = \dim S_{h,n}.
\end{aligned}
$$

Since Γ is projective it contains the projective cover P of Y as a summand. Now the equality $\dim \operatorname{Hom}_{R_{n\delta}}(Y,Y) = \dim \operatorname{Hom}_{R_{n\delta}}(\Gamma, Y)$ implies that

$$\dim \operatorname{Hom}_{R_{n\delta}}(Y,Y) = \dim \operatorname{Hom}_{R_{n\delta}}(P, Y).$$

This verifies the condition in Lemma 3.43, so Y is a projective $R_{n\delta}/\mathrm{Ann}_{R_{n\delta}}(Y)$-module.

As $\mathrm{Hom}_{R_{n\delta}}(\Gamma, Y)$ and $\mathrm{Hom}_{R_{n\delta}}(\Gamma, Z)$ have the same dimension, every $R_{n\delta}$-homomorphism from Γ to Z has image lying in Y . So since Y is a quotient of Γ, we can describe Y alternatively as the subspace of Z spanned by the images of all $R_{n\delta}$-homomorphisms from Γ to Z. This description implies that Y is stable under all $R_{n\delta}$-endomorphisms of Z. So, restriction gives a well-defined map $\mathrm{End}_{R_{n\delta}}(Z) \to \mathrm{End}_{R_{n\delta}}(Y)$. It is injective since we know from Theorem 3.66 and Proposition 3.65 that the homomorphism $\mathrm{End}_{R_{n\delta}}(Z) \to \mathrm{End}_{R_{n\delta}}\left(\bigoplus_{\nu \in X(h,n)} M_n \mathsf{y}_\nu\right)$ induced by restriction is injective. Since $\mathrm{End}_{R_{n\delta}}(Z) \cong S_{h,n}$ and $\mathrm{End}_{R_{n\delta}}(Y)$ has the same dimension as $S_{h,n}$, we deduce that $\mathrm{End}_{R_{n\delta}}(Y) \cong S_{h,n}$.

For $h \geq n$, the number of irreducible representations of $S_{h,n}$ is equal to $|X_+(n)|$. Combining what we have already proved with Fitting's lemma [34, 1.4], we deduce that Y has the same amount of non-isomorphic irreducible modules appearing in its head. It follows in particular that M has at least $|X_+(n)|$ non-isomorphic composition factors. Since n and i are arbitrary, we can now apply Corollary 3.10 to conclude that M has exactly $|X_+(n)|$ non-isomorphic composition factors, and we have proved (ii).

Since Y is a direct sum of submodules of M_n, the assumption on the number of composition factors now implies that every irreducible constituent of M_n appears in the head of Y. Hence every irreducible constituent of M_n appears in the head of the projective $R_{n\delta}$-module Γ. Now we know that every homomorphism from Γ to Z has image lying in Y, while every composition factor of Z/Y appears in the head of the projective module Γ. This shows $Z = Y$, and we have proved (i).

Further, observe that $M_n \cong Z^{(1^n)}$ is a summand of $Y = Z$, hence $\mathrm{Ann}_{R_{n\delta}}(Y) = \mathrm{Ann}_{R_{n\delta}}(M)$. In other words, $R_{n\delta}/\mathrm{Ann}_{R_{n\delta}}(Y) = \mathscr{S}_n$. We have already shown that Y is a projective $R_{n\delta}/\mathrm{Ann}_{R_{n\delta}}(Y)$-module, which means that Z and all its summands are projective $\mathscr{S}_n$-modules. Taking h large enough, this shows in particular that $I_\nu^n Z_\nu = Z^\nu$ is projective for each $\nu \vDash n$.

Finally, to show that every irreducible $\mathscr{S}_n$-module appears in the head of Z, note that M_n is a faithful $\mathscr{S}_n$-module, and so every irreducible $\mathscr{S}_n$-module appears as some composition factor of M_n, and we have seen that a copy of every composition factor of M_n does appear in the head of $Y = Z$. $\square$

Corollary 3.75. The functor I_ν^n sends $\mathscr{S}_\nu$-modules to $\mathscr{S}_n$-modules and the functor ${}^*I_\nu^n$ sends $\mathscr{S}_n$-modules to $\mathscr{S}_\nu$-modules.

Proof. We prove the first statement, the second statement is proved similarly. Since I_ν^n is exact, it suffices to check the first statement on projective $\mathscr{S}_\nu$-modules. In turn, since according to the parabolic analogue of Theorem 3.74, every indecomposable projective $\mathscr{S}_\nu$-module is a submodule of M_ν, we just need to check that $I_\nu^n M_\nu$ is a $\mathscr{S}_n$-module. But this is clear since $I_\nu^n M_\nu \cong M_n$ by Lemma 3.48. $\square$

In this chapter we begin to exploit the Morita context provided by Theorem 3.74. Throughout the chapter, except in §3.8, we assume that $\mathcal{O} = F$.

3.6. Morita equivalence

Let $h \geq n$, and $Z = \bigoplus_{\nu \in X(h,n)} Z^\nu$. We always regard Z as a $(\mathscr{S}_n, S_{h,n})$-bimodule, with $S_{h,n}$ acting as in Theorem 3.66. Define the functors

$$\alpha_{h,n} : \mathscr{S}_n\text{-mod} \to S_{h,n}\text{-mod}, \quad V \mapsto \mathrm{Hom}_{\mathscr{S}_n}(Z, V)$$
$$\beta_{h,n} : S_{h,n}\text{-mod} \to \mathscr{S}_n\text{-mod}, \quad W \mapsto Z \otimes_{S_{h,n}} W.$$

Proposition 3.76. The functors $\alpha_{h,n}$ and $\beta_{h,n}$ are mutually inverse equivalences of categories between $\mathscr{S}_n$-mod and $S_{h,n}$-mod.

Proof. This follows from the fact that Z is a projective generator for $\mathscr{S}_n$ proved in Theorem 3.74(iii). $\square$

Recall from Chapter II that $S_{h,n}$ is a quasi-hereditary algebra with weight poset $X_+(h,n)$ partially ordered by the dominance order $\leq$. We can identify $X_+(h,n)$ with the set $X_+(n)$ of partitions of n, since $h \geq n$. Also, for $\lambda \in X_+(n)$, the algebra $S_{h,n}$ has the irreducible module $L_h(\lambda)$, the standard module $\Delta_h(\lambda)$ and the costandard module $\nabla_h(\lambda)$. For all $\lambda \in X_+(n)$, define the (graded) $\mathscr{S}_n$-modules:

$$L(\lambda) := \beta_{h,n}(L_h(\lambda)), \tag{3.77}$$
$$\Delta(\lambda) := \beta_{h,n}(\Delta_h(\lambda)), \tag{3.78}$$
$$\nabla(\lambda) := \beta_{h,n}(\nabla_h(\lambda)). \tag{3.79}$$

Since $\beta_{h,n}$ is a Morita equivalence, the imaginary Schur algebra $\mathscr{S}_n$ is a quasi-hereditary algebra with weight poset $X_+(n)$ partially ordered by $\leq$. Moreover, $\{L(\lambda)\}$, $\{\Delta(\lambda)\}$ and $\{\nabla(\lambda)\}$ for all $\lambda \in X_+(n)$ give the irreducible, standard and costandard $\mathscr{S}_n$-modules. The following facts follow from Morita equivalence.

Lemma 3.80. Let $\lambda, \mu \in X_+(n)$. Then:

(i) $\Delta(\lambda)$ has simple head isomorphic to $L(\lambda)$, and all other composition factors are of the form $L(\nu)$ for $\nu < \lambda$.

(ii) $[\Delta(\lambda) : L(\mu)] = [\Delta_h(\lambda) : L_h(\mu)]$.

We next explain why the definitions (3.77)-(3.79) are independent of the choice of $h \geq n$. Take $h \geq l \geq n$ and, using Lemma 2.23, identify $S_{l,n}$ with the subalgebra $eS_{h,n}e$ of $S_{h,n}$, where e is the idempotent of (2.22). Recall an equivalence of categories from Lemma 2.26:

$$\mathrm{infl}_{S_{l,n}}^{S_{h,n}} : S_{l,n}\text{-mod} \to S_{h,n}\text{-mod} : M \mapsto S_{h,n}e \otimes_{eS_{h,n}e} M.$$

Lemma 3.81. The functors $\beta_{h,n} \circ \mathrm{infl}_{S_{l,n}}^{S_{h,n}}$ and $\beta_{l,n}$ from $S_{l,n}$-mod to $\mathscr{S}_n$-mod are isomorphic.

Proof. The module $\bigoplus_{\lambda \in X(l,n)} Z^\lambda$ is precisely the $(\mathscr{S}_n, S_{l,n})$-subbimodule Ze of Z. So $\beta_{l,n}$ is isomorphic to $Ze \otimes_{eS_{h,n}e} ?$. Now we have the functorial isomorphisms

$$Z \otimes_{S_{h,n}} (S_{h,n}e \otimes_{eS_{h,n}e} M) \cong (Z \otimes_{S_{h,n}} S_{h,n}e) \otimes_{eS_{h,n}e} M \cong Ze \otimes_{eS_{h,n}e} M$$

for any $M \in S_{l,n}$-mod. □

By Lemmas 2.26 and 2.27, we have that $L_h(\lambda) \cong \mathrm{infl}_{S_{l,n}}^{S_{h,n}} L_l(\lambda)$. Hence Lemma 3.81 yields $\beta_{l,n}(L_l(\lambda)) \cong \beta_{h,n}(L_h(\lambda))$ as $\mathscr{S}_n$-modules. So, the definition (3.77) is independent of the choice of $h \geq n$, and a similar argument gives independence of h for (3.78) and (3.79).

In conclusion of this section, we make a small detour to mixed imaginary tensor spaces of §1.2. For $\boldsymbol{n} = (n_1, \dots, n_l) \in \mathbb{Z}^l_{\geq 0}$, the corresponding mixed tensor space is defined as

$$M_{\boldsymbol{n}} := M_{n_1,1} \circ \dots \circ M_{n_l,l},$$

where $M_{n_i,i}$ is a colored space of color i for each $i \in I'$, which can be considered as a module over the (color i) imaginary Schur algebra $\mathscr{S}_{n_i,i} := R_{n_i\delta}/\operatorname{Ann}_{R_{n_i\delta}}(M_{n_i,i})$. Let $n = n_1 + \dots + n_l$, and define the *mixed imaginary Schur algebra*

$$\mathscr{S}_{\boldsymbol{n}} := R_{n\delta}/\operatorname{Ann}_{R_{n\delta}}(M_{\boldsymbol{n}}).$$

Moreover, for $h_i \geq n_i$ and $\nu \in X(n_i, h_i)$ we have defined modules Z_i^ν (previously denoted for brevity Z^ν since we had $i \in I'$ fixed), and

$$Z(i, n_i) = \bigoplus_{\nu \in X(n_i,n_i)} Z_i^\nu \qquad (i \in I').$$

We have functors $\alpha_{h_i,n_i,i} := \operatorname{Hom}_{\mathscr{S}_{n_i,i}}(Z(n_i,i), ?)$ and $\beta_{h_i,n_i,i} = Z(i,n_i) \otimes_{S_{h_i,n_i}} ?$. Set

$$Z_{\boldsymbol{n}} = Z(1, n_1) \circ \dots \circ Z(l, n_l).$$

The following result strengthens Theorem 2 from the Introduction.

Proposition 3.82. We have

(i) $Z_{\boldsymbol{n}}$ is a projective generator for $\mathscr{S}_{\boldsymbol{n}}$.

(ii) $\operatorname{End}_{\mathscr{S}_{\boldsymbol{n}}}(Z_{\boldsymbol{n}}) \cong \operatorname{End}_{\mathscr{S}_{n_1,1}}(Z(n_1,1)) \otimes \dots \otimes \operatorname{End}_{\mathscr{S}_{n_l,l}}(Z(n_l,l)) \cong S_{\boldsymbol{h},\boldsymbol{n}} := S_{h_1,n_1} \otimes \dots \otimes S_{h_l,n_l}$.

(iii) The functors $\alpha_{\boldsymbol{h},\boldsymbol{n}} := \mathrm{Hom}_{\mathscr{S}_{\boldsymbol{n}}}(Z_{\boldsymbol{n}}, ?) : \mathscr{S}_{\boldsymbol{n}}\text{-mod} \to S_{\boldsymbol{h},\boldsymbol{n}}\text{-mod}$ and $\beta_{\boldsymbol{h},\boldsymbol{n}} = Z_{\boldsymbol{n}} \otimes_{S_{\boldsymbol{h},\boldsymbol{n}}} ? : S_{\boldsymbol{h},\boldsymbol{n}}\text{-mod} \to \mathscr{S}_{\boldsymbol{n}}\text{-mod}$, are mutually inverse equivalences of categories between $\mathscr{S}_{\boldsymbol{n}}$-mod and $S_{\boldsymbol{h},\boldsymbol{n}}$-mod.

(iv) There is a functorial isomorphism

$$\beta_{\boldsymbol{h},\boldsymbol{n}}(W_1 \otimes \cdots \otimes W_l) \cong \beta_{h_1,n_1,1}(W_1) \circ \cdots \circ \beta_{h_l,n_l,l}(W_l)$$

for $W_1 \in S_{h_1,n_1}$-mod, $\dots, W_l \in S_{h_l,n_l}$-mod.

Proof. Part (i) follows from Theorem 3.74(iii). Part (ii) follows from Theorem 3.66 and Mackey Theorem. Part (iii) follows from part (i). Part (iv) follows from the definitions and transitivity of induction. □

3.61. Induction and Morita equivalence

In this section we prove a key result that our Morita equivalence 'intertwines' imaginary induction and tensor products for usual Schur algebras. We fix an integer $h \geq n$, and a composition $\nu = (n_1, \dots, n_a) \vDash n$. Recall the Morita equivalence $\beta_{h,n} : S_{h,n}\text{-mod} \to \mathscr{S}_n\text{-mod}$.

Theorem 3.83. We have an isomorphism of functors from $S_{h,n_1}\text{-mod} \times \cdots \times S_{h,n_a}\text{-mod}$ to $\mathscr{S}_n$-mod:

$$I_\nu^n(\beta_{h,n_1}? \boxtimes \cdots \boxtimes \beta_{h,n_a}?) \cong \beta_{h,n}(? \otimes \cdots \otimes ?).$$

Proof. Choose $\chi = (h_1, ..., h_a) \vDash h$ with $h_k \geq n_k$ for $k = 1, \dots, a$, and denote $S_{\chi,\nu} := S_{h_1,n_1} \otimes \cdots \otimes S_{h_a,n_a}$. Write $X(\chi, \nu)$ for the set of all compositions $\gamma = (g_1, \dots, g_h) \in$

$X(h,n)$ such that

$$\gamma_1{:=}\ (g_1,\ldots,g_{h_1}),\gamma_2{:=}\ (g_{h_1+1},\ldots,g_{h_1+h_2}),\ldots,\gamma_a{:=}\ (g_{h_1+\cdots+h_{a-1}+1},\ldots,g_h)$$

satisfy $\gamma_k\in X(h_k,n_k)$ for each $k=1,\ldots,a$.

Consider the set of triples:

$$\Omega=\{(\gamma,\delta,u)\mid \gamma,\delta\in X(\chi,\nu),\ u\in {}^{\gamma}\mathcal{D}_{\nu}^{\delta}\}.$$

For a triple $(\gamma,\delta,u)\in\Omega$, we have $\gamma_k,\delta_k\in X(h_k,n_k)$ for each $k=1,\ldots,a$, and $u=(u_1,\ldots,u_a)\in\mathfrak{S}_\nu=\mathfrak{S}_{n_1}\times\cdots\times\mathfrak{S}_{n_a}$ with each $u_k\in {}^{\gamma_k}\mathcal{D}_{n_k}^{\delta_k}$. So we have the element

$$\bar{\varphi}^u_{\gamma,\delta}:=\varphi^{u_1}_{\gamma_1,\delta_1}\otimes\cdots\otimes\varphi^{u_a}_{\gamma_a,\delta_a}\in S_{\chi,\nu}.$$

Then $\{\bar{\varphi}^u_{\gamma,\delta}\mid(\gamma,\delta,u)\in\Omega\}$ is a basis for $S_{\chi,\nu}$.

Recall the $(\mathscr{S}_n,S_{h,n})$-bimodule $Z=\bigoplus_{\lambda\in X(h,n)}I^n_\lambda Z_\lambda=\bigoplus_{\lambda\in X(h,n)}Z^\lambda$, and define also the $(\mathscr{S}_\nu,S_{\chi,\nu})$-bimodule

$$Z^{\chi,\nu}:=\bigoplus\nolimits_{\lambda\in X(\chi,\nu)}I^\nu_\lambda Z_\lambda\cong\left(\bigoplus\nolimits_{\lambda_1\in X(h_1,n_1)}Z^{\lambda_1}\right)\boxtimes\cdots\boxtimes\left(\bigoplus\nolimits_{\lambda_a\in X(h_a,n_a)}Z^{\lambda_a}\right).$$

Then $I^n_\nu Z^{\chi,\nu}$ is an $(\mathscr{S}_n,S_{\chi,\nu})$-bimodule in a natural way. Moreover, by transitivity of induction, we have $I^n_\nu Z^{\chi,\nu}\cong\bigoplus_{\lambda\in X(\chi,\nu)}Z^\lambda$, so $I^n_\nu Z^{\chi,\nu}$ can be identified with the summand Ze_ν of the $(\mathscr{S}_n,S_{h,n})$-bimodule Z, where e_ν is the idempotent

$$e_\nu:=\sum_{\lambda\in X(\chi,\nu)}e(\lambda)\in S_{h,n}.$$

Identifying $e_\nu S_{h,n} e_\nu$ with $\mathrm{End}_{\mathscr{S}_n}(Ze_\nu)$, we obtain an algebra embedding of $S_{\chi,\nu}$ into $e_\nu S_{h,n} e_\nu$. By definition of the actions of $S_{\chi,\nu}$ and $e_\nu S_{h,n} e_\nu$ on Ze_ν and Lemma 3.48(i), this embedding maps the basis element $\bar{\varphi}^u_{\gamma,\delta} \in S_{\chi,\nu}$ to $\varphi^u_{\gamma,\delta} \in e_\nu S_{h,n} e_\nu$, for all $(\gamma, \delta, u) \in \Omega$. In other words:

Claim 1. *Identifying $S_{\chi,\nu}$ with a subalgebra $e_\nu S_{h,n} e_\nu$ via the map $\bar{\varphi}^u_{\gamma,\delta} \mapsto \varphi^u_{\gamma,\delta}$, the $(\mathscr{S}_n, S_{\chi,\nu})$-bimodule $I^n_\nu Z^{\chi,\nu}$ is isomorphic to Ze_ν, regarding the latter as a $(\mathscr{S}_n, S_{\chi,\nu})$-bimodule by restricting the natural action of $e_\nu S_{h,n} e_\nu$ to $S_{\chi,\nu}$.*

Now let $S_{\chi,n}$ be the Levi subalgebra of $S_{h,n}$ as in (2.21). Then e_ν is the central idempotent of $S_{\chi,n}$ such that $e_\nu S_{\chi,n} e_\nu \cong S_{\chi,\nu}$. So in fact, the embedding of $S_{\chi,\nu}$ into $e_\nu S_{\chi,n} e_\nu$ from Claim 1 identifies $S_{\chi,\nu}$ with the summand $e_\nu S_{\chi,n} e_\nu$ of $S_{\chi,n}$. Making this identification, define the functor

$$I = S_{h,n} e_\nu \otimes_{e_\nu S_{\chi,n} e_\nu} ? : S_{\chi,\nu}\text{-mod} \to S_{h,n}\text{-mod}\,.$$

Using associativity of tensor product, the functor I can be thought of as the composite of the natural inflation functor $S_{\chi,n} e_\nu \otimes_{e_\nu S_{\chi,n} e_\nu} ? : S_{\chi,\nu}\text{-mod} \to S_{\chi,n}\text{-mod}$ followed by ordinary induction $\mathrm{ind}^{S_{h,n}}_{S_{\chi,n}} : S_{\chi,n}\text{-mod} \to S_{h,n}\text{-mod}$ as defined in Section 2.25. In view of this, the following fact follows immediately from Lemma 2.28:

Claim 2. *The following functors from $S_{h_1,n_1}\text{-mod} \times \cdots \times S_{h_a,n_a}\text{-mod}$ to $S_{h,n}\text{-mod}$ are isomorphic:*

$$I(? \boxtimes \cdots \boxtimes ?) \qquad \textit{and} \qquad \mathrm{infl}^{S_{h,n_1}}_{S_{h_1,n_1}} ? \otimes \cdots \otimes \mathrm{infl}^{S_{h,n_a}}_{S_{h_a,n_a}} ?.$$

Now note that from Claim 1 and associativity of tensor product we have the natural isomorphisms

$$I_\nu^n(Z^{\chi,\nu} \otimes_{S_{\chi,\nu}} N) \cong (I_\nu^n Z^{\chi,\nu}) \otimes_{S_{\chi,\nu}} N \cong Ze_\nu \otimes_{S_{\chi,\nu}} N$$
$$\cong Ze_\nu \otimes_{e_\nu S_{\chi,n} e_\nu} N \cong Z \otimes_{S_{h,n}} S_{h,n} e_\nu \otimes_{e_\nu S_{\chi,n} e_\nu} N,$$

which is $\beta_{h,n}(I(N))$. Thus we have proved:

Claim 3. *The functors $I_\nu^n \circ (Z^{\chi,\nu} \otimes_{S_{\chi,\nu}} ?)$ and $\beta_{h,n} \circ I$ from $S_{\chi,\nu}$-mod to $\mathscr{S}_n$-mod are isomorphic.*

We have the isomorphism of functors from S_{h_1,n_1}-mod $\times \cdots \times S_{h_a,n_a}$-mod to $\mathscr{S}_\nu$-mod:

$$\beta_{h_1,n_1} ? \boxtimes \cdots \boxtimes \beta_{h_a,n_a} ? \cong Z_{\chi,\nu} \otimes_{S_{\chi,\nu}} (? \boxtimes \cdots \boxtimes ?).$$

So, in view of Claims 2 and 3, we have

Claim 4. *There is an isomorphism of functors*

$$I_\nu^n(\beta_{h_1,n_1} ? \boxtimes \cdots \boxtimes \beta_{h_a,n_a} ?) \cong \beta_{h,n}(\mathrm{infl}_{S_{h_1,n_1}}^{S_{h,n_1}} ? \otimes \cdots \otimes \mathrm{infl}_{S_{h_a,n_a}}^{S_{h,n_a}} ?)$$

from S_{h_1,n_1}-mod $\times \cdots \times S_{h_a,n_a}$-mod to $\mathscr{S}_n$-mod.

Finally, by Lemma 2.26, the functors

$$\mathrm{infl}_{S_{h_k,n_k}}^{S_{h,n_k}} : S_{h_k,n_k}\text{-mod} \to S_{h,n_k}\text{-mod} \qquad (1 \leq k \leq a)$$

are equivalences of categories. By Lemma 3.81, the functors $\beta_{h,n_k} \circ \mathrm{infl}_{S_{h_k,n_k}}^{S_{h,n_k}}$ and β_{h,n_k} are isomorphic. The theorem follows on combining these statements and Claim 4. $\square$

As a first application, we get the commutativity of induction product on the category of imaginary representations:

Corollary 3.84. Let $M \in \mathscr{S}_m$-mod and $N \in \mathscr{S}_n$-mod. Then $M \circ N \cong N \circ M$.

Proof. For sufficiently large h we have $M = \beta_{h,m}(V)$, $N = \beta_{h,n}(W)$, $M \circ N = I^{m+n}_{(m,n)}(M \boxtimes N)$, and $N \circ M = I^{m+n}_{(n,m)}(N \boxtimes M)$. Now the result follows from $V \otimes W \cong W \otimes V$ and the theorem. $\square$

As a second application we establish a version of Steinberg Tensor Product Theorem. Let $p = \operatorname{char} F > 0$ and $\lambda \vdash n$. Considered λ as an element of $X_+(n, n)$. Recall from Section 2.21, that there exists a unique p-adic expansion

$$\lambda = \lambda(0) + p\lambda(1) + p^2\lambda(2) \dots$$

such that the partitions $\lambda(0) \vdash m_0, \lambda(1) \vdash m_1, \lambda(2) \vdash m_2, \dots$ are all p-restricted. With this notation we have:

Theorem 3.85. **(Imaginary Steinberg Tensor Product Theorem)** Let $n_r := p^r m_r$ for $r = 0, 1, 2, \dots$, and consider the composition

$$\nu = (n_0, n_1, n_2, \dots) \vDash n.$$

Then

$$L(\lambda) = I^n_\nu\big(L(\lambda(0)) \boxtimes L(p\lambda(1)) \boxtimes L(p^2\lambda(2)) \boxtimes \dots\big).$$

Proof. This comes from Theorem 3.83 and Lemma 2.20. $\square$

As a third application, we prove that imaginary induction and restriction respect standard and costandard filtrations. A filtration $0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_b = M$ of an $\mathscr{S}_n$-module M is called *standard* (resp. *costandard*) if for each $k = 1, \ldots, b$, the quotient M_k/M_{k-1} is isomorphic to $\Delta(\lambda)$ (resp. $\nabla(\lambda)$) for some $\lambda \vdash n$ (depending on k). Similarly, a filtration $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_b = M$ of an $\mathscr{S}_\nu$-module M is called *standard* (resp. *costandard*) if for each $k = 1, \ldots, b$, the quotient M_k/M_{k-1} is isomorphic to $\Delta(\lambda_1) \boxtimes \cdots \boxtimes \Delta(\lambda_a)$ (resp. $\nabla(\lambda_1) \boxtimes \cdots \boxtimes \nabla(\lambda_a)$) for some $\lambda_1 \vdash n_1, \ldots \lambda_a \vdash n_a$ (depending on k).

Theorem 3.86. We have

(i) The functor I_ν^n sends $\mathscr{S}_\nu$-modules with standard (resp. costandard) filtrations to $\mathscr{S}_n$-modules with standard (resp. costandard) filtrations.

(ii) The functor ${}^*I_\nu^n$ sends $\mathscr{S}_n$-modules with standard (resp. costandard) filtrations to $\mathscr{S}_\nu$-modules with standard (resp. costandard) filtrations.

Proof. (i) It suffices to check that $I_\nu^n(\Delta(\lambda_1) \boxtimes \cdots \boxtimes \Delta(\lambda_a))$ has a standard filtration, for arbitrary $\lambda_1 \vdash n_1, \ldots, \lambda_a \vdash n_a$. By Theorem 3.83,

$$I_\nu^n(\Delta(\lambda_1) \boxtimes \cdots \boxtimes \Delta(\lambda_a)) \cong \beta_{h,n}(\Delta_h(\lambda_1) \otimes \cdots \otimes \Delta_h(\lambda_a)).$$

So the result follows since $\Delta_h(\lambda_1) \otimes \cdots \otimes \Delta_h(\lambda_a)$ has a standard filtration as an $S_{h,n}$-module by Lemma 2.18. This proves (i) in the case of standard filtrations; the result for costandard filtrations is proved similarly.

(ii) We prove (ii) in the case of costandard filtrations; the analogous result for standard filtrations follows by dualizing. Take $N \in \mathscr{S}_n$-mod with a costandard filtration. Using the cohomological criterion for costandard filtrations [10, A2.2(iii)], we need to show that $\operatorname{Ext}^1_{\mathscr{S}_\nu}(M, {}^*I_\nu^n N) = 0$ for all $M \in \mathscr{S}_\nu$-mod with a standard filtration. For such M, by (i) and the cohomological criterion for costandard filtrations, we have $\operatorname{Ext}^1_{\mathscr{S}_\nu}(I_\nu^n M, N) = 0$. So the result follows from the following:

Claim. *For $M \in \mathscr{S}_\nu$-mod and $N \in \mathscr{S}_n$-mod, we have* $\mathrm{Ext}^1_{\mathscr{S}_\nu}(M, {}^*I^n_\nu N) \cong \mathrm{Ext}^1_{\mathscr{S}_\nu}(I^n_\nu M, N)$.

To prove the claim, the adjoint functor property gives us an isomorphism of functors $\mathrm{Hom}_{\mathscr{S}_\nu}(M, ?) \circ {}^*I^n_\nu \cong \mathrm{Hom}_{\mathscr{S}_n}(I^n_\nu M, ?)$. Since ${}^*I^n_\nu$ is exact and sends injectives to injectives (being adjoint to the exact functor I^n_ν), an application of [16, I.4.1(3)] completes the proof of the claim. □

Corollary 3.87. Let $\nu = (n_1, \dots, n_a) \vDash n$, $\lambda \vdash n$, and $\lambda_1 \vdash n_1, \dots, \lambda_a \vdash n_a$. Then both of

(i) the multiplicity of $\Delta(\lambda)$ in a standard filtration of $I^n_\nu(\Delta(\lambda_1) \boxtimes \cdots \boxtimes \Delta(\lambda_a))$,

(ii) the multiplicity of $\Delta(\lambda_1) \boxtimes \cdots \boxtimes \Delta(\lambda_a)$ in a standard filtration of ${}^*I^n_\nu \Delta(\lambda)$

are given by the Littlewood-Richardson rule.

Proof. The modules in (i) and (ii) have standard filtrations by Theorem 3.86. Now (i) follows from Theorem 3.83 and the classical fact about tensor product multiplicities over the Schur algebra, and (ii) follows from (i) and adjointness, together with the usual properties of standard and costandard filtrations. □

3.7. Alternative definitions of standard modules

Our goal now is to give two alternative definitions of the standard module $\Delta(\lambda)$ without reference to the Schur algebra and Morita equivalence. Recall from (2.15) and (2.16) the modules $Z^\nu(V_h)$ and $\Lambda^\nu(V_h)$ for the classical Schur algebra $S_{h,n}$.

Lemma 3.88. For $\nu \vDash n$, we have $Z^\nu \cong \beta_{h,n}(Z^\nu(V_h))$ and $\Lambda^\nu \cong \beta_{h,n}(\Lambda^\nu(V_h))$.

Proof. By Lemma 2.17(i), we have $Z^\nu(V_h) \cong S_{h,n} e(\nu)$. So, by Lemma 2.33, we get $\beta_{h,n}(S_{h,n} e(\nu)) \cong Z e(\nu)$, which is precisely the summand Z^ν of Z by the definition of the action from Theorem 3.66.

For the second statement, using the embedding κ from Lemma 2.7 and Lemma 2.17(ii), we have $\Lambda^\nu(V_h) \cong S_{h,n}\kappa(\mathtt{y}_\nu)$. So, by Lemma 2.33, we get $\beta_{h,n}(\Lambda^\nu(V_h)) \cong Z\kappa(\mathtt{y}_\nu)$. By definition of κ, together with Theorem 3.66, we have $Z\kappa(\mathtt{y}_\nu) = M_n\mathtt{sgn}(\mathtt{y}_\nu) = M_n\mathtt{x}_\nu = \Lambda^\nu$, as required. $\square$

In view of Lemma 3.20, the antiautomorphism τ of $R_{n\delta}$ factors through to give a (homogeneous) antiautomorphism

$$\tau : \mathscr{S}_n \to \mathscr{S}_n,$$

which leads to the notion of contravariant duality $\circledast$ on $\mathscr{S}_n$-mod. We have a (not necessarily degree zero) isomorphism $L(\lambda)^\circledast \cong L(\lambda)$ for each $\lambda \in X_+(n)$, since this is true even as $R_{n\delta}$-modules. We now prove a stronger result:

Lemma 3.89. For all $\lambda \in X_+(n)$ we have $L(\lambda)^\circledast \cong L(\lambda)$ and $\Delta(\lambda)^\circledast \cong \nabla(\lambda)$.

Proof. By Lemma 3.88, $M_n = Z^{(1^n)} \cong \beta_{h,n}(V_h^{\otimes n})$. So the only (graded) composition factors of M_n are of the form $L(\lambda)$ for $\lambda \in X_+(n)$ and each such $L(\lambda)$ occurs with some non-zero *graded* multiplicity $m_\lambda \in \mathbb{Z}_{>0}$. The formal characters of the modules $L(\lambda)$ are linearly independent. Hence, since $\mathrm{ch}_q M_n = \sum_{\lambda\in X_+(n)} m_\lambda \mathrm{ch}_q L(\lambda)$ is bar-invariant, by Lemma 3.20, we conclude that each $\mathrm{ch}_q L(\lambda)$ is bar-invariant, which immediately implies that $L(\lambda)^\circledast \cong L(\lambda)$. It follows from a general theory of quasi-hereditary algebras that $\Delta(\lambda)^\circledast$ is the costandard module $\nabla(\lambda)$ up to a degree shift, and now the first statement of the lemma implies the second one. $\square$

Now we obtain the desired characterizations of $\Delta(\lambda)$. Recall the element $u_\lambda \in \mathfrak{S}_n$ from Lemma 2.3.

Theorem 3.90. Let $\lambda \vdash n$. Then:

(i) $\mathrm{Hom}_{\mathscr{S}_n}(Z^\lambda, \Lambda^{\lambda^{\mathrm{tr}}}) \cong F$, and the image of any non-zero homomorphism in $\mathrm{Hom}_{\mathscr{S}_n}(Z^\lambda, \Lambda^{\lambda^{\mathrm{tr}}})$ is isomorphic to $\Delta(\lambda)$;

(ii) $\Delta(\lambda)$ is isomorphic to the submodule $Z^\lambda u_{\lambda^{\mathrm{tr}}} \mathtt{x}_{\lambda^{\mathrm{tr}}}$ of M_n.

Proof. (i) follows from Lemma 3.88, the definition (3.78), and Lemma 2.19, since $\beta_{h,n}$ is an equivalence of categories.

(ii) Note that Z^λ contains $M_n \mathtt{y}_\lambda$ as a submodule. Moreover, as M_n is a faithful $F\mathfrak{S}_n$-module and $\mathtt{y}_\lambda u_{\lambda^{\mathrm{tr}}} \mathtt{x}_{\lambda^{\mathrm{tr}}} \neq 0$ by Lemma 2.3, we conclude that $M_n \mathtt{y}_\lambda u_{\lambda^{\mathrm{tr}}} \mathtt{x}_{\lambda^{\mathrm{tr}}} \neq 0$. Hence $Z^\lambda u_{\lambda^{\mathrm{tr}}} \mathtt{x}_{\lambda^{\mathrm{tr}}} \neq 0$. Finally, observe that $Z^\lambda u_{\lambda^{\mathrm{tr}}} \mathtt{x}_{\lambda^{\mathrm{tr}}}$ is both a homomorphic image of Z^λ and a submodule of $\Lambda^{\lambda^{\mathrm{tr}}}$. So the result follows from (i). $\square$

We will write $\tilde{M}$ for the *right* $\mathscr{S}_n$-module obtained from $M \in \mathscr{S}_n$-mod by twisting the left action into a right action using the antiautomorphism τ of $\mathscr{S}_n$. In this way, we obtain right $\mathscr{S}_n$-modules $\tilde{L}(\lambda)$, $\tilde{\Delta}(\lambda)$, and $\tilde{\nabla}(\lambda)$.

Theorem 3.91. We have

(i) $\mathscr{S}_n$ has a filtration as a $(\mathscr{S}_n, \mathscr{S}_n)$-bimodule with factors isomorphic to $\Delta(\lambda) \otimes \tilde{\Delta}(\lambda)$, each appearing once for each $\lambda \vdash n$ and ordered in any way refining the dominance order on partitions so that factors corresponding to most dominant λ appear at the bottom of the filtration.

(ii) Z has a filtration as a $(\mathscr{S}_n, S_{h,n})$-bimodule with factors $\Delta(\lambda) \otimes \tilde{\Delta}_h(\lambda)$ appearing once for each $\lambda \vdash n$ and ordered in any way refining the dominance order so that factors corresponding to most dominant λ appear at the bottom of the filtration.

Proof. (i) This follows from the general theory of quasi-hereditary algebras, as is explained for example after [5, (1.2e)].

(ii) The functor $Z\otimes_{S_{h,n}}?$ can be viewed as an exact functor from the category of $(S_{h,n}, S_{h,n})$-bimodules to the category of $(\mathscr{S}_n, S_{h,n})$-bimodules. We have:

$$Z \otimes_{S_{h,n}} (\Delta_h(\lambda) \otimes \tilde{\Delta}_h(\lambda)) \cong (Z \otimes_{S_{h,n}} \Delta_h(\lambda)) \otimes \tilde{\Delta}_h(\lambda) \cong \Delta(\lambda) \otimes \tilde{\Delta}_h(\lambda).$$

So applying $Z\otimes_{S_{h,n}}?$ to the filtration of Lemma 2.13 gives the result. $\square$

3.8. Base change

Recall that $\mathcal{O}$ denotes the ground ring which is always assumed to be either $\mathbb{Z}$ or F. The algebras $S_{n,h}$, $\mathscr{S}_n$, and the modules M_n, Z^λ, etc. are all defined over $\mathbb{Z}$, although in many results proved in the previous sections we have assumed that $\mathcal{O}$ is a field. We now work over $\mathcal{O} = \mathbb{Z}$, and study the base change from $\mathbb{Z}$ to F. Throughout this section, it will also be convenient to denote by K a field of characteristic zero and use notation like $\mathscr{S}_{n,\mathbb{Z}}$, $\mathscr{S}_{n,F}$, $\mathscr{S}_{n,K}$, etc. to specify the ring over which the objects are considered.

Lemma 3.92. We have:

(i) $M_{n,\mathbb{Z}}$ is a $\mathbb{Z}$-free module of finite rank with $M_{n,\mathbb{Z}} \otimes_{\mathbb{Z}} F \cong M_{n,F}$;

(ii) $Z_{n,\mathbb{Z}}$ is a $\mathbb{Z}$-free module of finite rank with $Z_{n,\mathbb{Z}} \otimes_{\mathbb{Z}} F \cong Z_{n,F}$. Moreover, $Z_{n,\mathbb{Z}} = Y_{n,\mathbb{Z}} := R_{n\delta}(\mathbb{Z})\gamma_n M_{n,\mathbb{Z}}$.

Proof. (i) comes from the Lemma 3.21.

(ii) By (i), $M_{n,\mathbb{Z}}$ is a lattice in $M_{n,K}$, and by definition, we have $Z_{n,\mathbb{Z}} = Z_{n,K} \cap M_{n,\mathbb{Z}}$. So $Z_{n,\mathbb{Z}}$ is a lattice in $Z_{n,K}$, and also a direct summand of the $\mathbb{Z}$-module $M_{n,\mathbb{Z}}$. Hence the natural map

$$i : Z_{n,\mathbb{Z}} \otimes_{\mathbb{Z}} F \to M_{n,\mathbb{Z}} \otimes_{\mathbb{Z}} F = M_{n,F}$$

is injective. Since the action of $\mathfrak{S}_n$ on M_n is compatible with base change, we have $\operatorname{im} i \subseteq Z_{n,F}$.

Recall the submodule $Y_n = R_{n\delta}\gamma_n M_n$ from (3.67). Note that the natural (not necessarily injective) map from $Y_{n,\mathbb{Z}} \otimes_{\mathbb{Z}} F$ to $M_{n,F}$ has image $Y_{n,F}$. Now by Propsosition 3.57(iii), $Y_{n,\mathbb{Z}} \subseteq Z_{n,\mathbb{Z}}$, so $Y_{n,F} \subseteq \operatorname{im} i$. By Theorem 3.74(i), $Y_{n,F} = Z_{n,F}$, so by the previous paragraph, the map $i : Z_{n,\mathbb{Z}} \otimes_{\mathbb{Z}} F \to Z_{n,F}$ is an isomorphism.

Finally, the embedding $Y_{n,\mathbb{Z}} \to Z_{n,\mathbb{Z}}$ has to be an isomorphism, since otherwise for some field F the induced map $Y_{n,\mathbb{Z}} \otimes_{\mathbb{Z}} F \to Z_{n,\mathbb{Z}} \otimes_{\mathbb{Z}} F$ is not surjective, and so the composition

$$Y_{n,\mathbb{Z}} \otimes_{\mathbb{Z}} F \to Z_{n,\mathbb{Z}} \otimes_{\mathbb{Z}} F \to Z_{n,F} = Y_{n,F}$$

is not surjective, giving a contradiction. $\square$

Since induction commutes with base change, we deduce:

Corollary 3.93. Let $\nu \vDash n$. Then $Z^{\nu}_{\mathbb{Z}}$ is a $\mathbb{Z}$-free module of finite rank with $Z^{\nu}_{\mathbb{Z}} \otimes_{\mathbb{Z}} F \cong Z^{\nu}_{F}$.

Now we can define standard modules over $\mathbb{Z}$. For $\lambda \vdash n$, set

$$\Delta_{\mathbb{Z}}(\lambda) = Z^{\lambda}_{\mathbb{Z}} u_{\lambda^{\mathrm{tr}}} \mathtt{x}_{\lambda^{\mathrm{tr}}}.$$

Compare this to Theorem 3.90(ii), in which we worked over a field.

Theorem 3.94. $\Delta_{\mathbb{Z}}(\lambda)$ is $\mathbb{Z}$-free of finite rank with $\Delta_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}} F \cong \Delta_F(\lambda)$. Moreover, the formal characters of $\Delta_{\mathbb{Z}}(\lambda)$ and $\Delta_F(\lambda)$ are the same.

Proof. By definition $Z^{\lambda}_{\mathbb{Z}}$ is a submodule of $M_{n,\mathbb{Z}}$, and $\Delta_{\mathbb{Z}}(\lambda)$ is a submodule of the torsion free $\mathbb{Z}$-module $M_{n,\mathbb{Z}}$, so $\Delta_{\mathbb{Z}}(\lambda)$ is torsion free. There is a natural map $\Delta_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}} K \to M_{n,K}$, which is injective since K is flat over $\mathbb{Z}$. It is easy to check that the image of this map is precisely $\Delta_K(\lambda)$, proving that $\Delta_{\mathbb{Z}}(\lambda)$ is a $\mathbb{Z}$-lattice in $\Delta_K(\lambda)$. In particular, $\Delta_{\mathbb{Z}}(\lambda)$ has rank equal to $\dim \Delta_K(\lambda)$. By Theorem 3.91(ii) with $h = n$, we have

$$\sum_{\nu \in X(n,n)} \dim Z_K^\nu = \sum_{\lambda \vdash n} (\dim \Delta_K(\lambda))(\dim \Delta_{n,K}(\lambda)),$$

where $\Delta_{n,K}(\lambda)$ denotes the standard module for the Schur algebra $S_{n,n,K}$. In view of Corollary 3.93, $\dim Z_K^\nu = \dim Z_F^\nu$, and it is well-known that the dimensions of standard modules for the Schur algebra do not depend on the ground field. So

$$\sum_{\nu \in X(n,n)} \dim Z_F^\nu = \sum_{\lambda \vdash n} (\dim \Delta_K(\lambda))(\dim \Delta_{n,F}(\lambda)).$$

There is a natural map $i : \Delta_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}} F \to M_{n,F}$ with image $\Delta_F(\lambda)$ induced by the embedding $\Delta_{\mathbb{Z}}(\lambda) \to M_{n,\mathbb{Z}}$. So $\dim \Delta_K(\lambda) \geq \dim \Delta_F(\lambda)$. On the other hand, applying Theorem 3.91(ii) over F, we have that

$$\sum_{\nu \in X(n,n)} \dim Z_F^\nu = \sum_{\lambda \vdash n} (\dim \Delta_F(\lambda))(\dim \Delta_{n,F}(\lambda)).$$

Comparing with our previous expression, we conclude that $\dim \Delta_F(\lambda) = \dim \Delta_K(\lambda)$ for all $\lambda \vdash n$. Hence i is injective. The result about the characters is now clear. $\square$

We now show that the imaginary Schur algebra and its Morita equivalence with a classical Schur algebra are defined over $\mathbb{Z}$.

Theorem 3.95. We have:

(i) $\mathscr{S}_{n,\mathbb{Z}}$ is $\mathbb{Z}$-free of finite rank with $\mathscr{S}_{n,\mathbb{Z}} \otimes_{\mathbb{Z}} F \cong \mathscr{S}_{n,F}$;

(ii) $Z_{\mathbb{Z}}^\nu$ is a projective $\mathscr{S}_{n,\mathbb{Z}}$-module for each $\nu \vDash n$;

(iii) $\mathrm{End}_{\mathscr{S}_{n,\mathbb{Z}}} \left(\bigoplus_{\nu \in X(n,h)} Z_{\mathbb{Z}}^\nu \right) \cong S_{h,n,\mathbb{Z}}$;

(iv) $\bigoplus_{\nu\in X(n,h)} Z^\nu_\mathbb{Z}$ is a projective generator for $\mathscr{S}_{n,\mathbb{Z}}$, so $\mathscr{S}_{n,\mathbb{Z}}$ is Morita equivalent to $S_{h,n,\mathbb{Z}}$ for $h \geq n$.

Proof. (i) By definition, $\mathscr{S}_{n,\mathbb{Z}}$ is the $\mathbb{Z}$-submodule of $\mathrm{End}_\mathbb{Z}(M_{n,\mathbb{Z}})$ spanned by the images of the $\mathbb{Z}$-basis elements of $R_{n\delta,\mathbb{Z}}$ which are of the form $\psi_w y_1^{b_1}\dots y_d^{b_d} 1_{\boldsymbol{i}}$. Since the degree of each y_r is 2, all but finitely many such elements act as zero, so $\mathscr{S}_{n,\mathbb{Z}}$ is finitely generated over $\mathbb{Z}$, whence $\mathscr{S}_{n,\mathbb{Z}}$ is a lattice in $\mathscr{S}_{n,K}$.

The natural inclusion $\mathscr{S}_{n,\mathbb{Z}} \hookrightarrow \mathrm{End}_\mathbb{Z}(M_{n,\mathbb{Z}})$ yields a map

$$\mathscr{S}_{n,\mathbb{Z}} \otimes_\mathbb{Z} F \to \mathrm{End}_\mathbb{Z}(M_{n,\mathbb{Z}}) \otimes F \cong \mathrm{End}_F(M_n(F))$$

whose image is $\mathscr{S}_{n,F}$. This map is injective since $\dim \mathscr{S}_{n,K} = \dim \mathscr{S}_{n,F}$ by Theorems 3.91(i) and 3.94.

(ii) By Corollary 3.93, we have $Z^\nu_\mathbb{Z} \otimes_\mathbb{Z} F \cong Z^\nu_F$, which is a projective $\mathscr{S}_{n,F}$-module by Theorem 3.74. Therefore, in view of the Universal Coefficients Theorem, $Z^\nu_\mathbb{Z}$ is a projective $\mathscr{S}_{n,\mathbb{Z}}$-module.

(iii) Denote

$$E_\mathcal{O} := \mathrm{End}_{\mathscr{S}_{n,\mathcal{O}}}\big(\bigoplus_{\nu\in X(n,h)} Z^\nu_\mathcal{O}\big).$$

By Theorem 3.66, we have $E_F \cong S_{h,n,F}$. Moreover, $E_\mathbb{Z}$ is an $\mathcal{O}$-lattice in E_K, and there is a natural embedding $E_\mathbb{Z} \otimes_\mathbb{Z} F \to E_F$, cf. [34, Lemma 14.5]. The last embedding is an isomorphism by dimension. So we can identify $E_\mathbb{Z} \otimes K$ with E_K and $E_\mathbb{Z} \otimes_\mathbb{Z} F$ with E_F.

Now, the basis element $\varphi^u_{\mu,\lambda}$ of $E_K \cong S_{h,n,K}$ acts as zero on all summands except Z^μ_K, on which it is induced by the right multiplication by $s^u_{\mu,\lambda}$. By definition, $Z^\nu_\mathbb{Z} = Z^\nu_K \cap M_{n,\mathbb{Z}}$. Also, $s^u_{\mu,\lambda} \in \mathbb{Z}\mathfrak{S}_n$, therefore $s^u_{\mu,\lambda}$ stabilizes $M_{n,\mathbb{Z}}$. Hence $Z^\mu_\mathbb{Z} s_{\mu,\lambda} \subseteq Z^\lambda_\mathbb{Z}$, so each $\varphi^u_{\mu,\lambda} \in E_K$ restricts to give a well-defined element of $E_\mathbb{Z}$. We have constructed a isomorphic copy $S_{\mathbb{Z}}$ of $S_{h,n,\mathbb{Z}}$ in $E_{\mathbb{Z}}$, namely, the $\mathbb{Z}$-span of the standard basis elements $\varphi^u_{\mu,\lambda} \in S_{h,n,K}$.

It remains to show that $S_{\mathbb{Z}} = E_{\mathbb{Z}}$. We have a short exact sequence of $\mathbb{Z}$-modules:

$$0 \to S_{\mathbb{Z}} \to E_{\mathbb{Z}} \to Q_{\mathbb{Z}} \to 0,$$

and we need to prove that $Q_{\mathbb{Z}} = 0$, for which it suffices to prove that $Q_{\mathbb{Z}} \otimes_{\mathbb{Z}} F = 0$. Tensoring with F, we have an exact sequence

$$S_{\mathbb{Z}} \otimes_{\mathbb{Z}} F \xrightarrow{i} E_F \to Q_{\mathbb{Z}} \otimes_{\mathbb{Z}} F \to 0.$$

The map i sends $1 \otimes \varphi^u_{\mu,\lambda}$ to the corresponding endomorphism $\varphi^u_{\mu,\lambda}$ defined as in Theorem 3.66. Hence, i is injective, so i is an isomorphism by dimensions, and $Q_{\mathbb{Z}} \otimes_{\mathbb{Z}} F = 0$, as required.

(iv) By (ii), $\bigoplus_{\nu \in X(n,h)} Z^{\nu}_{\mathbb{Z}}$ is a projective $\mathscr{S}_{n,\mathbb{Z}}$-module. For $h \geq n$, it is a projective generator, because this is so on tensoring with F, using (i) and Theorem 3.74. $\square$

3.9. Ringel duality and double centralizer properties

Let S be a quasi-hereditary algebra with weight poset $(\Lambda_+, \leq)$ and standard modules $\Delta(\lambda)$. Recall that a (finite dimensional) S-module is called *tilting* if it has both a standard and a costandard filtrations. By [41], for each $\lambda \in \Lambda_+$, there exists a unique indecomposable tilting module $T(\lambda)$ such that $[T(\lambda) : \Delta(\lambda)] = 1$ and $[T(\lambda) : \Delta(\mu)] = 0$ unless $\mu \leq \lambda$. Furthermore, every tilting module is isomorphic to a direct sum of indecomposable tilting modules $T(\lambda)$. A *full tilting* module is a tilting module that contains every $T(\lambda)$, $\lambda \in \Lambda_+$, as a summand. Given a full tilting module T, the *Ringel dual* of S relative to T is the algebra $S^* := \mathrm{End}_S(T)^{\mathrm{op}}$. Writing endomorphisms on the right, T is naturally a right $\mathrm{End}_S(T)$-module, hence a left S^*-module. Ringel [41] showed that S^* is also a quasi-hereditary algebra with weight poset Λ_+, but ordered with the opposite order. We will need the following known result (for references see [5, Section 4.5]).

Lemma 3.96. Regarded as a left S^*-module, T is a full tilting module for S^*. Moreover, the Ringel dual $\mathrm{End}_{S^*}(T)^{\mathrm{op}}$ of S^* relative to T is isomorphic to S.

Applying Ringel's theorem first to the Schur algebra $S_{h,n}$, we obtain the indecomposable tilting modules $\{T_h(\lambda) \mid \lambda \in X_+(h,n)\}$ of $S_{h,n}$. For $h \geq n$, define

$$T(\lambda) := \beta_{h,n}(T_h(\lambda)) \qquad (\lambda \vdash n). \tag{3.97}$$

Since $\beta_{h,n}$ is Morita equivalence, $\{T(\lambda) \mid \lambda \vdash n\}$ are the indecomposable tilting modules for $\mathscr{S}_n$.

Lemma 3.98. The indecomposable tilting modules for $\mathscr{S}_n$ are precisely the indecomposable summands of Λ^ν for all $\nu \vDash n$. Furthermore, for $\lambda \vdash n$, the module $T(\lambda)$ occurs exactly once as a summand of $\Lambda^{\lambda^{\mathrm{tr}}}$, and if $T(\mu)$ is a summand of $\Lambda^{\lambda^{\mathrm{tr}}}$ for some $\mu \vdash \lambda$, then $\mu \leq \lambda$.

Proof. By [10, Section 3.3(1)], $T_h(\lambda)$ occurs exactly once as a summand of $\Lambda^{\lambda^{\mathrm{tr}}}(V_h)$, and if $T_h(\mu)$ is a summand of $\Lambda^{\lambda^{\mathrm{tr}}}(V_h)$ then $\mu \leq \lambda$. Now the result follows on applying the functor $\beta_{h,n}$ and using Lemma 3.88. $\square$

Corollary 3.99. For $\lambda \vdash n$, the module $T(\lambda)$ is the unique indecomposable summand of $\Lambda^{\lambda^{\mathrm{tr}}}$ containing a submodule isomorphic to $\Delta(\lambda)$.

Proof. By Theorem 3.90(i), $\Lambda^{\lambda^{\mathrm{tr}}}$ has a unique submodule isomorphic to $\Delta(\lambda)$. By Lemma 3.98, $\Lambda^{\lambda^{\mathrm{tr}}}$ has a unique summand isomorphic to $T(\lambda)$ and for any other summand M of $\Lambda^{\lambda^{\mathrm{tr}}}$, we have $\mathrm{Hom}_{\mathscr{S}_n}(\Delta(\lambda), M) = 0$. $\square$

Theorem 3.100. **(Imaginary Ringel Duality)** Let $h \geq n$. The $\mathscr{S}_n$-module $T := \bigoplus_{\nu \in X(h,n)} \Lambda^\nu$ is a full tilting module. Moreover, the Ringel dual $\mathscr{S}_n^*$ of $\mathscr{S}_n$ relative to T is precisely the algebra $S_{h,n}^{\mathrm{op}}$ where $S_{h,n}$ acts on T as in Theorem 3.64.

Proof. By Lemma 3.98, T is a full tilting module. The second statement is a restatement of Theorem 3.64. $\square$

So far we have only had 'halves' of imaginary Schur-Weyl and Howe dualities, namely: $\mathrm{End}_{\mathscr{S}_n}(M_n) \cong F\mathfrak{S}_n$ and $\mathrm{End}_{\mathscr{S}_n}\left(\bigoplus_{\nu \in X(h,n)} \Lambda^\nu\right) \cong S_{h,n}$. Now we can finally establish the 'second halves'.

Theorem 3.101. **(Double Centralizer Properties)** Let $h \geq n$. Then:

(i) $\mathrm{End}_{\mathscr{S}_n}\left(\bigoplus_{\nu \in X(h,n)} \Lambda^\nu\right) \cong S_{h,n}$ and $\mathrm{End}_{S_{h,n}}\left(\bigoplus_{\nu \in X(h,n)} \Lambda^\nu\right) \cong \mathscr{S}_n$, where the right $S_{h,n}$-action is as in Theorem 3.64;

(ii) $\mathrm{End}_{\mathscr{S}_n}(M_n) \cong F\mathfrak{S}_n$ and $\mathrm{End}_{F\mathfrak{S}_n}(M_n) \cong \mathscr{S}_n$ where the right $F\mathfrak{S}_n$-action is as in Theorem 3.44.

Proof. (i) Combine Theorem 3.100 with Lemma 3.96.

(ii) By Theorem 3.44, we know already that $F\mathfrak{S}_n \cong \mathrm{End}_{\mathscr{S}_n}(M_n)$. Let $e := e((1^n)) \in S_{h,n}$, see (2.10). We know that $T_h(\lambda)$ is a summand of $\Lambda^{\lambda^{\mathrm{tr}}}(V_h)$, for any $\lambda \in X_+(h,n)$. So, by Lemma 2.17(ii), $T_h(\lambda)$ is both a submodule and a quotient of the $S_{h,n}$-module $S_{h,n}e$. Moreover, $S_{h,n}e \cong V_n^{\otimes n}$, so $S_{h,n}e$ is self-dual. From this one deduces a (well-known) fact that every composition factor of both the socle and the head of $T_h(\lambda)$ belongs to the head of the projective $S_{h,n}$-module $S_{h,n}e$.

Now let $T := \bigoplus_{\nu \in X(h,n)} \Lambda^\nu$, and set $\tilde{T}$ to be the left $S_{h,n}$-module obtained from the right module T by twisting with τ. Then $\tilde{T}$ is a full tilting module for $S_{h,n}$ by Lemma 3.96 and Theorem 3.100. So, by the previous paragraph, every composition factor of the socle and the head of $\tilde{T}$ belongs to the head of $S_{h,n}e$. By Lemma 2.30, we deduce that $\mathrm{End}_{S_{h,n}}(\tilde{T}) \cong \mathrm{End}_{eS_{h,n}e}(e\tilde{T})$. Switching to right actions, and using (i), we have now shown that $\mathscr{S}_n \cong \mathrm{End}_{eS_{h,n}e}(Te)$. So, to prove (ii), it suffices to show that $\mathrm{End}_{F\mathfrak{S}_n}(M_n) \cong \mathrm{End}_{eS_{h,n}e}(Te)$.

As a left $\mathscr{S}_n$-module, $M_n \cong Te$. Recall the map κ from Lemma 2.7 and the action of $S_{h,n}$ on T from Theorem 3.64. One now easily checks that the $(\mathscr{S}_n, eS_{h,n}e)$-bimodule Te is isomorphic to the $(\mathscr{S}_n, F\mathfrak{S}_n)$-bimodule M_n, if we identify $F\mathfrak{S}_n$ with $eS_{h,n}e$, so that $w \mapsto \kappa(\mathtt{sgn}(w)w)$ for each $w \in \mathfrak{S}_n$. In view of this, we have $\mathrm{End}_{F\mathfrak{S}_n}(M_n) \cong \mathrm{End}_{eS_{h,n}e}(Te)$. □

We conclude this section with imaginary analogues of well-known results of Donkin and Mathiew-Papadopoulo. For $\lambda \vdash n$ we denote by $P(\lambda)$ the projective cover of $L(\lambda)$ in the category $\mathscr{S}_n$-mod. Similarly, for $\mu \in X_+(h,n)$, let $P_h(\mu)$ denote the projective cover of $L_h(\mu)$ in the category $S_{h,n}$-mod. Using Morita equivalence, we get:

$$P(\lambda) = \beta_{h,n}(P_h(\lambda)) \qquad (\lambda \vdash n). \tag{3.102}$$

Recall the generalized Kostka numbers $k_{\lambda,\mu}$ from (2.11).

Theorem 3.103. For $\nu \in X(h,n)$ we have:

(i) $Z^\nu \cong \bigoplus_{\lambda \vdash n} P(\lambda)^{\oplus k_{\lambda,\nu}}$;

(ii) $\Lambda^\nu \cong \bigoplus_{\lambda \vdash n} T(\lambda^{\mathrm{tr}})^{\oplus k_{\lambda,\nu}}$.

Proof. (i) By [9, Lemma 3.4(i)], we have $Z^\nu(V_h) \cong \bigoplus_{\lambda \vdash n} P_h(\lambda)^{\oplus k_{\lambda,\nu}}$. Now apply the equivalence of categories $\beta_{h,n}$, using (3.102) and Lemmas 3.88.

(ii) By [37, Corollary 2.3], we have $\Lambda^\nu(V_h) \cong \bigoplus_{\lambda\vdash n} T_h(\lambda^{\mathrm{tr}})^{\oplus k_{\lambda,\nu}}$. Now apply the equivalence of categories $\beta_{h,n}$, using (3.97) and Lemmas 3.88. $\square$

Throughout the chapter we assume that $\mathcal{O} = F$ unless otherwise stated, and $h \geq n$.

3.10. Characters of imaginary modules

Let $h \geq n$, $\lambda \in X_+(h,n)$, and $\mu \in X(h,n)$. Recall the notion of a column strict λ-tableau from Section 2.24. Denote

$$\mathrm{col}_{\lambda,\mu} := \sharp\{\text{column strict } \lambda\text{-tableaux of type } \mu\}. \tag{3.104}$$

Recalling the $S_{h,n}$-module $\Lambda^{\lambda^{\mathrm{tr}}}(V_h)$ defined in (2.16), the following equality for its weight multiplicities is clear:

$$\dim e(\mu)\Lambda^{\lambda^{\mathrm{tr}}}(V_h) = \mathrm{col}_{\lambda,\mu}.$$

The following combinatorial result is easy to check using the definition of $\mathrm{col}_{\lambda,\mu}$:

Lemma 3.105. Let $\lambda \in X_+(h,n)$, $\mu \in X(h,n)$. If the last column of the partition λ has height l, and $\bar{\lambda} \in X_+(h,n-l)$ is the partition obtained from λ by deleting this last column, then

$$\mathrm{col}_{\lambda,\mu} = \sum_{\substack{1\leq s_1<\cdots<s_l\leq h,\\ \mu-\varepsilon_{s_1}-\cdots-\varepsilon_{s_l}\in X(h,n)}} \mathrm{col}_{\bar{\lambda},\mu-\varepsilon_{s_1}-\cdots-\varepsilon_{s_l}}.$$

Recall the classical Kostka numbers from (2.12).

Lemma 3.106. Let $\lambda,\mu \vdash n$. Then $\mathrm{col}_{\lambda,\mu} = \sum_{\nu\vdash n} K_{\nu^{\mathrm{tr}},\mu} K_{\nu,\lambda^{\mathrm{tr}}}$.

Proof. We have the well-known fact that over $\mathbb{C}$ the module $\Lambda^{\lambda^{\mathrm{tr}}}(V_h)$ decomposes as $\Lambda^{\lambda^{\mathrm{tr}}}(V_h) = \bigoplus_{\nu \vdash n} \Delta_h(\nu^{\mathrm{tr}})^{\oplus K_{\nu,\lambda^{\mathrm{tr}}}}$. Passing to the dimensions of the μ-weight spaces in the last equality yields the lemma. $\square$

3.101. Gelfand-Graev words and shuffles

Recall from (3.17) that we have fixed an extremal word $\boldsymbol{i} = i_1 \dots i_e$ of L_δ. Recall that $i_1 = 0$ and $i_e = i$. As in (3.54), we also write $\boldsymbol{i}$ in the form

$$\boldsymbol{i} = j_1^{m_1} \dots j_r^{m_r} \qquad \text{(with } j_k \neq j_{k+1} \text{ for all } 1 \leq k < r\text{)}.$$

Note that always $m_1 = 1$. We define the *Gelfand-Graev words* (of *type* $\boldsymbol{i}$):

$$\boldsymbol{g}^{(n)} = \boldsymbol{g}_{\boldsymbol{i}}^{(n)} := i_1^n i_2^n \dots i_e^n = j_1^{m_1 n} \dots j_r^{m_r n} \in I_{n\delta}$$

for any $n \in \mathbb{Z}_{>0}$ and, more generally, for any composition $\mu \in X(h, n)$, set:

$$\boldsymbol{g}^\mu := \boldsymbol{g}^{(\mu_1)} \dots \boldsymbol{g}^{(\mu_h)} \in I_{n\delta} \tag{3.107}$$

Lemma 3.108. Let $n = l_1 + \dots + l_a$ for some $l_1, \dots, l_a \in \mathbb{Z}_{>0}$. Suppose that for each $1 \leq c \leq a$, we are given a word $\boldsymbol{j}^{(c)}$ of the imaginary tensor space M_{l_c}. Assume that a Gelfand-Graev word $\boldsymbol{g}^\mu$ of type $\boldsymbol{i}$ appears as a summand in the shuffle product $\boldsymbol{j}^{(1)} \circ \dots \circ \boldsymbol{j}^{(a)}$. Then $\boldsymbol{j}^{(1)}, \dots, \boldsymbol{j}^{(a)}$ are all Gefand-Graev words of type $\boldsymbol{i}$.

Proof. Clearly we may assume that $a = 2$. Then we may write $\boldsymbol{j}^{(1)} = \boldsymbol{k}^{(1)}\boldsymbol{l}^{(1)}$ and $\boldsymbol{j}^{(2)} = \boldsymbol{k}^{(2)}\boldsymbol{l}^{(2)}$, so that $\boldsymbol{g}^{(\mu_1)}$ appears in the shuffle product $\boldsymbol{k}^{(1)} \circ \boldsymbol{k}^{(2)}$. Recall that $\boldsymbol{g}^{(\mu_1)} = j_1^{m_1\mu_1} \dots j_r^{m_r\mu_1}$. It follows that $\boldsymbol{k}^{(1)} = j_1^{a_1} \dots j_r^{a_r}$, $\boldsymbol{k}^{(2)} = j_1^{b_1} \dots j_r^{b_r}$ with $a_k + b_k = m_k\mu_1$ for all $k = 1, \dots, r$. Note that $a_2 + b_2 = m_2\mu_1 = m_2(a_1 + b_1)$. We claim that

$a_2 = a_1m_2$ and $b_2 = b_1m_2$. Indeed, otherwise either $a_2 > a_1m_2$ or $b_2 > b_1m_2$. But the first inequality contradicts the fact that $\boldsymbol{j}^{(1)} = \boldsymbol{k}^{(1)}\boldsymbol{l}^{(1)}$ is a word of M_{l_1}, and the second inequality contradicts the fact that $\boldsymbol{j}^{(2)} = \boldsymbol{k}^{(2)}\boldsymbol{l}^{(2)}$ is a word of M_{l_2}. Continuing this way, we see that $a_k = m_ka_1$ and $b_k = m_kb_1$ for all $k = 1, \dots, r$. In other words, $\boldsymbol{k}^{(1)} = \boldsymbol{g}^{(a_1)}$ and $\boldsymbol{k}^{(2)} = \boldsymbol{g}^{(b_1)}$. Now the Gelfand-Graev word $\boldsymbol{g}^{(\mu_2)} \dots \boldsymbol{g}^{(\mu_h)}$ appears in the shuffle product $\boldsymbol{l}^{(1)} \circ \boldsymbol{l}^{(2)}$. Moreover, $\boldsymbol{l}^{(1)}$ and $\boldsymbol{l}^{(2)}$ are words of $M_{l_1-a_1}$ and $M_{l_2-b_1}$, respectively. So we can apply induction on the length of μ. $\square$

For $n \in \mathbb{Z}_{\geq 0}$, we denote

$$c(n) := \prod_{k=1}^{r} [m_kn]^!_{j_k} \in \mathscr{A},$$

Note by Lemma 2.61 that $\dim_q 1_{\boldsymbol{i}} L_\delta = c(1)$, since we have chosen $\boldsymbol{i}$ to be an extremal word in L_δ. For $\mu \in X(h, n)$, denote

$$c(\mu) := c(\mu_1) \dots c(\mu_h) = \prod_{m=1}^{h} \prod_{k=1}^{r} [m_k\mu_m]^!_{j_k} \in \mathscr{A}. \tag{3.109}$$

In the simply laced types all $m_k = 1$, and so $c(\mu) = ([\mu_1]^!_q \dots [\mu_h]^!_q)^e$.

Recall the numbers $\mathrm{col}_{\lambda,\mu}$ defined in (3.104).

Proposition 3.110. Let $\lambda \in X_+(h, n)$ and $\mu \in X(h, n)$. Set $\lambda^{\mathrm{tr}} = (l_1, \dots, l_a)$. Then $\boldsymbol{g}^\mu$ appears in the quantum shuffle product

$$\boldsymbol{i}^{l_1} \circ \dots \circ \boldsymbol{i}^{l_a} \tag{3.111}$$

with the coefficient $c(\mu)\mathrm{col}_{\lambda,\mu}/c(1)^n$.

Proof. We apply induction on a. If $a = 1$, then (3.111) is just the concatenation $\boldsymbol{i}^{l_1} = \boldsymbol{g}^{(1^n)}$, and the result is clear since $c((1^n)) = c(1)^n$ and $\mathrm{col}_{(1^n),(1^n)} = 1$. For the inductive step, let $a > 1$ and denote by $\bar{\lambda} \in X_+(h, n - l_a)$ the partition obtained from λ by deleting its last column. By the inductive assumption, for any $\nu \in X(h, n - l_a)$ we have that $\boldsymbol{g}^\nu$ appears in the quantum shuffle product

$$S := \boldsymbol{i}^{l_1} \circ \cdots \circ \boldsymbol{i}^{l_{a-1}}$$

with coefficient $c(\nu)\mathrm{col}_{\bar{\lambda},\nu}/c(1)^{n-l_a}$. Now (3.111) is $S \circ \boldsymbol{i}^{l_a}$. By Lemma 3.108, if the word $\boldsymbol{j}$ appearing in S has the property that some Gelfand-Graev word $\boldsymbol{g}^\mu$ appears in the shuffle product $\boldsymbol{j} \circ (\boldsymbol{i}^{(b)})^{l_a}$, then the word $\boldsymbol{j}$ must itself be Gelfand-Graev, i.e. $\boldsymbol{j} = \boldsymbol{g}^\nu$ for some $\nu \in X(h, n - l_a)$. Moreover, note that $\boldsymbol{g}^\mu$ appears in $\boldsymbol{g}^\nu \circ \boldsymbol{i}^{l_a}$ if and only if μ is of the form $\mu = \nu + \varepsilon_{s_1} + \cdots + \varepsilon_{s_{l_a}}$ for some $1 \leq s_1 < \cdots < s_{l_a} \leq h$, in which case $\boldsymbol{g}^\mu$ appears in $\boldsymbol{g}^\nu \circ \boldsymbol{i}^{l_a}$ with the coefficient $c(\mu)/c(\nu)c(1)^{l_a}$. Now the result follows in view of Lemma 3.105. $\square$

Recall Gelfand-Graev modules $\Gamma_n \cong \Gamma_{nm_1,j_1} \circ \cdots \circ \Gamma_{nm_r,j_r}$ from Section 3.51.

Lemma 3.112. Let $M \in R_{n\delta}$-mod and $\mu \in X(n, h)$. Then

$$\dim_q M_{\boldsymbol{g}^\mu} = c(\mu) \dim_q \mathrm{Hom}_{R_{n\delta}}(\Gamma_{\mu_1} \circ \cdots \circ \Gamma_{\mu_h}, M).$$

Proof. Let

$$\sigma = (\mu_1 m_1 \alpha_{j_1}, \ldots, \mu_1 m_r \alpha_{j_r}, \ldots, \mu_h m_1 \alpha_{j_1}, \ldots, \mu_h m_r \alpha_{j_r}).$$

Consider the irreducible module

$$L := L(j_1^{\mu_1 m_1}) \boxtimes \cdots \boxtimes L(j_r^{\mu_1 m_r}) \boxtimes \cdots \boxtimes L(j_1^{\mu_h m_1}) \boxtimes \cdots \boxtimes L(j_r^{\mu_h m_r})$$

over the parabolic R_σ. Note that $\dim_q M_{\boldsymbol{g}^\mu} = c(\mu)[\mathrm{Res}_\sigma M : L]_q$. Since $\Gamma_{m,i}$ is the projective cover of $L(i^m)$ by Proposition 3.53(i), $[\mathrm{Res}_\sigma M : L]_q$ equals

$$\dim_q \mathrm{Hom}_{R_\sigma}(\Gamma_{\mu_1 m_1, j_1} \boxtimes \cdots \boxtimes \Gamma_{\mu_1 m_r, j_r} \boxtimes \cdots \boxtimes \Gamma_{\mu_h m_1, j_1} \boxtimes \cdots \boxtimes \Gamma_{\mu_h m_r, j_r}, \mathrm{Res}_\sigma M),$$

and the result follows by the adjunction of Ind and Res. □

3.102. Gelfand-Graev fragment of the formal character of $\Delta(\lambda)$

Recall Gelfand-Graev words $\boldsymbol{g}^\mu = \boldsymbol{g}_{\boldsymbol{i}}^\mu$ from Section 3.101, scalars $c(n), c(\mu) \in \mathscr{A}$ from (3.109), and $R_{n\delta}$-modules $L(\lambda), \Delta(\lambda), T(\lambda)$ from (3.77), (3.78), (3.97).

Lemma 3.113. We have $L((1^n)) = \Delta((1^n)) = T((1^n)) = \Lambda_n$, the Gelfand Graev word $\boldsymbol{g}^{(1^n)} = \boldsymbol{i}^n$ appears in $\mathrm{ch}_q\,\Delta((1^n))$ with multiplicity $c((1^n)) = c(1)^n$, and $\boldsymbol{g}^\mu$ with $\mu \in X(n,n) \setminus \{(1^n)\}$ does not appear in $\mathrm{ch}_q\,\Delta((1^n))$.

Proof. It well-known for the usual Schur algebras that $T_n((1^n)) = \Delta_n((1^n)) = L_n(1^n)$. By applying the Morita equivalence $\beta_{n,n}$ and Lemma 3.98, we now obtain $L((1^n)) = \Delta((1^n)) = T((1^n)) = \Lambda^{(n)} = \Lambda_n$, and the first two statements follow from Lemma 3.59.

For the last statement, let $\mu \in X(n,n) \setminus \{(1^n)\}$. We have to prove that $\Delta((1^n))_{\boldsymbol{g}^\mu} = 0$. In view of Theorem 3.94, we may assume that F has characteristic zero. By Lemma 3.112, we need to prove that $\mathrm{Hom}_{R_{n\delta}}(\Gamma_{\mu_1} \circ \cdots \circ \Gamma_{\mu_n}, \Lambda_n) = 0$. We have by adjunction of Ind and Res and Lemma 3.48(ii):

$$\mathrm{Hom}_{R_{n\delta}}(\Gamma_{\mu_1} \circ \cdots \circ \Gamma_{\mu_n}, M_n) \cong \mathrm{Hom}_{R_{\mu,\delta}}(\Gamma_{\mu_1} \boxtimes \cdots \boxtimes \Gamma_{\mu_n}, M_\mu \otimes_{F\mathfrak{S}_\mu} F\mathfrak{S}_n).$$

By Proposition 3.57(iii), the latter space is isomorphic as a right $F\mathfrak{S}_n$-module to $\mathtt{sgn}_{\mathfrak{S}_\mu} \otimes_{F\mathfrak{S}_\mu} F\mathfrak{S}_n$. This module is annihilated by the (right) multiplication by $\mathtt{x}_n$. Therefore the right multiplication by $\mathtt{x}_n$ annihilates the image of any non-zero homomorphism from $\Gamma_{\mu_1} \circ \cdots \circ \Gamma_{\mu_n}$ to M_n. On the other hand, the right multiplication by $\mathtt{x}_n$ acts as an automorphism of $\Lambda_n = M_n \mathtt{x}_n$ since F has characteristic zero. This implies that $\mathrm{Hom}_{R_{n\delta}}(\Gamma_{\mu_1} \circ \cdots \circ \Gamma_{\mu_n}, \Lambda_n) = 0$. $\square$

Recall that for a composition $\mu \vDash n$ we denote by $\mu^+ \vdash n$ the unique partition obtained from μ by a permutation of its parts.

Corollary 3.114. Let $\lambda \vdash n$ and $\mu \vDash n$. Then $\boldsymbol{g}^\mu$ appears in $\mathrm{ch}_q \Lambda^{\lambda^{\mathrm{tr}}}$ with the coefficient $c(\mu)\mathrm{col}_{\lambda,\mu}$. In particular, $\boldsymbol{g}^\mu$ appears in $\mathrm{ch}_q \Lambda^{\lambda^{\mathrm{tr}}}$ with the same coefficient as $\boldsymbol{g}^{\mu^+}$.

Proof. Let $\lambda^{\mathrm{tr}} = (l_1, \dots, l_a)$. Then in view of Lemma 3.113, we have

$$\Lambda^{\lambda^{\mathrm{tr}}} = \Lambda_{l_1} \circ \cdots \circ \Lambda_{l_a} = \Delta(1^{l_1}) \circ \cdots \circ \Delta(1^{l_a}).$$

So if $\boldsymbol{g}^\mu$ appears in $\mathrm{ch}_q \Lambda^{\lambda^{\mathrm{tr}}}$, then $\boldsymbol{g}^\mu$ appears in the shuffle product $\boldsymbol{j}^{(1)} \circ \cdots \circ \boldsymbol{j}^{(a)}$, where $\boldsymbol{j}^{(c)}$ is a word of $\Delta(1^{l_c})$ for all $c = 1, \dots, a$. By Lemma 3.108, each $\boldsymbol{j}^{(c)}$ is a Gelfand-Graev word (of type $\boldsymbol{i}$). By Lemma 3.113, we have $\boldsymbol{j}^{(c)} = \boldsymbol{g}^{(1^{l_c})} = \boldsymbol{i}^{l_c}$ for $c = 1, \dots, a$. Now the result follows from Proposition 3.110. The second statement comes by noticing that $\mathrm{col}_{\lambda,\mu} = \mathrm{col}_{\lambda,\mu^+}$ and $c(\mu) = c(\mu^+)$. $\square$

Recall the Kostka numbers $K_{\lambda,\mu}$ from (2.12). The matrix $K := (K_{\lambda,\mu})_{\lambda,\mu \vdash n}$ is unitriangular, in particular it is invertible. Let

$$N = (N_{\lambda,\mu})_{\lambda,\mu \vdash n} := K^{-1}$$

be the inverse matrix.

Lemma 3.115. Let $\lambda \vdash n$. We have:

(i) $\operatorname{ch}_q \Lambda^{\lambda^{\mathrm{tr}}} = \sum_{\mu \vdash n} K_{\mu^{\mathrm{tr}},\lambda^{\mathrm{tr}}} \operatorname{ch}_q \Delta(\mu) = \operatorname{ch}_q \Delta(\lambda) + \sum_{\mu < \lambda} K_{\mu^{\mathrm{tr}},\lambda^{\mathrm{tr}}} \operatorname{ch}_q \Delta(\mu)$;

(ii) $\operatorname{ch}_q \Delta(\lambda) = \sum_{\mu \vdash n} N_{\mu^{\mathrm{tr}},\lambda^{\mathrm{tr}}} \operatorname{ch}_q \Lambda^{\mu^{\mathrm{tr}}} = \operatorname{ch}_q \Lambda^{\lambda^{\mathrm{tr}}} + \sum_{\mu < \lambda} N_{\mu^{\mathrm{tr}},\lambda^{\mathrm{tr}}} \operatorname{ch}_q \Lambda^{\mu^{\mathrm{tr}}}$.

Proof. By Lemma 3.113 and Theorem 3.94, the characters of Λ^{μ} and $\Delta(\mu)$ are independent of the ground field for all $\mu \vdash n$. So we may assume that $F = \mathbb{C}$, in which case $\Delta(\mu) = \nabla(\mu) = T(\mu) = L(\mu)$ for all $\mu \vdash n$. Now (i) follows from Theorem 3.103(ii), and (ii) follows from (i). $\square$

We can now determine the multiplicity of any Gelfand-Graev word in the standard module $\Delta(\lambda)$. We refer to this partial character information as the *Gelfand-Graev fragment* of the character.

Theorem 3.116. Let $\lambda \vdash n$ and $\mu \vdash n$, and $\nu \vDash n$. Then:

(i) $\dim_q \Delta(\lambda)_{\boldsymbol{g}^{\nu}} = \dim_q \Delta(\lambda)_{\boldsymbol{g}^{\nu+}}$;

(ii) $\dim_q \Delta(\lambda)_{\boldsymbol{g}^{\mu}} = c(\mu) K_{\lambda,\mu}$.

Proof. (i) By Lemma 3.115(ii), we have

$$\dim_q \Delta(\lambda)_{\boldsymbol{g}^{\nu}} = \sum_{\mu \vdash n} N_{\mu^{\mathrm{tr}},\lambda^{\mathrm{tr}}} \dim_q (\Lambda^{\mu^{\mathrm{tr}}})_{\boldsymbol{g}^{\nu}}.$$

So it suffices to prove that $\dim_q (\Lambda^{\mu^{\mathrm{tr}}})_{\boldsymbol{g}^{\nu}} = \dim_q (\Lambda^{\mu^{\mathrm{tr}}})_{\boldsymbol{g}^{\nu+}}$ for all $\mu \vdash n$. But this is contained in Corollary 3.114.

(ii) Using Lemma 3.115(ii), Corollary 3.114 and Lemma 3.106, we get:

$$\begin{aligned}\dim_q \Delta(\lambda)_{\boldsymbol{g}^\mu} &= \sum_{\nu\vdash n} N_{\nu^{\mathrm{tr}},\lambda^{\mathrm{tr}}} \dim_q (\Lambda^{\nu^{\mathrm{tr}}})_{\boldsymbol{g}^\mu} \\ &= \sum_{\nu\vdash n} N_{\nu^{\mathrm{tr}},\lambda^{\mathrm{tr}}}\, c(\mu)\mathrm{col}_{\nu,\mu} \\ &= \sum_{\nu,\kappa\vdash n} N_{\nu^{\mathrm{tr}},\lambda^{\mathrm{tr}}}\, c(\mu) K_{\kappa^{\mathrm{tr}},\mu} K_{\kappa,\nu^{\mathrm{tr}}} \\ &= \sum_{\kappa\vdash n} c(\mu) K_{\kappa^{\mathrm{tr}},\mu}\delta_{\kappa,\lambda^{\mathrm{tr}}} = c(\mu)K_{\lambda,\mu},\end{aligned}$$

where we have used that $N = K^{-1}$. $\square$

We can extend the above result to the Gelfand-Graev fragments of characters of other imaginary modules:

Corollary 3.117. Let $\lambda \vdash n$ and $\nu \vDash n$, and suppose that $h \geq n$, $W \in S_{h,n}$-mod, and $M = \beta_{h,n}(W) \in \mathscr{S}_n$-mod. Then

$$\dim_q M_{\boldsymbol{g}^\nu} = c(\nu) \dim e(\nu) W.$$

In particular, $\dim_q L(\lambda)_{\boldsymbol{g}^\nu} = c(\nu) k_{\lambda,\nu}$.

Proof. Note that $\{\mathrm{ch}_q \Delta_h(\mu) \mid \mu \vdash n\}$ is a linear basis in the character ring of modules in $S_{h,n}$-mod. So we can write $\mathrm{ch}\, W = \sum_{\mu\vdash n} n_\mu \,\mathrm{ch}\, \Delta_h(\mu)$ for some $n_\mu \in \mathbb{Z}$. Applying the Morita-equivalence $\beta_{h,n}$, we then also have

$$\mathrm{ch}_q M = \sum_{\mu\vdash n} n_\mu \mathrm{ch}_q \Delta(\mu).$$

So, applying Theorem 3.116,

$$\begin{aligned}\dim_q M_{\boldsymbol{g}^\nu} &= \sum_{\mu \vdash n} n_\mu \dim_q \Delta(\mu)_{\boldsymbol{g}^\nu} = \sum_{\mu \vdash n} n_\mu c(\nu) \dim e(\nu) \Delta_h(\mu) \\ &= c(\nu) \dim e(\nu) W,\end{aligned}$$

as required. □

3.103. Imaginary Jacobi-Trudy formula

The formal characters of standard modules $\Delta(\lambda)$ in terms of the characters of the modules $\Delta(1^m)$ can in principle be found from Lemma 3.115, since the modules Λ^ν are just $\Delta(1^{n_1}) \circ \dots \circ \Delta(1^{n_a})$ if $\nu = (n_1, \dots, n_a)$. A standard way of dealing with the inverse matrix $N = K^{-1}$ is through Jacobi-Trudy formulas.

Let $\lambda = (l_1, \dots, l_a) \vdash n$. Note by Corollary 3.84 that

$$\mathrm{ch}_q \Delta(1^k) \circ \mathrm{ch}_q \Delta(1^l) = \mathrm{ch}_q \Delta(1^l) \circ \mathrm{ch}_q \Delta(1^k) \qquad (k, l \in \mathbb{Z}_{>0}).$$

So we can use the quantum shuffle product to make sense of the following determinant as an element of $\mathscr{A} I_{n\delta}$:

$$\mathscr{D}(\lambda) := \det \left(\mathrm{ch}_q \Delta(1^{l_r - r + s}) \right)_{1 \le r, s \le a} \in \mathscr{A} I_{n\delta}.$$

where $\mathrm{ch}_q\,\Delta(1^0)$ is interpreted as (a multiplicative) identity, and $\mathrm{ch}_q\,\Delta(1^m)$ is interpreted as (a multiplicative) 0 if $m < 0$. For example, if $\lambda = (3, 1, 1)$, we get

$$\begin{aligned}\mathscr{D}((3,1,1)) =& \det\begin{pmatrix} \mathrm{ch}_q\,\Delta(1^3) & \mathrm{ch}_q\,\Delta(1^4) & \mathrm{ch}_q\,\Delta(1^5) \\ 1 & \mathrm{ch}_q\,\Delta(1) & \mathrm{ch}_q\,\Delta(1^2) \\ 0 & 1 & \mathrm{ch}_q\,\Delta(1) \end{pmatrix} \\ =& \mathrm{ch}_q\,\Delta(1^3) \circ \mathrm{ch}_q\,\Delta(1) \circ \mathrm{ch}_q\,\Delta(1) + \mathrm{ch}_q\,\Delta(1^5) \\ & - \mathrm{ch}_q\,\Delta(1^4) \circ \mathrm{ch}_q\,\Delta(1) - \mathrm{ch}_q\,\Delta(1^3) \circ \mathrm{ch}_q\,\Delta(1^2).\end{aligned}$$

Remark 3.118. The characters of the modules $\Delta(1^m)$ are well-understood in many situations. Let i be the color of the tensor space we are working with, and $\boldsymbol{i} = (i_1, \dots, i_e)$ so that $i_e = i$. Then $\boldsymbol{i}^m$ is a word of $\Delta(1^m) = L(1^m)$, see Lemma 3.113. Oftentimes the word $\boldsymbol{i}^m$ is homogeneous, and so $\Delta(1^m)$ is the homogeneous irreducible module associated to the connected component of $\boldsymbol{i}^m$ in the word graph. For example in Lie type $\mathtt{C} = \mathtt{A}_l^{(1)}$ this is *always* the case.

Theorem 3.119. **(Imaginary Jacobi-Trudy Formula)** Let $\lambda \vdash n$. Then $\mathrm{ch}_q\,\Delta(\lambda) = \mathscr{D}(\lambda^{\mathrm{tr}})$.

Proof. Let $\lambda^{\mathrm{tr}} = (l_1, \dots, l_a)$ and take $h \geq n$. It follows from the classical Jacobi-Trudy formula that in the Grothendieck group of $S_{h,n}$-mod we have

$$[\Delta_h(\lambda)] = \det([\Lambda^{(l_r - r + s)}(V_h)])_{1 \leq r,s \leq a}$$

with determinant defined using multiplication given by tensor product. Applying the equivalence of categories $\beta_{h,n}$, Theorem 3.83, and Lemmas 3.88, 3.113 we get

$$[\Delta(\lambda)] = \det([\Lambda_{l_r-r+s}])_{1\leq r,s\leq a} = \det([\Delta(1^{l_r-r+s})])_{1\leq r,s\leq a}$$

with determinant defined using multiplication given by induction product '$\circ$'. Passing to the formal characters, we obtain the required formula. $\square$

CHAPTER IV

STRATIFYING KLR ALGEBRAS OF SYMMETRIC AFFINE LIE TYPE

The work in this chapter has appeared in the articles [30, 28], which have been submitted for publication. It is co-authored with Alexander Kleshchev. We developed the results in the co-authored material jointly over many meetings, and, by the nature of collaborative mathematical work, it is difficult to attribute exact portions of the co-authored material to either Kleshchev or myself individually.

4.1. Stratification

Throughout this chapter, unless otherwise stated, k is an arbitrary field of characteristic $p \geq 0$. In this section, we mainly follow [33]. All notions we consider, such as algebras, modules, ideals, etc., are assumed to be ($\mathbb{Z}$-)graded.

4.11. Graded algebras

We recall some basics of graded representation theory, and introduce *Laurentian* algebras in this section. If H is a Noetherian (graded) k-algebra, we denote by H-mod the category of finitely generated graded left H-modules. The morphisms in this category are all homogeneous degree zero H-homomorphisms, which we denote $\hom_H(-,-)$. For $V \in H$-mod, let $q^d V$ denote its grading shift by d, so if V_m is the degree m component of V, then $(q^d V)_m = V_{m-d}$. For a polynomial $a = \sum_d a_d q^d \in \mathbb{Z}[q, q^{-1}]$ with non-negative coefficients, we set $aV := \bigoplus_d (q^d V)^{\oplus a_d}$.

For $U, V \in H$-mod, we set

$$\mathrm{Hom}_H(U,V) := \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_H(U,V)_d,$$

where $\mathrm{Hom}_H(U,V)_d := \mathrm{hom}_H(q^d U, V)$. We define $\mathrm{ext}^m_H(U,V)$ and $\mathrm{Ext}^m_H(U,V)$ similarly. Since U is finitely generated, $\mathrm{Hom}_H(U,V)$ can be identified with the set of all H-module homomorphisms ignoring the gradings. A similar result holds for $\mathrm{Ext}^m_H(U,V)$, since U has a resolution by finitely generated projective modules. Given $V, W \in H$-mod, we write $V \simeq W$ to indicate that $V \cong q^n W$ for some $n \in \mathbb{Z}$.

A vector space V is called *Laurentian* if its graded components V_m are finite dimensional and $V_m = 0$ for $m \ll 0$. Then the graded dimension $\dim_q V$ is a Laurent series. An algebra is called *Laurentian* if it is so as a vector space. In this case all irreducible H-modules are finite dimensional, there are only finitely many irreducible H-modules up to isomorphism and degree shift, and every finitely generated H-module has a projective cover, see [25, Lemma 2.2].

Let H be a Laurentian algebra. We fix a complete irredundant set $\{L(\pi) \mid \pi \in \Pi\}$ of irreducible H-modules up to isomorphism and degree shift. By above, the set Π is finite. For each $\pi \in \Pi$, we also fix a projective cover $P(\pi)$ of $L(\pi)$. Let

$$M(\pi) := \mathrm{rad}\, P(\pi) \qquad (\pi \in \Pi), \tag{4.1}$$

so that $P(\pi)/M(\pi) \cong L(\pi)$.

From now on we assume in addition that H is *Schurian*, i.e. $\mathrm{End}(L(\pi)) \cong k$ for all π. For any $V \in H$-mod and $\pi \in \Pi$, the *composition multiplicity* of $L(\pi)$ in V is defined as $[V : L(\pi)]_q := \dim_q \mathrm{Hom}(P(\pi), V) \in \mathbb{Z}((q))$.

4.12. Standard objects and stratification

We continue with the notation of the previous subsection. Let Σ be a subset of Π. An object X of the category $\mathbf{C} := H$-mod *belongs to* Σ if $[X : L(\sigma)]_q \neq 0$ implies $\sigma \in \Sigma$. Let $\mathbf{C}(\Sigma)$ be the full subcategory of $\mathbf{C}$ consisting of all objects which belong to Σ. The natural inclusion $\iota_\Sigma : \mathbf{C}(\Sigma) \to \mathbf{C}$ has left adjoint $\mathcal{Q}^\Sigma : \mathbf{C} \to \mathbf{C}(\Sigma)$ with $\mathcal{Q}^\Sigma(V) = V/\mathcal{O}^\Sigma(V)$, where $\mathcal{O}^\Sigma(V)$ is the unique minimal submodule among submodules $U \subseteq V$ such that V/U belongs to Σ. Let also $\mathcal{O}_\Sigma(V)$ be the unique maximal subobject of V which belongs to Σ.

Applying $\mathcal{O}^\Sigma$ to the left regular module H yields a (two-sided) ideal $\mathcal{O}^\Sigma(H) \trianglelefteq H$. By [33, Lemma 3.12], for $V \in H$-mod, we have $\mathcal{O}^\Sigma(H)V = \mathcal{O}^\Sigma(V)$. Set $H(\Sigma) := H/\mathcal{O}^\Sigma(H)$. Then we can regard $\mathcal{Q}^\Sigma(V)$ as an $H(\Sigma)$-module and identify $\mathbf{C}(\Sigma)$ and $H(\Sigma)$-mod. In this way, $\mathcal{Q}^\Sigma$ becomes a functor $\mathcal{Q}^\Sigma : H\text{-mod} \to H(\Sigma)\text{-mod}$.

We now suppose that there is a fixed surjection

$$\varrho : \Pi \to \Xi \tag{4.2}$$

for some set Ξ endowed with a *partial order* $\leq$. We then have a partial preorder $\leq$ on Π with $\pi \leq \sigma$ if and only if $\varrho(\pi) \leq \varrho(\sigma)$. For $\pi \in \Pi$ and $\xi \in \Xi$ we set

$$\Pi_{<\pi} := \{\sigma \in \Pi \mid \sigma < \pi\},\ \Pi_{\leq\pi} := \{\sigma \in \Pi \mid \sigma \leq \pi\},$$
$$\Pi_{<\xi} := \{\sigma \in \Pi \mid \varrho(\sigma) < \xi\},\ \Pi_{\leq\xi} := \{\sigma \in \Pi \mid \varrho(\sigma) \leq \xi\},$$

and write $\mathcal{O}^{\leq\pi} := \mathcal{O}^{\Pi_{\leq\pi}}$, $\mathcal{O}_{<\xi} := \mathcal{O}_{\Pi_{<\xi}}$, $\mathcal{Q}^{<\pi} := \mathcal{Q}^{\Pi_{<\pi}}$, $H_{\leq\xi} := H(\Pi_{\leq\xi})$, etc.

Recalling (4.1), we define for all $\pi \in \Pi$:

$$K(\pi) := \mathcal{O}^{\leq \pi}(P(\pi)) = \mathcal{O}^{\leq \pi}(M(\pi)), \qquad \bar{K}(\pi) := \mathcal{O}^{<\pi}(M(\pi)),$$

and

$$\Delta(\pi) := \mathcal{Q}^{\leq \pi}(P(\pi)) = P(\pi)/K(\pi), \qquad \bar{\Delta}(\pi) := P(\pi)/\bar{K}(\pi). \tag{4.3}$$

Note that $\bar{K}(\pi) \supseteq K(\pi)$, so $\bar{\Delta}(\pi)$ is naturally a quotient of $\Delta(\pi)$. Moreover, head $\Delta(\pi) \cong$ head $\bar{\Delta}(\pi) \cong L(\pi)$. We call the modules $\Delta(\pi)$ *standard* and the modules $\bar{\Delta}(\pi)$ *proper standard*. By [33, Lemma 3.10], $\Delta(\pi)$ is the projective cover of $L(\pi)$ in the category $\mathbf{C}_{\leq \pi}$. For $V \in \mathbf{C}$, a finite Δ-*filtration* (or a *standard filtration*) is a filtration $V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_N = 0$ such that each for $0 \leq n < N$ we have $V_n/V_{n+1} \simeq \Delta(\pi)$ for some $\pi \in \Pi$.

Let $\xi \in \Xi$. Then $\mathbf{C}_{<\xi}$ is a Serre subcategory of $\mathbf{C}_{\leq \xi}$, and the quotient category

$$\mathbf{C}_\xi := \mathbf{C}_{\leq \xi}/\mathbf{C}_{<\xi}, \tag{4.4}$$

is called the ξ-*stratum*. Up to isomorphism and degree shift, $\{L(\pi) \mid \varrho(\pi) = \xi\}$ is a complete family of simple objects in $\mathbf{C}_\xi$, and $P_\xi(\pi) := \Delta(\pi)/\mathcal{O}_{<\xi}(\Delta(\pi))$ is the projective cover of $L(\pi)$ in $\mathbf{C}_\xi$. Finally, setting

$$\Delta(\xi) := \bigoplus_{\pi \in \varrho^{-1}(\xi)} \Delta(\pi) \quad \text{and} \quad B_\xi := \mathrm{End}_H(\Delta(\xi))^{\mathrm{op}}, \tag{4.5}$$

by [33, Corollary 4.9], the stratum category $\mathbf{C}_\xi$ is graded equivalent to B_ξ-mod.

We have a natural exact projection functor $\mathcal{R}_\xi : \mathbf{C}_{\leq\xi} \to \mathbf{C}_\xi$. If we identify $\mathbf{C}_\xi$ and B_ξ-mod, the functor R_ξ becomes

$$\mathcal{R}_\xi = \mathrm{Hom}_{H_{\leq\xi}}(\Delta(\xi), -) : H_{\leq\xi}\text{-mod} \to B_\xi\text{-mod}\,.$$

Its left adjoint

$$\mathcal{E}_\xi = \Delta_\xi \otimes_{B_\xi} - : B_\xi\text{-mod} \to H_{\leq\xi}\text{-mod} \tag{4.6}$$

is called a *weak standardization functor*. By [33, Lemma 4.11], if $\varrho(\pi) = \xi$ then $\Delta(\pi) \cong \mathcal{E}_\xi(P_\xi(\pi))$ and $\bar{\Delta}(\pi) \cong \mathcal{E}_\xi(L(\pi))$. A weak standardization functor is called a *standardization functor* if it is exact. This is equivalent to the requirement that $\Delta(\xi)$ is flat as a B_ξ-module.

Definition 4.7. The algebra H as above is called *properly stratified* (with respect to the fixed preorder $\leq$) if the following properties hold:

`(Filt)` For every $\pi \in \Pi$, the object $K(\pi)$ has a finite Δ-filtration with quotients of the form $q^d\Delta(\sigma)$ for $\sigma > \pi$.

`(Flat)` For every $\xi \in \Xi$, the right B_ξ-module $\Delta(\xi)$ is finitely generated and flat.

4.13. Convex preorders

Recall the notion of convex preorders on the positive roots Φ_+. We denote by $\Psi := \Phi_+^{\mathrm{re}} \sqcup \{\delta\}$ the set of *indivisible positive roots*. In this section we prove some needed results on convex preorders.

Lemma 4.8. [39, Lemma 3.7] There is $w \in W'$ such that $p(\Phi_{\succ\delta}) = w\Phi'_+$.

Let $p(\Phi_{\succ\delta}) = w\Phi'_+$ according to the lemma. For $i \in I'$, we denote

$$\gamma_i := w\alpha_i \quad \text{and} \quad \gamma_i^{\pm} := \widehat{\pm\gamma_i}.$$

Then $\gamma_i^{\pm} \in \Phi_+^{\mathrm{re}}$, $\gamma_i^+ + \gamma_i^- = \delta$, and $\gamma_i^+ \succ \delta \succ \gamma_i^-$. Note that $\Delta_{\succ\delta} := \{\gamma_1, \ldots, \gamma_l\}$ is a base in Φ'.

Lemma 4.9. [39, Example 3.5] Let Δ be any base in Φ' and $\alpha \in \Delta$. There exists a convex preorder $\preceq$ on Φ_+ with the following three properties:

(i) $\Delta_{\succ\delta} = \Delta$;

(ii) $\hat{\alpha} \succ \hat{\alpha} + \delta \succ \hat{\alpha} + 2\delta \succ \cdots \succ \delta \cdots \succ \widehat{-\alpha} + 2\delta \succ \widehat{-\alpha} + \delta \succ \widehat{-\alpha}$;

(iii) Every root in Φ_+^{re}, which is not of the form $\widehat{\pm\alpha} + n\delta$, is either greater than $\hat{\alpha}$ or less than $\widehat{-\alpha}$.

In this subsection we write $\equiv$ for $\equiv \pmod{\mathbb{Z}\delta}$.

Lemma 4.10. Let $i \in I'$ and $\gamma_i^{\pm} = \eta^{\pm} + \theta^{\pm}$ for some $\eta^{\pm}, \theta^{\pm} \in Q_+$. Suppose that $\eta^{\pm}$ is a sum of positive roots $\preceq \gamma_i^{\pm}$, and $\theta^{\pm}$ is a sum of positive roots $\succeq \gamma_i^{\pm}$. Then $\eta^- + \eta^+ \neq \gamma_i^-$ unless $\eta^+ = \theta^- = 0$.

Proof. By assumption, θ^+ is a sum of positive roots $\succeq \gamma_i^+$. Since $\gamma_i^+ \succ \delta$, these positive roots are in $\Phi_{\succ\delta}$. So we can write $\theta^+ \equiv \sum_{m=1}^{l} c_m\gamma_m$ with coefficients $c_m \in \mathbb{Z}_{\geq 0}$. Furthermore, η^- is a sum of positive roots less than γ_i^-. As $\gamma_i^- \prec \delta$, these positive roots are in $\Phi_{\prec\delta}$. So we can write $\eta^- \equiv -\sum_{m=1}^{l} d_m\gamma_m$ with coefficients $d_m \in \mathbb{Z}_{\geq 0}$. Assume that $\eta^- + \eta^+ = \gamma_i^-$, in which case $\theta_- = \eta_+$. As $\{\delta, \gamma_1, \ldots, \gamma_l\}$ are linearly independent, we deduce that all d_m and c_m with $m \neq i$ are zero and $d_i + c_i = 2$.

If $d_i = 1$ then $\theta^- = \eta^+ \equiv 0$, which implies $\theta^- = \eta^+ = 0$ by height considerations. If $d_i = 0$, we have $\eta^- \equiv 0$ so $\eta^- = 0$ by heights, which implies $\theta^- = \eta^+ = \gamma_i^-$ and

$\theta_+ = \gamma_i^+ - \gamma_i^- \equiv 2\gamma_i$. Since $\mathrm{ht}(\theta^+) < \mathrm{ht}(\gamma_i^+)$, we deduce that θ^+ is a sum of positive roots which are strictly greater than γ_i^+. As $(\theta_+, \gamma_i^+) = 4$, any presentation of θ^+ as a sum of positive roots which are strictly greater than γ_i^+ must have at least four summands. Each of these summands is a non-negative linear combination of γ_m's. This contradicts $\theta^+ \equiv 2\gamma_i$. The case $d_i = 2$ is similar to the case $d_i = 0$. $\square$

Lemma 4.11. Let $n \in \mathbb{Z}_{>0}$ and $\delta = \theta_r^- + \theta_r^+$ for $r = 1, \dots, n$, with each θ_r^- being a sum of positive roots $\preceq \delta$ and each θ_r^+ being a sum of positive roots $\succeq \delta$. If $\sum_{r=1}^n \theta_r^\pm = n\gamma_i^\pm$, then $\theta_r^\pm = \gamma_i^\pm$ for all $r = 1, \dots n$.

Proof. For $1 \leq r \leq n$ we have $\theta_r^\pm \equiv \pm \sum_{j=1}^l c_{r,j}^\pm \gamma_j^+$ for some $c_{r,i}^\pm \in \mathbb{Z}_{\geq 0}$. So $\pm n\gamma_i \equiv \pm \sum_{r=1}^n \sum_{j=1}^l c_{r,j}^\pm \gamma_j^+$. Now linear independence of the $\gamma_j^\pm$ modulo $\mathbb{Z}\delta$ and considerations of height imply $c_{r,j}^\pm = \delta_{i,j}$ for all r. $\square$

4.14. Kostant partitions and root partitions

In this subsection we recall the definition of Kostant partitions and root partitions, modifying our notation slightly to emphasize the connection to stratification. Let $\theta \in Q_+$. A *Kostant partition of* θ is a tuple $\xi = (x_\beta)_{\beta \in \Psi}$ of non-negative integers such that $\sum_{\beta \in \Psi} x_\beta \beta = \theta$. If $\{\beta_1 \succ \cdots \succ \beta_r\} = \{\beta \in \Psi \mid x_\beta \neq 0\}$, then, denoting $x_u := x_{\beta_u}$, we also write ξ in the form $\xi = (\beta_1^{x_1}, \dots, \beta_r^{x_r})$. We denote by $\Xi(\theta)$ the set of all Kostant partitions of θ. Denoting the left (resp. right) lexicographic order on $\Xi(\theta)$ by $\leq_l$ (resp. $\leq_r$), we will always use the *bilexicographic partial order* $\leq$ on $\Xi(\theta)$, i.e. $\xi \leq \zeta$ if and only if $\xi \leq_l \zeta$ and $\xi \geq_r \zeta$.

Let $\alpha \in \Psi$. By convexity, the Kostant partition (α) is the unique minimal element in $\Xi(\alpha)$. A *minimal pair for* α is a minimal element in $\Xi(\alpha) \setminus \{(\alpha)\}$. A minimal pair for α exists, provided α is not a simple root, which we always assume when speaking of minimal pairs for α. Using the property (Con2) from [24, §3.1], it is easy to see that a minimal pair is always a Kostant partition of the form (β, γ) for $\beta, \gamma \in \Psi$ with $\beta > \gamma$. A minimal pair (β, γ) is called *real* if both β and γ are real.

Lemma 4.12. Let $\alpha \in \Phi_+^{\mathrm{re}}$. If α has no real minimal pair, then $\alpha = \gamma_i^{\pm} + n\delta$ for some $i \in I'$ and $n \in \mathbb{Z}_{>0}$, in which case $(\gamma_i^+ + (n-1)\delta, \delta)$ is a minimal pair for $\gamma_i^+ + n\delta$ and $(\delta, \gamma_i^- + (n-1)\delta)$ is a minimal pair for $\gamma_i^- + n\delta$.

Proof. The first half is [39, Lemma 12.4]. For the second half, if $(\gamma_i^+ + (n-1)\delta, \delta)$ is not a minimal pair for $\gamma_i^+ + n\delta$, then we would be able to write $\gamma_i^+ + n\delta = \beta + \gamma$ with $\beta, \gamma \in \Phi_{\succ\delta}$. But modulo δ both β and γ are positive sums of γ_j's, which leads to a contradiction. The argument for $\gamma_i^- + n\delta$ is similar. $\square$

If μ is a usual partition of n, we write $\mu \vdash n$ and $n = |\mu|$. An *l-multipartition* of n is a tuple $\underline{\mu} = (\mu^{(1)}, \dots, \mu^{(l)})$ of partitions such that $|\underline{\mu}| := |\mu^{(1)}| + \cdots + |\mu^{(l)}| = n$. The set of the all l-multipartitions of n is denoted by $\mathscr{P}_n$, and $\mathscr{P} := \sqcup_{n \geq 0} \mathscr{P}_n$.

A *root partition of* θ is a pair $(\xi, \underline{\mu})$, where $\xi = (x_\beta)_{\beta \in \Psi} \in \Xi(\theta)$ and $\underline{\mu} \in \mathscr{P}_{x_\delta}$. We write $\Pi(\theta)$ for the set of root partitions of θ. There is a natural surjection $\rho : \Pi(\theta) \to \Xi(\theta),\ (\xi, \underline{\mu}) \mapsto \xi$. The bilexicographic partial order $\leq$ on $\Xi(\theta)$ induces a partial preorder $\leq$ on $\Pi(\theta)$, i.e. $\pi \leq \sigma$ if and only if $\rho(\pi) \leq \rho(\sigma)$.

Let $(\xi, \underline{\mu}) \in \Pi(\theta)$ with $\xi = (x_\beta)_{\beta \in \Psi}$. As all but finitely many integers x_β are zero, there is a finite subset $\beta_1 \succ \cdots \succ \beta_s \succ \delta \succ \beta_{-t} \succ \cdots \succ \beta_{-1}$ of Ψ such that $x_\beta = 0$ for $\beta \in \Psi$ outside of this subset. Then, denoting $x_u := x_{\beta_u}$, we can write any root partition of θ in the form

$$(\xi, \underline{\mu}) = (\beta_1^{x_1}, \dots, \beta_s^{x_s}, \underline{\mu}, \beta_{-t}^{x_{-t}}, \dots, \beta_{-1}^{x_{-1}}), \tag{4.13}$$

where all $x_u \in \mathbb{Z}_{\geq 0}$, $\underline{\mu} \in \mathscr{P}$, and $|\underline{\mu}|\delta + \sum x_u \beta_u = \theta$.

4.15. Reduction modulo p

The KLR algebra R_θ is defined over an arbitrary commutative unital ring k, and if we need to to emphasize which k we are working with, we will use the notation $R_{\theta,k}$. Likewise in the notation for modules. Let p be a fixed prime number, and $F := \mathbb{Z}/p\mathbb{Z}$ be the prime field of characteristic p. We will use the p-modular system $(F, \mathcal{O}, K)$ with $F = \mathbb{F}_p$, $\mathcal{O} = \mathbb{Z}_p$ and $K = \mathbb{Q}_p$.

Let $k = K$ or F, and V_k be an $R_{\theta,k}$-module. An $R_{\theta,\mathcal{O}}$-module $V_\mathcal{O}$ is called an *$\mathcal{O}$-form of V_k* if every graded component of $V_\mathcal{O}$ is free of finite rank as an $\mathcal{O}$-module and, identifying $R_{\theta,\mathcal{O}} \otimes_\mathcal{O} k$ with $R_{\theta,k}$, we have $V_\mathcal{O} \otimes_\mathcal{O} k \cong V_k$ as $R_{\theta,k}$-modules. Every $V_K \in R_{\theta,K}$-mod has an $\mathcal{O}$-form: pick $R_{\theta,K}$-generators $v_1, \dots, v_r$ and define $V_\mathcal{O} := R_{\theta,\mathcal{O}} \cdot v_1 + \cdots + R_{\theta,\mathcal{O}} \cdot v_1$. We always can and will pick the generators which are homogeneous weight vectors. Let $V_K \in R_{\theta,K}$-mod and $V_\mathcal{O}$ be an $\mathcal{O}$-form of V_K. The $R_{\theta,F}$-module $V_\mathcal{O} \otimes_\mathcal{O} F$ is called a *reduction modulo* p of V_K. Reduction modulo p in general depends on the choice of $V_\mathcal{O}$. However, as explained in [22, Lemma 4.3], we have a generalization of the standard result for finite groups:

Lemma 4.14. If $V_K \in R_{\theta,K}$-mod and L_F is an irreducible $R_{\theta,F}$-module, then the multiplicity $[V_\mathcal{O} \otimes_\mathcal{O} F : L_F]_q$ is independent of the choice of an $\mathcal{O}$-form $V_\mathcal{O}$ of V_K.

Reduction modulo p commutes with induction and restriction [22, Lemma 4.5]:

Lemma 4.15. Let $\underline{\theta} = (\theta_1, \dots, \theta_m) \in Q_+^m$, $\theta = \theta_1 + \cdots + \theta_m$, $V_\mathcal{O} \in R_{\underline{\theta};\mathcal{O}}$-mod, and $W_\mathcal{O} \in R_{\theta,\mathcal{O}}$-mod. Then for any $\mathcal{O}$-algebra k, there are natural isomorphisms of $R_{\theta,k}$-modules

$$(\mathrm{Ind}_{\underline{\theta}} V_\mathcal{O}) \otimes_\mathcal{O} k \cong \mathrm{Ind}_{\underline{\theta}}(V_\mathcal{O} \otimes_\mathcal{O} k)$$

and of $R_{\underline{\theta};k}$-modules

$$(\mathrm{Res}_{\underline{\theta}} W_\mathcal{O}) \otimes_\mathcal{O} k \cong \mathrm{Res}_{\underline{\theta}}(W_\mathcal{O} \otimes_\mathcal{O} k).$$

In particular, reduction modulo p preserves formal characters. This fact together with linear independence of formal characters of irreducible modules has the following consequence:

Lemma 4.16. Let $V_1, \dots, V_r$ be $R_{\theta,K}$-modules such that $\mathrm{ch}_q V_1, \dots, \mathrm{ch}_q V_r$ are linearly independent. Let $L_1, \dots, L_s$ be a complete set of composition factors of reductions modulo p of the modules $V_1, \dots, V_r$. Then $s \geq r$.

4.2. Semicuspidal modules

The main goal of this section is to generalize some results of Chapter III and [39]. In Chapter III we assumed that the convex order was balanced, while [39] assumes that $p = 0$. We want to avoid both of these assumptions.

In this section we often work with a composition $\nu = (n_1, \dots, n_a)$ of n, the corresponding parabolic subalgebra $R_{\nu,\delta} := R_{n_1\delta,\dots,n_a\delta}$, and the corresponding induction and restriction functors $I_\nu^n := \mathrm{Ind}_{n_1\delta,\dots,n_a\delta}$ and ${}^*I_\nu^n := \mathrm{Res}_{n_1\delta,\dots,n_a\delta}$.

4.21. Semicuspidal modules

We fix a convex preorder $\preceq$ on Φ_+, an *indivisible positive root* α, and $n \in \mathbb{Z}_{>0}$. Following [39] (see also [32, 38, 24, 44]) an $R_{n\alpha}$-module V is called *semicuspidal* if $\theta, \eta \in Q_+$, $\theta + \eta = n\alpha$, and $\mathrm{Res}_{\theta,\eta} V \neq 0$ imply that θ is a sum of positive roots $\preceq \alpha$ and η is a sum of positive roots $\succeq \alpha$.

Weights $\boldsymbol{i} \in I^{n\alpha}$, which appear in some semicuspidal $R_{n\alpha}$-modules, are called *semicuspidal weights*. We denote by $I^{n\alpha}_{\mathrm{nsc}}$ the set of non-semicuspidal weights. Let

$$1_{\mathrm{nsc}} := \sum_{\boldsymbol{i} \in I^{n\alpha}_{\mathrm{nsc}}} 1_{\boldsymbol{i}}.$$

Following [39], define the *semicuspidal algebra*

$$C_{n\alpha} = C_{n\alpha,k} := R_{n\alpha}/R_{n\alpha}1_{\mathrm{nsc}}R_{n\alpha}. \tag{4.17}$$

Then the category of finitely generated semicuspidal $R_{n\alpha}$-modules is equivalent to the category $C_{n\alpha}$-mod.

Theorem 4.18. Let $\alpha \in \Phi_+^{\mathrm{re}}$ and $n \in \mathbb{Z}_{>0}$. There is a unique up to isomorphism irreducible $\circledast$-self-dual semicuspidal R_α-module. We denote it $L(\alpha)$. Moreover, $L(\alpha^n) := q^{n(n-1)/2}L(\alpha)^{\circ n}$ is the unique up to isomorphism irreducible $\circledast$-self-dual semicuspidal $R_{n\alpha}$-module.

Proof. Follows from the main results of [24], see also [44]. □

Lemma 4.19. Let $\alpha \in \Phi_+^{\mathrm{re}}$ and $n \in \mathbb{Z}_{>0}$. Then $L(\alpha^n)_F$ is a reduction modulo p of $L(\alpha^n)_K$.

Proof. See [24, Proposition 4.9] and [22, Lemma 4.6]. □

In the rest of this section we work with the imaginary case trying to understand the irreducible semicuspidal $R_{n\delta}$-modules.

4.22. Minuscule imaginary modules

The proof of the following lemma in [39, Lemma 12.3] seems to need the assumption $p = 0$, but the same result will later follow in general from Lemmas 4.19 and 4.21(iii).

Lemma 4.20. [39, Lemma 12.3] Assume that $p = 0$. Let $\alpha \in \Psi$, $L \in R_\alpha$-mod be a semicuspidal module, and (β, γ) be a minimal pair for α. Then all composition factors of $\mathrm{Res}_{\gamma,\beta} L$ are of the form $L_\gamma \boxtimes L_\beta$, where L_γ is an irreducible semicuspidal R_γ-module and L_β is an irreducible semicuspidal R_β-module.

By [24], there are exactly $|\mathscr{P}_1| = l$ isomorphism classes of self-dual irreducible semicuspidal R_δ-modules. These modules can be labeled canonically by the elements of I', see [24] for balanced convex orders, and [39, 44] in general. We now describe the approach of [39]. One needs to be careful to make sure that the assumption $p = 0$ made in [39] can be avoided. Recall the base $\Delta_{\succ\delta} = \{\gamma_1, \dots, \gamma_l\}$ in Φ'_+ from §4.13 and the roots $\gamma_i^\pm$. In characteristic zero, parts (i) and (ii) of the following result are contained in [39], and this will be used in the proof.

Lemma 4.21. Let $i \in I'$. Then the module $L(\gamma_i^-) \circ L(\gamma_i^+)$ has a simple $\circledast$-self-dual head. Moreover, denoting this simple module by $L_{\delta,i}$, we have the following:

(i) The R_δ-module $L_{\delta,i}$ is cuspdial, and $\{L_{\delta,i} \mid i \in I'\}$ is a complete and irredundant system of irreducible $\circledast$-self-dual semicuspidal R_δ-modules.

(ii) $\mathrm{Res}_{\gamma_i^-,\gamma_i^+} L_{\delta,i} \cong L(\gamma_i^-) \boxtimes L(\gamma_i^+)$, and $\mathrm{Res}_{\gamma_j^-,\gamma_j^+} L_{\delta,i} = 0$ if $i \neq j$.

(iii) Reduction modulo p of $L_{\delta,i,K}$ is $L_{\delta,i,F}$.

Proof. By Mackey's Theorem and Lemma 4.10, we have $\mathrm{Res}_{\gamma_i^-,\gamma_i^+}(L(\gamma_i^-) \circ L(\gamma_i^+)) \cong L(\gamma_i^-) \boxtimes L(\gamma_i^+)$. If L is a simple constituent of the head, then $L(\gamma_i^-) \boxtimes L(\gamma_i^+)$ appears in the socle of $\mathrm{Res}_{\gamma_i^-,\gamma_i^+} L$. Since Res is an exact functor and the multiplicity of the irreducible $\circledast$-self-dual module $L(\gamma_i^-) \boxtimes L(\gamma_i^+)$ in $\mathrm{Res}_{\gamma_i^-,\gamma_i^+}(L(\gamma_i^-) \circ L(\gamma_i^+))$ is 1, the head is simple and self-dual. The first part of (ii) also follows.

Now, we explain that in characteristic zero, (i) and (ii) are contained in [39]. Indeed, (i) is [39, Theorem 17.3]. To see the second part of (ii), in view of [39, Theorem 13.1] and Lemma 4.9, we may assume that (γ_j^+, γ_j^-) is a minimal pair for δ, in which case by Lemma 4.20, all composition factors of $\mathrm{Res}_{\gamma_j^-,\gamma_j^+} L_{\delta,i}$ are of the form

$L(\gamma_j^-) \boxtimes L(\gamma_j^+)$, in particular, $L(\gamma_j^-) \boxtimes L(\gamma_j^+)$ appears in the socle of $\mathrm{Res}_{\gamma_j^-,\gamma_j^+} L_{\delta,i}$, whence $L_{\delta,i}$ is a quotient of $L(\gamma_j^-) \circ L(\gamma_j^+)$, i.e. $L_{\delta,i} \cong L_{\delta,j}$, giving a contradiction.

Pick R-forms $L(\gamma_i^{\pm})_R$ of $L(\gamma_i^{\pm})_K$. By Lemmas 4.15 and 4.19, we have that $L(\gamma_i^-)_R \circ L(\gamma_i^+)_R$ is an R-form of $L(\gamma_i^-)_k \circ L(\gamma_i^+)_k$ for $k = K$ or F. We have a surjection $\varphi : L(\gamma_i^-)_K \circ L(\gamma_i^+)_K \to L_{\delta,i,K}$. Let $L_{\delta,i,R} := \varphi(L(\gamma_i^-)_R \circ L(\gamma_i^+)_R)$. Note that $L_{\delta,i,R}$ is an R-form of $L_{\delta,i,K}$. On the other hand, we have a surjection $L(\gamma_i^-)_F \circ L(\gamma_i^+)_F \to L_{\delta,i,R} \otimes_R F$. This implies that $L_{i,\delta,F}$ is a quotient of $L_{\delta,i,R} \otimes_R F$. As $L_{\delta,i,K}$ is semicuspidal by [39], it now follows that so is $L_{\delta,i,F}$.

Let $j \neq i$. By the characteristic zero result, we have $\mathrm{Res}_{\gamma_j^-,\gamma_j^+} L_{\delta,i,K} = 0$. It now follows that $\mathrm{Res}_{\gamma_j^-,\gamma_j^+} L_{\delta,i,F} = 0$, too, which completes the proof of (ii). By (ii), we conclude that $L_{\delta,i,F} \ncong L_{\delta,j,F}$. By counting, we complete the proof of (i).

To prove (iii), note by characters that all composition factors of $L_{\delta,i,R} \otimes_R F$ are semicuspidal. Now we can conclude that $L_{\delta,i,R} \otimes_R F \cong L_{\delta,i,F}$ using (ii). $\square$

Following the terminology of [24], we call the modules $L_{\delta,i}$ *minuscule modules*.

4.23. Imaginary Schur-Weyl duality

In this section we recall some results from the Chapter III, and generalize these results to the case of an arbitrary convex order in this symmetric Lie type situation.

Fix $i \in I'$. Recall the minuscule module $L_{\delta,i}$ from §4.22. Consider the $R_{n\delta}$-module $M_{n,i} := L_{\delta,i}^{\circ n}$ and the algebra $\mathscr{S}_{n,i} := R_{n\delta}/\operatorname{Ann}_{R_{n\delta}}(M_{n,i})$. Since i is fixed, we often suppress it from our notation and write $L_\delta = L_{\delta,i}$, $M_n = M_{n,i}$, $\mathscr{S}_n = \mathscr{S}_{n,i}$, etc.

We have the following result, the proof of which given in Chapter III does not use the fact that the convex order is balanced. We record it again here for reader convenience.

Theorem 4.22. Let $i \in I'$ and $n \in \mathbb{Z}_{>0}$. Then:

(i) $\mathrm{End}_{R_{n\delta}}(M_n)^{\mathrm{op}} \cong \mathrm{end}_{R_{n\delta}}(M_n)^{\mathrm{op}} \cong k\mathfrak{S}_n$.

(ii) M_n is a projective $\mathscr{S}_n$-module, and $M_n^{\circledast} \cong M_n$.

(iii) Assume that $p > n$ or $p = 0$. Then $\mathscr{S}_n$ is semisimple, M_n is a projective generator over $\mathscr{S}_n$, and $\mathscr{S}_n$ is Morita equivalent to $k\mathfrak{S}_n$.

By Theorem 4.22, if $p = 0$, the number of composition factors of M_n is equal to the number of partitions of n. Now using reduction modulo p argument involving Lemmas 4.21(iii) and 4.16, we deduce that the same is true in general:

Lemma 4.23. The number of composition factors of M_n, up to isomorphism and degree shift, is equal to the number of partitions of n.

Recall the roots γ_i^+ and γ_i^- from §4.13. As i is fixed we will denote $\gamma_\pm := \gamma_i^\pm$.

Lemma 4.24. We have $\mathrm{Res}_{n\gamma_-,n\gamma_+} M_n \cong L(\gamma_-^n) \boxtimes L(\gamma_+^n)$.

Proof. Follows using Mackey's Theorem and Lemmas 4.11, 4.21(ii). □

For $\alpha \in \Phi_+^{\mathrm{re}}$, we denote by $P(\alpha^n)$ the projective cover of the irreducible semicuspidal module $L(\alpha^n)$. We will use a special projective module, which we we refer to as a *Gelfand-Graev module*. Note that its definition is different from the one in Chapter III even for balanced orders:

$$\Gamma_n = \Gamma_{n,i} := P(\gamma_-^n) \circ P(\gamma_+^n).$$

Lemma 4.25. We have $\dim_q \mathrm{Hom}_{R_{n\delta}}(\Gamma_n, M_n) = 1$.

Proof. We have $\mathrm{Hom}_{R_{n\delta}}(\Gamma_n, M_n) \cong \mathrm{Hom}_{R_{n\gamma_-,n\gamma_+}}(P(\gamma_-^n) \boxtimes P(\gamma_+^n), \mathrm{Res}_{n\gamma_-,n\gamma_+} M_n)$. So the result follows from Lemma 4.24. □

Denote by $\mathbf{1}_{\mathfrak{S}_n}$ the trivial (right) $k\mathfrak{S}_n$-module. Note that $\mathrm{Hom}_{R_{n\delta}}(\Gamma_n, M_n)$ is naturally a right $k\mathfrak{S}_n$-module, since $\mathfrak{S}_n$ acts on M_n on the right in view of Theorem 4.22(i). Since this module is 1-dimensional by Lemma 4.25, it is either the trivial or the sign module. If it happens to be the sign module, we redefine the right action of $\mathfrak{S}_n$ on M_n by tensoring it with the sign representation. So we may assume without loss of generality that

$$\mathrm{Hom}_{R_{n\delta}}(\Gamma_n, M_n) \cong \mathbf{1}_{\mathfrak{S}_n}. \tag{4.26}$$

For a composition $\nu = (n_1, \dots, n_h) \in \Lambda(h, n)$, we define the $R_{\nu,\delta}$-modules

$$M_\nu := M_{n_1} \boxtimes \cdots \boxtimes M_{n_h}, \quad \Gamma_\nu := \Gamma_{n_1} \boxtimes \cdots \boxtimes \Gamma_{n_h}, \quad \text{and} \quad \Gamma^\nu := I_\nu^n \Gamma_\nu.$$

We have the parabolic analogue $\mathscr{S}_\nu$ of $\mathscr{S}_n$ defined as

$$\mathscr{S}_\nu := R_{\nu,\delta}/\mathrm{Ann}_{R_{\nu,\delta}}(M_\nu) \cong \mathscr{S}_{n_1} \otimes \cdots \otimes \mathscr{S}_{n_h}.$$

The functors ${}^*I_\nu^n$ and I_ν^n induce the functors between $\mathscr{S}_n$-mod and $\mathscr{S}_\nu$-mod.

Lemma 4.27. We have ${}^*I_\nu^n \Gamma_n \cong \Gamma_\nu \oplus X$, where X is a projective $R_{\nu,\delta}$-module with $\mathrm{Hom}_{R_{\nu,\delta}}(X, M_\nu) = 0$.

Proof. Mackey's Theorem yields a filtration of

$${}^*I_\nu^n \Gamma_n = \mathrm{Res}_{n_1\delta,\dots,n_h\delta} \mathrm{Ind}_{n\gamma_-,n\gamma_+} P(\gamma_i^n) \boxtimes P(\gamma_+^n)$$

with projective subquotients, one of which is Γ_ν (ignoring grading shifts for now). So we get a decomposition ${}^*I_\nu^n \Gamma_n \cong q^d \Gamma_\nu \oplus X$ where X is a projective module. It remains to notice that $\dim_q \mathrm{Hom}_{R_{\nu,\delta}}({}^*I_\nu^n\Gamma_n, M_\nu) = 1$, which is done using adjointness of Res and Coind. $\square$

4.24. Divided powers

We recall some results of Chapter III and extend to the arbitrary convex order case. Set $\mathtt{x}_n := \sum_{x\in\mathfrak{S}_n} x$. Set

$$X_n := M_n x_n \quad \text{and} \quad Z_n := \{v \in M_n \mid vx = v \text{ for all } x \in \mathfrak{S}_n\}.$$

Fixing a non-zero $R_{n\delta}$-homomorphism $f_n : \Gamma_n \to M_n$, we also set $Y_n := \operatorname{im} f_n \subseteq M_n$, cf. Lemma 4.25. Eventually we will prove that $Y_n = Z_n$. For now, it is only clear from (4.26) that $Y_n \subseteq Z_n$. From Chapter III, we have

Lemma 4.28.

(i) X_n is an irreducible $R_{n\delta}$-module.

(ii) $\mathrm{soc}\, Z_n = X_n$, and no composition factor of Z_n/X_n is isomorphic to a submodule of M_n.

From now on fix $h \geq n$. For $L = X, Z, Y$ and a composition $\nu = (n_1, \dots, n_h) \in \Lambda(h, n)$, we set $L_\nu := L_{n_1} \boxtimes \cdots \boxtimes L_{n_h}$, $L^\nu := I_\nu^n L_\nu$, and $L := \bigoplus_{\nu\in\Lambda(h,n)} L^\nu$. For the proof of the following results see §3.5.

Lemma 4.29. For $L = X, Z, Y$, we have ${}^*I_\nu^n L_n \cong L_\nu$.

Theorem 4.30. For $L = X$ or Z, there is an algebra isomorphism $\mathrm{End}_{R_{n\delta}}(L)^{\mathrm{op}} \cong S(h, n)$, where $S(h, n)$ is the classical Schur algebra.

Lemma 4.31. Let $\lambda, \mu \in \Lambda(h,n)$. Then

$$\begin{aligned}\dim_q \operatorname{Hom}_{R_{n\delta}}(Y^\lambda, Y^\mu) &= \dim_q \operatorname{Hom}_{R_{n\delta}}(\Gamma^\lambda, Y^\mu) \\ &= \dim_q \operatorname{Hom}_{R_{n\delta}}(\Gamma^\lambda, Z^\mu) = |\mathfrak{S}_\lambda \backslash \mathfrak{S}_n / \mathfrak{S}_\mu|.\end{aligned}$$

We give a slightly simpler proof of the following result compared to Theorem 3.74:

Theorem 4.32. We have:

(i) $Z = \bigoplus_{\nu \in \Lambda(h,n)} Z^\nu$ is a projective generator for $\mathscr{S}_n$.

(ii) $Z_n = Y_n$.

Proof. (i) As Y_ν is a non-zero submodule of Z_ν, it contains the simple socle X_ν of Z_ν, see Lemma 4.28. Applying I_ν^n to the embeddings $X_\nu \subseteq Y_\nu \subseteq Z_\nu$, we get embeddings $X^\nu \subseteq Y^\nu \subseteq Z^\nu$. By Lemma 4.31,

$$\begin{aligned}\dim_q \operatorname{Hom}_{R_{n\delta}}(Y, Y) &= \dim_q \operatorname{Hom}_{R_{n\delta}}(\Gamma, Y) = \dim_q \operatorname{Hom}_{R_{n\delta}}(\Gamma, Z) \\ &= \sum_{\lambda,\mu \in \Lambda(h,n)} |\mathfrak{S}_\lambda \backslash \mathfrak{S}_n / \mathfrak{S}_\mu| = \dim S(h,n),\end{aligned}$$

the last equality for the dimension of the classical Schur algebra being well-known.

In particular, this implies that Y is projective as an $R_{n\delta}/\operatorname{Ann}_{R_{n\delta}}(Y)$-module by the Schubert's criterion, see e.g. [5, Lemma 4.3.1]. But $M_n = Y^{(1^n)}$ is a summand of Y, so $\operatorname{Ann}_{R_{n\delta}}(Y) = \operatorname{Ann}_{R_{n\delta}}(M_n)$, and Y is a projective $\mathscr{S}_n$-module. By the classical theory [12], the number of isomorphism classes of irreducible $S(h,n)$-modules equals to the number of partitions of n. By Fittings' Lemma, the number of isomorphism classes of indecomposable summands of Y equals the number of isomorphism classes of irreducible modules over $\mathrm{End}_{R_{n\delta}}(Y) = S(h,n)$. We now deduce from Lemma 4.23 that Y is a projective generator for $\mathscr{S}_n$.

(ii) By (i), every irreducible $\mathscr{S}_n$-modules appears in the head of the projective $R_{n\delta}$-module Γ. As $\dim_q \mathrm{Hom}_{R_{n\delta}}(\Gamma, Y) = \dim_q \mathrm{Hom}_{R_{n\delta}}(\Gamma, Z)$, every homomorphism from Γ to Z has image lying in Y, and it follows that $Y = Z$. $\square$

4.25. Imaginary semicuspidal irreducible and Weyl modules

Recall that we have fixed $h \geq n$. By Theorem 4.32, we may regard Z as a $(\mathscr{S}_n, S(h,n))$-bimodule. Then by Morita theory, we have an equivalences of categories

$$\beta_n : S(h,n)\text{-mod} \to \mathscr{S}_n\text{-mod}, \quad W \mapsto Z \otimes_{S(h,n)} W.$$

By the classical theory [12], the Schur algebra $S(h,n)$ is quasihereditary with irreducible module $L_{\mathtt{cl}}(\lambda)$ and standard modules $W_{\mathtt{cl}}(\lambda)$ labeled by the partitions $\lambda \vdash n$. Recall that we are working with a fixed $i \in I'$. Define the $\mathscr{S}_n$-modules:

$$L(\lambda) = L_i(\lambda) := \beta_n(L_{\mathtt{cl}}(\lambda))$$
$$W(\lambda) = W_i(\lambda) := \beta_n(W_{\mathtt{cl}}(\lambda))$$

so that $\{L_i(\lambda) \mid \lambda \vdash n\}$ is a complete and irredundant family of irreducible modules over $\mathscr{S}_n = \mathscr{S}_{n,i}$ up to isomorphism and degree shift. By inflating, these are irreducible semicupsidal $R_{n\delta}$-modules. It is easy to see that $L_i(\lambda)^{\circledast} \cong L_i(\lambda)$.

Now we complete a classification of the irreducible semicuspidal $R_{n\delta}$-modules. To every multipartition $\underline{\mu} = (\mu^{(1)}, \dots, \mu^{(l)}) \in \mathscr{P}_n$, we associate the $R_{n\delta}$-module

$$L(\underline{\mu}) := L_1(\mu^{(1)}) \circ \cdots \circ L_l(\mu^{(l)}).$$

Theorem 4.33. Let $n \in \mathbb{Z}_{>0}$. Then

(i) $\{L(\underline{\mu}) \mid \underline{\mu} \in \mathscr{P}_n\}$ is a complete and irredundant set of $\circledast$-selfdual irreducible semicuspidal $R_{n\delta}$-modules up to isomorphism.

(ii) For $\underline{\lambda}, \underline{\mu} \in \mathscr{P}_n$ with $|\lambda^{(i)}| = |\mu^{(i)}| =: n_i$ for all $i = 1, \dots, l$, we have

$$[\mathrm{Res}_{n_1\delta,\dots,n_l\delta} L(\underline{\mu}) : L_1(\lambda^{(1)}) \boxtimes \cdots \boxtimes L_l(\lambda^{(l)})]_q = \delta_{\underline{\lambda},\underline{\mu}}.$$

Proof. The proof is the same as that of [24, Theorem 5.10, Lemma 5.11]. □

Theorems 1 and 2 from the Introduction, except for the reduction modulo p statement in Theorem 2, follow easily from the results obtained in this section together with Schubert's criterion [5, Lemma 4.3.1]. The part of Theorem 2 concerning reduction modulo p comes from Corollary 4.60 below.

4.3. Stratifying KLR algebras

Throughout the section $\alpha \in \Psi$, $\theta \in Q_+$ and $\pi \in \Pi(\theta)$.

4.31. Semicuspidal standard modules

For real α, we denote by $\Delta(\alpha^n)$ the projective cover of $L(\alpha^n)$ in the category $C_{n\alpha}$-mod. We also denote by $\Delta(\underline{\mu})$ the projective cover of $L(\underline{\mu})$ in the category $C_{n\delta}$-mod. Sometimes, we will also use a special notation $\Delta_{\delta,i}$ for the projective cover of $L_{\delta,i}$ in C_δ-mod, in other words $\Delta_{\delta,i} = \Delta(\underline{\mu}(i))$, where $\underline{\mu}(i)$ is the multipartition of 1 with the only non-empty component $\underline{\mu}(i)^{(i)} = (1)$.

Lemma 4.34. Let $\alpha \in \Psi$ and $V \in C_{n\alpha}$-mod. Denote $\Delta := \Delta(\alpha^n)$ if α is real, and $\Delta := \Delta(\underline{\mu})$ for any $\underline{\mu} \in \mathscr{P}_n$ if $\alpha = \delta$. Then $\mathrm{Ext}^1_{R_{n\alpha}}(\Delta, V) = 0$.

Proof. Any extension of Δ by V belongs to $C_{n\alpha}$-mod. Since Δ is a projective object in $C_{n\alpha}$-mod, the extension has to split. $\square$

Lemma 4.35. Let $\alpha \in \Phi_+^{\mathrm{re}}$, and $n = n_1 + \cdots + n_a$ for $n_1, \dots, n_a \in \mathbb{Z}_{\geq 0}$. Then:

(i) $\Delta(\alpha)^{\circ n} \cong q^{n(n-1)/2}[n]!\,\Delta(\alpha^n)$.

(ii) $\mathrm{Res}_{n_1\alpha,\dots,n_a\alpha}\Delta(\alpha^n) \cong \Delta(\alpha^{n_1}) \boxtimes \cdots \boxtimes \Delta(\alpha^{n_a})$.

Proof. (i) All composition factors of $\Delta(\alpha)^{\circ n}$ are of the form $L(\alpha^n)$, so it is an $C_{n\alpha}$-module. We claim that this $C_{n\alpha}$-module is projective. It suffices to prove that $\mathrm{Ext}^1_{C_{n\alpha}}(\Delta(\alpha)^{\circ n}, L(\alpha^n)) = 0$, which follows from $\mathrm{Ext}^1_{R_{n\alpha}}(\Delta(\alpha)^{\circ n}, L(\alpha^n))$. But

$$\mathrm{Ext}^1_{R_{n\alpha}}(\Delta(\alpha)^{\circ n}, L(\alpha^n)) \cong \mathrm{Ext}^1_{R_{\alpha,\dots,\alpha}}(\Delta(\alpha)^{\boxtimes n}, \mathrm{Res}_{\alpha,\dots,\alpha}L(\alpha^n)).$$

Now,

$$\mathrm{Res}_{\alpha,\dots,\alpha}L(\alpha^n) \cong [n]!L(\alpha)^{\boxtimes n}, \tag{4.36}$$

cf. [7, Lemma 2.11], so the claim follows from the Künneth formula and Lemma 4.34.

It follows from the previous paragraph that $\Delta(\alpha)^{\circ n} \cong m(q)\,\Delta(\alpha^n)$ for some $m(q) \in \mathbb{Z}[q, q^1]$. To prove that $m(q) = [n]!$ it suffices to observe using (4.36) that $\dim_q \mathrm{Hom}_{R_{n\alpha}}(\Delta(\alpha)^{\circ n}, L(\alpha^n)) = [n]!$.

(ii) follows from (i) and the computation of $\mathrm{Res}_{n_1\alpha,\dots,n_a\alpha}(\Delta(\alpha)^{\circ n})$, which is performed using Mackey's Theorem and convexity. $\square$

4.32. Standard modules

To a Kostant partition $\xi = (\beta_1^{x_1}, \dots, \beta_r^{x_r}) \in \Xi(\theta)$ we associate a parabolic subalgebra

$$R_\xi = R_{x_1\beta_1} \otimes \cdots \otimes R_{x_r\beta_r} \subseteq R_\theta$$

and the corresponding functors

$$\mathrm{Res}_\xi : R_\theta\text{-mod} \to R_\xi\text{-mod} \quad \text{and} \quad \mathrm{Ind}_\xi,\ \mathrm{Coind}_\xi : R_\xi\text{-mod} \to R_\theta\text{-mod}\,. \tag{4.37}$$

For every $\pi = (\xi, \underline{\mu}) \in \Pi(\theta)$ as in (4.13), we define the *proper standard module*

$$\bar{\Delta}(\pi) = L(\beta_1^{x_1}) \circ \cdots \circ L(\beta_s^{x_s}) \circ L(\underline{\mu}) \circ L(\beta_{-t}^{x_{-t}}) \circ \cdots \circ L(\beta_{-1}^{x_{-1}}) = \mathrm{Ind}_\xi L_\pi, \tag{4.38}$$

and the *standard module*

$$\Delta(\pi) = \Delta(\beta_1^{x_1}) \circ \cdots \circ \Delta(\beta_s^{x_s}) \circ \Delta(\underline{\mu}) \circ \Delta(\beta_{-t}^{x_{-t}}) \circ \cdots \circ \Delta(\beta_{-1}^{x_{-1}}) = \mathrm{Ind}_\xi \Delta_\pi, \tag{4.39}$$

where we have used the notation

$$L_\pi := L(\beta_1^{x_1}) \boxtimes \cdots \boxtimes L(\beta_s^{x_s}) \boxtimes L(\underline{\mu}) \boxtimes L(\beta_{-t}^{x_{-t}}) \boxtimes \cdots \boxtimes L(\beta_{-1}^{x_{-1}}),$$
$$\Delta_\pi := \Delta(\beta_1^{x_1}) \boxtimes \cdots \boxtimes \Delta(\beta_s^{x_s}) \boxtimes \Delta(\underline{\mu}) \boxtimes \Delta(\beta_{-t}^{x_{-t}}) \boxtimes \cdots \boxtimes \Delta(\beta_{-1}^{x_{-1}})$$

for modules over the parabolic subalgebra R_ξ. In Lemma 4.44 we will show that these definitions agree with general definitions from §4.12. Define also

$$\bar{\nabla}(\pi) := \mathrm{Coind}_\xi L_\pi \cong \bar{\Delta}(\pi)^{\circledast} \qquad (\pi \in \Pi(\theta)), \tag{4.40}$$

the isomorphism coming from Lemma 2.54.

Theorem 4.41. [24] Let $\theta \in Q_+$. We have:

(i) For every $\pi \in \Pi(\theta)$, the module $\bar{\Delta}(\pi)$ has simple head; denote it $L(\pi)$.

(ii) $\{L(\pi) \mid \pi \in \Pi(\theta)\}$ is a complete and irredundant system of irreducible R_θ-modules up to isomorphism and degree shift.

(iii) For every $\pi \in \Pi(\theta)$, we have $L(\pi)^{\circledast} \cong L(\pi)$.

(iv) Then in the Grothendieck group $[R_\theta\text{-mod}]$, we have $[\bar{\Delta}(\pi)] = [L(\pi)] + \sum_{\sigma<\pi} d_{\pi,\sigma}[L(\sigma)]$ for some $d_{\pi,\sigma} \in \mathbb{Z}[q,q^{-1}]$ (which depend on p).

(v) For all $\pi, \sigma \in \Pi(\theta)$, we have that $\mathrm{Res}_{\rho(\pi)} L(\pi) \cong L_\pi$ and $\mathrm{Res}_{\rho(\sigma)} L(\pi) \neq 0$ implies $\sigma \leq \pi$.

Corollary 4.42. Let $\theta \in Q_+$ and $\pi, \sigma \in \Pi(\theta)$.

(i) $\mathrm{Res}_{\rho(\sigma)} \bar{\Delta}(\pi) \neq 0$ implies $\sigma \leq \pi$, and $\mathrm{Res}_{\rho(\pi)} \bar{\Delta}(\pi) \cong L_\pi$.

(ii) $\mathrm{Res}_{\rho(\sigma)} \bar{\nabla}(\pi) \neq 0$ implies $\sigma \leq \pi$ and $\mathrm{Res}_{\rho(\pi)} \bar{\nabla}(\pi) \cong L_\pi$.

(iii) $\mathrm{Res}_{\rho(\sigma)} \Delta(\pi) \neq 0$ implies $\sigma \leq \pi$, and $\mathrm{Res}_{\rho(\pi)} \Delta(\pi) \cong \Delta_\pi$.

Proof. If $\mathrm{Res}_{\rho(\sigma)} \bar{\Delta}(\pi) \neq 0$, then $\mathrm{Res}_{\rho(\sigma)} L(\pi') \neq 0$ for some composition factor $L(\pi')$ of $\bar{\Delta}(\pi)$. So, using Theorem 4.41(v), we get $\sigma \leq \pi' \leq \pi$. The rest of (i) follows from the exactness of Res and Theorem 4.41(iv),(v). The proofs of (ii) and (iii) are similar. □

Proposition 4.43. Let $\theta \in Q_+$, $\pi, \sigma \in \Pi(\theta)$, and $m \in \mathbb{Z}_{\geq 0}$. Then

$$\mathrm{Ext}^m_{R_\theta}(\Delta(\pi), \bar{\nabla}(\sigma)) = 0$$

unless $\rho(\pi) = \rho(\sigma)$. Moreover, if $\rho(\pi) = \rho(\sigma)$, then $\mathrm{Ext}^1_{R_\theta}(\Delta(\pi), \bar{\nabla}(\sigma)) = 0$ and $\dim_q \mathrm{Hom}_{R_\theta}(\Delta(\pi), \bar{\nabla}(\sigma)) = \delta_{\sigma,\pi}$. In particular, head $\Delta(\pi) \cong L(\pi)$.

Proof. The proof follows that of [39, Proposition 24.3]. By adjointness of Coind and Res, we have

$$\mathrm{Ext}^m_{R_\theta}(\Delta(\pi), \bar{\nabla}(\sigma)) \cong \mathrm{Ext}^m_{R_{\rho(\sigma)}}(\mathrm{Res}_{\rho(\sigma)}\Delta(\pi), L_\sigma)$$

By Corollary 4.42(iii), $\mathrm{Res}_{\rho(\sigma)}\Delta(\pi) \neq 0$ implies $\sigma \leq \pi$. On the other hand, by adjointness of Ind and Res, we have

$$\mathrm{Ext}^m_{R_\theta}(\Delta(\pi), \bar{\nabla}(\sigma)) \cong \mathrm{Ext}^m_{R_{\rho(\pi)}}(\Delta_\pi, \mathrm{Res}_{\rho(\pi)}\bar{\nabla}(\sigma)).$$

By Corollary 4.42(ii), $\mathrm{Res}_{\rho(\pi)}\bar{\nabla}(\sigma) \neq 0$ implies $\pi \leq \sigma$. So we are reduced to $\rho(\pi) = \rho(\sigma)$, in which case, using Corollary 4.42(iii), we have

$$\mathrm{Ext}^m_{R_\theta}(\Delta(\pi), \bar{\nabla}(\sigma)) \cong \mathrm{Ext}^m_{R_{\rho(\sigma)}}(\mathrm{Res}_{\rho(\sigma)}\Delta(\pi), L_\sigma) \cong \mathrm{Ext}^m_{R_{\rho(\sigma)}}(\Delta_\pi, L_\sigma).$$

Now, the result follows from Künneth formula and Lemma 4.34. $\square$

Lemma 4.44. Let $\theta \in Q_+$ and $\pi \in \Pi(\theta)$.

(i) $\Delta(\pi)$ is the largest quotient of $P(\pi)$ all of whose composition factors $L(\sigma)$ satisfy $\sigma \leq \pi$.

(ii) $\bar{\Delta}(\pi)$ is the largest quotient of $P(\pi)$ which has $L(\pi)$ with multiplicity 1 and such that all its other composition factors $L(\sigma)$ satisfy $\sigma < \pi$.

(iii) Let $I(\pi)$ denote the injective hull of $L(\pi)$ in the category of all graded R_θ-modules. Then $\bar{\nabla}(\pi)$ is the largest submodule of $I(\pi)$ which has $L(\pi)$ with multiplicity 1 and all its other composition factors $L(\sigma)$ satisfy $\sigma < \pi$.

Proof. (i) Since $\mathrm{head}\,\Delta(\pi) \cong L(\pi)$, we have a short exact sequence $0 \to X \to P(\pi) \to \Delta(\pi) \to 0$, and it suffices to prove that $\mathrm{Hom}_{R_\theta}(X, L(\sigma)) = 0$ if $\sigma \leq \pi$. Using the long exact sequence which arises by applying $\mathrm{Hom}_{R_\theta}(-, L(\sigma))$ to the short exact sequence, we have to prove $\mathrm{Ext}^1_{R_\theta}(\Delta(\pi), L(\sigma)) = 0$ for $\sigma \leq \pi$. But

$$\mathrm{Ext}^1_{R_\theta}(\Delta(\pi), L(\sigma)) \cong \mathrm{Ext}^1_{R_{\rho(\pi)}}(\Delta_\pi, \mathrm{Res}_{\rho(\pi)} L(\sigma)).$$

In view of Theorem 4.41(v), we may assume that $\rho(\pi) = \rho(\sigma)$, in which case $\mathrm{Res}_{\rho(\pi)} L(\sigma) \cong L_\sigma$. Now, the result follows from Künneth formula and Lemma 4.34.

(ii) In view of Theorem 4.41(i), we have a short exact sequence $0 \to X \to P(\pi) \to \bar{\Delta}(\pi) \to 0$, and it suffices to prove that $\mathrm{Hom}_{R_\theta}(X, L(\sigma)) = 0$ if $\sigma < \pi$. Using the long exact sequence which arises by applying $\mathrm{Hom}_{R_\theta}(-, L(\sigma))$ to the short exact sequence, we have to prove $\mathrm{Ext}^1_{R_\theta}(\bar{\Delta}(\pi), L(\sigma)) = 0$ for $\sigma < \pi$. But $\mathrm{Ext}^1_{R_\theta}(\bar{\Delta}(\pi), L(\sigma)) \cong \mathrm{Ext}^1_{R_{\rho(\pi)}}(L_\pi, \mathrm{Res}_{\rho(\pi)} L(\sigma))$. An application of Theorem 4.41(v) completes the proof.

(iii) In this proof only, we will work in the larger category of all graded R_θ-modules. By (4.40), $\mathrm{soc}\, \bar{\nabla}(\pi) \cong L(\pi)$, so there is a short exact sequence $0 \to \bar{\nabla}(\pi) \to I(\pi) \to X \to 0$, and it suffices to prove that X does not have a submodule, all of whose irreducible subfactors are $\simeq L(\sigma)$ with $\sigma < \pi$. So it suffices to prove that X does not have a *finitely generated* submodule Y, all of whose composition factors are $\simeq L(\sigma)$ with $\sigma < \pi$. Otherwise apply $\mathrm{Hom}_{R_\theta}(Y, -)$ to the short exact sequence to get an exact sequence

$$\mathrm{Hom}_{R_\theta}(Y, I(\pi)) \to \mathrm{Hom}_{R_\theta}(Y, X) \to \mathrm{Ext}^1_{R_\theta}(Y, \bar{\nabla}(\pi)) \to 0.$$

Note that the middle term of this sequence is non-zero, while the first term is zero since the socle of $I(\pi)$ is $L(\pi)$. Finally, the third term is zero. Indeed,

$$\mathrm{Ext}^1_{R_\theta}(Y, \bar{\nabla}(\pi)) = \mathrm{Ext}^1_{R_\theta}(Y, \mathrm{Coind}_{\rho(\pi)} L_\pi) \cong \mathrm{Ext}^1_{R_{\rho(\pi)}}(\mathrm{Res}_{\rho(\pi)} Y, L_\pi) = 0,$$

since $\mathrm{Res}_{\rho(\pi)}Y = 0$ in view of Theorem 4.41(v). This a contradiction. $\square$

4.33. Standardization functor

We now want to check the condition `(Flat)` from Definition 4.7, which guarantees existence of standardization functor.

Proposition 4.45. Let $\pi, \sigma \in \Pi(\theta)$ satisfy $\rho(\pi) = \rho(\sigma) =: \xi$. Then the natural map $\mathrm{Hom}_{R_\xi}(\Delta_\pi, \Delta_\sigma) \to \mathrm{Hom}_{R_\theta}(\Delta(\pi), \Delta(\sigma))$ is an isomorphism.

Proof. By adjointness, we have $\mathrm{Hom}_{R_\theta}(\Delta(\pi), \Delta(\sigma)) \cong \mathrm{Hom}_{R_\xi}(\Delta_\pi, \mathrm{Res}_\xi \Delta(\sigma))$. By Corollary 4.42(iii), $\mathrm{Res}_\xi \Delta(\sigma) \cong \Delta_\sigma$, and the result follows. $\square$

Corollary 4.46. Let $\xi \in \Xi(\theta)$, $\Delta(\xi) = \bigoplus_{\pi \in \rho^{-1}(\xi)} \Delta(\pi)$, and $\Delta_\xi := \bigoplus_{\pi \in \rho^{-1}(\xi)} \Delta_\pi$. Then the natural map $\mathrm{End}_{R_\xi}(\Delta_\xi) \to \mathrm{End}_{R_\theta}(\Delta(\xi))$ is an isomorphism of algebras.

Theorem 4.47. Let $\theta \in Q_+$, $\xi \in \Xi(\theta)$, $\Delta(\xi) = \bigoplus_{\pi \in \rho^{-1}(\xi)} \Delta(\pi)$, and $B_\xi := \mathrm{End}_{R_\theta}(\Delta(\xi))^{\mathrm{op}}$. Then, as a right B_ξ-module, $\Delta(\xi)$ is finitely generated projective, in particular, finitely generated flat.

Proof. We write ξ in the form $\xi = (\beta_1^{x_1}, \dots, \beta_r^{x_r})$ for $\beta_1 \succ \cdots \succ \beta_r$. Note that $\mathrm{End}_{R_\xi}(\Delta_\xi)^{\mathrm{op}} \cong B_{\beta_1^{x_1}} \otimes \cdots \otimes B_{\beta_r^{x_r}}$. So by Corollary 4.46, we have $B_\xi \cong B_{\beta_1^{x_1}} \otimes \cdots \otimes B_{\beta_r^{x_r}}$. Moreover, each $\Delta(\beta_m^{x_m})$ is a projective generator in the category $C_{x_m \beta_m}$-mod. So, by Morita theory, $\Delta(\beta_m^{x_m})$ is finitely generated projective as a right module over its endomorphism algebra $B_{\beta_m^{x_m}}$. It follows that Δ_ξ is finitely generated projective as a right module over its endomorphism algebra B_ξ. Finally since R_θ is free of finite rank over R_ξ, it follows that $\Delta(\xi) = \mathrm{Ind}_\xi \Delta_\xi$ is finitely generated projective over B_ξ. $\square$

We have established the property `(Flat)` from Definition 4.7. The property `(Filt)` is more difficult to check. We are missing the equality $\mathrm{Ext}^2_{R_\theta}(\Delta(\pi), \bar{\nabla}(\sigma)) = 0$

if $\rho(\pi) = \rho(\sigma)$, which is needed for standard arguments as in [7, Theorem 3.13] yielding a Δ-filtration on $P(\pi)$. So we will have to proceed in a round about way using reduction modulo p and the results of McNamara [39] who has established the result in characteristic zero. For now, note using Proposition 4.43 and the Künneth formula, that it suffices to prove the following for all $n \in \mathbb{Z}_{>0}$:

$$\mathrm{Ext}^2_{R_{n\alpha}}(\Delta(\alpha^n), L(\alpha^n)) = \mathrm{Ext}^2_{R_{n\delta}}(\Delta(\underline{\lambda}), L(\underline{\mu})) = 0 \qquad (\alpha \in \Phi_+^{\mathrm{re}},\ \underline{\lambda}, \underline{\mu} \in \mathscr{P}_n).$$

4.34. Boundedness

Let $\theta = \sum_{i \in I} a_i \alpha_i$ and $n = \mathrm{ht}(\theta)$. Recalling that $I = \{0, 1, \dots, l\}$, pick a permutation $(i_0, \dots, i_l)$ of $(0, \dots, l)$ with $a_{i_0} > 0$, and define $\boldsymbol{i} := i_0^{a_{i_0}} \cdots i_l^{a_{i_l}} \in I^\theta$. Then the stabilizer of $\boldsymbol{i}$ in S_n is the standard parabolic subgroup $S_{\boldsymbol{i}} := S_{a_{i_0}} \times \cdots \times S_{a_{i_l}}$. Let $S^{\boldsymbol{i}}$ be a set of left coset representatives for $S_n / S_{\boldsymbol{i}}$. Then by [20, Theorem 2.9] or [42, Proposition 3.9], the element

$$z = z_{\boldsymbol{i}} := \sum_{w \in S^{\boldsymbol{i}}} (y_{w(1)} + \cdots + y_{w(a_{i_1})}) 1_{w \cdot \boldsymbol{i}} \tag{4.48}$$

is central of degree 2 in R_θ. Let R'_θ be the subalgebra of R_θ generated by

$$\{1_{\boldsymbol{i}} \mid \boldsymbol{i} \in I^\theta\} \cup \{\psi_r \mid 1 \le r < n\} \cup \{y_r - y_{r+1} \mid 1 \le r < n\}.$$

The restrictions from R_θ to R'_θ of modules $L(\pi), L_{\delta,i}, \Delta(\pi)$, etc. are denotes $L'(\pi), L'_{\delta,i}, \Delta'(\pi)$, etc.

Lemma 4.49. [4, Lemma 3.1] We have

(i) $\{(y_1 - y_2)^{m_1} \cdots (y_{n-1} - y_n)^{m_{n-1}} \tau_w 1_{\boldsymbol{i}} \mid m_r \in \mathbb{Z}_{\geq 0}, w \in S_n, \boldsymbol{i} \in I^\theta\}$ is a basis for R'_θ.

(ii) If $a_{i_0} \cdot 1_k \neq 0$ in k, then there is an algebra isomorphism $R_\theta \cong R'_\theta \otimes k[z]$.

For $\theta \in \Phi_+ \setminus \{n \cdot \delta \mid p|n\}$, and in particular for $\theta \in \Psi$, there always exists an index i_0 with $a_{i_0} \cdot 1_k \neq 0$. We always make this choice. Then:

Corollary 4.50. For $\alpha \in \Psi$, we have $R_\alpha \cong R'_\alpha \otimes k[z]$.

Let $\alpha \in \Psi$, and L be an irreducible R_α-module. Then z acts as zero on L, so the restriction L' is an irreducible R'_α-module by the corollary. For $\alpha \in \Phi_+$, we consider the module over $R_\alpha = R'_\alpha \otimes k[z]$:

$$\tilde{\Delta}(\alpha) := L'(\alpha) \otimes k[z]. \tag{4.51}$$

Eventually we will prove that $\tilde{\Delta}(\alpha) \cong \Delta(\alpha)$.

Lemmas 4.21(iii) and 4.19 show that the statement of Lemma 4.20 holds without the assumption $p = 0$. This statement and Lemma 4.21(ii) is all that is needed for the argument of [39, Theorem 15.5] to go through, so we get:

Lemma 4.52. [39, Theorem 15.5] Let $\alpha \in \Psi$. Then dimension of the graded components $\dim(C_\alpha)_d$ are bounded as a function of d.

Note that $\tilde{\Delta}(\alpha) \in C_\alpha$-mod and $F[z]$ acts on $\tilde{\Delta}(\alpha)$ freely, so the restriction of the natural surjection $\varphi : R_\alpha \to C_\alpha$ to $F[z]$ is injective, and its image gives us a central subalgebra $F[z] \subseteq C_\alpha$. Every projective C_α-module is free over the subalgebra $F[z]$, and by Lemma 4.52, it has to be free of finite rank. Moreover, we can write $C_\alpha = C'_\alpha \otimes F[z]$ for the finite dimensional algebra $C'_\alpha := \varphi(R'_\alpha)$. The same argument works for C_δ. Thus:

Corollary 4.53. Let $\alpha \in \Psi$. Every standard R_α-module is free of finite rank upon restriction to the subalgebra $F[z]$. Moreover, we can represent C_α as a tensor product of algebra $C_\alpha \cong C'_\alpha \otimes F[z]$ with finite dimensional C'_α.

It is now clear that $\Delta(\alpha) \cong P'(\alpha) \otimes F[z]$ and $\Delta_{\delta,i} \cong P'_{\delta,i} \otimes F[z]$, where $P'(\alpha)$ is the projective cover of $L'(\alpha)$ in C'_α-mod and $P'_{\delta,i}$ is the projective cover of $L'_{\delta,i}$ in C'_δ-mod. The following result in characteristic zero is obtained in [39]:

Lemma 4.54. Let $\alpha \in \Psi$ and $i \in I'$.

(i) If $\alpha \in \Phi_+$ and (β, γ) is a real minimal pair for α, then there exists a short exact sequence

$$0 \to q\,\Delta(\beta) \circ \Delta(\gamma) \to \Delta(\gamma) \circ \Delta(\beta) \to \Delta(\alpha) \to 0.$$

(ii) If $n > 1$ and $\alpha = \gamma_i^{\pm} + n\delta$, then, setting $\beta^{\pm} := \gamma_i^{\pm} + (n-1)\delta$, there exist short exact sequences of the form

$$0 \to \Delta(\beta^+) \circ \Delta_{\delta,i} \to \Delta_{\delta,i} \circ \Delta(\beta^+) \to (q+q^{-1})\Delta(\gamma_i^+ + n\delta) \to 0,$$
$$0 \to \Delta_{\delta,i} \circ \Delta(\beta^-) \to \Delta(\beta^-) \circ \Delta_{\delta,i} \to (q+q^{-1})\Delta(\gamma_i^- - n\delta) \to 0.$$

(iii) If $\alpha = \delta$, then there exists a short exact sequence

$$0 \to q^2\,\Delta(\gamma_i^+) \circ \Delta(\gamma_i^-) \to \Delta(\gamma_i^-) \circ \Delta(\gamma_i^+) \to \Delta_{\delta,i} \to 0.$$

Proof. (i) Lemma 4.52 and the central subalgebra $F[z] \subseteq C_\alpha$ are the main ingredients in the proof of [39, Lemma 16.1], which now goes through to give the short exact sequence

$$0 \to q\,\Delta(\beta) \circ \Delta(\gamma) \to \Delta(\gamma) \circ \Delta(\beta) \to Q \to 0,$$

where Q is a projective C_α-module. To prove that $Q \cong \Delta(\alpha)$ it suffices to prove that $\dim_q \operatorname{Hom}_{R_\alpha}(Q, L(\alpha)) = 1$. Applying $\operatorname{Hom}_{R_\alpha}(-, L(\alpha))$ to the short exact sequence and observing that $\operatorname{Hom}_{R_\alpha}(\Delta(\beta) \circ \Delta(\gamma), L(\alpha)) = 0$ by semicuspidality of $L(\alpha)$, we see that it suffices to prove that $\dim_q \operatorname{Hom}_{R_\alpha}(\Delta(\gamma) \circ \Delta(\beta), L(\alpha)) = 1$. By adjointness, this dimension equals the multiplicity $[\operatorname{Res}_{\gamma,\beta} : L(\gamma) \boxtimes L(\beta)]_q$. In view of Lemma 4.19, this multiplicity if independent of the characteristic of the ground field. Since the result is true in characteristic zero by [39], we deduce that it also holds in positive characteristic.

(ii) is proved analogously to (i).

(iii) In view of [39, Theorem 13.1] and Lemma 4.9, we may assume that (γ_j^+, γ_j^-) is a minimal pair for δ. As in (i), we have a short exact sequence

$$0 \to q^2\, \Delta(\gamma_i^+) \circ \Delta(\gamma_i^-) \to \Delta(\gamma_i^-) \circ \Delta(\gamma_i^+) \to Q \to 0,$$

where Q is a projective C_δ-module. To prove that $Q \cong \Delta_{\delta,i}$ it suffices to prove that $\dim_q \operatorname{Hom}_{R_\delta}(Q, L_{\delta,j}) = \delta_{i,j}$, which follows by applying $\operatorname{Hom}_{R_\alpha}(-, L_{\delta,j})$ to the short exact sequence and observing that $\operatorname{Hom}_{R_\delta}(\Delta(\gamma_i^+) \circ \Delta(\gamma_i^-), L_{\delta,j}) = 0$ by semicuspidality of $L_{\delta,j}$, while $\dim_q \operatorname{Hom}_{R_\delta}(\Delta(\gamma_i^-) \circ \Delta(\gamma_i^+), L_{\delta,j}) = \delta_{i,j}$ by Lemma 4.21(ii). $\square$

4.35. Stratification

Recall from the end of §4.33 that to prove that R_α is properly stratified we need some Ext-result. In this subsection we prove the missing result under an explicit restriction on p. Again, we follow [39] closely.

Lemma 4.55. Let $\Delta_\delta := \bigoplus_{i \in I'} \Delta_{\delta,i}$. Then $\Delta_\delta^{\circ n}$ is a projective $C_{n\delta}$-module. Moreover, if $p > n$ or $p = 0$, then $\Delta_\delta^{\circ n}$ is a projective generator in $C_{n\delta}$-mod.

Proof. To prove that $\Delta_\delta^{\circ n}$ is projective in $C_{n\delta}$-mod, it suffices to show that $\mathrm{Ext}^1_{C_{n\delta}}(\Delta_\delta^{\circ n}, L) = 0$ for any irreducible $C_{n\delta}$-module L, which would follow from $\mathrm{Ext}^1_{R_{n\delta}}(\Delta_\delta^{\circ n}, L) = 0$. But the latter Ext-group is isomorphic to $\mathrm{Ext}^1_{R_{\delta,\dots,\delta}}(\Delta_\delta \boxtimes \cdots \boxtimes \Delta_\delta, \mathrm{Res}_{\delta,\dots,\delta} L)$, which is indeed trivial by Künneth formula, since all composition factors of $\mathrm{Res}_{\delta,\dots,\delta} L$ are of the form $L_1 \boxtimes \cdots \boxtimes L_n$ with each L_r semicuspidal.

To show that $\Delta_\delta^{\circ n}$ is a projective generator, it now suffices to show that $\dim_q \mathrm{Hom}_{R_{n\delta}}(\Delta_\delta^{\circ n}, L(\underline{\mu})) \neq 0$ for any $\underline{\mu} \in \mathscr{P}_n$, which from Theorems 4.22(iii) and 4.33. □

Theorem 4.56. Let $\alpha \in \Psi$ and $n \in \mathbb{Z}_{>0}$.

(i) Let $\alpha = \delta$. If $p > n$ or $p = 0$, then for all $\underline{\lambda}, \underline{\mu} \in \mathscr{P}_n$, we have $\mathrm{Ext}^m_{R_{n\delta}}(\Delta(\underline{\lambda}), L(\underline{\mu})) = 0$ for all $m > 0$.

(ii) If α is real, then $\mathrm{Ext}^m_{R_{n\alpha}}(\Delta(\alpha^n), L(\alpha^n)) = 0$ for all $m > 0$.

Proof. (i) By Lemma 4.55, $\Delta_\delta^{\circ n}$ is a projective generator in $C_{n\delta}$-mod, so it suffices to prove that $\mathrm{Ext}^m_{R_{n\delta}}(\Delta_\delta^{\circ n}, L(\underline{\mu})) = 0$ for all $\underline{\mu} \in \mathscr{P}_n$. The last Ext group is isomorphic to $\mathrm{Ext}^m_{R_{\delta,\dots,\delta}}(\Delta_\delta^{\boxtimes n}, \mathrm{Res}_{\delta,\dots,\delta} L(\underline{\mu}))$. All composition factors of $\mathrm{Res}_{\delta,\dots,\delta} L(\underline{\mu})$ are of the form $L_1 \boxtimes \cdots \boxtimes L_n$ with each $L_r \in C_\delta$-mod, so by the Künneth formula, we may assume that $n = 1$, i.e. we need to prove $\mathrm{Ext}^m_{R_\delta}(\Delta_{\delta,i}, L_{\delta,j}) = 0$ for all $i, j \in I'$ and $m > 0$. But this follows by applying $\mathrm{Hom}_{R_\delta}(-, L_{\delta,j})$ to the short exact sequence in Lemma 4.54(iii), using Lemma 4.21(ii) and induction on the height.

(ii) In view of Theorem 4.18(i) and Lemma 4.35(i), we may assume that $n = 1$. To prove $\mathrm{Ext}^m_{R_\alpha}(\Delta(\alpha), L(\alpha)) = 0$, we apply $\mathrm{Hom}_{R_\alpha}(-, L(\alpha))$ to the short exact sequence in Lemma 4.54(i),(ii), and use (i) and induction on height. □

Taking into account the results of §§4.32,4.33, we now have:

Corollary 4.57. Let $\theta = \sum_{i\in I} n_i\alpha_i \in Q_+$ and assume that $p > \min\{n_i \mid i \in I\}$. For any convex preorder on Φ_+, the algebra R_α is properly stratified with standard modules $\{\Delta(\pi) \mid \pi \in \Pi(\theta)\}$ and proper standard modules $\{\bar{\Delta}(\pi) \mid \pi \in \Pi(\theta)\}$.

4.4. Reduction modulo p of irreducible and standard modules

4.41. Reduction modulo p of irreducible modules

We already know from Lemma 4.19 that reduction modulo p of a real semicuspidal module $L(\alpha^n)_K$ is $L(\alpha^n)_F$. We now look at reductions modulo p of some imaginary semicuspidal modules. For $\lambda, \mu \vdash n$, we denote by $d^p_{\mathtt{cl}}(\lambda,\mu) := [W_{\mathtt{cl}}(\lambda) : L_{\mathtt{cl}}(\lambda)]$ the decomposition numbers for the classical Schur algebra $S(n,n)$ in characteristic p. It is known that $d^p_{\mathtt{cl}}(\lambda,\lambda) = 1$ and $d^p_{\mathtt{cl}}(\lambda,\mu) = 0$ unless $\mu \trianglelefteq \lambda$ in the dominance order. For $\underline{\lambda}, \underline{\mu} \in \mathscr{P}_n$, we define $d^p(\underline{\lambda},\underline{\mu}) := \prod_{i\in I'} d^p_{\mathtt{cl}}(\lambda^{(i)},\mu^{(i)})$ if $|\lambda^{(i)}| = |\mu^{(i)}|$ for all $i \in I'$, and set $d^p(\underline{\lambda},\underline{\mu}) := 0$ otherwise. Again, $d^p(\underline{\lambda},\underline{\lambda}) = 1$ and $d^p(\underline{\lambda},\underline{\mu}) = 0$ unless $\underline{\mu} \trianglelefteq \underline{\lambda}$, which means by definition that $\mu^{(i)} \trianglelefteq \lambda^{(i)}$ for all $i \in I'$.

Lemma 4.58. Let $i \in I'$ and $\lambda, \mu \vdash n$. Then $W_i(\lambda)_F$ is reduction modulo of $W_i(\lambda)_K = L_i(\lambda)_K$. In particular, $[L_i(\lambda)_{\mathcal{O}} \otimes F : L_i(\mu)_F]_q = d^p_{\mathtt{cl}}(\lambda,\mu)$.

Proof. The first statement is proved exactly as Theorem 3.94. The second statement now follows by the Morita equivalence β_n from §4.25. □

Lemma 4.59. Let $\underline{\lambda}, \underline{\mu} \in \mathscr{P}_n$. Then $L(\underline{\lambda})_{\mathcal{O}} \otimes_{\mathcal{O}} F$ is semicuspidal, and $[L(\underline{\lambda})_{\mathcal{O}} \otimes_{\mathcal{O}} F : L(\underline{\mu})_F]_q = d^p(\underline{\lambda},\underline{\mu})$.

Proof. Induction and reduction modulo p commute by Lemma 4.15, so the result follows from Lemma 4.58 and Theorem 4.33(i). □

Corollary 4.60. For $\underline{\mu} \in \mathscr{P}_n$ and $p > n$, reduction modulo p of $L(\underline{\mu})_K$ is $L(\underline{\mu})_F$.

Let $\theta \in Q_+$ and $\pi \in \Pi(\theta)$. Let $1_F \in R_{\theta,F}$ be a primitive idempotent such that $R_{\theta,F}1_F \cong P(\pi)_F$. By an argument in [22, §4.1], there is an idempotent $1_{\mathcal{O}} \in R_{\theta,\mathcal{O}}$ with $1_F = 1_{\mathcal{O}} \otimes 1$. Let $P(\pi)_{\mathcal{O}} := R_{\theta,\mathcal{O}}1_{\mathcal{O}}$. Extending scalars to K we get a projective $R_{\theta,\mathcal{O}}$-module $P(\pi)_{\mathcal{O}} \otimes_{\mathcal{O}} K$. So we can decompose it as a direct sum of some projective indecomposable modules $P(\sigma)_K$.

Lemma 4.61. Let $\underline{\lambda} \in \mathscr{P}_n$ and $\pi = (\xi, \underline{\lambda}) \in \Pi(\theta)$. Then in the Grothendieck group $[R_{\theta,F}\text{-mod}]$, we have

$$[L(\pi)_{\mathcal{O}} \otimes_{\mathcal{O}} F] = [L(\pi)_F] + \sum_{\underline{\mu} \lhd \underline{\lambda}} d^p(\underline{\lambda}, \underline{\mu})[L((\xi, \underline{\mu}))_F] + \sum_{\sigma < \pi} a_{\pi,\sigma}[L(\sigma)_F]$$

for some bar-invariant Laurent polynomials $a_{\pi,\sigma} \in \mathbb{Z}[q, q^{-1}]$. Moreover,

$$P(\pi)_{\mathcal{O}} \otimes_{\mathcal{O}} K \cong P(\pi)_K \oplus \bigoplus_{\underline{\mu} \rhd \underline{\lambda}} d^p(\underline{\mu}, \underline{\lambda}) P((\xi, \underline{\mu}))_K \oplus \bigoplus_{\sigma > \pi} a_{\sigma,\pi} P(\sigma)_K.$$

Proof. Similar to the proof of [22, Lemma 4.8], but using Lemma 4.59. □

Corollary 4.62. All composition factors $L(\sigma)_F$ of a reduction modulo p of $\Delta(\pi)_K$ satisfy $\sigma \leq \pi$.

4.42. Reduction modulo p of standard modules

The proof of following result uses an idea from [43].

Lemma 4.63. Let $\pi \in \Pi(\theta)$. Then $\Delta(\pi)_F$ contains a submodule M such that $\Delta(\pi)_F/M$ is a reduction modulo p of $\Delta(\pi)_K$.

Proof. By Lemma 4.61, we can decompose $P(\pi)_{\mathcal{O}} \otimes_{\mathcal{O}} K \cong P(\pi)_K \oplus Q$ for some $R_{\theta,K}$-module Q. Since $\Delta(\pi)_K$ is a quotient of $P(\pi)_K$, there is an $R_{\theta,K}$-submodule $V_K \subseteq P(\pi)_{\mathcal{O}} \otimes_{\mathcal{O}} K$ with $P(\pi)_{\mathcal{O}} \otimes_{\mathcal{O}} K/V_K \cong \Delta(\pi)_K$. Let $V_{\mathcal{O}} = V_K \cap P(\pi)_{\mathcal{O}}$, where we consider $P(\pi)_{\mathcal{O}}$ as an $\mathcal{O}$-submodule of $P(\pi)_{\mathcal{O}} \otimes_{\mathcal{O}} K$ in a natural way. Note that $V_{\mathcal{O}}$ is a pure $R_{\theta,\mathcal{O}}$-invariant sublattice in $P(\pi)_{\mathcal{O}}$ and $P(\pi)_{\mathcal{O}}/V_{\mathcal{O}}$ is an $\mathcal{O}$-form of $\Delta(\pi)_K$. So $(P(\pi)_{\mathcal{O}}/V_{\mathcal{O}}) \otimes_{\mathcal{O}} F$, which is a reduction modulo p of $\Delta(\pi)_K$, is a quotient of $P(\pi)_F$. By Corollary 4.62, all composition factors $L(\sigma)_F$ of $(P(\pi)_{\mathcal{O}}/V_{\mathcal{O}}) \otimes_{\mathcal{O}} F$ satisfy $\sigma \leq \pi$, so by definition of $\Delta(\pi)_F$ as the largest quotient of $P(\pi)_F$ with such composition factors, $(P(\pi)_{\mathcal{O}}/V_{\mathcal{O}}) \otimes_{\mathcal{O}} F$ is a quotient of $\Delta(\pi)_F$. $\square$

Let $\alpha \in \Phi_+^{\mathrm{re}}$ and $n \in \mathbb{Z}_{>0}$. We have a semicuspidal standard module $\Delta(\alpha^n)_K$. Pick a generator $v \in \Delta(\alpha^n)_K$ which is a homogeneous weight vector. Consider the $R_{n\alpha,\mathcal{O}}$-invariant lattice $\Delta(\alpha^n)_{\mathcal{O}} := R_{n\alpha,\mathcal{O}} \cdot v$, and the reduction $\Delta(\alpha^n)_{\mathcal{O}} \otimes_{\mathcal{O}} F$.

By Lemma 4.21, $\mathrm{Res}_{\gamma_i^-,\gamma_i^+} L_{\delta,i} \cong L(\gamma_i^-) \boxtimes L(\gamma_i^+)$ and $\mathrm{Res}_{\gamma_i^-,\gamma_i^+} L_{\delta,j} = 0$ for $j \neq i$. So, picking a weight $\boldsymbol{j}^{\pm}$ of $L(\gamma_i^{\pm})$, we have a weight $\boldsymbol{j}^i := \boldsymbol{j}^- \boldsymbol{j}^+$ of $L_{\delta,i}$ such that $1_{\boldsymbol{j}^i} L_{\delta,j} = 0$ for all $j \neq i$. Pick a homogeneous generator $v \in \Delta_{\delta,i,K}$ *of weight* $\boldsymbol{j}^i$. Consider the invariant lattice $\Delta_{\delta,i,\mathcal{O}} := R_{\delta,\mathcal{O}} \cdot v$ and the reduction $\Delta_{\delta,i,\mathcal{O}} \otimes_{\mathcal{O}} F$.

Lemma 4.64. We have

(i) $\Delta(\alpha^n)_{\mathcal{O}} \otimes_{\mathcal{O}} F$ is a semicuspidal $R_{n\alpha,F}$-module with simple head $L(\alpha^n)_F$, and so it is a quotient of $\Delta(\alpha^n)_F$.

(ii) $\Delta_{\delta,i,\mathcal{O}} \otimes_{\mathcal{O}} F$ is a semicuspidal $R_{\delta,F}$-module with simple head $L_{\delta,i,F}$, and so it is a quotient of $\Delta_{\delta,i,F}$.

Proof. By Lemma 4.19, $L(\alpha^n)_{\mathcal{O}} \otimes_{\mathcal{O}} F \cong L(\alpha^n)_F$ is irreducible, so all composition factors of $\Delta(\alpha^n)_{\mathcal{O}} \otimes_{\mathcal{O}} F$ are isomorphic to $L(\alpha^n)_F$, i.e. this module is semicuspidal. By Lemma 4.21(iii), we see similarly that $\Delta_{\delta,i,\mathcal{O}} \otimes_{\mathcal{O}} F$ is also semicuspidal. In both situations, $v \otimes 1 \in \Delta_{\mathcal{O}} \otimes_{\mathcal{O}} F$ is a cyclic generator of $\Delta_{\mathcal{O}} \otimes_{\mathcal{O}} F$, and it remains to apply Lemma 2.56. $\square$

Now we can prove a stronger result:

Theorem 4.65. Let $\alpha \in \Phi_+^{\mathrm{re}}$ and $i \in I'$. Then $\Delta(\alpha)_F \cong \Delta(\alpha)_{\mathcal{O}} \otimes_{\mathcal{O}} F$ and $\Delta_{\delta,i,F} \cong \Delta_{\delta,i,\mathcal{O}} \otimes_{\mathcal{O}} F$.

Proof. Apply induction on $\mathrm{ht}(\alpha)$. The base being clear, and the inductive step is obtained from Lemmas 4.64 and 4.54 by character considerations. □

Corollary 4.66. If $\alpha \in \Phi_+^{\mathrm{re}}$, then $\Delta(\alpha) \cong \tilde{\Delta}(\alpha)$ and $\mathrm{End}_{R_\alpha}(\Delta(\alpha)) \cong F[z]$.

Proof. Since $L'(\alpha)$ is irreducible, we deduce by adjointness that $\tilde{\Delta}(\alpha)$ has simple head, whence it is a quotient of $\Delta(\alpha)$. Now compare the characters using [39, Theorem 18.3] in characteristic zero and Theorem 4.65. □

Corollary 4.67. If $\alpha \in \Phi_+^{\mathrm{re}}$ and $n \in \mathbb{Z}_{>0}$, then $\Delta(\alpha^n)_F \cong \Delta(\alpha^n)_{\mathcal{O}} \otimes_{\mathcal{O}} F$ and $\mathrm{End}_{R_\alpha}(\Delta(\alpha^n)) \cong F[z_1, \dots, z_n]^{\mathfrak{S}_n}$.

Proof. The first statement follows from Lemmas 4.64(i), 4.35 and Theorem 4.65 by induction on n. The second statement then follows using the fact that it is true in characteristic zero [39]. □

We can now prove that certain cuspidal algebras C_α are 'defined over integers'.

Corollary 4.68. Let $\alpha \in \Phi_+$ and $n \in \mathbb{Z}_{>0}$. Then $C_{n\alpha,\mathcal{O}}$ and $C_{\delta,\mathcal{O}}$ are free over $\mathcal{O}$, with $C_{n\alpha,k} \cong C_{n\alpha,\mathcal{O}} \otimes_{\mathcal{O}} k$ and $C_{\delta,k} \cong C_{\delta,\mathcal{O}} \otimes_{\mathcal{O}} k$ for and $k = F$ or K.

Proof. We explain the argument for C_δ, the argument for $C_{n\alpha}$ being similar. The isomorphisms $C_{\delta,k} \cong C_{\delta,\mathcal{O}} \otimes_{\mathcal{O}} k$ are clear, and it suffices to prove that $\dim_q C_{\delta,K} = \dim_q C_{\delta,F}$. But $\dim_q C_{\delta,k} = \sum_{i \in I'} (\dim_q L_{\delta,i})(\dim_q \Delta_{\delta,i})$, which, as we have now proved, is the same for $k = K$ and F. □

We conjecture that a similar statement is true in general. The part which remains open is:

Conjecture 4.69. Let $n \in \mathbb{Z}_{>0}$ and $k = F$ or K. Then $C_{n\delta,\mathcal{O}}$ is free over $\mathcal{O}$ and $C_{n\delta,k} \cong C_{n\delta,\mathcal{O}} \otimes_{\mathcal{O}} k$.

The only difficult thing here is to show that $C_{n\delta,\mathcal{O}}$ has no p-torsion. The following result implies that $C_{n\delta,\mathcal{O}}$ at least has no p-torsion if $p > n$.

Lemma 4.70. Let $n \in \mathbb{Z}_{>0}$, $\underline{\mu} \in \mathscr{P}_n$, and $p > n$. Then $\Delta(\underline{\mu})_F$ is a reduction modulo p of $\Delta(\underline{\mu})_K$.

Proof. Working over $k = F$ or K, by Lemma 4.55, $\Delta_\delta^{\circ n}$ is a projective generator in $C_{n\delta}$-mod. So, we can decompose $\Delta_\delta^{\circ n} = \bigoplus_{\underline{\mu} \in \mathscr{P}_n} m(\underline{\mu}) \Delta(\underline{\mu})_k$ with *non-zero* multiplicities $m(\underline{\mu})$, which a priori might depend on k. Moreover,

$$\begin{aligned} m(\underline{\mu}) &= \dim_q \operatorname{Hom}_{R_{n\delta}}(\Delta_\delta^{\circ n}, L(\underline{\mu})) = \dim_q \operatorname{Hom}_{R_{\delta,\dots,\delta}}(\Delta_\delta^{\boxtimes n}, \operatorname{Res}_{\delta,\dots,\delta} L(\underline{\mu})) \\ &= \dim_q \operatorname{Hom}_{R_{\delta,\dots,\delta}}((\bigoplus_{i \in I'} \Delta_{\delta,i})^{\boxtimes n}, \operatorname{Res}_{\delta,\dots,\delta} L(\underline{\mu})) \\ &= \sum_{i_1,\dots,i_n \in I'} \dim_q \operatorname{Hom}_{R_{\delta,\dots,\delta}}(\Delta_{\delta,i_1} \boxtimes \dots \boxtimes \Delta_{\delta,i_n}, \operatorname{Res}_{\delta,\dots,\delta} L(\underline{\mu})) \\ &= \sum_{i_1,\dots,i_n \in I'} [\operatorname{Res}_{\delta,\dots,\delta} L(\underline{\mu})) : L_{\delta,i_1} \boxtimes \dots \boxtimes L_{\delta,i_n}]_q. \end{aligned}$$

The last expression is independent of k by Corollary 4.60. It now follows from Lemma 4.63 by a character argument that $\operatorname{ch}_q \Delta(\underline{\mu})_F = \operatorname{ch}_q \Delta(\underline{\mu})_K$ and that $\Delta(\underline{\mu})_F$ is a reduction modulo p of $\Delta(\underline{\mu})_K$. □

4.5. Zigzag algebras

In this section we introduce the *affine zigzag algebra* $\mathsf{A}_n^{\mathrm{aff}}$, which is intended to describe the higher imaginary strata. Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a connected graph without loops or multiple edges. Eventually, we will need only the case where Γ is of finite ADE type, but for now do not need to assume that in this section. We also do not need to assume that k is a field. If $i, j \in \Gamma_0$, we use the notation $i \,\text{——}\, j$ to indicate that $\{i, j\} \in \Gamma_1$. In this case we say that i and j are *neighbors*.

4.51. Huerfano-Khovanov zigzag algebras

The *zigzag algebra* $\mathsf{A} := \mathsf{A}(\Gamma)$ *of type* Γ is defined in [14] as follows:

Definition 4.71. First assume that $|\Gamma_0| > 1$. Let $\overline{\Gamma}$ be the quiver obtained by doubling all edges between connected vertices and then orienting the edges so that if i and j are neighboring vertices in Γ, then there is an arrow $\mathsf{a}^{i,j}$ from j to i and an arrow $\mathsf{a}^{j,i}$ from i to j. For example, $\overline{\mathtt{A}}_\ell$ is the quiver in Figure 4.1. below.

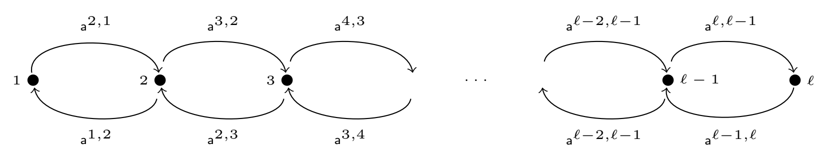


FIGURE 4.1. The quiver $\overline{\mathtt{A}}_\ell$.

Then $\mathsf{A}(\Gamma)$ is the path algebra of $\overline{\Gamma}$, generated by length-0 paths e_i for $i \in \Gamma_0$, and length-1 paths $\mathsf{a}^{i,j}$, modulo the following relations:

(i) All paths of length three or greater are zero.

(ii) All paths of length two that are not cycles are zero.

(iii) All length-two cycles based at the same vertex are equivalent.

If $|\Gamma_0| = 1$, i.e. $\Gamma = \mathtt{A}_1$, we merely decree that $\mathsf{A}(\Gamma) := k[\mathsf{c}]/(\mathsf{c}^2)$, where c is an indeterminate in degree 2.

For type $\Gamma \neq \mathtt{A}_1$, for every vertex i, let j be any neighbor of i, and write $\mathsf{c}^{(i)}$ for the cycle $\mathsf{a}^{i,j}\mathsf{a}^{j,i}$. The relations in $\mathsf{A}(\Gamma)$ imply that $\mathsf{c}^{(i)}$ is independent of choice of j.

Define $\mathsf{c} := \sum_{i\in\Gamma_0} \mathsf{c}^{(i)}$. Then $\mathsf{A}(\Gamma)$ has basis

$$\{\mathsf{a}^{i,j} \mid i \in \Gamma_0,\ j \text{ a neighbor of } i\} \cup \{\mathsf{c}^m \mathsf{e}_i \mid i \in \Gamma_0,\ m \in \{0,1\}\}.$$

Note that $\mathsf{A}(\Gamma)$ is graded by path length. The graded dimension is

$$\dim_q \mathsf{A}(\Gamma) = |\Gamma_0|(1+q^2) + 2|\Gamma_1|q. \tag{4.72}$$

4.52. Affine zigzag algebras

We define the *rank n affine zigzag algebra* $\mathsf{A}_n^{\mathrm{aff}}(\Gamma)$ as follows:

Definition 4.73. If $|\Gamma_0| > 1$, let $\mathsf{A}_n^{\mathrm{aff}}(\Gamma)$ be the graded k-algebra generated by the elements

$$\{\mathsf{e}_{\boldsymbol{i}} \mid \boldsymbol{i} \in \Gamma_0^n\} \cup \{\mathsf{s}_t \mid 1 \le t \le n-1\}$$

$$\cup \{\mathsf{a}_t^{i,j} \mid 1 \le t \le n,\ i,j \in \Gamma_0 \text{ with } i \text{ — } j\} \cup \{\mathsf{z}_t \mid 1 \le t \le n\}$$

in degrees $0,0,1,2$ respectively, subject only to the relations

$$\textstyle\sum_{\boldsymbol{i}\in\Gamma_0^n} \mathsf{e}_{\boldsymbol{i}} = 1, \qquad \mathsf{e}_{\boldsymbol{i}}\mathsf{e}_{\boldsymbol{j}} = \delta_{\boldsymbol{i},\boldsymbol{j}}\mathsf{e}_{\boldsymbol{i}}, \tag{4.74}$$

$$\mathsf{s}_t\mathsf{e}_{\boldsymbol{i}} = \mathsf{e}_{s_t\boldsymbol{i}}\mathsf{s}_t, \qquad \mathsf{z}_t\mathsf{e}_{\boldsymbol{i}} = \mathsf{e}_{\boldsymbol{i}}\mathsf{z}_t, \qquad \mathsf{a}_t^{i,j}\mathsf{e}_{\boldsymbol{i}} = \delta_{i_t,j}\mathsf{e}_{i_1,\dots,i_{t-1},i,i_{t+1},\dots,i_n}\mathsf{a}_t^{i,j}, \tag{4.75}$$

$$\mathsf{s}_t^2 = 1, \qquad \mathsf{s}_t\mathsf{s}_u = \mathsf{s}_u\mathsf{s}_t \text{ if } |t-u| > 1, \qquad \mathsf{s}_t\mathsf{s}_{t+1}\mathsf{s}_t = \mathsf{s}_{t+1}\mathsf{s}_t\mathsf{s}_{t+1}, \tag{4.76}$$

$$\mathsf{s}_t\mathsf{a}_u^{i,j} = \mathsf{a}_{s_t(u)}^{i,j}\mathsf{s}_t, \qquad \mathsf{a}_t^{i,j}\mathsf{a}_t^{j,i} = \mathsf{a}_t^{i,m}\mathsf{a}_t^{m,i}, \qquad \mathsf{a}_t^{i'',j''}\mathsf{a}_t^{i',j'}\mathsf{a}_t^{i,j} = 0, \tag{4.77}$$

$$(1-\delta_{t,u}\delta_{i,j'}\delta_{i',j})\mathsf{a}_t^{i,j}\mathsf{a}_u^{i',j'} = (1-\delta_{t,u})\mathsf{a}_u^{i',j'}\mathsf{a}_t^{i,j}, \tag{4.78}$$

$$\mathsf{z}_t\mathsf{z}_u = \mathsf{z}_u\mathsf{z}_t, \qquad \mathsf{z}_t\mathsf{a}_u^{i,j} = \mathsf{a}_u^{i,j}\mathsf{z}_t, \tag{4.79}$$

plus one final relation, which we give after introducing a convenient bit of notation. For $1 \le t \le n$ and $i \in \Gamma_0$, let j be any neighbor of i, and set $\mathsf{c}_t^{(i)} := \mathsf{a}_t^{i,j}\mathsf{a}_t^{j,i}$. This is well-defined by (4.77). Then set $\mathsf{c}_t := \sum_{i\in\Gamma_0} \mathsf{c}_t^{(i)}$. The final relation is

$$(\mathsf{s}_t\mathsf{z}_u - \mathsf{z}_{s_t(u)}\mathsf{s}_t)\mathsf{e}_{\boldsymbol{i}} = \begin{cases} (\delta_{u,t} - \delta_{u,t+1})(\mathsf{c}_t + \mathsf{c}_{t+1})\mathsf{e}_{\boldsymbol{i}} & i_t = i_{t+1}; \\ (\delta_{u,t} - \delta_{u,t+1})\mathsf{a}_t^{i_{t+1},i_t}\mathsf{a}_{t+1}^{i_t,i_{t+1}}\mathsf{e}_{\boldsymbol{i}} & i_t \text{ --- } i_{t+1}; \\ 0 & \text{otherwise.} \end{cases} \tag{4.80}$$

If $|\Gamma_0| = 1$, let $\mathsf{A}_n^{\mathrm{aff}}(\Gamma)$ be the graded k-algebra generated by the elements

$$\{\mathsf{s}_t \mid 1 \le t \le n-1\} \cup \{\mathsf{c}_t \mid 1 \le t \le n\} \cup \{\mathsf{z}_t \mid 1 \le t \le n\}$$

in degrees $0, 2, 2$ respectively, subject only to the relations

$$\mathsf{s}_t^2 = 1, \qquad \mathsf{s}_t\mathsf{s}_u = \mathsf{s}_u\mathsf{s}_t \ \text{ if } \ |t-u| > 1, \qquad \mathsf{s}_t\mathsf{s}_{t+1}\mathsf{s}_t = \mathsf{s}_{t+1}\mathsf{s}_t\mathsf{s}_{t+1}, \tag{4.81}$$

$$\mathsf{z}_t\mathsf{z}_u = \mathsf{z}_u\mathsf{z}_t, \quad \mathsf{c}_t\mathsf{c}_u = (1-\delta_{t,u})\mathsf{c}_u\mathsf{c}_t, \quad \mathsf{z}_t\mathsf{c}_u = \mathsf{c}_u\mathsf{z}_t, \quad \mathsf{s}_t\mathsf{c}_u = \mathsf{c}_{s_t(u)}\mathsf{s}_t, \tag{4.82}$$

$$(\mathsf{s}_t\mathsf{z}_u - \mathsf{z}_{s_t(u)}\mathsf{s}_t) = (\delta_{u,t} - \delta_{u,t+1})(\mathsf{c}_t + \mathsf{c}_{t+1}). \tag{4.83}$$

4.53. Basis Theorem

For $1 \le t \le n$, we will write $\mathsf{a}_t^{i,i} := 1$. If $\boldsymbol{i}, \boldsymbol{j} \in \Gamma_0^n$ such that $i_t = j_t$ or $i_t \text{ --- } j_t$ for all t, we write $\boldsymbol{i} \text{ --- } \boldsymbol{j}$, and define

$$\mathsf{a}^{\boldsymbol{i},\boldsymbol{j}} := \mathsf{a}_1^{i_1,j_1} \cdots \mathsf{a}_n^{i_n,j_n}.$$

The elements $\mathsf{s}_1, \ldots, \mathsf{s}_{n-1}$ satisfy the Coxeter relations of the symmetric group, so for a reduced decomposition $w = s_{r_1} \cdots s_{r_m} \in \mathfrak{S}_n$, we have well-defined elements $\mathsf{s}_w := \mathsf{s}_{r_1} \cdots \mathsf{s}_{r_m} \in \mathsf{A}_n^{\mathrm{aff}}(\Gamma)$. For $\boldsymbol{t} \in \mathbb{Z}_{\geq 0}^n$ we will write $\mathsf{z}^{\boldsymbol{t}} = \mathsf{z}_1^{t_1} \cdots \mathsf{z}_n^{t_n}$ and for $\boldsymbol{u} \in \{0,1\}^n$ we will write $\mathsf{c}^{\boldsymbol{u}} = \mathsf{c}_1^{u_1} \cdots \mathsf{c}_n^{u_n}$. Finally, it will be useful to write $\mathsf{e}_{1\cdots 1} := 1 \in \mathsf{A}_n^{\mathrm{aff}}(\Gamma)$ when $|\Gamma_0| = 1$, so that we may consider this case as part of the larger family of affine zigzag algebras.

Theorem 4.84. The following set is a k-basis for $\mathsf{A}_n^{\mathrm{aff}}(\Gamma)$:

$$\{\mathsf{z}^{\boldsymbol{t}}\mathsf{c}^{\boldsymbol{u}}\mathsf{a}^{\boldsymbol{i},w\boldsymbol{j}}\mathsf{s}_w\mathsf{e}_{\boldsymbol{j}}\}, \tag{4.85}$$

ranging over $w \in \mathfrak{S}_n$, $\boldsymbol{i}$——$w\boldsymbol{j} \in \Gamma_0^n$, $\boldsymbol{t} \in \mathbb{Z}_{\geq 0}^n$, and $\boldsymbol{u} \in \{0,1\}^n$ such that $u_m \leq \delta_{i_m,(w\boldsymbol{j})_m}$.

Proof. From the defining relations, one may easily see by induction that $\mathsf{A}_n^{\mathrm{aff}}(\Gamma)$ is spanned by the elements in (4.85). We show that these elements are linearly independent by constructing a faithful representation for $\mathsf{A}_n^{\mathrm{aff}}(\Gamma)$. Define

$$V := \bigoplus_{\substack{w \in \mathfrak{S}_n \\ \boldsymbol{i} — w\boldsymbol{j} \in \Gamma_0^n}} k[z_1, \ldots, z_n, x_1, \ldots, x_n]\langle \mathrm{sh}(\boldsymbol{i}, w\boldsymbol{j})\rangle / \big(x_m^{\delta_{i_m,(w\boldsymbol{j})_m}+1}\big)_{1 \leq m \leq n},$$

where $\mathrm{sh}(\boldsymbol{i}, w\boldsymbol{j}) := n - \delta_{i_1,(w\boldsymbol{j})_1} - \cdots - \delta_{i_n,(w\boldsymbol{j})_n}$, and z's and x's are indeterminates of degree 2. If a polynomial f belongs to the component corresponding to $w \in \mathfrak{S}_n$ and $\boldsymbol{i}$——$w\boldsymbol{j} \in \Gamma_0^n$, we will label it with subscripts, a la $f_{\boldsymbol{i},w,\boldsymbol{j}}$. Elements of $\mathfrak{S}_n$ act on z's and x's by place permutation.

For $b_1, b_2 \in \mathbb{Z}_{\geq 0}$, define $g_{r,b_1,b_2} \in k[z_r, z_{r+1}]$ to be zero if $b_1 = b_2$, and

$$g_{r,b_1,b_2} := \frac{b_1 - b_2}{|b_1 - b_2|}\big(z_r z_{r+1}\big)^{\min(b_1,b_2)} \left(\sum_{\kappa=0}^{|b_1-b_2|-1} z_r^{\kappa} z_{r+1}^{|b_1-b_2|-\kappa-1} \right)$$

otherwise. For $1 \le r < n$, $w \in \mathfrak{S}_n$, and $\boldsymbol{i}, \boldsymbol{j} \in \Gamma_0^n$, we also define

$$h_{r,\boldsymbol{i},w,\boldsymbol{j}} = \begin{cases} x_r + x_{r+1} & i_r = i_{r+1}; \\ (\delta_{i_r,(w\boldsymbol{j})_r} + \delta_{i_{r+1},(w\boldsymbol{j})_r} x_r)(\delta_{i_{r+1},(w\boldsymbol{j})_{r+1}} + \delta_{i_r,(w\boldsymbol{j})_{r+1}} x_{r+1}) & i_r \text{---} i_{r+1}; \\ 0 & \text{otherwise.} \end{cases}$$

Finally, if $m^{\boldsymbol{t}} = z_1^{t_1} \cdots z_n^{t_n} x_1^{t_{n+1}} \cdots x_n^{t_{2n}}$ is a monomial in some summand of V for $\boldsymbol{t} \in \mathbb{Z}_{\ge 0}^{2n}$, then for $1 \le r < n$ define $m^{\boldsymbol{t},r} := m^{\boldsymbol{t}} / z_r^{t_r} z_{r+1}^{t_{r+1}}$. With this notation out of the way, we describe a well-defined action of generators of $\mathsf{A}_n^{\mathrm{aff}}(\Gamma)$ on V:

$$\begin{aligned} \mathsf{e}_{\boldsymbol{k}} \cdot m_{\boldsymbol{i},w,\boldsymbol{j}}^{\boldsymbol{t}} &= \delta_{\boldsymbol{k},\boldsymbol{i}} m_{\boldsymbol{i},w,\boldsymbol{j}}^{\boldsymbol{t}}, \\ \mathsf{z}_r \cdot m_{\boldsymbol{i},w,\boldsymbol{j}}^{\boldsymbol{t}} &= (z_r m^{\boldsymbol{t}})_{\boldsymbol{i},w,\boldsymbol{j}}, \\ \mathsf{a}_r^{i,j} \cdot m_{\boldsymbol{i},w,\boldsymbol{j}}^{\boldsymbol{t}} &= \delta_{j,i_r} [(\delta_{j,(w\boldsymbol{j})_r} + \delta_{i,(w\boldsymbol{j})_r} x_r) m^{\boldsymbol{t}}]_{i_1 \cdots i_{r-1} i i_{r+1} \cdots i_n, w, \boldsymbol{j}}, \\ \mathsf{s}_r \cdot m_{\boldsymbol{i},w,\boldsymbol{j}}^{\boldsymbol{t}} &= [s_r(m^{\boldsymbol{t}})]_{s_r \boldsymbol{i}, s_r w, \boldsymbol{j}} + [h_{r,\boldsymbol{i},w,\boldsymbol{j}} g_{r,t_r,t_{r+1}} s_r(m^{\boldsymbol{t},r})]_{s_r \boldsymbol{i}, w, \boldsymbol{j}}. \end{aligned}$$

If $|\Gamma_0| = 1$, we additionally define the action

$$\mathsf{c}_r \cdot m_w^{\boldsymbol{t}} = (x_r m^{\boldsymbol{t}})_w.$$

Excluding (4.80) and (4.76) for now, one may directly check that this action obeys the defining relations of $\mathsf{A}_n^{\mathrm{aff}}(\Gamma)$. Then the relation (4.80) may be checked with the aid of the following easily verified fact:

$$g_{r,t_r+1,t_{r+1}} - z_{r+1} g_{r,t_r,t_{r+1}} = z_r^{t_r} z_{r+1}^{t_r+1} = z_r g_{r,t_r,t_{r+1}} - g_{r,t_r,t_{r+1}+1}.$$

This leaves the Coxeter relations. The relation $\mathsf{s}_r\mathsf{s}_u = \mathsf{s}_u\mathsf{s}_r$, for $|r-u|>1$, may be directly checked. Note that $\mathsf{s}_r^2\mathsf{z}_b = \mathsf{z}_b\mathsf{s}_r^2$ already holds as operators on V, by the relations already proven. Now we prove that $\mathsf{s}_r^2 = 1$ on V by induction on the z-degree $d_z(\boldsymbol{t}) := t_1 + \cdots + t_n$ of $m^{\boldsymbol{t}}_{\boldsymbol{i},w,\boldsymbol{j}}$. The base case $d_z(\boldsymbol{t}) = 0$ is easy to check. Then, for $d_z(\boldsymbol{t}) > 0$, we may write $m^{\boldsymbol{t}}_{\boldsymbol{i},w,\boldsymbol{j}} = (z_b\hat{m}^{\boldsymbol{t}})_{\boldsymbol{i},w,\boldsymbol{j}}$ for some $1 \leq b \leq n$. Then by the induction assumption,

$$\mathsf{s}_r^2 \cdot m^{\boldsymbol{t}}_{\boldsymbol{i},w,\boldsymbol{j}} = \mathsf{s}_r^2 \cdot (z_b\hat{m}^{\boldsymbol{t}})_{\boldsymbol{i},w,\boldsymbol{j}} = \mathsf{s}_r^2\mathsf{z}_b \cdot \hat{m}^{\boldsymbol{t}}_{\boldsymbol{i},w,\boldsymbol{j}} = \mathsf{z}_b\mathsf{s}_r^2 \cdot \hat{m}^{\boldsymbol{t}}_{\boldsymbol{i},w,\boldsymbol{j}} = \mathsf{z}_b \cdot \hat{m}^{\boldsymbol{t}}_{\boldsymbol{i},w,\boldsymbol{j}} = m^{\boldsymbol{t}}_{\boldsymbol{i},w,\boldsymbol{j}}.$$

The braid relation follows in a similar fashion, after noting that

$$(\mathsf{s}_r\mathsf{s}_{r+1}\mathsf{s}_r - \mathsf{s}_{r+1}\mathsf{s}_r\mathsf{s}_{r+1})\mathsf{z}_b = \mathsf{z}_{s_r s_{r+1} s_r b}(\mathsf{s}_r\mathsf{s}_{r+1}\mathsf{s}_r - \mathsf{s}_{r+1}\mathsf{s}_r\mathsf{s}_{r+1})$$

as operators on V by the relations already proven. Thus V is an $\mathsf{A}_n^{\mathrm{aff}}(\Gamma)$-module.

The elements in (4.85) act as linearly independent operators:

$$\mathsf{z}^{\boldsymbol{t}}\mathsf{c}^{\boldsymbol{u}}\mathsf{a}^{\boldsymbol{i},w\boldsymbol{j}}\mathsf{s}_w\mathsf{e}_{\boldsymbol{j}} \cdot 1_{\boldsymbol{k},\mathrm{id},\boldsymbol{k}} = \delta_{\boldsymbol{j},\boldsymbol{k}}(z_1^{t_1}\cdots z_n^{t_n}x_1^{u_1}\cdots x_n^{u_n})_{\boldsymbol{i},w,\boldsymbol{j}},$$

which proves the theorem. □

Corollary 4.86. We have

$$\dim_q \mathsf{A}_n^{\mathrm{aff}}(\Gamma) = n!\left(\frac{(1+q^2)|\Gamma_0| + 2q|\Gamma_1|}{1-q^2}\right)^n.$$

The affine zigzag algebra $\mathsf{A}_n^{\mathrm{aff}}(\Gamma)$ possesses an anti-involution τ which sends $\mathsf{a}_r^{i,j} \mapsto \mathsf{a}_r^{j,i}$, and is the identity on the other generators. By applying τ to Theorem 4.84 it follows that $\mathsf{A}_n^{\text{aff}}$ has bases similar to (4.85) with generators written in the opposite order. Thus we have

Corollary 4.87. $\mathsf{A}_n^{\text{aff}}(\Gamma)$ is free as both a left and right $k[\mathsf{z}_1, \ldots, \mathsf{z}_n]$-module.

4.54. Center of the affine zigzag algebra

It will be convenient to define some additional elements of $\mathsf{A}_n^{\text{aff}}(\Gamma)$. Write $\mathsf{c} := \sum_{t=1}^n \mathsf{c}_n$. For $1 \leq r < t \leq n$, set $\mathsf{x}_{t,r} := -\mathsf{c}_t - \mathsf{c}_r$ if $|\Gamma_0| = 1$, and if $|\Gamma_0| > 1$, set for any $\boldsymbol{i} \in \Gamma_0^n$:

$$\mathsf{x}_{t,r}\mathsf{e}_{\boldsymbol{i}} := \begin{cases} -(\mathsf{c}_t^{i_t} + \mathsf{c}_r^{i_r})\mathsf{e}_{\boldsymbol{i}} & i_t = i_r; \\ -\mathsf{a}_t^{i_r,i_t}\mathsf{a}_r^{i_t,i_r}\mathsf{e}_{\boldsymbol{i}} & i_t \text{---} i_r; \\ 0 & \text{otherwise,} \end{cases}$$

and then set $\mathsf{x}_{t,r} = \sum_{\boldsymbol{i} \in \Gamma_0^n} \mathsf{x}_{t,r}\mathsf{e}_{\boldsymbol{i}}$.

The following lemma follows by inductively applying relations (4.80) or (4.83).

Lemma 4.88. We have $\mathsf{s}_r\mathsf{z}_r^t\mathsf{z}_{r+1}^t = \mathsf{z}_r^t\mathsf{z}_{r+1}^t\mathsf{s}_r$ for all $t \in \mathbb{Z}_{\geq 0}$, and

$$\mathsf{s}_r\mathsf{z}_r^{t_r}\mathsf{z}_{r+1}^{t_{r+1}} - \mathsf{z}_r^{t_{r+1}}\mathsf{z}_{r+1}^{t_r}\mathsf{s}_r = \frac{t_{r+1}-t_r}{|t_{r+1}-t_r|}\mathsf{x}_{r+1,r}(\mathsf{z}_r\mathsf{z}_{r+1})^{\min(t_r,t_{r+1})} \sum_{\kappa=0}^{|t_{r+1}-t_r|-1} \mathsf{z}_r^{\kappa}\mathsf{z}_{r+1}^{|t_{r+1}-t_r|-\kappa-1},$$

for all $t_r \neq t_{r+1} \in \mathbb{Z}_{\geq 0}$.

Let C be the commutative subalgebra of $\mathsf{A}_n^{\text{aff}}(\Gamma)$ generated by all z's, c's and e's. For $\mathsf{x} \in \mathsf{C}$ and $1 \leq t \leq n$, let x_t be the unique element of the subalgebra generated by all z's, c's and e's excluding c_t, such that $\mathsf{x} - \mathsf{x}_t \in \mathsf{c}_t\mathsf{C}$.

Lemma 4.89. The center $Z(\mathsf{A}_n^{\text{aff}})$ of the affine zigzag algebra consists of all elements $\mathsf{x} \in \mathsf{C}$ such that

(i) x is invariant under the diagonal action of $\mathfrak{S}_n$ on

$$k[\mathsf{z}_1, \ldots, \mathsf{z}_n] \otimes k[\mathsf{c}_1, \ldots, \mathsf{c}_n] \otimes \left(\bigoplus_{\boldsymbol{i} \in \Gamma_0^n} k\mathsf{e}_{\boldsymbol{i}} \right).$$

(ii) For all $1 \leq t \leq n$ and every bijection $b : \Gamma_0 \to \Gamma_0$, x_t is invariant under the action $\mathsf{e}_{\boldsymbol{i}} \mapsto \mathsf{e}_{i_1, \ldots, i_{t-1}, b(i_t), i_{t+1}, \ldots, i_n}$ on $\bigoplus_{\boldsymbol{i} \in \Gamma_0^n} k\mathsf{e}_{\boldsymbol{i}}$.

Proof. First we show that $Z(\mathsf{A}_n^{\mathrm{aff}}) \subseteq \mathsf{C}$. Write X for the basis (4.85). For $\omega \in \mathfrak{S}_n$, write $X \supseteq X_\omega = \{\mathsf{z}^{\boldsymbol{t}} \mathsf{c}^{\boldsymbol{u}} \mathsf{a}^{\boldsymbol{i}, \omega \boldsymbol{j}} \mathsf{s}_\omega \mathsf{e}_{\boldsymbol{i}}\}$, over all admissible $\boldsymbol{t}, \boldsymbol{u}, \boldsymbol{i}, \boldsymbol{j}$. One may easily show by induction on the Bruhat order that if $\mathsf{x}_\omega \in kX_\omega$, then $(\mathsf{x}_\omega \mathsf{z}_r - \mathsf{z}_{\omega(r)} \mathsf{x}_\omega) \in k(\bigcup_{\omega' \prec \omega} X_{\omega'})$ for all $1 \leq r \leq n$ and $\omega \in \mathfrak{S}_n$.

Let $0 \neq \mathsf{x} \in Z(\mathsf{A}_n^{\mathrm{aff}})$. Then x may be uniquely written as $\sum_{\omega \in \mathfrak{S}_n} \mathsf{x}_\omega$, where each $\mathsf{x}_\omega \in kX_\omega$. Assume that $\sigma \in \mathfrak{S}_n$ is such that $\mathsf{x}_\sigma \neq 0$ and $\mathsf{x}_\omega = 0$ for all $\omega \in \mathfrak{S}_n$ with $\ell(\omega) > \ell(\sigma)$. Then for every $1 \leq r \leq n$ we have

$$\begin{aligned} 0 = (\mathsf{z}_r \mathsf{x} - \mathsf{x} \mathsf{z}_r) &= \sum_{\ell(\omega) = \ell(\sigma)} (\mathsf{z}_r \mathsf{x}_\omega - \mathsf{x}_\omega \mathsf{z}_r) + \sum_{\ell(\omega) < \ell(\sigma)} (\mathsf{z}_r \mathsf{x}_\omega - \mathsf{x}_\omega \mathsf{z}_r) \\ &= \sum_{\ell(\omega) = \ell(\sigma)} (\mathsf{z}_r - \mathsf{z}_{\omega(r)}) \mathsf{x}_\omega + \mathsf{y}, \end{aligned}$$

where $\mathsf{y} \in k(\bigcup_{\ell(\omega) < \ell(\sigma)} X_\omega)$, after applying the claim in the first paragraph. Since each $(\mathsf{z}_r - \mathsf{z}_{\omega(r)}) \mathsf{x}_\omega \in kX_\omega$, the basis theorem implies in particular that $(\mathsf{z}_r - \mathsf{z}_{\sigma(r)}) \mathsf{x}_\sigma = 0$, so by Corollary 4.87 we have $\sigma(r) = r$ for every r. Thus $\sigma = \mathrm{id}$, and $\mathsf{x} \in kX_{\mathrm{id}}$. Moreover, since centrality implies that $\mathsf{e}_{\boldsymbol{i}} \mathsf{x} \mathsf{e}_{\boldsymbol{j}} = 0$ for $\boldsymbol{i} \neq \boldsymbol{j}$, no a's may appear in x; we have that x is a k-linear combination of basis elements of the form $\mathsf{z}^{\boldsymbol{t}} \mathsf{c}^{\boldsymbol{u}} \mathsf{e}_{\boldsymbol{i}}$. Hence $\mathsf{x} \in \mathsf{C}$.

Now we show that $\mathsf{x} \in \mathsf{C}$ commutes with every $\mathsf{a}_r^{i,j}$ if and only if x satisfies (ii). Use the basis theorem to write $\mathsf{x} = \sum_{\boldsymbol{t},\boldsymbol{u},\boldsymbol{i}} \lambda_{\boldsymbol{t},\boldsymbol{u},\boldsymbol{i}} \mathsf{z}^{\boldsymbol{t}} \mathsf{c}^{\boldsymbol{u}} \mathsf{e}_{\boldsymbol{i}}$ for some $\lambda_{\boldsymbol{t},\boldsymbol{u},\boldsymbol{i}} \in k$. Then

$$\mathsf{x}\mathsf{a}_r^{i,j} - \mathsf{a}_r^{i,j}\mathsf{x} = \sum_{\substack{\boldsymbol{t},\boldsymbol{u},\boldsymbol{i} \\ u_r=0, i_r=j}} (\lambda_{\boldsymbol{t},\boldsymbol{u},\boldsymbol{i}} - \lambda_{\boldsymbol{t},\boldsymbol{u},i_1\cdots i_{r-1} i i_{r+1}\cdots i_n}) \mathsf{z}^{\boldsymbol{t}} \mathsf{c}^{\boldsymbol{u}} \mathsf{a}_r^{i,j} \mathsf{e}_{\boldsymbol{i}}.$$

Thus x commutes with every $\mathsf{a}_r^{i,j}$ if and only if $\lambda_{\boldsymbol{t},\boldsymbol{u},\boldsymbol{i}} = \lambda_{\boldsymbol{t},\boldsymbol{u},\boldsymbol{j}}$ whenever $u_r = 0$ and $i_t = j_t$ for all $t \neq r$. Since $\mathsf{x}_r = \sum_{\boldsymbol{t},\boldsymbol{u},\boldsymbol{i},u_r=0} \lambda_{\boldsymbol{t},\boldsymbol{u},\boldsymbol{i}} \mathsf{z}^{\boldsymbol{t}} \mathsf{c}^{\boldsymbol{u}} \mathsf{e}_{\boldsymbol{i}}$, the claim follows.

Finally, we show that $\mathsf{x} \in \mathsf{C}$ commutes with every s_r if and only if x satisfies (i). The 'if' direction follows easily from the defining relations and Lemma 4.88, so we will prove the 'only if' direction. For $\mathsf{x} \in \mathsf{A}_n^{\mathrm{aff}}(\Gamma)$, we say x *involves* $\mathsf{z}^{\boldsymbol{t}} \mathsf{c}^{\boldsymbol{u}} \mathsf{a}^{\boldsymbol{i},w\boldsymbol{j}} \mathsf{s}_w \mathsf{e}_{\boldsymbol{j}}$ if the basis expansion of x includes this term with a nonzero coefficient. Then for a tuple $\boldsymbol{t} \in \mathbb{Z}_{\geq 0}^n$, define $d(\boldsymbol{t}) = t_1 + \cdots + t_n$, and set

$$d(\mathsf{x}) := \max\{d(\boldsymbol{t}) \mid \mathsf{x} \text{ involves } \mathsf{z}^{\boldsymbol{t}} \mathsf{c}^{\boldsymbol{u}} \mathsf{a}^{\boldsymbol{i},w\boldsymbol{j}} \mathsf{s}_w \mathsf{e}_{\boldsymbol{j}} \text{ for some } \boldsymbol{u}, \boldsymbol{i}, w, \boldsymbol{j}\}.$$

Now assume $\mathsf{x} \in \mathsf{C}$ is central. We may use the basis theorem to write $\mathsf{x} = \mathsf{x}' + \mathsf{x}''$, where $\mathsf{x}' = \sum_{\boldsymbol{t},\boldsymbol{u},\boldsymbol{i},d(\boldsymbol{t})=d(\mathsf{x})} \lambda_{\boldsymbol{t},\boldsymbol{u},\boldsymbol{i}} \mathsf{z}^{\boldsymbol{t}} \mathsf{c}^{\boldsymbol{u}} \mathsf{e}_{\boldsymbol{i}}$, and $d(\mathsf{x}'') < d(\mathsf{x})$. For all $1 \leq r \leq n$ it follows from relations and Lemma 4.88 that $d(\mathsf{s}_r \mathsf{x}'' \mathsf{s}_r - \mathsf{x}'') < d(\mathsf{x})$, and

$$\mathsf{s}_r \mathsf{x}' \mathsf{s}_r - \mathsf{x}' = \sum_{\substack{\boldsymbol{t},\boldsymbol{u},\boldsymbol{i} \\ d(\boldsymbol{t})=d(\mathsf{x})}} (\lambda_{s_r\boldsymbol{t},s_r\boldsymbol{u},s_r\boldsymbol{i}} - \lambda_{\boldsymbol{t},\boldsymbol{u},\boldsymbol{i}}) \mathsf{z}^{\boldsymbol{t}} \mathsf{c}^{\boldsymbol{u}} \mathsf{e}_{\boldsymbol{i}} + \mathsf{y}$$

where $d(\mathsf{y}) < d(\mathsf{x})$. Thus, by the basis theorem and centrality of x, it follows that $\lambda_{s_r\boldsymbol{t},s_r\boldsymbol{u},s_r\boldsymbol{i}} = \lambda_{\boldsymbol{t},\boldsymbol{u},\boldsymbol{i}}$ for all $d(\boldsymbol{t}) = d(\mathsf{x})$. But then, by Lemma 4.88, we have that $\mathsf{y} = 0$, and therefore x' commutes with every s_r and satisfies (i). Thus $\mathsf{x}'' = \mathsf{x} - \mathsf{x}'$ commutes with every s_r, so by induction, x'' also satisfies (i), completing the proof. $\square$

Corollary 4.90. The algebra $\mathsf{A}_n^{\mathrm{aff}}(\Gamma)$ is indecomposable.

Proof. By Lemma 4.89, the only primitive central idempotent is 1, so the result follows. □

4.55. Cyclotomic zigzag algebras

By the basis theorem, the subalgebra of $\mathsf{A}_n^{\mathrm{aff}}(\Gamma)$ generated by all $\mathsf{e}_{\boldsymbol{i}}$'s and $\mathsf{a}_t^{i,j}$'s is naturally isomorphic to $\mathsf{A}(\Gamma)^{\otimes n}$ (if $|\Gamma_0| = 1$, interpret this as the subalgebra generated by c_t's). Moreover, the subalgebra of $\mathsf{A}_n^{\mathrm{aff}}(\Gamma)$ generated by all s_t's is isomorphic to $k\mathfrak{S}_n$. Together they generate the subalgebra isomorphic to the semidirect tensor product

$$\mathsf{A}_n(\Gamma) := \mathsf{A}(\Gamma)^{\otimes n} \otimes k\mathfrak{S}_n,$$

with the action of $\mathfrak{S}_n$ on the tensors in $\mathsf{A}(\Gamma)^{\otimes n}$ by place permutation. We refer to this algebra as the *finite zigzag algebra of rank n*. It makes an appearance in [45].

Define the *Murphy elements* in $\mathsf{A}_n(\Gamma)$ as follows:

$$\mathsf{l}_r := \sum_{t=1}^{r-1} \mathsf{x}_{t,r} \otimes (t, r) \qquad (r = 1, \dots, n)$$

where $(t, r) \in \mathfrak{S}_n$ is the transposition of t and r.

Lemma 4.91. For each $\boldsymbol{i} \in \Gamma_0^n$, let $\kappa_{\boldsymbol{i}} \in k$. Let $f : \mathsf{A}_n^{\mathrm{aff}}(\Gamma) \to \mathsf{A}_n(\Gamma)$ be the map which sends

$$\mathsf{z}_r \mapsto \mathsf{l}_r + \sum_{\boldsymbol{i} \in \Gamma_0^n} \kappa_{s_{r-1} \cdots s_2 s_1 \boldsymbol{i}} \mathsf{c}\mathsf{e}_{\boldsymbol{i}},$$

and is the identity on all other generators. Then f is a surjective homomorphism of graded k-algebras. The kernel of this homomorphism is the 2-sided ideal generated by $\mathsf{z}_1 - \sum_{\boldsymbol{i} \in \Gamma_0^n} \kappa_{\boldsymbol{i}} \mathsf{c} \mathsf{e}_{\boldsymbol{i}}$.

Proof. That f is a surjective homomorphism is easily checked. It is also easy to see from the defining relations that $\mathsf{A}_n^{\mathrm{aff}}$ is in fact generated by $\mathsf{z}' := \mathsf{z}_1 - \sum_{\boldsymbol{i} \in \Gamma_0^n} \kappa_{\boldsymbol{i}} \mathsf{c} \mathsf{e}_{\boldsymbol{i}}$, together with all the $\mathsf{e}_{\boldsymbol{i}}$'s, s_t's and $\mathsf{a}_t^{i,j}$'s (or c_t's in the case $|\Gamma_0| = 1$). Then $\mathsf{A}_n^{\mathrm{aff}}/\mathsf{A}_n^{\mathrm{aff}}\mathsf{z}'\mathsf{A}_n^{\mathrm{aff}}$ is generated by the images of these generators (excluding z'), and since $\mathsf{z}' \in \ker f$, the homomorphism f factors through to a surjection $\overline{f} : \mathsf{A}_n^{\mathrm{aff}}/\mathsf{A}_n^{\mathrm{aff}}\mathsf{z}'\mathsf{A}_n^{\mathrm{aff}} \to \mathsf{A}_n$. There also exists a surjection g in the other direction which sends the generators of A_n to their images in $\mathsf{A}_n^{\mathrm{aff}}/\mathsf{A}_n^{\mathrm{aff}}\mathsf{z}'\mathsf{A}_n^{\mathrm{aff}}$. Then $\overline{f}$ and g are mutual inverses, and the second statement follows. $\square$

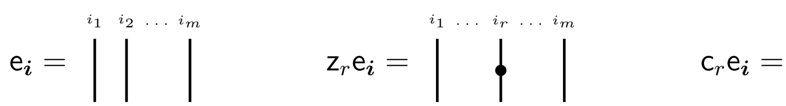
Let $m \in \mathbb{Z}_{>0}$, and let $\boldsymbol{\kappa} : \Gamma_0^n \to k^m$ be any function. We define the *cyclotomic zigzag algebra* $\mathsf{A}_n^{\boldsymbol{\kappa}}(\Gamma)$ to be the quotient of $\mathsf{A}_n^{\mathrm{aff}}(\Gamma)$ by the 2-sided ideal generated by the element

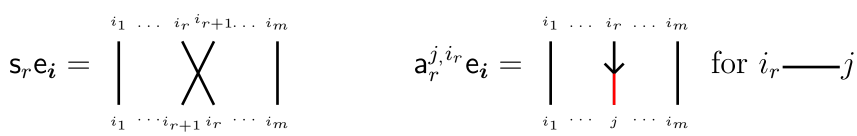
$$\sum_{\boldsymbol{i} \in \Gamma_0^n} \prod_{j=1}^{m} (\mathsf{z}_1 - \kappa(\boldsymbol{i})_j \mathsf{c}) \mathsf{e}_{\boldsymbol{i}}.$$

Since c is central and nilpotent it follows that z_1 is nilpotent in $\mathsf{A}_n^{\boldsymbol{\kappa}}(\Gamma)$, and thus—inductively applying the relation (4.80)—every z_i is nilpotent. Thus $\mathsf{A}_n^{\boldsymbol{\kappa}}(\Gamma)$ is finitely generated as a k-module by Theorem 4.84. By Lemma 4.91 we have $\mathsf{A}_n^{\boldsymbol{\kappa}}(\Gamma) \cong \mathsf{A}_n$ when $m = 1$.

4.56. Diagrammatics for the affine zigzag algebra

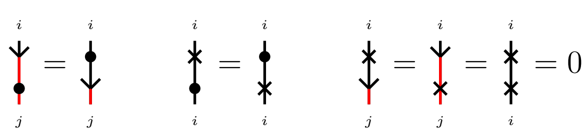
For reader convenience, we provide a diagrammatic description of the algebra $\mathsf{A}_n^{\mathrm{aff}}(\Gamma)$. We depict the (idempotented) generators as the following diagrams:

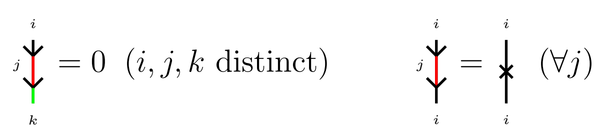


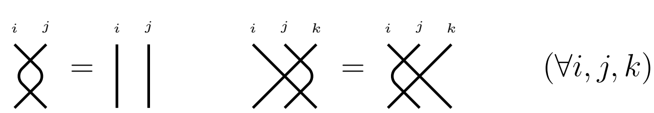


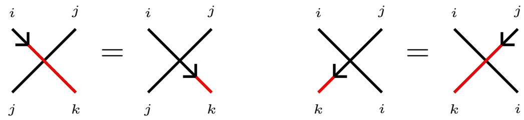
The red color is just intended to highlight that the label for the rth strand has changed. Then $\mathsf{A}_n^{\mathrm{aff}}(\Gamma)$ is spanned by planar diagrams that look locally like these generators, equivalent up to the usual isotopies (see [20]). Multiplication of diagrams is given by stacking vertically, and products are zero unless labels for strands match.

Then the defining local relations can be drawn as follows:

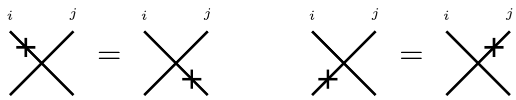




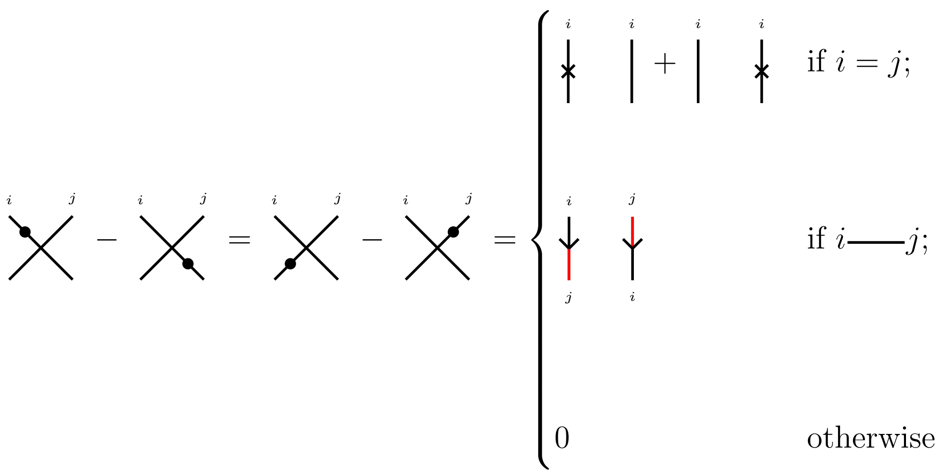




$(\forall i, j, k)$



$(\forall i, j)$



4.6. The minuscule imaginary stratum category

For the remainder of this chapter we assume $\preceq$ is a *balanced order*. We also assume that the graph Γ is the Dynkin diagram corresponding to the finite type Cartan matrix C', and write A for $\mathsf{A}(\Gamma)$, $\mathsf{A}_n^{\mathrm{aff}}$ for $\mathsf{A}_n^{\mathrm{aff}}(\Gamma)$, etc.

4.61. Special words

For each $i \in I'$, we choose a special word $\boldsymbol{b}^i \in I^\delta$:

$$\text{Type } \mathtt{A}_\ell^{(1)}: \quad \boldsymbol{b}^i := 012\cdots(i-1)\ell(\ell-1)(\ell-2)\cdots(i+1)i$$

$$\text{Type } \mathtt{D}_\ell^{(1)}: \quad \boldsymbol{b}^i := \begin{cases} 0234\cdots\ell(\ell-2)(\ell-3)\cdots(i+1)123\cdots i & \text{if } 1 \le i \le \ell-2; \\ 0234\cdots(\ell-2)\ell 123\cdots(\ell-1) & \text{if } i = \ell-1; \\ 0234\cdots(\ell-1)123\cdots(\ell-2)\ell & \text{if } i = \ell \end{cases}$$

$$\text{Type } \mathtt{E}_6^{(1)}: \quad \boldsymbol{b}^i := \begin{cases} 024354265431 & \text{if } i = 1; \\ 024354136542 & \text{if } i = 2; \\ 024354126543 & \text{if } i = 3; \\ 024354123654 & \text{if } i = 4; \\ 024354123465 & \text{if } i = 5; \\ 024354123456 & \text{if } i = 6 \end{cases}$$

$$\text{Type } \mathtt{E}_7^{(1)}: \quad \boldsymbol{b}^i := \begin{cases} 013425463542765431 & \text{if } i = 1; \\ 013425463541376542 & \text{if } i = 2; \\ 013425463541276543 & \text{if } i = 3; \\ 013425463541237654 & \text{if } i = 4; \\ 013425463541234765 & \text{if } i = 5; \\ 013425463541234576 & \text{if } i = 6; \\ 013425463541234567 & \text{if } i = 7 \end{cases}$$

$$\text{Type } \mathtt{E}_8^{(1)}: \quad \boldsymbol{b}^i := \begin{cases} 08765423143564257643542876543 1 & \text{if } i = 1; \\ 087654231435642576435413876542 & \text{if } i = 2; \\ 087654231435642576435412876543 & \text{if } i = 3; \\ 087654231435642576435412387654 & \text{if } i = 4; \\ 087654231435642576435412348765 & \text{if } i = 5; \\ 087654231435642576435412345876 & \text{if } i = 6; \\ 087654231435642576435412345687 & \text{if } i = 7; \\ 087654231435642576435412345678 & \text{if } i = 8. \end{cases}$$

Let $d = \mathrm{ht}(\delta)$. Following [31], for $1 \leq r < d$ and $\boldsymbol{i} \in I^\delta$, we say $s_r \in \mathfrak{S}_d$ is $\boldsymbol{i}$-admissible if $(\alpha_{i_r}, \alpha_{i_{r+1}}) = 0$. More generally, $s_{r_1} \cdots s_{r_t}$ is a reduced expression for $w \in \mathfrak{S}_d$ and each s_{r_k} is $(s_{r_{k+1}} \cdots s_{r_t}\boldsymbol{i})$-admissible, then we say w is $\boldsymbol{i}$-admissible. This property is independent of reduced expression for w. In addition, admissibility is preserved by products in the sense that if w is $\boldsymbol{i}$-admissible and w' is $(w\boldsymbol{i})$-admissible, then $w'w$ is $\boldsymbol{i}$-admissible. The *connected component* of $\boldsymbol{i}$ is $\mathrm{Con}(\boldsymbol{i}) := \{w\boldsymbol{i} \mid \boldsymbol{i}\text{-admissible } w \in \mathfrak{S}_d\}$. Clearly $\mathrm{Con}(\boldsymbol{i}) = \mathrm{Con}(\boldsymbol{j})$ if and only if $\boldsymbol{i} \in \mathrm{Con}(\boldsymbol{j})$. We will write $G^i := \mathrm{Con}(\boldsymbol{b}^i)$ and $G^\delta := \bigcup_{i \in I'} G^i$.

Definition 4.92. Let $\boldsymbol{i} \in I^\delta$. For $t \in \{1, \ldots, d\}$, define the *t-neighbor sequence* of $\boldsymbol{i}$ to be $\mathrm{nbr}_t(\boldsymbol{i}) := (n_1, \ldots, n_t) \in \{0, N, S\}^t$, where

$$n_r = \begin{cases} S, & \text{if } i_r = i_t; \\ N, & \text{if } (\alpha_{i_r}, \alpha_{i_t}) < 0; \\ 0, & \text{otherwise.} \end{cases}$$

Then $\underline{\mathrm{nbr}}_t(\boldsymbol{i})$, the *reduced t-neighbor sequence of $\boldsymbol{i}$*, is achieved by deleting all 0's from $\mathrm{nbr}_t(\boldsymbol{i})$.

Example 4.93. Take $\mathtt{C} = \mathtt{A}_7^{(1)}$. Then $\boldsymbol{i} = 01726354 \in G^4$, $\mathrm{nbr}_6(\boldsymbol{i}) = 000N0S$, and $\underline{\mathrm{nbr}}_6(\boldsymbol{i}) = NS$.

The following is clear:

Lemma 4.94. If s_r is $\boldsymbol{i}$-admissible, then $\underline{\mathrm{nbr}}_{s_r(t)}(s_r\boldsymbol{i}) = \underline{\mathrm{nbr}}_t(\boldsymbol{i})$.

Lemma 4.95. Let $i, j \in I'$ such that $(\alpha_i, \alpha_j) = -1$.

(i) If $\boldsymbol{i} \in G^\delta$, then $\boldsymbol{i}$ satisfies the homogeneity condition: if $i_r = i_s$ for some $r < s$, then there exist t, u with $r < t < u < s$ such that $(\alpha_{i_r}, \alpha_{i_t}) = (\alpha_{i_r}, \alpha_{i_u}) = -1$.

(ii) For all $\boldsymbol{i} \in G^i$, we have $i_1 = 0$, $i_d = i$, i_1 is a neighbor of i_2, and i_{d-1} is a neighbor of i_d.

(iii) If $\mathtt{C} \neq \mathtt{A}_1^{(1)}$ and $\boldsymbol{i} \in G^\delta$, then

$$\underline{\mathrm{nbr}}_t(\boldsymbol{i}) = \begin{cases} (NSN)^a NS, & \text{if } 1 < t < d; \\ (NSN)^a NNS, & \text{if } t = d, \end{cases} \tag{4.96}$$

for some $a \geq 0$.

(iv) If $\boldsymbol{i} \in G^\delta$ and $r < d - 1$, then $s_r\boldsymbol{i} \in G^\delta$ if and only if s_r is $\boldsymbol{i}$-admissible.

(v) There exists some $w_{i,j} \in \mathfrak{S}_d$ such that $w_{i,j}\boldsymbol{b}^j = \boldsymbol{b}^i$, and $w_{i,j} = w_1 s_{d-1} w_2$, where w_2 is $\boldsymbol{b}^j$-admissible and w_1 is $s_{d-1}w_2\boldsymbol{b}^j$-admissible.

(vi) For any $\boldsymbol{i}, \boldsymbol{i}' \in G^i$, there exists a unique $w_{\boldsymbol{i}',\boldsymbol{i}} \in \mathfrak{S}_d$ such that $w_{\boldsymbol{i}',\boldsymbol{i}}\boldsymbol{i} = \boldsymbol{i}'$ and $w_{\boldsymbol{i}',\boldsymbol{i}}$ is $\boldsymbol{i}$-admissible.

(vii) For any $\boldsymbol{i} \in G^i$ and $\boldsymbol{j} \in G^j$, there exists a unique $w_{\boldsymbol{i},\boldsymbol{j}} \in \mathfrak{S}_d$ such that $w_{\boldsymbol{i},\boldsymbol{j}}\boldsymbol{j} = \boldsymbol{i}$, and $w_{\boldsymbol{i},\boldsymbol{j}} = w_1 s_{d-1} w_2$, where w_2 is $\boldsymbol{i}$-admissible and w_1 is $s_{d-1}w_2\boldsymbol{i}$-admissible.

Proof. (i) It is straightforward to check that $\boldsymbol{b}^i$ satisfies the homogeneity condition. Thus by [31, Lemma 3.3], every $\boldsymbol{i} \in G^i$ satisfies this condition.

(ii) If $1 < r < d$, then $(\boldsymbol{b}^i)_r$ has a neighbor somewhere to the left and right in $\boldsymbol{b}^i$, so no $\boldsymbol{b}^i$-admissible element w may send r to 1 or d, so $i_1 = (\boldsymbol{b}^i)_1 = 0$, and $i_d = (\boldsymbol{b}^i)_d = i$ for every $\boldsymbol{i} \in G^j$. Moreover it cannot be that $i_{d-1} = i_d$ by (i), and if it were the case that $(\alpha_{i_d}, \alpha_{i_{d-1}}) = 0$, then we would have $s_{d-1}\boldsymbol{i} \in G^i$, but $(s_{d-1}\boldsymbol{i})_d \neq i$, a contradiction. Thus i_{d-1} and i_d are neighbors, and a similar argument proves the same for i_1 and i_2.

(iii) We have by part (ii) that s_1 and s_{d-1} are never admissible transpositions for $\boldsymbol{i} \in G^\delta$. Therefore, by Lemma 4.94, it is enough to check that that statement (iii) holds for the special words $\boldsymbol{b}^i$, which may be readily done.

(iv) The statement holds for $r = 1$ by part (ii), since s_1 is never $\boldsymbol{i}$-admissible, and $i_1 = 0$ for every $\boldsymbol{i} \in G^\delta$. Let $1 < r < d-1$. If s_r is not $\boldsymbol{i}$-admissible, then $(\alpha_{i_r}, \alpha_{i_{r+1}}) = -1$ by (i). By part (iii), $\underline{\mathrm{nbr}}_{r+1}(\boldsymbol{i}) = (NSN)^a NS$ for some $a \geq 0$. Then $\underline{\mathrm{nbr}}_r(s_r\boldsymbol{i}) = (NSN)^a S$. But then again by part (iii), $s_r\boldsymbol{i} \notin G^\delta$.

(v) In type $\mathtt{A}_\ell^{(1)}$, for $1 \leq i < \ell$ we may take $w_{i,i+1} = s_{d-1} \cdots s_{i+1}$, and in type $\mathtt{D}_\ell^{(1)}$, take $w_{i,i+1} = s_{d-1} \cdots s_{d-i}$ if $1 \leq i \leq \ell - 3$, $w_{\ell-2,\ell-1} = s_{d-1} \cdots s_{d-\ell-1}$, and $w_{\ell-2,\ell} = s_{d-1} \cdots s_{d-\ell-2}$. In both types, if $j > i$, we may simply take $w_{j,i}$ to be the inverse of $w_{i,j}$ defined above. This leaves the exceptional types $\mathtt{E}_\ell^{(1)}$, where the existence of such elements can be easily verified.

(vi) The existence of $w_{\boldsymbol{i}',\boldsymbol{i}}$ is guaranteed by the definition of $G^i = \mathrm{Con}(\boldsymbol{i})$, and we note that $\boldsymbol{i}$-admissible elements are in bijection with G^i since $\boldsymbol{i}$-admissible elements cannot transpose similar letters. This proves uniqueness.

(vii) We may take $w_{\boldsymbol{i},\boldsymbol{j}} = w_{\boldsymbol{i},\boldsymbol{b}^i} w_{i,j} w_{\boldsymbol{b}^j,\boldsymbol{j}}$ to show existence. Uniqueness follows as in part (vi) from consideration of the fact that no similar letters are transposed in this product. $\square$

4.62. Minuscule semicuspidal modules

Recall from §4.21 that, when k is a field, the irreducible semicuspidal R_δ-modules may be canonically labeled $L_{\delta,i}$, for $i \in I'$.

Lemma 4.97. [24, Lemma 5.1, Corollary 5.3] Let k be a field. For each $i \in I'$, $L_{\delta,i}$ can be characterized up to isomorphism and grading shift as the unique irreducible R_δ-module such that $i_1 = 0$ and $i_d = i$ for all words $\boldsymbol{i}$ of $L_{\delta,i}$.

Lemma 4.98. For each $i \in I'$, $\mathrm{ch}_q L_{\delta,i} = \sum_{\boldsymbol{i} \in G^i} \boldsymbol{i}$.

Proof. By Lemmas 4.95(i) and [31, Theorem 3.4], there exists a homogeneous irreducible R_δ-module with character $\sum_{\boldsymbol{i} \in G^i} \boldsymbol{i}$. By Lemmas 4.95(ii) and 4.97, this module must be $L_{\delta,i}$. $\square$

Therefore G^δ is a complete set of semicuspidal words in I^δ, and we have $C_\delta = R_\delta / R_\delta 1_{\mathrm{nsc}} R_\delta$, where $1_{\mathrm{nsc}} = \sum_{\boldsymbol{i} \in I^\delta \setminus G^\delta} 1_{\boldsymbol{i}}$.

Lemma 4.99. Assume $\mathtt{C} \neq \mathtt{A}_1^{(1)}$. For all $\boldsymbol{i} \in G^\delta$, $s_{d-1}\boldsymbol{i} \in G^{i_{d-1}}$.

Proof. By Lemma 4.95(ii), i_{d-1}, i_d are neighbors, and by Lemma 4.95(iii) $\boldsymbol{i}$ satisfies (4.96), so it follows that $s_{d-1}\boldsymbol{i}$ also satisfies (4.96). Moreover $s_{d-1}\boldsymbol{i}$ also satisfies the homogeneity condition in Lemma 4.95(i); if it did not there would be some t such that $\underline{\mathrm{nbr}}_t(s_{d-1}\boldsymbol{i})$ contains a subword SS or SNS, which is not the case. For every $1 < r < d-1$, i_r has a neighbor to the left and right in $\boldsymbol{i}$, so the same is true in $s_{d-1}\boldsymbol{i}$, and consideration of $\underline{\mathrm{nbr}}_d(\boldsymbol{i})$ shows that $(s_{d-1}\boldsymbol{i})_{d-1}$ also has neighbors to the left and right in $s_{d-1}\boldsymbol{i}$. Thus $j_1 = 0$ and $j_d = (s_{d-1}\boldsymbol{i})_d = i_{d-1}$ for every $\boldsymbol{j} \in \mathrm{Con}(s_{d-1}\boldsymbol{i})$.

Then there is an irreducible homogeneous module with character $\sum_{\boldsymbol{j}\in\mathrm{Con}(s_{d-1}\boldsymbol{i})}\boldsymbol{j}$, which by Lemma 4.97 is isomorphic to $L_{\delta,i_{d-1}}$. But by Lemma 4.98, we must have $\mathrm{Con}(s_{d-1}\boldsymbol{i}) = G^{i_{d-1}}$. $\square$

4.63. A spanning set for C_δ

For each $w \in \mathfrak{S}_d$, choose a distinguished reduced expression $w = s_{r_1}\cdots s_{r_t}$, and define $\psi_w = \psi_{r_1}\cdots\psi_{r_t} \in R_\delta$. In general, this element will depend on the choice of reduced expression, but as we will see, this is not the case in C_δ. We will write $\psi_{\boldsymbol{i},\boldsymbol{j}}$ (resp. $\psi_{i,j}$) for the element $\psi_{w_{\boldsymbol{i},\boldsymbol{j}}}$ (resp. $\psi_{w_{i,j}}$) defined in Lemma 4.95.

Lemma 4.100.

(i) The algebra C_δ is non-negatively graded.

(ii) The elements ψ_w are independent of reduced expression for w in C_δ.

(iii) In C_δ, $\psi_r y_t = y_{s_r(t)}\psi_r$, for all r, t.

Proof. All of these follow from Lemma 4.95(i). We have $1_{\boldsymbol{i}} = 0$ in C_δ if $i_r = i_{r+1}$ for some $1 \le r < d$. So there are no generators $\psi_r 1_{\boldsymbol{j}}$ in negative degrees, hence (i). Part (iii) also follows from that observation. Finally, semicuspidal words have no subwords of the form iji, so braid relations hold on the nose, hence (ii). $\square$

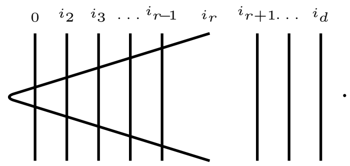
Lemma 4.101. The following facts hold in C_δ:

(i) $y_1 = \cdots = y_{d-1}$.

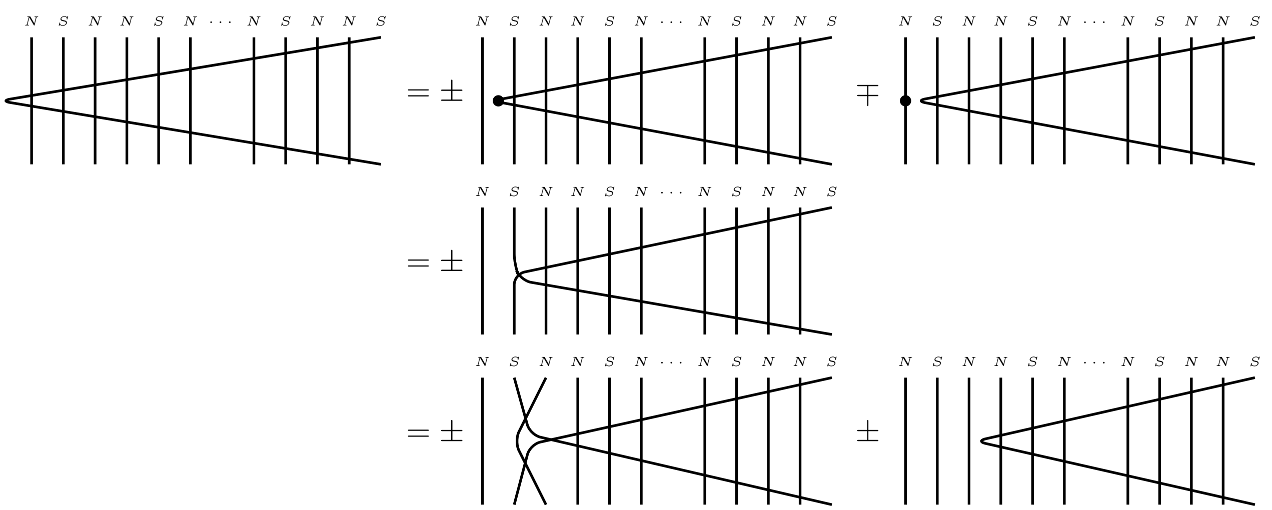
(ii) $(y_1 - y_d)^2 = 0$.

Proof. For this proof, it will be convenient to use the diagrammatic presentation for R_δ, see [20]. For now, we assume that $\mathtt{C} \neq \mathtt{A}_1^{(1)}$. We prove (i) first. Let $\boldsymbol{i} = 0i_2i_3\cdots i_d \in$

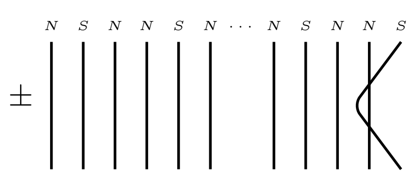
G^δ. Let $1 < r < d$. The following diagram is zero in C_δ since all semicuspidal words start with 0, and $i_r \neq 0$:



We will simplify this diagram using relations. Note that we may ignore strands to the right of i_r and strands whose colors do not neighbor i_r. Omitting such strands, and recalling from Lemma 4.95(iii) that $\underline{\mathrm{nbr}}_k(\boldsymbol{i}) = (NSN)^a NS$ for some $a \geq 0$, we have, using the relations in R_δ:

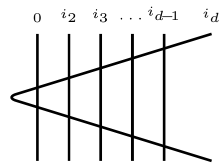


The first term in the last line involves an (S, S)-crossing and hence is zero in C_δ. We may continue on in this fashion, moving the S strand past NSN-triples, until we arrive at

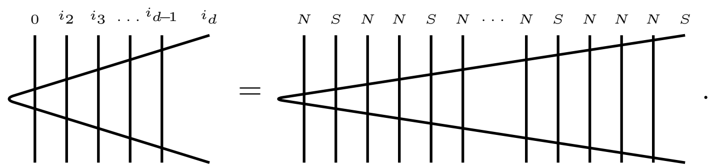


The (N,S) crossing opens, giving $\pm(y_s - y_r)1_{\boldsymbol{i}}$, for some $s < r$. Recalling that the initial diagram was zero, we have $y_s 1_{\boldsymbol{i}} = y_r 1_{\boldsymbol{i}}$. Applying induction on r, for every semicuspidal word $\boldsymbol{i}$, it follows that $y_1 = \cdots = y_{d-1}$ in C_δ.

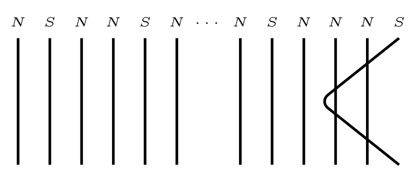
Now we prove (ii). Let $\boldsymbol{i} = 0 i_2 i_3 \cdots i_d \in G^\delta$. Again, this diagram is zero in C_δ:



As in the proof of (i), we omit non-neighbors of i_d, and use the fact that $\underline{\mathrm{nbr}}_d(\boldsymbol{i}) = (NSN)^a NNS$ from Lemma 4.95(iii) to write



We then move the S-strand past (NSN)-strands as in the first part, to arrive at



Applying the quadratic relation twice yields $\pm(y_t - y_d)(y_s - y_d)1_{\boldsymbol{i}}$, for some $t < s < d$. But then $y_t = y_s = y_1$ by (i), so we have $(y_1 - y_d)^2 1_{\boldsymbol{i}}$ for all semicuspidal words $\boldsymbol{i}$, which implies the result.

Finally, assume $\mathtt{C} = \mathtt{A}_1^{(1)}$. Then $d = 2$, $G^\delta = \{01\}$, and so claim (i) is trivial. Since $1_{10} = 0$ in C_δ, we get $0 = \psi_1 1_{10} \psi_1 = \psi_1^2 1_{01} = \pm(y_1 - y_2)^2 1_{01} = \pm(y_1 - y_2)^2$, proving claim (ii). $\square$

Lemma 4.102. Let $u \in \mathfrak{S}_d$. We have $\psi_u 1_{\boldsymbol{i}} = 0$ in C_δ unless:

(i) $\boldsymbol{i} \in G^\delta$, $u\boldsymbol{i} \in G^{i_d}$, and $u = w_{u\boldsymbol{i},\boldsymbol{i}}$, in which case $\deg(\psi_u 1_{\boldsymbol{i}}) = 0$, or;

(ii) $\boldsymbol{i} \in G^\delta$, $u\boldsymbol{i} \in G^j$ for some $j \in I'$ such that $(\alpha_j, \alpha_{i_d}) = -1$, and $u = w_{u\boldsymbol{i},\boldsymbol{i}}$, in which case $\deg(\psi_u 1_{\boldsymbol{i}}) = 1$.

Proof. Note that in type $\mathtt{A}_1^{(1)}$, we have $G^\delta = \{01\}$, so $\psi_1 1_{01} = 0$, and in this case the lemma is trivial. Thus we restrict our attention to the other cases.

Assume that $\psi_u 1_{\boldsymbol{i}} = e_{u\boldsymbol{i}} \psi_u 1_{\boldsymbol{i}} \neq 0$. Then it must be that $\boldsymbol{i}, u\boldsymbol{i} \in G^\delta$. We may write $u = w'w''$, where $w'' \in \mathfrak{S}_{d-1}$ and w' is a minimal length left coset representative of $\mathfrak{S}_{d-1}$ in $\mathfrak{S}_d$. By Lemma 4.100(ii), $\psi_u = \psi_{w'}\psi_{w''}$. By Lemma 4.95(iv), w'' must be $\boldsymbol{i}$-admissible. If $w' = \mathrm{id}$, then $w\boldsymbol{i}_d = i_d$, $\deg(\psi_w 1_{\boldsymbol{i}}) = 0$ and we are in case (i) by the uniqueness of Lemma 4.95(vi).

Assume $w' \neq \mathrm{id}$. Then for some r, $w' = s_r s_{r+1} \cdots s_{d-1}$ is a reduced expression for w'. By Lemma 4.100(ii), $\psi_u = \psi_r \psi_{r+1} \cdots \psi_{d-1} \psi_{w''}$ in C_δ. By Lemma 4.95(iv), $s_r s_{r+1} \cdots s_{d-2}$ is $s_{d-1}w''\boldsymbol{i}$-admissible. Further, $(\alpha_{i_{d-1}}, \alpha_{i_d}) = -1$ by Lemma 4.95(ii), so $\deg(\psi_u 1_{\boldsymbol{i}}) = 1$, and we are in case (ii) by the uniqueness of Lemma 4.95(vii). $\square$

Given a word $\boldsymbol{i} \in G^\delta$, define

$$W_{\boldsymbol{i}} = \{w_{\boldsymbol{j},\boldsymbol{i}} \in \mathfrak{S}_d \mid \boldsymbol{j} \in G^j \text{ for some } j \text{ such that } (\alpha_j, \alpha_{i_d}) \neq 0\}.$$

Note that by Lemma 4.95(vi) and (vii), $W_{\boldsymbol{i}}$ is in bijection with $\bigcup_{j \in I', (\alpha_j, \alpha_{i_d}) \neq 0} G^j$.

Lemma 4.103. If $\deg(\psi_w 1_{\boldsymbol{i}}) \geq 1$, then $(y_1 - y_d)\psi_w 1_{\boldsymbol{i}} = 0$ in C_δ.

Proof. By Lemma 4.102, we only need consider the case where $\boldsymbol{i} \in G^\delta$ and $w \in W_{\boldsymbol{i}}$. Since $\deg(\psi_w 1_{\boldsymbol{i}}) \geq 1$, it must be that $w\boldsymbol{i} \in G^j$, where $(\alpha_j, \alpha_{i_d}) = -1$, so $(w\boldsymbol{i})_d = j \neq i_d$ and $(w\boldsymbol{i})_1 = i_1 = 0$. Thus $w(1) = 1$ and $w(d) < d$, so by Lemma 4.100(iii), we have that

$$(y_1 - y_d)\psi_w 1_{\boldsymbol{i}} = \psi_w(y_1 - y_{w(d)})1_{\boldsymbol{i}},$$

but $y_1 - y_{w(d)} = 0$ in C_δ by Lemma 4.101(i). $\square$

Proposition 4.104. The following is a spanning set for C_δ:

$$X := \{y_1^b(y_1 - y_d)^m \psi_w 1_{\boldsymbol{i}} \mid \boldsymbol{i} \in G^\delta, w \in W_{\boldsymbol{i}}, m + \deg(\psi_w 1_{\boldsymbol{i}}) \leq 1, b \in \mathbb{Z}_{\geq 0}\}.$$

Proof. By the basis theorem [20, Theorem 2.5] or [42, Theorem 3.7], we have that

$$\{y_1^{b_1} \cdots y_{d-1}^{b_{d-1}}(y_1 - y_d)^{b_d}\psi_w 1_{\boldsymbol{i}} \mid \boldsymbol{i} \in I^\delta, w \in \mathfrak{S}_d, b_i \in \mathbb{Z}_{\geq 0}\}$$

spans R_δ. We get the spanning set X by throwing out elements of this set which are known to be zero or redundant in C_δ via Lemmas 4.101, 4.102 and 4.103. $\square$

4.64. A basis for C_δ

To prove linear independence of X, we construct a graded R_δ-module which descends to a faithful C_δ-module. Let

$$V := \left(\bigoplus_{\substack{\boldsymbol{i},\boldsymbol{j} \in G^\delta \\ i_d = j_d}} k[z,x]/(x^2) \right) \oplus \left(\bigoplus_{\substack{\boldsymbol{i},\boldsymbol{j} \in G^\delta \\ (\alpha_{i_d}, \alpha_{j_d}) = -1}} k[z,x]\langle 1 \rangle/(x) \right),$$

where z, x are indeterminates in degree 2. We will label polynomials $f(z,x)$ belonging to the $\boldsymbol{i},\boldsymbol{j}$-th summand of V with subscripts, a la $f_{\boldsymbol{i},\boldsymbol{j}}$. The seemingly extraneous indeterminate x in the second group of summands is included for convenience in describing the action on V.

Lemma 4.105. The vector space V is a graded R_δ-module, with the action of generators defined in types $\mathtt{C} \neq \mathtt{A}_1^{(1)}$ as follows:

$$1_{\boldsymbol{k}} \cdot f_{\boldsymbol{i},\boldsymbol{j}} = \delta_{\boldsymbol{k},\boldsymbol{i}} f_{\boldsymbol{i},\boldsymbol{j}}$$

$$y_r \cdot f_{\boldsymbol{i},\boldsymbol{j}} = \begin{cases} (zf)_{\boldsymbol{i},\boldsymbol{j}} & 1 \leq r \leq d-1; \\ (zf - xf)_{\boldsymbol{i},\boldsymbol{j}} & r = d \end{cases}$$

$$\psi_r \cdot f_{\boldsymbol{i},\boldsymbol{j}} = \begin{cases} f_{s_r \boldsymbol{i},\boldsymbol{j}} & s_r \text{ is } \boldsymbol{i}\text{-admissible}; \\ f_{s_{d-1}\boldsymbol{i},\boldsymbol{j}} & r = d-1 \text{ and } i_d = j_d; \\ \varepsilon_{i_d,j_d}(xf)_{s_{d-1}\boldsymbol{i},\boldsymbol{j}} & r = d-1 \text{ and } i_{d-1} = j_d; \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathtt{C} = \mathtt{A}_1^{(1)}$, the action of $1_{\boldsymbol{k}}$, y_r are as above, but $\psi_1 v = 0$ for all $v \in V$.

Proof. Note that this action is well-defined by Lemma 4.99. It is a straightforward check using the facts on semicuspidal words in Lemma 4.95 that this action obeys the defining relations of R_δ. □

The R_δ-module V descends to a C_δ-module since $1_{\text{nsc}} V = 0$. Moreover, the elements of our putative basis X act on V as linearly independent operators:

$$y_1^b (y_1 - y_d)^m \psi_w 1_{\boldsymbol{i}} \cdot 1_{\boldsymbol{j},\boldsymbol{j}} = \begin{cases} \delta_{\boldsymbol{i},\boldsymbol{j}} (z^b)_{w\boldsymbol{i},\boldsymbol{i}} & \deg(\psi_w 1_{\boldsymbol{i}}) = 1 \text{ (and thus } m = 0); \\ \delta_{\boldsymbol{i},\boldsymbol{j}} (z^b x^m)_{w\boldsymbol{i},\boldsymbol{i}} & \deg(\psi_w 1_{\boldsymbol{i}}) = 0. \end{cases}$$

This proves

Theorem 4.106. The set X of Proposition 4.104 is a basis for C_δ.

For each $\alpha \in Q_+$ and dominant weight Λ associated to $\mathtt{C}$, there is an important quotient R_α^Λ of R_α called the *cyclotomic KLR algebra* (see e.g. [20, 6]). Of relevance to the discussion at hand is the level-one case $R_\delta^{\Lambda_0}$; it is by definition the quotient of R_δ by the two-sided ideal generated by the elements $\{y_1^{\delta_{i_1,0}} 1_{\boldsymbol{i}} \mid \boldsymbol{i} \in I^\delta\}$. By [24, Lemma 5.1], when k is a field, $\{L_{\delta,i} \mid i \in I'\}$ is a full set of irreducible modules for $R_\delta^{\Lambda_0}$, so $1_{\boldsymbol{i}} = 0$ in $R_\delta^{\Lambda_0}$ unless $\boldsymbol{i} \in G^\delta$. Thus there is a natural surjection $C_\delta \twoheadrightarrow R_\delta^{\Lambda_0} \cong C_\delta / C_\delta y_1 C_\delta$.

We may also construct a map $R_\delta^{\Lambda_0} \to C_\delta$ by defining

$$1_{\boldsymbol{i}} \mapsto 1_{\boldsymbol{i}}, \qquad \psi_r \mapsto \psi_r, \qquad y_r \mapsto y_r - y_1.$$

This is a well-defined homomorphism of algebras which splits the natural surjection $C_\delta \twoheadrightarrow R_\delta^{\Lambda_0}$. Thus Theorem 4.106 has the following

Corollary 4.107. $C_\delta \cong k[y_1] \otimes R_\delta^{\Lambda_0}$ as graded k-algebras, and $R_\delta^{\Lambda_0}$, considered as a subalgebra of C_δ, has basis

$$\{(y_1 - y_d)^m \psi_w 1_{\boldsymbol{i}} \mid \boldsymbol{i} \in G^\delta, w \in W_{\boldsymbol{i}}, m + \deg(\psi_w 1_{\boldsymbol{i}}) \le 1\}.$$

4.65. Description of B_δ

The orthogonal idempotents $\{1_{\boldsymbol{i}} \mid \boldsymbol{i} \in G^\delta\}$ in C_δ are primitive, since $(1_{\boldsymbol{i}} C_\delta 1_{\boldsymbol{i}})_0$ is 1-dimensional. Setting $1_j := 1_{\boldsymbol{b}^j}$ and $1_\Delta = \sum_{j \in I'} 1_j$, we have that $C_\delta 1_j \cong \Delta_{\delta,j}$ and $\Delta_\delta = \bigoplus_{j \in I'} \Delta_{\delta,j} \cong C_\delta 1_\Delta$.

The following theorem establishes a Morita equivialence between the cyclotomic KLR algebra $R_\delta^{\Lambda_0}$ and the zigzag algebra A.

Theorem 4.108. $1_\Delta R_\delta^{\Lambda_0} 1_\Delta \cong \mathsf{A}$ as graded k-algebras.

Proof. By Corollary 4.107, $1_\Delta R_\delta^{\Lambda_0} 1_\Delta$, has basis

$$\{(y_1 - y_d)^m 1_j \mid j \in I', m \in \{0,1\}\} \cup \{\psi_{i,j} 1_j \mid i, j \in I', (\alpha_i, \alpha_j) = -1\}.$$

Color the vertices of C' with $+$'s and $-$'s in an alternating fashion. We define a linear map $f : 1_\Delta R_\delta^{\Lambda_0} 1_\Delta \to \mathsf{A}$ on the above basis.

$$f[(y_1 - y_d)^m 1_i] = \begin{cases} \mathsf{c}^m \mathsf{e}_i & \text{if color}(i) = + \\ -\mathsf{c}^m \mathsf{e}_i & \text{if color}(i) = - \end{cases}$$

$$f[\psi_{j,i} 1_i] = \begin{cases} \varepsilon_{ij} \mathsf{a}^{j,i} & \text{if color}(i) = + \\ \mathsf{a}^{j,i} & \text{if color}(i) = - \end{cases}$$

It is straightforward to check that f is an algebra homomorphism using the lemmas in §4.63, noting in particular that $\psi_{i,j}\psi_{j,i}1_i = \varepsilon_{ij}(y_1 - y_d)1_i$ in C_δ for neighboring i and j. As f is a bijection of bases, it is an isomorphism. $\square$

Since $1_\Delta C_\delta 1_\Delta \cong k[y_1] \otimes 1_\Delta R_\delta^{\Lambda_0} 1_\Delta$ by Corollary 4.107, we have the following

Corollary 4.109. $\mathrm{End}_{C_\delta}(\Delta_\delta) \cong 1_\Delta C_\delta 1_\Delta \cong k[\mathsf{z}] \otimes \mathsf{A}$ as graded algebras, where z is an indeterminate in degree 2.

There is an algebra isomorphism $\mathsf{A}^{\mathrm{op}} \to \mathsf{A}$ given by $\mathsf{a}^{i,j} \mapsto \mathsf{a}^{j,i}$, so we have

Corollary 4.110. $B_\delta = \mathrm{End}_{C_\delta}(\Delta_\delta)^{\mathrm{op}} \cong k[\mathsf{z}] \otimes \mathsf{A}$.

4.7. On the higher imaginary stratum categories

Now we will build on the previous section to explicitly describe the algebra $\mathrm{End}_{C_{n\delta}}(\Delta_\delta^{\circ n})$ for all n. We will show that $\mathrm{End}_{C_{n\delta}}(\Delta_\delta^{\circ n})$ is isomorphic to the rank n affine zigzag algebra $\mathsf{A}_n^{\mathrm{aff}}$, defined in §4.52 for Γ being the finite type Dynkin diagram of type C', giving a Morita equivalence between $B_{n\delta}$ and $\mathsf{A}_n^{\mathrm{aff}}$ when k is a field of characteristic $p = 0$ or $p > n$.

4.71. Endomorphisms of $\Delta_\delta^{\circ n}$

The following lemma follows from consideration of Theorem 4.106:

Lemma 4.111. For $i, j \in I'$, $\mathrm{Hom}_{C_\delta}(\Delta_{\delta,i}, \Delta_{\delta,j}) \cong 1_i C_\delta 1_j$, and $1_i C_\delta 1_j$ has basis

$$\{y_1^b (y_1 - y_d)^m 1_j \mid b \in \mathbb{Z}_{\geq 0}, m \in \{0, 1\}\} \text{ if } i = j,$$

and

$$\{y_1^b \psi_{i,j} 1_j \mid b \in \mathbb{Z}_{\geq 0}\} \text{ if } (\alpha_i, \alpha_j) = -1,$$

and is zero otherwise.

Lemma 4.112. For $n \in \mathbb{Z}_{\geq 0}$ we have

$$\dim_q \mathrm{End}_{C_{n\delta}}(\Delta_\delta^{\circ n}) = \frac{n!(\ell + 2(\ell - 1)q + \ell q^2)^n}{(1 - q^2)^n}.$$

Proof. By the Mackey Theorem, $\mathrm{Res}^{m\delta}_{\delta,\ldots,\delta}\mathrm{Ind}^{n\delta}_{\delta,\ldots,\delta}(\Delta_\delta^{\boxtimes n})$ has $n!$ subquotients isomorphic to $\Delta_\delta^{\boxtimes n}$. But $\Delta_\delta^{\boxtimes n}$ is projective as a $C_\delta \otimes \cdots \otimes C_\delta$-module, so these subquotients are in fact summands. Thus Frobenius Reciprocity gives

$$\begin{aligned}
\mathrm{End}_{C_{n\delta}}(\Delta_\delta^{\circ n}) &\cong \mathrm{Hom}_{C_\delta \cdots C_\delta}\left(\Delta_\delta^{\boxtimes n}, (\Delta_\delta^{\boxtimes n})^{\oplus n!}\right) \\
&\cong ((\mathrm{End}_{C_\delta}(\Delta_\delta))^{\otimes n})^{\oplus n!} \\
&\cong (k[\mathsf{z}] \otimes \mathsf{A})^{\otimes n})^{\oplus n!}
\end{aligned}$$

as vector spaces. The result follows by consideration of (4.72) above. $\square$

To avoid confusion, let v_i be the generating vector of word $\boldsymbol{b}^i$ in $\Delta_{\delta,i}$, so that v_i corresponds to 1_i via the equality $\Delta_{\delta,i} = C_\delta 1_i$. Let $v_\delta = \sum_{i \in I'} v_i \in \Delta_\delta$. Per the last section, we have the homomorphisms

$$\begin{array}{cccc} e_i : \Delta_\delta \to \Delta_{\delta,i} & z : \Delta_\delta \to \Delta_\delta & c : \Delta_\delta \to \Delta_\delta & a^{i,j} : \Delta_{\delta,j} \to \Delta_{\delta,i} \\ v_\delta \mapsto v_i & v_\delta \mapsto y_1 v_\delta & v_\delta \mapsto (y_1 - y_d) v_\delta & v_j \mapsto \psi_{j,i} v_i \end{array}$$

which generate $\mathrm{End}_{C_\delta}(\Delta_\delta)$ and satisfy the relations in the zigzag algebra. For $1 \leq r \leq n$, let $z_r, c_r, a_r^{i,j} \in \mathrm{End}_{C_{n\delta}}(\Delta_\delta^{\circ n})$ be defined by inserting the relevant map into the rth slot of $\Delta_\delta^{\boxtimes n}$ and inducing. Writing $\Delta_{\boldsymbol{i}} := \Delta_{\delta,i_1} \circ \cdots \circ \Delta_{\delta,i_n}$ for $\boldsymbol{i} \in (I')^n$, we have that $\Delta_\delta^{\circ n} = \bigoplus_{\boldsymbol{i} \in (I')^n} \Delta_{\boldsymbol{i}}$. Let $e_{\boldsymbol{i}}$ be the projection $\Delta_\delta^{\circ n} \to \Delta_{\boldsymbol{i}} \subseteq \Delta_\delta^{\circ n}$ induced from $e_{i_1} \otimes \cdots \otimes e_{i_n}$.

Now we describe a last family of endomorphisms of $\Delta_\delta^{\circ n}$. Let $i, j \in I'$. As explained in Chapter III, we have a nonzero degree-zero homomorphism $r^{i,j} : L_{\delta,i} \circ L_{\delta,j} \to L_{\delta,j} \circ L_{\delta,i}$. We will describe this map explicitly later in this section. We have $L_\delta^{\circ 2} = \bigoplus_{i,j \in I'} L_{\delta,i} \circ L_{\delta,j}$, so we may consider $r := \sum_{i,j \in I'} r^{i,j}$ as an endomorphism of $L_\delta^{\circ 2}$. More generally, for $1 \leq t < n$, we have an endomorphism r_t of $L_\delta^{\circ n}$ given by inserting r into the $(t, t+1)$-th slots and inducing. It can be seen that $r_1 \ldots, r_{n-1}$ satisfy Coxeter relations of the symmetric group $\mathfrak{S}_n$, and, together with projections to summands, generate a space T of dimension $\ell^n n!$ in $\mathrm{End}_{C_{n\delta}}(L_\delta^{\circ n}) = \mathrm{End}_{C_{n\delta}}(L_\delta^{\circ n})_0$.

Now, for $1 \leq t < n$, we lift r_t to some homogeneous $\hat{r}_t \in \mathrm{End}_{C_{n\delta}}(\Delta_\delta^{\circ n})_0$.

Lemma 4.113.

(i) The homogeneous lift of an endomorphism of $L_\delta^{\circ n}$ to $\Delta_\delta^{\circ n}$ is unique.

(ii) The elements $\hat{r}_1, \ldots, \hat{r}_{n-1}$ satisfy the Coxeter relations of $\mathfrak{S}_n$.

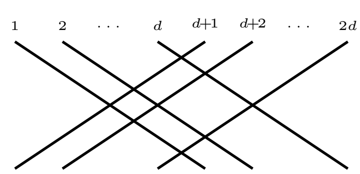
Proof. The space $T \subseteq \mathrm{End}_{C_{n\delta}}(L_\delta^{\circ n})_0$ has a basis which lifts to give $\ell^n n!$ linearly independent elements in $\mathrm{End}_{C_{n\delta}}(\Delta_\delta^{\circ n})_0$, so by Lemma 4.112, this is a basis for $\mathrm{End}_{C_{n\delta}}(\Delta_\delta^{\circ n})_0$. It follows that homogeneous lifts must be unique. Part (ii) follows from (i) and the fact that $r_1, \dots, r_{n-1}$ satisfy Coxeter relations. $\square$

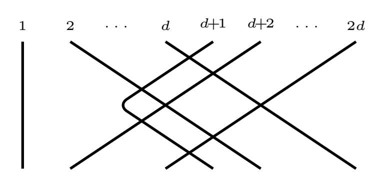
Define $\xi_1 \in \{\pm 1\}$ in all types as follows:

$$\xi_1 = \begin{cases} 1 & \mathtt{C} = \mathtt{A}_1^{(1)}; \\ \varepsilon_{10} \cdots \varepsilon_{\ell,\ell-1}\varepsilon_{0,\ell} & \mathtt{C} = \mathtt{A}_{\ell>1}^{(1)}; \\ (-1)^\ell & \mathtt{C} = \mathtt{D}_\ell^{(1)}; \\ -1 & \mathtt{C} = \mathtt{E}_\ell^{(1)}. \end{cases}$$

Then for all other $i \in I'$, define ξ_i such that $\xi_i \xi_j = -1$ if $(\alpha_i, \alpha_j) = -1$.

Let $\sigma, \sigma' \in R_{2\delta}$ be the following products of ψ's, displayed diagrammatically:

$\sigma :=$ 

, $\sigma' :=$ 

.

The labels in this case only indicate strand position and are not meant to color the strands.

In order to understand the multiplicative structure of $\mathrm{End}_{C_{n\delta}}(\Delta_\delta^{\circ n})$, we will need to describe the maps $\hat{r}_t$ more explicitly and examine commutation relations between these maps and the others detailed above. The following two lemmas are steps in this direction. Their proofs are straightforward but rather lengthy exercises in manipulating KLR diagrams. For this reason we defer the proofs until Chapter VI.

Lemma 4.114. Let $i, j \in I'$, and recall that $v_i \otimes v_j$ is a generator for $\Delta_{\delta,i} \circ \Delta_{\delta,j}$. Then we have

$$\sigma' v_i \otimes v_j = \begin{cases} \xi_i[y_d \otimes 1 + 1 \otimes (y_d - 2y_1)]v_i \otimes v_i & i = j; \\ \xi_i \varepsilon_{ij}(\psi_{j,i} \otimes \psi_{i,j})v_i \otimes v_j & (\alpha_i, \alpha_j) = -1; \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.115. Let $i, j, m \in I'$ with $(\alpha_i, \alpha_j) = -1$. Then we have

$$(\psi_{j,i} \otimes 1)\sigma v_m \otimes v_i = [\sigma(1 \otimes \psi_{j,i}) + \delta_{j,m}\xi_j(1 \otimes \psi_{j,i}) - \delta_{i,m}\xi_i(\psi_{j,i} \otimes 1)]v_m \otimes v_i.$$

Following [19], let x be an indeterminate in degree 2, and let $\iota : R_\delta \to k[x] \otimes R_\delta$ be the algebra homomorphism defined by $\iota(1_{\boldsymbol{i}}) = 1_{\boldsymbol{i}}$, $\iota(\psi_r) = \psi_r$, and $\iota(y_r) = y_r + x$. Let $L_{\delta,i,x} := k[x] \otimes L_{\delta,i}$ be the $k[x] \otimes R_\delta$-module with action twisted by ι. There is a homomorphism $r^{i,j}_{x,x'} : L_{\delta,i,x} \circ L_{\delta,j,x'} \to L_{\delta,j,x'} \circ L_{\delta,i,x}$ defined in terms of certain intertwining elements of R_δ. Then $r^{i,j}$ is equal to

$$r^{i,j} := [(x - x')^{-s} r^{i,j}_{x,x'}]_{x=x'=0},$$

where s is maximal such that $r^{i,j}_{x,x'}(L_{\delta,i,x} \circ L_{\delta,j,x'}) \subseteq (x - x')^s L_{\delta,j,x'} \circ L_{\delta,i,x}$.

For $i \in I'$, let $x_i \in L_{\delta,i}$ (resp. $x_j \in L_{\delta,j}$) be the image of v_i in the quotient $\Delta_{\delta,i} \twoheadrightarrow L_{\delta,i}$. It can be seen as that

$$r^{i,j}_{x,x'}(x_i \otimes x_j) = (x - x')^\kappa \sigma x_j \otimes x_i + (x - x')^{\kappa-1} \sigma' x_j \otimes x_i,$$

where $\kappa = \sum_{a=1}^{d}\sum_{b=1}^{d}\delta_{\boldsymbol{b}_a^i,\boldsymbol{b}_b^j}$. All $y's$ and ψ's of positive degree act as zero on $L_{\delta,i}$ and $L_{\delta,j}$, so, pushing the results of Lemma 4.114 through to $L_{\delta,j,x'}\circ L_{\delta_i,x}$ shows that $\sigma' x_j\otimes x_i = \delta_{i,j}\xi_i(x-x')x_j\otimes x_i$. Thus $r^{i,j}(x_i\otimes x_j) = (\sigma+\xi_i\delta_{i,j})x_j\otimes x_i$.

It may be seen via Theorem 4.106 and word considerations that $(1_{\boldsymbol{b}^i\boldsymbol{b}^j}\Delta_{\delta,j}\circ\Delta_{\delta,i})_0$ has basis $\{v_i\otimes v_i, \sigma v_i\otimes v_i\}$ if $i=j$, and $\{\sigma v_j\otimes v_i\}$ if $i\neq j$. Thus the lifting condition implies that $\hat{r}_1(v_i\otimes v_j) = (\sigma+\delta_{i,j}\xi_i)v_j\otimes v_i$. More generally, for $1\le t<n$ and $\boldsymbol{i}\in(I')^n$ we have that $\hat{r}_t(v_{i_1}\otimes\cdots\otimes v_{i_n})$ is equal to

$$(1\otimes\cdots\otimes(\sigma+\delta_{i_t,i_{t+1}}\xi_{i_t})\otimes\cdots\otimes 1)v_{i_1}\otimes\cdots\otimes v_{i_{t-1}}\otimes v_{i_{t+1}}\otimes v_{i_t}\otimes v_{i_{t+2}}\otimes\cdots\otimes v_{i_n},$$

where $\sigma+\delta_{i_t,i_{t+1}}\xi_{i_t}$ occupies the $(t,t+1)$-th slots.

Lemma 4.116. For $1\le t<n$, $1\le u\le n$, and $\boldsymbol{i}\in(I')^n$,

$$(\hat{r}_t a_u^{i,j} - a_{s_t(u)}^{i,j}\hat{r}_t)e_{\boldsymbol{i}} = 0, \qquad (\hat{r}_t c_u - c_{s_t(u)}\hat{r}_t)e_{\boldsymbol{i}} = 0,$$

$$(\hat{r}_t z_u - z_{s_t(u)}\hat{r}_t)e_{\boldsymbol{i}} = \begin{cases}(\delta_{u,t}-\delta_{u,t+1})\xi_{i_t}(c_t+c_{t+1})e_{\boldsymbol{i}} & i_t = i_{t+1};\\ (\delta_{u,t}-\delta_{u,t+1})\xi_{i_t}\varepsilon_{i_{t+1},i_t}a_t^{i_{t+1},i_t}a_{t+1}^{i_t,i_{t+1}}e_{\boldsymbol{i}} & (\alpha_{i_t},\alpha_{i_{t+1}}) = -1;\\ 0 & \text{otherwise.}\end{cases}$$

Proof. It is enough to check this in the case $n=2$.

$$\hat{r}_1 a_1^{i,j}(v_j\otimes v_m) = (\psi_{j,i}\otimes 1)(\sigma+\delta_{i,m}\xi_i)v_m\otimes v_i,$$
$$a_2^{i,j}\hat{r}_1(v_j\otimes v_m) = (\sigma+\delta_{j,m}\xi_j)(1\otimes\psi_{j,i})v_m\otimes v_i.$$

The first claim then follows for $u = 1$ by Lemma 4.115, and since $\hat{r}_1^2 = 1$, the claim also holds for $u = 2$, completing the proof of the first statement.

The second statement follows from the first when $\mathtt{C} \neq \mathtt{A}_1^{(1)}$ since c_t may be expressed in terms of a^{ij}'s. When $\mathtt{C} = \mathtt{A}_1^{(1)}$, we have

$$\hat{r}_1 c_1(v_1 \otimes v_1) = [(y_1 - y_2) \otimes 1](\sigma + 1)v_1 \otimes v_1$$
$$c_2 \hat{r}_1(v_1 \otimes v_1) = (\sigma + 1)[1 \otimes (y_1 - y_2)]v_1 \otimes v_1.$$

The equality of these expressions is easily verified.

For the final statement, let $\boldsymbol{i} = ji$ for $j, i \in I'$. Then for $u = 1$, we have

$$\begin{aligned}\hat{r}_1 z_1 e_{ji}(v_j \otimes v_i) &= (y_1 \otimes 1)(\sigma + \delta_{i,j}\xi_i)v_i \otimes v_j \\ &= \sigma(1 \otimes y_1)v_i \otimes v_j - \sigma' v_i \otimes v_j + \delta_{i,j}\xi_i(y_1 \otimes 1)v_i \otimes v_j,\end{aligned}$$

after applying a KLR braid relation. Since

$$z_2 \hat{r}_1 e_{ji}(v_j \otimes v_i) = (\sigma + \delta_{i,j}\xi_i)(1 \otimes y_1)v_i \otimes v_j,$$

the result follows from the definitions of the maps and Lemma 4.114. The case $u = 2$ follows from the first two statements. $\square$

For $w = s_{t_1} \cdots s_{t_m} \in \mathfrak{S}_n$, define $\hat{r}_w = \hat{r}_{t_1} \cdots \hat{r}_{t_m} \in \mathrm{End}_{C_{n\delta}}(\Delta_\delta^{\circ n})_0$. By Lemma 4.113(ii) this definition is independent of reduced expression for w. For convenience we will set $a_t^{i,i} = 1$.

Lemma 4.117. The algebra $\mathrm{End}_{C_{n\delta}}(\Delta_\delta^{\circ n})$ is generated by the elements $e_{\boldsymbol{i}}$, $\hat{r}_t$, z_u, and $a_u^{i,j}$ (and c_u in type $\mathtt{A}_1^{(1)}$). Moreover, $\mathrm{End}_{C_{n\delta}}(\Delta_\delta^{\circ n})$ has basis

$$\{z_1^{t_1}\cdots z_n^{t_n} c_1^{u_1}\cdots c_n^{u_n} a_1^{i_1,(w\boldsymbol{j})_1}\cdots a_n^{i_n,(w\boldsymbol{j})_n}\hat{r}_w e_{\boldsymbol{j}}\}, \tag{4.118}$$

ranging over $w \in \mathfrak{S}_n$, $t_m \in \mathbb{Z}_{\geq 0}$, $u_m \in \{0,1\}$, $u_m \leq \delta_{i_m,(w\boldsymbol{j})_m}$, and $\boldsymbol{i}, \boldsymbol{j} \in (I')^n$ such that $(\alpha_{i_m}, \alpha_{(w\boldsymbol{j})_m}) \neq 0$.

Proof. For $w \in \mathfrak{S}_n$, define the block permutation $\mathrm{bl}(w) \in \mathfrak{S}_{nd}$ by

$$\mathrm{bl}(w)(a) = w(\lceil a/d\rceil)d + (a-1 \bmod d) - d + 1.$$

Then we have

$$z_1^{t_1}\cdots z_n^{t_n} c_1^{u_1}\cdots c_n^{u_n} a_1^{i_1,(w\boldsymbol{j})_1}\cdots a_n^{i_n,(w\boldsymbol{j})_n}\hat{r}_w e_{\boldsymbol{j}}(v_{j_1}\otimes\cdots\otimes v_{j_n}) =$$

$$\psi_{\mathrm{bl}(w^{-1})}(y_1^{t_1}(y_1-y_d)^{u_1}\psi_{(w\boldsymbol{j})_1,i_1}\otimes\cdots\otimes y_n(y_1-y_d)^{u_n}\psi_{(w\boldsymbol{j})_n,i_n})v_{i_1}\otimes\cdots\otimes v_{i_n},$$

plus terms of the form $\psi_{w'}x_1\otimes\cdots\otimes x_n$, where $x_r \in \Delta_{i_r}$, and $w' \in \mathfrak{S}_{nd}$ is a minimal left coset representative for $\mathfrak{S}_{nd}/\mathfrak{S}_d\times\cdots\times\mathfrak{S}_d$ such that w' is lower than $\mathrm{bl}(w^{-1})$ in the Bruhat order. Thus, using Lemma 4.111 and induction on the Bruhat order, it can be shown that the elements (4.118) form a linearly independent set of endomorphisms. Now, comparing graded dimension with Lemma 4.112 proves the result. $\square$

4.72. Proof of the Main Theorem

Theorem 4.119. $\mathrm{End}_{R_{n\delta}}(\Delta_\delta^{\circ n}) \cong \mathrm{End}_{C_{n\delta}}(\Delta_\delta^{\circ n}) \cong \mathsf{A}_n^{\mathrm{aff}}(\mathtt{C}')$ as graded k-algebras.

Proof. We construct a map $f : \mathsf{A}_n^{\mathrm{aff}} \to \mathrm{End}_{C_{n\delta}}(\Delta_\delta^{\circ n})$ on generators:

$$\mathsf{e}_{\boldsymbol{i}} \mapsto e_{\boldsymbol{i}}, \qquad \mathsf{s}_t \mapsto \hat{r}_t, \qquad \mathsf{z}_t \mapsto z_t, \qquad \mathsf{a}_t^{i,j} \mapsto \begin{cases} \varepsilon_{ij} a_t^{i,j} & \xi_j = 1; \\ a_t^{i,j} & \xi_j = -1, \end{cases}$$

with $\mathsf{c}_t \mapsto c_t$ in type $\mathtt{A}_1^{(1)}$. By Theorem 4.108, Lemma 4.113, and Lemma 4.116, images of the generators obey the defining relations of $\mathsf{A}_n^{\mathrm{aff}}$, and hence f defines an algebra homomorphism. Moreover, f is a bijection (up to sign) of the basis elements of Lemma 4.118 and Lemma 4.84, so f is an isomorphism. □

Corollary 4.120. If k is a field of characteristic $p = 0$ or $p > n$, then $B_{n\delta}$ is Morita equivalent to $\mathsf{A}_n^{\mathrm{aff}}(\mathtt{C}')$.

Proof. In this situation the module $\Delta_\delta^{\circ n}$ is a projective generator for $B_{n\delta}$, so $B_{n\delta}$ is Morita equivalent to $\mathrm{End}_{C_{n\delta}}(\Delta_\delta^{\circ n})^{\mathrm{op}} \cong (\mathsf{A}_n^{\mathrm{aff}})^{\mathrm{op}}$. But the map $\mathsf{A}_n^{\mathrm{aff}} \to (\mathsf{A}_n^{\mathrm{aff}})^{\mathrm{op}}$ which sends $\mathsf{a}_t^{i,j} \mapsto \mathsf{a}_t^{j,i}$, and is the identity on other generators, is easily seen to be an algebra isomorphism, so the result follows. □

CHAPTER V

SKEW SPECHT MODULES AND REAL CUSPIDAL MODULES IN TYPE A

The work in this chapter has appeared in the article [40], which has been submitted for publication.

In this chapter we develop the theory of skew Specht modules in finite and affine type $\mathtt{A}$, and investigate their connection to the cuspidal systems developed in the previous chapters. First we briefly recall the Lie theoretic notation associated with these types.

5.1. Preliminaries

5.11. Lie theoretic notation

We use notation similar to [27], [24]. Let $e \in \{0, 2, 3, 4, \ldots\}$ and $I = \mathbb{Z}/e\mathbb{Z}$. Let Γ be the quiver with vertex set I and a directed edge $i \to j$ if $j = i - 1 \pmod{e}$. Thus Γ is a quiver of type $\mathtt{A}_\infty$ if $e = 0$ or $\mathtt{A}_{e-1}^{(1)}$ if $e > 0$. The corresponding *Cartan matrix* $\mathtt{C} = (a_{i,j})_{i,j \in I}$ is defined by

$$a_{i,j} := \begin{cases} 2 & \text{if } i = j; \\ 0 & \text{if } j \neq i, i \pm 1; \\ -1 & \text{if } i \to j \text{ or } i \leftarrow j; \\ -2 & \text{if } i \leftrightarrows j. \end{cases}$$

Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of $(a_{i,j})_{i,j \in I}$, with root system Φ, positive roots Φ_+, simple roots $\{\alpha_i \mid i \in I\}$, fundamental dominant weights $\{\Lambda_i \mid i \in I\}$, and normalized invariant form $(\cdot,\cdot)$ such that $(\alpha_i,\alpha_j)=a_{ij}$ and $(\Lambda_i,\alpha_j)=\delta_{i,j}$. Let P_+ be the set of dominant integral weights, and let $Q_+ := \bigoplus_{i\in I}\mathbb{Z}_{\geq 0}\alpha_i$ be the positive root lattice. For $\alpha=\sum_{i\in I} m_i\alpha_i \in Q_+$, define the *height* of α to be $\mathrm{ht}(\alpha)=\sum_{i\in I} m_i$. When $e>0$, we label the *null-root* $\delta=\sum_{i\in I}\alpha_i$. Finally fix a level $l\in\mathbb{Z}_{>0}$ and a *multicharge* $\kappa=(k_1,\dots,k_l)\in I^l$.

5.12. Words

Sequences of elements of I will be called *words*, and the set of all words is denoted $\langle I\rangle$. If $\boldsymbol{i}=i_1\cdots i_d\in\langle I\rangle$, then $|\boldsymbol{i}|:=\alpha_{i_1}+\cdots+\alpha_{i_d}\in Q_+$. For $\alpha\in Q_+$, denote

$$\langle I\rangle_\alpha := \{\boldsymbol{i}\in\langle I\rangle \mid |\boldsymbol{i}|=\alpha\}.$$

If α is of height d, then $\mathfrak{S}_d$ with simple transpositions $s_1,\dots,s_{d-1}$ has a left action on $\langle I\rangle_\alpha$ via place permutations.

5.13. Young diagrams

An l-multipartition $\boldsymbol{\lambda}$ of d is an l-tuple of partitions $(\lambda^{(1)},\dots,\lambda^{(l)})$ such that $\sum_{i=1}^{l}|\lambda^{(i)}|=d$. For $1\leq i\leq l$, let $n(\boldsymbol{\lambda},i)$ be the number of nonzero parts of $\lambda^{(i)}$. When $l=1$, we will usually write $\boldsymbol{\lambda}=\lambda=\lambda^{(1)}$. The *Young diagram* of the partition $\boldsymbol{\lambda}$ is

$$\{(a,b,m)\in\mathbb{Z}_{>0}\times\mathbb{Z}_{>0}\times\{1,\dots,l\} \mid 1\leq b\leq\lambda_a^{(m)}\}.$$

We call the elements of this set *nodes* of $\boldsymbol{\lambda}$. We will usually identify the multipartition with its Young diagram. To each node $A = (a, b, m)$ we associate its *residue*

$$\operatorname{res} A = \operatorname{res}^{\kappa} A = k_m + (b - a) \pmod{e}.$$

An *i-node* is a node of residue i. The *residue content of* $\boldsymbol{\lambda}$ is $\operatorname{cont}(\boldsymbol{\lambda}) := \sum_{A \in \boldsymbol{\lambda}} \alpha_{\operatorname{res} A} \in Q_+$. Denote

$$\mathscr{P}^{\kappa}_{\alpha} := \{\boldsymbol{\lambda} \in \mathscr{P}^{\kappa} \mid \operatorname{cont}(\boldsymbol{\lambda}) = \alpha\}, \qquad (\alpha \in Q_+).$$

and set $\mathscr{P}^{\kappa}_{d} := \bigcup_{\operatorname{ht}(\alpha)=d} \mathscr{P}^{\kappa}_{\alpha}$. For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}^{\kappa}_{d}$, we say $\boldsymbol{\lambda}$ *dominates* $\boldsymbol{\mu}$, and write $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$, if

$$\sum_{a=1}^{m-1} |\lambda^{(a)}| + \sum_{b=1}^{c} \lambda^{(m)}_b \geq \sum_{a=1}^{m-1} |\mu^{(a)}| + \sum_{b=1}^{c} \mu^{(m)}_b$$

for all $1 \leq m \leq l$ and $c \geq 1$.

A node $A \in \boldsymbol{\lambda}$ is *removable* if $\boldsymbol{\lambda} \backslash \{A\}$ is a Young diagram, and a node $B \notin \boldsymbol{\lambda}$ is *addable* if $\boldsymbol{\lambda} \cup \{B\}$ is a Young diagram. Define $\boldsymbol{\lambda}_A := \boldsymbol{\lambda} \backslash \{A\}$ and $\boldsymbol{\lambda}^B := \boldsymbol{\lambda} \cup \{B\}$.

Let $\boldsymbol{\lambda}' = (\lambda^{(l)'}, \dots, \lambda^{(1)'})$ signify the *conjugate partition* to $\boldsymbol{\lambda}$, where $\lambda^{(i)'}$ is obtained by swapping the rows and columns of $\lambda^{(i)}$.

5.14. Tableaux

Let $\boldsymbol{\lambda} \in \mathscr{P}^{\kappa}_{d}$. A $\boldsymbol{\lambda}$-tableau $\mathtt{T}$ is an injective map $\mathtt{T} : \{1, \dots, d\} \to \boldsymbol{\lambda}$, i.e. a labeling of the nodes of $\boldsymbol{\lambda}$ with the integers $1, \dots, d$. We also label the inverse of this bijection with $\mathtt{T}$; if $\mathtt{T}(r) = (a, b, m)$ we will also write $\mathtt{T}(a, b, m) = r$. We set $\operatorname{res}_{\mathtt{T}}(r) = \operatorname{res} \mathtt{T}(r)$.

The *residue sequence* of $\mathtt{T}$ is the word

$$\boldsymbol{i}(\mathtt{T}) = \boldsymbol{i}^k(\mathtt{T}) = \mathrm{res}_{\mathtt{T}}(1) \cdots \mathrm{res}_{\mathtt{T}}(d) \in \langle I \rangle.$$

A $\boldsymbol{\lambda}$-tableau is *row-strict* if $\mathtt{T}(a,b,m) < \mathtt{T}(a,c,m)$ when $b < c$, and *column-strict* if $\mathtt{T}(a,b,m) < \mathtt{T}(c,b,m)$ when $a < c$. We say $\mathtt{T}$ is standard if it is row- and column-strict. Let $\mathrm{Tab}(\boldsymbol{\lambda})$ (resp. $\mathrm{St}(\boldsymbol{\lambda})$) be the set of all (resp. standard) $\boldsymbol{\lambda}$-tableaux.

Let $\boldsymbol{\lambda} \in \mathscr{P}^\kappa$, $i \in I$, A be a removable i-node, and B be an addable i-node of $\boldsymbol{\lambda}$. We set

$$d_A(\boldsymbol{\lambda}) := \#\{\text{addable } i\text{-nodes strictly below } A\} - \#\{\text{removable } i\text{-nodes strictly below } A\}$$
$$d^B(\boldsymbol{\lambda}) := \#\{\text{addable } i\text{-nodes strictly above } B\} - \#\{\text{removable } i\text{-nodes strictly above } B\}.$$

In [3, Section 3.5], the *degree* of $\mathtt{T}$ is defined inductively as follows. If $d = 0$, then $\mathtt{T} = \varnothing$ and $\deg \mathtt{T} := 0$. For $d > 0$, let A be the node occupied by d in $\mathtt{T}$. Let $\mathtt{T}_{<d} \in \mathrm{St}(\boldsymbol{\lambda}_A)$ be the tableau obtained by removing this node, and set

$$\deg \mathtt{T} := d_A(\boldsymbol{\lambda}) + \deg \mathtt{T}_{<d}.$$

Similarly, define the dual notion of *codegree* of $\mathtt{T}$ by codeg $\varnothing = 0$ and

$$\mathrm{codeg}\ \mathtt{T} := d^A(\boldsymbol{\lambda}) + \mathrm{codeg}\ \mathtt{T}_{<d}.$$

The group $\mathfrak{S}_d$ acts on the set of $\boldsymbol{\lambda}$-tableaux on the left by acting on entries; considering $\mathtt{T}$ as a function $\boldsymbol{\lambda} \to \{1, \dots, d\}$, we have $w \cdot \mathtt{T} = w \circ \mathtt{T}$. Let $\mathtt{T}^{\boldsymbol{\lambda}}$ be the $\boldsymbol{\lambda}$-tableau in which the numbers $1, 2, \dots, d$ appear in order from left to right along the successive rows, starting from the top. Let $\mathtt{T}_{\boldsymbol{\lambda}} := (\mathtt{T}^{\boldsymbol{\lambda}})'$, where we define the conjugate tableau in the obvious way.

For each $\boldsymbol{\lambda}$-tableau $\mathtt{T}$, define permutations $w^{\mathtt{T}}$ and $w_{\mathtt{T}} \in \mathfrak{S}_d$ such that

$$w^{\mathtt{T}}\mathtt{T}^{\boldsymbol{\lambda}} = \mathtt{T} = w_{\mathtt{T}}\mathtt{T}_{\boldsymbol{\lambda}}.$$

5.15. Bruhat order

Let ℓ be the length function on $\mathfrak{S}_d$ with respect to the Coxeter generators $s_1, \dots, s_{d-1}$. Let $\trianglelefteq$ be the Bruhat order on $\mathfrak{S}_d$, so that $1 \trianglelefteq w$ for all $w \in \mathfrak{S}_d$. Define a partial order $\trianglelefteq$ on $\mathrm{St}(\boldsymbol{\lambda})$ as follows:

$$\mathtt{S} \trianglelefteq \mathtt{T} \iff w^{\mathtt{S}} \trianglelefteq w^{\mathtt{T}}.$$

5.16. Skew diagrams and tableaux

Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}^\kappa$, with $\boldsymbol{\mu} \subseteq \boldsymbol{\lambda}$ as Young diagrams. Then we call $\boldsymbol{\lambda}/\boldsymbol{\mu} := \boldsymbol{\lambda} \backslash \boldsymbol{\mu}$ a *skew diagram*. A (level one) skew diagram is called a *skew hook* if it is connected and does not have two nodes on the same diagonal. We may consider a Young diagram as a skew diagram with empty inner tableau. With $\boldsymbol{\mu}$ fixed, let $\mathscr{S}^\kappa_{\boldsymbol{\mu},d}$ be the of skew diagrams $\boldsymbol{\lambda}/\boldsymbol{\mu}$ such that $|\boldsymbol{\lambda}/\boldsymbol{\mu}| = d$. Let $\mathscr{S}^\kappa_{\boldsymbol{\mu}} = \bigcup \mathscr{S}^\kappa_{\boldsymbol{\mu},d}$. Residue and content for skew diagrams are defined as before; for example $\mathrm{cont}(\boldsymbol{\lambda}/\boldsymbol{\mu}) := \sum_{A \in \boldsymbol{\lambda}/\boldsymbol{\mu}} \alpha_{\mathrm{res} A} \in Q_+$. Denote

$$\mathscr{S}^\kappa_{\boldsymbol{\mu},\alpha} = \{\boldsymbol{\lambda}/\boldsymbol{\mu} \in \mathscr{S}^\kappa_{\boldsymbol{\mu}} \mid \mathrm{cont}(\boldsymbol{\lambda}/\boldsymbol{\mu}) = \alpha\}.$$

For $\boldsymbol{\lambda}/\boldsymbol{\mu}, \boldsymbol{\nu}/\boldsymbol{\mu} \in \mathscr{S}^\kappa_{\boldsymbol{\mu}}$, we say that $\boldsymbol{\lambda}/\boldsymbol{\mu}$ dominates $\boldsymbol{\nu}/\boldsymbol{\mu}$, or $\boldsymbol{\lambda}/\boldsymbol{\mu} \trianglerighteq \boldsymbol{\nu}/\boldsymbol{\mu}$, if $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\nu}$.

For $\boldsymbol{\lambda}/\boldsymbol{\mu} \in \mathscr{S}^{\kappa}_{\boldsymbol{\mu},d}$, a $\boldsymbol{\lambda}/\boldsymbol{\mu}$-tableau is a bijection $\mathtt{t} : \{1, \ldots, d\} \to \boldsymbol{\lambda}/\boldsymbol{\mu}$. Let $\mathrm{Tab}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ be the set of $\boldsymbol{\lambda}/\boldsymbol{\mu}$-tableaux. We define the residue sequence of $\boldsymbol{i}(\mathtt{t})$ in the same manner as for Young tableaux, and $\mathtt{t}^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ we define to be the $\boldsymbol{\lambda}/\boldsymbol{\mu}$-tableau in which the numbers $1, \ldots, d$ appear in order from left to right, starting from the top. We will write $\boldsymbol{i}^{\boldsymbol{\lambda}/\boldsymbol{\mu}} := \boldsymbol{i}(\mathtt{t}^{\boldsymbol{\lambda}/\boldsymbol{\mu}})$. For every $\boldsymbol{\lambda}/\boldsymbol{\mu}$-tableau $\mathtt{t}$, define a $\boldsymbol{\lambda}$-tableau $\mathtt{Y}(\mathtt{t})$ by setting $\mathtt{Y}(\mathtt{t})(a,b,m) = \mathtt{T}^{\boldsymbol{\mu}}(a,b,m)$ for $(a,b,m) \in \boldsymbol{\mu}$ and $\mathtt{Y}(\mathtt{t})(a,b,m) = \mathtt{t}(a,b,m) + |\boldsymbol{\mu}|$ for $(a,b,m) \in \boldsymbol{\lambda}/\boldsymbol{\mu}$. For example, if $l = 1$, $\lambda = (4,4,1)$, and $\mu = (2,1,1)$, then

$$\mathtt{t}^{\lambda/\mu} = \quad \begin{array}{cccc} & & \boxed{1} & \boxed{2} \\ & \boxed{3} & \boxed{4} & \boxed{5} \end{array}, \qquad \text{and} \qquad \mathtt{Y}(\mathtt{t}^{\lambda/\mu}) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 7 & 8 & 9 \\ \hline 4 \\ \cline{1-1} \end{array}.$$

Let $\mathrm{St}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ be the set of standard (i.e. row- and column-strict) $\boldsymbol{\lambda}/\boldsymbol{\mu}$-tableaux. For $\mathtt{t} \in \mathrm{St}(\boldsymbol{\lambda}/\boldsymbol{\mu})$, we define

$$\deg \mathtt{t} := \deg \mathtt{Y}(\mathtt{t}) - \deg \mathtt{T}^{\boldsymbol{\mu}}.$$

The symmetric group $\mathfrak{S}_d$ acts on $\mathrm{Tab}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ in the obvious fashion. For $\mathtt{t} \in \mathrm{Tab}(\boldsymbol{\lambda}/\boldsymbol{\mu})$, define $w^{\mathtt{t}}$ by $w^{\mathtt{t}}\mathtt{t}^{\boldsymbol{\lambda}/\boldsymbol{\mu}} = \mathtt{t}$. Define a partial order on $\mathrm{Tab}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ as follows:

$$\mathtt{s} \trianglelefteq \mathtt{t} \quad \text{if and only if} \quad w^{\mathtt{s}} \trianglelefteq w^{\mathtt{t}}.$$

Lemma 5.1.1. Let $\mathtt{s}, \mathtt{t} \in \mathrm{Tab}(\boldsymbol{\lambda}/\boldsymbol{\mu})$. Then $\mathtt{s} \trianglelefteq \mathtt{t}$ if and only if $\mathtt{Y}(\mathtt{s}) \trianglelefteq \mathtt{Y}(\mathtt{t})$.

Proof. Let $\widehat{w^{\mathtt{t}}}$ be the image of $w^{\mathtt{t}}$ under the 'right side' embedding $\mathfrak{S}_d \hookrightarrow \mathfrak{S}_{|\boldsymbol{\mu}|} \times \mathfrak{S}_d \hookrightarrow \mathfrak{S}_{|\boldsymbol{\lambda}|}$. Then $w^{\mathtt{Y}(\mathtt{t})} = \widehat{w^{\mathtt{t}}} w^{\mathtt{Y}(\mathtt{t}^{\boldsymbol{\lambda}/\boldsymbol{\mu}})}$, with $\ell(w^{\mathtt{Y}(\mathtt{t})}) = \ell(\widehat{w^{\mathtt{t}}}) + \ell(w^{\mathtt{Y}(\mathtt{t}^{\boldsymbol{\lambda}/\boldsymbol{\mu}})})$, and similarly for $w^{\mathtt{Y}(\mathtt{s})}$. Since $w^{\mathtt{s}} \trianglelefteq w^{\mathtt{t}}$ if and only if $\widehat{w^{\mathtt{s}}} \trianglelefteq \widehat{w^{\mathtt{t}}}$, the result follows. $\square$

Remark 5.1.2. In order to translate between the orders in the various papers cited, we provide the following dictionary. Our partial order on partitions and tableaux agrees with that of [3]. In [27] the order on tableaux (which we'll call $\trianglelefteq_U$) amounts to $\mathtt{S} \trianglelefteq_U \mathtt{T} \iff w^{\mathtt{S}'} \trianglelefteq w^{\mathtt{T}'}$. As is shown in [27, Lemma 2.18(ii)], when $\mathtt{S}, \mathtt{T} \in \mathrm{St}(\boldsymbol{\mu})$, we have $\mathtt{S} \trianglelefteq_U \mathtt{T} \iff \mathtt{S} \trianglerighteq \mathtt{T}$. In [36], the reverse Bruhat order $(1 \geq w)$ is used on elements of $\mathfrak{S}_d$, and the order on tableaux (which we'll call $\trianglelefteq_M$) is defined (on row-strict tableaux) by the shape condition in Lemma 5.1.5. Thus Lemma 5.1.5 will give $\mathtt{S} \trianglelefteq_M \mathtt{T} \iff \mathtt{S} \trianglerighteq \mathtt{T}$ when $\mathtt{S}, \mathtt{T}$ are row-strict.

For nodes A, B in $\boldsymbol{\lambda}/\boldsymbol{\mu}$, we say that A is *earlier* than B if $\mathtt{t}^{\boldsymbol{\lambda}/\boldsymbol{\mu}}(A) < \mathtt{t}^{\boldsymbol{\lambda}/\boldsymbol{\mu}}(B)$; i.e. A is above or directly to the left of B, or in an earlier component.

Let $\mathtt{T}$ be a $\boldsymbol{\lambda}$-tableau and suppose that $r = \mathtt{T}(a_1, b_1, m_1)$ and $s = \mathtt{T}(a_2, b_2, m_2)$. We write $r \nearrow_{\mathtt{T}} s$ if $m_1 = m_2$, $a_1 > a_2$ and $b_1 < b_2$; informally, if r and s are in the same component of $\boldsymbol{\lambda}$ and s is strictly to the northeast of r. We write $r \nearrow\!\!\!\nearrow_{\mathtt{T}} s$ if $r \nearrow_{\mathtt{T}} s$ or $m_1 > m_2$. Other rotations of the symbols $\nearrow_{\mathtt{T}}$ and $\nearrow\!\!\!\nearrow_{\mathtt{T}}$ have similarly obvious meanings.

The following lemmas are proved in [3] and [36] in the context of Young diagrams, but the proofs carry over to skew shapes without significant alteration. The first lemma is obvious.

Lemma 5.1.3. Let $\mathtt{t} \in \mathrm{St}(\boldsymbol{\lambda}/\boldsymbol{\mu})$. Then $s_r \mathtt{t} \in \mathrm{St}(\boldsymbol{\lambda}/\boldsymbol{\mu})$ if and only if $r \nearrow\!\!\!\nearrow_{\mathtt{T}} r+1$, or $r+1 \nearrow\!\!\!\nearrow_{\mathtt{T}} r$.

Lemma 5.1.4. Let $\mathtt{s}, \mathtt{t} \in \mathrm{Tab}(\boldsymbol{\lambda}/\boldsymbol{\mu})$. Then $\mathtt{s} \trianglelefteq \mathtt{t}$ if and only if $\mathtt{s} = (a_1 b_1) \cdots (a_r b_r)\mathtt{t}$ for some transpositions $(a_1 b_1), \ldots, (a_r b_r)$ such that for each $1 \leq n \leq r$ we have $a_n < b_n$ and b_n is in an earlier node in $(a_{n+1} b_{n+1}) \cdots (a_r b_r)\mathtt{t}$ than a_n.

Proof. This follows from applying Lemma 5.1.1 and [3, Lemma 3.4] to $\mathtt{Y}(\mathtt{s})$ and $\mathtt{Y}(\mathtt{t})$. □

Given $\boldsymbol{\lambda}/\boldsymbol{\mu} \in \mathscr{S}_{\boldsymbol{\mu}}^{\kappa}$ and a row-strict $\mathtt{t} \in \mathrm{Tab}(\boldsymbol{\lambda}/\boldsymbol{\mu})$, for all $1 \leq a \leq d$ define $\mathtt{t}_{\leq a}$ to be the tableau obtained by erasing all nodes occupied by entries greater than a.

Lemma 5.1.5. Let $\mathtt{s}, \mathtt{t}$ be row-strict $\boldsymbol{\lambda}/\boldsymbol{\mu}$-tableaux. Then $\mathtt{s} \trianglelefteq \mathtt{t}$ if and only if $\mathrm{sh}(\mathtt{Y}(\mathtt{s})_{\leq a}) \trianglerighteq \mathrm{sh}(\mathtt{Y}(\mathtt{t})_{\leq a})$ for each $a = |\boldsymbol{\mu}| + 1, \dots, |\boldsymbol{\mu}| + d$.

Proof. This follows from Lemma 5.1.1 and [36, Theorem 3.8]. $\square$

Lemma 5.1.6. Let $\boldsymbol{\lambda}/\boldsymbol{\mu} \in \mathscr{S}_{\boldsymbol{\mu}}^{\kappa}$ and $\mathtt{s}, \mathtt{t} \in \mathrm{St}(\boldsymbol{\lambda}/\boldsymbol{\mu})$, and $r \in \{1, \dots, d-1\}$ such that $r \downarrow_{\mathtt{t}} r+1$ or $r \rightarrow_{\mathtt{t}} r+1$. Then $\mathtt{s} \triangleleft s_r\mathtt{t}$ implies $\mathtt{s} \trianglelefteq \mathtt{t}$.

Proof. By [3, Lemma 3.7], $\mathtt{Y}(\mathtt{s}) \triangleleft \mathtt{Y}(s_r\mathtt{t}) = s_{r+|\boldsymbol{\mu}|}\mathtt{Y}(\mathtt{t})$ if and only if $\mathtt{Y}(\mathtt{s}) \trianglelefteq \mathtt{Y}(\mathtt{t})$, and the result follows by Lemma 5.1.1. $\square$

5.2. Manipulating elements of KLR algebras

Let $\alpha \in Q_+$ and $\mathrm{ht}(\alpha) = d$. For every $w \in \mathfrak{S}_d$, fix a *preferred reduced expression* $w = s_{r_1} \cdots s_{r_m}$, and define $\psi_w = \psi_{r_1} \cdots \psi_{r_m} \in R_\alpha$. In general ψ_w depends on the choice of reduced expression. When w is *fully commutative* however, i.e., when one can go from any reduced expression for w to any other using only the braid relations of the form $s_r s_t = s_t s_r$ for $|r - t| > 1$, the element ψ_w is independent of the choice of reduced expression.

For $\boldsymbol{\lambda}/\boldsymbol{\mu} \in \mathscr{S}_{\boldsymbol{\mu},\alpha}^{\kappa}$ and $\mathtt{t} \in \mathrm{Tab}(\boldsymbol{\lambda}/\boldsymbol{\mu})$, define

$$\psi^{\mathtt{t}} := \psi_{w^{\mathtt{t}}}.$$

Lemma 5.2.1. Let $\mathtt{t} \in \mathrm{St}(\boldsymbol{\lambda}/\boldsymbol{\mu})$. If $w^{\mathtt{t}} = s_{r_1} \cdots s_{r_m}$ is a reduced decomposition in $\mathfrak{S}_d$, then

$$\deg \mathtt{t} - \deg \mathtt{t}^{\boldsymbol{\lambda}/\boldsymbol{\mu}} = \deg(\psi_{r_1} \cdots \psi_{r_m} 1_{\boldsymbol{i}^{\boldsymbol{\lambda}/\boldsymbol{\mu}}}).$$

Proof. Write $c = |\boldsymbol{\mu}|$. Let $s_{t_1} \cdots s_{t_n}$ be a reduced decomposition for $w^{\mathtt{Y}(\mathtt{t}^{\boldsymbol{\lambda}/\boldsymbol{\mu}})}$. Then $\widehat{w^{\mathtt{t}}} = s_{r_1+c} \cdots s_{r_m+c}$ is reduced and $w^{\mathtt{Y}(\mathtt{t})} = \widehat{w^{\mathtt{t}}} w^{\mathtt{Y}(\mathtt{t}^{\boldsymbol{\lambda}/\boldsymbol{\mu}})} = s_{r_1+c} \cdots s_{r_m+c} s_{t_1} \cdots s_{t_n}$ is reduced. Then by [3, Corollary 3.13] we have

$$\begin{aligned}
\deg \mathtt{Y}(\mathtt{t}) - \deg \mathtt{T}^{\boldsymbol{\lambda}} &= \deg(\psi_{r_1+c} \cdots \psi_{r_m+c} \psi_{t_1} \cdots \psi_{t_n} 1_{\boldsymbol{i}^{\boldsymbol{\lambda}}}) \\
&= \deg(\psi_{r_1+c} \cdots \psi_{r_m+c} 1_{\boldsymbol{i}^{\mathtt{Y}(\mathtt{t}^{\boldsymbol{\lambda}/\boldsymbol{\mu}})}}) + \deg(\psi_{t_1} \cdots \psi_{t_n} 1_{\boldsymbol{i}^{\boldsymbol{\lambda}}}) \\
&= \deg(\psi_{r_1} \cdots \psi_{r_m} 1_{\boldsymbol{i}^{\boldsymbol{\lambda}/\boldsymbol{\mu}}}) + \deg(\psi_{t_1} \cdots \psi_{t_n} 1_{\boldsymbol{i}^{\boldsymbol{\lambda}}})
\end{aligned}$$

and

$$\deg \mathtt{Y}(\mathtt{t}^{\boldsymbol{\lambda}/\boldsymbol{\mu}}) - \deg \mathtt{T}^{\boldsymbol{\lambda}} = \deg(\psi_{t_1} \cdots \psi_{t_n} 1_{\boldsymbol{i}^{\boldsymbol{\lambda}}}),$$

which implies the result. □

Proposition 5.2.1. *Let $f(y) = f(y_1, \ldots, y_d) \in \mathcal{O}[y_1, \ldots y_d]$ be a polynomial in the generators y_r of R_α. Let $1 \le r_1, \ldots, r_m \le d-1$. Then*

(i) *$f(y)\psi_{r_1} \cdots \psi_{r_m} 1_{\boldsymbol{i}}$ is an $\mathcal{O}$-linear combination of elements of the form $\psi_{r_1}^{\epsilon_1} \cdots \psi_{r_m}^{\epsilon_m} g(y) 1_{\boldsymbol{i}}$, where $g(y) \in \mathcal{O}[y_1, \ldots, y_d]$, each $\epsilon_i \in \{0, 1\}$, and $s_{r_1}^{\epsilon_1} \cdots s_{r_m}^{\epsilon_m}$ is a reduced expression.*

(ii) *If $w = s_{r_1} \cdots s_{r_m}$ is reduced, and $s_{t_1} \cdots s_{t_m}$ is another reduced expression for w, then*

$$\psi_{r_1} \cdots \psi_{r_m} 1_{\boldsymbol{i}} = \psi_{t_1} \cdots \psi_{t_m} 1_{\boldsymbol{i}} + \sum_{u \lhd w} d_u \psi_u g_u(y) 1_{\boldsymbol{i}},$$

where each $d_u \in \mathcal{O}$, $g_u(y) \in \mathcal{O}[y_1, \ldots, y_d]$, and each u in the sum is such that $\ell(u) \leq m - 3$. Alternatively,

$$\psi_{r_1} \cdots \psi_{r_m} 1_{\boldsymbol{i}} = \psi_{t_1} \cdots \psi_{t_m} 1_{\boldsymbol{i}} + (*),$$

where $(*)$ is an $\mathcal{O}$-linear combination of elements of the form $\psi_{r_1}^{\epsilon_1} \cdots \psi_{r_m}^{\epsilon_m} g(y)$, where $g(y) \in \mathcal{O}[y_1, \ldots, y_d]$, $\epsilon_i \in \{0, 1\}$, $\epsilon_i = 0$ for at least three distinct i's, and $s_{r_1}^{\epsilon_1} \cdots s_{r_m}^{\epsilon_m}$ is a reduced expression.

Proof. This is proved in [3, Lemma 2.4] and [3, Proposition 2.5], for the case of cyclotomic KLR algebras, but the cyclotomic relation is not used in the proof. □

Theorem 5.2.2. [20, Theorem 2.5], [42, Theorem 3.7] Let $\alpha \in Q_+$. Then

$$\{\psi_w y_1^{m_1} \cdots y_d^{m_d} 1_{\boldsymbol{i}} \mid w \in \mathfrak{S}_d, m_1, \ldots, m_d \in \mathbb{Z}_{\geq 0}, \boldsymbol{i} \in \langle I \rangle_\alpha\}$$

is an $\mathcal{O}$-basis for R_α.

5.3. Skew Specht modules

In this section we define the graded skew Specht module $S^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$. In fact, the construction is exactly the same as the one given for graded (row) Specht modules (associated to Young diagrams) in [27], only applied in the more general context of skew diagrams. For the reader's convenience, and since the particulars will be put to use often in Section 5.4, we provide the construction of skew Specht modules here. The 'spanning' result [27, Prop. 5.14] and the proof given in that paper also generalize without much difficulty to the skew case.

5.31. Garnir skew tableaux

Let $A = (a, b, m)$ be a node of $\boldsymbol{\lambda}/\boldsymbol{\mu} \in \mathscr{S}^{\kappa}$. We say A is a *Garnir node* if $(a+1, b, m)$ is also a node of $\boldsymbol{\lambda}/\boldsymbol{\mu}$. The *A-Garnir belt* $\mathbf{B}^A$ is the set of nodes

$$\mathbf{B}^A = \{(a, c, m) \in \boldsymbol{\lambda}/\boldsymbol{\mu} \mid c \geq b\} \cup \{(a+1, c, m) \in \boldsymbol{\lambda}/\boldsymbol{\mu} \mid c \leq b\}.$$

The *A-Garnir tableau* is the $\boldsymbol{\lambda}/\boldsymbol{\mu}$-tableau $\mathtt{g}^A$ that is equal to $\mathtt{t}^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ outside the Garnir belt, and with numbers $\mathtt{t}^{\boldsymbol{\lambda}/\boldsymbol{\mu}}(a, b, m)$ through $\mathtt{t}^{\boldsymbol{\lambda}/\boldsymbol{\mu}}(a+1, b, m)$ inserted into the Garnir belt, in order from bottom left to top right.

Lemma 5.3.1. Suppose that $\boldsymbol{\lambda}/\boldsymbol{\mu} \in \mathscr{S}^{\kappa}_{\boldsymbol{\mu}}$, A is a Garnir node of $\boldsymbol{\lambda}/\boldsymbol{\mu}$, and $\mathtt{t} \in \mathrm{Tab}(\boldsymbol{\lambda}/\boldsymbol{\mu})$. If $\mathtt{t} \trianglelefteq \mathtt{g}^A$, then $\mathtt{t}$ agrees with $\mathtt{t}^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ outside the A-Garnir belt.

Proof. Since $w^{\mathtt{g}^A}$ fixes the entries outside the Garnir belt, $w^{\mathtt{t}} \leq w^{\mathtt{g}^A}$ must do the same. □

5.32. Bricks

Take $\boldsymbol{\lambda}/\boldsymbol{\mu} \in \mathscr{S}^{\kappa}_{\boldsymbol{\mu}}$ and Garnir node $A = (a, b, m) \in \boldsymbol{\lambda}/\boldsymbol{\mu}$. A brick is a set of nodes

$$\{(c, d, m'), (c, d+1, m'), \ldots, (c, d+e-1, m')\} \subseteq \mathbf{B}^A$$

such that $\mathrm{res}(c,d,m') = \mathrm{res}(A)$. Let k^A be the total number of bricks in $\mathbf{B}^A$, and let f^A be the number of bricks in row a of $\mathbf{B}^A$. Label the bricks $B_1^A, \ldots, B_{k^A}^A$ in order from left to right, beginning at the bottom left.

For $1 \le r \le k^A$, let $w_r^A \in \mathfrak{S}_d$ be the element that swaps B_r^A and B_{r+1}^A. Define the group of *brick permutations*

$$\mathfrak{S}^A := \langle w_1^A, \ldots, w_{k^A-1}^A \rangle \cong \mathfrak{S}_{k^A}.$$

This is the trivial group if $k^A = 0$, e.g. if $e = 0$.

Let Gar^A be the set of row-strict $\boldsymbol{\lambda}/\boldsymbol{\mu}$-tableaux which are are obtained by the action of $\mathfrak{S}^A$ on $\mathtt{g}^A$. All tableaux in Gar^A save $\mathtt{g}^A$ are standard. By Lemma 5.1.5, $\mathtt{g}^A$ is the unique maximal element of Gar^A, and there exists a unique minimal element $\mathtt{t}^A$, which has the bricks $B_1^A, \ldots, B_{f^A}^A$ in order from left to right in row a, and the remaining bricks in order from left to right in row $a+1$. By definition, if $\mathtt{t} \in \mathrm{Gar}^A$, then $\boldsymbol{i}(\mathtt{t}) = \boldsymbol{i}(\mathtt{g}^A)$. Define $\boldsymbol{i}^A$ as this common residue sequence.

Let $\mathscr{D}^A$ be the set of minimal length left coset representatives of $\mathfrak{S}_{f^A} \times \mathfrak{S}_{k^A - f^A}$ in $\mathfrak{S}^A$. We have

$$\mathrm{Gar}^A = \{w\mathtt{t}^A \mid w \in \mathscr{D}^A\}.$$

Lemma 5.3.2. Suppose that $\boldsymbol{\lambda}/\boldsymbol{\mu} \in \mathscr{S}_{\boldsymbol{\mu}}^{\kappa}$ and $A \in \boldsymbol{\lambda}/\boldsymbol{\mu}$ is a Garnir node. Then

$$\mathrm{Gar}^A \backslash \{\mathtt{g}^A\} = \{\mathtt{t} \in \mathrm{St}(\boldsymbol{\lambda}/\boldsymbol{\mu}) \mid \mathtt{t} \trianglelefteq \mathtt{g}^A \text{ and } \boldsymbol{i}(\mathtt{t}) = \boldsymbol{i}^A\}.$$

Moreover, $\deg \mathtt{t} = \deg s_r \mathtt{g}^A - a_{i_r, i_{r+1}}$ for all $\mathtt{t} \in \mathrm{Gar}^A \backslash \{\mathtt{g}^A\}$, where $r = \mathtt{g}^A(A) - 1$.

Proof. The first statement is clear from the preceding discussion and Lemma 5.3.1.

For the second statement, we instead prove that $\operatorname{codeg}(s_r \mathtt{g}^A) - \operatorname{codeg}(\mathtt{t}) = -a_{i_r,i_{r+1}}$, which is equivalent by [3, Lemma 3.12]. By the definition of codegree, and the fact that $\mathtt{Y}(\mathtt{t})$ and $\mathtt{Y}(s_r \mathtt{g}^A)$ agree outside of the bricks of the Garnir belt, it is enough to consider the case where $\boldsymbol{\lambda}/\boldsymbol{\mu}$ is a two-row Young diagram, with $\boldsymbol{\lambda} = \boldsymbol{\lambda}/\boldsymbol{\mu} = ((k^A e - 1, (k^A - f^A)e))$, and $A = (1, (k^A - f^A)e, 1)$. Each brick contributes 0 to $\operatorname{codeg}(\mathtt{t})$, and every brick contributes 0 to $\operatorname{codeg}(s_r \mathtt{g}^A)$, except for $B^A_{k^A - f^A}$ (the rightmost brick in the bottom row), which contributes 2 if $e = 2$ and 1 if $e > 2$. Thus $\operatorname{codeg}(\mathtt{t}) = 0$ and $\operatorname{codeg}(s_r \mathtt{g}^A) = -a_{i_r,i_{r+1}}$, and the result follows. □

5.33. The skew Specht module

Fix a Garnir node $A \in \boldsymbol{\lambda}/\boldsymbol{\mu}$. Define

$$\sigma_r^A := \psi_{w_r^A} 1_{\boldsymbol{i}^A} \quad \text{and} \quad \tau_r^A := (\sigma_r^A + 1) 1_{\boldsymbol{i}^A}.$$

Write $u \in \mathfrak{S}^A$ as a reduced product $w_{r_1}^A \cdots w_{r_a}^A$ of simple generators in $\mathfrak{S}^A$. If $u \in \mathscr{D}^A$, then u is fully commutative, and thus we have well-defined elements

$$\{\tau_u^A := \tau_{r_1}^A \cdots \tau_{r_a}^A \mid u \in \mathscr{D}^A\}.$$

For any $\mathtt{s} \in \operatorname{Gar}^A$, we may can write $w^{\mathtt{s}} = u^{\mathtt{s}} w^{\mathtt{t}^A}$ so that $\ell(w^{\mathtt{s}}) = \ell(u^{\mathtt{s}}) + \ell(w^{\mathtt{t}^A})$ and $u^{\mathtt{s}} \in \mathscr{D}^A$, and the elements $\psi_{u^{\mathtt{s}}}$, $\psi^{\mathtt{t}^A}$ and $\psi^{\mathtt{s}} = \psi_{u^{\mathtt{s}}} \psi^{\mathtt{t}^A}$ are all independent of the choice of reduced decomposition.

Definition 5.3.3. Let $\boldsymbol{\lambda}/\boldsymbol{\mu} \in \mathscr{S}^{\kappa}_{\boldsymbol{\mu},\alpha}$, and $A \in \boldsymbol{\lambda}/\boldsymbol{\mu}$ be a Garnir node. The *Garnir element* is

$$g^A := \sum_{u \in \mathscr{D}^A} \tau_u^A \psi^{\mathtt{t}^A} \in R_\alpha. \tag{5.1}$$

By Lemma 5.3.2, all summands on the right side of (5.1) are of the same degree.

Definition 5.3.4. Let $\alpha \in Q_+$, $d = \mathrm{ht}(\alpha)$, and $\boldsymbol{\lambda}/\boldsymbol{\mu} \in \mathscr{S}^{\kappa}_{\boldsymbol{\mu},\alpha}$. Define the *graded skew row permutation module* $M^{\boldsymbol{\lambda}/\boldsymbol{\mu}} = M^{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathcal{O})$ to be the graded R_α-module generated by the vector $m^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ in degree $\deg \mathtt{t}^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ and subject only to the following relations:

(i) $1_{\boldsymbol{j}} m^{\boldsymbol{\lambda}/\boldsymbol{\mu}} = \delta_{\boldsymbol{j},\boldsymbol{i}^{\boldsymbol{\lambda}/\boldsymbol{\mu}}} m^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ for all $\boldsymbol{j} \in \langle I \rangle_\alpha$;

(ii) $y_r m^{\boldsymbol{\lambda}/\boldsymbol{\mu}} = 0$ for all $r = 1, \dots, d$;

(iii) $\psi_r m^{\boldsymbol{\lambda}/\boldsymbol{\mu}} = 0$ for all $r = 1, \dots, d-1$ such that $r \to_{\mathtt{t}^{\boldsymbol{\lambda}/\boldsymbol{\mu}}} r+1$.

Definition 5.3.5. Let $\alpha \in Q_+$, $d = \mathrm{ht}(\alpha)$, and $\boldsymbol{\lambda}/\boldsymbol{\mu} \in \mathscr{S}^{\kappa}_{\boldsymbol{\mu},\alpha}$. We define the *graded skew Specht module* $S^{\boldsymbol{\lambda}/\boldsymbol{\mu}} = S^{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathcal{O})$ to be the graded R_α-module generated by the vector $z^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ in degree $\deg \mathtt{t}^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ and subject only to the following relations:

(i) $1_{\boldsymbol{j}} z^{\boldsymbol{\lambda}/\boldsymbol{\mu}} = \delta_{\boldsymbol{j},\boldsymbol{i}^{\boldsymbol{\lambda}/\boldsymbol{\mu}}} z^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ for all $\boldsymbol{j} \in \langle I \rangle_\alpha$;

(ii) $y_r z^{\boldsymbol{\lambda}/\boldsymbol{\mu}} = 0$ for all $r = 1, \dots, d$;

(iii) $\psi_r z^{\boldsymbol{\lambda}/\boldsymbol{\mu}} = 0$ for all $r = 1, \dots, d-1$ such that $r \to_{\mathtt{t}^{\boldsymbol{\lambda}/\boldsymbol{\mu}}} r+1$;

(iv) $g^A z^{\boldsymbol{\lambda}/\boldsymbol{\mu}} = 0$ for all Garnir nodes $A \in \boldsymbol{\lambda}/\boldsymbol{\mu}$.

In other words, $S^{\boldsymbol{\lambda}/\boldsymbol{\mu}} = (R_\alpha/J_\alpha^{\boldsymbol{\lambda}/\boldsymbol{\mu}})\langle \deg(\mathtt{t}^{\boldsymbol{\lambda}/\boldsymbol{\mu}})\rangle$, where $J_\alpha^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ is the homogeneous left ideal of R_α generated by the elements

(i*) $1_{\boldsymbol{j}} - \delta_{\boldsymbol{j},\boldsymbol{i}^{\boldsymbol{\lambda}/\boldsymbol{\mu}}}$ for all $\boldsymbol{j} \in \langle I \rangle_\alpha$;

(ii*) y_r for all $r = 1, \dots, d$;

(iii*) ψ_r for all $r = 1, \dots, d-1$ such that $r \to_{\mathtt{t}^{\boldsymbol{\lambda}/\boldsymbol{\mu}}} r+1$;

(iv*) g^A for all Garnir nodes $A \in \boldsymbol{\lambda}/\boldsymbol{\mu}$.

The elements (i*)-(iii*) generate a left ideal $K^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ such that $R_\alpha/K^{\boldsymbol{\lambda}/\boldsymbol{\mu}} \cong M^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$. So we have a natural surjection $M^{\boldsymbol{\lambda}/\boldsymbol{\mu}} \twoheadrightarrow S^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ with kernel $J^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ generated by the Garnir relations $g^A m^{\boldsymbol{\lambda}/\boldsymbol{\mu}} = 0$. This surjection maps $m^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ to $z^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ and $J^{\boldsymbol{\lambda}/\boldsymbol{\mu}} = J_\alpha^{\boldsymbol{\lambda}/\boldsymbol{\mu}} m^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$.

For $\mathtt{t} \in \mathrm{Tab}(\boldsymbol{\lambda}/\boldsymbol{\mu})$, we write

$$m^{\mathtt{t}} := \psi^{\mathtt{t}} m^{\boldsymbol{\lambda}/\boldsymbol{\mu}} \in M^{\boldsymbol{\lambda}/\boldsymbol{\mu}} \qquad \text{and} \qquad v^{\mathtt{t}} := \psi^{\mathtt{t}} z^{\boldsymbol{\lambda}/\boldsymbol{\mu}} \in S^{\boldsymbol{\lambda}/\boldsymbol{\mu}}.$$

5.34. A basis for $M^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$and a spanning set for $S^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$

Theorem 5.3.6. The elements of the set

$$\{m^{\mathtt{t}} \mid \mathtt{t} \in \mathrm{Tab}(\boldsymbol{\lambda}/\boldsymbol{\mu}) \text{ is row-strict}\}$$

form an $\mathcal{O}$-basis for $M^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$.

Proof. This is [27, Theorem 5.6] in the Young diagram case. But since $M^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ is a permutation module in the sense of [27, §3.6], the proof in the skew case also follows immediately from [27, Theorem 3.23]. □

Proposition 5.3.1. *The elements of the set*

$$\{v^{\mathtt{t}} \mid \mathtt{t} \in \mathrm{St}(\boldsymbol{\lambda}/\boldsymbol{\mu})\} \tag{5.2}$$

span $S^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ *over* $\mathcal{O}$. *Moreover, we have* $\deg(v^{\mathtt{t}}) = \deg(\mathtt{t})$ *for all* $\mathtt{t} \in \mathrm{St}(\boldsymbol{\lambda}/\boldsymbol{\mu})$.

Proof. Using Lemma 5.2.1,

$$\deg(v^{\mathtt{t}}) = \deg(\psi^{\mathtt{t}} 1_{\boldsymbol{i}^{\boldsymbol{\lambda}/\boldsymbol{\mu}}} z^{\boldsymbol{\lambda}/\boldsymbol{\mu}}) = \deg(\psi^{\mathtt{t}} 1_{\boldsymbol{i}^{\boldsymbol{\lambda}/\boldsymbol{\mu}}}) + \deg(z^{\boldsymbol{\lambda}/\boldsymbol{\mu}}) = \deg(\mathtt{t}) - \deg(\mathtt{t}^{\boldsymbol{\lambda}/\boldsymbol{\mu}}) + \deg(z^{\boldsymbol{\lambda}/\boldsymbol{\mu}}),$$

which proves the second statement, as $\deg(z^{\boldsymbol{\lambda}/\boldsymbol{\mu}}) = \deg(\mathtt{t}^{\boldsymbol{\lambda}/\boldsymbol{\mu}})$ by definition.

The proof of the first statement follows exactly as it does in the Young diagram case provided in [27, Proposition 5.14]—there are clear skew analogues of the results in [27, §5.5–5.6]—the only caveat is that our preferred partial order on standard tableaux is opposite that of [27], so one must swap the direction of '$\triangleleft$' signs when necessary, and make use of the analogous skew dominance results in Lemmas 5.1.1–5.1.6. □

5.4. Restrictions of Specht modules

In this section we show that for $\boldsymbol{\lambda} \in \mathscr{P}^{\kappa}_{\alpha+\beta}$, the $R_{\alpha,\beta}$-module $\mathrm{Res}_{\alpha,\beta} S^{\boldsymbol{\lambda}}$ has a filtration with subquotients isomorphic to $S^{\boldsymbol{\mu}} \boxtimes S^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$, with $\boldsymbol{\mu} \in \mathscr{P}^{\kappa}_{\alpha}$ and $\boldsymbol{\lambda}/\boldsymbol{\mu} \in \mathscr{S}^{\kappa}_{\boldsymbol{\mu},\beta}$. As a consequence, we get that (5.2) is an $\mathcal{O}$-basis for $S^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$. For the case of Young diagrams, this was shown in [27, Corollary 6.24]:

Theorem 5.4.1. Let $\boldsymbol{\lambda} \in \mathscr{P}^{\kappa}_{\alpha}$. Then $S^{\boldsymbol{\lambda}}$ has $\mathcal{O}$-basis $\{v^{\mathtt{T}} \mid \mathtt{T} \in \mathrm{St}(\boldsymbol{\lambda})\}$.

5.41. Submodules of $\mathrm{Res}_{\alpha,\beta} S^{\boldsymbol{\lambda}}$

Let $\alpha, \beta \in Q_+$ and $\mathrm{ht}(\alpha) = a, \mathrm{ht}(\beta) = b$. Let $\boldsymbol{\lambda} \in \mathscr{P}^{\kappa}_{\alpha+\beta}$, $\boldsymbol{\mu} \in \mathscr{P}^{\kappa}_{\alpha}$. By Theorem 5.4.1, $S^{\boldsymbol{\lambda}}_{\alpha,\beta} := \mathrm{Res}_{\alpha,\beta}(S^{\boldsymbol{\lambda}})$ has $\mathcal{O}$-basis $\{v^{\mathtt{T}} \mid \mathtt{T} \in B\}$, where

$$B = \{\mathtt{T} \in \mathrm{St}(\lambda) \mid \ \mathrm{cont}(\mathrm{sh}(\mathtt{T}_{\leq a})) = \alpha\},$$

since $1_{\alpha,\beta}v^{\mathtt{T}} = v^{\mathtt{T}}$ if and only if $\boldsymbol{i}(\mathtt{T}) = i_1 \cdots i_{a+b}$ has $\alpha_{i_1} + \cdots + \alpha_{i_a} = \alpha$, and is zero otherwise. Define

$$B_{\boldsymbol{\mu}} = \{\mathtt{T} \in B \mid \mathrm{sh}(\mathtt{T}_{\leq a}) \trianglerighteq \boldsymbol{\mu}\} \qquad \text{and} \qquad C_{\boldsymbol{\mu}} = \mathcal{O}\{v^{\mathtt{T}} \in S^{\boldsymbol{\lambda}} \mid \mathtt{T} \in B_{\boldsymbol{\mu}}\}.$$

Lemma 5.4.2. If $\mathtt{U}, \mathtt{T} \in B$, $\mathtt{U} \trianglelefteq \mathtt{T}$, and $\mathtt{T} \in B_{\boldsymbol{\mu}}$, then $\mathtt{U} \in B_{\boldsymbol{\mu}}$.

Proof. If $\mathtt{U} \trianglelefteq \mathtt{T}$, then by Lemma 5.1.5, $\mathrm{sh}(\mathtt{U}_{\leq a}) \trianglerighteq \mathrm{sh}(\mathtt{T}_{\leq a}) \trianglerighteq \boldsymbol{\mu}$. □

Lemma 5.4.3. $C_{\boldsymbol{\mu}}$ is an $R_{\alpha,\beta}$-submodule of $S^{\boldsymbol{\lambda}}_{\alpha,\beta}$.

Proof. We show that $C_{\boldsymbol{\mu}}$ is invariant under the action of generators of $R_{\alpha,\beta}$. For idempotents $1_{\boldsymbol{ij}}$ this is clear. Let $\mathtt{T} \in B_{\boldsymbol{\mu}}$.

(i) For $1 \leq j \leq a+b$, $y_j v^{\mathtt{T}}$ is an $\mathcal{O}$-linear combination of $v^{\mathtt{U}} \in B$ for $\mathtt{U} \triangleleft \mathtt{T}$, by [3, Lemma 4.8]. By Lemma 5.4.2, each $v^{\mathtt{U}}$ is in $C_{\boldsymbol{\mu}}$.

(ii) For $j \in \{1, \ldots, a-1, a+1, \ldots, a+b-1\}$, where $j \rightarrow_{\mathtt{T}} j+1$ or $j \downarrow_{\mathtt{T}} j+1$, then $\psi_j v^{\mathtt{T}}$ is a linear combination of $v^{\mathtt{U}} \in B$ for $\mathtt{U} \triangleleft \mathtt{T}$, by [3, Lemma 4.9], and the result follows by Lemma 5.4.2.

(iii) For $j \in \{1, \ldots, a-1, a+1, \ldots, a+b-1\}$, where $j \nearrow_{\mathtt{T}} j+1$, then $\psi_j v^{\mathtt{T}}$ is a linear combination of $v^{\mathtt{U}} \in B$ for $\mathtt{U} \triangleleft \mathtt{T}$, by [11, Lemma 2.14], and the result follows by Lemma 5.4.2.

(iv) Assume $j \in \{1, \ldots, a-1, a+1, \ldots, a+b-1\}$, and $j+1 \nearrow_{\mathtt{T}} j$. Then $s_j\mathtt{T} \triangleright \mathtt{T}$, and $s_j w^{\mathtt{T}} = w^{s_j\mathtt{T}}$, with $\ell(w^{s_j\mathtt{T}}) = \ell(w^{\mathtt{T}}) + 1$. Then by Lemma 5.2.1, $\psi_j v^{\mathtt{T}} = v^{s_j\mathtt{T}} + \sum_{\mathtt{U} \triangleleft s_j\mathtt{T}} c_{\mathtt{U}} v^{\mathtt{U}}$ for some constants $c_{\mathtt{U}} \in \mathcal{O}$. But $(s_j\mathtt{T})_{\leq a} = \mathtt{T}_{\leq a}$, so $s_j\mathtt{T} \in B_{\boldsymbol{\mu}}$ and the result follows by Lemma 5.4.2.

This exhausts the possibilities for $\mathtt{T}$ and completes the proof. □

Now define $B_{\rhd\boldsymbol{\mu}} = \bigcup_{\boldsymbol{\nu}\rhd\boldsymbol{\mu}} B_{\boldsymbol{\nu}} = \{\mathtt{T} \in B \mid \mathrm{sh}(\mathtt{T}_{\le a}) \rhd \boldsymbol{\mu}\}$. Then $C_{\rhd\boldsymbol{\mu}} := \sum_{\boldsymbol{\nu}\rhd\boldsymbol{\mu}} C_{\boldsymbol{\nu}} = \mathcal{O}\{v^{\mathtt{T}} \in S^{\boldsymbol{\lambda}} \mid \mathrm{sh}(\mathtt{T}_{\le a}) \rhd \boldsymbol{\mu}\}$ is an $R_{\alpha,\beta}$-submodule of $S^{\boldsymbol{\lambda}}_{\alpha,\beta}$. Define $N_{\boldsymbol{\mu}} = C_{\boldsymbol{\mu}}/C_{\rhd\boldsymbol{\mu}}$, and write

$$x^{\mathtt{T}} = v^{\mathtt{T}} + C_{\rhd\boldsymbol{\mu}} \in S^{\boldsymbol{\lambda}}_{\alpha,\beta}/C_{\rhd\boldsymbol{\mu}}$$

for $\mathtt{T} \in B$. To cut down on notational clutter in what follows, write $\boldsymbol{\xi}$ for $\boldsymbol{\lambda}/\boldsymbol{\mu}$, $\xi^{(i)}$ for the components $\lambda^{(i)}/\mu^{(i)}$ of $\boldsymbol{\lambda}/\boldsymbol{\mu}$, and $\xi^{(i)}_j$ for the jth row of nodes in $\xi^{(i)}$. Then for $\mathtt{T} \in \mathrm{Tab}(\boldsymbol{\mu})$, $\mathtt{t} \in \mathrm{Tab}(\boldsymbol{\xi})$, define $\mathtt{Tt} \in \mathrm{Tab}(\boldsymbol{\lambda})$ such that $(\mathtt{Tt})_{\le a} = \mathtt{T}$ and $\mathtt{Tt}(A) = \mathtt{Y}(\mathtt{t})(A)$ for nodes $A \in \boldsymbol{\xi}$. From the definition it is clear that $N_{\boldsymbol{\mu}}$ has homogeneous $\mathcal{O}$-basis

$$\{x^{\mathtt{T}} \mid \mathtt{T} \in \mathrm{St}(\boldsymbol{\lambda}), \mathrm{sh}(\mathrm{cont}(\mathtt{T}_{\le a})) = \boldsymbol{\mu}\} = \{x^{\mathtt{Tt}} \mid \mathtt{T} \in \mathrm{St}(\boldsymbol{\mu}), \mathtt{t} \in \mathrm{St}(\boldsymbol{\xi})\}.$$

Write $\mathtt{T}^{\boldsymbol{\mu}\boldsymbol{\xi}} := \mathtt{T}^{\boldsymbol{\mu}}\mathtt{t}^{\boldsymbol{\xi}} = \mathtt{Y}(\mathtt{t}^{\boldsymbol{\xi}})$, and write $x^{\boldsymbol{\mu}\boldsymbol{\xi}}$ for $x^{\mathtt{T}^{\boldsymbol{\mu}\boldsymbol{\xi}}}$.

5.42. Constructing a morphism $S^{\boldsymbol{\mu}} \boxtimes S^{\boldsymbol{\lambda}/\boldsymbol{\mu}} \to N_{\boldsymbol{\mu}}$

Define a (graded) morphism f from the free module $R_{\alpha,\beta}\langle \deg \mathtt{T}^{\boldsymbol{\mu}} + \deg \mathtt{t}^{\boldsymbol{\xi}}\rangle$ to $N_{\boldsymbol{\mu}}$ by $f : 1_{\alpha,\beta} \mapsto x^{\boldsymbol{\mu}\boldsymbol{\xi}}$.

Proposition 5.4.1. *The kernel of f contains the left ideal $K^{\boldsymbol{\mu}}_{\alpha} \otimes R_{\beta} + R_{\alpha} \otimes K^{\boldsymbol{\xi}}_{\beta}$.*

Proof. We show that the relevant generators of $K^{\boldsymbol{\mu}}_{\alpha} \otimes R_{\beta}$, given by (i*)–(iii*) in Definition 5.3.5 are sent to zero by f. The proof for $R_{\alpha} \otimes K^{\boldsymbol{\xi}}_{\beta}$ is similar.

(i*) First we consider idempotents.

$$f[(1_{\boldsymbol{j}} - \delta_{\boldsymbol{j},\boldsymbol{i}^{\mu}}) \otimes 1_\beta] = (1_{\boldsymbol{j},\beta} - \delta_{\boldsymbol{j},\boldsymbol{i}^{\mu}})x^{\boldsymbol{\mu\xi}} = \sum_{\boldsymbol{k}\in I^\beta} 1_{\boldsymbol{jk}}x^{\boldsymbol{\mu\xi}} - \delta_{\boldsymbol{j},\boldsymbol{i}^{\mu}}x^{\boldsymbol{\xi}}$$

$$= \sum_{\boldsymbol{k}\in I^\beta} \delta_{\boldsymbol{jk},\boldsymbol{i}^{\mu}\boldsymbol{i}^{\xi}}x^{\boldsymbol{\mu\xi}} - \delta_{\boldsymbol{j},\boldsymbol{i}^{\mu}}x^{\boldsymbol{\mu\xi}} = \delta_{\boldsymbol{ji}^{\xi},\boldsymbol{i}^{\mu}\boldsymbol{i}^{\xi}}x^{\boldsymbol{\mu\xi}} - \delta_{\boldsymbol{j},\boldsymbol{i}^{\mu}}x^{\boldsymbol{\mu\xi}} = 0.$$

(ii*) For $1 \le r \le a$, we have by [3, Lemma 4.8] that $f(y_r) = y_r \cdot x^{\boldsymbol{\mu\xi}}$ is an $\mathcal{O}$-linear combination of $x^{\mathtt{U}}$, where $\mathtt{U} \in B$ and $\mathtt{U} \triangleleft \mathtt{T}^{\boldsymbol{\mu\xi}}$. But $\mathtt{T}^{\boldsymbol{\mu\xi}}$ is minimal such that $\mathrm{sh}(\mathtt{T}_{\le a}) = \boldsymbol{\mu}$, so each $\mathtt{U} \in B_{\triangleright\boldsymbol{\mu}}$, and thus $f(y_r) = 0$.

(iii*) Note that $r \rightarrow_{\mathtt{T}^{\mu}} r+1$ implies $r \rightarrow_{\mathtt{T}^{\mu\xi}} r+1$, so by [3, Lemma 4.9] it follows that for $1 \le r \le a-1$, $f(\psi_r) = \psi_r x^{\boldsymbol{\mu\xi}}$ is an $\mathcal{O}$-linear combination of $x^{\mathtt{U}}$, where $\mathtt{U} \in B$ and $\mathtt{U} \triangleleft \mathtt{T}^{\boldsymbol{\mu\xi}}$. But then as in (2) this implies that $f(\psi_r) = 0$.

□

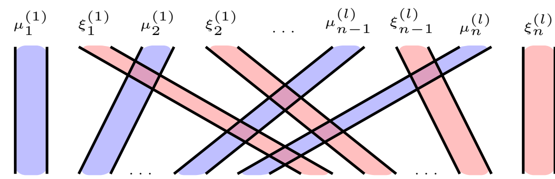
The goal in the rest of this section is to show that in fact, the kernel of f contains the the left ideal $J^{\boldsymbol{\mu}}_\alpha \otimes R_\beta + R_\alpha \otimes J^{\boldsymbol{\xi}}_\beta$, i.e., $g^{A_{\boldsymbol{\mu}}} \otimes 1_\beta$ (resp. $1_\alpha \otimes g^{A_{\boldsymbol{\xi}}}$) are sent to zero by f, for Garnir nodes $A_{\boldsymbol{\mu}} \in \boldsymbol{\mu}$ (resp. $A_{\boldsymbol{\xi}} \in \boldsymbol{\xi}$). As the proofs for $\boldsymbol{\mu}$ and $\boldsymbol{\xi}$ are similar (see Remark 5.4.9), we focus on the former and leave the latter for the reader to verify. We will occasionally need to make use of the following lemma, proved in [11, Lemma 2.16]:

Lemma 5.4.4. Suppose $\boldsymbol{\lambda} \in \mathscr{P}^\kappa_\alpha$, $\mathtt{T} \in \mathrm{St}(\boldsymbol{\lambda})$, $j_1, \ldots, j_r \in \{1, \ldots, d-1\}$, and that when $\psi_{j_1} \cdots \psi_{j_r} z^{\boldsymbol{\lambda}}$ is expressed as a linear combination of standard basis elements, $v^{\mathtt{T}}$ appears with non-zero coefficient. Then the expression $s_{j_1} \cdots s_{j_r}$ has a reduced expression for $w^{\mathtt{T}}$ as a subexpression.

Note that $w^{\boldsymbol{\mu}\boldsymbol{\xi}} := w^{\mathtt{T}^{\boldsymbol{\mu}\boldsymbol{\xi}}}$ is in $\mathscr{D}_{a,b}^{\mu_1^{(1)},\xi_1^{(1)},\ldots,\mu_{n(\boldsymbol{\lambda},l)}^{(l)},\xi_{n(\boldsymbol{\lambda},l)}^{(l)}}$, the set of minimal length double coset representatives for

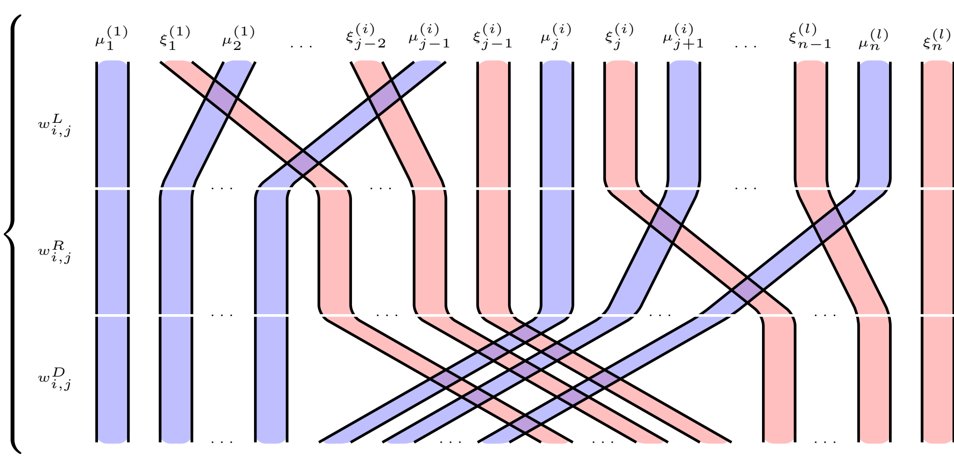
$$\mathfrak{S}_a \times \mathfrak{S}_b \backslash \mathfrak{S}_{a+b} / \mathfrak{S}_{\mu_1^{(1)}} \times \mathfrak{S}_{\xi_1^{(1)}} \times \cdots \times \mathfrak{S}_{\mu_{n(\boldsymbol{\lambda},l)}^{(l)}} \times \mathfrak{S}_{\xi_{n(\boldsymbol{\lambda},l)}^{(l)}},$$

and as such is fully commutative. Writing $n := n(\boldsymbol{\lambda}, l)$, in diagrammatic form we have

$w^{\boldsymbol{\mu}\boldsymbol{\xi}} =$  .

Here we are letting $\mu_i^{(j)}$ in the diagram stand for $(a_1, \ldots, a_k)$, where $a_1, \ldots, a_k$ are the entries (in order) in $\mathtt{T}^{\boldsymbol{\lambda}}$ of the nodes contained in the ith row of $\mu^{(j)}$, and similarly for $\xi_i^{(j)}$.

Let $1 \leq i \leq l$, $1 \leq j \leq n(\boldsymbol{\lambda}, i)$. It will be useful to write $w^{\boldsymbol{\mu}\boldsymbol{\xi}} = w_{i,j}^D w_{i,j}^R w_{i,j}^L$, the decomposition into fully commutative elements of $\mathfrak{S}_{a+b}$ given as follows:

$w^{\boldsymbol{\mu}\boldsymbol{\xi}} =$  .

Define $\psi^X_{i,j} := \psi_{w^X_{i,j}}$ for $X \in \{L, R, D\}$, and set

$$c_{i,j} = \sum_{\substack{1 \le h \le i-1 \\ 1 \le k \le n(\boldsymbol{\lambda},h)}} \mu^{(h)}_k + \sum_{1 \le k \le j-1} \mu^{(i)}_k,$$
$$d_{i,j} = \sum_{\substack{1 \le h \le i-1 \\ 1 \le k \le n(\boldsymbol{\lambda},h)}} \xi^{(h)}_k + \sum_{1 \le k \le j-1} \xi^{(i)}_k.$$

If $\Psi := \psi_{r_1} \cdots \psi_{r_s}$ for some $r_1, \dots, r_s$, then we will write $\Psi[c] := \psi_{r_1+c} \cdots \psi_{r_s+c}$ for admissible $c \in \mathbb{Z}$. The following lemma will aid us in translating between Garnir relations defining $S^{\boldsymbol{\lambda}}$ and those defining $S^{\boldsymbol{\mu}}$.

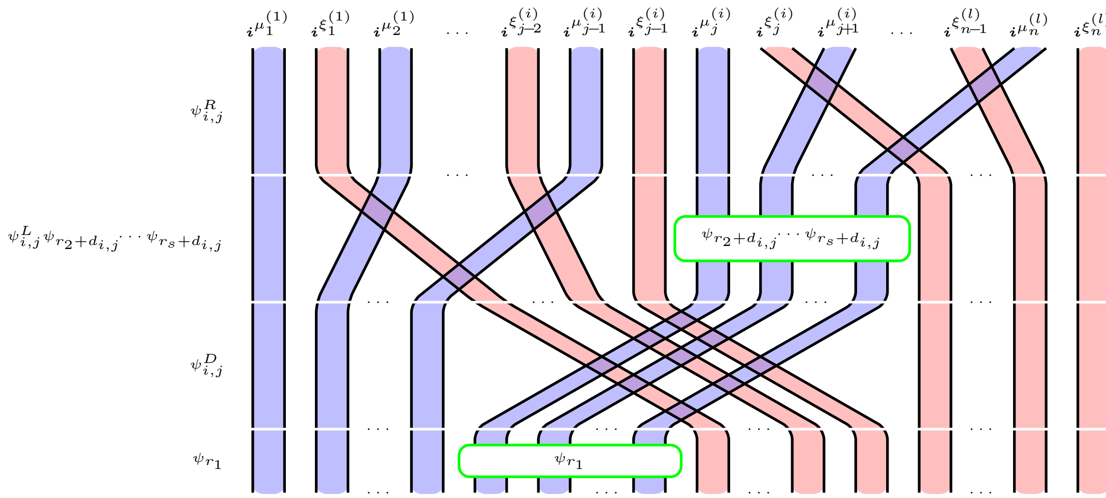
Lemma 5.4.5. Assume $r_1, \dots, r_s$ are such that $c_{i,j} + 1 \le r_1, \dots, r_s \le a - 1$, and $\Psi = \psi_{r_1} \cdots \psi_{r_s}$. Then

$$\Psi x^{\boldsymbol{\mu}\boldsymbol{\xi}} = \psi^D_{i,j} \psi^L_{i,j} \Psi[d_{i,j}] \psi^R_{i,j} x^{\boldsymbol{\lambda}}.$$

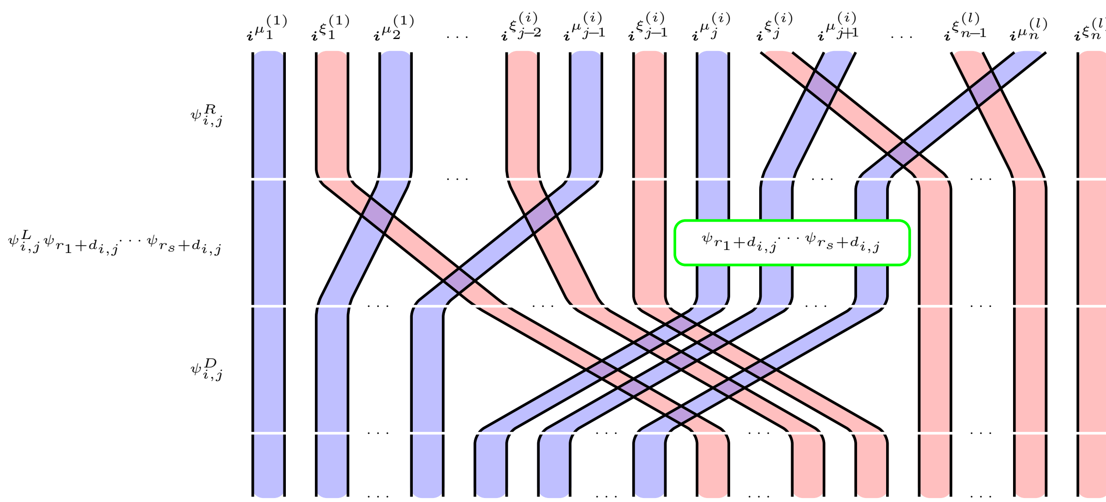
Proof. We go by induction on s, the base case $s = 0$ being trivial. By assumption we have

$$\Psi x^{\boldsymbol{\mu}\boldsymbol{\xi}} = \psi_{r_1} \cdots \psi_{r_s} x^{\boldsymbol{\mu}\boldsymbol{\xi}} = \psi_{r_1} \psi^D_{i.j} \psi^L_{i.j} \psi_{r_2+d_{i,j}} \cdots \psi_{r_s+d_{i,j}} \psi^R_{i,j} x^{\boldsymbol{\lambda}}.$$

Write $\boldsymbol{i}^{\mu^{(h)}_k}$ for the residue sequence associated with the nodes in $\mu^{(h)}_k$ in $\mathtt{T}^{\boldsymbol{\lambda}}$, and similarly for $\boldsymbol{i}^{\xi^{(h)}_k}$. In terms of Khovanov-Lauda diagrams, with the vector $x^{\boldsymbol{\lambda}}$ pictured as being at the top of the diagram, we must show that



is equal to



Let $\boldsymbol{j} = w^L_{i,j}w_{r_2+d_{i,j}}\cdots w_{r_s+d_{i,j}}w^R_{i,j}\boldsymbol{i}^{\boldsymbol{\lambda}}$. Since $s_{r_1}w^D_{i,j} = w^D_{i,j}s_{r_1+d_{i,j}}$ and $\ell(s_{r_1}w^D_{i,j}) = \ell(w^D_{i,j}) + \ell(s_{r_1})$, it follows from Lemma 5.2.1 that

$$\psi_{r_1}\psi^D_{i,j}1_{\boldsymbol{j}} = \psi^D_{i,j}\psi_{r_1+d_{i,j}}1_{\boldsymbol{j}} + \sum_{u \lhd w^D_{i,j}} c_u\psi_u\psi^{\epsilon_u}_{r_1+d_{i,j}}f_u(y)1_{\boldsymbol{j}} \tag{5.3}$$

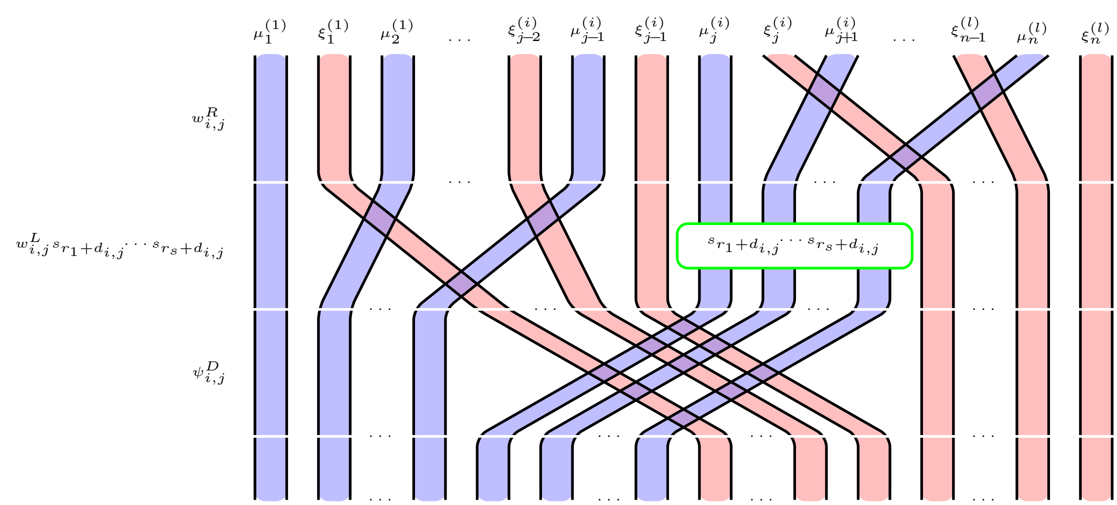
for some constants $c_u \in \mathcal{O}$, polynomials $f_u(y_1, \ldots, y_{b+d})$, and $\epsilon_u \in \{0,1\}$. Thus it remains to show that

$$\psi_u \psi_{r_1+d_{i,j}}^{\epsilon_u} f_u(y) \psi_{r_2+d_{i,j}} \cdots \psi_{r_s+d_{i,j}} \psi_{i,j}^R \psi_{i,j}^L x^{\boldsymbol{\lambda}} = 0 \in S^{\boldsymbol{\lambda}}/C_{\triangleright \boldsymbol{\mu}}$$

for all u in the sum in (5.3). Let $s_{t_1^R} \cdots s_{t_{N_R}^R}$ be the preferred reduced expression for $w_{i,j}^R$, and similarly for $w_{i,j}^L$. Pushing the y's to the right to act (as zero) on $x^{\boldsymbol{\lambda}}$, this is by lemma 5.2.1 an $\mathcal{O}$-linear combination of terms of the form

$$\psi_u \psi_{r_1+d_{i,j}}^{\epsilon_u} \psi_{r_2+d_{i,j}}^{\epsilon_2} \cdots \psi_{r_s+d_{i,j}}^{\epsilon_s} \psi_{t_1^R}^{\epsilon_{s+1}} \cdots \psi_{t_{N_R}^R}^{\epsilon_{s+N_R}} \psi_{t_1^L}^{\epsilon_{s+N_R+1}} \cdots \psi_{t_{N^L}^L}^{\epsilon_{s+N_R+N_L}} x^{\boldsymbol{\lambda}} \tag{5.4}$$

for some $\epsilon_i \in \{0,1\}$. Write Θ for the sequence of ψ's in (5.4). Assume $v^{\mathtt{U}}$ appears with nonzero coefficient when $\Theta v^{\boldsymbol{\lambda}}$ is expanded in terms of basis elements. Then it follows from Lemma 5.4.4 that one can write $w^{\mathtt{U}}$ diagrammatically by removing crossings from the diagram



and in particular, removing at least one crossing from $w_{i,j}^D$, the third row of the diagram, since $u \triangleleft w_{i,j}^D$. But in any case, this implies that there is a pink strand that ends to the left of a blue strand, i.e., some $t \leq a$ such that $(w^{\mathtt{U}})^{-1}(t)$ is in $\xi_k^{(h)}$ for some h, k. Then $\mathrm{sh}(\mathtt{U}_{\leq a}) \neq \boldsymbol{\mu}$. But since $N_{\boldsymbol{\mu}}$ is an $R_{\alpha,\beta}$-submodule, we must have $\mathtt{U} \in B_{\boldsymbol{\mu}}$. This implies that $\mathtt{U} \in B_{\rhd\boldsymbol{\mu}}$, and hence $x^{\mathtt{U}} = 0 \in S^{\lambda}/C_{\boldsymbol{\mu}}$. □

Let $A_{\boldsymbol{\mu}}$ be a Garnir node in $\boldsymbol{\mu}$. This is also a Garnir node of $\boldsymbol{\lambda}$, and when we consider it as such, we will label it with $A_{\boldsymbol{\lambda}}$. Let $\mathbf{B}^{A_{\boldsymbol{\lambda}}}$ be the Garnir belt associated with $A_{\boldsymbol{\lambda}}$, and let $\mathbf{B}^{A_{\boldsymbol{\mu}}}$ be the Garnir belt of nodes in $\boldsymbol{\mu}$. Assume $A_{\boldsymbol{\lambda}}$ is in row j of the ith component of $\boldsymbol{\lambda}$. We subdivide the sets of nodes of $\mu_j^{(i)}$, $\mu_{j+1}^{(i)}$ and $\xi_j^{(i)}$ in the following fashion:

(i) We subdivide $\mu_j^{(i)}$ into three sets:

  (a) Let $\mu^{A,1}$ be the nodes of $\mu_j^{(i)}$ not contained in $\mathbf{B}^{A_{\boldsymbol{\mu}}}$.

  (b) Let $\mu^{A,2}$ be the nodes of $\mu_j^{(i)}$ contained in bricks in $\mathbf{B}^{A_{\boldsymbol{\mu}}}$.

  (c) Let $\mu^{A,3}$ be the nodes of $\mu_j^{(i)}$ contained in $\mathbf{B}^{A_{\boldsymbol{\mu}}}$, but not contained in any brick.

(ii) We subdivide $\xi_j^{(i)}$ into three sets:

  (a) Let $\xi^{A,1}$ be the nodes of $\xi_j^{(i)}$ contained in a brick in $\mathbf{B}^{A_{\boldsymbol{\lambda}}}$ which contains nodes of $\boldsymbol{\mu}$.

  (b) Let $\xi^{A,2}$ be the nodes of $\xi_j^{(i)}$ contained in a brick in $\mathbf{B}^{A_{\boldsymbol{\lambda}}}$ which is entirely contained in $\boldsymbol{\xi}$.

  (c) Let $\xi^{A,3}$ be the nodes of $\xi_j^{(i)}$ contained in $\mathbf{B}^{A_{\boldsymbol{\lambda}}}$, but not contained in any brick.

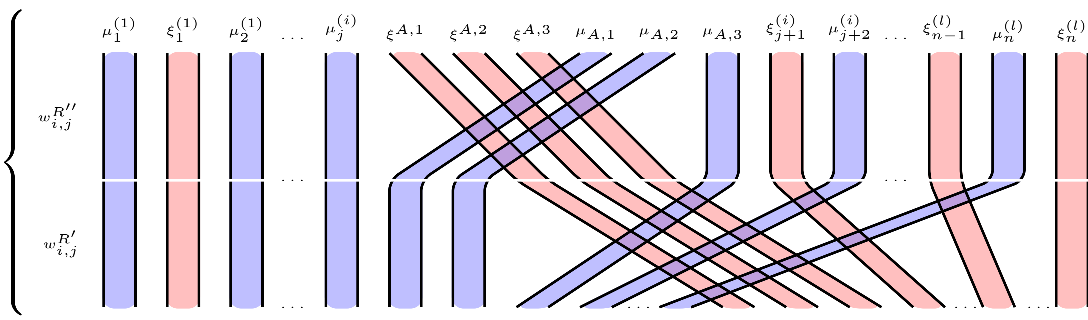
(iii) We subdivide $\mu_{j+1}^{(i)}$ into three sets:

(a) Let $\mu_{A,1}$ be the nodes of $\mu^{(i)}_{j+1}$ contained in $\mathbf{B}^{A\boldsymbol{\mu}}$ but not contained in any brick.

(b) Let $\mu_{A,2}$ be the nodes of $\mu^{(i)}_{j+1}$ contained in bricks in $\mathbf{B}^{A\boldsymbol{\mu}}$.

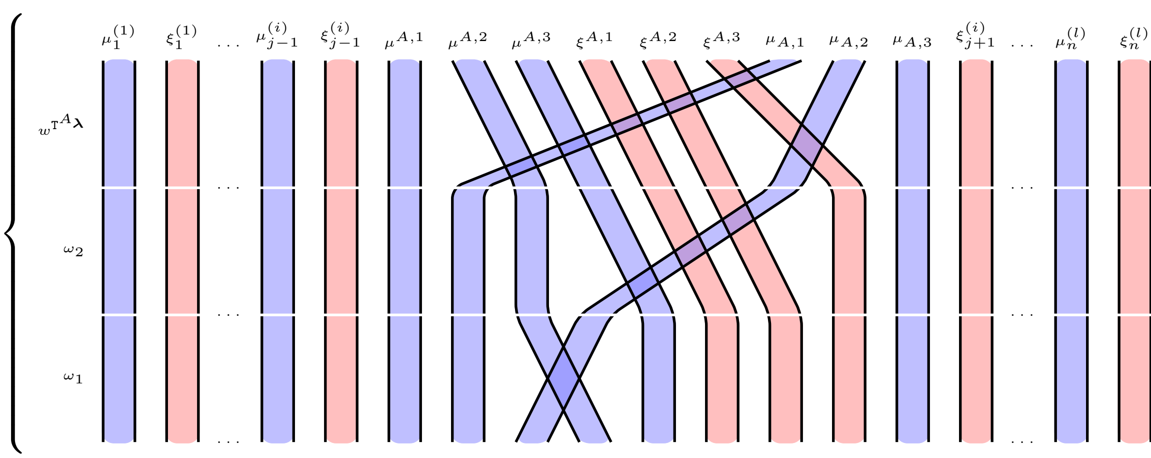
(c) Let $\mu_{A,3}$ be the nodes of $\mu^{(i)}_{j+1}$ not contained in $\mathbf{B}^{A\boldsymbol{\mu}}$.

Now write $w^R_{i,j} = w^{R'}_{i,j} w^{R''}_{i,j}$, where $w^{R'}_{i,j}$, $w^{R''}_{i,j}$ are given as follows:

$w^R_{i,j} =$ 

.

Let $\mathtt{G}^{A\boldsymbol{\lambda}} = \omega \mathtt{T}^{A\boldsymbol{\lambda}}$ and $\mathtt{G}^{A\boldsymbol{\mu}} = \zeta \mathtt{T}^{A\boldsymbol{\mu}}$, where $\omega \in \mathscr{D}^{A\boldsymbol{\lambda}}$ and $\zeta \in \mathscr{D}^{A\boldsymbol{\mu}}$. Then $\omega = \omega_2\omega_1$, where $\omega_1, \omega_2 \in \mathfrak{S}^{A\boldsymbol{\lambda}}$ are given as follows:

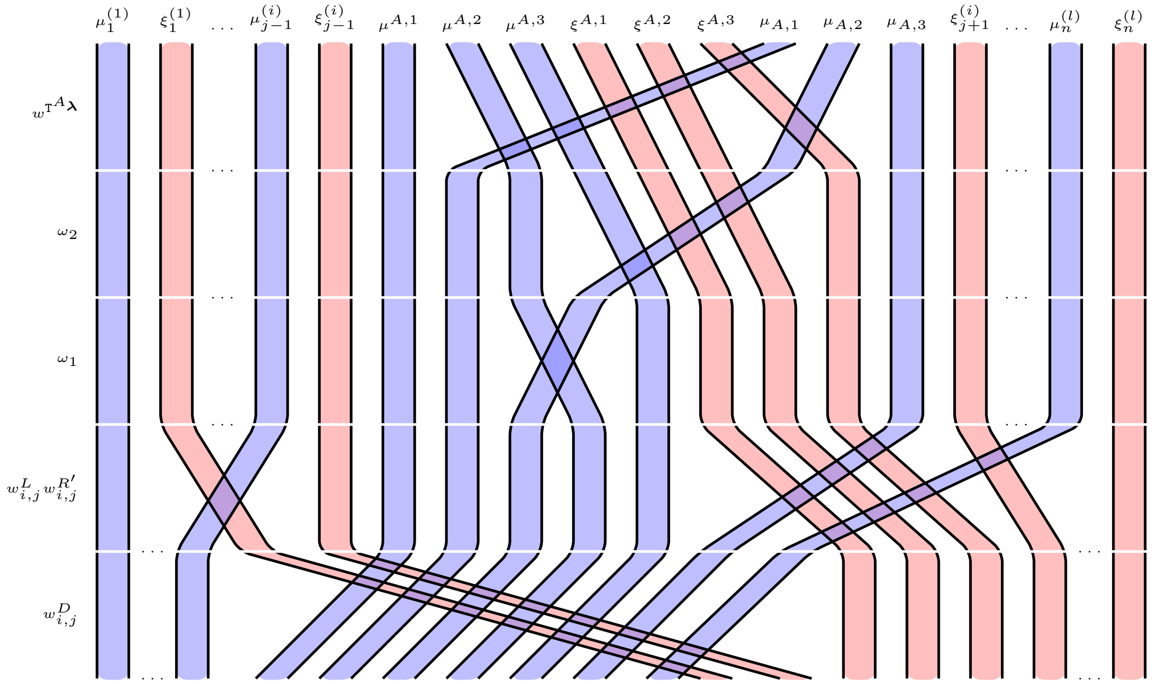
$w^{\mathtt{G}^{A\boldsymbol{\lambda}}} =$ 

.

Then

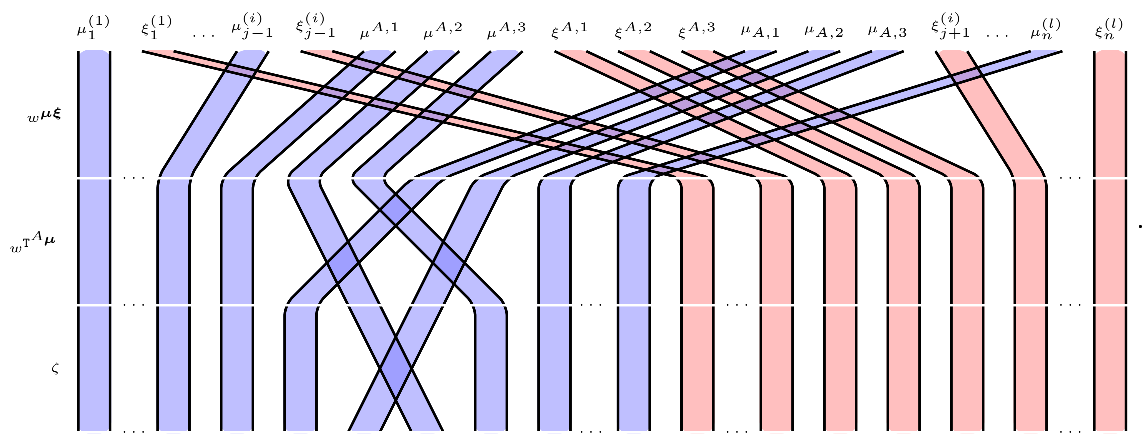
$$\zeta w^{\mathtt{T}^{A\boldsymbol{\mu}}} w^{\boldsymbol{\mu}\boldsymbol{\xi}} = w^{(\mathtt{G}^{A\boldsymbol{\mu}})\mathtt{t}^{\boldsymbol{\xi}}} = w^D_{i,j} w^L_{i,j} w^{R'}_{i,j} w^{\mathtt{G}^{A\boldsymbol{\lambda}}} = w^D_{i,j} w^L_{i,j} w^{R'}_{i,j} \omega_1 \omega_2 w^{\mathtt{T}^{A\boldsymbol{\lambda}}}. \tag{5.5}$$

This is best seen diagrammatically. On the right side of (5.5) we have



(5.6)

and pulling the $\mu^{A,2}$, $\mu^{A,3}$ strands to the left gives us the left side of (5.5):



Let $\omega_1 = w^{A_{\boldsymbol{\lambda}}}_{r^1_1} \cdots w^{A_{\boldsymbol{\lambda}}}_{r^1_{n_1}}$ and $\omega_2 = w^{A_{\boldsymbol{\lambda}}}_{r^2_1} \cdots w^{A_{\boldsymbol{\lambda}}}_{r^2_{n_2}}$ be reduced words for ω_1 and ω_2 in $\mathfrak{S}^{A_{\boldsymbol{\lambda}}}$. Now consider

$$w = (w^{A_{\boldsymbol{\lambda}}}_{r^1_1})^{\epsilon^1_1} \cdots (w^{A_{\boldsymbol{\lambda}}}_{r^1_{n_1}})^{\epsilon^1_{n_1}} (w^{A_{\boldsymbol{\lambda}}}_{r^2_1})^{\epsilon^2_1} \cdots (w^{A_{\boldsymbol{\lambda}}}_{r^2_{n_2}})^{\epsilon^2_{n_2}} \in \mathfrak{S}^{A_{\boldsymbol{\lambda}}}, \tag{5.7}$$

where each $\epsilon^h_k \in \{0,1\}$. In other words, w is achieved by deleting simple transpositions in $\mathfrak{S}^{A_\lambda}$ from ω.

Lemma 5.4.6. If $\epsilon^2_k = 0$ for some $1 \leq k \leq n_2$, then

$$\psi^D_{i,j}\psi^L_{i,j}\psi^{R'}_{i,j}(\sigma^{A_{\boldsymbol{\lambda}}}_{r^1_1})^{\epsilon^1_1} \cdots (\sigma^{A_{\boldsymbol{\lambda}}}_{r^1_{n_1}})^{\epsilon^1_{n_1}} (\sigma^{A_{\boldsymbol{\lambda}}}_{r^2_1})^{\epsilon^2_1} \cdots (\sigma^{A_{\boldsymbol{\lambda}}}_{r^2_{n_2}})^{\epsilon^2_{n_2}} \psi^{\mathtt{T}^{A_{\boldsymbol{\lambda}}}} x^{\boldsymbol{\lambda}} = 0.$$

Proof. By Lemma 5.2.1,

$$\psi^D_{i,j}\psi^L_{i,j}\psi^{R'}_{i,j}(\sigma^{A_{\boldsymbol{\lambda}}}_{r^1_1})^{\epsilon^1_1} \cdots (\sigma^{A_{\boldsymbol{\lambda}}}_{r^1_{n_1}})^{\epsilon^1_{n_1}} (\sigma^{A_{\boldsymbol{\lambda}}}_{r^2_1})^{\epsilon^2_1} \cdots (\sigma^{A_{\boldsymbol{\lambda}}}_{r^2_{n_2}})^{\epsilon^2_{n_2}} \psi^{\mathtt{T}^{A_{\boldsymbol{\lambda}}}} v^{\boldsymbol{\lambda}}$$

is an $\mathcal{O}$-linear combination of elements of the form $v^{\mathtt{T}}$, where a reduced expression for $w^{\mathtt{T}}$ appears as a subexpression in the (not necessarily reduced) concatenation of reduced expressions associated with

$$w^D_{i,j}w^L_{i,j}w^{R'}_{i,j}(w^{A_{\boldsymbol{\lambda}}}_{r^1_1})^{\epsilon^1_1} \cdots (w^{A_{\boldsymbol{\lambda}}}_{r^1_{n_1}})^{\epsilon^1_{n_1}} (w^{A_{\boldsymbol{\lambda}}}_{r^2_1})^{\epsilon^2_1} \cdots (w^{A_{\boldsymbol{\lambda}}}_{r^2_{n_2}})^{\epsilon^2_{n_2}} w^{\mathtt{T}^{A_{\boldsymbol{\lambda}}}}.$$

In other words, one can write $w^{\mathtt{T}}$ by removing crossings in (5.6), and in particular (since $\epsilon^2_k = 0$ for some k), removing at least one of the pink/blue crossings in the second row. In any case then, there is some pink strand that $w^{\mathtt{T}}$ sends to the left side, i.e., some $c \leq a$ such that $(w^{\mathtt{T}})^{-1}(c) \in \xi^{(h)}_k$ for some h, k. Then $\mathrm{sh}(\mathtt{T}_{\leq a}) \neq \boldsymbol{\mu}$. But since $w^{\mathtt{T}}$ is obtained by removing crossings in $w^{(\mathtt{G}^{A_{\boldsymbol{\mu}}})\mathtt{t}^{\boldsymbol{\xi}}}$, we have $\mathtt{T} \trianglelefteq \mathtt{G}^{A_{\boldsymbol{\mu}}}\mathtt{t}^{\boldsymbol{\xi}}$. If

s_r is the transposition such that $s_r\mathtt{G}^{A_{\boldsymbol{\mu}}}\mathtt{t}^{\boldsymbol{\xi}} \in \mathrm{St}(\boldsymbol{\lambda})$, then Lemma 5.1.6 implies that $\mathtt{T} \trianglelefteq s_r\mathtt{G}^{A_{\boldsymbol{\mu}}}\mathtt{t}^{\boldsymbol{\xi}} \in B_{\boldsymbol{\mu}}$, which in turn implies by Lemma 5.4.2 that $\mathtt{T} \in B_{\boldsymbol{\mu}}$. But then $\mathtt{T} \in B_{\triangleright\boldsymbol{\mu}}$, and thus $x^{\mathtt{T}} = 0 \in S^{\lambda}/C_{\triangleright\boldsymbol{\mu}}$. □

Every $w \in \mathscr{D}^{A_{\boldsymbol{\lambda}}}$ can be written as a reduced expression of the form (5.7) for some $\epsilon_k^h \in \{0,1\}$. If $\epsilon_k^2 = 0$ for some $1 \leq k \leq n_2$, or equivalently, if there is some node (a,b,m) in $\boldsymbol{\xi}$ such that $w\mathtt{T}^{A_{\boldsymbol{\lambda}}}(a,b,m) \neq \mathtt{G}^{A_{\boldsymbol{\lambda}}}(a,b,m)$, then the above lemma implies that

$$\psi_{i,j}^{D}\psi_{i,j}^{L}\psi_{i,j}^{R'}\tau_w^{A_{\boldsymbol{\lambda}}}\psi^{\mathtt{T}^{A_{\boldsymbol{\lambda}}}}x^{\boldsymbol{\lambda}} = 0,$$

and

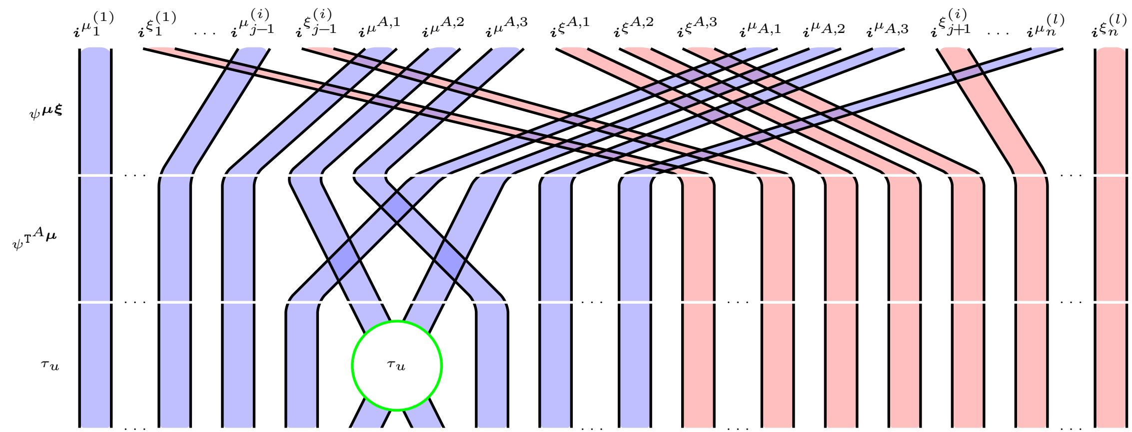
$$\psi_{i,j}^{D}\psi_{i,j}^{L}\psi_{i,j}^{R'}\tau_w^{A_{\boldsymbol{\lambda}}}\psi^{\mathtt{T}^{A_{\boldsymbol{\lambda}}}}x^{\boldsymbol{\lambda}} = \psi_{i,j}^{D}\psi_{i,j}^{L}\psi_{i,j}^{R'}(\sigma_{r_1^1}^{A_{\boldsymbol{\lambda}}})^{\epsilon_1^1}\cdots(\sigma_{r_{n_1}^1}^{A_{\boldsymbol{\lambda}}})^{\epsilon_{n_1}^1}\psi_{\omega_2}\psi^{\mathtt{T}^{A_{\boldsymbol{\lambda}}}}x^{\boldsymbol{\lambda}}$$

otherwise. Let $f^{A_{\boldsymbol{\lambda}}}$ and $f^{A_{\boldsymbol{\mu}}}$ denote the number of bricks in the top row of $\mathbf{B}^{A_{\boldsymbol{\lambda}}}$ and $\mathbf{B}^{A_{\boldsymbol{\mu}}}$ respectively. Note that $w = (w_{r_1^1}^{A_{\boldsymbol{\lambda}}})^{\epsilon_1^1}\cdots(w_{r_{n_1}^1}^{A_{\boldsymbol{\lambda}}})^{\epsilon_{n_1}^1}\omega_2$ is a reduced expression for an element in $\mathscr{D}^{A_{\boldsymbol{\lambda}}}$ if and only if $(w_{r_1^1}^{A_{\boldsymbol{\lambda}}})^{\epsilon_1^1}\cdots(w_{r_{n_1}^1}^{A_{\boldsymbol{\lambda}}})^{\epsilon_{n_1}^1}$ is a reduced expression for an element in $\mathscr{D}^{f^{A_{\boldsymbol{\mu}}},k^{A_{\boldsymbol{\lambda}}}-f^{A_{\boldsymbol{\lambda}}}}$. Since $k^{A_{\boldsymbol{\mu}}} = k^{A_{\boldsymbol{\lambda}}} - (f^{A_{\boldsymbol{\lambda}}} - f^{A_{\boldsymbol{\mu}}})$, this allows us to associate $\mathscr{D}^{A_{\boldsymbol{\mu}}}$ with $\mathscr{D}^{A_{\boldsymbol{\lambda}}}$ in the following way. Let $\widehat{\mathscr{D}}^{A_{\boldsymbol{\lambda}}}$ be the set of all $w \in \mathscr{D}^{A_{\boldsymbol{\lambda}}}$ such that $\epsilon_k^2 \neq 0$ for all k. Then there is a bijection between $\mathscr{D}^{A_{\boldsymbol{\mu}}}$ and $\widehat{\mathscr{D}}^{A_{\boldsymbol{\lambda}}}$ given by $u \mapsto u[d_{i,j}]\omega_2$.

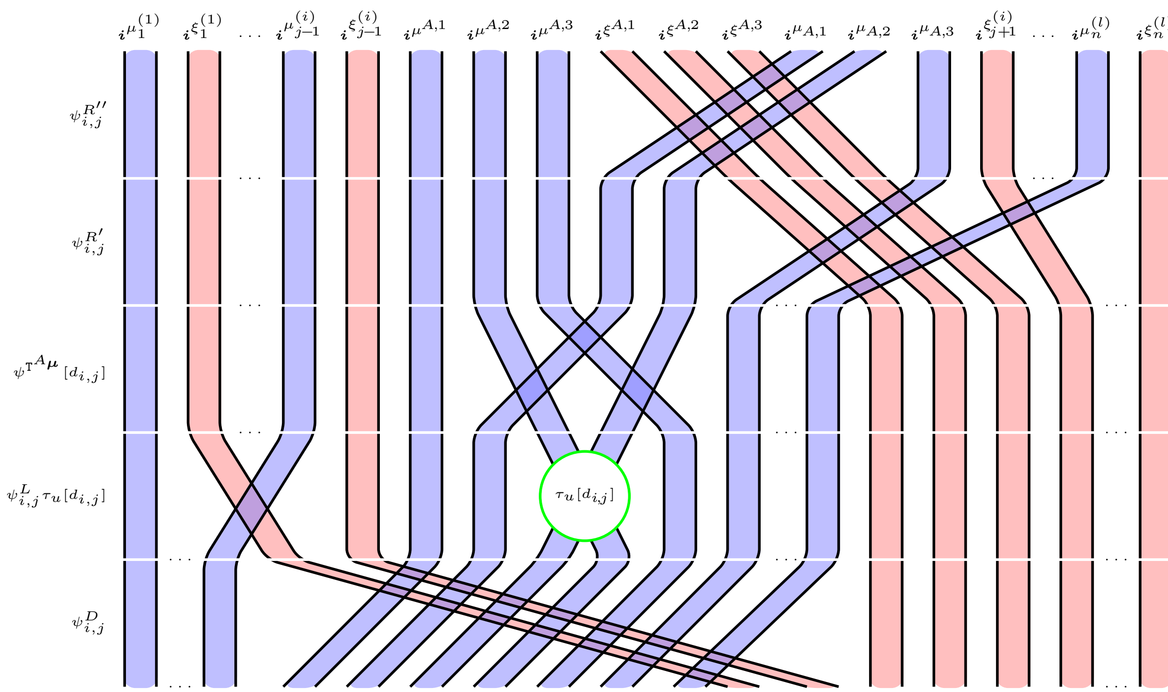
Lemma 5.4.7. For all $u \in \mathscr{D}^{A_{\boldsymbol{\mu}}}$,

$$\tau_u\psi^{\mathtt{T}^{A_{\boldsymbol{\mu}}}}\psi^{\boldsymbol{\mu}\boldsymbol{\xi}}x^{\boldsymbol{\lambda}} = \psi_{i,j}^{D}\psi_{i,j}^{L}\psi_{i,j}^{L'}\tau_u[d_{i,j}]\psi_{\omega_2}\psi^{\mathtt{T}^{A_{\boldsymbol{\lambda}}}}x^{\boldsymbol{\lambda}}.$$

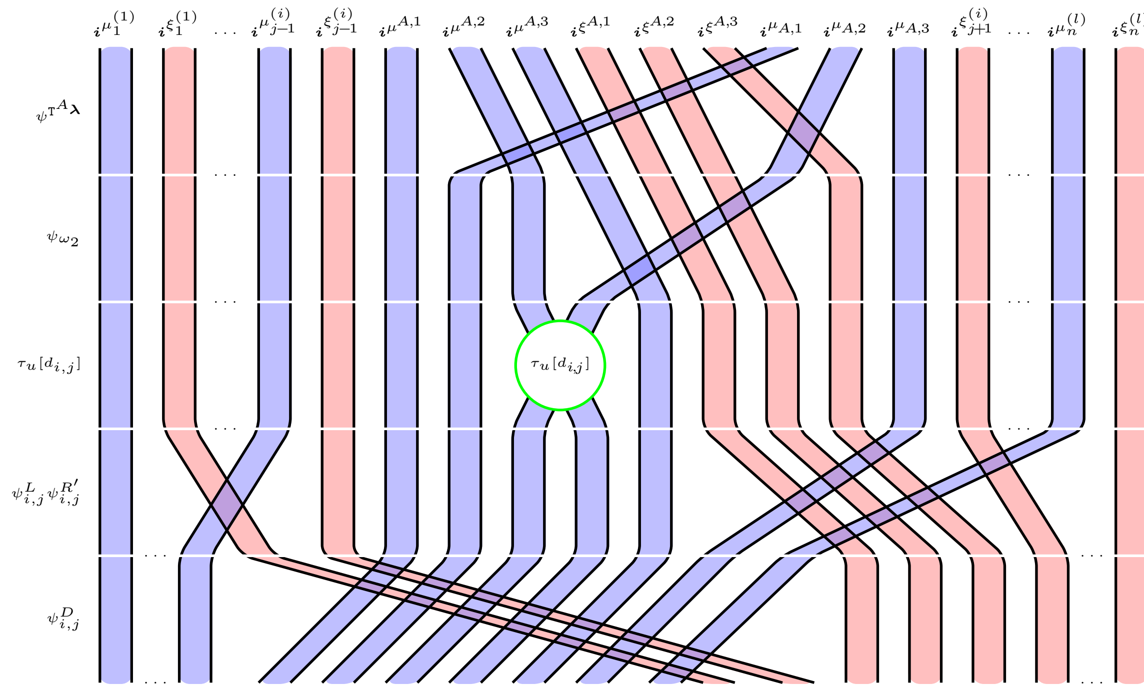
Proof. This is easily seen in terms of Khovanov-Lauda diagrams, with $x^{\boldsymbol{\lambda}}$ pictured as being at the top of the diagram. The left side:



is by Lemma 5.4.5 equal to



which, after an isotopy of strands, becomes the right side in the lemma statement:



completing the proof. ☐

Lemma 5.4.8. Let $A_{\boldsymbol{\mu}}$ be a Garnir node of $\boldsymbol{\mu}$. Then $f(g^{A_{\boldsymbol{\mu}}} \otimes 1_\beta) = 0$.

Proof. We make use of Lemma 5.4.7 and the bijection between $\mathscr{D}^{A_{\boldsymbol{\mu}}}$ and $\widehat{\mathscr{D}}^{A_{\boldsymbol{\lambda}}}$:

$$
\begin{aligned}
f(g^{A_{\boldsymbol{\mu}}} \otimes 1_\beta) = g^{A_{\boldsymbol{\mu}}} \cdot x^{\boldsymbol{\mu}\boldsymbol{\xi}} &= \left(\sum_{u \in \mathscr{D}^{A_{\boldsymbol{\mu}}}} \tau_u \psi^{\mathtt{T}^{A_{\boldsymbol{\mu}}}} \right) \psi^{\boldsymbol{\mu}\boldsymbol{\xi}} x^{\boldsymbol{\lambda}} \\
&= \sum_{u \in \mathscr{D}^{A_{\boldsymbol{\mu}}}} \psi^D_{i,j} \psi^L_{i,j} \psi^{R'}_{i,j} \tau_u [d_{i,j}] \psi_{\omega_2} \psi^{\mathtt{T}^{A_{\boldsymbol{\lambda}}}} x^{\boldsymbol{\lambda}} \\
&= \psi^D_{i,j} \psi^L_{i,j} \psi^{R'}_{i,j} \sum_{u \in \mathscr{D}^{A_{\boldsymbol{\mu}}}} \tau_u [d_{i,j}] \psi_{\omega_2} \psi^{\mathtt{T}^{A_{\boldsymbol{\lambda}}}} x^{\boldsymbol{\lambda}} \\
&= \psi^D_{i,j} \psi^L_{i,j} \psi^{R'}_{i,j} \sum_{w \in \mathscr{D}^{A_{\boldsymbol{\lambda}}}} \tau_w \psi^{\mathtt{T}^{A_{\boldsymbol{\lambda}}}} x^{\boldsymbol{\lambda}} \\
&= \psi^D_{i,j} \psi^L_{i,j} \psi^{R'}_{i,j} g^{A_{\boldsymbol{\lambda}}} x^{\boldsymbol{\lambda}} \\
&= 0.
\end{aligned}
$$

$\square$

Remark 5.4.9. Although we have focused on Garnir nodes in $\boldsymbol{\mu}$, there are obvious analogues (whose proofs are entirely analogous) of Lemmas 5.4.5, 5.4.6, and 5.4.7, which imply the analogue of Proposition 5.4.8:

$$f(1_\alpha \otimes g^{A_{\boldsymbol{\xi}}}) = 0$$

for Garnir nodes in $\boldsymbol{\xi}$.

Proposition 5.4.2. *The map* $f : R_{\alpha,\beta}\langle \deg \mathtt{T}^{\boldsymbol{\mu}} + \deg \mathtt{t}^{\boldsymbol{\xi}} \rangle \to N_{\boldsymbol{\mu}}$ *induces a graded isomorphism*

$$f : S^{\boldsymbol{\mu}} \boxtimes S^{\boldsymbol{\xi}} \xrightarrow{\sim} N_{\boldsymbol{\mu}}.$$

Proof. We have that f factors through to a map

$$R_{\alpha,\beta}/(J^{\boldsymbol{\mu}}_{\alpha} \otimes R_{\beta} + R_{\alpha} \otimes J^{\boldsymbol{\xi}}_{\beta})\langle \deg \mathtt{T}^{\boldsymbol{\mu}} + \deg \mathtt{t}^{\boldsymbol{\xi}} \rangle \to N_{\boldsymbol{\mu}}$$

by Lemmas 5.4.1, 5.4.8 and Remark 5.4.9. However, we also have

$$R_{\alpha,\beta}/(J^{\boldsymbol{\mu}}_{\alpha} \otimes R_{\beta} + R_{\alpha} \otimes J^{\boldsymbol{\xi}}_{\beta})\langle \deg \mathtt{T}^{\boldsymbol{\mu}} + \deg \mathtt{t}^{\boldsymbol{\xi}} \rangle \cong R_{\alpha}/J^{\boldsymbol{\mu}}_{\alpha}\langle \deg \mathtt{T}^{\boldsymbol{\mu}} \rangle \otimes R_{\beta}/J^{\boldsymbol{\xi}}_{\beta}\langle \deg \mathtt{t}^{\boldsymbol{\xi}} \rangle = S^{\boldsymbol{\mu}} \boxtimes S^{\boldsymbol{\xi}}.$$

Moreover, for all $\mathtt{T} \in \mathrm{St}(\boldsymbol{\mu})$, $\mathtt{t} \in \mathrm{St}(\boldsymbol{\xi})$,

$$f(v^{\mathtt{T}} \boxtimes v^{\mathtt{t}}) = f(\psi^{\mathtt{T}} v^{\boldsymbol{\mu}} \boxtimes \psi^{\mathtt{t}} v^{\boldsymbol{\xi}}) = \psi^{\mathtt{T}} \psi^{\mathtt{t}}[a] x^{\boldsymbol{\mu}\boldsymbol{\xi}} = \psi^{\mathtt{T}} \psi^{\mathtt{t}}[a] \psi^{\boldsymbol{\mu}\boldsymbol{\xi}} x^{\boldsymbol{\lambda}} = x^{\mathtt{Tt}} + \sum_{\mathtt{U} \lhd \mathtt{Tt}} d_{\mathtt{U}} x^{\mathtt{U}}$$

for some constants $d_{\mathtt{U}}$, by Lemma 5.2.1. Since $\{v^{\mathtt{T}} \boxtimes v^{\mathtt{t}} \mid \mathtt{T} \in \mathrm{St}(\boldsymbol{\mu}), \mathtt{t} \in \mathrm{St}(\boldsymbol{\xi})\}$ is a spanning set for $S^{\boldsymbol{\mu}} \boxtimes S^{\boldsymbol{\xi}}$ and $\{x^{\mathtt{Tt}} \mid \mathtt{T} \in \mathrm{St}(\boldsymbol{\mu}), \mathtt{t} \in \mathrm{St}(\boldsymbol{\xi})\}$ is a basis for $N_{\boldsymbol{\mu}}$, it follows that f is an isomorphism. □

5.43. A basis for $S^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ and a filtration for $\mathrm{Res}_{\alpha,\beta} S^{\boldsymbol{\lambda}}$

Proposition 5.4.2 in hand, we may now prove two theorems which complete the analogy with the definition (1.16) in the semisimple case, and justify our use of the term *skew Specht module* for $S^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$.

Theorem 5.4.10. Let $\boldsymbol{\lambda}/\boldsymbol{\mu} \in \mathscr{S}^{\kappa}_{\boldsymbol{\mu},\alpha}$. Then $S^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ has a homogeneous $\mathcal{O}$-basis

$$\{v^{\mathtt{t}} \mid \mathtt{t} \in \mathrm{St}(\boldsymbol{\lambda}/\boldsymbol{\mu})\}. \tag{5.8}$$

Proof. By Proposition 5.2, the set (5.8) spans $S^{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ over $\mathcal{O}$, and the set is linearly independent by Proposition 5.4.2. □

Theorem 5.4.11. Let $\boldsymbol{\lambda} \in \mathscr{P}^{\kappa}_{\alpha+\beta}$. Let $\{\boldsymbol{\mu}_1, \dots \boldsymbol{\mu}_k\} = \{\boldsymbol{\mu} \in \mathscr{P}^{\kappa}_{\alpha} \mid \boldsymbol{\mu} \subseteq \boldsymbol{\lambda}\}$ and assume the labels are such that $\boldsymbol{\mu}_i \rhd \boldsymbol{\mu}_j \implies i < j$. Write

$$V_i := \sum_{j=1}^{i} C_{\boldsymbol{\mu}_j} = \mathcal{O}\left\{v^{\mathtt{T}} \in S^{\boldsymbol{\lambda}} \mid \mathtt{T} \in \mathrm{St}(\boldsymbol{\lambda}), \mathrm{sh}(\mathtt{T}_{\leq a}) = \boldsymbol{\mu}_j \text{ for some } j < i\right\}$$

for all i. Then

$$0 = V_0 \leq V_1 \leq V_2 \leq \cdots \leq V_k = \mathrm{Res}_{\alpha,\beta} S^{\boldsymbol{\lambda}}$$

is a graded filtration of $\mathrm{Res}_{\alpha,\beta} S^{\boldsymbol{\lambda}}$ by $R_{\alpha,\beta}$-submodules, with subquotients

$$V_i/V_{i-1} \cong S^{\boldsymbol{\mu}_i} \boxtimes S^{\boldsymbol{\lambda}/\boldsymbol{\mu}_i}.$$

Proof. The fact that $V_k = \mathrm{Res}_{\alpha,\beta} S^{\boldsymbol{\lambda}}$ follows from the fact that $B = \bigcup_{j=1}^{k} B_{\boldsymbol{\mu}_j}$ and $\{v^{\mathtt{T}} \mid \mathtt{T} \in B\}$ is a basis for $\mathrm{Res}_{\alpha,\beta} S^{\boldsymbol{\lambda}}$. Since $C_{\boldsymbol{\mu}_i} \geq C_{\boldsymbol{\mu}_j}$ if $\boldsymbol{\mu}_j \trianglerighteq \boldsymbol{\mu}_i$, we have

$$V_i = \sum_{j=1}^{i} C_{\boldsymbol{\mu}_j} = C_{\boldsymbol{\mu}_i} \oplus \sum_{\substack{j \leq i-1 \\ \boldsymbol{\mu}_j \not\rhd \boldsymbol{\mu}_i}} C_{\boldsymbol{\mu}_j}$$

and

$$V_{i-1} = \sum_{j=1}^{i-1} C_{\boldsymbol{\mu}_j} = \sum_{\boldsymbol{\mu}_j \rhd \boldsymbol{\mu}_i} C_{\boldsymbol{\mu}_j} \oplus \sum_{\substack{j \leq i-1 \\ \boldsymbol{\mu}_j \not\rhd \boldsymbol{\mu}_i}} C_{\boldsymbol{\mu}_j} = C_{\rhd \boldsymbol{\mu}_i} \oplus \sum_{\substack{j \leq i-1 \\ \boldsymbol{\mu}_j \not\rhd \boldsymbol{\mu}_i}} C_{\boldsymbol{\mu}_j},$$

which implies that $V_i/V_{i-1} \cong C_{\boldsymbol{\mu}_i}/C_{\rhd \boldsymbol{\mu}_i} = N_{\boldsymbol{\mu}_i} \cong S^{\boldsymbol{\mu}_i} \boxtimes S^{\boldsymbol{\lambda}/\boldsymbol{\mu}_i}$. □

Remark 5.4.12. Theorem 5.4.11 may be compared with [15, Theorem 3.1], which gives a similar result for restrictions of classical Specht modules over the symmetric group algebra to Young subgroups. However, the connection between our skew Specht $R_\alpha(\mathbb{F})$-modules $S^{\lambda/\mu}$ and the skew Specht $\mathbb{F}\mathfrak{S}_n$-module $S^{\lambda/\mu}_{\mathbb{F}\mathfrak{S}_n}$ defined in [15] is not as direct as may be expected. Taking $e = \mathrm{ch}\ \mathbb{F}$, it is shown in [6] that there exists a surjection $R_n := \bigoplus_{\mathrm{ht}(\alpha)=n} R_\alpha \twoheadrightarrow R_n^{\Lambda_i} \cong \mathbb{F}\mathfrak{S}_n$. Inflating $S^{\lambda/\mu}_{\mathbb{F}\mathfrak{S}_n}$ along this map, we have an R_n-module $\mathrm{infl}\, S^{\lambda/\mu}_{\mathbb{F}\mathfrak{S}_n}$, and truncating $S^{\lambda/\mu}$ yields an $\mathbb{F}\mathfrak{S}_n$-module $\mathrm{pr}\, S^{\lambda/\mu}$. However it is not the case that $\mathrm{pr}\, S^{\lambda/\mu} \cong S^{\lambda/\mu}_{\mathbb{F}\mathfrak{S}_n}$ nor $\mathrm{infl}\, S^{\lambda/\mu}_{\mathbb{F}\mathfrak{S}_n} \cong S^{\lambda/\mu}$ in general, though the (ungraded) dimensions do agree in the latter case, and both statements hold when $\mu = \varnothing$ and $\kappa = (i)$.

For an explicit example of this difference, take $e = \mathrm{ch}\ \mathbb{F} > 0$, and $n > 1$. Let

$$\lambda = (ne - e + 1, ne - 2e + 2, ne - 3e + 3, \ldots, n),$$
$$\mu = (ne - e, ne - 2e + 1, ne - 3e + 2, \ldots, n - 1).$$

Then λ/μ consists of n disconnected nodes of some residue i (depending on κ). The $\mathbb{F}\mathfrak{S}_n$-module $S^{\lambda/\mu}_{\mathbb{F}\mathfrak{S}_n}$ is isomorphic to the regular module ${}_{\mathbb{F}\mathfrak{S}_n}\mathbb{F}\mathfrak{S}_n$, and thus $\mathrm{infl}\, S^{\lambda/\mu}_{\mathbb{F}\mathfrak{S}_n}$ is reducible. However, $S^{\lambda/\mu}$ as defined in this paper is irreducible—in fact it is the unique irreducible $R_{n\alpha_i}$-module (up to grading shift), see [24, §2.2]. Moreover, $\mathrm{pr}\, S^{\lambda/\mu} = 0$ as $\mathbb{F}\mathfrak{S}_n$ has no $n!$-dimensional irreducible modules.

5.44. Induction product of skew Specht modules

The following theorem was proved in [27, Theorem 8.2] in the context of Young diagrams, but the proof is applicable with no significant alteration to the more general case of skew diagrams.

Theorem 5.4.13. Suppose that $\boldsymbol{\lambda}/\boldsymbol{\mu} \in \mathscr{S}^\kappa_\alpha$. Then

$$S^{\boldsymbol{\lambda}/\boldsymbol{\mu}} \cong S^{\lambda^{(1)}/\mu^{(1)}} \circ \cdots \circ S^{\lambda^{(l)}/\mu^{(l)}} \langle d_{\boldsymbol{\lambda}/\boldsymbol{\mu}} \rangle,$$

as graded R_α-modules, where

$$d_{\boldsymbol{\lambda}/\boldsymbol{\mu}} = \deg(\mathtt{t}^{\boldsymbol{\lambda}/\boldsymbol{\mu}}) - \deg(\mathtt{t}^{\lambda^{(1)}/\mu^{(1)}}) - \cdots - \deg(\mathtt{t}^{\lambda^{(l)}/\mu^{(l)}}).$$

5.5. Joinable diagrams

In this section we present a useful, albeit rather technical, result regarding the graded characters of skew Specht modules whose associated component diagrams jibe with each other in a specific sense. This result, together with Theorem 5.4.13, will make it possible for us to identify cuspidal modules in §5.7 while operating solely at the level of characters.

Definition 5.5.1. Let $l = 2$, $\kappa = (k_1, k_2)$, and $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}) \in \mathscr{P}^\kappa$. Write $x_i := n(\boldsymbol{\lambda}, i)$, and $y_i := \lambda^{(i)}_1$. If $(x_1, 1, 1)$ (the bottom left node in $\lambda^{(1)}$) and $(1, y_2, 2)$ (the top right node in $\lambda^{(2)}$) are such that $\operatorname{res}(x_1, 1, 1) = \operatorname{res}(1, y_2, 2) + 1$, we call $\boldsymbol{\lambda}$ *joinable*.

In this section we will assume that $\boldsymbol{\lambda}$ is joinable. Define the one-part multicharges $\kappa^* := (k_2 + x_1)$ and $\kappa_* := (k_2 + x_1 - 1)$. We now define $\lambda^*/\mu^* \in \mathscr{S}^{\kappa^*}$ and $\lambda_*/\mu_* \in \mathscr{S}^{\kappa_*}$ by setting:

$$\lambda^* := (\lambda^{(1)} + y_2 - 1, \ldots, \lambda^{(1)}_{x_1} + y_2 - 1, \lambda^{(2)}_1, \ldots, \lambda^{(2)}_{x_2}), \qquad \mu^* := ((y_2 - 1)^{x_1}),$$

and

$$\lambda_* := (\lambda^{(1)} + y_2, \ldots, \lambda^{(1)}_{x_1} + y_2, \lambda^{(2)}_2, \ldots, \lambda^{(2)}_{x_2}), \qquad \mu_* := (y_2^{x_1 - 1}).$$

In other words, λ^*/μ^* is achieved by shifting the Young diagram associated with $\lambda^{(1)}$ until its bottom-left node lies directly above the top-right node of $\lambda^{(2)}$, and then viewing this as a one-part skew diagram. Similarly, λ_*/μ_* is achieved by shifting the Young diagram associated with $\lambda^{(1)}$ until its bottom-left node lies directly to the right of the top-right node of $\lambda^{(2)}$.

There is an obvious bijection τ^* (resp. τ_*) between nodes of $\boldsymbol{\lambda}$ and λ^*/μ^* (resp. λ_*/μ_*), given by

$$\begin{aligned} \lambda^{(1)} \ni (a, b, 1) &\xmapsto{\tau^*} (a, b + y_2 - 1) \in \lambda^*/\mu^* \\ \lambda^{(2)} \ni (a, b, 2) &\xmapsto{\tau^*} (a + x_1, b) \in \lambda^*/\mu^* \end{aligned}$$

and, respectively,

$$\begin{aligned} \lambda^{(1)} \ni (a, b, 1) &\xmapsto{\tau_*} (a, b + y_2) \in \lambda_*/\mu_* \\ \lambda^{(2)} \ni (a, b, 2) &\xmapsto{\tau_*} (a + x_1 - 1, b) \in \lambda_*/\mu_* \end{aligned}$$

Note that the charges κ^* and κ_* are chosen so that residues of nodes are preserved under this bijection. Let $\mathtt{T} \in \mathrm{St}(\boldsymbol{\lambda})$. Viewing the tableau as a function $\{1, \ldots, d\} \to \boldsymbol{\lambda}$, then composing with τ^* (resp. τ_*) gives a λ^*/μ^*-tableau (resp. λ_*/μ_*-tableau). Define

$$\mathtt{T}^* := \tau^* \circ \mathtt{T} \quad \text{and} \quad \mathtt{T}_* := \tau_* \circ \mathtt{T}.$$

Then we have bijections

$$\{\mathtt{T} \in \mathrm{St}(\boldsymbol{\lambda}) \mid \mathtt{T}(x_1, 1, 1) < \mathtt{T}(1, y_2, 2)\} \xrightarrow{\tau^*} \mathrm{St}(\lambda^*/\mu^*)$$

and

$$\{\mathtt{T} \in \mathrm{St}(\boldsymbol{\lambda}) \mid \mathtt{T}(x_1, 1, 1) > \mathtt{T}(1, y_2, 2)\} \xrightarrow{\mathcal{T}_*} \mathrm{St}(\lambda_*/\mu_*)$$

Example 5.5.1. Let $e = 3$, $\kappa = (0, 1)$, $\lambda^{(1)} = (3, 2, 2)$ and $\lambda^{(2)} = (2, 2)$. Then $\boldsymbol{\lambda}$ is joinable since $\mathrm{res}(3, 1, 1) = 1 = 0 + 1 = \mathrm{res}(1, 2, 2) + 1$. Then, with respect to the row- and column-leading tableaux, we have:

$$\mathtt{T}^{\boldsymbol{\lambda}} = \begin{array}{l} \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} \end{array} \\ \\ \begin{array}{|c|c|} \hline 8 & 9 \\ \hline 10 & 11 \\ \hline \end{array} \end{array} \qquad (\mathtt{T}^{\boldsymbol{\lambda}})^* = \begin{array}{cccc} & 1 & 2 & 3 \\ & 4 & 5 & \\ & 6 & 7 & \\ 8 & 9 & & \\ 10 & 11 & & \end{array} \qquad \mathtt{T}_{\boldsymbol{\lambda}} = \begin{array}{l} \begin{array}{|c|c|c|} \hline 5 & 8 & 11 \\ \hline 6 & 9 \\ \cline{1-2} 7 & 10 \\ \cline{1-2} \end{array} \\ \\ \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \end{array} \qquad (\mathtt{T}_{\boldsymbol{\lambda}})_* = \begin{array}{ccccc} & & 5 & 8 & 11 \\ & & 6 & 9 & \\ 1 & 3 & 7 & 10 & \\ 2 & 4 & & & \end{array}$$

Lemma 5.5.2. Let $\boldsymbol{\lambda} \in \mathscr{P}^\kappa$ be joinable, $\mathrm{res}(1, y_2, 2) = i$, and let $\mathtt{T} \in \mathrm{St}(\boldsymbol{\lambda})$. Then

$$\deg \mathtt{T}^* = \deg \mathtt{T} - \big(\Lambda_i, \mathrm{cont}(\lambda^{(1)})\big)$$

if $\mathtt{T}(x_1, 1, 1) < \mathtt{T}(1, y_2, 2)$, and

$$\deg \mathtt{T}_* = \deg \mathtt{T} - \big(\Lambda_{i+1}, \mathrm{cont}(\lambda^{(1)})\big)$$

if $\mathtt{T}(x_1, 1, 1) > \mathtt{T}(1, y_2, 2)$.

Proof. We prove the first statement. The second is similar. Let $\mathtt{U} = \mathtt{T}_{\leq t}$ for some t. We'll show that the claim holds for $\mathtt{U}$:

$$\deg(\mathtt{U}^*) = \deg(\mathtt{U}) - \big(\Lambda_i, \mathrm{cont}(\mathrm{sh}(\mathtt{U})^{(1)})\big)\,, \tag{5.9}$$

going by induction on the size of $\mathrm{sh}(\mathtt{U})$.

For the base case we have $\mathrm{sh}(\mathtt{U}) = (\varnothing, \varnothing)$, so that $\deg(\mathtt{U}) = 0 = \deg(\mathtt{U}^*) = 0$.

Now we attack the induction step. By the inductive definition of degree for tableaux, we just need to show that for every removable node A in $\mathtt{U}$,

$$d_{\tau^*(A)}(\mathrm{sh}(\mathtt{Y}(\mathtt{U}^*))) - d_A(\mathrm{sh}(\mathtt{U})) = \begin{cases} -1 & A \in \lambda^{(1)} \text{ and } \mathrm{res}(A) = i, \\ 0 & \text{otherwise.} \end{cases} \tag{5.10}$$

By the construction of $\mathtt{U}^*$, it is clear that for $1 \le r \le x_1 - 1$ and $j \in I$, the r-th row of $\mathrm{sh}(\mathtt{U})^{(1)}$ has an addable (resp. removable) j-node if and only if the corresponding r-th row in $\mathrm{sh}(\mathtt{Y}(\mathtt{U}^*))$ has an addable (resp. removable) j-node. Similarly, for $2 \le r \le x_2$, the r-th row of $\mathrm{sh}(\mathtt{U})^{(2)}$ has an addable (resp. removable) j-node if and only if the corresponding $(x_1 + r)$-th row in $\mathrm{sh}(\mathtt{Y}(\mathtt{U}^*))$ has an addable (resp. removable) j-node. Thus it remains to compare addable/removable nodes of rows $x_1, x_1 + 1$ in $\mathrm{sh}(\mathtt{U})^{(1)}$ and row 1 in $\mathrm{sh}(\mathtt{U})^{(2)}$ with the rows $x_1, x_1 + 1$ in $\mathrm{sh}(\mathtt{Y}(\mathtt{U}^*))$.

For simplicity, we label

- $B := (x_1 - 1, 1, 1)$, the bottom-left node in $\lambda^{(1)}$. Write $B^* := \tau^*(B)$. Both B and B^* have residue $i + 1$.
- $C := (1, y_2 - 1, 2)$, the node to the left of the top-right node in $\lambda^{(2)}$. Write $C^* := \tau^*(C)$. Both C and C^* have residue $i - 1$.
- $D := (1, y_2, 2)$, the top-right node in $\lambda^{(2)}$. Write $D^* := \tau^*(D)$. Both D and D^* have residue i.

There are five cases to consider.

(i) $\{B, C, D\} \cap \mathrm{sh}(\mathtt{U}) = \varnothing$.

- Row x_1 of $\mathrm{sh}(\mathtt{U})^{(1)}$ has addable node B iff B^* is addable in $\mathrm{sh}(\mathtt{Y}(\mathtt{U}^*))$. Row x_1 of $\mathrm{sh}(\mathtt{U})^{(1)}$ has no removable nodes.

- Row x_1+1 of $\mathrm{sh}(\mathtt{U})^{(1)}$ has no addable/removable nodes.
- Row 1 of $\mathrm{sh}(\mathtt{U})^{(2)}$ has an addable (resp. removable) j-node iff row x_1+1 of $\mathrm{sh}(\mathtt{Y}(\mathtt{U}^*))$ has an addable (resp. removable) j-node.
- Row x_1 of $\mathrm{sh}(\mathtt{Y}(\mathtt{U}^*))$ has a removable i-node (the bottom-right node of μ_* to be precise).

From this (5.10) follows.

(ii) $\{B, C, D\} \cap \mathrm{sh}(\mathtt{U}) = \{B\}$.

- Row x_1 of $\mathrm{sh}(\mathtt{U})^{(1)}$ has an addable (resp. removable) j-node iff row x_1 of $\mathrm{sh}(\mathtt{Y}(\mathtt{U}^*))$ has an addable (resp. removable) j-node.
- Row x_1+1 of $\mathrm{sh}(\mathtt{U})^{(1)}$ has an addable i-node, and no removable nodes.
- Row 1 of $\mathrm{sh}(\mathtt{U})^{(2)}$ has an addable (resp. removable) j-node iff row x_1+1 of $\mathrm{sh}(\mathtt{Y}(\mathtt{U}^*))$ has an addable (resp. removable) j-node.

From this (5.10) follows.

(iii) $\{B, C, D\} \cap \mathrm{sh}(\mathtt{U}) = \{C\}$.

- Row x_1 of $\mathrm{sh}(\mathtt{U})^{(1)}$ has addable node B iff B^* is addable in $\mathrm{sh}(\mathtt{Y}(\mathtt{U}^*))$. Row x_1 of $\mathrm{sh}(\mathtt{U})^{(1)}$ has no removable nodes.
- Row x_1+1 of $\mathrm{sh}(\mathtt{U})^{(1)}$ has no addable/removable nodes.
- Row 1 of $\mathrm{sh}(\mathtt{U})^{(2)}$ has an addable i-node D. Row 1 of $\mathrm{sh}(\mathtt{U})^{(2)}$ has removable node C iff C^* is removable in row x_1+1 of $\mathrm{sh}(\mathtt{Y}(\mathtt{U}^*))$.
- Row x_1 of $\mathrm{sh}(\mathtt{Y}(\mathtt{U}^*))$ has no removable nodes.
- Row x_1+1 of $\mathrm{sh}(\mathtt{Y}(\mathtt{U}^*))$ has no addable nodes.

From this (5.10) follows.

(iv) $\{B, C, D\} \cap \mathrm{sh}(\mathtt{U}) = \{B, C\}$.

- Row x_1 of $\mathrm{sh}(\mathtt{U})^{(1)}$ has an addable (resp. removable) j-node iff row x_1 of $\mathrm{sh}(\mathtt{Y}(\mathtt{U}^*))$ has an addable (resp. removable) j-node.
- Row $x_1 + 1$ of $\mathrm{sh}(\mathtt{U})^{(1)}$ has an addable i-node and no removable node.
- Row 1 of $\mathrm{sh}(\mathtt{U})^{(2)}$ has an addable (resp. removable) j-node iff row $x_1 + 1$ of $\mathrm{sh}(\mathtt{Y}(\mathtt{U}^*))$ has an addable (resp. removable) j-node.

From this (5.10) follows.

(v) $\{B, C, D\} \cap \mathrm{sh}(\mathtt{U}) = \{B, C, D\}$.

- Row x_1 of $\mathrm{sh}(\mathtt{U})^{(1)}$ has an addable (resp. removable) j-node to the right of B iff row x_1 of $\mathrm{sh}(\mathtt{Y}(\mathtt{U}^*))$ has an addable (resp. removable) j-node to the right of B^*. The $(i+1)$-node B is not removable in row x_1 of $\mathrm{sh}(\mathtt{U})^{(1)}$ iff row $x_1 + 1$ of $\mathrm{sh}(\mathtt{Y}(\mathtt{U}^*))$ has an addable $(i+1)$-node to the right of D^*.
- Row $x_1 + 1$ of $\mathrm{sh}(\mathtt{U})^{(1)}$ has an addable i-node and no removable node.
- Row 1 of $\mathrm{sh}(\mathtt{U})^{(2)}$ has an addable $(i+1)$-node. Row 1 of $\mathrm{sh}(\mathtt{U})^{(2)}$ has removable node D iff D^* is removable in row $x_1 + 1$ of $\mathrm{sh}(\mathtt{Y}(\mathtt{U}^*))$.

From this (5.10) follows.

Thus in all cases, (5.10) is satisfied, and the lemma follows by induction. □

Definition 5.5.3. We say that an arbitrary skew diagram $\boldsymbol{\lambda}/\boldsymbol{\mu}$ is *minimal* if $\mu_1^{(i)} < \lambda_1^{(i)}$ and $\mu_{n(\boldsymbol{\lambda},i)}^{(i)} = 0$ for all i. Less formally, a skew diagram is minimal if, in each component, it has nodes in the top row and in the leftmost column.

Definition 5.5.4. Let l=2. We say that $\boldsymbol{\lambda}/\boldsymbol{\mu} \in \mathscr{S}^{\kappa}$ is *joinable* if it is minimal and $\boldsymbol{\lambda}$ is joinable.

Assuming $\boldsymbol{\lambda}/\boldsymbol{\mu}$ is joinable, define $\kappa^*, \kappa_*, x_i, y_i$ as before, with respect to $\boldsymbol{\lambda}$. In the same vein as before we construct a skew tableau λ^*/μ^* by shifting the skew diagram $\lambda^{(1)}/\mu^{(1)}$ until the lower left node lies above the upper right node of $\lambda^{(2)}/\mu^{(2)}$, and we construct a skew tableau λ^*/μ^* by shifting the skew diagram $\lambda^{(1)}/\mu^{(1)}$ until the lower left node lies directly to the right of the upper right node of $\lambda^{(2)}/\mu^{(2)}$. Specifically, define $\lambda^*/\mu^* \in \mathscr{S}^{\kappa^*}$ and $\lambda_*/\mu_* \in \mathscr{S}^{\kappa_*}$ by setting:

$$\lambda^* := (\lambda_1^{(1)} + y_2 - 1, \ldots, \lambda_{x_1}^{(1)} + y_2 - 1, \lambda_1^{(2)}, \ldots, \lambda_{x_2}^{(2)}),$$
$$\mu^* := (\mu_1^{(1)} + y_2 - 1, \ldots, \mu_{x_1}^{(1)} + y_2 - 1, \mu_1^{(2)}, \ldots, \mu_{x_2}^{(2)}),$$

and

$$\lambda_* := (\lambda_1^{(1)} + y_2, \ldots, \lambda_{x_1}^{(1)} + y_2, \lambda_2^{(2)}, \ldots, \lambda_{x_2}^{(2)}),$$
$$\mu_* := (\mu_1^{(1)} + y_2, \ldots, \mu_{x_1}^{(1)} + y_2, \mu_2^{(2)}, \ldots, \mu_{x_2}^{(2)}).$$

With τ_* and τ^* defined as before with respect to $\boldsymbol{\lambda}$, we define

$$\mathrm{Tab}(\lambda^*/\mu^*) \ni \mathtt{t}^* := \tau^* \circ \mathtt{t} \qquad \text{and} \qquad \mathrm{Tab}(\lambda_*/\mu_*) \ni \mathtt{t}_* := \tau_* \circ \mathtt{t}$$

for $\mathtt{u} \in \mathrm{Tab}(\boldsymbol{\lambda}/\boldsymbol{\mu})$. We have bijections

$$\{\mathtt{t} \in \mathrm{St}(\boldsymbol{\lambda}/\boldsymbol{\mu}) \mid \mathtt{t}(x_1, 1, 1) < \mathtt{t}(1, y_2, 2)\} \xrightarrow{\tau^*} \mathrm{St}(\lambda^*/\mu^*)$$

and

$$\{\mathtt{t} \in \mathrm{St}(\boldsymbol{\lambda}/\boldsymbol{\mu}) \mid \mathtt{t}(x_1, 1, 1) > \mathtt{t}(1, y_2, 2)\} \xrightarrow{\tau_*} \mathrm{St}(\lambda^*/\mu^*)$$

Proposition 5.5.2. *Let $\boldsymbol{\lambda}/\boldsymbol{\mu} \in \mathscr{S}^\kappa$ be joinable, with the top right node in $\lambda^{(2)}$ having residue i. Let $\mathtt{t} \in \mathrm{St}(\boldsymbol{\lambda}/\boldsymbol{\mu})$. Then*

$$\deg \mathtt{t}^* = \deg \mathtt{t} - \big(\Lambda_i, \mathrm{cont}(\lambda^{(1)}/\mu^{(1)})\big)$$

if $\mathtt{t}(x_1, 1, 1) < \mathtt{t}(1, y_2, 2)$, *and*

$$\deg \mathtt{t}_* = \deg \mathtt{t} - \big(\Lambda_{i+1}, \mathrm{cont}(\lambda^{(1)}/\mu^{(1)})\big)$$

if $\mathtt{t}(x_1, 1, 1) > \mathtt{t}(1, y_2, 2)$.

Proof. We prove the first statement. The second is similar. Let $\nu = \lambda^* \backslash \mathrm{sh}((\mathtt{t}^{\boldsymbol{\lambda}})^*)$. Then by definition,

$$\deg \mathtt{t}^* = \deg \mathtt{Y}(\mathtt{t}^*) - \deg \mathtt{T}^{(\mu^*)} \tag{5.11}$$

$$\deg \mathtt{t} = \deg \mathtt{Y}(\mathtt{t}) - \deg \mathtt{T}^{\boldsymbol{\mu}} \tag{5.12}$$

$$\deg \mathtt{Y}(\mathtt{t})^* = \deg \mathtt{Y}(\mathtt{Y}(\mathtt{t})^*) - \deg \mathtt{T}^{\nu} \tag{5.13}$$

Lemma 5.5.2 gives us

$$\deg \mathtt{Y}(\mathtt{t})^* = \deg \mathtt{Y}(\mathtt{t}) - \big(\Lambda_i, \mathrm{cont}(\lambda^{(1)})\big) \tag{5.14}$$

Note that $\mathtt{Y}(\mathtt{Y}(\mathtt{t})^*)$ and $\mathtt{Y}(\mathtt{t}^*)$ agree outside of μ^*, so

$$\begin{aligned}
\deg \mathtt{Y}(\mathtt{t})^* + \deg \mathtt{T}^{\nu} - \deg \mathtt{Y}(\mathtt{t}^*) &= \deg \mathtt{Y}(\mathtt{Y}(\mathtt{t})^*) - \deg \mathtt{Y}(\mathtt{t}^*) \\
&= \deg \mathtt{Y}(\mathtt{Y}(\mathtt{t})^*)_{\leq|\mu^*|} - \deg \mathtt{Y}(\mathtt{t}^*)_{\leq|\mu^*|} \\
&= \deg \mathtt{Y}\left((\mathtt{T}^{\boldsymbol{\mu}})^*\right) - \deg \mathtt{T}^{(\mu^*)} \\
&= \deg(\mathtt{T}^{\boldsymbol{\mu}})^* + \deg \mathtt{T}^{\nu} - \deg \mathtt{T}^{(\mu^*)} \\
&= \deg \mathtt{T}^{\boldsymbol{\mu}} - \big(\Lambda_i, \mathrm{cont}(\mu^{(1)})\big) + \deg \mathtt{T}^{\nu} - \deg \mathtt{T}^{(\mu^*)},
\end{aligned} \tag{5.15}$$

using (5.9) in the last step. Combining equations (5.11)—(5.15) yields the result. □

Lemma 5.5.5. Let $\boldsymbol{\lambda}/\boldsymbol{\mu} \in \mathscr{S}^{\kappa}$ be a joinable skew diagram, and assume the top right node in $\lambda^{(2)}$ has residue i. With $\lambda^*/\mu^* \in \mathscr{S}^{\kappa^*}$ and $\lambda_*/\mu_* \in \mathscr{S}^{\kappa_*}$ defined as above, we have

$$\mathrm{ch}_{q\,q}(S^{\boldsymbol{\lambda}/\boldsymbol{\mu}}) = q^{d^*}\mathrm{ch}_{q\,q}\left(S^{\lambda^*/\mu^*}\right) + q^{d_*}\mathrm{ch}_{q\,q}\left(S^{\lambda_*/\mu_*}\right) = q^{d_{\boldsymbol{\lambda}/\boldsymbol{\mu}}}\mathrm{ch}_{q\,q}\left(S^{\lambda^{(1)}/\mu^{(1)}} \circ S^{\lambda^{(2)}/\mu^{(2)}}\right),$$

where

$$\begin{aligned}
d^* &= \big(\Lambda_i, \mathrm{cont}(\lambda^{(1)}/\mu^{(1)})\big) \\
d_* &= \big(\Lambda_{i+1}, \mathrm{cont}(\lambda^{(1)}/\mu^{(1)})\big) \\
d_{\boldsymbol{\lambda}/\boldsymbol{\mu}} &= \deg \mathtt{t}^{\boldsymbol{\lambda}/\boldsymbol{\mu}} - \deg \mathtt{t}^{\lambda^{(1)}/\mu^{(1)}} - \deg \mathtt{t}^{\lambda^{(2)}/\mu^{(2)}}.
\end{aligned}$$

Proof. The first equality follows from Corollary 5.4.10 and Proposition 5.5.2, via the bijections that τ^* and τ_* induce on basis elements. The second equality is Theorem 5.4.13. □

5.6. Cuspidal systems

Our primary motivation in developing the theory of skew Specht modules was to describe real cuspidal modules.

5.61. Main results

For reader convenience we recall the needed results on cuspidal systems in this section. Let $\alpha \in \Phi_+$. Given an R_α-module M, we say M is *semicuspidal* (resp. *cuspidal*) if $\mathrm{Res}^{\alpha}_{\beta,\gamma} M \neq 0$ implies that β is a sum of positive roots less than or equal to (resp. less than) α, and γ is a sum of positive roots greater than or equal to (resp. greater than) α. The following is proved in [24, 44], and Chapter III:

Theorem 5.6.1.

(i) For every $\alpha \in \Phi_+^{\mathrm{re}}$, there is a unique simple cuspidal R_α-module L_α.

(ii) For every $n > 0$, the simple semicuspidal $R_{n\delta}$-modules may be canonically labeled $\{L(\boldsymbol{\nu}) \mid \boldsymbol{\nu} \vdash n\}$, where $\boldsymbol{\nu} = (\nu^{(1)}, \dots, \nu^{(e-1)})$ ranges over $(e-1)$-multipartitions of n.

Let $\alpha \in Q_+$. Define the set $\Pi(\alpha)$ of *root partitions* of α to be the set of all pairs $(M, \boldsymbol{\nu})$, where $M = (n_\beta)_{\beta \in \Psi}$ is a tuple of nonnegative integers such that $\sum_{\beta \in \Psi} n_\beta \beta = \alpha$, and $\boldsymbol{\nu}$ is an $(e-1)$-multipartition of n_δ. There is a bilexicographic partial order $\leq$ on $\Pi(\alpha)$, see [24]. Given $(M, \boldsymbol{\nu}) \in \Pi(\alpha)$, define the *proper standard module*

$$\overline{\Delta}(M, \boldsymbol{\nu}) := L_{\beta_1}^{\circ n_{\beta_1}} \circ \cdots \circ L_{\beta_k}^{\circ n_{\beta_k}} \circ L(\boldsymbol{\nu}) \circ L_{\beta_{k+1}}^{\circ n_{\beta_{k+1}}} \circ \cdots \circ L_{\beta_t}^{\circ n_{\beta_t}} \langle \mathrm{shift}(M, \boldsymbol{\nu}) \rangle,$$

where $\beta_1, \dots, \beta_t$ are the real positive roots indexing nonzero entries in M, labeled such that $\beta_1 \succ \cdots \succ \beta_k \succ \delta \succ \beta_{k+1} \cdots \succ \beta_t$, and $\mathrm{shift}(M, \boldsymbol{\nu}) = \sum_{i=1 \neq t}(\beta_i, \beta_i) n_{\beta_i}(n_{\beta_i} - 1)/4$.

Theorem 5.6.2. [24, Main Theorem]

(i) For every root partition $(M, \boldsymbol{\nu})$, the proper standard module $\overline{\Delta}(M, \boldsymbol{\nu})$ has irreducible head, denoted $L(M, \boldsymbol{\nu})$.

(ii) $\{L(M, \boldsymbol{\nu}) \mid (M, \boldsymbol{\nu}) \in \Pi(\alpha)\}$ is a complete and irredundant system of irreducible R_α-modules up to isomorphism.

(iii) $[\overline{\Delta}(M, \boldsymbol{\nu}) : L(M, \boldsymbol{\nu})]_q = 1$, and $[\overline{\Delta}(M, \boldsymbol{\nu}) : L(M, \boldsymbol{\zeta})]_q \neq 0$ implies $(N, \boldsymbol{\nu}) \leq (M, \boldsymbol{\zeta})$.

(iv) $L(M, \boldsymbol{\nu})^{\circledast} \cong L(M, \boldsymbol{\nu})$.

5.62. Minuscule imaginary representations

The 'smallest' simple semicuspidal imaginary modules, those in R_δ-mod, are of particular importance. By the above, they are in bijection with $(e-1)$-multipartitions of 1. We label them $L_{\delta,i}$, for $i \in I \backslash \{0\}$.

Proposition 5.6.1. *For each $i \in I \backslash \{0\}$, $L_{\delta,i}$ can be characterized up to isomorphism as the unique irreducible R_δ-module such that $i_1 = 0$ and $i_e = i$ for all words $\boldsymbol{i}$ of $L_{\delta,i}$.*

Proof. This is [24, Lemma 5.1, Corollary 5.3]. □

5.63. Minimal pairs

Let $\rho \in \Phi_+^{\mathrm{re}}$. A pair of positive roots (β, γ) is called a *minimal pair* for ρ if

(i) $\beta + \gamma = \rho$ and $\beta \succ \rho$;

(ii) for any other pair (β', γ') satisfying (i) we have $\beta' \succ \beta$ or $\gamma' \prec \gamma$.

Lemma 5.6.3. Let $\rho \in \Phi_+^{\mathrm{re}}$ and (β, γ) be a minimal pair for ρ. If L is a composition factor of $\overline{\Delta}(\beta, \gamma) = L_\beta \circ L_\gamma$, then $L \cong L(\beta, \gamma)$ or $L \cong L_\rho$, up to shift.

Proof. This follows from the minimality of $(\beta, \gamma) \in \Pi(\rho)\backslash\{\rho\}$ and Theorem 5.6.2(iii). $\square$

One can be more precise in the case that (β, γ) be a *real* minimal pair for ρ; i.e., when $\beta, \gamma \in \Phi_+^{\mathrm{re}}$. Define

$$p_{\beta,\gamma} := \max\{n \in \mathbb{Z}_{\geq 0} \mid \beta - n\gamma \in \Phi_+\}. \tag{5.16}$$

Lemma 5.6.4. [24, Remark 6.5]. Let $\rho \in \Phi_+^{\mathrm{re}}$, and let (β, γ) be a real minimal pair for ρ. Then

$$[L_\beta \circ L_\gamma] = [L(\beta, \gamma)] + q^{p_{\beta,\gamma} - (\beta,\gamma)}[L_\rho],$$

and

$$[L_\gamma \circ L_\beta] = q^{-(\beta,\gamma)}[L(\beta, \gamma)] + q^{-p_{\beta,\gamma}}[L_\rho].$$

Lemmas 5.6.3 and 5.6.4 are useful in inductively constructing cuspidal modules.

5.64. Extremal words

Let $i \in I$. Define $\theta_i^* : \langle I \rangle \to \langle I \rangle$ by

$$\theta_i^*(\boldsymbol{j}) = \begin{cases} j_1 \cdots j_{d-1} & \text{if } j_d = i; \\ 0 & \text{otherwise.} \end{cases}$$

Extend θ_i^* linearly to a map $\theta_i^* : \mathscr{A}\langle I \rangle \to \mathscr{A}\langle I \rangle$. Let $x \in \mathscr{A}\langle I \rangle$, and define

$$\varepsilon_i(x) := \max\{k \geq 0 \mid (\theta_i^*)^k(x) \neq 0\}.$$

Definition 5.6.5. A word $i_1^{a_1} \cdots i_b^{a_b} \in \langle I \rangle$, with $a_1, \ldots, a_b \in \mathbb{Z}_{\geq 0}$, is called *extremal* for x if

$$a_b = \varepsilon_{i_b}(x), a_{b-1} = \varepsilon_{i_{b-1}}((\theta_{i_b}^*)^{a_b}(x)), \ldots, a_1 = \varepsilon_{i_1}((\theta_{i_2}^*)^{a_2} \cdots (\theta_{i_b}^*)^{a_b}(x)).$$

A word $\boldsymbol{i} \in \langle I \rangle$ is called *extremal* for $M \in R_\alpha$-mod if it is an extremal word for $\mathrm{ch}_{q\,q} M \in \mathscr{A}\boldsymbol{I}$.

The following lemma is useful in establishing multiplicity-one results for R_α-modules.

Lemma 5.6.6. [24, Lemma 2.28]. Let L be an irreducible R_α-module, and $\boldsymbol{i} = i_1^{a_1} \cdots i_b^{a_b} \in \langle I \rangle_\alpha$ be an extremal word for L. Then $\dim_q L_{\boldsymbol{i}} = [a_1]^! \cdots [a_b]^!$.

5.7. Cuspidal modules and skew hook Specht modules

Take a balanced convex preorder $\preceq$ on Φ_+. In this section we prove that the cuspidal modules L_ρ, for $\rho \in \Phi_+^{\mathrm{re}}$ are skew Specht modules associated to certain skew hook shapes, and provide an inductive process for identifying them as such.

5.71. Cuspidal modules for a balanced convex preorder

Throughout this section we work with Young diagrams and skew diagrams of level $l = 1$. Let $\kappa = (i)$. For $i \in I$, Let $\iota_i = (1) \in \mathscr{P}^{\kappa}_{\alpha_i}$. The following is clear:

Lemma 5.7.1. For $i \in I$, $L_{\alpha_i} \cong S^{\iota_i}$.

Let $\kappa = (0)$, and $\eta_i \in \mathscr{S}^{\kappa}_{\delta}$ be the hook partition of content δ with a node of residue i in the bottom row. Let $X_0 = 0$ and define $X_{i-1} := F\{v^{\mathtt{T}} \in S^{\eta_i} \mid \mathrm{res}_{\mathtt{T}}(e) = i-1\} \subseteq S^{\eta_i}$ for $1 < i \leq e-1$.

Lemma 5.7.2.

(i) X_{i-1} is a submodule of S^{η_i}.

(ii) $X_{i-1} \cong L_{\delta,i-1}\langle 1\rangle$ if $i > 1$.

(iii) $S^{\eta_i}/X_{i-1} \cong L_{\delta,i}$.

Proof. For $i > 1$, it is easy to see that

$$\{\mathtt{T} \in \mathrm{St}(\eta_i) \mid \mathrm{res}_{\mathtt{T}}(e) = i-1\} = \{\mathtt{T} \in \mathrm{St}(\eta_i) \mid \deg \mathtt{T} = 1\}$$

and

$$\{\mathtt{T} \in \mathrm{St}(\eta_i) \mid \mathrm{res}_{\mathtt{T}}(e) = i\} = \{\mathtt{T} \in \mathrm{St}(\eta_i) \mid \deg \mathtt{T} = 0\},$$

give a partition of $\mathrm{St}(\eta_i)$, and hence X_{i-1} is the span of degree 1 elements in S^{η_i}. As there are no repeated entries in words of S^{η_1}, it follows that every negatively-graded element of R_δ acts as zero on S^{η_1}, and hence X_{i-1} is a submodule. Moreover all words of X_{i-1} are of the form $(0, \dots, i-1)$, and all word spaces are 1-dimensional and in degree 1. Thus it follows from Proposition 5.6.1 that $X_{i-1} \cong L_{\delta,i-1}\langle 1\rangle$. Then all words of S^{η_i}/X_{i-1} are of the form $(0,\ldots,i)$ and all word spaces are 1-dimensional and in degree 0, so again it follows from Proposition 5.6.1 that $S^{\eta_i}/X_{i-1} \cong L_{\delta,i}$. $\square$

For $1 \le i \le e-1$, $m \in \mathbb{Z}_{\ge 0}$, let $\lambda^{m,i}/\mu^{m,i}$ be the skew hook diagram in $\mathscr{S}^{\kappa}_{m\delta+\alpha_i}$, where $l=1$, $\kappa = ((1-m)i \pmod{e})$,

$$\lambda^{m,i} = (mi+1, ((m-1)i+1)^{e-i}, \ldots, (i+1)^{e-i}, 1^{e-i})$$

and

$$\mu^{m,i} = (((m-1)i)^{e-i}, \ldots, (2i)^{e-i}, i^{e-i}).$$

Lemma 5.7.3. For $1 \le i \le e-1$, $m \in \mathbb{Z}_{\ge 0}$, $L(m\delta+\alpha_i) \cong S^{\lambda^{m,i}/\mu^{m,i}}$.

Proof. We prove this by induction on m. As $\lambda^{0,i}/\mu^{0,i} = \iota_i$, the claim follows by Lemma 5.7.1. Now assume that $L(m\delta+\alpha_i) \cong S^{\lambda^{m,i}/\mu^{m,i}}$. It is easy to see that $(m\delta+\alpha_i, \delta)$ is a minimal pair for $(m+1)\delta+\alpha_i$ (see [24, §6.1]). By Lemma 5.7.2, the factors of S^{η_i} are $L_{\delta,i}$ and $L_{\delta,i-1}\langle 1\rangle$. Thus by Lemma 5.6.3 the only possible factors (up to shift) of $S^{\lambda^{m,i}/\mu^{m,i}} \circ S^{\eta_i}$ are

$$L((m+1)\delta+\alpha_i) \qquad \text{and} \qquad L(m\delta+\alpha_i, \delta^{(j)}), \text{ for } j \in I\backslash\{0\}, \tag{5.17}$$

where we write $\delta^{(j)}$ for the $(e-1)$-multipartition of 1 which is (1) in the jth component and empty elsewhere.

Note that $\boldsymbol{\lambda}/\boldsymbol{\mu} := (\lambda^{m,i}, \eta_i)/(\mu^{m,i}, \varnothing)$ is joinable, with λ_*/μ_* (as defined in §5.5) equal to $\lambda^{m+1,i}/\mu^{m+1,i}$, so by Lemma 5.5.5, we have

$$\mathrm{ch}_{q\,q}(S^{\lambda^{m,i}/\mu^{m,i}} \circ S^{\eta_i}) = q^a \mathrm{ch}_{q\,q}(S^{\boldsymbol{\zeta}^{m+1,i}}) + q^b \mathrm{ch}_{q\,q}(S^{\lambda^*/\mu^*})$$

for some $a, b \in \mathbb{Z}$. By injectivity of the character map [20, Theorem 3.17], it follows that the only factors of $S^{\lambda^{m+1,i}/\mu^{m+1,i}}$ are those in (5.17), up to some shift. If $\mathtt{t} \in \mathrm{St}(\lambda^{m+1,i}/\mu^{m+1,i})$, with $\boldsymbol{i}(\mathtt{t}) = i_1 \cdots i_k$, note that $\alpha_{i_{k-e+1}} + \cdots + \alpha_{i_k} \neq \delta$, i.e., there is no sequence of removable nodes whose residues add up to δ, as is easily seen. Thus $\mathrm{Res}_{m\delta+\alpha_i,\delta} S^{\lambda^{m+1,i}/\mu^{m+1,i}} = 0$. But by adjointness and Theorem 5.6.2(i), $\mathrm{Res}_{m\delta+\alpha_i,\delta} L(m\delta + \alpha_i, \delta^{(j)}) \neq 0$ for all $j \in I \backslash \{0\}$. Hence $L(m\delta + \alpha_i, \delta^{(j)})$ is not a factor of $S^{\lambda^{m+1,i}/\mu^{m+1,i}}$ for any j, and the only possible factor is $L((m+1)\delta + \alpha_i)$ some number of times, with shifts.

Consider the extremal word

$$\boldsymbol{i} = 0^{m+1} 1^{m+1} \cdots (i-1)^{m+1} (e-1)^{m+1} \cdots (i+1)^{m+1} i^{m+2}$$

of $S^{\lambda^{m+1,i}/\mu^{m+1,i}}$. There are $((m+1)!)^{e-1}(m+2)!$ distinct $\mathtt{t} \in \mathrm{St}(\lambda^{m+1,i}/\mu^{m+1,i})$ such that $\boldsymbol{i}(\mathtt{t}) = \boldsymbol{i}$, so this is the (ungraded) dimension of the $\boldsymbol{i}$-word space of $S^{\lambda^{m+1,i}/\mu^{m+1,i}}$. By Lemma 5.6.6, the dimension of a module with extremal word $\boldsymbol{i}$ must be exactly $([m+1]^!)^{e-1}[m+2]^!$, which implies that $L((m+1)\delta + \alpha_i)$ can only appear once in $S^{\lambda^{m+1,i}/\mu^{m+1,i}}$, with some shift.

Let $\mathtt{t}^{\mathrm{top}} \in \mathrm{St}(\lambda^{m+1,i}/\mu^{m+1,i})$ be the tableau achieved by entering $1, \ldots, m$ in the 0-nodes of $\lambda^{m+1,i}/\mu^{m+1,i}$ from top to bottom, then $m+1, \ldots, 2m$ in the 1-nodes from top to bottom, and so forth, until the $(i-1)$-nodes are filled, then fill the nodes with residue $e-1, e-2, \ldots, i$ in the same fashion, working from top to bottom. Then $\mathtt{t}^{\mathrm{top}}$ has residue sequence $\boldsymbol{i}$, and

$$\deg \mathtt{t}^{\mathrm{top}} = \frac{(e-1)m(m+1)}{2} + \frac{(m+1)(m+2)}{2}.$$

Let $\mathtt{t}^{\mathrm{bot}}$ be constructed in the same fashion, except with nodes filled from bottom to top. Then

$$\deg \mathtt{t}^{\mathrm{bot}} = -\frac{(e-1)m(m+1)}{2} - \frac{(m+1)(m+2)}{2}.$$

As these degrees are the greatest and least in the expression $([m+1]^!)^{e-1}[m+2]^!$ it follows that $S^{\lambda^{m+1,i}/\mu^{m+1,i}}$ is symmetric with respect to grading, and hence $S^{\lambda^{m+1,i}/\mu^{m+1,i}} \cong L((m+1)\delta + \alpha_i)$ with no shift. □

For $1 \leq i \leq e-1$, $m \in \mathbb{Z}_{\geq 1}$, let $l = 1$, $\kappa = ((1-m)i \pmod{e})$, and let $\lambda_{m,i}/\mu_{m,i}$ be the skew hook diagram in $\mathscr{S}^\kappa$, where

$$\lambda_{m,i} = (mi, ((m-1)i+1)^{e-i}, \dots, (i+1)^{e-i}, 1^{e-i-1})$$
$$\mu_{m,i} = (((m-1)i)^{e-i}, \dots, (2i)^{e-i}, i^{e-i}).$$

Lemma 5.7.4. For $1 \leq i \leq e-1$, $m \in \mathbb{Z}_{\geq 1}$, $L(m\delta - \alpha_i) \cong S^{\lambda_{m,i}/\mu_{m,i}}\langle 1-m\rangle$.

Proof. We go by induction on m, and the proof proceeds in the same manner as Lemma 5.7.3. The base case is slightly different however. $S^{\lambda_{1,i}/\mu_{1,i}}$ is the hook partition with residue content δ, and upper right corner with residue $i-1$.

By [24, Lemma 5.2], $L(\delta-\alpha_i)$ factors through the cyclotomic quotient to become the unique irreducible $R^{\Lambda_0}_{\delta-\alpha_i}$-module. Consideration of the words of $S^{\lambda_{1,i}/\mu_{1,i}}$ shows that it factors through the cyclotomic quotient as well. Moreover, all of its word spaces are 1-dimensional and in degree 0, so it follows that $S^{\lambda_{1,i}/\mu_{1,i}} \cong L(\delta - \alpha_i)$.

The induction step proceeds as in Lemma 5.7.3, with $(\delta, m\delta - \alpha_i)$ used as a minimal pair for $(m+1)\delta - \alpha_i$. Considering the induction product $S^{\eta_i} \circ S^{\lambda_{1,i}/\mu_{1,i}}$ and using Lemma 5.5.5, one sees that the only possible factor of $S^{\lambda_{1,i}/\mu_{1,i}}$ is $L((m+1)\delta - \alpha_i)$, some number of times, with shifts. Consideration of the extremal word

$$\boldsymbol{i} = 0^{m+1}1^{m+1}\cdots(i-1)^{m+1}(e-1)^{m+1}\cdots(i+1)^{m+1}i^m$$

shows that $L((m+1)\delta-\alpha_i)$ appears but once as a factor of $S^{\lambda_{1,i}/\mu_{1,i}}$, with some shift. $L((m+1)\delta-\alpha_i)$ must have $\boldsymbol{i}$-word space of graded dimension $([m+1]^!)^{e-1}[m]^!$. As before, we define two standard $\lambda_{1,i}/\mu_{1,i}$-tableaux; $\mathtt{t}^{\mathrm{top}}$, where the nodes are filled in from top to bottom according to their order in $\boldsymbol{i}$, and $\mathtt{t}^{\mathrm{bot}}$, where the nodes are filled similarly from bottom to top. Then

$$\begin{aligned}
\deg \mathtt{t}^{\mathrm{top}} &= \left[\frac{(e-1)m(m+1)}{2} + \frac{(m-1)m}{2}\right] - m \\
\deg \mathtt{t}^{\mathrm{bot}} &= \left[-\frac{(e-1)m(m+1)}{2} - \frac{(m-1)m}{2}\right] - m.
\end{aligned}$$

On the right we have the greatest and least degrees in the expression $([m+1]^!)^{e-1}[m]^!$, shifted by $-m$, hence $L((m+1)\delta-\alpha_i) \cong S^{\lambda_{1,i}/\mu_{1,i}}\langle 1-(m+1)\rangle$, completing the proof. $\square$

5.72. Identifying cuspidal modules as skew hook Specht modules

We now present an inductive process for identifying cuspidal modules as skew hook Specht modules with a certain shift.

Proposition 5.7.1. *Let α be a real positive root, and assume that for all real positive roots β with $\mathrm{ht}(\beta) < \mathrm{ht}(\alpha)$, we have $L_\beta \cong S^{\lambda_\beta/\mu_\beta}\langle c_\beta\rangle$ for some skew hook diagram $\lambda_\beta/\mu_\beta \in \mathscr{S}^\kappa_\beta$, where $\kappa = (k)$ for some $k \in I$ and $c_\beta \in \mathbb{Z}$. Then the following process gives a skew hook diagram $\lambda_\alpha/\mu_\alpha$ and $c_\alpha \in \mathbb{Z}$ such that $L_\alpha \cong S^{\lambda_\alpha/\mu_\alpha}\langle c_\alpha\rangle$.*

(i) If $\alpha = m\delta + \alpha_i$ for some $m \in \mathbb{Z}_{\geq 0}$ and $i \in I\backslash\{0\}$, then $\lambda_\alpha/\mu_\alpha = \lambda^{m,i}/\mu^{m,i}$ and $c_\alpha = 0$.

(ii) If $\alpha = m\delta - \alpha_i$ for some $m \in \mathbb{Z}_{\geq 1}$ and $i \in I\backslash\{0\}$, then $\lambda_\alpha/\mu_\alpha = \lambda_{m,i}/\mu_{m,i}$ and $c_\alpha = 1 - m$.

(iii) Else there is a real minimal pair (β, γ) for α.

(a) If $\boldsymbol{\lambda}/\boldsymbol{\mu} := (\lambda_\beta, \lambda_\gamma)/(\mu_\beta, \mu_\gamma)$ is joinable, then $\lambda_\alpha/\mu_\alpha = \lambda_*/\mu_*$, and

$$c_\alpha = c_\beta + c_\gamma - p_{\beta,\gamma} + (\beta, \gamma) + d_* - d_{\boldsymbol{\lambda}/\boldsymbol{\mu}};$$

(b) else $\boldsymbol{\lambda}/\boldsymbol{\mu} := (\lambda_\gamma, \lambda_\beta)/(\mu_\gamma, \mu_\beta)$ is joinable, $\lambda_\alpha/\mu_\alpha = \lambda^*/\mu^*$, and

$$c_\alpha = c_\beta + c_\gamma + p_{\beta,\gamma} + d^* - d_{\boldsymbol{\lambda}/\boldsymbol{\mu}},$$

where d_*, d^* are as in Lemma 5.5.5, $d_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ as in Lemma 5.4.13, and $p_{\beta,\gamma}$ as in (5.16).

Proof. (1) and (2) are Lemmas 5.7.3 and 5.7.4, so assume we are in case (3). There exists a real minimal pair (β, γ) for α by [24, Lemma 6.9]. By assumption $L(\beta) \cong S^{\lambda_\beta/\mu_\beta}\langle c_\beta\rangle$, and $L(\gamma) \cong S^{\lambda_\gamma/\mu_\gamma}\langle c_\gamma\rangle$. We have $\beta = m\delta + (-1)^s(\alpha_i + \cdots + \alpha_j)$ for some $s \in \{0, 1\}$, $1 \leq i \leq j \leq e-1$ and $\gamma = m'\delta + (-1)^{s'}(\alpha_{i'} + \cdots + \alpha_{j'})$ for some $s' \in \{0, 1\}$, $1 \leq i' \leq j' \leq e-1$. Since $\beta + \gamma$ is a real root, one of the following must be true:

$$s = s', j + 1 = i' \quad \text{or} \quad s = s', j' + 1 = i \quad \text{or} \quad s = -s', j = j' \quad \text{or} \quad s = -s', i = i'.$$

Note that since λ_β/μ_β (resp. $\lambda_\gamma/\mu_\gamma$) is a skew hook diagram, $s = 0$ (resp. $s' = 0$) implies that the lower left node of λ_β/μ_β (resp. $\lambda_\gamma/\mu_\gamma$) has residue i (resp. i'), and the top right node has residue j (resp. j'). If $s = 1$ (resp. $s' = 1$), then the lower left node of λ_β/μ_β (resp. $\lambda_\gamma/\mu_\gamma$) has residue $j+1$ (resp. $j'+1$), and the top right node has residue $i-1$ (resp. $i'-1$). In any case then, we see that one of $(\lambda_\beta, \lambda_\gamma)/(\mu_\beta, \mu_\gamma)$ or $(\lambda_\gamma, \lambda_\beta)/(\mu_\gamma, \mu_\beta)$ must be joinable.

Assume the former, and set $\boldsymbol{\lambda}/\boldsymbol{\mu} := (\lambda_\beta, \lambda_\gamma)/(\mu_\beta, \mu_\gamma)$. Then, using Lemma 5.6.4,

$$[S^{\lambda_\beta/\mu_\beta} \circ S^{\lambda_\gamma/\mu_\gamma}] = q^{-c_\beta - c_\gamma}[L_\beta \circ L_\gamma] = q^{-c_\beta - c_\gamma}[L(\beta,\gamma)] + q^{-c_\beta - c_\gamma + p_{\beta,\gamma} - (\beta,\gamma)}[L_\alpha].$$

Using Lemma 5.5.5 and the fact that $\mathrm{ch}_{q\,q}$ is injective on $[R_\alpha\text{-mod}]$, we also have

$$[S^{\lambda_\beta/\mu_\beta} \circ S^{\lambda_\gamma/\mu_\gamma}] = q^{d^* - d_{\boldsymbol{\lambda}/\boldsymbol{\mu}}}[S^{\lambda^*/\mu^*}] + q^{d_* - d_{\boldsymbol{\lambda}/\boldsymbol{\mu}}}[S^{\lambda_*/\mu_*}].$$

Thus, L_α must be (a shift of) S^{λ^*/μ^*} or S^{λ_*/μ_*}. But $1_{\beta,\gamma} z^{\lambda^*/\mu^*} \neq 0$, so $\mathrm{Res}_{\beta,\gamma} S^{\lambda^*/\mu^*} \neq 0$, and thus the cuspidality property of $L(\alpha)$ implies it must be the latter, proving the validity of step (3)(a). If instead, $\boldsymbol{\lambda}/\boldsymbol{\mu} := (\lambda_\beta, \lambda_\gamma, \mu_\beta, \mu_\gamma)$ is joinable, we use the second statement in Lemma 5.6.4 and a similar argument to prove the validity of step (3)(b). □

Corollary 5.7.2. *For a balanced convex preorder, all real cuspidal modules of R_α are skew hook Specht modules up to some shift.*

Proof. Apply Proposition 5.7.1 inductively, with base case given by Lemma 5.7.1. □

Remark 5.7.5. In [32, §8.4], Kleshchev and Ram showed that in *finite* type $\mathtt{A}$, the cuspidal modules (associated to a convex lexicographic order) are Specht modules associated to hook partitions. Thus one can view Corollary 5.7.2 as an affine analogue of this fact.

5.73. Cuspidal modules for a special preorder

To give a complete picture, we explicitly describe the skew Specht modules corresponding to real positive roots in the case of a certain balanced *e-row preorder* on Φ_+, where the associated skew hook diagrams take on a very regular pattern. Take

(i) $m\delta + \alpha \succ m'\delta \succ m''\delta - \alpha$, for all $m \in \mathbb{Z}_{\geq 0}$, $m', m'' \in \mathbb{Z}_{\geq 1}$, $\alpha \in \Phi'_+$.

(ii) $m\delta + \alpha_i + \cdots + \alpha_j \succ m'\delta + \alpha_{i'} + \cdots + \alpha_{j'}$ if

$$i < i'; \qquad \text{or} \qquad i = i', m < m'; \qquad \text{or} \qquad i = i', m = m', j < j'.$$

(iii) $m\delta - \alpha_i - \cdots - \alpha_j \succ m'\delta - \alpha_{i'} - \cdots - \alpha_{j'}$ if

$$i > i'; \qquad \text{or} \qquad i = i', m > m'; \qquad \text{or} \qquad i = i', m = m', j < j'.$$

Under this preorder, it is easy to see that for any $\alpha \succ \delta$ not of the form $m\delta + \alpha_i$, the positive root $\beta \succ \alpha$ immediately preceding α in the order constitutes the lefthand side of a real minimal pair $(\beta, \alpha - \beta)$ for α. Similarly, for $\alpha \prec \delta$ not of the form $m\delta - \alpha_i$, the positive root $\alpha \succ \beta$ immediately succeeding α in the order constitutes the righthand side of a real minimal pair $(\alpha - \beta, \beta)$ for α. Then, applying the inductive process in Proposition 5.7.1, we arrive at:

(i) For $1 \leq i \leq j \leq e-1$ and $m \in \mathbb{Z}_{\geq 0}$, $L(m\delta + \alpha_i + \cdots + \alpha_j) \cong S^{\lambda/\mu}$, where λ/μ is the minimal skew hook diagram with residues shown on the left in Figure 5.1. below, with the 0-node appearing on the inner corners m times, and the i-node appearing on the outer corners $m+1$ times.

(ii) For $1 \leq i \leq j \leq e-1$ and $m \in \mathbb{Z}_{\geq 1}$, $L(m\delta - \alpha_i - \cdots - \alpha_j) \cong S^{\lambda/\mu}\langle 1-m \rangle$, where λ/μ is the minimal skew hook diagram with residues shown on the right in Figure 5.1. below, with the 0-node appearing on the inner corners m times, and the i-node appearing on the outer corners $m-1$ times.

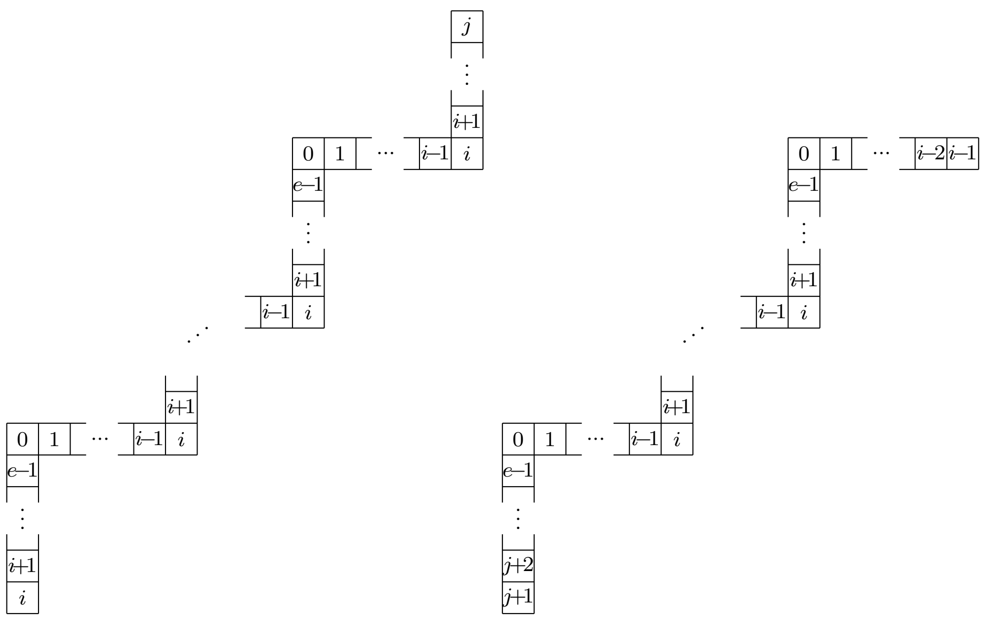


FIGURE 5.1. Skew hooks associated with real cuspidal modules.

CHAPTER VI

CALCULATIONS

The work in this chapter has appeared in the articles [29, 28]. It is co-authored with Alexander Kleshchev. We developed the results in the co-authored material jointly over many meetings, and, by the nature of collaborative mathematical work, it is difficult to attribute exact portions of the co-authored material to either Kleshchev or myself individually.

In this chapter we prove some necessary, but rather technical results cited in chapters III and IV.

6.1. Imaginary tensor space for non-simply-laced types

In this section we construct the minuscule R_δ-modules $L_{\delta,i}$ of color i for a Cartan matrix $\mathtt{C}$ of non-simply-laced type, along with the endomorphisms τ_r of $M_n = L_{\delta,i}^{\circ n}$ that satisfy the Coxeter relations of the symmetric group $\mathfrak{S}_n$.

6.11. Minuscule representations for non-simply-laced types

Write $R_\alpha^z = \mathcal{O}[z] \otimes_{\mathcal{O}} R_\alpha$, where z is an indeterminate element of degree 2. In the spirit of [19, Section 1.3], we construct in each case an R_δ^z-module $L_{\delta,i}^z$ with quotient isomorphic to $L_{\delta,i}$ as an R_δ-module. Kang, Kashiwara and Kim were able to construct this z-deformation of arbitrary modules for simply-laced types in general, but for the non-simply-laced types considered below we are forced to be more explicit. This approach allows us to construct the nonzero map τ out of the map R defined in [19, Chapter 1] which satisfies Coxeter relations, but unfortunately happens to be zero on $L_{\delta,i} \circ L_{\delta,i}$.

6.12. Construction of $L^z_{\delta,i}$ in type $\mathtt{B}_l^{(1)}$

We label the vertices of the diagram $\mathtt{B}_l^{(1)}$ as shown below in Figure 6.1..

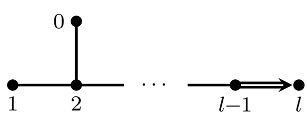


FIGURE 6.1. The diagram $\mathtt{B}_l^{(1)}$.

Fix $i \in I'$. Define

$$\boldsymbol{i} := \begin{cases} (0,2,3,\dots,l,l,l-1,\dots,i+1,1,2,\dots,i) & \text{if } i < l; \\ (0,2,3,\dots,l,1,2,\dots,l) & \text{if } i = l. \end{cases}$$

Let $C_{\boldsymbol{i}}$ be the connected component of $\boldsymbol{i}$ in the weight graph G_δ. For $\boldsymbol{j} \in C_{\boldsymbol{i}}$, $1 \leq k \leq 2l$, define constants

$$\xi_{\boldsymbol{j},r} = \begin{cases} 1 & \text{if } j_r = l \text{ and } j_t \neq l \text{ for all } t < r \\ -1 & \text{if } j_k = l \text{ and } j_t = l \text{ for some } t < r \\ 0 & \text{otherwise} \end{cases}$$

Let $L^z_{\delta,l}$ be the graded free $\mathcal{O}[z]$-module on basis

$$B = \{v_{\boldsymbol{j}} \mid \boldsymbol{j} \in C_{\boldsymbol{i}}\},$$

where each $v_{\boldsymbol{j}}$ is in degree 0. For $i \neq l$, let $L^z_{\delta,i}$ be the graded free $\mathcal{O}[z]$-module on basis

$$B = \{v^{(c)}_{\boldsymbol{j}} \mid c \in \{\pm 1\}, \boldsymbol{j} \in C_{\boldsymbol{i}}\},$$

where $v^{(c)}_{\boldsymbol{j}}$ is in degree c.

Define an action of generators of R^z_δ on $L^z_{\delta,l}$ as follows:

$$\begin{aligned}
1_{\boldsymbol{k}} v_{\boldsymbol{j}} &:= \delta_{\boldsymbol{j},\boldsymbol{k}} v_{\boldsymbol{j}} \\
y_r v_{\boldsymbol{j}} &:= \begin{cases} z^2 v_{\boldsymbol{j}} & \text{if } \xi_{\boldsymbol{j},r} = 0 \\ \xi_{\boldsymbol{j},r} z v_{\boldsymbol{j}} & \text{if } \xi_{\boldsymbol{j},r} \neq 0 \end{cases} \\
\psi_r v_{\boldsymbol{j}} &:= \begin{cases} v_{s_r \boldsymbol{j}} & \text{if } j_r \cdot j_{r+1} = 0 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

If $i \neq l$, define an action of generators of R^z_δ on $L^z_{\delta,i}$ as follows:

$$\begin{aligned}
1_{\boldsymbol{k}} v^{(c)}_{\boldsymbol{j}} &:= \delta_{\boldsymbol{j},\boldsymbol{k}} v^{(c)}_{\boldsymbol{j}} \\
y_r v^{(c)}_{\boldsymbol{j}} &:= \begin{cases} z^2 v^{(c)}_{\boldsymbol{j}} & \text{if } \xi_{\boldsymbol{j},r} = 0 \\ \xi_{\boldsymbol{j},r} z v^{(1)}_{\boldsymbol{j}} & \text{if } \xi_{\boldsymbol{j},r} \neq 0, c = 1 \\ -\xi_{\boldsymbol{j},r} \left(z v^{(-1)}_{\boldsymbol{j}} + v^{(1)}_{\boldsymbol{j}} \right) & \text{if } \xi_{\boldsymbol{j},r} \neq 0, c = -1 \end{cases} \\
\psi_r v^{(c)}_{\boldsymbol{j}} &:= \begin{cases} v^{(c)}_{s_r \boldsymbol{j}} & \text{if } j_r \cdot j_{r+1} = 0 \\ v^{(-1)}_{\boldsymbol{j}} & \text{if } c = 1, j_r = j_{r+1} = l \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Proposition 6.1. The formulas above define a graded action of R^z_δ on $L^z_{\delta,i}$, and $L^z_{\delta,i}$ is $\mathcal{O}[z]$-free on basis B.

Proof. Assume $i < l$. We check that the given action agrees with the algebra relations (2.42)–(2.48). That relations (2.42), (2.44), and (2.47) are satisfied is clear. For purposes of checking relations we will have cause to combine cases to write the action of y_k and ψ_k as

$$y_r v_{\boldsymbol{j}}^{(c)} = [(1-\delta_{j_r,l})z^2 + \xi_{\boldsymbol{j},r} cz]v_{\boldsymbol{j}}^{(c)} - \delta_{c,-1}\xi_{\boldsymbol{j},r} v_{\boldsymbol{j}}^{(1)}$$
$$\psi_r v_{\boldsymbol{j}}^{(c)} = \delta_{j_r\cdot j_{r+1},0} v_{s_r\boldsymbol{j}}^{(c)} + \delta_{c,1}\delta_{j_r,j_{r+1}}\delta_{j_r,l} v_{\boldsymbol{j}}^{(-1)}.$$

We omit idempotents $1_{\boldsymbol{j}}$ in the below, considering only $\boldsymbol{j} \in C_{\boldsymbol{i}}$, as other idempotents act as zero and thus cause the the remaining relations to be satisfied.

Relation (2.43). We have

$$\begin{aligned}
y_t y_r v_{\boldsymbol{j}}^{(c)} :=& y_t \left[[(1-\delta_{j_r,l})z^2 + \xi_{\boldsymbol{j},r} cz]v_{\boldsymbol{j}}^{(c)} - \delta_{c,-1}\xi_{\boldsymbol{j},r} v_{\boldsymbol{j}}^{(1)} \right] \\
=& [(1-\delta_{j_r,l})z^2 + \xi_{\boldsymbol{j},r} cz] \left[[(1-\delta_{j_t,l})z^2 + \xi_{\boldsymbol{j},t} cz]v_{\boldsymbol{j}}^{(c)} - \delta_{c,-1}\xi_{\boldsymbol{j},t} v_{\boldsymbol{j}}^{(1)} \right] \\
&\quad - \delta_{c,-1}\xi_{\boldsymbol{j},r} \left[[(1-\delta_{j_t,l})z^2 + \xi_{\boldsymbol{j},t} cz]v_{\boldsymbol{j}}^{(1)} \right] \\
=& [(1-\delta_{j_r,l})z^2 + \xi_{\boldsymbol{j},r} cz][(1-\delta_{j_t,l})z^2 + \xi_{\boldsymbol{j},t} cz]v_{\boldsymbol{j}}^{(c)} \\
&\quad - \delta_{c,-1} \left[(1-\delta_{j_r,l})\xi_{\boldsymbol{j},t} z^2 + (1-\delta_{j_t,l})\xi_{\boldsymbol{j},r} z^2 + 2\xi_{\boldsymbol{j},r}\xi_{\boldsymbol{j},t} \right] v_{\boldsymbol{j}}^{(1)},
\end{aligned}$$

which is invariant under the exchange of r and t, hence $y_t y_r v_{\boldsymbol{j}}^{(c)} = y_r y_t v_{\boldsymbol{j}}^{(c)}$.

Relation (2.45). Assume $j_r \cdot j_{r+1} = 0$. Then

$$y_t \psi_r v_{\boldsymbol{j}}^{(c)} = [(1-\delta_{(s_r\boldsymbol{j})_t,l})z^2 + \xi_{s_r\boldsymbol{j},t} cz]v_{s_r\boldsymbol{j}}^{(c)} - \delta_{c,-1}\xi_{s_r\boldsymbol{j},t} v_{s_r\boldsymbol{j}}^{(1)}$$
$$\psi_r y_{s_r t} v_{\boldsymbol{j}}^{(c)} = [(1-\delta_{j_{s_r t},l})z^2 + \xi_{\boldsymbol{j},s_r t} cz]v_{s_r\boldsymbol{j}}^{(c)} - \delta_{c,-1}\xi_{\boldsymbol{j},s_r t} v_{s_r\boldsymbol{j}}^{(1)}$$

We have $(s_r\boldsymbol{j})_t = j_{s_rt}$, and since $j_r \neq j_{r+1}$, we have $\xi_{\boldsymbol{j},s_rt} = \xi_{s_r\boldsymbol{j},t}$. Then

$$(y_t\psi_r - \psi_r y_{s_rt})v_{\boldsymbol{j}}^{(c)} = 0 = \delta_{j_r,j_{r+1}}(\delta_{t,r+1} - \delta_{t,r})v_{\boldsymbol{j}}^{(c)}.$$

Next assume $j_r = j_{r+1} = l$, $c = 1$. Then

$$\begin{aligned} y_t\psi_r v_{\boldsymbol{j}}^{(1)} &= [(1 - \delta_{j_t,l})z^2 - \xi_{\boldsymbol{j},t}]v_{\boldsymbol{j}}^{(-1)} - \xi_{\boldsymbol{j},t}v_{\boldsymbol{j}}^{(1)} \\ \psi_r y_{s_rt}v_{\boldsymbol{j}}^{(1)} &= [(1 - \delta_{j_{s_rt},l})z^2 + \xi_{\boldsymbol{j},s_rt}z]v_{\boldsymbol{j}}^{(-1)}. \end{aligned}$$

There are a few cases to consider:

(i) $t \neq r, r+1$. Then $\xi_{\boldsymbol{j},s_rt} = \xi_{\boldsymbol{j},t} = \delta_{j_t,l} = \delta_{j_{s_rt},l} = 0$, so

$$(y_t\psi_r - \psi_r y_{s_rt})v_{\boldsymbol{j}}^{(c)} = 0 = \delta_{j_r,j_{r+1}}(\delta_{t,r+1} - \delta_{t,r})v_{\boldsymbol{j}}^{(c)}.$$

(ii) $t = r$. Then $\xi_{\boldsymbol{j},t} = \delta_{j_t,l} = \delta_{j_{s_rt},l} = 1$, $\xi_{\boldsymbol{j},s_rt} = -1$, so

$$(y_t\psi_r - \psi_r y_{s_rt})v_{\boldsymbol{j}}^{(c)} = -v_{\boldsymbol{j}}^{(1)} = \delta_{j_r,j_{r+1}}(\delta_{t,r+1} - \delta_{t,r})v_{\boldsymbol{j}}^{(c)}.$$

(iii) $t = r+1$. Then $\xi_{\boldsymbol{j},s_rt} = \delta_{j_t,l} = \delta_{j_{s_rt},l} = 1$, $\xi_{\boldsymbol{j},t} = -1$, so

$$(y_t\psi_r - \psi_r y_{s_rt})v_{\boldsymbol{j}}^{(c)} = v_{\boldsymbol{j}}^{(1)} = \delta_{j_r,j_{r+1}}(\delta_{t,r+1} - \delta_{t,r})v_{\boldsymbol{j}}^{(c)}.$$

Now assume $j_r = j_{r+1} = l$, $c = -1$. Then

$$\begin{aligned} y_t\psi_r v_{\boldsymbol{j}}^{(-1)} &= 0 \\ \psi_r y_{s_rt}v_{\boldsymbol{j}}^{(-1)} &= -\xi_{\boldsymbol{j},s_rt}v_{\boldsymbol{j}}^{(-1)}. \end{aligned}$$

If $t \neq r, r+1$, then $\xi_{\boldsymbol{j},s_r t} = 0$. If $t = r$, then $\xi_{\boldsymbol{j},s_r t} = -1$, and $t = r+1$, then $\xi_{\boldsymbol{j},s_r t} = 1$. In all these cases (3.2.8) is satisfied.

This leaves $j_r \cdot j_{r+1} \neq 0$, with $j_r \neq 0$ or $j_{r+1} \neq 0$. Then $y_t \psi_r v_{\boldsymbol{j}}^{(c)} = \psi_r y_{s_r t} v_{\boldsymbol{j}}^{(c)} = 0$. For $\boldsymbol{j} \in C_{\boldsymbol{i}}$, $j_r = j_{r+1}$ implies $j_r = j_{r+1} = l$, so we have $j_r \neq j_{r+1}$. Then $\delta_{j_r, j_{r+1}}(\delta_{t,r+1} - \delta_{t,r}) v_{\boldsymbol{j}}^{(c)} = 0$, so in all cases (2.45) is satisfied.

Relation (2.46). If $j_r \cdot j_{r+1} \geq 0$, then this relation is clearly satisfied. So assume $j_r \cdot j_{r+1} < 0$. In this case $\psi_r^2 v_{\boldsymbol{j}}^{(c)} = 0$. Then we just check a few cases:

(i) $j_r, j_{r+1} \neq l$. Then

$$Q_{j_r,j_{r+1}}(y_r, y_{r+1}) v_{\boldsymbol{j}}^{(c)} = \epsilon_{j_r,j_{r+1}}(y_r - y_{r+1}) v_{\boldsymbol{j}}^{(c)} = \epsilon_{j_r,j_{r+1}}[z^2 v_{\boldsymbol{j}}^{(c)} - z^2 v_{\boldsymbol{j}}^{(c)}] = 0.$$

(ii) $j_r = l-1, j_{r+1} = l$. Then since $\boldsymbol{j} \in C_{\boldsymbol{i}}$, we have $\xi_{\boldsymbol{j},r} = 0, \xi_{\boldsymbol{j},r+1} = 1$, and

$$\begin{aligned} Q_{j_r,j_{r+1}}(y_r, y_{r+1}) v_{\boldsymbol{j}}^{(c)} &= \epsilon_{l-1,l}(y_r - y_{r+1}^2) v_{\boldsymbol{j}}^{(c)} \\ &= \epsilon_{l-1,l}[z^2 v_{\boldsymbol{j}}^{(c)} - y_{r+1}(czv_{\boldsymbol{j}}^{(c)} - \delta_{c,-1} v_{\boldsymbol{j}}^{(1)})] \\ &= \epsilon_{l-1,l}[z^2 v_{\boldsymbol{j}}^{(c)} - (cz(czv_{\boldsymbol{j}}^{(c)} - \delta_{c,-1} v_{\boldsymbol{j}}^{(1)}) - \delta_{c,-1} z v_{\boldsymbol{j}}^{(1)})] = 0. \end{aligned}$$

(iii) $j_r = l, j_{r+1} = l-1$. Then since $\boldsymbol{j} \in C_{\boldsymbol{i}}$, we have $\xi_{\boldsymbol{j},r} = -1, \xi_{\boldsymbol{j},r+1} = 0$, and

$$\begin{aligned} Q_{j_r,j_{r+1}}(y_r, y_{r+1}) v_{\boldsymbol{j}}^{(c)} &= \epsilon_{l,l-1}(y_r^2 - y_{r+1}) v_{\boldsymbol{j}}^{(c)} \\ &= \epsilon_{l,l-1}[y_r(-czv_{\boldsymbol{j}}^{(c)} + \delta_{c,-1} v_{\boldsymbol{j}}^{(1)}) - z^2 v_{\boldsymbol{j}}^{(c)}] \\ &= \epsilon_{l,l-1}[(-cz(-czv_{\boldsymbol{j}}^{(c)} + \delta_{c,-1} v_{\boldsymbol{j}}^{(1)}) - \delta_{c,-1} v_{\boldsymbol{j}}^{(1)}) - z^2 v_{\boldsymbol{j}}^{(c)}] = 0. \end{aligned}$$

Thus relation (2.46) is satisfied.

Relation (2.48). We have

$$
\begin{aligned}
\psi_r\psi_{r+1}\psi_r v_{\boldsymbol{j}}^{(c)} = \delta_{j_r\cdot j_{r+1},0}\delta_{j_r\dot{j}_{r+2},0}\delta_{j_{r+1}\cdot j_{r+2},0}v^{(c)}_{s_rs_{r+1}s_r\boldsymbol{j}} \\
+ \delta_{c,1}\delta_{j_r\cdot j_{r+1},0}\delta_{j_{r+1},j_{r+2}}\delta_{j_{r+1},l}v^{(-1)}_{s_{r+1}s_r\boldsymbol{j}} \\
+ \delta_{c,1}\delta_{j_r\cdot j_{r+1},0}\delta_{j_r,j_{r+2}}\delta_{j_r,l}v^{(-1)}_{\boldsymbol{j}} \\
+ \delta_{c,1}\delta_{j_r\cdot j_{r+2},0}\delta_{j_r,j_{r+1}}\delta_{j_r,l}v^{(-1)}_{s_rs_{r+1}\boldsymbol{j}} \\
\psi_{r+1}\psi_r\psi_{r+1} v_{\boldsymbol{j}}^{(c)} = \delta_{j_r\cdot j_{r+1},0}\delta_{j_r\dot{j}_{r+2},0}\delta_{j_{r+1}\cdot j_{r+2},0}v^{(c)}_{s_{r+1}s_rs_{r+1}\boldsymbol{j}} \\
+ \delta_{c,1}\delta_{j_r\cdot j_{r+1},0}\delta_{j_{r+1},j_{r+2}}\delta_{j_{r+1},l}v^{(-1)}_{s_{r+1}s_r\boldsymbol{j}} \\
+ \delta_{c,1}\delta_{j_r\cdot j_{r+1},0}\delta_{j_r,j_{r+2}}\delta_{j_r,l}v^{(-1)}_{\boldsymbol{j}} \\
+ \delta_{c,1}\delta_{j_r\cdot j_{r+2},0}\delta_{j_r,j_{r+1}}\delta_{j_r,l}v^{(-1)}_{s_rs_{r+1}\boldsymbol{j}}.
\end{aligned}
$$

So, since $s_rs_{r+1}s_r = s_{r+1}s_rs_{r+1}$, we have

$$(\psi_{r+1}\psi_r\psi_{r+1} - \psi_r\psi_{r+1}\psi_r)v_{\boldsymbol{j}}^{(c)} = 0.$$

Thus, if $j_r \neq j_{r+2}$, relation (2.48) is satisfied. If $\boldsymbol{j} \in C_{\boldsymbol{i}}$ is such that $j_r = j_{r+2}$, it follows that $j_r = j_{r+2} = l$, and $j_r \cdot j_{r+1} = 0$. Then $Q_{j_r,j_{r+1}} = 1$, and again (2.48) is satisfied.

The relations in case $i = l$ are more easily verified by similar computations. □

6.13. Construction of $L^z_{\delta,i}$ in type $\mathtt{C}_l^{(1)}$

We label the vertices of the diagram $\mathtt{C}_l^{(1)}$ as shown in Figure 6.2..

Fix $i \in I'$. Let $L^z_{\delta,i,0}$ be the graded free 1-dimensional $\mathcal{O}[z]$-module on generator x_0 (in degree 0). Define an action of $R^z_{\alpha_0}$ on $L^z_{\delta,i,0}$ by $1_{(0)}x_0 = x_0$, $y_1x_0 = z^2x_0$.

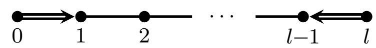


FIGURE 6.2. The diagram $\mathtt{C}_l^{(1)}$.

Define

$$\boldsymbol{j}^{(1)} = \begin{cases} (1, \dots, l-1, l, l-1, \dots i+1) & \text{if } i < l; \\ (1, \dots, l-1) & \text{if } i = l. \end{cases}$$

Let $L^z_{\delta,i,1}$ be the graded free 1-dimensional $\mathcal{O}[z]$-module on generator x_1 (in degree 0). Define an action of generators of $R^z_{\delta-\alpha_0-\dots-\alpha_i}$ as follows:

$$1_{\boldsymbol{k}} x_1 := \delta_{\boldsymbol{j}^{(1)},\boldsymbol{k}} x_1$$

$$y_r x_1 := \begin{cases} z x_1 & \text{if } r < l; \\ z^2 x_1 & \text{if } r = l; \\ -z x_1 & \text{if } r > l. \end{cases}$$

$$\psi_r x_1 := 0.$$

If $i > 1$, define

$$\boldsymbol{j}^{(2)} = (1, \dots, i-1),$$

and let $L^z_{\delta,i,2}$ be the graded free 1-dimensional $\mathcal{O}[z]$-module on generator x_2 (in degree 0). Define an action of generators of $R^z_{\alpha_1+\cdots+\alpha_{i-1}}$ on $L_{\delta,i,2}$ as follows:

$$1_{\boldsymbol{k}} x_2 := \delta_{\boldsymbol{j}^{(2)},\boldsymbol{k}} x_2$$
$$y_r x_2 := -z x_2$$
$$\psi_r x_2 := 0.$$

Let $L^z_{\delta,i,3}$ be the graded free 1-dimensional $\mathcal{O}[z]$-module on generator x_3 (in degree 0). Define an action of $R^z_{\alpha_i}$ on $L_{\delta,i,3}$ by

$$1_{(i)} x_3 = x_3$$
$$y_1 x_3 = \begin{cases} -z x_3 & \text{if } i < l; \\ z^2 x^3 & \text{if } i = l. \end{cases}$$

Proposition 6.2. The formulas above define a graded action of the algebras $R^z_{\alpha_0}$, $R^z_{\delta-\alpha_0-\cdots-\alpha_i}$, $R^z_{\alpha_1+\cdots+\alpha_{i-1}}$ and $R^z_{\alpha_i}$ on the modules $L^z_{\delta,i,0}$, $L^z_{\delta,i,1}$, $L^z_{\delta,i,2}$, and $L^z_{\delta,i,3}$ respectively, which are $\mathcal{O}[z]$-free on their respective bases.

Proof. It is easily checked that the given action agrees with the algebra relations (2.42)–(2.48). □

Now define the $R^z_{\alpha_0,\delta-\alpha_0-\alpha_i,\alpha_i}$-module

$$L^z_{\delta,i} = \begin{cases} L_{\delta,i,0} \boxtimes L_{\delta,i,1} \boxtimes L_{\delta,i,3} & \text{if } i = 1; \\ L_{\delta,i,0} \boxtimes L_{\delta,i,1} \circ L_{\delta,i,2} \boxtimes L_{\delta,i,3} & \text{if } i > 1. \end{cases}$$

If $i = 1$, write $x = x_0 \otimes x_1 \otimes x_3$ and $\boldsymbol{j} = (0, \boldsymbol{j}^{(1)}, 1)$. Otherwise write $x = x_0 \otimes x_1 \otimes x_2 \otimes x_3$ and $\boldsymbol{j} = (0, \boldsymbol{j}^{(1)}, \boldsymbol{j}^{(2)}, i)$. Then x is a word vector of word $\boldsymbol{j}$, and $L^z_{\delta,i}$ is $\mathcal{O}[z]$-free on basis

$$B = \{\psi_u x \mid u \in \mathcal{D}^{1,2l-i-1,i-1,1}_{1,2l-2,1}\}.$$

We are most of the way to defining an action of the standard generators of R^z_δ on $L^z_{\delta,i}$. The generators $1_{\boldsymbol{k}}, y_r$, and ψ_r (for $1 < r < 2l-1$) act as already prescribed by membership in the subalgebra $R^z_{\alpha_0, \delta-\alpha_0-\alpha_i, \alpha_i}$. It remains to define $\psi_1 v = \psi_{2l-1} v = 0$, and $1_{\boldsymbol{k}} v = 0$ (if $k_1 \neq 0$ or $k_{2n} \neq i$) for all $v \in L^z_{\delta,i}$.

Proposition 6.3. The description above defines a graded action of R^z_δ on $L^z_{\delta,i}$, and $L^z_{\delta,i}$ is $\mathcal{O}[z]$-free on basis B.

Proof. We check that the given action agrees with the algebra relations (2.42)–(2.48) for R^z_δ. By construction, all relations that do not involve ψ_1 or ψ_{2l-1} are easily seen to be satisfied due to the local nature of the relations and the fact that $L^z_{\delta,i,1} \circ L^z_{\delta,i,2}$ is an $R^z_{\delta-\alpha_0-\alpha_i}$-module. Additionally, all relations that involve $1_{\boldsymbol{k}}$ with $k_1 \neq 0$ or $k_{2l} \neq i$ are trivially satisfied. We check the rest of the relations that are not immediately obvious below, for basis vectors $\psi_u x \in B$.

Relation (2.45). There is no $u \in \mathcal{D}^{1,2l-i-1,i-1,1}_{1,2l-2,1}$ such that $(u\boldsymbol{j})_2 = 0$ or $(u\boldsymbol{j})_{2l-1} = i$, so we just need verify that the left side of the relation is always zero. Indeed, ψ_1 and ψ_{2l-1} always act as zero, and the action of y_1 and y_{2l} commute with all ψ_r's by construction, giving the result.

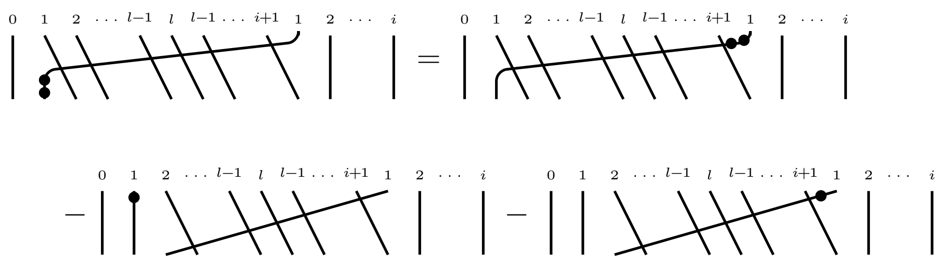
Relation (2.46). We have $\psi_1^2(\psi_u x) = 0$. Note that $(u\boldsymbol{j})_1 = 1$, so we check:

$$Q_{0,1}(y_1, y_2) \cdot \psi_u x = \epsilon_{01}(y_1 - y_2^2)\psi_u x = \epsilon_{01}(z^2 \psi_u x - y_2^2 \psi_u x).$$

Now either $u(2) = 2$, in which case $y_2^2\psi_u x = \psi_u y_2^2 x = z^2\psi_u x$, or $u(2) > 2$, in which case $\psi_u = \psi_u'\psi_2 \cdots \psi_{2l-i}x$ for some $\psi_{u'}$ that does not involve ψ_1 or ψ_2. Then

$$y_2^2\psi_u x = y_2^2\psi_u'\psi_2 \cdots \psi_{2l-i}x = \psi_{u'}y_2^2\psi_1 \cdots \psi_{2l-i}x.$$

Diagrammatically, $y_2^2\psi_2 \cdots \psi_{2l-i}x$ is of the form



where we picture the vector x as being at the top of each diagram. But then this is

$$z^2\psi_2 \cdots \psi_{2l-i}x - z\psi_3 \cdots \psi_{2l-i}x + z\psi_3 \cdots \psi_{2l-i}x = z^2\psi_2 \cdots \psi_{2l-i}x,$$

so $y_2^2\psi_u x = z^2\psi_u x$, and thus $Q_{0,1}(y_1, y_2) \cdot \psi_u x = 0$ in any case.

On the other side, we have $\psi_{2l-1}^2(\psi_u x) = 0$. Note that either $(u\boldsymbol{j})_{2l-1} = i - 1$ or $(u\boldsymbol{j})_{2l-1} = i + 1$. The case $i = l$ is handled similarly to the above argument. For simplicity assume $i \leq l - 2$ (again the case $i = l - 1$ is handled similarly). If $(u\boldsymbol{j})_{2l-1} = i - 1$, then ψ_u does not involve ψ_{2l-2} or ψ_{2l-1}, and thus

$$Q_{i-1,i}(y_{2l-1}, y_{2l}) \cdot \psi_u x = \epsilon_{i-1,i}(y_{2l-1} - y_{2l}) \cdot \psi_u x = \epsilon_{i-1,i}(y_{2l-1}\psi_u x + z\psi_u x),$$

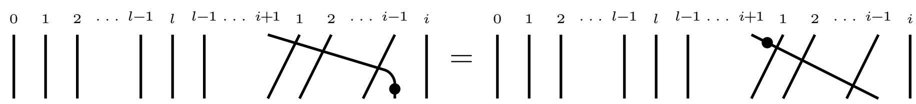
but $y_{2l-1}\psi_u x = \psi_u y_{2l-1}x = -z\psi_u x$, so this is zero. If $(u\boldsymbol{j})_{2l-1} = i + 1$, then

$$Q_{i+1,i}(y_{2l-1}, y_{2l}) \cdot \psi_u x = \epsilon_{i+1,i}(y_{2l-1} - y_{2l}) \cdot \psi_u x = \epsilon_{i+1,i}(y_{2l-1}\psi_u x + z\psi_u x),$$

and ψ_u can be written $\psi_{u'}\psi_{2l-2}\cdots\psi_{2l-i}$ for some $\psi_{u'}$ that does not involve ψ_{2l-2} or ψ_{2l-1}. Then

$$y_{2l-1}\psi_u x = y_{2l-1}\psi_{u'}\psi_{2l-2}\cdots\psi_{2l-i}x = \psi_{u'}y_{2l-1}\psi_{2l-2}\cdots\psi_{2l-i}x.$$

Diagrammatically, $y_{2l-1}\psi_{2l-2}\cdots\psi_{2l-i}x$ is of the form

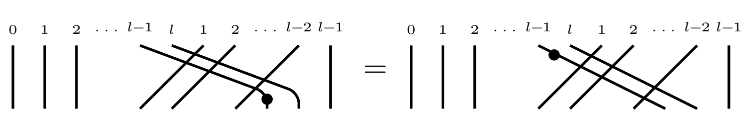


which is $-z\psi_{2l-2}\cdots\psi_{2l-i}x$, so $y_{2l-1}\psi_u x = -z\psi_u x$. Thus $Q_{i+1,i}(y_{2l-1}, y_{2l})\cdot\psi_u x = 0$.

Relation (2.48). The only case for which this relation is non-trivial is when $i = l-1$. Indeed, in all other cases there is no $\boldsymbol{k} = u\boldsymbol{j}$ where $k_1 = k_3$ or $k_{2l-2} = k_{2l}$. When $i = l-1$, the non-trivial case occurs when $(u\boldsymbol{j})_{2l-2} = l-1$, $(u\boldsymbol{j})_{2l-1} = l$. Then ψ_u can be written $\psi_{u'}\psi_{2l-3}\cdots\psi_l\psi_{2l-2}\cdots\psi_{l+1}$ for some $\psi_{u'}$ not involving ψ_{2l-3}, ψ_{2l-2} or ψ_{2l-1}. Then

$$\begin{aligned}
&\frac{Q_{l-1,l}(y_{2l}, y_{2l-1}) - Q_{l-1,l}(y_{2l-2}, y_{2l-1})}{y_{2l} - y_{2l-2}}\psi_u x \\
&= \epsilon_{l-1,l}\frac{(y_{2l}^2 - y_{2l-1}) - (y_{2l-2}^2 - y_{2l-1})}{y_{2l} - y_{2l-2}}\psi_u x \\
&= \epsilon_{l-1,l}(y_{2l} + y_{2l-2})\cdot\psi_u x \\
&= \epsilon_{l-1,l}(-z\psi_u x + y_{2l-2}\psi_u x) \\
&= \epsilon_{l-1,l}(-z\psi_u x + \psi_{u'}y_{2l-2}\psi_{2l-3}\cdots\psi_l\psi_{2l-2}\cdots\psi_{l+1}x)
\end{aligned}$$

Diagrammatically, $y_{2l-2}\psi_{2l-3}\cdots\psi_l\psi_{2l-2}\cdots\psi_{l+1}x$ is of the form



But this is $z\psi_{2l-3}\cdots\psi_l\psi_{2l-2}\cdots\psi_{l+1}x$, so $y_{2l-2}\psi_u x = z\psi_u x$, and we are done. $\square$

6.14. Construction of $L^z_{\delta,i}$ in type $\mathtt{F}_4^{(1)}$

We label the vertices of the diagram $\mathtt{F}_4^{(1)}$ as shown below in Figure 6.3..

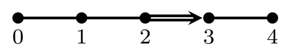


FIGURE 6.3. The diagram $\mathtt{F}_4^{(1)}$.

Fix $i \in I'$. If $i \in \{1,2\}$, let X be the set of the following tuples:

$$\begin{array}{lll}
(1,2,3,4,5,6), & (1,3,2,4,5,6), & (1,2,3,5,4,6), \\
(1,3,2,5,4,6), & (3,1,2,4,5,6), & (1,2,3,5,6,4), \\
(3,1,2,5,4,6), & (1,3,2,5,6,4), & (3,1,2,5,6,4), \\
(3,1,5,2,4,6), & (1,3,5,2,6,4), & (3,1,5,2,6,4), \\
(3,5,1,2,6,4), & (3,1,5,6,2,4), & (3,5,1,6,2,4), \\
(3,5,1,2,4,6), & (1,3,5,6,2,4), & (1,3,5,2,4,6).
\end{array}$$

If $i = 3$, let X be the set of the following tuples:

$$\begin{array}{lll}
(1,2,3,4,5,6), & (1,3,2,4,5,6), & (1,2,3,5,4,6), \\
(1,3,2,5,4,6), & (3,1,2,4,5,6), & (3,1,2,5,4,6), \\
(3,1,5,2,4,6), & (3,5,1,2,4,6), & (1,3,5,2,4,6).
\end{array}$$

If $i = 4$, let X be the set of the following tuples:

$$(1,2,3,4,6,5), \quad (1,3,2,4,6,5), \quad (3,1,2,4,6,5).$$

For $a \in \{1, \ldots, 6\}$, define

$$\omega_a = \begin{cases} 3, & a \in \{1, 3, 4, 6\} \\ 4, & a \in \{2, 5\} \end{cases}$$

$$\chi_a = \begin{cases} 1 & a \in \{1, 2, 4\} \\ -1 & a \in \{3, 5, 6\}. \end{cases}$$

For $\nu \in X$, define

$$\boldsymbol{i}_\nu = \begin{cases} (0, 1, 2, \omega_{\nu_1}, \omega_{\nu_2}, \omega_{\nu_3}, 2, \omega_{\nu_4}, \omega_{\nu_5}, \omega_{\nu_6}, 2, 1), & i = 1 \\ (0, 1, 2, \omega_{\nu_1}, \omega_{\nu_2}, \omega_{\nu_3}, 2, \omega_{\nu_4}, \omega_{\nu_5}, \omega_{\nu_6}, 1, 2), & i = 2 \\ (0, 1, 2, \omega_{\nu_1}, \omega_{\nu_2}, \omega_{\nu_3}, 2, \omega_{\nu_4}, \omega_{\nu_5}, 1, 2, \omega_{\nu_6}), & i = 3 \\ (0, 1, 2, \omega_{\nu_1}, \omega_{\nu_2}, \omega_{\nu_3}, 2, \omega_{\nu_4}, 1, 2, \omega_{\nu_5}, \omega_{\nu_6}), & i = 4. \end{cases}$$

Let $C_{\boldsymbol{i}_\nu}$ be the connected $\boldsymbol{i}_\nu$-component of the word graph G_δ. Let $L^z_{\delta,i}$ be the free graded $\mathcal{O}[z]$-module on basis

$$B = \{v_{\nu,\boldsymbol{j}} \mid \nu \in X, \boldsymbol{j} \in C_{\boldsymbol{i}_\nu}\}.$$

If $i \in \{1,2\}$, the grading is given by

$$\deg v_{\nu,\boldsymbol{j}} = \begin{cases} 3, & \nu = (1,3,2,5,4,6) \\ 2, & \nu \in \{(1,2,3,5,4,6),(1,3,2,4,5,6)\} \\ 1, & \nu \in \{(1,2,3,4,5,6),(3,1,2,5,4,6),(1,3,2,5,6,4),(1,3,5,2,4,6)\} \\ 0, & \nu \in \{(3,1,2,4,5,6),(1,2,3,5,6,4),(3,5,1,2,4,6),(1,3,5,6,2,4)\} \\ -1, & \nu \in \{(3,1,2,5,6,4),(3,1,5,2,4,6),(1,3,5,2,6,4),(3,5,1,6,2,4)\} \\ -2, & \nu \in \{(3,5,1,2,6,4),(3,1,5,6,2,4)\} \\ -3, & \nu = (3,1,5,2,6,4). \end{cases}$$

If $i = 3$, the grading is given by

$$\deg v_{\nu,\boldsymbol{j}} = \begin{cases} 2, & \nu = (1,3,2,5,4,6) \\ 1, & \nu \in \{(1,2,3,5,4,6),(1,3,2,4,5,6)\} \\ 0, & \nu \in \{(1,2,3,4,5,6),(3,1,2,5,4,6),(1,3,5,2,4,6)\} \\ -1, & \nu \in \{(3,1,2,4,5,6),(3,5,1,2,4,6)\} \\ -2, & \nu = (3,1,5,2,4,6). \end{cases}$$

If $i = 4$, the grading is given by

$$\deg v_{\nu,\boldsymbol{j}} = \begin{cases} 1, & \nu = (1,3,2,4,6,5) \\ 0, & \nu = (1,2,3,4,6,5) \\ -1, & \nu = (3,1,2,4,6,5). \end{cases}$$

For $\boldsymbol{j} \in C_{\boldsymbol{i}_\nu}$, let $\mu(\boldsymbol{j})$ be the list of positions in $\boldsymbol{j}$ occupied by 3 or 4, in increasing order. If $j_r \in \{3,4\}$, let $l(r,\boldsymbol{j}) \in \{1,\dots,6\}$ be such that $(\mu(\boldsymbol{j}))_{l(r,\boldsymbol{j})} = r$. For example, if $\boldsymbol{j} = (0,1,2,3,3,2,4,3,4,3,2,1)$, then $\mu(\boldsymbol{j}) = (4,5,7,8,9,10)$, and $l(8,\boldsymbol{j}) = 4$.

We now define an action of generators of R^z_δ on $L^z_{\delta,i}$:

$$1_{\boldsymbol{k}} v_{\nu,\boldsymbol{j}} = \delta_{\boldsymbol{j},\boldsymbol{k}} v_{\nu,\boldsymbol{j}}.$$

$$y_r v_{\nu,\boldsymbol{j}} = \begin{cases} z^2 v_{\nu,\boldsymbol{j}}, & j_r \in \{0,1,2\} \\ \chi_{\nu_{l(r,\boldsymbol{j})}}(z v_{\nu,\boldsymbol{j}} + Y(r,\nu,\boldsymbol{j})), & j_r \in \{3,4\} \end{cases}$$

$$\psi_r v_{\nu,\boldsymbol{j}} = \begin{cases} v_{\nu,s_r\boldsymbol{j}}, & j_r \cdot j_{r+1} = 0 \\ \Psi(r,\nu,\boldsymbol{j}), & j_r, j_{r+1} \in \{3,4\} \\ 0, & \text{otherwise.} \end{cases}$$

where

$$Y(r,\nu,\boldsymbol{j}) = \begin{cases} v_{\boldsymbol{j},(1,3,2,4,5,6)}, & \nu = (3,1,2,4,5,6),\ l(r,\boldsymbol{j}) \in \{1,2\} \\ v_{\boldsymbol{j},(1,2,3,5,4,6)}, & \nu = (1,2,3,5,6,4),\ l(r,\boldsymbol{j}) \in \{5,6\} \\ v_{\boldsymbol{j},(1,3,2,5,4,6)}, & \nu = (3,1,2,5,4,6),\ l(r,\boldsymbol{j}) \in \{1,2\} \\ v_{\boldsymbol{j},(1,3,2,5,4,6)}, & \nu = (1,3,2,5,6,4),\ l(r,\boldsymbol{j}) \in \{5,6\} \\ v_{\boldsymbol{j},(1,3,2,5,6,4)}, & \nu = (3,1,2,5,6,4),\ l(r,\boldsymbol{j}) \in \{1,2\} \\ v_{\boldsymbol{j},(3,1,2,5,4,6)}, & \nu = (3,1,2,5,6,4),\ l(r,\boldsymbol{j}) \in \{5,6\} \\ v_{\boldsymbol{j},(1,3,5,2,4,6)}, & \nu = (3,1,5,2,4,6),\ l(r,\boldsymbol{j}) \in \{1,2\} \\ v_{\boldsymbol{j},(3,1,2,5,4,6)}, & \nu = (3,1,5,2,4,6),\ l(r,\boldsymbol{j}) \in \{3,4\} \\ v_{\boldsymbol{j},(1,3,2,5,6,4)}, & \nu = (1,3,5,2,6,4),\ l(r,\boldsymbol{j}) \in \{3,4\} \\ v_{\boldsymbol{j},(1,3,5,2,4,6)}, & \nu = (1,3,5,2,6,4),\ l(r,\boldsymbol{j}) \in \{5,6\} \\ v_{\boldsymbol{j},(1,3,5,2,6,4)}, & \nu = (3,1,5,2,6,4),\ l(r,\boldsymbol{j}) \in \{1,2\} \end{cases}$$

and, continuing:

$$Y(r,\nu,\boldsymbol{j}) = \begin{cases} v_{\boldsymbol{j},(3,1,2,5,6,4)}, & \nu=(3,1,5,2,6,4),\ l(r,\boldsymbol{j})\in\{3,4\} \\ v_{\boldsymbol{j},(3,1,5,2,4,6)}, & \nu=(3,1,5,2,6,4),\ l(r,\boldsymbol{j})\in\{5,6\} \\ \epsilon_{34}v_{\boldsymbol{j},(1,2,3,5,6,4)}, & \nu=(3,5,1,2,6,4),\ l(r,\boldsymbol{j})\in\{1,2,3,4\} \\ v_{\boldsymbol{j},(3,5,1,2,4,6)}, & \nu=(3,5,1,2,6,4),\ l(r,\boldsymbol{j})\in\{5,6\} \\ v_{\boldsymbol{j},(1,3,5,6,2,4)}, & \nu=(3,1,5,6,2,4),\ l(r,\boldsymbol{j})\in\{1,2\} \\ \epsilon_{43}v_{\boldsymbol{j},(3,1,2,4,5,6)}, & \nu=(3,1,5,6,2,4),\ l(r,\boldsymbol{j})\in\{3,4,5,6\} \\ -v_{\boldsymbol{j},(1,2,3,4,5,6)}, & \nu=(3,5,1,6,2,4) \\ \epsilon_{34}v_{\boldsymbol{j},(1,2,3,5,4,6)}, & \nu=(3,5,1,2,4,6),\ l(r,\boldsymbol{j})\in\{1,2,3,4\} \\ \epsilon_{43}v_{\boldsymbol{j},(1,3,2,4,5,6)}, & \nu=(1,3,5,6,2,4),\ l(r,\boldsymbol{j})\in\{3,4,5,6\} \\ v_{\boldsymbol{j},(1,3,2,5,4,6)}, & \nu=(1,3,5,2,4,6),\ l(r,\boldsymbol{j})\in\{3,4\} \\ v_{\boldsymbol{j},(1,3,2,4,6,5)}, & \nu=(3,1,2,4,6,5),\ l(r,\boldsymbol{j})\in\{1,2\} \\ 0, & \text{otherwise.} \end{cases}$$

We also have

$$\Psi(r,\nu,\boldsymbol{j}) = \begin{cases} v_{s_{l(r,\boldsymbol{j})}\nu, s_r\boldsymbol{j}} & \nu_{l(r,\boldsymbol{j})} < \nu_{l(r,\boldsymbol{j})+1},\ s_{l(r,\boldsymbol{j})}\nu \in X \\ 2\epsilon_{43} z v_{s_r\boldsymbol{j},(1,2,3,4,5,6)}, & \nu = (1,3,2,4,5,6),\ l(r,\boldsymbol{j}) = 2 \\ 2\epsilon_{34} z v_{s_r\boldsymbol{j},(1,2,3,4,5,6)}, & \nu = (1,2,3,5,4,6),\ l(r,\boldsymbol{j}) = 4 \\ 2\epsilon_{43} z v_{s_r\boldsymbol{j},(1,2,3,5,4,6)}, & \nu = (1,3,2,5,4,6),\ l(r,\boldsymbol{j}) = 2 \\ 2\epsilon_{34} z v_{s_r\boldsymbol{j},(1,3,2,4,5,6)}, & \nu = (1,3,2,5,4,6),\ l(r,\boldsymbol{j}) = 4 \\ \epsilon_{34} v_{s_r\boldsymbol{j},(1,2,3,4,5,6)}, & \nu = (3,1,2,4,5,6),\ l(r,\boldsymbol{j}) = 2 \\ \epsilon_{43} v_{s_r\boldsymbol{j},(1,2,3,4,5,6)}, & \nu = (1,2,3,5,6,4),\ l(r,\boldsymbol{j}) = 4 \\ \epsilon_{34} v_{s_r\boldsymbol{j},(1,2,3,5,4,6)}, & \nu = (3,1,2,5,4,6),\ l(r,\boldsymbol{j}) = 2 \\ 2\epsilon_{34} z v_{s_r\boldsymbol{j},(3,1,2,4,5,6)}, & \nu = (3,1,2,5,4,6),\ l(r,\boldsymbol{j}) = 4 \\ 2\epsilon_{43} z v_{s_r\boldsymbol{j},(1,2,3,5,6,4)}, & \nu = (1,3,2,5,6,4),\ l(r,\boldsymbol{j}) = 2 \\ \epsilon_{43} v_{s_r\boldsymbol{j},(1,3,2,4,5,6)}, & \nu = (1,3,2,5,6,4),\ l(r,\boldsymbol{j}) = 4 \\ \epsilon_{34} v_{s_r\boldsymbol{j},(1,2,3,5,6,4)}, & \nu = (3,1,2,5,6,4),\ l(r,\boldsymbol{j}) = 2 \\ \epsilon_{43} v_{s_r\boldsymbol{j},(3,1,2,4,5,6)}, & \nu = (3,1,2,5,6,4),\ l(r,\boldsymbol{j}) = 4 \\ \epsilon_{43} v_{s_r\boldsymbol{j},(3,1,2,4,5,6)}, & \nu = (3,1,5,2,4,6),\ l(r,\boldsymbol{j}) = 4 \\ \epsilon_{34} v_{s_r\boldsymbol{j},(1,2,3,5,6,4)}, & \nu = (1,3,5,2,6,4),\ l(r,\boldsymbol{j}) = 2, \end{cases}$$

and, continuing:

$$\Psi(r,\nu,\boldsymbol{j}) = \begin{cases} \epsilon_{34}(2zv_{s_r\boldsymbol{j},(3,1,5,2,6,4)} \\ \quad +v_{s_r\boldsymbol{j},(1,3,5,2,6,4)} \\ \qquad +v_{s_r\boldsymbol{j},(3,1,2,5,6,4)}), & \nu=(3,5,1,2,6,4),\ l(r,\boldsymbol{j})=2 \\ \epsilon_{43}(2zv_{s_r\boldsymbol{j},(3,1,5,2,6,4)} \\ \quad +v_{s_r\boldsymbol{j},(3,1,5,2,4,6)} \\ \qquad +v_{s_r\boldsymbol{j},(3,1,2,5,6,4)}), & \nu=(3,1,5,6,2,4),\ l(r,\boldsymbol{j})=4 \\ 2\epsilon_{34}zv_{s_r\boldsymbol{j},(3,1,5,6,2,4)} \\ \quad +\epsilon_{34}v_{s_r\boldsymbol{j},(1,3,5,6,2,4)} \\ \qquad -v_{s_r\boldsymbol{j},(3,1,2,4,5,6)}, & \nu=(3,5,1,6,2,4),\ l(r,\boldsymbol{j})=2 \\ 2\epsilon_{43}zv_{s_r\boldsymbol{j},(3,5,1,2,6,4)} \\ \quad +\epsilon_{43}v_{s_r\boldsymbol{j},(3,5,1,2,4,6)} \\ \qquad -v_{s_r\boldsymbol{j},(1,2,3,5,6,4)}, & \nu=(3,5,1,6,2,4),\ l(r,\boldsymbol{j})=4 \end{cases}$$

and, continuing:

$$\Psi(r,\nu,\boldsymbol{j}) = \begin{cases} \epsilon_{34}(2zv_{s_r\boldsymbol{j},(3,1,5,2,4,6)} & \\ \quad +v_{s_r\boldsymbol{j},(1,3,5,2,4,6)} & \\ \qquad +v_{s_r\boldsymbol{j},(3,1,2,5,4,6)}), & \nu=(3,5,1,2,4,6),\ l(r,\boldsymbol{j})=2 \\ -v_{s_r\boldsymbol{j},(1,2,3,4,5,6)}, & \nu=(3,5,1,2,4,6),\ l(r,\boldsymbol{j})=4 \\ -v_{s_r\boldsymbol{j},(1,2,3,4,5,6)}, & \nu=(1,3,5,6,2,4),\ l(r,\boldsymbol{j})=2 \\ \epsilon_{43}(2zv_{s_r\boldsymbol{j},(1,3,5,2,6,4)} & \\ \quad +v_{s_r\boldsymbol{j},(1,3,2,5,6,4)} & \\ \qquad +v_{s_r\boldsymbol{j},(1,3,5,2,4,6)}), & \nu=(1,3,5,6,2,4),\ l(r,\boldsymbol{j})=4 \\ \epsilon_{34}v_{s_r\boldsymbol{j},(1,2,3,5,4,6)}, & \nu=(1,3,5,2,4,6),\ l(r,\boldsymbol{j})=2 \\ \epsilon_{43}v_{s_r\boldsymbol{j},(1,3,2,4,5,6)}, & \nu=(1,3,5,2,4,6),\ l(r,\boldsymbol{j})=4 \\ 2\epsilon_{43}zv_{s_r\boldsymbol{j},(1,2,3,4,6,5)}, & \nu=(1,3,2,4,6,5),\ l(r,\boldsymbol{j})=2 \\ \epsilon_{34}v_{s_r\boldsymbol{j},(1,2,3,4,6,5)}, & \nu=(3,1,2,4,6,5),\ l(r,\boldsymbol{j})=2 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 6.4. The formulas above define a graded action of R^z_δ on $L^z_{\delta,i}$, and $L^z_{\delta,i}$ is $\mathcal{O}[z]$-free on basis B.

Proof. That the given action agrees with the algebra relations (2.42)–(2.48) for R^z_δ has been checked via computer. □

6.15. Construction of $L^z_{\delta,i}$ in type $\mathtt{G}_2^{(1)}$

We label the vertices of the diagram $\mathtt{G}_2^{(1)}$ as shown below in Figure 6.4.. Fix

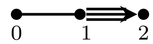


FIGURE 6.4. The diagram $\mathtt{G}_2^{(1)}$.

$i \in I'$. If $i = 1$, let X be the set of the following tuples:

$$(1,2,3), \quad (1,3,2), \quad (2,1,3), \quad (2,3,1), \quad (3,1,2), \quad (3,2,1).$$

If $i = 2$, let X be the set of the following tuples:

$$(1,2,3), \quad (2,1,3).$$

For the type $\mathtt{G}_2$ case, we assume $\mathcal{O} = \mathbb{C}$ when defining $L^z_{\delta,i}$. For $a \in \{1,2,3\}$, define

$$\chi_a = \begin{cases} 1 & a = 1 \\ \xi & a = 2 \\ \xi^2 & a = 3. \end{cases}$$

For $\nu \in X$, define

$$\boldsymbol{i} = \begin{cases} (0,1,2,2,2,1), & i = 1 \\ (0,1,2,2,1,2), & i = 2. \end{cases}$$

Let L^z_δ be the free graded $\mathcal{O}[z]$-module on basis

$$B = \{v_\nu \mid \nu \in X\}.$$

If $i = 1$, the grading is given by

$$\deg v_\nu = \begin{cases} 3, & \nu = (1,2,3) \\ 1, & \nu \in \{(2,1,3),(1,3,2)\} \\ -1, & \nu \in \{(2,3,1),(3,1,2)\} \\ -3, & \nu = (3,2,1). \end{cases}$$

If $i = 2$, the grading is given by

$$\deg v_\nu = \begin{cases} 1, & \nu = (1,2,3) \\ -1, & \nu = (2,1,3). \end{cases}$$

We now define an action of generators of R^z_δ on L^z_δ:

$$1_{\boldsymbol{k}} v_\nu = \delta_{\boldsymbol{k},\boldsymbol{i}} v_\nu.$$

$$y_r v_\nu = \begin{cases} z^3 v_\nu, & j_r \in \{0,1\} \\ \chi_{\nu_{r-2}} z v_\nu + Y(r-2,\nu), & r \in \{3,4\} \\ \chi_{\nu_3} z v_\nu + Y(3,\nu), & r = 5, i = 1 \\ \xi^2 z v_\nu, & r = 6, i = 2 \end{cases}$$

$$\psi_r v_\nu = \begin{cases} v_{s_{r-2}\nu}, & r \in \{3,4\}, \nu_{r-2} < \nu_{r-1}, s_{r-2}\nu \in X \\ 0, & \text{otherwise,} \end{cases}$$

where

$$Y(t,\nu) = \begin{cases} (-1)^t v_{(1,2,3)} & \nu = (2,1,3), t \in \{1,2\} \\ (-1)^{t+1} v_{(1,2,3)} & \nu = (1,3,2), t \in \{2,3\} \\ -v_{(1,3,2)} & \nu = (2,3,1), t = 1 \\ -v_{(2,1,3)} & \nu = (2,3,1), t = 2 \\ v_{(2,1,3)} + v_{(1,3,2)} & \nu = (2,3,1), t = 3 \\ -v_{(1,3,2)} - v_{(2,1,3)} & \nu = (3,1,2), t = 1 \\ v_{(1,3,2)} & \nu = (3,1,2), t = 2 \\ v_{(2,1,3)} & \nu = (3,1,2), t = 3 \\ -v_{(2,3,1)} & \nu = (3,2,1), t = 1 \\ -v_{(3,1,2)} + v_{(2,3,1)} & \nu = (3,2,1), t = 2 \\ v_{(3,1,2)} & \nu = (3,2,1), t = 3 \end{cases}$$

Proposition 6.5. The formulas above define a graded action of R^z_δ on $L^z_{\delta,i}$, and $L^z_{\delta,i}$ is $\mathcal{O}[z]$-free on basis B.

Proof. That the given action agrees with the algebra relations (2.42)–(2.48) for R^z_δ has been checked via computer. □

6.16. Construction of $L_{\delta,i}$ for non-simply-laced types

Let $\mathtt{C}$ be a Cartan matrix of type $\mathtt{B}_l^{(1)}, \mathtt{C}_l^{(1)}, \mathtt{F}_4^{(1)}$, or $\mathtt{G}_2^{(1)}$, and fix $i \in I'$.

Proposition 6.6. Viewed as an R_δ-module, $L^z_{\delta,i}$ has submodule $W = \mathcal{O}\{z^k b \mid k > 0, b \in B\}$, and $L^z_{\delta,i}/W \cong L_{\delta,i}$.

Proof. R_δ^z has a two-sided ideal $R_\delta^z z$, with $R_\delta^z / R_\delta^z z \cong R_\delta$ as $\mathcal{O}$-algebras. $L_{\delta,i}^z$ is free as an $\mathcal{O}$-module with basis $\{z^k b \mid k \in \mathbb{Z}_{\geq 0}, b \in B\}$, and has an R_δ^z-submodule $R_\delta^z z L_{\delta,i}^z = \mathcal{O}\{z^k b \mid k \in \mathbb{Z}_{>0}, b \in B_i\}$. Then $L'_{\delta,i} := L_{\delta,i}^z / R_\delta^z z L_{\delta,i}^z$ is an $R_\delta^z / R_\delta^z z \cong R_\delta$-module which is free as an $\mathcal{O}$-module with basis $\{\bar{b} \mid b \in B\}$.

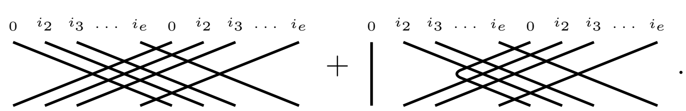
All words $\boldsymbol{j}$ of $L'_{\delta,i}$ have $j_1 = 0$, $j_e = i$ by construction. Thus all composition factors are $L_{\delta,i}$ by [24, Corollary 5.3]. By considering the graded dimension of any extremal word space in $L'_{\delta,i}$, it is clear that $L_{\delta,i}$ has composition multiplicity one in $L'_{\delta,i}$, hence $L'_{\delta,i} = L_{\delta,i}$. □

Note that in the case of $\mathtt{G}_2$, where we previously assumed $\mathcal{O} = \mathbb{C}$, the coefficients of the action of R_δ on the basis of $L_{\delta,i}$ are integral, so we may consider instead the $\mathbb{Z}$-form and extend scalars to construct $L_{\delta,i}$ for arbitrary $\mathcal{O}$.

6.17. The endomorphism $\tau_r : M_n \to M_n$ for non-simply-laced types

Let z, z' be algebraically independent, and write $R_\alpha^{z,z'}$ for $\mathcal{O}[z,z'] \otimes_{\mathcal{O}} R_\alpha$. Then $L_{\delta,i}^{z'} \circ L_{\delta,i}^z$ and $L_{\delta,i}^z \circ L_{\delta,i}^{z'}$ are $R_{2\delta}^{z,z'}$-modules that are free as $\mathcal{O}[z,z]$-modules. Recall the map $R_{L_{\delta,i}^{z'}, L_{\delta,i}^z} : L_{\delta,i}^{z'} \circ L_{\delta,i}^z \to L_{\delta,i}^z \circ L_{\delta,i}^{z'}$, and the intertwining elements φ defined in [19].

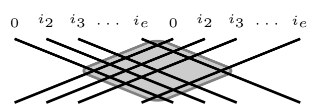
Proposition 6.7. Let $v^z \in L_{\delta,i}^z$ be a word vector of word $\boldsymbol{i}$, with $y_k v^z = c_k z^{a_k} v^z$ for all admissible k. Then $R_{L_{\delta,i}^{z'}, L_{\delta,i}^z}(v^{z'} \otimes v^z)$ is equal to



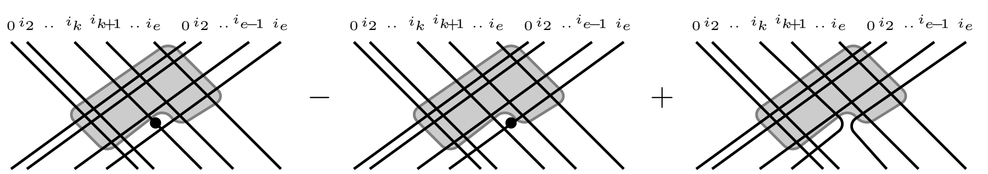
$$\prod_{\substack{k,m \\ i_k = i_m}} (c_k z^{a_k} - c_m z'^{a_m}) \quad + \quad \prod_{\substack{k,m>1 \\ i_k = i_m}} (c_k z^{a_k} - c_m z'^{a_m}) .$$

When we diagrammatically describe module elements, we always picture the vector as being at the top of the diagram, the algebra elements below and acting upwardly, and $\mathcal{O}[z, z']$-coefficients at the bottom.

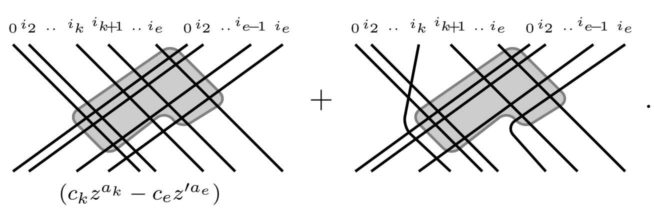
Proof. Note that $i_1 = 0$, and $i_k \neq 0$ for all $k > 1$. Let w_1 be the block permutation of the tensor factors as in (3.23). Then by definition $R_{L^{z'}_{\delta,i}, L^{z}_{\delta,i}}(v^{z'} \otimes v^{z}) = \varphi_{w_1}(v^{z} \otimes v^{z'})$. We write this diagrammatically as



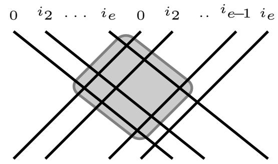
where any strand crossings in the gray section are understood to represent φ's instead of ψ's. Assume k is the smallest such that $i_k = i_e$. Then the above is equal to



By the commuting properties of the block intertwiner (see [19, Lemma 1.3.1]), we can move the beads up to act on the tensor factors, and in the third term the i_k-strand can be pulled to the left, giving



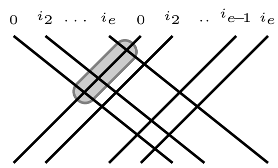
The latter term is zero however; $\psi_1 L^z_{\delta,i} = 0$ in all cases. We get a similar result for the next smallest k' such that $i_{k'} = i_e$, and we work our way up the i_e-strand to get



.

$$\prod_{i_k = i_e} (c_k z^{a_k} - c_e z'^{a_e})$$

Then we apply the same arguments to the i_{e-1}-strand, and work our way up the strands recursively, until we have



.

$$\prod_{\substack{k,m>1 \\ i_k = i_m}} (c_k z^{a_k} - c_m z'^{a_m})$$

Then applying the definition of φ to the lone $(0,0)$-crossing gives the result. $\square$

6.18. The map $R_{L^{z'}_{\delta,i}, L^z_{\delta,i}}$ in type $\mathtt{B}^{(1)}_l$

Fix $i \in I'$. Recall the construction of $L^z_{\delta,i}$ from section 6.12. Define the word

$$\boldsymbol{i} := \begin{cases} (0,2,3,\dots,l-1,l,l,l-1,\dots,i+1,1,2,\dots,i) & \text{if } i < l; \\ (0,2,3,\dots,l,1,2,\dots,l) & \text{if } i = l. \end{cases} \tag{6.8}$$

Then $\boldsymbol{i}$ is an extremal word for $L^z_{\delta,i}$. Let

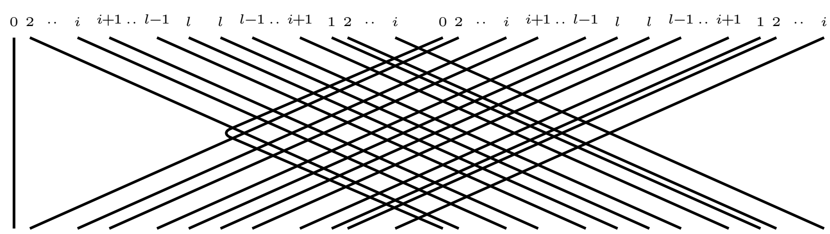
$$v^z_1 := v^{(1)}_{\boldsymbol{i}} \;\; (i < l), \qquad v^z_1 := v_{\boldsymbol{i}} \;\; (i = l).$$

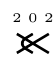
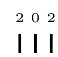
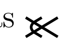
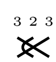
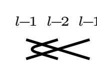
Then v_1^z is a vector of word $\boldsymbol{i}$ that generates $L^z_{\delta,i}$.

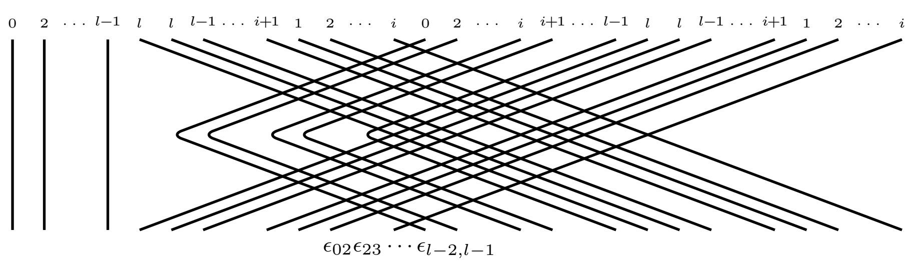
Proposition 6.9.

$$R(v_1^{z'} \otimes v_1^z) \in (z^2 - z'^2)^{4l-4}\left[\sigma + (-1)^{l+i} + (z - z')R^z_\delta \otimes R^{z'}_\delta\right](v_1^z \otimes v_1^{z'}).$$

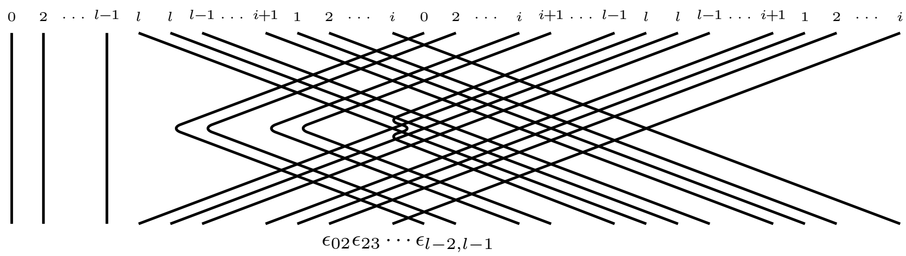
Proof. First, assume $i < l$. In view of Proposition 6.7, we focus on rewriting the term

$C =$ 

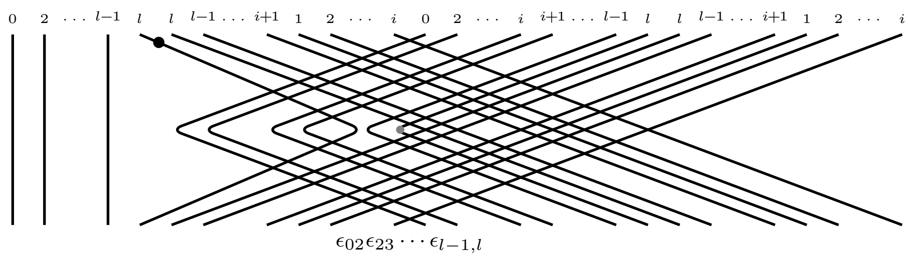
The algebra braid relation implies that  opens to become , since if the 0-strand is moved to the right side of the $(2, 2)$-crossing, it slides up to induce a $(0, 2)$-crossing in the second factor or a $(1, 2)$-crossing in the first factor, both of which are zero. In the future we will simply say that a shape such as  'opens' if the the  term in the associated algebra relation induces such strand crossings that act as zero on either of the tensor factors. Next, the braids  through  open in succession, giving

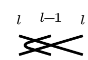
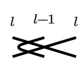
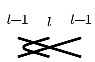
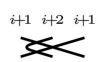
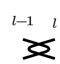
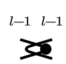
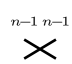


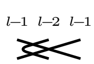
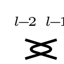
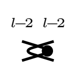
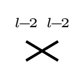
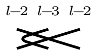
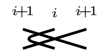
This is equal to $A+B$, where

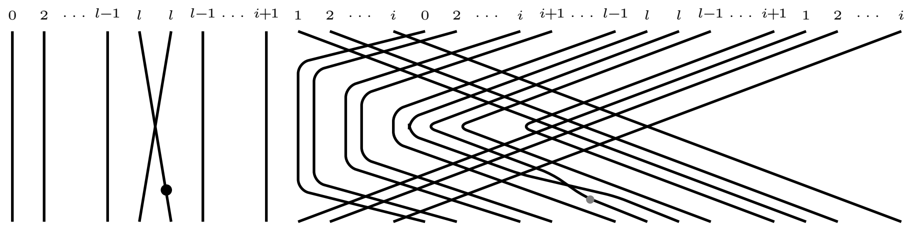
$A =$ 

and

$B =$ 

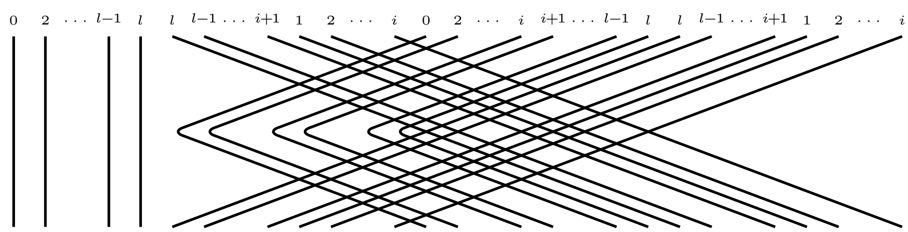
The latter diagram is meant to stand for a sum of two terms; one taken with the black bead in place and one with the gray bead. We focus on A for now. The upper  opens, giving beads on the l-strands which move up to act on the tensor factors, introducing a factor of $(-z+z')$. Next, the lower  braid opens. Pulling strands to the right, the braids  through  open in succession. We now apply the quadratic algebra relation to open . One term in this relation is zero, and the other has , which becomes .

Next,  opens, and we apply the quadratic relation to . Again one term in this relation is zero, and the other has , which becomes . This process repeats for braids  through , giving

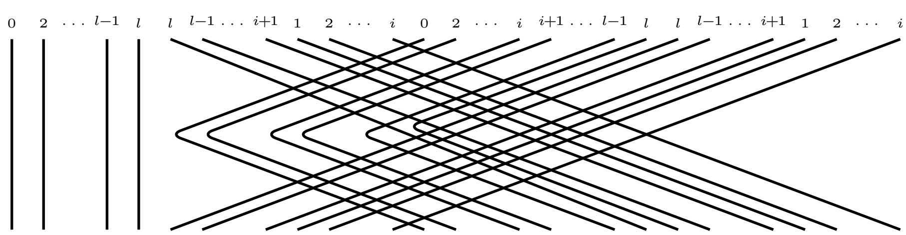
$A =$ 

.

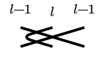
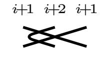
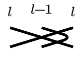
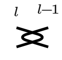
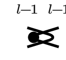
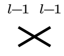
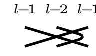
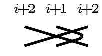
Now, focusing on B and moving the beads up to act on the tensor factors, we have that $B = B_1 + B_2$, where

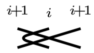
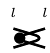
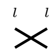
$B_1 =$ 

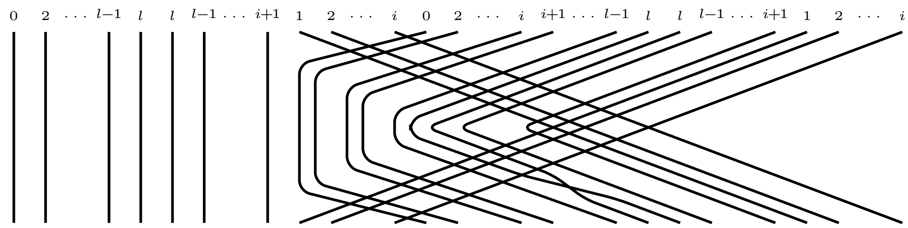
and

$B_2 =$ 

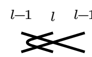
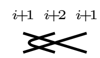
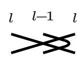
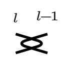
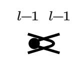
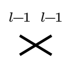
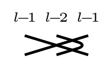
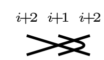
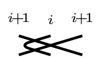
.

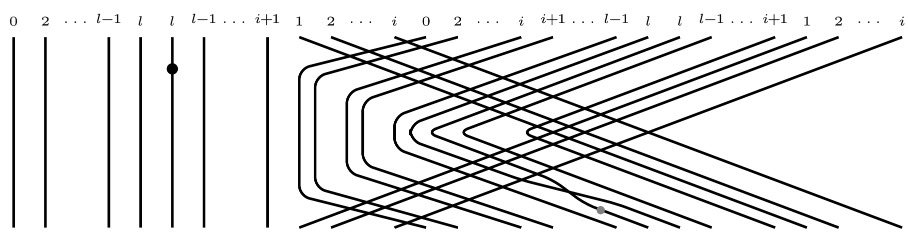
We focus on B_1. The braids  through  open in succession. Then  opens, introducing a quadratic term  which opens, introducing , which becomes . This process repeats with the braids  through .

Then  opens, leaving , which becomes , and we have

$B_1 =$  .

$(-1)^{l+i}\epsilon_{02}\epsilon_{23}\cdots\epsilon_{i-1,i}(z+z')$

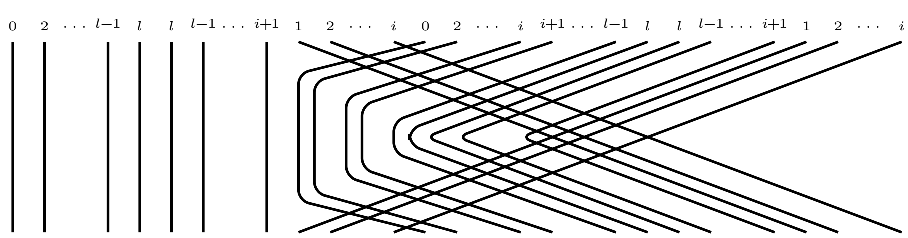
Now we consider B_2. The braids  through  open in succession. Then  opens, leading to a quadratic term  which opens to give  which becomes . This process repeats with braids  through . Finally, the  braid opens, giving

$B_2 =$ 

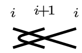
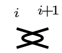
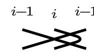
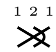
$(-1)^{l+i}\epsilon_{02}\epsilon_{23}\cdots\epsilon_{i-1,i}$

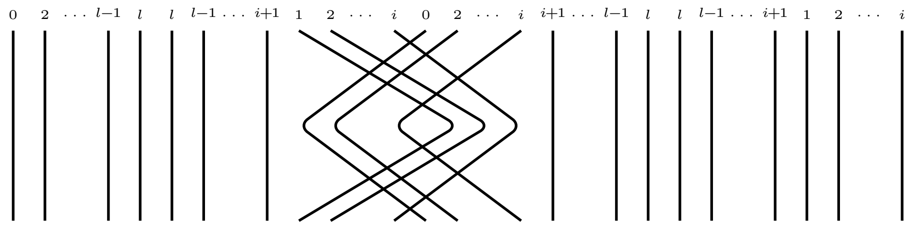
Now we prove the following claim.

Claim.

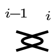
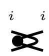
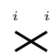
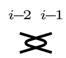
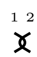
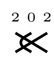
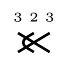
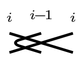


is equal to $(-1)\epsilon_{02}\epsilon_{23}\cdots\epsilon_{i-1,i}(z^2 - z'^2)v_1^z \otimes v_1^{z'}$.

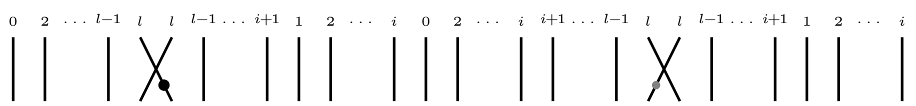
Proof of Claim. The braid  opens, then the quadratic factor  opens, introducing a factor of $(z^2 - z'^2)$. Then the braids  through  open in succession, giving



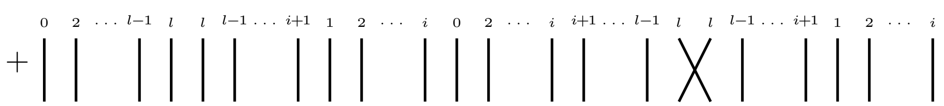
$(-1)\epsilon_{12}\cdots\epsilon_{i-1,i}(z^2 - z'^2)$

Next  opens, introducing , which becomes . The process repeats with the quadratic factors  through . Next the braids  and  through  open in succession, proving the claim.

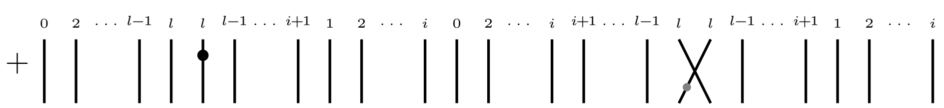
Then, applying the claim, we have that $C = A + B_1 + B_2$ is equal to



$(-1)^{l+i+1}(z^2 - z'^2)(z - z')$



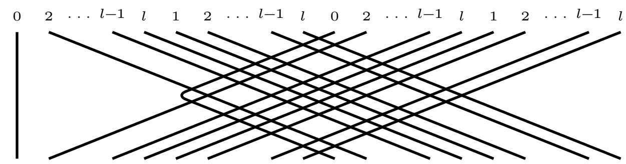
$(-1)^{l+i+1}(z^2 - z'^2)(z + z')$

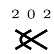
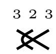
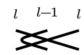


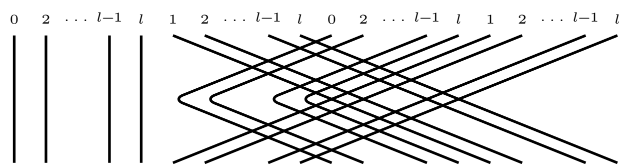
$(-1)^{l+i+1}(z^2 - z'^2)$

Finally, simplifying and applying Proposition 6.7 gives the result.

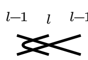
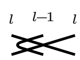
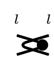
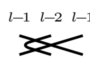
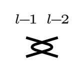
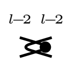
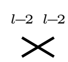
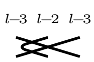
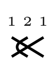
Now we assume $i = l$. Let

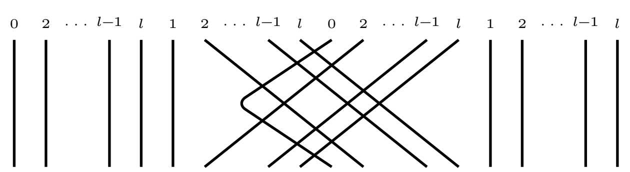
$C =$ .

We now attempt to simplify this term. The braids  and  through  open in succession. This last relation introduces a factor of $(z + z')$, giving

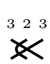
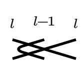
.

$\epsilon_{02}\epsilon_{23}\cdots\epsilon_{l-1,l}(z+z')$

Now the braids  and  open, introducing a factor of , which becomes . Then the braid  opens, introducing a quadratic factor , which opens, yielding a factor , which becomes . This process repeats for the braids  through , giving

.

$(-1)\epsilon_{02}\epsilon_{23}\cdots\epsilon_{l-1,l}(z+z')$

Now, the braids  and  through  open in succession. This last relation introduces a factor of $(-z + z')$, so we have $C = (z^2 - z'^2)v_1^z \otimes v_1^{z'}$, and applying Proposition 6.7 completes the proof. $\square$

6.19. The map $R_{L^{z'}_{\delta,i},L^{z}_{\delta,i}}$ in type $\mathtt{C}_l^{(1)}$

Fix $i \in I'$. Recall the construction of $L^z_{\delta,i}$ from section 6.13. Take

$$\boldsymbol{i} = (0,1,1,2,2,\dots,i-1,i-1,i,i+1,\dots,l-1,l,l-1,\dots,i). \tag{6.10}$$

Then $\boldsymbol{i}$ is an extremal word for $L^z_{\delta,i}$. Let

$$v_1^z := (\psi_{2i-1}\cdots\psi_{2l-2})\cdots(\psi_{2k+1}\cdots\psi_{2l-i+k-1})\cdots(\psi_3\cdots\psi_{2l-i})x.$$

Then v_1^z is a vector of word $\boldsymbol{i}$ that generates $L^z_{\delta,i}$. To see this, note that

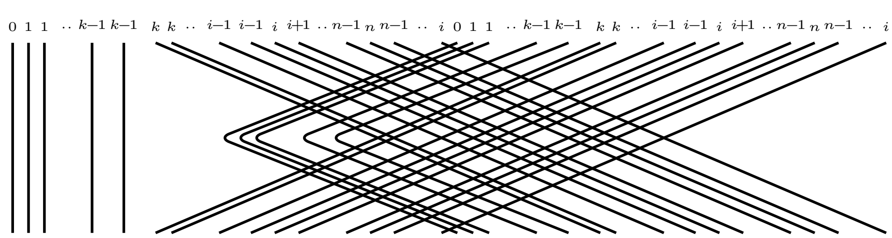
$$(\psi_{2i-3}\psi_{2i-2})\cdots(\psi_5\psi_6)(\psi_3\psi_4)v_1^z = \psi_a 1_{\boldsymbol{j}} x,$$

where $\boldsymbol{j} = (0,1,\dots,n{-}1,n,n{-}1,\dots,i{+}1,1,2,\dots,i)$ and $\psi_a = \psi_{a_k}\cdots\psi_{a_1}$ is such that $\psi_{a_m}1_{a_{m-1}\cdots a_1\boldsymbol{j}}$ is of degree zero for all $1 \le m \le k$. But this implies that $\psi_b\psi_a 1_{\boldsymbol{j}} = 1_{\boldsymbol{j}}$ for some b, so x is in the R^z_δ-span of v_1^z, and x generates $L^z_{\delta,i}$, proving the claim.

Proposition 6.11.

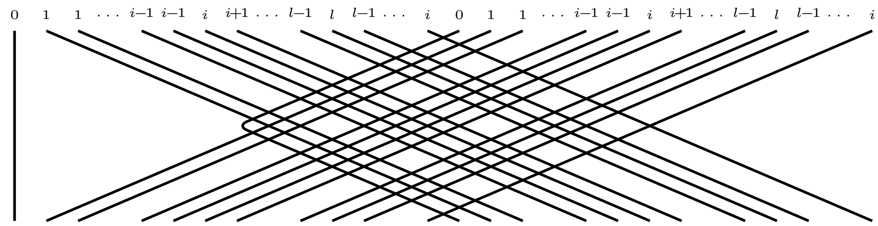
$$R(v_1^{z'} \otimes v_1^z) \in (z^2 - z'^2)^{l+1}\left[\sigma + (-1)^{l+i+1} + (z-z')R^z_\delta \otimes R^{z'}_\delta\right](v_1^z \otimes v_1^{z'}).$$

Proof. For $1 \le k \le i$, let

$A_k =$ 

.

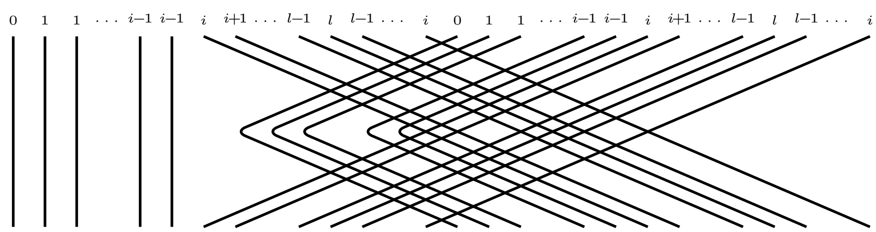
We now focus on simplifying the term A_1, or



.

This is in $A_2 + (z - z')R^z_\delta \otimes R^{z'}_\delta A_2$, by a direct calculation, and $A_k \in A_{k+1} + (z - z')R^z_\delta \otimes R^{z'}_\delta A_{k+1}$ for all $2 \leq k \leq i-1$, which one sees by 'pulling the $(k-1)$-strands to the right'. Using this fact recursively, we have that $A_1 \in A_i + (z-z')R^z_\delta \otimes R^{z'}_\delta A_i$, where

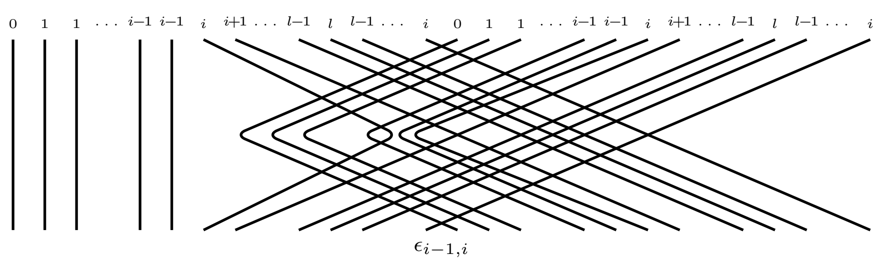
$A_i =$



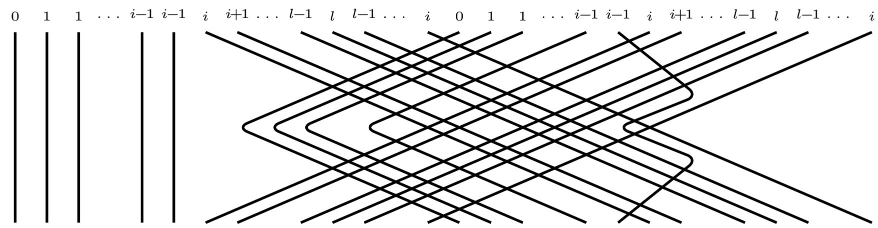
.

Now, via $(i, i-1, i)$-braid relations, moving crossed strands up to act on the individual tensor factors as zero when possible, we have that $A_i = B_1 + B_2$, where

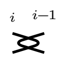
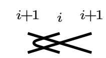
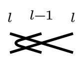
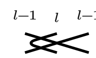
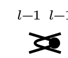
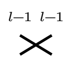
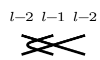
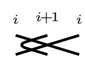
$B_1 =$

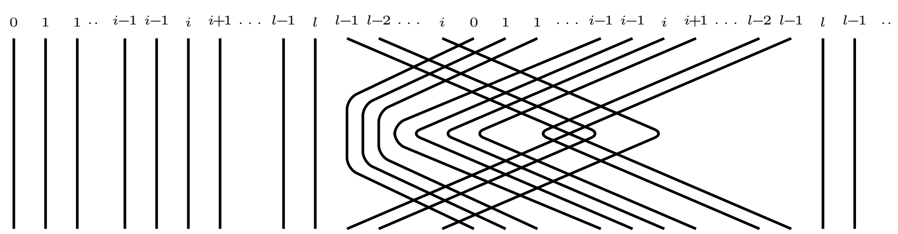


and

$B_2 =$ 

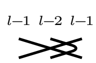
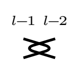
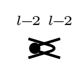
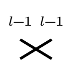
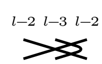
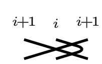
.

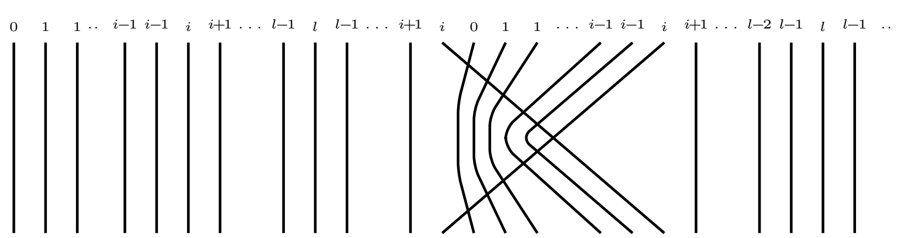
We focus on B_1.  opens, introducing a $(z - z')$ factor. Next the braids  through  open in succession. Then  opens, yielding  which becomes . Next, the braids  through  open in succession, giving

$B_1 =$ 

.

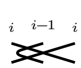
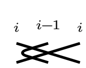
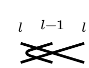
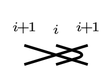
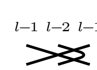
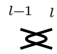
$(-1)^{n+i+1}(z - z')$

Next,  opens, leading to , which opens, yielding  which becomes . This process repeats for braids  through , giving

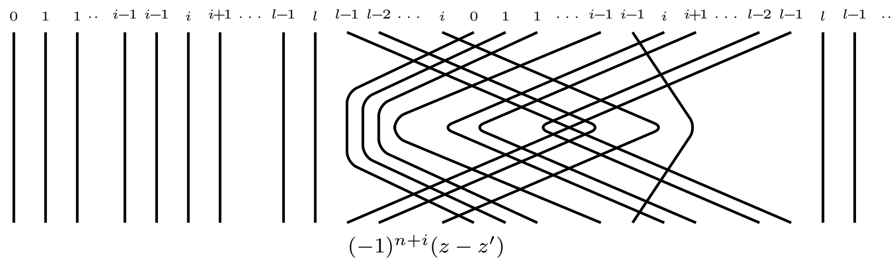
$B_1 =$ 

.

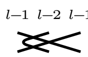
$(-1)^{n+i+1}(z - z')$

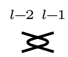
Which simplifies to $(-1)^{n+i+1}(z - z')^2(v_1^z \otimes v_2^z)$. Now we focus on B_2. The rightmost  opens, then on the left side the braids  through  open in succession. Then from the right, the braids  through  open in succession. Then 

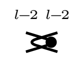
opens, introducing a factor of $(-z+z')$, giving

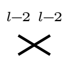
$B_2 =$ 

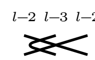
.

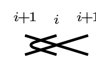
Now 

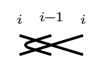
opens, giving 

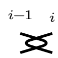
, which opens yielding 

, which becomes 

. This process repeats with braids 

through 

. Then 

opens, yielding 

which opens, introducing a factor of $(-2z')$, so we have $B_2 = (-1)^{n+i+1}(z-z')(2z')(v_1^z \otimes v_1^{z'})$. Then $A_i = B_1 + B_2 = (-1)^{n+i+1}(z^2 - z'^2)(v_1^z \otimes v_1^{z'})$, so $A_1 \in (z^2 - z'^2)\left[(-1)^{n+i+1} + (z-z')R_\delta^z \otimes R_\delta^{z'}\right](v_1^z \otimes v_1^{z'})$. Then Proposition 6.7 provides the result. $\square$

6.110. The map $R_{L_{\delta,i}^{z'}, L_{\delta,i}^z}$ in type $\mathtt{F}_4$

Fix $i \in I'$. Recall the construction of $L_{\delta,i}^z$ from section 6.14. Let

$$\boldsymbol{i} = \begin{cases} (0,1,2,3,3,2,4,4,3,3,2,1), & \text{if } i = 1; \\ (0,1,2,3,3,2,4,4,3,3,1,2), & \text{if } i = 2; \\ (0,1,2,3,4,3,2,3,4,1,2,3), & \text{if } i = 3; \\ (0,1,2,3,4,3,2,3,1,2,3,4), & \text{if } i = 4. \end{cases} \tag{6.12}$$

Then $\boldsymbol{i}$ is an extremal word for $L^z_{\delta,i}$. Let

$$v^z_1 = \begin{cases} v_{(1,3,2,5,4,6),\boldsymbol{i}} & \text{if } i \in \{1,2\} \\ v_{(1,2,3,4,5,6),\boldsymbol{i}} & \text{if } i = 3 \\ v_{(1,2,3,4,6,5),\boldsymbol{i}} & \text{if } i = 4 \end{cases}$$

Then v^z_1 is a vector of word $\boldsymbol{i}$, and it is easily seen from the action of generators that v^z_1 generates $L^z_{\delta,i}$.

Proposition 6.13.

$$R(v^{z'}_1 \otimes v^z_1) \in (z^2 - z'^2)^{24} \left[\sigma + c + (z - z')R^z_\delta \otimes R^{z'}_\delta\right] (v^z_1 \otimes v^{z'}_1),$$

where $c = 1$ if $i \in \{1,4\}$ and $c = -1$ if $i \in \{2,3\}$.

Proof. The proof of this proposition is a straightforward but lengthy calculation made with the aid of a computer, using Proposition 6.7 and the same techniques as in the $\mathtt{B}^{(1)}_l$ and $\mathtt{C}^{(1)}_l$ cases. □

6.111. The map $R_{L^{z'}_{\delta,i}, L^z_{\delta,i}}$ in type $\mathtt{G}^{(1)}_2$

Fix $i \in I'$. Recall the construction of $L^z_{\delta,i}$ from section 6.15. Let

$$\boldsymbol{i} = \begin{cases} (0,1,2,2,2,1), & \text{if } i = 1; \\ (0,1,2,2,1,2), & \text{if } i = 2. \end{cases} \tag{6.14}$$

Then $\boldsymbol{i}$ is an extremal word for $L^z_{\delta,i}$. Let

$$v^z_1 := v_{(1,2,3)}.$$

Then v^z_1 is a vector of word $\boldsymbol{i}$ that generates $L^z_{\delta,i}$.

Proposition 6.15.

$$R_{L^{z'}_{\delta,i},L^z_{\delta,i}}(v^{z'}_1 \otimes v^z_1) \in (z^3 - z'^3)^8 \left[\sigma + (-1)^i + (z-z')R^z_\delta \otimes R^{z'}_\delta\right](v^z_1 \otimes v^{z'}_1).$$

Proof. As in the $\mathtt{F}^{(1)}_4$ case, the proof of this proposition is a straightforward but lengthy calculation made with the aid of a computer, using Proposition 6.7 and the same techniques as in the $\mathtt{B}^{(1)}_l$ and $\mathtt{C}^{(1)}_l$ cases. □

6.112. Constructing τ_r from $R_{L^{z'}_{\delta,i},L^z_{\delta,i}}$

Let $\mathtt{C}$ be a Cartan matrix of type $\mathtt{B}^{(1)}_l, \mathtt{C}^{(1)}_l, \mathtt{F}^{(1)}_4$, or $\mathtt{G}^{(1)}_2$. Fix $i \in I'$. In Sections 6.18-6.111 we have chosen a word vector v^z_1 of extremal word $\boldsymbol{i}$ that generates $L^z_{\delta,i}$, and shown that

$$R_{L^{z'}_{\delta,i},L^z_{\delta,i}}(v^{z'}_1 \otimes v^z_1) \in f(z,z')\left[\sigma + c + (z-z')R^z_\delta \otimes R^{z'}_\delta\right](v^z_1 \otimes v^{z'}_1),$$

for some $c = \pm 1$ and nonzero $f(z,z') \in \mathcal{O}[z,z']$. As $v^{z'}_1 \otimes v^z_1$ generates $L^{z'}_{\delta,i} \circ L^z_{\delta,i}$ and $R_{L^{z'}_{\delta,i},L^z_{\delta,i}}$ is an $R^{z,z'}_{2\delta}$-linear endomorphism, this implies that every element in the image of $R_{L^{z'}_{\delta,i},L^z_{\delta,i}}$ is divisible by f. Let $\pi : L^z_{\delta,i} \circ L^{z'}_{\delta,i} \twoheadrightarrow L_{\delta,i} \circ L_{\delta,i}$ and $\pi' : L^{z'}_{\delta,i} \circ L^z_{\delta,i} \twoheadrightarrow L_{\delta,i} \circ L_{\delta,i}$ be the quotient maps given by setting $z = z' = 0$. Then since $L^{z'}_{\delta,i} \circ L^z_{\delta,i}$ is free as an $\mathcal{O}[z,z']$-module, there is a well-defined map $\tilde{\tau} := \pi \circ f^{-1} R_{L^{z'}_{\delta,i},L^z_{\delta,i}}$ which factors through π' to give a $R_{2\delta}$-homomorphism

$$\tau : L_{\delta,i} \circ L_{\delta,i} \to L_{\delta,i} \circ L_{\delta,i}$$

such that $\tau(v_2) = (\sigma + c)v_2$, where $v_2 = \pi(v_1^z \otimes v_1^{z'}) = \pi'(v_1^{z'} \otimes v_1^z)$.

Write v_1 for the image of v_1^z under the quotient $L_{\delta,i}^z \twoheadrightarrow L_{\delta,i}$ of Proposition 6.6. Then v_1 spans the 1-dimensional top degree component $(1_{\boldsymbol{i}} L_{\delta,i})_N$ of the (extremal) word space $1_{\boldsymbol{i}} L_{\delta,i}$ in $L_{\delta,i}$, and $v_2 = v_1 \otimes v_1$. Write v_n for $v_1 \otimes \cdots \otimes v_1 \in M_n = L_{\delta,i}^{\circ n}$. Inserting the endomorphism τ into the r-th and $(r+1)$-th positions in M_n yields endomorphisms $\tau_r : M_n \to M_n$, $v_n \mapsto (\sigma_r + c)v_n$.

Proposition 6.16. The endomorphisms τ_r satisfy the usual Coxeter relations of the standard generators of the symmetric group $\mathfrak{S}_n$, i.e., $\tau_r^2 = 1$, $\tau_r\tau_s = \tau_s\tau_r$ for $|r-s| > 1$ and $\tau_r\tau_{r+1}\tau_r = \tau_{r+1}\tau_r\tau_{r+1}$.

Proof. The braid relations follow from the fact that the map R satisfies these relations (see [19, Chapter 1]). By Proposition 6.7 we have that $f(z, z') = \prod_{\substack{k,m \\ i_k = i_m}} (c_k z^{a_k} - c_m z'^{a_m})$, where $y_k v_1^z = c_k^{a_k} v_1^z$. Then it follows from [19, Lemma 1.3.1] that $R_{L_{\delta,i}^z, L_{\delta,i}^{z'}} \circ R_{L_{\delta,i}^{z'}, L_{de,i}^z}(v_1^{z'} \otimes v_1^z) = f^2(v_1^{z'} \otimes v_1^z)$, so $\tau^2(v_1 \otimes v_1) = v_1 \otimes v_1$, and thus $\tau_r^2 = 1$. $\square$

6.2. Proofs of zigzag relations

This section is devoted to proving Lemmas 4.114 and 4.115, which are crucial in determining the commutation relations among generating endomorphisms of $\Delta_\delta^{\circ n}$. In all cases, the approach to proving these lemmas is similar:

(i) Every element of $\Delta_{\delta,i} \circ \Delta_{\delta,j}$ should be written as a linear combination of terms of the form $\psi_w(x_1 \otimes x_2)v_i \otimes v_j$, where $x_1, x_2 \in R_\delta$, and w is a minimal left coset representative for $\mathfrak{S}_{2d}/\mathfrak{S}_d \times \mathfrak{S}_d$. Diagrammatically speaking, this is a matter of moving beads, and crossings of strands which originate from the same side, to the top of the diagram by applying KLR relations.

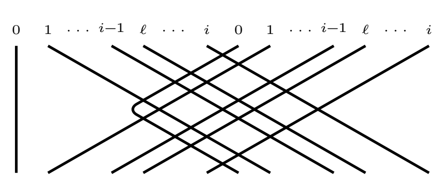
(ii) Once all terms are rewritten as in (i), use Lemmas 4.100 through 4.103 to simplify the expressions $(x_1 \otimes x_2)v_i \otimes v_j$, rewriting these elements of $\Delta_{\delta,i} \boxtimes \Delta_{\delta,j}$ in the form of the basis in Theorem 4.106.

We have written a Sage program which performs steps (i) and (ii), and have used this algorithm to verify Lemmas 4.114 and 4.115 in the exceptional cases of type $\mathtt{E}_\ell^{(1)}$. This program is available upon request. In the following proofs we assume $\mathtt{C}$ is of type $\mathtt{A}_\ell^{(1)}$ or $\mathtt{D}_\ell^{(1)}$.

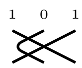
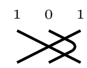
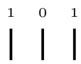
Lemma 4.114. *Let $i, j \in I'$, and recall that $v_i \otimes v_j$ is a generator for $\Delta_{\delta,i} \circ \Delta_{\delta,j}$. Then we have*

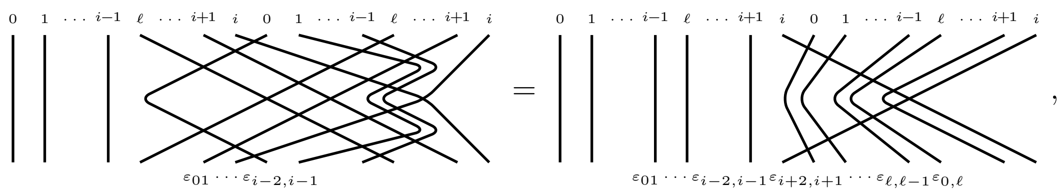
$$\sigma' v_i \otimes v_j = \begin{cases} \xi_i[y_d \otimes 1 + 1 \otimes (y_d - 2y_1)]v_i \otimes v_i & i = j; \\ \xi_i \varepsilon_{ij}(\psi_{j,i} \otimes \psi_{i,j})v_i \otimes v_j & (\alpha_i, \alpha_j) = -1; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Case $i = j$, $\mathtt{C} = \mathtt{A}_\ell^{(1)}$. If $\ell = 1$, the result is easily checked. Assume $\ell \geq 2$. We depict $\sigma' v_i \otimes v_i$ diagrammatically, where $v_i \otimes v_i$ is conceived to be at the top of the diagram:

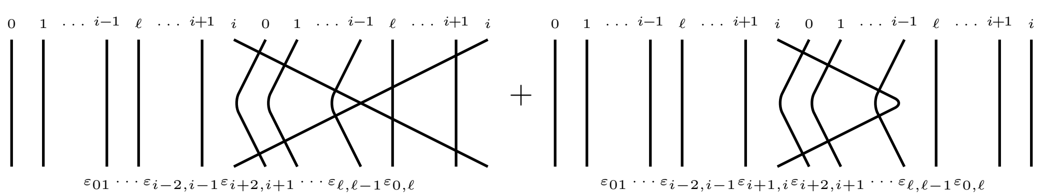


.

We now move crossings up, when possible, to act on the individual factors $\Delta_{\delta,i}$, and use Lemmas 4.95 and 4.102 to recognize when these terms are zero. Applying the braid relation to the  braid, we see that the  term allows for the $(0,1)$-crossing to move up to act on $\Delta_{\delta,i}$ as zero, leaving only the remainder term ε_{01} . This behavior will occur frequently enough that we will merely say that the $(i,i+1,i)$-braid 'opens'. Indeed, the $(1,0,1)$- through $(i-1,i-2,i-1)$-braids open in succession, giving:



after the $(\ell,0,\ell)$-braid opens, followed by the $(\ell-1,\ell,\ell-1)$- through $(i+2,i+1,i+2)$-braids in succession. Now, applying the $(i,i+1,i)$-braid relation, this is equal to



(6.17)

In the left term in (6.17), the $(i,i-1,i)$-braid opens, introducing an $(i+1,i)$-double crossing, which opens to give

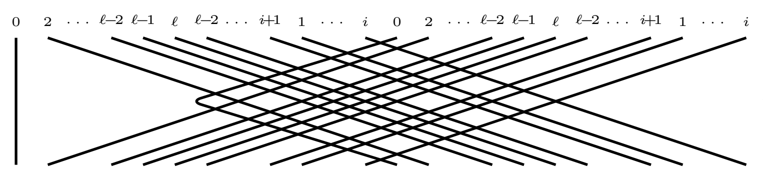
$$-\xi_i[1 \otimes (y_{d-1} - y_d)]v_i \otimes v_i = -\xi_i[1 \otimes (y_1 - y_d)]v_i \otimes v_i.$$

In the right term in (6.17), the $(i, i-1)$-double crossing opens to give

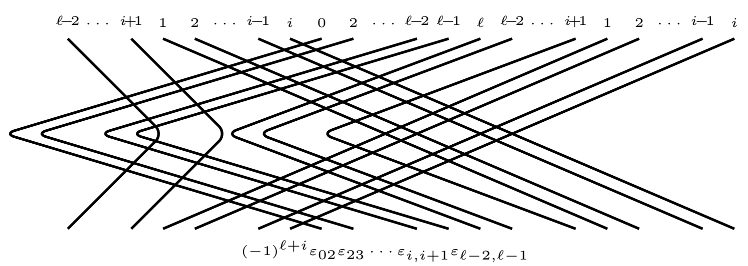
$$\xi_i[y_d \otimes 1 - 1 \otimes y_i]v_i \otimes v_i = \xi_i[y_d \otimes 1 - 1 \otimes y_1]v_i \otimes v_i,$$

proving the claim.

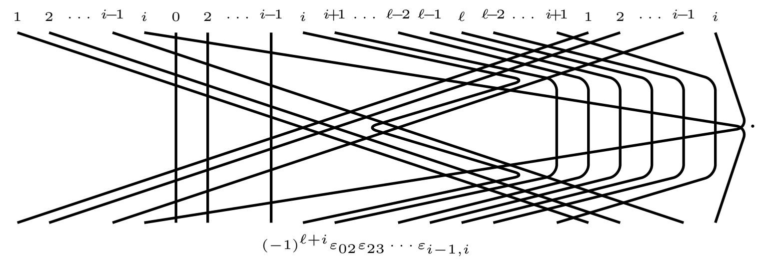
<u>Case $i = j$, $\mathtt{C} = \mathtt{D}_\ell^{(1)}$, $1 \le i \le \ell - 2$</u>. We depict $\sigma' v_i \otimes v_i$ diagrammatically:



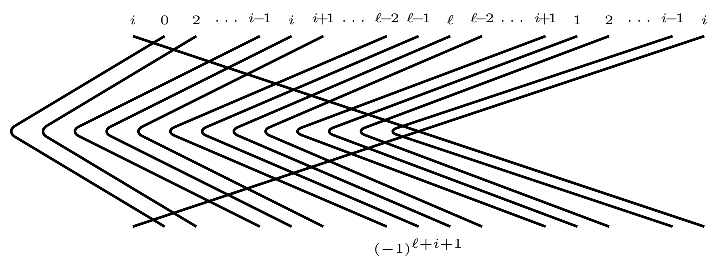
We begin dragging the 0-strand to the right to simplify the diagram. The $(2, 0, 2)$-braid opens, followed by the $(3, 2, 3)$- through $(\ell-1, \ell-2, \ell-1)$-braids in succession. Then the $(\ell-2, \ell, \ell-2)$-braid opens, followed by the $(\ell-3, \ell-2, \ell-3)$- through $(i+1, i+2, i+1)$-braids in succession, giving (excluding straight strands on the left):



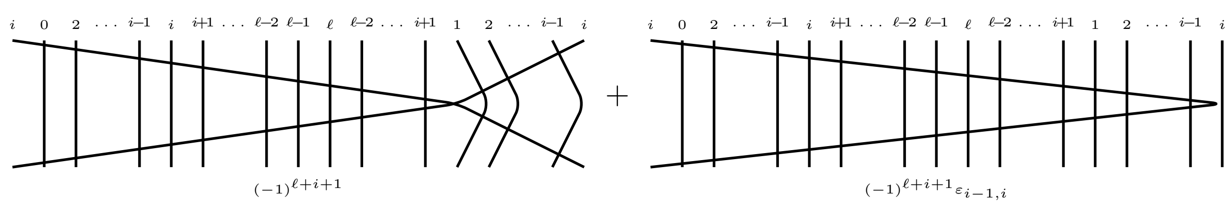
Now the $(\ell-2, \ell-1)$-double crossing opens, introducing a $(\ell-2, \ell-3, \ell-2)$-braid which opens, followed by a $(\ell-2, \ell-3)$-double crossing which opens. This sequence repeats until the $(i+2, i+1, i+2)$-braid opens, followed by $(i+2, i+1)$-double crossing which opens. Finally, the $(i+1, i, i+1)$-braid opens, giving:



Now the central $(i-1, i, i-1)$- through $(1, 2, 1)$-braids open in succession, and then $(i-2, i-1)$- through $(1, 2)$-double crossings open in succession. Then the $(2, 0, 2)$-braid opens, followed by the $(3, 2, 3)$- through $(i-1, i-2, i-1)$-braids opening in succession, giving (omitting straight strands on the left):



Now, applying the braid relation to the $(i, i-1, i)$-braid gives



In the term on the left, the $(i, i+1, i)$-braid opens, introducing $(i-1, i)$- and $(i, i+1)$-double crossings which open, finally introducing an $(i, i-1, i)$-braid which opens, giving

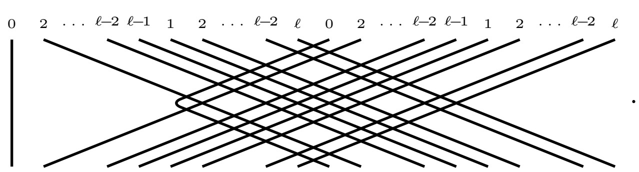
$$(-1)^{\ell+i}[1\otimes(y_{d-1}-y_d)]v_i\otimes v_i=(-1)^{\ell+i}[1\otimes(y_1-y_d)]v_i\otimes v_i$$

In the term on the right, the $(i,i-1)$-double crossing opens, followed by an $(i,i+1)$-crossing opening, finally introducing an $(i,i-1,i)$-braid which opens, giving

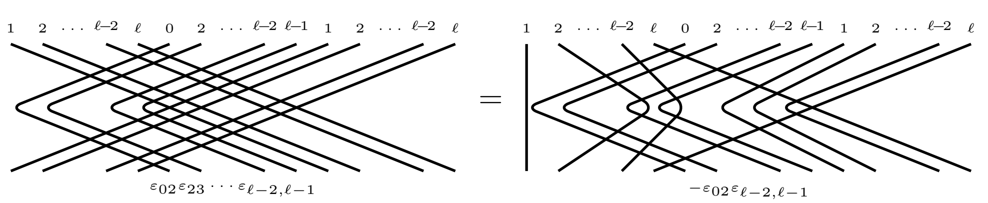
$$(-1)^{\ell+i+1}[y_d\otimes 1-1\otimes y_1]v_i\otimes v_i,$$

proving the statement.

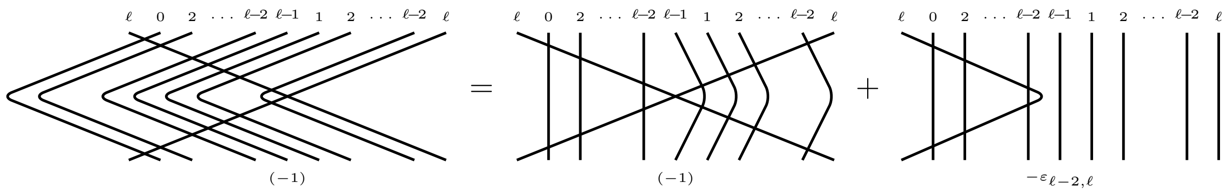
<u>Case $i=j$, $\mathtt{C}=\mathtt{D}_\ell^{(1)}$, $i=\ell,\ell-1$.</u> We'll check the $i=\ell$ case, the other case being similar. We depict $\sigma' v_i\otimes v_i$ diagrammatically:



As in the last case, we begin by pulling the 0-strand to the right. The $(2,0,2)$-braid opens, then the $(3,2,3)$- through $(\ell-1,\ell-2,\ell-1)$-braids open in succession, giving (omitting straight strands on the left):



after the $(1,2,1)$-braid opens, followed by the $(2,1,2)$- through $(\ell-2,\ell-\ell-3,\ell-2)$-braids. Now the $(2,3)$- through $(\ell-2,\ell-1)$-braids open in succession. Then $(2,0,2)$-braid opens, followed by the $(3,2,3)$- through $(\ell-2,\ell-3,\ell-2)$- braids in succession, giving (omitting straight strands on the left):



after applying the braid relation to the $(\ell, \ell-2, \ell)$-braid. In the left term, the $(\ell, \ell-2, \ell)$-braid opens, then the $(\ell-2, \ell)$-double crossing opens, giving

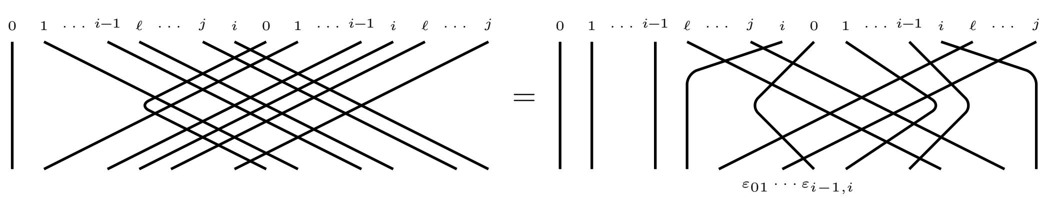
$$-(1 \otimes (y_{d-1} - y_d))v_i \otimes v_i = -(1 \otimes (y_1 - y_d))v_i \otimes v_i.$$

In the right term, the $(\ell, \ell-2)$-double crossing opens, giving

$$(y_d \otimes 1 - 1 \otimes y_{\ell-2})v_i \otimes v_i = (y_d \otimes 1 - 1 \otimes y_1)v_i \otimes v_i,$$

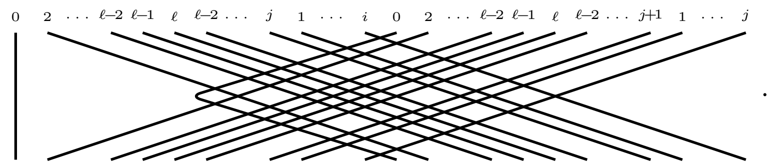
proving the claim in this case.

Case $(\alpha_i, \alpha_j) = -1$, $\mathtt{C} = \mathtt{A}_\ell^{(1)}$. We have $j = i+1$ or $j = i-1$. We will prove the statement in the former case; the latter is similar. We write $\sigma' v_i \otimes v_j$ diagrammatically:

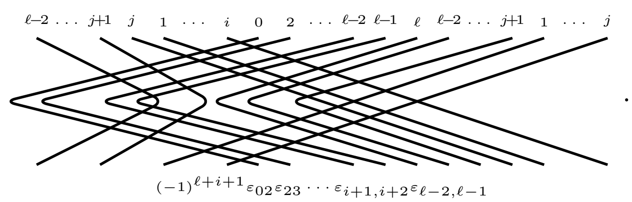


after the $(1, 0, 1)$- through $(i, i-1, i)$-braids open in succession. Now the $(\ell, 0, \ell)$-braid opens, followed by the $(\ell-1, \ell, \ell-1)$- through $(j, j+1, j)$-braids in succession, giving $\xi_i \varepsilon_{ij}(\psi_{j,i} \otimes \psi_{i,j})v_i \otimes v_j$, as desired.

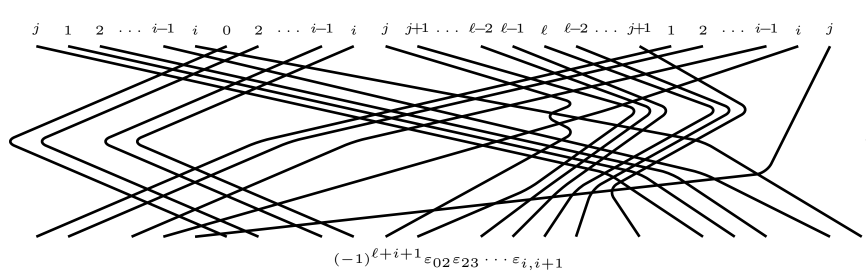
Case $(\alpha_i, \alpha_j) = -1$, $\mathtt{C} = \mathtt{D}_\ell^{(1)}$, $1 \le i, j \le \ell - 2$. We have $j = i+1$ or $j = i-1$. We will prove the statement in the former case; the latter is similar. We write $\sigma' v_i \otimes v_j$ diagrammatically:



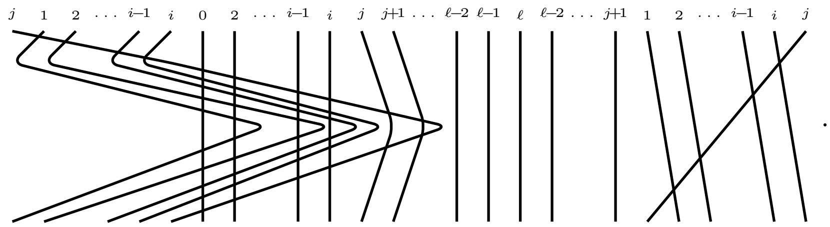
Dragging the 0-strand to the right, the $(2, 0, 2)$-braid opens, then the $(3, 2, 3)$- through $(\ell-1, \ell-2, \ell-1)$-braids open in succession. Then $(\ell, \ell-2, \ell)$- and $(\ell-2, \ell, \ell-2)$-braids open, followed by $(\ell-3, \ell-2, \ell-2)$- through $(j+1, j+2, j+1)$-braids opening in succession. This gives (omitting straight strands on the left):



Now the $(\ell-2, \ell-1)$-double crossing opens. The $(\ell-2, \ell-3, \ell-2)$-braid opens, which introduces an $(\ell-2, \ell-3)$-double crossing which opens. This sequence repeats, until the $(j+2, j+1, j+1)$-braid opens, introducing a $(j+2, j+1)$-double crossing which opens. Finally, a $(j+1, j, j+1)$-braid opens, giving (omitting straight strands on the left):

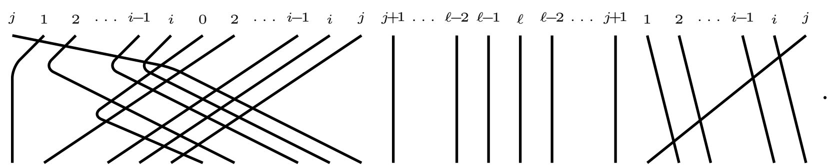


Now the (i, j, i)-braid opens, and then the $(i-1, i, i-1)$-through $(1, 2, 1)$-braids open in succession. Finally, the $(j, j+1, j)$-braid opens, giving:



$(-1)^{\ell+i}\varepsilon_{02}\varepsilon_{12}\varepsilon_{i+2,i+1}$

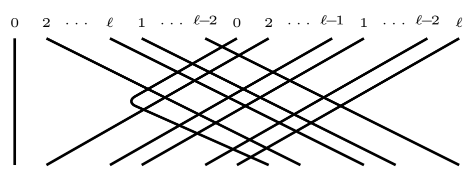
Now the $(j, j+1)$-double crossing opens, followed by the $(i-1, i)$- through $(1, 2)$-double crossings in succession, giving (omitting strands on the right):



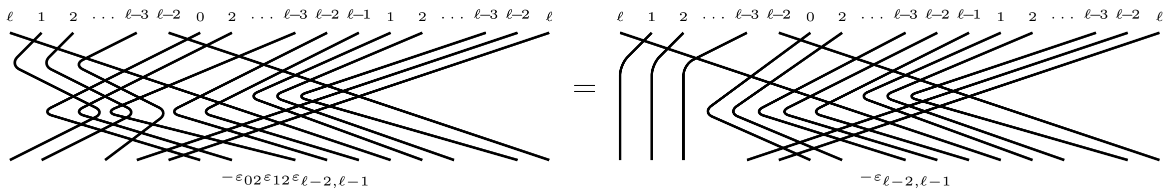
$(-1)^{\ell+i+1}\varepsilon_{02}\varepsilon_{23}\varepsilon_{i-1,i}$

Now the $(2, 0, 2)$-braid opens, followed by the $(3, 2, 3)$- through (j, i, j)-braids in succession, giving $(-1)^{\ell+i+1}\varepsilon_{i,i+1}(\psi_{j,i} \otimes \psi_{i,j})v_i \otimes v_j$, as desired.

Case $(\alpha_i, \alpha_j) = -1$, $\mathtt{C} = \mathtt{D}_\ell^{(1)}$, $\ell - 2 \leq i, j \leq \ell$. We will check the case $i = \ell - 2$, $j = \ell$. The other cases are similar. We write $\sigma' v_i \otimes v_j$ diagrammatically:



Dragging the 0-strand to the right, the $(2, 0, 2)$-braid opens, then the $(3, 2, 3)$- through $(\ell-1, \ell-2, \ell-1)$-braids open in succession. The $(\ell-3, \ell-2, \ell-2)$-braid opens, and then the $(\ell-4, \ell-3, \ell-4)$- through $(1, 2, 1)$-braids open in succession, giving (omitting straight strands on the left):



after the $(\ell-4,\ell-3)$- through $(1,2)$-double crossings open in succession, followed by the $(2,0,2)$- and $(3,2,3)$- through $(\ell-3,\ell-4,\ell-3)$-braids opening in succession. Now the $(\ell-2,\ell-3,\ell-2)$-braid opens, introducing an $(\ell-2,\ell-1)$-double crossing which opens, followed by $(\ell-2,\ell-3,\ell-2)$- and $(\ell,\ell-2,\ell)$-braids opening, which gives $-\varepsilon_{ij}(\psi_{j,i}\otimes\psi_{i,j})v_i\otimes v_j$, as desired.

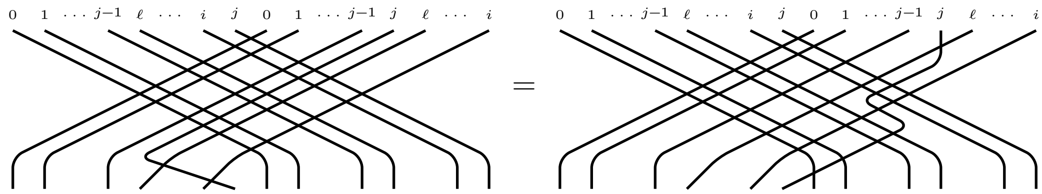
Case $(\alpha_i,\alpha_j)=0$, all types. By the usual manipulations of KLR elements (cf. [3, §2.6]), we may write $\sigma' v_i\otimes v_j$ as a sum of terms of the form $1_{\boldsymbol{b}^j\boldsymbol{b}^i}\psi_w x_i\otimes x_j$, where $x_i\in\Delta_{\delta,i}$ and $x_j\in\Delta_{\delta,j}$, and $w\triangleleft\sigma'$ (where we consider σ' as an element of $\mathfrak{S}_{2d}$) is a minimal left coset representative for $\mathfrak{S}_{2d}/\mathfrak{S}_d\times\mathfrak{S}_d$. Since $(\boldsymbol{b}^j\boldsymbol{b}^i)_1=(\boldsymbol{b}^j\boldsymbol{b}^i)_{d+1}=0$ and $i_1=0$ for every word $\boldsymbol{i}$ of $\Delta_{\delta,i}$ and $\Delta_{\delta,j}$, it follows that $w=\mathrm{id}$. But $1_j\Delta_{\delta,i}=0$ by Lemma 4.111, so $\sigma' v_i\otimes v_j=0$. □

Lemma 4.115. *Let $i,j,m\in I'$ with $(\alpha_i,\alpha_j)=-1$. Then we have*

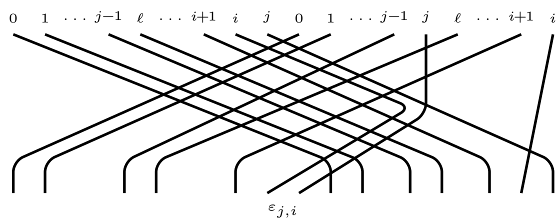
$$(\psi_{j,i}\otimes 1)\sigma v_m\otimes v_i=[\sigma(1\otimes\psi_{j,i})+\delta_{j,m}\xi_j(1\otimes\psi_{j,i})-\delta_{i,m}\xi_i(\psi_{j,i}\otimes 1)]v_m\otimes v_i.$$

Proof. Case $m=j$, $\mathtt{C}=\mathtt{A}_\ell^{(1)}$. Since i and j are neighbors, either $j=i-1$ or $j=i+1$. We will prove the claim in the former case; the latter is similar. We depict

$(\psi_{j,i} \otimes 1)\sigma v_j \otimes v_i$ diagrammatically:



Applying the braid relation to the (i, j, i)-braid, we have $\sigma(1 \otimes \psi_{j,i})v_j \otimes v_i$, plus the error term:



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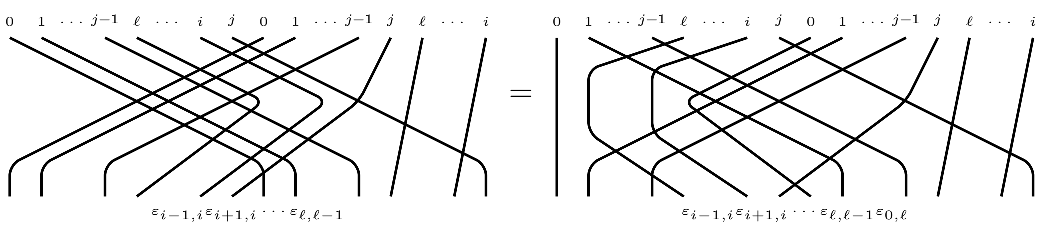
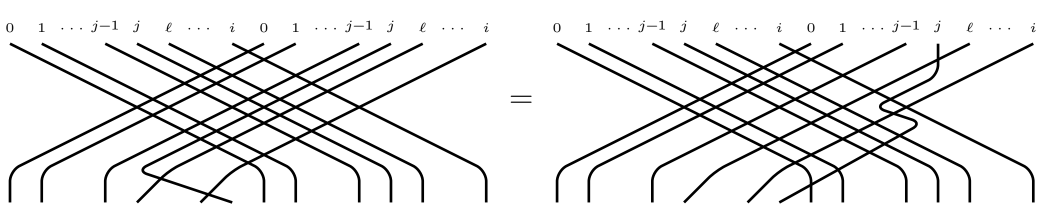
the $(i, i+1, i)$- through $(\ell, \ell-1, \ell)$-braids open in succession, giving



,

after the $(0, \ell, 0)$-braid opens. Now, the $(1, 0, 1)$- through $(j, j-1, j)$-braids open in succession, giving $\xi_j(1 \otimes \psi_{j,i})v_j \otimes v_i$, as desired.

Case $m = i$, $\mathtt{C} = \mathtt{A}_\ell^{(1)}$. We show that the claim holds in the case $i = j + 1$; the case $i = j - 1$ is similar. We depict $(\psi_{j,i} \otimes 1)\sigma v_i \otimes v_i$ diagrammatically:



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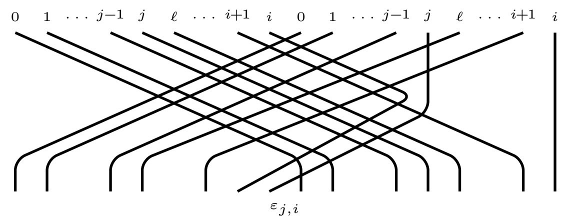
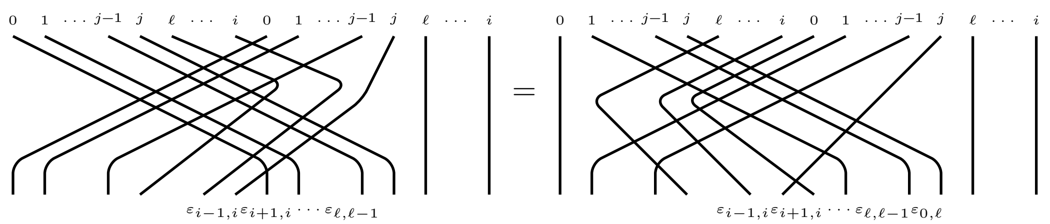
Applying the braid relation to the (i, j, i)-braid, we get $\sigma v_i \otimes \psi_{j,i} v_i$ plus the error term:

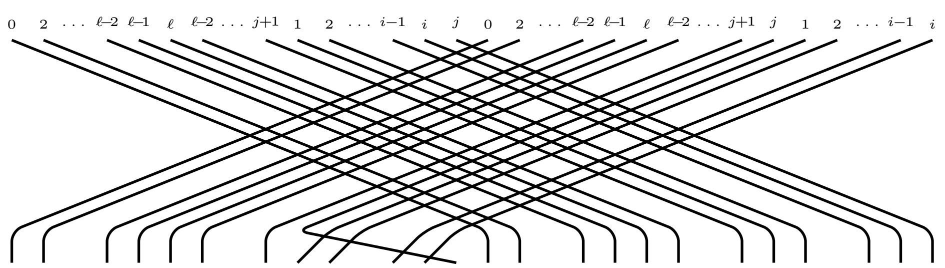


For this term, the $(i+1, i, i+1)$- through $(\ell, \ell-1, \ell)$-braids open, giving

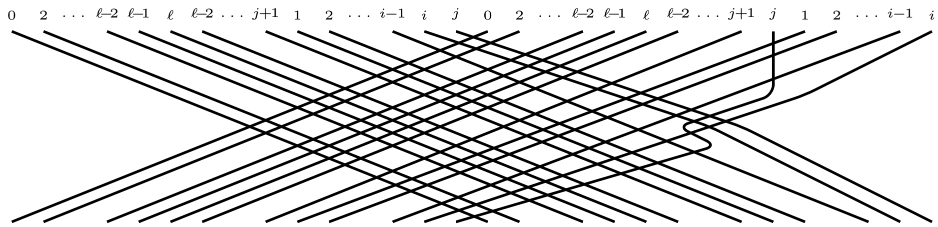


after the $(0, \ell, 0)$-braid opens. Now the $(1, 0, 1)$- through $(j, j-1, j)$-braids open in succession, giving $-\xi_i(\psi_{j,i} \otimes 1)v_i \otimes v_i$, as desired.

<u>Case $m = j$, $\mathtt{C} = \mathtt{D}_\ell^{(1)}$, $1 \le i, j \le \ell - 2$.</u> We check that (4.115) holds in the case $j = i+1$. The case $j = i-1$ is similar. We depict $(\psi_{j,i} \otimes 1)\sigma v_j \otimes v_i$ diagrammatically, with $v_j \otimes v_i$ at the top of the diagram:

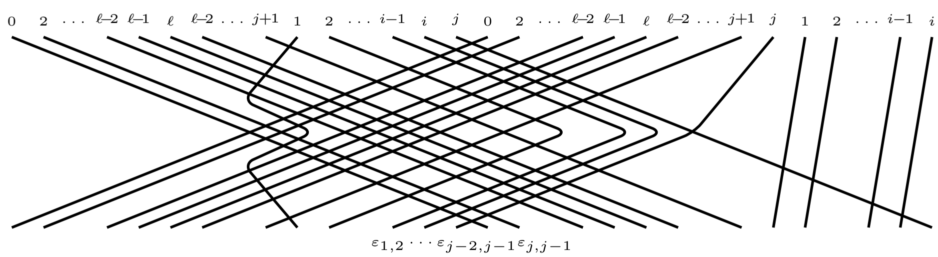


The j-strand moves past the first (i,i)-crossing, as the open term in the (i,j,i)-braid relation is zero. This gives



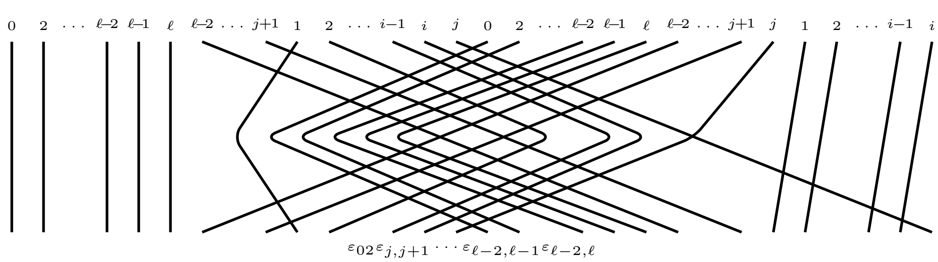
.

Applying the braid relation to the (i,j,i)-braid, we have $\sigma(1 \otimes \psi_{j,i})v_j \otimes v_j$, plus a remainder term. Now we simplify the remainder term. The (i,j,i)-braid opens, followed by the $(i-1,i,i-1)$- through $(1,2,1)$-braids opening in succession. This gives



.

Now the $(2,1,2)$- and $(0,2,0)$-braids open, followed by the $(3,2,3)$- through $(\ell-1,\ell-2,\ell-1)$-braids and the $(\ell,\ell-2,\ell)$-braid, giving



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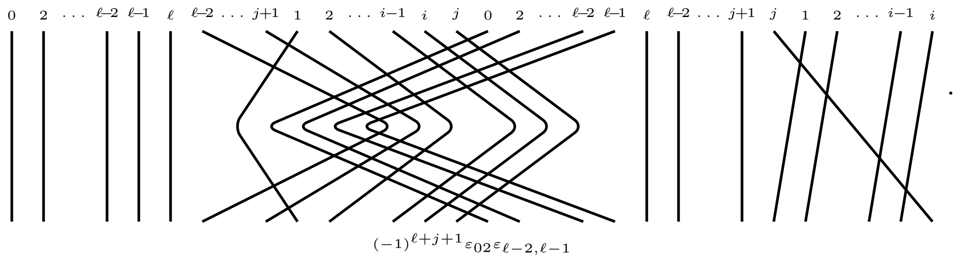
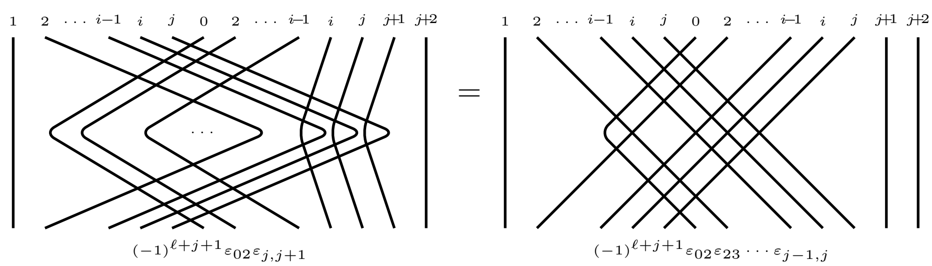
Now the $(\ell-2, \ell, \ell-2)$-braid opens, followed by the $(\ell-3, \ell-2, \ell-3)$- through $(j, j+1, j)$-braids opening in succession, giving



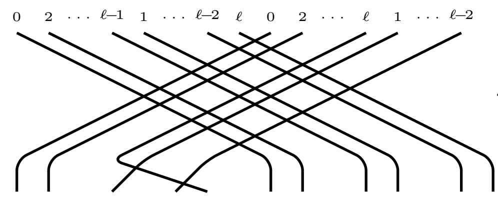
Now the $(\ell-2, \ell-1)$-double crossing opens. Then the $(\ell-2, \ell-3, \ell-2)$-braid opens, followed by the $(\ell-3, \ell-2)$-double crossing. This pattern repeats until the $(j+2, j+1, j+2)$-braid opens, followed by the $(j+1, j+2)$-double crossing. Then the $(j+1, j, j+1)$-braid opens, which gives (omitting strands outside the central area)



after the $(2,3)$- through $(j, j+1)$-double crossings open. Now the $(2,0,2)$-crossing opens, followed by the $(3,2,3)$- through $(j, j-1, j)$-braids, giving $(-1)^{\ell+j+1}(1 \otimes \psi_{j,i})v_j \otimes v_i$, as desired.

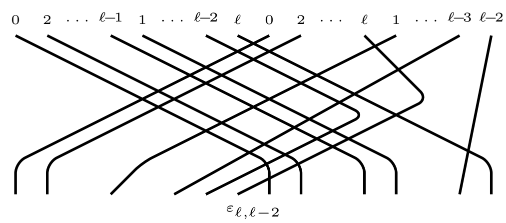
Case $m = j$, $\mathtt{C} = \mathtt{D}_\ell^{(1)}$, $\ell - 2 \le i, j \le \ell$. We check that the claim holds in the case $i = \ell - 2, j = \ell$. The other cases are similar. We depict $(\psi_{j,i} \otimes 1)\sigma v_j \otimes v_i$

diagrammatically:



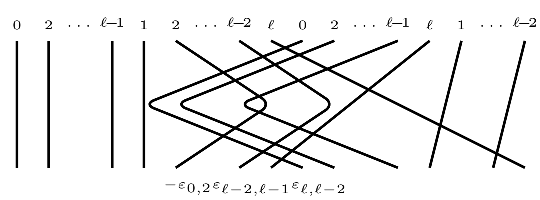
.

The ℓ-strand moves past the first $(\ell-2, \ell-2)$-crossing, and applying the braid relation to the next $(\ell-2, \ell, \ell-2)$-braid gives $(1 \otimes \psi_{\ell,\ell-2})v_\ell \otimes v_{\ell-2}$ plus an error term:



.

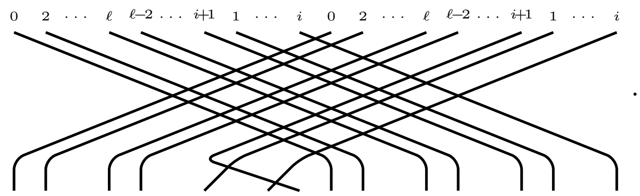
Now the $(\ell-3, \ell-2, \ell-3)$- through $(1, 2, 1)$-braids open in succession. Then the $(2, 1, 2)$-braid and $(0, 2, 0)$-braids open, followed by the $(3, 2, 3)$- through $(\ell-1, \ell-2, \ell-1)$-braids opening in succession, giving:



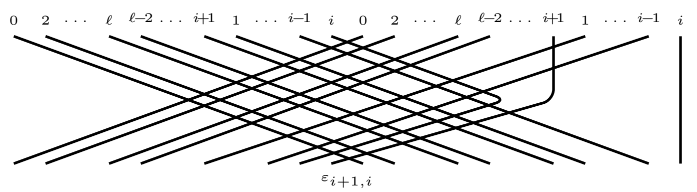
.

Now the $(2, 3)$- through $(\ell-2, \ell-1)$-double crossings open, introducing a $(2, 0, 2)$-braid, which opens. Then the $(3, 2, 3)$- through $(\ell-2, \ell-3, \ell-2)$-braids open, followed by a $(\ell, \ell-2, \ell)$-braid which opens, giving $\xi_\ell(1 \otimes \psi_{\ell,\ell-2})v_\ell \otimes v_{\ell-2}$, as desired.

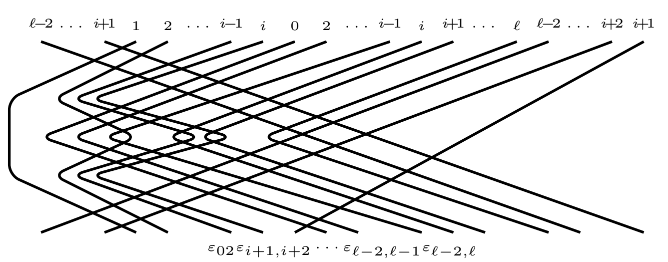
Case $m = i$, $\mathtt{C} = \mathtt{D}_\ell^{(1)}$, $1 \le i, j \le \ell - 2$. We show that the claim holds in the case $i = j + 1$; the case $i = j - 1$ is similar. We depict $(\psi_{j,i} \otimes 1)\sigma v_i \otimes v_i$ diagrammatically:



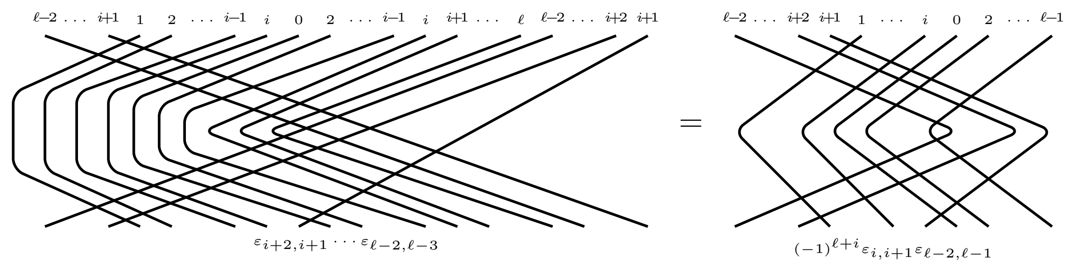
Now the $(i+1)$-strand moves up to the right past the first (i, i)-crossing. Applying the braid relation to the next $(i, i+1, i)$-braid gives $(1 \otimes \psi_{j,i})\sigma v_i \otimes v_i$, plus a remainder term:



Dragging the i-strand to the left, the $(i-1, i, i-1)$- through $(1, 2, 1)$-braids open in succession, followed by the $(2, 1, 2)$- and $(0, 2, 0)$-braids. Then the $(3, 2, 3)$- through $(\ell-1, \ell-2, \ell-1)$-braids open in succession, followed by the $(\ell, \ell-2, \ell)$-braid, giving (omitting straight strands outside the central area):

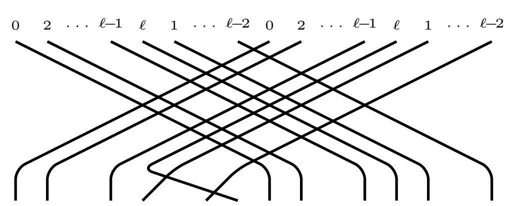


Now the $(2,3)$- through $(i,i+1)$-double crossings open, introducing a $(2,0,2)$-braid which opens, followed by $(3,2,3)$- through $(i,i-1,i)$-braids which open, giving:

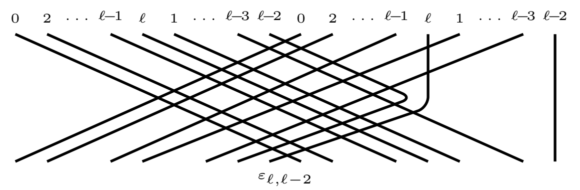


after the $(\ell-2,\ell,\ell-2)$-braid opens, followed by the $(\ell-3,\ell-2,\ell-3)$-through $(i+1,i+2,i+1)$-braids in succession. Now the $(\ell-2,\ell)$-double crossing opens. The $(\ell-2,\ell-3,\ell-2)$-braid opens, followed by an $(\ell-2,\ell-3)$-double crossing which opens. This sequence repeats until the $(i+2,i+1,i+2)$-braid opens, followed by an $(i+2,i+1)$-double crossing which opens. Finally, the $(i+1,i,i+1)$-braid opens, giving $-\xi_i(\psi_{j,i}\otimes 1)v_i\otimes v_i$, as desired.

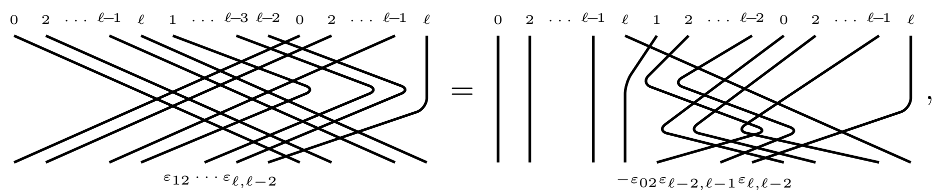
<u>Case $m=i$, $\mathtt{C}=\mathtt{D}_\ell^{(1)}$, $\ell-2\leq i,j\leq\ell$.</u> We show that the claim holds in the case $i=\ell-2, j=\ell$; the other cases are similar. We depict $(\psi_{j,i}\otimes 1)\sigma v_i\otimes v_i$ diagrammatically:



The ℓ-strand moves up past the first $(\ell-2,\ell-2)$-crossing. Applying the braid relation to the next $(\ell-2,\ell,\ell-2)$-braid gives $\sigma(1\otimes\psi_{\ell,\ell-2})v_{\ell-2}\otimes v_{\ell-2}$, plus an error term:



Now we simplify this error term. The $(\ell-3,\ell-2,\ell-3)$- through $(1,2,1)$-braids open in succession, giving (omitting straight strands to the right):



after the $(2,1,2)$- and $(0,2,0)$-braids open, followed by the $(3,2,3)$- through $(\ell-1,\ell-2,\ell-1)$-braids opening in succession. Now, the $(2,3)$- through $(\ell-2,\ell-1)$-braids open in succession. Then the $(2,0,2)$- braid opens, followed by the $(3,2,3)$- through $(\ell-2,\ell-3,\ell-2)$-braids opening in succession. Finally the $(\ell,\ell-2,\ell)$-braid opens, giving $(\psi_{\ell,\ell-2}\otimes 1)v_{\ell-2}\otimes v_{\ell-2}$, as desired.

Case $j\neq m\neq i$, all types. We may write

$$(\psi_{j,i}\otimes 1)\sigma v_m\otimes v_i=\sigma(1\otimes\psi_{j,i})v_m\otimes v_i+(*),$$

where $(*)$ is a linear combination of terms of the form $1_{\boldsymbol{b}^j\boldsymbol{b}^m}\psi_w x_1\otimes x_2$, where $x_1\in\Delta_{\delta,m}$, $x_2\in\Delta_{\delta,i}$, and $w\lhd\sigma$ is a minimal left coset representative for $\mathfrak{S}_{2d}/\mathfrak{S}_d\times\mathfrak{S}_d$. As in the similar case in Lemma 4.114, it follows that $\psi_w=1$. Thus x_1 is a vector of word $\boldsymbol{b}^j$ and x_2 is a vector of word $\boldsymbol{b}^m$. Hence by Lemma 4.111, it follows that $(*)$ is zero unless m neighbors both j and i. But since i neighbors j by assumption, this cannot be the case. $\square$

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