# MOTIVES OF LOG SCHEMES

by

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A DISSERTATION

Presented to the Department of Mathematics
and the Graduate School of the University of Oregon
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

June 2017

DISSERTATION APPROVAL PAGE

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Title: Motives of Log Schemes

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Degree awarded June 2017

DISSERTATION ABSTRACT

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Doctor of Philosophy

Department of Mathematics

June 2017

Title: Motives of Log Schemes

This thesis introduces two notions of motive associated to a log scheme. We introduce a category of log motives *à la* Voevodsky, and prove that the embedding of Voevodsky motives is an equivalence, in particular proving that any homotopy-invariant cohomology theory of schemes extends uniquely to log schemes. In the case of a log smooth degeneration, we give an explicit construction of the motivic Albanese of the degeneration, and show that the Hodge realization of this construction gives the Albanese of the limit Hodge structure.

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ACKNOWLEDGEMENTS

I am very grateful to my family and friends for their support during this long (and ongoing) project. Special thanks are due to my mother, Carol, and father, Charles, for their tremendous investment in my education.

I am thankful for those friends I worked with: Riley, Amber and Prashant, Cordelia and Xiaowen and Nathan, Peter, Max and Bronson and Rob, Christin and Justin. I now understand how much more enjoyable mathematics is as a social sport.

My friends from the sciences and engineering I thank for enriching my life with their perspectives: Alex, Heather, and Kristin; Brent, Bursaw, David, Kat, Max and Rahel; Steve, Jaimie, and Matt; Dan, Ron, Pete, and Derrick; Balu, and Mike; and Samira. Without you, I would know so little of the world outside of mathematics.

There are also many people who have helped to keep my life grounded and balanced; without them, I would have succumbed to stress long ago: in addition to all of the above, I thank David', Lane, Amelia and Kevin, Demetre and Adam, Katelyn, and towering above them all, Kathleen.

I owe much to the department staff, especially Jessica, for handling all of my bureaucratic trouble-making. I am also very thankful for the many people who made my trips to the Higher School of Economics in 2016 both possible and hospitable. During these trips I completed and revised a great deal of this work.

The debt I owe my advisor, Vadim, is immense and impossible to repay. Without his patience and persistence, my degree would be worth much less. I could not have hoped for a better teacher or friend in an advisor.

To my friends and enemies,

and most especially

to those who can't decide.

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CHAPTER I

INTRODUCTION

This thesis is part of a joint project with Vadim Vologodsky.

## 1.1. For a General Audience

**1.1.1.** Consider a family $X$ of smooth proper genus-2 curves degenerating to a union $X_0 = \mathbb{P}^1 \cup E$ of $\mathbb{P}^1$ and a genus-1 curve creating two nodes. In topology, or even differential geometry, we can recover the genus-2 curve by removing the two points and gluing the collars. The precise way in which the gluing is performed can be described as follows.

**1.1.2.** Let $p$ be one of the singular points. A choice of local coordinate $t$ for the degeneration gives a trivialization

$$dt_p : T_pX = N(\mathbb{P}^1|X)|_p \times N(E|X)|_p \to \mathbb{C}$$

Taking the fiber over 1 gives a choice of identification of punctured normal bundles

$$N(D|X)|_p^\circ \simeq N(E|X)|_p^\circ \tag{1.1.2.1}$$

**1.1.3.** Since $N(D|X) \cap N(E|X) = 0$ we must have $N(\mathbb{P}^1|X) \subset TE$ so that

$$N(\mathbb{P}^1|X)|_p = (TE \times_X \mathbb{P}^1)|_p = N(p|E),$$

and similarly

$$N(E|X)|_p = (T\mathbb{P}^1 \times_X E)|_p = N(p|\mathbb{P}^1).$$

**1.1.4.** Now apply the tubular neighborhood theorem from differential geometry to identify neighborhoods $TN(p|-)$ of $p$ with the normal bundle $N(p|-)$ at $p$:

$$N(E|X)|_p = N(p|\mathbb{P}^1) \cong TN(p|\mathbb{P}^1) \subset \mathbb{P}^1$$
$$N(\mathbb{P}^1|X)|_p = N(p|E) \cong TN(p|E) \subset E$$

Applying (1.1.2.1) gives identifications of the *punctured* tubular neighborhoods $TN(p|-) \setminus p$, allowing us to glue.

**1.1.5.** In algebraic geometry, there is no tubular neighborhood theorem, so we cannot glue along punctured tubular neighborhoods. There is a substitute, though, if we are willing to work with *stable homotopy types*: the stable homotopy theory of schemes has a *punctured* tubular neighborhood theorem, and we can use this to glue in the stable motivic homotopy category.

Working with motivic homotopy types gives access to the rich algebraic and holomorphic data common in algebraic geometry: Galois representations, periods, motives, Hodge structure, etc.

**1.1.6.** The key in the construction of this algebraic punctured tubular neighborhood theorem is remembering the choice of coordinate along the special fiber. This vanishes on restriction to the special fiber, so to preserve it will require equipping the special fiber with additional structure.

The precise structure required is a special case of a more general object called a virtual log scheme; this thesis is a part of a broader work, Conjecture 1.2.6, to show that all (nice) virtual log schemes have stable motivic homotopy types.

**1.1.7.** In the thesis, we prove several weakened versions of the conjecture.

The first result is that, in the case of a (nice) degeneration of smooth projective algebraic varieties, the cohomology degenerates "algebraically;" see 1.2.14 for a concrete application to limits of integrals on Calabi-Yau varieties.

The second result is that virtual log schemes have Voevodsky $\mathbb{Q}$-motives, which are a notion of algebraic cohomology theories with rational coefficients.

## 1.2. For a Technical Audience

**1.2.1.** A log structure on a scheme $T$ is a (Zariski-)sheaf of (commutative) monoids $M_T$ with a map $\alpha : M_T \to (\mathcal{O}_T, \cdot)$ inducing an isomorphism on invertible elements $M_T^* \xrightarrow{\sim} \mathcal{O}_T^*$. A **log scheme** is a pair $(T, M_T)$. (See chapter II.)

We restrict to a nice subcategory of all log schemes (the **fine** log schemes), which in particular have the canonical abelianization map $M_T \to M_T^+$ an injection and $M_T^+/\mathcal{O}_T^*$ finitely generated constructible. The locus $T^* \subset T$ where $M_T = M_T^*$ is called the **trivial locus**, and is open.

Log structure can be pulled back along morphisms of schemes: if $f : T \to V$, set $f^*M_V = f^{-1}M_V \oplus_{f^{-1}\mathcal{O}_V^*} \mathcal{O}_T^*$.

**Example 1.2.2.** Let $X \to S$ be a proper family over the disk with local coordinate $t$, smooth away from $t = 0$ and with fiber over 0 a divisor with normal crossings.

If the map $f : X \to S$ is given by

$$\begin{array}{ccccc} X^* & \xhookrightarrow{\ j\ } & X & \xhookleftarrow{\ i\ } & X_0 \\ \big\downarrow{\scriptstyle \mathrm{sm}} & & \big\downarrow{\scriptstyle \mathrm{pr}} & & \big\downarrow \\ S^* & \hookrightarrow & S & \hookleftarrow & 0 \end{array}$$

with $S^* = S \setminus 0$, we choose the log structure on $X$ given by $j_* \mathcal{O}^*_{X^*} \cap \mathcal{O}_X$; it induces a log structure on $X_0$ given by $M_{X_0} = i^* M_X$. Observe that the function $t$ gives a global section of $M_{X_0}$, and that $\alpha(t) = 0$.

**1.2.3.** There is an extension of the analytification functor to log schemes:

$$(T, M_T) \mapsto (T, M_T)^{\text{log-an}} = \left\{ (x, x^\flat) : x \in X^{\text{an}}, x^* M_T/\mathbb{R}_+ \overset{x^\flat}{\underset{\twoheadrightarrow}{\dashleftarrow}} x^* M_T/\mathbb{C}^* \right\}$$

the resulting space is called the Kato-Nakayama space[KN99] of $(T, M_T)$ (or just "of $T$" where there is no ambiguity). It has a proper map to $T^{\text{an}}$, and the fibers are subgroups of compact tori.

In the setting of Example 1.2.2, $(X, M_X)^{\text{log-an}}$ is homotopy-equivalent to $(X^*)^{\text{an}}$, and $(X_0, M_{X_0})^{\text{log-an}}$ has the homotopy type of a punctured tubular neighborhood of $X_0$ in $X$. The log scheme $(X, M_X)$ is a special case of a **regular** log scheme; regular log schemes always have Kato-Nakayama space homotopy-equivalent to the analytification of their trivial locus.

**1.2.4.** The Kato-Nakayama construction depends only on the abelianization $M^+$ of $M$, so we introduce the notion of of a **virtual log scheme** (see chapter III):

A virtual log structure on $T$ is an extension of sheaves of abelian groups $\mathcal{O}^*_T \subset M^+_T \twoheadrightarrow \Lambda_T$. If $T$ is noetherian and $\Lambda_T$ is constructible with finitely generated fibers, then $(T, M^+_T)$ is called a **fine** virtual log scheme. We restrict ourself to these.

**1.2.5.** The Morel-Voevodsky category (see [MV99]) $\mathrm{Ho}(\mathrm{Sm}/T)$ of $\mathbb{A}^1$**-homotopy types** of schemes over $T$ is a homotopy category of simplicial sheaves on $\mathrm{Sm}/T$ where $\mathbb{A}^1$ plays the role of the interval; if $T$ is defined over $\mathbb{C}$, $\mathrm{Ho}(\mathrm{Sm}/T)$ has a topological realization functor to homotopy types over $T^{\text{an}}$. Much of the machinery of homotopy theory works in $\mathrm{Ho}(\mathrm{Sm}/T)$, including stabilization: there is a distinction between the suspension operators $\Sigma_s$, smash with the simplicial constant sheaf with fiber $S^1$, and $\Sigma_t$, smash with the zero-dimensional simplicial sheaf representing $\mathbb{G}_m$.

This thesis is part of a project to prove

**Conjecture 1.2.6.** *Let $(T, M_T^+)$ be a fine virtual log scheme. Then*

*(a) the topological homotopy type $\Sigma[T^{\text{log-an}}]_+ \in \mathrm{Ho}(\mathrm{Top}/T^{\mathrm{an}})_+$ (see [Jam95]) is the topological realization of a motivic homotopy type $[\Sigma_s T_+^{\log}] \in Ho(\mathrm{Sm}/T)_+$, and*

*(b) in the case that $(T, M_T^+)$ comes from a regular log scheme, the canonical map*

$$\Sigma_s[T^*]_+ \to [\Sigma_s T_+^{\log}]$$

*in $H(\mathrm{Sm}/T)_+$ is an $\mathbb{A}^1$-homotopy equivalence.*

**Example 1.2.7.** Consider the setting of Example 1.2.2, and suppose that $X_0 = D$ is smooth. Then $X_0^{\text{log-an}}$ is the sphere bundle of the normal bundle to $X_0$ in $X$, and is homotopy-equivalent to the analytification of the punctured normal bundle $N(D|X)^\circ$; thus $[X_0^{\log}] = [N(D|X)^\circ]$ is even a smooth scheme over $X_0$, stronger than the conjecture.

Even further, since $D$ is cut out by the global function $t$, $N(D|X)$ is canonically trivialized, so that $X_0^{\log} = X_0 \times \mathbb{G}_m$ and $X_0^{\text{log-an}} = X_0^{\mathrm{an}} \times S^1$.

**Remark 1.2.8.** We might hope to strengthen Conjecture 1.2.6 by omitting the point and the suspension; however consider Example 1.2.2 with $X$ a genus-1 curve with $X_0 = p$ a single point. If an unstable motivic homotopy type $[X^{\log}]$ existed, we would have a map

$$[\mathbb{G}_{m,p}] = [X_0^{\log}] \to [X^{\log}] = [X^*].$$

Standard results in $\mathbb{A}^1$-homotopy theory imply there are no such maps over $X$.

**1.2.9.** After pointing and suspension in the $\mathbb{A}^1$-homotopy category, we *do* have a map

$$\Sigma_s[N(D|X)^\circ]_+ \to \Sigma_s[X \setminus D]_+:$$

a fundamental result of the $\mathbb{A}^1$-homotopy theory is that

$$\mathrm{cone}([N(D|X)^\circ] \to [N(D|X)]) \simeq \mathrm{cone}([X \setminus D] \to [X]).$$

Coning over $D$ as well gives

$$\mathrm{cone}([N(D|X)^\circ] \sqcup [D] \to [N(D|X)]) \simeq \mathrm{cone}([X \setminus D] \sqcup [D] \to [X]).$$

We can identify the former with $\Sigma_s[N(D|X)^\circ]_+$, and by functoriality have a map

$$\Sigma_s[N(D|X)^\circ]_+ \simeq \mathrm{cone}([X \setminus D] \sqcup [D] \to [X]) \to \Sigma_s[X \setminus D]_+.$$

**Example 1.2.10.** If instead $X_0 = D_1 \cup D_2$ is the union of two smooth components with normal crossings $D_{12}$, then the punctured tubular neighborhoods of the two divisors must be "glued" along their restrictions to the double intersection. We only have the gluing maps after suspension, so we obtain for $[\Sigma_s(X_0^{\log})_+]$

$$\mathrm{hocolim}(\Sigma_s[N(D_{12}|D_1)^\circ]_+ \rightrightarrows \Sigma_s[N(D_1|X)^\circ]_+ \sqcup \Sigma_s[N(D_2|X)^\circ]_+)$$

**1.2.11.** If we are willing to work in the abelian setting and $T$ is a field, we have a satisfactory result: let $R$ be a coefficient ring with $\operatorname{char} T$ inverted, and let $\operatorname{Log}/T$ and $v\operatorname{Log}/T$ be the categories of fine log schemes over $T$ and fine virtual log schemes over $T$, respectively.

For $\mathcal{L} = \operatorname{Log}/T$ or $v\operatorname{Log}/T$, let $DA(\mathcal{L}; R)$ be the quotient of the derived category of presheaves of $R$-modules on $\mathcal{L}$ by the ideal generated by

- $\mathbb{A}^1$-homotopy
- cdh covers on underlying schemes which induce isomorphisms on the abelianization $M^+$
- the complex $[\mathbb{G}_m] \to [\mathbb{A}^1(\log 0)]$

**Theorem 1.2.12.** *The functor $DA(\mathrm{Sch}/T; R) \to DA(\mathcal{L}; R)$ is an equivalence, with quasi-inverse $(X, M_X) \mapsto S^{\dim X} M^+_{X^{\mathrm{sat}}} \otimes R$.*

**1.2.13.** One interpretation of the theorem is that any Weil cohomology theory uniquely extends from $\mathrm{Sch}/T$ to $\mathcal{L}$.

**1.2.14.** An application of these ideas is the compatibility of nearby cycles with Hodge realization.

Original constructions of limit motives and limit stable motivic homotopy types are due to Ayoub (see [Ayo08], [AIS]), as is the proof of compatibility with the $\ell$-adic and rigid-analytic realization, but results on compatibility with Hodge realization are not present in the literature.

**1.2.15.** For $H$ an anti-effective mixed Hodge structure (e.g. the cohomology of a complex variety), the **1-Hodge substructure** $H_{(1)}$ is the largest sub-Hodge structure containing $W_1 H$ and the Tate part of $\mathrm{Gr}_2^W H$. If $H$ is the cohomology of a curve, $H = H_{(1)}$.

**Theorem 1.2.16.** *Assume the situation of Example 1.2.2 over a field $k \subset \mathbb{C}$; then*

$$Y \mapsto R\Gamma_{\mathrm{cdh}}(Y \times X_0^{\mathrm{sat}}; \mathrm{pr}_2^* M^+) \otimes \mathbb{Q}$$

*is a 1-motive (see [Bar07]) over $\mathbb{G}_{m,k}$ with Hodge realization dual to the* 1*-Hodge substructure of the family of limit Hodge structures of the cohomology, $H(X_\infty)_{(1)} \otimes \mathbb{Q}$.*

**Remark 1.2.17.** This theorem can be interpreted as the Deligne conjecture on 1-motives (see 5.1.6, or [Del74]) for limit Hodge structures.

**1.2.18.** Since Voevodsky motives satisfy cdh descent, they do not provide information in infinitesimal families: if $i : S_0 \hookrightarrow S$ is a nilimmersion, then any motive takes $i$ to a quasi-isomorphism.

The construction in Theorem 1.2.16 admits an obvious weakening, where the cdh topology is replaced by the Zariski or Nisnevich topologies; this construction *does* provide infinitesimal information.

**Example 1.2.19.** Let $f : X \to S*$ be a maximal degeneration of Calabi-Yau varieties, of relative dimension $d$ over a punctured disk $S^*$. Maximality here means that the monodromy $T = 1 + N$ is unipotent of maximal index $n$ (so that $N^n \neq 0$ but $N^{n+1} = 0$).

There are canonical sections (up to scaling) $\delta_1$ and $\delta_2$ of $R^d f_*^{\mathrm{an}} \mathbb{Q}_X$: a generator of $\ker N^n$ and a preimage of the generator of $\ker N^{n-1} / \ker N^n$ (see chapter IV).

The Poincaré pairing over $\mathbb{C}$ of these two sections gives a multivalued function on $S*$, which descends to a function $q(s) = \exp(2\pi i \langle \delta_1, \delta_2 \rangle / \langle \delta_1, \delta_1 \rangle)$ called the **canonical coordinate**.

Expanding in $s$ to obtain a power series $q(s) = \sum a_i s^i$, we obtain complex numbers $a_i$. The constant term, $a_0$, is the period of the limit 1-Hodge structure $H^n(\lim X; \mathbb{Q})_{(1)} = \mathrm{Ext}^1_{MHS}(\mathbb{Q}(-1), \mathbb{Q}) = \mathbb{C}/2\pi i \mathbb{Q}$.

If the degeneration is defined over a field $k \subset \mathbb{C}$, theorem 1.2.16 implies that this class is in the image of the Hodge realization of $\mathrm{Ext}^1_{DM}(\mathbb{Q}(-1), \mathbb{Q}) \cong k^* \otimes \mathbb{Q}$, as predicted by [Vol07].

A future work will explain that $q(s)$ actually lives in $k((t)) \otimes \mathbb{Q}$.

## 1.3. Organization

This thesis is divided into two parts. The first part consists of several chapters of background material, much of which can be found in the literature: chapter II reviews the necessary preliminaries on log geometry, chapter III introduces the new notion of virtual log geometry, chapters VI and V provide background on Voevodsky motives and Deligne 1-motives, and chapter IV reviews the necessary background in Hodge theory.

The second part is divided into two chapters, the proof and applications of theorem 1.2.12, and the proof and applications of theorem 1.2.16.

# Background

CHAPTER II

# LOG GEOMETRY

**2.0.1.** Let $E \subset X$ be a divisor with normal crossings. The cohomology of $X \setminus E$ can be computed as the hypercohomology of a certain complex of vector bundles on $X$ known as the **logarithmic de Rham complex**,

$$\Omega^\bullet_X(\log E) = \{\alpha : \operatorname{ord}_E \alpha \geq -1\} \subset \Omega^\bullet_{\eta_X}$$

where $\Omega^\bullet_{\eta_X}$ is the complex of rational differential forms and $\operatorname{ord}_E(\alpha)$ is the order of pole of $\alpha$ along $E$.

Log geometry is a formalism where spaces can be "tagged" so as to remember that they have been obtained by compactification.

## 2.1. Monoids

**Definition 2.1.1.** A **(commutative) monoid** is a set $M$ with an associative (commutative) unital binary operation $\cdot : M \times M \to M$. A morphism of monoids is a unital morphism preserving the operation. The category of monoids is denoted Mon.

All monoids will be commutative.

**Example 2.1.2.** Any cone in an abelian group is a monoid. There is a left adjoint to the forgetful functor $\mathrm{Set} \leftarrow \mathrm{Mon}$ taking a set $S$ to $\mathbb{N}^S$.

**Proposition 2.1.3.** *There is an adjoint triple*

$$\mathrm{for} : \mathrm{Ab} \;\begin{matrix} \longleftarrow \\ \xrightarrow{\;\perp\;} \\ \xleftarrow[\;\perp\;]{} \end{matrix}\; \mathrm{Mon} : \mathrm{for}, \mathrm{un}$$

*respectively sending a monoid $M$ to the free abelian group* $\operatorname{ab} M = M^+$ *generated by it, forgetting the inverses in a group, and taking a monoid $M$ to its invertible elements* $\operatorname{un} M = M^*$.

*Proof.* Standard. ∎

**Definition 2.1.4.** A monoid $M$ is called **integral** if the unit map $M \to \operatorname{for}\operatorname{ab} M$ is an injection, **finitely generated** if it admits a surjection from $\mathbb{N}^n$ for some $n \in \mathbb{N}$, and **fine** if it is both integral and finitely generated.

**Example 2.1.5.** Consider the pointed monoid $\mathbb{F}_1 = \{0, 1\}$ with obvious multiplication law. Then $\mathbb{F}_1^+$ has 0 invertible, so $[0] = [0]^2$ implies $[1] = [0]$, and $\mathbb{F}_1^+$ is the trivial group: $\mathbb{F}_1$ is not integral.

**Proposition 2.1.6.** *The category of monoids admits finite colimits.*

*Proof.* Finite coproducts is evident. Coequalizers $f, g : M \rightrightarrows N$ can be obtained by taking the equivalence relation $n \sim n'$ iff $n = f(m)$ and $n' = g(m)$ for some $m \in M$, and extending the relation to respect the monoid operation: $n \sim' n'$ iff there exist $x, a, a' \in N$ such that $n = xa$, $n' = xa'$, and $a \sim a'$. The set $N/\sim'$ has a canonical monoid structure coequalizing $(f, g)$.

□

**Example 2.1.7.** Not all coequalizers can be written as quotients: the coequalizer $2, 3 : \mathbb{N} \rightrightarrows \mathbb{N}$, i.e., $\mathbb{N}/\langle 2 \sim 3\rangle$, has $[2] = [3]$, but since neither 2 nor 3 is invertible, we cannot write a submonoid of $\mathbb{N}$ to quotient by.

**Definition 2.1.8.** A monoid $P$ is **saturated** if it is integral and $P = P \cap (P^+ \otimes \mathbb{Q})$. It is **fs** if it is fine and saturated.

**Proposition 2.1.9.** *The forgetful functor* $\mathrm{Mon} \leftarrow \mathrm{Mon}^{\mathrm{sat}} : \mathrm{for}$ *has a left adjoint* $\mathrm{sat} : \mathrm{Mon} \to \mathrm{Mon}^{\mathrm{sat}}$.

*Proof.* Standard.

□

**Example 2.1.10.** $P = 2^{\mathbb{N}}3^{\mathbb{N}} \subset \mathbb{N}$ is not saturated; its saturation is $\mathbb{N}$.

### 2.2. Logarithmic Structure on Rings

**Proposition 2.2.1.** *Let $A$ be a (commutative unital) ring. The functor from $A$-algebras to monoids which forgets addition,* $\mathrm{Mon} \leftarrow A\text{-}\mathrm{Alg} : \mathrm{for}$, *admits a left adjoint* $A[-] : \mathrm{Mon} \to A\text{-}\mathrm{Alg}$ *("monoid algebra").*

*Proof.* Standard.

□

**Definition 2.2.2.** Let $A$ be a ring. A **prelog structure** on $A$ is a morphism of monoids $\alpha : P \to$ for $A$. It is a **log structure** if it induces an isomorphism on invertible elements: $\alpha|_{P^*} : P^* \xrightarrow{\sim} A^*$.

A log ring is a ring equipped with log structure $A = (\underline{A}, \alpha_A)$. A morphism $\phi$ of log rings is a map $\phi^\flat$ of monoids and a map $\underline{\phi}$ of rings such that $\alpha\phi^\flat = \underline{\phi}\alpha$.

**Proposition 2.2.3.** *The forgetful functor from log structures to prelog structures has a left adjoint sending $P \to A$ to $P \oplus_{P^*} A^* \to A$.*

*The forgetful functor from log rings to rings has a fully faithful left adjoint sending a ring $\underline{A}$ to $\underline{A}^* \subset \underline{A}$; we abusively write $\underline{A}$ for the image.*

*Proof.* Standard.

□

**Proposition 2.2.4.** *Let $A$ be a log ring. The forgetful functor $A\text{-}\mathrm{Alg} \to \underline{A}\text{-}\mathrm{Alg}$ from log rings over $A$ to rings over $\underline{A}$ admits a left adjoint, called the pullback log structure.*

*Proof.* Let $\phi : \underline{B} \to \underline{A}$; then the pullback log structure on $\underline{B}$ is $B = (\underline{B}, \phi^*\alpha_A)$ where $\phi^*\alpha_A$ is the map

$$\phi \circ \alpha_A : M_A \oplus_{A^*} B^* \to B.$$

□

### 2.3. Logarithmic Structure on Spaces

**Definition 2.3.1.** Let $(X, \mathcal{O}_X)$ be a locally ringed space. A prelog structure on $X$ is a morphism of sheaves of monoids $\alpha_X : M_X \to \mathcal{O}_X$; $\alpha_X$ is a log structure if over every open set it is a log structure.

If $f : X \to Y$ is a morphism of locally ringed spaces and $\alpha_Y : M_Y \to \mathcal{O}_Y$ is a log structure, We write $f^*\alpha_Y : f^*M_Y \to \mathcal{O}_X$ for the pullback log structure.

A log space is a pair $X = (\underline{X}, \alpha_X)$ of a locally ringed space equipped with log structure. A morphism of log spaces $f : X \to Y$ is a pair $\underline{f} : \underline{X} \to \underline{Y}$ and $f^\flat : M_X \leftarrow f^*M_Y$ compatible with $\alpha_X$ and $f^*(\alpha_Y)$.

**Example 2.3.2.** (i) Locally ringed spaces embed into log spaces by taking $M_X = \mathcal{O}_X^*$. The image is the **trivial** log spaces.

(ii) If $X$ is a smooth proper variety and $Z \subset X$ a smooth closed subvariety, then (setting $j : U \to X$ to be the inclusion of the complement) $X(\log Z) = (X, j_*\mathcal{O}_U^* \cap \mathcal{O}_X)$ is a log structure. This is the **compactification** log structure of $X$ along $Z$, or of $U$ in $X$.

(iii) If $X$ is a log space and $i : \underline{Z} \hookrightarrow \underline{X}$ is a sub-locally ringed space of $\underline{X}$, then the inverse image sheaf $i^{-1}M_X$ is a prelog structure on $\underline{Z}$. The associated log structure is the **restriction** log structure on $\underline{Z}$.

(iv) If $i : Z = V(t) \to X$ is cut out by $t \in \Gamma(\mathcal{O}_X)$, then $i^{-1}M_{X(\log Z)}$ is a prelog structure on $Z$, and $i^*M_{X(\log Z)}$ is non-canonically $\mathcal{O}_Z^* t^{\mathbb{N}}$, where $t$ is a function cutting out $Z$.

(v) Let $k$ be a field; the log space $\mathbf{pt}_k = \operatorname{Spec}(t^{\mathbb{N}} \to k)$ (or just $\mathbf{pt}$ if the field is understood) is the **log point**.

(vi) Any monoid $P$ has monoid algebra $\mathbb{Z}[P]$; $\mathbb{A}_P = \operatorname{Spec} \mathbb{Z}[P]$ has a canonical log structure $P \oplus_{P^*} \mathcal{O}_{\mathbb{A}_P}^*$, where $P$ and $P^*$ are written for their respective constant sheaves on $\mathbb{A}_P$. $\mathbb{A}_P$ is then a toric variety, with the locus of trivial log structure identified with the dense torus $\operatorname{Spec} \mathbb{Z}[P^*]$.

(vii) If $A$ is a ring and $P \to$ for $A$ is a morphism of monoids, then $\operatorname{Spec}(P \to A)$ is the log scheme whose underlying space is $\operatorname{Spec} A$ with log structure $P \oplus_{P^*} \mathcal{O}_{\operatorname{Spec} A}^*$.

**Definition 2.3.3.** Let $X$ be a log space. A **chart** for $X$ is a map $\phi : X \to \mathbb{A}_P$ such that $M_X = \phi^* P$. $X$ is said to be **quasi-coherent** if it locally admits a chart, and **coherent**, or **integral**, **fine**, **saturated**, or **fs** if it locally admits a chart to $\mathbb{A}_P$ with $P$ finitely generated, or integral, fine, saturated, or fs.

**Example 2.3.4.** $\mathbb{P}^1(\log 0, \infty)$ is a fine log scheme, with charts $P = t^{\mathbb{N}}$ and $Q = t^{-\mathbb{N}}$.

**Example 2.3.5.** Let $P = 2^{\mathbb{N}} 3^{\mathbb{N}} \subset \mathbb{N}$. (See example 2.1.10.) Then $\operatorname{Spec} k[P]$, the cuspidal cubic, is not saturated. The saturation is $P^{\mathrm{sat}} = \mathbb{N}$, and the unit of the adjunction induces a map $\operatorname{Spec} k[\mathbb{N}] \to \operatorname{Spec} k[P]$: the normalization of the cusp.

**Proposition 2.3.6.** *If $X$ is coherent, then $M_X/\mathcal{O}_X^*$ is constructible.*

*Proof.* The property is local, in which case it follows from the fact that $M_X/\mathcal{O}_X^* \cong \phi^{-1}(P/P^*)$, where $\phi : X \to \mathbb{A}_P$ is a chart.

□

**Definition 2.3.7.** A morphism $f : T \to T'$ of integral log schemes is a **thickening** if $\underline{f}$ is an infinitesimal thickening on underlying schemes. A morphism $g : X \to Y$ of integral log schemes is **formally smooth** if it admits left lifting with respect to thickenings, and **smooth** if $X$ and $Y$ are coherent and $\underline{f}$ is locally of finite presentation.

**Example 2.3.8.** Let $P$ be a fs monoid; then $\mathbb{A}_P \to \mathrm{Spec}(\mathbb{N} \to \mathbb{Z})$ and $\mathbb{A}_P \to \mathrm{Spec}\,\mathbb{Z}$ are smooth.

**Definition 2.3.9.** Let $T$ be a log scheme. A **one-parameter degeneration** over $T$ is a smooth morphism of log schemes $X \to T \times \mathbf{pt}$.

**Remark 2.3.10.** Usually $T$ is a field.

**Example 2.3.11.** Let $X \to S$ be a semistable degeneration, with $X_s \to s$ the special fiber and $X_\sigma \to \sigma$ the generic fiber. Then $X \times_S S(\log s) \to S(\log s)$ is a smooth morphism of log schemes, and so $X_s \times_s \mathbf{pt}_s \to \mathbf{pt}_s$ is a one-parameter degeneration.

**Remark 2.3.12.** Compare with the notion of **logarithmic deformation**, where the log structure on $X_s$ is locally given by compactification log structure, in [Ste95].

### 2.4. Cohomology of Log Spaces

There are many cohomology theories defined on log schemes in the literature: singular, de Rham, étale, and crystalline cohomologies all have extensions to log schemes. See Theorem 1.2.12 for a proof that all homotopy-invariant Weil cohomology theories extend uniquely to log schemes.

### 2.41. Kato-Nakayama Space and Singular Cohomology

**Definition 2.4.1.** The **log analytic point** is the log analytic space $\mathrm{pt}^{\text{log-an}} = \mathrm{Spec}(\mathbb{R}_{\geq 0} \times S^1 \to \mathbb{C})$ where the map is given by multipication.

Let $X$ be a log scheme. The **Kato-Nakayama space**, or **log analytification**, of $X$, is the set $X(\mathrm{pt}^{\text{log-an}})$ equipped with the topology generated by the projection map $X(\mathrm{pt}^{\text{log-an}}) \to \underline{X}^{\mathrm{an}}$ and for every section $m \in \Gamma(U, M_X)$ the map $\mathrm{pr}_{S^1} \circ \mathrm{ev}_m : X(\mathrm{pt}^{\text{log-an}}) \to S^1$ (where $S^1$ is equipped with the Euclidean topology).

**Proposition 2.4.2.** *Let $X$ be a fine log scheme. Then $X^{\text{log-an}}$ is a topological space over $\underline{X}^{\mathrm{an}}$, with fiber over $x \in \underline{X}^{\mathrm{an}}$ given by* $\mathrm{Hom}(x^* M^+/\mathbb{C}^*, S^1)$.

*Proof.* Immediate.

□

**Example 2.4.3.** $\mathbb{A}_{\mathbb{N}}^{\text{log-an}} \to (\mathbb{A}^1)^{\text{an}}$ is the "oriented real blowup" of the complex plane at the origin: the fiber over 0 is $S^1$, and other fibers are a single point. In particular, $\operatorname{Spec}(\mathbb{N} \to \mathbb{C})^{\text{log-an}} = S^1$.

**Theorem 2.4.4** ([NO10])**.** *Let $f : X \to S$ be a smooth proper morphism of fine log schemes. Then $f^{\text{log-an}}$ is a fiber bundle.*

*Proof.* See [NO10], theorem 5.1.

□

**Remark 2.4.5.** Note that the underlying morphism $\underline{X} \to \underline{S}$ need *not* be smooth.

**Definition 2.4.6.** Let $X$ be a log space over $\mathbb{C}$ and $\Lambda$ a coefficient ring. The **singular cohomology** of $X$ is the singular cohomology of $X^{\text{log-an}}$, $H^{\cdot}_{\text{sing}}(X;\Lambda) = H^{\cdot}_{\text{sing}}(X^{\text{log-an}};\Lambda)$.

### 2.42. de Rham Cohomology

**Definition 2.4.7.** Let $\phi : B \to A$ be a morphism of log rings, and let $E$ be a $\underline{A}$-module. A **derivation** of $\phi$ with values in $E$ is a pair of maps $\delta : M_A \to E$ and $d : \underline{A} \to E$ such that $d$ is a $\underline{B}$-derivation over $\underline{A}$ with values in $E$, $\delta$ is a homomorphism of monoids $M_A \to (E, +)$ with $\delta(\phi^\flat M_B) = 0$, and $d(\alpha(p)) = \alpha(p)\delta(p)$.

**Example 2.4.8.** Let $P \to A = t^{\mathbb{N}} \times k^* \subset k[t]$ and $Q \to B = k^* \subset k$. Then $d : k[t] \to k[t] \cdot dt/t$ and $d\log : t^{\mathbb{N}} \times k^* \to k[t] \cdot dt/t$, defined by $d(t) = t \cdot dt/t$, $d\log(t) = dt/t$, is a derivation.

**Proposition 2.4.9.** *Let $\phi : B \to A$ be a morphism of log rings. There is a universal $B$-derivation $(d, d\log) : A \to \Omega^1_\phi$, with presentation*

$$\Omega^1_{\underline{\phi}} \oplus A\{d\log p : p \in (P/\phi^\flat(Q))^+\}/K$$

*where $K$ is generated by expressions of the form*

$$\alpha(p)d\log(p) - d\alpha(p) \text{ and } d\log(pp') - d\log(p) - d\log(p').$$

*Proof.* Similar to the classical result.

□

**Definition 2.4.10.** Let $f : X \to Y$ be a morphism of log schemes. The Kähler differentials $\Omega^1_{X/Y}$ of $X$ over $Y$ is the sheaf on $X$ associated to $U \mapsto \Omega^1_{(f_U^*, f_U^\flat)}$.

**Proposition 2.4.11.** *Let $f : X \to Y$ be a morphism of fine log schemes with underlying morphism of schemes finitely presented. Then $\Omega^1_f$ is coherent.*

*Proof.* Similar to the classical result.

□

**Definition 2.4.12.** Let $f : X \to Y$ be a morphism of log schemes. The **de Rham complex** of $f$ is the complex $\Omega^\cdot_f = (\bigwedge^\cdot \Omega^1_f, d)$, and the **de Rham cohomology** of $f$ is $\mathbb{R}f_*\Omega^\cdot_f$.

**Example 2.4.13.** Let $X = \operatorname{Spec}(t^{\mathbb{N}} \to \mathbb{C}[t^{\pm}])$ over $\mathbb{C}$; then $X^* = \mathbb{G}_m$ and $\Omega^\cdot_X$ is the log de Rham complex for $\mathbb{G}_m \subset \mathbb{A}^1$, so $H^\cdot_{dR}(X) = H^\cdot_{dR}(X^*)$.

**Theorem 2.4.14.** *Let $X/\mathbb{C}$ be an fs log scheme admitting an atlas of the form $\underline{X} \to V(\Sigma) \subset \mathbb{A}_P$, where $\Sigma \subset P$ is an ideal, such that $\underline{X} \to V(\Sigma)$ is smooth.*

*Then there is a canonical isomorphism $H^\cdot_{dR}(X) \cong H^\cdot_{\text{sing}}(X; \mathbb{C})$.*

*Proof.* See [KN99].

□

## 2.5. Blow-ups of Log Schemes

**Definition 2.5.1.** Let $P$ be a monoid. An **ideal** of $P$ is a subset $\Sigma$ closed under multiplication by $P$, i.e., $P \cdot \Sigma = \Sigma$.

**Definition 2.5.2.** Let $X$ be a quasicoherent log scheme. A **quasicoherent sheaf of ideals** on $X$ is a sheaf of ideals $\mathcal{I} \subset M$ such that locally there exists a chart $\pi : U \to \mathbb{A}_P$ such that $\mathcal{I}_U \cong \pi^\flat J$ for some ideal $J \subset P$. An **idealized** quasicoherent log scheme is such a pair $(X, \mathcal{I})$.

**Definition 2.5.3.** Let $(X, \mathcal{I})$ be an idealized fs log scheme. The **blowup** $\operatorname{Bl}_{\mathcal{I}} X$ of $X$ along $\mathcal{I}$ is the log scheme over $X$ given on a chart $X \to \mathbb{A}_P$ by pulling back the saturation of the classical blow-up $\operatorname{Bl}_{k[P]I} \operatorname{Spec}(k[P])$ equipped with log structure given locally by charts $U_a = \mathbb{A}_{P[I/a]}$.

**Example 2.5.4.** Let $\mathcal{I} = (x)$ on $\mathbb{A}^1(\log 0)$; then $\operatorname{Bl}_{\mathcal{I}} X = \mathbb{A}^1(\log 0)$. Let $\mathcal{I} = (x, y)$ on $\operatorname{Spec}(x^{\mathbb{N}} y^{\mathbb{N}} \to k)$; then $\operatorname{Bl}_{\mathcal{I}} X = \mathbb{P}^1(\log\{0, \infty\})$.

**Lemma 2.5.5.** *Suppose $t : X \to S$ is a log-smooth 1-parameter degeneration. After log blowups at ideals supported along the special fiber, the degeneration is semistable.*

*Proof.* This is due to [Niz06], theorem 5.10. □

CHAPTER III

# VIRTUAL LOG GEOMETRY

We introduce the notion of virtual log structure and cohomology of virtual log spaces, and compare these notions with log geometry.

## 3.1. Virtual Log Structure on Schemes

**Definition 3.1.1.** Let $\underline{X}$ be a locally ringed space. A **virtual log structure** on $\underline{X}$ is an extension $L$ of an abelian sheaf $\Lambda$ (called the **characteristic** of $L$) by $\mathcal{O}^*_{\underline{X}}$. A **virtual log space** is a pair $X = (\underline{X}, L_X)$ of a space and virtual log structure.

Virtual log structure can be pulled back along a morphism of spaces; if $Y$ is a virtual log space and $f : \underline{X} \to \underline{Y}$ is a map of locally ringed spaces the pullback virtual log structure $f^*L_Y$ on $\underline{X}$ is given by

$$\begin{array}{ccccc} f^{-1}\mathcal{O}^*_{\underline{Y}} & \hookrightarrow & f^{-1}L_Y & \twoheadrightarrow & f^{-1}\Lambda_Y \\ \downarrow{\scriptstyle f^*} & \ulcorner & \downarrow & & \| \\ \mathcal{O}^*_{\underline{X}} & \hookrightarrow & f^*L_Y & \twoheadrightarrow & f^{-1}\Lambda_Y \end{array}$$

A morphism $X \to Y$ of virtual log spaces is a pair $f = (\underline{f}, f^\flat)$ with $\underline{f} : \underline{X} \to \underline{Y}$ and $f^\flat : L_X \leftarrow \underline{f}^* L_Y$ extending $\underline{f}^* : \mathcal{O}^*_X \leftarrow \underline{f}^{-1}\mathcal{O}^*_Y$.

**Proposition 3.1.2.** *The forgetful functor* for : vLog $\to$ Sch *admits a fully faithful left adjoint by taking* $\underline{X}$ *to* $(\underline{X}, \mathcal{O}^*_X)$*. We abusively write* $\underline{X}$ *for the image.*

*Proof.* Standard.

□

**Proposition 3.1.3.** *The category of virtual log spaces admits finite fiber products: if* $X, Y \to Z$ *then* $X \times_Z Y = (\underline{X} \times_{\underline{Z}} \underline{Y}, \mathrm{pr}_1^* L_X \oplus_{\mathrm{pr}^* L_Z} \mathrm{pr}_2^* L_Y)$.

*Proof.* Standard.

□

**3.1.4.** The embedding Sch $\to$ vLog extends to a "virtualization" functor $(-)^+ :$ Log $\to v\,$Log taking a log structure $M$ on $\underline{X}$ to its group completion $L = M^+$; this is *not* an embedding, even when restricted to $\mathrm{Log}^{fs}$.

**Definition 3.1.5.** A virtual log space $X$ is said to be

– **quasi-coherent** if $\Lambda$ is quasi-constructible (i.e., $\underline{X}$ admits a stratification such that $\Lambda$ is constant on the open strata; see [Ogu16, def. 2.5.1]);

– **coherent** if furthermore $\Lambda$ has finitely generated stalks;

– **saturated** (respectively, **finite**) if furthermore $\Lambda$ is torsion-free (respectively, torsion).

**Example 3.1.6.** If $X$ is a quasi-coherent (respectively, coherent, saturated, finite) log scheme then its virtualization $X^+$ is quasi-coherent (coherent, saturated, finite).

**Proposition 3.1.7.** *Let $X$ be a coherent virtual log scheme with $\underline{X}$ noetherian. Then $\Lambda_X$ is noetherian.*

*Proof.* If $\dim X = 0$, then topologically $\underline{X}$ is a finite number of points (as $\underline{X}$ is noetherian), and since $X$ is coherent $\Lambda_X = \bigoplus_{x\in\underline{X}} \Lambda_{X,x}$ is a finite sum of finitely generated abelian groups at each point, thus noetherian.

If $\Lambda_Y$ is noetherian for every coherent noetherian $Y$ of dimension at most $d$ and $\dim X = d+1$, then let $j : X^{(0)} \hookrightarrow X$ be the inclusion of the points of $X$ of codimension-0, and observe that $j^{-1}\Lambda_X$ is noetherian (by the same argument as in the dimension-0 case).

Let $\underline{Y} = \operatorname{supp}(\ker \Lambda_X \to j_*j^{-1}\Lambda_X)$ (with inclusion $\underline{i} : \underline{Y} \hookrightarrow \underline{X}$) and $Y = (\underline{Y}, i^* \ker(L_X \to j_*j^{-1}\Lambda_X))$. Then $Y$ is coherent (since $\underline{Y} \subset \underline{X}$ is closed, and $\Lambda_Y$ is a subsheaf of $i^{-1}\Lambda_X$) and has dimension at most $d$, so that $i_*\Lambda_Y = \ker(\Lambda_X \to j_*j^{-1}\Lambda_X)$ is noetherian. Since $\Lambda_X$ is an extension of $j_*j^{-1}\Lambda_X$ by $\Lambda_Y$, it also is noetherian.

□

### 3.2. Splittings of Virtual Log Structure

One of the key differences between log schemes and virtual log schemes is the possibility of splittings:

**Definition 3.2.1.** A **splitting** of the virtual log structure on a virtual log space $X$ is a splitting of the extension, equivalent to a section of the canonical morphism $X \to \underline{X}$. A **splitting family** for a virtual log space $X$ is a space $\underline{T}$ with a map of virtual log spaces $\underline{T} \to X$; the pullback virtual log structure along such a map is canonically split.

The category of splitting families of $X$, with obvious morphisms, is denoted $\mathrm{Split}(X)$. We say that $X$ is **splittable** if there is a section of the canonical map $X \to \underline{X}$, and admits a **splitting space** $X^{\mathrm{sp}}$ if there is a final object $X^{\mathrm{sp}} \in \mathrm{Split}(X)$

**Example 3.2.2.** Let $E$ be an elliptic curve with compactification log structure at a point $p$. Then any splitting space is birational with $\underline{E}$, but over $\underline{p}$ the fiber is $\mathbb{G}_m$, a contradiction.

**Definition 3.2.3.** An affine scheme $T$ is **seminormal** if every map to the cusp $C$ lifts along the normalization $\mathbb{A}^1 \to C$:

$$\begin{array}{ccc} & & \mathbb{A}^1 \\ & \nearrow & \downarrow \\ T & \longrightarrow & C \end{array}$$

A scheme is seminormal if its local schemes are seminormal, i.e., every $\mathrm{Spec}\,\mathcal{O}_{T,t} \subset T$ is seminormal. See [LV81] for more on semi-normal varieties.

**Theorem 3.2.4.** *Let $X$ be a finite virtual log scheme. Then the functor $v\,\mathrm{Log}(-, X) : \mathrm{Sch}^{sn} \to \mathrm{Set}$ from seminormal schemes to sets is representable. (We say "$X$ has a semi-normal splitting space.")*

We first prove the theorem in two special cases.

**Lemma 3.2.5.** *Let $X$ be an finite virtual log scheme with $\Lambda$ constant cyclic on an open subset $\underline{U} \subset \underline{X}$ and vanishing on $\underline{X} \setminus \underline{U}$.*

*Then $v\,\mathrm{Log}(-, X) : \mathrm{Sch}^{sn} \to \mathrm{Set}$ is representable.*

For the proof, we need the following machinery of Kollár:

**Definition 3.2.6** ([Kól11], definition 2)**.** Let $X$ and $R$ be reduced $S$-schemes. A morphism $\sigma : R \to X \times_S X$ is a **set-theoretic equivalence relation** on $X$ if every geometric point $\overline{s} : \mathrm{Spec}\,\overline{k} \to S$ gives an equivalence relation on $\overline{s}$-points. It is **finite** if the compositions with the projections $\sigma_1$ and $\sigma_2$ are finite.

**Definition 3.2.7** ([Kól11], definition 4)**.** Given two finite morphisms $\sigma_1, \sigma_2 : R \rightrightarrows X$, a morphism $X \to Y$ is the **geometric quotient** of $X$ by $R$ if is the coequalizer of $\sigma_1, \sigma_2$, is finite, and the geometric fibers are the set-theoretic equivalence classes.

**Lemma 3.2.8** ([Kól11], lemma 17)**.** *Let $S$ be a Noetherian scheme. Assume that $X$ is finite over $S$, and let $p_1, p_2 : R \rightrightarrows X$ be a finite, set-theoretic equivalence relation over $S$. Then the geometric quotient $X/R$ exists.*

*Proof.* See [Kól11], lemma 17.

□

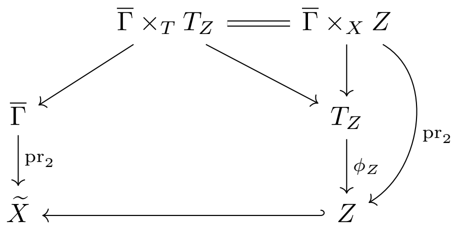
*Proof (of lemma 3.2.5).* Let $\lambda$ be the generator of $\Gamma(U, \Lambda)$, or order $n$, and let $v \in \Gamma(U, L)$ be a preimage. Let $V$ be the pullback

$$\begin{array}{ccc} V & \xrightarrow{t} & \mathbb{G}_m \\ \downarrow & \lrcorner & \downarrow{\scriptstyle n:1} \\ \underline{U} & \xrightarrow{m^n} & \mathbb{G}_m \end{array}$$

and set $\overline{V} = \mathrm{Nm}_V \, \underline{X}$. Let $Z = X \setminus U$ and $R = \Delta_X(\overline{V}) \cup (\overline{V} \times_X \overline{V}) \times_X Z \subset \overline{V} \times_X \overline{V}$, a finite set-theoretic equivalence relation on $\overline{V}$ (in the sense of [Kól11, def. 2]).

Applying [Kól11, lem. 17], the quotient $\widetilde{X} = \overline{V}/R$ exists. We will show that $\widetilde{X}^{sn}$ is the seminormal splitting scheme of $X$.

Let $\phi : T \to X$ be a seminormal splitting family; in particular, $\phi_U : T_U \to U$ factors as $T_U \xrightarrow{\phi_V} V \to U$. Set $\Gamma = \mathrm{graph}(\phi_V)$, closed in $T_U \times_X V$, and let $\overline{\Gamma}$ be its closure in $T \times_X \widetilde{X}$. Then we have a commutative diagram




Since $\overline{\Gamma} \sqcup T_Z \to T$ is an $h$-cover (indeed, $\overline{\Gamma}_Z \to \overline{\Gamma} \to T$ and $\overline{\Gamma}_Z \to T_Z \to T$ are agree), this gives a section of $\mathrm{Sh}^h(LT, L\widetilde{X})$; but $T$ is seminormal, so we obtain (see [Voe93, prop. 3.2.10]) a canonical map $\widetilde{\phi} : T \to \widetilde{X}$, factoring $\phi$. The factoring is unique, as any section of $\mathrm{Sh}^h(LT, L\widetilde{X})$ is determined by its restrictions to $\overline{\Gamma} \sqcup T_Z$, in turn determined by the restriction to $\Gamma_U \sqcup T_Z = \mathrm{graph}(\phi_V) \sqcup \phi_Z$. Again, $T$ is seminormal, so $\widetilde{\phi}$ factors uniquely through $\widetilde{X}^{sn}$.

□

**Lemma 3.2.9.** *Let $X$ be a virtual log scheme with characteristic sheaf $\Lambda_X$ supported on $Z \subset X$ and generated by a nontrivial section $\lambda \in \Gamma(V, \Lambda)$ of order $n$.*

*Then $v\,\mathrm{Log}(-, X) : \mathrm{Sch}^{sn} \to \mathrm{Sets}$ is representable.*

*Proof.* Let $U = \underline{X} \setminus \underline{Z}$ and let $m \in \Gamma(V, L)$ be a preimage of $\lambda$. Set $W$ as the pullback

$$\begin{array}{ccc} W & \xrightarrow{t} & \mathbb{G}_m \\ {\scriptstyle \mathrm{pr}_1}\downarrow & \lrcorner & \downarrow{\scriptstyle n:1} \\ V & \xrightarrow{m^n} & \mathbb{G}_m \end{array}$$

Let $D\left(\mathrm{pr}_1^\flat m|_{W_U} - t\big|_{W_U}\right)$ be the locus where $\mathrm{pr}_1^\flat m$ is a function and disagrees with $t$, and set

$$\widetilde{V} = W \setminus D\left(\mathrm{pr}_1^\flat m|_{W_U} - t\big|_{W_U}\right)$$

Write $\overline{V} = \mathrm{Nm}_{\widetilde{V}} X$ and $R = \Delta_X(\overline{V}) \cup \left(\overline{V} \times_X \overline{V} \times_X (X \setminus V)\right) \subset \overline{V} \times_X \overline{V}$. This is a finite set-theoretic equivalence relation on $\overline{V}$, so that the quotient $\widetilde{X} = \overline{V}/R$ exists (again, by [Kól11, lem. 17]).

We reduce to the case $V = X, \widetilde{V} = \widetilde{X}$ by applying Lemma 3.2.5, observing that $\widetilde{V} \to V$ is finite.

The virtual log structure on $W$ is canonically split by $\mathrm{sp} : \underline{W} \to W$ sending $\mathrm{pr}_1^\flat m$ to $t$. Let $s = \phi^\flat(m) \in \Gamma(T, \mathcal{O}_T^*)$; then $f = (\phi, s)$ gives a unique map $T \to W$. Since $f^*(\mathrm{pr}_1^\flat(m)) = \phi^\flat(m) = s = f^*(t)$, everywhere on $T$, we see that $f$ factors through $\widetilde{V}$. Since $T$ is seminormal, there is even a unique factorization through $\widetilde{V}^{sn}$.

□

**3.2.10.** *Proof (of Theorem 3.2.4).* Let $\lambda \in \Gamma(U, \Lambda_X)$ be a non-trivial section, and $j : U \hookrightarrow X$ the inclusion; the virtual log scheme $X_\lambda = (\underline{X}, \ker(L_X \to \Lambda_X / j_! \lambda^{\mathbb{Z}}))$ meets the hypotheses of Lemma 3.2.9, so that there is a seminormal splitting scheme $\widetilde{X}_\lambda^{sn} \to X_\lambda$. We then have the following diagram:

$$\begin{array}{ccc} X^{\mathrm{sp}(\lambda)} & \longrightarrow & \underline{\widetilde{X}}_\lambda^{sn} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ X & \longrightarrow & X_\lambda \end{array}$$

The virtual log structure on $X^{\mathrm{sp}(\lambda)}$ is

$$L_{X^{\mathrm{sp}(\lambda)}} = \underline{\pi}^{-1} L_X \oplus_{\underline{\pi}^{-1} L_{X_\lambda}} \mathcal{O}^*_{\underline{\widetilde{X}}_\lambda};$$

in particular, any lift of $\lambda$ to $L_X$ pulls back to an invertible function on $X^{\mathrm{sp}(\lambda)}$.

Any seminormal splitting family of $X$ by composition gives a seminormal splitting family of $X_\lambda$, and thus factors through $\underline{\widetilde{X}_\lambda^{sn}}$ to give a factoring through $X^{\mathrm{sp}(\lambda)}$.

Replacing $X$ by $X^{\mathrm{sp}(\lambda)}$ we obtain a new virtual log scheme, with $\Lambda_X$ of strictly shorter length. Since $\Lambda_X$ is noetherian, after finitely many steps $\Lambda_X = 0$, i.e. we obtain a scheme $\underline{\widetilde{X}}$. Let $X^{\mathrm{sp}}$ be the seminormalization of this scheme equipped with the pullback virtual log structure from $X$. Then any seminormal splitting family of $X$ factors through $\widetilde{X}$ and thus $X^{\mathrm{sp}}$.

□

**Definition 3.2.11.** Let $X$ be a noetherian virtual log space.

– The **finite part** of $X$ is the finite virtual log scheme $X^{\mathrm{fin}} = (\underline{X}, \ker(L_X \to \Lambda_X \otimes \mathbb{Q}))$. There is a canonical map $X \to X^{\mathrm{fin}}$ corresponding to the inclusion $L_X \supset L_{X^{\mathrm{fin}}}$.

– Assume $X^{\mathrm{fin}}$ admits a splitting space. The **partial saturation** of $X$ is $X^{p\,\mathrm{sat}} = \big(\underline{X^{\mathrm{fin,sp}}}, \pi^* L_X\big)$. Here $\pi : X^{\mathrm{fin,sp}} \to X^{\mathrm{fin}}$ is the projection. There is a canonical map $X^{p\,\mathrm{sat}} \to X^{\mathrm{fin,sp}}$ corresponding to the pullback of the inclusion $L_X \supset L_{X^{\mathrm{fin}}}$.

– Assume $X^{\mathrm{fin}}$ admits a splitting space. The **saturation** of $X$ is the pullback

$$\begin{array}{ccc} X^{\mathrm{sat}} & \longrightarrow & \underline{X^{\mathrm{fin,sp}}} \\ \big\downarrow & \lrcorner & \big\downarrow \\ X^{p\,\mathrm{sat}} & \longrightarrow & X^{\mathrm{fin,sp}} \end{array}$$

**3.2.12.** $X^{p\,\mathrm{sat}}$ abstractly admits an action of $\mathcal{H}\mathrm{om}(\Lambda^{\mathrm{tor}}, \mathcal{O}^*)$, which is a finitely generated quotient of $\mu_\infty$. The action does not fix the splitting, however: maps $X^{p\,\mathrm{sat}} \to X$ form a $\mathcal{H}\mathrm{om}(\Lambda^{\mathrm{tor}}, \mathcal{O}^*)$-torsor, and unless it is trivialized, the action does not descend to $X^{\mathrm{sat}}$.

## 3.3. Cohomology of Virtual Log Schemes

### 3.301. Kato-Nakayama Space and Singular Cohomology

**Definition 3.3.1.** Let $X$ be a virtual log space over $\mathbb{C}$. The **Kato-Nakayama space** $X^{\text{log-an}}$ of $X$ is, as a set,

$$\left\{ (x, \sigma) : x \in \underline{X}(\mathbb{C}), x^* L_X / \mathbb{R}_+ \underset{\arg \circ x^\flat}{\overset{\sigma}{\rightleftarrows}} x^* L_X / \mathbb{C}^* \right\}$$

with topology generated by the projection $X^{\text{log-an}} \to \underline{X}^{\text{an}}$ and for $m \in \Gamma(U, L_X)$ evaluation maps $\mathrm{ev}_m : U^{\text{log-an}} \to S^1$, $\mathrm{ev}_m(x, \sigma) = \sigma(x^* m)$.

**Proposition 3.3.2.** *Let $X$ be a virtual log space. Then $X^{\text{log-an}} = (X^+)^{\text{log-an}}$, i.e. they are canonically homeomorphic.*

*Proof.* We may assume $\underline{X} = \operatorname{Spec} \mathbb{C}$.

Splittings of $M^+/\mathbb{R}_+ \to \Lambda$ are the same as splittings $\mathbb{C}^*/\mathbb{R}_+ \subset M^+/\mathbb{R}_+$, aka maps $S^1 \to M^+/\mathbb{R}_+$.

Maps $x^\flat : M \leftarrow \mathbb{R}_{\geq 0} \times S^1$ over $x^* = \mathrm{id} : \mathbb{C}^* \leftarrow \mathbb{C}^*$ are by divisibility the same as maps $M/\mathbb{C}^* \leftarrow 0 \times S^1$; by adjunction this is the same as maps $M^+/\mathbb{C}^* \leftarrow S^1$.

The topology on $(X^+)^{\text{log-an}}$ is at least as fine as on $X^{\text{log-an}}$; given $m - n \in \Gamma(U, M^+)$ with $m, n \in \Gamma(U, M)$ we have $\mathrm{ev}_{m-n}(x, \sigma)\, \mathrm{ev}_m(x, \sigma)/\, \mathrm{ev}_n(x, \sigma)$; since division is continuous on $S^1$, this is continuous in the topology generated by $\mathrm{ev}_m$ and $\mathrm{ev}_n$.

□

**Definition 3.3.3.** If $X$ is a virtual log analytic space defined over $\mathbb{C}$, then the **virtual log Betti cohomology** of $X$ is the singular cohomology of the space $X^{\text{log-an}}$.

### 3.302. de Rham Cohomology

**Definition 3.3.4.** Let $X$ be a virtual log space with $\underline{X}$ integral. The sheaf of **Kähler forms** on $X$, denoted $\Omega^1_X$, is given by the quotient of $\overline{\Omega}^1_X$ by sections supported in positive codimension, where $\overline{\Omega}^1_X$ is given by

$$\begin{array}{ccccc} \mathcal{O}^* \otimes \mathcal{O} & \hookrightarrow & L \otimes \mathcal{O} & \twoheadrightarrow & \Lambda \otimes \mathcal{O} \\ \downarrow{\scriptstyle d\log \otimes \mathcal{O}} & \ulcorner & \downarrow{\scriptstyle \delta \otimes \mathcal{O}} & & \| \\ \Omega^1_{\underline{X}} & \hookrightarrow & \overline{\Omega}^1_X & \twoheadrightarrow & \Lambda \otimes \mathcal{O} \end{array}$$

The **de Rham complex** $\Omega^\cdot_X$ of $X$ is $\bigwedge^\cdot \Omega^1_X$

The quotient $\overline{\Omega}^1 \to \Omega^1$ compensates for the lack of a map $\alpha$ found in log geometry.

**Example 3.3.5.** Let $X = \mathbb{A}^1(\log 0)^+$; then $\overline{\Omega}^1_X = \Omega^1_{\underline{X}} \oplus_{\mathcal{O}^*} (\mathcal{O}^* t^{\mathbb{Z}} \otimes \mathcal{O})$. Let $i : 0 \to \underline{X}$ and $j : \mathbb{G}_m \to \underline{X}$ be the obvious inclusions. We have a global section $\delta(t)$ such that $j^*\delta(t) = d\log(t)$; in particular, $tj^*\delta(t) = dt$, so that $j^*(t\delta(t) - dt) = 0$. Since $t$ is not invertible on all of $\underline{X}$, however, there are no global relations between $\delta(t)$ and $dt$: $t\delta(t) - dt$ is supported at the origin.

The image in $\Omega^1_X$, however, does satisfy the global relation $t\delta(t) = dt$.

**Proposition 3.3.6.** *Let $X$ be a quasi-coherent log scheme with $\underline{X}$ integral and locally noetherian. Then $\Omega^1_X \cong \Omega^1_{X^+}$.*

*Proof.* The canonical map $\overline{\Omega}^1_{X^+} \to \Omega^1_{X^+}$ factors through $\Omega^1_X$: if $m \in M^+$ then $\alpha(m)d\log(m) - d\alpha(m) = 0$ in $\Omega^1_X$, while $\alpha(m)\delta(m) - d\alpha(m)$ in $\overline{\Omega}^1_{X^+}$ is supported on $V(\alpha(m))$, i.e. in positive codimension.

For the same reason, the map $\overline{\Omega}^1_{X^+} \to \Omega^1_X$ descends to $\Omega^1_{X^+}$.

□

**Proposition 3.3.7.** *Let $X$ be an fs log scheme admitting an atlas of the form $X \to V(\Sigma) \subset \mathbb{A}_P$ where $\Sigma \subset P$ is an ideal, and $\underline{X} \to V(\Sigma)$ is smooth. Then $H^{\cdot}_{dR}(X^+) \cong H^{\cdot}_{\mathrm{sing}}(X^+)$.*

*Proof.* Immediate from the results of Chapter II.

□

CHAPTER IV

HODGE THEORY

## 4.1. Pure, Mixed, and Polarized Hodge Structures

For the general theory of mixed Hodge structures, we refer the reader to [Del71] and [Del74]. For polarization and the period mapping, see [Gri68a], [Gri68b], and [Gri70].

**Definition 4.1.1.** Let $n$ be an integer. A **pure Hodge structure** of weight $n$ is a finitely generated abelian group $V = V_{\mathbb{Z}}$ and a decomposition

$$V_{\mathbb{C}} = V \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=n} V_{\mathbb{C}}^{pq}$$

such that $V_{\mathbb{C}}^{pq} = \overline{V}^{qp}$. (Here the conjugation is with respect to the image of $V_{\mathbb{Z}} \to V_{\mathbb{C}}$.) We write $h^{pq}$ for the **Hodge numbers** $\dim V_{\mathbb{C}}^{pq}$.

A morphism of such Hodge structures is a map $f_{\mathbb{Z}} : V_{\mathbb{Z}} \to W_{\mathbb{Z}}$ which preserves the decomposition. The category of pure Hodge structures of weight $n$ is denoted $HS_n$.

**Example 4.1.2.** Let $X$ be a smooth projective variety over $\mathbb{C}$. Then $H^n(X; \mathbb{Q})$ has a canonical pure Hodge structure of weight $n$, with $F^\cdot$ induced by the brutal filtration on $\Omega^\cdot_{X/\mathbb{C}}$.

**Remark 4.1.3.** A pure Hodge structure of weight $n$ gives a flag $F^\cdot$ on $V_{\mathbb{C}}$, with $F^p = \bigoplus_{p' \geq p} V^{p(n-p)}$, satisfying the condition $F^p \cap \overline{F}^{n-p+1} = 0$.

Given such a flag, we can recover the decomposition by taking $V_{\mathbb{C}}^{pq} = F^p \cap \overline{F}^{n-p}$.

**Definition 4.1.4.** A **mixed Hodge structure** with is a finitely generated abelian group $V = V_{\mathbb{Z}}$ with an increasing filtration $W_\cdot$ on $V_{\mathbb{Q}}$ and a decreasing filtration on $F^\cdot$ on $V_{\mathbb{C}}$ inducing a pure Hodge structure of weight $n$ on $\mathrm{Gr}_n^W V_{\mathbb{Q}}$.

A morphism of mixed Hodge structures is a map $V_{\mathbb{Z}} \to W_{\mathbb{Z}}$ preserving the two filtrations.

**Example 4.1.5.** Let $X$ be an algebraic variety over $\mathbb{C}$. Then $H^n(X; \mathbb{Q})$ has a canonical mixed Hodge structure. See [Del71] for the smooth case, and [Del74] for singular varieties.

**4.1.6.** The category of mixed Hodge structures is an abelian rigid tensor category of global dimension 1. It has neither enough projectives nor enough injectives. For this reason, we introduce (see [Bei86])

**Definition 4.1.7.** A **mixed Hodge complex** is a tuple

$$\left(V_{\mathbb{Z}}, \alpha_{\mathbb{Q}} : V_{\mathbb{Z}} \otimes \mathbb{Q} \xrightarrow{\sim} V_{\mathbb{Q}}, \alpha_{\mathbb{C}} : W^{\mathbb{Q}} V_{\mathbb{Q}} \otimes \mathbb{C} \xrightarrow{\sim} W^{\mathbb{C}} V_{\mathbb{C}}\right)$$

with $V_{\mathbb{Z}}$ a complex of abelian groups, $(W^{\mathbb{Q}} V_{\mathbb{Q}})$ a filtered complex of $\mathbb{Q}$-vector spaces, and $(W^{\mathbb{C}} F_{\mathbb{C}} V_{\mathbb{C}})$ a biregular filtered complex of $\mathbb{C}$-vector spaces, and $\alpha_{\mathbb{Q}}$ a quasi-isomorphism and $\alpha_{\mathbb{C}}$ a filtered quasi-isomorphism.

Morphisms and weak equivalences in the category of mixed Hodge complexes are evident. The **derived category of mixed Hodge complexes** is $D\,\mathrm{MHC} = \mathrm{MHC}[\mathrm{qis}^{-1}]$.

**Theorem 4.1.8.** *The obvious functor $D\,\mathrm{MHS} \to D\,\mathrm{MHC}$ is a derived equivalence.*

**Definition 4.1.9.** Let $S$ be a complex analytic space. A **mixed Hodge complex of sheaves** on $S$ is a tuple

$$\left(V_{\mathbb{Z}}, \alpha_{\mathbb{Q}} : V_{\mathbb{Z}} \otimes \mathbb{Q} \xrightarrow{\sim} V_{\mathbb{Q}}, \alpha_{\mathbb{C}} : W^{\mathbb{Q}} V_{\mathbb{Q}} \otimes \mathbb{C} \xrightarrow{\sim} W^{\mathbb{C}} V_{\mathbb{C}}\right)$$

with $V_{\mathbb{Z}}$ a complex of abelian sheaves, $(W^{\mathbb{Q}} V_{\mathbb{Q}})$ a filtered complex of $\mathbb{Q}$-vector spaces, and $(W^{\mathbb{C}} F_{\mathbb{C}} V_{\mathbb{C}})$ a biregular filtered complex of $\mathbb{C}$-vector spaces, and $\alpha_{\mathbb{Q}}$ a quasi-isomorphism and $\alpha_{\mathbb{C}}$ a filtered quasi-isomorphism.

Morphisms and weak equivalences in the category of mixed Hodge complexes of sheaves are evident. There is an exact functor $R\Gamma D\,\mathrm{MHC}(S) \to D\,\mathrm{MHC}$ taking $(V_{\mathbb{Z}}, W^{\mathbb{Q}} V_{\mathbb{Q}}, W^{\mathbb{C}} F_{\mathbb{C}} V_{\mathbb{C}})$ to $(R\Gamma V_{\mathbb{Z}}, \mathrm{Dec}(W^{\mathbb{Q}}) R\Gamma V_{\mathbb{Q}}, \mathrm{Dec}(W^{\mathbb{C}}) F_{\mathbb{C}} R\Gamma V_{\mathbb{C}})$. The filtration $\mathrm{Dec}(W)$ is the *filtration decaleé* (see [Del71]), $\mathrm{Dec}(W)_k R^n\Gamma V = R^n \Gamma W_{k-n} V$.

**Definition 4.1.10.** A mixed Hodge structure $H$ is **(homologically) effective** if $F^0 H_{\mathbb{C}} = 0$, and **cohomologically effective** if $F^0 H_{\mathbb{C}} = H_{\mathbb{C}}$. The subcategory of (co)homologically effective mixed Hodge structures is denoted $\mathrm{MHS}_{\mathrm{eff}}$ ($\mathrm{MHS}^{\mathrm{eff}}$).

**Example 4.1.11.** The (co)homology of an algebraic variety is (co)homologically effective. The dual of a homologically effective mixed Hodge structure is cohomologically effective, and vice-versa.

**Definition 4.1.12.** Let $V$ be a pure Hodge structure of weight $n$. A **polarization** of $V$ is a graded-symmetric bilinear form $S(-,-)$ on $V$ whose complexification descends to a pairing $V_{\mathbb{C}}^{pq} \otimes V_{\mathbb{C}}^{qp} \to \mathbb{C}$ satisfying $i^{p-q} S(v, \overline{v}) > 0$ for $v \in V_{\mathbb{C}}^{pq}$.

**Example 4.1.13.** Let $X$ be a smooth projective variety, and $H$ a hyperplane section. Then $H^n(X)$ has a canonical polarization given as follows: let $\alpha, \beta \in H^n(X)$; write them as $\alpha = [H]^{n-a}\alpha'$ and $\beta = [H]^{n-b}\beta'$, with $a$ and $b$ minimal. Then

$$S(\alpha, \beta) = (-1)^{\binom{a+b}{2}} \int_X \alpha \cup \beta.$$

**4.1.14.** Let $\pi : X \to C$ be a smooth projective family of complex algebraic varieties over a curve. To each $t \in C$ we can assign a pure Hodge structure of weight $n$, $H^n(X_t)$; by the work of Griffiths ([Gri68a], [Gri68b], [Gri70]), this can be arranged into a vector bundle $\mathcal{V} = R^n\pi_*\mathbb{Q}_X$ with filtered complexification $\mathcal{F}^{\cdot}\mathcal{V}_{\mathbb{C}}$; the hyperplane class induces a polarization $S$ on $\mathcal{V}$. This data is the prototypical example of a **polarized variation of Hodge structure** of weight $n$.

### 4.2. Limit Hodge Structure

Let $\mathcal{V}$ be a polarized variation of Hodge structure of weight $n$ over the puncture disk $D^*$. There is a canonical extension of the vector bundle over $D$; the new fiber is called the **nearby cycles** of the degeneration. Schmid developed a technique to canonically (up to choice of coordinate on $D$) endow the new fiber with *mixed* Hodge structure.

The idea is to exhibit the flag $\mathcal{F}$ as obtained from a map to a flag variety, and compute a limit in this flag variety.

**Definition 4.2.1** (Griffiths)**.** Let $V_{\mathbb{Z}}$ be a lattice, $h^{pq}$ integers such that $\sum_{p+q=n} h^{pq} = \operatorname{rk} V$, and $S$ a bilinear form $V^{\otimes 2} \to \mathbb{Q}$. The **polarized flag variety** for $(V, S)$ is $\check{\mathcal{D}} = O(V_{\mathbb{C}}, S)/B(V_{\mathbb{C}}, S)$ (where $B(V_{\mathbb{C}}, S)$ is a Borel subgroup of $O(V_{\mathbb{C}}, S)$, the elements of $\operatorname{Aut}(V_{\mathbb{C}})$ preserving the form).

The **Hodge flag variety** $\mathcal{D}$ is the collection of $F^{\cdot} \subset \check{\mathcal{D}}$ such that $(V, F^{\cdot}, S)$ is a polarized variation of Hodge structure of weight $n$; this is a closed condition, so $\mathcal{D} \subset \check{\mathcal{D}}$ is a closed algebraic variety. The **period domain** is $\mathcal{P} = \mathcal{D}\,/O(V_{\mathbb{Z}}, S)$.

**Definition 4.2.2.** Let $T$ be a smooth complex variety. A polarized variation of Hodge structure on $T$ is a locally liftable map $\Phi : T \to \mathcal{D}$ such that the induced filtered vector bundle with lattice satisfies

$$\nabla\mathcal{F}^n \subset \mathcal{F}^{n-1} \otimes \Omega^1.$$

The map $\Phi$ is called the period mapping.

**4.2.3.** Assume that the polarized variation of Hodge structure $\mathcal{V}$ comes from a smooth projective family of algebraic varieties over the puncture disk $D^*$. The existence of local liftings gives a map $\widetilde{\Phi} : \mathfrak{h} \to \mathcal{D}$ lifting $\Phi$. The deck transformation group exhibits $T$ as translation by 1 in $\mathfrak{h}$, and this induces an action (also denoted $T$) on $\mathcal{D}$.

The monodromy operator $T$ is quasi-unipotent (due to a theorem of Borel); by base-changing if necessary, we may assume it is unipotent. Then the monodromy logarithm

$$N = \log T = \sum (-1)^k (1-T)^k / k \in \mathfrak{g} = \operatorname{Lie}\operatorname{Aut}(V_{\mathbb{C}}, S)$$

is nilpotent. Define

$$\widetilde{\Psi}(s) = e^{-sN} \widetilde{\Phi}(s).$$

This new map descends to a map $\Psi : D^* \to \mathcal{P}$.

**Theorem 4.2.4** (Schmid)**.** *The map $\Psi$ extends over $D$, and $(N, W(N)[n], \Psi(0))$ (where $W(N)[n]_i = \ker N^{n-i}$) gives a polarized mixed Hodge structure on $V_{\mathbb{Z}}$.*

**4.2.5.** Steenbrink gives an alternate construction (see [Ste76a]), using the sheaf of nearby cycles.

**4.2.6.** Let $f : X \to S$ be a proper 1-parameter degeneration, with special fiber $X_0 \to 0$. Equip $S$ with compactification log structure along 0, and $X \to S$ with the pullback log structure.

**Definition 4.2.7.** Assume $S$ is an analytic disk, $j : \sigma \to S$ the embedding of $\sigma = S \setminus 0$, and let $\overline{j} : \langle \to \sigma$ be the universal cover; write $k = \overline{j} \circ j$.

The **nearby cycles** of a sheaf $F$ of abelian groups on $X_\sigma$ is the sheaf $\psi_f F$ of abelian groups on $X_0$ given by $i^* R k_* k^* F$.

$$\begin{array}{ccccccc}
X_\langle & \xrightarrow{\overline{j}} & X_\sigma & \xhookrightarrow{j} & X & \xhookleftarrow{i} & X_0 \\
\downarrow & & \downarrow & & \downarrow f & & \downarrow \\
\langle & \xrightarrow{\overline{j}} & \sigma & \xhookrightarrow{j} & S & \xhookleftarrow{i} & 0
\end{array}$$

**Lemma 4.2.8.** *(i) $\mathbb{H}^{\cdot} \psi_f \mathbb{Z} \simeq H^{\cdot} \mathbb{Z}_{X_s}$ for $s \neq 0$.*

*(ii)* $\psi_f \mathbb{Q} \simeq S^{\cdot}(\mathcal{O} \otimes \mathbb{Q} \to M_f \otimes \mathbb{Q})$

*(iii)* $\psi_f \mathbb{C} \simeq \Omega^{\cdot}_{X/S}(\log X_0/0)$

*Proof.* Part i is obvious, and the others follow from theorem 2.4.4 and theorem 2.4.14.

□

**Definition 4.2.9.** The **limit mixed Hodge complex** of $f$ is

$$R\Gamma(R\psi_f\mathbb{Z}, S^{\cdot}(\mathcal{O}\otimes\mathbb{Q}\to M_f\otimes\mathbb{Q}), \Omega^{\cdot}_{X/S}(\log X_0/0))$$

.

## 4.3. Mixed Hodge Modules

**4.3.1.** The theory of mixed Hodge modules lifts mixed Hodge structures to the relative setting: one should think of a mixed Hodge module as a "sheaf of mixed Hodge structures."

**4.3.2.** We will not give a formal review of the theory here, or even the complete definition; we will only state some basic properties that we need. For the original sources, see [Sai86a], [Sai86b]; for a (relatively) concise introduction, see [Sch14].

For every scheme $X/\mathbb{C}$ there is a category $MHM(X)$, and a six functors plus vanishing cycle formalism. The category $MHM(\operatorname{Spec}\mathbb{C})$ is identified with MHS($\mathbb{Q}$), and Deligne's canonical mixed Hodge structure on the cohomology of a scheme $X$ is canonically $Rf_*f^*\mathbb{Q}$. The constructions of Schmid and Steenbrink are realized using the vanishing cycle formalism. If $X$ is smooth, then $\underline{\operatorname{Ext}}^1(f^*\mathbb{Q}, f^*\mathbb{Q}(1))$ is canonically identified with $\mathcal{O}_X^*\otimes\mathbb{Q}$.

## 4.4. 1-Hodge Structures

**Definition 4.4.1.** A (co)homologically effective mixed Hodge structure $H$ is called **(co)homological 1-Hodge** if $F^{-1}H_{\mathbb{C}} = H_{\mathbb{C}}$ ($F^2H_{\mathbb{C}} = 0$). The full subcategory of (co)homological 1-Hodge structures is denoted $\mathrm{MHS}_1$ ($\mathrm{MHS}^1$).

**Proposition 4.4.2.** *The inclusions* $\mathrm{MHS}^1\subset\mathrm{MHS}^{\mathrm{eff}}$ *and* $\mathrm{MHS}_1\subset\mathrm{MHS}_{\mathrm{eff}}$ *admit left adjoints, denoted* $\mathrm{Alb}^\vee$ *and* Alb*, called the cohomological and homological Albanese, respectively.*

*Proof.* Standard.

□

CHAPTER V

# 1-MOTIVES

**5.0.1.** In [Del74], Deligne introduces an abelian category of mixed motives generated by curves, along with Hodge, étale, and Betti realizations. After Voevodsky's construction of a triangulated category of mixed motives, Orgogozo [Org04] gave an embedding of the derived category of Deligne 1-motives into the Voevodsky category.

## 5.1. 1-motives

In this section, $k$ is a field with separable closure $\overline{k}$ and $\Lambda$ a ring of coefficients.

**Definition 5.1.1.** A **semi-abelian** variety over $k$ is an extension $G$ of an abelian variety by an algebraic torus. A **lattice** $L$ over $k$ is a finitely generated torsion-free commutative group scheme over $k$.

A 1-motive over $k$ is a tuple $(L, G, u)$ with $L$ a lattice and $G$ a semi-abelian variety over $k$, such that $u : L(\overline{k}) \to G(\overline{k})$ is a homomorphism of groups. We usually denote the tuple $[L \xrightarrow{u} G]$.

The category of 1-motives over $k$, with obvious morphisms, is denoted $\mathcal{M}_1(k)$; because cokernels can be torsion, it is not abelian. Tensoring the morphisms with $\mathbb{Q}$ gives the category of 1-motives over $k$ up to isogeny, denoted $\mathcal{M}_1(k; \mathbb{Q})$.

**Example 5.1.2.** Let $C$ be a smooth projective curve over an algebraically closed field $k$. Then the **motivic cohomology** of $C$ is $H^0_{\mathcal{M}}(C) = [\operatorname{Spec} \mathcal{O}(C) \to 0]$, $H^1_{\mathcal{M}}(C) = [0 \to \operatorname{Pic}^0(C)]$, and $H^2_{\mathcal{M}}(C) = [0 \to \mathbb{G}_m]$. If $C = \overline{C} \setminus S$ with $\overline{C}$ smooth and projective, then $H^1_{\mathcal{M}}(C) = [\mathbb{Z}^S \to \operatorname{Pic}^0(\overline{C})]$, where $u : \mathbb{Z}^S \to \operatorname{Pic}^0(\overline{C})$ is the kernel of the map $\mathbb{Z}^S \to \operatorname{Pic}(\overline{C}) \to \mathbb{Z}$.

**Theorem 5.1.3.** *Let $k$ be a perfect field. Then the functor $D\mathcal{M}_1(k; \mathbb{Q}) \to DM^{\text{ét}}(k; \mathbb{Q})$ given by $[L \to G] \mapsto \operatorname{Sch}(-; L) \to \operatorname{Sch}(-; G)$ is fully faithful with essential image the thick subcategory generated by motives of curves.*

*Proof.* See [Org04] or [Bar07].

□

**5.1.4.** We thus have realization functors for $\mathcal{M}_1(k; \mathbb{Q})$. Deligne gives independent definitions for these; they are verified to agree in the case of Hodge realization ([Vol12]).

**Theorem 5.1.5.** *The functor* $R_{\text{Hodge}} : \mathcal{M}_1(\mathbb{C};\mathbb{Q}) \to MHS_1(\mathbb{Q})$ *is an equivalence.*

*Proof.* Using Deligne's definition of Hodge realization, see [Del74].

□

**Conjecture 5.1.6** (Deligne)**.** *Let $H$ be a 1-mixed Hodge structure "of geometric origin;" then $H$ can be obtained as the Hodge realization of a 1-motive over $k$.*

**Remark 5.1.7.** Theorem 1.2.16 can be interpreted as a proof of the conjecture for $H$ the Albanese of the limit Hodge structure.

# CHAPTER VI

# MOTIVES

**6.0.1.** In topology, Brown representability implies that every Eilenberg-Steenrod cohomology theory is determined by its value over a point; the (triangulated) category of cohomology theories is thus $D\,\mathrm{Ab}$.

For Weil cohomology theories of schemes, the situation is somewhat more complicated. We briefly describe the background required for later chapters.

## 6.1. Homotopy-Invariant and Transfer for Sheaves

**Definition 6.1.1.** Let $S$ be a scheme. A presheaf $F$ on $\mathrm{Sm}/S$ is **homotopy-invariant** if the canonical map $\mathrm{pr}^* : F(- \times \mathbb{A}^1) \leftarrow F(-)$ is an isomorphism of sheaves.

**Definition 6.1.2.** Let $k$ be a field and $X, Y$ schemes over $S$. An **elementary finite correspondence** is a integral subscheme $\gamma$ of $X \times Y$ such that $\mathrm{pr}_X : \gamma \to X$ is an isomorphism and $\mathrm{pr}_Y : \gamma \to Y$ is finite.

The **finite correspondences** $\mathrm{Cor}_k(X, Y)$ from $X$ to $Y$ are the free abelian group generated by the elementary finite correspondences. Given $\gamma \in \mathrm{Cor}_k(X, Y)$ and $\delta \in \mathrm{Cor}_k(Y, Z)$ we can form the composition $\gamma \circ \delta = \mathrm{pr}_{(X \times Z)*}(\mathrm{pr}^*_{X \times Y}\,\gamma \cap \mathrm{pr}^*_{Y \times Z}\,\delta)$.

The category whose objects are smooth schemes and morphisms are finite correspondences is denoted $\mathrm{Cor}_k$.

A **presheaf with transfers** on $\mathrm{Sm}/k$ is a presheaf on the category $\mathrm{Cor}_k$.

**Proposition 6.1.3.** *$\mathcal{O}^*$ has a natural structure of homotopy-invariant sheaf with transfers.*

*Proof.* Nm gives the transfer structure; homotopy-invariance is obvious. □

## 6.2. Nisnevich Topology

**Definition 6.2.1.** Let $X$ be a scheme. A **Nisnevich covering space** of $X$ is a morphims $Y \to X$ such that $Y \to X$ is étale and every fiber $Y_x \to x$ admits a section.

**Example 6.2.2.** Let $X$ be over an algebraically closed field $k$. Then Nisnevich covers and étale covers are the same.

**Proposition 6.2.3.** *A Nisnevich covering space admits a constructible section, i.e., a stratification with sections over the open strata.*

*Proof.* Obvious from the definition. ☐

**Proposition 6.2.4.** *Let $X$ be a smooth scheme over a perfect field and $F$ a homotopy-invariant Nisnevich sheaf with transfers. Then $H^\cdot(X_{\mathrm{Zar}}; F) \cong H^\cdot(X_{\mathrm{Nis}}; F)$.*

*Proof.* See [MVW06], 13.9. ☐

### 6.3. cdh Topology

**Definition 6.3.1.** An **abstract blow-up** is a proper birational morphism.

**Definition 6.3.2.** The **cdh topology** is the Grothendieck topology generated by abstract blow-ups and Nisnevich covers.

**Proposition 6.3.3** ([MVW06] 13.27)**.** *Let $X$ be a smooth scheme over a perfect field and $F$ a homotopy-invariant Nisnevich sheaf with transfers. Then $H^\cdot(X_{\mathrm{Nis}}; F) \cong H^\cdot(X_{\mathrm{cdh}}; F^{\mathrm{cdh}})$.*

### 6.4. Voevodsky Motives

**Definition 6.4.1.** Let $k$ be a perfect field and $\Lambda$ a ring. The category of effective Voevodsky (étale) motives is $DM^{\mathrm{eff}(,\text{ét})}(k;\Lambda) = D\,\mathrm{Sh}(\mathrm{Cor}_{k,\mathrm{Nis}\,(\text{ét})};\Lambda)/\left\langle F \leftarrow F(- \times \mathbb{A}^1)\right\rangle$.

The category of effective (étale) Voevodsky motives without transfers is $DA^{\mathrm{eff}(,\text{ét})}(k;\Lambda) = D\,\mathrm{Sh}(\mathrm{Sm}_{k,\mathrm{Nis}\,(\text{ét})};\Lambda)/\left\langle F \leftarrow F(- \times \mathbb{A}^1)\right\rangle$.

**Proposition 6.4.2.** *The Yoneda maps $\mathrm{Cor}_k \to DM^{\mathrm{eff}(,\text{ét})}$ and $\mathrm{Sm}_k \to DA^{\mathrm{eff}(,\text{ét})}$ gives a collection of compact generators for each; in particular, every motive $F$ has a resolution by representables.*

*Further, the Yoneda map is compatible with the monoidal structure, and this induces on $DM^{\mathrm{eff}(,\text{ét})}$ and $DA^{\mathrm{eff}(,\text{ét})}$ a triangulated tensor structure.*

*Proof.* See [MVW06] chapter 8 and its appendix 8A. ☐

**Proposition 6.4.3.** *If $\Lambda$ contains $\mathbb{Q}$, the "add transfers" map $DA^{\mathrm{eff},\text{ét}} \to DM^{\mathrm{eff},\text{ét}}$ is an equivalence.*

*Proof.* See [Ayo] appendix B. ∎

**Definition 6.4.4.** Let $D^{\text{eff}} = DM^{\text{eff}(,\text{ét})}$ or $DA^{\text{eff}(,\text{ét})}$. Then the Voevodsky, or non-effective Voevodsky, or stable Voevodsky, (étale) motives are the category $D = DM^{(\text{ét})}$ or $DA^{(\text{ét})}$ given by inverting the Tate motive $\operatorname{cone}(\text{pt} \to \mathbb{G}_m)$ under the tensor product.

**Proposition 6.4.5.** *Let $X = \operatorname{Spec} k[P]$ be a fs toric variety and $j : T = \operatorname{Spec} k[P^+] \hookrightarrow X$ the inclusion of its dense torus. Then $(j_*\mathcal{O}^*)^{\text{cdh}} \simeq Rj_*^{\text{cdh}}\mathcal{O}^*$.*

*Proof.* We may replace $X$ with its germ at a point in the toric boundary; $X$ is then the spectrum of a smooth local ring and $T$ is its generic point. The result is then immediate from the fact the $\mathcal{O}^*$ is homotopy-invariant with transfers, so $H^i(T_{\text{Zar}}; \mathcal{O}^*) \cong H^i(T_{\text{Nis}}; \mathcal{O}^*)$, which is zero for $i > 0$. ∎

### 6.5. Realizations

There are many functors from $DM$ or $DA$, corresponding to the various familiar Weil cohomology theories.

**Definition 6.5.1.** Let $D(\mathbb{C}; \Lambda)$ be a category of Voevodsky motives. The **Betti realization** is the unique tensor-triangulated functor $R_{\text{Betti}} : D(\mathbb{C}; \Lambda) \to D(\Lambda - \text{Mod})$ satisfying $R_{\text{Betti}}([X]) = C_\cdot^{\text{sing}}(X(\mathbb{C}))$ for smooth $X$.

**Definition 6.5.2.** Let $D(\mathbb{C}; \Lambda)$, with $\Lambda \subset \mathbb{C}$, be a category of Voevodsky motives. The **Hodge realization** is the unique tensor-triangulated functor $R_{\text{Hodge}} : D(\mathbb{C}; \Lambda) \to DMHS(\Lambda)$ taking $[X]$ to the canonical mixed Hodge complex on its homology.

**Definition 6.5.3.** Let $D(k; \Lambda)$, with $k$ perfect and $\Lambda$ torsion, be a category of Voevodsky motives. The **étale realization** is the unique tensor-triangulated functor $R_{\text{ét}} : D(k; \Lambda) \to D\operatorname{Rep}(\operatorname{Gal}(k); \Lambda)$ taking $[X]$ to the canonical chain complex computing étale homology with $\Lambda$ coefficients.

# Main Results

CHAPTER VII

# LOG MOTIVES

## 7.1. The Category of Log Motives

**Definition 7.1.1.** Let $k$ be a field, and $\mathrm{Log}/k$ the category of coherent log schemes over $k$. The category of **log motives** with $\Lambda$-coefficients over $k$, written $DA^{\log}(k;\Lambda)$, is the quotient of $D\,\mathrm{pSh}(\mathrm{Log}/k;\Lambda)$ by the thick subcategory $I$ generated by

(cdh) Mayer-Vietoris relations in the cdh topology: if $U \twoheadrightarrow T$ is a cdh cover, then

$$[U \times_T U] \to [U] \to [T]$$

is in $I$, where $[-]$ is the representable $\Lambda$-presheaf.

($\mathbb{A}^1$) $\mathbb{A}^1$-homotopy relation: $F(- \times \mathbb{A}^1) \to F$ is in $I$ for every $F \in \mathrm{pSh}(\mathrm{Log}/k;\Lambda)$.

(**pt**) log relation: $\mathbb{G}_m \to \mathbb{A}^1(\log 0)$ is in $I$, where $\mathbb{A}^1(\log 0) = \mathrm{Spec}(x^{\mathbb{N}} \to k[x])$.

(log) change of log structure relation: if $T \to S$ is a map of log schemes over $X$ and $S' \to S$ is an isomorphism on underlying schemes, then $[T'] \to [T] \oplus [S'] \to [S]$ is in $I$.

**Remark 7.1.2.** Replacing Log with the category of fine or fs log schemes, or with coherent or saturated virtual log schemes, gives the same category.

**7.1.3.** There is an obvious functor $\Phi : DA(k;\Lambda) \to DA^{\log}(k;\Lambda)$.

**Theorem 7.1.4.** *Assume $\Lambda$ contains $\mathbb{Q}$. Then the functor $\Phi$ is an equivalence.*

*Proof.* We must show

(a) $\Phi$ is fully faithful, and

(b) $\Phi$ is essentially surjective.

In fact, (a) implies (b), by induction on dimension: if it is zero dimensional, then $M^+ = k^*\mathbb{Z}^n$ so the corresponding Voevodsky motive is $[\mathbb{G}_m^n]$.

In higher dimension, $M^+$ is generically constant, so we may induct using cdh blowup triangles. Here fully faithfulness is assumed so as to ensure the maps between objects in the essential image (in this triangle) are also in the image.

To prove (a), it suffices to notice the category of Voevodsky motives is rigid, and to show that $R\,\mathrm{Hom}(-, \Lambda(n))$ on $DA(k;\Lambda)$ extends to $DA^{\log}(k;\Lambda)$.

We give the construction for any log scheme $X$ with smooth underlying scheme; these generate $DA^{\log}$. Consider $M^+$ as a motivic sheaf over $\underline{X}$, so that it can be evaluated on any scheme smooth over $\underline{X}$. Then compute $R\Gamma(\underline{X}_{\mathrm{Nis}}, S^n M^+)$; that this agrees in the case of trivial log structure follows from the fact that $S^n[X] = [S^n X]$.

□

**Remark 7.1.5.** If we had an understanding of the meaning of exterior algebra over the integers, we could remove the condition that $\mathbb{Q} \subset \Lambda$.

**Conjecture 7.1.6.** *Let $\underline{X}$ be a toric variety, and let $X$ be the $\underline{X}$ equipped with the compactification log structure induced by the toric boundary.*

*Let $F$ be a log motive. Then the restriction map $F(X) \to F(T)$ is a quasi-isomorphism.*

**Remark 7.1.7.** The conjecture becomes true if $F$ is instead a Kummer étale $\ell$-adic sheaf; see [Ill02].

The conjecture holds in the case $X = \mathbb{A}_{\mathbb{N}}$ by the relation (**pt**), or if $X$ is fs and $F = M^+_{\mathrm{cdh}}$ by proposition 6.4.5.

**Lemma 7.1.8.** *Let $F$ be a log motive over $k$, and assume Conjecture 7.1.6 for $F$.*

*Then $F$ satisfies* ***acyclicity of log blow-ups****: if $\widetilde{S} \to S$ is a log blowup of a log smooth f.s. log scheme over $k$, then $F(\widetilde{S}) \leftarrow F(S)$ is acyclic.*

*Proof.* Using the Mayer-Vietoris relations, we are reduced to the case that $\widetilde{S} \to S$ admits a global chart, in which case it can be expressed as a pullback square

$$\begin{array}{ccc} \widetilde{S} & \longrightarrow & \mathrm{Bl}_I \mathbb{A}_P \\ \downarrow & \lrcorner & \downarrow \\ S & \longrightarrow & \mathbb{A}_P \end{array}$$

where $P$ is a f.s. monoid and $I$ is an ideal of $P$.

Factoring this square vertically into strict and change-of-log structure, we obtain:

$$\begin{array}{ccc} \widetilde{S} & \longrightarrow & \mathrm{Bl}_I \mathbb{A}_P \\ \big\downarrow & & \big\downarrow \\ (\widetilde{S})_S & \longrightarrow & (\mathrm{Bl}_I \mathbb{A}_P)_{\mathbb{A}_P} \\ \big\downarrow & & \big\downarrow \\ S & \longrightarrow & \mathbb{A}_P \end{array}$$

The upper square is change-of-log structure, so has acyclic image under $F$. The lower is by Nisnevich descent equivalent to

$$\begin{array}{ccc} (\widetilde{Z})_Z & \longrightarrow & (\mathrm{Bl}_I \mathbb{A}_P)_{\mathbb{A}_P} \\ \big\downarrow & & \big\downarrow \\ Z & \longrightarrow & \mathbb{A}_P \end{array}$$

where $Z$ is the center of the log blowup; this latter is an abstract blowup square, and has acyclic image under $F$. We are thus reduced to showing that $\mathrm{Bl}_I \mathbb{A}_P \to \mathbb{A}_P$ has acyclic image under $F$.

We mimic the proof for Kummer étale sheaves given in [Ill02]: let $X = \mathbb{A}_P$, $\widetilde{X} = \mathrm{Bl}_I \mathbb{A}_P$, with blow-down map $\pi$. Set $K = \mathrm{cone}(\pi_* F_{\widetilde{X}} \to F_X)$. We wish to show $K \simeq 0$.

By induction on $\mathrm{rk}\, P$ we may assume that $K \simeq 0$ away from the vertex $0 = V(P)$, so that $R\Gamma(X; K) \simeq R\Gamma(0; K_0)$. Applying the conjecture, we have that $R\Gamma(X; K) = R\Gamma(X^*; K_{X^*})$; over $X^*$ we know $K$ is zero.

□

**Corollary 7.1.9.** *Let $X$ be a log motive. Then $T \mapsto R\Gamma_{\mathrm{cdh}}(T \times X, \pi_2^* M_X^+)$ satisfies acyclicity under log blowups.*

**Corollary 7.1.10.** $R\Gamma(-; M_{\psi_t X}) \cong R\Gamma(-; M_{\psi_t \widetilde{X}})$.

# CHAPTER VIII

# 1-MOTIVES OF VIRTUAL LOG SCHEMES

In this section, all schemes are over $\mathbb{C}$ and all mixed Hodge modules are algebraic. For $f : X \to \mathbb{A}^1$ a regular function, $\psi_f$ always refers to the functor of nearby cycles of mixed Hodge modules or perverse sheaves; in particular, $\psi_f^{\mathrm{Ch(Perv)}} = R\psi_f^{\mathrm{Ch(Sh)}}[-1]$.

## 8.1. Limit 1-motives

**8.1.1.** Let $\tau : X \to \mathbf{pt}$ be a log smooth morphism. We write $\psi X$ for the pullback

$$\begin{array}{ccc} \psi X & \longrightarrow & X \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathbf{pt}^{\mathrm{sp}} & \xrightarrow{\mathrm{sp}} & \mathbf{pt} \end{array}$$

Choosing an isomorphism $t : \mathbf{pt} \xrightarrow{\sim} \mathrm{Spec}(\epsilon^{\mathbb{N}} \xrightarrow{0} k)$ gives an isomorphism $\mathbf{pt}^{\mathrm{sp}} \xrightarrow{\sim} \mathbb{G}_m$, sending the coordinate on $\mathbb{G}_m$ to $(\mathrm{sp} \circ t)^\flat(\epsilon)$, and a fiber $\psi_t X$.

**Definition 8.1.2.** Let $f : X \to \mathbb{A}^1(\log 0)$ be the virtualization of a log smooth morphism. The **limit 1-motive** (up to isogeny) of $f$ is the complex

$$\psi^{(1)}[X] = R\Gamma_{\mathrm{cdh}}(\psi^{\mathrm{sat}} X \times -, M) \otimes \mathbb{Q}$$

equipped with the finite monodromy, the action of $\hat{\mathbb{Z}}(1)$ lifted from $R\Gamma_{\mathrm{cdh}}(\psi^{p\,\mathrm{sat}} X \times -, M)$, and unipotent monodromy logarithm $N : \psi^{(1)}[X](1) \to \psi^{(1)}[X]$ defined by

$$\begin{aligned}
&R\Gamma_{\mathrm{cdh}}(\psi^{\mathrm{sat}} X \times -, M)(1) \otimes \mathbb{Q} \\
&\simeq R\Gamma_{\mathrm{cdh}}(\psi^{p\,\mathrm{sat}} X \times -, M)(1) \otimes \mathbb{Q} \\
&\xrightarrow{\overline{\delta}(1)} R\Gamma_{\mathrm{cdh}}(\psi^{p\,\mathrm{sat}} X \times -, \mu[1]) \otimes \mathbb{Q} \\
&\to R\Gamma_{\mathrm{cdh}}(\psi^{p\,\mathrm{sat}} X \times -, \mathcal{O}^*) \otimes \mathbb{Q} \\
&\to R\Gamma_{\mathrm{cdh}}(\psi^{p\,\mathrm{sat}} X \times -, M) \otimes \mathbb{Q} \\
&\simeq R\Gamma_{\mathrm{cdh}}(\psi^{\mathrm{sat}} X \times -, M) \otimes \mathbb{Q}.
\end{aligned}$$

Let $1^{p\,\mathrm{sat}} : \psi^{p\,\mathrm{sat}} X \to X^{p\,\mathrm{sat}}$ be the obvious morphism. Here $\overline{\delta}$ is the composition of the connecting homomorphism in

$$\mathbb{Q} \xrightarrow{t} M_{(\psi^{p\,\mathrm{sat}} X)_{X^{p\,\mathrm{sat}}}} \otimes \mathbb{Q} \xrightarrow{\left(1^{p\,\mathrm{sat}}\right)^{\flat}} M_{\psi^{p\,\mathrm{sat}} X} \otimes \mathbb{Q} \xrightarrow{\delta} \mathbb{Q}[1],$$

with the projection $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$.

## 8.2. Limit Mixed Hodge Modules

**8.2.1.** Let $M$ be a mixed Hodge module on $X$ and $f \in \mathcal{O}_X$ a function; let $X_0 = V(f)$. Then the mixed Hodge module of nearby cycles $\psi_f M$ in $MHM(X_0)$ is equipped with the monodromy operator: the map $N(1)$ gives a nilpotent endomorphism of $\psi_f M$. Terms of the weight filtration on $\psi_f M$ are given by the kernels of powers of $N(1)$.

If $M = \mathbb{Q}_X$ and $f : X \to \mathbb{A}^1$ is a semistable degeneration, then Steenbrink ([Ste76b]) has computed that

$$\operatorname{rat} \ker N(1)^i \psi_f M = \tau^{\mathrm{Ch(Sh)}}_{\leq -\dim X_0 + i} \operatorname{rat} \psi_f M,$$

so

$$\operatorname{rat} \operatorname{Gr}^W_{2i} \psi_f M = \bigwedge^i \left(\overline{\Lambda}[-1]\right) [\dim X_0]. \tag{8.2.1.1}$$

where $\overline{\Lambda} = \mathrm{Nm}_{X_0 *} \mathbb{Q}_{\mathrm{Nm}\, X_0} / \mathbb{Q}_{X_0}$.

**8.2.2.** Let $M$ be the variation of mixed Hodge structure over $\mathbb{G}_m$ given by $[t] \in \mathcal{O}^*_{\mathbb{G}_m} \cong \mathcal{E}\mathrm{xt}^1_{\mathbb{G}_m}(\mathbb{Z}(-1), \mathbb{Z})$, so that $F^1 M_t = (v_0 + t v_2)\mathbb{C}$, where $v_{2i}$ generates $\mathbb{Z}(-1)$. The monodromy operator is given by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

By [Sch73], in this case the nilpotent orbit is actually constant, so the limit Hodge filtration $\psi_t M$ is simply

$$F^\bullet \psi_t M = (e^{-t \log T} \cdot F^\bullet M_t)|_{t=0}$$

so that $F^1 \psi_t M = (v_0 + v_2)\mathbb{C}$, which gives a splitting of $\psi_t M$.

## 8.3. Main Result

**Theorem 8.3.1.** *Let $t : X \to \mathbb{A}^1(\log 0)$ be a proper log smooth degeneration over $\mathbb{C}$.*

*Then the Hodge realization of $\psi_t^{(1)}[X^+]$ agrees with the 1-Hodge component of the limit Hodge structure $\mathrm{Alb}^\vee R\Gamma\psi_t\mathbb{Q}_X(1)$ of [Ste76b].*

*Proof.* First we show that $\mathrm{sat}_* M_{\psi_t^{\mathrm{sat}}X}[-1] \cong R\mathcal{H}\mathrm{om}(\mathbb{Q}(-1), \psi_t\mathbb{Q})$. We abuse notation, occasionally writing $T$ for $\underline{T}$ where it will not cause confusion.

**8.3.2.** We reduce to the case that $t$ is semistable and $\psi_t X$ is saturated: by lemma 2.5.5, after log blow-up we obtain a semistable family. But lemma 7.1.9 implies the 1-motive of the two degenerations are the same. The morphism $\psi_t^{\mathrm{sat}}X \to \psi_t X$ induces an isomorphism on limit 1-motives by definition, and the corresponding map on limit Hodge structures induces an isomorphism, also by definition; see [Ste76b] for details.

**8.3.3.** We show that $R\mathcal{H}\mathrm{om}(\mathbb{Q}(-1), \psi_t\mathbb{Q})$ is constructibly concentrated in degree-1.

Let $X_0 = V(t)$, with trivial log structure. Then we have a short exact sequence of mixed Hodge modules on $X_0$

$$\psi_t\mathbb{Q}_X \hookrightarrow \phi_t\mathbb{Q}_X \twoheadrightarrow \mathbb{Q}_{X_0} \tag{8.3.3.1}$$

Applying $R\mathcal{H}\mathrm{om}(\mathbb{Q}(-1), -)$ with (8.2.1.1) to the sequence (8.3.3.1) we obtain

$$\mathbb{Q}_{X_0}[-1] \to R\mathcal{H}\mathrm{om}(\mathbb{Q}(-1), \psi_t\mathbb{Q}_X) \to \overline{\Lambda}[\dim X_0 - 1](-1) \to$$

This expresses $R\mathcal{H}\mathrm{om}(\mathbb{Q}(-1), \psi_t\mathbb{Q}_X)$ (a perverse sheaf) as constructibly concentrated in degree-$(\dim X_0 - 1)$.

**8.3.4.** Next we construct a canonical map

$$R\Gamma\overline{\phi} : \psi_t^{(1)}\left[X^+\right] \to R\Gamma R\mathcal{H}\mathrm{om}(\mathbb{Q}(-1), \psi_t\mathbb{Q}_X).$$

Consider first the map

$$\begin{array}{llcl} \phi_X : & \mathcal{O}_X^*[-1] \xrightarrow{\simeq} R\mathcal{H}\mathrm{om}(\mathbb{Q}_X(-1), \mathbb{Q}_X) & \longrightarrow & R\mathcal{H}\mathrm{om}(\mathbb{Q}_X(-1), j_*j^*\mathbb{Q}_X) \\ & s & \longmapsto & [s] \end{array}$$

where $j$ is the open embedding of $U = X \setminus X_0$ into $X$, which takes an invertible function $f$ and uses it to parameterize an extension $\mathcal{E}\mathrm{xt}^1(\mathbb{Q}, \mathbb{Q}(1))$ over $U$, then extends to $X$.

We can compose the restriction along $i : X_0 \hookrightarrow X$, $i^{-1}\phi_X$, with the canonical map $j_* j^* \mathbb{Q}_X \to \psi_t \mathbb{Q}_X$:

$$\phi : i^{-1}\mathcal{O}_X^*[-1] \to R\mathcal{H}\mathrm{om}(\mathbb{Q}_X(-1), \psi_t \mathbb{Q}_X)$$

If $s$ restricts to 1 over $X_0$, then $s + z \cdot (1 - s)$ gives a section of

$$\mathcal{E}\mathrm{xt}^1(\mathbb{Q}_{X \times \mathbb{A}^1}(-1), \mathrm{pr}_1^* \psi_t \mathbb{Q}_X).$$

The section gives a family of variations of Hodge structure parameterized by $\mathbb{A}^1$; all such are constant, so the fibers $z = 0$ (given by $\phi(s)$) and $z = 1$ (given by $\phi(1)$) are canonically identified. Thus $\phi(s)$ is split, and $\phi$ factors through $M_{X_0} = i^{-1}\mathcal{O}_X^* / \ker i^*$.

Furthermore, $\phi(t) = \phi(1) = \mathbb{Q}(-1) \oplus \psi_t j_* j^* \mathbb{Q}_X$, as $\phi(t) \in \mathcal{E}\mathrm{xt}^1(\mathbb{Q}(-1), j^*\mathbb{Q}_X)$ is the pullback from $\phi_{\mathbb{A}^1}(t) \in \mathcal{E}\mathrm{xt}^1_{\mathbb{G}_m}(\mathbb{Q}(-1), \mathbb{Q})$, so $\psi_t \phi(t)$ is the pullback of $\psi_t \phi_{\mathbb{A}^1}(t)$, which is split by 8.2.2.

Thus $\phi$ factors through a map

$$\overline{\phi} : M_{\psi_t X} \to R\mathcal{H}\mathrm{om}(\mathbb{Q}_X(-1), \psi_t \mathbb{Q}_X).$$

Applying sheaf cohomology and extending in the obvious way to $\mathrm{Sch}_{\mathbb{C}}$ gives the desired map $R\Gamma\overline{\phi}$.

**8.3.5.** Now we show that $\overline{\phi}$ is an isomorphism.

We have the diagram

$$\begin{array}{ccc}
\mathcal{O}_{X_0}^* & \xrightarrow{\sim} & \mathcal{E}\mathrm{xt}^1(\mathbb{Q}_{X_0}(-1), \mathbb{Q}_{X_0}) \\
\downarrow & & \downarrow \\
M_{\psi_t X} & \xrightarrow{\overline{\phi}} & \mathcal{E}\mathrm{xt}^1(\mathbb{Q}_{X_0}(-1), \psi_t \mathbb{Q}_X) \\
\downarrow & & \downarrow \\
\Lambda_{\psi_t X} & \xrightarrow{\overline{\pi}} & \mathcal{E}\mathrm{xt}^1(\mathbb{Q}_{X_0}(-1), \phi_t \mathbb{Q}_X)
\end{array}$$

We are thus reduced to showing that $\overline{\pi}$ is an isomorphism. But this is immediate: $\phi_t \mathbb{Q}_X$ is a direct sum of pure Tate modules, so by weights $\mathcal{E}\mathrm{xt}^1(\mathbb{Q}(-1), \phi_t \mathbb{Q}_X) = \mathcal{E}\mathrm{xt}^1(\mathbb{Q}(-1), \overline{\Lambda}(-1)[-1]) = \overline{\Lambda}$. Any section $s = (s_i)_{1 \le i \le r}$ of $\overline{\Lambda}$ can be realized from $M_{\psi_t X}$ as $f^s = f_1^{s_1} \cdots f_r^{s_r}$, where $t = f_1 \cdots f_r$. $\square$

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