HOMOLOGICAL PROPERTIES OF STANDARD KLR MODULES

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DISSERTATION ABSTRACT

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Khovanov-Lauda-Rouquier algebras, or KLR algebras, are a family of algebras known to categorify the upper half of the quantized enveloping algebra of a given Lie algebra. In finite type, these algebras come with a family of standard modules, which correspond to PBW bases under this categorification. In this thesis, we show that there are no non-zero homomorphisms between distinct standard modules and that all non-zero endomorphisms of standard modules are injective. We then apply this result to obtain applications to the modular representation theory of KLR algebras. Restricting our attention to finite type A, we are then able to compute explicit projective resolutions of all standard modules. Finally, in finite type A when α is a positive root, we let Δ be the direct sum of all distinct standard modules and compute the algebra structure on $\mathrm{Ext}^\bullet(\Delta, \Delta)$.

This dissertation includes unpublished co-authored material.

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Chapter I

INTRODUCTION

This thesis contains unpublished material coauthored by the author and his advisor, Alexander Kleshchev. In particular, Dr. Kleshchev has contributed to Chapters II and III. The material from Chapter II will appear in a paper accepted by Compositio Mathematica.

1.1. Background

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $U_q(\mathfrak{g})$ its quantized enveloping algebra, an associative $\mathbb{Q}(q)$-algebra for an indeterminate q. Let $U_q^+(\mathfrak{g}) \subset U_q(\mathfrak{g})$ be the subalgebra corresponding to a maximal nilpotent subalgebra of $\mathfrak{g}$. Then $U_q^+(\mathfrak{g})$ comes equipped with several interesting bases. Among them are Lusztig's canonical basis and various PBW bases, one for each choice of convex order on Φ_+, the set of positive roots. Dualizing with respect to Lusztig's form on $U_q^+(\mathfrak{g})$ also gives rise to Lusztig's dual canonical basis and various dual PBW bases.

In 2008 Khovanov, Lauda, and Rouquier defined a family of algebras, which have since been studied extensively and are now known as KLR algebras. Letting Q_+ denote the $\mathbb{Z}_{\geq 0}$-span of the positive roots of $\mathfrak{g}$, for each $\theta \in Q_+$ there exists a corresponding KLR algebra H_θ. Their direct sum $H := \bigoplus_{\theta \in Q_+} H_\theta$ is a locally unital algebra. Khovanov, Lauda, and Rouquier were able to show that the representation theory of H provides a categorification of $U_q^+(\mathfrak{g})$. To be more precise, let $\mathrm{Proj}(H)$ denote the category of finitely generated projective H-modules, and

[Proj(H)] its split Grothendieck group. Then, [Proj(H)] is a $\mathbb{Z}[q,q^{-1}]$-algebra, with multiplication induced by the induction product on H-modules and the q-action induced by upward degree shift. Khovanov, Lauda, and Rouquier's categorification theorem states that there is a unique algebra isomorphism

$$\gamma : U_q^+(\mathfrak{g}) \to \mathbb{Q}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} [\mathrm{Proj}(H)], e_i \to [H_{\alpha_i}],$$

where e_i is the generator of $U_q^+(\mathfrak{g})$ corresponding to the simple root α_i, and $[H_{\alpha_i}]$ is the class of the regular H_{α_i}-module. Moreover, in simply laced types it is known that γ maps Lusztig's canonical basis to the classes of the projective indecomposable modules [VV].

The categorification theorem above has led to much interest in the representation theory of KLR algebras, see for example [BK2], [BKM], [Ka], [M], [VV]. In [BKM], KLR algebras are shown to possess *affine quasihereditary* structures. Affine quasihereditary algebras are a generalization of Cline, Parshall and Scott's notion of quasihereditary algebras [CPS] that include infinite dimensional algebras and share many of their properties. In particular, KLR algebras come with distinguished collections of *standard* modules and *proper standard* modules. Standard modules are known to correspond to the PBW bases of $U_q^+(\mathfrak{g})$ under (a generalization of) the categorification theorem [BKM], and proper standard modules correspond to dual PBW bases. The standard modules for H_θ are labeled by *Kostant partitions* of θ. We denote the set of Kostant partitions by $\mathrm{KP}(\theta)$, and for $\pi \in \mathrm{KP}(\theta)$ we denote by

$\Delta(\pi)$ the corresponding standard module. The standard KLR modules will be the primary objects of interest in this thesis.

1.2. Overview of Results

In Chapter II we study homomorphisms between standard modules in arbitrary finite Lie type. Theorem A is the main result from Section 2.1.

Theorem A. *Let $\Delta(\pi)$ and $\Delta(\sigma)$ be standard H_θ-modules. If π and σ are distinct, then*

$$\mathrm{Hom}_{H_\alpha}(\Delta(\pi), \Delta(\sigma)) = 0.$$

Moreover, every nonzero endomorphism of $\Delta(\pi)$ is injective.

It is a surprising result because this phenomenon seems to be unique to the *affine* quasihereditary setting. For nonsemisimple, finite dimensional quasihereditary algebras, there will always exist a nontrivial homomorphism between some distinct standard modules. This result might be compared with [BCGM], where something similar occurs in an entirely different setting.

In Section 2.2 we turn our attention to the modular representation theory of KLR algebras. Note that KLR algebras are defined over an arbitrary base ring $\Bbbk$, and when we wish to specify this base ring, we use the notation $H_{\theta,\Bbbk}$. Using the p-modular system (F, R, K) with $F = \mathbb{Z}/p\mathbb{Z}$, $R = \mathbb{Z}_p$ and $K = \mathbb{Q}_p$, we can reduce modulo p any finitely generated $H_{\alpha,K}$ module. An application of Theorem A and the Universal Coefficient Theorem then yields the following result.

Theorem B. *Let $\theta \in Q_+$ and $\pi, \sigma \in \mathrm{KP}(\theta)$. Then the R-module*

$$\mathrm{Ext}^1_{H_{\theta,R}}(\Delta(\pi)_R, \Delta(\sigma)_R)$$

is torsion-free. Moreover,

$$\dim_q^F \mathrm{Ext}^1_{H_{\theta,F}}(\Delta(\pi)_F, \Delta(\sigma)_F) = \dim_q^K \mathrm{Ext}^1_{H_{\theta,K}}(\Delta(\pi)_K, \Delta(\sigma)_K)$$

if and only if the R-module $\mathrm{Ext}^2_{H_{\theta,R}}(\Delta(\pi)_R, \Delta(\sigma)_R)$ is torsion-free.

An important problem in the study of modular representation theory is to determine when the reductions modulo p of irreducible modules remain irreducible. As a final application, for each $\pi \in \mathrm{KP}(\theta)$ we use a universal extension procedure to construct an R-form, $Q(\pi)_R$, of the projective indecomposable modules $P(\pi)_K$. The following result then gives a reformulation of this problem.

Theorem C. *Let $\theta \in Q_+$. Then reductions modulo p of all irreducible $H_{\alpha,K}$-modules are irreducible if and only if one of the following equivalent conditions holds:*

(i) *$Q(\pi)_R \otimes_R F$ is a projective $H_{\theta,F}$-module for all $\pi \in \mathrm{KP}(\theta)$;*

(ii) *$\mathrm{Ext}^1_{H_{\theta,F}}(Q(\pi)_R \otimes_R F, \Delta(\sigma)_F) = 0$ for all $\pi, \sigma \in \mathrm{KP}(\theta)$;*

(iii) *$\mathrm{Ext}^2_{H_{\theta,R}}(Q(\pi)_R, \Delta(\sigma)_R)$ is torsion-free for all $\pi, \sigma \in \mathrm{KP}(\theta)$.*

While the results in Chapter II are valid for arbitrary finite Lie type, in Chapters III and IV we restrict our attention to the finite type A case. In Chapter III we construct explicit projective resolutions of all standard KLR modules. To achieve this, note that each $\pi \in \mathrm{KP}(\alpha)$ can be written uniquely in the form $\pi = (\alpha^m, \beta^n, \dots)$ for positive roots $\alpha > \beta > \cdots$ and integers $m, n, \dots$ such that $m\alpha + n\beta + \cdots = \theta$. It is shown in [BKM] that $\Delta(\pi)$ can then be constructed as an induction product

$$\Delta(\pi) \simeq \Delta(\alpha^m) \circ \Delta(\beta^n) \circ \cdots .$$

Thus, many properties of arbitrary standard modules can be deduced from properties of the *semicuspidal* standard modules $\Delta(\alpha^m)$. In particular, it suffices to construct a projective resolution of $\Delta(\alpha^m)$. This is done in Chapter III, where we construct a resolution of the form

$$\ldots \longrightarrow P_{n+1} \xrightarrow{d_n} P_n \longrightarrow \ldots \longrightarrow P_0 \longrightarrow \Delta(\alpha^m) \longrightarrow 0.$$

Each projective module P_n in the resolution of the form $P_n = \bigoplus_{\lambda \in \Lambda(n)} q^{s_\lambda} H_{\alpha^m} e_\lambda$ for some explicitly defined degree shifts $s_\lambda \in \mathbb{Z}$ and idempotents $e_\lambda \in H_{\alpha^m}$. These are indexed by certain compositions $\lambda \in \Lambda(n)$. Moreover, the differential $d_n : P_{n+1} \to P_n$ is given by right multiplication with a $\Lambda(n+1) \times \Lambda(n)$ matrix $D = (d_n^{\mu,\lambda})$ with entries $d_n^{\mu,\lambda} \in e_\mu H_{\alpha^m} e_\lambda$.

In Chapter IV, we demonstrate the usefulness of these resolutions by way of example. By restricting our attention to the case H_α for α a positive root in type A, we are able to compute all extension groups between standard modules. Moreover, setting $\Delta := \bigoplus_{\pi \in \mathrm{KP}(\alpha)} \Delta(\pi)$, we compute the algebra structure on $\mathrm{Ext}^\bullet(\Delta, \Delta)$. For α of height l and $\pi \in \mathrm{KP}(\alpha)$, we define an explicit ideal $J^\pi \leq \Bbbk[y_1, \dots, y_l]$ and observe that there is a natural refinement operation on the set of Kostant partitions. For $n \in \mathbb{Z}_{\geq 0}$ we define the notion of an n-refinement of any Kostant partition $\pi \in \mathrm{KP}(\alpha)$, and denote by $\mathrm{Ref}_n(\pi) \subseteq \mathrm{KP}(\alpha)$ the collection of all such n-refinements of π. With this set up, we first prove the following:

Theorem D. *Let $\pi, \sigma \in KP(\alpha)$. Then, $\mathrm{Ext}^n_{H_\alpha}(\Delta(\pi), \Delta(\sigma)) = 0$ unless σ is an n-refinement of π. If σ is an n-refinement of π, then there is a graded vector space isomorphism*

$$\mathrm{Ext}^n_{H_\alpha}(\Delta(\pi), \Delta(\sigma)) \simeq q^{-n}\Bbbk[y_1, \ldots, y_l]/J^\pi,$$

where each y_j is of degree 2.

To compute the algebra structure on $\mathrm{Ext}^\bullet(\Delta, \Delta)$, we let $\mathcal{A}$ be the vector space of $\mathrm{KP}(\alpha) \times \mathrm{KP}(\alpha)$ upper triangular matrices, $M = (f_{\sigma,\pi})_{\sigma,\pi \in \mathrm{KP}(\alpha)}$, with entries $f_{\sigma,\pi} \in q^{-n}\Bbbk[y_1, \ldots, y_l]/J^\pi$ if $\sigma \in \mathrm{Ref}_n(\pi)$, and $f_{\sigma,\pi} = 0$ otherwise. We observe that for $\sigma \in \mathrm{Ref}_n(\pi)$, there is a natural surjection

$$\Bbbk[y_1, \ldots, y_l]/J^\sigma \xrightarrow{\rho_{\sigma,\pi}} \Bbbk[y_1, \ldots, y_l]/J^\pi.$$

For $\sigma \in \mathrm{Ref}_n(\pi)$ and $\tau \in \mathrm{Ref}_m(\sigma)$, this allows us to define products by the rule

$$f_{\tau,\sigma} f_{\sigma,\pi} = (-1)^{mn} \rho_{\sigma,\pi}(f_{\tau,\sigma}) f_{\sigma,\pi} \in q^{-(m+n)}\Bbbk[y_1, \ldots, y_n]/J^\pi,$$

for any $f_{\tau,\sigma} \in q^{-m}\Bbbk[y_1, \ldots, y_l]/J^\sigma$ and $f_{\sigma,\pi} \in q^{-n}\Bbbk[y_1, \ldots, y_l]/J^\pi$. Extending this product to usual matrix multiplication then gives $\mathcal{A}$ the structure of a graded, associative algebra. This allows us to state our final result.

Theorem E. *Let α be a positive root in type A. There is an isomorphism of graded, associative algebras* $\mathrm{Ext}^\bullet(\Delta,\Delta) \simeq \mathcal{A}$.

1.3. Preliminaries

In this section we recall some known results and develop notational conventions that will remain throughout the remainder of this thesis unless otherwise stated.

1.3.1. KLR algebras

We follow closely the set up of [BKM]. In particular, Φ is an irreducible root system with simple roots $\{\alpha_i \mid i \in I\}$ and the corresponding set of positive roots Φ_+. Denote by Q the root lattice and by $Q_+ \subset Q$ the set of $\mathbb{Z}_{\geq 0}$-linear combinations of simple roots, and write $\mathrm{ht}(\theta) = \sum_{i\in I} c_i$ for $\theta = \sum_{i\in I} c_i\alpha_i \in Q_+$. The standard symmetric bilinear form $Q \times Q \to \mathbb{Z}$, $(\alpha,\beta) \mapsto \alpha\cdot\beta$ is normalized so that $d_i := (\alpha_i \cdot \alpha_i)/2$ is equal to 1 for the short simple roots α_i. We also set $d_\beta := (\beta\cdot\beta)/2$ for all $\beta \in \Phi_+$. The Cartan matrix is $\mathtt{C} = (\mathtt{c}_{i,j})_{i,j\in I}$ with $\mathtt{c}_{i,j} := (\alpha_i \cdot \alpha_j)/d_i$.

Fix a commutative unital ring $\Bbbk$ and an element $\theta \in Q_+$ of height n. The symmetric group $\mathfrak{S}_n$ with simple transpositions $s_r := (r\, r{+}1)$ acts on the set $I^\theta = \{\boldsymbol{i} = i_1 \cdots i_n \in I^n \mid \sum_{j=1}^n \alpha_{i_j} = \theta\}$ by place permutations. Choose signs $\varepsilon_{i,j}$ for all $i,j \in I$ with $\mathtt{c}_{ij} < 0$ so that $\varepsilon_{i,j}\varepsilon_{j,i} = -1$. With this data, Khovanov-Lauda [KL1, KL2] and Rouquier [Ro] define the $\Bbbk$-algebra H_θ with unit 1_θ, called the *KLR algebra*, given by generators

$$\{1_{\boldsymbol{i}} \mid \boldsymbol{i} \in I^\theta\} \cup \{y_1, \ldots, y_n\} \cup \{\psi_1, \ldots, \psi_{n-1}\}$$

subject only to the following relations:

- $y_r y_s = y_s y_r$;
- $1_{\boldsymbol{i}} 1_{\boldsymbol{j}} = \delta_{\boldsymbol{i},\boldsymbol{j}} 1_{\boldsymbol{i}}$ and $\sum_{\boldsymbol{i} \in I^\theta} 1_{\boldsymbol{i}} = 1_\theta$;
- $y_r 1_{\boldsymbol{i}} = 1_{\boldsymbol{i}} y_r$ and $\psi_r 1_{\boldsymbol{i}} = 1_{s_r \cdot \boldsymbol{i}} \psi_r$;
- $(y_t \psi_r - \psi_r y_{s_r(t)}) 1_{\boldsymbol{i}} = \delta_{i_r, i_{r+1}} (\delta_{t,r+1} - \delta_{t,r}) 1_{\boldsymbol{i}}$;
- $\psi_r^2 1_{\boldsymbol{i}} = \begin{cases} 0 & \text{if } i_r = i_{r+1}, \\ \varepsilon_{i_r, i_{r+1}} \big(y_r{}^{-\mathsf{c}_{i_r, i_{r+1}}} - y_{r+1}{}^{-\mathsf{c}_{i_{r+1}, i_r}} \big) 1_{\boldsymbol{i}} & \text{if } \mathsf{c}_{i_r, i_{r+1}} < 0, \\ 1_{\boldsymbol{i}} & \text{otherwise;} \end{cases}$
- $\psi_r \psi_s = \psi_s \psi_r$ if $|r - s| > 1$;
- $(\psi_{r+1} \psi_r \psi_{r+1} - \psi_r \psi_{r+1} \psi_r) 1_{\boldsymbol{i}} =$
$$\begin{cases} \displaystyle\sum_{r+s=-1-\mathsf{c}_{i_r, i_{r+1}}} \varepsilon_{i_r, i_{r+1}} y_r^r y_{r+2}^s 1_{\boldsymbol{i}} & \text{if } \mathsf{c}_{i_r, i_{r+1}} < 0 \text{ and } i_r = i_{r+2}, \\ 0 & \text{otherwise.} \end{cases}$$

The last relation is called the *braid relation*. Note that H_θ is graded with $\deg 1_{\boldsymbol{i}} = 0$, $\deg(y_r 1_{\boldsymbol{i}}) = 2d_{i_r}$ and $\deg(\psi_r 1_{\boldsymbol{i}}) = -\alpha_{i_r} \cdot \alpha_{i_{r+1}}$.

For each element $w \in \mathfrak{S}_n$, fix a reduced expression $w = s_{r_1} \cdots s_{r_l}$ and set $\psi_w = \psi_{r_1} \cdots \psi_{r_l} \in H_\theta$ (this element depends in general on the choice of a reduced decomposition).

Theorem 1.3.1. (Basis Theorem) *[KL1, Theorem 2.5] The sets*

$$\{\psi_w y_1^{a_1} \cdots y_n^{a_n} 1_{\boldsymbol{i}}\} \quad \textit{and} \quad \{y_1^{a_1} \cdots y_n^{a_n} \psi_w 1_{\boldsymbol{i}}\}, \tag{3.1}$$

with w running over $\mathfrak{S}_n$, a_r running over $\mathbb{Z}_{\geq 0}$, and $\boldsymbol{i}$ running over I^θ, are $\Bbbk$-bases for H_θ.

It follows that H_θ is Noetherian if so is $\Bbbk$, which we always assume from now on. It also follows that for any $1 \leq r \leq n$, the subalgebra $\Bbbk[y_r] \subseteq H_\theta$, generated by y_r, is isomorphic to the polynomial algebra $\Bbbk[y]$—this fact will be often used without further comment. Moreover, for each $\boldsymbol{i} \in I^\theta$, the subalgebra $\mathcal{P}(\boldsymbol{i}) \subseteq 1_{\boldsymbol{i}} H_\alpha 1_{\boldsymbol{i}}$ generated by $\{y_r 1_{\boldsymbol{i}} \mid 1 \leq r \leq n\}$ is isomorphic to a polynomial algebra in n variables. By defining $\mathcal{P}_\theta := \bigoplus_{\boldsymbol{i} \in I^\theta} \mathcal{P}(\boldsymbol{i})$, we obtain a linear action of $\mathfrak{S}_n$ on $\mathcal{P}_\theta$ given by

$$w \cdot y_1^{a_1} \cdots y_n^{a_n} 1_{\boldsymbol{i}} = y_{w(1)}^{a_1} \cdots y_{w(n)}^{a_n} 1_{w \cdot \boldsymbol{i}},$$

for any $w \in \mathfrak{S}_n$, $\boldsymbol{i} \in I^\theta$ and $a_1, \ldots, a_n \in \mathbb{Z}_{\geq 0}$. Setting $\Lambda(\theta) := \mathcal{P}_\theta^{\mathfrak{S}_n}$, we have:

Theorem 1.3.2. *[KL1, Theorem 2.9] $\Lambda(\theta)$ is the center of H_θ.*

If H is a Noetherian graded $\Bbbk$-algebra, we denote by H-mod the category of finitely generated graded left H-modules. The morphisms in this category are all homogeneous degree zero H-module homomorphisms, which we denote $\hom_H(-,-)$. For $V \in H$-mod, let $q^d V$ denote its grading shift by d, so if V_m is the degree m component of V, then $(q^d V)_m = V_{m-d}$. More generally, for a Laurent polynomial

$a = a(q) = \sum_d a_d q^d \in \mathbb{Z}[q, q^{-1}]$ with non-negative coefficients, we set $aV := \bigoplus_d (q^d V)^{\oplus a_d}$.

For $U, V \in H$-mod, we set $\mathrm{Hom}_H(U,V) := \bigoplus_{d \in \mathbb{Z}} \mathrm{Hom}_H(U,V)_d$, where

$$\mathrm{Hom}_H(U,V)_d := \mathrm{hom}_H(q^d U, V) = \mathrm{hom}_H(U, q^{-d} V).$$

We define $\mathrm{ext}^m_H(U,V)$ and $\mathrm{Ext}^m_H(U,V)$ similarly. Since U is finitely generated, $\mathrm{Hom}_H(U,V)$ can be identified in the obvious way with the set of all H-module homomorphisms ignoring the gradings. A similar result holds for $\mathrm{Ext}^m_H(U,V)$, since U has a resolution by finitely generated projective modules. We use $\cong$ to denote an isomorphism in H-mod and $\simeq$ an isomorphism up to a degree shift.

Let q be a variable, and $\mathbb{Z}((q))$ be the ring of Laurent series. The quantum integers $[n] = (q^n - q^{-n})/(q - q^{-1})$ and expressions like $1/(1-q^2)$ are always interpreted as elements of $\mathbb{Z}((q))$. From now on until the end of Section 2.1, we assume that $\Bbbk$ is a field. A graded $\Bbbk$-vector space $V = \bigoplus_{m \in \mathbb{Z}} V_m$ is called *Laurentian* if the graded components V_m are finite dimensional for all $m \in \mathbb{Z}$ and and $V_m = 0$ for $m \ll 0$. The *graded dimension* of a Laurentian vector space V is

$$\dim_q V := \sum_{m \in \mathbb{Z}} (\dim V_m) q^m \in \mathbb{Z}((q)).$$

We always work in the category H_θ-mod. Note that H_θ is Laurentian as a vector space. Therefore so is any $V \in H_\theta$-mod, and then so are all $1_{\boldsymbol{i}} V$ for $\boldsymbol{i} \in I^\theta$. The *formal character* of $V \in H_\theta$-mod is an element of $\bigoplus_{\boldsymbol{i}\in I^\theta} \mathbb{Z}((q))\cdot \boldsymbol{i}$ defined as follows:

$$\mathrm{ch}_q\, V := \sum_{\boldsymbol{i}\in I^\theta} (\dim_q 1_{\boldsymbol{i}} V)\cdot \boldsymbol{i}.$$

Note that $\mathrm{ch}_q\,(q^d V) = q^d \mathrm{ch}_q\,(V)$, where the first q^d means the degree shift. We refer to $1_{\boldsymbol{i}}V$ as the *$\boldsymbol{i}$-weight space* of V and to its vectors as *vectors of weight $\boldsymbol{i}$*.

There is an anti-automorphism of H_θ which fixes all the generators. Given $V \in H_\theta$-mod, we denote $V^\circledast := \mathrm{Hom}_\Bbbk(V, \Bbbk)$ viewed as a left H_θ-module via this anti-automorphism. Note that in general $V^\circledast$ is not finitely generated as an H_θ-module, but we will apply $\circledast$ only to finite dimensional modules. In that case, we have $\mathrm{ch}_q\, V^\circledast = \overline{\mathrm{ch}_q\, V}$, where the bar means the *bar-involution*, i.e. the automorphism of $\mathbb{Z}[q,q^{-1}]$ that swaps q and q^{-1} extended to $\bigoplus_{\boldsymbol{i}\in I^\theta} \mathbb{Z}[q,q^{-1}]\cdot \boldsymbol{i}$.

Let $\gamma_1, \dots, \gamma_m \in Q_+$ and $\theta = \gamma_1 + \cdots + \gamma_m$. Consider the set of concatenations

$$I^{\gamma_1,\dots,\gamma_m} := \{\boldsymbol{i}^1 \cdots \boldsymbol{i}^m \mid \boldsymbol{i}^1 \in I^{\gamma_1}, \dots, \boldsymbol{i}^m \in I^{\gamma_m}\} \subseteq I^\theta.$$

There is a natural (non-unital) algebra embedding

$$\iota_{\gamma_1,\dots,\gamma_m} : H_{\gamma_1} \otimes \cdots \otimes H_{\gamma_m} \to H_\theta, \tag{3.2}$$

which sends the unit $1_{\gamma_1} \otimes \cdots \otimes 1_{\gamma_m}$ to the idempotent

$$1_{\gamma_1,\dots,\gamma_m} := \sum_{\boldsymbol{i}\in I^{\gamma_1,\dots,\gamma_m}} 1_{\boldsymbol{i}} \in H_\theta. \tag{3.3}$$

We have an exact induction functor

$$\mathrm{Ind}^{\theta}_{\gamma_1,\dots,\gamma_m} : (H_{\gamma_1} \otimes \cdots \otimes H_{\gamma_m})\text{-mod} \to H_\theta\text{-mod}$$

defined by $\mathrm{Ind}^{\theta}_{\gamma_1,\dots,\gamma_m} = H_\theta 1_{\gamma_1,\dots,\gamma_m} \otimes_{H_{\gamma_1}\otimes\cdots\otimes H_{\gamma_m}} -$.

For $V_1 \in H_{\gamma_1}$-mod, $\dots, V_m \in H_{\gamma_m}$-mod, we denote by $V_1 \boxtimes \cdots \boxtimes V_m$ the vector space $V_1 \otimes \cdots \otimes V_m$, considered naturally as an $(H_{\gamma_1} \otimes \cdots \otimes H_{\gamma_m})$-module, and set

$$V_1 \circ \cdots \circ V_m := \mathrm{Ind}^{\theta}_{\gamma_1,\dots,\gamma_m} V_1 \boxtimes \cdots \boxtimes V_m.$$

1.3.2. Standard modules

The KLR algebras H_θ are known to be *affine quasihereditary* in the sense of [Kl3], see [Ka, BKM, KlL]. Central to this theory is the notion of *standard modules*, whose definition depends on a choice of a certain partial order. We first fix a *convex order* on Φ_+, i.e. a total order such that whenever α, β, and $\alpha+\beta$ all belong to Φ_+, $\alpha \leq \beta$ implies $\alpha \leq \alpha+\beta \leq \beta$. By [P], there is a one-to-one correspondence between convex orders on Φ_+ and reduced decompositions of the longest element in the corresponding Weyl group.

A *Kostant* partition of $\theta \in Q_+$ is a tuple $\pi = (\pi_1, \dots, \pi_r)$ of positive roots $\pi_1 \geq \pi_2 \geq \cdots \geq \pi_r$ such that $\pi_1 + \cdots + \pi_r = \theta$. Let $\mathrm{KP}(\theta)$ denote the set of all Kostant partitions of θ and for π as above define $\pi'_m = \pi_{r-m+1}$. Now, we have a *bilexicographical partial order* on $\mathrm{KP}(\theta)$, also denoted by $\leq$, i.e. if $\pi = (\pi_1, \ldots, \pi_r), \sigma = (\sigma_1, \ldots, \sigma_s) \in \mathrm{KP}(\theta)$ then $\pi < \sigma$ if and only if the following two conditions are satisfied:

- $\pi_1 = \sigma_1, \ldots, \pi_{l-1} = \sigma_{l-1}$ and $\pi_l < \sigma_l$ for some l;
- $\pi_1' = \sigma_1', \ldots, \pi_{m-1}' = \sigma_{m-1}'$ and $\pi_m' > \sigma_m'$ for some m.

To every $\pi \in \mathrm{KP}(\theta)$, McNamara [M] (cf. [KlR, Theorem 7.2]) associates an absolutely irreducible finite dimensional $\circledast$-self-dual H_θ-module $L(\pi)$ so that $\{L(\pi) \mid \pi \in \mathrm{KP}(\theta)\}$ is a complete, irredundant set of irreducible H_θ-modules, up to isomorphism and degree shift. Since $L(\pi)$ is $\circledast$-self-dual, its formal character is bar-invariant. The key special case is where $\alpha \in \Phi_+$ and $\pi = (\alpha)$, in which case $L(\pi) = L(\alpha)$ is called a *cuspidal irreducible module*. For $m \in \mathbb{Z}_{>0}$, we write (α^m) for the Kostant partition $(\alpha, \ldots, \alpha) \in \mathrm{KP}(m\alpha)$, where α appears m times. The corresponding simple module $L(\alpha^m)$ is called *semicuspidal*. The cuspidal modules have the following nice property:

Lemma 1.3.3. *[M, Lemma 3.4] (cf. [KlR, Lemma 6.6]). For any $\alpha \in \Phi_+$ and $m \in \mathbb{Z}_{>0}$, we have $L(\alpha^m) \simeq L(\alpha)^{\circ m}$.*

If $\pi = (\pi_1, \ldots, \pi_r) \in \mathrm{KP}(\theta)$, the *reduced standard module* is defined to be

$$\bar{\Delta}(\pi) := q^{s(\pi)} L(\pi_1) \circ \cdots \circ L(\pi_m) \tag{3.4}$$

for a specific degree shift $s(\pi)$, whose description will not be important. Note that the Grothendieck group of finite dimensional graded H_θ-modules can be considered as a $\mathbb{Z}[q, q^{-1}]$-module with $q[V] = [qV]$. By [M, Theorem 3.1] (cf. [KlR, 7.2, 7.4]), the H_θ-module $\bar{\Delta}(\pi)$ has simple head $L(\pi)$, and in the Grothendieck group, we have

$$[\bar{\Delta}(\pi)] = [L(\pi)] + \sum_{\sigma<\pi} d_{\pi,\sigma}[L(\sigma)] \tag{3.5}$$

for some coefficients $d_{\pi,\sigma} \in \mathbb{Z}[q, q^{-1}]$, called the *(graded) decomposition numbers*. The decomposition numbers depend on the characteristic of the ground field $\Bbbk$.

Let $P(\pi)$ denote a projective cover of $L(\pi)$ in H_θ-mod. For $V \in H_\theta$-mod we define the (graded) composition multiplicity

$$[V : L(\pi)]_q := \dim_q \operatorname{Hom}(P(\pi), V) \in \mathbb{Z}((q)).$$

The *standard module* $\Delta(\pi)$ is defined as the largest quotient of $P(\pi)$ all of whose composition factors are of the form $L(\sigma)$ with $\sigma \leq \pi$, see [Ka, Corollary 4.13], [BKM, Corollary 3.16], [Kl3, (4.2)]. We note that while the irreducible modules $L(\pi)$ are all finite dimensional, the standard modules $\Delta(\pi)$ are always infinite dimensional. The standard modules have the usual nice properties:

Theorem 1.3.4. [BKM, §3] *Let $\theta \in Q_+$ and $\pi, \sigma \in \mathrm{KP}(\theta)$.*

(i) *$\Delta(\pi)$ has a simple head $L(\pi)$, and $[\Delta(\pi) : L(\sigma)]_q \neq 0$ implies $\sigma \leq \pi$.*

(ii) *We have $\operatorname{Hom}_{H_\theta}(\Delta(\pi), \Delta(\sigma)) = 0$ unless $\pi \leq \sigma$.*

(iii) *For $m \geq 1$, we have $\operatorname{Ext}^m_{H_\theta}(\Delta(\pi), \Delta(\sigma)) = 0$ unless $\pi < \sigma$.*

(iv) *The module $P(\pi)$ has a* finite *filtration $P(\pi) = P_0 \supset P_1 \supset \cdots \supset P_N = 0$ such that $P_0/P_1 \cong \Delta(\pi)$ and for $r = 1, 2, \dots, N-1$ we have $P_r/P_{r+1} \simeq \Delta(\sigma^{(r)})$ for some $\sigma^{(r)} > \pi$.*

(v) *Denoting the graded multiplicities of the factors in a Δ-filtration of $P(\pi)$ by $(P(\pi) : \Delta(\sigma))_q$, we have $(P(\pi) : \Delta(\sigma))_q = d_{\sigma,\pi}(q)$.*

To construct the standard modules more explicitly, let us first assume that $\alpha \in \Phi_+$ and explain how to construct the *cuspidal standard module* $\Delta(\alpha)$. Put $q_\alpha := q^{\alpha\cdot\alpha/2}$. By [BKM, Lemma 3.2], for each $m \in \mathbb{Z}_{>0}$, there exists a unique, up to isomorphism, indecomposable H_α-module $\Delta_m(\alpha)$ such that there are short exact sequences

$$0 \to q_\alpha^{2(m-1)} L(\alpha) \to \Delta_m(\alpha) \to \Delta_{m-1}(\alpha) \to 0,$$
$$0 \to q_\alpha^{2} \Delta_{m-1}(\alpha) \to \Delta_m(\alpha) \to L(\alpha) \to 0,$$

where we use the convention $\Delta_0(\alpha) = 0$. This yields an inverse system

$$\cdots \to \Delta_2(\alpha) \to \Delta_1(\alpha) \to \Delta_0(\alpha),$$

and we have $\Delta(\alpha) \cong \varprojlim \Delta_m(\alpha)$, see [BKM, Corollary 3.16].

Let $m \in \mathbb{Z}_{>0}$. An explicit endomorphism $e_m \in \mathrm{End}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m})^{\mathrm{op}}$ is defined in [BKM, Section 3.2], and then

$$\Delta(\alpha^m) \cong q_\alpha^{m(m-1)/2} \Delta(\alpha)^{\circ m} e_m. \tag{3.6}$$

Finally, for an arbitrary $\theta \in Q_+$ and $\pi \in \mathrm{KP}(\theta)$, gather together the equal parts of π to write $\pi = (\pi_1^{m_1}, \dots, \pi_s^{m_s})$, with $\pi_1 > \cdots > \pi_s$. Then by [BKM, (3.5)],

$$\Delta(\pi) \cong \Delta(\pi_1^{m_1}) \circ \cdots \circ \Delta(\pi_s^{m_s}). \tag{3.7}$$

Thus, cuspidal standard modules are building blocks for arbitrary standard modules. We will need some of their additional properties. Let $\alpha \in \Phi_+$. If $\pi \in \mathrm{KP}(\alpha)$ is minimal such that $\pi > (\alpha)$, then by [BKM, Lemma 2.6], $\pi = (\beta, \gamma)$ for positive roots $\beta > \alpha > \gamma$. In this case, (β, γ) is called a *minimal pair* for α and we write $\mathrm{mp}(\alpha)$ for the set of all such. The following result proved in [BKM, §§3.1,4.3] describes some of the important properties of $\Delta(\alpha)$.

Theorem 1.3.5. *Let $\alpha \in \Phi_+$. Then:*

(i) $[\Delta(\alpha) : L(\alpha)]_q = 1/(1-q_\alpha^2)$ *and* $[\Delta(\alpha) : L(\pi)]_q = 0$ *for* $\pi \neq (\alpha)$.

(ii) Let $\mathbf{C}_\alpha$ be the category of all modules in H_α-mod all of whose composition factors are $\simeq L(\alpha)$. Any $V \in \mathbf{C}_\alpha$ is a finite direct sum of copies of the indecomposable modules $\simeq \Delta_m(\alpha)$ and $\simeq \Delta(\alpha)$. Moreover, $\Delta(\alpha)$ is a projective cover of $L(\alpha)$ in $\mathbf{C}_\alpha$. Furthermore, $\mathrm{Ext}^m_{H_\alpha}(\Delta(\alpha), V) = 0$ *for $m \geq 1$ and $V \in \mathbf{C}_\alpha$.*

(iii) $\mathrm{End}_{H_\alpha}(\Delta(\alpha)) \cong \Bbbk[y]$ *for y in degree $2d_\alpha$.*

(iv) There is a short exact sequence $0 \to q_\alpha^2 \Delta(\alpha) \to \Delta(\alpha) \to L(\alpha) \to 0$.

(v) For $(\beta, \gamma) \in \mathrm{mp}(\alpha)$ there is a short exact sequence

$$0 \to q^{-\beta\cdot\gamma}\Delta(\beta) \circ \Delta(\gamma) \xrightarrow{\varphi} \Delta(\gamma) \circ \Delta(\beta) \to [p_{\beta,\gamma}+1]\Delta(\alpha) \to 0,$$

where $p_{\beta,\gamma}$ is the largest integer p such that $\beta - p\gamma$ is a root.

Corollary 1.3.6. *Let $\alpha \in \Phi_+$. The dimensions of the graded components $\Delta(\alpha)_d$ are 0 for $d \ll 0$ and are bounded above by some $N > 0$ independent of d.*

Proof. By Theorem 1.3.5(i), we have $\dim_q \Delta(\alpha) = \frac{1}{1-q_\alpha^2} \dim_q L(\alpha)$, which implies the result since $L(\alpha)$ is finite dimensional. □

1.3.3. Endomorphisms of standard modules

We shall denote by y_α the degree $2d_\alpha$ endomorphism of $\Delta(\alpha)$ which corresponds to y under the algebra isomorphism $\mathrm{End}_{H_\alpha}(\Delta(\alpha)) \cong \Bbbk[y]$ in Theorem 1.3.5(iii).

Lemma 1.3.7. *Let $\alpha \in \Phi_+$. Then every non-zero H_α-endomorphism of $\Delta(\alpha)$ is injective, and every submodule of $\Delta(\alpha)$ is equal to $y_\alpha^s(\Delta(\alpha)) \cong q_\alpha^{2s}\Delta(\alpha)$ for some $s \in \mathbb{Z}_{\geq 0}$.*

Proof. It follows from the construction of y_α in [BKM, Theorem 3.3] that y_α is injective and $y_\alpha(\Delta(\alpha)) \cong q_\alpha^2\Delta(\alpha)$. This in particular implies the first statement.

Let $V \subseteq \Delta(\alpha)$ be a submodule and $f : V \to \Delta(\alpha)$ be the natural inclusion. First, assume that V is indecomposable. By Theorem 1.3.5(ii), up to degree shift, V is isomorphic to $\Delta(\alpha)$ or $\Delta_m(\alpha)$ for some $m \geq 1$. If $V \simeq \Delta_m(\alpha)$ then $\Delta(\alpha)/V$ is infinite dimensional and has a simple head, so by Theorem 1.3.5(ii) again, $\Delta(\alpha)/V \simeq \Delta(\alpha)$. Then the short exact sequence

$$0 \to V \to \Delta(\alpha) \to \Delta(\alpha)/V \to 0$$

splits by projectivity in Theorem 1.3.5(ii), contradicting indecomposability of $\Delta(\alpha)$. If instead $V \simeq \Delta(\alpha)$, consider the composition

$$\Delta(\alpha) \xrightarrow{\sim} V \xrightarrow{f} \Delta(\alpha).$$

This produces a graded endomorphism of $\Delta(\alpha)$, so that $V = y_\alpha^s(\Delta(\alpha))$ for some $s \geq 0$. Since there are inclusions $\Delta(\alpha) \supset y_\alpha \Delta(\alpha) \supset y_\alpha^2 \Delta(\alpha) \supset \cdots$, the general case follows from the case when V is indecomposable.

$\square$

Let again $\alpha \in \Phi_+$. We next consider the standard modules of the form $\Delta(\alpha^m)$. By functoriality, the endomorphism $\mathrm{id}^{\otimes(r-1)} \otimes y_\alpha \otimes \mathrm{id}^{\otimes m-r}$ of the $H_\alpha^{\otimes m}$-module $\Delta(\alpha)^{\boxtimes m}$ induces an endomorohism Y_r of the $H_{m\alpha}$-module $\Delta(\alpha)^{\circ m}$. The endomorphisms $Y_1, \dots, Y_m \in \mathrm{End}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m})$ commute. Moreover, in [BKM, Section 3.2], some additional endomorphisms $\partial_1, \dots, \partial_{m-1} \in \mathrm{End}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m})$ are constructed, and it is proved in [BKM, Lemmas 3.7–3.9] that the algebra $\mathrm{End}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m})^{\mathrm{op}}$ is isomorphic to the nilHecke algebra NH_m, with $\partial_1, \dots, \partial_{m-1}$ and (appropriately scaled) $Y_1, \dots, Y_m$ corresponding to the standard generators of NH_m. The element e_m used in (3.6) is an explicit idempotent in NH_m. Consider the algebra of symmetric functions

$$\Lambda_{\alpha,m} := \Bbbk[Y_1, \dots, Y_m]^{\mathfrak{S}_m} = Z(NH_m),$$

with the variables Y_r in degree $2d_\alpha$. Note that $\dim_q \Lambda_{\alpha,m} = 1/\prod_{r=1}^m (1-q_\alpha^{2r})$. It is known, see e.g. [KLM, Theorem 4.4(iii)], that

$$e_m NH_m e_m = e_m \Lambda_{\alpha,m} \cong \Lambda_{\alpha,m}. \tag{3.8}$$

Theorem 1.3.8. *Let $\alpha \in \Phi_+$ and $m \in \mathbb{Z}_{>0}$. Then:*

(i) For any $\pi \in \mathrm{KP}(m\alpha)$, we have $[\Delta(\alpha^m) : L(\pi)]_q = \delta_{\pi,(\alpha^m)}/\prod_{r=1}^m (1-q_\alpha^{2r})$.

(ii) The module $\Delta(\alpha^m)$ is a projective cover of $L(\alpha^m)$ in the category of all modules in H_α-mod all of whose composition factors are $\simeq L(\alpha^m)$.

(iii) $\mathrm{End}_{H_\alpha}(\Delta(\alpha)) \cong \Lambda_{\alpha,m}$.

(iv) Every submodule of $\Delta(\alpha^m)$ is isomorphic to $q^d\Delta(\alpha^m)$ for some $d \in \mathbb{Z}_{\geq 0}$, and every non-zero $H_{m\alpha}$-endomorphism of $\Delta(\alpha^m)$ is injective.

Proof. Part (i) is [BKM, Lemma 3.10], and part (ii) follows from [Kl3, Lemma 4.11], since (α^m) is minimal in $\mathrm{KP}(\alpha)$ by convexity. By (i) and (ii), we have that $\dim_q \mathrm{End}_{H_{m\alpha}}(\Delta(\alpha^m)) = 1/\prod_{r=1}^m (1-q_\alpha^{2r})$.

(iii) We have that $NH_m = \mathrm{End}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m})^{\mathrm{op}}$ acts naturally on $\Delta(\alpha)^{\circ m}$ on the right, and so $\Lambda_{\alpha,m} = Z(NH_m)$ acts naturally on $\Delta(\alpha^m) = \Delta(\alpha)^{\circ m} e_m$. This defines an embedding $\Lambda_{\alpha,m} \to \mathrm{End}_{H_{m\alpha}}(\Delta(\alpha^m))$. This embedding must be an isomorphism by dimensions.

(iv) In view of Lemma 1.3.7, every non-zero

$$f \in \Bbbk[Y_1, \dots, Y_m] \subseteq NH_m = \mathrm{End}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m})^{\mathrm{op}}$$

acts as an injective linear operator on $\Delta(\alpha)^{\circ m}$. The result now follows from (3.8) and (ii). $\square$

Finally, we consider a general case. Let $\theta \in Q_+$ and $\pi = (\pi_1^{m_1}, \dots, \pi_s^{m_s}) \in \mathrm{KP}(\theta)$ with $\pi_1 > \cdots > \pi_s$. By functoriality of induction, we have a natural embedding

$$\Lambda_{\pi_1,m_1} \otimes \cdots \otimes \Lambda_{\pi_s,m_s} \to \mathrm{End}_{H_\theta}(\Delta(\pi)),\ f_1 \otimes \cdots \otimes f_s \mapsto f_1 \circ \cdots \circ f_s. \quad (3.9)$$

Theorem 1.3.9. *Let $\theta \in Q_+$ and $\pi = (\pi_1^{m_1}, \dots, \pi_s^{m_s}) \in \mathrm{KP}(\theta)$ with $\pi_1 > \cdots > \pi_s$. Then*

$$\mathrm{End}_{H_\theta}(\Delta(\pi)) \cong \Lambda_{\pi_1,m_1} \otimes \cdots \otimes \Lambda_{\pi_s,m_s}$$

via (3.9), and every non-zero H_θ-endomorphism of $\Delta(\pi)$ is injective.

Proof. It is easy to see from Theorem 1.3.8(iv) that every non-zero endomorphism in the image of the embedding (3.9) is injective. To see that there are no other endomorphisms, we first use adjointness of End and Res to see that $\mathrm{End}_{H_\theta}(\Delta(\pi))$ is isomorphic to

$$\mathrm{Hom}_{H_{m_1\pi_1} \otimes \cdots \otimes H_{m_s\pi_s}}(\Delta(\pi_1^{m_1}) \boxtimes \cdots \boxtimes \Delta(\pi_s^{m_s}), \mathrm{Res}^{\theta}_{m_1\pi_1,\dots,m_s\pi_s} \Delta(\pi)),$$

and then note that by the Mackey Theorem, as in [M, Lemma 3.3], we we have $\mathrm{Res}^{\theta}_{m_1\pi_1,\dots,m_s\pi_s}\Delta(\pi)\cong\Delta(\pi_1^{m_1})\boxtimes\cdots\boxtimes\Delta(\pi_s^{m_s})$. $\square$

Chapter II

PROOF OF THEOREM A AND APPLICATIONS

This chapter contains material coauthored by the author and his thesis advisor, Alexander Kleshchev. While both parties offered significant contributions, it is impossible to distinguish their exact inputs.

2.1. Proof of Theorem A

We give the proof of Theorem A based on the recent work of Kashiwara-Park [KP]. Our original proof was different and relied on some unpleasant computation for non-simply-laced types. For simply laced types however, our original proof is very simple and elementary, and so we give it later in this section, too.

2.1.1. Proof of Theorem A modulo a hypothesis

The following hypothesis concerns a fundamental property of cuspidal standard modules and is probably true beyond finite Lie types:

Hypothesis 2.1.1. Let α be a positive root of height n and $1 \leq r \leq n$. Then upon restriction to the subalgebra $\Bbbk[y_r] \subseteq H_\alpha$, the module $\Delta(\alpha)$ is free of finite rank.

The goal of this subsection is to prove Theorem A assuming the hypothesis. In §2.1.2 the hypothesis will be proved using results of

Kashiwara and Park, while in §2.1.3 we will give a more elementary proof for simply laced types.

Lemma 2.1.2. *Hypothesis 2.1.1 is equivalent to the property that* $y_1, \ldots, y_n$ *act by injective linear operators on* $\Delta(\alpha)$.

Proof. The forward direction is clear. For the converse, assume that y_r acts injectively on $\Delta(\alpha)$. We construct a finite basis for $1_{\boldsymbol{i}}\Delta(\alpha)$ as a $\Bbbk[y_r]$-module for every $\boldsymbol{i} \in I^\alpha$. Let $m := \deg(y_r 1_{\boldsymbol{i}})$. For every $a = 0, 1, \ldots, m-1$, let d_a be minimal with $d_a \equiv a \pmod{m}$ and $1_{\boldsymbol{i}}\Delta(\alpha)_{d_a} \neq 0$. Pick a linear basis of $\oplus_{a=0}^{m-1} 1_{\boldsymbol{i}}\Delta(\alpha)_{d_a}$ and note that the $\Bbbk[y_r]$-module generated by the elements of this basis is free. Factor out this $\Bbbk[y_r]$-submodule, and repeat. The process will stop after finitely many steps, thanks to Corollary 1.3.6. □

While Hypothesis 2.1.1 claims that every $\Bbbk[y_r]$ acts freely on $\Delta(\alpha)$, no $\Bbbk[y_r, y_s]$ does:

Lemma 2.1.3. *Let* $\alpha \in \Phi_+$ *be a root of height* $n > 1$. *Then, for every vector* $v \in \Delta(\alpha)$, *and distinct* $r, s \in \{1, \cdots, n\}$, *there is a polynomial* $f \in \Bbbk[x, y]$ *such that* $f(y_r, y_s)v = 0$.

Proof. We may assume v is a homogenous weight vector. By Corollary 1.3.6, the dimensions of the graded components of $\Delta(\alpha)$ are uniformly bounded. The result follows, as the number of linearly independent degree d monomials in x, y grows without bound. □

One can say more about the polynomial f in the lemma, see for example Proposition 2.1.11.

Now, let $\theta \in Q_+$ be arbitrary of height n, and $\pi = (\pi_1 \geq \cdots \geq \pi_l) \in \mathrm{KP}(\theta)$. Setting $\mathfrak{S}_\pi := \mathfrak{S}_{\mathrm{ht}(\pi_1)} \times \cdots \times \mathfrak{S}_{\mathrm{ht}(\pi_l)} \subset \mathfrak{S}_n$, integers $r, s \in \{1, \dots, n\}$ are called π-*equivalent*, written $r \sim_\pi s$, if they belong to the same orbit of the action of $\mathfrak{S}_\pi$ on $\{1, \dots, n\}$. Finally, recalling the idempotents (3.3), we set

$$1_\pi := 1_{\pi_1, \dots, \pi_l}.$$

Lemma 2.1.4. *Let $\theta \in Q_+$, $n = \mathrm{ht}(\theta)$, and $\pi \not\geq \sigma$ be elements of* $\mathrm{KP}(\theta)$. *If $w \in \mathfrak{S}_n$ satisfies $1_\pi \psi_w 1_\sigma \neq 0$ then there exists some $1 \leq r < n$ such that $r \sim_\pi r+1$, but $w^{-1}(r) \not\sim_\sigma w^{-1}(r+1)$.*

Proof. Write $\pi = (\pi_1 \geq \cdots \geq \pi_l)$ and $\sigma = (\sigma_1 \geq \cdots \geq \sigma_m)$. The assumption $1_\pi \psi_w 1_\sigma \neq 0$ implies that $\boldsymbol{i}^\pi = w \cdot \boldsymbol{i}^\sigma$ for some $\boldsymbol{i}^\pi \in I^{\pi_1, \dots, \pi_l}$ and $\boldsymbol{i}^\sigma \in I^{\sigma_1, \dots, \sigma_m}$. Write $\boldsymbol{i}^\pi := \boldsymbol{i}_1^\pi \cdots \boldsymbol{i}_l^\pi$ with $\boldsymbol{i}_a^\pi \in I^{\pi_a}$ for all a, and $\boldsymbol{i}^\sigma := \boldsymbol{i}_1^\sigma \cdots \boldsymbol{i}_m^\sigma$ with $\boldsymbol{i}_b^\sigma \in I^{\sigma_b}$ for all b. Assume for a contradiction that for every $1 \leq r < n$ we have $r \sim_\pi r+1$ implies that $w^{-1}(r) \sim_\sigma w^{-1}(r+1)$. Then there is a partition $\{1, \dots, l\} = \bigsqcup_{b=1}^m A_b$ such that $\sigma_b = \sum_{a \in A_b} \pi_a$ for all $b = 1, \dots, m$. By convexity, cf. [BKM, Lemma 2.4], we have $\min\{\pi_a \mid a \in A_b\} \leq \sigma_b \leq \max\{\pi_a \mid a \in A_b\}$. This implies $\pi \geq \sigma$. $\square$

Theorem 2.1.5. *Let $\theta \in Q_+$ and $\pi, \sigma \in \mathrm{KP}(\theta)$. If $\pi \neq \sigma$, then*

$$\mathrm{Hom}_{H_\theta}(\Delta(\pi), \Delta(\sigma)) = 0.$$

Proof. Let $n = \mathrm{ht}(\theta)$ and write $\pi = (\pi_1 \geq \cdots \geq \pi_l)$ and $\sigma = (\sigma_1 \geq \cdots \geq \sigma_m)$. It suffices to prove that

$$\mathrm{Hom}_{H_\theta}(\Delta(\pi_1) \circ \cdots \circ \Delta(\pi_l), \Delta(\sigma_1) \circ \cdots \circ \Delta(\sigma_m)) = 0.$$

If not, let φ be a nonzero homomorphism. By Theorem 1.3.4(ii), we may assume that $\pi < \sigma$. Using Lemma 2.1.3, pick a generator $v \in \Delta(\pi_1) \circ \cdots \circ \Delta(\pi_l)$ such that $v = 1_\pi v$ and for any $r \sim_\pi r+1$, there is a non-zero polynomail $f \in \Bbbk[x, y]$ with $f(y_r, y_{r+1})v = 0$. Then $f(y_r, y_{r+1})\varphi(v) = 0$ as well.

Denote by $\mathcal{D}^\sigma$ the set of shortest length coset representatives for $\mathfrak{S}_n/\mathfrak{S}_\sigma$. Then, we can write $\varphi(v) = \sum_{w \in \mathcal{D}^\sigma} \psi_w \otimes v_w$ for some $v_w \in \Delta(\sigma_1) \otimes \cdots \otimes \Delta(\sigma_m)$. Since $\varphi(v) = 1_\pi \varphi(v)$ and $1_\sigma v_w = v_w$, we have that $1_\pi \psi_w 1_\sigma \neq 0$ whenever $v_w \neq 0$. In particular, if $u \in \mathcal{D}^\sigma$ is an element of maximal length such that $v_u \neq 0$, then by Lemma 2.1.4, $r \sim_\pi r+1$ and $u^{-1}(r) \not\sim_\sigma u^{-1}(r+1)$ for some $1 \leq r < n$.

Now, we have:

$$\begin{aligned} f(y_r, y_{r+1})\varphi(v) &= f(y_r, y_{r+1}) \sum_{w \in \mathcal{D}^\sigma} \psi_w \otimes v_w \\ &= f(y_r, y_{r+1})\psi_u \otimes v_u + \sum_{w \neq u} f(y_r, y_{r+1})\psi_w \otimes v_w \\ &= \psi_u \otimes f(y_{u^{-1}(r)}, y_{u^{-1}(r+1)})v_u + (*), \end{aligned}$$

where $(*)$ is a sum of elements of the form $\psi_w \otimes v'_w$ with $v'_w \in \Delta(\sigma_1) \otimes \cdots \otimes \Delta(\sigma_m)$ and $w \in \mathcal{D}^\sigma \setminus \{u\}$. The last equality follows because in H_θ for all $1 \leq t \leq n$ and $w \in \mathfrak{S}_n$, we have that $y_t \psi_w = \psi_w y_{w^{-1}(t)} + (**)$, where

$(**)$ is a linear combination of elements of the form ψ_y with $y \in \mathfrak{S}_n$ being Bruhat smaller than w.

Since $u^{-1}(r) \not\sim_\sigma u^{-1}(r+1)$, there are distinct integers $a, b \in \{1, \dots, m\}$ and integers $1 \leq c \leq \mathrm{ht}(\sigma_a)$ and $1 \leq d \leq \mathrm{ht}(\sigma_b)$ such that for any pure tensor $v = v^1 \otimes \cdots \otimes v^m \in \Delta(\sigma_1) \otimes \cdots \otimes \Delta(\sigma_m)$, and $s, t \in \mathbb{Z}_{\geq 0}$, we have

$$y^s_{u^{-1}(r)} y^t_{u^{-1}(r+1)} v = v^1 \otimes \cdots \otimes y^s_c v^a \otimes \cdots \otimes y^t_d v^b \otimes \cdots \otimes v^m.$$

By Hypothesis 2.1.1, $f(y_{u^{-1}(r)}, y_{u^{-1}(r+1)}) v_u \neq 0$. Hence $f(y_r, y_{r+1}) \varphi(v) \neq 0$ giving a contradiction. $\square$

2.1.2. Proof of Hypothesis 2.1.1 using Kashiwara-Park Lemma

We begin with a key lemma which follows immediately from the results of [KP]:

Lemma 2.1.6. *Let $\alpha \in \Phi_+$, $n = \mathrm{ht}(\alpha)$ and $i \in I$. Define*

$$\mathfrak{p}_{i,\alpha} := \sum_{\boldsymbol{i} \in I^\alpha} \left(\prod_{r \in [1,n], i_r = i} y_r \right) 1_{\boldsymbol{i}}.$$

Then $\mathfrak{p}_{i,\alpha} \Delta(\alpha) \neq 0$.

Proof. This follows from [KP, Definition 2.2(b)] and [KP, Proposition 3.5]. $\square$

Theorem 2.1.7. *Let $\alpha \in \Phi_+$ have height n. Then, $y_r^m v \neq 0$ for all $1 \leq r \leq n$, $m \in \mathbb{Z}_{\geq 0}$, and nonzero $v \in \Delta(\alpha)$. In particular, Hypothesis 2.1.1 holds.*

Proof. The 'in particular' statement follows from Lemma 2.1.2.

We may assume that v is a weight vector of some weight $\boldsymbol{i}$. Let $i = i_r$. The element $\mathfrak{p}_{i,\alpha}$ defined in Lemma 2.1.6 is central by Theorem 1.3.2. By Lemma 2.1.6 and Theorem 1.3.9, the multiplication with $\mathfrak{p}_{i,\alpha}$ on $\Delta(\alpha)$ is injective, so multiplication with $\mathfrak{p}_{i,\alpha}^m$ is also injective. But $\mathfrak{p}_{i,\alpha}^m$ involves $y_r 1_{\boldsymbol{i}}$, so $0 \neq \mathfrak{p}_{i,\alpha}^m v = h y_r^m v$ for some $h \in H_\alpha$, and the theorem follows. $\square$

2.1.3. Elementary proof of the Hypothesis for simply laced types

Throughout this subsection, we assume that the root system Φ is of (finite) ADE type. Let $\theta = a_1\alpha_1 + \cdots + a_l\alpha_l \in Q_+$ and $n = \mathrm{ht}(\theta) = a_1 + \cdots + a_l$. Pick a permutation $(i_1, \dots, i_l)$ of $(1, \dots, l)$ with $a_{i_1} > 0$, and define $\boldsymbol{i} := i_1^{a_{i_1}} \cdots i_l^{a_{i_l}} \in I^\theta$. Then the stabilizer of $\boldsymbol{i}$ in $\mathfrak{S}_n$ is the standard parabolic subgroup $\mathfrak{S}_{\boldsymbol{i}} := \mathfrak{S}_{a_{i_1}} \times \cdots \times S_{a_{i_l}}$. Let $\mathcal{D}^{\boldsymbol{i}}$ be a set of coset representatives for $\mathfrak{S}_n/\mathfrak{S}_{\boldsymbol{i}}$. Then by Theorem 1.3.2, the element

$$z = z_{\boldsymbol{i}} := \sum_{w \in \mathcal{D}^{\boldsymbol{i}}} (y_{w(1)} + \cdots + y_{w(a_{i_1})}) 1_{w \cdot \boldsymbol{i}} \tag{1.1}$$

is central of degree 2 in H_θ. For any $1 \leq r \leq n$, note that

$$a_{i_1} y_r = z - \sum_{w \in \mathcal{D}^{\boldsymbol{i}}} ((y_{w(1)} - y_r) + \cdots + (y_{w(a_{i_1})} - y_r)) 1_{w \cdot \boldsymbol{i}}. \tag{1.2}$$

Let H'_θ be the subalgebra of H_θ generated by

$$\{1_{\boldsymbol{i}} \mid \boldsymbol{i} \in I^\theta\} \cup \{\psi_r \mid 1 \le r < n\} \cup \{y_r - y_{r+1} \mid 1 \le r < n\}.$$

For the reader's convenience, we reprove a lemma from [BK1, Lemma 3.1]:

Lemma 2.1.8. *Let θ, $\boldsymbol{i}$, and z be as above. Then:*

(i) $\{(y_1 - y_2)^{m_1} \cdots (y_{n-1} - y_n)^{m_{n-1}} \psi_w 1_{\boldsymbol{i}} \mid m_r \in \mathbb{Z}_{\ge 0}, w \in \mathfrak{S}_n, \boldsymbol{i} \in I^\theta\}$ *is a basis for H'_θ.*

(ii) *If $a_{i_1} \cdot 1_{\Bbbk} \neq 0$ in $\Bbbk$, then there is an algebra isomorphism*

$$H_\theta \cong H'_\theta \otimes \Bbbk[z]. \tag{1.3}$$

Proof. In view of the basis (3.1), part (i) follows on checking that the span of the given monomials is closed under multiplication, which follows from the defining relations. For (ii), note using (1.2) that the natural multiplication map $\Bbbk[z] \otimes H'_\theta \to H_\theta$ is surjective. It remains to observe that the two algebras have the same graded dimension. □

Let α now be a positive root. Then one can always find an index i_1 with $a_{i_1} \cdot 1_{\Bbbk} \neq 0$, so in this case we always have (1.3) for an appropriate choice of $\boldsymbol{i}$. We always assume that this choice has been made. Following [BK1], we can now present another useful description of the cuspidal standard module $\Delta(\alpha)$. Denote by $L'(\alpha)$ the restriction of the cuspidal irreducible module $L(\alpha)$ from H_α to H'_α.

Lemma 2.1.9. *Let $\alpha \in \Phi_+$.*

(i) *$L'(\alpha)$ is an irreducible H'_α-module.*

(ii) *$\Delta(\alpha) \cong H_\alpha \otimes_{H'_\alpha} L'(\alpha)$.*

(iii) *The element z acts on $\Delta(\alpha)$ freely.*

Proof. Note that z acts as zero on $L(\alpha)$, which implies (i) in view of (1.3). Moreover, it is now easy to see that $H_\alpha \otimes_{H'_\alpha} L'(\alpha)$ has a filtration with the subfactors isomorphic to $q^{2d}L(\alpha)$ for $d = 0, 1, \dots$. Furthermore, by Frobenius Reciprocity and (i), the module $H_\alpha \otimes_{H'_\alpha} L'(\alpha)$ has simple head $L(\alpha)$. Now (ii) follows from Theorem 1.3.5(ii). Finally, (iii) follows from (ii) and (1.3). $\square$

Using the description of $\Delta(\alpha)$ from Lemma 2.1.9(ii), we can now establish Hypothesis 2.1.1:

Theorem 2.1.10. *Let $\alpha \in \Phi_+$ and $\{v_1, \dots, v_N\}$ be a $\Bbbk$-basis of $L'(\alpha)$. Then the $\Bbbk[y_r]$-module $\Delta(\alpha)$ is free with basis $\{1 \otimes v_1, \dots, 1 \otimes v_N\}$. In particular, Hypothesis 2.1.1 holds for simply laced types.*

Proof. By (1.2), we can write $y_r = \frac{1}{a_{i_1}} z + (*)$, where $(*)$ is an element of H'_α. For each $1 \le m \le N$, we have

$$y_r^b(1 \otimes v_m) = \left(\frac{1}{a_{i_1}}\right)^b z^b \otimes v_m + (**),$$

where $(**)$ is a linear combination of terms of the form $z^c \otimes v_t$ with $c < b$. So $\{1 \otimes v_1, \dots, 1 \otimes v_N\}$ is a basis of the free $\Bbbk[y_r]$-module $\Delta(\alpha)$. $\square$

The following strengthening of Lemma 2.1.3 is not needed for the proof of Theorem A, but we include it for completeness.

Proposition 2.1.11. *Let $\alpha \in \Phi_+$ and $n = \mathrm{ht}(\alpha)$. For any $1 \le r, s \le n$, there is $d \in \mathbb{Z}_{>0}$ such that $(y_r - y_s)^d$ annihilates $\Delta(\alpha)$.*

Proof. Pick d such that $(y_r - y_s)^d$ annihilates $L(\alpha)$. Since $\Delta(\alpha) = H_\alpha \otimes_{H'_\alpha} L'(\alpha)$ is spanned by vectors of the form $z^m \otimes v'$ with $m \in \mathbb{Z}_{\ge 0}$ and $v' \in L'(\alpha)$, it suffices to note that $(y_r - y_s)^d(z^m \otimes v') = z^m \otimes (y_r - y_s)^d v' = 0$. $\square$

2.2. Reduction Modulo p

Let p be a fixed prime number, and F be the prime field of characteristic p. We will use the p-modular system (F, R, K) with $R = \mathbb{Z}_p$ and $K = \mathbb{Q}_p$. Note that $R/pR = F$.

From now on, we will work with different ground rings, so our notation needs to become more elaborate. Recall that the KLR algebra H_θ is defined over an arbitrary commutative unital ring $\Bbbk$, and to emphasize which $\Bbbk$ we are working with, we will use the notation $H_{\theta,\Bbbk}$ or $H_{\theta;\Bbbk}$. In all our notation we will now use the corresponding index. For example, let $\Bbbk$ be a field. We now denote the irreducible cuspidal modules over $H_{\theta,\Bbbk}$ by $L(\theta)_\Bbbk$. We now write $\dim^\Bbbk V$ for the dimension of a $\Bbbk$-vector space V, and $\dim_q^\Bbbk V$ for the graded dimension of a graded $\Bbbk$-vector space V.

If V is a finitely generated R-module, we write $\mathrm{d}^R V := \dim^{R/pR}(V/pV)$, which, by Nakayama's Lemma, equals the number of generators in any minimal generating set of V. If V is a graded R-

module with finitely generated graded components V_m such that $V_m = 0$ for $m \ll 0$, we set

$$\mathrm{d}_q^R\, V := \sum_{m\in\mathbb{Z}} (\mathrm{d}^R\, V_m) q^m \in \mathbb{Z}((q)).$$

Let $\Bbbk \in \{F, R, K\}$, and B be a connected positively graded $\Bbbk$-algebra, so that $B/B_{>0} \cong \Bbbk$. If V is a finitely generated graded B-module we define

$$\mathrm{d}_q^B\, V := \mathrm{d}_q^{B/B_{>0}}(V/B_{>0}V) \in \mathbb{Z}[q, q^{-1}].$$

By Nakayama's Lemma, if $\{v_1, \dots, v_r\}$ is a minimal set of homogeneous generators of the B-module V, then $\mathrm{d}_q^B\, V = q^{\deg(v_1)} + \cdots + q^{\deg(v_r)}$.

2.2.1. Changing scalars

In this subsection we develop the usual formalism of modular representation theory for KLR algebras. There will be nothing surprising here, but we need to exercise care since we work with infinite dimensional algebras and often with infinite dimensional modules.

Recall from Section 1.3 that for a left Noetherian graded algebra R, we denote by R-mod the category of finitely generated graded R-modules, for which we have the groups $\mathrm{ext}^i_R(V, W)$ and $\mathrm{Ext}^i_R(V, W)$. To deal with change of scalars in Ext groups, we will use the following version of the Universal Coefficient Theorem:

Theorem 2.2.1. (Universal Coefficient Theorem) *Let V_R, W_R be modules in $H_{\theta,R}$-*mod*, free as R-modules, and $\Bbbk$ be an R-algebra. Then* *for every $j \in \mathbb{Z}_{\geq 0}$ there is an exact sequence of (graded) $\Bbbk$-modules*

$$0 \to \mathrm{Ext}^j_{H_{\theta,R}}(V_R, W_R) \otimes_R \Bbbk \to \mathrm{Ext}^j_{H_{\theta,\Bbbk}}(V_R \otimes_R \Bbbk, W_R \otimes_R \Bbbk)$$
$$\to \mathrm{Tor}^R_1\left(\mathrm{Ext}^{j+1}_{H_{\theta,R}}(V_R, W_R), \Bbbk\right) \to 0.$$

In particular,

$$\mathrm{Ext}^j_{H_{\theta,R}}(V_R, W_R) \otimes_R K \cong \mathrm{Ext}^j_{H_{\theta,K}}(V_R \otimes_R K, W_R \otimes_R K).$$

Proof. The standard proof for the ungraded modules works in our setting. First, apply $\mathrm{Hom}_{H_{\theta,R}}(-, W_R)$ to a free resolution of V_R to get a complex $C_\bullet$ of free (graded) R-modules with finitely many generators in every graded degree. Then follow the proof of [R, Theorem 8.22]. The second statement follows from the first since K is a flat R-module. □

We need another standard result, whose proof is omitted.

Lemma 2.2.2. *Let $\Bbbk = K$ or F, $V_R, W_R \in H_{\theta,R}$-mod be free as R-modules, and*

$$0 \to W_R \xrightarrow{\iota} E_R \xrightarrow{\rho} V_R \to 0$$

be the extension corresponding to a class $\xi \in \mathrm{Ext}^1_{H_{\theta,R}}(V_R, W_R)$. Identifying $\mathrm{Ext}^1_{H_{\theta,R}}(V_R, W_R) \otimes_R \Bbbk$ with a subgroup of $\mathrm{Ext}^1_{H_{\theta,\Bbbk}}(V_R \otimes_R \Bbbk, W_R \otimes_R \Bbbk)$, we have that

$$0 \to W_R \otimes_R \Bbbk \xrightarrow{\iota \otimes \mathrm{id}_\Bbbk} E_R \otimes_R \Bbbk \xrightarrow{\rho \otimes \mathrm{id}_\Bbbk} V_R \otimes_R \Bbbk \to 0$$

is the extension corresponding to a class $\xi \otimes 1_\Bbbk \in \mathrm{Ext}^1_{H_{\theta,R}}(V_R, W_R) \otimes_R \Bbbk$.

Let $\Bbbk = K$ or F, and $V_\Bbbk$ be an $H_{\theta,\Bbbk}$-module. We say that an $H_{\theta,R}$-module V_R is an *R-form of $V_\Bbbk$* if every graded component of V_R is free of finite rank as an R-module and, identifying $H_{\theta,R} \otimes_R \Bbbk$ with $H_{\theta,\Bbbk}$, we have $V_R \otimes_R \Bbbk \cong V_\Bbbk$ as $H_{\theta,\Bbbk}$-modules. If $\Bbbk = K$, by a *full lattice* in V_K we mean an R-submodule V_R of V_K such that every graded component $V_{d,R}$ of V_R is a finite rank free R-module which generates the graded component $V_{d,K}$ as a K-module. If V_R is an $H_{\theta,R}$-invariant full lattice in V_K, it is an R-form of V_K. Now we can see that every $V_K \in H_{\theta,K}$-mod has an R-form: pick $H_{\theta,K}$-generators $v_1, \dots, v_r$ and define $V_R := H_{\theta,R} \cdot v_1 + \dots + H_{\theta,R} \cdot v_1$.

The projective indecomposable modules over $H_{\theta,F}$ have projective R-forms. Indeed, any $P(\pi)_F$ is of the form $H_{\theta,F}e_{\pi,F}$ for some *degree zero* idempotent $e_{\pi,F}$. By the Basis Theorem, the degree zero component $H_{\theta,F,0}$ of $H_{\theta,F}$ is defined over R; more precisely, we have $H_{\theta,\Bbbk,0} = H_{\theta,R,0} \otimes_R \Bbbk$ for $\Bbbk = K$ or F. Since $H_{\theta,F,0}$ is finite dimensional, by the classical theorem on lifting idempotents [CR, (6.7)], there exists an idempotent $e_{\pi,R} \in H_{\theta,R,0}$ such that $e_{\pi,F} = e_{\pi,R} \otimes 1_F$, and $P(\pi)_R := H_{\theta,R}e_{\pi,R}$ is an R-form of $P(\pi)_F$. The notation $P(\pi)_R$ will be reserved only for this specific R-form of $P(\pi)_F$. Note that, while the $H_{\theta,R}$-module $P(\pi)_R$ is indecomposable, it is not in general true that $P(\pi)_R \otimes_R K \cong P(\pi)_K$, see Lemma 2.2.8 for more information.

Let $V_K \in H_{\theta,K}$-mod and V_R be an R-form of V_K. The $H_{\theta,F}$-module $V_R \otimes_R F$ is called a *reduction modulo p* of V_K. Reduction modulo p in general depends on the choice of V_R. However, as usual, we have:

Lemma 2.2.3. *If* $V_K \in H_{\theta,K}$-mod *and* V_R *is an* R*-form of* V_K*, then for any* $\pi \in \mathrm{KP}(\theta)$*, we have*

$$[V_R \otimes_R F : L(\pi)_F]_q = \dim_q^K \mathrm{Hom}_{H_{\theta,K}}(P(\pi)_R \otimes_R K, V_K).$$

In particular, the composition multiplicities $[V_R \otimes_R F : L(\pi)_F]_q$ *are independent of the choice of an* R*-form* V_R*.*

Proof. We have $[V_R \otimes_R F : L(\pi)_F]_q = \dim_q^F \mathrm{Hom}_{H_{\theta,F}}(P(\pi)_F, V_R \otimes_R F)$. By the Universal Coefficient Theorem,

$$\mathrm{Hom}_{H_{\theta,F}}(P(\pi)_F, V_R \otimes_R F) \cong \mathrm{Hom}_{H_{\theta,R}}(P(\pi)_R, V_R) \otimes_R F.$$

Moreover, note that $\mathrm{Hom}_{H_{\theta,R}}(P(\pi)_R, V_R)$ is R-free of (graded) rank equal to $\dim_q^{\Bbbk} \mathrm{Hom}_{H_{\theta,R}}(P(\pi)_R, V_R) \otimes_R \Bbbk$ for $\Bbbk = F$ or K. Now, by the Universal Coefficient Theorem again, we have that

$$\dim_q^K \mathrm{Hom}_{H_{\theta,R}}(P(\pi)_R, V_R) \otimes_R K = \dim_q^K \mathrm{Hom}_{H_{\theta,K}}(P(\pi)_R \otimes_R K, V_R \otimes_R K),$$

which completes the proof, since $V_R \otimes_R K \cong V_K$. □

Our main interest is in reduction modulo p of the irreducible $H_{\theta,K}$-modules $L(\pi)_K$. Pick a non-zero homogeneous vector $v \in L(\pi)_K$ and define $L(\pi)_R := H_{\theta,R} \cdot v$. Then $L(\pi)_R$ is an $H_{\theta,R}$-invariant full lattice in $L(\pi)_K$, and reducing modulo p, we get an $H_{\theta,F}$-module $L(\pi)_R \otimes_R F$. In general, $L(\pi)_R \otimes_R F$ is not $L(\pi)_F$, although this happens 'often', for example for cuspidal modules:

Lemma 2.2.4. *[Kl1, Proposition 3.20] Let $\alpha \in \Phi_+$. Then $L(\alpha)_R \otimes_R F \cong L(\alpha)_F$.*

To generalize this lemma to irreducible modules of the form $L(\alpha^m)$, we need to observe that induction and restriction commute with extension of scalars. More precisely, for $\gamma_1, \dots, \gamma_m \in Q_+$, $\theta = \gamma_1 + \dots + \gamma_m$, and any ground ring $\Bbbk$, we denote by $H_{\gamma_1,\dots,\gamma_m;\Bbbk}$ the algebra $H_{\gamma_1,\Bbbk} \otimes_{\Bbbk} \dots \otimes_{\Bbbk} H_{\gamma_m,\Bbbk}$ identified as usual with a (non-unital) subalgebra of $H_{\theta,\Bbbk}$. Now, the following lemma is immediate:

Lemma 2.2.5. *Let $V_R \in H_{\gamma_1,\dots,\gamma_m;R}$-mod and $W_R \in H_{\theta,R}$-mod. Then for any R-algebra $\Bbbk$, there are natural isomorphisms of $H_{\theta,\Bbbk}$-modules*

$$(\mathrm{Ind}^{\theta}_{\gamma_1,\dots,\gamma_m} V_R) \otimes_R \Bbbk \cong \mathrm{Ind}^{\theta}_{\gamma_1,\dots,\gamma_m}(V_R \otimes_R \Bbbk)$$

and of $H_{\gamma_1,\dots,\gamma_m;\Bbbk}$-modules

$$(\mathrm{Res}^{\theta}_{\gamma_1,\dots,\gamma_m} W_R) \otimes_R \Bbbk \cong \mathrm{Res}^{\theta}_{\gamma_1,\dots,\gamma_m}(W_R \otimes_R \Bbbk).$$

Let $\alpha \in \Phi_+$ and $m \in \mathbb{Z}_{>0}$. If $\Bbbk$ is a field, by Lemma 1.3.3, we have $L(\alpha^m)_{\Bbbk} \simeq L(\alpha)_{\Bbbk}^{\circ m}$. By Lemma 2.2.5, $L(\alpha^m)_R := (L(\alpha)_R)^{\circ m}$ satisfies $L(\alpha^m)_R \otimes_R \Bbbk \simeq L(\alpha^m)_{\Bbbk}$ for $\Bbbk = K$ or F. Taking into account Lemmas 2.2.3 and 2.2.4, we get:

Lemma 2.2.6. *Let $\alpha \in \Phi_+$ and $m \in \mathbb{Z}_{>0}$. Then reduction modulo p of $L(\alpha^m)_K$ is $L(\alpha^m)_F$.*

It was conjectured in [KlR, Conjecture 7.3] that reduction modulo p of $L(\pi)_K$ is always $L(\pi)_F$, but counterexamples are given in [Wi] (see also [BKM, Example 2.16]). Still, it is important to understand when we have $L(\pi)_R \otimes_R F \cong L(\pi)_F$:

Problem 2.2.7. Let $\theta \in Q_+$.

(i) If $\pi \in \mathrm{KP}(\theta)$, determine when $L(\pi)_R \otimes_R F \cong L(\pi)_F$.

(ii) We say that *James' Conjecture has positive solution* (for θ) if the isomorphism in (i) holds for all $\pi \in \mathrm{KP}(\theta)$. Determine the minimal lower bound p_θ on $p = \operatorname{char} F$ so that James' Conjecture has positive solution for all $p \geq p_\theta$.

At least, we always have:

Lemma 2.2.8. *Let $\theta \in Q_+$ and $\pi \in \mathrm{KP}(\theta)$. Then in the Grothendieck group of finite dimensional $H_{\theta,F}$-modules we have*

$$[L(\pi)_R \otimes_R F] = [L(\pi)_F] + \sum_{\sigma<\pi} a_{\pi,\sigma}[L(\sigma)_F] \tag{2.1}$$

for some bar-invariant Laurent polynomials $a_{\pi,\sigma} \in \mathbb{Z}[q, q^{-1}]$. Moreover,

$$P(\pi)_R \otimes_R K \cong P(\pi)_K \oplus \bigoplus_{\sigma>\pi} a_{\sigma,\pi} P(\sigma)_K.$$

Proof. Let $\Bbbk = K$ or F and consider the reduced standard module $\bar{\Delta}(\pi)_\Bbbk$, see (3.4). In view of (3.5), we can write

$$[L(\pi)_\Bbbk] := [\bar{\Delta}(\pi)_\Bbbk] + \sum_{\sigma<\pi} f^{\Bbbk}_{\pi,\sigma}[\bar{\Delta}(\sigma)_\Bbbk]$$

for some $f^{\Bbbk}_{\pi,\sigma} \in \mathbb{Z}[q, q^{-1}]$. Using Lemmas 2.2.5, 2.2.4 and induction on the bilexicographical order on $\mathrm{KP}(\pi)$, we now deduce that the equation

(2.1) holds for some, not necessarily bar-invariant, coefficients $a_{\pi,\sigma} \in \mathbb{Z}[q, q^{-1}]$. Then we also have

$$\mathrm{ch}_q\,(L(\pi)_R \otimes_R F) = \mathrm{ch}_q\,(L(\pi)_F) + \sum_{\sigma<\pi} a_{\pi,\sigma}\mathrm{ch}_q\,(L(\sigma)_F).$$

Since reduction modulo p preserves formal characters, the left hand side is bar-invariant. Moreover, every $\mathrm{ch}_q\,(L(\sigma)_F)$ is bar-invariant. This implies that the coefficients $a_{\pi,\sigma}$ are also bar-invariant, since by [KL1, Theorem 3.17], the formal characters $\{\mathrm{ch}_q\, L(\nu)_F \mid \nu \in \mathrm{KP}(\theta)\}$ are linearly independent.

Finally, for any $\sigma \in \mathrm{KP}(\theta)$, we have

$$a_{\sigma,\pi} = \dim_q^K \mathrm{Hom}_{H_{\theta,K}}(P(\pi)_R \otimes_R K, L(\sigma)_K),$$

thanks to by Lemma 2.2.3. This implies the second statement. □

Remark 2.2.9. For $\Bbbk = K$ and F, denote by $d^{\Bbbk}_{\pi,\sigma}$, the corresponding decomposition numbers, see (3.5), and consider the *decomposition matrices* $D^{\Bbbk} := (d^{\Bbbk}_{\pi,\sigma})_{\pi,\sigma\in\mathrm{KP}(\theta)}$. Setting $A := (a_{\pi,\sigma})_{\pi,\sigma\in\mathrm{KP}(\theta)}$, we have $D^F = D^K A$. So the matrix A plays the role of the *adjustment matrix* in the classical James' Conjecture [J]. Note that James' Conjecture has positive solution in the sense of Problem 2.2.7 if and only if A is the identity matrix.

2.2.2. Integral forms of standard modules

Our next goal is to construct some special R-forms of standard modules. We call an $H_{\theta,R}$-module $\Delta(\pi)_R$ a *universal R-form of a* *standard module* if it is an R form for both $\Delta(\pi)_K$ and $\Delta(\pi)_F$. We show how to construct these for all π.

By Theorem 1.3.4(i), for any field $\Bbbk$ and $\alpha \in \Phi_+$, the standard module $\Delta(\alpha^m)_\Bbbk$ has simple head $L(\alpha^m)_\Bbbk$. Pick a homogeneous generator $v \in \Delta(\alpha^m)_K$ and consider the R-form $\Delta(\alpha^m)_R := H_{m\alpha,R} \cdot v$ of $\Delta(\alpha^m)_K$. Further, for any $\theta \in Q_+$ and $\pi = (\pi_1^{m_1}, \ldots, \pi_s^{m_s}) \in \mathrm{KP}(\theta)$ with $\pi_1 > \cdots > \pi_s$, we define the following R-form of $\Delta(\pi)_K$ (cf. Lemma 2.2.5):

$$\Delta(\pi)_R := \Delta(\pi_1^{m_1})_R \circ \cdots \circ \Delta(\pi_s^{m_s})_R.$$

Let $1_{(\pi),R} := 1_{m_1\pi_1,\ldots,m_s\pi_s;R}$. Then, for an appropriate set $\mathcal{D}^{(\pi)}$ of coset representatives in a symmetric group, we have that $\{\psi_w 1_{(\pi),R} \mid w \in \mathcal{D}^{(\pi)}\}$ is a basis of $H_{\theta,R}1_{(\pi),R}$ considered as a right $H_{m_1\pi_1,\ldots,m_s\pi_s;R}$-module. So

$$\Delta(\pi)_R = \bigoplus_{w \in \mathcal{D}^{(\pi)}} \psi_w 1_{(\pi),R} \otimes \Delta(\pi_1^{m_1})_R \otimes \cdots \otimes \Delta(\pi_s^{m_s})_R.$$

In particular, choosing $v_t \in \Delta(\pi_t^{m_t})_K$ with $\Delta(\pi_t^{m_t})_R = H_{m_t\pi_t,R} \cdot v_t$ for all $1 \leq t \leq s$ and setting $v := 1_{(\pi),K} \otimes v_1 \otimes \cdots \otimes v_s$, we have

$$\Delta(\pi)_R = H_{\theta,R} \cdot v \tag{2.2}$$

Now we show that $\Delta(\pi)_R$ is a universal R-form:

Lemma 2.2.10. *Let $\theta \in Q_+$, and $\pi \in \mathrm{KP}(\theta)$. Then $\Delta(\pi)_R \otimes_R F \cong \Delta(\pi)_F$.*

Proof. In view of (3.7) and Lemma 2.2.5, we may assume that π is of the form (β^m) for a positive root β so that $\theta = m\beta$. By Lemma 2.2.3, we have for any $\sigma \in \mathrm{KP}(\theta)$:

$$[\Delta(\beta^m)_R \otimes_R F : L(\sigma)_F]_q = \dim_q^K \mathrm{Hom}_{H_{\theta,K}}(P(\sigma)_R \otimes_R K, \Delta(\beta^m)_K).$$

By convexity, (β^m) is a minimal element of $\mathrm{KP}(\theta)$. So Lemma 2.2.8 implies that all composition factors of $\Delta(\beta^m)_R \otimes_R F$ are $\simeq L(\beta^m)_F$. Moreover,

$$[\Delta(\beta^m)_R \otimes_R F : L(\beta^m)_F]_q = [\Delta(\beta^m)_K : L(\beta^m)_K]_q = [\Delta(\beta^m)_F : L(\beta^m)_F]_q.$$

By construction, $\Delta(\beta^m)_R$ is cyclic, hence so is $\Delta(\beta^m)_R \otimes_R F$. So, $\Delta(\beta^m)_R \otimes_R F$ is a module with simple head and belongs to the category of all modules in $H_{\theta,F}$-mod with composition factors $\simeq L(\beta^m)_F$. Since (β^m) is minimal in $\mathrm{KP}(\theta)$, we have that $\Delta(\beta^m)_F$ is the projective cover of $L(\beta^m)_F$ in this category, see [Kl3, Lemma 4.11]. So there is a surjective homomorphism from $\Delta(\beta^m)_F$ onto $\Delta(\beta^m)_R \otimes_R F$. This has to be an isomorphism since we have proved that the two modules have the same composition multiplicities. $\square$

From now on, the notation $\Delta(\pi)_R$ is reserved for a *universal* R-form. We begin with a rather obvious consequence of what we have proved so far:

Proposition 2.2.11. *Let $\theta \in Q_+$ and $\pi, \sigma \in \mathrm{KP}(\theta)$.*

(i) *If $\pi \neq \sigma$, then* $\mathrm{Hom}_{H_{\theta,R}}(\Delta(\pi)_R, \Delta(\sigma)_R) = 0$.

(ii) *For any R-algebra $\Bbbk$, we have*

$$\mathrm{End}_{H_{\theta,R}}(\Delta(\pi)_R) \otimes_R \Bbbk = \mathrm{End}_{H_{\theta,\Bbbk}}(\Delta(\pi)_R \otimes_R \Bbbk).$$

(iii) *If $\pi \not< \sigma$, then $\mathrm{Ext}^j_{H_{\theta,R}}(\Delta(\pi)_R, \Delta(\sigma)_R) = 0$ for all $j \geq 1$.*

Proof. By the Universal Coefficient Theorem, for any $j \geq 0$, we can embed $\mathrm{Ext}^j_{H_{\theta,R}}(\Delta(\pi)_R, \Delta(\sigma)_R) \otimes_R F$ into $\mathrm{Ext}^j_{H_{\theta,F}}(\Delta(\pi)_F, \Delta(\sigma)_F)$. So (i) follows from Theorem A, and (iii) follows from Theorem 1.3.4(iii). Now (ii) also follows from the Universal Coefficient Theorem, since $\mathrm{Ext}^1_{H_{\theta,R}}(\Delta(\pi)_R, \Delta(\pi)_R) = 0$ by (iii), which makes the Tor_1-term trivial. $\square$

It turns out that torsion in the Ext groups between $\Delta(\pi)_R$'s bears some importance for Problem 2.2.7, see Remark 2.2.15. So we try to make progress in understanding this torsion. Given an R-module V, denote by $V^{\mathtt{Tors}}$ its *torsion submodule*. Of especial importance for us will be the torsion in Ext-groups: $\mathrm{Ext}^j_{H_{\theta,R}}(\Delta(\pi)_R, \Delta(\sigma)_R)^{\mathtt{Tors}}$. The following result was surprising for us:

Theorem 2.2.12. *Let $\theta \in Q_+$ and $\pi, \sigma \in \mathrm{KP}(\theta)$. Then the R-module*

$$\mathrm{Ext}^1_{H_{\theta,R}}(\Delta(\pi)_R, \Delta(\sigma)_R)$$

is torsion-free.

Proof. By Proposition 2.2.11, we may assume that $\pi < \sigma$. By the Universal Coefficient Theorem, there is an exact sequence

$$0 \to \mathrm{Hom}_{H_{\theta,R}}(\Delta(\pi)_R, \Delta(\sigma)_R) \otimes_R F \to \mathrm{Hom}_{H_{\theta,F}}(\Delta(\pi)_F, \Delta(\sigma)_F)$$
$$\to \mathrm{Tor}_1^R(\mathrm{Ext}^1_{H_{\theta,R}}(\Delta(\pi)_R, \Delta(\sigma)_R), F) \to 0.$$

By Theorem A, the middle term vanishes, so the third term also vanishes, which implies the theorem. □

We will need the following generalization:

Corollary 2.2.13. *Let $\theta \in Q_+$, $\sigma \in \mathrm{KP}(\theta)$, and V be an $H_{\theta,R}$-module with a finite Δ-filtration, all of whose subfactors are of the form $\simeq \Delta(\pi)_R$ for $\pi \neq \sigma$. Then the R-module $\mathrm{Ext}^1_{H_{\theta,R}}(V, \Delta(\sigma)_R)$ is torsion-free.*

Proof. Apply induction on the length of the Δ-filtration, the induction base coming from Theorem 2.2.12. If the filtration has length greater than 1, we have an exact sequence

$$0 \to V_1 \to V \to V_2 \to 0,$$

such that the inductive assumption apples to V_1, V_2. Then we get a long exact sequence

$$\mathrm{Hom}_{H_{\theta,R}}(V_1, \Delta(\sigma)_R) \to \mathrm{Ext}^1_{H_{\theta,R}}(V_2, \Delta(\sigma)_R)$$
$$\to \mathrm{Ext}^1_{H_{\theta,R}}(V, \Delta(\sigma)_R) \to \mathrm{Ext}^1_{H_{\theta,R}}(V_1, \Delta(\sigma)_R).$$

By Theorem A, the first term vanishes. By the inductive assumption, the second and fourth terms are torsion-free. Hence so is the third term. $\square$

While we have just proved that there is no torsion in $\mathrm{Ext}^1_{H_{\theta,R}}(\Delta(\pi)_R, \Delta(\sigma)_R)$, the following result reveals the importance of torsion in Ext^2-groups.

Corollary 2.2.14. *Let $\theta \in Q_+$ and $\pi, \sigma \in \mathrm{KP}(\theta)$. We have*

$$\begin{aligned} &\dim_q^F \mathrm{Ext}^1_{H_{\theta,F}}(\Delta(\pi)_F, \Delta(\sigma)_F) \\ = \ &\dim_q^K \mathrm{Ext}^1_{H_{\theta,K}}(\Delta(\pi)_K, \Delta(\sigma)_K) + \mathrm{d}_q^R \mathrm{Ext}^2_{H_{\theta,R}}(\Delta(\pi)_R, \Delta(\sigma)_R)^{\mathtt{Tors}}. \end{aligned}$$

In particular,

$$\dim_q^F \mathrm{Ext}^1_{H_{\theta,F}}(\Delta(\pi)_F, \Delta(\sigma)_F) = \dim_q^K \mathrm{Ext}^1_{H_{\theta,K}}(\Delta(\pi)_K, \Delta(\sigma)_K)$$

if and only if the R-module $\mathrm{Ext}^2_{H_{\theta,R}}(\Delta(\pi)_R, \Delta(\sigma)_R)$ is torsion-free.

Proof. By the Universal Coefficient Theorem, there is an exact sequence

$$\begin{aligned} 0 \to \mathrm{Ext}^1_{H_{\theta,R}}(\Delta(\pi)_R, \Delta(\sigma)_R) \otimes_R F \to \mathrm{Ext}^1_{H_{\theta,F}}(\Delta(\pi)_F, \Delta(\sigma)_F) \\ \to \mathrm{Tor}_1^R(\mathrm{Ext}^2_{H_{\theta,R}}(\Delta(\pi)_R, \Delta(\sigma)_R), F) \to 0 \end{aligned}$$

and an isomorphism

$$\mathrm{Ext}^1_{H_{\theta,R}}(\Delta(\pi)_R, \Delta(\sigma)_R) \otimes_R K \cong \mathrm{Ext}^1_{H_{\theta,K}}(\Delta(\pi)_K, \Delta(\sigma)_K).$$

The last isomorphism and Theorem 2.2.12 imply

$$\dim_q^K \operatorname{Ext}^1_{H_{\theta,K}}(\Delta(\pi)_K, \Delta(\sigma)_K) = \mathrm{d}_q^R \operatorname{Ext}^1_{H_{\theta,R}}(\Delta(\pi)_R, \Delta(\sigma)_R).$$

On the other hand,

$$\mathrm{d}_q^R \operatorname{Ext}^2_{H_{\theta,R}}(\Delta(\pi)_R, \Delta(\sigma)_R)^{\mathtt{Tors}} = \dim_q^F \operatorname{Tor}_1^R(\operatorname{Ext}^2_{H_{\theta,R}}(\Delta(\pi)_R, \Delta(\sigma)_R), F),$$

so the result now follows from the exactness of the first sequence. □

Remark 2.2.15. By Theorem 2.2.12, lack of torsion in the group $\operatorname{Ext}^2_{H_{\theta,R}}(\Delta(\pi)_R, \Delta(\sigma)_R)$ is equivalent to the fact that the extension groups $\operatorname{Ext}^1_{H_\theta}(\Delta(\pi), \Delta(\sigma))$ have the same graded dimension in characteristic 0 and p. This is relevant for Problem 2.2.7. However, we do not understand the *precise* connection between Problem 2.2.7 and lack of torsion in the groups $\operatorname{Ext}^2_{H_{\theta,R}}(\Delta(\pi)_R, \Delta(\sigma)_R)$. For example, we do not know if such lack of torsion for all π, σ implies (or is equivalent to) James' Conjecture having positive solution. In the next section we establish a different statement of that nature. Set

$$\Delta_{\Bbbk} := \bigoplus_{\pi \in \mathrm{KP}(\theta)} \Delta(\pi)_{\Bbbk}.$$

By the Universal Coefficient Theorem, all groups $\operatorname{Ext}^j_{H_{\theta,R}}(\Delta(\pi)_R, \Delta(\sigma)_R)$ are torsion free if and only if the dimension of the $\Bbbk$-algebras $\operatorname{Ext}^\bullet_{H_{\theta,\Bbbk}}(\Delta_{\Bbbk}, \Delta_{\Bbbk})$ is the same for $\Bbbk = K$ and $\Bbbk = F$, and

$$\operatorname{Ext}^\bullet_{H_{\theta,\Bbbk}}(\Delta_{\Bbbk}, \Delta_{\Bbbk}) \cong \operatorname{Ext}^\bullet_{H_{\theta,R}}(\Delta_R, \Delta_R) \otimes_R \Bbbk$$

for $\Bbbk = K$ and F. We do not know if James' Conjecture has positive solution under the assumption that *all* groups $\mathrm{Ext}^j_{H_{\theta,R}}(\Delta(\pi)_R, \Delta(\sigma)_R)$ are torsion-free.

2.2.3. Integral forms of projective modules in characteristic zero

Recall that by lifting idempotents, we have constructed projective R-forms $P(\pi)_R$ of the projective indecomposable modules $P(\pi)_F$. Our next goal is to construct some interesting R-forms of the projective modules $P(\pi)_K$. As we cannot denote them $P(\pi)_R$, we will have to use the notation $Q(\pi)_R$. We will construct $Q(\pi)_R$ using the usual 'universal extension procedure' applied to universal R-forms of the standard modules, but in our 'infinite dimensional integral situation' we need to be rather careful. We begin with some lemmas.

Lemma 2.2.16. *Let $\Bbbk$ be a field and $V \in H_{\theta,\Bbbk}$-*mod *have the following properties:*

(i) *V is indecomposable;*

(ii) *V has a finite Δ-filtration with the top factor $\Delta(\pi)_\Bbbk$;*

(iii) $\mathrm{Ext}^1_{H_{\theta,\Bbbk}}(V, \Delta(\sigma)_\Bbbk) = 0$ *for all $\sigma \in \mathrm{KP}(\theta)$.*

Then $V \cong P(\pi)_\Bbbk$.

Proof. We have a short exact sequence $0 \to M \to P \to V \to 0$, where P is a finite direct sum of indecomposable projective modules. By [Kl3, Corollary 7.10(i)], M has a finite Δ-filtration. Now, by property (iii), the short exact sequence splits. Hence V is projective. As it is indecomposable, it must be of the form $q^d P(\sigma)$. By the property (ii), $\pi = \sigma$ and $d = 0$. $\square$

For $\pi \in \mathrm{KP}(\theta)$ and $\Bbbk \in \{F, K, R\}$, we denote by $B_{\pi,\Bbbk}$ the endomorphism algebra $\mathrm{End}_{H_{\theta,\Bbbk}}(\Delta(\pi)_\Bbbk)^{\mathrm{op}}$. Then $\Delta(\pi)_\Bbbk$ is naturally a right $B_{\pi,\Bbbk}$-module. We will need to know that this $B_{\pi,\Bbbk}$-module is finitely generated. In fact, we will prove that it is finite rank free. First of all, this is known over a field:

Lemma 2.2.17. *Let $\pi \in \mathrm{KP}(\theta)$ and $\Bbbk$ be a field. Then:*

(i) *$B_{\pi,\Bbbk}$ is a commutative polynomial algebra in finitely many variables of positive degrees.*

(ii) *Let $N_{\pi,\Bbbk}$ be the ideal in $B_{\pi,\Bbbk}$ spanned by all monomials of positive degree, and $M := \Delta(\pi)_\Bbbk N_{\pi,\Bbbk}$. Then $\Delta(\pi)_\Bbbk / M \cong \bar{\Delta}(\pi)_\Bbbk$, see the notation (3.4).*

(iii) *Let $v_1, \dots, v_N \in \Delta(\pi)_\Bbbk$ be such that $\{v_1 + M, \dots, v_N + M\}$ is a $\Bbbk$-basis of $\Delta(\pi)_\Bbbk / M$. Then $\{v_1, \dots, v_N\}$ is a basis of $\Delta(\pi)_\Bbbk$ as a $B_{\pi,\Bbbk}$-module.*

Proof. For (i) see Theorem 1.3.9. For (ii) and (iii), see [Kl3, Proposition 5.7]. $\square$

The following general lemma, whose proof is omitted, will help us to transfer the result of Lemma 2.2.17 from K and F to R:

Lemma 2.2.18. *Let B_R be an R-algebra and V_R be a B_R-module. Assume that B_R and V_R are free as R-modules. If $v_1, \dots, v_N \in V_R$ are* *such that* $\{v_1 \otimes 1_{\Bbbk}, \dots, v_N \otimes 1_{\Bbbk}\}$ *is a basis of* $V_R \otimes_R \Bbbk$ *as a* $B_R \otimes_R \Bbbk$*-module for* $\Bbbk = K$ *and* F*, then* $\{v_1, \dots, v_N\}$ *is a basis of* V_R *as a* B_R*-module.*

Lemma 2.2.19. *Let* $\pi \in \mathrm{KP}(\theta)$*. As a* $B_{\pi,R}$*-module,* $\Delta(\pi)_R$ *is finite rank free.*

Proof. Let $\pi = (\pi_1^{m_1}, \dots, \pi_s^{m_s})$ for positive roots $\pi_1 > \dots > \pi_s$. Choose $v = 1_{(\pi),K} \otimes v_1 \otimes \dots \otimes v_s$ as in (2.2). There is a submodule $M \subset \Delta(\pi)_K$ with $\Delta(\pi)_K/M \cong \bar{\Delta}(\pi)_K$. Pick $h_1, \dots, h_N \in H_{\theta,R}$ such that $\{h_1 v + M, \dots, h_N v + M\}$ is an R-basis of $\bar{\Delta}(\pi)_R = H_{\theta,R} \cdot (v + M)$. By Lemma 2.2.17, $\{h_1 v \otimes 1_{\Bbbk}, \dots, h_N v \otimes 1_{\Bbbk}\}$ is a $B_{\pi,\Bbbk}$-basis of $\Delta(\pi)_R \otimes_R \Bbbk$ for $\Bbbk = K$ or F. Now apply Proposition 2.2.11(ii) and Lemma 2.2.18. □

Corollary 2.2.20. *Let* $\Bbbk \in \{F, K, R\}$*,* $V \in H_{\theta,\Bbbk}\text{-mod}$*,* $\pi \in \mathrm{KP}(\theta)$ *and* $j \in \mathbb{Z}_{\geq 0}$*. Then* $\mathrm{Ext}^j_{H_{\theta,\Bbbk}}(V, \Delta(\pi)_{\Bbbk})$ *is finitely generated as a* $B_{\pi,\Bbbk}$*-module.*

Proof. Since $H_{\theta,\Bbbk}$ is Noetherian, V has a resolution by finite rank free modules over $H_{\theta,\Bbbk}$. Applying $\mathrm{Hom}_{H_{\theta,\Bbbk}}(-, \Delta(\pi)_{\Bbbk})$ to this resolution, we get a complex with terms being finite direct sums of modules $\simeq \Delta(\pi)_{\Bbbk}$, which are finite rank free over $B_{\pi,\Bbbk}$, thanks to Lemmas 2.2.17 and 2.2.19. As $B_{\pi,\Bbbk}$ is Noetherian, the cohomology groups of the complex are finitely generated $B_{\pi,\Bbbk}$-modules. □

Remark 2.2.21. It is a more subtle issue to determine whether $\mathrm{Ext}^j_{H_{\theta,\Bbbk}}(\Delta(\pi)_{\Bbbk}, V)$ is finitely generated as a $B_{\pi,\Bbbk}$-module. We do not know if this is always true.

Lemma 2.2.22. (Universal Extension Procedure) *Let* $\Bbbk \in \{F, K, R\}$*,* $\sigma \in \mathrm{KP}(\theta)$*, and* $V_{\Bbbk}$ *be an indecomposable* $H_{\theta,\Bbbk}$*-module with a* *finite Δ-filtration, all of whose subfactors are of the form $\simeq \Delta(\pi)_{\Bbbk}$ for $\pi \not\geq \sigma$. If $\Bbbk = R$, assume in addition that $V_R \otimes_R K$ is indecomposable. Let*

$$r(q) := \mathrm{d}_q^{B_{\sigma,\Bbbk}} \operatorname{Ext}^1_{H_{\theta,\Bbbk}}(V_{\Bbbk}, \Delta(\sigma)_{\Bbbk}) \in \mathbb{Z}[q, q^{-1}].$$

Then there exists an $H_{\theta,\Bbbk}$-module $E(V_{\Bbbk}, \Delta(\sigma)_{\Bbbk})$ with the following properties:

(i) *$E(V_{\Bbbk}, \Delta(\sigma)_{\Bbbk})$ is indecomposable;*

(ii) $\operatorname{Ext}^1_{H_{\theta,\Bbbk}}(E(V_{\Bbbk}, \Delta(\sigma)_{\Bbbk}), \Delta(\sigma)_{\Bbbk}) = 0$*;*

(iii) *there is a short exact sequence*

$$0 \to \overline{r(q)}\Delta(\sigma)_{\Bbbk} \to E(V_{\Bbbk}, \Delta(\sigma)_{\Bbbk}) \to V_{\Bbbk} \to 0.$$

Proof. In this proof we drop $H_{\theta,\Bbbk}$ from the indices and write Ext^1 for $\operatorname{Ext}^1_{H_{\theta,\Bbbk}}$, etc. Also, when this does not cause a confusion, we drop $\Bbbk$ from the indices. Let $\xi_1, \dots, \xi_r$ be a minimal set of homogeneous generators of $\operatorname{Ext}^1(V, \Delta(\sigma))$ as a B_σ-module, and $d_s := \deg(\xi_s)$ for $s = 1, \dots, r$, so that $r(q) = \sum_s q^{d_s}$. The extension

$$0 \to q^{-d_1}\Delta(\sigma) \to E_1 \to V \to 0,$$

corresponding to ξ_1, yields the long exact sequence

$$\operatorname{Hom}(q^{-d_1}\Delta(\sigma), \Delta(\sigma)) \xrightarrow{\varphi} \operatorname{Ext}^1(V, \Delta(\sigma)) \xrightarrow{\psi} \operatorname{Ext}^1(E_1, \Delta(\sigma)) \to 0.$$

We have used that $\operatorname{Ext}^1(q^{-d_1}\Delta(\sigma), \Delta(\sigma)) = 0$, see Proposition 2.2.11(iii). Note that $q^{-d_1}\Delta(\sigma) = \Delta(\sigma)$ as H_θ-modules but with degrees shifted down by d_1. So we can consider the identity map $\mathrm{id} : q^{-d_1}\Delta(\sigma) \to \Delta(\sigma)$, which has degree d_1. The connecting homomorphism φ maps this identity map to ξ_1. It follows that $\operatorname{Ext}^1(E_1, \Delta(\sigma))$ is generated as a B_σ-module by the elements $\bar{\xi}_2 := \psi(\xi_2), \ldots, \bar{\xi}_r := \psi(\xi_r)$. Repeating the argument $r-1$ more times, we get an extension

$$0 \to q^{-d_1}\Delta(\sigma) \oplus \cdots \oplus q^{-d_r}\Delta(\sigma) = \overline{r(q)}\Delta(\sigma) \to E \to V \to 0$$

such that in the corresponding long exact sequence

$$\begin{aligned} \operatorname{Hom}(E, \Delta(\sigma)) &\xrightarrow{\chi} \operatorname{Hom}(\overline{r(q)}\Delta(\sigma), \Delta(\sigma)) \\ &\xrightarrow{\varphi} \operatorname{Ext}^1(V, \Delta(\sigma)) \to \operatorname{Ext}^1(E, \Delta(\sigma)) \to 0, \end{aligned}$$

for all $s = 1, \ldots, r$, we have $\varphi(\rho_s) = \xi_s$, where ρ_s is the (degree d_s) projection onto the sth summand. In particular, φ is surjective, and $\operatorname{Ext}^1(E, \Delta(\sigma)) = 0$.

It remains to prove that E is indecomposable. We first prove this when $\Bbbk$ is a field. In that case, if $E = E' \oplus E''$, then both E' and E'' have finite Δ-filtrations, see [Kl3, Corollary 7.10]. Since $\operatorname{Ext}^1(\Delta(\sigma), \Delta(\pi)) = 0$ for $\pi \not> \sigma$, there is a partition $J' \sqcup J'' = \{1, \ldots, r\}$ such that there are submodules

$$M' \cong \oplus_{j \in J'} q^{d_j}\Delta(\sigma) \subseteq E', \quad M'' \cong \oplus_{j \in J''} q^{d_j}\Delta(\sigma) \subseteq E'',$$

and E'/M', M''/E'' have Δ-filtrations. Since $\mathrm{Hom}(\Delta(\sigma), V) = 0$, we now deduce that $V \cong E'/M' \oplus E''/M''$. As V is indecomposable, we may assume that $E'/M' = 0$. Then some projection ρ_s lifts to a homomorphism $E \to \Delta(\sigma)$, which shows that this ρ_s is in the image of χ, and hence in the kernel of φ, which is a contradiction.

Now let $\Bbbk = R$. Note that V and E are free as R-modules since so are all $\Delta(\nu)_R$'s. If E_R is decomposable, then so is $E_R \otimes K$, so it suffices to prove that $E_R \otimes K$ is indecomposable. In view of Corollary 2.2.13, the $B_{\sigma,K}$-module

$$\mathrm{Ext}^1(V_R, \Delta(\sigma)_R) \otimes_R K \cong \mathrm{Ext}^1(V_R \otimes_R K, \Delta(\sigma)_K)$$

is minimally generated by $\xi_{1,R} \otimes 1_K, \dots, \xi_{r,R} \otimes 1_K$. It follows, using Lemma 2.2.2, that $E_R \otimes_R K \cong E_K$, where E_K is constructed using the universal extension procedure starting with the indecomposable module $V_K := V_R \otimes_R K$ as in the first part of the proof of the lemma. By the field case established in the previous paragraph, E_K is indecomposable. $\square$

Let $\pi \in \mathrm{KP}(\theta)$. For $\Bbbk \in \{R, K, F\}$, we construct a module $Q(\pi)_\Bbbk$ starting with $\Delta(\pi)_\Bbbk$, and repeatedly applying the universal extension procedure. To simplify notation we drop some of the indices $\Bbbk$ if this does not lead to a confusion. Given Laurent polynomials $r_0(q), r_1(q), \dots, r_m(q) \in \mathbb{Z}[q, q^{-1}]$ with non-negative coefficients and Kostant partitions $\pi^0, \pi^1, \dots, \pi^m \in \mathrm{KP}(\theta)$, we will use the notation

$$V = r_0(q)\Delta(\pi^0) \mid r_1(q)\Delta(\pi^1) \mid \cdots \mid r_m(q)\Delta(\pi^m)$$

to indicate that the H_θ-module V has a filtration $V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_{m+1} = (0)$ such that $V_s/V_{s+1} \cong r_s(q)\Delta(\pi^s)$ for $s = 0, 1 \ldots, m$.

If $\mathrm{Ext}^1_{H_\theta}(\Delta(\pi), \Delta(\sigma)) = 0$ for all $\sigma \in \mathrm{KP}(\theta)$, we set $Q(\pi)_{\Bbbk} := \Delta(\pi)_{\Bbbk}$. Otherwise, let $\pi^{1,\Bbbk} \in \mathrm{KP}(\theta)$ be minimal with $\mathrm{Ext}^1_{H_\theta}(\Delta(\pi), \Delta(\pi^{1,\Bbbk})) \neq 0$. Note that this $\pi^{1,\Bbbk}$ might indeed depend on the ground ring $\Bbbk$, hence the notation. Also notice $\pi^{1,\Bbbk} > \pi$. Let

$$E(\pi, \pi^{1,\Bbbk})_{\Bbbk} := E(\Delta(\pi), \Delta(\pi^{1,\Bbbk})),$$

see Lemma 2.2.22. By construction, we have

$$E(\pi, \pi^{1,\Bbbk})_{\Bbbk} = \Delta(\pi) \mid \overline{r_{1,\Bbbk}(q)}\Delta(\pi^{1,\Bbbk}),$$

where

$$r_{1,\Bbbk}(q) = \mathrm{d}_q^{B_{\pi^{1,\Bbbk}}} \mathrm{Ext}^1_{H_\theta}(\Delta(\pi), \Delta(\pi^{1,\Bbbk})).$$

This Laurent polynomial might depend on $\Bbbk$, hence the notation. If

$$\mathrm{Ext}^1_{H_\theta}(E(\pi, \pi^{1,\Bbbk}), \Delta(\sigma)) = 0$$

for all $\sigma \in \mathrm{KP}(\theta)$, we set $Q(\pi)_{\Bbbk} := E(\pi, \pi^{1,\Bbbk})_{\Bbbk}$. Otherwise, let $\pi^{2,\Bbbk} \in \mathrm{KP}(\theta)$ be minimal with $\mathrm{Ext}^1_{H_\theta}(E(\pi, \pi^{1,\Bbbk}), \Delta(\pi^{2,\Bbbk})) \neq 0$. Note that $\pi^{2,\Bbbk} > \pi$ and $\pi^{2,\Bbbk} \neq \pi^{1,\Bbbk}$. Let

$$E(\pi, \pi^{1,\Bbbk}, \pi^{2,\Bbbk})_{\Bbbk} := E(E(\pi, \pi^{1,\Bbbk}), \Delta(\pi^{2,\Bbbk})).$$

By construction, we have

$$E(\pi, \pi^{1,\Bbbk}, \pi^{2,\Bbbk})_{\Bbbk} = \Delta(\pi) \mid \overline{r_{1,\Bbbk}(q)}\Delta(\pi^{1,\Bbbk}) \mid \overline{r_{2,\Bbbk}(q)}\Delta(\pi^{2,\Bbbk}),$$

where

$$r_{2,\Bbbk}(q) = \mathrm{d}_q^{B_{\pi^{2,\Bbbk}}} \operatorname{Ext}^1_{H_\theta}(E(\pi, \pi^{1,\Bbbk}), \Delta(\pi^{2,\Bbbk})).$$

If $\operatorname{Ext}^1_{H_\theta}(E(\pi, \pi^{1,\Bbbk}, \pi^{2,\Bbbk}), \Delta(\sigma)) = 0$ for all $\sigma \in \mathrm{KP}(\theta)$, we set

$$Q(\pi)_{\Bbbk} := E(\pi, \pi^{1,\Bbbk}, \pi^{2,\Bbbk}).$$

Since on each step we will have to pick $\pi^{t,\Bbbk} > \pi$, which does not belong to $\{\pi, \pi^{1,\Bbbk}, \dots, \pi^{t-1,\Bbbk}\}$, the process will stop after finitely many steps, and we will obtain a module

$$E(\pi, \pi^{1,\Bbbk}, \dots, \pi^{m_{\Bbbk},\Bbbk})_{\Bbbk} = \Delta(\pi) \mid \overline{r_{1,\Bbbk}(q)}\Delta(\pi^{1,\Bbbk}) \mid \cdots \mid \overline{r_{m_{\Bbbk},\Bbbk}(q)}\Delta(\pi^{m_{\Bbbk},\Bbbk}),$$

where

$$r_{t,\Bbbk}(q) = \mathrm{d}_q^{B_{\pi^{t,\Bbbk}}} \operatorname{Ext}^1_{H_{\theta,\Bbbk}}(E(\pi, \pi^{1,\Bbbk}, \dots, \pi^{t-1,\Bbbk})_{\Bbbk}, \Delta(\pi^{t,\Bbbk})_{\Bbbk}) \tag{2.3}$$

for all $1 \le t \le m_{\Bbbk}$, and such that

$$\operatorname{Ext}^1_{H_{\theta,\Bbbk}}(E(\pi, \pi^{1,\Bbbk}, \dots, \pi^{m_{\Bbbk},\Bbbk})_{\Bbbk}, \Delta(\sigma)_{\Bbbk}) = 0$$

for all $\sigma \in \mathrm{KP}(\theta)$. We set

$$Q(\pi)_{\Bbbk} := E(\pi, \pi^{1,\Bbbk}, \dots, \pi^{m_{\Bbbk},\Bbbk})_{\Bbbk}.$$

Theorem 2.2.23. *Let $\theta \in Q_+$ and $\pi \in \mathrm{KP}(\theta)$.*

(i) *For $\Bbbk = K$ or F, we have $Q(\pi)_\Bbbk \cong P(\pi)_\Bbbk$.*

(ii) *For $\Bbbk = K$ or F, the Laurent polynomial $r_{t,\Bbbk}(q)$ from (2.3) equals the decomposition number $d^{\Bbbk}_{\pi^{t,\Bbbk},\pi}$ for all $1 \le t \le m_\Bbbk$, and $d^{\Bbbk}_{\sigma,\pi} = 0$ for $\sigma \notin \{\pi^{t,\Bbbk} \mid 1 \le t \le m_\Bbbk\}$.*

(iii) *$m_R = m_K$; setting $m := m_R$, we may choose $\pi^{1,R} = \pi^{1,K}, \dots, \pi^{m,R} = \pi^{m,K}$ and then $r_{t,R}(q) = r_{t,K}(q)$ for all $1 \le t \le m$.*

(iv) *$Q(\pi)_R \otimes_R K \cong P(\pi)_K$.*

Proof. Part (i) follows from the construction and Lemma 2.2.16. Part (ii) follows from part (i), the construction, and Theorem 1.3.4(v).

To show (iii) and (iv), we prove by induction on $t = 0, 1, \dots$ that we can choose $\pi^{t,R} = \pi^{t,K}$, $r_{t,R}(q) = r_{t,K}(q)$ and

$$E(\pi, \pi^{1,R}, \dots, \pi^{t,R})_R \otimes_R K \cong E(\pi, \pi^{1,K}, \dots, \pi^{t,K})_K. \tag{2.4}$$

The induction base is simply the statement $\Delta(\pi)_R \otimes_R K \cong \Delta(\pi)_K$. For the induction step, assume that $t > 0$ and the claim has been proved for all $s < t$.

Let $\xi_{1,R}, \dots, \xi_{r,R}$ be a minimal set of generators of the $B_{\pi^{t,R},R}$-module

$$\mathrm{Ext}^1_{H_{\theta,R}}(E(\pi, \pi^{1,R}, \dots, \pi^{t-1,R})_R, \Delta(\pi^{t,R})_R),$$

so that

$$r_{t,R}(q) = \deg(\xi_{1,R}) + \cdots + \deg(\xi_{r,R})$$

Using Corollary 2.2.13 and the Universal Coefficient Theorem, we deduce that $\pi^{t,K}$ can be chosen to be $\pi^{t,R}$ and the $B_{\pi^{t,R},K}$-module

$$\mathrm{Ext}^1(\Delta(\pi)_R, \Delta(\pi^{t,R})_R) \otimes_R K \cong \mathrm{Ext}^1(V_R \otimes_R K, \Delta(\pi^{t,R})_K)$$

is minimally generated by $\xi_{1,R} \otimes 1_K, \ldots, \xi_{r,R} \otimes 1_K$, so that $r_{t,K}(q) = r_{t,R}(q)$. Finally (2.4) comes from Lemma 2.2.2. $\square$

In view of Theorem 2.2.23(i), $Q(\pi)_R$ in general is not an R-form of $Q(\pi)_F \cong P(\pi)_F$. For every $\pi \in \mathrm{KP}(\theta)$, define the $H_{\theta,F}$-module

$$X(\pi) := Q(\pi)_R \otimes F.$$

Theorem 2.2.24. *James' Conjecture has positive solution for θ if and only if one of the following equivalent conditions holds:*

(i) $X(\pi)$ *is projective;*

(ii) $X(\pi) \cong P(\pi)_F$ *for all* $\pi \in \mathrm{KP}(\theta)$*;*

(iii) $\mathrm{Ext}^1_{H_{\theta,F}}(X(\pi), \Delta(\sigma)_F) = 0$ *for all* $\pi, \sigma \in \mathrm{KP}(\theta)$*;*

(iv) *the R-module* $\mathrm{Ext}^2_{H_{\theta,R}}(Q(\pi)_R, \Delta(\sigma)_R)$ *is torsion-free for all* $\pi, \sigma \in \mathrm{KP}(\theta)$.

Proof. (i) and (ii) are equivalent by an argument involving formal characters and Lemma 2.2.8. Furthermore, (i) and (iii) are equivalent by Lemma 2.2.16. Since since $\mathrm{Ext}^1_{H_{\theta,R}}(Q(\pi)_R, \Delta(\sigma)_R) = 0$ for all σ, (iii) is equivalent to (vi) by the Universal Coefficient Theorem. Finally, we prove that (ii) is equivalent to James' Conjecture having positive solution. If $X(\pi) \cong P(\pi)_F$ for all π, then they have the same graded dimension, so the R-modules $Q(\pi)_R$ and $P(\pi)_R$ have the same graded R-rank, whence the K-modules $P(\pi)_K \cong Q(\pi)_R \otimes_R K$ and $P(\pi)_R \otimes_R K$ have the same graded dimension, therefore $P(\pi)_R \otimes_R K \cong P(\pi)_K$ for all π, see Lemma 2.2.8, whence James' Conjecture has positive solution.

Conversely, assume James' Conjecture has positive solution. This means that $d^K_{\sigma,\pi} = d^F_{\sigma,\pi}$ for all $\sigma, \pi \in \mathrm{KP}(\theta)$. By Theorem 2.2.23(ii), on every step of our universal extension process, we are going to have the same dimension of the Ext^1-group over K and F, so, by Theorem 2.2.23(iii), on every step of our universal extension process, we are also going to have the same rank of the appropriate Ext^1-groups over R and F. Now, use Lemma 2.2.2 as in the proof of Theorem 2.2.23(iv) to show that $Q(\pi)_R \otimes_R F \cong P(\pi)_F$. □

Remark 2.2.25. We conjecture that $P(\pi)_F$ has an X-filtration with the top quotient $X(\pi)$ and $X(\sigma)$ appearing $a_{\sigma,\pi}(q)$ times. On the level of Grothendieck groups, this is true thanks to Lemma 2.2.8. But it seems not obvious even that $X(\pi)$ is a quotient of $P(\pi)_F$.

Chapter III

PROJECTIVE RESOLUTIONS

This chapter contains material coauthored by the author and his thesis advisor, Alexander Kleshchev. While both parties offered significant contributions, it is impossible to distinguish their exact inputs.

3.1. Preliminaries

We begin by introucing notation and recalling some known results that we rely on for the remainder of the chapter. For any $i, j \in I$, we fix a choice of signs $\varepsilon_{i,j} = \mathrm{sgn}(j - i)$ in the definition of H_θ for concreteness. We also work only in finite type A throughout the chapter. For $d \in \mathbb{Z}_{\geq 0}$, a *composition* of d is an element $\lambda = (\lambda_1, \cdots, \lambda_m) \in \mathbb{Z}_{\geq 0}^m$ with $\sum_{k=1}^m \lambda_k = d$. For such λ, we define the parabolic subgroup

$$\mathfrak{S}_\lambda := \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_m} \leq \mathfrak{S}_d.$$

If $\mu = \{\mu_1, \ldots, \mu_n\} \in \mathbb{Z}_{\geq 0}^n$ is also a composition of d, then we set ${}^\mu\mathcal{D}^\lambda$ to be a choice of minimal length double coset representatives for $\mathfrak{S}_\mu \backslash \mathfrak{S}_d / \mathfrak{S}_\lambda$. Similarly, we set $\mathcal{D}^\lambda$ and ${}^\lambda\mathcal{D}$ to be a choice of left and right minimal length coset representatives for $\mathfrak{S}_d / \mathfrak{S}_\lambda$ and $\mathfrak{S}_\lambda \backslash \mathfrak{S}_d$, respectively.

Given $w \in \mathfrak{S}_d$ we denote by $\ell(w)$ its *length*, i.e. the minimal $\ell \in \mathbb{Z}_{\geq 0}$ such that there exists a reduced expression $w = s_{i_1} \cdots s_{i_\ell}$. We say that

$w \in \mathfrak{S}_d$ is *fully commutative* if any reduced expression for w can be obtained from any other by means of braid relations that only involve commuting generators. Note that if $w \in \mathfrak{S}_d$ is fully commutative and $\theta \in Q_+$ has height d, then the element $\psi_w \in H_\theta$ is well defined and does not depend on the choice of reduced expression for w. The following is well-known and can be deduced from [DJ, Lemma 1.6].

Lemma 3.1.1. *Let λ, μ be compositions of d and $w \in {}^{\mu}\mathcal{D}$. Then there exist unique elements $u \in {}^{\mu}\mathcal{D}^{\lambda}$ and $v \in \mathfrak{S}_\lambda$ such that $w = uv$ and $\ell(w) = \ell(u) + \ell(v)$.*

For $\theta \in Q_+$ with height d and any $1 \leq r < s \leq d$, we denote the cycle $(s, s-1, \dots, r) \in \mathfrak{S}_d$ by $(s \to r)$ and consider the corresponding element

$$\psi_{s \to r} := \psi_{(s \to r)} = \psi_{s-1} \dots \psi_r \in H_\theta. \tag{1.1}$$

Since cycles have unique reduced decompositions, $\psi_{s \to r}$ is well defined. The following lemma easily follows from the defining relations of H_θ.

Lemma 3.1.2. *Let $1 \leq r < s \leq d$, $t \in (r, s)$, $u \in [r, s)$, and $\boldsymbol{i} \in I^\theta$. Then in H_θ we have:*

(i) $1_{\boldsymbol{i}} \psi_{s \to r} \psi_t = 1_{\boldsymbol{i}} \psi_{t-1} \psi_{s \to r}$ *unless* $i_s = i_{t-1} = i_t \pm 1$;

(ii) $1_{\boldsymbol{i}} \psi_{s \to r} y_{u+1} = 1_{\boldsymbol{i}} y_u \psi_{s \to r}$ *unless* $i_s = i_u$.

Recalling the parabolic embeddings introduced in 3.2, we will also need the following result, which is easily seen using this embedding along with Theorem 1.3.1 and Lemma 3.1.1.

Lemma 3.1.3. *Let $\theta \in Q_+$ and $\boldsymbol{i}, \boldsymbol{j} \in I^\theta$. Then, $H_\theta 1_{\boldsymbol{i}} \circ H_\eta 1_{\boldsymbol{j}} \cong H_{\theta+\eta} 1_{\boldsymbol{ij}}$.*

3.1.1. Divided power idempotents

Fix $i \in I$ and $m \in \mathbb{Z}_{\geq 0}$ and denote by $w_{0,m}$ the longest element of $\mathfrak{S}_m$. Also, set

$$y_{0,m} := \prod_{r=1}^{m} y_r^{r-1}, \quad y'_{0,m} := \prod_{r=1}^{m} y_r^{m-r}.$$

Lemma 3.1.4. *We have:*

(i) $\psi_{w_{0,m}} y \psi_{w_{0,m}} = 0$ *for any* $y \in F[y_1, \dots, y_m]$ *of degree less than* $m(m-1)/2$

(ii) $\psi_{w_{0,m}} y_{0,m} \psi_{w_{0,m}} = \psi_{w_{0,m}}$.

(iii) $\psi_{w_{0,m}} y'_{0,m} \psi_{w_{0,m}} = (-1)^{m(m-1)/2} \psi_{w_{0,m}}$.

Proof. This is noted in [KL1, §2.2]. $\square$

As in [KL1], taking onto account Lemma 3.1.4(ii), we have an idempotent

$$1_{i^{(m)}} := \psi_{w_{0,m}} y_{0,m} \in H_{m\alpha_i}.$$

Let $\theta \in Q_+$. We define I^θ_{div} to be the set of all expressions of the form $i_1^{(m_1)} \cdots i_r^{(m_r)}$ with $m_1, \dots, m_r \in \mathbb{Z}_{\geq 0}$, $i_1, \dots, i_r \in I$ and $m_1\alpha_{i_1} + \cdots + m_r\alpha_{i_r} = \theta$. We refer to such expressions as *divided power words.* We identify I^θ with the subset of I^θ_{div} which consists of all divided power words as above with all $m_k = 1$. We use the same notation for concatenation of divided power words as for concatenation of words.

Fix $\boldsymbol{i} = i_1^{(m_1)} \cdots i_r^{(m_r)} \in I_{\text{div}}^{\theta}$. We have the *divided power idempotent*

$$1_{\boldsymbol{i}} = 1_{i_1^{(m_1)} \cdots i_r^{(m_r)}} := \iota_{m_1\alpha_{i_1},\dots,m_r\alpha_{i_r}}(1_{i_1^{(m_1)}} \otimes \cdots \otimes 1_{i_r^{(m_r)}}) \in H_\theta.$$

Define $\boldsymbol{i}! := [m_1]! \cdots [m_r]!$ and

$$\langle \boldsymbol{i} \rangle := \sum_{k=1}^{r} m_k(m_k - 1)/2. \tag{1.2}$$

Set

$$\hat{\boldsymbol{i}} := i_1^{m_1} \cdots i_r^{m_r} \in I^\theta. \tag{1.3}$$

Note that $1_{\boldsymbol{i}} 1_{\hat{\boldsymbol{i}}} = 1_{\hat{\boldsymbol{i}}} 1_{\boldsymbol{i}} = 1_{\boldsymbol{i}}$. The following technical lemmas will be needed in what is to follow.

Lemma 3.1.5. [KL1, §2.5] *Let U (resp. W) be a left (resp. right) H_θ-module, free of finite rank as a $\mathbb{Z}$-module. For $\boldsymbol{i} \in I_{\text{div}}^{\theta}$, we have*

$$\dim_q(1_{\boldsymbol{i}} U) = \boldsymbol{i}! q^{\langle \boldsymbol{i} \rangle} \dim_q(1_{\hat{\boldsymbol{i}}} U) \quad \textit{and} \quad \dim_q(W 1_{\boldsymbol{i}}) = \boldsymbol{i}! q^{-\langle \boldsymbol{i} \rangle} \dim_q(W 1_{\hat{\boldsymbol{i}}}).$$

Lemma 3.1.6. *Let $\alpha = \alpha_a + \alpha_{a+1} + \cdots + \alpha_b \in \Phi_+$ be a positive root, $m \in \mathbb{Z}_{>0}$, $\theta = m\alpha$, and $\boldsymbol{i} = a^{(m)}(a+1)^{(m)} \cdots b^{(m)} \in I_{\text{div}}^{\theta}$. Then*

$$\dim_q 1_{\boldsymbol{i}} L(\alpha^m) = q^{-(b-a+1)m(m-1)/2}.$$

Proof. We have that $L(\alpha)^{\circ m} \cong q^{m(m-1)/2} L(\alpha)^{\circ m}$, see [KlR, Lemma 6.6] or [M, Lemma 3.4]. So the result follows from Lemma 3.1.5 and [Kl2, Lemma 2.10]. $\square$

Lemma 3.1.7. *In the algebra $H_{m\alpha_i}$, we have*

(i) $1_{i^r i^{(s)} i^t} 1_{i^{(m)}} = 1_{i^{(m)}}$ *if* $r+s+t=m$.

(ii) $1_{i^{(m)}} \psi_{m \to 1} 1_{i\, i^{(m-1)}} = \psi_{m \to 1} 1_{i\, i^{(m-1)}}$.

Proof. (i) follows from Lemma 3.1.4(ii) since we can write

$$1_{i^{(m)}} = \iota_{r\alpha_i, s\alpha_i, t\alpha_i}(1_{r\alpha_i} \otimes \psi_{w_{0,s}} \otimes 1_{t\alpha_i}) \psi_u y_{0,m}$$

for some $u \in \mathfrak{S}_m$.

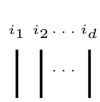
(ii) We have

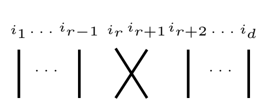
$$\begin{aligned}
&1_{i^{(m)}} \psi_{m-1} \psi_{m-2} \dots \psi_1 1_{i\, i^{(m-1)}} \\
=&1_{i^{(m)}} \psi_{m-1} \psi_{m-2} \dots \psi_1 \iota_{\alpha_i, (m-1)\alpha_i}(1_{\alpha_i} \otimes \psi_{w_{0,m-1}} y_{0,m-1}) \\
=&1_{i^{(m)}} \psi_{w_{0,m}} \iota_{\alpha_i, (m-1)\alpha_i}(1_{\alpha_i} \otimes y_{0,m-1}) \\
=&\psi_{w_{0,m}} \iota_{\alpha_i, (m-1)\alpha_i}(1_{\alpha_i} \otimes y_{0,m-1}) \\
=&\psi_{m-1} \psi_{m-2} \dots \psi_1 \iota_{\alpha_i, (m-1)\alpha_i}(1_{\alpha_i} \otimes \psi_{w_{0,m-1}} y_{0,m-1}) \\
=&\psi_{m-1} \psi_{m-2} \dots \psi_1 1_{i\, i^{(m-1)}},
\end{aligned}$$

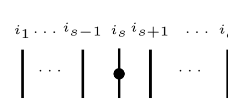
where we have used Lemma 3.1.4(ii) for the third equality. $\square$

3.1.2. Diagrammatic notation

Let $\theta \in Q_+$ with $\mathrm{ht}(\theta) = d$. We will use the Khovanov-Lauda [KL1] diagrammatic notation for elements of H_θ. In particular, for $\boldsymbol{i} = i_1 \cdots i_d \in I^\theta$, $1 \le r < d$ and $1 \le s \le d$, we denote

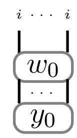
$1_{\boldsymbol{i}} =$ 

, $1_{\boldsymbol{i}} \psi_r =$ 

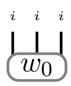
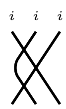
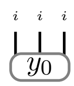
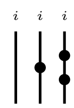
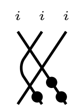
, $1_{\boldsymbol{i}} y_s =$ 

.

We speak of $\deg(1_{\boldsymbol{i}}\psi_r)$ as the *degree of a crossing* in the middle picture above. In addition, we denote

$$\psi_{w_{0,d}} =: \boxed{w_0}, \quad y_{0,d} =: \boxed{y_0}, \quad 1_{i^{(d)}} =$$  $$=: \boxed{i \cdots i} = \boxed{i^d}.$$

For example, if $d = 3$, we have

$$1_{i^3}\psi_{w_0} =$$  $=$  $, \ 1_{i^3}y_0 =$  $=$  $, \ 1_{i^{(3)}} = \boxed{i\ i\ i} =$  $.$

More generally, we denote

$$1_{i_1^{(m_1)}\cdots i_r^{(m_r)}} =: \boxed{i_1^{m_1}} \cdots \boxed{i_r^{m_r}}.$$

3.2. Semicuspidal Resolution

Throughout the subsection, we fix $m \in \mathbb{Z}_{>0}$, $a, b \in \mathbb{Z}$ with $a \le b$, and set $l := b + 2 - a$, $d := lm$. We denote

$$\alpha := \alpha_a + \cdots + \alpha_{b+1} \in \Phi_+$$

and $\theta := m\alpha$. Note that $l = \mathrm{ht}(\alpha)$ and $d = \mathrm{ht}(\theta)$. Our goal is to construct a resolution $P_\bullet = P_\bullet^{\alpha^m}$ of the semicuspidal standard module $\Delta(\alpha^m)$.

3.2.1. Combinatorics

We consider the set of compositions

$$\Lambda = \Lambda^{\alpha^m} := \{\lambda = (\lambda_a, \dots, \lambda_b) \mid \lambda_a, \dots, \lambda_b \in [0, m]\}.$$

For $\lambda \in \Lambda$, we denote $|\lambda| := \lambda_a + \cdots + \lambda_b$, and for $n \in \mathbb{Z}_{\geq 0}$, we set

$$\Lambda(n) = \Lambda^{\alpha^m}(n) := \{\lambda \in \Lambda \mid |\lambda| = n\}.$$

Let $a \leq i \leq b$. We set

$$\mathsf{e}_i := (0, \dots, 0, 1, 0, \dots, 0) \in \Lambda(1),$$

with 1 in the i^{th} position.

Let $\lambda \in \Lambda$. Set

$$\begin{aligned}
\boldsymbol{j}^\lambda &:= a^{m-\lambda_a}(a+1)^{m-\lambda_{a+1}} \cdots b^{m-\lambda_b}(b+1)^m b^{\lambda_b} \cdots a^{\lambda_a} \in I^\theta, \\
\boldsymbol{i}^\lambda &:= a^{(m-\lambda_a)}(a+1)^{(m-\lambda_{a+1})} \dots b^{(m-\lambda_b)}(b+1)^{(m)} b^{(\lambda_b)} \dots a^{(\lambda_a)} \in I^\theta_{\text{div}}.
\end{aligned}$$

Note that $\widehat{\boldsymbol{i}^\lambda} = \boldsymbol{j}^\lambda$. We also associate to λ a composition ω_λ of d with $2n + 1$ non-negative parts:

$$\omega_\lambda := (m - \lambda_a, m - \lambda_{a+1}, \dots, m - \lambda_b, m, \lambda_b, \lambda_{b-1}, \dots, \lambda_a).$$

Let $i \in [a, b]$. We denote

$$r_i^-(\lambda) := \sum_{s=a}^{i}(m - \lambda_s), \quad r_i^+(\lambda) := d - \sum_{s=a}^{i-1} \lambda_s, \quad l_i^{\pm}(\lambda) := r_{i\pm 1}^{\pm}(\lambda) + 1,$$

where $r_{a-1}^-(\lambda)$ is interpreted as 0, and $r_{b+1}^+(\lambda)$ is interpreted as $d - \sum_{s=a}^{b} \lambda_s$. Moreover, denote

$$r_{b+1}(\lambda) := d - \sum_{s=a}^{b} \lambda_s, \quad l_{b+1}(\lambda) := r_{b+1}(\lambda) - m + 1.$$

Define

$$\begin{aligned} U_i^{\pm}(\lambda) &:= [l_i^{\pm}(\lambda), r_i^{\pm}(\lambda)],\ U_i(\lambda) := U_i^-(\lambda) \sqcup U_i^+(\lambda), \\ U_{b+1}(\lambda) &:= [l_{b+1}(\lambda), r_{b+1}(\lambda)]. \end{aligned}$$

Observe that for all $j \in [a, b+1]$, we have

$$U_j(\lambda) = \{s \in [1, d] \mid \boldsymbol{j}_s^{\lambda} = j\}.$$

We also consider the sets of multicompositions

$$\begin{aligned} \boldsymbol{\Lambda} &:= (\Lambda^{\alpha})^m = \{\boldsymbol{\delta} = (\delta^{(1)}, \dots, \delta^{(m)}) \mid \delta^{(1)}, \dots, \delta^{(m)} \in \Lambda^{\alpha}\}, \\ \boldsymbol{\Lambda}(n) &:= \{\boldsymbol{\delta} = (\delta^{(1)}, \dots, \delta^{(m)}) \in \boldsymbol{\Lambda} \mid |\delta^{(1)}| + \dots + |\delta^{(m)}| = n\}. \end{aligned}$$

Note that by definition all $\delta_i^{(r)} \in \{0, 1\}$. For $1 \le r \le m$ and $a \le i \le b$, we define $\mathbf{e}_i^r \in \boldsymbol{\Lambda}(1)$ to be the multicomposition whose rth component is e_i and whose other components are zero. For $\boldsymbol{\delta} \in \boldsymbol{\Lambda}$, denote

$$\lambda^{\boldsymbol{\delta}} := \delta^{(1)} + \cdots + \delta^{(m)} \in \Lambda.$$

Fix $\boldsymbol{\delta} = (\delta^{(1)}, \dots, \delta^{(m)}) \in \boldsymbol{\Lambda}$. Define

$$\boldsymbol{j}^{\boldsymbol{\delta}} := \boldsymbol{j}^{\delta^{(1)}} \dots \boldsymbol{j}^{\delta^{(m)}} \in I^{\theta}.$$

For $i \in [a, b]$ we define

$$\begin{aligned} U_i^{(\pm)}(\boldsymbol{\delta}) &:= \{l(r-1) + u \mid r \in [1, m],\, u \in U_i^{(\pm)}(\delta^{(r)})\}, \\ U_i(\boldsymbol{\delta}) &:= U_i^{+}(\boldsymbol{\delta}) \sqcup U_i^{-}(\boldsymbol{\delta}), \\ U_{b+1}(\boldsymbol{\delta}) &:= \{l(r-1) + u \mid r \in [1, m],\, u \in U_{b+1}(\delta^{(r)})\}. \end{aligned}$$

Observe that for any $j \in [a, b+1]$, we have

$$U_j^{(\boldsymbol{\delta})} = \{s \in [1, d] \mid \boldsymbol{j}_s^{\boldsymbol{\delta}} = j\} \quad \text{and} \quad |U_j^{(\pm)}(\lambda^{\boldsymbol{\delta}})| = |U_j^{(\pm)}(\boldsymbol{\delta})|.$$

For $\lambda \in \Lambda$, $\boldsymbol{\delta} \in \boldsymbol{\Lambda}$, and $i \in [a, b]$, we define some signs:

$$\begin{aligned} \mathtt{sgn}_{\lambda;i} &:= (-1)^{\sum_{j=a}^{i-1} \lambda_j}, \qquad \mathtt{sgn}_{\boldsymbol{\delta};r,i} := (-1)^{\sum_{s=1}^{r-1} |\delta^{(s)}| + \sum_{j=a}^{i-1} \delta_j^{(r)}}, \\ t_{\boldsymbol{\delta}} &:= \sum_{\substack{1 \le r < s \le m \\ a \le j < i \le b}} \delta_i^{(r)} \delta_j^{(s)}, \qquad \chi_{\boldsymbol{\delta}} := (-1)^{t_{\boldsymbol{\delta}}}, \\ \xi_\lambda &:= (-1)^{\sum_{i=a}^{b} \lambda_i(\lambda_i - 1)/2}, \qquad \xi_{\boldsymbol{\delta}} := \chi_{\boldsymbol{\delta}} \xi_{\lambda^{\boldsymbol{\delta}}}. \end{aligned}$$

Lemma 3.2.1. *Let* $\boldsymbol{\delta} = (\delta^{(1)}, \dots, \delta^{(m)}) \in \boldsymbol{\Lambda}$, *$i \in [a,b]$ and $r \in [1,m]$. If $\delta_i^{(r)} = 0$, then*

$$\mathtt{sgn}_{\lambda^{\boldsymbol{\delta}};i}\chi_{\boldsymbol{\delta}} = \chi_{\boldsymbol{\delta}+\mathbf{e}_i^r}\mathtt{sgn}_{\boldsymbol{\delta};r,i}(-1)^{\sum_{s=1}^{r-1}\delta_i^{(s)}}.$$

Proof. Let $\lambda := \lambda^{\boldsymbol{\delta}}$ and $\boldsymbol{\gamma} := \boldsymbol{\delta} + \mathbf{e}_i^r$. Writing '$\equiv$' for '$\equiv \pmod 2$', we have to prove

$$\sum_{j=a}^{i-1}\lambda_j + \sum_{t<s,\ k>j}\delta_k^{(t)}\delta_j^{(s)} \equiv \sum_{t<s,\ k>j}\gamma_k^{(t)}\gamma_j^{(s)} + \sum_{s=1}^{r-1}|\delta^{(s)}| + \sum_{j=a}^{i-1}\delta_j^{(r)} + \sum_{s=1}^{r-1}\delta_i^{(s)}.$$

Note that

$$\sum_{t<s,\ k>j}\gamma_k^{(t)}\gamma_j^{(s)} = \sum_{t<s,\ k>j}\delta_k^{(t)}\delta_j^{(s)} + \sum_{s>r,\ j<i}\delta_j^{(s)} + \sum_{s<r,\ j>i}\delta_j^{(s)}, \qquad (2.1)$$

so the required comparison boils down to

$$\sum_{j=a}^{i-1}\lambda_j \equiv \sum_{s>r,\ j<i}\delta_j^{(s)} + \sum_{s<r,\ j>i}\delta_j^{(s)} + \sum_{s=1}^{r-1}|\delta^{(s)}| + \sum_{j=a}^{i-1}\delta_j^{(r)} + \sum_{s=1}^{r-1}\delta_i^{(s)},$$

which is easy to see. $\square$

Lemma 3.2.2. *Let* $\boldsymbol{\gamma} = (\gamma^{(1)}, \dots, \gamma^{(m)}) \in \boldsymbol{\Lambda}$, *$i \in [a,b]$ and $r \in [1,m]$. If $\gamma_i^{(r)} = 1$, then*

$$\xi_{\boldsymbol{\gamma}}\mathtt{sgn}_{\lambda^{\boldsymbol{\gamma}};i} = \mathtt{sgn}_{\boldsymbol{\gamma}-\mathbf{e}_i^r;r,i}\xi_{\boldsymbol{\gamma}-\mathbf{e}_i^r}(-1)^{\sum_{s=r+1}^{m}\gamma_i^{(s)}\}}.$$

Proof. Let $\mu := \lambda^{\gamma}$, $\boldsymbol{\delta} := \boldsymbol{\gamma} - \mathbf{e}_i^r$, and $\lambda := \lambda^{\boldsymbol{\delta}}$. Writing '$\equiv$' for '$\equiv \pmod 2$', we have to prove the comparison

$$\begin{aligned}&\sum_{t<s,\ k>j} \gamma_k^{(t)}\gamma_j^{(s)} + \sum_{j=a}^{b} \mu_j(\mu_j-1)/2 + \sum_{j=a}^{i-1} \mu_j \\ &\equiv \sum_{s=1}^{r-1} |\delta^{(s)}| + \sum_{j=a}^{i-1} \delta_j^{(r)} + \sum_{t<s,\ k>j} \delta_k^{(t)}\delta_j^{(s)} + \sum_{j=a}^{b} \lambda_j(\lambda_j-1)/2 + \sum_{s=r+1}^{m} \gamma_i^{(s)}.\end{aligned}$$

Note that

$$\sum_{j=a}^{b} \mu_j(\mu_j-1)/2 - \sum_{j=a}^{b} \lambda_j(\lambda_j-1)/2 = \lambda_i.$$

So, using also (2.1), the required comparison boils down to

$$\sum_{s>r,\ j<i} \delta_j^{(s)} + \sum_{s<r,\ j>i} \delta_j^{(s)} + \sum_{j=a}^{i} \lambda_j \equiv \sum_{s=1}^{r-1} |\delta^{(s)}| + \sum_{j=a}^{i-1} \delta_j^{(r)} + \sum_{s=r+1}^{m} \delta_i^{(s)},$$

which is easy to see. □

3.2.2. The resolution $P_\bullet$

Let $\lambda \in \Lambda$. Recalling divided power word $\boldsymbol{i}^\lambda \in I^\theta_{\mathrm{div}}$ from §3.2.1, we set

$$s_\lambda := -\frac{lm(m-1)}{2} + (m+1)n - \sum_{i=a}^{b} \lambda_i^2 \in \mathbb{Z},$$

$$e_\lambda := 1_{\boldsymbol{i}^\lambda} \in H_\theta, \quad P_\lambda = P_\lambda^{\alpha^m} := q^{s_\lambda} H_\theta e_\lambda.$$

In particular, P_λ is a projective left H_θ-module. Further, set for any $n \in \mathbb{Z}_{\geq 0}$:

$$P_n = P_n^{\alpha^m} := \bigoplus_{\lambda \in \Lambda(n)} P_\lambda.$$

Note that $P_n = 0$ for $n > d - m$. The projective resolution $P_\bullet = P_\bullet^{\alpha^m}$ of $\Delta(\alpha^m)$ will be of the form

$$\ldots \longrightarrow P_{n+1} \xrightarrow{d_n} P_n \longrightarrow \ldots \longrightarrow P_0 \longrightarrow \Delta(\alpha^m) \longrightarrow 0. \tag{2.2}$$

To describe the boundary maps d_n, we first consider a more general situation. Suppose we are given two sets of idempotents $\{e_a \mid a \in A\}$ and $\{f_b \mid b \in B\}$ in an algebra H. An $A \times B$ matrix $D := (d^{a,b})_{a\in A, b\in B}$ with every $d^{a,b} \in e_a H f_b$ then yields the homomorphism between the projective H-modules

$$\rho_D : \bigoplus_{a\in A} He_a \to \bigoplus_{b\in B} Hf_b,\ (r_a e_a)_{a\in A} \mapsto \big(\sum_{a\in A} r_a d^{a,b}\big)_{b\in B}, \tag{2.3}$$

which we refer to as the *right multiplication with D.*

We now define a $\Lambda(n+1) \times \Lambda(n)$ matrix D_n with entries $d_n^{\mu,\lambda} \in e_\mu H_\theta e_\lambda$. Let $\lambda \in \Lambda(n)$ and $a \le i \le b$ be such that $\lambda_i < m$. Recalling (1.1), define

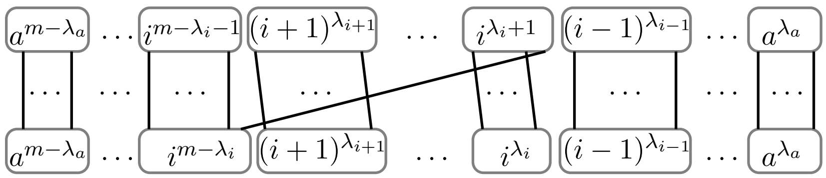
$$\psi_{\lambda;i} := \psi_{r_i^+(\lambda+\mathtt{e}_i)\to r_i^-(\lambda)}.$$

Note that $\psi_{\lambda;i} 1_{\boldsymbol{j}^\lambda} = 1_{\boldsymbol{j}^{\lambda+\mathtt{e}_i}} \psi_{\lambda;i}$. Recalling the sign $\mathtt{sgn}_{\lambda;i}$ from §3.2.1, we now set

$$d_n^{\mu,\lambda} := \begin{cases} \mathtt{sgn}_{\lambda;i} e_\mu \psi_{\lambda;i} e_\lambda & \text{if } \mu = \lambda + \mathtt{e}_i \text{ for some } a \le i \le b, \\ 0 & \text{otherwise.} \end{cases}$$

Diagrammatically, for $\mu = \lambda + \mathsf{e}_i$ as above, we have

$$d_n^{\mu,\lambda} = \pm$$



We now set the boundary map d_n to be the right multiplication with D_n:

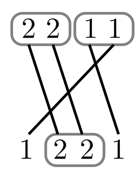
$$d_n := \rho_{D_n}.$$

Example 3.2.3. Let $a = b = 1$ and $m = 2$. Then the resolution $P_\bullet$ is

$$0 \to H_\theta 1_{2^{(2)}1^{(2)}} \xrightarrow{d_1} H_\theta 1_{12^{(2)}1} \xrightarrow{d_0} q^{-2} H_\theta 1_{1^{(2)}2^{(2)}} \longrightarrow \Delta((\alpha_1+\alpha_2)^2) \longrightarrow 0,$$

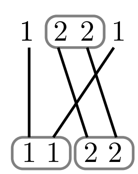
where d_1 is a right multiplication with

$$1_{2^{(2)}1^{(2)}}\psi_3\psi_2\psi_1 1_{12^{(2)}1} =$$



and d_0 is a right multiplication with

$$1_{12^{(2)}1}\psi_3\psi_2 1_{1^{(2)}2^{(2)}} =$$



It is far from clear that $\ker d_n = \operatorname{im} d_{n+1}$ but at least the following is easy to see:

Lemma 3.2.4. *The homomorphisms d_n are homogeneous of degree 0 for all n.*

Proof. Let $\lambda \in \Lambda(n)$ be such that $\lambda_i < m$ for some $a \leq i \leq b$, so that $\mu := \lambda + \mathsf{e}_i \in \Lambda(n+1)$. The homomorphism $d_n = \rho_{D_n}$ is a right multiplication with the matrix D_n. Its (μ, λ)-component is a homomorphism $P_\mu \to P_\lambda$ obtained by the right multiplication with $\pm e_\mu \psi_{\lambda;i} e_\lambda$. Recall that $P_\mu = q^{s_\mu} H_\theta e_\mu$ and $P_\lambda = q^{s_\lambda} H_\theta e_\lambda$. So we just need to show that $s_\mu = s_\lambda + \deg(e_\mu \psi_{\lambda;i} e_\lambda)$. This is an easy computation using the fact that by definition we have $\deg(e_\mu \psi_{\lambda;i} e_\lambda) = m - 2\lambda_i$. $\square$

3.2.3. The resolution $Q_\bullet$

In order to check that $P_\bullet$ is a resolution of $\Delta(\alpha^m)$, we show that it is a direct summand of a known resolution $Q_\bullet$ of $q^{m(m-1)/2}\Delta(\alpha)^{\circ m}$. To describe the latter resolution, let us first consider the special case $m = 1$.

Lemma 3.2.5. *We have that $P_\bullet^\alpha$ is a resolution of $\Delta(\alpha)$.*

Proof. This is a special case of [BKM, Theorem 4.12], corresponding to the standard choice of $(\alpha_{i+1} + \cdots + \alpha_j, \alpha_i)$ as the minimal pair for an arbitrary positive root $\alpha_i + \cdots + \alpha_j$ in the definition of $\boldsymbol{i}_{\alpha,\sigma}$, see [BKM, §4.5]. $\square$

Let $Q_\bullet$ be the resolution $q^{m(m-1)/2}(P_\bullet^\alpha)^{\circ m}$. To describe $Q_\bullet$ more explicitly, let $n \in \mathbb{Z}_{\geq 0}$ and $\boldsymbol{\delta} = (\delta^{(1)}, \dots, \delta^{(m)}) \in \boldsymbol{\Lambda}(n)$. Recalling the definitions of §3.2.1, we set

$$e_{\boldsymbol{\delta}} := 1_{\boldsymbol{j}^{\boldsymbol{\delta}}} \in H_\theta, \quad Q_{\boldsymbol{\delta}} := q^{n+m(m-1)/2} H_\theta e_{\boldsymbol{\delta}}.$$

Further for any $n \in \mathbb{Z}_{\geq 0}$, we set

$$Q_n := \bigoplus_{\boldsymbol{\delta} \in \boldsymbol{\Lambda}(n)} Q_{\boldsymbol{\delta}}.$$

The projective resolution $Q_\bullet$ is

$$\dots \longrightarrow Q_{n+1} \xrightarrow{c_n} Q_n \longrightarrow \dots \longrightarrow Q_0 \longrightarrow q^{m(m-1)/2} \Delta(\alpha)^{\circ m} \longrightarrow 0,$$

where c_n is the right multiplication with the $\boldsymbol{\Lambda}(n+1) \times \boldsymbol{\Lambda}(n)$ matrix $C = (c_n^{\gamma,\boldsymbol{\delta}})$ defined as follows. If $\boldsymbol{\delta} + \mathbf{e}_i^r \in \boldsymbol{\Lambda}$ for some $r \in [1, m]$ and $i \in [a, b]$, i.e. $\delta_i^{(r)} = 0$, we set

$$\psi_{\boldsymbol{\delta};r,i} := \iota_{\alpha,\dots,\alpha}(1^{\otimes(r-1)} \otimes \psi_{\delta^{(r)};i} \otimes 1^{\otimes(m-r)}).$$

Recalling the signs defined in §3.2.1, for $\boldsymbol{\delta} \in \boldsymbol{\Lambda}(n)$ and $\boldsymbol{\gamma} \in \boldsymbol{\Lambda}(n+1)$, we now define

$$c_n^{\gamma,\boldsymbol{\delta}} = \begin{cases} \mathtt{sgn}_{\boldsymbol{\delta};r,i} e_{\boldsymbol{\gamma}} \psi_{\boldsymbol{\delta};r,i} e_{\boldsymbol{\delta}} & \text{if } \boldsymbol{\gamma} = \boldsymbol{\delta} + \mathbf{e}_i^r \text{ for } 1 \leq r \leq m \text{ and } a \leq i \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

The fact that $Q_\bullet$ is indeed isomorphic to the resolution $q^{m(m-1)/2}(P_\bullet^\alpha)^{\circ m}$ is easily checked using the isomorphism $H_\theta e_{\boldsymbol{\delta}} \cong H_\theta 1_{\boldsymbol{j}^{\delta(1)}} \circ \dots \circ H_\theta 1_{\boldsymbol{j}^{\delta(m)}}$, which comes from Lemma 3.1.3.

Example 3.2.6. Let $a = b = 1$ and $m = 2$. Then the resolution $Q_\bullet$ is

$$0 \to q^3 H_\theta 1_{2121} \xrightarrow{c_1} q^2 H_\theta 1_{2112} \oplus q^2 H_\theta 1_{1221} \xrightarrow{c_0} q H_\theta 1_{1212} \longrightarrow$$
$$q\Delta(\alpha_1 + \alpha_2)^{\circ 2} \longrightarrow 0,$$

where c_1 is a right multiplication with the matrix $(-1_{2121}\psi_3 \;\; 1_{2121}\psi_1)$, and c_0 is a right multiplication with the matrix $\begin{pmatrix} 1_{2112}\psi_1 \\ 1_{1221}\psi_3 \end{pmatrix}$.

3.2.4. Comparison maps

We now construct what will end up being a pair of chain maps $f : P_\bullet \to Q_\bullet$ and $g : Q_\bullet \to P_\bullet$ with $g \circ f = \mathrm{id}$. As usual, f_n and g_n will be given as right multiplications with certain matrices F_n and G_n, respectively.

Let $\lambda \in \Lambda$. Recall the definitions of §3.2.1. We denote by w_0^λ the longest element of the parabolic subgroup $\mathfrak{S}_{\omega_\lambda} \leq \mathfrak{S}_d$. We also denote

$$y^\lambda := \iota_{\omega_\lambda}(1_{(m-\lambda_a)\alpha_a} \otimes \cdots \otimes 1_{(m-\lambda_b)\alpha_b} \otimes y_{0,m} \otimes 1_{\lambda_b\alpha_b} \otimes \cdots \otimes 1_{\lambda_a\alpha_a}).$$

Let $\boldsymbol{\delta} = (\delta^{(1)}, \ldots, \delta^{(m)}) \in \boldsymbol{\Lambda}$. We define $u(\boldsymbol{\delta}) \in \mathfrak{S}_d$ as follows: for all $i = a, \ldots, b$, the permutation $u(\boldsymbol{\delta})$ maps:

(i) the elements of $U_i^{\pm}(\lambda^{\boldsymbol{\delta}})$ increasingly to the elements of $U_i^{\pm}(\boldsymbol{\delta})$;

(ii) the elements of $U_{b+1}(\lambda^{\boldsymbol{\delta}})$ increasingly to the elements of $U_{b+1}(\boldsymbol{\delta})$.

Set $w(\boldsymbol{\delta}) := u(\boldsymbol{\delta})^{-1}$. Then $w(\boldsymbol{\delta})$ can also be characterized as the element of $\mathfrak{S}_d$ which for all $i = a, \ldots, b$, maps the elements of $U_i^{(\pm)}(\boldsymbol{\delta})$ increasingly to the elements of $U_i^{(\pm)}(\lambda^{\boldsymbol{\delta}})$ and the elements of $U_{b+1}(\boldsymbol{\delta})$ increasingly to the elements of $U_{b+1}(\lambda^{\boldsymbol{\delta}})$.

Recall the signs $\chi_{\boldsymbol{\delta}}$ and $\xi_{\boldsymbol{\delta}}$ defined in §3.2.1. We now define F_n as the $\Lambda(n) \times \boldsymbol{\Lambda}(n)$-matrix with the entries $f_n^{\lambda,\boldsymbol{\delta}}$ defined for any $\lambda \in$

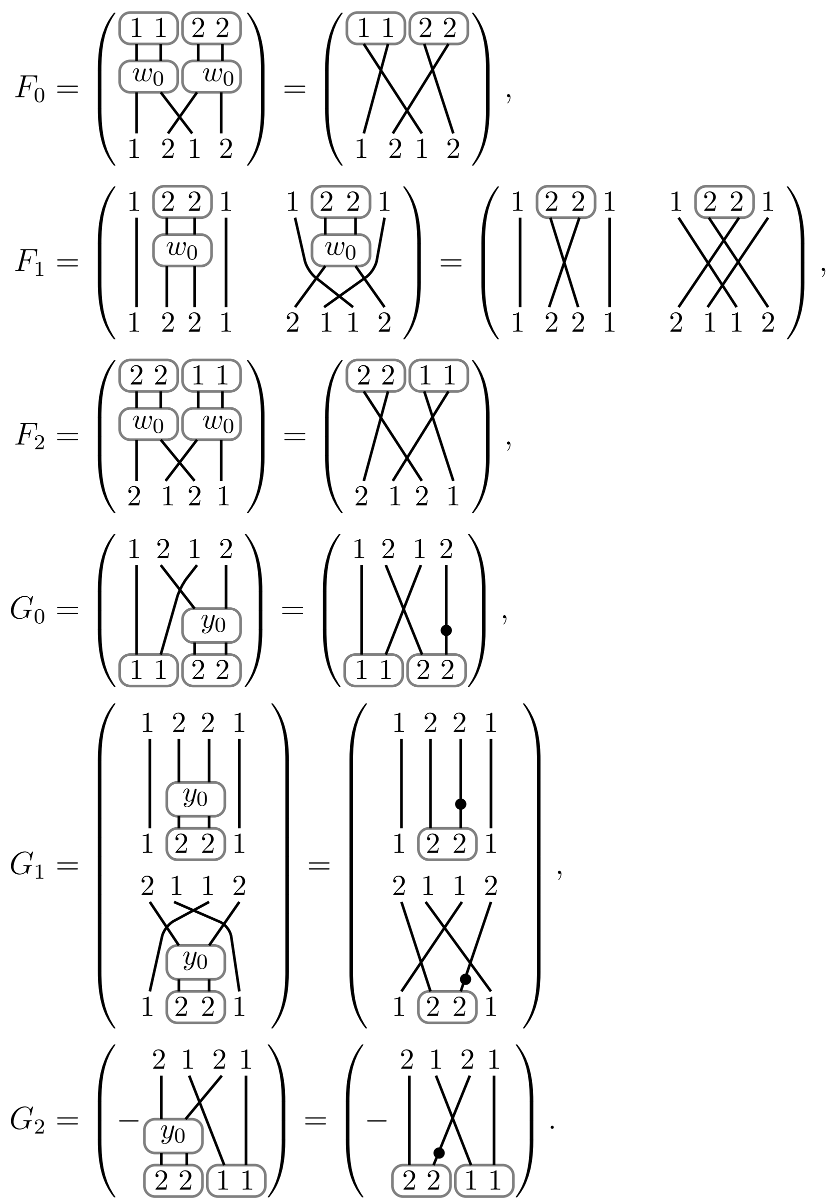
$\Lambda(n), \boldsymbol{\delta} \in \boldsymbol{\Lambda}(n)$ as follows:

$$f_n^{\lambda,\boldsymbol{\delta}} := \begin{cases} \chi_{\boldsymbol{\delta}} e_\lambda \psi_{w_0^\lambda} \psi_{w(\boldsymbol{\delta})} e_{\boldsymbol{\delta}} & \text{if } \lambda = \lambda^{\boldsymbol{\delta}}, \\ 0 & \text{otherwise.} \end{cases}$$

We define G_n as the $\boldsymbol{\Lambda}(n) \times \Lambda(n)$-matrix with the entries $g_n^{\boldsymbol{\delta},\lambda}$ defined for any $\boldsymbol{\delta} \in \boldsymbol{\Lambda}(n), \lambda \in \Lambda(n)$ as follows:

$$g_n^{\boldsymbol{\delta},\lambda} := \begin{cases} \xi_{\boldsymbol{\delta}} e_{\boldsymbol{\delta}} \psi_{u(\boldsymbol{\delta})} y^\lambda e_\lambda & \text{if } \lambda = \lambda^{\boldsymbol{\delta}}, \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.2.7. Let $m = d = 2$ as in Examples 3.2.3 and 3.2.6. Then:



Lemma 3.2.8. *Let $\boldsymbol{\delta} \in \boldsymbol{\Lambda}(n)$ and $\lambda = \lambda^{\boldsymbol{\delta}}$. Then:*

(i) $\deg(\psi_{w(\boldsymbol{\delta})} e_{\boldsymbol{\delta}}) = \frac{m(m-1)(l-1)}{2} - mn + \sum_{i=a}^{b} \lambda_i^2$,

(ii) $\deg(f_n^{\lambda,\boldsymbol{\delta}}) = -\frac{m(m-1)(l+1)}{2} + mn - \sum_{i=a}^{b} \lambda_i^2$,

(iii) $\deg(g_n^{\boldsymbol{\delta},\lambda}) = \frac{m(m-1)(l+1)}{2} - mn + \sum_{i=a}^{b} \lambda_i^2$.

Proof. (i) We prove this by induction on m. Denote the right hand side by $R(m)$ and the left hand side by $L(m)$. If $m = 1$ then $w(\boldsymbol{\delta}) = 1$, so $L(1) = 0$. Moreover,

$$R(1) = -n + \sum_{i=a}^{b} \lambda_i^2 = -n + \sum_{i=a}^{b} (\delta_i^{(1)})^2 = -n + \sum_{i=a}^{b} \delta_i^{(1)} = 0.$$

Let $m > 1$. It suffices to prove that $R(m) - R(m-1) = L(m) - L(m-1)$. Let $\delta^{(m)} = (\varepsilon_a, \dots, \varepsilon_b)$. Then, since all ε_i are 0 or 1, we have

$$\begin{aligned} R(m) - R(m-1) = {} & \frac{m(m-1)(l-1)}{2} - mn + \sum_{i=a}^{b} \lambda_i^2 \\ & - \frac{(m-1)(m-2)(l-1)}{2} \\ & + (m-1)(n - \sum_{i=a}^{b} \varepsilon_i) - \sum_{i=a}^{b} (\lambda_i - \varepsilon_i)^2 \\ = {} & (m-1)(l-1) - n - m \sum_{i=a}^{b} \varepsilon_i + 2 \sum_{i=a}^{b} \lambda_i \varepsilon_i. \end{aligned}$$

On the other hand, consider the Khovanov-Lauda diagram of $\psi_{w(\boldsymbol{\delta})} e_{\boldsymbol{\delta}}$. The bottom positions of the diagram correspond to to the letters of the word $\boldsymbol{j}^{\boldsymbol{\delta}}$, and so the rightmost l bottom positions of this diagram correspond to the letters of $\boldsymbol{j}^{\delta^{(m)}}$. In other words, counting from the right, the sequence of colors of these positions is $a^{\varepsilon_a}, \dots, b^{\varepsilon_b}, b+1, b^{1-\varepsilon_b}, \dots, a^{1-\varepsilon_a}$. Note that the strings which originate in these positions do not intersect each other, so $L(m) - L(m-1)$ equals the sum of the degrees of the intersections of these strings with the other strings of the diagram, i.e.

$$
\begin{aligned}
L(m) - L(m-1) = & \sum_{i=a+1}^{b} \varepsilon_i(\lambda_{i-1} - \varepsilon_{i-1}) + \lambda_b - \varepsilon_b \\
& + \sum_{i=a+1}^{b} (1-\varepsilon_i)(m-1-2(\lambda_i - \varepsilon_i) + \lambda_{i-1} - \varepsilon_{i-1}) \\
& + (1-\varepsilon_a)(m-1-2(\lambda_a - \varepsilon_a)),
\end{aligned}
$$

which is easily seen to equal the expression for $R(m)-R(m-1)$ obtained above.

(ii) This follows from (i) since $\deg(f_n^{\lambda,\boldsymbol{\delta}}) = \deg(\psi_{w(\boldsymbol{\delta})}e_{\boldsymbol{d}}) + \deg(e_\lambda \psi_{w_0^\lambda})$ and

$$
\deg(e_\lambda \psi_{w_0^\lambda}) = -m(m-1) - \sum_{i=a}^{b} (\lambda_i(\lambda_i - 1) + (m-\lambda_i)(m-\lambda_i-1)).
$$

(iii) This follows from (i) since

$$
\deg(g_n^{\boldsymbol{\delta},\lambda}) = \deg(e_{\boldsymbol{\delta}}\psi_{u(\boldsymbol{\delta})}) + \deg(y^\lambda) = \deg(\psi_{w(\boldsymbol{\delta})}e_{\boldsymbol{\delta}}) + m(m-1).
$$

□

Corollary 3.2.9. *The homomorphisms f_n and g_n are homogeneous of degree 0 for all n.*

Proof. Let $\boldsymbol{\delta} \in \boldsymbol{\Lambda}(n)$ and $\lambda = \lambda^{\boldsymbol{\delta}}$. The homomorphism f_n is a right multiplication with the matrix F_n. Its $(\lambda, \boldsymbol{\delta})$-component is a homomorphism $P_\lambda \to Q_{\boldsymbol{\delta}}$ obtained by the right multiplication with $f_n^{\lambda,\boldsymbol{\delta}}$. Recall that $P_\lambda = q^{s_\lambda} H_\theta e_\lambda$ and $Q_{\boldsymbol{\delta}} = q^{n+m(m-1)/2} H_\theta e_{\boldsymbol{\delta}}$. So we just need to show that $s_\lambda = n+m(m-1)/2+\deg(f_n^{\lambda,\boldsymbol{\delta}})$, which easily follows from Lemma 3.2.8(ii).

The homomorphism g_n is a right multiplication with the matrix G_n. Its $(\boldsymbol{\delta},\lambda)$-component is a homomorphism $Q_{\boldsymbol{\delta}} \to P_\lambda$ obtained by the right multiplication with $g_n^{\boldsymbol{\delta},\lambda}$. So we just need to show that $n+m(m-1)/2 = s_\lambda + \deg(g_n^{\boldsymbol{\delta},\lambda})$, which easily follows from Lemma 3.2.8(iii). $\square$

Corollary 3.2.10. *Suppose $\boldsymbol{\delta},\boldsymbol{\varepsilon} \in \boldsymbol{\Lambda}(n)$ are such that $\lambda^{\boldsymbol{\delta}} = \lambda^{\boldsymbol{\varepsilon}}$. Then* $\deg(\psi_{w(\boldsymbol{\delta})}e_{\boldsymbol{\delta}}) = \deg(\psi_{w(\boldsymbol{\varepsilon})}e_{\boldsymbol{\varepsilon}})$.

3.2.5. Independence of reduced decompositions

Throughout this subsection we fix $\boldsymbol{\delta} \in \boldsymbol{\Lambda}(n)$ and set $\lambda := \lambda^{\boldsymbol{\delta}}$.

Recall that in general the element $\psi_w \in H_\theta$ depends on a choice of a reduced decomposition of $w \in \mathfrak{S}_d$. While it is clear from the form of the braid relations in the KLR algebra that $e_\lambda\psi_{w_0^\lambda}$ does not depend on a choice of a reduced decomposition of w_0^λ, it is not obvious that a similar statement is true for $\psi_{w(\boldsymbol{\delta})}e_{\boldsymbol{\delta}}$ and $e_{\boldsymbol{\delta}}\psi_{u(\boldsymbol{\delta})}$. So a priori the elements $f_n^{\lambda,\boldsymbol{\delta}} = \pm e_\lambda\psi_{w_0^\lambda}\psi_{w(\boldsymbol{\delta})}e_{\boldsymbol{\delta}}$ and $g_n^{\boldsymbol{\delta},\lambda} = \pm e_{\boldsymbol{\delta}}\psi_{u(\boldsymbol{\delta})}y^\lambda e_\lambda$ might depend on choices of reduced decompositions of $w(\boldsymbol{\delta})$ and $u(\boldsymbol{\delta})$. In this section we will prove that this is not the case, and so in this sense the maps f_n and g_n are canonical.

Recall the composition ω_λ and the words $\boldsymbol{j}^\lambda, \boldsymbol{j}^{\boldsymbol{\delta}}$ from §3.2.1.

Lemma 3.2.11. *The element $w(\boldsymbol{\delta})$ is the unique element of ${}^{\omega_\lambda}\mathcal{D}^{(l^m)}$ with $w(\boldsymbol{\delta}) \cdot \boldsymbol{j}^{\boldsymbol{\delta}} = \boldsymbol{j}^\lambda$.*

Proof. That $w(\boldsymbol{\delta}) \in {}^{\omega_\lambda}\mathcal{D}^{(l^m)}$ and $w(\boldsymbol{\delta}) \cdot \boldsymbol{j}^{\boldsymbol{\delta}} = \boldsymbol{j}^\lambda$ follows from the definitions. To prove the uniqueness statement, let $w \in {}^{\omega_\lambda}\mathcal{D}^{(l^m)}$ and

$w \cdot \boldsymbol{j}^{\boldsymbol{\delta}} = \boldsymbol{j}^{\lambda}$. Since $w \in {}^{\omega_\lambda}\mathcal{D}$, it maps the elements of $U_{b+1}(\boldsymbol{\delta})$ increasingly to the elements of $U_{b+1}(\lambda)$. By definition, $\boldsymbol{j}^{\boldsymbol{\delta}} = \boldsymbol{j}^{\delta^{(1)}} \dots \boldsymbol{j}^{\delta^{(m)}}$. For every $r \in [1, m]$, the entries of $\boldsymbol{j}^{\delta^{(r)}}$ have the following properties: (1) each $i \in [a, b+1]$ appears among them exactly once; (2) the entries that precede $b+1$ appear in the increasing order; (3) the entries that succeed $b+1$ appear in the decreasing order. Since $w \in \mathcal{D}^{(l^m)}$, it maps the positions corresponding to the entries in (2) to the positions which are to the left of the positions occupied with $b+1$ in $\boldsymbol{j}^{\lambda}$, and it maps the positions corresponding to the entries in (3) to the positions which are to the right of the positions occupied with $b+1$ in $\boldsymbol{j}^{\lambda}$. In other words, for all $i \in [a, b]$, the permutation w maps the elements of $U_i^{\pm}(\boldsymbol{\delta})$ to the elements of $U_i^{\pm}(\lambda)$. As $w \in {}^{\omega_\lambda}\mathcal{D}$, it now follows that for every $i \in [a, b]$, the permutation w maps the elements of $U_i^{\pm}(\boldsymbol{\delta})$ to the elements of $U_i^{\pm}(\lambda)$ increasingly. We have shown that $w = w(\boldsymbol{\delta})$. □

Lemma 3.2.12. *Let $\varepsilon, \delta \in \Lambda^{\alpha}$, and $\boldsymbol{j}^{\varepsilon} = w \cdot \boldsymbol{j}^{\delta}$ for some $w \in \mathfrak{S}_l$. Then either $\varepsilon = \delta$ and $w = 1$, or $\deg(\psi_w 1_{\boldsymbol{j}^{\delta}}) > 0$.*

Proof. Since every $i \in [a, b+1]$ appears in $\boldsymbol{j}^{\delta}$ exactly once, $\varepsilon = \delta$ implies $w = 1$. On the other hand, if $\varepsilon \neq \delta$, let i be maximal with $\varepsilon_i \neq \delta_i$. Then the strings colored i and $i+1$ in the Khovanov-Lauda diagram D for $\psi_w 1_{\boldsymbol{j}^{\delta}}$ intersect (for any choice of a reduced decomposition of w), which contributes a degree 1 crossing to D. On the other hand, since every $j \in [a, b+1]$ appears in $\boldsymbol{j}^{\delta}$ exactly once, D has no same color crossings, which are the only possible crossings of negative degree. The lemma follows. □

Lemma 3.2.13. *Suppose that $w \in {}^{(l^m)}\mathcal{D}^{\omega_\lambda}$ and $w \cdot \boldsymbol{j}^\lambda$ is of the form $\boldsymbol{i}^{(1)} \dots \boldsymbol{i}^{(m)}$ with $\boldsymbol{i}^{(1)}, \dots, \boldsymbol{i}^{(m)} \in I^\alpha$. Then $w \cdot \boldsymbol{j}^\lambda = \boldsymbol{j}^{\boldsymbol{\varepsilon}}$ for some $\boldsymbol{\varepsilon} \in \boldsymbol{\Lambda}$ with $\lambda^{\boldsymbol{\varepsilon}} = \lambda$.*

Proof. Ler $r \in [1, m]$. By assumption, the entries $i_1^{(r)}, \dots, i_l^{(r)}$ of $\boldsymbol{i}^{(r)}$ have the following properties: (1) each $i \in [a, b+1]$ appears among them exactly once; (2) the entries that precede $b+1$ appear in the increasing order; (3) the entries that succeed $b+1$ appear in the decreasing order. The result follows. $\square$

Lemma 3.2.14. *Let $P = \{w \in {}^{\omega_\lambda}\mathcal{D} \mid w \cdot \boldsymbol{j}^{\boldsymbol{\delta}} = \boldsymbol{j}^\lambda\}$. Then $w(\boldsymbol{\delta}) \in P$ and $\deg(\psi_{w(\boldsymbol{\delta})} e_{\boldsymbol{\delta}}) < \deg(\psi_w e_{\boldsymbol{\delta}})$ for any $w \in P \setminus \{w(\boldsymbol{\delta})\}$.*

Proof. It is clear that $w(\boldsymbol{\delta}) \in P$. On the other hand, by Lemma 3.1.1, an arbitrary $w \in P$ can be written uniquely in the form $w = xy$ with $x \in {}^{\omega_\lambda}\mathcal{D}^{(l^m)}$, $y \in \mathfrak{S}_{(l^m)}$ and $\ell(xy) = \ell(x) + \ell(y)$.

Since $xy \cdot \boldsymbol{j}^{\boldsymbol{\delta}} = \boldsymbol{j}^\lambda$, we have $y \cdot \boldsymbol{j}^{\boldsymbol{\delta}} = x^{-1} \cdot \boldsymbol{j}^\lambda$. As $x^{-1} \in {}^{(l^m)}\mathcal{D}^{\omega_\lambda}$, it follows from Lemma 3.2.13 that $y \cdot \boldsymbol{j}^{\boldsymbol{\delta}} = x^{-1} \cdot \boldsymbol{j}^\lambda$ is of the form $\boldsymbol{j}^{\boldsymbol{\varepsilon}}$ for some $\boldsymbol{\varepsilon} \in \boldsymbol{\Lambda}$ with $\lambda^{\boldsymbol{\varepsilon}} = \lambda$. By Lemma 3.2.11, $x = w(\boldsymbol{\varepsilon})$. If $w \neq w(\boldsymbol{\delta})$, then $y \neq 1$ and we have

$$
\begin{aligned}
\deg(\psi_w e_{\boldsymbol{\delta}}) &= \deg(\psi_{w(\boldsymbol{\varepsilon})} e_{\boldsymbol{\varepsilon}}) + \deg(\psi_y e_{\boldsymbol{\delta}}) \\
&= \deg(\psi_{w(\boldsymbol{\delta})} e_{\boldsymbol{\delta}}) + \deg(\psi_y e_{\boldsymbol{\delta}}) > \deg(\psi_{w(\boldsymbol{\delta})} e_{\boldsymbol{\delta}}),
\end{aligned}
$$

where we have used Corollary 3.2.10 for the second equality and Lemma 3.2.12 for the inequality. $\square$

Lemma 3.2.15. *The element $f_n^{\lambda,\boldsymbol{\delta}} = e_\lambda \psi_{w_0^\lambda} \psi_{w(\boldsymbol{\delta})} e_{\boldsymbol{\delta}}$ is independent of the choice of reduced expressions for w_0^λ and $w(\boldsymbol{\delta})$.*

Proof. It is clear from the form of the braid relations in the KLR algebra that $e_\lambda \psi_{w_0^\lambda}$ is independent of the choice of a reduced expression for w_0^λ. On the other hand, if $\psi_{w(\boldsymbol{\delta})} e_\delta$ and $\psi'_{w(\boldsymbol{\delta})} e_{\boldsymbol{\delta}}$ correspond to different reduced expressions of $w(\boldsymbol{\delta})$, it follows from the defining relations of the KLR algebra and Theorem 1.3.1 that $\psi_{w(\boldsymbol{\delta})} e_\delta - \psi'_{w(\boldsymbol{\delta})} e_{\boldsymbol{\delta}}$ is a linear combination of elements of the form $\psi_u y e_{\boldsymbol{\delta}}$ with $u \in \mathfrak{S}_d$, $y \in \mathbb{Z}[y_1, \ldots, y_d]$ such that $\deg(\psi_u y e_{\boldsymbol{\delta}}) = \deg(\psi_{w(\boldsymbol{\delta})} e_\delta)$ and $u \boldsymbol{j}^{\boldsymbol{\delta}} = \boldsymbol{j}^\lambda$. We have to prove $e_\lambda \psi_{w_0^\lambda} \psi_u y e_{\boldsymbol{\delta}} = 0$. Suppose otherwise.

Since we are using any preferred reduced decompositions for u, we may assume in addition that $u \in {}^{\omega_\lambda}\mathcal{D}$, since otherwise $e_\lambda \psi_{w_0^\lambda} \psi_u = 0$. Now by Lemma 3.2.14, $\deg(\psi_u e_{\boldsymbol{\delta}}) > \deg(\psi_{w(\boldsymbol{\delta})} e_\delta)$, whence $\deg(\psi_u y e_{\boldsymbol{\delta}}) > \deg(\psi_{w(\boldsymbol{\delta})} e_{\boldsymbol{\delta}})$, giving a contradiction. □

Lemma 3.2.16. *The element $g_n^{\boldsymbol{\delta},\lambda} = \xi_{\boldsymbol{\delta}} e_{\boldsymbol{\delta}} \psi_{u(\boldsymbol{\delta})} y^\lambda e_\lambda$ is independent of the choice of a reduced expression for $u(\boldsymbol{\delta})$.*

Proof. The argument is similar to that of the previous lemma. If $e_{\boldsymbol{\delta}} \psi_{u(\boldsymbol{\delta})}$ and $e_{\boldsymbol{\delta}} \psi'_{u(\boldsymbol{\delta})}$ correspond to different reduced expressions of $u(\boldsymbol{\delta})$, then $e_{\boldsymbol{\delta}} \psi_{u(\boldsymbol{\delta})} - e_{\boldsymbol{\delta}} \psi'_{u(\boldsymbol{\delta})}$ is a linear combination of elements of the form $e_{\boldsymbol{\delta}} y \psi_w$ with $w \in \mathfrak{S}_d$, $y \in \mathbb{Z}[y_1, \ldots, y_d]$ such that, $\deg(e_{\boldsymbol{\delta}} y \psi_w) = \deg(e_{\boldsymbol{\delta}} \psi_{u(\boldsymbol{\delta})})$ and $w^{-1} \boldsymbol{j}^{\boldsymbol{\delta}} = \boldsymbol{j}^\lambda$. Moreover, in the Khovanov-Lauda dialgram of $\psi_w 1_{\boldsymbol{j}^\lambda}$ the strings colored $b+1$ do not cross each other, since this was the case for the diagram of $\psi_{u(\boldsymbol{\delta})} 1_{\boldsymbol{j}^\lambda}$. Hence, if $e_{\boldsymbol{\delta}} y \psi_w y^\lambda e_\lambda \neq 0$, we may assume in addition that $w^{-1} \in {}^{\omega_\lambda}\mathcal{D}$. Now, using Lemma 3.2.14, we conclude that $\deg(e_{\boldsymbol{\delta}} y \psi_w) > \deg(e_{\boldsymbol{\delta}} \psi_{u(\boldsymbol{\delta})})$, getting a contradiction. □

3.2.6. Splitting

In this subsection, we aim to show that $g \circ f = \mathrm{id}$. We fix $\lambda \in \Lambda(n)$ throughout the subsection. We need to prove $\sum_{\boldsymbol{\delta} \in \boldsymbol{\Lambda}(n)} f_n^{\lambda,\boldsymbol{\delta}} g_n^{\boldsymbol{\delta},\lambda} = e_\lambda$. Since $f_n^{\lambda,\boldsymbol{\delta}} = 0$ unless $\lambda^{\boldsymbol{\delta}} = \lambda$, this is equivalent to

$$\sum_{\boldsymbol{\delta} \in \boldsymbol{\Lambda}(n),\ \lambda^{\boldsymbol{\delta}} = \lambda} f_n^{\lambda,\boldsymbol{\delta}} g_n^{\boldsymbol{\delta},\lambda} = e_\lambda.$$

Let $\boldsymbol{\delta} \in \boldsymbol{\Lambda}(n)$ with $\lambda^{\boldsymbol{\delta}} = \lambda$. We say that $\boldsymbol{\delta}$ is *initial* if a preceeds $a+1$ in $\boldsymbol{j}^{\delta^{(r)}}$ for $r \in [1, m-\lambda_a]$ and a succeeds $a+1$ in $\boldsymbol{j}^{\delta^{(r)}}$ for $r \in (m-\lambda_a, m]$. In other words, $\boldsymbol{\delta}$ is initial if $\delta_a^{(r)} = 0$ for $r \in [1, m-\lambda_a]$ and $\delta_a^{(r)} = 1$ for $r \in (m-\lambda_a, m]$.

Let $w \in \mathfrak{S}_d$ and $1 \le r, s \le d$. We say that (r,s) is an *inversion pair for* w if $r < s$, $w(r) > w(s)$, and $\boldsymbol{j}_s^\lambda - \boldsymbol{j}_r^\lambda = \pm 1$.

Lemma 3.2.17. *Let $\boldsymbol{\delta} \in \boldsymbol{\Lambda}(n)$ be initial with $\lambda^{\boldsymbol{\delta}} = \lambda$. Set $\bar{\alpha} = \alpha_{a+1} + \cdots + \alpha_b$, $\bar{\theta} = m\bar{\alpha}$, $\bar{\lambda} = (\lambda_{a+1}, \dots, \lambda_b)$, $\bar{n} := \lambda_{a+1} + \cdots + \lambda_b$, and $\bar{\boldsymbol{\delta}} = (\bar{\delta}^{(1)}, \dots, \bar{\delta}^{(m)})$, where $\bar{\delta}^{(r)} = (\delta_{a+1}^{(r)}, \dots, \delta_b^{(r)})$ for all $r \in [1, m]$. Then*

$$f_n^{\lambda,\boldsymbol{\delta}} g_n^{\boldsymbol{\delta},\lambda} = \iota_{(m-\lambda_a)\alpha_a, \bar{\theta}, \lambda_a\alpha_a}(1_{a^{(m-\lambda_a)}} \otimes f_{\bar{n}}^{\bar{\lambda},\bar{\boldsymbol{\delta}}} g_{\bar{n}}^{\bar{\boldsymbol{\delta}},\bar{\lambda}} \otimes 1_{a^{(\lambda_a)}}). \tag{2.4}$$

Proof. By definition,

$$f_n^{\lambda,\boldsymbol{\delta}} g_n^{\boldsymbol{\delta},\lambda} = (-1)^{\sum_{i=a}^b \lambda_i(\lambda_i-1)/2} e_\lambda \psi_{w_0^\lambda} \psi_{w(\boldsymbol{\delta})} \psi_{u(\boldsymbol{\delta})} y^\lambda e_\lambda.$$

Throughout the proof, 'inversion pair' means 'inversion pair for $w(\boldsymbol{\delta})$'. Recall that $w(\boldsymbol{\delta}) = u(\boldsymbol{\delta})^{-1}$. Since $\boldsymbol{\delta}$ is initial, in the Khovanov-Lauda diagram for $\psi_{w(\boldsymbol{\delta})}$ (for any choice of reduced expression) no strings of color a cross each other. We want to apply quadratic relations on pairs of strings, one of which has color a and the other has color $a+1$. These correspond to inversion pairs (r,s) with $r \in U_a^-(\lambda), s \notin U_a^-(\lambda)$ or $s \in U_a^+(\lambda), r \notin U_a^+(\lambda)$.

Note that there are exactly $r-1$ inversion pairs of the form (r,s) when $r \in U_a^-(\lambda)$ and $d-s$ inversion pairs of the form (r,s) when $s \in U_a^+(\lambda)$. Applying the corresponding quadratic relations, we see that $f_n^{\lambda,\boldsymbol{\delta}} g_n^{\boldsymbol{\delta},\lambda}$ equals

$$(-1)^{\lambda_a(\lambda_a-1)/2} e_\lambda \psi_{w_0^\lambda} \iota_{(m-\lambda_a)\alpha_a,\bar{\theta},\lambda_a\alpha_a}(y_{0,m-\lambda_a} \otimes f_{\bar{n}}^{\bar{\lambda},\bar{\boldsymbol{\delta}}} g_{\bar{n}}^{\bar{\boldsymbol{\delta}},\bar{\lambda}} \otimes y'_{0,\lambda_a}) y^\lambda e_\lambda + (*), \tag{2.5}$$

where (*) a sum of elements of the form

$$e_\lambda \psi_{w_0^\lambda} \iota_{(m-\lambda_a)\alpha_a,\bar{\theta},\lambda_a\alpha_a}(Y^- \otimes X \otimes Y'^+) y^\lambda e_\lambda,$$

with $X \in H_{\bar{\theta}}$, $Y^\pm$ a polynomial in the variables y_r with $r \in U_a^\pm(\lambda)$, and $\deg Y^- + \deg Y^+ < \deg y_{0,m-\lambda_a} + \deg y'_{0,\lambda_a}$. By Lemma 3.1.4(i), we have $(*) = 0$. So by Lemma 3.1.4(iii), the expression (2.5) equals the right hand side of (2.4). $\square$

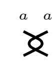
Define $\boldsymbol{\delta}_\lambda = (\delta_\lambda^{(1)}, \cdots, \delta_\lambda^{(m)})$ to be the unique element of $\boldsymbol{\Lambda}(n)$ such that for each $a \le i \le b$ we have:

- i preceeds $b+1$ in $\boldsymbol{j}^{\delta_\lambda^{(r)}}$ for $1 \le r \le m - \lambda_i$;
- i succeeds $b+1$ in $\boldsymbol{j}^{\delta_\lambda^{(r)}}$ for $m - \lambda_i < r \le m$.

Note that $\lambda^{\boldsymbol{\delta}_\lambda} = \lambda$ but $\boldsymbol{\delta}_{\lambda^{\boldsymbol{\delta}}}$ in general differs from $\boldsymbol{\delta}$.

Lemma 3.2.18. *Let $\boldsymbol{\delta} \in \boldsymbol{\Lambda}$ satisfy $\lambda^{\boldsymbol{\delta}} = \lambda$. Then*

$$f_n^{\lambda,\boldsymbol{\delta}} g_n^{\boldsymbol{\delta},\lambda} = \begin{cases} e_\lambda & \text{if } \boldsymbol{\delta} = \boldsymbol{\delta}_\lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $\boldsymbol{\delta} = \boldsymbol{\delta}_\lambda$, the result follows by induction on $\mathrm{ht}(\alpha)$ using Lemma 3.2.17. If $\boldsymbol{\delta} \neq \boldsymbol{\delta}_\lambda$, we may assume, using Lemma 3.2.17, that $\boldsymbol{\delta}$ is not initial. This implies that for some $r \in [1, m)$, we have $\delta_a^{(r)} = 1$ and $\delta_a^{(r+1)} = 0$, i.e. the last entry of the word $\boldsymbol{j}^{\delta^{(r)}}$ and the first entry of the word $\boldsymbol{j}^{\delta^{(r+1)}}$ are both equal to a. It follows that 

is a sub-diagram of a Khovanov-Lauda diagram for $\psi_{w(\boldsymbol{\delta})}\psi_{u(\boldsymbol{\delta})}y^\lambda e_\lambda$, so $f_n^{\lambda,\boldsymbol{\delta}} g_n^{\boldsymbol{\delta},\lambda} = \pm e_\lambda \psi_{w_0^\lambda} \psi_{w(\boldsymbol{\delta})}\psi_{u(\boldsymbol{\delta})} y^\lambda e_\lambda = 0$. □

Corollary 3.2.19. *For any $n \in \mathbb{Z}_{\geq 0}$, we have $g_n \circ f_n = \mathrm{id}$.*

3.2.7. Proof of the main theorem, assuming f and g are chain maps

In sections 3.3 and 3.4, we will prove that f and g constructed above are chain maps. The goal of this subsection is to demonstrate that this is sufficient to establish out main result:

Theorem 3.2.20. *If f and g are chain maps then $P_\bullet = P_\bullet^{\alpha^m}$ is a projective resolution of $\Delta(\alpha^m)$.*

Proof. The modules are projective by construction. By Corollary 3.2.19, $P_\bullet$ is a complex, isomorphic to a direct summand of the complex $Q_\bullet$. Since $Q_\bullet$ is a resolution of $q^{m(m-1)/2}\Delta(\alpha)^{\circ m} \cong [m]!\Delta(\alpha)$, it follows from the assumptions that $P_\bullet$ is exact in positive degrees, and its 0^{th}

cohomology is a direct summand of

$$q^{m(m-1)/2}\Delta(\alpha)^{\circ m} \cong q^{m(m-1)/2}[m]!\Delta(\alpha^m).$$

We deduce that the 0^{th} cohomology of $P_\bullet$ is isomorphic to a direct sum of modules of the form $q^s\Delta(\alpha^m)$. To check that it is just $\Delta(\alpha^m)$, observe that $\dim_q \operatorname{Hom}_{H_\theta}(P_0, L(\alpha^m))$ equals

$$\dim_q \operatorname{Hom}_{H_\theta}(q^{-lm(m-1)/2}H_\theta 1_{a^{(m)}\cdots(b+1)^{(m)}}, L(\alpha^m)).$$

Finally, Lemma 3.1.6 implies

$$\dim_q \operatorname{Hom}_{H_\theta}(q^{-lm(m-1)/2}H_\theta 1_{a^{(m)}\cdots(b+1)^{(m)}}, L(\alpha^m)) = \\ q^{lm(m-1)/2}\dim_q 1_{a^{(m)}\cdots(b+1)^{(m)}}L(\alpha^m) = 1,$$

completing the proof. □

3.3. Verification That f Is a Chain Map

We continue with the running assumptions of the previous section. In addition, throughout the section we fix $n \in \mathbb{Z}_{\geq 0}$, $\mu = (\mu_a, \ldots, \mu_b) \in \Lambda(n+1)$ and $\boldsymbol{\delta} = (\delta^{(1)}, \ldots, \delta^{(m)}) \in \boldsymbol{\Lambda}(n)$.

3.3.1. Special reduced expressions

Recall the notation of §3.2.1. Let $\lambda \in \Lambda^{\alpha^m}$ and $\delta \in \Lambda^{\alpha}$ be such that $\bar{\lambda} := \lambda - \delta \in \Lambda^{\alpha^{m-1}}$. For $i \in [a, b+1]$, we denote by p_i the position occupied by i in $\boldsymbol{j}^\delta$ and set

$$q_i := \begin{cases} l_i^-(\lambda) & \text{if } i \neq b+1 \text{ and } \delta_i^{(1)} = 0, \\ l_i^+(\lambda) & \text{if } i \neq b+1 \text{ and } \delta_i^{(1)} = 1, \\ l_{b+1}(\lambda) & \text{if } i = b+1. \end{cases} \tag{3.1}$$

Let $Q := \{q_a, \dots, q_{b+1}\}$. Note that $\{p_a, \dots, p_{b+1}\} = [1, l]$. Define $x_\delta^\lambda \in \mathfrak{S}_d$ to be the permutation which maps p_i to q_i for all $i \in [a, b+1]$, and maps the elements of $[l+1, d]$ increasingly to the elements of $[1, d] \setminus Q$. It is easy to see that x_δ^λ is fully commutative, so $\psi_{x_\delta^\lambda}$ is well defined, and $1_{\boldsymbol{j}^\lambda} \psi_{x_\delta^\lambda} = \psi_{x_\delta^\lambda} 1_{\boldsymbol{j}^\delta \boldsymbol{j}^{\bar{\lambda}}}$.

Now let $\lambda := \lambda^{\boldsymbol{\delta}}$, and $r \in [1, m]$. Define

$$\boldsymbol{\delta}^{\geq r} := (\delta^{(r)}, \dots, \delta^{(m)}) \in (\Lambda^\alpha)^{m-r+1}, \quad \lambda^{\geq r} := \lambda^{\boldsymbol{\delta}^{\geq r}} \in \Lambda^{\alpha^{m-r+1}}.$$

Define $x(\boldsymbol{\delta}, 1) := x_{\delta^{(1)}}^\lambda \in \mathfrak{S}_d$. More generally, define permutations

$$x(\boldsymbol{\delta}, r) := (1_{\mathfrak{S}_{(r-1)l}}, x_{\delta^{(r)}}^{\lambda^{\geq r}}) \in \mathfrak{S}_{(r-1)l} \times \mathfrak{S}_{(m-r+1)l} \leq \mathfrak{S}_d$$

for all $r = 1, \dots, m-1$.

Recall the element $w(\boldsymbol{\delta}) \in \mathfrak{S}_d$ defined in §3.2.4. The following lemma follows from definitions:

Lemma 3.3.1. *We have* $w(\boldsymbol{\delta}) = x(\boldsymbol{\delta}, 1) \cdots x(\boldsymbol{\delta}, m-1)$ *and* $\ell(w(\boldsymbol{\delta})) = \ell(x(\boldsymbol{\delta}, 1)) + \cdots + \ell(x(\boldsymbol{\delta}, m-1))$.

In view of the lemma, when convenient, we will always choose reduced decompositions so that

$$\psi_{w(\boldsymbol{\delta})} = \psi_{x(\boldsymbol{\delta},1)} \dots \psi_{x(\boldsymbol{\delta},m-1)}. \tag{3.2}$$

By Lemma 3.2.15, we then have

$$f_n^{\lambda,\boldsymbol{\delta}} = \chi_{\boldsymbol{\delta}} e_\lambda \psi_{w_0^\lambda} \psi_{x(\boldsymbol{\delta},1)} \dots \psi_{x(\boldsymbol{\delta},m-1)}. \tag{3.3}$$

3.3.2. A commutation lemma

It will be convenient to use the following notation. Let $\lambda \in \Lambda(n)$. Consider the parabolic (non-unital) subalgebra

$$H^\lambda := H_{(m-\lambda_a)\alpha_a,\dots,(m-\lambda_b)\alpha_b,m\alpha_{b+1},\lambda_b\alpha_b,\dots,\lambda_a\alpha_a} \subseteq H_\theta.$$

We have

$$H^\lambda \cong H_{(m-\lambda_a)\alpha_a} \otimes \dots \otimes H_{(m-\lambda_b)\alpha_b} \otimes H_{m\alpha_{b+1}} \otimes H_{\lambda_b\alpha_b} \otimes \dots \otimes H_{\lambda_a\alpha_a}.$$

The natural (unital) embeddings of the algebras

$$H_{(m-\lambda_a)\alpha_a}, \dots, H_{(m-\lambda_b)\alpha_b}, H_{m\alpha_{b+1}}, H_{\lambda_b\alpha_b}, \dots, H_{\lambda_a\alpha_a}$$

into H^λ allow us to consider them as (non-unital) subalgebras of H_θ. We denote the corresponding (non-unital) algebra embeddings by

$$\iota^\lambda_{a;-}, \dots, \iota^\lambda_{b;-}, \iota^\lambda_{b+1}, \iota^\lambda_{b;+}, \dots, \iota^\lambda_{a;+}.$$

For example, setting

$$\psi^{\lambda}_{w_0}(i;-) := \iota^{\lambda}_{i;-}(\psi_{w_0;m-\lambda_i}),$$

$$\psi^{\lambda}_{w_0}(i;+) := \iota^{\lambda}_{i;+}(\psi_{w_0;\lambda_i}),$$

$$\psi^{\lambda}_{w_0}(b+1) := \iota^{\lambda}_{b+1}(\psi_{w_0;m}),$$

for all $i \in [a,b]$, we can write $1_{\boldsymbol{j}^\lambda}\psi_{w_0^\lambda}$ as a commuting product

$$1_{\boldsymbol{j}^\lambda}\psi_{w_0^\lambda} = \psi^{\lambda}_{w_0}(a;-)\dots\psi^{\lambda}_{w_0}(b;-)\psi^{\lambda}_{w_0}(b+1)\psi^{\lambda}_{w_0}(b;+)\dots\psi^{\lambda}_{w_0}(a;+). \quad (3.4)$$

Lemma 3.3.2. *Let $i \in [a,b]$ with $\mu_i > 0$, and $\lambda := \mu - \mathsf{e}_i$. Then*

$$1_{\boldsymbol{j}^\mu}\psi_{\lambda;i}\psi_{w_0^\lambda} = 1_{\boldsymbol{j}^\mu}\psi_{w_0^\mu}\psi_{l_i^+(\mu)\to l_i^-(\lambda)}.$$

Proof. We have

$$\begin{aligned}
1_{\boldsymbol{j}^\mu}\psi_{\lambda;i}\psi_{w_0^\lambda} &= 1_{\boldsymbol{j}^\mu}\psi_{\lambda;i}1_{\boldsymbol{j}^\lambda}\psi_{w_0^\lambda} \\
&= 1_{\boldsymbol{j}^\mu}\psi_{\lambda;i}\psi^{\lambda}_{w_0}(b+1)\prod_{j\in[a,b]}\psi^{\lambda}_{w_0}(j;-)\psi^{\lambda}_{w_0}(j;+) \\
&= 1_{\boldsymbol{j}^\mu}\psi^{\mu}_{w_0}(b+1)\Big[\prod_{j\in[a,b]\setminus\{i\}}\psi^{\mu}_{w_0}(j;-)\psi^{\mu}_{w_0}(j;+)\Big]\times \\
&\qquad \psi_{\lambda;i}\psi^{\lambda}_{w_0}(i;-)\psi^{\lambda}_{w_0}(i;+) \\
&= 1_{\boldsymbol{j}^\mu}\psi^{\mu}_{w_0}(b+1)\Big[\prod_{j\in[a,b]}\psi^{\mu}_{w_0}(j;-)\psi^{\mu}_{w_0}(j;+)\Big]\psi_{l_i^+(\mu)\to l_i^-(\lambda)} \\
&= 1_{\boldsymbol{j}^\mu}\psi_{w_0^\mu}\psi_{l_i^+(\mu)\to l_i^-(\lambda)},
\end{aligned}$$

where we have used (3.4) for the second equality, Lemma 3.1.2(i) for the third equality, and the definition of $\psi_{\lambda,i}$ as an explicit cycle element for the fourth equality. $\square$

3.3.3. Proof that f is a chain map

If $\lambda = \lambda^{\boldsymbol{\delta}}$, let $q_a, \dots, q_{b+1}$ be defined as in (3.1). For $j \in [a, b+1]$, we consider the cycle

$$c_j = \begin{cases} r_j^-(\mu) \to l_j^-(\mu) & \text{if } j \neq b+1 \text{ and } \delta_j^{(1)} = 0, \\ r_j^+(\lambda) \to l_j^+(\lambda) & \text{if } j \neq b+1 \text{ and } \delta_j^{(1)} = 1, \\ r_{b+1} \to l_{b+1}(\lambda) & \text{if } j = b+1. \end{cases}$$

Let c be the commuting product of cycles:

$$c := c_a c_{a+1} \dots c_{b+1}.$$

Lemma 3.3.3. *Suppose $i \in [a, b]$ is such that $\mu_i > 0$ and $\lambda := \mu - \mathsf{e}_i = \lambda^{\boldsymbol{\delta}}$. Set $\bar{\theta} = (m-1)\alpha$, $\bar{\mu} := \mu - \delta^{(1)}$, and $\bar{\lambda} := \lambda - \delta^{(1)}$. Then $1_{\boldsymbol{j}^\mu} \psi_{\lambda;i} e_\lambda \psi_{w_0^\lambda} \psi_{x(\boldsymbol{\delta},1)}$ equals*

$$\begin{cases} 1_{\boldsymbol{j}^\mu} \psi_{w_0^\mu} \psi_{x(\boldsymbol{\delta}+\mathbf{e}_i^1,1)} \psi_{\boldsymbol{\delta};1,i} + 1_{\boldsymbol{j}^\mu} \psi_c \psi_{x^\mu_{\delta^{(1)}}} \iota_{\alpha,\bar{\theta}}(1_\alpha \otimes 1_{\boldsymbol{j}^{\bar{\mu}}} \psi_{\bar{\lambda};i} e_{\bar{\lambda}} \psi_{w_0^{\bar{\lambda}}}) & \text{if } \delta_i^{(1)} = 0, \\ -1_{\boldsymbol{j}^\mu} \psi_c \psi_{x^\mu_{\delta^{(1)}}} \iota_{\alpha,\bar{\theta}}(1_\alpha \otimes 1_{\boldsymbol{j}^{\bar{\mu}}} \psi_{\bar{\lambda};i} e_{\bar{\lambda}} \psi_{w_0^{\bar{\lambda}}}) & \text{if } \delta_i^{(1)} = 1. \end{cases}$$

Proof. By Lemma 3.1.4(ii), we have $e_\lambda \psi_{w_0^\lambda} = 1_{\boldsymbol{j}^\lambda} \psi_{w_0^\lambda}$. So using also Lemma 3.3.2, we get

$$1_{\boldsymbol{j}^\mu} \psi_{\lambda;i} e_\lambda \psi_{w_0^\lambda} \psi_{x(\boldsymbol{\delta},1)} = 1_{\boldsymbol{j}^\mu} \psi_{\lambda;i} 1_{\boldsymbol{j}^\lambda} \psi_{w_0^\lambda} \psi_{x(\boldsymbol{\delta},1)} = 1_{\boldsymbol{j}^\mu} \psi_{w_0^\mu} \psi_{l_i^+(\mu) \to l_i^-(\lambda)} \psi_{x(\boldsymbol{\delta},1)}.$$

As usual, we denote by p_j the position occupied by j in $\delta^{(1)}$ and q_j be defined as in (3.1). By Lemma 3.3.1, the permutation $x(\boldsymbol{\delta}, 1)$ maps p_j to q_j for all $j \in [a, b+1]$, and the elements of $[l+1, d]$ increasingly to the elements of $[1, d] \setminus Q$. We consider two cases.

Case 1: $\delta_i^{(1)} = 0$. In this case we have $q_i = l_i^-(\lambda)$. So the KLR diagram D of $1_{\boldsymbol{j}^\mu}\psi_{l_i^+(\mu)\to l_i^-(\lambda)}\psi_{x(\boldsymbol{\delta},1)}$ has an i-string S from the position p_i in the bottom to the position $l_i^+(\mu)$ in the top, and the only (i,i)-crossings in D will be the crossings of the string S with the $m-\mu_i$ strings which originate in the positions $L_i^-(\mu)$ in the top. The $(i+1)$-string T in D connecting p_{i+1} in the bottom to q_{i+1} in the top is to the right of all (i,i) crossings. Pulling T to the left produces error terms, which arise from opening (i,i)-crossings, but all of them, except the last one, amount to zero when multiplied on the left by $1_{\boldsymbol{j}^\mu}\psi_{w_0^\mu}$. The last error term is equal to $\psi_{x^\mu_{\delta^{(1)}}}\iota_{\alpha,\bar{\theta}}(1_\alpha \otimes \psi_{l_i^+(\bar{\mu})\to l_i^i(\bar{\lambda})})$), and the result of pulling T past all (i,i)-crossings gives $\psi_{x(\boldsymbol{\delta}+\mathbf{e}_i^1,1)}\psi_{\boldsymbol{\delta};1,i}$. Multiplying on the left by $1_{\boldsymbol{j}^\mu}\psi_{w_0^\mu}$, gives

$$1_{\boldsymbol{j}^\mu}\psi_{w_0^\mu}\psi_{x(\boldsymbol{\delta}+\mathbf{e}_i^1,1)}\psi_{\boldsymbol{\delta};1,i} + 1_{\boldsymbol{j}^\mu}\psi_{w_0^\mu}\psi_{x^\mu_{\delta^{(1)}}}\iota_{\alpha,\bar{\theta}}(1_\alpha \otimes \psi_{l_i^+(\bar{\mu})\to l_i^i(\bar{\lambda})}),$$

and it remains to observe using Lemma 3.1.2 that

$$\psi_{w_0^\mu}\psi_{x^\mu_{\delta^{(1)}}}\iota_{\alpha,\bar{\theta}}(1_\alpha \otimes \psi_{l_i^+(\bar{\mu})\to l_i^i(\bar{\lambda})}) = 1_{\boldsymbol{j}^\mu}\psi_c\psi_{x^\mu_{\delta^{(1)}}}\iota_{\alpha,\bar{\theta}}(1_\alpha \otimes 1_{\boldsymbol{j}^{\bar{\mu}}}\psi_{\bar{\lambda};i}e_{\bar{\lambda}}\psi_{w_0^{\bar{\lambda}}}).$$

Case 2: $\delta_i^{(1)} = 1$. Let D be the KLR diagram of $1_{\boldsymbol{j}^\mu}\psi_{l_i^+(\mu)\to l_i^-(\lambda)}\psi_{x(\boldsymbol{\delta},1)}$. Let S be the i-string originating in the position $l_i^+(\mu)$ in the top row of D, and T be the $(i+1)$-string originating in the position p_{i+1} in the bottom row of D. The quadratic relation on these strings produces a difference of two terms, one with a dot on S and the other with a dot on T. The term with a dot on T equals 0 after multiplying on the left by $1_{\boldsymbol{j}^\mu}\psi_{w_0^\mu}$. The term with a dot on S, when multiplied on the left by $1_{\boldsymbol{j}^\mu}\psi_{w_0^\mu}$, yields

$$-1_{\boldsymbol{j}^\mu}\psi_{w_0^\mu}\psi_{x^\mu_{\delta^{(1)}}}\iota_{\alpha,\bar\theta}(1_\alpha\otimes\psi_{l_i^+(\bar\mu)\to l_i^-(\bar\lambda)}) = -1_{\boldsymbol{j}^\mu}\psi_c\psi_{x^\mu_{\delta^{(1)}}}\iota_{\alpha,\bar\theta}(1_\alpha\otimes 1_{\boldsymbol{j}^{\bar\mu}}\psi_{\bar\lambda;i}e_{\bar\lambda}\psi_{w_0^{\bar\lambda}}),$$

where we have used Lemma 3.1.2 for the last equality. $\square$

Corollary 3.3.4. *If $\mu_i > 0$ for some $i \in [a,b]$ and $\lambda := \mu - \mathsf{e}_i = \lambda^{\boldsymbol{\delta}}$, then*

$$1_{\boldsymbol{j}^\mu}\psi_{\lambda;i}e_\lambda\psi_{w_0^\lambda}\psi_{w(\boldsymbol{\delta})} = \sum_{r\in[1,m]:\ \delta_i^{(r)}=0} (-1)^{\sum_{s=1}^{r-1}\delta_i^{(s)}} 1_{\boldsymbol{j}^\mu}\psi_{w_0^\mu}\psi_{w(\boldsymbol{\delta}+\mathsf{e}_i^r)}\psi_{\boldsymbol{\delta};r,i}. \tag{3.5}$$

Proof. The proof is by induction on m, the induction base $m=1$ being obvious. Let $\bar{\boldsymbol{\delta}} := (\delta^{(2)},\dots,\delta^{(m)})$, $\bar\theta = (m-1)\alpha$, $\bar\mu := \mu - \delta^{(1)}$, and $\bar\lambda := \lambda - \delta^{(1)}$. By (3.2), we have

$$1_{\boldsymbol{j}^\mu}\psi_{\lambda;i}e_\lambda\psi_{w_0^\lambda}\psi_{w(\boldsymbol{\delta})} = 1_{\boldsymbol{j}^\mu}\psi_{\lambda;i}e_\lambda\psi_{w_0^\lambda}\psi_{x(\boldsymbol{\delta},1)}\psi_{x(\boldsymbol{\delta},2)}\dots\psi_{x(\boldsymbol{\delta},m-1)}. \tag{3.6}$$

Now we apply Lemma 3.3.3. We consider the case $\delta_i^{(1)} = 0$, the case $\delta_i^{(1)} = 1$ being similar. Then we get the following expression for the right hand side of (3.6):

$$\big(1_{\boldsymbol{j}^\mu}\psi_{w_0^\mu}\psi_{x(\boldsymbol{\delta}+\mathbf{e}_i^1,1)}\psi_{\boldsymbol{\delta};1,i}+1_{\boldsymbol{j}^\mu}\psi_c\psi_{x^\mu_{\delta(1)}}\iota_{\alpha,\bar\theta}(1_\alpha\otimes 1_{\boldsymbol{j}^{\bar\mu}}\psi_{\bar\lambda;i}e_{\bar\lambda}\psi_{w_0^{\bar\lambda}})\big)\times$$
$$\big(\psi_{x(\boldsymbol{\delta},2)}\dots\psi_{x(\boldsymbol{\delta},m-1)}\big).$$

Opening parentheses, we get two summands S_1+S_2. Note that

$$\begin{aligned}S_1&=1_{\boldsymbol{j}^\mu}\psi_{w_0^\mu}\psi_{x(\boldsymbol{\delta}+\mathbf{e}_i^1,1)}\psi_{\boldsymbol{\delta};1,i}\psi_{x(\boldsymbol{\delta},2)}\dots\psi_{x(\boldsymbol{\delta},m-1)}\\&=1_{\boldsymbol{j}^\mu}\psi_{w_0^\mu}\psi_{x(\boldsymbol{\delta}+\mathbf{e}_i^1,1)}\psi_{x(\boldsymbol{\delta},2)}\dots\psi_{x(\boldsymbol{\delta},m-1)}\psi_{\boldsymbol{\delta};1,i}\\&=1_{\boldsymbol{j}^\mu}\psi_{w_0^\mu}\psi_{w(\boldsymbol{\delta}+\mathbf{e}_i^1)}\psi_{\boldsymbol{\delta};1,i}.\end{aligned}$$

Moreover, using the inductive assumption for the third equality below, we see that S_2 equals

$$\begin{aligned}&1_{\boldsymbol{j}^\mu}\psi_c\psi_{x^\mu_{\delta(1)}}\iota_{\alpha,\bar\theta}(1_\alpha\otimes 1_{\boldsymbol{j}^{\bar\mu}}\psi_{\bar\lambda;i}e_{\bar\lambda}\psi_{w_0^{\bar\lambda}})\psi_{x(\boldsymbol{\delta},2)}\dots\psi_{x(\boldsymbol{\delta},m-1)}\\=&1_{\boldsymbol{j}^\mu}\psi_c\psi_{x^\mu_{\delta(1)}}\iota_{\alpha,\bar\theta}\big(1_\alpha\otimes 1_{\boldsymbol{j}^{\bar\mu}}\psi_{\bar\lambda;i}e_{\bar\lambda}\psi_{w_0^{\bar\lambda}}\psi_{x(\bar{\boldsymbol{\delta}},1)}\dots\psi_{x(\bar{\boldsymbol{\delta}},m-2)}\big)\\=&1_{\boldsymbol{j}^\mu}\psi_c\psi_{x^\mu_{\delta(1)}}\iota_{\alpha,\bar\theta}\big(1_\alpha\otimes 1_{\boldsymbol{j}^{\bar\mu}}\psi_{\bar\lambda;i}e_{\bar\lambda}\psi_{w_0^{\bar\lambda}}\psi_{w(\bar{\boldsymbol{\delta}})}\big)\\=&1_{\boldsymbol{j}^\mu}\psi_c\psi_{x^\mu_{\delta(1)}}\iota_{\alpha,\bar\theta}\Big(1_\alpha\otimes\sum_{r\in[2,m]:\ \delta_i^{(r)}=0}(-1)^{\sum_{s=2}^{r-1}\delta_i^{(s)}}1_{\boldsymbol{j}^{\bar\mu}}\psi_{w_0^{\bar\mu}}\psi_{w(\bar{\boldsymbol{\delta}}+\mathbf{e}_i^{r-1})}\psi_{\bar{\boldsymbol{\delta}};r-1,i}\Big).\end{aligned}$$

As $\delta_i^{(1)}=0$, we have $\sum_{s=2}^{r-1}\delta_i^{(s)}=\sum_{s=1}^{r-1}\delta_i^{(s)}$. So

$$
\begin{aligned}
&\iota_{\alpha,\bar{\theta}}\big(1_\alpha\otimes\sum_{r\in[2,m]:\ \delta_i^{(r)}=0}(-1)^{\sum_{s=2}^{r-1}\delta_i^{(s)}}1_{\boldsymbol{j}^{\bar{\mu}}}\psi_{w_0^{\bar{\mu}}}\psi_{w(\bar{\boldsymbol{\delta}}+\mathbf{e}_i^{r-1})}\psi_{\bar{\boldsymbol{\delta}};r-1,i}\big)\\
&=\sum_{r\in[2,m]:\ \delta_i^{(r)}=0}(-1)^{\sum_{s=1}^{r-1}\delta_i^{(s)}}\iota_{\alpha,\bar{\theta}}\big(1_\alpha\otimes 1_{\boldsymbol{j}^{\bar{\mu}}}\psi_{w_0^{\bar{\mu}}}\Big(\prod_{t=1}^{m-2}\psi_{x(\bar{\boldsymbol{\delta}}+\mathbf{e}_i^{r-1},t)}\Big)\psi_{\bar{\boldsymbol{\delta}};r-1,i}\big)\\
&=\sum_{r\in[2,m]:\ \delta_i^{(r)}=0}(-1)^{\sum_{s=1}^{r-1}\delta_i^{(s)}}\iota_{\alpha,\bar{\theta}}\big(1_\alpha\otimes 1_{\boldsymbol{j}^{\bar{\mu}}}\psi_{w_0^{\bar{\mu}}}\big)\Big(\prod_{t=2}^{m-1}\psi_{x(\boldsymbol{\delta}+\mathbf{e}_i^{r},t)}\Big)\psi_{\boldsymbol{\delta};r,i}.
\end{aligned}
$$

Moreover,

$$
\psi_{x^{\mu}_{\delta^{(1)}}}\prod_{t=2}^{m-1}\psi_{x(\boldsymbol{\delta}+\mathbf{e}_i^{r},t)}=\psi_{x(\boldsymbol{\delta}+\mathbf{e}_i^{r},1)}\prod_{t=2}^{m-1}\psi_{x(\boldsymbol{\delta}+\mathbf{e}_i^{r},t)}=\psi_{w(\boldsymbol{\delta}+\mathbf{e}_i^{r})}.
$$

So S_2 equals

$$
\begin{aligned}
&\sum_{\substack{r\in[2,m]:\\ \delta_i^{(r)}=0}}(-1)^{\sum_{s=1}^{r-1}\delta_i^{(s)}}1_{\boldsymbol{j}^{\mu}}\psi_c\psi_{x^{\mu}_{\delta^{(1)}}}\iota_{\alpha,\bar{\theta}}(1_\alpha\otimes 1_{\boldsymbol{j}^{\bar{\mu}}}\psi_{w_0^{\bar{\mu}}})\Big(\prod_{t=2}^{m-1}\psi_{x(\boldsymbol{\delta}+\mathbf{e}_i^{r},t)}\Big)\psi_{\boldsymbol{\delta};r,i}\\
&=\sum_{r\in[2,m]:\ \delta_i^{(r)}=0}(-1)^{\sum_{s=1}^{r-1}\delta_i^{(s)}}1_{\boldsymbol{j}^{\mu}}\psi_{w_0^{\mu}}\psi_{x^{\mu}_{\delta^{(1)}}}\Big(\prod_{t=2}^{m-1}\psi_{x(\boldsymbol{\delta}+\mathbf{e}_i^{r},t)}\Big)\psi_{\boldsymbol{\delta};r,i}\\
&=\sum_{r\in[2,m]:\ \delta_i^{(r)}=0}(-1)^{\sum_{s=1}^{r-1}\delta_i^{(s)}}1_{\boldsymbol{j}^{\mu}}\psi_{w_0^{\mu}}\psi_{w(\boldsymbol{\delta}+\mathbf{e}_i^{r})}\psi_{\boldsymbol{\delta};r,i},
\end{aligned}
$$

where we have used Lemma 3.1.2(i) to see the first equality. Thus S_1+S_2 equals the right hand side of (3.5). $\square$

The following statement means that f is chain map:

Proposition 3.3.5. *Let $\mu \in \Lambda(n+1)$ and $\boldsymbol{\delta} \in \boldsymbol{\Lambda}(n)$. Then*

$$\sum_{\lambda \in \Lambda(n)} d_n^{\mu,\lambda} f_n^{\lambda,\boldsymbol{\delta}} = \sum_{\boldsymbol{\gamma} \in \boldsymbol{\Lambda}(n+1)} f_{n+1}^{\mu,\boldsymbol{\gamma}} c_n^{\boldsymbol{\gamma},\boldsymbol{\delta}}.$$

Proof. By definition, $d_n^{\mu,\lambda} = 0$ unless $\lambda = \mu - \mathtt{e}_i$ for some $i \in [a,b]$, and $f_n^{\lambda,\boldsymbol{\delta}} = 0$ unless $\lambda = \lambda^{\boldsymbol{\delta}}$. On the other hand, $f_n^{\mu,\boldsymbol{\gamma}} = 0$ unless $\mu = \lambda^{\boldsymbol{\gamma}}$, and $c_n^{\boldsymbol{\gamma},\boldsymbol{\delta}} = 0$, unless $\boldsymbol{\delta} = \boldsymbol{\gamma} - \mathbf{e}_i^r$ for some $i \in [a,b]$ and $r \in [1,m]$. So we may assume that $\mu = \lambda^{\boldsymbol{\delta}+\mathbf{e}_i^r}$ for some $i \in [a,b]$ and $r \in [1,m]$ such that $\delta_i^{(r)} = 0$. In this case, letting $\lambda := \lambda^{\boldsymbol{\delta}}$, we have to prove

$$d_n^{\mu,\lambda} f_n^{\lambda,\boldsymbol{\delta}} = \sum_{r \in [1,m]:\ \delta_i^{(r)}=0} f_{n+1}^{\mu,\boldsymbol{\delta}+\mathbf{e}_i^r} c_n^{\boldsymbol{\delta}+\mathbf{e}_i^r,\boldsymbol{\delta}}.$$

By definition of the elements involved, this means

$$\begin{aligned}
&(\mathtt{sgn}_{\lambda;i} e_\mu \psi_{\lambda;i} e_\lambda)(\chi_{\boldsymbol{\delta}} e_\lambda \psi_{w_0^\lambda} \psi_{w(\boldsymbol{\delta})} e_{\boldsymbol{\delta}}) \\
= &\sum_{r \in [1,m]:\ \delta_i^{(r)}=0} (\chi_{\boldsymbol{\delta}+\mathbf{e}_i^r} e_\mu \psi_{w_0^\mu} \psi_{w(\boldsymbol{\delta}+\mathbf{e}_i^r)} e_{\boldsymbol{\delta}+\mathbf{e}_i^r})(\mathtt{sgn}_{\boldsymbol{\delta};r,i} e_{\boldsymbol{\delta}+\mathbf{e}_i^r} \psi_{\boldsymbol{\delta};r,i} e_{\boldsymbol{\delta}}).
\end{aligned}$$

Equivalently, we need to prove

$$\mathtt{sgn}_{\lambda;i} \chi_{\boldsymbol{\delta}} e_\mu \psi_{\lambda;i} e_\lambda \psi_{w_0^\lambda} \psi_{w(\boldsymbol{\delta})} = \sum_{r \in [1,m]:\ \delta_i^{(r)}=0} \chi_{\boldsymbol{\delta}+\mathbf{e}_i^r} \mathtt{sgn}_{\boldsymbol{\delta};r,i} e_\mu \psi_{w_0^\mu} \psi_{w(\boldsymbol{\delta}+\mathbf{e}_i^r)} \psi_{\boldsymbol{\delta};r,i},$$

which, in view of Corollary 3.3.4, is equivalent to the statement that

$$\mathtt{sgn}_{\lambda;i} \chi_{\boldsymbol{\delta}} = \chi_{\boldsymbol{\delta}+\mathbf{e}_i^r} \mathtt{sgn}_{\boldsymbol{\delta};r,i} (-1)^{\sum_{s=1}^{r-1} \delta_i^{(s)}}$$

for all $r \in [1,m]$ such that $\delta_i^{(r)} = 0$. But this is Lemma 3.2.1. $\square$

3.4. Verification That g Is a Chain Map

We continue with the running assumptions of Section 3.2. In addition, throughout the section we fix $n \in \mathbb{Z}_{\geq 0}$, $\lambda \in \Lambda(n)$ and $\boldsymbol{\gamma} = (\gamma^{(1)}, \dots, \gamma^{(m)}) \in \boldsymbol{\Lambda}(n+1)$.

3.4.1. Special reduced expressions and a commutation lemma

Recall the notation of §3.2.1. Let $\mu \in \Lambda^{\alpha^m}$ and $\gamma \in \Lambda^{\alpha}$ be such that $\bar{\mu} := \mu - \gamma \in \Lambda^{\alpha^{m-1}}$. For $i \in [a, b+1]$, we denote

$$p^i := (m-1)l + (\text{the position occupied by } i \text{ in } \boldsymbol{j}^{\delta}), \tag{4.1}$$

$$q^i := \begin{cases} r_i^-(\mu) & \text{if } i \neq b+1 \text{ and } \gamma_i^{(m)} = 0, \\ r_i^+(\mu) & \text{if } i \neq b+1 \text{ and } \gamma_i^{(m)} = 1, \\ r_{b+1}(\mu) & \text{if } i = b+1. \end{cases} \tag{4.2}$$

Let $Q := \{q^a, \dots, q^{b+1}\}$. Note that $\{p^a, \dots, p^{b+1}\} = (d-l, d]$.

Define $z_\mu^\gamma \in \mathfrak{S}_d$ to be the permutation which maps q^i to p^i for all $i \in [a, b+1]$, and maps the elements of $[1, d] \setminus Q$ increasingly to the elements of $[1, d-l]$. It is easy to see that z_μ^γ is fully commutative, so $\psi_{z_\mu^\gamma}$ is well defined, and $\psi_{z_\mu^\gamma} 1_{\boldsymbol{j}^\mu} = 1_{\boldsymbol{j}^{\bar{\mu}} \boldsymbol{j}^\gamma} \psi_{z_\mu^\gamma}$.

Now let $\mu := \lambda^{\boldsymbol{\gamma}}$, and $r \in [1, m]$. Define

$$\boldsymbol{\gamma}^{\leq r} := (\gamma^{(1)}, \dots, \gamma^{(r)}) \in \boldsymbol{\Lambda}^{\alpha^r}, \quad \mu^{\leq r} := \lambda^{\boldsymbol{\gamma}^{\leq r}} \in \Lambda^{\alpha^r}.$$

Define $z(\boldsymbol{\gamma}, m) := z_\mu \gamma^{(m)} \in \mathfrak{S}_d$. More generally, define for all $r = 2, \dots, m$, the permutations

$$z(\boldsymbol{\gamma}, r) := (x_{\mu^{\leq r}}^{\gamma^{(r)}}, 1_{\mathfrak{S}_{(m-r)l}}) \in \mathfrak{S}_{rl} \times \mathfrak{S}_{(m-r)l} \times \leq \mathfrak{S}_d.$$

Recall the element $u(\boldsymbol{\gamma}) \in \mathfrak{S}_d$ defined in §3.2.4. The following lemma follows from definitions:

Lemma 3.4.1. *We have* $u(\boldsymbol{\gamma}) = z(\boldsymbol{\gamma}, 2) \cdots z(\boldsymbol{\gamma}, m)$ *and* $\ell(u(\boldsymbol{\gamma})) = \ell(z(\boldsymbol{\gamma}, 1)) + \cdots + \ell(z(\boldsymbol{\gamma}, m))$.

In view of the lemma, when convenient, we will always choose reduced decompositions so that

$$\psi_{u(\boldsymbol{\gamma})} = \psi_{z(\boldsymbol{\gamma},2)} \dots \psi_{z(\boldsymbol{\gamma},m)}. \tag{4.3}$$

In view of Lemma 3.2.15, we have

$$g_{n+1}^{\boldsymbol{\gamma},\mu} = \xi_{\boldsymbol{\gamma}} \psi_{z(\boldsymbol{\gamma},2)} \dots \psi_{z(\boldsymbol{\gamma},m)} y^\mu e_\mu. \tag{4.4}$$

Lemma 3.4.2. *Let* $i \in [a, b]$ *with* $\lambda_i < m$, *and* $\mu := \lambda + \mathsf{e}_i$. *Then*

$$y^\mu e_\mu \psi_{\lambda;i} e_\lambda = \psi_{\lambda;i} y^\lambda e_\lambda.$$

Proof. Recalling the notation of §3.3.2, we have that $y^\mu e_\mu \psi_{\lambda;i} e_\lambda$ equals

$$
\begin{aligned}
&\Big[\prod_{j\in[a,b]} \iota^\mu_{j,-}(1_{j^{(m-\mu_j)}})\iota^\mu_{j,+}(1_{j^{(\mu_j)}})\Big]\iota^\mu_{b+1}(y_{0,m}1_{(b+1)^{(m)}})\psi_{\lambda;i}e_\lambda\\
=&\iota^\mu_{i,-}(1_{i^{(m-\mu_i)}})\iota^\mu_{i,+}(1_{i^{(\mu_i)}})\Big[\prod_{j\in[a,b]\setminus\{i\}} \iota^\mu_{j,-}(1_{j^{(m-\mu_j)}})\iota^\mu_{j,+}(1_{j^{(\mu_j)}})\Big]\\
&\times \iota^\mu_{b+1}(y_{0,m}1_{(b+1)^{(m)}})\psi_{\lambda;i}e_\lambda\\
=&\iota^\mu_{i,-}(1_{i^{(m-\mu_i)}})\iota^\mu_{i,+}(1_{i^{(\mu_i)}})\psi_{\lambda;i}\Big[\prod_{j\in[a,b]\setminus\{i\}} \iota^\lambda_{j,-}(1_{j^{(m-\lambda_j)}})\iota^\lambda_{j,+}(1_{j^{(\lambda_j)}})\Big]\\
&\times \iota^\lambda_{b+1}(y_{0,m})\iota^\lambda_{b+1}(1_{(b+1)^{(m)}})e_\lambda\\
=&\iota^\mu_{i,-}(1_{i^{(m-\mu_i)}})\iota^\mu_{j,+}(1_{j^{(\mu_j)}})\psi_{\lambda;i}y^\lambda e_\lambda = \psi_{\lambda;i}y^\lambda e_\lambda,
\end{aligned}
$$

where we have used Lemma 3.1.2 for the second equality and Lemma 3.1.7 for the last equality. □

3.4.2. Proof that g is a chain map

Recall that we have fixed $\lambda \in \Lambda(n)$ and $\boldsymbol{\gamma} \in \boldsymbol{\Lambda}(n+1)$.

Lemma 3.4.3. *Suppose that $i \in [a,b]$ is such that $\lambda_i < m$ and $\mu := \lambda + \mathsf{e}_i = \lambda^{\boldsymbol{\gamma}}$. Set $\bar\theta = (m-1)\alpha$, $\bar\mu := \mu - \gamma^{(m)}$, and $\bar\lambda := \lambda - \gamma^{(m)}$. Then $\psi_{z(\boldsymbol{\gamma},m)} y^\mu e_\mu \psi_{\lambda;i} e_\lambda$ equals*

$$
\begin{cases}
\begin{aligned}&\psi_{\boldsymbol{\gamma}-\mathbf{e}_i^m;m,i}\psi_{z(\boldsymbol{\gamma}-\mathbf{e}_i^m,m)}y^\lambda e_\lambda\\ &-\iota_{\bar\theta,\alpha}(y^{\bar\mu}e_{\bar\mu}\psi_{\bar\lambda;i}e_{\bar\lambda}\otimes 1_\alpha)\psi_{z_\lambda^{\gamma^{(m)}}}y^{m-1}_{r_{b+1}(\lambda)}e_\lambda\end{aligned} & \text{if } \gamma_i^{(m)} = 1,\\
\iota_{\bar\theta,\alpha}(y^{\bar\mu}e_{\bar\mu}\psi_{\bar\lambda;i}e_{\bar\lambda}\otimes 1_\alpha)\psi_{z_\lambda^{\gamma^{(m)}}}y^{m-1}_{r_{b+1}(\lambda)}e_\lambda & \text{if } \gamma_i^{(m)} = 0.
\end{cases}
$$

Proof. Using Lemma 3.4.2, we get

$$\psi_{z(\boldsymbol{\gamma},m)}y^{\mu}e_{\mu}\psi_{\lambda;i}e_{\lambda} = \psi_{z(\boldsymbol{\gamma},m)}\psi_{\lambda;i}y^{\lambda}e_{\lambda}.$$

Recalling (4.1) and (4.2), the permutation $z(\boldsymbol{\gamma}, m)$ maps q^j to p^j for all $j \in [a, b+1]$, and the elements of $[1, d] \setminus Q$ increasingly to the elements of $[1, d-l]$. We consider two cases.

Case 1: $\gamma_i^{(m)} = 1$. In this case we have $q^i = r_i^+(\lambda)$. So the KLR diagram D of $\psi_{z(\boldsymbol{\gamma},m)}\psi_{\lambda;i}1_{\boldsymbol{j}^{\lambda}}$ has an i-string S from the position $r_i^-(\lambda)$ in the bottom to the position p^i in the top, and the only (i,i)-crossings in D will be the crossings of the string S with the λ_i strings which originate in the positions $H_i^+(\lambda)$ in the bottom. The $(i+1)$-string T in D originating in the position p_{i+1} in the top is to the left of all (i,i) crossings. Pulling T to the right produces error terms, which arise from opening (i,i)-crossings, but all of them, except the last one, amount to zero, when multiplied on the right by $y^{\lambda}e_{\lambda}$. The last error term is equal to $-\iota_{\bar{\theta},\alpha}(\psi_{\bar{\lambda};i} \otimes 1_{\alpha})\psi_{z_{\lambda}^{\gamma^{(m)}}}$, and the result of pulling T past all (i,i)-crossings gives $\psi_{\boldsymbol{\gamma}-\mathbf{e}_i^m;m,i}\psi_{z(\boldsymbol{\gamma}-\mathbf{e}_i^m,m)}$. Multiplying on the left by $y^{\lambda}e_{\lambda}$ gives

$$-\iota_{\bar{\theta},\alpha}(\psi_{\bar{\lambda};i} \otimes 1_{\alpha})\psi_{z_{\lambda}^{\gamma^{(m)}}}y^{\lambda}e_{\lambda} + \psi_{\boldsymbol{\gamma}-\mathbf{e}_i^m;m,i}\psi_{z(\boldsymbol{\gamma}-\mathbf{e}_i^m,m)}y^{\lambda}e_{\lambda}.$$

It remains to observe, using Lemma 3.1.2, that

$$-\iota_{\bar{\theta},\alpha}(\psi_{\bar{\lambda};i} \otimes 1_{\alpha})\psi_{z_{\lambda}^{\gamma^{(m)}}}y^{\lambda}e_{\lambda} = -\iota_{\bar{\theta},\alpha}(y^{\bar{\mu}}e_{\bar{\mu}}\psi_{\bar{\lambda};i}e_{\bar{\lambda}} \otimes 1_{\alpha})\psi_{z_{\lambda}^{\gamma^{(m)}}}y_{r_{b+1}(\lambda)}^{m-1}e_{\lambda}.$$

Case 2: $\gamma_i^{(m)} = 0$. Let D be the KLR diagram of $\psi_{z(\boldsymbol{\gamma},m)}\psi_{\lambda;i}1_{\boldsymbol{j}^\lambda}$. Let S be the i-string originating in the position $r_i^-(\lambda)$ in the bottom row of D, and T be the $(i+1)$-string originating in the position p^{i+1} in the top row of D. The quadratic relation on these strings produces a linear combination of two diagrams, one with a dot on S and the other with a dot on T. The term containing a dot on T produces a term which is zero after multiplying on the right by $y^\lambda e^\lambda$. The term containing a dot on S, when multiplied on the right by $y^\lambda e^\lambda$, yields

$$\iota_{\bar{\theta},\alpha}(\psi_{\bar{\lambda};i}\otimes 1_\alpha)\psi_{z_\lambda^{\gamma^{(m)}}}y^\lambda e_\lambda = \iota_{\bar{\theta},\alpha}(y^{\bar{\mu}}e_{\bar{\mu}}\psi_{\bar{\lambda};i}e_{\bar{\lambda}}\otimes 1_\alpha)\psi_{z_\lambda^{\gamma^{(m)}}}y_{r_{b+1}(\lambda)}^{m-1}e_\lambda,$$

where we have used Lemmas 3.1.2 and 3.1.7 to deduce the last equality.

□

Corollary 3.4.4. *Suppose that $i \in [a,b]$ is such that $\lambda_i < m$ and $\mu := \lambda + \mathsf{e}_i = \lambda^{\boldsymbol{\gamma}}$. Then*

$$\psi_{u(\boldsymbol{\gamma})}y^\mu e_\mu \psi_{\lambda;i}e_\lambda = \sum_{r\in[1,m]:\ \gamma_i^{(r)}=1} (-1)^{\sum_{s=r+1}^m \gamma_i^{(s)}}\psi_{\boldsymbol{\gamma}-\mathbf{e}_i^r;r,i}\psi_{u(\boldsymbol{\gamma}-\mathbf{e}_i^r)}y^\lambda e_\lambda.$$

Proof. The proof is by induction on m, the induction base $m=1$ being obvious. Let $\bar{\gamma} := (\gamma^{(1)},\dots,\gamma^{(m-1)})$, $\bar{\theta} = (m-1)\alpha$, $\bar{\mu} := \mu - \gamma^{(m)}$, and $\bar{\lambda} := \lambda - \gamma^{(m)}$. By (4.3), we have

$$\psi_{u(\boldsymbol{\gamma})}y^\mu e_\mu \psi_{\lambda;i}e_\lambda = \psi_{z(\boldsymbol{\gamma},2)}\dots\psi_{z(\boldsymbol{\gamma},m)}y^\mu e_\mu\psi_{\lambda;i}e_\lambda. \tag{4.5}$$

Now we apply Lemma 3.4.3. We consider the case $\gamma_i^{(m)} = 1$, the case $\gamma_i^{(m)} = 0$ being similar. Then we get the following expression for the right hand side of (4.5):

$$\psi_{z(\boldsymbol{\gamma},2)} \dots \psi_{z(\boldsymbol{\gamma},m-1)} \big(\psi_{\boldsymbol{\gamma}-\mathbf{e}_i^m;m,i} \psi_{z(\boldsymbol{\gamma}-\mathbf{e}_i^m,m)} y^{\lambda} e_{\lambda}$$
$$-\iota_{\bar{\theta},\alpha}(y^{\bar{\mu}} e_{\bar{\mu}} \psi_{\bar{\lambda};i} e_{\bar{\lambda}} \otimes 1_{\alpha}) \psi_{z_{\lambda}^{\gamma(m)}} y_{r_{b+1}(\lambda)}^{m-1} e_{\lambda}\big).$$

Opening parentheses, we get two summands $S_1 + S_2$. Note that

$$\begin{aligned} S_1 &= \psi_{z(\boldsymbol{\gamma},2)} \dots \psi_{z(\boldsymbol{\gamma},m-1)} \psi_{\boldsymbol{\gamma}-\mathbf{e}_i^m;m,i} \psi_{z(\boldsymbol{\gamma}-\mathbf{e}_i^m,m)} y^{\lambda} e_{\lambda} \\ &= \psi_{\boldsymbol{\gamma}-\mathbf{e}_i^m;m,i} \psi_{z(\boldsymbol{\gamma},2)} \dots \psi_{z(\boldsymbol{\gamma},m-1)} \psi_{z(\boldsymbol{\gamma}-\mathbf{e}_i^m,m)} y^{\lambda} e_{\lambda} \\ &= \psi_{\boldsymbol{\gamma}-\mathbf{e}_i^m;m,i} \psi_{u(\boldsymbol{\gamma}-\mathbf{e}_i^m)} y^{\lambda} e_{\lambda}. \end{aligned}$$

Moreover, using the inductive assumption, for the third equality below, S_2 equals

$$
\begin{aligned}
&-\psi_{z(\boldsymbol{\gamma},2)}\dots\psi_{z(\boldsymbol{\gamma},m-1)}\iota_{\bar{\theta},\alpha}(y^{\bar{\mu}}e_{\bar{\mu}}\psi_{\bar{\lambda};i}e_{\bar{\lambda}}\otimes 1_{\alpha})\psi_{z_{\lambda}^{\gamma^{(m)}}}y^{m-1}_{r_{b+1}(\lambda)}e_{\lambda}\\
&=-\iota_{\bar{\theta},\alpha}(\psi_{z(\bar{\boldsymbol{\gamma}},2)}\dots\psi_{z(\bar{\boldsymbol{\gamma}},m-1)}y^{\bar{\mu}}e_{\bar{\mu}}\psi_{\bar{\lambda};i}e_{\bar{\lambda}}\otimes 1_{\alpha})\psi_{z_{\lambda}^{\gamma^{(m)}}}y^{m-1}_{r_{b+1}(\lambda)}e_{\lambda}\\
&=-\iota_{\bar{\theta},\alpha}(\psi_{u(\bar{\boldsymbol{\gamma}})}y^{\bar{\mu}}e_{\bar{\mu}}\psi_{\bar{\lambda};i}e_{\bar{\lambda}}\otimes 1_{\alpha})\psi_{z_{\lambda}^{\gamma^{(m)}}}y^{m-1}_{r_{b+1}(\lambda)}e_{\lambda}\\
&=-\iota_{\bar{\theta},\alpha}\Big[\sum_{\substack{r\in[1,m-1]:\\ \gamma_i^{(r)}=1}}(-1)^{\sum_{s=r+1}^{m-1}\gamma_i^{(s)}}\psi_{\bar{\boldsymbol{\gamma}}-\mathbf{e}_i^r;r,i}\psi_{u(\bar{\boldsymbol{\gamma}}-\mathbf{e}_i^r)}y^{\bar{\lambda}}e_{\bar{\lambda}}\otimes 1_{\alpha}\Big]\psi_{z_{\lambda}^{\gamma^{(m)}}}y^{m-1}_{r_{b+1}(\lambda)}e_{\lambda}\\
&=\sum_{r\in[1,m-1]:\ \gamma_i^{(r)}=1}(-1)^{\sum_{s=r+1}^{m}\gamma_i^{(s)}}\psi_{\boldsymbol{\gamma}-\mathbf{e}_i^r;r,i}\psi_{z(\boldsymbol{\gamma}-\mathbf{e}_i^r,2)}\dots\psi_{z(\boldsymbol{\gamma}-\mathbf{e}_i^r,m-1)}\\
&\quad\times\iota_{\bar{\theta},\alpha}\Big[y^{\bar{\lambda}}e_{\bar{\lambda}}\otimes 1_{\alpha}\Big]\psi_{z(\boldsymbol{\gamma}-\mathbf{e}_i^r,m)}y^{m-1}_{r_{b+1}(\lambda)}e_{\lambda}\\
&=\sum_{r\in[1,m-1]:\ \gamma_i^{(r)}=1}(-1)^{\sum_{s=r+1}^{m}\gamma_i^{(s)}}\psi_{\boldsymbol{\gamma}-\mathbf{e}_i^r;r,i}\psi_{z(\boldsymbol{\gamma}-\mathbf{e}_i^r,2)}\dots\psi_{z(\boldsymbol{\gamma}-\mathbf{e}_i^r,m)}\\
&\quad\times\iota_{\bar{\theta},\alpha}\Big[y^{\bar{\lambda}}e_{\bar{\lambda}}\otimes 1_{\alpha}\Big]y^{m-1}_{r_{b+1}(\lambda)}e_{\lambda}\\
&=\sum_{r\in[1,m-1]:\ \gamma_i^{(r)}=1}(-1)^{\sum_{s=r+1}^{m}\gamma_i^{(s)}}\psi_{\boldsymbol{\gamma}-\mathbf{e}_i^r;r,i}\psi_{u(\boldsymbol{\gamma}-\mathbf{e}_i^r)}y^{\lambda}e_{\lambda}.
\end{aligned}
$$

Thus S_1+S_2 equals the right hand side of the expression in the corollary.

□

The following statement means that g is chain map:

Proposition 3.4.5. *Let $\lambda\in\Lambda(n)$ and $\boldsymbol{\gamma}\in\boldsymbol{\Lambda}(n+1)$. Then*

$$\sum_{\mu\in\Lambda(n+1)}g_{n+1}^{\boldsymbol{\gamma},\mu}d_n^{\mu,\lambda}=\sum_{\boldsymbol{\delta}\in\boldsymbol{\Lambda}(n)}c_n^{\boldsymbol{\gamma},\boldsymbol{\delta}}g_n^{\boldsymbol{\delta},\lambda}.$$

Proof. By definition, $d_n^{\mu,\lambda} = 0$ unless $\mu = \lambda + \mathtt{e}_i$ for some $i \in [a,b]$, and $g_{n+1}^{\boldsymbol{\gamma},\mu} = 0$ unless $\mu = \lambda^{\boldsymbol{\gamma}}$. On the other hand, $g_n^{\boldsymbol{\delta},\lambda} = 0$ unless $\lambda = \lambda^{\boldsymbol{\delta}}$, and $c_n^{\boldsymbol{\gamma},\boldsymbol{\delta}} = 0$, unless $\boldsymbol{\delta} = \boldsymbol{\gamma} - \mathbf{e}_i^r$ for some $i \in [a,b]$ and $r \in [1,m]$. So we may assume that there exists $i \in [a,b]$ such that $\lambda^{\boldsymbol{\gamma}} = \lambda + \mathtt{e}_i$, in which case, setting $\mu := \lambda^{\boldsymbol{\gamma}}$, we have to prove

$$g_{n+1}^{\boldsymbol{\gamma},\mu} d_n^{\mu,\lambda} = \sum_{r\in[1,m]:\ \gamma_i^{(r)}=1} c_n^{\boldsymbol{\gamma},\boldsymbol{\gamma}-\mathbf{e}_i^r} g_n^{\boldsymbol{\gamma}-\mathbf{e}_i^r,\lambda}.$$

By definition of the elements involved, this means

$$\begin{aligned}&(\xi_{\boldsymbol{\gamma}} e_{\boldsymbol{\gamma}} \psi_{u(\boldsymbol{\gamma})} y^{\mu} e_\mu)(\mathtt{sgn}_{\lambda;i} e_\mu \psi_{\lambda;i} e_\lambda)\\ &= \sum_{r\in[1,m]:\ \gamma_i^{(r)}=1} (\mathtt{sgn}_{\boldsymbol{\gamma}-\mathbf{e}_i^r;r,i} e_{\boldsymbol{\gamma}} \psi_{\boldsymbol{\gamma}-\mathbf{e}_i^r;r,i} e_{\boldsymbol{\gamma}-\mathbf{e}_i^r})(\xi_{\boldsymbol{\gamma}-\mathbf{e}_i^r} e_{\boldsymbol{\gamma}-\mathbf{e}_i^r} \psi_{u(\boldsymbol{\gamma}-\mathbf{e}_i^r)} y^{\lambda} e_\lambda).\end{aligned}$$

Equivalently, we need to prove

$$\xi_{\boldsymbol{\gamma}} \mathtt{sgn}_{\lambda;i} \psi_{u(\boldsymbol{\gamma})} y^{\mu} e_\mu \psi_{\lambda;i} e_\lambda = \sum_{\substack{r\in[1,m]:\\ \gamma_i^{(r)}=1}} \mathtt{sgn}_{\boldsymbol{\gamma}-\mathbf{e}_i^r;r,i} \xi_{\boldsymbol{\gamma}-\mathbf{e}_i^r} \psi_{\boldsymbol{\gamma}-\mathbf{e}_i^r;r,i} \psi_{u(\boldsymbol{\gamma}-\mathbf{e}_i^r)} y^{\lambda} e_\lambda,$$

which, in view of Corollary 3.4.4, is equivalent to the statement that

$$\xi_{\boldsymbol{\gamma}} \mathtt{sgn}_{\lambda;i} = \mathtt{sgn}_{\boldsymbol{\gamma}-\mathbf{e}_i^r;r,i} \xi_{\boldsymbol{\gamma}-\mathbf{e}_i^r} (-1)^{\sum_{s=r+1}^m \gamma_i^{(s)}}$$

for all $r \in [1,m]$ such that $\gamma_i^{(r)} = 1$. But this is Lemma 3.2.2. $\square$

Chapter IV

EXTENSION ALGEBRAS

Throughout this chapter we will assume that $\alpha = \alpha_a + \cdots + \alpha_{b+1}$ is a positive root in finite type A of height $l := b - a + 2$. For $i, j \in I$ we also fix a choice of signs $\varepsilon_{i,j} = \mathrm{sgn}(j - i)$ for concreteness. In this setting, we shall explicitly compute all extensions between standard modules. Moreover, setting $\Delta := \bigoplus_{\pi \in \mathrm{KP}(\alpha)} \Delta(\pi)$, we compute the algebra structure on $\mathrm{Ext}^{\bullet}(\Delta, \Delta)$.

4.1. The Resolution $P_{\bullet}(\pi)$

For $i \leq j \in \mathbb{Z}_{>0}$ and $\beta := \alpha_i + \alpha_{i+1} \cdots + \alpha_j \in \Phi_+$, we define the tuple $\boldsymbol{i}_\beta := i(i+1)\cdots j$ and corresponding idempotent $e(\beta) := 1_{\boldsymbol{i}_\beta}$. For arbitrary $\pi = (\pi_1, \cdots \pi_m) \in \mathrm{KP}(\alpha)$ with $\pi_1 > \cdots > \pi_m$ let $\boldsymbol{i}_\pi$ be the concatenation $\boldsymbol{i}_\pi := \boldsymbol{i}_{\pi_1} \cdots \boldsymbol{i}_{\pi_m}$ and also set $e(\pi) := 1_{\boldsymbol{i}_\pi}$.

We now define a *refinement operator* on $\mathrm{KP}(\alpha)$. Let $\pi = (\pi_1, \cdots \pi_m) \in \mathrm{KP}(\alpha)$, and suppose for some $1 \leq t \leq m$ we have $\pi_t = \alpha_i + \cdots + \alpha_j$ for $a \leq i < j \leq b + 1$. Then, for any $i \leq k < j$ we define $\mathrm{ref}_k(\pi) := (\sigma_1, \cdots, \sigma_{m+1}) \in \mathrm{KP}(\alpha)$ to be the unique Kostant partition such that:

- $\sigma_s = \pi_s$ for $1 \leq s \leq t - 1$;
- $\sigma_t = \alpha_{k+1} + \cdots + \alpha_j$ and $\sigma_{t+1} = \alpha_i + \cdots + \alpha_k$;
- $\sigma_s = \pi_{s-1}$ for $t + 2 \leq s \leq m + 1$.

In the case $\sigma = \mathrm{ref}_k(\pi)$ as above, we also define $s(\sigma, \pi) = m - t \in \mathbb{Z}_{\geq 0}$. If instead, $\sigma = \mathrm{ref}_{k_1}\mathrm{ref}_{k_2} \cdots \mathrm{ref}_{k_n}(\pi)$ for some $n \geq 1$, we will call σ an *n-refinement* of π and will write $\mathrm{Ref}_n(\pi)$ for the set of all n-refinements of π. We call σ a *refinement* of π if $\sigma \in \mathrm{Ref}_n(\pi)$ for some $n > 0$.

For any $\pi \in \mathrm{KP}(\alpha)$, we define a projective resolution $P_\bullet(\pi) \to \Delta(\pi)$ as follows. We set $P_n(\pi) := \bigoplus_{\sigma \in \mathrm{Ref}_n(\pi)} q^n H_\alpha e(\sigma)$. The boundary map $d_n : P_{n+1}(\pi) \to P_n(\pi)$ will be defined via right multiplication with a matrix D as in 2.3. For $\sigma, \tau \in \mathrm{KP}(\alpha)$, define $w(\sigma, \tau) \in \mathfrak{S}_l$ to be the unique permutation with $e(\sigma)\psi_{w(\sigma,\tau)} = \psi_{w(\sigma,\tau)}e(\tau)$. The matrix $D = \left(d_n^{\sigma,\tau}\right)_{\sigma \in \mathrm{Ref}_{n+1}(\pi), \tau \in \mathrm{Ref}_n(\pi)}$ is then defined via

$$d_n^{\sigma,\tau} := \begin{cases} (-1)^{s(\sigma,\tau)}\psi_{w(\sigma,\tau)}, \text{ if } \tau \in \mathrm{Ref}_n(\pi) \text{ and } \sigma \in \mathrm{Ref}_1(\tau); \\ 0 \text{ otherwise.} \end{cases}$$

Remark 4.1.1. In the case $\pi = (\alpha)$, the resolution $P_\bullet(\pi)$ should not be confused with the resolution $P_\bullet^{\alpha^m}$ with $m = 1$ as defined in 2.2. These resolutions are distinct, but it will be shown below that they are isomorphic. We choose to work with $P_\bullet(\alpha)$ since it will be convenient for us when computing the algebra structure on $\mathrm{Ext}^\bullet(\Delta, \Delta)$ in Section 4.2.

Remark 4.1.2. If $\pi = (\pi_1, \cdots, \pi_m) \in \mathrm{KP}(\alpha)$ with $\pi_1 > \cdots > \pi_m$, it is easily seen that $P_\bullet(\pi)$ is isomorphic to $P_\bullet(\pi_1) \circ \cdots \circ P_\bullet(\pi_m)$ using isomorphisms from Lemma 3.1.3.

We will show the resolution $P_\bullet(\pi)$ is isomorphic to the one constructed in [BKM, Theorem 4.12], whose definition we now recall. To define the resolution from [BKM], let $\sigma \in \mathrm{KP}(\alpha)$ be minimal such that $\sigma \geq (\alpha)$. By [BKM, 2.6], this implies $\sigma = (\beta, \gamma)$ with $\beta \geq \alpha \geq \gamma$. For $i \in I$ and the empty tuple ε, let $\boldsymbol{i}_{\alpha_i,\varepsilon} := i$. Now suppose α has height $n \geq 2$ and that γ has height m. For $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{n-1}) \in \{0,1\}^{n-1}$, let $|\varepsilon| = \varepsilon_1 + \cdots \varepsilon_{n-1}$, $\varepsilon_{<m} := (\varepsilon_1, \dots, \varepsilon_{m-1})$ and $\varepsilon_{>m} := (\varepsilon_{m+1}, \dots, \varepsilon_{n-1})$. Then, define $\boldsymbol{i}_{\alpha,\varepsilon} \in I^\alpha$ recursively from

$$\boldsymbol{i}_{\alpha,\varepsilon} := \begin{cases} \boldsymbol{i}_{\gamma,\varepsilon_{<m}} \boldsymbol{i}_{\beta,\varepsilon_{>m}}, & \text{if } \varepsilon_m = 0, \\ \boldsymbol{i}_{\beta,\varepsilon_{>m}} \boldsymbol{i}_{\gamma,\varepsilon_{<m}}, & \text{if } \varepsilon_m = 1. \end{cases}$$

The resolution from [BKM] is then defined by

$$Q_r = Q_r(\alpha) = \bigoplus_{\varepsilon \in \{0,1\}^n, |\varepsilon| = r} q^r H_\alpha 1_{\boldsymbol{i}_{\alpha,\varepsilon}}.$$

Moreover, the differential $\partial_r : Q_r(\alpha) \to Q_{r-1}(\alpha)$ is given by right multiplication with a matrix $D' = (\partial^r_{\varepsilon,\delta})_{|\varepsilon|=r, |\delta|=r-1}$. Each $\partial^r_{\varepsilon,\delta}$ is zero unless the tuples ε and δ differ in just one entry. If ε and δ differ in just the t^{th} entry, then

$$\partial^r_{\varepsilon,\delta} = (-1)^{\varepsilon_1 + \cdots + \varepsilon_{t-1}} \psi_{w(\varepsilon,\delta)},$$

where $w(\varepsilon, \delta) \in \mathfrak{S}_l$ is the unique permutation with $1_{\boldsymbol{i}_{\alpha,\varepsilon}} \psi_{w(\varepsilon,\delta)} = \psi_{w(\varepsilon,\delta)} 1_{\boldsymbol{i}_{\alpha,\delta}}$. Finally, for arbitrary $\pi = (\pi_1, \dots, \pi_t) \in \mathrm{KP}(\alpha)$, the resolution $Q_\bullet(\pi)$ is given by the total complex of the tensor product of the complexes $Q_\bullet(\pi_1), \dots, Q_\bullet(\pi_t)$. The following lemma follows from the above definitions and an easy induction argument.

Lemma 4.1.3. *Let $a \leq k \leq b$ and $\varepsilon \in \{0,1\}^n$ for some $n \in \mathbb{Z}_{\geq 0}$. Then, k appears to the right of $(k+1)$ in the tuple $\boldsymbol{i}_{\alpha,\varepsilon}$ if and only if $\varepsilon_r = 1$.*

The following result shows that the complex $P_\bullet(\pi)$ gives a resolution of $\Delta(\pi)$ and should be compared to Lemma 3.2.5.

Lemma 4.1.4. *Let $\pi \in \mathrm{KP}(\alpha)$. There is an isomorphism of chain complexes $f_\bullet : P_\bullet(\pi) \xrightarrow{\sim} Q_\bullet(\pi)$.*

Proof. It follows from Remark 4.1.2 and the definition of $Q_\bullet(\pi)$ that it suffices to prove the lemma in the case $\pi = (\alpha)$. For each $r \in \mathbb{Z}_{\geq 0}$, there is a bijection between $\mathrm{Ref}_r(\alpha)$ and the set $\{\varepsilon \in \{0,1\}^{l-1} \mid |\varepsilon| = r\}$ defined as follows. For a root $\beta = \alpha_i + \cdots + \alpha_j$ with $i \leq j$, we define tuples $\varepsilon_\beta := (0^{j-i}, 1)$ and $\varepsilon'_\beta := (0^{j-i})$. Note that $\varepsilon_\beta \in \{0,1\}^{j-i+1}$, $\varepsilon'_\beta \in \{0,1\}^{j-i}$, $|\varepsilon_\beta| = 1$, and $|\varepsilon'_\beta| = 0$. Then, for any $\sigma = (\sigma_1, \ldots, \sigma_{r+1}) \in \mathrm{Ref}_r(\alpha)$ with $\sigma_1 > \cdots > \sigma_{r+1}$ we define the tuple ε_σ to be the concatenation $\varepsilon_\sigma := \varepsilon_{\sigma_{r+1}} \varepsilon_{\sigma_r} \cdots \varepsilon_{\sigma_2} \varepsilon'_{\sigma_1}$. Note that $\varepsilon_\sigma \in \{0,1\}^{l-1}$ and since $\sigma \in \mathrm{Ref}_r(\alpha)$, $|\varepsilon_\sigma| = r$. The assignment $\sigma \to \varepsilon_\sigma$ then gives the desired bijection. As an example for the reader, if $\alpha = \alpha_1 + \cdots + \alpha_5$ and $\sigma = (\alpha_3 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_2)$, then $\varepsilon_\sigma = (0,1,0,0)$.

Let $w_\sigma \in \mathfrak{S}_l$ be the unique permutation so that $e(\sigma)\psi_{w_\sigma} = \psi_{w_\sigma} 1_{\boldsymbol{i}_{\alpha,\varepsilon_\sigma}}$. Let $a \leq t \leq b$. It follows from definitions that t will appear to the right of $(t+1)$ in the tuple $\boldsymbol{i}_\sigma$ if and only if $\varepsilon_{\sigma t} = 1$. Since we are working in type A and α is a root, Lemma 4.1.3 implies that right multiplication by ψ_{w_σ} defines an isomorphism $q^r H_\alpha e(\sigma) \to q^r H_\alpha 1_{\boldsymbol{i}_{\alpha,\varepsilon_\sigma}}$. We then define $f_r : P_r(\pi) \to Q_r(\pi)$ so that its restriction to each summand $q^r H_\alpha e(\sigma) \subseteq P_r(\pi)$ is given by right multiplication by ψ_{w_σ}.

We still must show that $f_\bullet$ is a chain map. Suppose $\sigma \in \mathrm{Ref}_r(\alpha)$ and $\sigma = \mathrm{ref}_s(\tau)$ for some $\tau \in \mathrm{Ref}_{t-1}(\alpha)$. Since the braid relations hold exactly in H_α, we have $e(\sigma)\psi_{w(\sigma,\tau)}e(\tau)\psi_\tau 1_{\boldsymbol{i}_{\alpha,\varepsilon_\tau}} = e(\sigma)\psi_w$, where $w \in \mathfrak{S}_l$ is the unique permutation with $1_\sigma\psi_w = \psi_w 1_{\boldsymbol{i}_{\alpha,\varepsilon_\tau}}$. Similarly, $1_\sigma\psi_\sigma 1_{\boldsymbol{i}_{\alpha,\varepsilon_\sigma}}\psi_{w(\varepsilon_\sigma,\varepsilon_\tau)}1_{\boldsymbol{i}_{\alpha,\varepsilon_\tau}} = e(\sigma)\psi_w$. Also, signs have been defined so that $s(\sigma,\tau) = (-1)^{\varepsilon_{\sigma 1}+\cdots+\varepsilon_{\sigma r-1}}$. Thus, $f_{r-1}d_r = \partial_r f_r$, and this completes the proof. $\square$

4.2. The Algebra $\mathrm{Ext}^\bullet(\Delta,\Delta)$

Let $\pi = (\pi_1,\cdots,\pi_m) \in \mathrm{KP}(\alpha)$. If $a \le r, s \le b+1$, we call the pair (r,s) *π-equivalent* if there is some $1 \le n \le m$ and $i \le r, s \le j$ with $\pi_n = \alpha_i + \cdots + \alpha_j$. We will call an element $w \in \mathfrak{S}_l$ a *π-shuffle* if for all π-equivalent pairs (r,s), $r < s$ implies $w(r) < w(s)$. The following lemma is well known, but its proof is included for completeness.

Lemma 4.2.1. *Let $\pi = (\pi_1,\cdots,\pi_k) \in KP(\alpha)$. For each $1 \le r \le k$, let c_m be the minimal index with $\pi_m = \alpha_{c_m} + \cdots$. The H_α module $\Delta(\pi)$ is cyclic, generated by a vector v_π of degree zero and weight $\boldsymbol{i}_\pi$, and has basis*

$$\{\psi_w y_{c_1}^{a_1}\cdots y_{c_k}^{a_k} e(\pi)\cdot v_\pi \mid w \in \mathfrak{S}_l \textit{ is a } \pi\textit{-shuffle}, a_i \in \mathbb{Z}_{\ge 0} \textit{ for } 1 \le i \le k\}.$$

Moreover, if (i,j) is π-equivalent, then $y_i \cdot v_\pi = y_j \cdot v_\pi$, and $\psi_w \cdot v_\pi = 0$ for any $w \in \mathfrak{S}_l$ that is not a π-shuffle.

Proof. To simplify notation, we may assume $\alpha = \alpha_1 + \cdots + \alpha_l$. First, suppose $\pi = (\alpha)$. Consider the surjection $p^\alpha : P_0(\alpha) \to \Delta(\alpha)$ provided by Lemma 4.1.4. If $v_{(\alpha)} := p^\alpha(e(\pi))$, then $v_{(\alpha)}$ is a cyclic generator of $\Delta(\alpha)$ of degree zero and weight $\boldsymbol{i}_\pi$. Recall that $P_0(\alpha) = H_\alpha e(\pi)$ has basis

$$\{\psi_w y_1^{a_1} \cdots y_l^{a_l} e(\pi) \mid w \in \mathfrak{S}_l, a_i \in \mathbb{Z}_{\geq 0} \text{ for } 1 \leq i \leq l\}.$$

Choose any $1 \leq i < l$, and let $\sigma := \mathrm{ref}_i(\pi)$. Consider the restriction of the boundary map to

$$d_0^{\sigma,\pi} : qH_\alpha e(\sigma) \to H_\alpha e(\pi).$$

Let $w_i \in \mathfrak{S}_l$ be the unique permutation with $e(\sigma)\psi_{w_i} = \psi_{w_i} e(\pi)$. Then, for any basis element $\psi_w y_1^{a_1} \cdots y_l^{a_l} e(\sigma) \in qH_\alpha e(\sigma)$,

$$\begin{aligned} d_0^{\sigma,\pi}(\psi_w y_1^{a_1} \cdots y_l^{a_l} e(\sigma)) &= \psi_w y_1^{a_1} \cdots y_l^{a_l} e(\sigma)\psi_{w_i} \\ &= \psi_w \psi_{w_i} y_{w(1)}^{a_1} \cdots y_{w(l)}^{a_l} e(\pi) \in \ker p^\alpha. \end{aligned}$$

Since $(i, i+1)$ is the unique pair of consecutive integers inverted by w_i and the braid relations hold exactly in H_α, we have

$$\psi_w \psi_{w_i} e(\pi) = \begin{cases} \pm\psi_{ww_i}(y_i - y_{i+1})e(\pi), & \text{if } w \text{ inverts } (i, i+1); \\ \psi_{ww_i} e(\pi), \text{otherwise}. \end{cases}$$

Now, let $t \in \mathfrak{S}_l$ with $t \neq 1$. If t inverts $(i, i+1)$ then there exists $w \in \mathfrak{S}_l$ such that $t = ww_i$ and w does not invert $(i, i+1)$. It follows that $\mathrm{Im}(d_0)$

has basis

$$\{(y_i - y_{i+1})y_1^{a_1} \cdots y_l^{a_l} e(\pi) \mid 1 \le i \le l-1, a_j \in \mathbb{Z}_{\ge 0}\} \cup$$
$$\{\psi_t y_1^{a_1} \cdots y_l^{a_l} e(\pi) \mid 1 \ne t \in \mathfrak{S}_l, a_j \in \mathbb{Z}_{\ge 0}\}.$$

In particular, this implies $\psi_t \cdot v_{(\alpha)} = 0$ for any $1 \ne t \in \mathfrak{S}_l$. Since $\Delta(\pi) \simeq H_\alpha e(\pi)/\mathrm{Im}(d_0)$, it has basis

$$\{y_1^{a_1} \cdot v_{(\alpha)} | a_1 \in \mathbb{Z}_{\ge 0}\}.$$

Finally, if $\pi = (\pi_1, \dots, \pi_k) \in \mathrm{KP}(\alpha)$ is arbitrary, then since $\alpha \in \Phi_+$,

$$\Delta(\pi) \simeq \Delta(\pi_1) \circ \cdots \circ \Delta(\pi_k).$$

Since we have already computed a basis for each $\Delta(\pi_i)$, the general case follows. $\square$

For $\boldsymbol{i} \in I^\alpha$ and $M \in H_\alpha$-mod, recall that there is an isomorphism $\mathrm{Hom}_{H_\alpha}(H_\alpha 1_{\boldsymbol{i}}, M) \xrightarrow{\sim} 1_{\boldsymbol{i}} M$. The Hom functor will also take maps defined by right multiplication to their dual maps defined by left multiplication, and will convert positive degree shifts to negative degree shifts. Applying these observations to Lemma 4.1.4 yields the following.

Lemma 4.2.2. *Let $\pi \in KP(\alpha)$, and $M \in H_\alpha$-mod. Then,* $\mathrm{Ext}^r(\Delta(\pi), M)$ *is the r^{th} cohomology group of the complex*

$$0 \to e(\pi)M \xrightarrow{d_0} \bigoplus_{\sigma \in Ref_1(\pi)} q^{-1} 1_\sigma M \xrightarrow{d_1} \bigoplus_{\sigma \in Ref_2(\pi)} q^{-2} 1_\sigma M \to \cdots.$$

Here, the m^{th} coboundary map d_m is given by left multiplication by the matrix $D = (d_m^{\tau,\sigma})$, where $d_m^{\tau,\sigma} : q^{-m}e(\tau)M \to q^{-m-1}e(\sigma)M$ for $\tau \in Ref_m(\pi)$ and $\sigma \in Ref_{m+1}(\pi)$. Moreover, $d_m^{\tau,\sigma}$ is zero unless $\sigma \in Ref_1(\tau)$, and in this case $d_m^{\tau,\sigma}$ is given by left multiplication by $(-1)^{s(\sigma,\tau)}\psi_{w(\sigma,\tau)}$.

Now we turn our attention to the case where $M = \Delta(\sigma)$. The following lemmas are needed for the proof of Theorem 4.2.5.

Lemma 4.2.3. *Let $\sigma, \tau \in KP(\alpha)$. Then, $e(\tau)\Delta(\sigma) = 0$ unless $\sigma \in Ref_t(\tau)$ for some $t \in \mathbb{Z}_{\geq 0}$.*

Proof. Since σ and τ are refinements of (α), there are unique integers $r_1, \ldots, r_n$ and $s_1, \ldots, s_m$ such that $m, n \in \mathbb{Z}_{>0}$ are minimal with $\sigma = \mathrm{ref}_{r_1} \cdots \mathrm{ref}_{r_n}\alpha$ and $\tau = \mathrm{ref}_{s_1} \cdots \mathrm{ref}_{s_m}\alpha$. If σ is not a refinement of τ then there is some k such that $s_k \neq r_i$ for all $a \leq i \leq b$. It follows that the pair $(k, k+1)$ is σ-equivalent but is not τ-equivalent. Let $w(\tau, \sigma) \in \mathfrak{S}_l$ be the unique element such that $e(\tau)\psi_{w(\tau,\sigma)}e(\sigma) \neq 0$. Then, $w(\tau, \sigma)$ is not a σ-shuffle, and so the result follows from Lemma 4.2.1. $\square$

Lemma 4.2.4. *Let $\pi, \sigma, \tau \in KP(\alpha)$. Suppose that $\sigma \in Ref_t(\pi)$ for some $t > 0$ and that $\pi = ref_r(\tau)$ for some $1 \leq r < l$. Then,*

$$\psi_{w(\pi,\tau)}\psi_{w(\tau,\sigma)}e(\sigma) = \psi_{w(\pi,\sigma)}(y_i - y_j)e(\sigma), \tag{2.1}$$

where $i, j \in I^\alpha$ are given by $(\boldsymbol{i}_\sigma)_i = r$, $(\boldsymbol{i}_\sigma)_j = r + 1$.

.

Proof. Since $\pi = \mathrm{ref}_r(\tau)$, r and $r + 1$ are the only neighboring entries of $\boldsymbol{i}_\tau$ that are permuted by $w(\pi, \tau)$. Since $\sigma \in \mathrm{Ref}_t(\pi)$, the permutation

$w(\tau, \sigma)$ also permutes the entries r and $r+1$ in $\boldsymbol{i}_\sigma$. Since α is a root in type A, the braid relations hold exactly in H_α Moreover, for each $\boldsymbol{i} \in I^\alpha$ and $1 \leq k \leq l$, we have the relation

$$\psi_k^2 1_{\boldsymbol{i}} = \begin{cases} (y_k - y_{k+1}) 1_{\boldsymbol{i}}, \text{ if } |i_k - i_{k+1}| = 1, \\ 1_{\boldsymbol{i}}, \text{ otherwise.} \end{cases}$$

The left hand side of 2.1 can be simplified using this relation and the braid relations. Moreover, each of these simplifications can be made using the relation $\psi_k^2 1_{\boldsymbol{i}} = 1_{\boldsymbol{i}}$ except for exactly one, corresponding to the permutation of the entries r and $r+1$. In this case, a factor $(y_j - y_i)$ is introduced, giving the result. □

We introduce some temporary notation used in the proof of the next theorem. Suppose $\sigma, \pi \in \mathrm{KP}(\alpha)$ and $\sigma = (\sigma_1, \ldots, \sigma_n)$. For each $1 \leq i, j \leq n$ we will write $i \sim_\sigma^\pi j$ if there exist some $k_0, k_1, \ldots, k_m \in \mathbb{Z}$ such that $\sigma_i + \sigma_j + \sigma_{k_1} + \cdots + \sigma_{k_m} = \pi_{k_0}$.

Theorem 4.2.5. *Suppose α is a root of height l and $\pi, \sigma \in KP(\alpha)$. Then, $\mathrm{Ext}^r_{H_\alpha}(\Delta(\pi), \Delta(\sigma)) = 0$ unless σ is an r-refinement of π. If σ is an r-refinement of π, then there is a graded vector space isomorphism*

$$\mathrm{Ext}^r_{H_\alpha}(\Delta(\pi), \Delta(\sigma)) \simeq q^{-r} \Bbbk[y_1, \ldots, y_l]/J^\pi,$$

where each y_j is of degree 2, and J^π is the ideal generated by all $y_j - y_k$ such that j and k are π-equivalent. In particular,

$$\mathrm{Hom}_{H_\alpha}(\Delta(\pi), \Delta(\sigma)) \simeq \begin{cases} 0, \ \textit{if } \pi \neq \sigma, \\ \\ \Bbbk[y_1, \ldots, y_l]/J^\pi, \ \textit{if } \pi = \sigma. \end{cases}$$

Proof. Again we may assume $\alpha = \alpha_1 + \cdots + \alpha_l$ for simplicity. Taking $M = \Delta(\sigma)$ in the Lemma 4.2.2 yields the sequence

$$0 \to e(\pi)\Delta(\sigma) \xrightarrow{d_0} \bigoplus_{\tau \in \mathrm{Ref}_1(\pi)} q^{-1} e(\tau)\Delta(\sigma) \xrightarrow{d_1} \bigoplus_{\tau \in \mathrm{Ref}_2(\pi)} q^{-2} e(\tau)\Delta(\sigma) \to \cdots .$$

By Lemma 4.2.3, $e(\tau)\Delta(\sigma) = 0$ unless σ is a refinement of τ. If σ is a refinement of τ, let $w(\tau, \sigma) \in \mathfrak{S}_l$ be the unique σ-shuffle with $e(\tau)\psi_{w(\tau,\sigma)} = \psi_{w(\tau,\sigma)} e(\sigma)$. Lemma 4.2.1 implies there is a vector space isomorphism $e(\tau)\Delta(\sigma) \simeq k[y_1, \ldots, y_n]$, where n is given by $\sigma = (\sigma_1, \ldots, \sigma_n)$. The isomorphism is defined by sending a basis vector $\psi_{w(\tau,\sigma)} y_{c_1}^{a_1} \cdots y_{c_n}^{a_n} 1_\sigma \cdot v_\sigma \in \Delta(\sigma)$ from Lemma 4.2.1 to the monomial $y_1^{a_1} \cdots y_n^{a_n} \in k[y_1, \ldots y_n]$.

Now suppose that σ is a refinement of both τ and ρ, and that $\rho = \mathrm{ref}_t(\tau)$. If τ is an s-refinement of π, we can consider the restriction of the boundary map

$$d_s^{\tau,\rho} : q^{-s} e(\tau)\Delta(\sigma) \to q^{-(s+1)} e(\rho)\Delta(\sigma).$$

By Lemma 4.2.2., it is defined by left multiplication by $(-1)^{s(\rho,\tau)}\psi_{w(\rho,\tau)}$. By Lemma 4.2.4 one has the equality

$$\psi_{w(\rho,\tau)}\psi_{w(\tau,\sigma)}e(\sigma) = \psi_{w(\rho,\sigma)}(y_i - y_j)e(\sigma),$$

where i, j are defined by $(\boldsymbol{i}_\sigma)_i = t$, $(\boldsymbol{i}_\sigma)_j = t+1$.

On the other hand, using the vector space isomorphism described above we can now consider $d_s^{\tau,\rho}$ as a map

$$d_t^{\tau,\rho} : q^{-s}\Bbbk[y_1, \ldots, y_n] \to q^{-(s+1)}\Bbbk[y_1, \ldots, y_n].$$

Under this identification, $d_k^{\tau,\rho}$ is multiplication by $(-1)^{s(\tau,\rho)}(y_{k_1} - y_{k_2})$ if $\sigma_{k_1} = \alpha_t + \cdots$ and $\sigma_{k_2} = \alpha_{t+1} + \cdots$. Now, notice that if $\sigma \in \mathrm{Ref}_r(\pi)$, then there are exactly $\binom{r}{m}$ skew shapes, τ, such that $\tau \in \mathrm{Ref}_m(\pi)$ and σ is a p-refinement of τ for some $p \geq 0$. Thus, the above complex is isomorphic to

$$\begin{aligned} 0 \to \Bbbk[y_1, \ldots, y_n] \to q^{-1}\Bbbk[y_1, \ldots, y_n]^{\oplus r} \to q^{-2}\Bbbk[y_1, \ldots y_n]^{\oplus\binom{r}{2}} \to \cdots \\ \to q^{-r}\Bbbk[y_1, \ldots, y_n] \to 0, \end{aligned}$$

with maps as described above. Recognizing this complex as a Koszul implies that $\mathrm{Ext}^p_{H_\alpha}(\Delta(\pi), \Delta(\sigma)) = 0$ for $p \neq r$, while

$$\mathrm{Ext}^r_{H_\alpha}(\Delta(\pi), \Delta(\sigma)) \simeq q^{-r}k[y_1, \ldots, y_n]/J,$$

where J is the ideal generated by all $y_{k_1} - y_{k_2}$ such that $k_1 \sim_\sigma^\pi k_2$. Now the result follows, since

$$q^{-r} k[y_1, \ldots, y_n]/J \simeq k[y_1, \ldots, y_l]/J^\pi.$$

See [E, Corollary 17.12] for more details on Koszul complexes. $\square$

Recall that $\Delta := \bigoplus_{\pi \in \mathrm{KP}(\alpha)} \Delta(\pi)$. To compute the algebra structure on $\mathrm{Ext}^\bullet(\Delta, \Delta)$, it will be useful to recall an equivalent definition of the Ext groups. Let R be any ring and M, N be R-modules. Choose projective resolutions $P_\bullet$ and $Q_\bullet$ of M and N, respectively. Then, we define the graded vector space $\mathrm{Hom}(P_\bullet, Q_\bullet)$ to have graded components

$$\mathrm{Hom}^i(P_\bullet, Q_\bullet) = \bigoplus_{n \in \mathbb{N}} \mathrm{Hom}_R(P_n, Q_{n-i}).$$

Define a degree $+1$ boundary map ∂ on $\mathrm{Hom}(P_\bullet, Q_\bullet)$ via $\partial(g) = dg - (-1)^{|g|} gd$, where we abuse notation by allowing d to represent the boundary maps on both $P_\bullet$ and $Q_\bullet$. It is well known [W, Cor. 10.7.5] that

$$\mathrm{Ext}^i(M, N) \simeq H^i(\mathrm{Hom}(P_\bullet, Q_\bullet)). \tag{2.2}$$

This isomorphism is defined by lifting an element $\bar{f} \in \mathrm{Ext}^i(M, N)$ to a map $f : P_i \to Q_0$. This map then lifts to a cycle $f_\bullet \in \mathrm{Hom}^i(P_\bullet, Q_\bullet)$, which is unique modulo the image of ∂.

Let σ and π be Kostant partitions of α. If σ is an s-refinement of π, then $q^s P_\bullet(\sigma)$ is naturally a graded subspace of $P_\bullet(\pi)$. Let $\tilde{p}_\sigma^\pi : P_\bullet(\pi) \to$

$P_\bullet(\sigma)$ be the natural surjection and define $p^\pi_\sigma : P_\bullet(\pi) \to P_\bullet(\sigma)$ via

$$p^\pi_\sigma|_{P_r(\pi)} := \begin{cases} (-1)^{(r-s)}\tilde{p}^\pi_\sigma, & \text{if } s \text{ is odd,} \\ \tilde{p}^\pi_\sigma, & \text{if } s \text{ is even.} \end{cases}$$

The signs have been chosen so that $\partial(p^\pi_\sigma) = 0$.

Given a polynomial $f \in \Bbbk[y_1, \ldots, y_l]$, we may consider f as an element of H_α by identifying the variables $y_1, \ldots, y_l$ with the the generators of H_α with the same name. Likewise, each such f may be considered an element of $\mathrm{Hom}^0(P_\bullet(\sigma), P_\bullet(\sigma))$ by defining

$$f|_{q^r H_\alpha 1_\tau} : q^r H_\alpha 1_\tau \to q^r H_\alpha 1_\tau$$

to be given by right multiplication by f, for each $r \in \mathbb{Z}_{\geq 0}$ and $\tau \in \mathrm{Ref}_r(\sigma)$. The composition $f \circ p^\pi_\sigma$ is also a cycle belonging to $\mathrm{Hom}^s(P_\bullet(\pi), P_\bullet(\sigma))$. The following lemma follows directly from these definitions.

Lemma 4.2.6. *Let $\pi, \sigma, \tau \in \mathrm{KP}(\alpha)$. Suppose that $\sigma \in Ref_s(\pi)$ and $\tau \in Ref_r(\sigma)$. Then, $p^\sigma_\tau p^\pi_\sigma = (-1)^{rs} p^\pi_\sigma$.*

Recall that we define $\Delta := \bigoplus_{\pi \in \mathrm{KP}(\alpha)} \Delta(\pi)$. We are now ready to compute the algebra structure on $\mathrm{Ext}^\bullet(\Delta, \Delta)$. Toward this end, recall the ideals J^π from Theorem 4.2.5, and let $\mathcal{A}_\alpha$ be the vector space of $\mathrm{KP}(\alpha) \times \mathrm{KP}(\alpha)$ upper triangular matrices, $M = (f_{\sigma,\pi})_{\sigma,\pi \in \mathrm{KP}(\alpha)}$, with entries $f_{\sigma,\pi} \in q^{-n}\Bbbk[y_1, \ldots, y_l]/J^\pi$ if $\sigma \in \mathrm{Ref}_n(\pi)$, and $f_{\sigma,\pi} = 0$ otherwise.

Notice that for $\sigma \in \mathrm{Ref}_n(\pi)$, $J^\sigma \subseteq J^\pi$, and so there is a natural surjection

$$\Bbbk[y_1, \dots, y_l]/J^\sigma \xrightarrow{\rho_{\sigma,\pi}} \Bbbk[y_1, \dots, y_l]/J^\pi.$$

For $\sigma \in \mathrm{Ref}_n(\pi)$ and $\tau \in \mathrm{Ref}_m(\sigma)$, this allows us to define products by the rule

$$f_{\tau,\sigma} f_{\sigma,\pi} = (-1)^{mn} \rho_{\sigma,\pi}(f_{\tau,\sigma}) f_{\sigma,\pi} \in q^{-(m+n)} \Bbbk[y_1, \dots, y_n]/J^\pi,$$

for any $f_{\tau,\sigma} \in q^{-m}\Bbbk[y_1, \dots, y_l]/J^\sigma$ and $f_{\sigma,\pi} \in q^{-n}\Bbbk[y_1, \dots, y_l]/J^\pi$. Extending this product to usual matrix multiplication then gives $\mathcal{A}_\alpha$ the structure of a graded, associative algebra. This allows us to state our final result.

Theorem 4.2.7. *Let α be a positive root in type A. There is an isomorphism of graded, associative algebras* $\mathrm{Ext}^\bullet(\Delta, \Delta) \simeq \mathcal{A}_\alpha$.

Proof. By Theorem 4.2.5, there is an obvious vector space isomorphism $\mathrm{Ext}^\bullet(\Delta, \Delta) \to \mathcal{A}_\alpha$. When σ is an s-refinement of π, it is defined by sending an element $\overline{f} \in \mathrm{Ext}^\bullet(\Delta(\pi), \Delta(\sigma)) \simeq \Bbbk[y_1, \dots, y_l]/J^\pi$ to the corresponding element $f_{\pi,\sigma} \in \Bbbk[y_1, \dots, y_l]/J^\pi \subset \mathcal{A}_\alpha$. It is left to compute the algebra structure on $\mathrm{Ext}^\bullet(\Delta, \Delta)$.

Indeed, let $\overline{f} \in \mathrm{Ext}^s(\Delta(\pi), \Delta(\sigma)) \simeq \Bbbk[y_1, \dots, y_l]/J^\pi$ and lift $\overline{f}$ to a polynomial $f \in \Bbbk[y_1, \dots, y_l]$. Under the isomorphism from Equation 2.2, $fp^\pi_\sigma \in \mathrm{Hom}^s(P_\bullet(\pi), P_\bullet(\sigma))$ is a cochain representative of $\overline{f}$. Similarly, if τ is an r-refinement of σ, let $\overline{g} \in \mathrm{Ext}^r(\Delta(\sigma), \Delta(\tau))$ and lift $\overline{g}$ to $gp^\sigma_\tau \in \mathrm{Hom}^r(P_\bullet(\sigma), P_\bullet(\tau))$. Since the product on $\mathrm{Ext}^\bullet(\Delta, \Delta)$ is induced by the composition, Lemma 4.2.6 implies that

$$\overline{g}\overline{f} = \overline{gp^{\sigma}_{\tau} \circ fp^{\pi}_{\sigma}} = (-1)^{rs}\overline{gfp^{\pi}_{\tau}} \in \mathrm{Ext}^{r+s}(\Delta(\pi), \Delta(\tau)).$$

This agrees with the product structure on $\mathcal{A}_{\alpha}$, which completes the proof. $\square$

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