GLUING MANIFOLDS WITH BOUNDARY AND BORDISMS OF POSITIVE SCALAR CURVATURE METRICS

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DISSERTATION ABSTRACT

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This thesis presents two main results on analytic and topological aspects of scalar curvature. The first is a gluing theorem for scalar-flat manifolds with vanishing mean curvature on the boundary. Our methods involve tools from conformal geometry and perturbation techniques for nonlinear elliptic PDE. The second part studies bordisms of positive scalar curvature metrics. We present a modification of the Schoen-Yau minimal hypersurface technique to manifolds with boundary which allows us to prove a hereditary property for bordisms of positive scalar curvature metrics. The main technical result is a convergence theorem for stable minimal hypersurfaces with free boundary in bordisms with long collars which may be of independent interest.

This dissertation includes unpublished co-authored material.

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CHAPTER I

INTRODUCTION

1.1. Background

Let M be a smooth compact oriented n-dimensional manifold. If its boundary is non-empty, we will denote it by ∂M. Let $\mathrm{Riem}(M)$ denote the space of smooth Riemannian metrics on M. For a metric $g \in \mathrm{Riem}(M)$, we will study its scalar curvature $R_g : M \to \mathbb{R}$ and the boundary's mean curvature $H_g : \partial M \to \mathbb{R}$ with respect to the outward-pointing normal vector. In this manuscript, we will explore issues related to the impact of the topology of M on various properties of these two curvature functions.

For 2-dimensional manifolds, this relationship is well-studied. If (Σ, g) is a Riemannian surface, the Gauss-Bonnet theorem relates R_g and H_g to the Euler characteristic $\chi(\Sigma)$ of the underlying surface according to the formula

$$\int_\Sigma R_g d\mu_g + 2\int_{\partial\Sigma} H_g d\sigma_g = 4\pi\chi(\Sigma). \tag{1.1}$$

In this equation, $d\mu_g$ denotes the volume element associated to g and $d\sigma_g$ denotes the induced volume element on the boundary. Let us consider some first examples.

Example 1.1.1. The torus T^2 has Euler characteristic $\chi(T^2) = 0$. Since T^2 has no boundary, equation (1.1) implies that the average value of the scalar curvature of any metric must be 0. We can conclude that T^2 does not admit a metric of strictly positive or strictly negative scalar curvature. Similar logic implies that the sphere S^2 is the only orientable closed surface which admits a metric of positive scalar curvature.

Example 1.1.2. Consider the surface with boundary $\Sigma = S^2 \setminus \{B_1, B_2\}$ obtained by deleting two disjoint balls from the 2-sphere and notice that $\chi(\Sigma) = 0$. Unlike the case of the torus, there is no obstruction to Σ admitting a metric of positive scalar curvature since the boundary term in (1.1) may be non-zero. We can find an example of this phenomenon by considering the restriction of the usual round metric on S^2. However, we can ask whether or not Σ admits a positive scalar curvature metric with *minimal boundary conditions* i.e. a metric which satisfies $H_g \equiv 0$. Clearly, (1.1) implies this is impossible. In fact, the only compact orientable surface with non-trivial boundary which admits a positive scalar curvature metric with minimal boundary conditions is the disk.

The interaction between metrics satisfying certain curvature conditions and the topology of the underlying manifold is an old and well-studied field in differential geometry. For conditions on the scalar curvature of a Riemannian manifold, there is the classical result of Kazdan-Warner which provides some clarity. Consider the following three classes of closed manifolds:

(A) Those which admit a metric with non-negative and non-vanishing scalar curvature.

(B) Those which admit a metric with vanishing scalar curvature and are not in class (A).

(C) Those which are not in classes (A) or (B).

It is worth noting that, in dimensions $n \geq 2$, the classes (A), (B), and (C) are all non-empty. For $n \geq 2$, (A) contains S^n, (B) contains the n-torus T^n, and (C) contains hyperbolic manifolds i.e. manifolds admitting metrics of constant negative sectional curvature. The following fact, called the Trichotomy Theorem, can be thought of as describing how flexible the above classes are in terms of their possible scalar curvatures.

Theorem 1.1.1. *[1] Let M be a closed connected manifold of dimension at least* 3*.*

1. *If M is in class (A), any function can be realized as the scalar curvature of some metric.*

2. *If M is in class (B), $f \in C^\infty(M)$ is the scalar curvature of a metric if and only if either $f(x) < 0$ somewhere, or $f \equiv 0$. Moreover, if the scalar curvature vanishes identically, the manifold is Ricci-flat.*

3. *If M is in class (C), a function is the scalar curvature of some metric if and only if it is negative somewhere.*

Example 1.1.3. Since the standard round metric on the 3-sphere lies has positive scalar curvature, S^3 lies in class (A). Theorem 1.1.1 shows that S^3 also admits a metric of negative scalar curvature and a metric with identically vanishing scalar curvature. Similar logic implies that any manifold of dimension greater than 2 admitting a metric of positive scalar curvature also admits a metric of vanishing scalar curvature and a metric of negative scalar curvature.

In light of the above remarks, requiring a manifold to admit a metric of positive scalar curvature, which we will abbreviate by *psc* from now on, is the strongest condition one can impose on the sign of the scalar curvature of a metric. We also see that requiring a manifold to admit a scalar flat metric is a weaker, yet still non-trivial condition. Finally, we see that requiring a manifold to admit a metric of negative scalar curvature is no condition at all, at least in dimensions $n \geq 3$.

1.2. Tools for Studying Scalar Curvature Conditions

In dimensions 2 and 3, there is a complete answer to the question of which closed manifolds admit a psc metric. In dimensions $n \geq 4$, however, it is an open problem to find smooth-topology invariants which answer this question. There are three main approaches which we will briefly mention.

(1) If $n = 4$ and M has a spin^c-structure with a non-trivial Seiberg-Witten invariant, then M cannot admit a psc metric. The Seiberg-Witten invariant is well-defined only under certain extra assumptions on M such as $b_2^+(M) \geq 2$.

(2) If $n \geq 5$ and M is simply connected, there is a complete answer: M admits a psc metric so long as M is not spin with non-zero $\hat{A}$-genus [2].

(3) If, for every metric g on M, there is a 2-sided, stable-minimal hypersurface which does not admit a psc metric, an argument due to Schoen-Yau [3] implies that M itself cannot be psc. This is most useful if the dimension of M is less than 8 and the integral homology group $H_{n-1}(M; \mathbb{Z})$ is non-trivial.

Each of the above items represents a topological obstruction to a manifold admitting a psc metric. They often overlap – for instance, both (2) and (3) can be used to show that T^n has no psc metric for $n \leq 7$.

In addition to the above obstructive tools, there are two main *constructive* techniques for studying scalar curvature that are relevant to us. The first comes from conformal geometry and originates from the Yamabe problem. It states that a manifold admits a psc metric if and only if it admits a metric whose conformal Laplacian is positive definite. This technique is described later in Section 1.4. The second constructive tool is a large class of results which could all be described as *gluing constructions*. One of the most fundamental gluing constructions is due to

Gromov-Lawson [2] which we will now briefly describe. Suppose we are given two n-dimensional manifolds M_0 and M_1 each containing an embedded k-dimensional sphere with trivial normal bundle. One can then remove neighborhoods about the sphere in M_0 and M_1 and identify the resulting boundaries to form a new manifold M. If M_0 and M_1 both admit psc metrics one can ask whether or not M also admits such a metric. If the embedded sphere has codimension $n - k \geq 3$, then one can glue the psc metrics on M_0 and M_1 to produce a third one on M. We will discuss a similar construction in Section 1.4.

1.3. The Structure of This Thesis

This thesis consists of two main parts – composing two separate chapters – each presenting generalizations of techniques mentioned in Section 1.2. The remainder of this chapter is devoted to introducing these two parts and stating the main results contained in each. In Part One, we will present a new gluing result for scalar-flat manifolds with minimal boundary conditions. This chapter will also serve as an introduction to the Yamabe problem on manifolds with boundary and other boundary value PDE we will study in the rest of this thesis. The material in Part One appears in a preprint written by the current author. Part Two begins by establishing a novel version of the Schoen-Yau minimal hypersurface technique (see Section 1.2) which is adapted to manifolds with boundary. We will then introduce the notion of *positive scalar curvature bordism* and use our new minimal hypersurface technique to study them. The material in Part Two appears in a preprint co-authored by the present author and Boris Botvinnik. The Appendix contains some details on Geometric Measure Theory and technicalities of representing hypersurfaces as graphs. The material in the Appendix is primarily used in Part Two.

1.4. Introduction to Part I

The main result of Part One can be described as a gluing construction for the Yamabe problem on manifolds with boundary. To present this result in context, let us first recall some basic background for the Yamabe problem.

Given an n-dimensional manifold M equipped with a Riemannian metric g, the *conformal class* of g is the set of metrics given by

$$[g] = \{\tilde{g} \in \mathrm{Riem}(M) \colon \tilde{g} = f \cdot g \text{ for some function } f : M \to \mathbb{R}_+\}.$$

The function f in the above definition is called a *conformal factor*. Given a conformal class of Riemannian metrics $C = [g]$ on a closed manifold M, the classical Yamabe problem asks if there is a metric in C of constant scalar curvature. Such metrics are critical points of the *Einstein-Hilbert functional* restricted to the class C

$$C \to \mathbb{R}, \quad g \mapsto \frac{c_n \int_M R_g d\mu_g}{\mathrm{Vol}_g(M)^{\frac{n-2}{n}}} \tag{1.2}$$

where $c_n = \frac{n-2}{4(n-1)}$ is a dimensional constant. Critical points of this functional are called *Yamabe metrics*.

Finding critical points of 1.2 is equivalent to solving a non-linear partial differential equation (PDE) called the Yamabe equation. In dimension $n = 2$, the problem of finding critical points of the total scalar curvature functional (in this setting we remove the dimensional constant c_n from (1.2)) is equivalent to the Uniformization problem which has long since been solved. For $n \geq 3$, however, finding a Yamabe metric in a general class C on a closed manifold M was a long-standing problem for decades, eventually solved by R. Schoen in [4] by showing that the infimum of the functional (1.2) is always achieved by some metric in C. When the solution of this problem was nearly a decade old, J. Escobar introduced and solved generalizations of this question to compact manifolds M with non-empty boundary.

1.41. The Yamabe Problem on manifolds with boundary

The natural functional to consider in the context of a non-trivial boundary is the *total scalar curvature plus total mean curvature*; see [5] for a detailed study of this functional. In order to make this quantity scale-invariant, it must be renormalized. In the case of the classical Yamabe problem for closed manifolds, this is accomplished in equation (1.2) by dividing the total scalar curvature by $\mathrm{Vol}_g(M)^{\frac{n-2}{n}}$. For manifolds with boundary, however, one may choose to renormalize with respect to the volume of the interior, the boundary, or some combination of the two volumes.

In [6], Escobar studies the following family of functionals

$$C \to \mathbb{R}, \quad g \mapsto \frac{c_n \int_M R_g d\mu_g + 2c_n \int_{\partial M} H_g d\sigma_g}{a\mathrm{Vol}_g(M)^{\frac{n-2}{n}} + (1-a)\mathrm{Vol}_g(\partial M)^{\frac{n-2}{n-1}}} \tag{1.3}$$

where a is a parameter in the interval $[0,1]$. For any fixed value of a, critical points of this functional are metrics of constant scalar curvature with constant mean curvature on the boundary. For $a = 1$, critical points of (1.3) are scalar-flat and for $a = 0$ critical points have vanishing mean curvature on the boundary. These extremal cases are studied, respectively, in [6] and [7, 8] where critical points are found for a large class of M and C by showing the infemum of 1.2 is achieved. Similar analysis of (1.3) for general values of a was carried out in [9]. Notice that scalar-flat metrics with vanishing mean curvature on the boundary are critical points of this functional for any value of a. Conformal classes which contain such metrics are called *Yamabe-null*.

Let us introduce the objects and notations we will require. For a smooth Riemannian n-dimensional manifold (M, g) with boundary ∂M, we will write Ric_g for its Ricci tensor and A_g for the second fundamental form of the boundary with respect to the outward unit normal vector ν. The scalar curvature of (M, g) is given by $R_g = \mathrm{tr}_g \mathrm{Ric}_g$ and its boundary mean curvature is $H_g = \mathrm{tr}_g A_g$. Notice that H_g is the sum of the principle curvatures at a point $p \in \partial M$, as opposed to their average (usually denoted by h_g) which is used in Escobar's original work.

As usual, the class of metrics conformal to g will be denoted by $[g]$. We will often write the conformal factor in the form $f = \psi^{\frac{4}{n-2}}$. A standard computation shows that the scalar curvature of $\tilde{g} = \psi^{\frac{4}{n-2}} g$ is given by

$$R_{\tilde{g}} = \frac{L_g \psi}{c_n \psi^{\frac{n+2}{n-2}}} \tag{1.4}$$

where L_g is the *conformal Laplacian* defined by $L_g = -\Delta_g + c_n R_g$. The mean curvature of the boundary with respect to $\tilde{g}$ is given by

$$H_{\tilde{g}} = \frac{B_g \psi}{2 c_n \psi^{\frac{n}{n-2}}} \tag{1.5}$$

where the first-order boundary operator B_g is given by $B_g = \partial_\nu + 2c_n H_g$ on ∂M.

In [6] Escobar studied and addressed the following question: Does a given conformal class $[g]$ contain a scalar-flat metric with constant boundary mean curvature? Inspecting formulas (1.4) and (1.5), this task is equivalent to solving the following elliptic problem with non-linear boundary conditions

$$\begin{cases} \Delta_g \psi = c_n R_g \psi & \text{in } M \\ \partial_\nu \psi = 2c_n(\lambda \psi^{\frac{n}{n-2}} - H_g \psi) & \text{on } \partial M \end{cases} \tag{1.6}$$

where λ is a constant. If ψ is a smooth solution to (1.6), then $\tilde{g} = \psi^{\frac{4}{n-2}} g$ has vanishing scalar curvature and constant boundary mean curvature equal to λ. As mentioned above, equation (1.6) can be viewed as the Euler-Lagrange equations for the total scalar curvature plus total mean curvature functional, renormalized with respect to the volume of the boundary and restricted to the class $[g]$. In terms of the conformal factor ψ, this functional takes the form

$$Q(\psi) = \frac{\int_M (|\nabla \psi|_g^2 + c_n R_g \psi^2) d\mu_g + 2c_n \int_{\partial M} H_g \psi^2 d\sigma_g}{(\int_{\partial M} |\psi|^{\frac{2(n-1)}{n-2}} d\sigma)^{\frac{n-2}{n-1}}}$$

where $d\mu_g$ and $d\sigma_g$ denote the Riemannian measure on M and ∂M induced by g.

1.42. Connected sum constructions

Let (M_1, g_1) and (M_2, g_2) be two n-dimensional closed Riemannian manifolds with constant scalar curvature. If M_1 and M_2 share a common embedded k-dimensional submanifold K with isomorphic normal bundles, one can form the *generalized connected sum* along K by deleting small neighborhoods around K and identifying the two boundaries

$$M_1 \#_K M_2 := (M_1 \setminus K) \sqcup (M_2 \setminus K)/\sim .$$

One can ask to what extent the metrics g_1 and g_2 can be used to produce a third constant scalar curvature metric on $M := M_1 \#_K M_2$ and how the signs of R_{g_1} and R_{g_2} effect the sign of this new scalar curvature. Such gluing constructions have a rich history in geometric analysis, too extensive to satisfactorily survey here.

For the new construction we will present in Part One, we will adopt a particular scheme first introduced by Mazzieri in [10] for gluing closed manifolds with non-zero constant scalar curvature. His work generalizes results of Joyce [11] on connected sums of closed manifolds of non-zero constant scalar curvature (see also [12]). Later, in [13], Mazzieri considers the delicate problem of gluing two closed scalar flat manifolds to produce another scalar-flat manifold. In general, this process may be obstructed if one of the two original manifolds is Ricci-flat. In particular, he proves the following.

Theorem 1.4.1. *[13] Let M be the generalized connected sum of two closed Riemannian scalar flat, non Ricci flat manifolds (M_1, g_1) and (M_2, g_2) along a common isometrically embedded submanifold (K, g_K) of codimension $n-k \geq 3$. Then there exists a number $\varepsilon_0 > 0$ and a family of metrics $\{\bar{g}_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0)}$ such that $\bar{g}_\varepsilon$ is scalar flat and $\bar{g}_\varepsilon \to g_i$ on compact subsets of $M_i \setminus K$ for $i = 1, 2$ in the C^2-topology as $\varepsilon \to 0$.*

Example 1.4.1. Let us show that it is necessary to assume both of the starting manifolds (M_1, g_1) and (M_2, g_2) are non Ricci-flat. Consider the case where (M_1, g_1) and (M_2, g_2) are both the flat 2-dimensional torus T^2. These are closed, scalar-flat Riemannian manifolds, but their connected sum is the surface of genus 2 and has negative Euler characteristic. Hence, by (1.1), $M_1 \# M_2$ cannot admit a metric with vanishing scalar curvature. The same result also holds for higher dimensional tori.

The main result of Part One is an analog of Theorem 1.4.1 for manifolds with boundary. Our generalization not only allows for M_1 and M_2 to have non-trivial boundary, but we also consider situations where K is embedded into the interiors of these manifolds, their boundaries, and even when K itself has a non-trivial boundary. See Figure 1.1.. Each of these three settings requires geometric modifications to the gluing argument which are quite technical. For this reason, we will first state our analog informally and provide three separate and precise statements later.

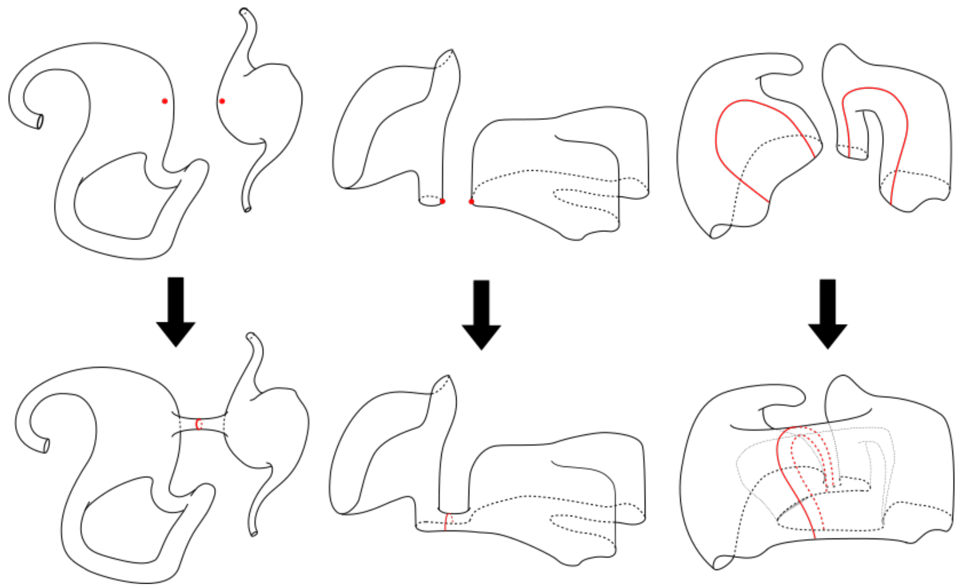

FIGURE 1.1. The generalized connected sum construction for interior, boundary, and relative embeddings.

Theorem 1.4.2. *Let (M_1, g_1) and (M_2, g_2) be two compact n-dimensional manifolds with boundary, each scalar-flat with minimal boundary conditions. Assume neither (M_1, g_1) nor (M_2, g_2) are Ricci-flat with vanishing second fundamental form of the boundary. Let M be the generalized connected sum of M_1 and M_2 along a common isometrically embedded submanifold (K, g_K) of codimension $n - k \geq 3$. Then there exists a number $\varepsilon_0 > 0$ and a family of metrics $\{\bar{g}_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0)}$ such that $\bar{g}_\varepsilon$ is scalar-flat with minimal boundary conditions and $\bar{g}_\varepsilon \to g_i$ on compact subsets of $M_i \setminus K$ for $i = 1, 2$ in the C^2-topology as $\varepsilon \to 0$.*

1.5. Introduction to Part II

In Part Two of this manuscript, we will first present a modification of the classical Schoen-Yau minimal hypersurface technique mentioned in Section 1.2 to manifolds with boundary. Next, we present an application of this technique to the study of psc-bordisms. Before we state our new results, let us first recall the details of the Schoen-Yau technique.

Theorem 1.5.1. [3, Proof of Theorem 1] *Let (Y, g) be a compact Riemannian manifold with $R_g > 0$, and $\dim Y = n \geq 3$. Let $X \subset Y$ be a smoothly embedded stable minimal hypersurface with trivial normal bundle. Then X admits a metric $\tilde{h}$ with $R_{\tilde{h}} > 0$. Furthermore, the metric $\tilde{h}$ can be chosen to be conformal to the restriction $g|_X$.*

We note that Theorem 1.5.1 is proven by analyzing the conformal Laplacian of the hypersurface X. It it crucial that X is stable minimal. For arbitrary (Y, g) it is a non-trivial problem to find a stable minimal hypersurface. For instance, if (Y, g) has positive Ricci curvature, then it cannot support a stable minimal hypersurfaces. However, in low dimensions, geometric measure theory can provide a source of stable minimal hypersurfaces.

Theorem 1.5.2. (See [14, Chapter 8], [15, Theorem 5.4.15]) *Let (Y, g) be a compact orientable Riemannian manifold with $3 \leq \dim Y = n \leq 7$. Assume $\alpha \in H_{n-1}(Y; \mathbb{Z})$ is a nontrivial element. Then there exists a smoothly embedded hypersurface $X \subset Y$ such that*

(i) up to multiplicity, X represents the class α;

(ii) X minimizes volume among all hypersurfaces which represent α up to multiplicity. In particular, the hypersurface X is stable minimal.

There are several important results based on Theorems 1.5.1 and 1.5.2. In particular, this gives a geometric proof that the torus T^n does not admit a metric of positive scalar curvature for $n \leq 7$; see [3]. This method was also crucial to provide the first counterexample to the Gromov-Lawson-Rosenberg conjecture; see [16].

1.51. The Schoen-Yau technique for manifolds with boundary

Let $(M, \bar{g})$ be a Riemannian manifold with non-empty boundary ∂M and $W \subset M$ be an embedded hypersurface. We say that a hypersurface W is *properly embedded* if, in addition, $\partial W = \partial M \cap W$. Such a hypersurface $W \subset M$ is *stable minimal with free boundary* if W is a local minimum of the volume functional among properly embedded hypersurfaces, see Section 3.11. We establish the following analogue of Theorem 1.5.1 for manifolds with boundary in Section 3.13.

Theorem 1.5.3. *Let $(M, \bar{g})$ be a compact Riemannian manifold with non-empty boundary ∂M, $R_{\bar{g}} > 0$, $H_{\bar{g}} \equiv 0$, and $\dim M = n + 1 \geq 3$. Let $W \subset M$ be an embedded stable minimal hypersurface with free boundary and trivial normal bundle. Then W admits a metric $\tilde{h}$ with $R_{\tilde{h}} > 0$ and $H_{\tilde{h}} \equiv 0$. Furthermore, the metric $\tilde{h}$ could be chosen to be conformal to the restriction $\bar{g}|_W$.*

The proof of Theorem 1.5.3 is similar to the case of closed manifolds. In particular, we have to analyze the conformal Laplacian on W with *minimal boundary conditions*. This boundary condition works well with the free boundary stability assumption.

For a compact oriented $(n+1)$-dimensional manifold M, we consider the relative integral homology group $H_n(M, \partial M; \mathbb{Z})$. Let $\bar{\alpha} \in H_n(M, \partial M; \mathbb{Z})$ be a non-trivial class which we may assume to be represented by a properly embedded hypersurface $W \subset M$. We notice that the boundary ∂W (which may possibly be empty) represents the class $\partial(\bar{\alpha}) \in H_{n-1}(\partial M; \mathbb{Z})$, where ∂ is the connecting homomorphism in the exact sequence

$$\cdots \to H_n(\partial M; \mathbb{Z}) \to H_n(M; \mathbb{Z}) \to H_n(M, \partial M; \mathbb{Z}) \xrightarrow{\partial} H_{n-1}(\partial M; \mathbb{Z}) \to \cdots \tag{1.7}$$

There is an analog of Theorem 1.5.2 which relies on a different regularity result, see Appendix A.3 for more details.

Theorem 1.5.4. (See [17, Theorem 5.2]) *Let $(M, \bar{g})$ be a compact orientable Riemannian manifold with non-empty boundary ∂M and $3 \leq \dim M = n + 1 \leq 7$. Assume $\bar{\alpha} \in H_n(M, \partial M; \mathbb{Z})$ is a nontrivial element. Then there exists a smooth properly embedded hypersurface $W \subset M$ such that*

(i) up to multiplicity, W represents the class $\bar{\alpha}$;

(ii) W minimizes volume with respect to $\bar{g}$ among all hypersurfaces which represent $\bar{\alpha}$ up to multiplicity. In particular, W is stable minimal with free boundary.

1.52. Positive scalar curvature bordism and minimal hypersurfaces

The main result of this paper is an application of Theorems 1.5.3 and 1.5.4 to provide new obstructions for psc-metrics to be psc-bordant.

Definition 1.5.1. Let (Y_0, g_0) and (Y_1, g_1) be closed oriented n-dimensional manifolds with psc-metrics. Then (Y_0, g_0) and (Y_1, g_1) are *psc-bordant* if there is a compact oriented $(n + 1)$-dimensional manifold $(Z, \bar{g})$ such that

- the manifold Z is an oriented bordism between Y_0 and Y_1, i.e., $\partial Z = Y_0 \sqcup -Y_1$;
- $\bar{g}$ is a psc-metric which restricts to $g_i + dt^2$ near the boundary $Y_i \subset \partial Z$ for $i = 0, 1$.

We write $(Z, \bar{g}) : (Y_0, g_0) \rightsquigarrow (Y_1, g_1)$ for a psc-bordism as above.

Sometimes we consider bordisms $(Z, \bar{g}) : (Y_0, g_0) \rightsquigarrow (Y_1, g_1)$ as above where the metrics do not necessarily have positive scalar curvature. However, we always assume that the metric $\bar{g}$ restricts to a product metric near the boundary.

Now we would like to enrich the psc-bordism relation with an extra structure, namely with a choice of homology classes $\alpha_i \in H_{n-1}(Y_i;\mathbb{Z})$, $i = 0,1$. Recall the following elementary observation. Let $\alpha \in H_{n-1}(Y;\mathbb{Z})$, where Y is an oriented closed n-dimensional manifold. Then the cohomology class $D\alpha \in H^1(Y;\mathbb{Z})$ Poincare-dual to α can be represented by a smooth map $\gamma : Y \to B\mathbb{Z} = S^1$. Furthermore, we can assume that a given point $s_0 \in S^1$ is a regular value for γ. It is easy to see that the inverse image $X_\gamma := \gamma^{-1}(s_0) \subset Y$ is an embedded hypersurface which represents the homology class α.

If M is an oriented $(n+1)$-dimensional manifold with a map $\bar{\gamma} : M \to S^1$, let $\gamma : \partial M \to S^1$ be the restriction $\bar{\gamma}|_{\partial M}$. There is a simple relation between the classes $[\bar{\gamma}] \in H^1(M;\mathbb{Z})$ and $[\gamma] \in H^1(\partial M;\mathbb{Z})$:

Lemma 1.5.5. *Let $\bar{\alpha} \in H_n(M,\partial M;\mathbb{Z})$ and $\alpha \in H_{n-1}(\partial M;\mathbb{Z})$ be Poincare dual to the classes $[\bar{\gamma}] \in H^1(M;\mathbb{Z})$ and $[\gamma] \in H^1(\partial M;\mathbb{Z})$. Then $\partial(\bar{\alpha}) = \alpha$, where*

$$\partial : H_n(M,\partial M;\mathbb{Z}) \to H_{n-1}(\partial M;\mathbb{Z})$$

is the connecting homomorphism. In particular, if $W = \bar{\gamma}^{-1}(s_0) \subset M$ is a smooth properly embedded hypersurface representing $\bar{\alpha}$, then the boundary ∂W represents the class α.

Definition 1.5.2. Let (Y_0, g_0) and (Y_1, g_1) be closed oriented n-dimensional Riemannian manifolds with given maps $\gamma_0 : Y_0 \to S^1$ and $\gamma_1 : Y_1 \to S^1$. We say that the triples (Y_0, g_0, γ_0) and (Y_1, g_1, γ_1) are *bordant* if there exists a bordism $(Z, \bar{g}) : (Y_0, g_0) \rightsquigarrow (Y_1, g_1)$ and a map $\bar{\gamma} : Z \to S^1$ such that $\bar{\gamma}|_{Y_i} = \gamma_i$ for $i = 0, 1$.

If the metrics g_0, g_1 and $\bar{g}$ are psc-metrics, we say that the triples (Y_0, g_0, γ_0) and (Y_1, g_1, γ_1) are *psc-bordant*. In both cases we use the notation

$$(Z, \bar{g}, \bar{\gamma}) : (Y_0, g_0, \gamma_0) \rightsquigarrow (Y_1, g_1, \gamma_1)$$

for such a bordism.

Theorem 1.5.6. *Let (Y_0, g_0) and (Y_1, g_1) be closed oriented connected n-dimensional manifolds with psc-metrics, $3 \leq n \leq 7$, and maps $\gamma_0 : Y_0 \to S^1$ and $\gamma_1 : Y_1 \to S^1$. Assume that (Y_0, g_0, γ_0) and (Y_1, g_1, γ_1) are psc-bordant.*

Then there exists a psc-bordism $(Z, \bar{g}, \bar{\gamma}) : (Y_0, g_0, \gamma_0) \rightsquigarrow (Y_1, g_1, \gamma_1)$ and a properly embedded hypersurface $W \subset Z$ such that

(i) the hypersurface W represents the class $\bar{\alpha} \in H_n(Z, \partial Z; \mathbb{Z})$ Poincare-dual to $[\bar{\gamma}] \in H^1(Z; \mathbb{Z})$;

(ii) the hypersurface $X_i := \partial W \cap Y_i \subset Y_i$ represents the class $\alpha_i \in H_{n-1}(Y_i; \mathbb{Z})$ Poincare-dual to $[\gamma_i] \in H^1(Y_i; \mathbb{Z})$, $i = 0, 1$;

(iii) there exists a metric $\bar{h}$ on W such that $R_{\bar{h}} > 0$ and $H_{\bar{h}} \equiv 0$ along ∂W, and $R_{h_i} > 0$, where $h_i = \bar{h}|_{X_i}$, in particular, $(W, \bar{h}) : (X_0, h_0) \rightsquigarrow (X_1, h_1)$ is a psc-bordism;

(iv) the metric $\bar{h}$ on W could be chosen to be conformal to the restriction $\bar{g}|_W$.

Remark 1.5.1. The psc-bordism $(Z, \bar{g}, \bar{\gamma})$ and hypersurface W may be chosen so that ∂W is arbitrarily C^k-close to a desired homologically volume minimizing representative of $\alpha_0 - \alpha_1$ for any k and $i = 0, 1$.

Recall a few definitions. We say that a conformal class C of metrics is *positive* if it contains a metric with positive scalar curvature. It is equivalent to the condition that the Yamabe constant $Y(X;C) > 0$. Now let W be a bordism with $\partial W = X_0 \sqcup X_1$, and C_0, C_1 be positive conformal classes on X_0, X_1 respectively. Then we say that the conformal manifolds (X_0, C_0) and (X_1, C_1) are positively conformally cobordant if the relative Yamabe invariant $Y(W, X_0 \sqcup X_1; C_0 \sqcup C_1) > 0$, see Section 3.3 for details. In these terms, the remark following Theorem 1.5.6 can be used to show the following:

Corollary 1.5.1. Let (Y_0, g_0, γ_0) and (Y_1, g_1, γ_1) be as in Theorem 1.5.6. Assume $X_i \subset Y_i$ are volume minimizing hypersurfaces representing homology classes Poincarè-dual to $[\gamma_i] \in H^1(X_i; \mathbb{Z})$, $i = 0, 1$. Then the conformal manifolds $(X_0, [g_0|_{X_0}])$ and $(X_1, [g_1|_{X_1}])$ are positively conformally cobordant.

The first step in the proof of Theorem 1.5.6 is to apply Theorem 1.5.4 to $\bar{\alpha}$, obtaining a minimal representative W. The main difficulty is that ∂W is, in general, not a minimal representative of $\partial\bar{\alpha}$ and so we may not apply Theorem 1.5.1 to conclude that ∂W even admits a psc-metric. However, in Section 3.2 we prove the Main Lemma, which states that ∂W becomes closer to minimizing $\partial\bar{\alpha}$ as longer collars are attached to the psc-bordism Z. This is the key step which allows us to produce the bordism in Theorem 1.5.6.

CHAPTER II

PART I: GLUING SCALAR-FLAT MANIFOLDS WITH BOUNDARY

In this chapter, we will more explicitly state and prove Theorem 1.4.2. Let us describe the main result, first in the case where gluing occurs along a submanifold embedded away from the boundary which we call an *interior embedding.* Let (M_1, g_1) and (M_2, g_2) be n-dimensional compact manifolds which are scalar-flat and have vanishing boundary mean curvatures. Moreover, suppose that each is equipped with an isometric embedding of a closed k-dimensional manifold (K, g_K), denoted by $\iota_* : K \to \mathring{M}_*$ $(* = 1, 2)$. Assuming that the isometry $\iota_1 \circ \iota_2^{-1}$ extends to an isomorphism of the normal bundles of K, we may form $M := M_1 \#_K M_2$, the generalized connected sum along K by removing small tubular neighborhoods and using the bundle isomorphism to identify annular regions (see Figure 1.1.).

In Section 2.1, we begin the construction by producing and studying a 1-parameter family of metrics g_ε on M transitioning between g_1 and g_2 on a neighborhood of the surgery site. The metrics g_ε can be thought of as attaching M_1 and M_2 by a thin, short K-shaped tube which becomes thinner as ε decreases. This family serves as a starting point for an iterative construction which produces a family of metrics conformal to g_ε, each scalar flat and of constant boundary mean curvature. More formally, we prove the following.

Theorem 2.0.7. *Let (M_1, g_1), (M_2, g_2) be compact n-dimensional manifolds with non-empty boundaries. Assume that*

$$R_{g_1} \equiv 0, \quad H_{g_1} \equiv 0, \quad R_{g_2} \equiv 0, \quad H_{g_2} \equiv 0, \quad \textit{and} \;\; \mathrm{Vol}_{g_1}(\partial M_1) = \mathrm{Vol}_{g_2}(\partial M_2).$$

Given isometric embeddings $\iota_1 : K \to \mathring{M}_1$, $\iota_2 : K \to \mathring{M}_2$ of a closed k-dimensional manifold (K, g_K) of codimension $m := n - k \geq 3$ with isomorphic normal bundles, there exists a family of scalar-flat metrics $\{\tilde{g}_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0)}$ (for some $\varepsilon_0 > 0$) on $M = M_1 \#_K M_2$ with constant boundary mean curvature $|H_{\tilde{g}_\varepsilon}| = \mathcal{O}(\varepsilon^{m-2})$. Moreover, for each ε, $\tilde{g}_\varepsilon$ is conformal to g_* away from a fixed tubular neighborhood of $\iota_*(K)$ in M_* and $\tilde{g}_\varepsilon \to g_*$ on compact sets of $M_* \setminus \iota_*(K)$ in the $\mathcal{C}^2$ topology as $\varepsilon \to 0$ for $* = 1, 2$.

The above codimension restriction allows spheres in fibers of the normal bundles to carry curvature, which will be required in our construction. If neither of the original manifolds (M_1, g_1), (M_2, g_2) are Ricci-flat with vanishing second fundamental form of the boundary, more can be accomplished – we may alter this construction in an ε-small non-conformal manner, so that the resulting metrics have vanishing boundary mean curvature.

Theorem 2.0.8. *Assume, in addition to the conditions in Theorem 2.0.7, that both manifolds (M_1, g_1) and (M_2, g_2) are not Ricci-flat with vanishing second fundamental form of their boundaries. Then there exists a second family of scalar-flat metrics $\{\hat{g}_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0)}$ on $M = M_1 \#_K M_2$ with vanishing boundary mean curvature. Moreover, $\hat{g}_\varepsilon \to g_*$ on compact sets of $M_* \setminus \iota_*(K)$ in the $\mathcal{C}^2$ topology as $\varepsilon \to 0$ for $* = 1, 2$.*

As mentioned earlier, we additionally consider gluing along boundaries i.e. when the embedding of K has a non-trivial intersection with ∂M_1 and ∂M_2. Carrying out the construction in this case requires substantial changes and new estimates. It is convenient to break into two further cases: that in which K is closed and embedded into the boundaries ∂M_* and that in which K itself has a boundary ∂K with $\mathring{K}$ and ∂K embedded into $\mathring{M}_*$ and ∂M_*, respectively. We will refer to the former as a *boundary embedding* and the latter as a *relative embedding*.

For boundary embeddings, we naturally require that the isometry $\iota_2 \circ \iota_1^{-1}$ extends to an isomorphism of the boundary normal bundles $\mathcal{N}(\iota_*(K)) \subset T\partial M_*$. Under this assumption, there is well-defined boundary connected sum along K, still denoted by $M = M_1 \#_K M_2$.

Theorem 2.0.9. *Let (M_1, g_1), (M_2, g_2) be as in Theorem 2.0.7 and suppose (K, g_K) is a closed manifold with isometric embeddings $\iota_1 : K \to \partial M_1$, $\iota_2 : K \to \partial M_2$ with $m = n - k \geq 3$. Assume that $\iota_2 \circ \iota_1^{-1}$ extends to an isomorphism of the normal bundles $\mathcal{N}(\iota_*(K)) \subset T\partial M_*$. Then there exists a family of scalar-flat metrics $\{\tilde{g}_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0)}$ with constant boundary mean curvature $H_{\tilde{g}_\varepsilon} = \mathcal{O}(\varepsilon^{m-2})$. Moreover, the metrics $\tilde{g}_\varepsilon$ are conformal to g_* away from a fixed tubular neighborhood of $\iota_*(K)$ in M_* and converge to the original metrics on compact sets of $M_* \setminus \iota_*(K)$ in the $\mathcal{C}^2$ topology as $\varepsilon \to 0$ for $* = 1, 2$.*

Theorem 2.0.10. *Assume, in addition to the conditions in Theorem 2.0.9, that both manifolds (M_1, g_1) and (M_2, g_2) are not Ricci-flat with vanishing second fundamental form of their boundaries. Then there exists a second family of scalar-flat metrics $\{\hat{g}\}_{\varepsilon \in (0,\varepsilon_0)}$ on $M = M_1 \#_K M_2$ with vanishing boundary mean curvature. Moreover, $\hat{g}_\varepsilon \to g_*$ on compact sets of $M_* \setminus \iota_*(K)$ in the $\mathcal{C}^2$ topology as $\varepsilon \to 0$ for $* = 1, 2$.*

The construction for a relative embedding, however, is a bit more delicate and we require additional assumptions on the embeddings ι_*.

Definition 2.0.3. We say that the isometric embeddings $\iota_* : K \to M_*$, $* = 1, 2$, are *surgery-ready* if

(i) ι_* is a proper embedding, i.e., $\iota_*(\mathring{K}) \subset \mathring{M}_*$ and $\iota_*(\partial K) \subset \partial M_*$;

(ii) the mean curvature H_{g_K} vanishes;

(iii) there is a neighborhood, $V \subset K$, of ∂K such that the embedding $\iota_*(K)$ agrees with the g_*-exponential map on $\iota_*(\partial K)$ (see Figure 2.4.);

(iv) the map $\iota_2 \circ \iota_1^{-1}$ extends to an isomorphism of the normal bundles $\mathcal{N}_1(K)$, $\mathcal{N}_2(K)$ which restricts to an isomorphism of the boundary normal bundles $\mathcal{N}_1(\partial K)$, $\mathcal{N}_2(\partial K)$.

Assuming the embeddings $\iota_* : K \to M_*$ are surgery-ready, there is a well-defined generalized connected sum $M = M_1 \#_K M_2$ along K, see Section 2.1 for details. Precisely, we have the following pair of theorems.

Theorem 2.0.11. *Let (M_1, g_1), (M_2, g_2) be as in Theorem 2.0.7 and (K, g_K) be a compact manifold with boundary. Assume $\iota_1 : K \to M_1$, $\iota_2 : K \to M_2$ are surgery ready isometric embeddings as above with $m = n - k \geq 3$. Then there exists a family of scalar-flat metrics $\{\tilde{g}_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0)}$ on $M = M_1 \#_K M_2$ with constant boundary mean curvature $H_{\tilde{g}_\varepsilon} = \mathcal{O}(\varepsilon^{m-2})$. Moreover, the metrics $\tilde{g}_\varepsilon$ are conformal to g_* away from a fixed tubular neighborhood of $\iota_*(K)$ in M_* and converge to the original metrics on compact sets of $M_* \setminus \iota_*(K)$ in the $\mathcal{C}^2$ topology as $\varepsilon \to 0$ for $* = 1, 2$.*

Theorem 2.0.12. *Assume, in addition to the conditions in Theorem 2.0.11, that both manifolds (M_1, g_1) and (M_2, g_2) are not Ricci-flat with vanishing second fundamental form of their boundaries. Then there exists a second family of scalar-flat metrics $\{\hat{g}_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0)}$ on $M = M_1 \#_K M_2$ with vanishing boundary mean curvature. Moreover, $\hat{g}_\varepsilon \to g_*$ on compact sets of $M_* \setminus \iota_*(K)$ in the $\mathcal{C}^2$ topology as $\varepsilon \to 0$ for $* = 1, 2$.*

2.1. Construction of g_ε and Main Technical Results

In this section, we construct the generalized connected sum $M = M_1 \#_K M_2$ and define a family of metrics $\{g_\varepsilon\}_{\varepsilon \in (0, \frac{1}{2})}$ on M. At this point, it is convenient to consider the cases of interior, boundary, and relative embeddings separately. The next step is to give pointwise and integral estimates for the scalar and boundary mean curvatures of the new metrics $\{g_\varepsilon\}_{\varepsilon\in(0,\frac{1}{2})}$. Finally, we study the family of operators Δ_{g_ε}, giving a local a priori estimate for solutions of the Δ_{g_ε}-Poisson equation.

In Section 2.11, we describe the process for interior embeddings, revisiting the construction in [10]. In this case, the g_*-exponential map identifies, for some small $r > 0$, the distance neighborhood

$$V_*^r := \{y \in M_* \colon \mathrm{dist}_{g_*}(y, \iota_*(K)) < r\}$$

with the portion of the normal bundle $\{w \in \mathcal{N}_*(K) \colon ||w||_{g_*} < r\}$. On V_*^r, these Fermi coordinates yield good asymptotic expressions for the metric tensor g_*. These local expressions are then used to transition from g_1 to g_2 on annular regions about $\iota_1(K)$ and $\iota_2(K)$, in turn yielding a globally-defined metric g_ε on the sum, M, for each $\varepsilon \in (0, \frac{1}{2})$.

In the case of boundary and relative embeddings, however, there are two sorts of geodesics which must be used to visit all of the neighborhood V_*^r from $\iota_*(K)$ – those of g_* and those of $g_*|_{\partial M_*}$. This complicates matters and we must provide new geometric constructions and estimates for a Poisson problem with mixed Dirichlet-Neuman boundary conditions. This analysis for boundary and relative embeddings is carried out in sections 2.12 and 2.13, respectively.

2.11. Interior embeddings

Throughout this section we will only consider the case of interior embeddings; when K is closed and embedded entirely within the interior $\mathring{M}_*$. By uniformly rescaling the metrics g_1 and g_2, we may assume that

$$\exp^{g_*}: \{w \in \mathcal{N}_*(K)\colon ||w||_{g_*} < 1\} \to M_*$$

is a diffeomorphism onto its image. For a fixed $\varepsilon \in (0, \frac{1}{2})$, we will give a local description of a gluing metric g_ε on the disjoint union

$$\left(M_1 \setminus V_1^{\varepsilon^2}\right) \sqcup \left(M_2 \setminus V_2^{\varepsilon^2}\right).$$

This description will, in fact, immediately yield a globally defined metric g_ε on the above disjoint union. We will then construct the connected sum $M_1 \#_K M_2$ in such a way so that the metric g_ε descends to it.

Let $U \subset K$ be a trivializing neighborhood for the normal bundles $\mathcal{N}_1(K)$ and $\mathcal{N}_2(K)$ with local coordinates $z = (z^1, \ldots, z^k)$. Denote the open unit m-ball by

$$D^m = \{x = (x^1, \ldots, x^m) \in \mathbb{R}^m\colon |x| < 1\}.$$

The map

$$F_*: U \times D^m \to M_*, \quad F_*(z, x) := \exp^{g_*}_{\iota_*(z)}(x)$$

gives Fermi coordinates (z, x) on a neighborhood of $\iota_*(U)$ in M_* for $* = 1, 2$. Abusing notations, we write (z, x) for the coordinates on both M_1, M_2 and suppress the use of the bundle isomorphism in identifying the trivializations over U. These coordinates give the following local expression for the metric g_*

$$g_* = g^{(*)}_{ij} dz^i dz^j + g^{(*)}_{i\alpha} dz^i dx^\alpha + g^{(*)}_{\alpha\beta} dx^\alpha dx^\beta$$

with the well-known expansions

$$g_{ij}^{(*)}(z,x) = g_{ij}^K(z) + \mathcal{O}(|x|), \quad g_{i\alpha}^{(*)}(z,x) = \mathcal{O}(|x|), \quad g_{\alpha\beta}^{(*)}(z,x) = \delta_{\alpha\beta} + \mathcal{O}(|x|^2).$$

Setting $x = \varepsilon e^{-t}\theta$ on M_1 and $x = \varepsilon e^t \theta$ on M_2, we introduce modified polar coordinates (z,t,θ) on a neighborhood about $\iota_*(U)$ in M_* for $* = 1,2$ where $\theta = (\theta^1, \ldots, \theta^{m-1})$ are spherical coordinates for the unit sphere S^{m-1} and $t \in (\log\varepsilon, -\log\varepsilon)$. Notice that t ranges between the values $\log\varepsilon$ and $-\log\varepsilon$ as $|x|$ ranges between ε^2 and 1. We define two functions $u_\varepsilon^{(1)}, u_\varepsilon^{(2)} : (\log\varepsilon, -\log\varepsilon) \to \mathbb{R}$ by

$$u_\varepsilon^{(1)}(t) := \varepsilon^{\frac{m-2}{2}} e^{-\frac{m-2}{2}t} \quad \text{and} \quad u_\varepsilon^{(2)}(t) := \varepsilon^{\frac{m-2}{2}} e^{\frac{m-2}{2}t}.$$

Using the coordinates (z,t,θ), the local expression for g_* can be reorganized in the form

$$\begin{aligned} g_* = \;& g_{ij}^{(*)} dz^i dz^j + \left(u_\varepsilon^{(*)}\right)^{\frac{4}{m-2}} \left(g_{tt}^{(*)} dt^2 + g_{\lambda\mu}^{(*)} d\theta^\lambda d\theta^\mu + g_{t\lambda}^{(*)} dt d\theta^\lambda\right) \\ & + g_{it}^{(*)} dz^i dt + g_{i\lambda}^{(*)} dz^i d\theta^\lambda. \end{aligned}$$

The asymptotics now take the form

$$\begin{array}{lll} g_{ij}^{(*)}(z,t,\theta) = g_{ij}^K(z) + \mathcal{O}(|x|), & g_{\lambda\mu}^{(*)}(z,t,\theta) = g_{\lambda\mu}^{(\theta)}(\theta) + \mathcal{O}(|x|), & g_{tt}^{(*)}(z,t,\theta) = 1 + \mathcal{O}(|x|^2) \\ g_{i\lambda}^{(*)}(z,t,\theta) = \mathcal{O}(|x|^2), & g_{it}^{(*)}(z,t,\theta) = \mathcal{O}(|x|^2), & g_{i\lambda}^{(*)}(z,t,\theta) = \mathcal{O}(|x|^2) \end{array}$$

where $g_{\lambda\mu}^{(\theta)}$ denotes a component of the standard round metric on the unit sphere S^{m-1} in the spherical coordinates $(\theta^1, \ldots, \theta^{m-1})$.

We are now ready to perform the interpolation between g_1 and g_2. Fix a cut-off smooth function $\xi : (\log\varepsilon, -\log\varepsilon) \to [0,1]$ which is non-increasing and takes the value 1 on $(\log\varepsilon, -1]$ and 0 on $[1, -\log\varepsilon)$. Similarly, let $\eta : (\log\varepsilon, -\log\varepsilon) \to [0,1]$ be a non-increasing, smooth function which takes the value 1 on $(\log\varepsilon, -\log\varepsilon - 1]$ and the value 0 on $(-\log\varepsilon - \frac{1}{2}, -\log\varepsilon)$.

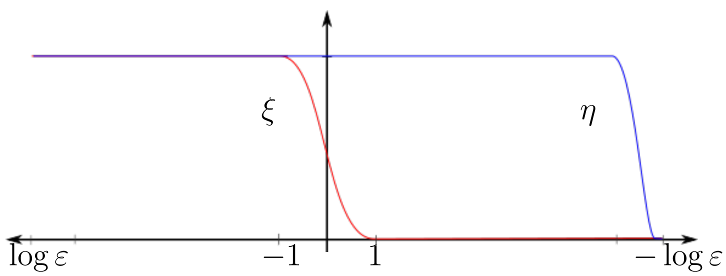

FIGURE 2.1. The cut-off functions ξ and η

Define a function $u_\varepsilon : (\log\varepsilon, -\log\varepsilon) \to \mathbb{R}$ by

$$u_\varepsilon(t) = \eta(t)u_\varepsilon^{(1)} + \eta(-t)u_\varepsilon^{(2)}.$$

Finally, for each $\varepsilon \in (0, \frac{1}{2})$, define a metric g_ε by

$$\begin{aligned} g_\varepsilon(z,t,\theta) = \quad & (\xi g_{ij}^{(1)} + (1-\xi)g_{ij}^{(2)})dz^i dz^j + u_\varepsilon^{\frac{4}{n-2}}\Big((\xi g_{tt}^{(1)} + (1-\xi)g_{tt}^{(2)})dt^2 \\ & +(\xi g_{\lambda\mu}^{(1)} + (1-\xi)g_{\lambda\mu}^{(2)})d\theta^\lambda d\theta^\mu + (\xi g_{t\lambda}^{(1)} + (1-\xi)g_{t\lambda}^{(2)})dtd\theta^\lambda]\Big) \\ & +(\xi g_{it}^{(1)} + (1-\xi)g_{it}^{(2)})dz^i dt + (\xi g_{i\lambda}^{(1)} + (1-\xi)g_{i\lambda}^{(2)})dz^i d\theta^\lambda. \end{aligned}$$

This defines a metric g_ε on the tubular annuli

$$V_*^1 \setminus \overline{V_*^{\varepsilon^2}} = \{y \in M_* | \varepsilon^2 < \mathrm{dist}_{g_*}(y, \iota_*(K)) < 1\}$$

for $* = 1, 2$. We set $g_\varepsilon = g_*$ on $M_* \setminus \overline{V_*^1}$. This gives well-defined metric g_ε on the disjoint union $(M_1 \setminus V_*^{\varepsilon^2}) \sqcup (M_2 \setminus V_*^{\varepsilon^2})$.

Now we are ready to describe the generalized connected sum $M = M_1 \#_K M_2$. See Figure 2.3. for a picture in the boundary embedding case. Let $\Phi : \mathcal{N}_1(K) \to \mathcal{N}_2(K)$ be the isomorphism of the normal bundles given in the hypothesis of Theorem 2.0.7.

For each $\varepsilon \in (0, \frac{1}{2})$, consider the auxiliary fiber-wise mapping Ψ_ε given by

$$\Psi_\varepsilon : (\mathcal{N}_1(K) \setminus \{0\}) \sqcup (\mathcal{N}_2(K) \setminus \{0\}) \to (\mathcal{N}_1(K) \setminus \{0\}) \sqcup (\mathcal{N}_2(K) \setminus \{0\})$$

$$\Psi_\varepsilon(z, t, \theta) := \begin{cases} \Phi(z, -t, \theta) & \text{if } (z, t, \theta) \in \mathcal{N}_1(K) \\ \Phi^{-1}(z, -t, \theta) & \text{if } (z, t, \theta) \in \mathcal{N}_2(K). \end{cases}$$

Notice that, in the Fermi coordinates (z, x), this mapping can be expressed as $\Psi_\varepsilon(z, x) = \Phi_\varepsilon(z, \frac{\varepsilon^2}{|x|^2} x)$. We define

$$M_\varepsilon := \Big((M_1 \setminus V_*^{\varepsilon^2}) \sqcup (M_2 \setminus V_*^{\varepsilon^2}) \Big) \, / \sim_\varepsilon$$

where we introduce the equivalence relation $\sim_\varepsilon$ on the disjoint union

$$\Big(V_1^1 \setminus \overline{V_1^{\varepsilon^2}} \Big) \sqcup \Big(V_2^1 \setminus \overline{V_2^{\varepsilon^2}} \Big)$$

as follows: If $y \in V_1^1 \setminus \overline{V_1^{\varepsilon^2}}$, then $y \sim_\varepsilon (F_2 \circ \Psi_\varepsilon \circ F_1^{-1})(y)$.

Observing that g_ε is invariant under Ψ_ε, the metric descends to M_ε. We will continue to denote this metric by g_ε. Since its diffeomorphism type does not depend on ε, we will drop the subscript when referring to the generalized connected sum and simply write $M = M_\varepsilon$. This finishes the definition of the family of Riemannian manifolds (M, g_ε). The coordinates (z, t, θ) which were originally used on M_1 will continue to be used as coordinates on M. We will require a piece of notation for certain subsets of the gluing region in M: For each $\varepsilon > 0$ and $a, b \geq 0$, we denote by

$$T^\varepsilon(a, b) = \{(z, t, \theta) \in M \colon \ \log \varepsilon + a \leq t \leq -\log \varepsilon - b\}.$$

Before we approach the problem of producing a solution to the system (1.6) on (M, g_ε), we will require two geometrical properties of the family $\{g_\varepsilon\}_{\varepsilon\in(0,\frac{1}{2})}$. In the present case of interior embeddings, these properties are identical to those found in [10]. Propositions 2.1.1 and 2.1.2 summarize the results of [10, Section 4].

Proposition 2.1.1. (cf. [10, Proposition 2]) *There is a constant $C > 0$ such that*

$$|R_{g_\varepsilon}| \le C\varepsilon^{-1}\cosh^{1-m}(t)$$

on $T^\varepsilon(0,0)$ and

$$\int_M |R_{g_\varepsilon}| d\mu_{g_\varepsilon} = \mathcal{O}(\varepsilon^{m-2}).$$

Moreover, the constant C depends only on $(K, g_K), (M_1, g_1),$ and (M_2, g_2).

The other feature of g_ε we will need is an ε-uniform a priori estimate for solutions of the Δ_{g_ε}-Poisson equation on the neck. Indeed, the family of operators $\{\Delta_{g_\varepsilon}\}_{\varepsilon\in(0,\varepsilon_0)}$ is not uniformly elliptic and the estimate is tailor made for the family of metrics g_ε. To state it, we will fix a family of weighting functions $\psi_\varepsilon : M \to \mathbb{R}$ satisfying

$$\psi_\varepsilon = \begin{cases} \varepsilon\cosh(t) & \text{on } T^\varepsilon(1,1) \\ 1 & \text{on } M \setminus T^\varepsilon(0,0) \end{cases}$$

and varying smoothly between the values on $T^\varepsilon(0,0)\setminus T^\varepsilon(1,1) \subset M$ (see Figure 2.2.). For a given parameter $\gamma \in (0, m-2)$ consider the following weighted Banach spaces

$$\mathcal{C}^0_\gamma(M) := \{v \in \mathcal{C}^0(M) \colon ||v||_{\mathcal{C}^0_\gamma(M)} := \sup_M |\psi_\varepsilon^\gamma v| < \infty\}.$$

Note that, for fixed ε, γ, the two norms $||\cdot||_{\mathcal{C}^0_\gamma(M)}$ and $\sup_M |\cdot|$ are equivalent, though the equivalence is not uniform in ε.

Proposition 2.1.2. (cf. [10, Proposition 4]) *Given* $\gamma \in (0, m-2)$, *there are constants* $\alpha_1, \alpha_2 > 0$ *and* $C > 0$ *satisfying the following statement for all* $\varepsilon \in (0, e^{-\max\{\alpha_1,\alpha_2\}})$. *If* $v, f \in \mathcal{C}^0(T^\varepsilon(\alpha_1, \alpha_2))$ *satisfy* $\Delta_{g_\varepsilon} v = f$, *then*

$$v \leq C\psi_\varepsilon^{-\gamma}\left(\sup_{T^\varepsilon(\alpha_1,\alpha_2)} |\psi_\varepsilon^{\gamma+2} f| + \sup_{\partial T^\varepsilon(\alpha_1,\alpha_2)} |\psi_\varepsilon^\gamma v|\right)$$

pointwise on $T^\varepsilon(\alpha_1, \alpha_2)$ *and*

$$||v||_{\mathcal{C}^0_\gamma(T^\varepsilon(\alpha_1,\alpha_2))} \leq C\left(||f||_{\mathcal{C}^0_{\gamma+2}(T^\varepsilon(\alpha_1,\alpha_2))} + ||v||_{\mathcal{C}^0_\gamma(\partial T^\varepsilon(\alpha_1,\alpha_2))}\right).$$

Moreover, the constants α_1, α_2, *and* C *depend only on* $\gamma, (K, g_K), (M_1, g_1)$, *and* (M_2, g_2).

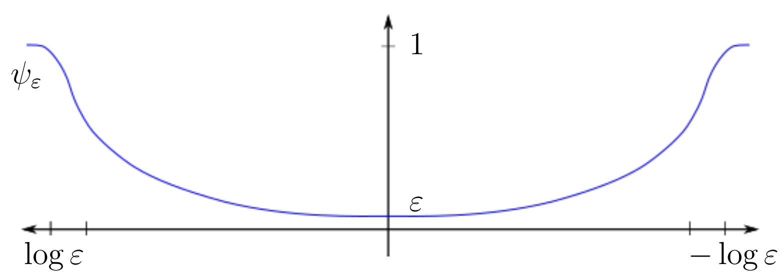

FIGURE 2.2. The weighting function ψ_ε

2.12. Boundary embeddings

In this section, we consider the setting of Theorems 2.0.9 and 2.0.10 – when $\iota_*(K)$ lies entirely within ∂M_*. As in Section 2.11, we begin by defining the family of metrics $\{g_\varepsilon\}_{\varepsilon\in(0,\frac{1}{2})}$. After uniformly rescaling the metrics g_1 and g_2, we may assume that both

$$\begin{aligned}\exp^{g_*|_{\partial M_*}} :&\{w \in \mathcal{N}^\partial_*(K) \colon ||w||_{g_*} < 1\} \to \partial M_* \\ \exp^{g_*} :&\{w \in \mathcal{N}(\partial M_*) \colon ||w||_{g_*} < 1\} \to M_*\end{aligned}$$

are diffeomorphisms onto their images for $* = 1, 2$.

Let $U \subset K$ be a trivializing neighborhood for the bundles $\mathcal{N}^\partial_1(K)$ and $\mathcal{N}^\partial_2(K)$ with local coordinates $z = (z^1, \ldots, z^k)$. The map

$$F'_* : U \times D^{m-1} \to \partial M_*, \quad F'_*(z, x') := \exp^{g_*|_{\partial M_*}}_{\iota_*(z)}(x')$$

gives Fermi coordinates (z, x') for the boundary ∂M_*. We denote the upper unit m-ball by

$$D^m_+ := \{(x', x^m) \in D^{m-1} \times \mathbb{R} \colon |(x', x^m)| < 1 \text{ and } x^m \geq 0\}.$$

We identify the last component of D^m_+ with the inward normal $\mathcal{N}(\partial M_*)$. Now the map

$$F_* : U \times D^m_+ \to M_*, \quad F(z, x', x^m) := \exp^{g_*}_{F'_*(z,x')}(x^m)$$

gives coordinates (z, x', x^m) on a neighborhood of $\iota_*(U)$ in M_* for $* = 1, 2$. We will write $x = (x', x^m)$ and $|x| := \sqrt{|x'|^2 + |x^m|^2}$. In the coordinates (z, x), the metric can be written as

$$g_* = g^{(*)}_{ij} dz^i dz^j + g^{(*)}_{k\gamma} dz^k dx^\gamma + g^{(*)}_{\alpha\beta} dx^\alpha dx^\beta$$

with the following well-known expansions

$$g^{(*)}_{ij}(z,x) = g^K_{ij}(z) + \mathcal{O}(|x|), \quad g^{(*)}_{k\gamma}(z,x) = \mathcal{O}(|x|), \quad g^{(*)}_{\alpha\beta}(z,x) = \delta_{\alpha\beta} + \mathcal{O}(|x|).$$

We again introduce modified polar coordinates (z,t,θ) by setting $x = \varepsilon e^{-t}\theta$ on M_1 and $x = \varepsilon e^{t}\theta$ on M_2. Here $\theta = (\theta^1, \ldots, \theta^{m-1})$ are spherical coordinates on the unit upper hemisphere

$$S^{m-1}_+ := \{\theta \in S^{m-1} \colon 0 \le \theta^1 \le \frac{\pi}{4}\}$$

and $t \in (\log\varepsilon, -\log\varepsilon)$. Notice that the boundary ∂S^{m-1}_+ can be identified with the set $\{\theta \in S^{m-1} \colon \theta^1 = \frac{\pi}{4}\}$. Using the coordinates (z,t,θ), the local expression for g_* can be reorganized in the form

$$\begin{aligned} g_* = \; & g^{(*)}_{ij} dz^i dz^j + \left(u^{(*)}_\varepsilon\right)^{\frac{4}{m-2}} \left(g^{(*)}_{tt} dt^2 + g^{(*)}_{\lambda\mu} d\theta^\lambda d\theta^\mu + g^{(*)}_{t\lambda} dt d\theta^\lambda\right) \\ & + g^{(*)}_{it} dz^i dt + g^{(*)}_{i\lambda} dz^i d\theta^\lambda \end{aligned}$$

where $u^{(*)}_\varepsilon$ are defined as in Section 2.11. The asymptotics now take the form

$$\begin{array}{lll} g^{(*)}_{ij}(z,t,\theta) = g^K_{ij}(z) + \mathcal{O}(|x|), & g^{(*)}_{\lambda\mu}(z,t,\theta) = g^{(\theta)}_{\lambda\mu}(\theta) + \mathcal{O}(|x|), & g^{(*)}_{tt}(z,t,\theta) = 1 + \mathcal{O}(|x|) \\ g^{(*)}_{i\lambda}(z,t,\theta) = \mathcal{O}(|x|), & g^{(*)}_{it}(z,t,\theta) = \mathcal{O}(|x|), & g^{(*)}_{i\lambda}(z,t,\theta) = \mathcal{O}(|x|) \end{array}$$

where $g^{(\theta)}_{\lambda\mu}$ denotes a component of the standard round metric on the upper unit hemisphere S^{m-1}_+ in the spherical coordinates $(\theta^1, \ldots, \theta^{m-1})$.

Using the same cutoff functions ξ and η we introduced in the case of interior embeddings, define the function u_ε as in Section 2.11. For each $\varepsilon \in (0, \frac{1}{2})$, set

$$\begin{aligned} g_\varepsilon(z,t,\theta) = \quad & (\xi g^{(1)}_{ij} + (1-\xi) g^{(2)}_{ij}) dz^i dz^j + u_\varepsilon^{\frac{4}{n-2}} \Big((\xi g^{(1)}_{tt} + (1-\xi) g^{(2)}_{tt}) dt^2 \\ & + (\xi g^{(1)}_{\lambda\mu} + (1-\xi) g^{(2)}_{\lambda\mu}) d\theta^\lambda d\theta^\mu + (\xi g^{(1)}_{t\lambda} + (1-\xi) g^{(2)}_{t\lambda}) dt d\theta^\lambda] \Big) \\ & + (\xi g^{(1)}_{it} + (1-\xi) g^{(2)}_{it}) dz^i dt + (\xi g^{(1)}_{i\lambda} + (1-\xi) g^{(2)}_{i\lambda}) dz^i d\theta^\lambda. \end{aligned}$$

This defines a metric g_ε on the tubular annuli $V^1_* \setminus \overline{V^{\varepsilon^2}_*}$ for $* = 1, 2$. We set $g_\varepsilon = g_*$ on $M_* \setminus \overline{V^1_*}$. This gives well-defined metric g_ε on the disjoint union $(M_1 \setminus V^{\varepsilon^2}_1) \sqcup (M_2 \setminus V^{\varepsilon^2}_2)$.

Now we are ready to describe the generalized connected sum $M = M_1 \#_K M_2$. See Figure 2.3. for a visual description. Let $\Phi : \mathcal{N}_1(K) \to \mathcal{N}_2(K)$ be the isomorphism of the normal bundles given in the hypothesis of Theorem 2.0.9. For each $\varepsilon \in (0, \frac{1}{2})$, consider mapping Ψ_ε given by

$$\Psi_\varepsilon : (\mathcal{N}_1(K) \setminus \{0\}) \sqcup (\mathcal{N}_2(K) \setminus \{0\}) \to (\mathcal{N}_1(K) \setminus \{0\}) \sqcup (\mathcal{N}_2(K) \setminus \{0\})$$

$$\Psi_\varepsilon(z,t,\theta) := \begin{cases} \Phi(z, -t, \theta) & \text{if } (z,t,\theta) \in \mathcal{N}_1(K) \\ \Phi^{-1}(z, -t, \theta) & \text{if } (z,t,\theta) \in \mathcal{N}_2(K). \end{cases}$$

We define

$$M := \Big((M_1 \setminus V^{\varepsilon^2}_*) \sqcup (M_2 \setminus V^{\varepsilon^2}_*) \Big) / \sim_\varepsilon$$

where we introduce equivalence relation $\sim_\varepsilon$ on the disjoint union

$$\Big(V^1_1 \setminus \overline{V^{\varepsilon^2}_1} \Big) \sqcup \Big(V^1_2 \setminus \overline{V^{\varepsilon^2}_2} \Big)$$

as follows: If $y \in V^1_1 \setminus \overline{V^{\varepsilon^2}_1}$, then $y \sim_\varepsilon (F_2 \circ \Psi_\varepsilon \circ F_1^{-1})(y)$.

Observing that g_ε is invariant under Ψ_ε, the metric descends to M. This finishes the definition of the family of Riemannian manifolds (M, g_ε).

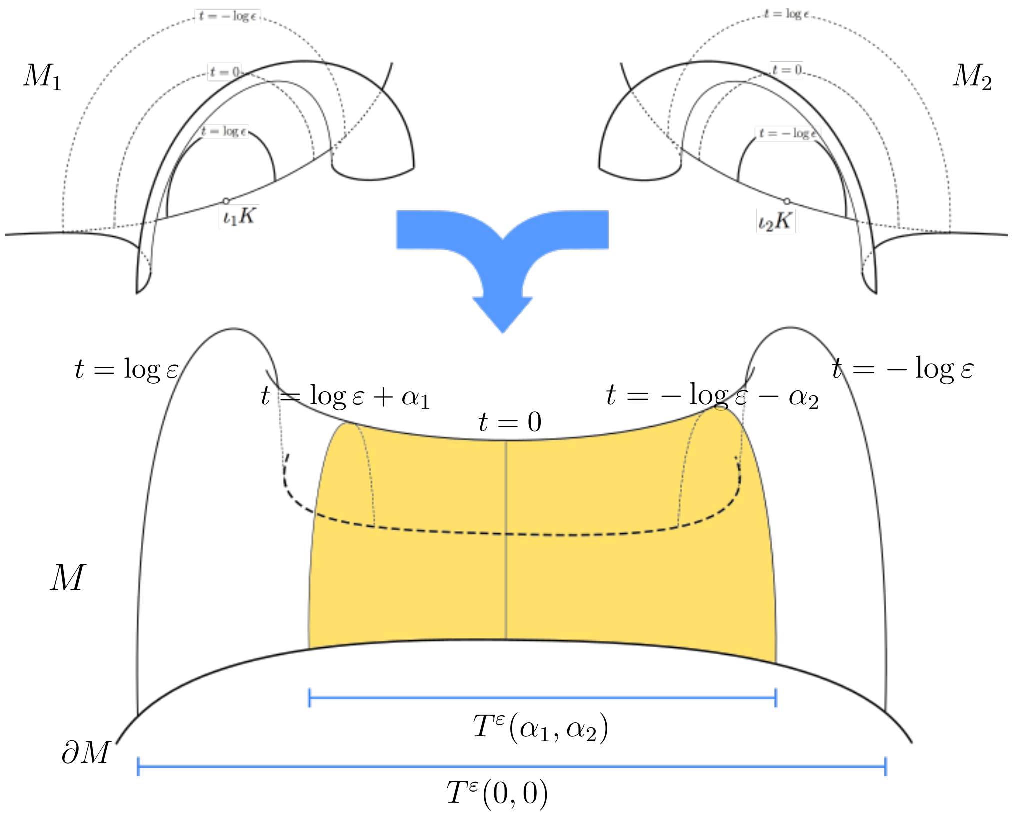

FIGURE 2.3. The construction of (M, g_ε) and the neck region $T^\varepsilon(\alpha_1, \alpha_2)$

2.121. The scalar and boundary mean curvatures of g_ε

The next step is to produce analogs of Propositions 2.1.1 and 2.1.2 for the case of boundary embeddings. In addition to the estimate for the scalar curvature R_{g_ε}, we will require a similar estimate for the boundary mean curvature H_{g_ε}.

Proposition 2.1.3. *There is a constant $C > 0$, independent of ε, such that*

$$|R_{g_\varepsilon}| \leq C\varepsilon^{-1}\cosh^{1-m}(t), \quad |H_{g_\varepsilon}| \leq C\cosh^{2-m}(t)$$

on $T^\varepsilon(0,0)$ *and*

$$\int_M |R_{g_\varepsilon}| d\mu_{g_\varepsilon} = \mathcal{O}(\varepsilon^{m-2}), \quad \int_{\partial M} |H_{g_\varepsilon}| d\sigma_{g_\varepsilon} = \mathcal{O}(\varepsilon^{m-2}).$$

Proof. The estimate on R_{g_ε} can be obtained by an argument identical to the one found in [10] so we will only present the estimate on H_{g_ε}.

Let us first restrict our attention to the portion of $T^\varepsilon(0,0)$ where $\log\varepsilon + 1 \leq t \leq -1$. On this portion of the neck the cut off function ξ takes take the value 1 and g_ε take the form

$$\begin{aligned} g_\varepsilon(z,x) =& g^{(1)}_{ij}(z,x)dz^i dz^j + (1+\varepsilon^{m-2}|x|^{2-m})^{\frac{4}{m-2}} g^{(1)}_{\alpha\beta}(z,x)dx^\alpha dx^\beta \\ &+ g^{(1)}_{i\gamma}(z,x)dz^i dx^\gamma. \end{aligned}$$

We will drop the upper indices and write $g_{ij} = g^{(1)}_{ij}$, unless otherwise mentioned.

It will be useful to introduce a new formal parameter $\phi > 0$ and introduce the following two metrics on the neck $T^\varepsilon(0,0)$

$$\begin{aligned} g(z,x,\phi) &= g^{(1)}_{ij}(z,x)dz^i dz^j + (1+\phi)^{\frac{4}{m-2}} g^{(1)}_{\alpha\beta}(z,x)dx^\alpha dx^\beta + g^{(1)}_{i\gamma}(z,x)dz^i dx^\gamma \\ \tilde{g}(z,\phi) &= g^K_{ij}(z)dz^i dz^j + (1+\phi)^{\frac{4}{m-2}}\delta_{\alpha\beta}dx^\alpha dx^\beta \end{aligned}$$

If we choose $\phi = \varepsilon^{m-2}|x|^{2-m}$ in the formula for $g(z,x,\phi)$, observe that we recover the gluing metric g_ε. Furthermore, we obtain the original metric g_1 if we take $\phi = 0$ in the formula for $g(z,x,\phi)$. Our goal is to compute the boundary mean curvatures of the product metrics $\tilde{g}(z,\phi)$ and $\tilde{g}(z,0)$ then compare them to the corresponding curvatures of $g(z,x,\phi)$ and $g(z,x,0)$ in order to arrive at the desired estimate.

The Taylor expansions for the metric components now take the form

$$g_{ij}(z,x,\phi) = \tilde{g}_{ij}(z,\phi) + \mathcal{O}(|x|), \quad g_{\alpha\beta}(z,x,\phi) = \tilde{g}_{\alpha\beta}(z,\phi) + \mathcal{O}(|x|),$$

$$g_{i\alpha}(z,x,\phi) = \mathcal{O}(|x|)$$

Inspired by [10], it will be convenient to adopt the following variant of big-o notation.

Definition 2.1.1. Let $a \in \mathbb{N}_0$ and let f be a function of z, x, and ϕ. We say f belongs to the class $\mathcal{A}_a$ if

$$|f(z,x,\phi)| \leq C|x|^a \quad \text{and} \quad |f(z,x,\phi) - f(z,x,0)| \leq C|x|^a|\phi|$$

for some constant $C > 0$.

Notice that the product of an $\mathcal{A}_a$ function with an $\mathcal{A}_b$ function lies in the class $\mathcal{A}_{a+b}$. For the coefficients of the inverse of g_ϕ, we may write

$$g^{ij}(z,x,\phi) = \tilde{g}^{ij}(z,\phi) + \mathcal{A}_1, \quad g^{\alpha\beta}(z,x,\phi) = \tilde{g}^{\alpha\beta}(z,\phi) + \mathcal{A}_1, \quad g^{i\alpha}(z,x,\phi) = \mathcal{A}_1.$$

Continuing, for any derivative of a component of $g(z,x,\phi)$, we have

$$\partial_a g_{rs}(z,x,\phi) = \partial_a \tilde{g}_{rs}(z,\phi) + \mathcal{A}_0 + |\nabla\phi|\mathcal{A}_1$$

where $g_{rs}(z,x,\phi)$ may be any component of $g(z,x,\phi)$ in the coordinates (z,x) and ∂_a may be any derivative with respect to z^i $(i = 1,\ldots,k)$ or x^α $(\alpha = 1,\ldots,m)$. Writing Γ for a Christoffel symbol of $g(z,x,\phi)$ and $\tilde{\Gamma}$ for the corresponding symbol of $\tilde{g}(z,x)$, one may use the above computation with the Kozul formula to find

$$\Gamma = \tilde{\Gamma} + \mathcal{A}_0 + |\nabla\phi|\mathcal{A}_1.$$

Now consider the product metric $\tilde{g}(z,\phi)$. We have $H_{\tilde{g}(z,0)} = 0$ since the boundary mean curvature of $(B^m_+(0), \delta_{\alpha\beta})$ vanishes. Using the formula for boundary mean curvature under conformal change,

$$\begin{aligned} H_{\tilde{g}(z,\phi)} &= \frac{1}{2c_n}(1+\phi)^{\frac{-m}{m-2}}\partial_\nu\phi \\ &= -\frac{m-2}{2c_n}\lim_{x^m\to 0}(1+\phi)^{\frac{-m}{m-2}}\varepsilon^{m-2}|x|^{-m}(x^m) \\ &= 0, \end{aligned}$$

where x^m is the last coordinate of x. Next we compute $H_{g(z,x,\phi)}$ in terms of $H_{\tilde{g}(z,\phi)}$ using the above expressions for the Christoffel symbols

$$\begin{aligned} H_{g(z,x,\phi)} &= g^{rs}(z,x,\phi)\Gamma^l_{rs}g_{lm}(z,x,\phi) \\ &= (\tilde{g}^{rs}(z,\phi) + \mathcal{A}_1)(\tilde{\Gamma}^l_{rs} + \mathcal{A}_0 + |\nabla\phi|\mathcal{A}_1)(\tilde{g}_{lm}(z,\phi) + \mathcal{A}_1) \\ &= H_{\tilde{g}(z,\phi)} + \mathcal{A}_0 + |\nabla\phi|\mathcal{A}_1. \end{aligned}$$

Taking $\phi = 0$ in the above equation and subtracting from $H_{g(z,x,\phi)}$ yields

$$|H_{g(z,x,\phi)} - H_{g(z,x,0)}| \leq |H_{\tilde{g}(z,\phi)} - H_{\tilde{g}(z,0)}| + C_1(|\phi| + |X||\nabla\phi|)$$

for some positive constant C_1 independent of ε, coming from the definition of $\mathcal{A}_0$ and $\mathcal{A}_1$. Now setting $\phi = \varepsilon^{m-2}|x|^{2-m}$ and recalling that $H_{\tilde{g}(z,\phi)}$ and $H_{\tilde{g}(z,0)}$ both vanish, we find

$$|H_{g_\varepsilon} - H_{g_1}| \leq C_1 e^{(m-2)t}$$

concluding our work for $t \in (\log \varepsilon + 1, -1)$.

Next, we move on to the portion $\{\log \varepsilon \leq t \leq \log \varepsilon + 1\}$. On this part of the neck ξ is still constant, but the normal conformal factor u_ε is effected by the cutoff function η. However, since η and its derivatives are uniformly bounded, it is straightforward to check that the estimate $|H_{g_\varepsilon}| \leq C_2 e^{(m-2)t}$ holds here, where C_2 is a constant independent of epsilon.

On the portion of the neck $\{-1 \leq t \leq 0\}$, η vanishes and now the cutoff function ξ effects all components of g_ε. However, we can still write

$$\begin{aligned} g_\varepsilon(z,t,\theta) =& (g^{(1)}_{ij} + \mathcal{O}(|x|))dz^i dz^j + (1 + \varepsilon^{m-2}|x|^{2-m})^{\frac{4}{m-2}}(g^{(1)}_{\alpha\beta} + \mathcal{O}(|x|))dx^\alpha dx^\beta \\ &+ (g^{(1)}_{k\gamma} + \mathcal{O}(|x|))dz^k dx^\gamma. \end{aligned}$$

In general, if two metrics are related by $g' = g + \mathcal{O}(|X|)$, we have $\Gamma' = \Gamma + \mathcal{O}(1)$ for any Christoffel symbol Γ' of g' and corresponding symbol Γ of g. Hence the boundary mean curvatures satisfy $|H_{g'} - H_g| = \mathcal{O}(1)$. Applying this fact to compare g_ε and g_1, we find that the mean curvature H_{g_ε} is uniformly bounded in ε. Since t is small in absolute value on this portion of the neck, we may choose $C_3 > 0$, independent of ε, so that

$$|H_{g_\varepsilon} - H_{g_1}| \leq C_3 e^{(m-2)t},$$

To summarize our efforts, for $t \in (\log \varepsilon, 0]$ and taking $C_4 = \max(C_1, C_2, C_3)$, we have

$$|H_{g_\varepsilon} - H_{g_1}| \leq C_4 e^{(m-2)t}.$$

Repeating these computations for the portion of the neck $\{0 \le t \le -\log \varepsilon\}$, one can show that there is a constant C_5, independent of ε, satisfying

$$|H_{g_\varepsilon} - H_{g_2}| \le C_5 e^{(2-m)t}$$

for such t. Recalling that $H_{g_*} \equiv 0$ for $* = 1, 2$, these two inequalities give the pointwise estimate claimed in Proposition 2.1.3 where the constant is given by $C = \max(C_4, C_5)$.

We conclude the proof by using our pointwise estimate to obtain the L^1 estimate on the boundary mean curvature

$$\begin{aligned}
\int_{\partial M} |H_{g_\varepsilon}| d\sigma_{g_\varepsilon} &\le C \cdot \int_{\partial M \cap T^\varepsilon(0,0)} \cosh^{2-m}(t) d\sigma_{g_\varepsilon} \\
&= C \cdot \mathrm{Vol}_{g_K}(K) \omega_{m-2} \varepsilon^{(m-2)} \int_{\log(\varepsilon)}^{-\log(\varepsilon)} e^{(2-m)t} \cosh^{(2-m)}(t) dt \\
&\le C' \cdot \mathrm{Vol}_{g_K}(K) \omega_{m-2} \varepsilon^{m-2}
\end{aligned}$$

where ω_{m-2} denotes the volume of the unit sphere S^{m-2} and C' is another positive constant independent of ε. $\square$

2.122. Local Expression for Δ_{g_ε} and the Barrier Function ϕ_δ

Before we can state our analogue of the a priori estimate Proposition 2.1.2 for the boundary embedding case, we will need to construct a particular barrier function. First we define a function on the unit upper hemisphere S^{m-1}_+ in spherical coordinates $\beta(\theta) := (L+1) - L\cos(\theta^1)$ where $L > 0$ is a constant to be determined. Notice that β satisfies

$$\begin{cases} \Delta_\theta \beta(\theta) = -(m-1) L \cos(\theta^1) & \text{in } S^{m-1}_+ \\ \partial_{\theta^1} \beta(\theta) = \beta(\theta) & \text{on } \partial S^{m-1}_+ \end{cases}$$

and $1 \leq \beta(\theta) \leq L+1$ in S_+^{m-1}. Now, for a fixed parameter $\delta \in (\frac{2-m}{2}, \frac{m-2}{2})$, we define the function on the gluing region by

$$\phi_\delta(z,t,\theta) := \begin{cases} \frac{\cosh^\delta(t)}{u_\varepsilon(t)}\beta(\theta) & \text{if } \delta \leq 0 \\ \frac{\cosh(\delta t)}{u_\varepsilon(t)}\beta(\theta) & \text{if } \delta \geq 0 \end{cases}$$

which is a version of the barrier function used in [10], modified for the present case of boundary embeddings. The following lemma states the key properties of ϕ_δ which we will need for the a priori estimate.

Lemma 2.1.4. *Let $\delta \in (\frac{2-m}{2}, \frac{m-2}{2})$. There exists a choice of parameters $\alpha_1, \alpha_2 > 1$, $L > 0$, and a constant $C > 0$ so that*

$$\begin{aligned} \Delta_{g_\varepsilon}\phi_\delta &\leq -Cu_\varepsilon^{\frac{-4}{m-2}}\phi_\delta && \text{in } T^\varepsilon(\alpha_1,\alpha_2) \\ \partial_\nu\phi_\delta &\geq \tfrac{1}{2}u_\varepsilon^{\frac{-2}{m-2}}\phi_\delta && \text{on } \partial M \cap T^\varepsilon(\alpha_1,\alpha_2) \end{aligned}$$

is satisfied for all $\varepsilon \in (0, e^{-\max(\alpha_1,\alpha_2)})$.

Proof. Our first step is to obtain a useful local expression for the g_ε-Laplacian. We will only need to consider the portion of the neck $T^\varepsilon(1,1)$ where the cut off function η is constant and the components of g_ε take the form

$$\begin{aligned} g_{ij}^\varepsilon &= g_{ij}^K + \mathcal{O}(|x|), & g_{it}^\varepsilon &= \mathcal{O}(|x|^2) \\ g_{i\lambda}^\varepsilon &= \mathcal{O}(|x|^2), & g_{tt}^\varepsilon &= u_\varepsilon^{\frac{4}{m-2}}(1+\mathcal{O}(|x|)) \\ g_{t\lambda}^\varepsilon &= u_\varepsilon^{\frac{4}{m-2}}\mathcal{O}(|x|), & g_{\lambda\mu}^\varepsilon &= u_\varepsilon^{\frac{4}{m-2}}(g_{\lambda\mu}^{(\theta)} + \mathcal{O}(|x|)) \end{aligned}$$

where $g_{\lambda\mu}^{(\theta)}$ denotes a component of the standard round metric on the upper unit hemisphere S_+^{m-1} in spherical coordinates $\theta = (\theta^1, \ldots, \theta^{m-1})$. As for the volume form, we have

$$\sqrt{g_\varepsilon} = \sqrt{g_K}\sqrt{g_\theta}\, u_\varepsilon^{\frac{2m}{m-2}}(1 + \mathcal{O}(|x|))$$

where we write $\sqrt{g_\theta} = \sqrt{\det\left(g^{(\theta)}_{\lambda\mu}\right)}$. One can use the above expressions with Cramer's rule to compute the following expansions for components of the inverse matrix g_ε^{-1}

$$\begin{aligned}
g_\varepsilon^{ij} &= g_K^{ij} + \mathcal{O}(|x|), & g_\varepsilon^{it} &= \mathcal{O}(|x|^2)\\
g_\varepsilon^{i\lambda} &= \mathcal{O}(|x|^2), & g_\varepsilon^{tt} &= u_\varepsilon^{\frac{-4}{m-2}}(1 + \mathcal{O}(|x|))\\
g_\varepsilon^{t\lambda} &= u_\varepsilon^{\frac{-4}{m-2}}\mathcal{O}(|x|), & g_\varepsilon^{\lambda\mu} &= u_\varepsilon^{\frac{-4}{m-2}}(g^{\lambda\mu}_{(\theta)} + \mathcal{O}(|x|)).
\end{aligned}$$

Recall the following general fact: for a local coordinate system $y = (y^1, \dots, y^n)$ of a Riemannian manifold (N, g), the g-Laplacian can be expressed as $\Delta_g \cdot = \frac{1}{\sqrt{g}}\partial_{y^a}(\sqrt{g}\; g^{ab}\partial_{y^b}\cdot)$. Using this, a straight-forward computation gives us the following expression

$$\Delta_{g_\varepsilon} = u_\varepsilon^{\frac{-4}{m-2}}\left(\partial_t^2 + (m-2)\tanh\left(\frac{m-2}{2}t\right)\partial_t + \Delta_\theta + u_\varepsilon^{\frac{4}{m-2}}\Delta_K + \mathcal{O}(|x|)\Phi_1\right)$$

where Δ_θ is the Laplace operator of the standard round metric on S^{m-1}, Δ_K is the Laplace operator of (K, g_K), and Φ_1 is a linear second-order operator with ε-uniformly bounded coefficients. Now notice that one can conjugate Δ_{g_ε} by u_ε to find

$$\Delta_{g_\varepsilon}\cdot = u_\varepsilon^{-\frac{m+2}{m-2}}\mathcal{D}_\varepsilon(u_\varepsilon\cdot) \tag{2.1}$$

where $\mathcal{D}_\varepsilon$ is an operator of the form

$$\mathcal{D}_\varepsilon = \partial_t^2 - \left(\frac{m-2}{2}\right)^2 + \Delta_\theta + u_\varepsilon^{\frac{4}{m-2}}\Delta_K + \mathcal{O}(|x|)\Phi_2.$$

In the above, Φ_2 is another linear second order operator with ε-uniformly bounded coefficients.

Let us first consider the case $\delta \in (\frac{2-m}{2}, 0)$. One can use the conjugation formula (2.1) to find

$$\begin{aligned}\Delta_{g_\varepsilon}\phi_\delta &= u_\varepsilon^{-\frac{m+2}{m-2}}\mathcal{D}_\varepsilon(\cosh^\delta(t)\beta(\theta)) \\ &= u_\varepsilon^{\frac{-4}{m-2}}\phi_\delta\left(\delta^2 - \left(\frac{m-2}{2}\right)^2 + \frac{(m-1)L\cos(\theta^1)}{\beta(\theta)} + \mathcal{O}(|x|) + (\delta-\delta^2)\cosh^{-2}(t)\right).\end{aligned}$$

Evidently, we have $\delta - \delta^2 \leq 0$. If we choose the positive constant $L := \frac{\left(\frac{m-2}{2}\right)^2 - \delta^2}{m}$, then the inequality

$$\begin{aligned}\delta^2 - \left(\frac{m-2}{2}\right)^2 + \frac{(m-1)L\cos(\theta^1)}{\beta(\theta)} &\leq \delta^2 - \left(\frac{m-2}{2}\right)^2 + (m-1)L \\ &< 0\end{aligned}$$

for all θ. Now, in order to deal with the above $\mathcal{O}(|x|)$ term in the expression for $\Delta_{g_\varepsilon}\phi_\delta^\partial$, observe that we can find α_1, α_2 such that

$$\delta^2 - \left(\frac{m-2}{2}\right)^2 + \frac{(m-1)L\cos(\theta^1)}{\beta(\theta)} + \mathcal{O}(|X|) \leq \frac{1}{2}\left(\delta^2 - \left(\frac{m-2}{2}\right)^2 + (m-1)L\right)$$

on $T^\varepsilon(\alpha_1, \alpha_2)$ for all $\varepsilon \in (0, e^{-\max(\alpha_1,\alpha_2)})$. Now setting $C := \frac{1}{2}\left(\delta^2 - \left(\frac{m-2}{2}\right)^2 + (m-1)L\right)$,

$$\Delta_{g_\varepsilon}\phi_\delta \leq -Cu_\varepsilon^{\frac{-4}{m-2}}\phi_\delta$$

on $T^\varepsilon(\alpha_1, \alpha_2)$. As similar argument for $\delta \in (0, \frac{m-2}{2})$ yields the desired estimate for $\Delta_{g_\varepsilon}\phi_\delta$.

Next, we consider the outward normal derivative of ϕ_δ. Recall the following general fact: if $\{\partial_{y^1}, \ldots, \partial_{y^{n-1}}\}$ span the boundary tangent space of a Riemannian manifold (N, g) and ∂_{y^n} points outwards, then the outward normal unit vector to ∂N with respect to g is given by the formula $\frac{g^{na}\partial_{y^a}}{\sqrt{g^{nn}}}$. In our present situation, observe that $\{\partial_{z^1}, \ldots, \partial_{z^k}, \partial_t, \partial_{\theta^1}, \ldots, \partial_{\theta^{m-2}}\}$ span the tangent space of $\partial M \cap T^\varepsilon(1,1)$ and ∂_{θ^1} points outwards. Using this formula with the expressions for components of g_ε^{-1}, observe that the outward normal derivative on $\partial M \cap T^\varepsilon(1,1)$ with respect to g_ε can be written as

$$\partial_\nu = u_\varepsilon^{\frac{2}{m-2}} (u_\varepsilon^{-\frac{4}{m-2}} \partial_{\theta^1} + \mathcal{O}(|X|)\Phi_3)$$

where Φ_3 is a linear first-order differential operator on $\partial M \cap T^\varepsilon(1,1)$ with ε-uniformly bounded coefficients. Applying this to the barrier function ϕ_δ, we have

$$\partial_\nu \phi_\delta = \phi_\delta u_\varepsilon^{-\frac{2}{m-2}} (1 + \mathcal{O}(|x|)).$$

By choosing yet larger α_1, α_2, we may assume that the above term satisfies $1+\mathcal{O}(|x|) \geq \frac{1}{2}$. we may assume

$$\partial_\nu \phi_\delta \geq \frac{1}{2} u_\varepsilon^{-\frac{2}{m-2}} \phi_\delta$$

on $\partial M \cap T^\varepsilon(\alpha_1, \alpha_2)$ for all $\varepsilon \in (0, e^{-\max(\alpha_1,\alpha_2)})$, as claimed. $\square$

2.123. The local a priori estimate

In order to state the a priori estimate, we will decompose the boundary of the region $T^\varepsilon(\alpha_1, \alpha_2)$ into two portions $\partial T^\varepsilon(\alpha_1, \alpha_2) = \partial_1 T^\varepsilon(\alpha_1, \alpha_2) \cup \partial_2 T^\varepsilon(\alpha_1, \alpha_2)$ where

$$\begin{aligned}
\partial_1 T^\varepsilon(\alpha_1, \alpha_2) &= \{(z, t, \theta) \in T^\varepsilon(\alpha_1, \alpha_2) \colon t = \log \varepsilon + \alpha_1 \text{ or } t = -\log \varepsilon - \alpha_2\} \\
\partial_2 T^\varepsilon(\alpha_1, \alpha_2) &= \{(z, t, \theta) \in T^\varepsilon(\alpha_1, \alpha_2) \colon \theta^1 = \frac{\pi}{2}\}.
\end{aligned}$$

Note that $\partial_1 T^\varepsilon(\alpha_1, \alpha_2) \subset M$, $\partial_2 T^\varepsilon(\alpha_1, \alpha_2) \subset \partial M$, and the two meet at a corner.

Proposition 2.1.5. *Given $\gamma \in (0, m-2)$ there are ε-uniform constants $\alpha_1, \alpha_2 > 1$ and $C > 0$ satisfying the following statement for all $\varepsilon \in (0, e^{-\max\{\alpha_1, \alpha_2\}})$. If $v, f \in \mathcal{C}^0(T^\varepsilon(\alpha_1, \alpha_2))$ satisfy $\Delta_{g_\varepsilon} v = f$, then*

$$v \leq C \psi_\varepsilon^{-\gamma} \left(\sup_{T^\varepsilon(\alpha_1,\alpha_2)} |\psi_\varepsilon^{\gamma+2} f| + \sup_{\partial_1 T^\varepsilon(\alpha_1,\alpha_2)} |\psi_\varepsilon^{\gamma} v| + \sup_{\partial_2 T^\varepsilon(\alpha_1,\alpha_2)} |\psi_\varepsilon^{\gamma+1} \partial_\nu v| \right)$$

pointwise on $T^\varepsilon(\alpha_1, \alpha_2)$ and

$$||v||_{\mathcal{C}^0_\gamma(T^\varepsilon(\alpha_1,\alpha_2))} \leq C \left(||f||_{\mathcal{C}^0_{\gamma+2}(T^\varepsilon(\alpha_1,\alpha_2))} + ||v||_{\mathcal{C}^0_\gamma(\partial_1 T^\varepsilon(\alpha_1,\alpha_2))} + ||\partial_\nu v||_{\mathcal{C}^0_{\gamma+1}(\partial_2 T^\varepsilon(\alpha_1,\alpha_2))} \right).$$

Proof. Set $\delta = \gamma - \frac{m-2}{2}$ and let C', α_1, α_2 be the constants given by Lemma 2.1.4. Now consider the function

$$\tilde{v} = a\phi_\delta - v$$

where the constant $a > 0$ is given by

$$\begin{aligned}
a := \ &\max(2, C'^{-1}) \Big(\sup_{T^\varepsilon(\alpha_1,\alpha_2)} |u_\varepsilon^{\frac{4}{m-2}} \phi_\delta^{-1} f| + \sup_{\partial_1 T^\varepsilon(\alpha_1,\alpha_2)} |\phi_\delta^{-1} v| \\
&+ \sup_{\partial_2 T^\varepsilon(\alpha_1,\alpha_2)} |u_\varepsilon^{\frac{2}{m-2}} \phi_\delta^{-1} \partial_\nu v| \Big).
\end{aligned}$$

Our goal is to show that $\tilde{v} \geq 0$. First note that $\tilde{v}$ is superharmonic – applying the inequalities of Lemma 2.1.4, we have

$$\begin{aligned}\Delta_{g_\varepsilon}\tilde{v} &\leq -aC' u_\varepsilon^{\frac{-4}{m-2}}\phi_\delta - f\\ &\leq -u_\varepsilon^{\frac{-4}{m-2}}\phi_\delta \sup_{T_\alpha^\varepsilon} |u_\varepsilon^{\frac{4}{m-2}}\phi_\delta^{-1} f| u - f\\ &\leq 0.\end{aligned}$$

Also observe that $\tilde{v} \geq 0$ on $\partial_1 T^\varepsilon(\alpha_1, \alpha_2)$. So far, we have found

$$\begin{aligned}\Delta_{g_\varepsilon}\tilde{v} &\leq 0 \quad \text{in } T^\varepsilon(\alpha_1,\alpha_2)\\ \tilde{v} &\geq 0 \quad \text{on } \partial_1 T^\varepsilon(\alpha_1,\alpha_2).\end{aligned}$$

The maximum principle for Δ_{g_ε} tells us the minimum of $\tilde{v}$ occurs somewhere on the boundary of $T^\varepsilon(\alpha_1, \alpha_2)$. Suppose the minimum of $\tilde{v}$ occurs at a point $y_0 \in \partial_2 T^\varepsilon(\alpha_1, \alpha_2)$. We may then apply the Hopf lemma and the estimate on $\partial_\nu \phi_\delta$ from Lemma 2.1.4 to obtain a contradiction

$$\begin{aligned}0 &> \partial_\nu \tilde{v}(y_0)\\ &\geq aC'\phi_\delta u_\varepsilon^{\frac{-2}{m-2}} - \partial_\nu v(y_0)\\ &\geq 0.\end{aligned}$$

We conclude that the minimum of $\tilde{v}$ must occur on $\partial_1 T^\varepsilon(\alpha_1, \alpha_2)$. Since $\tilde{v}$ is non-negative there, $\tilde{v} \geq 0$ on all of $T^\varepsilon(\alpha_1, \alpha_2)$. In other words,

$$\begin{aligned} v \leq \max(2, C'^{-1})\phi_\delta \Big(\sup_{T^\varepsilon(\alpha_1,\alpha_2)} |u_\varepsilon^{\frac{4}{m-2}} \phi_\delta^{-1} f| + \sup_{\partial_1 T^\varepsilon(\alpha_1,\alpha_2)} |\phi_\delta^{-1} v| \\ + \sup_{\partial_2 T^\varepsilon(\alpha_1,\alpha_2)} |u_\varepsilon^{\frac{2}{m-2}} \phi_\delta^{-1} \partial_\nu v| \Big) \end{aligned} \tag{2.2}$$

on $T^\varepsilon(\alpha_1, \alpha_2)$.

One can repeat the above argument, replacing $\tilde{v}$ with $a\phi_\delta + v$, to arrive at a similar lower bound on v. Together, we arrive at

$$\begin{aligned} \sup_{T^\varepsilon(\alpha_1,\alpha_2)} |\phi_\delta^{-1} v| \leq C' \Big(\sup_{T^\varepsilon(\alpha_1,\alpha_2)} |u_\varepsilon^{\frac{4}{m-2}} \phi_\delta^{-1} f| + \sup_{\partial_1 T^\varepsilon(\alpha_1,\alpha_2)} |\phi_\delta^{-1} v| \\ + \sup_{\partial_2 T^\varepsilon(\alpha_1,\alpha_2)} |u_\varepsilon^{\frac{2}{m-2}} \phi_\delta^{-1} \partial_\nu v| \Big), \end{aligned} \tag{2.3}$$

noting that the constant $\max(2, C'^{-1})$ is independent of ε.

To phrase our estimate in terms of the weighted Banach spaces $\mathcal{C}^0_\gamma$, we need to compare the functions u_ε and ϕ_δ to the weighting functions ψ_ε. Recall the following basic fact of the hyperbolic cosine function: For every $\lambda > 0$, there is a positive constant C_λ so that

$$C_\lambda^{-1} \cosh^\lambda(s) \leq \cosh(\lambda s) \leq C_\lambda \cosh^\lambda(s)$$

holds for all $t \in \mathbb{R}$. For instance, recalling that $\psi_\varepsilon = \varepsilon \cosh(t)$ on $T^\varepsilon(\alpha_1, \alpha_2)$, there is a constant C_δ depending only on δ such that

$$C_\delta^{-1} \psi_\varepsilon^{\frac{m-2}{2} - \delta} \leq \varepsilon^\delta \phi_\delta^{-1} \leq C_\delta \psi_\varepsilon^{\frac{m-2}{2} - \delta}.$$

Recalling that $\gamma = \frac{m-2}{2} - \delta$, one may replace ϕ_δ^∂ and u_ε with appropriate powers of ψ_ε to reorganize the estimates (2.2) and (2.3) to the one claimed in Proposition 2.1.5 where $C = \max(2, C'^{-1}, C_\delta)$. $\square$

2.13. The relative embedding

We will now consider the relative embedding case. Now K itself has non-empty boundary ∂K. Let $U \to \partial K$ be a coordinate chart for the boundary of K with coordinates $z' = (z^1, \dots, z^{k-1})$ and, letting $z^k \in [0,1]$ be the inward normal direction, form Fermi coordinates $z = (z', z^k)$ on a neighborhood of U in K. We will split the chart $U \times [0,3]$ into three parts

$$U^- := U \times [0,1], \quad U^T := U \times [1,2], \quad U^+ := U \times [2,3].$$

On U^+, we give Fermi coordinates given by

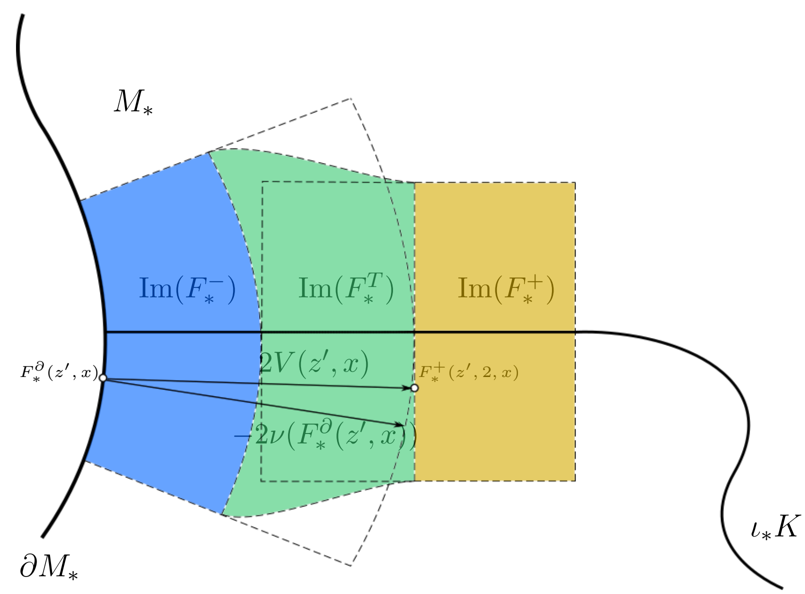

FIGURE 2.4. The coordinate charts F_*^-, F_*^T, F_*^+ and the vector fields V and $-\nu$

$$F_*^+ : U^+ \times D^m \to M_*, \quad (z, x) \mapsto \exp^{g_*}_{\iota_*(z)}(x)$$

which we originally saw in the interior embedding case from Section 2.11. As for U^-, we first have boundary Fermi coordinates (z', x) for ∂M_* given by

$$F_*^\partial : U \times \{0\} \times D^m \to M_*, \quad (z', x) \mapsto \exp^{g_*|_{\partial M_*}}_{\iota_*(z')}(x).$$

Now, similar to the boundary embedding construction from Section 2.12, we get coordinates on M_* by the mapping

$$F_*^- : U^- \times D^m \to M_*, \quad (z', z^k, x) \mapsto \exp^{g_*}_{F_*^\partial(z',x)}(-z^k \nu),$$

where ν is the outward-pointing normal vector to ∂M_* with respect to g_*. In order to transition between the two coordinate systems F_*^- and F_*^+, we first define a vector $V(z', x) \in T_{F_*^\partial(z',x)} M_*$ by solving the equation

$$\exp^{g_*}_{F_*^\partial(z',x)}(2V(z', x)) = F_*^+(z', 2, x).$$

Now we fix a non-increasing cutoff function $\alpha : [0, 3] \to [0, 3]$ which takes the value 1 on $[0, 1]$ and 0 on $[2, 3]$ and form a transitioning normal vector by

$$\overline{\nu}(z', z^k, x) := -\nu(F_*^\partial(z', x))\alpha(z^k) + (1 - \alpha(z^k))V(z', x).$$

The coordinate system on U^T is given by the mapping

$$F_*^T : U^T \times B^m \to M_*, \quad (z', z^k, x) \mapsto \exp^{g_*}_{F_*^\partial(z',x)}(z^k \overline{\nu}(z', z^k, x)).$$

Noting that $F_*^+ = F_*^T$ when $z^k = 2$, $F_*^- = F_*^T$ when $z^k = 1$, and $z = (z', z^k)$, we have well-defined coordinates (z, x) on a neighborhood of the boundary of $\iota_*(K)$ in M_* (see Figure 2.4.). As for an interior neighborhood of $\iota_*(K)$, we have the Fermi coordinates from Section 2.11 and refer to both coordinate systems with (z, x).

On either interior or boundary charts, we introduce the coordinates (z, t, θ) by setting $x = \varepsilon e^{-t}\theta$ on M_1 and $x = \varepsilon e^{t}\theta$ on M_2. Here $\theta = (\theta^1, \ldots, \theta^{m-1})$ are spherical coordinates on the unit sphere S^{m-1} and $t \in (\log\varepsilon, -\log\varepsilon)$. The metric g_* can be expressed in the form

$$
\begin{aligned}
g_* = \ & g_{ij}^{(*)} dz^i dz^j + \left(u_\varepsilon^{(*)}\right)^{\frac{4}{m-2}} \left(g_{tt}^{(*)} dt^2 + g_{\lambda\mu}^{(*)} d\theta^\lambda d\theta^\mu + g_{t\lambda}^{(*)} dt d\theta^\lambda\right) \\
& \qquad + g_{it}^{(*)} dz^i dt + g_{i\lambda}^{(*)} dz^i d\theta^\lambda
\end{aligned}
$$

where $u_\varepsilon^{(*)}$ is defined as in Section 2.11. The asymptotics now take the form

$$
\begin{array}{lll}
g_{ij}^{(*)}(z,t,\theta) = g_{ij}^K(z) + \mathcal{O}(|x|), & g_{\lambda\mu}^{(*)}(z,t,\theta) = g_{\lambda\mu}^{(\theta)}(\theta) + \mathcal{O}(|x|), & g_{tt}^{(*)}(z,t,\theta) = 1 + \mathcal{O}(|x|) \\
g_{i\lambda}^{(*)}(z,t,\theta) = \mathcal{O}(|x|), & g_{it}^{(*)}(z,t,\theta) = \mathcal{O}(|x|), & g_{i\lambda}^{(*)}(z,t,\theta) = \mathcal{O}(|x|)
\end{array}
$$

where $g_{\lambda\mu}^{(\theta)}$ denotes a component of the standard round metric on S^{m-1} in the spherical coordinates $(\theta^1, \ldots, \theta^{m-1})$.

Using the same cutoff functions ξ and η we introduced in the case of interior embeddings, define the function u_ε as in Section 2.11. For each $\varepsilon \in (0, \frac{1}{2})$, set

$$
\begin{aligned}
g_\varepsilon(z,t,\theta) = \ & (\xi g_{ij}^{(1)} + (1-\xi) g_{ij}^{(2)}) dz^i dz^j + u_\varepsilon^{\frac{4}{n-2}} \Big((\xi g_{tt}^{(1)} + (1-\xi) g_{tt}^{(2)}) dt^2 \\
& + (\xi g_{\lambda\mu}^{(1)} + (1-\xi) g_{\lambda\mu}^{(2)}) d\theta^\lambda d\theta^\mu + (\xi g_{t\lambda}^{(1)} + (1-\xi) g_{t\lambda}^{(2)}) dt d\theta^\lambda] \Big) \\
& + (\xi g_{it}^{(1)} + (1-\xi) g_{it}^{(2)}) dz^i dt + (\xi g_{i\lambda}^{(1)} + (1-\xi) g_{i\lambda}^{(2)}) dz^i d\theta^\lambda .
\end{aligned}
$$

This defines a metric g_ε on the tubular annuli $V_*^1 \setminus \overline{V_*^{\varepsilon^2}}$ for $* = 1, 2$. We set $g_\varepsilon = g_*$ on $M_* \setminus \overline{V_*^1}$. This gives well-defined metric g_ε on the disjoint union $(M_1 \setminus V_*^{\varepsilon^2}) \sqcup (M_2 \setminus V_*^{\varepsilon^2})$.

Let $\Phi : \mathcal{N}_1(K) \to \mathcal{N}_2(K)$ be the isomorphism of the normal bundles given in the hypothesis of Theorem 2.0.11. For each $\varepsilon \in (0, \frac{1}{2})$, consider mapping Ψ_ε given by

$$\Psi_\varepsilon : (\mathcal{N}_1(K) \setminus \{0\}) \sqcup (\mathcal{N}_2(K) \setminus \{0\}) \to (\mathcal{N}_1(K) \setminus \{0\}) \sqcup (\mathcal{N}_2(K) \setminus \{0\})$$

$$\Psi_\varepsilon(z, t, \theta) := \begin{cases} \Phi(z, -t, \theta) & \text{if } (z, t, \theta) \in \mathcal{N}_1(K) \\ \Phi^{-1}(z, -t, \theta) & \text{if } (z, t, \theta) \in \mathcal{N}_2(K). \end{cases}$$

For each $\varepsilon \in (0, \frac{1}{2})$, we construct the generalized connected sum

$$M = \Big((M_1 \setminus V_1^{\varepsilon^2}) \sqcup (M_2 \setminus V_2^{\varepsilon^2}) \Big) / \sim_\varepsilon$$

where we introduce a relation $\sim_\varepsilon$ on the annuli $(V_1^1 \setminus V_1^{\varepsilon^2}) \sqcup (V_2^1 \setminus V_2^{\varepsilon^2})$: If $y \in V_1^1 \setminus \overline{V_1^{\varepsilon^2}}$, then $y \sim_\varepsilon (F_2 \circ \Psi_\varepsilon \circ F_1^{-1})(y)$. Observing that g_ε is invariant under Ψ_ε, the metric descends to M. This finishes the definition of the family of Riemannian manifolds (M, g_ε).

Recalling that we assume the mean curvature H_{g_K} vanishes on ∂K, the proof of the following proposition is very similar to argument in Proposition 2.1.3 and so we omit it.

Proposition 2.1.6. *There is a constant $C > 0$, independent of ε, such that*

$$|R_{g_\varepsilon}| \leq C\varepsilon^{-1} \cosh^{1-m}(t), \quad |H_{g_\varepsilon}| \leq C \cosh^{2-m}(t)$$

on $T^\varepsilon(0,0)$ *and*

$$\int_M |R_{g_\varepsilon}| d\mu_{g_\varepsilon} = \mathcal{O}(\varepsilon^{m-2}), \quad \int_{\partial M} |H_{g_\varepsilon}| d\sigma_{g_\varepsilon} = \mathcal{O}(\varepsilon^{m-2}).$$

As for the local a priori estimate, we will need to again decompose the boundary of $\partial T^\varepsilon(\alpha_1, \alpha_2)$ into two pieces

$$\begin{aligned}
\partial_1 T^\varepsilon(\alpha_1, \alpha_2) =& \{(z, t, \theta) \in T^\varepsilon(\alpha_1, \alpha_2) \colon t = \log \varepsilon + \alpha_1 \text{ or } t = -\log \varepsilon - \alpha_2\} \\
\partial_2 T^\varepsilon(\alpha_1, \alpha_2) =& \{(z, t, \theta) \in T^\varepsilon(\alpha_1, \alpha_2) \colon z \in \partial K\}.
\end{aligned}$$

We will use the same notation for $\partial_1 T^\varepsilon(\alpha_1, \alpha_2)$ and $\partial_2 T^\varepsilon(\alpha_1, \alpha_2)$ as we did in the case of boundary embeddings. There is also an analogue of the estimates in Propositions 2.1.2 and 2.1.5 for the present case of relative embeddings. Its proof is very similar to that of Proposition 2.1.5 and we leave it to the reader.

Proposition 2.1.7. *Given* $\gamma \in (0, m-2)$ *there are* ε*-uniform constants* $\alpha_1, \alpha_2 > 1$ *and* $C > 0$ *satisfying the following statement for all* $\varepsilon \in (0, e^{-\max\{\alpha_1, \alpha_2\}})$. *If* $v, f \in \mathcal{C}^0(T^\varepsilon(\alpha_1, \alpha_2))$ *satisfy* $\Delta_{g_\varepsilon} v = f$, *then*

$$v \le C \psi_\varepsilon^{-\gamma} \left(\sup_{T^\varepsilon(\alpha_1, \alpha_2)} |\psi_\varepsilon^{\gamma+2} f| + \sup_{\partial_1 T^\varepsilon(\alpha_1, \alpha_2)} |\psi_\varepsilon^{\gamma} v| + \sup_{\partial_2 T^\varepsilon(\alpha_1, \alpha_2)} |\psi_\varepsilon^{\gamma+1} \partial_\nu v| \right)$$

pointwise on $T^\varepsilon(\alpha_1, \alpha_2)$ *and*

$$||v||_{\mathcal{C}^0_\gamma(T^\varepsilon(\alpha_1, \alpha_2))} \le C \left(||f||_{\mathcal{C}^0_{\gamma+2}(T^\varepsilon(\alpha_1, \alpha_2))} + ||v||_{\mathcal{C}^0_\gamma(\partial_1 T^\varepsilon(\alpha_1, \alpha_2))} + ||\partial_\nu v||_{\mathcal{C}^0_{\gamma+1}(\partial_2 T^\varepsilon(\alpha_1, \alpha_2))} \right).$$

2.2. The Linear Analysis

Now that we have constructed the generalized connected sum (M, g_ε), we will turn our attention to equation (1.6). At this point, there is no need to consider the interior, boundary, and relative embedding cases independently as we did in Section 2.1. Unless otherwise mentioned, **from now on we will speak of all three cases simultaneously.**

Our first task will be to study the family of linear operators $(\Delta_{g_\varepsilon}, \partial_\nu)$ for $\varepsilon \in (0, \frac{1}{2})$. Before we continue, now is a good time to make some informal remarks. The first non-zero Steklov eigenvalue of $(\Delta_{g_\varepsilon}, \partial_\nu)$, which we write as λ_ε, is the smallest number such that the following equation admits a non-constant solution f

$$\begin{cases} \Delta_{g_\varepsilon} f = 0 & \text{on } M \\ \partial_\nu f = \lambda_\varepsilon f & \text{on } \partial M. \end{cases}$$

In general, $\lambda_\varepsilon \to 0$ as $\varepsilon \to 0$. For this reason, there is no general result which would provide us a useful ε-uniform $\mathcal{C}^0(M)$ estimate for our linear problem.

This in mind, we take two measures to combat this degeneracy. In addition to working in the weighted Banach spaces $\mathcal{C}^0_\gamma(M)$ we introduced in Section 2.1, we will initially solve (with estimates) a modification of the linear problem. Speaking informally, this auxiliary problem is formulated by projecting the linear problem along a hand-made model for the first non-constant eigenfunction. This model is a function denoted by β_ε which takes the values 1 on $M_1 \setminus V_1^\varepsilon$, -1 on $M_2 \setminus V_2^\varepsilon$, and interpolates between them on the neck so that $\int_M \beta_\varepsilon d\mu_{g_\varepsilon} = 0$ (see Section 2.21).

Given $\gamma \in (0, m-2)$ and suitable functions $f \in \mathcal{C}^0_{\gamma+2}(M)$, $\ell \in \mathcal{C}^0_{\gamma}(\partial M)$, we will produce a function $u \in \mathcal{C}^0_{\gamma}(M)$ satisfying

$$\begin{cases} \Delta_{g_\varepsilon} u = f & \text{on } M \\ \partial_\nu u = \ell - \lambda \beta_\varepsilon & \text{on } \partial M \end{cases} \tag{2.4}$$

where λ is a real number depending on f and ℓ. Notice that the functions f, ℓ must satisfy

$$\int_M f d\mu_{g_\varepsilon} = \int_{\partial M} \ell d\sigma_{g_\varepsilon}, \tag{2.5}$$

which is simply Green's formula applied to u. We will refer to (2.5) as the orthogonality condition of equation (2.4). As we produce this solution, we also obtain an ε-uniform $\mathcal{C}^0_{\gamma}$-norm a priori estimate for u using standard elliptic estimates on (M_*, g_*) with the local a priori estimate of Propositions 2.1.2, 2.1.5, and 2.1.7.

2.21. The linear problem I

For each $\alpha_1, \alpha_2 > 1$, let us fix ρ_1 and ρ_2, two smooth functions on $M_1 \sqcup M_2$ satisfying

$$\rho_1 = \begin{cases} 1 & \text{on } M_1 \setminus T^\varepsilon(\alpha_1, 0) \\ 0 & \text{on } M_2 \setminus T^\varepsilon(0, -2\log\varepsilon - \alpha_1 - 1) \end{cases}$$

$$\rho_2 = \begin{cases} 1 & \text{on } M_2 \setminus T^\varepsilon(0, \alpha_2) \\ 0 & \text{on } M_1 \setminus T^\varepsilon(-2\log\varepsilon - \alpha_2 - 1, 0) \end{cases}$$

and $\partial_\nu \rho_1 \equiv 0$, and $\partial_\nu \rho_2 \equiv 0$ on $\partial M_1 \sqcup \partial M_2$. Understanding that ρ_1 and ρ_2 descend to the connected sum M, we then define $\beta_\varepsilon : M \to \mathbb{R}$ by $\beta_\varepsilon := \rho_1 - \rho_2$.

In the case of interior embeddings, where we have not altered the original metrics on the boundary, it is immediate that

$$\int_{\partial M} \beta_\varepsilon d\sigma_{g_\varepsilon} = 0$$

since we assume $\mathrm{Vol}_{g_1}(\partial M_1) = \mathrm{Vol}_{g_2}(\partial M_2)$. To arrange for β_ε to have vanishing average value on the boundary in the case of boundary and relative embeddings (where $d\sigma_{g_\varepsilon}$ is affected by the gluing), we may have to choose α_1 and α_2 differently. However, notice that this can always be achieved by only increasing either α_1 or α_2. Since the estimates of Propositions 2.1.2, 2.1.5, and 2.1.7 also hold for these larger parameters, **from now on we will assume that α_1 and α_2 have been chosen so that Propositions 2.1.2, 2.1.5, and 2.1.7 apply and $\int_{\partial M} \beta_\varepsilon d\sigma_{g_\varepsilon} = 0$.**

In this section we build an approximate solution to (2.4) which is straight-forward to estimate, but accumulates many error terms in a gluing process. This construction is summarized in the following lemma which will subsequently be applied iteratively to establish a genuine solution to the linear problem (2.4), with estimates.

Lemma 2.2.1. *Let $\gamma \in (0, m-2)$ and $B \in (0,1)$. There is an $\varepsilon_0 > 0$ such that the following statement is satisfied for all $\varepsilon \in (0, \varepsilon_0)$: Suppose $f \in \mathcal{C}^0_{\gamma+2}(M)$ and $\ell \in \mathcal{C}^0_{\gamma+1}(\partial M)$ satisfy*

$$\int_M f d\mu_{g_\varepsilon} = \int_{\partial M} \ell d\sigma_\varepsilon.$$

Then there is $\lambda \in \mathbb{R}$, a function $u \in \mathcal{C}^0_\gamma(M)$, and an error term $E \in \mathcal{C}^0_{\gamma+2}(M)$ satisfying

$$\begin{cases} \Delta_{g_\varepsilon} u = f + E & \text{in } M \\ \partial_\nu u = \ell - \lambda\beta_\varepsilon & \text{on } \partial M \\ \int_M u d\mu_{g_\varepsilon} = 0 \end{cases}$$

Moreover, u, λ, and E satisfy the following estimates

$$\begin{aligned} ||u||_{\mathcal{C}^0_\gamma(M)} &\leq C(||f||_{\mathcal{C}^0_{\gamma+2}(M)} + ||\ell||_{\mathcal{C}^0_{\gamma+1}(\partial M)}) \\ |\lambda| &\leq C(||f||_{\mathcal{C}^0_{\gamma+2}(M)} + ||\ell||_{\mathcal{C}^0_{\gamma+1}(\partial M)}) \\ ||E||_{\mathcal{C}^0_{\gamma+2}(M)} &\leq C\varepsilon^{B\gamma}(||f||_{\mathcal{C}^0_{\gamma+2}(M)} + ||\ell||_{\mathcal{C}^0_{\gamma+1}(\partial M)}) \end{aligned}$$

where the constant $C > 0$ is independent of ε and B.

Proof. First we let $\rho_T := 1 - \rho_1 - \rho_2$ so that $\{\rho_1, \rho_T, \rho_2\}$ forms a partition of unity on M. We decompose f and ℓ with respect to this partition, writing

$$f_1 = f\rho_1, \quad f_T = f\rho_T, \quad f_2 = f\rho_2,$$

$$\ell_1 = \ell\rho_1, \quad \ell_T = \ell\rho_T, \quad \ell_2 = \ell\rho_2.$$

Next, we produce an approximate solution on the neck $T^\varepsilon(\alpha_1, \alpha_2)$.

Claim 2.2.2. *For the parameters γ, B and functions f, ℓ in Lemma 2.2.1, there is a unique function $\tilde{u}_T \in \mathcal{C}^0_\gamma(T^\varepsilon(\alpha_1, \alpha_2))$ satisfying*

$$\begin{cases} \Delta_{g_\varepsilon} \tilde{u}_T = f_T & \text{in } T^\varepsilon(\alpha_1, \alpha_2) \\ \tilde{u}_T = 0 & \text{on } \partial_1 T^\varepsilon(\alpha_1, \alpha_2) \\ \partial_\nu \tilde{u}_T = \ell_T & \text{on } \partial_2 T^\varepsilon(\alpha_1, \alpha_2). \end{cases} \tag{2.6}$$

Moreover, there is a constant $C_T > 0$, independent of ε, such that

$$||\tilde{u}_T||_{\mathcal{C}^0_\gamma(T^\varepsilon(\alpha_1,\alpha_2))} \leq C_T \left(||f_T||_{\mathcal{C}^0_{\gamma+2}(T^\varepsilon(\alpha_1,\alpha_2))} + ||\ell_T||_{\mathcal{C}^0_{\gamma+1}(\partial_2 T^\varepsilon(\alpha_1,\alpha_2))} \right).$$

Proof. Notice that $T^\varepsilon(\alpha_1, \alpha_2)$ is a compact manifold with corners. This allows us to apply the regularity theory in [18] – by [18, Theorem 1], there is a unique function

$$\tilde{u}_T \in \mathcal{C}^2\left(T^\varepsilon(\alpha_1,\alpha_2) \cup \partial_2 T^\varepsilon(\alpha_1,\alpha_2)\right) \cap \mathcal{C}^0(\overline{T^\varepsilon(\alpha_1,\alpha_2)})$$

solving equation (2.6). We may then apply Proposition 2.1.2, 2.1.5, and 2.1.7 with the parameter γ from the hypothesis of Lemma 2.2.1 and the function $\tilde{u}_T$ to arrive at the estimates in the claim. $\square$

We extend the domain of $\tilde{u}_T$ to all of M, which we will continue to call $\tilde{u}_T$, by declaring $\tilde{u}_T = 0$ on $M \setminus T^\varepsilon(\alpha_1, \alpha_2)$. While $\tilde{u}_T$ may not be differentiable on $\partial_1 T^\varepsilon(\alpha_1, \alpha_2)$, the function $u_T := \rho_T \tilde{u}_T$ is differentiable since the support of ρ_T is contained in $T^\varepsilon(\alpha_1 + 1, \alpha_2 + 1)$. One can compute

$$\Delta_{g_\varepsilon} u_T = f_T - q_1 - q_2$$
$$\partial_\nu u_T = \ell_T - q_1^\partial - q_2^\partial$$

where $q_* := \Delta_{g_\varepsilon}(\rho_* \tilde{u}_T)$ and $q_*^\partial := \partial_\nu(\rho_* \tilde{u}_T)$. The quantities q_* and q_*^∂ will be accounted for in the next step.

We now turn to the pieces of M which come from the original manifolds M_*. We define λ according to the formula

$$\lambda := \frac{1}{\int_{\partial M} (\rho_1 + \rho_2) d\sigma_{g_\varepsilon}} \left(\int_{\partial M} (\ell \beta_\varepsilon + q_1^\partial - q_2^\partial) d\sigma_{g_\varepsilon} - \int_M (f \beta_\varepsilon + q_1 - q_2) d\mu_{g_\varepsilon} \right), \quad (2.7)$$

which can be interpreted as the projection of f and ℓ along β_ε. Observe that, for $* = 1, 2$, this choice of λ implies

$$\int_M (f_* + q_*) d\mu_{g_\varepsilon} - \int_{\partial M} (\ell\rho_* + q_*^\partial + (-1)^* \lambda\rho_*) d\sigma_{g_\varepsilon} = 0, \tag{2.8}$$

which we will use later.

Using standard elliptic techniques [19][20], we may consider a distributional solution $\tilde{u}_*$ to the following system

$$\begin{cases} \Delta_{g_*} \tilde{u}_* = f_* + q_* + b_* \delta_{\iota_*} & \text{in } M_* \\ \partial_\nu \tilde{u}_* = \ell_* + q_*^\partial + (-1)^* \lambda \rho_* & \text{on } \partial M_* \\ \int_M \tilde{u}_* d\mu_{g_*} = 0 \end{cases}$$

where δ_{ι_*} denotes the Dirac distribution supported on the submanifold $\iota_*(K)$. Applying Green's theorem to $\tilde{u}_*$, the constant b_* is forced to be

$$b_* = \frac{1}{\mathrm{Vol}_{g_K}(K)} \left(\int_{\partial M_*} (\ell_* + q_*^\partial + (-1)^* \lambda\rho_*) d\sigma_{g_*} - \int_{M_*} (f_* + q_*) d\mu_{g_*} \right).$$

Claim 2.2.3. *There is a constant $C' > 0$ independent of ε such that*

$$|\tilde{u}_*| \leq C'(||f||_{\mathcal{C}^0(M)} + ||\ell||_{\mathcal{C}^0(\partial M)})$$

on $M_* \setminus V_*^1$,

$$|\tilde{u}_*| \leq C'|x|^{2-m}(||f||_{\mathcal{C}^0_{\gamma+2}(M)} + ||\ell||_{\mathcal{C}^0_{\gamma+1}(\partial M)})$$

on V_*^1, and

$$|\lambda| \leq C'(||f||_{\mathcal{C}^0_{\gamma+2}(M)} + ||\ell||_{\mathcal{C}^0_{\gamma+1}}).$$

Proof. To estimate $\tilde{u}_*$, it will be useful to consider the decomposition $\tilde{u}_* = \overline{u}_* + \hat{u}_*$ where

$$\begin{cases} \Delta_{g_*}\overline{u}_* = f_* + q_* + \mathrm{Vol}_{g_K}(K)b_* & \text{in } M_* \\ \partial_\nu \overline{u}_* = \ell_* + q_*^\partial + (-1)^* \lambda \rho_* & \text{on } \partial M_* \\ \int_{M_*} \overline{u}_* d\mu_{g_*} = 0 \end{cases}$$

$$\begin{cases} \Delta_{g_*}\hat{u}_* = -\mathrm{Vol}_{g_K}(K)b_* + b_*\delta_{\iota_*} & \text{in } M_* \\ \partial_\nu \hat{u}_* = 0 & \text{on } \partial M_* \\ \int_{M_*} \hat{u}_* d\mu_{g_*} = 0 \end{cases}$$

One can think of $\overline{u}_*$ and $\hat{u}_*$ as the finite and Green's function parts of $\tilde{u}_*$, respectively. Near the submanifold $\iota_*(K)$, one can use the Green's function construction presented in [19] to see that $\hat{u}_*$ takes the form

$$\hat{u}_* = \frac{b_*}{(m-2)\omega_{m-1}} \left(|x|^{2-m} + \mathcal{O}(|x|^{3-m}) \right)$$

where ω_{m-1} is the volume of unit sphere S^{m-1} and the term $\mathcal{O}(|x|^{3-m})$ depends only on the geometry of (M_*, g_*). It follows that there is a constant C_0, independent of ε, such that

$$|\hat{u}_*| \leq C_0 b_* |x|^{m-2} \tag{2.9}$$

on V_*^1.

Next, we consider $\overline{u}_*$. By taking $p = n$ and $k = 0$ in the L^p estimate (A.1) from Theorem A.1.1 in the Appendix applied to $\overline{u}_*$, there is a constant $C_1 > 0$ so that

$$\begin{aligned} ||\overline{u}_*||_{W^{2,n}(M_*,g_*)} \leq & C_1 \Big(\left\| f_* + q_* - \frac{Vol_{g_K}(K)}{Vol_{g_*}(M_*)} b_* \right\|_{L^n(M_*,g_*)} \\ & + ||\ell_* + q_*^\partial + (-1)^* \lambda \rho_*||_{W^{1,n}_\partial(M_*,g_*)} \Big) \end{aligned}$$

for $* = 1, 2$ where C_1 depends only on n and the geometry of $(M_1, g_1), (M_2, g_2)$. Now we may use the Sobolev Embedding Theorem [19, Theorem 2.30] and the Trace Theorem [20, Theorem B.10] to obtain the following $\mathcal{C}^0$ estimate

$$\begin{aligned} ||\overline{u}_*||_{\mathcal{C}^0(M_*)} \leq & C_2 \Big(\left\| f_* + q_* - \frac{Vol_{g_K}(K)}{Vol_{g_*}(M_*)} b_* \right\|_{\mathcal{C}^0(M_*)} \\ & + ||\ell_* + q_*^\partial + (-1)^* \lambda \rho_*||_{\mathcal{C}^0(\partial M_*)} \Big) \end{aligned} \tag{2.10}$$

where C_2 is a constant depending only on n and the geometry of $(M_1, g_1), (M_2, g_2)$.

To finish the proof of the claim, it suffices to estimate b_*, q_*, and q_*^∂. It will be convenient to consider the cases $* = 1, 2$ separately – in what follows, the statements will be made for $* = 1$, though analogous arguments hold for $* = 2$ and this is left to the reader. Subtracting (2.8) from b_1 shows

$$\begin{aligned} b_1 = & \frac{1}{\mathrm{Vol}_{g_K}(K)} \Big(\int_{\partial_2 T^\varepsilon(0,0) \setminus \partial_2 T^\varepsilon(\alpha_1,0)} (\ell_1 + q_1^\partial - \lambda \rho_1) \left(\frac{\sqrt{g_1^\partial} - \sqrt{g_\varepsilon^\partial}}{\sqrt{g_1^\partial}} \right) d\sigma_{g_1} \\ & - \int_{T^\varepsilon(0,0) \setminus T^\varepsilon(\alpha_1,0)} (f_1 + q_1) \left(\frac{\sqrt{g_1} - \sqrt{g_\varepsilon}}{\sqrt{g_1}} \right) d\mu_{g_1} \Big) \end{aligned} \tag{2.11}$$

where $\sqrt{g_1^\partial}$ and $\sqrt{g_\varepsilon^\partial}$ denote the Riemannian measures of $g_1|_{\partial M_1}$ and $g_\varepsilon|_{M_1}$, respectively. Notice that we only integrate over $T^\varepsilon(0,0) \setminus T^\varepsilon(\alpha_1, 0)$ since it contains the supports $\mathrm{spt}(\rho_1) \cap \mathrm{spt}(\sqrt{g_1} - \sqrt{g_\varepsilon})$. We will inspect each term in the expression (2.11).

On $T^\varepsilon(0,0) \setminus T^\varepsilon(\alpha_1+1,0)$, notice that $\sqrt{g_1} - \sqrt{g_\varepsilon} = \mathcal{O}(\varepsilon^{m-2})$ and on this portion of the boundary of M we have $\sqrt{g_1^\partial} - \sqrt{g_\varepsilon^\partial} = \mathcal{O}(\varepsilon^{m-2})$. Using this, we can find a constant C_3 which depends on γ and α_1, though not on ε, such that the following inequalities hold

$$\int_{\partial_2 T^\varepsilon(0,0)\setminus \partial_2 T^\varepsilon(\alpha_1,0)} \left| \ell_1 \left(\frac{\sqrt{g_1^\partial} - \sqrt{g_\varepsilon^\partial}}{\sqrt{g_1^\partial}} \right) \right| d\sigma_{g_1} \leq C_3 \varepsilon^{m-2} ||\ell||_{\mathcal{C}^0_{\gamma+1}(\partial M)}$$
$$\int_{T^\varepsilon(0,0)\setminus T^\varepsilon(\alpha_1,0)} \left| f_1 \left(\frac{\sqrt{g_1} - \sqrt{g_\varepsilon}}{\sqrt{g_1}} \right) \right| d\mu_{g_1} \leq C_3 \varepsilon^{m-2} ||f||_{\mathcal{C}^0_{\gamma+2}(M)}.$$

Next we require pointwise bounds on q_1 and q_1^∂ in order to estimate (2.10). By definition of q_1 and q_1^∂, we have the expressions

$$q_1 = (\Delta_{g_\varepsilon}\rho_1)\tilde{u}_T + 2g_\varepsilon(\nabla\rho_1, \nabla\tilde{u}_T) + \rho_1(\Delta_{g_\varepsilon}\tilde{u}_T) \quad \text{and} \quad q_1^\partial = \rho_1 \partial_\nu \tilde{u}_T$$

where we have used the fact that $\partial_\nu \rho_1 \equiv 0$ on ∂M. It is worthwhile to note that the support of $\nabla\rho_1$ satisfies

$$\mathrm{spt}(\nabla\rho_1) \subset \{y \in M_1 \colon e^{-\alpha_1 - 1} \leq \mathrm{dist}_{g_1}(y, \iota_1(K)) \leq 1\},$$

which we emphasize does not depend on ε. With this and the pointwise estimates of g_ε in mind, notice that, for any α_1 and α_2, we may assume that ρ_1 has been chosen so that both $|\Delta_{g_\varepsilon}\rho_1|$ and $|\nabla\rho_1|^2_{g_\varepsilon}$ are uniformly bounded in ε. Using this observation and the estimates of Propositions 2.1.2, 2.1.5, and 2.1.7, one can show

$$||(\Delta_{g_\varepsilon}\rho_1)\tilde{u}_T||_{\mathcal{C}^0_\gamma(M)} \leq C_4(||f||_{\mathcal{C}^0_{\gamma+2}(M)} + ||\ell||_{\mathcal{C}^0_{\gamma+1}(\partial M)})$$

for some C_4 independent of ε. Inspecting (2.6), we can find a constant C_5, depending on γ and α_1 but not ε, so that

$$||\rho_1 \Delta_{g_\varepsilon} \tilde{u}_T||_{\mathcal{C}^0(M)} \leq C_5 ||f||_{\mathcal{C}^0_{\gamma+2}(M)}$$

$$||\rho_1 \partial_\nu \tilde{u}||_{\mathcal{C}^0(\partial M)} \leq C_5 ||\ell||_{\mathcal{C}^0_{\gamma+1}(\partial M)}.$$

The final term we need to estimate is $g_\varepsilon(\nabla \rho_1, \nabla \tilde{u}_p)$. Let us define

$$D_{\alpha_1} := T^\varepsilon(\alpha_1, 0) \setminus T^\varepsilon(\alpha_1 + 1, 0).$$

Since $\tilde{u}_p$ is a solution to a Poisson equation on the region D_{α_1}, we may apply the classical gradient estimate [19], along with the pointwise estimates of g_ε above, to find an ε-uniform constant C_6 satisfying

$$|\nabla \tilde{u}_T|^2_{g_\varepsilon}(y) \leq \frac{C_6}{\mathrm{dist}_{g_1}(y, \partial D_{\alpha_1})} (||\tilde{u}_T||_{\mathcal{C}^0(D_{\alpha_1})} + ||f_T||_{\mathcal{C}^0(D_{\alpha_1})})$$

for all $y \in D_{\alpha_1}$. Using this estimate with the Cauchy-Schwarz inequality, we can estimate the final term in the expression for q_1

$$|g_\varepsilon(\nabla \rho_1, \nabla \tilde{u}_T)|(y) \leq C_7(||\tilde{u}_T||_{\mathcal{C}^0_\gamma(D_{\alpha_1})} + ||f_T||_{\mathcal{C}^0_{\gamma+2}(D_{\alpha_1})})$$

for another ε-uniform constant C_7.

Summarizing our work so far, we have found a constant C_8, independent of ε, such that

$$q_1(y) \leq C_8(||f||_{\mathcal{C}^0_{\gamma+2}(M)} + ||\ell||_{\mathcal{C}^0_{\gamma+1}(\partial M)}) \tag{2.12}$$

$$q_1^\partial(y) \leq C_8(||f||_{\mathcal{C}^0_{\gamma+2}(M)} + ||\ell||_{\mathcal{C}^0_{\gamma+1}(\partial M})$$

for all $y \in D_{\alpha_1}$. Notice that C_8 depends only on the geometry of $(M_1, g_1), (K, g_K)$, γ, and α_1. Integrating (2.12) yields the desired estimate of λ from the statement of the lemma. In turn, this estimate on λ, (2.12), and the expression (2.11) gives an estimate of the form

$$|b_1| \leq e^{(m-2)t} C_9 (||f||_{\mathcal{C}^0_{\gamma+2}(M)} + ||\ell||_{\mathcal{C}^0_{\gamma+1}(\partial M)})$$

Finally, recalling (2.9) and (2.10), we have arrived at the desired estimate of $|\tilde{u}_1|$. $\square$

Now we chose cut-off functions which will be used to glue together the functions $\tilde{u}_1, u_T$, and $\tilde{u}_2$ from the above claims. For the parameter $B \in (0,1)$ from the hypothesis of Lemma 2.2.1, let $\phi_1, \phi_2 : M \to [0,1]$ be smooth functions satisfying

$$\phi_1 = \begin{cases} 1 & \text{on } M_1 \setminus T^\varepsilon(-B \log \varepsilon, 0) \\ 0 & \text{on } M_2 \setminus T^\varepsilon(0, -(2-B)\log \varepsilon - 1) \end{cases}$$

$$\phi_2 = \begin{cases} 1 & \text{on } M_1 \setminus T^\varepsilon(0, -B \log \varepsilon) \\ 0 & \text{on } M_2 \setminus T^\varepsilon(-(2-B)\log \varepsilon - 1, 0) \end{cases}$$

which are monotone in t and have vanishing normal derivatives $\partial_\nu \phi_* \equiv 0$. ϕ_1 and ϕ_2 are not to be confused with the barrier functions ϕ_δ used in Section 2.12. Since $\varepsilon \in (0, e^{-\max(\alpha_1, \alpha_2)})$, we may have $\operatorname{spt}(\nabla \phi_*) \subset T^\varepsilon(\alpha_1, \alpha_2)$. Next, we will define the approximate solution

$$u := \phi_1 \tilde{u}_1 + u_T + \phi_2 \tilde{u}_2.$$

Observe that the above claims, along with the choice of ϕ_*, imply the estimate on $||u||_{\mathcal{C}^0_\gamma(M)}$ in Lemma 2.2.1. Our final task will be to inspect the error term.

Since the cut-off functions have vanishing normal derivative, we have

$$\partial_\nu u = \ell_1 + \ell_T + \ell_2 = \ell$$

and so we have accumulated no error term on the boundary. Moving on the the laplacian of u, it is straight-forward to compute (keeping the support of $\nabla\phi_*$ in mind)

$$\begin{aligned}\Delta_{g_\varepsilon} u =& \Delta_{g_\varepsilon}(\phi_1\tilde{u}_1) + \Delta_{g_\varepsilon} u_p + \Delta_{g_\varepsilon}(\phi_2\tilde{u}_2)\\ =& \Delta_{g_\varepsilon}(\phi_1)\tilde{u}_1 + g_\varepsilon(\nabla\phi_1, \nabla\tilde{u}_1) + \phi_1 f\rho_1 + \phi_1 q_1 + \phi_1 b_1 \delta_{\iota_1(K)}\\ &+ \Delta_{g_\varepsilon}(\phi_2)\tilde{u}_2 + g_\varepsilon(\nabla\phi_2, \nabla\tilde{u}_2) + \phi_2 f\rho_2 + \phi_2 q_2 + \phi_2 b_2 \delta_{\iota_2(K)}\\ &+ f\rho_T - q_1 - q_2\\ =& f + E_1 + E_2\end{aligned}$$

where $E_* = (\Delta_{g_\varepsilon}\phi_*)\tilde{u}_* + g_\varepsilon(\nabla\phi_*, \nabla\tilde{u}_*)$. And so the error in the statement of Lemma 2.2.1 is given by $E := E_1 + E_2$.

By symmetry, it suffices to estimate the term E_1. Observe that E_1 is supported in the annular region

$$\{(z, t, \theta) \in T^\varepsilon(0,0) : t \in [(1-B)\log\varepsilon, (1-B)\log\varepsilon + 1]\}.$$

By a careful choice of ϕ_1 and applying the same gradient estimate used above (see [19] and [18]), one can find a constant C_{10}, independent of ε, such that

$$||E_1||_{\mathcal{C}^0_{\gamma+2}(M)} \leq C_{10}\varepsilon^{B\gamma}(||f||_{\mathcal{C}^0_{\gamma+2}(M)} + ||\ell||_{\mathcal{C}^0_{\gamma+1}(\partial M)}).$$

This finishes the proof of Lemma 2.2.1 $\square$

2.22. The linear problem II

Lemma 2.2.1 can be refined by solving (2.4) without accumulating the error term E.

Lemma 2.2.4. *Let $\gamma \in (0, m-2)$. There exists a choice of parameters $\alpha_1, \alpha_2 > 1$, $\varepsilon_0 > 0$, and a constant $C > 0$ such that the following statement is satisfied for all $\varepsilon \in (0, \varepsilon_0)$. Given $f \in \mathcal{C}^0_{\gamma+2}(M)$ and $\ell \in \mathcal{C}^0_{\gamma+1}(\partial M)$ satisfying $\int_M f d\mu_{g_\varepsilon} = \int_{\partial M} \ell d\sigma_{g_\varepsilon}$, there is a constant $\lambda = \lambda(f, \ell) \in \mathbb{R}$ and a function $u \in \mathcal{C}^0_\gamma(M)$ satisfying*

$$\begin{cases} \Delta_{g_\varepsilon} u = f & \text{in } M \\ \partial_\nu u = \ell - \lambda \beta_\varepsilon & \text{on } \partial M \\ \int_M u \; d\mu_{g_\varepsilon} = 0 \end{cases}$$

with the estimates

$$\begin{aligned} ||u||_{\mathcal{C}^0_\gamma(M)} &\leq C(||f||_{\mathcal{C}^0_{\gamma+2}(M)} + ||\ell||_{\mathcal{C}^0_{\gamma+1}}) \\ |\lambda| &\leq C(||f||_{\mathcal{C}^0_{\gamma+2}(M)} + ||\ell||_{\mathcal{C}^0_{\gamma+1}}) \end{aligned}$$

Moreover, the constant $C > 0$ depends only on $(M_1, g_1), (M_2, g_2), (K, g_K), \gamma$.

Proof. We will iteratively construct sequences

$$f^{(j)} \in \mathcal{C}^0_{\gamma+2}(M), \quad \ell^{(j)} \in \mathcal{C}^0_{\gamma+1}(\partial M), \quad u^{(j)} \in \mathcal{C}^0_\gamma,$$

$$\lambda^{(j)} \in \mathbb{R}, \quad E^{(j)} \in \mathcal{C}^0_{\gamma+2}(M)$$

and show they converge in appropriate senses. Setting $f^{(0)} := f$ and $\ell^{(0)} := \ell$, Lemma 2.2.1 supplies a triple $u^{(0)}, \lambda^{(0)}$, and $E^{(0)}$ solving

$$\begin{cases} \Delta_{g_\varepsilon} u^{(0)} = f^{(0)} + E^{(0)} & \text{on } M \\ \partial_\nu u^{(0)} = \ell^{(0)} - \lambda^{(0)} \beta_\varepsilon & \text{on } \partial M \end{cases}$$

with estimates. Observe the assumption on f, ℓ implies that $\int_M E^{(0)} d\mu_{g_\varepsilon} = 0$.

Next set $f^{(1)} := -E^{(0)}$, $\ell^{(1)} := 0$ and again apply Lemma 2.2.1 to obtain $u^{(1)}, \lambda^{(1)}$, and $E^{(1)}$ satisfying the appropriate equations and estimates. In general, for $j \geq 1$, apply Lemma 2.2.1 with $f^{(j)} = -E^{(j-1)}$, $\ell^{(j)} = 0$, and $B \in (0,1)$ (to be chosen later) to obtain functions $u^{(j)}, \lambda^{(j)}$, and $E^{(j)}$ upon noting that $\int_M E^{(j-1)} d\mu_{g_\varepsilon} = 0$. In other words, for each $j \geq 1$, we have

$$\begin{cases} \Delta_{g_\varepsilon} u^{(j)} = f^{(j)} + E^{(j)} & \text{in } M \\ \partial_\nu u^{(j)} = -\lambda^{(j)} \beta_\varepsilon & \text{on } \partial M \end{cases}$$

along with a constant $C > 0$, independent of ε and j, such that

$$\begin{array}{rcccl} ||u^{(j)}||_{\mathcal{C}^0_\gamma(M)} & \leq & C||f^{(j)}||_{\mathcal{C}^0_{\gamma+2}(M)} & \leq & C(C\varepsilon^{B\gamma})^{j-1}(||f||_{\mathcal{C}^0_{\gamma+2}(M)} + ||\ell||_{\mathcal{C}^0_{\gamma+1}}) \\ |\lambda^{(j)}| & \leq & C||f^{(j)}||_{\mathcal{C}^0_{\gamma+2}(M)} & \leq & C(C\varepsilon^{B\gamma})^{j-1}(||f||_{\mathcal{C}^0_{\gamma+2}(M)} + ||\ell||_{\mathcal{C}^0_{\gamma+1}}) \\ ||E^{(j)}||_{\mathcal{C}^0_{\gamma+2}(M)} & \leq & C\varepsilon^{B\gamma}||f^{(j)}||_{\mathcal{C}^0_{\gamma+2}(M)} & \leq & (C\varepsilon^{B\gamma})^{j}(||f||_{\mathcal{C}^0_{\gamma+2}(M)} + ||\ell||_{\mathcal{C}^0_{\gamma+1}}) \end{array}$$

Now consider the partial sums

$$v^{(N)} := \sum_{j=0}^{N} u^{(j)}, \quad \mu^{(N)} := \sum_{j=0}^{N} \lambda^{(j)}$$

and observe that only one error term remains when computing $\Delta_{g_\varepsilon} v^{(N)}$

$$\begin{cases} \Delta_{g_\varepsilon} v^{(N)} = f + E^{(N)} & \text{in } M \\ \partial_\nu v^{(N)} = \ell - \mu^{(N)} \beta_\varepsilon & \text{on } \partial M. \end{cases}$$

Now choose $B \in (0,1)$ so that $C\varepsilon^{B\gamma}$ for all $\varepsilon \in (0, \varepsilon_0)$. One can inspect the above estimates from Lemma 2.2.1 and conclude that the partial sums $v^{(N)}$, $\mu^{(N)}$ form Cauchy sequences in their respective Banach spaces. In fact, the error term vanishes as we take $j \to \infty$

$$||E^{(N)}||_{\mathcal{C}^0_{\gamma+2}(M)} \leq (C\varepsilon^{B\gamma})^j (||f||_{\mathcal{C}^0_{\gamma+2}(M)} + ||\ell||_{\mathcal{C}^0_{\gamma+1}}) \to 0.$$

This gives us a real number λ and a function $u \in \mathcal{C}^0_\gamma$ such that

$$E^{(N)} \to 0, \quad v^{(N)} \to u, \quad \mu^{(N)} \to \lambda,$$

the convergence being in the appropriate space. As for the estimates of u and λ, observe that

$$\begin{aligned} ||v^{(N)}||_{\mathcal{C}^0_{\gamma+2}(M)} &\leq \sum_{j=0}^{N} ||u^{(j)}||_{\mathcal{C}^0_{\gamma+2}(M)} \\ &\leq \sum_{j=0}^{N} C(C\varepsilon^{B\gamma}(||f||_{\mathcal{C}^0_{\gamma+2}(M)} + ||\ell||_{\mathcal{C}^0_{\gamma+1}}) \\ &\to \frac{C}{1 - C\varepsilon^{B\gamma}}(||f||_{\mathcal{C}^0_{\gamma+2}(M)} + ||\ell||_{\mathcal{C}^0_{\gamma+1}}), \end{aligned}$$

which gives the estimate in Lemma 2.2.4. The desired bound on λ follows from a similar computation. $\square$

2.3. Returning to the Fixed Point Problem

The aim of the next two sections is to finish the proofs of Theorems 2.0.7, 2.0.9, and 2.0.9 by producing a function $\psi \in C^\infty(M)$ which solves the equation (1.6) on (M, g_ε) for each $\varepsilon \in (0, \varepsilon_0)$. Since we are seeking a small conformal change to g_ε, we will write the conformal factor as $\psi = 1 + u$. In terms of u, equation (1.6) becomes

$$\begin{cases} \Delta_{g_\varepsilon} u = F_\varepsilon(u) & \text{in } M \\ \partial_\nu u = F_\varepsilon^\partial(u) & \text{on } \partial M \end{cases} \tag{2.13}$$

where we have introduced the sort-hand notation

$$\begin{aligned} F_\varepsilon(u) &:= c_n R_{g_\varepsilon}(1+u) \\ F_\varepsilon^\partial(u) &:= 2c_n(Q(1+u)^{\frac{n}{n-2}} - H_{g_\varepsilon}(1+u)) \end{aligned}$$

for some constant Q. The convergence statements in Theorems 2.0.7, 2.0.9, and 2.0.11 will follow as consequences of our construction of u. Upon producing a solution u to (2.13), observe that $(1+u)^{\frac{4}{n-2}} g_\varepsilon$ will be scalar-flat and have constant boundary mean curvature Q.

In what follows, for a given $\gamma \in (0, m-2)$, we will restrict our attention to $u \in \mathcal{C}^0_\gamma(M)$ which lie in the ball of radius $r_\varepsilon := \varepsilon^{2\gamma}$ about $0 \in \mathcal{C}^0_\gamma(M)$. We will denote this ball by $B^\gamma_{r_\varepsilon}$. Let us suppose for a moment that we have in hand a solution $u \in B^\gamma_{r_\varepsilon}$ to (2.13). Integrating by parts will tell us the mean curvature of the resulting conformal metric

$$Q = \frac{\frac{1}{2}\int_M R_{g_\varepsilon}(1+u)d\mu_{g_\varepsilon} + \int_{\partial M} H_{g_\varepsilon}(1+u)d\sigma_{g_\varepsilon}}{\int_{\partial M}(1+u)^{\frac{n}{n-2}}d\sigma_{g_\varepsilon}}.$$

Using the L^1 estimates on R_{g_ε} and H_{g_ε} from Propositions 2.1.1, 2.1.3, and 2.1.6, one finds $|Q| = \mathcal{O}(\varepsilon^{m-2})$.

Before we solve (2.13), we will first use our linear analysis to establish a solution to the following projected version of the problem

$$\begin{cases} \Delta_{g_\varepsilon} u = F_\varepsilon(u) & \text{in } M \\ \partial_\nu u = F_\varepsilon^\partial(u) - \lambda_{F_\varepsilon(u)}\beta_\varepsilon & \text{on } \partial M. \end{cases} \tag{2.14}$$

Later, we will arrange for the vanishing of term $\lambda_{F_\varepsilon(u)}$, giving a genuine solution to (2.13).

To phrase (2.14) as a fixed point problem, we introduce the following maps

$$\begin{aligned} F_\varepsilon &: \mathcal{C}^0_\gamma(M) \to \mathcal{C}^0_{\gamma+2}(M) \times \mathcal{C}^0_{\gamma+1}(\partial M), \quad v \mapsto (F_\varepsilon(v), F_\varepsilon^\partial(v)) \\ G_\varepsilon &: \mathcal{C}^0_{\gamma+2}(M) \times \mathcal{C}^0_{\gamma+1}(\partial M) \to \mathcal{C}^0_\gamma(M), \quad (v, w) \mapsto G_\varepsilon(v, w) \end{aligned}$$

where $G_\varepsilon(v, w)$ is the solution to the boundary problem

$$\begin{cases} \Delta_{g_\varepsilon} G_\varepsilon(v, w) = v & \text{in } M \\ \partial_\nu G_\varepsilon(v, w) = w - \lambda_{G_\varepsilon(v,w)}\beta_\varepsilon & \text{on } \partial M, \end{cases}$$

whose existence is given by Lemma 2.2.4. Evidently, solving (2.14) is equivalent to finding a fixed point of the composition

$$P_\varepsilon : \mathcal{C}^0_\gamma(M) \to \mathcal{C}^0_\gamma(M), \quad v \mapsto G_\varepsilon(F_\varepsilon(v), F_\varepsilon^\partial(v))$$

for some γ.

Proposition 2.3.1. *Let $\gamma \in (0, \frac{1}{2})$. There is an $\varepsilon_0 > 0$ such that $P_\varepsilon(B^\gamma_{r_\varepsilon}) \subset B^\gamma_{r_\varepsilon}$ for all $\varepsilon \in (0, \varepsilon_0)$.*

Proof. As usual, C_k for $k = 1, 2, 3\ldots$ will denote positive constants independent of ε. For $v \in B^\gamma_{r_\varepsilon}$, we may apply Lemma 2.2.4 with the functions $F_\varepsilon(v), F^\partial_\varepsilon(v))$ to get a solution, $P_\varepsilon(v)$, of the linear problem along with the estimate

$$||P_\varepsilon(v)||_{\mathcal{C}^0_\gamma(M)} \leq C_1 \left(||F_\varepsilon(v)||_{\mathcal{C}^0_{\gamma+2}(M)} + ||F^\partial_\varepsilon(v)||_{\mathcal{C}^0_{\gamma+1}(\partial M)} \right).$$

It is suffices to dominate $||F_\varepsilon(v)||_{\mathcal{C}^0_{\gamma+2}(M)}$ and $||F^\partial_\varepsilon(v)||_{\mathcal{C}^0_{\gamma+1}(\partial M)}$ by the product of r_ε and some positive power of ε.

We begin with the first summand. Applying Propositions 2.1.1, 2.1.3, and 2.1.6 and the definition of ψ_ε,

$$\begin{aligned} |F_\varepsilon(v)\psi_\varepsilon^{\gamma+2}| &\leq C_2(|R_{g_\varepsilon}|\psi_\varepsilon^{\gamma+2} + |R_{g_\varepsilon}| \cdot |v|\psi_\varepsilon^{\gamma+2}) \\ &\leq C_3(\varepsilon^{m-2} + r_\varepsilon \varepsilon^{m-2}) \\ &\leq C_4 r_\varepsilon \varepsilon^{m-2-2\gamma}. \end{aligned}$$

For the second summand in the estimate, we have

$$\begin{aligned} |F^\partial_\varepsilon(v)|\psi_\varepsilon^{\gamma+1} &\leq C_5(\psi_\varepsilon^{\gamma+1}|Q|(1+v)^{\frac{n}{n-2}} - \psi_\varepsilon^{\gamma+1}|H_{g_\varepsilon}|(1+v)) \\ &\leq C_6 \varepsilon^{m-2} r_\varepsilon. \end{aligned}$$

Together, we have shown

$$||P_\varepsilon||_{\mathcal{C}^0_\gamma(M)} \leq C_7 r_\varepsilon \varepsilon^{m-2-2\gamma},$$

as claimed. $\square$

It is a good time to observe a fact we will use later – the proofs in this section hold if $|Q|$ was only $\mathcal{O}(\varepsilon^{\frac{m-2}{2}})$, so long as we restrict ourselves to $\gamma \in (0, \frac{1}{4})$. Now we are ready to solve (2.14).

Proposition 2.3.2. *Let $\gamma \in (0, \frac{1}{2})$. There exists an $\varepsilon_0 > 0$ so that, for each $\varepsilon \in (0, \varepsilon_0)$, (2.14) has a smooth solution $u \in B^{\gamma}_{r_\varepsilon}$.*

Proof. We will proceed by showing that the mapping P_ε is contractive on the ball $B^{\gamma}_{r_\varepsilon}$. In other words, we will show that there is a $\varepsilon_0 > 0$ so that

$$||P_\varepsilon(u) - P_\varepsilon(v)||_{\mathcal{C}^0_\gamma(M)} \leq K||u - v||_{\mathcal{C}^0_\gamma(M)}$$

for all $\varepsilon \in (0, \varepsilon_0)$ and $u, v \in B^{\gamma}_{r_\varepsilon}$. We begin by applying Lemma 2.2.4t

$$||P_\varepsilon(u) - P_\varepsilon(v)||_{\mathcal{C}^0_\gamma(M)} \leq C\left(||F_\varepsilon(u) - F_\varepsilon(v)||_{\mathcal{C}^0_{\gamma+2}(M)} + ||F^\partial_\varepsilon(u) - F^\partial_\varepsilon(v)||_{\mathcal{C}^0_{\gamma+1}(\partial M)}\right),$$

where $C > 0$ is independent of ε. By Proposition 2.3.1, all involved terms lie in $B^{\gamma}_{r_\varepsilon}$ for small ε.

For the first summand, keeping in mind the pointwise estimate on $|R_{g_\varepsilon}|$ from Propositions 2.1.1, 2.1.3, and 2.1.6, and the restriction on m, we find

$$\begin{aligned}
\psi_\varepsilon^{\gamma+2}|F_\varepsilon(u) - F_\varepsilon(v)| &\leq C_8\psi_\varepsilon^{\gamma+2}|R_{g_\varepsilon}(u - v)| \\
&\leq C_9\varepsilon\cosh^{3-m}(t)||u - v||_{\mathcal{C}^0_\gamma(M)} \\
&\leq C_9\varepsilon||u - v||_{\mathcal{C}^0_\gamma(M)}.
\end{aligned}$$

We can perform a similar estimate for the boundary term

$$
\begin{aligned}
\psi_\varepsilon^{\gamma+1}|F_\varepsilon^\partial(u) - F_\varepsilon^\partial(v)| &= C_{10}\varepsilon\cosh(t)\psi_\varepsilon^\gamma|Q((1+u)^{\frac{n}{n-2}} - (1+v)^{\frac{n}{n-2}}) - H_{g_\varepsilon}(u-v)| \\
&\leq C_{11}|Q|\cdot||u-v||_{\mathcal{C}_\gamma^0(\partial M)} + C_{12}\varepsilon\cosh(t)|H_{g_\varepsilon}|\cdot||u-v||_{\mathcal{C}_\gamma^0(\partial M)} \\
&\leq ||u-v||_{\mathcal{C}_\gamma^0}(C_{13}\varepsilon^{m-2} + C_{14}\varepsilon).
\end{aligned}
$$

Since all the constants C_i are independent of ε, we can find an $\varepsilon_0 > 0$ which makes P_ε a contractive mapping on $B_{r_\varepsilon}^\gamma$ for $\varepsilon < \varepsilon_0$.

The Banach fixed point theorem applied to P_ε on $B_{r_\varepsilon}^\gamma$ gives a fixed point of P_ε, which we call u_ε. Evidently, u_ε is a solution to equation (2.14), concluding the proof of Proposition 2.3.2. $\square$

2.4. Vanishing of $\lambda_{F_\varepsilon(v)}$

In the last section we found, for all sufficiently small ε, a solution $u_\varepsilon \in \mathcal{C}_\gamma^0(M)$ to

$$
\begin{cases}
\Delta_{g_\varepsilon}u_\varepsilon = F_\varepsilon(u_\varepsilon) & \text{in } M \\
\partial_\nu u_\varepsilon = F_\varepsilon^\partial(u_\varepsilon) - \lambda_{F_\varepsilon(u_\varepsilon)}\beta_\varepsilon & \text{on } \partial M.
\end{cases}
$$

The corresponding conformal metric $(1+u_\varepsilon)^{\frac{4}{n-2}}g_\varepsilon$ will be scalar flat, but will have boundary mean curvature equal to

$$
Q - \frac{1}{2c_n}(1+u)^{\frac{-n}{n-2}}\lambda_{F_\varepsilon(u_\varepsilon)}\beta_\varepsilon
$$

which is non-constant. Next, we will show that ε-small conformal changes can be made to the original metrics g_1 and g_2 before applying the gluing procedure such that, after applying the above construction and fixed point argument, the new projection term $\lambda_{\tilde{F}_\varepsilon(u_\varepsilon)}$ will vanish.

Fix $\tilde{w}_1$ and $\tilde{w}_2$, two non-zero smooth functions supported on the interiors of $M_1 \setminus V_1^\varepsilon$ and $M_2 \setminus V_2^\varepsilon$, respectively. For real parameters a_* ($* = 1, 2$) which will be chosen later, we consider the functions

$$w_* := a_* \varepsilon^{\frac{m-2}{2}} \tilde{w}_*$$

and use them to deform the original metrics

$$\tilde{g}_* := (1 + w_*)^{\frac{4}{n-2}} g_*.$$

Replacing g_1 and g_2 with $\tilde{g}_1$ and $\tilde{g}_2$ in the geometric gluing construction presented in Section 2.1, we produce a new family of metrics $\tilde{g}_\varepsilon$ on the generalized connected sum M. Of course, $\tilde{g}_\varepsilon$ only differs from g_ε on the supports of w_1 and w_2. Keeping in mind that $\sup_M |w_*| = \mathcal{O}(\varepsilon^{\frac{n-2}{2}})$, all of the analysis we have done on the family of linear operators $(\Delta_{g_\varepsilon}, \partial_\nu)$ also holds for the new family $(\Delta_{\tilde{g}_\varepsilon}, \partial_\nu)$. Namely, the proof of the a priori estimate in Lemma (2.2.4) also works for the metrics $\tilde{g}_\varepsilon$. As usual, we will assume that α_1 and α_2 have be chosen so that $\int_{\partial M} \beta_\varepsilon d\sigma_{\tilde{g}_\varepsilon} = 0$.

Next, we need to gather information about the new scalar curvature and boundary mean curvature. Notice that the support of $R_{\tilde{g}_\varepsilon}$ has three disjoint components – $T^\varepsilon(0,0)$ and the supports of w_*. Since $R_{\tilde{g}_\varepsilon}$ agrees with R_{g_ε} on $T^\varepsilon(0,0)$, we still have the estimate of Propositions 2.1.1, 2.1.3, and 2.1.6 there. On the support of w_*, the formula for scalar curvature under conformal change reads

$$R_{\tilde{g}_\varepsilon} = R_{\tilde{g}_*} = -\frac{1}{c_n}(1 + w_*)^{-\frac{n+2}{n-2}} \Delta_{g_*} w_*$$

and we conclude that $R_{\tilde{g}_\varepsilon} = \mathcal{O}(\varepsilon^{\frac{m-2}{2}})$ on the supports of w_*. Hence, there is a constant $C > 0$ such that

$$|R_{\tilde{g}_\varepsilon}| \leq C\varepsilon^{\frac{m-2}{2}} \psi_\varepsilon^{1-m}(t).$$

As for the mean curvature of the boundary, $H_{\tilde{g}_\varepsilon}$ does not differ from H_{g_ε} since w_* is supported away from the boundary.

Now, upon restricting our choice of γ to the interval $(0, \frac{1}{4})$, we may apply the fixed point argument from Section 2.3 to produce a solution $\tilde{u}_\varepsilon \in B^\gamma_{r_\varepsilon} \subset \mathcal{C}^0_\gamma(M)$ to

$$\begin{cases} \Delta_{\tilde{g}_\varepsilon} \tilde{u}_\varepsilon = \tilde{F}_\varepsilon(\tilde{u}_\varepsilon) & \text{in } M \\ \partial_\nu \tilde{u}_\varepsilon = \tilde{F}^\partial_\varepsilon(\tilde{u}_\varepsilon) - \lambda_{\tilde{F}_\varepsilon(\tilde{u}_\varepsilon)} \beta_\varepsilon & \text{on } \partial M \end{cases}$$

where $\tilde{F}_\varepsilon(u) := c_n R_{\tilde{g}_\varepsilon}(1+u)$ and $\tilde{F}^\partial_\varepsilon(u) := 2c_n(\widetilde{Q}(1+u)^{\frac{n}{n-2}} - H_{\tilde{g}_\varepsilon}(1+u))$. Once this is achieved, the conformal metric $(1+\tilde{u}_\varepsilon)^{\frac{4}{n-2}}\tilde{g}_\varepsilon$ will be scalar flat and have boundary mean curvature equal to

$$\widetilde{Q} - \frac{1}{2c_n}(1+\tilde{u}_\varepsilon)^{-\frac{n}{n-2}} \lambda_{\tilde{F}_\varepsilon(\tilde{u}_\varepsilon)} \beta_\varepsilon$$

where the constant $\widetilde{Q}$ can be computed by integrating by parts

$$\widetilde{Q} = \frac{\frac{1}{2}\int_M R_{\tilde{g}_\varepsilon}(1+\tilde{u}_\varepsilon) d\mu_{\tilde{g}_\varepsilon} + \int_{\partial M} H_{\tilde{g}_\varepsilon}(1+\tilde{u}_\varepsilon) d\sigma_{\tilde{g}_\varepsilon}}{\int_{\partial M}(1+\tilde{u}_\varepsilon)^{\frac{n}{n-2}} d\sigma_{\tilde{g}_\varepsilon}}.$$

As before, the projection term $\lambda_{\tilde{F}_\varepsilon(\tilde{v}_\varepsilon)}$ may be non-zero, though it now (continuously) depends on the parameters a_*. We will exploit this to establish the following proposition, concluding the proof of Theorems 2.0.7, 2.0.9, and 2.0.11. The following properties of the metrics $\tilde{g}_*$ will be useful in our computations later this section

$$\Delta_{\tilde{g}_*}\cdot = -\frac{1}{2c_n}(1+w_*)^{-\frac{n+2}{n-2}} g_*(\nabla w_*, \nabla \cdot) + (1+w_*)^{\frac{-4}{n-2}}\Delta_{g_*}\cdot$$
$$d\mu_{\tilde{g}_\varepsilon} = (1+w_*)^{\frac{2n}{n-2}} d\mu_{g_\varepsilon}.$$

Proposition 2.4.1. *For small ε, there is a choice of the real parameters a_1 and a_2 such that the resulting rough projection $\lambda_{\tilde{F}_\varepsilon(\tilde{u}_\varepsilon)}$ vanishes.*

Proof. It suffices to show that the sign of $\lambda_{\tilde{F}_\varepsilon(\tilde{u}_\varepsilon)}$ can be changed by manipulating a_1 and a_2. From the proof of Lemma 2.2.4, we may regard $\lambda_{\tilde{F}_\varepsilon(u_\varepsilon)}$ as the following sum

$$\lambda_{\tilde{F}_\varepsilon(u_\varepsilon)} = \sum_{j=0}^{\infty} \lambda^{(j)}_{\tilde{F}_\varepsilon(u_\varepsilon)}$$

where each term has estimate

$$|\lambda^{(j)}_{\tilde{F}_\varepsilon(\tilde{v}_\varepsilon)}| \leq C(C\varepsilon^{B\gamma})^j(||\tilde{F}_\varepsilon(\tilde{u}_\varepsilon)||_{\mathcal{C}^0_\gamma(M)} + ||\tilde{F}^\partial_\varepsilon(\tilde{u}_\varepsilon)||_{\mathcal{C}^0_{\gamma+1}(\partial M)}),$$

where $C > 0$ is uniform in ε. From this expression we see that the sign of $\lambda_{\tilde{F}_\varepsilon(u_\varepsilon)}$, for small ε and an appropriate choice of B, is determined by the first term in the sum. We will need to recall the formula for $\lambda^{(0)}$ from the proof of Lemma 2.2.4

$$\begin{aligned}\lambda^{(0)} :=& \frac{1}{\int_{\partial M}(\rho_1+\rho_2)d\sigma_{\tilde{g}_\varepsilon}}\Big(\int_M \tilde{F}_\varepsilon(\tilde{u}_\varepsilon)\beta_\varepsilon d\mu_{\tilde{g}_\varepsilon} - \int_{\partial M}\tilde{F}^\partial_\varepsilon(\tilde{u}_\varepsilon)\beta_\varepsilon d\sigma_{\tilde{g}_\varepsilon} + \\ &+ \int_M (\Delta_{\tilde{g}_\varepsilon}(\rho_1\tilde{u}_T) - \Delta_{\tilde{g}_\varepsilon}(\rho_2\tilde{u}_T))d\mu_{\tilde{g}_\varepsilon} - \int_{\partial M}(\partial_{\tilde{\nu}}(\rho_1\tilde{u}_T) - \partial_{\tilde{\nu}}(\rho_2\tilde{u}_T))d\sigma_{\tilde{g}_\varepsilon}\Big)\end{aligned}$$

where $\tilde{u}_T$ is the solution to

$$\begin{cases} \Delta_{\tilde{g}_\varepsilon} \tilde{u}_T = \tilde{F}_\varepsilon(\tilde{u}_\varepsilon)\rho_T & \text{on } T^\varepsilon(\alpha_1, \alpha_2) \\ \tilde{u}_T \equiv 0 & \text{on } \partial_1 T^\varepsilon(\alpha_1, \alpha_2) \\ \partial_\nu \tilde{u}_T = \tilde{F}^\partial_\varepsilon(\tilde{u}_\varepsilon)\rho_T & \text{on } \partial_2 T^\varepsilon(\alpha_1, \alpha_2) \end{cases}$$

which originally appeared in the first step in the proof of Lemma 2.2.4. Next, we will inspect each of the terms in this expression for $\lambda^{(0)}$.

Unpacking the notations in the first term, we have

$$\int_M \tilde{F}_\varepsilon(\tilde{v}_\varepsilon)\beta_\varepsilon d\mu_{\tilde{g}_\varepsilon} = c_n \int_M (R_{\tilde{g}_1} + R_{g_\varepsilon} + R_{\tilde{g}_2})(1 + \tilde{u}_\varepsilon)(\rho_1 - \rho_2) d\mu_{\tilde{g}_\varepsilon}.$$

Recalling that $\tilde{u}_\varepsilon$ lies in $B^\gamma_{r_\varepsilon} \subset \mathcal{C}^0_\gamma(M)$ and applying the pointwise estimate of R_{g_ε}, it is straightforward to show

$$\int_M R_{g_\varepsilon}(1 + \tilde{u}_\varepsilon)\rho_* d\mu_{\tilde{g}_\varepsilon} = -4m\mathrm{Vol}(K)\omega_{m-1} + \mathcal{O}(e^{-\alpha_*}\varepsilon^{m-2})$$

and

$$\int_M R_{\tilde{g}_*}\rho_* d\mu_{\tilde{g}_\varepsilon} = \frac{1}{c_n}\int_{M_*} |\nabla w_*|^2_{g_*} d\mu_{g_*}$$

where ω_{m-1} denotes the volume of the unit $(m-1)$-sphere.

After integrating by parts, the remaining piece of the first term can be written as

$$\int_M R_{\tilde{g}_*}\tilde{u}_\varepsilon d\mu_{\tilde{g}_\varepsilon} = \int_{M_*} w_* \Delta_{g_*}\tilde{u}_\varepsilon d\mu_{g_*} + \int_{\partial M_*} w_* \partial_\nu \tilde{u}_\varepsilon d\sigma_{g_*} + \mathcal{O}(\varepsilon^{m-2+\gamma}).$$

Now we Taylor expand and rearrange the above expression for $\Delta_{\tilde{g}_*}$ and $\partial_{\tilde{\nu}}$

$$\begin{aligned}\Delta_{g_*}\tilde{u}_\varepsilon &= \left(1+\frac{4}{n-2}w_* + \mathcal{O}(\varepsilon^{m-2})\right)\Delta_{\tilde{g}_*}\tilde{u}_\varepsilon - 2g_*(\nabla w_*, \nabla \tilde{u}_\varepsilon) + \\ &\quad + 2w_* g_*(\nabla w_*, \nabla\tilde{u}_\varepsilon) + \mathcal{O}(\varepsilon^{m-2+2\gamma}) \\ \partial_\nu \tilde{u}_\varepsilon &= \left(1+\frac{2}{n-2}w_* + \mathcal{O}(\varepsilon^{m-2})\right)\partial_{\tilde{\nu}}\tilde{u}_\varepsilon\end{aligned}$$

and multiply by w_* to find

$$\begin{aligned}\int_M R_{\tilde{g}_*}\tilde{u}_\varepsilon d\mu_{\tilde{g}_\varepsilon} &= \int_{M_*} w_*\tilde{F}_\varepsilon(\tilde{u}_\varepsilon)d\mu_{g_*} + \int_{\partial M} w_*(\tilde{F}^\partial_\varepsilon(\tilde{u}_\varepsilon) - \lambda_{\tilde{F}_\varepsilon(\tilde{u}_\varepsilon)}\beta_\varepsilon)d\sigma_{g_\varepsilon} + \mathcal{O}(\varepsilon^{m-2+\gamma}) \\ &= \int_M |\nabla w_*|^2_{g_*} d\mu_{g_*} - \lambda_{\tilde{F}_\varepsilon(\tilde{u}_\varepsilon)}\mathcal{O}(\varepsilon^{\frac{m-2}{2}}) + \mathcal{O}(\varepsilon^{m-2+\gamma})\end{aligned}$$

where we have used the formula for $R_{\tilde{g}_\varepsilon}$ in the expression for $\tilde{F}_\varepsilon(\tilde{u}_\varepsilon)$ and integrated by parts. To summarize our efforts so far, we have found

$$\begin{aligned}\int_M \tilde{F}_\varepsilon(\tilde{v}_\varepsilon)\beta_\varepsilon d\mu_{\tilde{g}_\varepsilon} = (c_n - 1)\left(\int_{M_1} |\nabla w_1|^2_{g_1} d\mu_{g_1} - \int_{M_2} |\nabla w_2|^2_{g_2} d\mu_{g_2}\right) - \lambda_{\tilde{F}_\varepsilon(\tilde{u}_\varepsilon)}\mathcal{O}(\varepsilon^{\frac{m-2}{2}}) + \\ + \mathcal{O}(e^{-\max(\alpha_1,\alpha_2)}\varepsilon^{m-2}).\end{aligned} \tag{2.15}$$

Moving along to the next term in the expression for $\lambda^{(0)}$, we have

$$\int_{\partial M}\tilde{F}^\partial_\varepsilon(\tilde{u}_\varepsilon)\beta_\varepsilon d\sigma_{\tilde{g}_\varepsilon} = 2c_n\int_{\partial M}(\widetilde{Q}(1+\tilde{u}_\varepsilon)^{\frac{n}{n-2}} - H_{\tilde{g}_\varepsilon}(1+\tilde{u}_\varepsilon))(\rho_1-\rho_2)d\sigma_{\tilde{g}_\varepsilon}.$$

Now since $H_{\tilde{g}_\varepsilon} \equiv H_{g_\varepsilon}$, we have

$$\int_{\partial M} H_{\tilde{g}_\varepsilon}(1+\tilde{u}_\varepsilon)\rho_* d\sigma_{\tilde{g}_\varepsilon} = \mathcal{O}(e^{-\alpha_*}\varepsilon^{m-2})$$

which can be seen by computing H_{g_ε} on this portion of the neck, noting that the cut off functions ξ and η both take the value of 1 on the support of ρ_1.

Now is a good time to comment on the convergence statements in the main theorems. As we have mentioned already, we may apply the pointwise estimate of $R_{\tilde{g}_\varepsilon}$ and the $\mathcal{C}^0_\gamma$-norm of $\tilde{v}_\varepsilon$ to find that $\tilde{Q}$ satisfies the estimate

$$|\widetilde{Q}| = \mathcal{O}(\varepsilon^{\frac{m-2}{2}}).$$

Evidently, $\tilde{F}_\varepsilon(\tilde{u}_\varepsilon) = \mathcal{O}(\varepsilon^{\frac{m-2}{2}})$ on the support of w_* and $\lambda_{\tilde{F}_\varepsilon(\tilde{u}_\varepsilon)} = \mathcal{O}(\varepsilon^{(m-2)/2})$. Using the computations made in this section, one can inspect the formula for $\widetilde{Q}$ and improve our estimate to $|\widetilde{Q}| = \mathcal{O}(\varepsilon^{m-2})$, as claimed in Theorems 2.0.7. 2.0.9. and 2.0.11. This can be used to estimate the remaining term in the expression for $\int_{\partial M} \tilde{F}_\varepsilon(\tilde{u}_\varepsilon)\beta_\varepsilon d\sigma_{\tilde{g}_\varepsilon}$ and conclude

$$\int_{\partial M} \tilde{F}^{\partial}_\varepsilon(\tilde{v}_\varepsilon)\beta_\varepsilon d\sigma_{\tilde{g}_\varepsilon} = \mathcal{O}(e^{-\max(\alpha_1,\alpha_2)}\varepsilon^{m-2}). \tag{2.16}$$

The final two integrals in the expression for $\lambda^{(0)}$ will be treated together. Integrating by parts, we have

$$\begin{aligned}\int_M \Delta_{g_\varepsilon}(\rho_*\tilde{u}_T)d\mu_{\tilde{g}_\varepsilon} - \int_{\partial M} \partial_\nu(\rho_*\tilde{u}_T)d\sigma_{\tilde{g}_\varepsilon} &= \int_M (\rho_*\Delta_{g_\varepsilon}\tilde{u}_T + 2(g_\varepsilon(\nabla\rho_*, \nabla\tilde{u}_T) + \tilde{u}_T\Delta_{g_\varepsilon}\rho_*) - \\ &\qquad \tilde{u}_T\Delta_{g_\varepsilon}\rho_*)d\mu_{g_\varepsilon} - \int_{\partial M} \rho_*\partial_\nu\tilde{u}_T d\sigma_{g_\varepsilon} \\ &= \int_M \rho_*\rho_T\tilde{F}_\varepsilon(\tilde{u}_\varepsilon) - \tilde{u}_T\Delta_{g_\varepsilon}\rho_* d\mu_{g_\varepsilon} - \\ &\qquad \int_{\partial M} \rho_*\rho_T\tilde{F}^{\partial}_\varepsilon(\tilde{u}_\varepsilon)d\sigma_{g_\varepsilon}\end{aligned}$$

where we have used the fact that $\partial_\nu \rho_* \equiv 0$. In order to proceed, we will need the pointwise estimate of Propositions 2.1.2, 2.1.5, and 2.1.7:

$$\begin{aligned}\tilde{u}_T &\leq C\psi_\varepsilon^\gamma \left(||\tilde{F}_\varepsilon(\tilde{u}_\varepsilon)||_{\mathcal{C}^0_{\gamma+2}(T^\varepsilon(\alpha_1,\alpha_2))} + ||\tilde{F}^\partial_\varepsilon(\tilde{u}_\varepsilon)||_{\mathcal{C}^0_{\gamma+1}(\partial_2 T^\varepsilon(\alpha_1,\alpha_2))} \right) \\ &\leq C'\varepsilon^{m-2}\psi_\varepsilon^\gamma\end{aligned}$$

for $C, C' > 0$ independent of ε. Keeping in mind that $\rho_*\rho_T$ and $\Delta_{\tilde{g}_\varepsilon}(\rho_*)$ vanish outside of $T^\varepsilon(\alpha_1, -2\log\varepsilon - \alpha_1 - 1)$ if $* = 1$ and $T^\varepsilon(-2\log\varepsilon - \alpha_2 - 1, \alpha_2)$ if $* = 2$, one can use the pointwise estimate on $\tilde{u}_T$ to find

$$\int_M \Delta_{g_\varepsilon}(\rho_*\tilde{u}_T)d\mu_{\tilde{g}_\varepsilon} - \int_{\partial M} \partial_\nu(\rho_*\tilde{u}_T)d\sigma_{\tilde{g}_\varepsilon} = \mathcal{O}(e^{-\alpha_*}\varepsilon^{m-2}).$$

Combining the above estimates, we have

$$\begin{aligned}\lambda^{(0)} =&(c_n - 1)\left(\int_{M_1} |\nabla w_1|^2_{g_1} d\mu_{g_1} - \int_{M_2} |\nabla w_2|^2_{g_2} d\mu_{g_2} \right) - \\ &\lambda^{(0)}\mathcal{O}(\varepsilon^{\frac{m-2}{2}}) + \mathcal{O}(e^{-\max(\alpha_1,\alpha_2)}\varepsilon^{m-2}).\end{aligned}$$

Since $||\nabla w_*||_{L^2} = a_*\mathcal{O}(\varepsilon^{m-2})$, we can choose α_1, α_2 so that the term $||\nabla w_1||_{L^2} - ||\nabla w_2||_{L^2}$ dominates the rest of the expression for $\lambda^{(0)}$. Evidently, one can vary the parameters a_1 and a_2 so that the sign of $\lambda^{(0)}$ – and hence the sign of $\lambda_{\tilde{F}_\varepsilon(\tilde{v}_\varepsilon)}$ – changes. As we previously noted, $\lambda_{\tilde{F}_\varepsilon(\tilde{v}_\varepsilon)}$ depends continuously on a_1 and a_2, so we conclude that there are suitable values of a_1 and a_2 for which the projection term $\lambda_{\tilde{F}_\varepsilon(\tilde{v}_\varepsilon)}$ vanishes. This finishes the proof of Theorems 2.0.7, 2.0.9, and 2.0.11. □

2.5. The Non-critical Case

So far, we have produced a family of metrics $(1+\tilde{u}_\varepsilon)^{\frac{4}{n-2}}\tilde{g}_\varepsilon$ on M, each scalar-flat and having constant boundary mean curvature of size $\mathcal{O}(\varepsilon^{m-2})$. In this section we will prove Theorems 2.0.8, 2.0.10, and 2.0.12, where we arrange for this mean curvature to vanish entirely. To achieve this, we will need yet another alteration to the above construction. From now on, we assume that neither of the original manifolds are Ricci-flat with totally geodesic boundary , i.e. we assume that

$$\max(\sup_{M_*}|ric_{g_*}|, \sup_{\partial M_*}|A_{g_*}|) > 0$$

for both $*=1$ and 2.

Let S_* be a positive-definite symmetric 2-tensor with

$$\operatorname{spt}(S_*) \subset \left((M_* \setminus \iota^1_*) \cap (\operatorname{spt}(\operatorname{Ric}_{g_*}) \cup \operatorname{spt}(A_{g_*}))\right).$$

For a real parameter $\tilde{r}_*$, set $r_* := \tilde{r}_*\varepsilon^{m-2}$ and consider the following variation of g_ε

$$\tilde{g}_\varepsilon := g_\varepsilon + r_1 S_1 + r_2 S_2, \quad \tilde{g}_* := g_* + r_* S_*.$$

We apply the constructions of the previous two section to $\tilde{g}_\varepsilon$ in order to produce a family of solutions, $v = v_\varepsilon(r_1, r_2) \in B^\gamma_{r_\varepsilon}$ to

$$\begin{cases} \Delta_{\tilde{g}_\varepsilon} v = \tilde{F}_\varepsilon(v, r_1, r_2) & \text{in } M \\ \partial_\nu v = \tilde{F}^\partial_\varepsilon(v, r_1, r_2) - \lambda_{\tilde{F}_\varepsilon(v,r_1,r_2)}\beta_\varepsilon & \text{on } \partial M \end{cases}$$

where

$$\tilde{F}_\varepsilon(v, r_1, r_2) := c_n R_{\tilde{g}_\varepsilon}(1 + v)$$

is defined as usual, but

$$\tilde{F}^\partial_\varepsilon(v, r_1, r_2) := -2c_n H_{\tilde{g}_\varepsilon}(1 + v)$$

has been altered so that, supposing we can arrange for $\lambda_{\tilde{F}_\varepsilon(v,r_1,r_2)} = 0$, the boundary mean curvature of $(1 + v)^{\frac{4}{n-2}}\tilde{g}_\varepsilon$ is exactly 0. As before, we will assume that $\int_{\partial M} \beta_\varepsilon d\sigma_{\tilde{g}_\varepsilon} = 0$, which can be achieved for any r_1 and r_2 by an appropriate choice of α_1 and α_2.

Notice that our choice of r_* ensures $R_{\tilde{g}_\varepsilon}$ satisfies the same pointwise bounds as in the previous sections. This will allow us to apply the results of Section 2.3 with trivial modifications once we verify

$$\int_M \tilde{F}_\varepsilon(v, r_1, r_2) d\mu_{\tilde{g}_\varepsilon} = \int_{\partial M} \tilde{F}^\partial_\varepsilon(v, r_1, r_2) d\sigma_{\tilde{g}_\varepsilon}. \tag{2.17}$$

The second and final step is to arrange for the vanishing of $\lambda_{\tilde{F}_\varepsilon(v,r_1,r_2)}$.

Let us take a moment to explain why simultaneous vanishing of the Ricci tensor and second fundamental form can potentially be an obstruction to achieving the conclusions of Theorems 2.0.8, 2.0.10, and 2.0.12. Briefly, (M, g_ε) may be in the same conformal class as an Einstein metric with Neumann boundary conditions in the sense of [21] and the total scalar curvature plus mean curvature functional $Q(g_\varepsilon)$ may stable under even non-conformal perturbations. For the metric $\tilde{g}_\varepsilon$, we can follow the calculations of [5] to compute

$$
\begin{aligned}
Q(\tilde{g}_\varepsilon) &= Q(g_\varepsilon) + 2c_n \sum_{*=1}^{2} r_* \left(\int_M g_*(S_*, \mathrm{Ric}_{g_*}) d\mu_{g_*} - \int_{\partial M} g_*(S_*, A_{g_*}) d\sigma_{g_*} \right) + \\
&\qquad \mathcal{O}(r_1^2) + \mathcal{O}(r_2^2) \\
&= Q(g_\varepsilon) + 2c_n \sum_{*=1}^{2} r_* \left(\int_M K_* d\mu_{g_*} - \int_{\partial M} K_*^\partial d\sigma_{g_*} \right) + \mathcal{O}(\varepsilon^{2(m-2)})
\end{aligned}
$$

where we have introduced the notation $K_* := g_*(S_*, \mathrm{Ric}_{g_*})$ and $K_*^\partial := g_*(S_*, A_{g_*})$.

From this formula, we can see that if both Ric_{g_*} and A_{g_*} vanish identically for $* = 1$ and 2, the first variation of $Q(g_\varepsilon)$ vanishes for all choices of S_* and we will be unable to correct the term $F(g_\varepsilon)$ with a small (relative to ε) perturbation of g_ε away from the gluing locus to achieve the desired vanishing mean curvature. This reasoning heuristically explains why our construction may fail to produce scalar-flat metrics with vanishing boundary mean curvature on M without assumptions on the Ricci tensor and second fundamental form.

2.51. Achieving the orthogonality condition

In this subsection, we will give a description of the values r_1 and r_2 for which (2.17) is satisfied.

Proposition 6. For small ε and $v \in B^\gamma_{r_\varepsilon}$, there is a smooth function f_v defined on a neighborhood $\overline{U}$ of $\frac{\varepsilon^{m-2}}{2}$ such that

$$
\int_M \tilde{F}_\varepsilon(v, r_1, f_v(r_1)) d\mu_{\tilde{g}_\varepsilon} = \int_{\partial M} \tilde{F}_\varepsilon^\partial(v, r_1, f_v(r_1)) d\sigma_{\tilde{g}_\varepsilon}
$$

for all $r_1 \in \overline{U}$.

Proof. For any $v \in B^{\gamma}_{r_\varepsilon} \subset \mathcal{C}^0_\gamma(M)$, we introduce the function

$$
\begin{aligned}
G_{v,\varepsilon}(r_1, r_2) :=& \frac{1}{c_n}\Big(\int_M \tilde{F}_\varepsilon(v, r_1, r_2) d\mu_{\tilde{g}_\varepsilon} - \int_{\partial M} \tilde{F}^{\partial}_\varepsilon(v, r_1, r_2) d\sigma_{\tilde{g}_\varepsilon}\Big) \\
=& \int_M R_{g_\varepsilon} d\mu_{g_\varepsilon} + 2\int_{\partial M} H_{g_\varepsilon} d\sigma_{g_\varepsilon} + \sum_{*=1,2} r_* \left(\int_{M_1} K_* d\mu_{g_*} - \int_{\partial M_*} K^{\partial}_* d\sigma_{g_*}\right) \\
&+ L_v(r_1, r_2) + Q_v(r_1, r_2)
\end{aligned}
$$

where we have introduced the notation

$$
\begin{aligned}
L_v(r_1, r_2) :=& \int_M v R_{g_\varepsilon} d\mu_{g_\varepsilon} + \sum_{*=1,2} r_* \left(\int_{M_*} v K_* d\mu_{g_*} - \int_{\partial M_*} v K^{\partial}_* d\sigma_{g_*}\right) - 2\int_{\partial M_*} v H_{g_\varepsilon} d\sigma_{g_\varepsilon} \\
Q_v(r_1, r_2) :=& \sum_{*=1,2} \int_{M_*} R_{\tilde{g}_*}(1+v) d\mu_{g_*} - 2\int_{\partial M_*} H_{\tilde{g}_\varepsilon}(1+v) \\
&- r_* \int_{M_*} K_*(1+v) d\mu_{g_*} + r_* \int_{\partial M_*} K^{\partial}_*(1+v) d\sigma_{g_*} + \mathcal{O}(\varepsilon^{2(m-2)}).
\end{aligned}
$$

L_v and Q_v can be interpreted as the linear and quadratic parts, respectively, of $G_{v,\varepsilon}$. We also introduce the function $H_\varepsilon(r_1, r_2) := G_{v,\varepsilon}(r_1, r_2) - L_v(r_1, r_2) - Q_v(r_2, r_2)$.

For simplicity, we will pick S_* to satisfying the following conditions. We assume that S_* has been chosen so that $\int_M K_* d\mu_{g_*} - \int_{\partial M_*} K^{\partial}_* d\sigma_{g_*} = 1$ and we will only consider the case when

$$
\int_M R_{g_\varepsilon} d\mu_{g_\varepsilon} + 2\int_{\partial M} H_{g_\varepsilon} d\sigma_{g_\varepsilon} < 0,
$$

though the argument is very similar if this quantity is positive. Since this term is $\mathcal{O}(\varepsilon^{m-2})$, we will scale the metric g_ε so that it is equal to $-\varepsilon^{m-2}$. Now H_ε takes the form

$$
H_\varepsilon(r_1, r_2) = -\varepsilon^{m-2} + r_1 + r_2
$$

and the vanishing locus of $H_\varepsilon(r_1, r_2)$ is given by $\{(r_1, r_2) : r_1 + r_2 = \varepsilon^{m-2}\}$. We will see that the zero set of $G_{v,\varepsilon}(r_1, r_2)$ is uniformly close to this set.

It is straight forward to check that there is a constant $C > 0$, independent of ε and $v \in B^\gamma_{r_\varepsilon}$, such that

$$L_v(r_1, r_2), Q_v(r_1, r_2) \leq C_1 \varepsilon^{m-2+\gamma}.$$

So, for any $\eta > 0$, there is sufficiently small ε so that

$$|L_v(r_1, r_2)|, |Q_v(r_1, r_2)| \leq \frac{\eta}{2} \varepsilon^{m-2}.$$

It follows that

$$\begin{aligned}\{G_{v,\varepsilon}(r_1, r_2) = 0\} &= \{(r_1, r_2) : r_1 + r_2 = \varepsilon^{m-2} - L_v(r_1, r_2) - Q_v(r_1, r_2)\} \\ &\subset \{(r_1, r_2) : (1-\eta)\varepsilon^{m-2} \leq r_1 + r_2 \leq (1+\eta)\varepsilon^{m-2}\} =: Z_\varepsilon.\end{aligned}$$

From these remarks, we can find many zeroes of $G_{v,\varepsilon}$. For instance, setting $r_1' := \varepsilon^{m-2}/2$, for any $v \in B^\gamma_{r_\varepsilon}$, there is a number $r_2' = r_2'(v)$ with $(r_1', r_2'(v)) \in Z_\varepsilon$ and $G_{v,\varepsilon}(r_1', r_2'(v)) = 0$. However, we will still need a degree of freedom to arrange for $\lambda_{\tilde{F}_\varepsilon} = 0$ in the next subsection. Fortunately, for each $v \in B^\gamma_{r_\varepsilon}$ we will find a 1-parameter family of solutions near (r_1', r_2') by applying the implicit function theorem to $G_{v,\varepsilon}$.

Computing the derivatives of $G_{\varepsilon,v}$,

$$\begin{aligned}\left|\frac{\partial}{\partial r_*}G_{\varepsilon,v}(0,0)\right| &= \left|\int_{M_*} K_*(1+v)d\mu_{g_*} - \int_{\partial M_*} K_*^{\partial}(1+v)d\sigma_{g_*}\right| \\ &\geq \left|\int_{M_*} K_* d\mu_{g_*} - \int_{\partial M_*} K_*^{\partial} d\sigma_{g_*}\right| \\ &\qquad - ||v||_{\mathcal{C}^0(M)}\left(\int_{M_*}|K_*|d\mu_{g_*} + \int_{\partial M_*}|K_*^{\partial}|d\sigma_{g_*}\right) \\ &\geq \frac{1}{2}\end{aligned}$$

for $* = 1, 2$ and all $v \in B^{\gamma}_{r_\varepsilon}$. From this we can find a radius $R > 0$, uniform in ε and $v \in B^{\gamma}_{r_\varepsilon}$, so that that $\left|\frac{\partial}{\partial r_*}G_{v,\varepsilon}\right| \geq \frac{1}{4}$ on $B_R(0) \subset \mathbb{R}^2$.

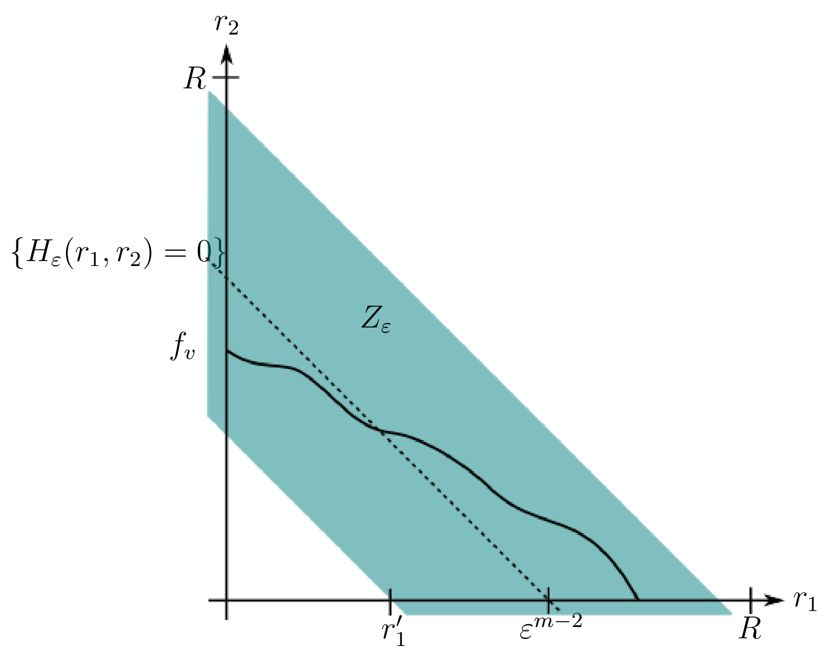

FIGURE 2.5. The region Z_ε and the function f_v in the r_1r_2-plane.

After perhaps restricting to smaller ε, the set $Z_\varepsilon \cap \{r_1, r_2 \geq 0\}$ is contained in $B_R(0)$. We may now apply the implicit function theorem on $G_{v,\varepsilon}$ about the points $(r_1', r_2'(v))$ to obtain, for every $v \in B^{\gamma}_{r_\varepsilon}$, open neighborhoods $U(v)$ and $V(v)$ containing r_1' and $r_2'(v)$, respectively, and a function $f_v : U(v) \to V(v)$ so that $G_{v,\varepsilon}(r_1, f_v(r_1)) = 0$

for all $r \in U(v)$ (see Figure 2.5.). In fact, we know apriori that f_v can be extended to the interval $(0,(1-\eta)\varepsilon^{m-2})$, and so we may choose open sets U and V which are independent of $v \in B^{\gamma}_{r_\varepsilon}$. Since the graph of each f_v lies in Z_ε they may be extended to $f_v : \overline{U} \to \overline{V}$. □

Before we continue, we will need one more property of the family $\{f_v\}_{v\in B^{\gamma}_{r_\varepsilon}}$. By construction, we have

$$f_v(r_1) = \int_M R_{g_\varepsilon} d\mu_{g_\varepsilon} + 2\int_{\partial M} H_{g_\varepsilon}(1+v)d\sigma_{g_\varepsilon} - r_1 + L_v(r_1, f_v(r_1)) + Q_v(r_1, r_2).$$

From this one can see, for small ε and any $r_1, r_1' \in \overline{U}$, that

$$|f_v(r_1) - f_v(r_1')| \leq 4|r_1 - r_1'|.$$

Now Ascoli-Arzela tells us that $\{f_v\}_{v\in B^{\gamma}_{r_\varepsilon}}$ is precompact in the $\mathcal{C}^0(\overline{U})$ norm. This function f will have the same Lipschitz norm bound.

2.52. Vanishing of the rough projection

Paralleling Section 2.3, we introduce the map $\tilde{P}_\varepsilon : \mathcal{C}^0_\gamma(M) \to \mathcal{C}^0_\gamma(M)$ sending a function v to the solution of

$$\begin{cases} \Delta_{\tilde{g}_\varepsilon}\tilde{P}_\varepsilon(v) = \tilde{F}_\varepsilon(v, r_1', f_v(r_1')) & \text{in } M \\ \partial_\nu \tilde{P}_\varepsilon(v) = \tilde{F}^{\partial}_\varepsilon(v, r_1', f_v(r_1')) - \lambda_{\tilde{F}_\varepsilon(v,r_1',f_v(r_1'))}\beta_\varepsilon & \text{on } \partial M \end{cases}$$

The arguments of that section can be repeated to show $\tilde{P}_\varepsilon$ is also a contraction mapping on $B^{\gamma}_{r_\varepsilon}$ for small ε and $\gamma \in (0, \frac{1}{4})$. This shows that $\{(\tilde{P}_\varepsilon)^j(0)\}_{j=1}^{\infty}$ converges to a fixed point $\tilde{v}_\varepsilon \in B^{\gamma}_{r_\varepsilon}$ with respect to the $\mathcal{C}^0_\gamma$-norm. From the previous section, after passing to a subsequence, the functions $f_{(\tilde{P}_\varepsilon)^j(0)}$ also converge to a continuous function $f : \overline{U} \to \overline{V}$ which verifies the orthogonality condition for $\tilde{v}_\varepsilon$. We conclude that, for any $r_1 \in \overline{U}$, we have

$$\begin{cases} \Delta_{\tilde{g}_\varepsilon} \tilde{v}_\varepsilon = \tilde{F}_\varepsilon(\tilde{v}_\varepsilon, r_1, f(r_1)) & \text{in } M \\ \partial_\nu \tilde{v}_\varepsilon = \tilde{F}^\partial_\varepsilon(\tilde{v}_\varepsilon, r_1, f(r_1)) - \lambda_{\tilde{F}_\varepsilon(\tilde{v}_\varepsilon, r_1, f(r_1))}\beta_\varepsilon & \text{on } \partial M. \end{cases} \tag{2.18}$$

The following proposition will complete the proof of Theorems 2.0.8, 2.0.10, and 2.0.12.

Proposition 7. There exists an $\varepsilon_0 > 0$ so that for all $\varepsilon \in (0, \varepsilon_0)$ there is a choice of $r_1 \in \overline{U}$ for which $\lambda_{\tilde{F}_\varepsilon(\tilde{v}_\varepsilon, r_1, f(r_1))}$ vanishes where $\tilde{v}_\varepsilon$ is given by (2.18).

Proof. Since $\lambda_{\tilde{F}_\varepsilon(\tilde{v}_\varepsilon, r_1, f(r_1))}$ is continuous in r_1, it suffices to show that its sign can be controlled by $r_1 \in \overline{U}$. Following Section 2.5, for small ε, the sign of $\lambda_{\tilde{F}_\varepsilon(\tilde{v}_\varepsilon, r_1, f(r_1))}$ is controlled by the sign of

$$\begin{aligned} \lambda^{(0)} = \frac{1}{\int_{\partial M}(\rho_1 + \rho_2) d\sigma_{\tilde{g}_\varepsilon}} \Big(& \int_M \tilde{F}_\varepsilon(\tilde{v}_\varepsilon, r_1, f(r_1))\beta_\varepsilon d\mu_{\tilde{g}_\varepsilon} \\ & - \int_{\partial M} \tilde{F}^\partial_\varepsilon(\tilde{v}_\varepsilon, r_1, f(r_1))\beta_\varepsilon d\sigma_{\tilde{g}_\varepsilon} + \int_M (\Delta_{\tilde{g}_\varepsilon}(\rho_1 \tilde{u}_T) \\ & - \Delta_{\tilde{g}_\varepsilon}(\rho_2 \tilde{u}_T)) d_{\tilde{g}_\varepsilon} - \int_{\partial M} \partial_\nu(\rho_1 \tilde{u}_T) - \partial_\nu(\rho_2 \tilde{U}_T) d\sigma_{\tilde{g}_\varepsilon} \Big). \end{aligned}$$

As before, we have

$$\int_M \Delta_{\tilde{g}_\varepsilon}(\rho_* \tilde{u}^\varepsilon_p) d\mu_{\tilde{g}_\varepsilon} + \int_{\partial M} \partial_\nu(\rho_* \tilde{u}_T) d\sigma_{\tilde{g}_\varepsilon} = \mathcal{O}(e^{-\alpha_*}\varepsilon^{m-2})$$

for $* = 1, 2$. For the first term appearing in the above expression for $\lambda^{(0)}$, we have

$$\begin{aligned}
\frac{1}{c_n}\int_M \tilde{F}_\varepsilon(\tilde{v}_\varepsilon, r_1, f(r_1))d\mu_{\tilde{g}_\varepsilon} &= r_1\int_{M_1} K_1 d\mu_{g_1} - f(r_1)\int_{M_2} K_2 d\mu_{g_2} \\
&\quad + \int_M R_{g_\varepsilon}\beta_\varepsilon dvol_{g_\varepsilon} + \mathcal{O}(\varepsilon^{m-2+2\gamma}) \\
&= r_1\int_{M_1} K_1 d\mu_{g_1} - f(r_1)\int_{M_2} K_2 d\mu_{g_2} + \mathcal{O}(e^{-\min(\alpha_1,\alpha_2)}\varepsilon^{m-2})
\end{aligned}$$

The boundary term has a similar estimate

$$\begin{aligned}
\frac{1}{c_n}\int_{\partial M} \tilde{F}^\partial_\varepsilon(\tilde{v}_\varepsilon, r_1, f(r_1))\beta_\varepsilon d\sigma_{\tilde{g}_\varepsilon} = &- r_1\int_{\partial M_1} K_1^\partial d\sigma_{g_1} + f(r_1)\int_{\partial M_2} K_2^\partial d\sigma_{g_2} \\
&+ \mathcal{O}(e^{-\min(\alpha_1,\alpha_2)}\varepsilon^{m-2}).
\end{aligned}$$

Summing these three expressions together gives us the expression we are looking for

$$\lambda^{(0)} = r_1 - f(r_1) + \mathcal{O}(e^{-\max(\alpha_1,\alpha_2)}\varepsilon^{m-2}).$$

Hence, we can choose large α_1 and α_2 so that the sign of $\lambda_{\tilde{F}_\varepsilon(\tilde{v}_\varepsilon, r_1, f(r_1))}$ is controlled by $r_1 - f(r_1)$. Evidently, the graph of f must intersect the line $\{r_1 = r_2\}$ in Z_ε (see Figure 2.5.) and we conclude that the sign of $r_1 - f(r_1)$ changes as r_1 varies over $\overline{U}$, finishing the proof of Proposition 2.52. $\square$

CHAPTER III

PART II: MINIMAL HYPERSURFACES WITH FREE BOUNDARY AND PSC-BORDISM

This chapter contains material which appears in a preprint written by the present author and Boris Botvinnik. The present author and Boris Botvinnik worked collaboratively on the content and exposition of all sections in this chapter.

3.1. Preliminaries and Theorem 1.5.3

In this section, we will prove Theorem 1.5.3. Before we begin, let us prepare by recalling the notion of stable minimality and the impact of the non-trivial boundary of M.

3.11. Stable minimal hypersurfaces with free boundary

Let $(M, \bar{g})$ be a compact oriented $(n+1)$-dimensional Riemannian manifold with nonempty boundary ∂M. Assume $W \subset M$ is a properly embedded hypersurface.

Let $\bar{h}$ denote the restriction metric $\bar{h} = \bar{g}|_W$ and fix a unit normal vector field ν^W on W which is compatible with the orientation. This determines the second fundamental form A^W on W given by the formula $A^W_{\bar{g}}(X,Y) = \bar{g}(\nabla_X Y, \nu^W)$ for vector fields X and Y tangential to W. The trace of $A^W_{\bar{g}}$ with respect to the metric $\bar{h}$ gives the mean curvature $H^W_{\bar{g}} = \mathrm{tr}_{\bar{h}} A^W_{\bar{g}}$. We will often omit the sub- and super-scripts, writing ν, A, and H if there is no risk of ambiguity.

Definition 3.1.1. Let $W \subset M$ be a properly embedded hypersurface. A *variation* of the hypersurface $W \subset M$ is a smooth one-parameter family $\{F_t\}_{t\in(-\varepsilon,\varepsilon)}$ of proper embeddings $F_t : W \to M$, $t \in (-\varepsilon, \varepsilon)$ such that F_0 coincides with the inclusion

$W \subset M$. A variation $\{F_t\}_{t\in(-\varepsilon,\varepsilon)}$ is said to be *normal* if the curve $t \mapsto F_t(x)$ meets W orthogonally for each $x \in W$.

The vector field $X = \frac{d}{dt}F_t|_{t=0}$ is called the *variational vector field* associated to $\{F_t\}_{t\in(-\varepsilon,\varepsilon)}$. For normal variations, the associated variational vector field takes the form $\phi \cdot \nu^W$ for some function $\phi \in C^\infty(W)$. Clearly, a variation $\{F_t\}_{t\in(-\varepsilon,\varepsilon)}$ gives a smooth function $t \mapsto \mathrm{Vol}(F_t(W))$.

Definition 3.1.2. A properly embedded hypersurface $W \subset (M, \bar{g})$ is *minimal with free boundary* if

$$\tfrac{d}{dt}\mathrm{Vol}(F_t(W))\big|_{t=0} = 0$$

for all variations $\{F_t\}_{t\in(-\varepsilon,\varepsilon)}$.

More notation: we denote by $d\sigma$ and $d\mu$ the volume forms of $(W, \bar{h})$ and $(\partial W, h)$, where $h = \bar{h}|_{\partial W}$ is the induced metric. We denote the outward-pointing unit length normal to ∂M by ν^∂. Below, Lemmas 3.1.1 and 3.1.2 contain well-known formulas, see [22].

Lemma 3.1.1. *Let $(M, \bar{g})$ be an oriented Riemannian manifold and let $W \subset M$ be a properly embedded hypersurface. If $\{F_t\}_{t\in(-\varepsilon,\varepsilon)}$ is a variation of W with variational vector field X, then*

$$\frac{d}{dt}\mathrm{Vol}(F_t(W))\bigg|_{t=0} = -\int_W H^W \bar{g}(X, \nu^W)d\mu + \int_{\partial W} \bar{g}(X, \nu^{\partial M})d\sigma. \tag{3.1}$$

In particular, a hypersurface W is minimal with free boundary if and only if $H^W_{\bar{g}} \equiv 0$ and W meets the boundary ∂M orthogonally.

Definition 3.1.3. A properly embedded minimal hypersurface with free boundary W is *stable* if

$$\frac{d^2}{dt^2}\mathrm{Vol}(F_t(W))\Big|_{t=0} \geq 0$$

for all variations $\{F_t\}_{t\in(-\varepsilon,\varepsilon)}$.

If a hypersurface W is minimal with free boundary, then any variational vector field must be parallel to ν^W on ∂W since the variation must go through proper embeddings. Hence, it is enough to consider only normal variations to analyze the second variation of the volume functional.

Lemma 3.1.2. *Let $(M, \bar{g})$ be an oriented Riemannian manifold and let $W \subset M$ be a properly embedded minimal hypersurface with free boundary. Let $\{F_t\}_{t\in(-\varepsilon,\varepsilon)}$ be a normal variation with variational vector field $\phi \cdot \nu^W$. Then*

$$\begin{aligned}\frac{d^2}{dt^2}\mathrm{Vol}(F_t(W))|_{t=0} = &\int_W \left(|\nabla\phi|^2 - \phi^2(\mathrm{Ric}_{\bar{g}}(\nu^W,\nu^W) + |A^W|^2)\right) d\mu \\ &- \int_{\partial W} \phi^2 A^{\partial M}(\nu^W,\nu^W) d\sigma, \end{aligned} \tag{3.2}$$

where $\mathrm{Ric}_{\bar{g}}$ *denotes the Ricci tensor of* $(M, \bar{g})$.

It will be useful to rewrite equation (3.2). The Gauss-Codazzi equations for a minimal hypersurface $W \subset M$ imply

$$R^M_{\bar{g}} = R^W_{\bar{h}} + 2\mathrm{Ric}_{\bar{g}}(\nu^W,\nu^W) + |A^W|^2$$

on W. Here $R^M_{\bar{g}}$ and $R^W_{\bar{h}}$ are the scalar curvatures of $(M, \bar{g})$ and $(W, \bar{h})$, respectively. It follows that the inequality $\frac{d^2}{dt^2}\mathrm{Vol}(F_t(W))\Big|_{t=0} \geq 0$ is equivalent to

$$\int_W |\nabla\phi|^2 d\mu \geq \int_W \frac{1}{2}\phi^2\left(R^M_{\bar{g}} - R^W_{\bar{h}} + |A^W|^2\right) d\mu - \int_{\partial W} \phi^2 A^{\partial M}(\nu^W,\nu^W) d\sigma. \tag{3.3}$$

3.12. Conformal Laplacian with minimal boundary conditions

The proof of Theorem 1.5.3 will rely on some basic facts about the conformal Laplacian on manifolds with boundary. Let $(W, \bar{h})$ be an n-dimensional manifold with non-empty boundary $(\partial W, h)$ where $h = \bar{h}|_{\partial W}$. We consider the following pair of operators acting on $C^\infty(W)$:

$$\begin{cases} L_{\bar{h}} &= -\Delta_{\bar{h}} + c_n R^W_{\bar{h}} \quad \text{in } W \\ B_{\bar{h}} &= \partial_\nu + 2c_n H^{\partial W}_{\bar{h}} \quad \text{on } \partial W, \end{cases}$$

where ν is the outward pointing normal vector to ∂W and $c_n = \frac{n-2}{4(n-1)}$.

Recall that if $\phi \in C^\infty(W)$ is a positive function, then the scalar and boundary mean curvatures of the conformal metric $\tilde{h} = \phi^{\frac{4}{n-2}}\bar{h}$ are given by

$$\begin{cases} R_{\tilde{h}} &= c_n^{-1}\phi^{-\frac{n+2}{n-2}} \cdot L_{\bar{h}}\phi \quad \text{in } W \\ H_{\tilde{h}} &= \frac{1}{2}c_n^{-1}\phi^{-\frac{n}{n-2}} \cdot B_{\bar{h}}\phi \quad \text{on } \partial W. \end{cases} \tag{3.4}$$

We consider a relevant Rayleigh quotient and take the infimum:

$$\lambda_1 = \inf_{\phi \not\equiv 0 \in H^1(W)} \frac{\int_W \left(|\nabla\phi|^2 + c_n R^W_{\bar{h}}\phi^2\right) d\mu + 2c_n \int_{\partial W} H^{\partial W}_{\bar{h}}\phi^2 d\sigma}{\int_W \phi^2 d\mu}. \tag{3.5}$$

According to standard elliptic PDE theory, we obtain an elliptic boundary problem, denoted by $(L_{\bar{h}}, B_{\bar{h}})$, and the infimum $\lambda_1 = \lambda_1(L_{\bar{h}}, B_{\bar{h}})$ is the *principal eigenvalue of the minimal boundary problem* $(L_{\bar{h}}, B_{\bar{h}})$. The corresponding Euler-Lagrange equations are the following:

$$\begin{cases} L_{\bar{h}}\phi &= \lambda_1\phi \quad \text{in } W \\ B_{\bar{h}}\phi &= 0 \quad \text{on } \partial W. \end{cases} \tag{3.6}$$

This problem was first studied by Escobar [6] in the context of the Yamabe problem on manifolds with boundary.

Let ϕ be a solution of (3.6). It is well-known that the eigenfunction ϕ is smooth and can be chosen to be positive. A straight-forward computation shows that the conformal metric $\tilde{h} = \phi^{\frac{4}{n-2}}\bar{h}$ has the following scalar and mean curvatures:

$$\begin{cases} R_{\tilde{h}} &= \lambda_1 \phi_1^{-\frac{4}{n-2}} \quad \text{in } W \\ H_{\tilde{h}} &\equiv 0 \quad\quad\quad\quad \text{on } \partial W. \end{cases} \tag{3.7}$$

In particular, the sign of the eigenvalue λ_1 is a conformal invariant, see [6, 9].

3.13. Proof of Theorem 1.5.3

Let $(M, \bar{g})$ and $W \subset M$ be as in Theorem 1.5.3. From the assumption $H^{\partial M} \equiv 0$, one can use the Gauss equations to show that $A^{\partial M}(\nu, \nu) = -H^{\partial W}$ where $H^{\partial W}$ is the mean curvature of ∂W as a hypersurface of W. Now, using the condition $R^M_{\bar{g}} > 0$, the stability inequality (3.3) implies

$$\int_W \left(|\nabla \phi|^2 + \frac{1}{2} R^W_{\bar{h}} \right) d\mu + \int_{\partial W} \phi^2 H^{\partial W} d\sigma \geq 0 \tag{3.8}$$

for all functions $\phi \in H^1(W)$ with strict inequality if $\phi \not\equiv 0$. By simple manipulation, the inequality (3.8) may be written as

$$\int_W \left(|\nabla \phi|^2 + c_n R^W_{\bar{h}} \right) d\mu + 2c_n \int_{\partial W} \phi^2 H^{\partial W} d\sigma > (1 - 2c_n) \int_W |\nabla \phi|^2 d\mu \tag{3.9}$$

for all $\phi \not\equiv 0 \in H^1(W)$. The right hand side of (3.9) is non-negative since $1 - 2c_n = \frac{n}{2(n-1)} > 0$. Furthermore, the left hand side of (3.9) coincides with the numerator of the Rayleigh quotient in equation (3.5). We conclude that the principal eigenvalue

$\lambda_1 = \lambda_1(L_{\bar{h}}, B_{\bar{h}})$ is positive. Let ϕ be an eigenfunction corresponding to λ_1. Then, according to (3.7), the metric $\tilde{h} = \phi^{\frac{4}{n-2}}\bar{h}$ has positive scalar curvature and zero mean curvature on the boundary. This completes the proof of Theorem 1.5.3.

3.2. Cheeger-Gromov Convergence of Minimizing Hypersurfaces

Here we introduce the notion of smooth convergence of hypersurfaces we require for the proof of Theorem 1.5.6. First, we consider the case when the hypersurfaces are embedded in the same ambient $(n+1)$-dimensional manifold M. Below we use coordinate charts $\Phi_j : U_j \to M$, where U_j is an open subset of $\mathbb{R}^{n+1}_+ = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \colon x_{n+1} \geq 0\}$.

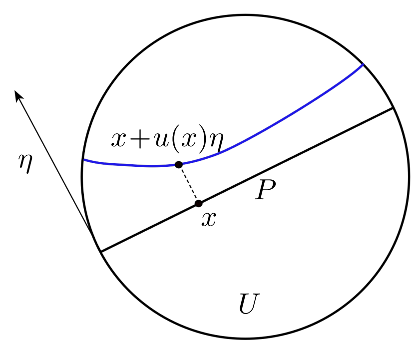

FIGURE 3.1. The hypersurface graph(u)

Let $P \subset \mathbb{R}^{n+1}$ be a hyperplane equipped with a normal unit vector η, and $U \subset \mathbb{R}^{n+1}_+$ be an open subset. Then for a function $u : P \cap U \to \mathbb{R}$, we denote by graph$(u)$ its *graph*, see Fig. 3.1.:

$$\mathrm{graph}(u) = \{x + u(x)\eta \mid x \in P \cap U\}.$$

Definition 3.2.1. Let $k \geq 1$ be an integer. Let $(M, \bar{g})$ be an $(n+1)$-dimensional compact Riemannian manifold and let $\{\Sigma_i\}_{i=1}^{\infty}$ be a sequence of smooth, properly embedded hypersurfaces. Then we say that the sequence $\{\Sigma_i\}_{i=1}^{\infty}$ converges to a smooth embedded hypersurface Σ_∞ *C^k-locally as graphs* if there exist

(i) coordinate charts $\Phi_j : U_j \to M$ for $j = 1, \ldots, N$;

(ii) hyperplanes $P_j \subset \mathbb{R}^{n+1}$ equipped with unit normal vectors η_j for $j = 1, \ldots, N$;

(iii) smooth functions $u_{i,j} : P_j \cap U_j \to \mathbb{R}$ for $j = 1, \ldots, N$, $i = 1, 2, \ldots$, and $i = \infty$,

which satisfy the following conditions:

(a) $\displaystyle\bigcup_{j=1}^{N} \Phi_j(\mathrm{graph}(u_{i,j}) \cap U_j) = \Sigma_i$ for $i = 1, 2, \ldots$ and $i = \infty$;

(b) for each $j = 1, \ldots, N$, $u_{i,j} \to u_{\infty,j}$ in the $C^k(P_j \cap U_j)$ topology as $i \to \infty$.

We say the sequence $\{\Sigma_i\}_{i=1}^{\infty}$ converges to a smooth embedded hypersurface Σ_∞ *smoothly locally as graphs* if it converges C^k-locally as graphs for all $k = 1, 2, \ldots$.

Next, we consider a sequence $\{(M_i, \Sigma_i, \bar{g}_i, \mathsf{S}_i)\}_{i=1}^{\infty}$, where $(M_i, \bar{g}_i)$ is a Riemannian manifold, $\Sigma_i \subset M_i$ is a properly embedded smooth hypersurface, and $\mathsf{S}_i \subset M_i$ a compact subset, playing a role of a base-point or a finite collection of base points.

Definition 3.2.2. Let $k \geq 1$ be an integer, and $\{(M_i, \Sigma_i, \bar{g}_i, \mathsf{S}_i)\}_{i=1}^{\infty}$ be a sequence as above, where $\dim M_i = n + 1$. We say that $\{(M_i, \Sigma_i, \bar{g}_i, \mathsf{S}_i)\}_{i=1}^{\infty}$ C^k-converges to $(M_\infty, \Sigma_\infty, \bar{g}_\infty, \mathsf{S}_\infty)$ if there is an exhaustion of M_∞ by precompact open sets

$$\mathsf{S}_\infty \subset \mathsf{U}_1 \subset \mathsf{U}_2 \subset \cdots \subset M_\infty, \quad M_\infty = \bigcup_{i=1}^{\infty} \mathsf{U}_i$$

and maps $\Psi_i : \mathsf{U}_i \to M_i$ which are diffeomorphisms onto their images for each $i = 1, 2, \ldots$, such that

(1) $\mathrm{dist}_H^{M_\infty}(\mathsf{S}_\infty, \Psi_i^{-1}(\mathsf{S}_i)) \to 0$ as $i \to \infty$, where $\mathrm{dist}_H^{M_\infty}$ is the Hausdorff distance for subsets of the manifold M_∞;

(2) the sequence $\{\Psi_i^* \bar{g}_i\}$ converges to $\bar{g}_\infty$ in the $C^k(\mathsf{U}_i)$-topology as $i \to \infty$;

(3) the sequence of hypersurfaces $\{\Psi_j^{-1}(\Sigma_i)\}_{i=1}^{\infty}$ converges C^k-locally as graphs in the manifold M_∞ to $\Sigma_\infty \cap \mathsf{U}_j$ as $i \to \infty$ for each $j = 1, \dots, N$.

Remark 3.2.1. We notice that the conditions (1) and (2) imply that the sequence $\{(M_i, \bar{g}_i, \mathsf{S}_i)\}_{i=1}^{\infty}$ C^k-converges to $(M_\infty, \bar{g}_\infty, \mathsf{S}_\infty)$ in the Cheeger-Gromov topology.

We say that $\{(M_i, \Sigma_i, \bar{g}_i, \mathsf{S}_i)\}_{i=1}^{\infty}$ *smoothly converges* to $(M_\infty, \Sigma_\infty, \bar{g}_\infty, \mathsf{S}_\infty)$ if it C^k-converges for all $k \geq 1$. Then we say that $\{(M_i, \Sigma_i, \bar{g}_i, \mathsf{S}_i)\}_{i=1}^{\infty}$ *sub-converges* to $(M_\infty, \Sigma_\infty, \bar{g}_\infty, \mathsf{S}_\infty)$ if it has a subsequence which converges to $(M_\infty, \Sigma_\infty, \bar{g}_\infty, \mathsf{S}_\infty)$. In this case we write

$$(M_i, \Sigma_i, \bar{g}_i, \mathsf{S}_i) \longrightarrow (M_\infty, \Sigma_\infty, \bar{g}_\infty, \mathsf{S}_\infty).$$

3.21. Main convergence result

We are ready to set the stage for the main result of this section. Let (Y, g) be a closed, oriented n-dimensional Riemannian manifold with a homology class $\alpha \in H_{n-1}(Y; \mathbb{Z})$. As we discussed in Section 1.5, the class α gives the Poincarè dual class $D\alpha = [\gamma] \in H^1(Y; \mathbb{Z})$ represented by some map $\gamma : Y \to S^1$. Furthermore, we assume that there is a bordism

$$(M, \bar{g}, \bar{\gamma}) : (Y, g, \gamma) \rightsquigarrow (Y', g', \gamma') \tag{3.10}$$

for some triple (Y', g', γ'). In the above, $\bar{\gamma} : M \to S^1$ represents a class $[\bar{\gamma}] \in H^1(M; \mathbb{Z})$ Poincarè dual to a class $\bar{\alpha} \in H_n(M, \partial M; \mathbb{Z})$.

Recall that $Y \subset \partial M$ and $\bar{g} = g + dt^2$ near Y. For a real number $L \geq 0$, we consider the following Riemannian manifold

$$(M_L, \bar{g}_L) := (M \cup_{Y \times \{-L\}} (Y \times [-L, 0]), \bar{g}_L),$$

where $\bar{g}_L$ restricts to $\bar{g}$ on M and to the product-metric $g + dt^2$ on $Y \times [-L, 0]$. We obtain another bordism

$$(M_L, \bar{g}_L, \bar{\gamma}_L) : (Y, g, \gamma) \rightsquigarrow (Y', g', \gamma'), \tag{3.11}$$

where $[\bar{\gamma}_L]$ is the image of $[\bar{\gamma}]$ under the isomorphism $H^1(M; \mathbb{Z}) \cong H^1(M_L; \mathbb{Z})$. We refer to the bordism $(M_L, \bar{g}_L, \bar{\gamma}_L)$ as the *L-collaring of* $(M, \bar{g}, \bar{\gamma})$. Below we will take L be an integer $i = 1, 2, \ldots$, and write $\bar{\alpha}_L \in H_n(M, \partial M; \mathbb{Z})$ for the class Poincarè dual to $[\bar{\gamma}_L]$.

Main Lemma. Let $(M, \bar{g}, \bar{\gamma}) : (Y, g, \gamma) \rightsquigarrow (Y', g', \gamma')$ be a bordism as in (3.10) and denote by $(M_i, \bar{g}_i, \bar{\gamma}_i)$ the i-collaring of $(M, \bar{g}, \bar{\gamma})$ as in (3.11) for $i = 0, 1, 2, \ldots$. Fix a basepoint in each component of Y, denote their union by S, and let S_i be the image of S under the inclusion

$$Y \cong Y \times \{0\} \subset Y \times [-i, 0] \subset M_i.$$

Assume $W_i \subset M_i$ is an oriented homologically volume minimizing representative of $\bar{\alpha}_i$ for $i = 0, 1, 2, \ldots$. If $X \subset Y$ is an embedded hypersurface which is the only volume minimizing representative of $\alpha \in H_{n-1}(Y; \mathbb{Z})$, then there is smooth subconvergence

$$(M_i, W_i, \bar{g}_i, \mathsf{S}_i) \longrightarrow (Y \times (-\infty, 0], X \times (-\infty, 0], g + dt^2, \mathsf{S}_\infty)$$

as $i \to \infty$ where $\mathsf{S}_\infty \subset Y \times \{0\}$ is the inclusion of S.

Remark 3.2.2. In Main Lemma, we allow the manifold Y' to be empty.

3.22. Proof of the Main Lemma: outline

Consider the limiting space $Y \times (-\infty, 0]$, with the exhaustive sequence $\mathsf{U}_i = Y \times (-i-1, 0]$ and maps $\Psi_i : \mathsf{U}_i \to M_i$ taking U_i identically onto $Y \times (-i-1, 0] \subset M_i$. Our choice of U_i and Ψ_i satisfy the conditions (1) and (2) from Definition 3.2.2 for obvious reasons.

It will be useful to equip M with a height function $F : M \to [-1, 0]$ satisfying $Y = F^{-1}(0)$ and $Y' = F^{-1}(-1)$. Extend this function to M_i by

$$F_i(x) = \begin{cases} t & \text{if } x = (y, t) \in Y \times [-i, 0] \\ F(x) - i & \text{if } x \in M. \end{cases}$$

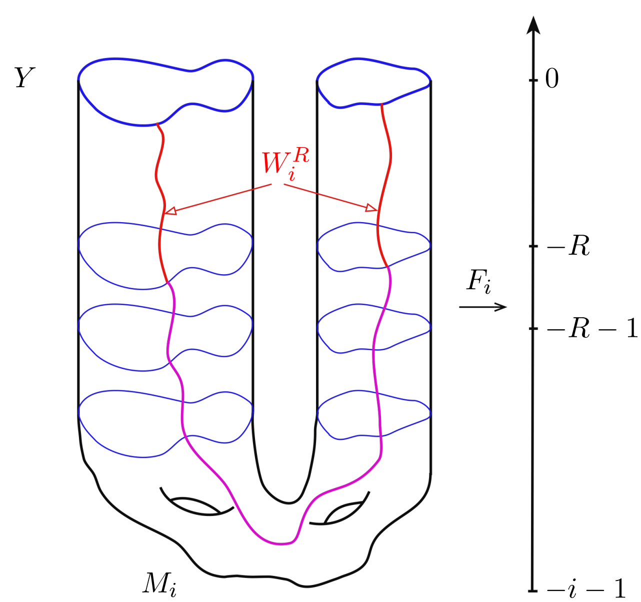

FIGURE 3.2. The hypersurface $W_i^R \hookrightarrow M_i$. In this figure, $Y' = \emptyset$.

For any positive integer i and heights $0 \leq R < R' \leq i$, we write

$$W_i^R = F_i^{-1}([-R, 0]) \quad \text{and} \quad W_i[-R', -R] = F_i^{-1}([-R', -R]).$$

Let $\alpha \in H_{n-1}(Y; \mathbf{Z})$ be the class from the statement of Main Lemma. For $L > 0$ let

$$\alpha \times [-L, 0] \in H_n(Y \times [-L, 0], Y \times \{-L, 0\}; \mathbf{Z})$$

be the product of α and the fundamental class of $([-L, 0], \{-L, 0\})$. We will break up the proof of Main Lemma into three claims.

Claim 3.2.1. *Let $L > 0$. The hypersurface $X \times [-L, 0] \subset Y \times [-L, 0]$ is the only homologically volume-minimizing representative of $\alpha \times [-L, 0]$.*

Claim 3.2.2. *For each $R > 0$, $\mathrm{Vol}(W_i^R) \to R \cdot \mathrm{Vol}(X)$ as $i \to \infty$.*

Claim 3.2.3. *For each $R > 0$, there is a sequence $\{a_i^R\}_{i=1}^{\infty}$ such that, for each $j = 1, 2, \ldots$, the hypersurfaces $\{\Psi_j^{-1}(W_{a_i^R}^R)\}_{i=1}^{\infty}$ converge smoothly locally as graphs in $Y \times (-\infty, 0]$.*

Now we show how Main Lemma follows from Claims 3.2.1, 3.2.2, and 3.2.3. Indeed, by Claim 3.2.3, for each $k = 1, 2, \ldots$, there is a sequence $\{a_i^k\}_{i=1}^{\infty}$ such that, for each $j = 1, 2, \ldots$, the hypersurfaces $\{\Psi_j^{-1}(W_{a_i^k}^k)\}_{i=1}^{\infty}$ converges smoothly locally as graphs to some hypersurface

$$W_{\infty,k} \subset Y \times (-\infty, 0].$$

We notice that the hypersurface $W_{\infty,k}$ is contained in $Y \times [-k, 0]$ and represents the class $\alpha \times [-k, 0]$. Since the convergence is smooth, we have

$$\mathrm{Vol}(\Psi_j^{-1}(W_{\infty,k})) = \lim_{i\to\infty} \mathrm{Vol}(\Psi_j^{-1}(W^k_{a^k_i})) = k \cdot \mathrm{Vol}(X),$$

where the last equality follows from Claim 3.2.2. However, according to Claim 3.2.1, the only volume minimizing representative of $\alpha \times [-k, 0]$ is the hypersurface $X \times [-k, 0]$ which has the volume $k \cdot \mathrm{Vol}(X)$. Thus $W_{\infty,k}$ must be $X \times [-k, 0]$. Evidently, the diagonal sequence $\{\Phi_j^{-1}(W_{a^i_i})\}_{i=1}^{\infty}$ has the property that, for each $k > 0$, $\Phi_j^{-1}(W^k_{a^i_i})$ converges smoothly locally as graphs to $X \times [-k, 0]$. This then completes the proof of Main Lemma.

3.23. Proof of Claim 3.2.1

Let $\Sigma \subset Y \times [-L, 0]$ be a properly embedded hypersurface representing the class $\alpha \times [-L, 0]$. Consider the projection function $P : \Sigma \to [-L, 0]$. The coarea formula [23, Theorem 5.3.9] applied to P yields

$$\int_\Sigma |\nabla P| d\mu = \int_{-L}^{0} \mathcal{H}^{n-1}(P^{-1}(t))dt \ , \tag{3.12}$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure associated to the metric $h + dt^2$ on $Y \times [-L, 0]$. Notice that P is weakly contractive in the sense that

$$|P(x) - P(y)| \leq \mathrm{dist}^\Sigma(x, y)$$

for all $x, y \in \Sigma$. Thus we have the pointwise bound $|\nabla P| \leq 1$. Furthermore, since $P^{-1}(t)$ represents the class $\alpha \in H_{n-1}(Y \times \{t\}; \mathbb{Z})$ for each $t \in [-L, 0]$,

$$\mathcal{H}^{n-1}(P^{-1}(t)) \geq \mathrm{Vol}(X)$$

with equality if and only if $P^{-1}(t)$ is X. Combining this observation with (3.12), we conclude

$$\mathrm{Vol}(\Sigma) \geq L \cdot \mathrm{Vol}(X)$$

with equality if and only if $\Sigma = X \times [-L, 0]$. This completes the proof of Claim 3.2.1.

3.24. Proof of Claim 3.2.2

Before we begin, we will construct particular (in general, non-minimizing) properly embedded hypersurfaces $N_L \subset M_L$ representing α_L with which to compare $\mathrm{Vol}(W_L)$ against.

Let $X \subset Y$ and $W_0 \subset M_0$ be as in Main Lemma. Since $\partial W_0 \cap Y$ and X represent the same homology class, they are bordant via a smooth, properly embedded hypersurface $\iota : U \hookrightarrow Y \times [0, 1]$. We identify $[0, 1] \cong [-L, -L + 1]$ to obtain the embedding

$$\iota_L : U \hookrightarrow []\iota Y \times [0, 1] \cong Y \times [-L, -L + 1] \hookrightarrow M_L.$$

Clearly the embedding $\iota : U \hookrightarrow Y \times [0, 1]$ may be chosen so that

$$N_L := W_0 \cup_{\partial W_0} U_L \cup (X \times [-L + 1, 0]),$$

where $U_L = \iota_L(U)$, is a smooth properly embedded hypersurface of M_L.

Evidently, $\mathrm{Vol}(N_L) = \mathrm{Vol}(W_0) + \mathrm{Vol}(U_L) + (L-1)\mathrm{Vol}(X)$ and N_L represents the same homology class as W_L. Since W_L is homologically area-minimizing, we have $\mathrm{Vol}(W_L) \le \mathrm{Vol}(N_L)$. In other words, we obtain the inequality

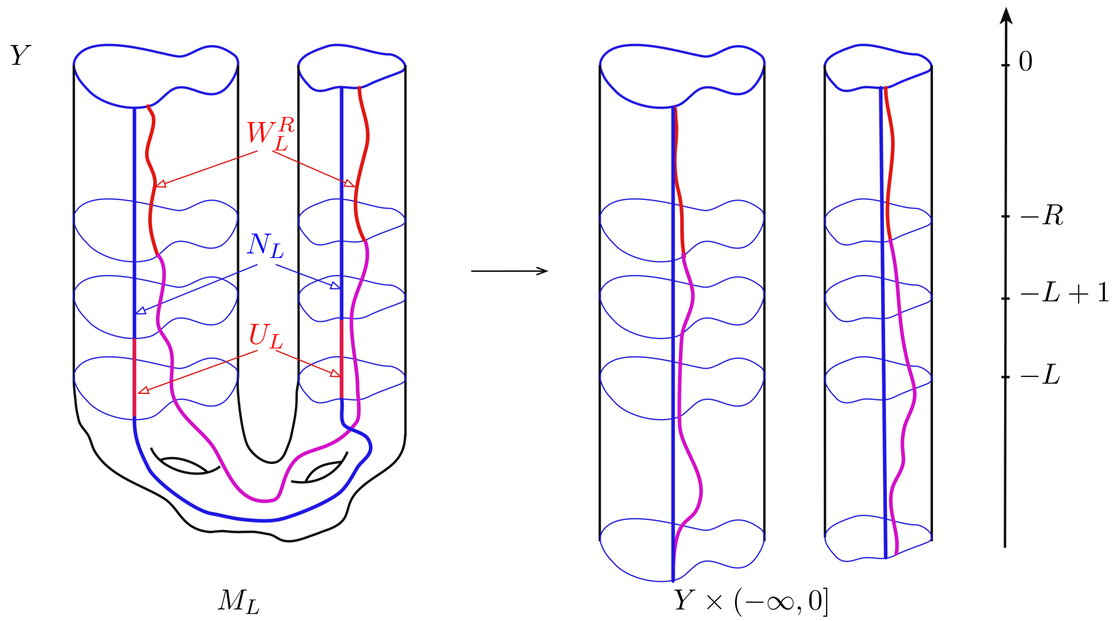

FIGURE 3.3. The hypersurface $N_L \hookrightarrow M_L$.

$$\mathrm{Vol}(W_L^R) + \mathrm{Vol}(W_L \setminus W_L^R) \le \mathrm{Vol}(W_0) + \mathrm{Vol}(U_L) + (L-1)\mathrm{Vol}(X) \tag{3.13}$$

for any $0 < R < L-1$.

Now we are ready to prove Claim 3.2.2. Assume it fails. Then there exist $\varepsilon_0, R_0 > 0$ and an increasing sequence of whole numbers $\{a_i\}_{i=1}^{\infty}$ such that the inequality

$$\mathrm{Vol}(W_{a_i}^{R_0}) > R_0 \cdot \mathrm{Vol}(X) + \varepsilon_0 \tag{3.14}$$

holds for all i. Combining the inequality (3.13) with the assumption (3.14), we have

$$\mathrm{Vol}(W_0) + \mathrm{Vol}(U_{a_i}) + (a_i - 1)\mathrm{Vol}(X) > \mathrm{Vol}(W_{a_i} \setminus W_{a_i}^{R_0}) + \varepsilon_0 + R_0\mathrm{Vol}(X). \tag{3.15}$$

Now we will inspect the first term in the right hand side of (3.15):

$$\begin{aligned}\mathrm{Vol}(W_{a_i} \setminus W_{a_i}^{R_0}) &= \mathrm{Vol}(W_{a_i}[a_{i-1} - a_i, -R_0]) + \mathrm{Vol}(W_{a_i}[-a_i - 1, a_{i-1} - a_i]) \\ &\geq (a_i - a_{i-1} - R_0)\mathrm{Vol}(X) + \mathrm{Vol}(W_{a_{i-1}}) \\ &> (a_i - a_{i-1})\mathrm{Vol}(X) + \varepsilon_0 + \mathrm{Vol}(W_{a_{i-1}} \setminus W_{a_{i-1}}^{R_0}). \end{aligned} \tag{3.16}$$

Here we use Claim 3.2.1 in the first inequality and the assumption (3.14) in the second.

Combining (3.15) with (3.16), we obtain

$$\mathrm{Vol}(W_0)+\mathrm{Vol}(U_{a_i})+(a_i-1)\mathrm{Vol}(X) > (a_i-a_{i-1}+R_0)\mathrm{Vol}(X)+2\varepsilon_0+\mathrm{Vol}(W_{a_{i-1}}\setminus W_{a_{i-1}}^{R_0}).$$

We iterate the argument to find

$$\mathrm{Vol}(W_0) + \mathrm{Vol}(U_{a_i}) + (a_1 - R_0 - 1)\mathrm{Vol}(X) > i \cdot \varepsilon_0 + \mathrm{Vol}(W_{a_1}) \tag{3.17}$$

for every $i = 1, 2, \ldots$. Since the left hand side of (3.17) is independent of i, we arrive at a contradiction by taking i to be sufficiently large.

3.25. Proof of Claim 3.2.3

While the proof of Claim 3.2.3 is rather technical, it is essentially a consequence of standard tools used in the study of stable minimal hypersurfaces. For instance, see [24] for a similar result in a 3-dimensional context. We divide the proof into three steps, referring to Appendix A when necessary.

To begin, we require the following straight-forward volume bound.

Step 1. For each $R > 0$, there is a constant $V_R > 0$ such that

$$\mathrm{Vol}(W_i[-\lambda - R, -\lambda]) \le V_R$$

holds for all i and all $\lambda \in [0, i - R]$. In particular, $\mathrm{Vol}(W_i \cap B_R^{M_i}(x)) \le V_R$ for all i and $x \in M_i$.

The next key ingredient is the following uniform bound on the second fundamental form A^{W_L}.

Step 2. There is a constant $C_1 > 0$, depending only on the geometry of $(M, \bar{g})$, such that

$$\sup_{x \in W_L} |A^{W_L}(x)|^2 \le C_1 \quad \text{for } L \ge 0.$$

Step 2 is a consequence of [25, Corollary 1.1]. See Appendix, Section A.32 for more details.

Step 3. For each $R > 0$ and $j = 1, 2, \ldots$, the sequence of hypersurfaces $\Psi_j^{-1}(W_i^R)$ sub-converges smoothly locally as graphs as $i \to \infty$.

Proof of Step 3. We restrict our attention to the tail of the sequence $\{W_i^R\}_{i=1}^{\infty}$, where $i \ge R + 1$. This allows us to consider each W_i^R and W_i^{R+1} as hypersurfaces of $Y \times (-\infty, 0]$ which is where we will show the convergence. By rescaling the original metric $\bar{g}$, we will assume that $\mathrm{inj}_g \ge 1$ and the bounds

$$\sup_{x \in B_1(y)} |\bar{g}_{ij}(x) - \delta_{ij}| \le \mu_0, \quad \sup_{x \in B_1(y)} \left| \frac{\partial \bar{g}_{ij}}{\partial x^k}(x) \right| \le \mu_0$$

hold for $1 \le i, j, k \le n + 1$ in geodesic normal coordinates centered about any $y \in Y \times (-\infty, 0]$ where μ_0 is the constant from Lemma A.2.1. Let $r = \min(\frac{1}{24}, \frac{1}{6\sqrt{20 C_0}})$ where C_0 is the constant from Step 2.

We cover $Y \times [-R, 0]$ by a finite collection of open balls $\mathcal{U} = \{B_r(y_l)\}_{l=1}^N$. Notice that each $B_r(y_l) \subset\subset Y \times [-R-1, 0]$. Consider a ball $B_r(y_l)$ in $\mathcal{U}$ with the property that

$$W_i^{R+1} \cap B_r(y_l) \neq \emptyset$$

for infinitely many i. Unless explicitly stated, we will continue to denote all subsequences by W_i^{R+1}. Our next goal is to show that the sequence of hypersurfaces $\{W_i^R \cap B_r(y_l)\}_{i=1}^\infty$ sub-converges smoothly locally as graphs.

We choose a subsequence of W_i^{R+1} and points $x_i \in W_i^{R+1} \cap B_r(y_l)$ which converge to some point $x_\infty \in \overline{B_r(y_l)}$. Now it will be convenient to work in the tangent space to the point x_∞. We use the short-hand notation $\phi = \exp_{x_\infty}^{\bar{g}}$ and let

$$B = \phi^{-1}(B_1(x_\infty)) \subset T_{x_\infty}(Y \times [-L-1, 0]).$$

Consider the properly embedded hypersurfaces $\Sigma_i \subset B$ with base points $p_i \in \Sigma_i$, given by

$$\Sigma_i = \phi^{-1}(B_1(x_\infty) \cap W_i^R), \quad p_i = \phi^{-1}(x_i).$$

We also write $Z = \phi^{-1}(y_l)$ Since $W_i^R \subset M_i$ are minimal, Σ_i are minimal hypersurfaces in B with respect to the metric $\bar{g}_B = (\phi^{-1})^*(\bar{g})$.

Notice that the choice of r allows us to apply Corollary A.2.1 to each $\Sigma_i \subset B$ at p_i with $s = 3r$. For each $i = 1, 2, \ldots$, we obtain an open subset $U_i \subset T_{p_i}\Sigma_i \cap B$, a unit normal vector $\eta_i \perp T_{p_i}\Sigma_i$, and a function $u_i : U_i \to \mathbb{R}$ satisfying the bounds (A.5) and such that $\operatorname{graph}(u_i) = B_{6r}^{\Sigma_i}(p_i)$. Moreover, the connected component of $B_{3r}^{\bar{g}_B}(p_i) \cap \Sigma_i$ containing x_0 lies in $B_{6r}^{\Sigma_i}(p_i)$.

We use compactness of S^n and pass to a subsequence so that the vectors η_i converge to some vector $\eta_\infty \in S^n$. Let $P_\infty \subset T_{x_\infty}(Y \times [-L-1, 0])$ be the hyperplane perpendicular to η_∞. For large enough i, we may translate and rotate the sets U_i to obtain open subsets $U_i' \subset P_\infty$ and functions $u_i' : U_i' \to \mathbb{R}$ such that

1. $\mathrm{graph}(u_i') = B_{4r}^{\Sigma_i}(p_i)$;
2. the ball $B_{2r}^{P_\infty}(0) \subset U_i'$;
3. for each $k \geq 1$ and $\alpha \in (0,1)$ there is a constant $C' > 0$, depending only on n, k, α, and the geometry of g, such that

$$||u_i'||_{C^{k,\alpha}(U_i')} \leq C',$$

see Fig. 3.4.. In particular, writing $u_i'' = u_i'|_{B_{2r}^{P_\infty}(0)}$, the sequence $\{u_i''\}_i$ is uniformly bounded in $C^{k,\alpha}(B_{2r}^{P_\infty}(0))$. Moreover, the connected component of $B_{2r}(p_i) \cap \Sigma_i$ containing p_i is contained in $\mathrm{graph}(u_i'')$. It follows that Σ_i', the connected component of $B_r(Z) \cap \Sigma_i$ containing p_i, lies in $\mathrm{graph}(u_i'')$.

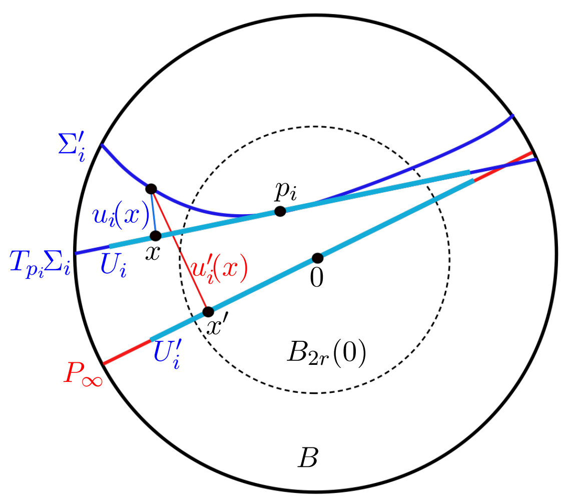

FIGURE 3.4. The functions u_i' and hypersurfaces Σ_i'

By Arzela-Ascoli, one can find a subsequence of u_i'' converging in $C^k(B_{2r}^{P_\infty}(0))$ to a function $u_\infty : B_{2r}^{P_\infty}(0) \to \mathbb{R}$. In particular, u_∞ is a strong solution to the minimal graph equation on $B_{2r}^{P_\infty}(0)$ with respect to $\bar{g}_B$ and Σ_i' converge as graphs to graph(u_∞). To summarize our current progress, the components of $W_i^{R+1} \cap B_r(y_l)$ containing x_i sub-converge smoothly to $\phi(\text{graph}(u_\infty))$. This finishes our work with the hypersurfaces Σ_i'.

Now suppose that there is a second sequence of connected components within $W_i^{R+1} \cap B_r(y_l)$. We can repeat the above process to obtain a second limiting hypersurface. Observe that the number of components of $W_i^{R+1} \cap B_r(y_l)$ uniformly bounded in i. Indeed, using the notation above, for any component $\bar{\Sigma}_i \subset W_i^{R+1} \cap B_r(y_l)$, we have

$$\mathrm{Vol}_{\bar{g}_B}(\Sigma_i') \geq \mathrm{Vol}_{\bar{g}_B}(B_r^{P_\infty}(0)),$$

which is uniformly bounded below in terms of r and the geometry of g. However, Step 1 implies that $\mathrm{Vol}(W_i^R \cap B_r(y_l))$ is bounded above uniformly in i so the number of connected components $W_i^R \cap B_r(y_l)$ is uniformly bounded in i. Hence the above process terminates after finitely many iterations. We conclude that the sequence $\{W_i^R \cap B_r(y_l)\}_{i=1}^\infty$ sub-converges smoothly locally as graphs to a minimal hypersurface $\Sigma_{\infty,l}$.

Now, restricting to this subsequence, we turn our attention to another ball $B_r(y_{l'})$ in the cover $\mathcal{U}$. We repeat the above argument to obtain a further subsequence and limiting minimal hypersurface $\Sigma_{\infty,l'}$. Repeating this process for each element of $\mathcal{U}$ produces a subsequence converging to a minimal hypersurface $W_\infty^R = \bigcup_{l=0}^N \Sigma_{\infty,l}$ smoothly locally as graphs. This completes the proof of Claim 3.2.3, and consequently, the proof of Main Lemma. $\square$

3.3. Proof of Theorem 1.5.6

In order to prove Theorem 1.5.6, we have to use fundamental facts relating conformal geometry and psc-bordism. We briefly recall necessary results, following the conventions in [26]. Let Y be a compact closed manifold with $\dim Y = n$ given together with a conformal class C of Riemannian metrics. Then the *Yamabe constant* of (Y, C) is defined as

$$Y(Y; C) = \inf_{g \in C} \frac{\int_Y R_g d\mu_g}{\mathrm{Vol}_g(Y)^{\frac{n-2}{n}}}.$$

We say that a conformal class C is *positive* if $Y(Y; C) > 0$. It is well-known that C is positive if and only if there exists a psc-metric $g \in C$.

Now let $Z : Y_0 \rightsquigarrow Y_1$ be a bordism between closed manifolds Y_0 and Y_1. Suppose we are given conformal classes C_0 and C_1 on Y_0 and Y_1, respectively. Let $\bar{C}$ be a conformal class on Z, such that $\bar{C}|_{Y_0} = C_0$ and $\bar{C}|_{Y_1} = C_1$, i.e. $\partial \bar{C} = C_0 \sqcup C_1$. Denote by $\bar{C}^0 = \{\bar{g} \in \bar{C} \colon H_{\bar{g}} \equiv 0\}$ the subclass of those metrics with vanishing mean curvature of the boundary. Then the *relative Yamabe constant* of $((Z, \bar{C}), (Y_0 \sqcup Y_1, C_0 \sqcup C_1))$ is defined as

$$Y_{\bar{C}}(Z, Y_0 \sqcup Y_1; C_0 \sqcup C_1) = \inf_{\bar{g} \in \bar{C}^0} \frac{\int_Z R_{\bar{g}} d\mu_{\bar{g}}}{\mathrm{Vol}_{\bar{g}}(Z)^{\frac{n-2}{n}}}.$$

This gives the *relative Yamabe invariant*

$$Y(Z, Y_0 \sqcup Y_1; C_0 \sqcup C_1) = \sup_{\bar{C},\, \partial\bar{C} = C_0 \sqcup C_1} Y_{\bar{C}}(Z, Y_0 \sqcup Y_1; C_0 \sqcup C_1).$$

Now we assume that the conformal classes C_0 and C_1 are positive. Then we say that positive conformal manifolds (Y_0, C_0) and (Y_1, C_1) are *positive-conformally bordant* if there exists a conformal manifold $(Z, \bar{C})$ and a bordism $Z : Y_0 \rightsquigarrow Y_1$ between Y_0 and

Y_1 such that $\partial\bar{C} = C_0 \sqcup C_1$ and $Y_{\bar{C}}(Z, Y_0 \sqcup Y_1; C_0 \sqcup C_1) > 0$. In this case, we write $(Z, \bar{C}) : (Y_0, C_0) \rightsquigarrow (Y_1, C_1)$.

We need the following result which relates the above notions to psc-bordisms.

Theorem 3.3.1. *[26, Corollary B] Let Y_0 and Y_1 be closed manifolds of dimension $n \geq 3$, $Z : Y_0 \rightsquigarrow Y_1$ be a bordism between Y_0 and Y_1, and g_0 and g_1 be psc-metrics on Y_0 and Y_1, respectively. Then $Y(Z, Y_0 \sqcup Y_1; [g_0] \sqcup [g_1]) > 0$ if and only if the boundary metric $g_0 \sqcup g_1$ on $Y_0 \sqcup Y_1$ may be extended to a psc-metric $\bar{g}$ on Z such that $\bar{g} = g_j + dt^2$ near Y_j for $j = 0, 1$.*

3.31. Long collars

We are ready to prove Theorem 1.5.6 for $n \leq 6$. The adjustments required to adapt the following proof to the case $n = 7$ are provided in Appendix A.33.

Let (Y_0, g_0, γ_0) and (Y_1, g_1, γ_1) be the manifolds from Theorem 1.5.6 and let $\alpha_0 \in H_{n-1}(Y_0; \mathbf{Z})$ and $\alpha_1 \in H_{n-1}(Y_1; \mathbf{Z})$ be the classes Poincarè dual to γ_0 and γ_1, respectively. It is convenient to use the notation [1] $Y = Y_0 \sqcup -Y_1$ and

$$\alpha = (\iota_0)_* \alpha_0 - (\iota_1)_* \alpha_1 \in H_{n-1}(Y; \mathbf{Z}),$$

where $\iota_j : Y_j \hookrightarrow Y$ is the inclusion map for $j = 0, 1$. Then we consider hypersurfaces $X_0 \subset Y_0$ and $X_1 \subset Y_0$ which are homologically volume minimizing representatives of the classes α_0 and $-\alpha_1$. The existence of such smooth X_0 and X_1 is guaranteed in this range of dimensions, see [3]. Notice that, by a small conformal change which does not effect the assumptions on (Y_j, g_j, γ_j), we may assume that X_j is the only

[1] Here we emphasize a proper orientation on Y_0 and Y_1

representative of α_j with minimal volume for $j = 0, 1$, see [27, Lemma 1.3]. We write (X, h_X) for the Riemannian manifold $(X_0 \sqcup X_1, g_0|_{X_0} \sqcup g_1|_{X_1})$.

Now we choose a psc-bordism $(Z, \bar{g}, \bar{\gamma}) : (Y_0, g_0, \gamma_0) \rightsquigarrow (Y_1, g_1, \gamma_1)$. We will use $(Z, \bar{g}, \bar{\gamma})$ to construct a psc-bordism which satisfies the conclusion of Theorem 1.5.6. We denote by $\bar{\alpha} \in H_n(Z; \mathbf{Z})$ the homology class Poincarè dual to $\bar{\gamma}$. Then $\partial\bar{\alpha} = \alpha$, see Lemma 1.5.5.

Now for each $i = 1, 2, \ldots$, we consider the i-collaring of the bordism $(Z, \bar{g}, \bar{\gamma})$, denoted by $(Z_i, \bar{g}_i, \bar{\gamma}_i)$, as in Section 3.21. By Theorem 1.5.4, there exists properly embedded hypersurfaces $W_i \subset Z_i$ which are homologically volume minimizing and represents $\bar{\alpha}_i$. The restrictions of $\bar{g}_i$ to W_i and ∂W_i are denoted by $\bar{h}_i$ and h_i, respectively.

In preparation to apply Main Lemma, we fix basepoints $x_j \in X_j$ for each $j = 0, 1$ and set $\mathsf{S} = \{x_0, x_1\} \subset X$. Naturally, the set S is identified with the subsets S_i in $(X \times \{0\}) \subset \partial Z_i$ for $i = 1, 2, \ldots$ and with S_∞ in the boundary of the cylinder $(X \times \{0\}) \subset (Y \times (-\infty, 0])$. According to Main Lemma we may find a subsequence $\{a_i\}_{i=1}^\infty$ such that

$$(Z_{a_i}, W_{a_i}, \bar{g}_{a_i}, \mathsf{S}_{a_i}) \longrightarrow (Y \times (-\infty, 0], X \times (-\infty, 0], g + dt^2, \mathsf{S}_\infty)$$

smoothly as $i \to \infty$ and the Riemannian manifolds $(\partial W_{a_i}, h_{a_i})$ converge to (X, h_X) in the smooth Cheeger-Gromov topology as $i \to \infty$.

Remark 3.3.1. We note that the manifolds $(\partial W_{a_i}, h_{a_i})$, (X, h_X) are compact and so there is no need to specify base points for this convergence.

The following is a special case of a much more general fact on the behavior of elliptic eigenvalue problems under smooth Cheeger-Gromov convergence (see [28]).

Lemma 3.3.2. *Let $\{(M_i, g_i')\}_{i=1}^{\infty}$ be a sequence of compact Riemannian manifolds smoothly converging to a compact Riemannian manifold (M_∞, g_∞') in the Cheeger-Gromov sense. If $Y(M_\infty; [g_\infty']) > 0$, then, upon passing to a subsequence, $Y(M_i; [g_i']) > 0$ for all sufficiently large i.*

Proof. For each $i = 1, 2, \ldots$, we denote by $\lambda_{1,i} = \lambda_1(L_{g_i'})$ the principal eigenvalue of the conformal Laplacian on (M_i, g_i'). Let $\phi_i \in C^\infty(M_i)$ be the eigenfunction satisfying

$$L_{g_i'}\phi_i = \lambda_{1,i}\phi_i, \qquad \sup_{M_i} \phi_i = 1. \tag{3.18}$$

Since $\{(M_i, g_i')\}_{i=1}^{\infty}$ is converging in the Cheeger-Gromov topology to a compact manifold, the coefficients of the operator $L_{g_i'}$ are bounded in the C^1-norm uniformly in i. In particular, there is a constant $C_1 > 0$, independent of i, such that $|R_{g_i'}| \leq C_1$ on M_i. An obvious estimate on the Rayleigh quotient (3.5) shows that the sequence $\{\lambda_{1,i}\}_{i=1}^{\infty}$ is uniformly bounded above and below.

This allows us to apply the Schauder estimate Theorem A.1.2 to ϕ_i uniformly in i. Using Arzelá-Ascoli, we can find a subsequence, still denoted by $\{(M_i, g_i')\}_{i=1}^{\infty}$, $\{\phi_i\}_{i=1}^{\infty}$, and $\{\lambda_{1,i}\}_{i=1}^{\infty}$, a function $\phi_\infty \in C^\infty(M_\infty)$, and a number $\lambda_{1,\infty}$ such that

$$\phi_i \to \phi_\infty \quad \lambda_{1,i} \to \lambda_{1,\infty}$$

where the former convergence is in the $C^{2,\alpha}$-topology. This allows us to take the limit of equation (3.18) as $i \to \infty$. Namely, ϕ_∞ is a non-zero solution of the equation

$$L_{g_\infty}\phi_\infty = \lambda_{1,\infty}\phi_\infty$$

and so $\lambda_{1,\infty} \geq \lambda_1(L_{g_\infty})$. On the other hand, we have assumed that $\lambda_1(L_{g_\infty}) > 0$. Hence $\lambda_{1,i} > 0$ for all sufficiently large i. $\square$

Now we return to the proof of Theorem 1.5.6. Since X is a stable minimal hypersurface of Y with trivial normal bundle, Theorem 1.5.1 implies that $Y(X, [g_X]) > 0$. Now we may apply Lemma 3.3.2 to find $Y(\partial W_{a_i}, [h_{a_i}]) > 0$ for sufficiently large i. Fix such an i and let $h'_{a_i} \in [h_{a_i}]$ be a psc metric on ∂W_{a_i}. Since each W_{a_i} is a stable minimal hypersurface with free boundary and trivial normal bundle, Theorem 1.5.3 states that $Y(W_{a_i}, \partial W_{a_i}; [\bar{h}_{a_i}]) > 0$ for all $i \in \mathbb{N}$. Finally, we use Theorem 3.3.1 to find a psc-metric $\tilde{h}_{a_i}$ on W_{a_i} which restricts to $h'_{a_i} + dt^2$ near ∂W_{a_i}. This completes the proof of Theorem 1.5.6 for $n \leq 6$.

APPENDIX

GEOMETRIC ANALYSIS BACKGROUND

Let us first describe the structure of this Appendix. In Section A.1, we state the basic regularity facts for solutions of elliptic equations both with and without boundary conditions. These facts are used in both Part One and Part Two. In Section A.2, we recall relevant facts on the minimal graph equation and provide the Schauder estimates we use in the proof of Main Lemma from Part Two. Section A.3 is dedicated to Theorem 1.5.4. Here we recall necessary results on currents and state well-known facts on their compactness and regularity, adapted to our setting. Section A.31 describes a simple doubling method which is a convenient technical tool in the remaining sections. In Section A.32, we justify Step 2 from the proof of Claim 3.2.3. In Section A.33, we discuss regularity issues in dimension 8 and prove Theorem 1.5.6 for $n = 7$.

A.1. Elliptic Estimates

In Part One, we use a regularity result for solutions of linear elliptic problems which is suited for the linear analysis in Section 2.21. The following theorem is a version of elliptic L^p estimate, tailored to the Neumann problem.

Theorem A.1.1. cf. [20, Theorem 3.2] *Let* (N, g_N) *be a compact Riemannian manifold with boundary* ∂N*. Assume that* $v \in W^{k+2,p}(N, g_N)$ *for some* $k, p \in \mathbb{N}_0$ *satisfies* $\int_N v \, d\mu_{g_N} = 0$*. Then there is a constant* C *depending only on the geometry of* (N, g_N)*,* k*, and* p *such that*

$$||v||_{W^{k+2,p}(N,g_N)} \leq C \left(||\Delta_{g_N} v||_{W^{k,p}(N,g_N)} + ||\partial_\nu v||_{W_\partial^{k+1,p}(N,g_N)} \right). \tag{A.1}$$

where the norm $||\cdot||_{W^{k,p}_{\partial}(N,g_N)}$ *is defined by*

$$||F||_{W^{k,p}_{\partial}(N,g_N)} := \inf\{||G||_{W^{k,p}(N,g_N)} \colon G \in W^{k,p}(N,g_N), G|_{\partial N} = F\}.$$

In Part Two, we will need the following standard Schauder estimate for solutions of linear elliptic propblems.

Theorem A.1.2. *[29, Corollary 6.3] Let* $\Omega \subset \mathbb{R}^n$ *be an open set and let* $\alpha \in (0,1)$*. Suppose* $u \in C^{2,\alpha}(\Omega)$ *satisfies a uniformly elliptic equation*

$$Lu = a^{ij}(x)u_{ij} + b^i(x)u_i + c(x)u = 0$$

with $a^{ij}, b^i, c \in C^\alpha(\Omega)$ *and ellipticity constant* $\lambda > 0$*. If* $\Omega' \subset\subset \Omega$ *with* $\mathrm{dist}^\Omega(\Omega', \partial\Omega) = d$*, then there is a constant* $C > 0$*, depending on* $d, \lambda, ||a^{ij}||_{C^\alpha(\Omega)}, ||b^i||_{C^\alpha(\Omega)}, ||c||_{C^\alpha(\Omega)}, n,$ *and* α*, such that*

$$||u||_{C^{2,\alpha}(\Omega')} \leq C||u||_{C^0(\Omega)}. \tag{A.2}$$

A.2. The Minimal Graph Equation

This section is concerned with local properties of hypersurfaces in Riemannian manifolds. Throughout this section we will consider the unit ball in Euclidian space $B = B_1(0) \subset \mathbb{R}^{n+1}$ equipped with a Riemannian metric g and a hypersurface $\Sigma^n \subset B$. The balls of radius $s > 0$ centered at $x \in \Sigma$ induced by g and $g|_\Sigma$ are denoted by $B^g_s(x) \subset B$ and $B^\Sigma_s(x) \subset \Sigma$, respectively. Assume there is a point $x_0 \in \Sigma \cap B_{1/4}(0)$.

The following straight-forward Riemannian version of [30, Lemma 2.4] allows us to consider Σ locally as a graph over $T_{x_0}\Sigma$.

Lemma A.2.1. *There is a constant $\mu_0 > 0$ so that if g satisfies*

$$\sup_{x\in B} |g_{ij}(x) - \delta_{ij}| \leq \mu_0, \quad \sup_{x\in B} \left|\tfrac{\partial g_{ij}}{\partial x^k}(x)\right| \leq \mu_0 \tag{A.3}$$

for $1 \leq i, j, k \leq n+1$ in standard Euclidian coordinates, then the following holds: If $s > 0$ satisfies

$$\mathrm{dist}^{\Sigma}(x_0, \partial\Sigma) \geq 3s, \quad \sup_{\Sigma} |A_g|^2 \leq \tfrac{1}{20s^2},$$

then there is an open subset $U \subset T_{x_0}\Sigma \subset \mathbb{R}^{n+1}$, a unit vector η normal to $T_{x_0}\Sigma$, and a function $u : U \to \mathbb{R}$ such that

1. $\mathrm{graph}(u) = B^{\Sigma}_{2s}(x_0)$;
2. $|\nabla u| \leq 1$ *and* $|\nabla\nabla u| \leq \frac{1}{s\sqrt{2}}$ *hold pointwise.*

Moreover, the connected component of $B^g_s(x_0) \cap \Sigma$ containing x_0 lies in $B^{\Sigma}_{2s}(x_0)$.

Now we will give a useful expression for the mean curvature of a graph. Let $U \subset \mathbb{R}^n$ be an open set with standard coordinates $x' = (x^1, \ldots, x^n)$ and let g be a Riemannian metric on $U \times \mathbb{R} \subset \mathbb{R}^{n+1}$. For a function $u : U \to \mathbb{R}$, consider its graph

$$\mathrm{graph}(u) = \{(x', u(x')) \in \mathbb{R}^{n+1} \colon x' \in U\}.$$

For $i = 1, \ldots, n$, we have the tangential vector fields $E_i = \frac{\partial}{\partial x^i} + \frac{\partial u}{\partial x^i}\frac{\partial}{\partial x^{n+1}}$ and the upward-pointing unit vector field ν normal to $\mathrm{graph}(u)$. Writing $h_{ij} = g(E_i, E_j)$ for the restriction metric, the mean curvature of graph(u) can be written

$$\begin{aligned} H_g &= h^{ij} g(\nu, \nabla_{E_i} E_j) \\ &= \left(g^{ij} - \tfrac{\nabla^i u \nabla^j u}{1+|\nabla u|^2}\right) \Big[\tfrac{\partial^2 u}{\partial x_i \partial x_j} + \Gamma^{n+1}_{ij} + \tfrac{\partial u}{\partial x_i} \Gamma^{n+1}_{n+1\ j} + \tfrac{\partial u}{\partial x_j} \Gamma^{n+1}_{n+1\ i} + \tfrac{\partial u}{\partial x_i} \tfrac{\partial u}{\partial x_j} \Gamma^{n+1}_{n+1\ n+1} \\ &\qquad - \tfrac{\partial u}{\partial x_r} \left(\Gamma^r_{ij} + \tfrac{\partial u}{\partial x_i} \Gamma^r_{n+1\ j} + \tfrac{\partial u}{\partial x_j} \Gamma^r_{i\ n+1} + \tfrac{\partial u}{\partial x_i} \tfrac{\partial u}{\partial x_j} \Gamma^r_{n+1\ n+1} \right) \Big], \end{aligned} \tag{A.4}$$

see [30, Section 7.1] for a detailed exposition in the 3-dimensional case.

Next, we will apply the Schauder estimates to the geometric setting in Section 3.2.

Corollary A.2.1. Suppose the unit ball $B = B_1(0) \subset \mathbb{R}^{n+1}$ is equipped with a Riemannian metric g satisfying

$$\sup_{x \in B} |g_{ij}(x) - \delta_{ij}| \leq \mu_0, \quad \sup_{x \in B} \left| \frac{\partial g_{ij}}{\partial x^k}(x) \right| \leq \mu_0$$

in Euclidian coordinates for all $1 \leq i, j, k \leq n+1$ where μ_0 is the constant from Lemma A.2.1. Let $C > 0$ be given and set $r = \min(\frac{1}{8}, \frac{1}{\sqrt{80C}})$. Assume that $\Sigma \subset B$ is a properly embedded minimal hypersurface with respect to g such that $\sup_B |A^g|^2 \leq C$ and there is a point $x_0 \in B_r(0) \cap \Sigma$. Then there is a smooth function $u : U \to \mathbb{R}$ on $U \subset T_{x_0}\Sigma$ and a unit normal vector to $T_{x_0}\Sigma$ such that

1. graph$(u) = B^{\Sigma}_{2r}(x_0)$;
2. $|\nabla u| \leq 1$ and $|\nabla \nabla u| \leq \frac{1}{s\sqrt{2}}$ hold pointwise;

3. for each $k \geq 1$ and $\alpha \in (0,1)$ there is a constant $C' > 0$, depending only on n, k, α, and $||g||_{C^{k,\alpha}(B)}$, such that

$$||u||_{C^{k,\alpha}(U)} \leq C'. \tag{A.5}$$

Moreover, the connected component of $B_r(x_0) \cap \Sigma$ containing x_0 is contained in $B^{\Sigma}_{2r}(x_0)$.

Proof. The choice of radius r allows us to apply Lemma A.2.1 to obtain an open subset $U \subset T_{x_0}\Sigma \subset \mathbb{R}^{n+1}$, a unit vector η normal to $T_{x_0}\Sigma$, and a smooth function $u : U \to \mathbb{R}$ such that $\mathrm{graph}(u) = B^{\Sigma}_{2s}(x_0)$, $|\nabla u| \leq 1$, and $|\nabla\nabla u| \leq \frac{1}{s\sqrt{2}}$ on U. Since Σ is minimal, u solves equation $H = 0$. Now since $||u||_{C^{1,\alpha}(U)}$ is bounded for any fixed $\alpha \in (0,1)$, one can inspect the expression A.4 to see that u solves a linear elliptic equation with coefficients bounded in C^{α} in terms of μ_0 and r. This allows us to apply Theorem A.1.2 to obtain the estimate $||u||_{C^{2,\alpha}(U')} \leq C||u||_{C^0(U)}$ for some $C > 0$ depending only on μ_0 and r. Standard elliptic estimates [29, Section 6] give a similar estimate in the $C^{k,\alpha}$-norm for any k. $\square$

A.3. Details on Theorem 1.5.4

Let us recall some basic notions from theory of integer multiplicity currents. The main reference for this material is [15, Chapter 4].

For an open subset $U \subset \mathbb{R}^{n+k}$, let $\Omega^n(U)$ denote the space of all n-forms on $\mathbb{R}^{n+k}$ with compact support in U. An *n-current* on U is a continuous linear functional $T : \Omega^n(U) \to \mathbb{R}$ and collection of such T for a vector space $\mathcal{D}_n(U)$. The *boundary* of an n-current T is the $(n-1)$-current ∂T defined by

$$(\partial T)(\omega) = T(d\omega), \qquad \omega \in \Omega^{n-1}(U).$$

The *mass* of $T \in \mathcal{D}_n(U)$ is given by $\mathbf{M}(T) = \sup\{T(\omega) : \omega \in \Omega^n(U), |\omega| \leq 1\}$. For example, if T is given by integration along a smooth oriented submanifold M, then $\mathbf{M}(T) = \mathrm{Vol}(M)$.

Let $\mathcal{H}^n$ denote the n-dimensional Hausdorff measure on $\mathbb{R}^{n+k}$. A current $T \in \mathcal{D}_n(U)$ is called *integer multiplicity rectifiable* (or simply *rectifiable*) if it takes the form

$$T(\omega) = \int_M \omega(\xi(x))\theta(x)d\mathcal{H}^n(x), \qquad \omega \in \Omega^n(U), \quad \text{where} \tag{A.6}$$

1. $M \subset U$ is $\mathcal{H}^n$-measurable and countably n-rectifiable, see [15, Section 3.2.14];

2. $\theta : M \to \mathbf{Z}$ is locally $\mathcal{H}^n$-integrable;

3. for $\mathcal{H}^n$-almost every $x \in M$, $\xi : M \to \Lambda^n T\mathbb{R}^{n+k}$ takes the form $\xi(x) = e_1 \wedge \ldots \wedge e_n$ where $\{e_i\}_{i=1}^n$ form an orthonormal basis for the approximate tangent space $T_x M$, see [15, Section 3.2.16].

Remark A.3.1. The above definition of integer multiplicity rectifiable currents can also be extended to Riemannian manifolds (M, g) – one defines the mass of a current using the Hausdorff measure given by the metric g.

The *regular* set $\mathrm{reg}(T)$ of a rectifiable n-current T is given by the set of points $x \in \mathrm{spt}(T)$ for which there exists an oriented n-dimensional oriented C^1-submanifold $M \subset U$, $r > 0$, and $m \in \mathbf{Z}$ satisfying

$$T|_{B_r(x)}(\omega) = m \cdot \int_{M \cap B_r(x)} \omega, \qquad \forall \omega \in \Omega^n(U).$$

The *singular* set $\operatorname{sing}(T)$ is given by $\operatorname{spt}(T) \setminus \operatorname{reg}(T)$. The abelian group of n-dimensional *integral flat chains* on U is given by

$$\mathcal{F}_n(U) = \{R + \partial S \colon R \in \mathcal{D}_n(U) \text{ and } S \in \mathcal{D}_{n+1}(U) \text{ are rectifiable}\}.$$

Now we consider subsets $B \subset A \subset U$. We have the group of *integral flat cycles*

$$\mathcal{C}_n(A, B) = \{T \in \mathcal{F}_n(U) \colon \operatorname{spt}(T) \subset A, \operatorname{spt}(\partial T) \subset B, \text{ or } n = 0\}$$

and the subgroup of *integral flat boundaries*

$$\mathcal{B}_n(A, B) = \{T + \partial S \colon T \in \mathcal{F}_n(U), \operatorname{spt}(T) \subset B, S \in \mathcal{F}_{n+1}(U), \operatorname{spt}(S) \subset A\}.$$

The quotient groups $\mathbf{H}_n(A, B) = \mathcal{C}_n(A, B)/\mathcal{B}_n(A, B)$ are the n-dimensional *integral current homology groups*.

There is a natural transformation between the integral singular homology functor and the integral current homology functor which induces an isomorphism $H_n(A, B; \mathbf{Z}) \cong \mathbf{H}_n(A, B)$ in the category of local Lipschitz neighborhood retracts, see [15, Section 4.4.1]. This isomorphism can be combined with a basic compactness result for rectifiable currents to find volume minimizing representatives of homology classes.

Lemma A.3.1. *Let $(M, \bar{g})$ be a compact $(n+1)$-dimensional Riemannian manifold with boundary and consider an integral homology class $\alpha \in H_n(M, \partial M; \mathbf{Z})$. Let $\tilde{\alpha} \in \mathbb{H}_n(M, \partial M)$ be the image of α under the isomorphism $H_n(M, \partial M; \mathbf{Z}) \to \mathbf{H}_n(M, \partial M)$. Then there exists a homologically volume minimizing integer multiplicity rectifiable current $T \in \tilde{\alpha}$.*

Proof. By the Nash embedding theorem there is an isometric embedding $\iota : M \to \mathbb{R}^{n+k}$ for some sufficiently large k. Let $\hat{M}$ be the image of this embedding and set $\hat{\alpha} = \iota_* \tilde{\alpha} \in \mathbf{H}_n(\hat{M}, \partial \hat{M})$. Applying the compactness result in [15, Section 5.1.6], we obtain a homologically volume minimizing current $\hat{T} \in \mathcal{C}_n(\hat{M}, \partial \hat{M})$ representing $\hat{\alpha}$. Since ι is an isometry, $(\iota^{-1})_* \hat{T}$ is the desired current. □

Since Lemma A.3.1 guarantees the existence of homologically volume minimizing representative for the homology class α from the hypothesis of Theorem 1.5.4, the final ingredient is regularity theory for volume minimizing rectifiable currents with free boundary. The following is a regularity theorem due to M. Grünter [17, Theorem 4.7] adapted to the context of an ambient Riemannian metric. See [18, 22, 31] for Riemannian adaptations of similar results.

Theorem A.3.2. *Let $S \subset \mathbb{R}^{n+1}$ be an n-dimensional smooth submanifold, $U \subset \mathbb{R}^{n+1}$ an open set with $\partial S \cap U = \emptyset$, and g a Riemannian metric on U with bounded injectivity radius and sectional curvature. Suppose $T \in \mathcal{F}_n(U)$ with $\operatorname{spt}(\partial T) \subset S$ satisfies $\mathbf{M}_g(T) \leq \mathbf{M}_g(T+R)$ for all open $W \subset\subset U$ and all $R \in \mathcal{F}_n(U)$ with $\operatorname{spt}(R) \subset W$ and $\operatorname{spt}(\partial R) \subset S$. Then we have*

- $\operatorname{sing}(T) = \emptyset$ *if* $n \leq 6$
- $\operatorname{sing}(T)$ *is discrete for* $n = 7$
- $\dim_{\mathcal{H}}(\operatorname{sing}(T)) \leq n - 7$ *if* $n > 7$

where $\dim_{\mathcal{H}}(A)$ *denotes the Hausdorff dimension of a subset* $A \subset U$.

We will briefly explain how Theorem 1.5.4 follows from Theorem A.3.2. Let T be the volume minimizing representative of $\bar{\alpha}$ from Theorem 1.5.4. For a point

$x \in \operatorname{spt}(T)$, set $\phi = \exp_x^{\bar{g}}$ and consider

$$U = \phi^{-1}(B_{r'}^{\bar{g}}(x)) \subset T_xM, \quad S = \phi^{-1}(\partial M \cap B_r^{\bar{g}}(x)),$$

$$T' = (\phi^{-1})_* T \in \mathcal{D}_n(U), \quad g = (\phi^{-1})_* \bar{g},$$

where $0 < r' < r \leq \operatorname{inj}(\bar{g})$. By Theorem A.3.2, the singular set of T' is empty and so there is a neighborhood V of $0 \in U$ such that $T'|_V$ is given by an integer multiple of integration along a C^1-submanifold $M \subset V$. Locally, M can be written as the graph of a C^1-function which weakly solves the minimal surface equation. Standard elliptic PDE methods imply that M is smooth, see, for instance the proof of Lemma A.3.3 below.

A.31. Doubling minimal hypersurfaces with free boundary

In this section we consider the reflection of a free boundary stable minimal hypersurface over its boundary. To fix the setting, let $(M, \bar{g})$ be an $(n+1)$-dimensional compact oriented Riemannian manifold with boundary ∂M and restriction metric $g = \bar{g}|_{\partial M}$. Assume that there is a neighborhood of the boundary on which $\bar{g} = g_{\partial M} + dt^2$. The *double* of $(M, \bar{g})$ is the smooth closed manifold $M_{\mathcal{D}}$ given by $M_{\mathcal{D}} = M \cup_{\partial M} (-M)$. Notice that the double $M_{\mathcal{D}}$ comes equipped with an involution $\iota : M_{\mathcal{D}} \to M_{\mathcal{D}}$ which interchanges the two copies of M and fixes the doubling locus $\partial M \subset M_{\mathcal{D}}$. Since $\bar{g}$ splits as a product near the boundary, one can also form the smooth doubling of $\bar{g}$, denoted by $\bar{g}_{\mathcal{D}}$, by setting $\bar{g}_{\mathcal{D}} = \bar{g}$ on M and $\bar{g}_{\mathcal{D}} = \iota_* \bar{g}$ on $-M$.

Lemma A.3.3. *Let $(M, \bar{g})$ be a compact oriented Riemannian manifold with boundary with $\bar{g} = g + dt^2$ near ∂M. If $\Sigma \subset M$ be a properly embedded minimal* *hypersurface with free boundary, then double of Σ, given by $\Sigma_{\mathcal{D}} = \Sigma \cup_{\partial\Sigma} \iota(\Sigma)$ is a smooth minimal hypersurface of $(M_{\mathcal{D}}, \bar{g}_{\mathcal{D}})$. Moreover, if Σ is stable, then so is $\Sigma_{\mathcal{D}}$.*

Proof. First, we will show that $\Sigma_{\mathcal{D}}$ is a smooth hypersurface. Clearly, $\Sigma_{\mathcal{D}}$ is smooth away from the doubling locus $\partial\Sigma \subset M_{\mathcal{D}}$. Let $x_0 \in \partial\Sigma$ and let $r > 0$ be less than the injectivity radius of $\bar{g}_{\mathcal{D}}$. Set $\phi = \exp_{x_0}^{\bar{g}_{\mathcal{D}}}$ and consider

$$\hat{\Sigma} = \phi^{-1}(\Sigma \cap B_r(x_0)), \quad \hat{\Sigma}_{\mathcal{D}} = \phi^{-1}(\Sigma_{\mathcal{D}} \cap B_r(x_0)), \quad \hat{g} = \phi^* \bar{g}_{\mathcal{D}}$$

and ν, the unit normal vector field to $\hat{\Sigma}$ with respect to $\hat{g}$. Evidently, $\hat{\Sigma}$ is a minimal hypersurface in $T_{x_0}M_{\mathcal{D}}$ with free boundary contained in $T_{x_0}\partial M \subset T_{x_0}M_{\mathcal{D}}$ with respect to $\hat{g}$. We choose an orthonormal basis for $T_{x_0}M_{\mathcal{D}}$ so that, writing $x \in T_{x_0}M$ as $(x^1, \dots, x^{n+1})$ in this basis,

1. $T_{x_0}\partial\hat{\Sigma} = \{(x^1, \dots, x^{n-1}, 0, 0)\}$;

2. $T_{x_0}\hat{\Sigma} = \{(x^1, \dots, x^n, 0)\}$;

3. $T_{x_0}\partial M = \{(x^1, \dots, x^{n-1}, 0, x^{n+1})\}$.

This can be accomplished since Σ meets ∂M orthogonally. In these coordinates, the involution ι now takes the form $(x^1, \dots, x^n, x^{n+1}) \mapsto (x^1, \dots, -x^n, x^{n+1})$. Notice that, because the second fundamental form of ∂M vanishes, $\phi^{-1}(\partial M \cap B_r(x_0))$ is contained in the hyperplane $\{(x^1, \dots, x^{n+1}) \colon x^n = 0\}$.

For a radius $r' < r$, we consider the n-dimensional ball

$$B^n_{r'}(0) = \{x \in T_{x_0}M \colon x^{n+1} = 0, ||x|| < r'\},$$

the n-dimensional half-ball $B^n_{r',+}(0) = \{x \in B^n_{r'}(0)\colon x^n \geq 0\}$, and the cylinder

$$C_{r'}(0) = \{x \in T_{x_0}M\colon (x^1, \ldots, x^n, 0) \in B^n_{r'}(0)\}.$$

For small enough r', we may write $\hat{\Sigma} \cap C_{r'}(0)$ as the graph of a function

$$u : B^n_{r',+}(0) \to \mathbb{R}, \quad \operatorname{graph}(u) = \hat{\Sigma} \cap C_{r'}(0)$$

where $\operatorname{graph}(u) = \{(x^1, \ldots, x^n, u(x^1, \ldots, x^n))\colon (x^1, \ldots, x^n, 0) \in B^n_{r'}(0)\}$. Now we may form the doubling of u to a function $u_{\mathcal{D}} : B^n_{r'}(0) \to \mathbb{R}$, setting

$$u_{\mathcal{D}}(x^1, \ldots, x^n) = \begin{cases} u(x^1, \ldots, x^n) & \text{if } x^n \geq 0 \\ u(x^1, \ldots, x^{n-1}, -x^n) & \text{if } x^n < 0. \end{cases}$$

To show $\Sigma_{\mathcal{D}}$ is smooth at x_0, it suffices to show that $u_{\mathcal{D}}$ is smooth along $\{x \in B^n_{r'}(0)\colon x^n = 0\}$.

From the free boundary condition, we have $\frac{\partial u}{\partial x^n} \equiv 0$ on $\{x^n = 0\}$ and so $u_{\mathcal{D}}$ has a continuous derivative on all of $B^n_{r'}(0)$. Since $\hat{\Sigma}$ is smooth and minimal, $u_{\mathcal{D}}$ is smooth and solves the minimal graph equation (A.4) with respect to the metric $\hat{g}_{\mathcal{D}}$ in the strong sense on $\{x \in B^n_{r'}(0)\colon x^n \neq 0\}$. Moreover, it follows from $\frac{\partial u}{\partial x^n} \equiv 0$ on $\{x^n = 0\}$ and the ι-invariance of $\bar{g}_{\mathcal{D}}$ that $u_{\mathcal{D}}$ solves the minimal graph equation weakly on the entire ball $B^n_{r'}(0)$.

From this point, the smoothness of $u_{\mathcal{D}}$ is a standard application of tools from nonlinear elliptic PDE theory, so we will be brief (see [30, Lemma 7.2]). Standard estimates for minimizers implies $u_{\mathcal{D}} \in H^2(B^n_{r'}(0))$ (see [32, Section 8.3.1]). Writing the equation (A.4) in divergence form, we have

$$\frac{\partial}{\partial x^i}\left(a^{ij}\frac{\partial u_{\mathcal{D}}}{\partial x^j}+b^i u_{\mathcal{D}}\right)=0 \tag{A.7}$$

where the coefficients a^{ij} and b^i depend on $u_{\mathcal{D}}$ and are only differentiable. Since $u_{\mathcal{D}}$ weakly solves equation (A.7),

$$\int_{B^n_{r'}(0)}\left(a^{ij}\frac{\partial u_{\mathcal{D}}}{\partial x^j}+b^i u_{\mathcal{D}}\right)\frac{\partial \psi}{\partial x^i}dx=0,$$

for any test function $\psi \in C_0^\infty(B^n_{r'}(0))$ Taking ψ to be of the form $-\frac{\partial w}{\partial x^k}$ for some function w and integrating by parts, one finds $\frac{\partial u_{\mathcal{D}}}{\partial x^k}$ is a weak solution of a uniformly elliptic linear equation with L^∞ coefficients for each $k=1,\ldots,n$.

Now we may apply the DeGiorgi-Nash theorem (see [29, Theorem 8.24]) to conclude that, for each $r''<r'$ there is an $\alpha\in(0,1)$ such that $\frac{\partial u_{\mathcal{D}}}{\partial x^k}\in C^{0,\alpha}(B^n_{r''}(0))$ for each $k=1,\ldots,n$. Now $u^D\in C^{1,\alpha}(B^n_{r''}(0))$ and the functions $\frac{\partial u_{\mathcal{D}}}{\partial x^k}$ solve a uniformly elliptic linear equation with Hölder coefficients. The Schauder estimates from Theorem A.1.2 allow us to conclude that $\frac{\partial u_{\mathcal{D}}}{\partial x^k}\in C^{2,\alpha}(B_{r'}(0))$. This argument may be iterated, see [29, Section 8], to conclude $u_{\mathcal{D}}\in C^{k,\alpha}(B^n_{r''}(0))$ for any k. This finishes the proof that $u_{\mathcal{D}}$ is a smooth solution to the mean curvature equation across the doubling locus $\{x^n=0\}$ and hence $\Sigma_{\mathcal{D}}$ is a smooth minimal hypersurface.

The last step is to show that $\Sigma_{\mathcal{D}}$ is stable. Let $\phi\in C^\infty(\Sigma_{\mathcal{D}})$ define a normal variation and write $\phi=\phi_0+\phi_1$ where ϕ_0 is invariant under the involution and ϕ_1 is anti-invariant under the involution. Now we will consider the second variation of the volume of $\Sigma_{\mathcal{D}}$ with respect to ϕ.

$$\begin{aligned}
\delta^2_\phi(\Sigma_{\mathcal{D}}) &= \int_{\Sigma_{\mathcal{D}}} |\nabla\phi|^2 - \phi^2(\mathrm{Ric}(\nu,\nu) + |A|^2) d\mu \\
&= \int_{\Sigma_{\mathcal{D}}} |\nabla\phi_0|^2 + 2g(\nabla\phi_0, \nabla\phi_1) + |\nabla\phi_1|^2 - (\phi_0^2 + 2\phi_0\phi_1 + \phi_1^2)(\mathrm{Ric}(\nu,\nu) + |A|^2) d\mu \\
&= \delta^2_{\phi_0}(\Sigma_{\mathcal{D}}) + \delta^2_{\phi_1}(\Sigma_{\mathcal{D}}) + \int_{\Sigma_{\mathcal{D}}} 2g(\nabla\phi_0, \nabla\phi_1) - 2\phi_0\phi_1(\mathrm{Ric}(\nu,\nu) + |A|^2) d\mu \\
&= 2\delta^2_{\phi_0|_\Sigma}(\Sigma) + 2\delta^2_{\phi_1|_\Sigma}(\Sigma) \geq 0
\end{aligned}$$

where the last equality follows from the fact that $g(\nabla\phi_0, \nabla\phi_1)$ and $\phi_0\phi_1$ are anti-invariant under the involution. This completes the proof of Lemma A.3.3. □

A.32. Second fundamental form bounds

In this section, we will prove Step 2 in Section 3.25. Let $(M_i, \bar{g}_i)$ and W_i be as in Main Lemma. The uniform second fundamental form bounds for the stable minimal hypersurfaces $W_i \subset M_i$ can be reduced to a classical estimate due to Schoen-Simon [25] for stable minimal hypersurfaces in Riemannian manifolds. In the following, $(M, \bar{g})$ is a complete $(n+1)$-dimensional Riemannian manifold, $x_0 \in M$, $\rho_0 \in (0, \mathrm{inj}_{\bar{g}}(x_0))$, and μ_1 is a constant satisfying

$$\sup_{B_\rho(0)} \left|\frac{\partial \bar{g}_{ij}}{\partial x^k}\right| \leq \mu_1, \quad \sup_{B_\rho(0)} \left|\frac{\partial^2 \bar{g}_{ij}}{\partial x^k \partial x^l}\right| \leq \mu_1^2, \tag{A.8}$$

on the metric ball $B_{\rho_0}(x_0)$ in geodesic normal coordinates $(x^1, \ldots, x^{n+1})$ centered at x_0.

Theorem A.3.4 (Corollary 1 [25]). *Suppose Σ is an oriented embedded C^2-hypersurface in an $(n+1)$-dimensional Riemannian manifold $(M, \bar{g})$ with $x_0 \in \overline{\Sigma}$, μ_1 satisfies* (A.8)*, and μ satisfies the bound $\rho_0^{-n}\mathcal{H}^n(\Sigma \cap B_{\rho_0}(x_0)) \leq \mu$. Assume that*

$\mathcal{H}^n(\Sigma \cap B_{\rho_0}(x_0)) < \infty$ *and* $\mathcal{H}^{n-2}(\mathrm{sing}(\Sigma) \cap B_{\rho_0}(x_0)) = 0$. *If* $n \leq 6$ *and* Σ *is stable in* $B_{\rho_0}(x_0)$, *then*

$$\sup_{B_{\rho_0}(x_0)} |A^\Sigma| \leq \frac{C}{\rho_0},$$

where C *depends only on* n, μ, *and* $\mu_1 \rho_0$.

Proof of Step 2. By Lemma A.3.3, the doubling $(W_i)_{\mathcal{D}}$ is a smooth stable minimal hypersurface of $(M_i)_{\mathcal{D}}$. In particular, the singular set of $(W_i)_{\mathcal{D}}$ is empty. Moreover, the manifolds $(M_i)_{\mathcal{D}}$ have uniformly bounded geometry so that the injectivity radius is uniformly bounded from below by some $\rho_0 > 0$, and there is a constant μ_1 so that the bounds (A.8) hold in normal coordinates about any $x \in (M_i)_{\mathcal{D}}$, any $\rho \in (0, \rho_0)$, and all $i = 1, 2. \ldots$. According to Step 1, there is a constant μ such that

$$\rho_0^{-n} \mathrm{Vol}(W_i \cap B_\rho(x)) \leq \mu$$

for all $i = 1, 2, \ldots$. Hence, we may uniformly apply Theorem A.3.4 on any ball $B_{\rho_0}(x_0) \subset (M_i)_{\mathcal{D}}$ intersecting W_i to obtain the bound in Step 2. □

A.33. Generic regularity in dimension 8

It is well known that codimension one volume minimizing currents, in general, have singularities if the ambient space is of dimension 8 or larger. However, in [27] N. Smale developed a method for removing these singularities in 8-dimensional Riemannian manifolds by making arbitrarily small conformal changes. In this section, we will describe the modifications necessary to adapt his method to the case of Theorem 1.5.6 with $n = 7$.

First, we will describe the perturbation result we will use. Let M be a compact $(n+1)$-dimensional manifold. For $k = 3, 4, \ldots$, let $\mathcal{M}_0^k$ denote the class of C^k metrics on M which split isometrically as a product on some neighborhood of ∂M. Fix a relative homology class $\alpha \in H_n(M, \partial M; \mathbb{Z})$. We will show the following.

Theorem A.3.5. *Let $g_0 \in \mathcal{M}_0^k$ and $n = 7$. For $\varepsilon > 0$, there exists a metric $g \in \mathcal{M}_0^k$ and a g_0-volume minimizing current T representing α such that $||g - g_0||_{C^k} < \varepsilon$ and* $\mathrm{spt}(T)$ *is smooth.*

The proof of Theorem A.3.5 follows by showing the constructions in [27] can be performed on the doubled manifold $M_{\mathcal{D}}$ (see Appendix A.31) in an involution-invariant manner. We proceed in two lemmas. The first lemma holds in any dimension.

Lemma A.3.6. *Let $g_0 \in \mathcal{M}_0^k$ and suppose T is a homologicly g_0-volume minimizing current representing α. For $\varepsilon > 0$, there is a metric $g \in \mathcal{M}_0^k$ such that $||g - g_0||_{C^k} < \varepsilon$ and T is the only g-volume minimizing current representative of α.*

Proof. Let A, $d\mu = \theta d\mathcal{H}^n$, and ξ be the underlying rectifiable set, measure, and choice of orientation for the approximate tangent space of A associated to the current T (see Section A.3). We may write $A = \cup_{j=1}^N A_j$ where each A_j are connected. Choose $p_j \in \mathrm{reg}(A_j) \setminus \partial M$ and $\rho > 0$ so that

$$(B_\rho(p_j) \cap A_j) \subset (\mathrm{reg}(A) \setminus \partial M), \quad j = 1, \dots, N.$$

Perhaps restricting to smaller ρ, let $x = (x^1, \dots, x^n)$ be geodesic normal coordinates for $B_\rho(p_j) \cap A_j$ and let t be the signed distance on $B_\rho(p_j)$ from A_j determined by ξ. This gives Fermi coordinates (t, x) on $B_\rho(p_j)$. Now fix a bump function $\eta : A \to [0, 1]$ satisfying

$$\eta(x) = \begin{cases} 1 & \text{for } x \in B_{\rho/2}(p_j) \cap A_j \\ 0 & \text{for } x \in B_\rho(p_j) \setminus B_{3\rho/4}(p_j) \end{cases}$$

for each $j = 1, \ldots N$. Also fix a smooth function $\phi : \mathbb{R} \to \mathbb{R}$ with $\text{spt}(\phi) \subset [-3/4, 3/4]$,

$$\phi(t) \geq 0 \text{ on } [-1,1], \phi(0) = 1, \text{ and } \phi(r) < 1 \text{ if } r \neq 0.$$

Consider the function $\phi_{\bar{\varepsilon}} : M \to \mathbb{R}$ given by

$$\phi_{\bar{\varepsilon}}(y) = \begin{cases} 1 - \bar{\varepsilon}^{k+1}\phi(t/\bar{\varepsilon})\eta(x) & \text{if } y = (x,t) \in B_\rho(p_j) \text{ for some } j \\ 1 & \text{otherwise} \end{cases}$$

for $\bar{\varepsilon} > 0$ satisfying $\text{spt}(\phi_{\bar{\varepsilon}}) \subset \cup_{j=1}^N B_{3\rho/4}(p_j)$. We have the perturbed metrics $g_{\bar{\varepsilon}} = \phi_{\bar{\varepsilon}}^{\frac{2}{n}} g_0 \in \mathcal{M}_0^k$. It is straight-forward to show that there exists $\varepsilon_1 \in (0, \varepsilon)$ such that, for any $\bar{\varepsilon} \in (0, \varepsilon_1]$, T is the only $g_{\bar{\varepsilon}}$-volume minimizing representative of α (see [27]). Perhaps restricting to smaller values of $\bar{\varepsilon}$, we may also arrange for $||g - g_{\bar{\varepsilon}}||_{C^k} < \varepsilon$. This completes the proof of Lemma A.3.6. □

Lemma A.3.7. *Let $n = 7$, $k \geq 3$, $g_0 \in \mathcal{M}^k$, and $\varepsilon > 0$. Suppose T is the only g_0-volume minimizing representative of α, then there exists $g \in \mathcal{M}^k$ such that $||g - g_0||_{C^k} < \varepsilon$ and α may be represented (up to multiplicity) by a smooth g-volume minimizing hypersurface.*

Proof. Following [27], we construct a conformal factor which will slide the minimizing current off itself in one direction and appeal to a perturbation result for isolated singularities which allows us to conclude that this new current has no singularity. Write $(M_{\mathcal{D}}, g_{0,\mathcal{D}})$ for the doubling of (M, g_0) (see Section A.31) with involution $\iota : M_{\mathcal{D}} \to M_{\mathcal{D}}$. The current T may also be doubled to obtain an involution-invariant current $T_{\mathcal{D}}$ on $M_{\mathcal{D}}$. Similarly to Section A.31, $T_{\mathcal{D}}$ is locally $g_{0,\mathcal{D}}$-volume minimizing. Let $A = \cup_{j=1}^N A_j$, $d\mu = \theta d\mathcal{H}^7$, and ξ be the underlying set, measure, and orientation associated to T, as in the proof of Lemma A.3.6.

Let $\rho_0 > 0$ and fix a smooth function $\phi : \mathbb{R} \to \mathbb{R}$ satisfying

1. $\phi(-t) = -\phi(t)$,
2. $\phi(t) \geq 0$ for $t \geq 0$,
3. $\phi(t) = t$ for $t \in [0, \frac{\rho_0}{4}]$,
4. $\phi(t) = \frac{\rho_0}{2}$ for $t \in [\frac{\rho_0}{2}, \frac{3\rho_0}{4}]$,
5. $\phi(t) = 0$ for $t \geq \rho_0$.

Let $\{B_\rho(p_j)\}_{j=1}^N$ be a collection of disjoint metric balls in $\mathring{M}$ centered at regular points $p_j \in A_j$. Choose $\rho_0 > 0$ small enough to ensure that, in Fermi coordinates (t, x) for A_j with ξ pointing into the side corresponding to $t > 0$, the function $(t, x) \mapsto \phi(t)\eta(x)$ is supported in $\cup_{j=1}^N B_\rho(p_j)$. For a fixed $s \in (0, 1)$ and a parameter $\bar{\varepsilon} \in (0, 1)$, consider the functions $u_{\bar{\varepsilon}} : \Sigma \to \mathbb{R}$ given by

$$u_{\bar{\varepsilon}}(y) = \begin{cases} 1 - \bar{\varepsilon}^s \phi(t)\eta(x) & \text{if } y = (t, x) \in \cup_{j=1}^N B_\rho(p_j) \\ 1 & \text{otherwise.} \end{cases}$$

The conformal metrics $g_{\bar{\varepsilon}} = u_{\bar{\varepsilon}}^{\frac{2}{n}} g_0$ will be used to find the desired smooth representative. Since $g_{\bar{\varepsilon}}$ splits as a product near ∂M, we may consider the corresponding ι-invariant metric $g_{\bar{\varepsilon}, \mathcal{D}}$ on $M_{\mathcal{D}}$.

For sake of contradiction, suppose that there is a sequence $\bar{\varepsilon}_i \to 0$ and homologically $g_{\bar{\varepsilon}_i}$-volume minimizing currents T_i representing α with $\operatorname{sing}(T_i) \neq \emptyset$ for all $i = 1, 2, \ldots$. Since $\mathbf{M}(T_i)$ is uniformly bounded in i, T_i weakly converges to some homologically g_0-volume minimizing current T_∞ which also represents α. Since T is assumed to be the unique such current, we must have $T_\infty = T$. Write P_i, $d\mu_i$, and ξ_i for the set, measure, and orientation corresponding to T_i for $i = 1, 2, \ldots$. Let Q_i be a connected component of P_i with $\mathrm{sing}(Q_i) \neq \emptyset$ for each $i = 1, 2, \ldots$. Now Q_i converges in the Hausdorff sense to some sheet Q of T. By the Allard regularity theorem [33], this convergence is smooth away from $\mathrm{sing}(Q)$. Hence, after passing to a subsequence, y_i converges to some $y \in \mathrm{sing}(Q)$.

In terms of the doubled manifold, the ι-invariant currents $T_{i,\mathcal{D}}$ are homologically $g_{\bar{\varepsilon}_i,\mathcal{D}}$-volume minimizing, $T_{i,\mathcal{D}}$ weakly converge to $T_{0,\mathcal{D}}$, and the doubled sets $Q_{i,\mathcal{D}}$ converge to $Q_{\mathcal{D}}$ smoothly away from $\mathrm{sing}(Q_{\mathcal{D}})$. Now let $\mathcal{N} \subset M_{\mathcal{D}}$ be a small distance neighborhood of $Q_{\mathcal{D}}$ so that $\mathcal{N} \setminus Q_{\mathcal{D}}$ consists of two disjoint, open sets $\mathcal{N}_-$ and $\mathcal{N}_+$ on which the signed distance to $Q_{\mathcal{D}}$ is negative and positive, respectively. In the doubled manifold, we may directly apply the following results from [27].

Lemma A.3.8. *[27, Proposition 1.6] For large i, we have*

1. $Q_{i,\mathcal{D}} \cap \mathcal{N}_- = \emptyset$
2. $Q_{i,\mathcal{D}} \cap \mathcal{N}_+ \setminus \mathrm{spt}(\phi_{\varepsilon_i}\eta)_{\mathcal{D}} \neq \emptyset$.

In light of Lemma A.3.8, the Simon maximum principle [34] shows

$$(Q_{i,\mathcal{D}} \setminus \mathrm{spt}(\phi_i\eta)_{\mathcal{D}}) \subset (\mathcal{N}_+ \setminus \mathrm{spt}(\phi_i\eta)_{\mathcal{D}})$$

for each $i = 1, 2, \ldots$. Recalling that $Q_{i,\mathcal{D}}$ converges to $Q_{\mathcal{D}}$ in the Hausdorff distance, we may apply the perturbation result [35, Theorem 5.6] to conclude that $Q_{i,\mathcal{D}}$ is smooth for sufficiently large i. This contradiction finishes the proof of Lemma A.3.7. $\square$

Theorem A.3.5 follows by first applying Lemma A.3.6 to approximate g_0 with a metric g_1 supporting a unique minimizing representative of α then applying

Lemma A.3.7 to approximate g_1 with a metric g_2 and obtain a g_2-volume minimizing representative of α.

Proof of Theorem 1.5.6 for $n = 7$. We will closely follow the argument presented in Section 3.3. Let $(Z, \bar{g}, \bar{\gamma}) : (Y_0, g_0, \gamma_0) \rightsquigarrow (Y_1, g_1, \gamma_1)$ be a psc-bordism and let $(Z_i, \bar{g}_i, \bar{\gamma}_i)$ be the corresponding i-collaring for $i = 1, 2, \ldots$. As usual, we denote by $\bar{\alpha}_i \in H_7(Z_i, \partial Z_i; \mathbb{Z})$ the Poincaré dual to $\bar{\gamma}_i$.

For each $i = 1, 2, \ldots$, we apply Theorem A.3.5 to obtain a metric $\hat{g}_i$ on Z_i so that

$$||\hat{g}_i - \bar{g}_i||_{C^i_{\bar{g}_i}} \leq \frac{1}{i}$$

and $\bar{\alpha}_i$ can be represented by a smooth $\hat{g}_i$-volume minimizing hypersurface W_i. It follows from the proofs of Lemmas A.3.6 and A.3.7 that $\hat{g}_i$ can and will be chosen so that $\{\hat{g}_i \neq \bar{g}_i\} \subset M_1 \subset M_i$ for $i = 1, 2, \ldots$. Indeed, the perturbations required to form $\hat{g}_i$ are supported on balls centered about chosen regular points of $\bar{g}_i$-volume minimizing currents and one can always find regular points of minimizers of $\bar{\alpha}_i$ in $M_1 \subset M_i$. Evidently, $\hat{g}_i$ has positive scalar curvature for all sufficiently large i. Since $\hat{g}_i = \bar{g}_i$ on $Y \times [-i, 0] \subset Z_i$, the proof of the Main Lemma shows that there is a subconvergence

$$(Z_i, W_i, \hat{g}_i, \mathsf{S}_i) \rightarrow (Y \times (-\infty, 0], X \times (-\infty, 0], g + dt^2, \mathsf{S})$$

where Y, X, g, S_i, and S_∞ are defined as in Section 3.3. One can now directly apply the argument from 3.31 to finish the proof of Theorem 1.5.6 for $n = 7$. $\square$

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