DANCING IN THE STARS: TOPOLOGY OF NON-K-EQUAL CONFIGURATION SPACES OF GRAPHS

by

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DISSERTATION ABSTRACT

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We prove that the non-k-equal configuration space of a graph has a discretized model, analogous to the discretized model for configurations on graphs. We apply discrete Morse theory to the latter to give an explicit combinatorial formula for the ranks of homology and cohomology of configurations of two points on a tree. We give explicit presentations for homology and cohomology classes as well as pairings for ordered and unordered configurations of two and three points on a few simple trees, and show that the first homology group of ordered and unordered configurations of two points in any tree is generated by the first homology groups of configurations of two points in three particular graphs, $K_{1,3}$, $K_{1,4}$, and the trivalent tree with 6 vertices and 2 vertices of degree 3, via graph embeddings.

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TABLE OF CONTENTS

Chapter Page

LIST OF FIGURES

Figure Page

LIST OF TABLES

CHAPTER I

INTRODUCTION

Configuration spaces are an area of current research with a long history at the intersection of topology, geometry, and combinatorics. The ordered configuration space of n points in a topological space X is

$$\mathrm{Conf}_n(X) := \{(x_1, \ldots, x_n) \in X^n | x_i \neq x_j \text{ if } i \neq j\}$$

and the unordered configuration space is

$$\overline{\mathrm{Conf}}_n(X) := \mathrm{Conf}_n(X)/\Sigma_n.$$

Configuration spaces naturally arise in a number of contexts. The most famous is in the study of iterated loop spaces, of which [CLM76] is the definitive reference. The space $\mathrm{Conf}_n(\mathbb{R}^d)$ is homotopy equivalent to the nth-level object of the little d-disks operad. By composition with the Kunneth map, the homology of $\mathrm{Conf}_*(\mathbb{R}^d)$ inherits an operad structure, and so the homology of a d-fold loop space is an algebra over this operad, which is in fact the degree d Poisson operad.

The case $d = 2$ is special, in that $\overline{\mathrm{Conf}}_n(\mathbb{R}^2)$ is a $K(\pi, 1)$ and its fundamental group is B_n, the braid group on n strands. Therefore, the group (co)homology of the braid groups can be calculated from these spaces [FN62a, FN62b].

Clearly $\mathrm{Conf}_n(\mathbb{R}^2)$ is equivalent to the complement of the union of the pairwise diagonals in $\mathbb{R}^{2n} \cong \mathbb{C}^n$. Arnol'd proved foundational results in the study of hyperplane arrangements in [Arn69] by showing that $\mathrm{Conf}_n(\mathbb{C})$ is a $\mathrm{K}(\pi, 1)$ space and giving a simple presentation for its cohomology ring.

The study of configuration spaces can be organized based on the type of underlying space. Configurations on Euclidean space have an extensive literature [FH00]. The Leray spectral sequence corresponding to the inclusion $\mathrm{Conf}_n(X) \hookrightarrow X^n$ converges to $H^*(\mathrm{Conf}_n(X))$, which is equivalent to the spectral sequence described on [CT78], and this collapses after the E_2 page if X is a smooth complex projective variety [Tot96]. If M is the interior of a manifold with boundary, McDuff proved that $\overline{\mathrm{Conf}}_n(M)$ satisfies integral homological stability [McD75]. More recently, results have extended to a notion of stability for the rational (co)homology of ordered configuration spaces on orientable manifolds [Chu12].

In this dissertation, we focus on configuration spaces on graphs. Configuration spaces on manifolds have been classically studied by means of fibrations, such as $\pi : \mathrm{Conf}_n(M) \to M$ where $\pi(x_1, \ldots, x_n) = x_1$. Maps such as these are no longer fibrations when considering configurations on graphs, and so different techniques are necessary. Abrams [Abr00] introduced a discretized model for the configuration space of n points on a graph, which has the advantage of inheriting a nice combinatorial structure from the vertices and edges of the graph. The (co)homology of configurations spaces can then be studied using discrete Morse theory, which was laid out by Forman in [For98]. These discrete models are very large cubical complexes, but a discrete Morse flow preserves their homotopy type while 'collapsing' onto a much smaller subset of critical cells. In [Far06a], Farley defined a discrete Morse flow on discretized unordered configuration spaces of graphs, which gives a computation for the integral homology of unordered configuration spaces of trees.

In chapter II, I extend the Farley result to ordered configuration spaces of graphs, and prove the following:

When $n = 2$ and the graph is a tree T, there is only one non-trivial differential in the resulting Morse chain complex M_*, $\partial_1 : M_1 \to M_0$, and so

$$H_1(\mathrm{Conf}_2(T)) \simeq H^1(\mathrm{Conf}_2(T)) \simeq \mathbb{Z}^k$$

$$\text{where} \qquad k = -1 + \sum_{\substack{v \text{ s.t.} \\ \mu(v) \geq 3}} (\mu(v) - 1)(\mu(v) - 2)$$

$\mu(v)$ is the valence of the vertex v, and all other (co)homology groups are trivial. I also give an alternate proof for the homology equivalence of $\mathrm{Conf}_n(\Gamma)$ with Abrams' discretized model as a demonstration of methods used in later chapters.

Chapter III introduces a foundational tool in my exploration of graph configuration spaces: explicit cycle/cocycle descriptions of configurations on trees. These give presentations for the (co)homology of ordered and unordered configurations of two points on a tree, which is a new result. I show that the first homology groups of ordered and unordered configurations of two points in any tree is generated by the first homology groups of configurations of two points in three particular graphs, $K_{1,3}$, $K_{1,4}$, and the trivalent tree with 6 vertices and 2 vertices of degree 3, under graph embeddings. As part of the setup, I give explicit presentations along with relations and evaluation pairings for configurations of two or three points on a few simple trees, which allows me to describe the actions of the forgetful and transfer maps on (co)homology classes. Finally, we make the connection with detection of right-angled Artin groups.

Recently, there has been new interest in non-k-equal configurations, sometimes called non-k-overlapping configurations. The ordered configuration space of n points, where no k are equal, in a topological space X is

$$\mathrm{Conf}_{n,k}(X) \coloneqq X^n \backslash \{x_{i_1} = \ldots = x_{i_k} \text{ for some } k\text{-set of indices } 1 \leq i_1 < \ldots < i_k \leq n\}$$

and the unordered non-k-equal configuration space is

$$\overline{\mathrm{Conf}}_{n,k}(X) \coloneqq \mathrm{Conf}_{n,k}(X)/\Sigma_n.$$

$\mathrm{Conf}_{n,k}(\mathbb{R})$ and $\mathrm{Conf}_{n,k}(\mathbb{C})$ are equivalent to complements of subspace arrangements, and their cohomology groups can be expressed combinatorially [BW95]. $\mathrm{Conf}_{*,k}(\mathbb{R}^d)$ form a bimodule over ordered configurations in $\mathbb{R}^d$. Their (co)homology was recently calculated and described in terms of this geometric structure in [DT], which promises applications to the description of the homology of d-fold loop spaces of fat wedges. Non-k-equal configurations on graphs are a new subarea of these developments that has not previously been explored, though it promises applications to a broader class of problems than the standard configurations on a graph.

An overarching theme to this dissertation is that of discretization through open covers by stars. In chapter IV, I prove a discretized model for non-k-equal configurations on a graph in the spirit of Abrams, using such a cover to decompose the space into contractible sets. The proof extensively utilizes the new methods illustrated in chapter II. Chapter V goes into depth with calculations of non-3-equal configurations on a few simple trees, using code that was run on SageMathCloud to simplify calculations. Appendix 1 provides this code.

CHAPTER II

ORDERED CONFIGURATION SPACES OF GRAPHS

2.1. Discrete Model of Configuration Spaces of Graphs

Abrams [Abr00] introduced a discretized model for $\mathrm{Conf}_n(\Gamma)$, which we call $\mathcal{D}_n(\Gamma)$. $\mathcal{D}_n(\Gamma)$ is a cubical complex with a combinatorial structure inherited from the vertices and edges of the graph. We give an alternate proof of the homology equivalence between $\mathrm{Conf}_n(\Gamma)$ and $\mathcal{D}_n(\Gamma)$ as an illustration of the techniques that will be used in the non-k-equal setting.

We will work exclusively with finite, connected graphs, as the general case follows directly from the Kunneth formula. We reduce to the case where every graph Γ has no loops and at least one vertex of valence (or degree) ≥ 2. Given a graph Γ, we call $V(\Gamma)$ the set of vertices and $E(\Gamma)$ the set of open edges. Γ is a 1-dimensional CW complex, whose 0-cells are $V(\Gamma)$ and 1-cells are $E(\Gamma)$ with attaching maps $\partial_e : \{0,1\} \to V(\Gamma)$. Let $\partial_i(e)$ denote the image $\partial_e(i)$. Each edge has an orientation-preserving homeomorphism with the unit interval, so Γ inherits a metric, the 'shortest path' between any two points in Γ. We construct a new graph Γ', which is a subdivison of Γ. For any edge e in $E(\Gamma)$ such that $\partial_1(e) \neq \partial_0(e)$, $e \in E(\Gamma')$ and $\partial_1(e), \partial_0(e) \in V(\Gamma')$. For any edge e in $E(\Gamma)$ such that $\partial_1(e) = \partial_0(e)$, we subdivide this edge by adding a vertex v to $V(\Gamma')$ and two edges a, b to $E(\Gamma')$ such that $\partial_1(a) = \partial_1(e)$, $\partial_0(a) = v$, $\partial_1(b) = v$, and $\partial_0(b) = \partial_1(e)$. If Γ has no vertices of valence ≥ 2, then Γ is the graph with two vertices and a single edge connecting them, and we subdivide the unique edge in Γ, so that Γ' is the graph with three vertices and two edges connecting two pairs of vertices. If Γ' is a subdivision of Γ, then the two graphs are homeomorphic (as topological spaces), and so $\mathrm{Conf}_n(\Gamma)$ is invariant under subdivisions of

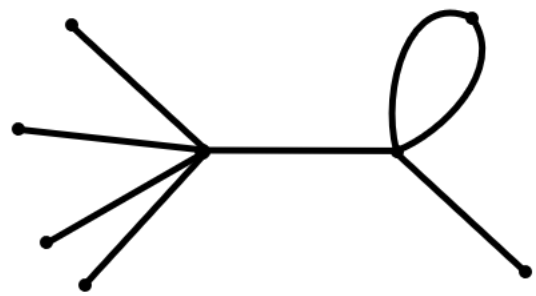

FIGURE 1 A graph Γ.

the graph. $\mathrm{Conf}_n(\Gamma)$ also inherits a metric as a subspace of the n-fold product of a metric space.

Let C_Γ be the set of vertices of Γ of valence ≥ 2. Let $\mathcal{V} = \{V_1, V_2, \ldots, V_s\}$ be an open cover of Γ where each V_j is an open set containing a single vertex from C_Γ and all of the edges connected to this vertex, except their opposite endpoint if it is another vertex in C_Γ. Then each V_j is an open star in Γ, and we call the single vertex of valence ≥ 2 the **central** vertex of V_j. An example of such an open covering is given in Figures 1 & 2.

Let σ be a choice of open star in $\mathcal{V}$ for each of the n configuration points. We say a configuration **conforms** to this choice if points in the configuration which are assigned to V_j by σ are in V_j and are closer to the central vertex of V_j than any point assigned to some other star, along an edge in which they are both located. If there is at least one point assigned to V_j, then we say that V_j is **occupied**. Notice that not all V_j must be occupied. Furthermore, if $V_j \cap V_k \neq \emptyset$, points assigned to V_j may move onto its intersection with V_k. Let ω be a choice of one point in each occupied V_j. We call this point the **designated** point

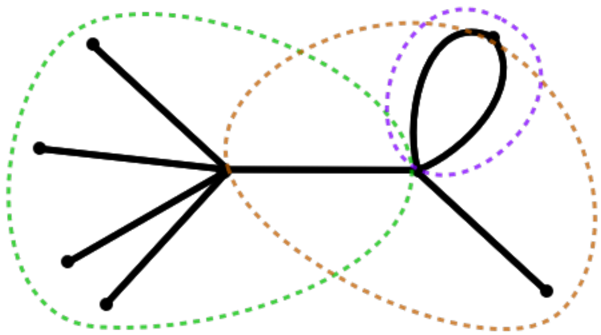

FIGURE 2 The open cover $\mathcal{V}$ of Γ by three open stars

of V_j. We call λ a **condition** if it is a choice of σ and a compatible ω, and then U_λ is the family of configurations conforming to σ where in each V_j the designated point is always the closest to the central vertex along whichever edge of the star it is located on. In particular, no point other than the designated point may be located at the central vertex, which we consider as part of every edge of the star. Let $\mathcal{U} = \{U_\lambda\}$, and if Λ is a set of conditions, we define $U_\Lambda = \bigcap_{\lambda\in\Lambda} U_\lambda$. In particular, if c is a configuration which satisfies $\lambda_1, \lambda_2, \ldots, \lambda_m$ then $c \in U_{\{\lambda_1,\lambda_2,\ldots,\lambda_m\}}$. We define $\zeta_b = \dfrac{\coprod_b [0,1)}{0 \sim 0}$ the star with b open edges.

Proposition 2.1.1. *$\mathcal{U}$ is an open, finite cover of Conf$_n(\Gamma)$.*

Proof. Given a configuration $c \in U_\lambda$, let ϵ_1 be the minimum distance between any two points in the configuration, and let ϵ_2 be the minimum distance from any non-designated point to any central vertex. Then the family of configurations $\left\{ c' \,\middle|\, d(c', c) < \dfrac{\min(\epsilon_1, \epsilon_2)}{3} \right\}$ is also in U_λ, so U_λ is open. Each configuration in $\mathrm{Conf}_n(\Gamma)$ is in at least one U_λ, so $\mathcal{U}$ covers $\mathrm{Conf}_n(\Gamma)$. Finally, Γ is a finite graph, so it can be covered by a finite number of stars, and there are a finite number of ways to distribute n points into the stars and choose a designated point in each occupied star. $\square$

Theorem 2.1.2. *Each U_Λ is the disjoint union of contractible spaces.*

Proof. First we assume that the designated point of each V_j is the same for all $\lambda \in \Lambda$. There is a map $G : U_\Lambda \times [0,1] \to U_\Lambda$ which pulls each designated point onto its central vertex: if x_{ω_j} is the designated point of V_j, at a distance of d_{ω_j} from the central vertex, then $G(c,t)$ is the identity on all points in the configuration, except x_{ω_j} is a distance $(1-t)d_{\omega_j}$ from the central vertex. This is a straight-line homotopy in U_Λ, because x_{ω_j} has no points in the configuration between it and the central vertex of V_j. Therefore G is continuous, and a homotopy. Then $G(U_\Lambda, 1)$ is a family of configurations in U_Lambda with some points fixed at central vertices of Γ and the rest constrained to move along some edge. If there are b_m non-designated points on an edge, regardless of whether or not they are all assigned to the same star, the space of configurations of these b_m points is homeomorphic to $\{(d_1, \ldots, d_m) | 0 < d_1 < \ldots < d_m < 1\}$. But this is exactly $\overset{o}{\Delta}^{b_m}$, the open b_m-simplex. Then a connected component of $G(U_\Lambda, 1)$ is homeomorphic to $\prod_m \overset{o}{\Delta}^{b_m}$, where m is the mth edge of Γ. This is contractible, so any connected component of U_Λ is contractible. Each component of U_λ must be disjoint, because we cannot change the ordering of points on an edge or move any point besides the x_{ω_j} through the central vertices. Therefore U_Λ is the disjoint union of contractible spaces.

Finally, we consider the case where there are some conditions which pick different designated points for at least one V_j. If there are at least two conditions which pick different designated points for the same V_j, then any configuration in U_Λ cannot have both points on the same edge. In particular, neither one can move onto the central vertex of V_j. Then there is a subset (perhaps empty) of the designated points which are always the designated point of the same V_j for all $\lambda \in \Lambda$, which we call E. There is a map

$G' : U_\Lambda \times [0,1] \to U_\Lambda$ which pulls the points in E onto their associated central vertex, defined analogously to G: each point $x_k \in E$ is at a distance of $(1-t)d_k$ at time t and G' is the identity on the remaining points. This is similarly a straight-line homotopy, and a connected component of $G'(U_\Lambda, 1)$ is again homeomorphic to a product of open simplices. Each component of U_Λ is disjoint, because we cannot change the ordering of points on an edge or move any point besides the points in E through the central vertices. Therefore U_Λ is the disjoint union of contractible spaces.

□

Remark 2.1.3. *If Γ is covered by s stars, then $|\mathcal{U}| = \sum_{r=1}^{\min(n,s)} \binom{n}{r} \frac{s!}{(s-r)!} r^{n-r}$. If only a single star with b edges is occupied and $|\Lambda| = \ell$, then U_Λ has $\frac{(n-\ell+b-1)!}{(b-1)!}$ connected components.*

Corollary 2.1.4. *$\mathcal{U}$ is a good open cover of $Conf_n(\Gamma)$, and so its nerve is an abstract simplicial model for $Conf_n(\Gamma)$.*

Now we subdivide Γ futher, so that there are n segments between any two central vertices in Γ and n segments between any central vertex and any root vertex in Γ. An edge still refers to an edge $e \in \Gamma$ from before this subdivision, between two central vertices or between a central and a root vertex. The subdivision means that each edge will have vertices at distance of multiples of $1/n$ from central vertices. Then Γ^n has a natural cubical structure, with cells $\tau = (\tau_1, \ldots, \tau_n)$ such that $\tau_i \in V(\Gamma)$ or $E(\Gamma)$. We define $\mathcal{D}_n(\Gamma) = \{\tau \in \Gamma^n | \overline{\tau}_i \cap \overline{\tau}_j = \emptyset$ for any $i \neq j\}$ and $D_\Lambda = \mathcal{D}_n(\Gamma) \cap U_\Lambda$. It follows immediately that $\mathcal{D} = \{D_\lambda\}$ is an open, finite cover of $\mathcal{D}_n(\Gamma)$.

Proposition 2.1.5. *Let M be some subset of the points $x_1, \ldots, x_n$ and let ς be an order on the points in M, and $|M| = m$. Let $(0,1)$ be subdivided into n equal segments, with a vertex* *at each multiple of* $1/n \in (0,1)$. *Let* $\mathcal{D}_m\frac{(0,1)}{n} = \{\tau \in (0,1)^m | \overline{\tau}_i \cap \overline{\tau}_j = \emptyset$ *for any* $i \neq j\}$, *which we can think of as the space of* $\frac{1}{n}$*-discrete configurations of* m *points in* $(0,1)$. *Let* Δ_ς^m *be the subset of* $\mathcal{D}_m\frac{(0,1)}{n}$ *where the order of the points is given by* ς. *Then* Δ_ς^m *is contractible.*

Proof. Let c_ς be the configuration of points with order ς such that the first point is at the vertex at $1/n$, the second at $2/n$, etc. There is a map $H : \Delta_\varsigma^m \times [0,1] \to \Delta_\varsigma^m$ which pulls every configuration onto c_ς: from $0 \leq 1/n$, $H(c,t)$ moves the first point in ς onto the vertex at $1/n$, from $1/n \leq t \leq 2/n$, it moves the second, and so on. $H(c,t) \in \Delta_\varsigma^m$ for all c and t, because the ith point in c_ς cannot be at any location $\frac{i-1}{n}$ or less along $(0,1)$. H is then a concatenation of straight-line homotopies in Δ_ς^m. Therefore Δ_ς^m is contractible. $\square$

Theorem 2.1.6. *Each* D_Λ *is likewise the disjoint union of contractible sets.*

Proof. First we assume that the designated point of each V_j is the same for all $\lambda \in \Lambda$. Then there is a map F which moves the non-designated points at least $1/n$ away from the central vertices and then moves the designated points onto their associated central vertex. It is necessary to first move the non-designated points away, so that the function remains in $\mathcal{D}_\Lambda$, which does not allow a point to occupy a vertex if there is another point on a segment abutting that vertex. Let $F : D_\Lambda \times [0,1] \to D_\Lambda$ and $c \in D_\Lambda$. If there is some point $x_i \in c$ a distance $d_i < 1/n$ from a central vertex, and x_i is not the designated point associated to that central vertex, x_i must be on a different edge than the designated point. Then when $0 \leq t \leq 1/2$, we move all such x_i a distance of $(2(1-t)-1)d_i + 2t/n$ from their central vertex. x_i can move up to $1/n$ from the central vertex and stay within D_λ, because all other points on the same edge must be at a distance of $2/n$ or greater from the central vertex. For all other points, F is the identity when $0 \leq t \leq 1/2$. Then on $1/2 \leq t \leq 1$, each designated point x_{k_j}, which is at a distance d_{k_j} at $t = 0$, moves a distance of $2(1-t)d_{k_j}$ from the central vertex of V_j. Since there can be no points other than the designated point on the segments closest to the central vertex, this function stays within D_Λ. It is also a straight-line homotopy on each designated point and the identity or a straight-line homotopy on each non-designated point, and therefore F is a homotopy. $F(D_\Lambda, 1)$ is a family of configurations where some points are fixed at central vertices, and each remaining point is constrained to move along some edge of the graph. Then a connected component of $F(D_\Lambda, 1)$ is homotopic to a product over the occupied edges of the graph of $\Delta_\varsigma^{b_m}$, where b_m is the number of non-designated points on the mth edge and ς is their order. Each component of D_Λ is disjoint and Prop 2.1.5 shows that the $\Delta_\varsigma^{b_m}$ are contractible, so D_Λ is the disjoint union of contractible sets.

Finally, we consider the case where there are some conditions which pick different designated points for at least one V_j. If there are at least two conditions which pick different designated points for the same V_j, then any configuration in D_Λ cannot have both points on the same edge, and neither one can move onto the central vertex of V_j. Then there is the subset E (perhaps empty) of the designated points which are always the designated point of the same V_j for all $\lambda \in \Lambda$. There is a map $F' : D_\Lambda \times [0,1] \to D_\Lambda$ which moves points not in E at least $1/n$ away from the central vertices and pulls the points in E onto their associated central vertex, defined analogously to F: on $0 \leq t \leq 1/2$, if x_i is point not in E at a distance of less than $1/n$ from a central vertex, we move it a distance of $(2(1-t)-1)d_i + 2t/n$ from the central vertex. Then on $1/2 \leq t \leq 1$, each point x_k in E moves a distance of $2(1-t)d_k$ from the central vertex of V_j. This is similarly a straight-line homotopy, and a connected component of $F'(D_\Lambda, 1)$ is again homeomorphic to a product of $\Delta_\varsigma^{b_m}$. Each component of D_Λ is disjoint, because we cannot change the ordering of points on an edge or move any point besides the points in E through the central vertices. Therefore D_Λ is the disjoint union of contractible spaces. $\square$

Theorem 2.1.7. *The inclusion $i : D_\Lambda \hookrightarrow U_\Lambda$ induces a homology equivalence for all Λ.*

Proof. Each component of D_Λ includes into exactly one component of U_Λ by definition. Given a configuration of points on an edge e, there is a homeomorphism of e that fixes its endpoints and sends the configuration to a discrete configuration: send the point in the interior of e closest to $\partial_0(e)$ a distance of $1/n$ from $\partial_0(e)$, the second-closest point a distance of $2/n$, etc. Then given a configuration in U_Λ, there is a homeomorphism of the graph which sends it to a configuration in D_Λ (though this choice of homeomorphism is obviously not unique or continuous). The space of homeomorphisms of an edge is contractible, so there is a path in U_Λ between the configurations, and the discrete configuration is in the same component of U_Λ as the starting configuration. Therefore there is at least one component of D_Λ in each component of U_Λ. Finally, there are exactly as many components of D_Λ as there are of U_Λ, because a component is determined by the number and order of non-designated points in each edge of the graph. Therefore the inclusion induces a bijection of components, and as all components are contractible, this is a homology equivalence for all choices of Λ. $\square$

Here we take a small digression to set up the machinery for a Mayer-Vietoris spectral sequence in homology.

Proposition 2.1.8 (Prop 15.2 of [BT82])**.** *Let $S_q(X)$ denote the singular q-chains in X and $S_q^{\mathcal{U}}(X)$ the singular q-chains subordinate to the countable open cover $\mathcal{U} = \{U_a\}_{a\in J}$. Let $U_{a_0\ldots a_p} = U_{a_0} \cap \ldots \cap U_{a_p}$. Let δ be the Čech boundary operator*

$$\delta : \bigoplus_{a_0<\ldots<a_p} S_q(U_{a_0\ldots a_p}) \to \bigoplus_{a_0<\ldots<a_{p-1}} S_q(U_{a_0\ldots a_{p-1}})$$

$$(\delta c)_{a_0\ldots a_{p-1}} = \sum_a c_{aa_0\ldots a_{p-1}}$$

where interchanging two indices introduces a minus sign, and

$$\epsilon : \bigoplus_a S_q(U_a) \to S_q(X)$$

is the sum. Then:

- *the homology of* $(S_*^{\mathcal{U}}(X), \partial)$ *is isomorphic to* $H_*(X)$ *in each dimension.*
- *the following sequence is exact:*

$$0 \leftarrow S_q^{\mathcal{U}}(X) \xleftarrow{\epsilon} \bigoplus_{a_0} S_q(U_{a_0}) \xleftarrow{\delta} \bigoplus_{a_0<a_1} S_q(U_{a_0a_1}) \xleftarrow{\delta} \cdots$$

Proposition 2.1.9. *Let* X *be a topological space. The Mayer-Vietoris Spectral Sequence associated to the decomposition of* X *by a countable open cover* $\mathcal{U}$ *is given by the double complex* $E^1_{p,q} = \bigoplus_{|I|=p} H_q(U_I)$ *for* $p \geq 0, q \geq 0$*, and*

$$d_1 = \delta_* : \bigoplus_{a_0<\ldots<a_p} H_q(U_{a_0\ldots a_p}) \to \bigoplus_{a_0<\ldots<a_{p-1}} H_q(U_{a_0\ldots a_{p-1}})$$

This spectral sequence converges to the homology of X*.*

Proof. There is a spectral sequence associated to the double complex with $E^0_{p,q}$ the singular q-chains in a p-fold intersection of open sets in $\mathcal{U}$, vertical differentials the usual boundary operator ∂, and horizontal differentials δ as in Prop 2.1.8. If we first take differentials horizontally, the spectral sequence collapses immediately in all but the first column because of exactness of the rows. Then the first column with vertical differentials is exactly the chain complex $(S_*^{\mathcal{U}}(X), \partial)$. Therefore the spectral sequence converges, and to the desired homology by Prop 2.1.8. If we take differentials vertically first, the spectral sequence becomes the Mayer-Vietoris spectral sequence in homology, which therefore also converges to the desired homology. □

Theorem 2.1.10. $H_*(Conf_n(\Gamma)) \cong H_*(\mathcal{D}_n(\Gamma))$

Proof. The Mayer-Vietoris decompositions of $\mathrm{Conf}_n(\Gamma)$ by $\mathcal{U}$ and $\mathcal{D}_n(\Gamma)$ by $\mathcal{D}$ give two spectral sequences as described in Prop 2.1.9. The inclusions of Prop 2.1.7 induce an equivalence of the spectral sequences, as the E^1 pages are isomorphic in each entry and the differentials are given by inclusions. Therefore both spectral sequences must converge to the same thing, with i^* giving the equivalence. □

2.2. Discrete Morse Theory

Now that the homology equivalence to a cubical complex has been established, the (co)homology of configuration spaces can be studied using discrete Morse theory, which was laid out by Forman in [For98]. The following definitions cleave to [Far06a], which differs slightly from [For98] in detail but not in spirit.

Let X be a finite CW complex, and K_i the set of its open i-cells. For $\alpha, \beta \in K$, we write $\alpha \leq \beta$ if $\alpha = \beta$ or $\alpha < \beta$ and we write $\alpha < \beta$ if $\alpha \neq \beta$ and $\alpha \subseteq \overline{\beta}$. We call α a **regular** face of β if $\alpha^{(p)} < \beta^{(p+1)}$, α is homeomorphic to its preimage under the characteristic map for β, and the closure of its preimage is a closed p-ball.

Definition 2.2.1. *A* ***discrete vector field*** W *on* X *is a sequence of partial functions* $W_i : K_i \to K_{i+1}$ *such that*

1. *Each* W_i *is injective*

2. *If* $W_i(\alpha) = \beta$*, then* α *is a regular face of* β

3. $im(W_i) \cap dom(W_{i+1}) = \emptyset$

Note that W_i is only defined on some subset of K_i. A **W-path of dimension p** is a sequence of p-cells $\alpha_0, \alpha_1, \ldots, \alpha_r$ such that if $W(\alpha_i)$ is undefined, $\alpha_{i+1} = \alpha_i$ and otherwise $\alpha_{i+1} \neq \alpha_i$ and $\alpha_{i+1} < W(\alpha_i)$. The W-path is **non-stationary** if $\alpha_1 \neq \alpha_0$ and **closed** if $\alpha_r = \alpha_0$. A discrete vector field W is a **discrete gradient vector field** if W has no non-stationary closed paths. The **critical** cells are those which are not in the range nor the doman of W, the **collapsible** cells are those in the range of W, and the **redundant** cells are those in the domain of W. It is worth noting that a discrete gradient vector field is equivalent to a discrete Morse function ([For98]), but we will not need to work with such functions.

Proposition 2.2.2 ([Far06a])**.** *Let $C_*(X)$ denote the cellular chain complex of X, with ∂ the standard boundary map. We may extend W_i to a map $C_i(X) \to C_{i+1}(X)$ that preserves our definitions of critical, collapsible, or redundant.*

1. *$f = 1 + \partial W + W\partial$ is a chain map $f : C_*(X) \to C_*(X)$ called the **discrete flow** corresponding to W, and for each chain there is some i such that $f^{\circ i} : C_*(X) \to C_*(X)$ stabilizes, so there is a well-defined chain map f^∞*

2. *The complexes $(C_*(X), \partial)$ and $(M_*(X), \pi\partial f^\infty)$ have isomorphic homology groups, where $M_i(X)$ is the free abelian group on critical i-cells, and $\pi : C_i(X) \to M_i(X)$ is the projection*

In [FS05], Farley and Sabalka defined a discrete gradient vector field (and associated flow) V on $\overline{\mathcal{D}}_n(\Gamma)$, the discretized model of $\overline{\mathrm{Conf}}_n(\Gamma)$, and showed that when Γ is a tree T, $H_i(\overline{\mathrm{Conf}}_n(T))$ is free abelian with rank equal to the number of critical i-cells with respect to the flow. We use the finite cover $\mathcal{D}_n(\Gamma) \to \overline{\mathcal{D}}_n(\Gamma)$ to lift this field to a discrete gradient vector field on $\mathrm{Conf}_n(\Gamma)$. We could not find the following result in the standard literature, so we prove it for ourselves.

Proposition 2.2.3. *If X is a finite cubical complex, with finite covering map $\pi : \widetilde{X} \to X$, and V is a discrete gradient vector field on X, then there exists a discrete gradient vector field $\widetilde{V}$ on $\widetilde{X}$ which is the lift of V.*

Proof. Let K_i be the set of open i-cells of X as before, and let $\widetilde{A}_i$ be the subset of $\widetilde{K}_i$ on which $V_i \circ \pi$ is defined. Then for any $\widetilde{\alpha} \in \widetilde{A}_i$, $V_i \circ \pi(\widetilde{\alpha})$ is an open cell β of which α is a regular face. Since the closed $(i+1)$-ball is simply connected, there is a unique lift of the attaching map of β such that $\widetilde{\alpha}$ is in its image. Then we define $\widetilde{V}_i(\widetilde{\alpha})$ to be the restriction of this lifted map to the open $(i+1)$-ball, which is some $\widetilde{\beta} \in \widetilde{K}_{i+1}$.

Then $\widetilde{V}$ is a sequence of partial functions $\widetilde{V}_i$. Each $\widetilde{V}_i$ must be injective, because if two regular faces of a cell both map to that cell in the cover, this implies that they map to the same cell in the base, so they must be the same face. Similarly, $\widetilde{V}_i(\widetilde{\alpha}) = \widetilde{\beta}$ implies $\widetilde{\alpha}$ is a regular face of $\widetilde{\beta}$, and $im(\widetilde{V}_i) \cap dom(\widetilde{V}_{i+1}) = \emptyset$. Therefore $\widetilde{V}$ is the discrete gradient vector field on $\widetilde{X}$ which is the lift of V. $\square$

The discrete gradient vector field V from [FS05] and [Far06a] is defined in the following manner: choose a maximal tree T in Γ. Embed this tree in $\mathbb{R}^2$, and pick a vertex $\star$ of valence 1 in T to be the root. Starting at $\star$, walk along the tree, following the leftmost branch at any intersection, and consecutively number the vertices in the order in which they are first encountered. Turn around when you reach a vertex of valence 1 and continue numbering from the next leftmost branch. Let $i(e)$ and $t(e)$ denote the endpoints of the edge e, where $i(e) \geq t(e)$ in the ordering on vertices. For a vertex $v \neq \star$, let $e(v)$ denote the unique edge of T incident with v and closest to $\star$.

Let $\tau = \{\tau_1, \ldots, \tau_{n-1}, v\}$ be a cell in $\overline{\mathcal{D}}_n(\Gamma)$, which we can think of as a set of edges and vertices because the configuration space is unordered. If $e(v) \cap \tau_i = \emptyset$ for $i = 1, \ldots, n-1$, then define the cell $\{\tau_1, \ldots, \tau_{n-1}, e(v)\}$ to be the **elementary reduction** of τ from v, and we say that v is **unblocked** in τ. Otherwise, there is some τ_i with $e(v) \cap \tau_i \neq \emptyset$, and we say

v is **blocked** in τ by τ_i. If v is the smallest unblocked vertex of τ in the order on vertices, then the reduction from v is **principal**.

We define V on $\overline{\mathcal{D}}_n(\Gamma)$ inductively: if τ is a 0-cell, let $V_0(\tau)$ be the principal reduction of τ if it exists. For $i > 0$, let $W_i(\tau)$ be its principal reduction if it exists and $\tau \notin imW_{i-1}$.

An edge $e \in \tau$ is called **order-respecting** in τ if $e \in T$ and $e(v) \cap e = t(e)$ implies $v > i(e)$ for every vertex $v \in \tau$. An order-respecting edge in τ is **minimal** if $i(e)$ is minimal among the initial vertices of order-respecting edges in τ.

Theorem 2.2.4 (Thm 3.6 of [FS05]).

- *A cell is critical if and only if it contains no order-respecting edges and all of its vertices are blocked*
- *A cell is redundant if and only if it*
  - *it contains no order-respecting edges and at least one of its vertices is unblocked OR*
  - *it contains an order-respecting edge and there is some unblocked vertex v such that $v < i(e)$*
- *A cell is collapsible if and only if it contains an order-respecting edge and, for any $v < i(e)$, v is blocked*

We then have a classification of cells in $\mathcal{D}_n(\Gamma)$ with respect to $\widetilde{V}$, where $\widetilde{\tau}$ is critical, redundant, or collapsible if and only if τ is. For the embedded tree T with ordered vertices in Figure 3, examples of a critical, redundant, and collapsible cell with respect to $\widetilde{V}$ are shown in Figures 4-6.

For the remainder of the section, we assume that $n = 2$ and Γ is a tree T.

Proposition 2.2.5. *Let $M_i(Conf_2(T))$ be the critical i-cells of $\mathcal{D}_n(\Gamma)$ with respect to $\widetilde{V}$, $M_i(\overline{Conf_2}(T))$ the critical i-cells of $\overline{\mathcal{D}}_n(\Gamma)$ with respect to V. Then they are both trivial for $i \geq 2$.*

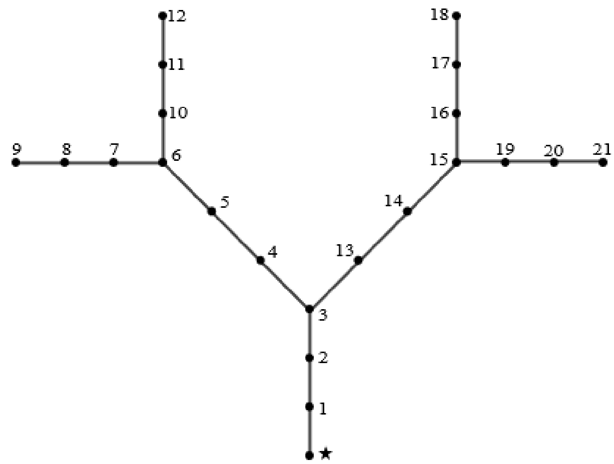

FIGURE 3 A tree T, embedded in $\mathbb{R}^2$ with ordered vertices

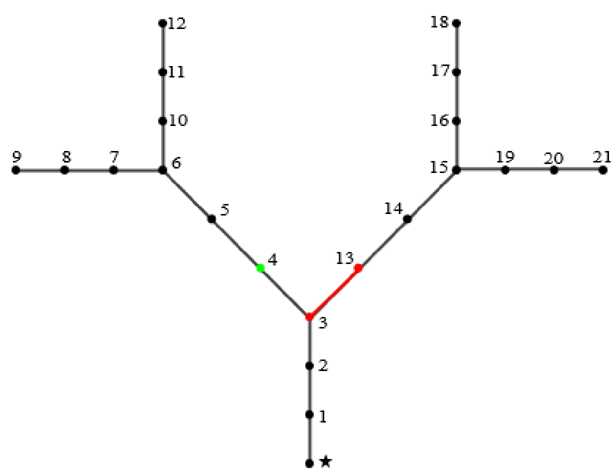

FIGURE 4 The critical cell $(v_4, [v_3, v_{13}])$. The edge $[v_3, v_{13}]$ is not order-respecting in this cell, and the vertex v_4 is blocked.

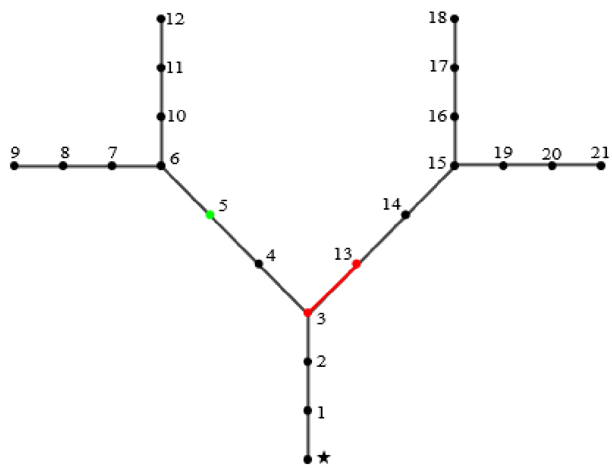

FIGURE 5 The redundant cell $(v_5, [v_3, v_{13}])$. The edge $[v_3, v_{13}]$ is not order-respecting in this cell, but the vertex v_5 is unblocked.

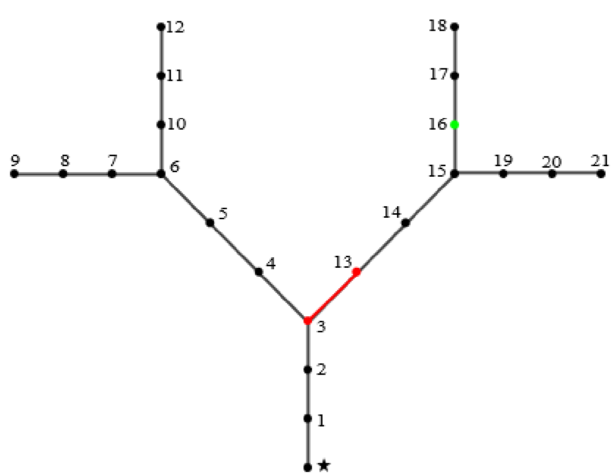

FIGURE 6 The collapsible cell $(v_{16}, [v_3, v_{13}])$. The edge $[v_3, v_{13}]$ is order-respecting in this cell, and $v_{16} > i([v_3, v_{13}])$.

Proof. There are no critical cells of dimension > 2, because there are no cells of degree > 2. All the cells of degree 2 are collapsible, because there are no vertices in the cell, so all edges are automatically order-respecting. $\square$

Proposition 2.2.6. *$M_0(Conf_2(T)) \cong \mathbb{Z}[(v_\star, v_1)] \oplus \mathbb{Z}[(v_1, v_\star)]$ and $M_0(\overline{Conf}_2(T)) \cong \mathbb{Z}[\{v_\star, v_1\}]$*

Proof. There are only two critical 0-cells in the ordered case, because there are only two ways for the vertices to be blocked in a cell with no edges; one of the points must be at the root and the other must be at the first vertex after the root. In the unordered case there is a single such configuration.

$\square$

Proposition 2.2.7. *$M_1(Conf_2(T)) \cong \mathbb{Z}^k$ where $k = \sum_{\substack{v \text{ s.t.} \\ \mu(v)\geq 3}} (\mu(v) - 1)(\mu(v) - 2)$ and $M_1(\overline{Conf}_2(T)) \cong \mathbb{Z}^{k'}$ where $k' = \frac{k}{2}$*

Proof. To construct a critical 1-cell, we need an essential vertex v', an edge e which terminates at that vertex, and another vertex v which we can place so that e is not order-respecting and v is also blocked in the cell. If the essential vertex has valence μ, then there are $(\mu - 1)(\mu - 2)$ critical cells resulting from arrangements around that vertex: there are $\mu - 1$ edges which terminate at v', and one of these is necessarily order-respecting because it is the first edge to the left from v'. Therefore we must pick from the remaining $\mu - 2$ edges to make a critical cell. If we pick the second edge from the left, there is one place to put v so that the second edge is not order-respecting and v is also blocked, which is at the initial vertex of the first edge. If we pick the third edge, there are now two places to put v so that the cell is critical, and so forth. These are triangular numbers, and the number of choices for e and v is the $(\mu - 2)$th triangular number, which is $\dfrac{(\mu-2)(\mu-1)}{2}$. There are two critical cells for each choice of e and v in the ordered case, so there are $\displaystyle\sum_{\substack{v \text{ s.t.} \\ \mu(v)\geq 3}} (\mu(v) - 1)(\mu(v) - 2)$ critical 1-cells in $\mathcal{D}_n(\Gamma)$ and half as many in $\overline{\mathcal{D}}_n(\Gamma)$. $\square$

Theorem 2.2.8. *When $n = 2$ and the graph is a tree T, there is exactly one non-trivial differential in the resulting Morse chain complex M_*, $d_1 : M_1 \to M_0$, and so*

$$H_1(\mathit{Conf}_2(T)) \simeq H^1(\mathit{Conf}_2(T)) \simeq \mathbb{Z}^k$$

$$\textit{where} \qquad k = -1 + \sum_{\substack{v \ s.t. \\ \mu(v) \geq 3}} (\mu(v) - 1)(\mu(v) - 2)$$

$$H_1(\overline{\mathit{Conf}}_2(T)) \simeq H^1(\overline{\mathit{Conf}}_2(T)) \simeq \mathbb{Z}^{k'}$$

$$\textit{where } k' = \frac{1}{2} \sum_{\substack{v \ s.t. \\ \mu(v) \geq 3}} (\mu(v) - 1)(\mu(v) - 2)$$

$\mu(v)$ is the valence of the vertex v, and all other (co)homology groups are trivial.

Proof. The result for unordered configurations follows directly from the fact that every critical 1-cell must be in the kernel of d_1, and the universal coefficient theorem. For ordered configurations, M_1 and M_0 are free abelian groups, and the image of d_1 is $\mathbb{Z}[(v_\star, v_1) - (v_1, v_\star)]$, so the kernel of d_1 must have rank one lower than M_1, which implies the theorem above. □

Though this is a new approach to the above result, it is not the first. The theorem follows from the conclusions of [BF09], and agrees with the results of [Far06a] and [CD14].The paper [MS16] also uses the decomposition of a tree graph into star subgraphs to obtain a closed-form formula for the ranks of $H_i(\overline{\mathrm{Conf}}_n(T))$.

CHAPTER III

TOPOLOGY OF CONFIGURATIONS OF TWO OR THREE POINTS

For the bulk of this chapter, we will focus on trees with a single central vertex and three or four branches, which we call Y and X or the tree with two essential vertices, each of valence 3, which we call H. We call any of these trees T nonspecifically, and we consider configurations of two or three points on these graphs, with a more general result for configurations of two points on any tree proven at the very end. For configurations of two points, there are no (co)homology classes in degrees higher than 1 by Thm 2.2.8. The branches of the trees Y, X, or H are labeled A through C, A through D, or A through E respectively.

3.1. Homology Classes

There are three basic paths in the configuration space which represent homology classes. In the ordered case, start with two points somewhere along the A branch, with the first point further out than the second. Let V_{ABC} be the image of $S^1 \hookrightarrow \mathrm{Conf}_2(T)$ which shuffles the two points around the central vertex from the A branch to the B to the C and back to the A, so that they switch their order, and then shuffling again from A to B to C, so that they return to their starting configuration. This path is illustrated in Figure 7. Then $\gamma_{ABC} = [V_{ABC}] \in H_1(\mathrm{Conf}_2(T))$. The unordered version of this class, denoted $\overline{\gamma}_{ABC}$, is similar, except the points only need to be shuffled once around the central vertex to return to where they started. Notice that $\gamma_{ABC} = \gamma_{BCA} = \gamma_{CAB}$ and $\gamma_{ABC} = -\gamma_{ACB}$, and similar equalities hold for the ordered version. When we have at least four branches around a central vertex, we can do the same shuffling around on four branches, and in the case of ordered configurations we will only need to go around the central vertex once to return

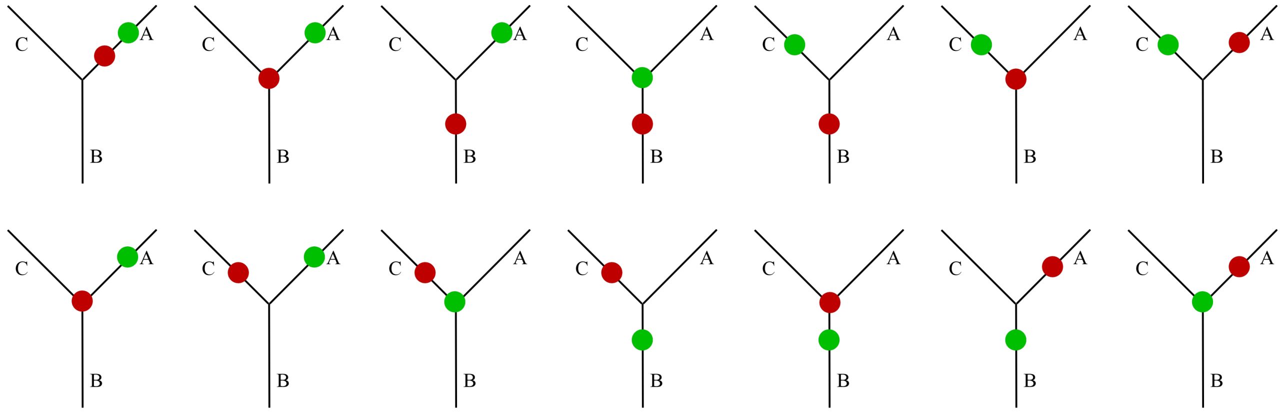

FIGURE 7 A path in $\mathrm{Conf}_2(T)$ represeting the class γ_{ABC}. The path starts in the upper left corner and continues clockwise.

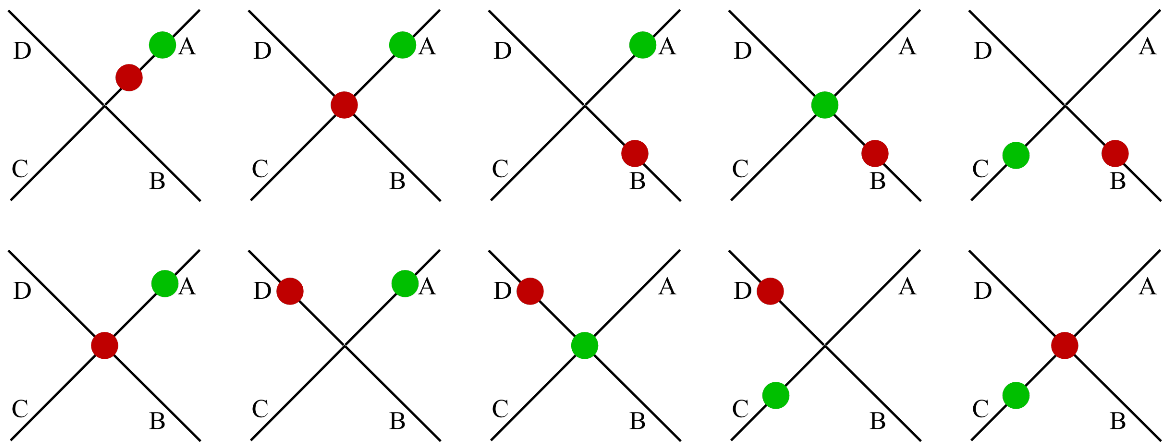

FIGURE 8 A path in $\mathrm{Conf}_2(T)$ represeting the class μ_{ABCD}. The path starts in the upper left corner and continues clockwise.

to the starting configuration. If we start from a configuration with both points on the A branch and the first point farther from the center than the second, and shuffle them from A onto B, C, D and return to A then this path represents the class $\mu_{ABCD} \in H_1(\mathrm{Conf}_2(T))$, illustrated in Figure 8. Notice that $\overline{\mu}_{ABCD} = \overline{\gamma}_{ABC} + \overline{\gamma}_{ACD}$.

When there are at least two essential vertices with an edge between them, we can shuffle the two points back and forth along this edge. Let $T_{AB,ED}$ be the image of $S^1 \hookrightarrow \mathrm{Conf}_2(T)$ which moves the first point from branch A to E, then the second point from B to D, then the first back to A, then the second back to B as in the sequence of pictures in Figure 9. Then $\tau_{AB,DE} = [T_{AB,DE}] \in H_1(\overline{\mathrm{Conf}}_2(T))$. The unordered version of this class, denoted $\overline{\tau}_{AB,ED}$ is the same path where the labels of the points have been forgotten.

3.2. Cohomology Classes

$\mathrm{Conf}_n(T)$ is not, in general, a manifold, but it is a polytope, and so it has a filtration $\emptyset = X_{-1} \subset X_0 \subset \ldots \subset X_n = \mathrm{Conf}_n(T)$ by closed subspaces such that $X_i \backslash X_{i-1}$ is a disjoint union of i-dimensional manifolds. We call the closure of one of these manifolds a **facet**. We say that two subspaces of a polytope are **transversal** if the sum of their tangent spaces at every point in their intersection gives the tangent space of the facet containing that point.

We will represent a cohomology class with a union of codimension one co-oriented submanifolds of facets that are transversal to the homology classes defined in the previous section. Let W be a collection of codimensional one submanifolds (with boundary) of facets such that for every submanifold V in the collection, and for every component of ∂V which does not intersect the boundary of the polytope, then for every facet f which contains a component and every facet F with $f \in \partial F$, there is a codimensional one submanifold V' of F with $\partial V' \cap f = \partial V \cap f$. Then W is a closed, possibly singular subspace and we call it a **codimension one subvariety**.

Proposition 3.2.1. *A codimension one subvariety of a polytope X is a cocyle on the subspace of chains transverse to each stratum of the polytope.*

Proof. Let W be such a subvariety. It is enough to show that for every transverse $\sigma : \Delta^2 \to X$, $\sigma^{-1}(W)$ is the disjoint union of piecewise smooth one-manifolds.

Any point x in $X_n \backslash X_{n-1}$ has a neighborhood V such that $W \cap V$ is a codimension one submanifold of V. Therefore by transversality $\sigma^{-1}(X \cap V)$ is a codimension one submanifold

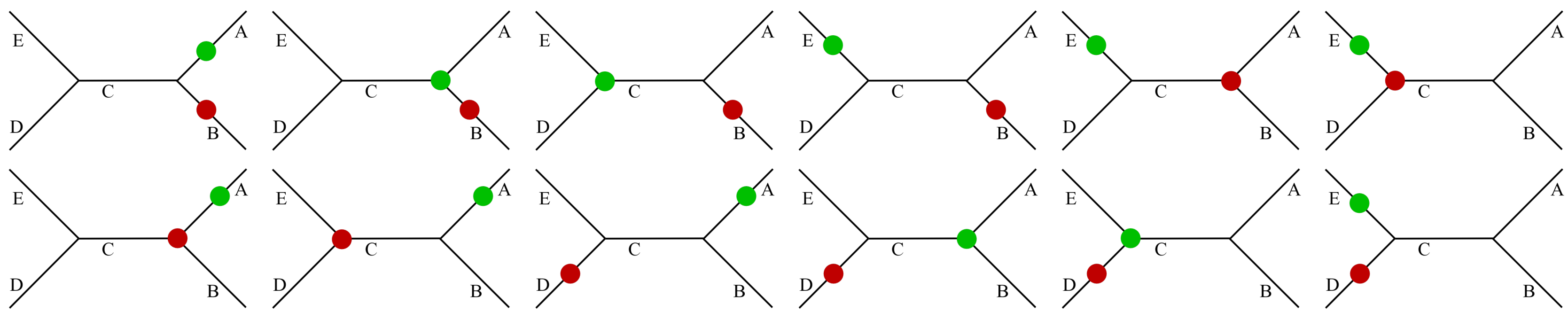

FIGURE 9 A path in $\mathrm{Conf}_2(T)$ represeting the class $\tau_{AB,ED}$. The path starts in the upper left corner and continues clockwise.

of Δ^2. For a point in $X_{n-1} \cap W$, we will show that it abuts exactly two codimension one submanifolds.

First note that $\sigma^{-1}(X_{n-1} \cap W)$ is the intersection of $\sigma^{-1}(X_{n-1})$ and $\sigma^{-1}(W)$. By transversality, $\sigma^{-1}(X_{n-1} \backslash X_{n-2})$ is a codimension one submanifold of Δ^2. Its complement is preimages of the codimension zero facets, and thus are codimension zero submanifold of Δ^2. At any point in $\sigma^{-1}(X_{n-1})$, only two such manifolds can have that point in their closure. Thus $\sigma^{-1}(X_{n-1} \cap W) \subset \sigma^{-1}(W)$ is a collection of points, each of which is the boundary of exactly two one-manifolds, the preimages of W intersected with the two abutting facets. Together, this makes $\sigma^{-1}(W)$ a piecewise smooth one-manifold. $\square$

Now we define a basic codimension one subvariety of $\mathrm{Conf}_n(T)$, which we call a detector:

Let e_i and e_j be edges in T which share a common endpoint. Let $W_{e_ie_j}$ be the subspace of $\mathrm{Conf}_2(T)$ such that the first point is on the edge e_i, the second point is on the edge e_j, and they are both equidistant from the common endpoint. This is a codimension one subvariety, and so with a co-orientation it represents a cohomology class. The normal bundle to $W_{e_ie_j}$ inside $\mathrm{Conf}_2(T)$ is oriented by assigning a positive orientation to configurations where the first point is moving closer to the center vertex faster than the second point, or where the second point is moving further from the center vertex faster than the first, and the negative orientation to the opposite scenarios. Then let $\eta_{e_ie_j} \in H^1(\mathrm{Conf}_2(T))$ be the resulting cohomology class represented by $W_{e_ie_j}$ with this orientation. The unordered version is denoted $\overline{\eta}_{e_ie_j}$. By convention $\overline{\eta}_{e_ie_j} = -\overline{\eta}_{e_je_i}$, as both classes refer to the subspace with a point on e_i and a point on e_j but with opposite orientations on the normal bundle, while $\eta_{e_ie_j}$ and $\eta_{e_je_i}$ refer to different underlying subspaces.

3.3. Presentations for Unordered Configurations

These are new results for the homology of unordered configuration spaces of the graph H. Presentations for the cohomology ring $H^*(\overline{\mathrm{Conf}}_n(T))$ in terms of 'cloud pictures' were given in [Far06b], but they are in terms of a different basis, and we don't yet understand how the two are related. By calculating the oriented intersection of homology and cohomology classes, we can demonstrate pairings which give a lower bound for the ranks of these groups. We obtain an upper bound from Theorem 2.2.8. In this section, the given group presentations realize this upper bound unless otherwise noted.

$\overline{\mathrm{Conf}}_2(Y)$ is homotopy equivalent to S^1, and going around this circle is equivalent to our three-branch-shuffling [Ghr01], so $H_1(\overline{\mathrm{Conf}}_2(Y)) \cong \mathbb{Z}$ is generated by $\overline{\gamma}_{ABC}$ and by choosing the appropriate orientation we have the pairing

$$\langle \overline{\eta}_{AB}, \overline{\gamma}_{ABC} \rangle = 1$$

The four inclusions $Y \hookrightarrow X$ induce homology maps whose images span all of $H_1(\overline{\mathrm{Conf}}_2(X))$, with the relation

$$\overline{\gamma}_{ABC} + \overline{\gamma}_{ACD} = \overline{\gamma}_{ABD} + \overline{\gamma}_{BCD}$$

There are six cohomology classes, whose pairings with the given homology classes are given in Table 1. Each of these pairings is straightforward to verify by considering the oriented intersection of the submanifolds representing $\overline{\gamma}$ and $\overline{\eta}$. In particular, $\langle \overline{\eta}_{AC}, \overline{\gamma}_{ABC} \rangle = -1$ because the point on the branch C is approaching the central vertex of X while the point on A is moving away when they are both equidistant from the central vertex. Relations

TABLE 1 Evaluation Pairings for $\overline{\mathrm{Conf}}_2(X)$

	$\overline{\gamma}_{ABC}$	$\overline{\gamma}_{ABD}$	$\overline{\gamma}_{ACD}$	$\overline{\gamma}_{BCD}$
$\overline{\eta}_{AB}$	1	1	0	0
$\overline{\eta}_{AC}$	-1	0	1	0
$\overline{\eta}_{AD}$	0	-1	-1	0
$\overline{\eta}_{BC}$	1	0	0	1
$\overline{\eta}_{BD}$	0	1	0	-1
$\overline{\eta}_{CD}$	0	0	1	1

between the cohomology classes are as follows:

$$\overline{\eta}_{AB} + \overline{\eta}_{AC} + \overline{\eta}_{AD} = 0$$

$$\overline{\eta}_{BA} + \overline{\eta}_{BC} + \overline{\eta}_{BD} = 0$$

$$\overline{\eta}_{CA} + \overline{\eta}_{CB} + \overline{\eta}_{CD} = 0$$

$$\overline{\eta}_{DA} + \overline{\eta}_{DB} + \overline{\eta}_{DC} = 0$$

The fourth equation is linearly dependent on the first three, so this is a presentation for $H^1(\overline{\mathrm{Conf}}_2(X)) \cong \mathbb{Z}^3$ and the pairings realize this rank. Paths in the configuration space which represent non-trivial homology classes are made up of sequences of branch-switching moves, such as when one point is stationary on the A branch, and the other point starts on the B branch and moves off, either onto the C or D branch, and these relations arise precisely from such moves.

The two inclusions $Y \hookrightarrow H$ induce homology maps whose images span all of $H_1(\overline{\mathrm{Conf}}_2(H))$, with the relation

$$\overline{\gamma}_{ABC} - \overline{\gamma}_{CDE} = \overline{\tau}_{AB,ED}$$

TABLE 2 Evaluation Pairings for $\overline{\mathrm{Conf}}_2(H)$

	$\overline{\gamma}_{ABC}$	$\overline{\gamma}_{CDE}$	$\overline{\tau}_{AB,ED}$
$\overline{\eta}_{AB}$	1	0	1
$\overline{\eta}_{AC}$	-1	0	-1
$\overline{\eta}_{BC}$	1	0	1
$\overline{\eta}_{CD}$	0	1	-1
$\overline{\eta}_{CE}$	0	-1	1
$\overline{\eta}_{DE}$	0	1	-1

There are six cohomology classes, whose pairings with the given homology classes are given by Table 2. Relations for cohomology classes are as follows:

$$\overline{\eta}_{AB} = \overline{\eta}_{BC} = -\overline{\eta}_{AC}$$

$$\overline{\eta}_{CD} = \overline{\eta}_{DE} = -\overline{\eta}_{CE}$$

so these cohomology classes are span $H^1(\overline{\mathrm{Conf}}_2(H) \cong \mathbb{Z}^2$ and the pairings realize this rank.

We define maps $\overline{\mathrm{Conf}}_2(T) \to \overline{\mathrm{Conf}}_3(T)$ by adding a new point to a configuration at one of the extremal vertices of the graph, after first 'pushing in' any points from the configuration on that edge. In the case of $\overline{\mathrm{Conf}}_3(Y)$, the three places to add in a new point induce three maps on homology, and the three images of $\overline{\gamma}_{ABC}$, which look like $\overline{\gamma}_{ABC}$ with an extra point 'parked' near the end of one of the three branches, span $H_1(\overline{\mathrm{Conf}}_3(Y))$ linearly independently. We will denote these classes $\overline{\gamma}_{A,ABC}$ $\overline{\gamma}_{B,ABC}$ and $\overline{\gamma}_{C,ABC}$ for the edge at which the extra point is parked. Let $U_{A,AB}$ be the codimension one subvariety of $\mathrm{Conf}_3(Y)$ such that two of the points are equidistant from the central vertex along the A and B branches, and the last point is somewhere further out along the A branch. Orient the normal bundle by assigning a positive orientation to configurations where, of the two nearly equidistant points, the one on the A branch is moving closer to the center vertex faster than the point on the B branch, or where the point on B is moving further from the center vertex faster than the (nearly equidistant) point on A, and the negative orientation to the opposite scenarios. Denote by $\overline{\eta}_{A,AB} \in H^1(\overline{\text{Conf}}_3(Y)$ the cohomology class represented by $U_{A,AB}$ with this orientation. Similarly, we define $\overline{\eta}_{B,AB}$ $\overline{\eta}_{B,BC}$ $\overline{\eta}_{C,BC}$ etc. We have the pairings as given in Table 3. Relations for cohomology classes are as follows:

TABLE 3 Evaluation Pairings for $\overline{\text{Conf}}_3(Y)$

	$\overline{\gamma}_{A,ABC}$	$\overline{\gamma}_{B,ABC}$	$\overline{\gamma}_{C,ABC}$
$\overline{\eta}_{A,AB}$	1	0	0
$\overline{\eta}_{B,AB}$	0	1	0
$\overline{\eta}_{A,AC}$	-1	0	0
$\overline{\eta}_{C,AC}$	0	0	-1
$\overline{\eta}_{B,BC}$	0	-1	0
$\overline{\eta}_{C,BC}$	0	0	1

$$\overline{\eta}_{A,AB} + \overline{\eta}_{A,AC} = \overline{\eta}_{B.AB} + \overline{\eta}_{B,BC} = \overline{\eta}_{C,AC} + \overline{\eta}_{C,BC} = 0$$

so these classes span $H^1(\overline{\text{Conf}}_3(Y)) \simeq \mathbb{Z}^3$ and their pairings with homology realize this rank.

The four extremal points which we can add to configurations in X induce four maps on homology $H_1(\overline{\text{Conf}}_2(X)) \to H_1(\overline{\text{Conf}}_3(X))$ which span $H_1(\overline{\text{Conf}}_3(X))$, and the image of the relation in $H_1(\overline{\text{Conf}}_2(X))$ is also a relation in $H_1(\overline{\text{Conf}}_3(X))$, so there are at most 12 linearly independent homology classes. Notice that this is the same image as the maps induced by graph inclusion $H_1(\overline{\text{Conf}}_3(Y)) \to H_1(\overline{\text{Conf}}_3(X))$. In addition, we have

$$\overline{\gamma}_{A,ABC} - \overline{\gamma}_{A,ABD} + \overline{\gamma}_{A,ACD} - \overline{\gamma}_{B,BCD} + \overline{\gamma}_{B,BCA} - \overline{\gamma}_{B,BDA} + \overline{\gamma}_{C,CDA} - \overline{\gamma}_{C,CDB} + \overline{\gamma}_{C,CBA} - \ldots = 0$$

which spans $H_1(\overline{\text{Conf}}_3(X)) \simeq \mathbb{Z}^{11}$. Let $R_{*,AB}$ be the codimension one subvariety of $\overline{\text{Conf}}_3(X)$ such that two of the points are equidistant from the central vertex along the A and B branches, respectively, and the third point is somewhere along the C or D branches or at the central vertex or along the A or B branches closer to the central vertex than the equidistant points. Orient the normal bundle by assigning a positive orientation to configurations where, of two nearly equidistant points along the A and B branches, the one on the A branch is moving closer to the center vertex faster than the point on the B branch, or where the point on B is moving further from the center vertex faster than the (nearly equidistant) point on A, and the negative orientation to the opposite scenarios. Denote by $\overline{\eta}_{*,AB} \in H^1(\overline{\mathrm{Conf}_3}(X))$ the cohomology class represented by $R_{*,AB}$ with this orientation. We similarly define the classes $\overline{\eta}_{*,AC}$ $\overline{\eta}_{*,BC}$ etc as well as the classes of the form $\overline{\eta}_{A,AB}$ as described previously. We have the relations

$$\overline{\eta}_{A,AB} + \overline{\eta}_{A,AC} + \overline{\eta}_{A,AD} = 0$$

$$\overline{\eta}_{B,AB} + \overline{\eta}_{B,BC} + \overline{\eta}_{B,BD} = 0$$

$$\overline{\eta}_{C,AC} + \overline{\eta}_{C,BC} + \overline{\eta}_{C,CD} = 0$$

$$\overline{\eta}_{D,AD} + \overline{\eta}_{D,BD} + \overline{\eta}_{D,CD} = 0$$

as well as

$$\overline{\eta}_{*,AB} + \overline{\eta}_{*,AC} + \overline{\eta}_{*,AD} + \overline{\eta}_{B,AB} + \overline{\eta}_{C,AC} + \overline{\eta}_{D,AD} = 0$$

and its three permutations by switching the branch labels, though the last is linearly dependent on the other relations. Then 18 detector classes minus 7 relations gives a subgroup of rank at most 11. The classes pair as given in Table 4, and so the cohomology classes span $H^1(\overline{\mathrm{Conf}_3}(X)) \simeq \mathbb{Z}^{11}$ and we have presentations for the homology and cohomology groups which realize this rank.

3.4. Presentations for Ordered Configurations

These are new results for (co)homology of ordered configuration spaces of the graphs Y, X, and H. As in the previous section, we can demonstrate pairings which give a lower bound for the ranks of these groups, with an upper bound for configurations of two points from Theorem 2.2.8. The group presentations in this section realize this upper bound unless otherwise noted.

$\mathrm{Conf}_2(Y)$ is homotopy equivalent to S^1, and going around the circle is equivalent to our ordered three-branch shuffling, so $H_1(\mathrm{Conf}_2(Y)$ is generated by γ_{ABC} and we have the pairing

$$\langle \eta_{AB}, \gamma_{ABC} \rangle = 1$$

The four inclusions $Y \hookrightarrow X$ induce homology maps, but their images do not span all of $H_1(\mathrm{Conf}_2(X))$. Instead, the classes μ_{ABCD}, μ_{ACBD}, μ_{ACDB}, μ_{BCDA}, μ_{CBDA}, μ_{CDBA} span with the relations

$$\mu_{ABCD} - \mu_{ACBD} + \mu_{ACDB} = \mu_{BCDA} - \mu_{CBDA} + \mu_{CDBA}$$

$$\mu_{ABCD} - \mu_{ACBD} - \mu_{CDBA} = \gamma_{ABC}$$

$$\mu_{ABCD} + \mu_{CBDA} - \mu_{CDBA} = \gamma_{ABD}$$

$$\mu_{ABCD} + \mu_{ACDB} + \mu_{CBDA} = \gamma_{ACD}$$

$$\mu_{BCDA} - \mu_{CBDA} + \mu_{CDBA} = \gamma_{BCD}$$

$$\gamma_{ABC} + \gamma_{ACD} = \gamma_{ABD} + \gamma_{BCD}$$

where the last relation is a linear combination of the first five, so we have a subgroup of rank at most 5. The 12 η cohomology classes with all their permutations span $H^1(\mathrm{Conf}_2(X)) \simeq \mathbb{Z}^5$ and pair with the homology classes as in Table 5. We have the relations

$$\eta_{AB} + \eta_{AC} + \eta_{AD} = 0$$

and its seven permutations by switching branch labeling and point order. The eighth relation is linearly dependent on the first seven, so likewise these cohomology classes span$H^1(\overline{\mathrm{Conf}}_2(X)) \cong \mathbb{Z}^5$.

The two inclusions (up to automorphism) of $Y \hookrightarrow H$ induce homology maps, but their images do not span all of $H_1(\mathrm{Conf}_2(H))$. Instead, the classes $\tau_{AB,DE}$, $\tau_{AB,ED}$, $\tau_{BA,DE}$, and $\tau_{BA,ED}$ along with γ_{ABC} and γ_{CDE} span with the relations

$$\gamma_{ABC} = \tau_{AB,ED} + \tau_{BA,ED}$$

$$\gamma_{ABC} = \tau_{AB,DE} + \tau_{BA,DE}$$

$$\gamma_{CDE} = \tau_{BA,ED} - \tau_{BA,DE}$$

$$\gamma_{CDE} = \tau_{AB,DE} - \tau_{AB,ED}$$

The last relation is linearly dependent on the first three, so these form a subgroup of rank at most 3. The 12 η cohomology classes span $H^1(\mathrm{Conf}_2(H))$ and pair with homology as in Table 6.

We have the relations

$$\eta_{AB} + \eta_{AC} = 0$$

and its eleven permutations by switching branch labeling and point order. The last nine are dependent on the first three, so these classes span $H^1(\mathrm{Conf}_2(H)) \cong \mathbb{Z}^3$ and the (co)homology classes with relations realize the correct rank.

Adding a new labeled point at an extremal vertex to a configuration in Y gives us nine maps $H_1(\mathrm{Conf}_2(Y)) \to H_1(\mathrm{Conf}_3(Y))$ for each of the three branches of the graph and each of the three ways to give the new point a number. These are analogous to the stability maps for configurations in an open manifold, where new points are 'pushed in' from infinity. The images of γ_{ABC}, which we will denote $\gamma_{1A,ABC}$, $\gamma_{1B,ABC}$, etc and refer to collectively as the image of stability maps, do not span $H_1(\mathrm{Conf}_3(Y))$.

The motivation for a different map $H_1(\mathrm{Conf}_2(Y)) \to H_1(\mathrm{Conf}_3(Y))$ comes from the little-1-disks operad: given a configuration of two points in Y, take a neighborhood around one of the points, remove it and glue in a configuration of two points in $(0,1)$. If we are careful when a point is at the central vertex to pick a neighborhood that coincides with the movement of the point through the central vertex, then we will get a path in $\mathrm{Conf}_3(Y)$ which comes from the path in $\mathrm{Conf}_2(Y)$, where now one of the points is shuffling around the central vertex with a pair of points that move as a unit, always keeping some fixed distance between them.

Remark 3.4.1. *The space of framed embeddings of* $\sqcup_n[0,1]$ *into* Γ *is homotopy equivalent to* $\mathit{Conf}_n(\Gamma)$*.*

Specifically, let $S_{1,23ABC}$ be a path in $\mathrm{Conf}_3(Y)$ which starts with all three points on the A branch, with 3 the closest to the central vertex, then 2, then 1, and shuffles 1 with 2 and 3 as a unit, maintaining the distance between them, as in the path representing γ_{ABC}. Then $\gamma_{1,23ABC} = [S_{1,23ABC}] \in H_1(\mathrm{Conf}_3(Y))$. We can think of this class as γ_{ABC} except with two points 'inserted' in the place of the old second point. By convention, $\gamma_{1,23BCA}$ is represented by a path which starts with all three points on the B branch, and then shuffles

1 with 2 and 3 as a unit, so unlike in previous cases there is no reason a priori to believe that $\gamma_{1,23ABC}$ and $\gamma_{1,23BCA}$ are the same class. There are 18 of these classes, which we refer to collectively as insertion classes, and together the parking and insertion classes span $H_1(\mathrm{Conf}_3(Y))$. We have the relations

$$\gamma_{1,23ABC} + \gamma_{1A,ABC} + \gamma_{3B,ABC} + \gamma_{2A,ABC} + \gamma_{3C,ABC} = \gamma_{2,13ABC}$$

and its five permutations,

$$\gamma_{1,23ABC} + \gamma_{1A,ABC} + \gamma_{1.32ABC} = \gamma_{1,23BCA} + \gamma_{1B,BCA} + \gamma_{1,32BCA} = \gamma_{1,23CAB} + \gamma_{1C,CAB} + \gamma_{1,32CAB}$$

and its two permutations,

$$\gamma_{2,13ABC} + \gamma_{1,32CAB} = \gamma_{2,13BCA}$$

and its five permutations. The last four are linearly dependent on previous relations, so 27 classes with $6 + 6 + 6 - 4 = 14$ relations give a subgroup of rank at most 13. There are 36 cohomology detector classes, as there are six in $H^1(\overline{\mathrm{Conf}_3}(Y))$ and six ways to assign labels to each one. The intersection matrix is too large to show in full, so we show a block of the matrix in Table 7 that determines the rest of the matrix via permutation of the labels on the points and automorphisms of the graph Y.

We have the relations

$$\eta_{1A,AB} + \eta_{1A,AC} = 0$$

$$\eta_{1A,BA} + \eta_{1A,CA} = 0$$

and their 16 permutations by switching branch labeling and point order, and

$$\eta_{1A,AB}+\eta_{1A,BA}+\eta_{2A,AB}+\eta_{2A,BA} = \eta_{1B,AB}+\eta_{1B,BA}+\eta_{2B,AB}+\eta_{2B,BA} = \eta_{1C,AC}+\eta_{1C,CA}+\eta_{2C,AC}+\eta_{2C,CA}$$

and its two permutations by switching point order. Only five of these relations are linearly independent, so $H^1(\mathrm{Conf}_3(Y)) \simeq \mathbb{Z}^{13}$ is spanned by the cohomology detector classes, as $H_1(\mathrm{Conf}_3(Y))$ is spanned by the image of the stability and insertion maps, with the given relations realizing the correct rank.

The four directions to push in a new point from the boundary of X induce 12 maps on homology $H_1(\mathrm{Conf}_2(X)) \to H_1(\mathrm{Conf}_3(X))$ and the images of the relations in $H_1(\mathrm{Conf}_2(X))$ are relations in $H_1(\mathrm{Conf}_3(X))$, so we have at most $12 \times 5 = 60$ linearly independent classes from the image of the stability maps. Even with the addition of 'insertion' classes, these do not span $H_1(\mathrm{Conf}_3(X))$. The code in A.2 calculates the rank of $H_1(\mathrm{Conf}_3(X))$ to be 61, and we were able to extract homology classes that are not in the span of the images of stability or insertion maps (according to Sage's linear algebra package). These new homology classes look like elements of the graph braid group on 3 strands: the path starts with the three points on different edges of X, and then moves one point at a time onto the unoccupied edge in cyclic order until the configuration comes around to the starting position again.

We have the relations

$$\mu_{1A,ABCD} + \mu_{2C,CDAB} = \mu_{1,23ABCD}$$

and its permutations which tells us all the 'insertion' classes are sums of the 'parking' classes. There are 108 cohomology detector classes, as there are 18 in $H^1(\overline{\mathrm{Conf}_3}(X))$, but with relations

$$\eta_{1A,AB} + \eta_{1A,AC} + \eta_{1A,AD} = 0$$

$$\eta_{1A,BA} + \eta_{1A,CA} + \eta_{1A,DA} = 0$$

and all their permutations we have at most 84 linearly independent cohomology classes.

$$\eta_{*,AB} + \eta_{*,AC} + \eta_{*,AD} + \eta_{1B,AB} + \eta_{1C,AC} + \eta_{1D,AD} = 0$$

and its 23 permutations, though only 18 of these are linearly independent, so the η classes sit inside $H^1(\mathrm{Conf}_3(X))$ as a group of at most $84-18=66$ linearly independent cohomology classes. We have a block of the intersection matrix as in Table 8 and similar intersection data for all permutations of the labels on the points and automorphisms of the graph X. These form a block of rank at most 60 inside of the intersection matrix.

3.5. Forgetful and Transfer Maps

A first application of explicit representations for homology and cohomology classes are chain and cochain level calculations. By 'forgetful', we mean $f : \mathrm{Conf}_n(T) \to \overline{\mathrm{Conf}}_n(T)$ which forgets the labels on a configuration of points. By 'transfer', we mean $t_* : H_1(\overline{\mathrm{Conf}}_n(T)) \to H_1(\mathrm{Conf}_n(T))$ which sums at the chain level over all possible labels of a configuration.

Since $f_*(\gamma_{ABC}) = 2 \cdot \overline{\gamma}_{ABC}$, we see that $f_* : H_1(\mathrm{Conf}_2(Y)) \simeq \mathbb{Z} \to H_1(\overline{\mathrm{Conf}}_2(Y)) \simeq \mathbb{Z}$ is multiplication by 2.

Since $f^*(\overline{\eta}_{AB})(\gamma_{ABC}) = 2$, $f^* : H^1(\overline{\mathrm{Conf}}_2(Y)) \to H^1(\mathrm{Conf}_2(Y))$ must also be the times 2 map.

Since $f_*(\mu_{ABCD}) = \overline{\gamma}_{ABC} + \overline{\gamma}_{ACD}$ (and similarly for the other μ and μ' classes), we see that the image of $f_* : H_1(\mathrm{Conf}_2(X)) \to H_1(\overline{\mathrm{Conf}}_2(X))$ is generated by $\overline{\gamma}_{ABC} + \overline{\gamma}_{ACD}$, $\overline{\gamma}_{ACB} + \overline{\gamma}_{ABD}$, and $\overline{\gamma}_{ACD} + \overline{\gamma}_{ADB}$. Notice that this is not surjective, but it is a full-rank embedding with cokernel $\mathbb{Z}_2$.

Since

$$f^*(\overline{\eta}_{AB})(\mu_{ABCD}) = f^*(\overline{\eta}_{AB})(\mu'_{ACDB}) = 1$$

while

$$f^*(\overline{\eta}_{AB})(\mu_{ACDB}) = f^*(\overline{\eta}_{AB})(\mu'_{ABCD}) = -1$$

and otherwise evaluates to zero, we see that $f^* : H^1(\overline{\mathrm{Conf}}_2(X)) \to H^1(\mathrm{Conf}_2(X))$ maps $\overline{\eta}_{AB}$ to $\eta_{AB} + \eta_{BA}$, and similarly for other $\overline{\eta}$ classes. Then the image of f^* is generated by these sums of η classes.

Obviously $f_*(\gamma_{A,ABC}) = 2 \cdot \overline{\gamma}_{A,ABC}$ (and similarly for the other permutations), while

$$f_*(\gamma_{1,23ABC}) = 2 \cdot \overline{\gamma}_{A,ABC} + \overline{\gamma}_{B,ABC} + \overline{\gamma}_{C,ABC}$$

$$f_*(\gamma_{1,23BCA}) = \overline{\gamma}_{A,BCA} + 2 \cdot \overline{\gamma}_{B,BCA} + \overline{\gamma}_{C,BCA} = \overline{\gamma}_{A,ABC} + 2 \cdot \overline{\gamma}_{B,ABC} + \overline{\gamma}_{C,ABC}$$

$$f_*(\gamma_{1,23CAB}) = \overline{\gamma}_{A,CAB} + \overline{\gamma}_{B,CAB} + 2 \cdot \overline{\gamma}_{C,CAB} = \overline{\gamma}_{A,ABC} + \overline{\gamma}_{B,ABC} + 2 \cdot \overline{\gamma}_{C,ABC}$$

Therefore the image of $f_* : H_1(\mathrm{Conf}_3(Y)) \to H_1(\overline{\mathrm{Conf}}_3(Y))$ is not surjective, but is generated by $\overline{\gamma}_{A,ABC} + \overline{\gamma}_{B,ABC}$, $\overline{\gamma}_{B,ABC} + \overline{\gamma}_{C,ABC}$, $\overline{\gamma}_{A,ABC} + \overline{\gamma}_{C,ABC}$, $2 \cdot \overline{\gamma}_{A,ABC}$, $2 \cdot \overline{\gamma}_{B,ABC}$, and $2 \cdot \overline{\gamma}_{C,ABC}$.

Since $f^*(\overline{\eta}_{A,AB}) = \eta_{1A,AB} + \eta_{1A,BA} + \eta_{2A,AB} + \eta_{2A,BA} + \eta_{3A,AB} + \eta_{3A,BA}$, and similarly for its permutations, the image of $f^* : H^1(\overline{\mathrm{Conf}}_3(Y)) \to H^1(\mathrm{Conf}_3(Y))$ is generated by such sums.

Since $f_*(\mu_{1A,ABCD}) = f_*(\mu'_{1A,ABCD}) = \overline{\gamma}_{A,ABC} + \overline{\gamma}_{A,ACD}$ and similarly for the other permutations, we have that the image of the parking classes under $f_* : H_1(\mathrm{Conf}_3(X)) \to H_1(\overline{\mathrm{Conf}}_3(X))$ is generated by $\overline{\gamma}_{A,ABC} + \overline{\gamma}_{A,ACD}$, $\overline{\gamma}_{A,ACD} - \overline{\gamma}_{A,ABD}$, $-\overline{\gamma}_{A,ABC} + \overline{\gamma}_{A,ABD}$, and their permutations by switching branch labeling.

Since $f^*(\overline{\eta}_{*,AB}) = \eta_{1*,AB} + \eta_{1*,BA} + \eta_{2*,AB} + \eta_{2*,BA} + \eta_{3*,AB} + \eta_{3*,BA}$, and similarly for its other permutations, the image of $f^* : H^1(\overline{\mathrm{Conf}}_3(X)) \to H^1(\mathrm{Conf}_3(X))$ is generated by expressions of this form as well as $\eta_{1A,AB} + \eta_{1A,BA} + \eta_{2A,AB} + \eta_{2A,BA} + \eta_{3A,AB} + \eta_{3A,BA}$ and its permutations.

Since $t_*(\overline{\gamma}_{ABC}) = \gamma_{ABC}$, we see that $t_* : H_1(\overline{\mathrm{Conf}}_2(Y)) \simeq \mathbb{Z} \to H_1(\mathrm{Conf}_2(Y)) \simeq \mathbb{Z}$ is an isomorphism, which means that $f_* \circ t_*$ is the times two map, which corresponds to the double covering of unordered two-point configuration space by ordered two-point configuration space.

Since $t^*(\eta_{AB})(\overline{\gamma}_{ABC}) = 1$, $t^*(\eta_{AB}) = \overline{\eta}_{AB}$ and $t^* : H^1(\mathrm{Conf}_2(Y)) \simeq \mathbb{Z} \to H^1(\overline{\mathrm{Conf}}_2(Y)) \simeq \mathbb{Z}$ is an isomorphism.

Since $t_*(\overline{\gamma}_{ABC}) = \gamma_{ABC} = \mu_{ABCD} + \mu_{ACBD} - \mu'_{ACDB}$, and similarly for its permutations, with a bit of manipulation we have that the image of $t_* : H_1(\overline{\mathrm{Conf}}_2(X)) \to H_1(\mathrm{Conf}_2(X))$ is generated by $\mu_{ABCD} + \mu'_{ABCD}$, and its permutations.

Since $t^*(\eta_{AB})(\overline{\gamma}_{ABC}) = 1$ and otherwise zero, and similarly for its permutations, we have that $t^*(\eta_{AB}) = \overline{\eta}_{AB}$ and $t^* : H^1(\mathrm{Conf}_2(X)) \to H^1(\overline{\mathrm{Conf}}_2(X))$ is surjective.

Since $t_*(\overline{\gamma}_{A,ABC}) = \gamma_{1A,ABC} + \gamma_{2A,ABC} + \gamma_{3A,ABC}$, and similarly for its permutations, we have that the image of $t_* : H_1(\overline{\mathrm{Conf}}_3(Y)) \to H_1(\mathrm{Conf}_3(Y))$ is generated by such sums. Then $f_* \circ t_*$ is the times three map, which corresponds to the triple covering of unordered three-point configuration space by ordered three-point configuration space.

We have that $t^*(\eta_{1A,AB}) = t^*(\eta_{2A,AB}) = t^*(\eta_{3A,AB}) = \overline{\eta}_{A,AB}$ and similarly for its permutations, so $t^* : H^1(\mathrm{Conf}_3(Y)) \to H^1(\overline{\mathrm{Conf}}_3(Y))$ is surjective.

Since

$$t_*(\overline{\gamma}_{A,ABC}) = \gamma_{1A,ABC} + \gamma_{2A,ABC} + \gamma_{3A,ABC} = \mu_{1A,ABCD} + \mu_{1A,ACBD} - \mu'_{ACDB} + \dots$$

and similarly for its permutations, with a bit of manipulation we have that the image of $t_* : H_1(\overline{\mathrm{Conf}}_3(X)) \to H_1(\mathrm{Conf}_3(X))$ is generated by sums of the form

$$\mu_{1A,ABCD} + \mu'_{1A,ABCD} + \mu_{2A,ABCD} + \mu'_{2A,ABCD} + \mu_{3A,ABCD} + \mu'_{3A,ABCD}$$

Since $t^*(\eta_{1A,AB}) = \overline{\eta}_{A,AB}$ and $t^*(\eta_{1*,AB}) = \overline{\eta}_{*,AB}$ and similarly for their permutations, we have that $t^* : H^1(\mathrm{Conf}_3(X)) \to H^1(\overline{\mathrm{Conf}}_3(X))$ is surjective.

3.6. Generators for the Topology of Configurations of Two Points

In this section, let $n = 2$ and Γ be any tree T. Embeddings of the graphs Y, X, and H into T induce inclusions of homology classes.

Theorem 3.6.1. *$H_1(\overline{Conf}_2(T))$ is generated by the homology classes induced by all possible embeddings of $Y \hookrightarrow T$ and $H^1(\overline{Conf}_2(T))$ by all detectors $\overline{\eta}_{e_ie_j}$ for branches $e_i, e_j \in T$ which share a common endpoint.*

Proof. Since there is no torsion, the universal coefficient theorem gives us $H_1(\overline{\mathrm{Conf}}_2(T)) \cong H^1(\overline{\mathrm{Conf}}_2(T))$. We must show that such induced homology classes pair with detectors to give subgroups of full rank. We proceed by induction on the number of essential vertices.

If T has a single essential vertex v, then $H_1(\overline{\mathrm{Conf}}_2(T)) \cong \mathbb{Z}^{\frac{1}{2}(\mu(v)-1)(\mu(v)-2)}$ by Theorem 2.2.8. T must be planar, so we embed T in $\mathbb{R}^2$, start on an edge of T with endpoint v and label the edges attached to v in clockwise order $e_1, \ldots, e_{\mu(v)}$. The cases where $\mu(v) \leq 4$ are covered in the calculations above. Otherwise, consider the set of homology classes $\{\overline{\gamma}_{e_1e_ie_j}\}$. There are $\frac{1}{2}(\mu(v)-1)(\mu(v)-2)$ classes in this set, as there are $(\mu(v)-1)(\mu(v)-2)$ to pick the e_i, e_j, but $\overline{\gamma}_{e_1e_ie_j} = \overline{\gamma}_{e_1e_je_i}$. Therefore we may write this set as $\{\overline{\gamma}_{e_1e_ie_j} | i < j\}$ and this pairs with the set of $\{\overline{\eta}_{e_ie_j} | i < j \text{ and } i, j \neq 1\}$ so that the intersection matrix contains the block shown in Table 9. Then we have constructed a diagonal block of rank $\frac{1}{2}(\mu(v)-1)(\mu(v)-2)$, which is the full rank, by Thm 2.2.8.

Therefore $H_1(\overline{\mathrm{Conf}_2}(T))$ is generated by all the embeddings $Y \hookrightarrow T$ and $H^1(\overline{\mathrm{Conf}_2}(T))$ by all detectors $\overline{\eta}_{e_i e_j}$.

Now we assume that this is true for all trees with fewer than m essential vertices. For any tree T with m essential vertices, we can write it as the union of two trees $T_1 \cup T_2$, each with strictly fewer than m essential vertices, such that $T_1 \cap T_2$ is an open edge of T. Then from the Mayer-Vietoris sequence, we get that the rank of $H_1(\overline{\mathrm{Conf}_2}(T))$ is exactly the rank of $H_1(\overline{\mathrm{Conf}_2}(T_1)) \oplus H_1(\overline{\mathrm{Conf}_2}(T_2))$, which is the rank we expect. Therefore the homology classes from T_1 and T_2 along with the detectors generate subgroups of $H_1(\mathrm{Conf}_2(T))$ and $H^1(\overline{\mathrm{Conf}_2}(T))$ of full rank. $\square$

The results of [MS16] imply the first half of Theorem 3.6.1, that the first homology group of unordered configurations is generated by switching around the essential vertices of a tree.

Theorem 3.6.2. *$H_1(\mathit{Conf}_2(T))$ is generated by the homology classes induced by all possible embeddings of $Y, X, H \hookrightarrow T$ and $H^1(\mathit{Conf}_2(T))$ by all detectors $\eta_{e_i e_j}$ for branches $e_i, e_j \in T$ which share a common endpoint.*

Proof. Since there is no torsion, the universal coefficient theorem gives us $H_1(\mathrm{Conf}_2(T)) \cong H^1(\mathrm{Conf}_2(T))$. We must show that such induced homology classes pair with detectors to give subgroups of full rank. We proceed by induction on the number of essential vertices in T.

If T has a single essential vertex v, then $H_1(\mathrm{Conf}_2(T)) \cong \mathbb{Z}^{(\mu(v)-1)(\mu(v)-2)-1}$ by Theorem 2.2.8. T must be planar, so we embed T in $\mathbb{R}^2$, start on an edge of T with endpoint v and label the edges attached to v in clockwise order $e_1, \ldots, e_{\mu(v)}$. If $\mu(v) \leq$

4 then the result follows from the calculations above. Otherwise, consider the set of cohomology classes $\{\eta_{e_ie_j} | i,j \neq 1 \text{ and if } i = 2 \text{ then } j \neq \mu(v)\} = E$. There are $(\mu(v)-1)(\mu(v)-2)-1$ classes in this set, and the Table 10 demonstrates a block of the intersection matrix (we write e_μ instead of $e_{\mu(v)}$ for simplicity). Then we have constructed an upper-triangular block of rank $(\mu(v)-1)(\mu(v)-2)-1$, which is the full rank by Thm 2.2.8. Now we assume that this is true for all trees with fewer than m essential vertices. For any tree T with m essential vertices, we can write it as the union of two trees $T_1 \cup T_2$, each with strictly fewer than m essential vertices, such that $T_1 \cap T_2$ is an open edge of T. Then from the Mayer-Vietoris sequence, we get that the rank of $H_1(\mathrm{Conf}_2(T))$ is one more than the rank of $H_1(\mathrm{Conf}_2(T_1)) \oplus H_1(\mathrm{Conf}_2(T_2))$, which is exactly the rank we expect. Since T_1 and T_2 share a single edge, there is some embedding $H \hookrightarrow T$ that sends one of the essential vertices of H to an essential vertex of T_1 and the other to an essential vertex of T_2. Label the images of the branches of H in T A through E according to their preimages. Then we have the pairings given by the $\tau_{AB,DE}$ column of Table 6. Therefore $\tau_{AB,DE}$ along with the homology classes from T_1 and T_2 generate a subgroup of $H_1(\mathrm{Conf}_2(T))$ of full rank, and similarly with the cohomology classes. □

Corollary 3.6.3 (Detection Theorem)**.** $H^*(PB_2(T)) \overset{\oplus res}{\hookrightarrow} H^*(PB_2(Y)) \oplus H^*(PB_2(X)) \oplus H^*(PB_2(H))$ *and* $H^*(B_2(T)) \overset{\oplus res}{\hookrightarrow} H^*(B_2(Y))$

The second statement is a detection theorem for right-angled Artin groups, as it was shown in [FS05] that $B_2(T)$ is a RAAG for any T. However, it is still an open question whether the same is true for pure braid groups.

TABLE 4 Evaluation Pairings for $\overline{\mathrm{Conf}}_3(X)$

	$\overline{\gamma}_{A,ABC}$	$\overline{\gamma}_{B,ABC}$	$\overline{\gamma}_{C,ABC}$	$\overline{\gamma}_{A,ABD}$	$\overline{\gamma}_{B,ABD}$	$\overline{\gamma}_{D,ABD}$	$\overline{\gamma}_{A,ACD}$	$\overline{\gamma}_{C,ACD}$	$\overline{\gamma}_{D,ACD}$	$\overline{\gamma}_{B,BCD}$	$\overline{\gamma}_{C,BCD}$	$\overline{\gamma}_{D,BCD}$
$\overline{\eta}_{A,AB}$	1	0	0	1	0	0	0	0	0	0	0	0
$\overline{\eta}_{B,AB}$	0	1	0	0	1	0	0	0	0	0	0	0
$\overline{\eta}_{*,AB}$	0	0	1	0	0	1	0	0	0	0	0	0
$\overline{\eta}_{A,AC}$	-1	0	0	0	0	0	1	0	0	0	0	0
$\overline{\eta}_{C,AC}$	0	0	-1	0	0	0	0	1	0	0	0	0
$\overline{\eta}_{*,AC}$	0	-1	0	0	0	0	0	0	1	0	0	0
$\overline{\eta}_{A,AD}$	0	0	0	-1	0	0	-1	0	0	0	0	0
$\overline{\eta}_{D,AD}$	0	0	0	0	0	-1	0	0	-1	0	0	0
$\overline{\eta}_{*,AD}$	0	0	0	0	-1	0	0	-1	0	0	0	0
$\overline{\eta}_{B,BC}$	0	1	0	0	0	0	0	0	0	1	0	0
$\overline{\eta}_{C,BC}$	0	0	1	0	0	0	0	0	0	0	1	0
$\overline{\eta}_{*,BC}$	1	0	0	0	0	0	0	0	0	0	0	1
$\overline{\eta}_{B,BD}$	0	0	0	0	1	0	0	0	0	-1	0	0
$\overline{\eta}_{D,BD}$	0	0	0	0	0	1	0	0	0	0	0	-1
$\overline{\eta}_{*,BD}$	0	0	0	1	0	0	0	0	0	0	-1	0
$\overline{\eta}_{C,CD}$	0	0	0	0	0	0	0	1	0	0	1	0
$\overline{\eta}_{D,CD}$	0	0	0	0	0	0	0	0	1	0	0	1
$\overline{\eta}_{*,CD}$	0	0	0	0	0	0	1	0	0	1	0	0

TABLE 5 Evaluation Pairings for $\mathrm{Conf}_2(X)$

	μ_{ABCD}	μ_{ACBD}	μ_{ACDB}	μ_{BCDA}	μ_{CBDA}	μ_{CDBA}	γ_{ABC}	γ_{ABD}	γ_{ACD}	γ_{BCD}
η_{AB}	1	0	-1	0	0	0	1	1	0	0
η_{BA}	0	0	0	1	0	-1	1	1	0	0
η_{AC}	0	1	1	0	0	0	-1	0	1	0
η_{CA}	0	0	0	0	1	1	-1	0	1	0
η_{AD}	-1	-1	0	0	0	0	0	-1	-1	0
η_{DA}	0	0	0	-1	-1	0	0	-1	-1	0
η_{BC}	0	-1	0	1	0	0	1	0	0	1
η_{CB}	1	0	0	0	-1	0	1	0	0	1
η_{BD}	0	1	0	0	0	-1	0	1	0	-1
η_{DB}	0	0	-1	0	1	0	0	1	0	-1
η_{CD}	1	0	0	0	0	1	0	0	1	1
η_{DC}	0	0	1	1	0	0	0	0	1	1

TABLE 6 Evaluation Pairings for $\mathrm{Conf}_2(H)$

	γ_{ABC}	γ_{CDE}	$\tau_{AB,ED}$	$\tau_{AB,DE}$	$\tau_{BA,ED}$	$\tau_{BA,DE}$
η_{AB}	1	0	1	1	0	0
η_{BA}	1	0	0	0	1	1
η_{AC}	-1	0	-1	-1	0	0
η_{CA}	-1	0	0	0	-1	-1
η_{BC}	1	0	0	0	1	1
η_{CB}	1	0	1	1	0	0
η_{CD}	0	1	-1	0	1	0
η_{DC}	0	1	0	1	0	1
η_{CE}	0	-1	0	-1	0	-1
η_{EC}	0	-1	1	0	-1	0
η_{DE}	0	1	0	1	0	1
η_{ED}	0	1	-1	0	1	0

TABLE 7 Block of Evaluation Pairings for $\mathrm{Conf}_3(Y)$

	$\gamma_{1A,ABC}$	$\gamma_{1,23ABC}$
$\eta_{1A,AB}$	1	1
$\eta_{1A,BA}$	1	0
$\eta_{1A,AC}$	-1	-1
$\eta_{1A,CA}$	-1	0
$\eta_{2A,BA}$	0	1
$\eta_{2A,CA}$	0	-1
$\eta_{3C,AC}$	0	1
$\eta_{3B,BC}$	0	-1
$\eta_{3B,CB}$	0	1
$\eta_{3C,BC}$	0	-1

TABLE 8 Block of Evaluation Pairings for $\mathrm{Conf}_3(X)$

	$\mu_{1A,ABCD}$	$\mu_{2A,ABCD}$	$\mu_{3A,ABCD}$
$\eta_{1A,AB}$	1	0	0
$\eta_{1A,AD}$	-1	0	0
$\eta_{2A,AB}$	0	1	0
$\eta_{2A,AD}$	0	-1	0
$\eta_{3A,AB}$	0	0	1
$\eta_{3A,AD}$	0	0	-1
$\eta_{1*,CB}$	-1	0	0
$\eta_{1*,CD}$	1	0	0
$\eta_{2*,CB}$	0	-1	0
$\eta_{2*,CD}$	0	1	0
$\eta_{3*,CB}$	0	0	-1
$\eta_{3*,CD}$	0	0	1

TABLE 9 Block of Evaluation Pairings for $\overline{\mathrm{Conf}}_2(T)$

	$\overline{\gamma}_{e_1e_2e_3}$	$\overline{\gamma}_{e_1e_2e_4}$	$\overline{\gamma}_{e_1e_2e_5}$	$\cdots$	$\overline{\gamma}_{e_1e_3e_4}$	.....
$\overline{\eta}_{e_2e_3}$	1	0	0	$\cdots$	0	$\cdots$
$\overline{\eta}_{e_2e_4}$	0	1	0	$\cdots$	0	$\cdots$
$\vdots$	$\vdots$	0	$\ddots$		0	$\cdots$
$\overline{\eta}_{e_3e_4}$	0	$\vdots$	0		1	
$\vdots$	0	0	$\vdots$	0		$\ddots$

TABLE 10 Block of Evaluation Pairings for $\mathrm{Conf}_2(T)$

	$\mu_{e_1e_3e_2e_\mu}$	$\mu_{e_1e_4e_2e_\mu}$	$\cdots$	$\mu_{e_1e_{\mu-1}e_2e_\mu}$	$\gamma_{e_1e_2e_3}$	$\mu_{e_1e_2e_3e_\mu}$	$\mu_{e_1e_4e_3e_\mu}$	$\cdots$	$\gamma_{e_1e_2e_4}$	$\mu_{e_1e_2e_4e_\mu}$	$\mu_{e_1e_3e_4e_\mu}$	$\cdots$
$\eta_{e_2e_3}$	1	0	$\cdots$	0	-1	0	$\cdots$		0	$\cdots$		
$\eta_{e_2e_4}$	0	1	0	$\cdots$	0	$\cdots$		0	-1	0	$\cdots$	
$\vdots$		0	1	0	$\cdots$				0			
$\eta_{e_2e_{\mu-1}}$	$\vdots$		0	1	0	0	$\cdots$					
$\eta_{e_3e_2}$		$\vdots$		0	1	-1	0	0		$\cdots$		
$\eta_{e_3e_\mu}$					0	1	1	1	0			
$\eta_{e_3e_4}$			$\vdots$			0	-1	0				
$\vdots$					$\vdots$		0	-1	0	0	$\cdots$	
$\eta_{e_4e_2}$							$\vdots$	0	1	-1	0	0
$\eta_{e_4e_\mu}$	0	0	0	0	0	0	0	0	0	1	1	1
$\eta_{e_4e_3}$										0	-1	0
$\vdots$	$\vdots$				$\vdots$				$\vdots$		0	$\ddots$

CHAPTER IV

NON-K-EQUAL CONFIGURATION SPACES OF GRAPHS

We begin by defining an open cover of the non-k-equal configuration space of n points on a graph. As before, we may reduce to the case of a connected graph Γ with no loops and at least one vertex of valence ≥ 2. Let $\mathcal{V} = \{V_j\}$ be the cover of Γ by open stars as defined in Section 2.1.

Let σ be a choice of open star for each of the n configuration points. A configuration **conforms** to this choice if points which are assigned to V_j by σ are in V_j and are strictly closer to the central vertex of V_j than any point assigned to some other star, along an edge in which they are both located. If there is at least one point assigned to V_j, then we say that V_j is **occupied**. Notice that not all V_j must be occupied. Furthermore, if $V_j \cap V_k \neq \emptyset$, then points assigned to V_j may move onto edges of V_k (but not the central vertex) and vice versa. Let ω be a partial order on the points of the configuration, so that it is a total order on the points assigned to each V_j by σ. When referring to points assigned to a specific V_j, we simply refer to their order as ω, so $d_{\omega(i)}$ is the distance of the ith point in the order from the central vertex of V_j. We call λ a **condition** if it is a choice of σ and a compatible ω. Then W_λ is the family of configurations conforming to σ such that for each V_j and for each point of order $i \geq k$ assigned to V_j, $d_{\omega(i)} > d_{\omega(i')}$ for any point of order $i' \leq i-(k-1)$ on the same edge. In particular, no point of order $i \geq k$ may be located at the central vertex, which we consider as part of every edge of the star.

Proposition 4.0.4. *$\mathcal{W}$ is an open, finite cover of $Conf_{n,k}(\Gamma)$*

Proof. Given a configuration $c \in W_\Lambda$, let ϵ_1 be the minimum distance between any two non-equal points in the configuration, and let ϵ_2 be the minimum distance from any point of order k or greater to any central vertex. Then the family of configurations

$\left\{c' \,\middle|\, d(c', c) < \dfrac{\min(\epsilon_1, \epsilon_2)}{3}\right\}$ is also in W_λ, so W_λ is open. There are finitely many ways to assign n points to open stars in Γ, and for each assignment there are finitely many ways to order the points in each V_j, so $\mathcal{W}$ is finite. Given a configuration in $\mathrm{Conf}_{n,k}(\Gamma)$, count outward from each vertex in C_Γ, so that the closest point is 'first' in that open star, and so forth. If two or more points are equidistant from a central vertex, we may put them in any order. If a point could be counted in more than one open star, we may assign it to either one, but any points in the same spot must be assigned to the same star. When all the points in an open star have been ordered (including possibly as part of the ordering of a different open star), we may stop counting from that central vertex. This process gives a partial order ω and therefore a condition λ which the configuration satisfies, and so the configuration is contained in W_λ. Therefore, $\mathcal{W}$ is a cover of $\mathrm{Conf}_{n,k}(\Gamma)$. $\square$

If Λ is a set of conditions, then $W_\Lambda = \cap_{\lambda \in \Lambda} W_\lambda$. If $|\Lambda| \geq 1$, we define a new binary relation $\prec$ on the n points in a configuration. For all partial orders ω of the conditions in Λ and all j indexing the open stars V_j, then $x_{\omega_j(t)} \prec x_{\omega_j(s)}$ when $s - t \geq k - 1$. Note that $\prec$ is not transitive nor total. If $x_j \prec x_i$, we say that x_j **precedes** x_i

Then by definition W_Λ is the family of configurations conforming to all σ of the conditions of Λ such that $d_i > d_j$ for any point x_i along the same edge as a point x_j which precedes it. Note that if both $x_i \prec x_j$ and $x_j \prec x_i$, then x_i and x_j must be on different edges of an open star with neither point allowed to occupy the central vertex. If there are no x_i such that $x_i \prec x_j$, then we say that x_j **has no precedent**.

Proposition 4.0.5. *Let e be an edge of Γ and A be a connected component of W_Λ. We define $\pi_e : A \to Conf_{m,k}(0,1)$ as the map which first forgets the points of a configuration which have no precedent and then projects a configuration on the graph onto a configuration on the edge e. Then $\pi_e(A)$ is contractible for all edges of Γ and components of W_Λ.*

Proof. The number of points which have at least one precedent (so they cannot move off the edge they are on) and are constrained to move on the edge e is consistent across A, and this is exactly m. An edge is covered by at most two open stars, so any points in e have either been assigned to V_j by all σ, or V_ℓ, or sometimes V_j and sometimes V_ℓ. In order for two points x_i and x_ι to collide in W_Λ (and by extension, $\pi_e(A)$), it must be true that $\sigma(x_i) = \sigma(x_\iota)$ for all σ of the conditions in Λ.

First we assume that all points in a configuration in $\pi_e(A)$ are assigned to the same star by every σ, though which star that is may change. A configuration $p \in \pi_e(A)$ then defines a homeomorphism $f_p : \pi_e(A) \to (0,1)^m$ where each factor is the distance of a point along e or between it and some preceeding point: the edge e has an orientation and a configuration on e then defines an order ς on the points, where the 'first' point is the point closest to $\partial_0(e)$ and so forth, and if at least two points are equal in p then their order is determined by the order of their labels in $\mathbb{N}$. Then $f_p(c)$ on the first factor is the distance between $\partial_0(e)$ and $x_{\varsigma(1)}$; on the ith factor, we find the largest j such that $x_{\varsigma(j)} \prec x_{\varsigma(i)}$ (if such a j does not exist we use $\partial_0(e)$ in place of $x_{\varsigma(j)}$). Then $f_p(c)$ on the ith factor is the distance between $x_{\varsigma(j)}$ and $x_{\varsigma(i)}$ as a proportion of the distance from $x_{\varsigma(j)}$ to $\partial_1(e)$. This function is continuous and bijective with a continuous inverse, so $\pi_e(A)$ in this case is contractible.

Now if not all the assignments are the same, we can partition the points into maximal subsets where they are. Points from different subsets cannot collide, because there is at least one condition which assigns them to different open stars. Therefore $\pi_e(A)$ is homeomorphic to a product of configurations of points in these maximal subsets, which reduces to a product of contractible spaces by the previous paragraph. $\square$

Theorem 4.0.6. *Each W_Λ is the disjoint union of contractible sets*

Proof. First we assume that the set of the first $k-1$ points in each V_j is the same (though they can be in any order) for all $\lambda \in \Lambda$. Given any $\lambda \in \Lambda$, there is a map $H_\lambda : W_\Lambda \times [0,1] \to W_\Lambda$ which pulls the first $k-1$ points of each open star onto their associated central vertex, one at a time: let ϖ be the partial order of λ, so that $x_{\varpi_j(i)}$ is the ith point of ϖ in V_j. Then when $0 \leq t \leq \frac{1}{k-1}$, $H_\lambda(c,t)$ moves each $x_{\varpi_j(1)}$ a distance of $(k-1)\left(\frac{1}{k-1} - t\right) d_{\varpi_j(i)}$ from the central vertex of each occupied V_j, exactly as the function G from Theorem 2.1.2 moved the designated points. In the non-k-equal case, even though $x_{\varpi_j(1)}$ may not be the 'first' point in V_j for all conditions, there are strictly fewer than k points between it and the central vertex by definition, and it has no precedents because its order must be strictly less than k for each condition in Λ. Then when $\frac{1}{k-1} \leq t \leq \frac{2}{k-1}$, $H_\lambda(c,t)$ moves each $x_{\varpi_j(2)}$ a distance of $(k-1)\left(\frac{2}{k-1} - t\right) d_{\varpi_j(2)}$ from the central vertex of V_j, and so forth for each $x_{\varpi_j(i)}$. Therefore H_λ is a continuous function and the concatenation of straight-line homotopies in W_Λ, so H_λ itself is a homotopy. $H_\lambda(W_\Lambda, 1)$ is a family of configurations in W_Λ with some points fixed at central vertices and the rest constrained to move along some edge. Therefore a connected component A of $H_\lambda(W_\Lambda, 1)$ is homeomorphic to a product over the occupied edges of $\pi_e(A)$, which are all contractible by 4.0.5. Therefore W_Λ is the disjoint union of contractible sets. In particular, if there are two different conditions λ and λ' such that $W_\lambda = W_{\lambda'}$, then the set of the first $k-1$ points must be the same, so $H_\lambda(W_\lambda, 1) = H_{\lambda'}(W_{\lambda'}, 1)$ and the homotopy type of intersections of sets in $\mathcal{W}$ does not depend on which condition we choose to define the same underlying set.

Now we consider the case where the set of the first $k-1$ points in each V_j may be different for different conditions. In this case, let κ be the set of points which have no precedent. We know that there are no more than $k-1$ of these points assigned to any open star, because there are at most $k-1$ points in any V_j which do not have $x_{\omega_j(1)}$ as a precedent for any ω, and by adding conditions we can only increase the number of precedents. There is a map $H'_\lambda : W_\Lambda \times [0,1] \to W_\Lambda$ which pulls the points in κ onto their associated central vertex, one by one, defined analogously to H_λ: on $\left[\dfrac{i-1}{k-1}, \dfrac{i}{k-1}\right]$ for $1 \leq i \leq k-1$, each $x_{\varpi_j(i)}$ which has no precedents moves a distance of $(k-1)\left(\frac{i}{k-1} - t\right) d_{\varpi_j(i)}$ from the central vertex of V_j and H'_λ is the identity on the $x_{\varpi_j(i)}$ not in κ as well as the other points. There are always strictly fewer than k points between any $x_{\varpi_j(i)}$ in κ and the central vertex by definition, and any point which is allowed to occupy a central vertex is moved onto it by this map. Then H'_λ is also a concatenation of straight-line homotopies, and a connected component of $H'_\lambda(W_\Lambda, 1)$ is again homeomorphic to a product over the occupied edges of $\pi_e(A)$, so W_Λ is the disjoint union of contractible spaces. $\square$

Now we subdivide Γ so that there are at least n segments between any two central vertices in Γ and at least n segments between any central vertex and any root vertex. An edge still refers to an edge $e \in \Gamma$ from before this subdivision, between two central vertices or between a central and a root vertex. The subdivision means that each edge will have vertices at distance of multiples of $1/n$ from central vertices. Then Γ^n has the structure of a cubical complex, with cells $\tau = (\tau_1, \ldots, \tau_n)$ such that $\tau_i \in V(\Gamma)$ or $E(\Gamma)$. We define

$$\mathcal{D}_{n,k}(\Gamma) = \{\tau \in \Gamma^n | \overline{\tau}_{i_1} \cap \ldots \cap \overline{\tau}_{i_k} = \emptyset \text{ for any } k\text{-set of indices } 1 \leq i_1 < \ldots < i_k \leq n\}$$

Then $\mathcal{D}_{n,k}(\Gamma)$ is a subcomplex of Γ^n. We define $\mathcal{D}_\lambda = \mathcal{D}_{n,k}(\Gamma) \cap W_\lambda$ and $\mathcal{D}_\Lambda = \cap_{\lambda \in \Lambda} \mathcal{D}_\lambda$.

Proposition 4.0.7. *Let M be some subset of the points $x_1, \ldots, x_n$ and $|M| = m$. Let Λ be a set of conditions, and $\prec$ be the resultant binary relation on $x_1, \ldots, x_n$. Let $(0,1)$ be subdivided into n equal segments, and let $\mathcal{D}_{m,k}\frac{(0,1)}{n}$ be the space of $\frac{1}{n}$-discrete non-k-equal configurations of m points in $(0,1)$. Let Δ^m_Λ be the subset of $\mathcal{D}_{m,k}\frac{(0,1)}{n}$ where if $x_i \prec x_j$ for $x_i, x_j \in M$, then $d_i < d_j$ along $(0,1)$. Then if Δ^m_Λ is nonempty, it is contractible.*

Proof. There is a map $H' : \Delta_\Lambda^m \times [0,1] \to \Delta_\Lambda^m$ which pulls every configuration of the points in M onto a 'lowest weight' configuration: for $0 \le t \le \frac{1}{m-1}$, $H'(c,t)$ moves the point x_i closest to 0 onto the vertex at $1/n$ (if there is more than one closest point, their order is determined by the order of their labels in $\mathbb{N}$). For $\frac{1}{m-1} \le t \le \frac{2}{m-1}$, it moves the next closest point x_j to $1/n$ unless $x_i \prec x_j$, in which case it moves x_j to $2/n$. For $\frac{i-1}{m-1} \le t \le \frac{i}{m-1}$, it moves the ith point to i/n. $H'(c,t)$ stays inside Δ_Λ^m and H is a concatenation of straight-line homotopies. Therefore Δ_Λ^m is contractible. $\square$

Theorem 4.0.8. *Each $\mathcal{D}_\Lambda$ is likewise the disjoint union of contractible sets.*

Proof. First we assume that the set of the first $k-1$ points in each V_j is the same (though they can be in any order) for all $\lambda \in \Lambda$. Given any $\lambda \in \Lambda$, there is a map $J_\lambda : \mathcal{D}_\Lambda \times [0,1] \to \mathcal{D}_\Lambda$ which first moves any points of order k or above at least $1/n$ from each central vertex, and then pulls the first $k-1$ points of each open star onto their associated central vertex, one at a time: if there is some point $x_i \in c$ which is not one of the first $k-1$ points in an open star V_j and its distance d_i from the central vertex of V_j is less than $1/n$, then when $0 \le t \le 1/k$, $J_\lambda(c,t)$ moves x_i a distance of $k(1-t)d_i + \frac{kt}{n}$ away from the central vertex, exactly as the function F from Theorem 2.1.6 moves the non-designated points. The x_i are able to move in this fashion, because if a point is on the interior of a segment in a discrete configuration, then the configuration where the point has moved to one of the endpoints of that segment must also be a discrete configuration by definition.

Let ϖ be the partial order of λ. Then when $1/k \le t \le 2/k$, $J_\lambda(c,t)$ moves each $x_{\varpi_j(1)}$ a distance of $k\left(\frac{2}{k} - t\right) d_{\varpi_j(1)}$ from the central vertex of V_j. Even though $x_{\varpi_j(1)}$ may not be the 'first' point in V_j for all conditions, there are strictly fewer than k points between it and the central vertex by definition, and it has no precedents because its order must be strictly less than k for each condition in Λ. Therefore the function remains in $\mathcal{D}_\Lambda$. Similarly, on $\left[\frac{i}{k}, \frac{i+1}{k}\right]$ for $2 \le i \le k-1$, J_λ moves $x_{\varpi_j(i)}$ a distance of $k\left(\frac{i+1}{k} - t\right) d_{\varpi_j(i)}$ from the central vertex of V_j. Therefore J_λ is a continuous function and the concatenation of straight-line homotopies in $\mathcal{D}_\Lambda$, so J_λ itself is a homotopy. $J_\lambda(\mathcal{D}_\Lambda, 1)$ is a family of configurations in $\mathcal{D}_\Lambda$ with some points fixed at central vertices and the rest constrained to move along some edge. Therefore a connected component A of $J_\lambda(\mathcal{D}_\Lambda, 1)$ is homeomorphic to a product over the occupied edges of Δ_Λ^m, which are all contractible by 4.0.7. Therefore $\mathcal{D}_\Lambda$ is the disjoint union of contractible sets. In particular, if there are two different conditions λ and λ' such that $\mathcal{D}_\lambda = \mathcal{D}_{\lambda'}$, then the set of the first $k-1$ points must be the same, so $J_\lambda(\mathcal{D}_\lambda, 1) = J_{\lambda'}(\mathcal{D}_{\lambda'}, 1)$ and the homotopy type of intersections of sets in $\mathcal{D}_n(\Gamma)$ does not depend on which condition we choose to define the same underlying set.

Now we consider the case where the set of the first $k-1$ points in each V_j may be different for different conditions. In this case, let κ be the set of points which have no precedent. We know that there are no more than $k-1$ of these points assigned to any open star, because there are at most $k-1$ points in any V_j which do not have $x_{\omega_j(1)}$ as a precedent for any ω, and by adding conditions we can only increase the number of precedents. There is a map $J'_\lambda : \mathcal{D}_\Lambda \times [0,1] \to \mathcal{D}_\Lambda$ which pulls the points in κ onto their associated central vertex, one by one, defined analogously to J_λ: on $0 \le t \le 1/k$, if x_i is a point not in κ at a distance of less than $1/n$ from a central vertex, $J'_\lambda(c,t)$ moves x_i a distance of $k(1-t)d_i + \frac{kt}{n}$ away from the central vertex, as J_λ did for all points of order k or greater. Then on $\left[\frac{i}{k}, \frac{i+1}{k}\right]$ for $1 \le i \le k-1$, J'_λ moves each $x_{\varpi_j(i)}$ in κ a distance of $k\left(\frac{i+1}{k} - t\right) d_{\varpi_j(i)}$ from the central vertex of V_j, just as J_λ moved the points of order $k-1$ or less. J'_λ is does not move any points not in κ when $t \in \left[\frac{1}{k}, 1\right]$. There are always strictly fewer than k points between any $x_{\varpi_j(i)}$ in κ and the central vertex by definition, and any point which is allowed to occupy a central vertex is moved onto it by this map. So J'_λ is similarly a concatenation of straight-line homotopies, and a connected component of $J'_\lambda(\mathcal{D}_\Lambda, 1)$ is again homeomorphic to a product over the occupied edges of $\Delta_\Lambda^{m_1} \times \Delta_\Lambda^{m_2} \times \ldots$, one factor for each maximal set of points constrained to an edge that are always assigned to the same open star (points which are assigned to different stars by at least one condition cannot collide). Therefore $\mathcal{D}_\Lambda$ is the disjoint union of contractible spaces. $\square$

Theorem 4.0.9. *The inclusion* $i : \mathcal{D}_\Lambda \hookrightarrow W_\Lambda$ *induces a homology equivalence for all* Λ*.*

Proof. Each component of D_Λ includes into exactly one component of U_Λ by definition. Given a configuration of points on an edge e, there is a homeomorphism of e that fixes its endpoints and sends the configuration to a discrete configuration: send the point (or points, if they coincide) in the interior of e closest to $\partial_0(e)$ a distance of $1/n$ from $\partial_0(e)$, the second-closest point(s) a distance of $2/n$, etc. Then given a configuration in U_Λ, there is a homeomorphism of the graph which sends it to a configuration in D_Λ (though this choice of homeomorphism is obviously not unique or continuous). The space of homeomorphisms of an edge is contractible, so there is a path in U_Λ between the configurations, and the discrete configuration is in the same component of U_Λ as the starting configuration. Therefore there is at least one component of D_Λ in each component of U_Λ. Finally, there are exactly as many components of D_Λ as there are of U_Λ, because a component is determined by the binary relation $\prec$ and the assignment of points with at least one precedent to some edge of the graph. Therefore the inclusion induces a bijection of components, and as all components are contractible, this is a homology equivalence for all choices of Λ. $\square$

Proposition 4.0.10. $H_*(Conf_{n,k}(\Gamma)) \cong H_*(\mathcal{D}_{n,k}(\Gamma))$

Proof. The Mayer-Vietoris decompositions of $\mathrm{Conf}_n(\Gamma)$ by $\mathcal{U}$ and $\mathcal{D}_n(\Gamma)$ by $\mathcal{D}$ give two spectral sequences as described in Prop 2.1.9. The inclusions of Prop 2.1.7 induce an equivalence of the spectral sequences, as the E^1 pages are isomorphic in each entry and the differentials are given by inclusions. Therefore both spectral sequences must converge to the same thing, with i^* giving the equivalence. $\square$

CHAPTER V

NON-3-EQUAL CONFIGURATION SPACES ON TREES

In this section we focus on non-3-equal configurations of three points on the graphs $[0,1] = I$, Y and X. The branches of Y are labeled A through C, and the branches of X are labeled A through D.

5.1. $\mathbf{Conf}_{3,3}(I)$

The ranks of the homology groups of $\mathrm{Conf}_{3,3}(I) \cong I^3 \backslash \{(x,x,x)\} \simeq S^1$ are computed in [BW95] and replicated by executing the code in A.3 of the Appendix: $H_1(\mathrm{Conf}_{3,3}(I)) \cong \mathbb{Z}$ and $H_i(\mathrm{Conf}_{3,3}(I)) \cong 0$ for $i > 1$. Let B be a path in $\mathrm{Conf}_{3,3}(I)$ which starts with the first point at 0 and the second and third points at 1, sends the second point to 0, the first point to 1, the third point to 0, the second point to 1, the first point to 0, and the third point to 1. This path is highlighted in Figure 10. We can think of this path as the collapse onto I of the braid on three colored strands given by alternating half-twists (of which there must be 6 in all to return to the starting point). Then $\beta = [B] \in H_1(\mathrm{Conf}_{3,3}(I))$.

Consider the hyperplane $x_2 = \frac{x_1}{2} + \frac{x_3}{2} \in I^3$. Let $E_{3>1}$ be the subspace of $\mathrm{Conf}_{3,3}(I)$ given by the intersection of this hyperplane with $\{x_3 > x_1\}$. There is a detector class $\eta_{3>1} \in H^1(\mathrm{Conf}_{3,3}(I))$ represented by intersection with $E_{3>1}$ where the normal bundle to $E_{3>1}$ in $\mathrm{Conf}_{3,3}(I)$ is oriented so that configurations where x_2 is approaching $\frac{x_1}{2} + \frac{x_3}{2}$ from above or where $\frac{x_1}{2} + \frac{x_3}{2}$ is approaching x_2 from below are positive and the opposite scenarios are negative. We have the pairing

$$\langle \eta_{3>1}, \beta \rangle = 1$$

so this gives a presentation for $H_1(\mathrm{Conf}_{3,3}(I))$ and $H^1(\mathrm{Conf}_{3,3}(I))$.

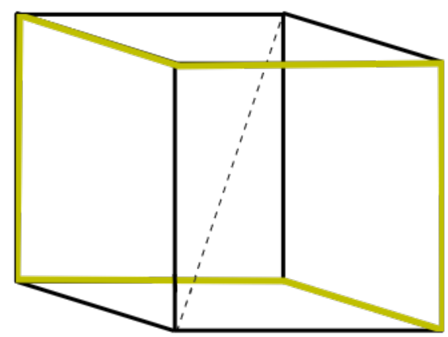

FIGURE 10 A path in $\mathrm{Conf}_{3,3}(I)$ representing the class β.

5.2. $\mathbf{Conf}_{3,3}(Y)$

$H_i(\mathrm{Conf}_{3,3}(Y)) \cong 0$ for $i > 1$ and $H_1(\mathrm{Conf}_{3,3}(Y)) \cong \mathbb{Z}^{13}$ are given by executing the code in A.4 of the Appendix. We demonstrate a subgroup of (co)homology classes along with pairings for $H_1(\mathrm{Conf}_{3,3}(Y))$, building off the bases demonstrated for $H_1(\mathrm{Conf}_2(Y))$ in Section 3.4.

Adding a new labeled point to a configuration so that it doubles a point already in the configuration gives us three maps $H_1(\mathrm{Conf}_2(Y)) \to H_1(\mathrm{Conf}_{3,3}(Y))$ for the three ways to give the new point a label. There are three images of $\gamma_{ABC} \in H_1(\mathrm{Conf}_{3,3}(Y))$, which we denote $\gamma_{1,23ABC}$, etc and refer to collectively as insertion classes. Adding a new labeled point to the central vertex of Y gives us three maps $H_1(\mathrm{Conf}_2(Y)) \to H_1(\mathrm{Conf}_{3,3}(Y))$ for the three ways to give the new point a label, and we call the images $\gamma_{1*,23ABC}$, etc. Embedding $I \hookrightarrow Y$ induces a map $H_1(\mathrm{Conf}_{3,3}(I)) \to H_1(\mathrm{Conf}_{3,3}(Y))$, and by abuse of notation we also call this image β.

Let W_{ABB} be the subspace of $\text{Conf}_{3,3}(Y)$ such that the first point is on the A branch, the second and third points are on the B branch, and $d_1 = d_2/2 + d_3/2$. The normal bundle to W_{ABB} inside $\text{Conf}_{3,3}(Y)$ is oriented by assigning a positive orientation to configurations where the first point is approaching the average of x_2 and x_3 from above or the average is approaching x_1 from below and the negative orientation to the opposite scenarios. Then let $\omega_{ABB} = [W_{ABB}] \in H^1(\text{Conf}_2(T))$ be the resulting cohomology class represented by W_{ABB} with this orientation. We have the pairings

$$\langle \omega_{ABB}, \gamma_{1,23ABC} \rangle = 1 \qquad \langle \omega_{BAA}, \gamma_{1,23ABC} \rangle = 1$$

$$\langle \omega_{ABA}, \gamma_{2,13ABC} \rangle = 1 \qquad \langle \omega_{BAB}, \gamma_{2,13ABC} \rangle = 1$$

$$\langle \omega_{AAB}, \gamma_{3,12ABC} \rangle = 1 \qquad \langle \omega_{BBA}, \gamma_{3,12ABC} \rangle = 1$$

$$\langle \omega_{BAB}, \gamma_{1*,23ABC} \rangle = 1 \quad \langle \omega_{AAB}, \gamma_{1*,23ABC} \rangle = 1 \quad \langle \omega_{BBA}, \gamma_{1*,23ABC} \rangle = 1 \quad \langle \omega_{ABA}, \gamma_{1*,23ABC} \rangle = 1$$

$$\langle \omega_{ABB}, \gamma_{2*,13ABC} \rangle = 1 \quad \langle \omega_{AAB}, \gamma_{2*,13ABC} \rangle = 1 \quad \langle \omega_{BBA}, \gamma_{2*,13ABC} \rangle = 1 \quad \langle \omega_{BAA}, \gamma_{2*,13ABC} \rangle = 1$$

$$\langle \omega_{ABA}, \gamma_{3*,12ABC} \rangle = 1 \quad \langle \omega_{ABB}, \gamma_{3*,12ABC} \rangle = 1 \quad \langle \omega_{BAA}, \gamma_{3*,12ABC} \rangle = 1 \quad \langle \omega_{BAB}, \gamma_{3*,12ABC} \rangle = 1$$

with other pairings within the subgroup zero.

5.3. $\mathbf{Conf}_{3,3}(X)$

$H_i(\text{Conf}_{3,3}(X)) \cong 0$ for $i > 1$ and $H_1(\text{Conf}_{3,3}(X)) \cong \mathbb{Z}^{49}$ are given by executing the code in A.5 of the Appendix. We demonstrate a subgroup of (co)homology classes along with pairings for $H_1(\text{Conf}_{3,3}(X))$, building off the bases demonstrated for $H_1(\text{Conf}_2(X))$ in Section 3.4.

Adding a new labeled point to a configuration so that it doubles a point already in the configuration gives us three maps $H_1(\mathrm{Conf}_2(X)) \to H_1(\mathrm{Conf}_{3,3}(X))$ for the three ways to give the new point a label. There are three images of $\mu_{ABCD} \in H_1(\mathrm{Conf}_{3,3}(X))$, which we denote $\mu_{1,23ABCD}$, etc and refer to collectively as insertion classes. Adding a new labeled point to the central vertex of X gives us three maps $H_1(\mathrm{Conf}_2(X)) \to H_1(\mathrm{Conf}_{3,3}(X))$ for the three ways to give the new point a label, and we call the images of $\mu_{1*,23ABCD}$, etc. Embedding $I \hookrightarrow X$ induces a map $H_1(\mathrm{Conf}_{3,3}(I)) \to H_1(\mathrm{Conf}_{3,3}(X))$, and by abuse of notation we also call this image β.

We have the pairings

$$\langle \omega_{ABB}, \mu_{1,23ABCD} \rangle = 1 \qquad \langle \omega_{BAB}, \mu_{2,13ABCD} \rangle = 1 \qquad \langle \omega_{BBA}, \mu_{3,12ABCD} \rangle = 1$$

$$\langle \omega_{BAA}, \mu_{1,23BCDA} \rangle = 1 \qquad \langle \omega_{ABA}, \mu_{2,13BCDA} \rangle = 1 \qquad \langle \omega_{AAB}, \mu_{3,12BCDA} \rangle = 1$$

$$\langle \omega_{AAB}, \mu_{1*,23ABCD} \rangle = 1 \quad \langle \omega_{BAB}, \mu_{1*,23ABCD} \rangle = 1$$

$$\langle \omega_{AAB}, \mu_{2*,13ABCD} \rangle = 1 \quad \langle \omega_{ABB}, \mu_{2*,13ABCD} \rangle = 1$$

$$\langle \omega_{ABA}, \mu_{3*,12ABCD} \rangle = 1 \quad \langle \omega_{ABB}, \mu_{3*,12ABCD} \rangle = 1$$

$$\langle \omega_{ABA}, \mu_{1*,23BCDA} \rangle = 1 \quad \langle \omega_{BBA}, \mu_{1*,23BCDA} \rangle = 1$$

$$\langle \omega_{BAA}, \mu_{2*,13BCDA} \rangle = 1 \quad \langle \omega_{BBA}, \mu_{2*,13BCDA} \rangle = 1$$

$$\langle \omega_{BAA}, \mu_{3*,12BCDA} \rangle = 1 \quad \langle \omega_{BAB}, \mu_{3*,12BCDA} \rangle = 1$$

with other pairings within the subgroup zero.

APPENDIX

SAGE CODE

To simplify calculations with the cubical complexes in this dissertation, we executed the following code in SageMathCloud.

A.1. $\mathbf{Conf}_n(Y)$

```
def generate_tree(n):
    lookup = [(0,n-t-1,0) for t in range(n-1)]
    lookup.append((0,0,0))
    lookup.extend([(t,0,0) for t in range(1,n)])
    lookup.extend([(0,0,t) for t in range(1,n)])
    tree = []
    for point in range(n-1):
        tree.append([point+1])
    tree.append([n, 2*n-1])
    for point in range(n, 2*n-2):
        tree.append([point+1])
    tree.append([])
    for point in range(2*n-1, 3*n-3):
        tree.append([point+1])
    tree.append([])
    return (lookup, tuple(tree))


def iterate_over_conf(T, n):
```

```
    iterator_list = [xrange(len(T))] * n
    for point_config in xmrange_iter(iterator_list):
        if len(set(point_config)) == n: # points all distinct
            yield point_config

def downstream_moves(point_config, T):
    output = []
    pointconfig_set = set(point_config)
    for p in point_config:
        downstream = [u for u in T[p] if u not in pointconfig_set]
        if downstream == []:
            output.append([None])
        else:
            output.append(downstream)
    return output

def downstream_cubes(point_config, T, lookup):
    cubes = []
    moves = downstream_moves(point_config, T)
    for combination in xmrange_iter(moves):
        new_cube = []
        for (coordinate, point) in zip(combination, point_config):
            embedded_coords = lookup[point]
            if coordinate == None:  # this point doesn't move in this cube
                intervals = [[u,u] for u in embedded_coords]
```

```
            else:
                downstream_embedded = lookup[coordinate]
                intervals = [sorted([u,v]) for (u,v) in zip(embedded_coords, d
            new_cube.extend(intervals)
        cubes.append(new_cube)
    return cubes


def the_complex(n):
    (lookup, T) =  generate_tree(n)
    cubes = []
    for point_config in iterate_over_conf(T, n):
        cubes.extend(downstream_cubes(point_config, T, lookup))
    return CubicalComplex(cubes)
```

Then for any number n,

```
the_complex(n).homology()
```

returns a list of the ranks of the reduced homology groups of $\mathrm{Conf}_n(Y)$.

A.2. $\mathbf{Conf}_n(X)$

The only difference from the previous section is in the definition of `generate_tree(n)`:

```
def generate_tree(n):
    lookup = [(0,n-t-1,0, 0) for t in range(n-1)]
    lookup.append((0,0,0, 0))
    lookup.extend([(t,0,0, 0) for t in range(1,n)])
    lookup.extend([(0,0,t, 0) for t in range(1,n)])
```

```
    lookup.extend([(0,0,0,t) for t in range(1,n)])
    tree = []
    for point in range(n-1):
        tree.append([point+1])
    tree.append([n, 2*n-1, 3*n-2])
    for point in range(n, 2*n-2):
        tree.append([point+1])
    tree.append([])
    for point in range(2*n-1, 3*n-3):
        tree.append([point+1])
    tree.append([])
    for point in range(3*n-2, 4*n-4):
        tree.append([point+1])
    tree.append([])
    return (lookup, tuple(tree))
```

A.3. $\mathbf{Conf}_{n,k}(I)$

```
def generate_interval(n):
    lookup = [(0,n-t+1,0) for t in range(n+1)]
    lookup.append((0,0,0))
    interval = []
    for point in range(n+1):
        interval.append([point+1])
    interval.append([])
    return (lookup, tuple(interval))
```

```
def no_k_equal(point_config,k):
    count_list = [point_config.count(i)<k for i in point_config]
    return count_list.count(False)==0


def iterate_over_conf(I, n, k):
    iterator_list = [xrange(len(I))] * n
    for point_config in xmrange_iter(iterator_list):
        if no_k_equal(point_config,k): # no k points are equal
            yield point_config


def downstream_moves(point_config, I,k):
    locationlist = config_to_locationlist(point_config,I)
    output = [None]*len(locationlist)
    multiplicity = [len(u) for u in locationlist]
    for p in set(point_config):
        available_points = multiplicity[p]
        downstream_points = [u for u in I[p] if multiplicity[u] < k-1]
        downstream = []
        for point in downstream_points:
            vacant_spots = k - multiplicity[point] - 1
            downstream.append((min(available_points, vacant_spots), point))
        if downstream == []:
            pass
        else:
```

```
            output[p]=downstream
    return output


def config_to_locationlist(config, I):
    locationlist = []
    for vertex in range(len(I)):
        locationlist.append([])
    for point,location in enumerate(config):
        locationlist[location].append(point)
    return locationlist



def locationlist_to_config(mult, I, n):
    config = [None] * n
    for location, pointlist in enumerate(mult):
        for point in pointlist:
            config[point] = location
    return config


def simultaneous_moves(point_config, I, k):
    down_moves = downstream_moves(point_config, I, k)
    locationlist = config_to_locationlist(point_config,I)
    moves = []
    for start in range(len(I)):
        possible_moves=[]
```

```
        if down_moves[start] == None:
            pass
        else:
            for (number, place) in down_moves[start]:
                for t in Combinations(locationlist[start], number).list():
                    possible_moves.append([t, place])
            moves.append(possible_moves)
    return [i for i in xmrange_iter(moves)]


def downstream_cubes(point_config, I, lookup, k):
    cubes=[]
    sim_moves=simultaneous_moves(point_config, I, k)
    for move in sim_moves:
        coord_list=[None]*len(point_config)
        for [points, place] in move:
            for point in points:
                coord_list[point]=place
        new_cube=[]
        for (coordinate, point) in zip(coord_list, point_config):
            embedded_coords = lookup[point]
            if coordinate == None:  # this point doesn't move in this cube
                intervals = [[u,u] for u in embedded_coords]
            else:
                downstream_embedded = lookup[coordinate]
                intervals = [sorted([u,v]) for (u,v) in zip(embedded_coords, d
```

```
            new_cube.extend(intervals)
        cubes.append(new_cube)
    return cubes
def the_complex(n,k):
    (lookup, I) =  generate_interval(n)
    cubes = []
    for point_config in iterate_over_conf(I, n,k):
        cubes.extend(downstream_cubes(point_config, I, lookup,k))
    return CubicalComplex(cubes)
```

A.4. $\mathbf{Conf}_{n,k}(Y)$

The only difference from the previous section is in the definition of `generate_interval(n)` and `the_complex(n,k)`:

```
def generate_tree(n):
    lookup = [(0,n-t-1,0) for t in range(n-1)]
    lookup.append((0,0,0))
    lookup.extend([(t,0,0) for t in range(1,n)])
    lookup.extend([(0,0,t) for t in range(1,n)])
    tree = []
    for point in range(n-1):
        tree.append([point+1])
    tree.append([n, 2*n-1])
    for point in range(n, 2*n-2):
        tree.append([point+1])
    tree.append([])
```

```
    for point in range(2*n-1, 3*n-3):
        tree.append([point+1])
    tree.append([])
    return (lookup, tuple(tree))


def the_complex(n,k):
    (lookup, T) =  generate_tree(n)
    cubes = []
    for point_config in iterate_over_conf(T, n,k):
        cubes.extend(downstream_cubes(point_config, T, lookup,k))
    return CubicalComplex(cubes)
```

A.5. $\mathbf{Conf}_{n,k}(X)$

The only difference from the previous section is in the definition of `generate_interval(n)`:

```
def generate_tree(n):
    lookup = [(0,n-t-1,0, 0) for t in range(n-1)]
    lookup.append((0,0,0, 0))
    lookup.extend([(t,0,0, 0) for t in range(1,n)])
    lookup.extend([(0,0,t, 0) for t in range(1,n)])
    lookup.extend([(0,0,0,t) for t in range(1,n)])
    tree = []
    for point in range(n-1):
        tree.append([point+1])
    tree.append([n, 2*n-1, 3*n-2])
```

```
for point in range(n, 2*n-2):
    tree.append([point+1])
tree.append([])
for point in range(2*n-1, 3*n-3):
    tree.append([point+1])
tree.append([])
for point in range(3*n-2, 4*n-4):
    tree.append([point+1])
tree.append([])
return (lookup, tuple(tree))
```

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