ON THE SUBREGULAR J-RING OF COXETER SYSTEMS

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DISSERTATION ABSTRACT

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Let (W, S) be an arbitrary Coxeter system, and let J be the asymptotic Hecke algebra associated to (W, S) via Kazhdan-Lusztig polynomials by Lusztig. We study a subalgebra J_C of J corresponding to the subregular cell C of W. We prove a factorization theorem that allows us to compute products in J_C without inputs from Kazhdan-Lusztig theory, then discuss two applications of this result. First, we describe J_C in terms of the Coxeter diagram of (W, S) in the case (W, S) is simply-laced, and deduce more connections between the diagram and J_C in some other cases. Second, we prove that for certain specific Coxeter systems, some subalgebras of J_C are free fusion rings, thereby connecting the algebras to compact quantum groups arising in operator algebra theory.

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To my teachers.

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CHAPTER I

INTRODUCTION

Hecke algebras of Coxeter systems are classical objects of study in representation theory because of their rich connections with finite groups of Lie type, Lie algebras, quantum groups, and the geometry of flag varieties (see, for example, [Cur87], [CIK71], [DJ86], [GP00], [KL79], [Lus84]). Let (W, S) be a Coxeter system, and let H be its Hecke algebra defined over the ring $\mathbb{Z}[v, v^{-1}]$. Using Kazhdan-Lusztig polynomials, Lusztig constructed the *asymptotic Hecke algebra* J of (W, S) from H in [Lus87a]. The algebra J can be viewed as a limit of H as the parameter v goes to infinity, and its representation theory is closely related to that of H (see [Lus87a], [Lus87b], [Lus89], [Lus14], [Gec98]). In particular, upon suitable extensions of scalars, J admits a natural homomorphism from H, hence representations of J induce representations of H ([Lus14]).

The asymptotic Hecke algebra J has several interesting features. First, given a Coxeter system (W, S), J is defined to be the free abelian group $J = \oplus_{w \in W} \mathbb{Z} t_w$, with multiplication of the basis elements declared by

$$t_x t_y = \sum_{z \in W} \gamma_{x,y,z^{-1}} t_z$$

where the coefficients $\gamma_{x,y,z^{-1}}$ $(x, y, z \in W)$ are nonnegative integers extracted from the structure constants of the *Kazhdan-Lusztig basis* of the Hecke algebra H of (W, S). The non-negativity of its structure constants makes J a $\mathbb{Z}_+$-*ring*, and the basis elements satisfy additional conditions which make J a *based ring* in the sense of [Lus87c] and [EGNO15] (see Section 4.3).

Another interesting feature of J is that for any *2-sided Kazhdan-Lusztig cell* E of W, the subgroup

$$J_E = \oplus_{w \in E} \mathbb{Z} t_w$$

of J is a subalgebra of J and also a based ring. Here, as the notation suggests, a 2-sided Kazhdan-Lusztig cell is a subset of W. The cells of W are defined using the Kazhdan-Lusztig basis of its associated Hecke algebra H and form a partition of W. Further, the subalgebra J_E is in fact a direct summand of J for each 2-sided cell E, and J admits the direct sum decomposition

$$J = \oplus_{E \in \mathcal{C}} J_E,$$

where $\mathcal{C}$ denotes the collection of all 2-sided cells of W (see Section 4.2). It is therefore natural to study J by first studying its direct summands corresponding to the cells.

In this paper, we focus on a particular 2-sided cell C of W known as the *subregular cell* and study the based ring J_C. We also study subalgebras J_s of J_C that correspond to the generators $s \in S$ of W. Thanks to a result of Lusztig in [Lus83], the cell C can be characterized as the set of elements in W with unique reduced expressions, and the main theme of the paper is to exploit this combinatorial characterization and study J_C and $J_s (s \in S)$ without reference to Kazhdan-Lusztig polynomials. This is desirable since a main obstacle in understanding J for arbitrary Coxeter systems lies in the difficulty of understanding Kazhdan-Lusztig polynomials.

A third important feature of the algebra J is that it has very interesting *categorification*. Here by categorification we mean the process of adding an extra layer of structure to an algebraic object to produce an interesting category which allows one to recover the object; more specially, we mean J appears as the Grothendieck ring of a *tensor category* $\mathcal{J}$ (see [EGNO15] for the definition of a tensor category, [Lus14] for the construction of $\mathcal{J}$). A well-known example of categorification is the categorification of the Hecke algebra H by the *Soergel category* $\mathcal{SB}$, which was used to prove the "positivity properties" of the *Kazhdan-Lusztig basis* of H in [EW14].

Just as the algebra J is constructed from H, the category $\mathcal{J}$ is constructed from the category $\mathcal{SB}$, also by Lusztig ([Lus14]). Further, just as the algebra J has a subalgebra of the form J_E for each 2-sided cell E and a subalgebra J_s for each generator $s \in S$, the category $\mathcal{J}$ has a subcategory $\mathcal{J}_E$ for each 2-sided cell E and a subcategory $\mathcal{J}_s$ for each $s \in S$. Moreover, $\mathcal{J}_E$ categorifies J_E for each 2-sided cell E, and $\mathcal{J}_E$ is a *multifusion category* in the sense of [EGNO15] whenever E is finite, which can happen for suitable cells even when the ambient group W is infinite. Similarly, J_s is a *fusion category* whenever J_s has finite rank. Multifusion and fusion categories have rich connections with quantum groups ([Kas95]), conformal field theory ([MS89]), quantum knot invariants ([Tur10]) and topological quantum field theory ([BK01]), so the categories $\mathcal{J}_E$ (in particular, $\mathcal{J}_C$) and $\mathcal{J}_s$ are interesting since they can potentially provide new examples of multifusion and fusion categories.

Historically, the intimate connection between the algebra J and its categorification $\mathcal{J}$ has been a major tool in the study of both objects. For Weyl groups and an affine Weyl groups, Lusztig ([Lus89], [Lus97]) and Bezrukanikov et al. ([Bez04], [BO04], [BFO09]) showed that there is a bijection between the two-sided cells in the group and unipotent conjugacy classes of an algebraic group, and that the subcategories of $\mathcal{J}$ corresponding to the cells can be described geometrically, as categories of vector bundles on a square of a finite set equivariant with respect to an algebraic group. Using categorical results, they computed the structure constants in J explicitly. For other Coxeter systems, however, the nature of J or $\mathcal{J}$ seems largely unknown, partly because there is no known recourse to advanced geometry. In this context, this dissertation may be viewed as an attempt to understand the subalgebra J_C of J for arbitrary Coxeter systems from a more combinatorial point of view. We hope to understand the structure of J_C by examining the multiplication rule in J_C, then, in some cases, use our knowledge of J to deduce the structure of $\mathcal{J}$. This idea is further discussed in Section 7.1.

The main results of the dissertation fall into two sets. First, we describe some connections between the *Coxeter diagram* G of an arbitrary Coxeter system (W, S) and the algebra J_C associated to (W, S). The first result in this spirit describes J_C in terms of G for all *simply-laced* Coxeter systems. Recall that given any vertex s in G, the fundamental group $\Pi_s(G)$ of G based at s is the group consisting of all homotopy equivalence classes of walks in G starting and ending at s, equipped with concatenation as the group operation. One may generalize this notion to define the *fundamental groupoid* $\Pi(G)$ of G as the set of homotopy equivalence classes of all walks on G, equipped with concatenation as a partial binary operation that is defined between two classes when their concatenation makes sense. We define the groupoid algebra of $\mathbb{Z}\Pi(G)$ of $\Pi(G)$ by mimicking the construction of a group algebra from a group, and we prove the following theorem.

Theorem A. *Let (W, S) be an any simply-laced Coxeter system, and let G be its Coxeter diagram. Let $\Pi(G)$ be the fundamental groupoid of G, let $\Pi_s(G)$ be the fundamental group of G based at s for any $s \in S$, let $\mathbb{Z}\Pi(G)$ be the groupoid algebra of $\Pi(G)$, and let $\mathbb{Z}\Pi_s(G)$ be the group algebra of $\Pi_s(G)$. Then $J_C \cong \mathbb{Z}\Pi(G)$ as based rings, and $J_s \cong \mathbb{Z}\Pi_s(G)$ as based rings for all $s \in S$.*

The key idea behind the theorem is to find a correspondence between basis elements of J_C and classes of walks on G. The correspondence then yields explicit formulas for the claimed isomorphisms.

In our second result, we study the case where G is *oddly-connected*. Here by oddly-connected we mean that each pair of distinct vertices in G are connected by a path involving only edges of odd weights.

Theorem B. *Let (W, S) be an oddly-connected Coxeter system. Then*

(a) $J_s \cong J_t$ as based rings for all $s, t \in S$.

(b) $J_C \cong \mathrm{Mat}_{S\times S}(J_s)$ as based rings for all $s \in S$. In particular, J_C is Morita equivalent to J_s for all $s \in S$.

Once again, we will provide explicit isomorphisms between the algebras using G.

In a third result, we describe all *fusion rings* that appear in the form J_s for some Coxeter system (W, S) and some choice of $s \in S$. We show that any such fusion ring is isomorphic to a ring J_s associated to a dihedral system, which is in turn always isomorphic to the *odd part* of a *Verlinde algebra* associated to the Lie group $SU(2)$ (see Definition 5.3.3).

Theorem C. *Let (W, S) be a Coxeter system and let $s \in S$. Suppose J_s is a fusion ring for some $s \in S$. Then there exists a dihedral Coxeter system (W', S') such that $J_t \cong J_{s'}$ as based rings for all $t \in S$ and for both $s' \in S'$.*

In our second set of results, we focus on certain specific Coxeter systems (W, S) whose Coxeter diagram involves edges of weight ∞, and show that for suitable choices of $s \in S$, J_s is isomorphic to a *free fusion ring* in the sense of [BV09]. A free fusion ring can be described in terms of the data of its underlying *fusion set*, and we describe these data explicitly for each free fusion ring J_s in our examples.

Furthermore, each free fusion ring we discuss is isomorphic to the Grothendieck rings of the category Rep($\mathbb{G}$) of representations of a known *partition quantum groups* $\mathbb{G}$, and we will identify the group $\mathbb{G}$ in all cases. Our main theorems appear as Theorem D and Theorem E in sections 7.3 and 7.4, but we omit their technical statements for the moment. We also highlight a common feature of rings of the form J_s and free free fusion rings, namely, that the products of basis elements for both types of rings are controlled "locally and inductively". We will explain this more precisely and study its consequence for an example in Section 7.5.

All the results mentioned above rely heavily on the following theorem, which says that a combinatorial "factorization" of a reduced word of an element into its *dihedral segments* (see Definition 2.4.2) carries over to a factorization of basis elements in J_C.

Theorem F. (Dihedral factorization) *Let x be the reduced word of an element in C, and let $x_1, x_2, \cdots, x_l$ be the dihedral segments of x. Then*

$$t_x = t_{x_1} \cdot t_{x_2} \cdot \cdots \cdot t_{x_l}.$$

The rest of the article is organized as follows. We review some preliminaries about Coxeter systems in Section 2.1 - 2.3 and study the the subregular cell in Section 2.4. In Chapter 3, we review some basic facts about Hecke algebras. In particular, we recall various facts about Hecke algebras of dihedral groups in Section 3.3. In Chapter 4, we define the algebras J, J_C and $J_s (s \in S)$ and explain how J_C and $J_s(s \in S)$ appear as based rings. We prove Theorem F in Chapter 5 and demonstrate how it can be used to compute products of basis elements in J_C. In Chapter 6, we prove our results on the connections between J_C and Coxeter diagrams. Finally, we discuss our second set of results in Chapter 7, where we prove that certain rings J_s are free fusion rings and highlight a common feature shared by rings of the form J_s and free fusion rings.

CHAPTER II

COXETER SYSTEMS

In this chapter we review the basic theory of Coxeter systems relevant to this article. Out main references are [BB05] and [Lus14].

2.1. Basic Notions

A Coxeter group is a group with a special form of presentation: for any set S and any map $m : S \times S \to \{1, 2, \cdots, \infty\}$ such that $m(s, s) = 1$ and $m_{s,t} = m_{t,s} \geq 2$ for all distinct elements $s, t \in S$, we may define a group W by the presentation

$$W = \langle S \mid (st)^{m(s,t)} = 1,\ \forall s, t \in S\rangle. \tag{2.1.1}$$

Any group arising this way is called a *Coxeter group*, and the pair (W, S) is called a *Coxeter system*. Throughout this article, we shall assume the generating set S is finite for all our Coxeter systems. If W is a finite group, we say (W, S) is a *finite* Coxeter system.

Example 2.1.1. (Dihedral groups) Let $n \in \mathbb{Z}_{\geq 3}$ and let (W, S) be the Coxeter system with $S = \{s, t\}$ and $W = \langle s, t \mid s^2 = t^2 = (st)^n = 1\rangle$. Then W is isomorphic to the dihedral group D_n of order $2n$, the group of symmetries of a regular n-gon P. To see this, let c be the center of P, let d a vertex of P, and let e be the midpoint of an edge incident to d. Let s' and t' be reflections with respect to the two lines going through c, d and through c, e, respectively. Then s', t' are involutions since they are reflections. Since the two lines form an angle of π/n, $s't'$ is rotation at an angle of

$2\pi/n$, hence $(s't')^n = 1$. It follows that the map $s \mapsto s', t \mapsto t'$ extends uniquely to a group homomorphism $\varphi : W \to D_n$. The map is surjective since s', t' generate D_n, and a moment's thought reveals that $|W| \leq 2n = |D_n|$, therefore φ must be an isomorphism.

Example 2.1.2. (Symmetric groups) Let $n \in \mathbb{Z}_{\geq 2}$, $S = \{s_1, s_2, \cdots, s_{n-1}\}$, and let W be the Coxeter group generated by S subject to the relations $s_i^2 = 1$ for all i, $(s_i s_j)^3 = 1$ for all i, j with $|i - j| = 1$, and $(s_i s_j)^2 = 1$ for all i, j with $|i - j| > 1$. Then W is isomorphic to the symmetric group S_n. More precisely, let s_i' be the i-th basic transposition $(i, i+1)$ in S_n, then it is straightforward to check that the map $s_i \mapsto s_i'$ extends to a group isomorphism from W to S_n.

Example 2.1.3. (Weyl groups) The Weyl group of a *root system* ([Hum90]) is a Coxeter group. As we shall see, Weyl groups constitute the majority of finite Coxeter groups (see [BB05]).

Remark 2.1.4. Dihedral groups may be viewed as the simplest interesting Coxeter groups in that their generating set contains only 2—the smallest interesting number of—generators. Consequently their theory, including that of their Hecke algebras, are relatively easy to understand. A main theme of this article is to extrapolate our knowledge about dihedral groups to other general Coxeter groups. Symmetric groups are also particularly nice Coxeter groups, thanks to their simple combinatorial realizations. We will frequently come back to these dihedral groups and symmetric groups to illustrate the general theory of Coxeter groups.

The data of a Coxeter system (W, S) can be efficiently encoded via a *Coxeter diagram* G. By definition, G is the loopless, weighted, undirected graph (V, E) with vertex set $V = S$ and with edges E given as follows. For any distinct $s, t \in S$,

$\{s,t\}$ forms an edge in G exactly when $m(s,t) \geq 3$, whence the weight of the edge is $m(s,t)$. When drawing a Coxeter graph, it is conventional to leave edges of weight 3 unlabeled. We call edges of weight 3 *simple*, and we say (W,S) is *simply-laced* if all edges of G are simple.

We call a Coxeter system (W,S) *irreducible* if its Coxeter graph G is connected. This terminology comes from the following fact. If G is not connected, then each connected component of G encodes a Coxeter system. Since $m(s,t) = 2$ for any vertices s,t selected from different connected components of G, and since $s^2 = t^2 = 1$ now that $m(s,s) = m(t,t) = 1$, we have $st = ts$. This means that the Coxeter groups corresponding to the connected components commute with each other, so W is isomorphic to the direct product of these Coxeter groups and hence "reducible". That said, for most purposes we may study only irreducible Coxeter systems.

We say two Coxeter systems (W,S) and (W',S') are isomorphic if their Coxeter graphs are isomorphic, i.e., if there is a bijection $\varphi : S \to S'$ such that $m(s,t) = m(\varphi(s),\varphi(t))$ for all $s,t \in S$. It is possible to have $W \cong W'$ as abstract groups without the systems (W,S) and (W',S') being isomorphic. For example, with $S = \{s,t\}$ and $m(s,t) = 6$ we get a Coxeter presentation of the dihedral group D_6, but D_6 is isomorphic to the direct product of the symmetric S_3 and the group C_2 of order 2, so it has a Coxeter presentation with $S' = \{s,t,u\}$ and $m(s,t) = 3$ and $m(s,u) = m(t,u) = 2$. This means that when speaking of a Coxeter system, one should technically specify not only the group W but also the generating set S. However, when S is tacitly understood, we often only mention W. This will be the case throughout the article.

There is a well-known classification of all finite irreducible Coxeter systems. Their Coxeter graphs are shown in Figure 2.1.1. Note that the finite Coxeter groups are exactly all the Weyl groups plus the groups H_3, H_4 and the dihedral groups $I_2(m)$ where $m = 5$ or $m \geq 7$.

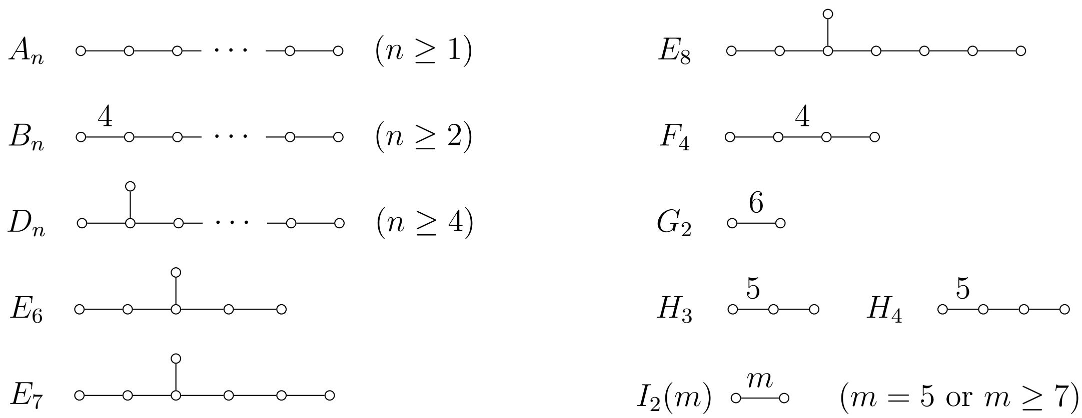

Figure 2.1.1: Classification of finite irreducible Coxeter systems.

We end the section with a few more definitions. Let (W, S) be a Coxeter system, and let $\langle S \rangle$ be the free monoid generated by S. It is natural to think of elements in W as represented by elements of $\langle S \rangle$, or *words* or *expressions* in the alphabet S. For $w \in W$, we define the *length* of w in W, written $l(w)$, to be the minimal length of a word representing w, and we call any such minimal-length word a *reduced word* or *reduced expression* of w. As we shall soon see, reduced words lie at the heart of the combinatorics of Coxeter groups.

Example 2.1.5. Let W be the symmetric group S_n. Denote each element $w \in W$ by its one-line notation $w = [w(1), w(2), \cdots, w(n)]$, define an inversion of w to be a pair of numbers $i, j \in [n]$ such that $i < j$ but $w(i) > w(j)$, and let $\mathrm{inv}(w)$ denote the number of inversions of w. Then it is well known that $\mathrm{inv}(w) = l(w)$. One may prove this by induction on $\mathrm{inv}(w)$.

2.2. Three Representations

Let (W, S) be a Coxeter system. We review some basic facts about words in W in this section. The main theme of the section is that representations of Coxeter groups often play a large role in establishing these facts. This will be the case even for the fact that distinct letters of S are distinct as elements in W, which one would certainly hope is true!

To describe the first representation, let $C_2 = \{-1, +1\} = \langle -1 \rangle$, the cyclic group of order 2. It is clear from Equation 2.1.1 that the map sending every $s \in S$ to -1 extends to a unique group homomorphism $\mathrm{sgn} : W \to C_2$ with $\mathrm{sgn}(w) = (-1)^{l(w)}$. Viewing sgn as a map from W to $\mathrm{GL}_1(\mathbb{R})$, we call sgn the *sign representation* of W. Note that for $s \in S$ and $w \in W$, we have $l(sw) \neq l(w)$ as $\mathrm{sgn}(sw) \neq \mathrm{sgn}(w)$. Meanwhile, by definition of l, $l(sw)$ and $l(w)$ can differ by at most 1. We have just proved the following (the proof for the second claim is similar).

Proposition 2.2.1. *For any $s \in S$ and $w \in W$, either $l(sw) = l(w) - 1$ or $l(sw) = l(w) + 1$; similarly, either $l(ws) = l(w) - 1$ or $l(ws) = l(ws) + 1$.*

A second representation of W is the *geometric representation*. It refers to the map σ in the following theorem.

Proposition 2.2.2 ([Lus14], Proposition 1.3)**.** *Let E be the $\mathbb{R}$-vector space with basis $(e_s)_{s \in S}$. For each $s \in S$, define a linear map $\sigma_s : E \to E$ by $\sigma_s(e_t) = e_t + 2\cos\frac{\pi}{m(s,t)} e_s$ for all $t \in S$. Then*

(a) There is a unique homomorphism $\sigma : W \to \mathrm{GL}(E)$ with $\sigma(s) = \sigma_s$ for all $s \in S$. In particular, $s \neq 1$ for any $s \in S$.

(b) If $s \neq t$ in S, then st has order $m(s,t)$ in W. In particular, $s \neq t$ in W.

Here, one may roughly think of E as endowed with a certain geometry where each pair $e_s, e_{s'}$ of basis vectors form an "angle" of $\pi - \frac{\pi}{m(s,s')}$, and think of σ_s as "reflection" across the hyperplane perpendicular to e_s for each $s \in S$, so that $\sigma_s \sigma_{s'}$ becomes rotation at an angle of $\frac{2\pi}{m(s,s')}$, much like the rotation $s't'$ in Example 2.1.1. (There are some subtleties with this argument; see [Hum90], Section 5.3.)

In light of Proposition 2.2.2, we shall henceforth identify S with a subset of W. We call S the set of *simple reflections* in W. Since $m(s,s) = 1$ by definition, $s^2 = 1$ by Equation 2.1.1. For $s, t \in S$, the defining relation $(st)^{m(s,t)}$ is then equivalent to

$$sts \cdots = tst \cdots , \tag{2.2.2}$$

where both sides are words that alternate in s and t and have length $m(s,t)$. We call such a relation a *braid relation*. The relation means that whenever one side of Equation 2.2.2 appears consecutively in a word representing an element in W, we may replace it with the other side of the equation and obtain a different expression of the same element. We call such a move a *braid move*. Thus, if two words can be obtained from each other by braid moves, then they express the same element in W. It turns out that the converse is also true:

Proposition 2.2.3 (Matsumoto's Theorem; see, e.g., [Lus14], Theorem 1.9)**.** *Any two reduced words of a same element in W can be obtained from each other by performing a finite sequence of braid moves.*

Example 2.2.4. Let W be the dihedral group with Coxeter generators $S = \{1, 2\}$ and $m(1,2) = M$ for some $M \geq 3$. For $0 \leq k \leq M$, let 1_k and 2_k be the alternating words $121 \cdots$ and $212 \cdots$ of length k, respectively. In particular, set $1_0 = 2_0 = 1_W$, the identity element of W. By Proposition 2.2.2 and the braid relations, if $M < \infty$, then W consists of the $2M$ elements $1_k, 2_k$ where $0 \leq k \leq M$, and they are all distinct except the equalities $1_0 = 2_0$ and $1_M = 2_M$; if $M = \infty$, then W consists of the elements $1_k, 2_k$ for all $k \in \mathbb{Z}_{\geq 0}$, and they are all distinct except for $1_0 = 2_0$. Moreover, it is clear that $l(1_k) = l(2_k) = k$ for all $0 \leq k \leq M$.

A third representation of W is the *permutation representation*. For its definition, see Section 1.3 of [BB05]. The representation serves as a key ingredient in the proof of the important *Strong Exchange Property* of Coxeter groups, which is in turn a key ingredient in the proof of Proposition 2.2.3. To state the property, define the set of *reflections* of W as the set T given by

$$T = \{wsw^{-1} : s \in S, w \in W\}. \tag{2.2.3}$$

Proposition 2.2.5 (Strong Exchange Property, [BB05], Theorem 1.4.3)**.** *Let $w = s_1 s_2 \cdots s_k$ ($s_i \in S$) and $t \in T$. If $l(tw) < l(w)$, then $tw = s_1 \cdots \hat{s}_i \cdots s_k$ for some $1 \leq i \leq k$, where $s_i \cdots \hat{s}_i \cdots s_k$ stands for the expression obtained by deleting s_i from $s_1 s_2 \cdots s_k$.*

For $x \in W$, define the *left descent set* and *right descent set* of x to be the sets

$$\mathcal{L}(x) = \{s \in S : l(sx) < l(x)\},$$

$$\mathcal{R}(x) = \{s \in S : l(xs) > l(x)\},$$

respectively. Then the Strong Exchange Property allows us to characterize the descent sets in terms of reduced words:

Proposition 2.2.6 (Descent criterion, [BB05], Corollary 1.4.6)**.** *Let $s \in S$ and $x \in W$. Then* *(a) $s \in \mathcal{L}(x)$ if and only if x has a reduced word beginning with s;*

(b) $s \in \mathcal{R}(x)$ if and only if x has a reduced word ending with s.

Proof. We first prove (a): the "if" implication is obvious; conversely, let $s_1 s_2 \cdots s_k$ be a reduced expression for w. By the Strong Exchange Property, we have $sw = ss_1 \cdots \hat{s}_i \cdots s_k$ for some $1 \leq i \leq k$. Hence $w = ss_1 \cdots \hat{s}_i \cdots s_k$, and the expression on the right must be reduced, so we are done. The proof of (b) is similar. $\square$

Example 2.2.7. Let $W = S_n$ as in Example 2.1.5. Note that any $1 \leq i \leq n-1$, multiplying an element by s_i on the left swaps the values i and $i+1$ in its one-line notation, while multiplying an element by s_i on the right swaps the values at i-th and $(i+1)$-th positions in the one-line notation. In light of Example 2.1.5, it follows that $s_i \in \mathcal{L}(w)$ if and only if $i+1$ appears to the left of i in the one-line notation of w, and $s_i \in \mathcal{R}(w)$ if and only if the value at the i-th position of w is larger than the value at the $(i+1)$-th position.

2.3. The Bruhat Order

Let (W, S) be a Coxeter system, and recall the definition of the set T of reflections from Equation 2.2.3. We may define a binary relation $\prec$ on W by declaring that $x \prec y$ for $x, y \in W$ if and only if $x = ty$ and $l(x) < l(y)$ for some $t \in T$, then take the reflexive and transitive closure of $\prec$ to form a partial order on W. This partial order is the important *Bruhat order* of W; we denote it by $\leq$. Note that by definition and Corollary 2.2.6, we have $sw < w$ for $s \in S, w \in W$ if and only if $l(sw) = l(w) - 1$, if and only if $s \in \mathcal{L}(w)$.

Remark 2.3.1. The name of the Bruhat order originates from the fact that when a Coxeter group W is the Weyl group of a complex, simply-connected, semisimple

Lie group G, W indexes the *Bruhat cells* in the flag variety associated to G, and the Bruhat order on W governs the containment of Bruhat cells inside Schubert varieties (closures of Bruhat cells) in the flag variety ([Che94]).

Example 2.3.2. Let W be the symmetric group S_n. Reflections in W, being conjugates of the basic transpositions, are just transpositions. Thus, in light of Example 2.2.7, $u \prec w$ for $u, w \in W$ if and only if the one-line notation of u can be obtained by swapping two values in the one-line notation of w that appear "out of order", i.e., with the larger value appearing to the left of the smaller value. Consequently, we have $u \leq w$ if and only if u can be obtained from w by a sequence of such moves that "rectify" inversions.

The Bruhat order has a convenient characterization in terms of reduced words: define a *subword* of any word $s_1 s_2 \cdots s_k \in S^*$ to be a word of the form $s_{i_1} s_{i_2} \cdots s_{i_l}$ where $1 \leq i_1 < i_2 < \cdots < i_l \leq k$, then:

Proposition 2.3.3 (Subword Property; [BB05], Corollary 2.2.3)**.** *Let $x, y \in W$. Then the following are equivalent:*

(a) $x \leq y$;

(b) every reduced word for y contains a subword that is a reduced word for x;

(c) some reduced word for y contains a subword that is a reduced word for x.

This immediately implies the following:

Corollary 2.3.4 ([BB05], Corollary 2.2.5)**.** *The map $w \mapsto w^{-1}$ on W is an automorphism of the Bruhat order, i.e., $u \leq w$ if and only if $u^{-1} \leq w^{-1}$.*

2.4. The Subregular Cell

Let (W, S) be an arbitrary Coxeter system. We study a particular subset of W called the subregular cell in this section. As mentioned in the introduction, the main object of study in this article will be a subalgebra of the asymptotic Hecke algebra of (W, S) corresponding to this set.

Definition 2.4.1. We define the *subregular cell* of W to be the set of all non-identity elements in W with a unique reduced words. We denote this cell by C and call its elements the *subregular elements* of W. For each $s \in S$, we denote by Γ_s the set of subregular elements of W whose reduced word ends in s (i.e., whose rightmost letter is s).

The word "cell" in the definition refers to the fact that C is a *Kazhdan-Lusztig cell.* We will elaborate on this fact in Section 3.2.

Observe that by Proposition 2.2.3, an element $w \in W$ has a unique reduced word if and only if one cannot apply any braid moves on any given reduced word of w. We make this more precise below.

Definition 2.4.2 (Dihedral segments)**.** For any word $x \in \langle S \rangle$ where no letter $s \in S$ appears consecutively, we define the *dihedral segments* of x to be the maximal contiguous subwords of x involving two letters.

For example, suppose $S = \{1, 2, 3\}$ and $x = 121313123$, then x has dihedral segments $x_1 = 121, x_2 = 13131, x_3 = 12, x_4 = 23$. We may think of breaking a word into its dihedral segments as a "factorization" process.

Clearly, the dihedral segments of a word must alternate in two letters and take the form $sts\cdots$ for some $s, t \in S$. It is thus convenient to have the following notation.

Definition 2.4.3. For $s, t \in S$ and $k \in \mathbb{N}$, let $(s,t)_k$ denote the alternating word $sts\cdots$ of length k. In particular, we take $(s,t)_0$ to be the empty word $\emptyset$ and $(s,t)_1$ to be the word s.

Definition 2.4.4. For $s, t \in S$ and $k \in \mathbb{N}$, we call $(s,t)_k$ *saturated* if $k \geq m(s,t)$.

Thus, we may apply a braid move on a word in $\langle S \rangle$ whenever the word contains a saturated dihedral segment. The following is now clear by Proposition 2.2.3.

Proposition 2.4.5 (Subregular Criterion)**.** *Let $x \in \langle S \rangle$. Then x is the reduced word of an element in C if and only no letter in S appears consecutively in x and no dihedral segment of x is saturated.*

Let G be the Coxeter diagram of (W,S), and recall that a walk on a graph or directed graph is a sequence of vertices $(v_1, v_2, \cdots, v_k)$ such that $\{v_i, v_{i+1}\}$ or (v_i, v_{i+1}) is a edge in the graph for all $1 \leq i \leq q-1$, respectively. Now, let $w = s_1 s_2 \cdots s_q$ be the reduced word of any element in C, then Proposition 2.4.5 implies that $m(s_i, s_{i+1}) \geq 3$, hence $\{s_i, s_{i+1}\}$ forms an edge in G, for all $1 \leq i \leq q-1$. We may therefore naturally visualize w as a walk $P(w) := (s_1, s_2, \cdots, s_q)$ on G.

Besides C, we will also be interested in the sets of the form $\Gamma_s \cap \Gamma_s^{-1}$ where $s \in S$. The visualizations mentioned above allows us to easily discover when C and $\Gamma_s \cap \Gamma_s^{-1}$ $(s \in S)$ are finite in terms of Coxeter diagrams.

Theorem 2.4.6. *Let (W,S) be an irreducible Coxeter system, and let $G = (V,E)$ be its Coxeter diagram. Then the following conditions are equivalent.*

(a) G is a tree, and at most one edge in G has a weight greater than 3.

(b) The subregular cell C is finite.

(c) The set $\Gamma_s \cap \Gamma_s^{-1}$ is finite for all $s \in S$.

(d) *The set* $\Gamma_s \cap \Gamma_s^{-1}$ *is finite for some* $s \in S$.

Proof. Clearly (b) $\Rightarrow$ (c) $\Rightarrow$ (d), so it suffices to show (a) $\Rightarrow$ (b) and (d) $\Rightarrow$ (a).

Assume (a). Let w be the reduced word of a subregular element, and let $P(w)$ be its corresponding walk on G as before. We will show C is finite by examining what $P(w)$ could be like. We will use some standard graph-theoretical terminology such as *paths*, *descendants* and *ancestors*; their definitions can all be found in Section 6.1 of [BM08].

Let $\{s, t\}$ be an edge of maximal weight in G (so either all edges in G have weight 3, or $\{s, t\}$ is the only edge in G with weight $m(s, t) > 3$). It is well-known that since G is a tree, removing the edge $\{s, t\}$ from G results in a graph with two connected components where one component is a tree containing s and the other a tree containing t. Call the first tree the s-tree and the second tree the t-tree, and view each of the trees as a rooted tree with root s or t. By assumption, an edge $\{u, v\}$ in either tree must be of weight $m(u, v) = 3$. Now, since w cannot contain any saturated dihedral segments by Proposition 2.4.5, $P(w)$ cannot contain any contiguous subsequence of the form (u, v, u) where one of u or v is not in the set $\{s, t\}$, therefore $P(w)$ must be one of the following forms.

(1) $P(w)$ lies entirely in the s-tree or the t-tree. In this case, $P(w)$ must be a path from a vertex to one of its *descendants* in its tree or a path from a vertex to one of its *ancestors* in its tree. Let us call the first type of path a *downward* path and the second type an *upward* path.

(2) $P(w)$ starts with a vertex in one of the two trees, travels along an upward path in its tree to the tree's root, travels back and forth along the edge $\{s, t\}$ a finite number of times, then finally travels along a downward path in one of

the trees and terminates. Here, $P(w)$ can travel back and forth the edge $\{s,t\}$ at most $m(s,t)-2$ times since w cannot contain $(s,t)_{m(s,t)}$ as a contiguous subword.

Since both the s-tree and t-tree contain only finitely many distinct paths, it follows that there are finitely many possibilities for $P(w)$, therefore C is finite.

It remains to show that (d) implies (a). Let $s \in S$ be such that $\Gamma_s \cap \Gamma_s^{-1}$ is finite. Since (W,S) is assumed to be irreducible, G is connected, so to show G is a tree it suffices to show that it contains no cycles. Now, suppose G contains a cycle $C = (v_1, v_2, \cdots, v_k)$ with $v_1 = v_k$, then we may pick a shortest path $P = (s = u_1, u_2, \cdots, u_l)$ among the paths connecting s to a vertex in C. Say $u_l = v_i$ for some $1 \le i \le k$. For each $n > 1$, define $(P \circ C^n \circ P^{-1})$ to be the walk

$$(s = u_1, u_2, \cdots, u_l = v_i, v_{i+1}, \cdots, v_k, v_1, \cdots, v_i, \cdots, v_i = u_l, u_{l-1}, \cdots, u_1 = s) \tag{2.4.4}$$

where C is traversed n times between the first and last appearance of v_i. Then for each n, $(P \circ C^n \circ P^{-1})$ must be of the form $P(w_n)$ where w_n is the reduced word of a subregular element in $\Gamma_s \cap \Gamma_s^{-1}$. This contradicts the assumption that $\Gamma_s \cap \Gamma_s^{-1}$ is finite, therefore G must be a tree.

Similarly, G cannot contain two edges of weight greater than 3 since otherwise, G must contain distinct vertices s_1, s_2, s_3, s_4 for which $m(s_1, s_2) > 3$, $m(s_3, s_4) > 3$ and for which G contains a path of the form $P = (s_1 = v_1, s_2 = v_2, v_3, \cdots, v_{k-1} = s_3, v_k = s_4)$. In this case, let P be a shortest path from s to a vertex in this path, consider the circuit

$$C = (v_1, v_2, v_3, \cdots, v_{k-1}, v_k, v_{k-1}, v_{k-2}, \cdots, v_2, v_1),$$

and define the walks $P \circ C^n \circ P^{-1}$ as in Equation 2.4.4. Then for each n, $P \circ C^n \circ P^{-1}$ must be of the form $P(w_n)$ where w_n is the reduced word of a subregular element in $\Gamma_s \cap \Gamma_s^{-1}$. This again contradicts the assumption that $\Gamma_s \cap \Gamma_s^{-1}$ is finite, so we are done. $\square$

In addition to thinking of C as walks on G, we may encode C with a directed graph. We describe the construction of the graph below for future use. To begin, associate to any word $w = s_1 s_2 \cdots s_q \in \langle S \rangle$ a sequence of words $w_i := s_1 s_2 \cdots s_i$ for $1 \leq i \leq q$, and let d_i be the last dihedral segment of w_i. Then by Proposition 2.4.5, w is the reduced word of a subregular element if and only if d_i is not saturated for any $1 \leq i \leq q$. In other words, to write down the reduced word of a subregular element we just need to ensure that we do not create any saturated dihedral segment as we write down the letters successively. To keep track of the d_i's, note that to complete the traversal of the walk $P(w_{i+1})$ after the walk $P(w_i)$ for some $1 \leq i \leq q-1$, we either (i) travel along the same edge last traversed in $P(w_i)$, so that d_{i+1} is an extension of d_i, or (ii) travel along an edge different from the last edge traversed in $P(w_i)$, so that d_{i+1} involves a different set of letters than those involved in d_i. This motivates the following definition.

Definition 2.4.7 (Subregular graph)**.** Let $F, L : S^* \setminus \{\emptyset\} \to S$ be the functions that send any nonempty word $w = s_1 s_2 \cdots s_k \in \langle S \rangle$ to its first and last letter s_1 and s_k, respectively, and let $D = (V, E)$ be the directed graph where

(a) $V = \{(s,t)_k : s, t \in S, 0 < k < m(s,t)\}$,

(b) E consists of directed edges (v, w) pointing from v to w, where

  (i) either $v = (s,t)_{k-1}$ and $w = (s,t)_k$ for some $s, t \in S$, $0 < k < m(s,t)$,

(ii) or v and w are alternating words that each involves two letters, they contain different sets of letters, but $L(v) = F(w)$.

We call the graph D the *subregular graph* of (W, S).

By the paragraph preceding Definition 2.4.7, the map $w \mapsto Q(w) := (d_1, \cdots, d_q)$ establishes a bijection between C and walks on D that start with a vertex of the form $s \in S$, with the restriction $k < m(s,t)$ in (a) ensuring no d_i is saturated.

Example 2.4.8. Let (W, S) be the Coxeter system whose Coxeter diagram is the triangle shown in Figure 2.4.2. The subregular graph D of (W, S) is the directed graph in Figure 2.4.3. Elements of C correspond to walks on D starting with one of the three top vertices.

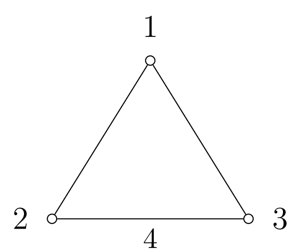

Figure 2.4.2: A Coxeter system (W, S).

Note that we can produce D algorithmically in the following way. First, draw the vertices $s \in S$. Second, find the edges emanating from the vertices just drawn using the definition of E from Definition 2.4.7, draw the target vertices of these edges if they are not already drawn, and draw these edges. If all the target vertices have been drawn, draw the edges and halt. Third, repeat the second step.

Certain subgraphs of the subregular graph can be used to study the sets $\Gamma_s \cap \Gamma_s^{-1}$ $(s \in S)$. By Definition 2.4.1, this set consists of the subregular elements whose

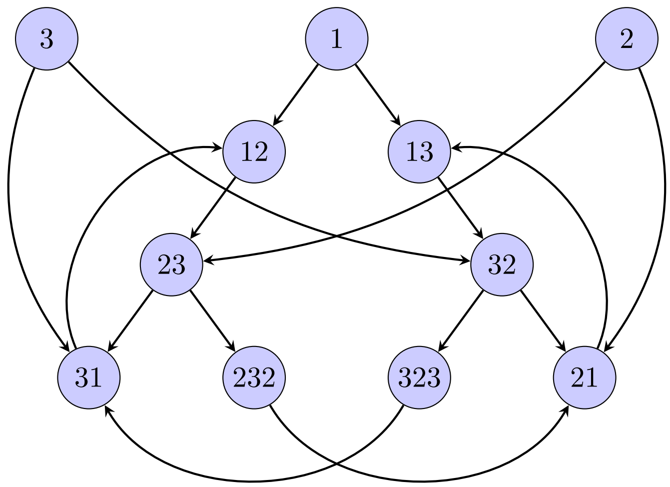

Figure 2.4.3: The subregular graph D of (W, S).

reduced words both start and end with s, so the set $\Gamma_s \cap \Gamma_s^{-1}$ corresponds bijectively with the walks in D of the form $(v_1, \cdots, v_k)$ where $v_1 = s$ and v_k ends with s. Let us call such a walk an *s-walk*. Since s-walks must start at the vertex s of D, to find all s-walks in D it suffices to consider the subgraph of D produced by using the algorithm described in the last paragraph but starting with drawing only s in the first step. We denote this subgraph by D_s.

Example 2.4.9. Let (W, S) be the Coxeter system in Example 2.4.8. The directed graphs D_1 and D_2 are shown in Figure 2.4.4 and Figure 2.4.5 (D_3 is similar to D_2 by the symmetry of the Coxeter diagram of (W, S)). For $s \in \{1, 2\}$, the elements of $\Gamma_s \cap \Gamma_s^{-1}$ correspond bijectively to the paths on D_s that start at the top diamond-shaped vertex and ends at one of the other diamond-shaped vertices.

Remark 2.4.10. In Section 3.7 of the paper [Lus83], where he first showed that the sets C and $\Gamma_s (s \in S)$ form KL cells, Lusztig associated a direct graph Γ'_s to the left cell Γ_s for each $s \in S$. Our treatment of the graphs D and D_s can be viewed as

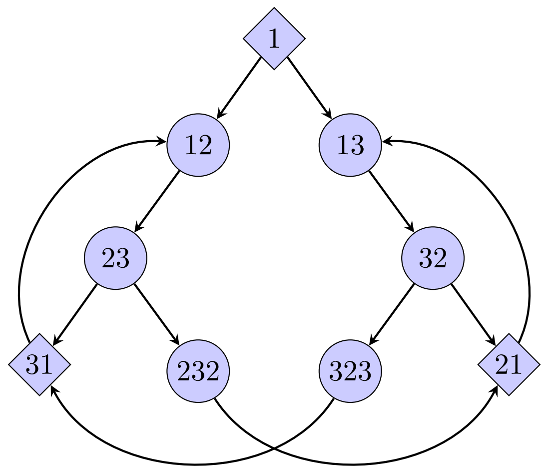

Figure 2.4.4: The graph D_1 for (W, S).

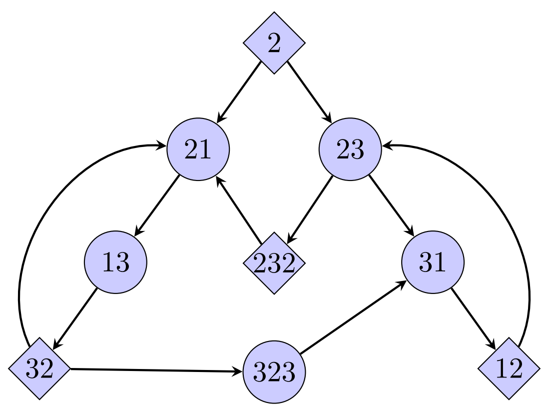

Figure 2.4.5: The graph D_2 for (W, S).

a reformulation and expansion of the same idea.

CHAPTER III

HECKE ALGEBRAS

In this chapter we review some basic facts about Hecke algebras relevant to this article. Out main reference is [Lus14]. In particular, we define the Hecke algebras over the ring $\mathbb{Z}[v, v^{-1}]$ and use the normalization seen in [Lus14], where the quadratic relations are $(T_s - v)(T_s + v^{-1}) = 0$ for all simple reflections s.

Throughout this chapter, let (W, S) be an arbitrary Coxeter system unless otherwise specified, and let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$.

3.1. Hecke Algebras and Their Kazhdan-Lusztig Bases

Following [Lus14], we define the *Iwahori-Hecke algebra* (or simply the *Hecke algebra* for short) of (W, S) to be the unital $\mathcal{A}$-algebra H generated by the set $\{T_s : s \in S\}$ subject to the relations

$$(T_s - v)(T_s + v^{-1}) = 0 \tag{3.1.1}$$

for all $s \in S$ and the relations

$$T_s T_t T_s \cdots = T_t T_s T_t \cdots \tag{3.1.2}$$

for all $s, t \in S$, where both sides have $m(s, t)$ factors. Note that when we set $v = 1$, the quadratic relation reduces to $T_s^2 = 1$, so H is isomorphic to the group algebra $\mathbb{Z}W$ of W by the braid relations in W and (3.1.2). In this sense, H is often called a *deformation* of $\mathbb{Z}W$.

Let $x \in W$, let $s_1 s_2 \cdots s_k$ be any reduced word of x, and set $T_x := T_{s_1} \cdots T_{s_k}$.

Thanks to Proposition 2.2.3 and Equation 3.1.2, this is well-defined, i.e., different reduced words of x produce the same element in H. The following is well known.

Proposition 3.1.1 ([Lus14], Proposition 3.3). *The set $\{T_x : x \in W\}$ is an $\mathcal{A}$-basis of H.*

The basis $\{T_x : x \in W\}$ is called the *standard basis* of H. Note that T_s is invertible, with $T_s^{-1} = T_s - (v - v^{-1})$, for all $s \in S$. Further, it is easy to check that the map sending $v^n \mapsto v^{-n}$ for all $n \in \mathbb{Z}$ and sending each T_s to T_s^{-1} extends uniquely to a $\mathcal{A}$-semilinear ring homomorphism $\bar{\ }: H \to H$. Now let

$$\mathcal{A}_{<n} = \oplus_{m:m<n}\mathbb{Z}v^m, \quad \mathcal{A}_{\leq n} = \oplus_{m:m\leq n}\mathbb{Z}v^m$$

for all $n \in \mathbb{Z}$, and let

$$H_{<0} = \oplus_{w\in W}\mathcal{A}_{<0}T_w, \quad H_{\leq 0} = \oplus_{w\in W}\mathcal{A}_{\leq 0}T_w.$$

We have:

Theorem 3.1.2 ([KL79], Theorem 1.1; [Lus14], Theorem 5.2). *(a) For any $w \in W$, there exists a unique element $c_w \in H_{\leq 0}$ such that $\bar{c}_w = c_w$ and $c_w = T_w \bmod H_{<0}$.*
(b) The set $\{c_w : w \in W\}$ is an $\mathcal{A}_{\leq 0}$-basis of $H_{\leq 0}$ and an $\mathcal{A}$-basis of H.

The set $\{c_w : w \in W\}$ is the famous *Kazhdan-Lusztig basis* of H. The transition matrices between the two bases give rise to the *Kazhdan-Lusztig polynomials*. By definition, they are the elements $p_{x,y} \in \mathcal{A}$ for which

$$c_y = \sum_{x\in W} p_{x,y}T_x \tag{3.1.3}$$

for all $x, y \in W$.

Notation 3.1.3. From now on we will mention the phrase "Kazhdan-Lusztig" numerous times. We will often abbreviate it to "KL".

Example 3.1.4. Let $s \in S$. The element $T_s + v^{-1}$ satisfies

$$\overline{T_s + v^{-1}} = T_s^{-1} + v = T_s - (v - v^{-1}) + v = T_s + v^{-1},$$

hence it follows from the characterization of the KL basis that $c_s = T_s + v^{-1}$. Consequently, $p_{e,s} = v^{-1}$ for the identity element e of W.

We should mention that many different normalizations of Hecke algebras exist in the literature. In particular, in the paper [KL79] where the KL basis was first introduced, the Hecke algebra H is defined over the ring $\mathbb{Z}[q, q^{-1}]$ and the defining quadratic relations for H takes the form

$$(T_s - q)(T_s + 1) = 0 \qquad (s \in S). \tag{3.1.4}$$

In this setting, the involution $^{-}\colon H \to H$, the KL basis, and the KL polynomials can still be defined, and the KL polynomials would actually be elements of the polynomial ring $\mathbb{Z}[q]$. In our setting, however, the KL polynomials turn out to be elements of $\mathbb{Z}[v^{-1}]$.

KL bases and KL polynomials enjoy remarkable positivity properties. For $x, y, z \in W$, let $p_{y,w}$ be the KL polynomial defined in Equation 3.1.3, and let $h_{x,y,z}$ be the unique elements in $\mathcal{A}$ such that

$$c_x c_y = \sum_{z \in W} h_{x,y,z} c_z. \tag{3.1.5}$$

Then:

Theorem 3.1.5 (Positivity of the KL basis and polynomials; [EW14], Corollary 1.2)**.**

(a) $p_{x,y} \in \mathbb{Z}_{\geq 0}[v^{-1}]$ *for all* $x, y \in W$.

(b) $h_{x,y,z} \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$ *for all* $x, y, z \in W$.

As mentioned in the introduction, these facts, along with the *Kazhdan-Lusztig conjecture* (Conjecture 1.5 of [KL79]), were proved only recently by Elias and Williamson as a consequence of their proof of Soergel's conjecture. We refer the reader to their paper [EW14] and sections 8.5, 8.7 and 8.9 of [Hum08] for detailed account of the fascinating history of the theorem.

Remark 3.1.6. It is well-known that KL polynomials can be computed recursively with the aid of the so-called *R-polynomials.* This is explained in sections 4 and 5 of [Lus14], and example computations can be found in Chapter 5 of [BB05]. However, the computation is often very difficult to carry out in practice, even for computers, and the computation algorithm does not seem adequate for a proof of part (a) of the above theorem. See Section 8.3 of [Hum08] for more comments on computations.

We end this section by recalling a multiplication formula for KL basis elements in H. The formula will play an important role in the proof of Lemma 5.2.4 and Theorem F.

Proposition 3.1.7 (Multplication of KL-basis; [Lus14], Theorem 6.6, Corollary

6.7). *Let $x \in W$, $s \in S$, and let $\leq$ be the Bruhat order on W. Then*

$$c_s c_y = \begin{cases} (v+v^{-1})c_y & \text{if} \quad sy < y \\ c_{sy} + \sum_{x: sx<x<y} \mu_{x,y} c_x & \text{if} \quad sy > y \end{cases}$$

and

$$c_y c_s = \begin{cases} (v+v^{-1})c_y & \text{if} \quad ys < y \\ c_{ys} + \sum_{x: xs<x<y} \mu_{x^{-1},y^{-1}} c_x & \text{if} \quad ys > y \end{cases},$$

where $\mu_{w,w'}$ denotes the coefficient of v^{-1} in $p_{w,w'}$ for any $w, w' \in W$.

Here, the coefficients of the form $\mu_{w,w'}$ are called *μ-coefficients*. The μ-coefficients can be used to define representations of H via *W-graphs* ([KL79]). Since the elements c_s clearly generate H, the proposition means that essentially the μ-coefficients also govern the multiplication of KL basis elements in H.

Remark 3.1.8. It is straightforward to check that the map $T_s \mapsto T_s (s \in S)$ induces a unique $\mathcal{A}$-linear anti-involution $^\flat : H \to H$ that sends T_w to $T_{w^{-1}}$ for all $w \in W$, and that the anti-involution commutes with the involution $^-$ from Section 3.1. From this fact and Corollary 2.3.4, it follows that $c_w^\flat = c_{w^{-1}}$, $p_{y,w} = p_{y^{-1},w^{-1}}$, $\mu_{y,w} = \mu_{y^{-1},w^{-1}}$ and $h_{x,y,z} = h_{y^{-1},x^{-1},z^{-1}}$ for all $x, y, z, w \in W$. That said, we may obtain either formula in Proposition 3.1.7 from the other via a change of variable $y = w^{-1}$. These phenomena, along with Corollary 2.3.4, may be viewed as those of "left-right symmetry" in Coxeter groups and Hecke algebras (for w^{-1} is simply the left-right reverse of w for any word w).

3.2. Kazhdan-Lusztig Cells

We recall the definition of the *Kazhdan-Lusztig cells* of W and some relevant facts in this section.

For each $x \in W$, let $D_x : H \to \mathcal{A}$ be the linear map such that

$$D_x(c_y) = \delta_{x,y}$$

for all $y \in W$, where $\delta_{x,y}$ is the Kronecker delta symbol. For $x, y \in W$, write $x \prec_L y$ if $D_x(c_s c_y) \neq 0$ for some $s \in S$, and write $x \prec_R y$ if $D_x(c_y c_s) \neq 0$ for some $s \in S$. Define $\leq_L$ and $\leq_R$ to be the transitive and reflexive closures of $\prec_L$ and $\prec_R$, respectively, and define another partial order $\leq_{LR}$ by declaring that $x \leq_{LR} y$ if there exists a sequence $x = z_1, \cdots, z_n = y$ in W such that $z_i \prec_L z_{i+1}$ or $z_i \prec_R z_{i+1}$ for all $1 \leq i \leq n-1$. Finally, define $\sim_L$ to be the equivalence relations such that $x \sim_L y$ if and only if we have both $x \leq_L y$ and $y \leq_L x$, and define $\sim_R, \sim_{LR}$ similarly. The equivalence classes of $\sim_L, \sim_R$ and $\sim_L R$ are called the *left (Kazhdan-Lusztig) cells*, *right (Kazhdan-Lusztig) cells* and *2-sided (Kazhdan-Lusztig) cells* of W, respectively. Clearly, each 2-sided KL cell is a union of left cells as well as a union of right cells.

Since the elements c_s $(s \in S)$ generate H as an $\mathcal{A}$-algebra, the following is clear:

Proposition 3.2.1 ([Lus14], Lemma 8.2)**.** *Let $y \in W$. Then*

(a) The set $H_{\leq_L y} := \oplus_{x: x \leq_L y} \mathcal{A} c_x$ is a left ideal of H.

(b) The set $H_{\leq_R y} := \oplus_{x: x \leq_R y} \mathcal{A} c_x$ is a right ideal of H.

(c) The set $H_{\leq_{LR} y} := \oplus_{x: x \leq_{LR} y} \mathcal{A} c_x$ is a 2-sided ideal of H.

The following is also immediate by inspection of Proposition 3.1.7.

Proposition 3.2.2. *Let $x, y \in W$. Then*

(a) $x \prec_L y$ precisely when $x = y$, when $x = sy > y$ for some $s \in S \setminus \mathcal{L}(y)$, or when $x < y, \mathcal{L}(x) \not\subseteq \mathcal{L}(y)$ and $\mu_{x,y} \neq 0$.

(b) $x \prec_R y$ precisely when $x = y$, when $x = ys > y$ for some $s \in S \setminus \mathcal{R}(y)$, or when $x < y, \mathcal{R}(x) \not\subseteq \mathcal{R}(y)$ and $\mu_{x^{-1},y^{-1}} \neq 0$.

Given this alternative characterization of $\prec$, Corollary 2.3.4 now implies that $x \leq_L y$ if and only if $x^{-1} \leq_R y^{-1}$. We have just proved the following.

Proposition 3.2.3 ([Lus14], Section 8.1)**.** *The map $x \mapsto x^{-1}$ takes left cells in W to right cells, right cells to left cells, and 2-sided cells to 2-sided cells.*

The following theorem of Lusztig also uses the description of $\prec$ in Proposition 3.2.2 in its proof. It justifies the name "cell" for the set C in Definition 2.4.1.

Theorem 3.2.4 ([Lus83], Theorem 3.8)**.** *The set C is a 2-sided Kazhdan-Lusztig cell of W, and Γ_s is a left Kazhdan-Lusztig cell of W for each $s \in S$.*

We recall one more fact about cells for later use.

Proposition 3.2.5 ([Lus14], Section 14.1)**.** *For any $x \in W$, we have $x \sim_{LR} x^{-1}$.*

3.3. The Dihedral Case

To illustrate the theory developed in this chapter and for future reference, we discuss the KL basis, KL polynomials, and KL cells for dihedral Coxeter systems in this section. Most of the material in this section can be found in Chapter 7 of [Lus14].

Let (W, S) be a dihedral Coxeter system as in Example 2.2.4, with $S = \{1, 2\}$ and $m(s, t) = M$ for some $3 \leq M \leq \infty$. Recall that 1_k and 2_k stand for the alternating words $121\cdots$ and $212\cdots$ of length k for any $0 \leq k \leq M$, respectively, and note that $y < w$ in W if and only if $l(y) < l(w)$ by Proposition 2.3.3.

We first describe the KL basis W. Recall from Example 3.1.4 that $c_s = T_s + v^{-1}$ for any $s \in S$. Let

$$\gamma_w = \sum_{y: y \leq w} v^{l(y)-l(w)} T_y \tag{3.3.6}$$

for all $w \in W$. It is clear that $\gamma_e = 1 = c_e$, and it is straightforward to check that

$$\gamma_1 = c_1\gamma_e, \quad \gamma_2 = c_2\gamma_e, \quad \gamma_{12} = c_1\gamma_2, \quad \gamma_{21} = c_2\gamma_1, \tag{3.3.7}$$

$$\gamma_{1_{k+1}} = c_1\gamma_{2_k} - \gamma_{1_{k-1}}, \qquad \gamma_{2_{k+1}} = c_2\gamma_{1_k} - \gamma_{2_{k-1}} \tag{3.3.8}$$

for all $2 \leq k < M$. These equations allows us to show by induction that γ_w is invariant under the involution $^-$ and that $\gamma_w = T_w \mod H_{<0}$, therefore $\gamma_w = c_w$ for all $w \in W$.

From Equation 3.3.6, it now follows that

$$p_{y,w} = v^{l(y)-l(w)}$$

for any $y, w \in W$ such that $y < w$.

Let ${}_k1 := (1_k)^{-1} = \cdots$ and ${}_k2 = (2_k)^{-1}$ for each $0 \leq k \leq M$, and write $x \to y$ or $y \leftarrow x$ if $y \prec_L x$. Then equations (3.3.7) and (3.3.8) implies that the $\prec$ relations in W are as follows if $M < \infty$.

$${}_01 \to {}_11 \leftrightarrows {}_21 \leftrightarrows {}_31 \leftrightarrows \cdots \leftrightarrows {}_{M-1}1 \to {}_M1, \tag{3.3.9}$$

$${}_02 \to {}_12 \leftrightarrows {}_22 \leftrightarrows {}_32 \leftrightarrows \cdots \leftrightarrows {}_{M-1}2 \to {}_M2. \tag{3.3.10}$$

It follows that the sets $L_1 = \{e\}, L_2 = \{{}_k1 : 0 < k < M\}, L_3 = \{{}_k2 : 0 < k < M\}$ and $L_4 = \{{}_M1\}$ are the left cells of W if $M < \infty$. Combined with similar computations for the $\prec_R$ relations, we may similarly see that $L_1, L_2' = \{1_k : 0 < k < M\}, L_3' = \{2_k : 0 < k < M\}$ and L_4 are the right cells of W and that $L_1, L_2 \cup L_3, L_4$ are the 2-sided cells of W. Of course, we may see this directly from Theorem 3.2.4 once we note that $L_2 = \Gamma_1$, $L_3 = \Gamma_2$ and that ${}_M1$ must form its own cell by Proposition 3.1.7. The KL cells of W can be described similarly if $M = \infty$: L_1, L_2, L_3 are the left cells, L_1, L_2', L_3' are the right cells, and $L_1, L_2 \cup L_3$ are the 2-sided cells.

Finally, we record a fact that we will use in Section 5.3. We need some notation: if an alternating word i_k ends in the letter j for some $i, j \in \{1, 2\}, k \in \mathbb{Z}_{\geq 0}$ and we wish to emphasize this fact, we shall write i_kj for the word. Similarly, we write i_kj for ${}_kj$ if the latter starts with the letter i.

Proposition 3.3.1. *Suppose $x = h_ki$ and $y = i_lj$ for some $h, i, j \in \{1, 2\}$ and $0 < k, l < M$. For $d \in \mathbb{Z}$, let $\phi(d) = k + l - 1 - 2d$. Then*

$$c_xc_y = c_{h_ki}c_{i_lj} = (v + v^{-1}) \sum_{d=\max(k+l-M,0)}^{\min(k,l)-1} c_{h_{\phi(d)}j} + \varepsilon$$

in H, where $\varepsilon = f \cdot c_{1_M}$ for some $f \in \mathcal{A}$ if $M < \infty$ and $\varepsilon = 0$ otherwise.

Proof. We sketch a possible proof: use induction on k. The base case $k = 1$ follows from Proposition 3.1.7. For the inductive step, use equations (3.3.7) and (3.3.8) to write c_x as $c_h(c_{k-1}i)$ or $c_h(c_{k-1i}) - (c_{{}_{k-2}i})$, use earlier cases of the induction to multiply the parenthesized part(s) with c_y and use Equation (3.3.8) again to multiply c_h onto the relevant products, then simplify the results to the desired form. □

We will decipher the sum in the proposition more carefully in Section 5.3.

CHAPTER IV

THE SUBREGULAR J-RING

In this chapter, we describe Lusztig's construction of the asymptotic Hecke algebra J of a Coxeter system and recall some basic properties of J. We show how KL cells in W give rise to subalgebras of J, then shift our focus to a particular subalgebra J_C of J corresponding to the subregular cell of W. We also recall the definition of a based ring and explain why J_C is a based ring.

Throughout the section, suppose (W, S) is an arbitrary Coxeter system with $S = [n] = \{1, 2, \cdots, n\}$ unless otherwise stated. Let H be the Iwahori-Hecke algebra of (W, S), and let $\{T_w : w \in W\}, \{c_w : w \in W\}$ and $\{p_{y,w} : y, w \in W\}$ be the standard basis, KL basis and KL polynomials in H, respectively.

4.1. The Asymptotic Hecke algebra J

Consider the elements $h_{x,y,z} \in \mathbb{Z}[v, v^{-1}]$ $(x, y, z \in W)$ from Equation 3.1.5. Lusztig showed in [Lus14] that for any $z \in W$, there exists a unique integer $\mathbf{a}(z) \geq 0$ that satisfies the conditions

(a) $h_{x,y,z} \in v^{\mathbf{a}(z)}\mathbb{Z}[v^{-1}]$ for all $x, y \in W$,

(b) $h_{x,y,z} \notin v^{\mathbf{a}(z)-1}\mathbb{Z}[v^{-1}]$ for some $x, y \in W$.

Define $\gamma_{x,y,z^{-1}}$ to the non-negative integer such that

$$h_{x,y,z} = \gamma_{x,y,z^{-1}} v^{\mathbf{a}(z)} \mod v^{\mathbf{a}(z)-1}\mathbb{Z}[v^{-1}],$$

and define multiplication on the free abelian group $J = \oplus_{w \in W} \mathbb{Z}w$ by

$$t_x t_y = \sum_{z \in W} \gamma_{x,y,z^{-1}} t_z$$

for all $x, y \in W$. It is known in that this product is well-defined (i.e., $\gamma_{x,y,z^{-1}} = 0$ for all but finitely many $z \in W$ for all $x, y \in W$), and the multiplication defined above is associative, making J a ring (see [Lus14], 18.3). We call J the *asymptotic Hecke algebra* or simply the *J-ring* of (W, S).

Note that the J-ring is naturally equipped with an anti-involution, thanks to the symmetry mentioned in Remark 3.1.8:

Proposition 4.1.1. *The $\mathbb{Z}$-linear map with $t_x \mapsto t_{x^{-1}}$ is an anti-homomorphism of J.*

Proof. Recall from Remark 3.1.8 that $h_{x,y,z} = h_{y^{-1},x^{-1},z^{-1}}$ for all $x, y, z \in W$. It follows that $\gamma_{x,y,z} = \gamma_{y^{-1},x^{-1},z^{-1}}$ for all $x, y, z \in W$. The result now follows from the definition of J. □

4.2. Subalgebras of J

For each $x \in W$, let $\Delta(x)$ be the unique non-negative integer such that

$$p_{1,x} \in n_x v^{-\Delta(x)} + v^{-\Delta(x)-1}\mathbb{Z}[v^{-1}]$$

for some $n_x \neq 0$. Let

$$\mathcal{D} = \{x \in W : \mathbf{a}(x) = \Delta(x)\}.$$

It is known that $d^2 = 1$ for all $d \in \mathcal{D}$ (see Chapter 14 of [Lus14]), and $\mathcal{D}$ is called the set of *distinguished involutions*. There are many intricate connections between $\mathcal{D}$, the coefficients $\gamma_{x,y,z}$, and KL cells in W. The connections would lead us to many subalgebras of J that are indexed by cells and have units provided by the distinguished involutions. We highlight the relevant facts below.

Proposition 4.2.1 ([Lus14], Conjectures 14.2)**.** *Let $x, y, z \in W$. Then*

(1) $\gamma_{x,y,z} = \gamma_{y,z,x}$.

(2) If $\gamma_{x,y,z} \neq 0$, then $x \sim_L y^{-1}, y \sim_L z^{-1}$ and $z \sim_L x^{-1}$.

(3) If $\gamma_{x,y,d} \neq 0$ for some $d \in \mathcal{D}$, then $y = x^{-1}$ and $\gamma_{x,y,d} = 1$. Further, for each $x \in W$ there is a unique element $d \in \mathcal{D}$ such that $\gamma_{x,x^{-1},d} = 1$.

(4) Each left KL cell Γ of W contains a unique element d from $\mathcal{D}$. Further, for this elements d, we have $\gamma_{x^{-1},x,d} = 1$ for all $x \in \Gamma$.

Remark 4.2.2. In the paper [Lus87a], where Lusztig first defined the asymptotic Hecke algebra J, Proposition 4.2.1 is proved for Coxeter systems satisfying certain mild conditions. The conditions can be found in Section 1.1 of the paper, the four parts of the proposition appear in Theorem 1.8, Corollary 1.9, Proposition 1.4 and Theorem 1.10 of the paper, respectively. For arbitrary Coxeter systems, the statements of the proposition, as well as the statement in Proposition 3.2.5, appear only as conjectures in Chapter 14 of [Lus14]. However, [Lus14] studies Hecke algebras in a more general setting, namely, with possibly *unequal parameters*, and the statements are known to be true in the setting of this paper, which is called the *equal parameter* or the *split* case in the book. The proofs of the statements rely heavily on Theorem 3.1.5; see Chapter 15 of [Lus14].

Definition 4.2.3. For any subset X of W, define $J_X := \oplus_{w \in X} \mathbb{Z} t_w$.

Corollary 4.2.4 ([Lus14], Section 18.3)**.**

(a) Let Γ be any left KL cell in W, say with $\Gamma \cap \mathcal{D} = \{d\}$. Then the subgroup $J_{\Gamma \cap \Gamma^{-1}}$ is actually a unital subalgebra of J; its unit is t_d.

(b) For any 2-sided cell E in W, the subgroup J_E is a subalgebra of J. Further, we have a direct sum decomposition $J = \oplus_{E \in \mathcal{C}} J_E$ of algebras, where $\mathcal{C}$ is the collection of all 2-sided KL cells of W.

(c) If E is a 2-sided cell such that $E \cap \mathcal{D}$ is finite, then J_E is a unital algebra with unit element $\sum_{d \in E \cap \mathcal{D}} t_d$.

(d) If $\mathcal{D}$ is finite, then J is a unital algebra with unit $\sum_{d \in \mathcal{D}} t_d$.

Proof. We will repeatedly use Proposition 4.2.1. When we say part (i), we will mean part (i) of the proposition.

(a) Let $x, y \in \Gamma \cap \Gamma^{-1}$, and suppose $\gamma_{x,y,z^{-1}} \neq 0$ for some $z \in W$. Then by part (2), $z = (z^{-1})^{-1} \sim_L y \in \Gamma$, and $z^{-1} \sim_L x^{-1}$ so that $z \sim_R x \in \Gamma^{-1}$ by Proposition 3.2.3. Thus, $z \in \Gamma \cap \Gamma^{-1}$. It follows that $J_{\Gamma \cap \Gamma^{-1}}$ is a subalgebra of J.

It remains to show that $t_x t_d = t_x = t_d t_x$ for all $x \in \Gamma \cap \Gamma^{-1}$. For $y \in \Gamma \cap \Gamma^{-1}$, since $\gamma_{d,x,y} = \gamma_{x,y,d}$ by Part (1), $\gamma_{d,x,y} \neq 0$ only if $y = x^{-1}$ by Part (3), and in this case $\gamma_{d,x,y} = \gamma_{d,x,x^{-1}} = \gamma_{x,x^{-1},d} = 1$. This implies $t_d t_x = t_x$. Similarly, $\gamma_{x,d,y} = \gamma_{y,x,d} \neq 0$ for some $y \in \Gamma \cap \Gamma^{-1}$ only if $y = x^{-1}$, whence $\gamma_{x,d,y} = \gamma_{x,d,x^{-1}} = \gamma_{x^{-1},x,d} = 1$ by Part (4). This implies $t_x t_d = t_x$.

(b) Let $x, y \in E$, and suppose $\gamma_{x,y,z^{-1}} \neq 0$ for some $z \in W$. Let Γ be the left cell containing y. Then $\Gamma \subseteq E$. By part (2), $z \sim_L y$, therefore $z \in \Gamma \subseteq E$ as well, hence J_E is a subalgebra.

Now suppose $x, y \in W$ belong in different 2-sided cells E and E', say with $x \in E$ and $y \in E'$. Then $y^{-1} \in E'$ by Proposition 3.2.5, hence $x \not\sim_L y^{-1}$. Part (2) now implies that $\gamma_{x,y,z^{-1}} = 0$ for all $z \in W$, therefore $t_x t_y = 0$. It follows that $J = \oplus_{E \in \mathcal{C}} J_E$.

(c) By part (4) of Proposition 4.2.1, the fact that $E \cap \mathcal{D}$ is finite implies E is a disjoint union of finitely many left cells $\Gamma_1, \cdots, \Gamma_k$. Suppose $\Gamma_i \cap \mathcal{D} = \{d_i\}$ for each $i \in [k]$, and let $x \in E$, say with $x \in \Gamma_i$ and $x^{-1} \in \Gamma_{i'}$ for some $i, i' \in [k]$. Then by parts (1), (2) and (3), $\gamma_{x,d_j,y} = \gamma_{y,x,d_j} \neq 0$ for some $y \in E, j \in [k]$ only if $d_j \sim_L x$ and $y = x^{-1}$. In this case, $j = i$ and $\gamma_{x,d_j,y} = \gamma_{y,x,d_i} = 1$ by part (4). Consequently,

$$t_x \left(\sum_{j=1}^{k} t_{d_j} \right) = t_x t_{d_i} = t_x.$$

Similarly, $\gamma_{d_j,x,y} = \gamma_{x,y,d_j}$ for some $y \in E, j \in [k]$ only if $d_j \sim_L x^{-1}$ and $y = x^{-1}$, in which case $j = i'$ and $\gamma_{d_j,x,y} = \gamma_{x,x^{-1},d_{i'}} = 1$. Consequently,

$$\left(\sum_{j=1}^{k} t_{d_j} \right) t_x = t_{d_{i'}} t_x = t_x.$$

It follows that $\sum_{d \in E \cap \mathcal{D}} t_d = \sum_{j=1}^{k} t_{d_k}$ is the unit of J_E, as claimed.

(d) Let $x \in W$, and let d_1, d_2 be the unique distinguished involution in the left cell of x and x^{-1}, respectively. To show $\sum_{d \in \mathcal{D}} t_d$ is the unit of J, it suffices to show that

$$t_x \left(\sum_{d \in \mathcal{D}} t_d \right) = t_x t_{d_1} = t_x = t_{d_2} t_x = \left(\sum_{d \in \mathcal{D}} t_d \right).$$

This can be proved in a similar way to the last part. $\square$

Remark 4.2.5. In part (3) of the corollary, we dealt with the case where $\mathcal{D}$ is finite. When $\mathcal{D}$ is infinite, J only has a generalized unit element in the sense that the elements $t_d (d \in \mathcal{D})$ satisfy $t_d t_{d'} = \delta_{d,d'}$ and $\sum_{d,d' \in \mathcal{D}} t_d J t_{d'} = J$. Lusztig also showed that even when $\mathcal{D}$ is not finite, J can be naturally imbedded into a certain unital algebra ([Lus14], 18.13). We will not need these technicalities, though.

4.3. The Subregular J-ring

Recall the definition of the subregular cell C and the left cells Γ_s $(s \in S)$ from Section 2.4. By Corollary 4.2.4, the sets J_C and $J_{\Gamma_s \cap \Gamma_s^{-1}}$ are subalgebras of the asymptotic Hecke algebra J. From now on, we shall call J_C the *based ring of the subregular cell* of (W, S), or simply the *subregular J-ring* of (W, S). For each $s \in S$, we shall write $J_s := J_{\Gamma_s \cap \Gamma_s^{-1}}$. This article is devoted to the study of the algebras J_C and $J_s (s \in S)$. These algebras naturally possess the additional structures of a *based ring*. We explain this below.

The following three definitions are taken from Chapter 3 of [EGNO15].

Definition 4.3.1 ($\mathbb{Z}_+$-rings)**.** Let A be a ring which is free as a $\mathbb{Z}$-module.

(a) A $\mathbb{Z}_+$-*basis* of A is a basis $B = \{t_i\}_{i \in I}$ such that for all $i, j \in I$, $t_i t_j = \sum_{k \in I} c_{ij}^k t_k$ where $c_{ij}^k \in \mathbb{Z}_{\geq 0}$ for all $k \in I$.

(b) A $\mathbb{Z}_+$-*ring* is a ring with a fixed $\mathbb{Z}_+$-basis and with an identity that is a nonnegative linear combination of the basis elements.

(c) A *unital* $\mathbb{Z}_+$-ring is a $\mathbb{Z}_+$ ring such that the identity 1 is a basis element.

Let A be a $\mathbb{Z}_+$-ring, and let I_0 be the set of $i \in I$ such that t_i occurs in the decomposition of 1. We call the elements of I_0 the *distinguished index set*. Let

$\tau : A \to \mathbb{Z}$ denote the group homomorphism defined by

$$\tau(t_i) = \begin{cases} 1 & \text{if} \quad i \in I_0, \\ \\ 0 & \text{if} \quad i \notin I_0. \end{cases}$$

Definition 4.3.2 (Based rings)**.** A $\mathbb{Z}_+$-ring A with a basis $\{t_i\}_{i\in I}$ is called a *based ring* if there exists an involution $i \mapsto i^*$ on I such that the induced map

$$a = \sum_{i\in I} c_i t_i \mapsto a^* := \sum_{i\in I} c_i t_{i^*}, c_i \in \mathbb{Z}$$

is an anti-involution of the ring A, and

$$\tau(t_i t_j) = \begin{cases} 1 & \text{if} \quad i = j^*, \\ \\ 0 & \text{if} \quad i \neq j^*. \end{cases} \tag{4.3.1}$$

Example 4.3.3. Here are two familiar examples of based rings.

(a) For each $n \geq \mathbb{Z}_{\geq 1}$, the ring of matrices $\mathrm{Mat}_{n\times n}(\mathbb{Z})$ is a based ring where $I = \{(i,j) : 1 \leq i,j \leq n\}$, the basis consists of the elementary matrices $E_{(i,j)} := E_{ij}(1 \leq i,j \leq n)$, $(i,j)^* = (j,i)$, and $I_0 = \{(i,i) : 1 \leq i \leq n\}$. The ring is unital only when $n = 1$.

(b) For any group G, the group ring $\mathbb{Z}G$ is a unital based ring, with the basis being the group elements and $g^* = g^{-1}$ for each $g \in G$.

For more examples of based rings, see Example 3.1.9 of [EGNO15].

We now use results from Section 4.2 to show that under certain finiteness conditions, all the subalgebras of J introduced in that section are based rings.

Proposition 4.3.4. *(a) Let E be any 2-sided KL cell in W that contains finitely many distinguished involutions. Then the algebra J_E is a based ring with basis $\{t_x\}_{x\in I}$ with index set $I = E$, with distinguished index set $I_0 = E \cap \mathcal{D}$, and with the map $^* : I \to I$ given by $x^* = x^{-1}$.*

(b) Let Γ be any left KL cell in W, and let d be the unique element in $\Gamma\cap\mathcal{D}$. Then $J_{\Gamma\cap\Gamma^{-1}}$ is a unital based ring with index set $I = \Gamma\cap\Gamma^{-1}$, with distinguished index set $I_0 = \{d\}$, and with $^* : I \to I$ given by $x^* = x^{-1}$.

Proof. (1) The set $\{t_x\}_{x\in E}$ forms a $\mathbb{Z}_+$-basis of J_E by the definition of J_E, and J_E is $\mathbb{Z}_+$-ring with distinguished index set $E\cap\mathcal{D}$ since the its unit is $\sum_{d\in E\cap\mathcal{D}} t_d$ by Part (c) of Corollary 4.2.4. J_E. The fact that $x \mapsto x^{-1}$ induces an anti-involution on J_E follows from Proposition 4.1.1. Finally, Equation (4.3.1) holds by parts (3) and (4) of Proposition 4.2.1. We have now shown that J_E is a based ring.

(2) The proof is similar to the previous part, with the only difference being that $J_{\Gamma\cap\Gamma^{-1}}$ is unital with $I_0 = \{d\}$ since t_d is its unit by Part (a) of Corollary 4.2.4. □

Corollary 4.3.5. *Let (W, S) be a Coxeter system where S is finite (this will be the case for all Coxeter systems in this paper). Let C, Γ_s, J_C and J_s be as before. Then*

(a) J_C is a based ring with index set $I = C$, distinguished index set $I_0 = S$ and anti-involution induced by the map $^* : I \to I$ with $x^* = x^{-1}$.

(b) For each $s \in S$, J_s is a based ring with index set $I = \Gamma_s \cap \Gamma_s^{-1}$, distinguished index set $I_0 = \{s\}$ and anti-involution induced by the map $^* : I \to I$ with $x^* = x^{-1}$.

Proof. This is immediate from Proposition 4.3.4 and Theorem 3.2.4 once we show that for each $s \in S$, the unique distinguished involution in Γ_s is exactly s. So it suffices to show that $s \in \mathcal{D}$ for each $s \in S$. This is well-known (we will also see this from the proof of Corollary 5.1.2, where we show $\mathbf{a}(s) = \Delta(s) = 1$ for all $s \in S$). $\square$

As mentioned in the introduction, rings of finite rank are of particular interest:

Definition 4.3.6 (Multifusion rings and fusion rings)**.** A *multifusion ring* is a based ring of finite rank. A *fusion ring* is a unital based ring of finite rank.

In view of Corollary 4.3.5, Theorem 2.4.6 now immediately implies the following.

Theorem 4.3.7. *Let (W, S) be an irreducible Coxeter system, and let $G = (V, E)$ be its Coxeter diagram. Then the following conditions are equivalent.*

(a) G is a tree, and at most one edge in G has a weight greater than 3.

(b) The based ring J_C is a multifusion ring.

(c) The based ring J_s is a fusion ring for all $s \in S$.

(d) The based ring J_s is a fusion ring for some $s \in S$.

We will study the structure of all fusion rings of the form J_s in Section 6.3.

To end the section, let us formulate the notion of an isomorphism of based rings for future use. Naturally, we define it to be a ring isomorphism that respects all the additional defining structures of a based ring.

Definition 4.3.8 (Isomorphism of Based Rings)**.** Let A be a based ring $\{t_i\}_{i\in I}$ with index set I, distinguished index set I_0 and anti-involution * induced by a map $^* : I \to I$. Let B be a based ring $\{t_j\}_{j\in J}$ with index set J, distinguished index set J_0 and anti-involution * induced by a map $^* : J \to J$. We define an *isomorphism of based rings* from A to B to be a unit-preserving ring isomorphism $\Phi : A \to B$ such that $\Phi(t_i) = t_{\phi(i)}$ for all $i \in I$ where ϕ is a bijection from I to J with $\phi(I_0) = J_0$ and such that $\Phi(t_i^*) = (\Phi(t_i))^*$ for all $i \in I$.

CHAPTER V

PRODUCTS IN J_C

The notations from the previous sections remain in force. In particular, we assume (W, S) is an arbitrary Coxeter system with $S = [n]$ for some $n \in \mathbb{N}$, and we use C to denote the subregular cell.

In this chapter, we develop the tools to study the algebra J_C. The notion of the *dihedral segments* of a word from Section 2.4 plays a central role. The main theorem of the chapter is Theorem F, which reduces the study of a basis element t_w in J_C to only the basis elements corresponding to its dihedral segments. We explain how to use Theorem F to compute the products of arbitrary basis elements in J_C.

5.1. The a-value Characterization of C

Recall that in Section 2.4, we used the notion of dihedral segments to characterize the subregular cell C in terms of reduced words. We give another characterization of the subregular cell C in this section, this time in terms of the $\mathbf{a}$-function defined in Section 4.1. To start, we recall some properties of $\mathbf{a}$.

Proposition 5.1.1 ([Lus14], 13.7, 14.2)**.** *Let $x, y \in W$. Then*

(a) $\mathbf{a}(x) \geq 0$*, where* $\mathbf{a}(x) = 0$ *if and only if x equals the identity element of W.*

(b) $\mathbf{a}(x) \leq \Delta(x)$*.*

(c) If $x \leq_{LR} y$, then $\mathbf{a}(x) \geq \mathbf{a}(y)$*. Hence, if $x \sim_{LR} y$, then* $\mathbf{a}(x) = \mathbf{a}(y)$*.*

(d) If $x \leq_L y$ and $\mathbf{a}(x) = \mathbf{a}(y)$*, then $x \sim_L y$.*

(e) If $x \leq_R y$ and $\mathbf{a}(x) = \mathbf{a}(y)$*, then $x \sim_R y$.*

(f) If $x \leq_{LR} y$ and $\mathbf{a}(x) = \mathbf{a}(y)$, then $x \sim_{LR} y$.

Corollary 5.1.2. $C = \{x \in W : \mathbf{a}(x) = 1\}$.

Proof. Let $s \in S$. Then $a(s) \geq 1$ by Part (1) of the proposition. On the other hand, recall from Example 3.1.4 that $c_s = T_s + v^{-1}$ ([Lus14], § 5), therefore $\Delta(s) = 1$ by the definition of Δ and $\mathbf{a}(s) \leq 1$ by part (2) of the proposition. It follows that $a(s) = 1$. Since s is clearly in C, Part (3) implies that $a(x) = 1$ for all $x \in C$.

Now let $x \in W \setminus C$. Then either x is the group identity and $\mathbf{a}(x) = 0$, or x has a reduced expression $x = s_1 s_2 \cdots s_k$ with $k > 1$ and each $s_i \in S$. In the latter case, $x \leq_L s_k$ by Proposition 3.1.7, so $\mathbf{a}(x) \geq \mathbf{a}(s_k) = 1$. Meanwhile, since $x \notin C$, $x \not\sim_{LR} s_k$, so $\mathbf{a}(x) \neq \mathbf{a}(s_k)$ by part (6) of Proposition 5.1.1. It follows that $\mathbf{a}(x) > 1$, and we are done. ☐

The $\mathbf{a}$-value characterization of C leads to a shortcut for studying products in J_C. To see how, consider the filtration

$$\cdots \subset H_{\geq 2} \subset H_{\geq 1} \subset H_{\geq 0} = H.$$

of the Hecke algebra H where

$$H_{\geq a} = \oplus_{x:\mathbf{a}(x)\geq a} \mathcal{A} c_x$$

for each $a \in \mathbb{N}$. By parts (3)-(6) of Proposition 5.1.1 and Proposition 3.2.1, this may be viewed as a filtration of submodules when we view H as its regular left module. It induces the left modules

$$H_a := H_{\geq a}/H_{\geq a+1}, \tag{5.1.1}$$

where H_a is spanned by images of the elements $\{c_x : \mathbf{a}(x) = a\}$. In particular, H_1 is spanned by the images of $\{c_x : x \in C\}$. By the construction of J, to compute a product $t_x \cdot t_y$ in J_c, it then suffices to consider the product $c_x \cdot c_y$ in H_1. More precisely, we have arrived at the following shortcut.

Corollary 5.1.3. *Let $x, y \in C$. Suppose*

$$c_x c_y = \sum_{z \in W} h_{x,y,z} c_z$$

for $h_{x,y,z} \in \mathcal{A}$. Then

$$t_x t_y = \sum_{z \in T} \gamma_{x,y,z^{-1}} t_z$$

in J_C, where $T = \{z \in C : h_{x,y,z} \in n_z v + \mathbb{Z}[v^{-1}] \text{ for some } n_z \neq 0\}$.

The corollary plays a key role in the proof of Lemma 5.2.4. A simple application of it reveals the following, which we will use repeatedly in the next section.

Corollary 5.1.4. *Let $x = s_1 s_2 \cdots s_k$ be the reduced word of an element in C. Then*

$$t_{s_1} t_x = t_x = t_x t_{s_k}.$$

Proof. This follows immediately from Corollary 5.1.3 and Proposition 3.1.7. $\square$

5.2. The Dihedral Factorization Theorem

Recall the definition of dihedral segments from Definition 2.4.2. This subsection is dedicated to the proof of Theorem F. We restate it below.

Theorem F. (Dihedral factorization) *Let x be the reduced word of an element in C, and let $x_1, x_2, \cdots, x_l$ be the dihedral segments of w. Then*

$$t_x = t_{x_1} \cdot t_{x_2} \cdot \cdots \cdot t_{x_l}.$$

Before we prove the theorem, let us make an assumption and a definition. First, since no simple reflection can appear consecutively in a reduced word of any element in W, we make the following assumption.

Assumption 5.2.1. *Henceforth in this article, whenever we speak of a word in a Coxeter system, we assume that no simple reflection appears consecutively in the word.*

Second, note that the process of factoring a reduced word (satisfying the above assumption) into its dihedral segments can be easily reversed, that is, we may recover a word from its dihedral segments by taking a proper "product". This motivates the following definition.

Definition 5.2.2 (Glued product)**.** For any two words $x_1, x_2 \in \langle S \rangle$ such that x_1 ends with the same letter that x_2 starts with, say $x_1 = \cdots st$ and $x_2 = tu \cdots$, we define their *glued product* to be the word $x_1 \circ x_2 := \cdots stu \cdots$ obtained by concatenating x_1 and x_2 then deleting one occurrence of the common letter.

The operation $\circ$ is obviously associative. Further, if $x_1, x_2, \cdots, x_k$ are the dihedral segments of x, then

$$x = x_1 \circ x_2 \circ \cdots \circ x_k. \tag{5.2.2}$$

For example, in the example following Definition 2.4.2, we saw that the word $x =$

121313123 has dihedral segments $x_1 = 121, x_2 = 13131, x_3 = 12, x_4 = 23$, and now we have $x_1 \circ x_2 \circ x_3 \circ x_4 = 121 \circ 13131 \circ 12 \circ 23 = 121313123 = x$.

To prove the theorem, we need to examine products in H and apply Corollary 5.1.3. To exploit the uniqueness of reduced expressions of elements of C, we need the following well-known fact.

Proposition 5.2.3 ([KL79], Statement 2.3.e)**.** *Let $x, y \in W, s \in S$ be such that $x < y, sy < y, sx > x$. Then $\mu(x,y) \neq 0$ if and only if $x = sy$; further, in this case, $\mu(x,y) = 1$.*

Lemma 5.2.4. *Let $x = s_1 s_2 s_3 \cdots s_k$ be the reduced word of an element in C. Let $x' = s_2 s_3 \cdots s_k$ and $x'' = s_3 \cdots s_k$ be the sequences obtained by removing the first letter and first two letters from x, respectively. Then in H_1, we have*

$$c_{s_1} c_{x'} = \begin{cases} c_{x''} & \textit{if } s_1 \neq s_3; \\ c_x + c_{x''} & \textit{if } s_1 = s_3. \end{cases}$$

Proof. By Proposition 3.1.7 and Corollary 5.1.2, in H_1 we have

$$c_s c_{x'} = c_x + \sum_P \mu_{z,x'} c_z$$

where $P = \{z \in C : s_1 z < z < x'\}$. Let $z \in P$. Then by Propositions 2.3.3 and 2.2.6, z has a unique reduced expression that is a proper subword of x' and starts with s_1. Since $s_1 \neq s_2$ now that x is reduced, we have $\mathcal{L}(z) = \{s_1\}$, therefore $s_2 x' < x'$ while $s_2 z > z$. Now, if $l(z) < l(x') - 1$, then $z \neq s_2 x$, so $\mu(z, x') = 0$ by Lemma 5.2.3. If $l(z) = l(x') - 1$, then we must have $s_3 = s_1$ and $z = x'' = s_2 x'$, for otherwise $s_2 \neq s_1, s_3 \neq s_1$, and any subword of $x' = s_2 s_3 \cdots s_k$ that starts with s_1

must have length smaller than $l(x') - 1$. This implies $\mu(z, x') = 1$ by Lemma 5.2.3. The lemma now follows. $\square$

Remark 5.2.5. We may derive Lemma 5.2.4 from Lemma 6.2 of [Gre07]. However, that lemma involves the notion of *star operations*, and we choose not to discuss it here as we will not need it anywhere else..

We are ready to prove Theorem F.

Proof of Theorem F. We use induction on l. The base case where $l = 1$ is trivially true. If $l > 1$, let y be the glued product $y = x_2 \circ x_3 \circ \cdots \circ x_l$ so that, by induction, it suffices to show

$$t_x = t_{x_1} \cdot t_y. \tag{5.2.3}$$

Suppose y starts with some $t \in S$. Note that the construction of the dihedral segments guarantees that x_1 contains at least two letters and is of the alternating form $w_1 = \cdots tst$ for some $s \in S$, while x_2, hence also y, is of the form $tu \cdots$ for some $u \in S \setminus \{s, t\}$.

We prove Equation (5.2.3) by induction on the length $k = l(x_1)$ of x_1. For the base case $k = 2$, Proposition 3.1.7 and Lemma 5.2.4 imply that

$$c_{x_1} c_y = c_{st} c_{tu\cdots} = c_s c_t c_{tu\cdots} = (v + v^{-1}) c_{stu\cdots} = (v + v^{-1}) c_{x_1 \circ y}$$

in H_1. Equation (5.2.3) then follows by Corollary 5.1.3. Now suppose $k > 2$, write $x_1 = s_1 s_2 s_3 \cdots s_k$, and let $x_1' = s_2 s_3 \cdots s_k$ and $x_1'' = s_3 \cdots s_k$. Since the letters $s_1, s_2, \cdots, s_k$ alternate between s_1 and s_2, Proposition 3.1.7 and Lemma 5.2.4 imply that

$$c_{s_1 s_2} \cdot c_{x_1'} = c_{s_1} c_{s_2} c_{x_1'} = (v + v^{-1}) c_{s_1} c_{x_1'} = (v + v^{-1})(c_{x_1} + c_{x_1''})$$

and similarly

$$c_{s_1 s_2} \cdot c_{x_1' \circ y} = (v + v^{-1})(c_{x_1 \circ y} + c_{x_1'' \circ y}).$$

From the last two equations, it follows that

$$\begin{aligned} t_{s_1 s_2} t_{x_1'} &= t_{x_1} + t_{x_1''} , \\ t_{s_1 s_2} t_{x_1' \circ y} &= t_{x_1 \circ y} + t_{x_1'' \circ y} , \end{aligned}$$

therefore

$$t_{x_1} t_y = (t_{s_1 s_2} t_{x_1'} - t_{x_1''}) t_y = t_{s_1 s_2} t_{x_1' \circ y} - t_{x_1'' \circ y} = t_{x_1 \circ y} + t_{x_1'' \circ y} - t_{x_1'' \circ y} = t_{x_1 \circ y} = t_x,$$

where the second equality holds by the inductive hypothesis now that $l(x_1') < l(x_1)$. This completes our proof. $\square$

To interpret the theorem, consider the following definition.

Definition 5.2.6. (Dihedral elements) We define a *dihedral element* in J_C to be a basis element of the form t_x, where x appears as a dihedral segment of some $y \in C$.

In light of the definition, Theorem F means that dihedral elements generate J_C. The theorem also means that the combinatorial factorization of an element into its dihedral segments in Equation 5.2.2 carries over to an algebraic one in J_C.

5.3. Products of Dihedral Elements

Now that Theorem F allows us to factor any basis element in J_C into dihedral elements, to understand products of the form $t_x t_y$ for $x, y \in W$, it is natural to first study products of dihedral elements. We do so in this section; the products $t_x t_y$ for arbitrary $x, y \in C$ will be treated in the next section.

As we shall see in the next section, the only interesting products of dihedral elements we need to study are those of the form $t_x t_y$ where x and y are generated by the same set of two simple reflections, say $\{i, j\} \subseteq S$, and x ends with the same letter that y starts with. This brings us to exactly the situation in Proposition 3.3.1. Let $M = m(i, j)$ as usual.

Proposition 5.3.1. *Suppose $x = h_k i$ and $y = i_l j$ for some $h, i, j \in \{1, 2\}$ and $0 < k, l < M$. For $d \in \mathbb{Z}$, let $\phi(d) = k + l - 1 - 2d$. Then in J_C, we have*

$$t_x t_y = t_{h_k i} t_{i_l j} = \sum_{d=\max(k+l-M,0)}^{\min(k,l)-1} t_{h_{\phi(d)} j}.$$

Proof. This follows immediately from proposition 3.3.1 and 5.2.4. □

Let us decipher the formula from Proposition 5.3.1. It says that the product $t_x t_y$ is the linear combination of the terms t_z, all with coefficient 1, where z runs through the elements in C whose reduced words begin with the same letter as x, end with the same letter as y, and have lengths from the set $A \setminus B$ where

$$A = \{|k - l| + 2j + 1 : 0 \leq j \leq \min(k, l) - 1\}$$

and

$$B = \{j : j \in A, j \geq M\} \cup \{2M - j : j \in A, j \geq M\}.$$

Note that when $k = 1$, this agrees with Corollary 5.1.4.

Example 5.3.2 (Product of dihedral elements)**.** Let $s, t \in S$.

(a) Suppose $m(s,t) = 7, x = stst$ and $y = tst$. Then by Proposition 5.3.1,

$$t_x t_y = t_{st} + t_{stst} + t_{ststst}.$$

(b) Suppose $m(s,t) = 7, x = stst$ and $y = tsts$. Then by Proposition 5.3.1,

$$t_x t_y = t_s + t_{sts} + t_{ststs} + \cancel{t_{stststs}} = t_s + t_{sts} + t_{ststs}.$$

(c) Suppose $m(s,t) = 7, x = tst$ and $y = tststs$. Then by Proposition 5.3.1,

$$t_x t_y = t_{tsts} + \cancel{t_{tststs}} + \cancel{t_{tstststs}} = t_{tsts}.$$

The rule we described before the example to get the list of lengths for the z's is well-known; it is the *truncated Clebsch-Gordan rule*. It governs the multplication of the basis elements of the *Verlinde algebra of the Lie group $SU(2)$*, which appears as the Grothendieck ring of certain fusion categories (see [EK95] and Section 4.10 of [EGNO15]). Since it will cause no confusion, we will also refer to this algebra simply as the *Verlinde algebra*.

Definition 5.3.3 (The Verlinde algebra, [EK95])**.** Let $M \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$. The *$M$-th Verlinde algebra* is the free abelian group $\mathrm{Ver}_M = \oplus_{1 \leq k \leq M-1} \mathbb{Z} L_k$, with multiplication defined by

$$L_k L_l = \sum_{d=\max(k+l-M,0)}^{\min(k,l)-1} L_{k+l-1-2d}.$$

We call the $\mathbb{Z}$-span of the elements L_k where k is an odd integer the *odd part* of Ver_M, and denote it by Ver_M^{odd}.

Note that by the multiplication formula, Ver_M^{odd} is clearly a subalgebra of Ver_M.

Indeed, suppose (W, S) is a dihedral system, say with $S = \{1, 2\}$ and $m(1, 2) = M$ for some $M \in \mathbb{Z}_{\geq 3} \cup \{\infty\}$, then we claim that the subalgebra J_1 of J_C is isomorphic to Ver_M^{odd}. To see this, recall that J_1 is given by the $\mathbb{Z}$-span of all t_{1_k} where k is odd, $0 < k < M$, and 1_k is the alternating word $121 \cdots 1$ containing k letters. Since the multiplication of such basis elements are governed by the truncated Clebsch-Gordan rule in Proposition 5.3.1, the map $t_{1_k} \mapsto L_k$ induces an isomorphism. Furthermore, it is easy to check that both Ver_M and Ver_M^{odd} are unital based rings with L_1 as the unit and with the identity map as the anti-involution, so this isomorphism is actually an isomorphism of based rings. By a similar argument, J_2 is isomorphic to Ver_M^{odd} as based rings as well. We discuss incarnations of Ver_M^{odd} for some small values of M below.

Example 5.3.4. Let (W, S) be a dihedral system with $S = \{1, 2\}$ and $M = m(1, 2)$.

(a) Suppose $M = 5$. Then $J_1 = \mathbb{Z}t_1 \oplus \mathbb{Z}t_{121}$, where t_1 is the unit and

$$t_{121}t_{121} = t_1,$$

so J_1, hence Ver_5^{odd}, is isomorphic to the *Ising fusion ring* that arises from the Ising model of statistical mechanics (see [EGNO15], Example 3.1.9).

(b) Suppose $M = 6$. Then $J_1 = \mathbb{Z}t_1 \oplus \mathbb{Z}t_{121} \oplus \mathbb{Z}t_{12121}$, where t_1 is the unit and

$$t_{121}t_{121} = t_1 + t_{121} + t_{12121}, \quad t_{121}t_{12121} = t_{12121}t_{121} = t_{121}, \quad t_{12121}t_{12121} = t_1.$$

On the other hand, the category $\mathcal{C}$ of complex representations of the symmetric group S_3 has three non-isomorphic simple objects 1 (the trivial representation),

χ (the sign representation) and V satisfying

$$1 \otimes \chi = \chi \otimes 1 = \chi, \quad 1 \otimes V = V \otimes 1 = V,$$

$$V \otimes V = 1 \oplus V \oplus \chi, \quad V \otimes \chi = \chi \otimes V = V, \quad \chi \otimes \chi = 1,$$

so J_1, hence Ver_6^{odd}, is isomorphic to the Grothendieck ring $\mathrm{Gr}(\mathcal{C})$ of $\mathcal{C}$.

5.4. Products of Arbitrary Elements

Let $x, y \in C$. We now describe the product $t_x t_y$ of two arbitrary basis elements in J_C. For convenience, we shall abuse notation slightly and not distinguish between an element in C and its unique reduced word. More precisely, we make the following assumption.

Assumption 5.4.1. *From now on, whenever we write $x \in C$, we assume not only that x is an element of the subregular cell, but also that x is the unique reduced word of the element.*

Recall the definition of J_X for $X \subseteq W$ from Definition 4.2.3 and the definition of Γ_s ($s \in S$) from the beginning of Section 4.3. Here is a simple fact about $t_x t_y$:

Proposition 5.4.2. *Let $a, b, c, d \in S$, let $x \in \Gamma_a^{-1} \cap \Gamma_b$, and let $y \in \Gamma_c^{-1} \cap \Gamma_d$. Then $t_x t_y = 0$ if $b \neq c$, and $t_x t_y \in J_{\Gamma_a^{-1} \cap \Gamma_d}$ if $b = c$.*

Proof. Recall that for any $s \in S$, Γ_s is a left KL cell in W that consists of the elements in C whose reduced word ends in s. Consequently, Γ_s^{-1} is a right KL cell by Proposition 3.2.3 and it consists of the elements in C whose reduced word starts with s. That said, the statement follows from part (2) of Proposition 4.2.1 in the following way. If $b \neq c$, then $x \in \Gamma_b$ while $y^{-1} \in \Gamma_c$, so $x \not\sim_L y^{-1}$. This implies $\gamma_{x,y,z^{-1}} = 0$ for all $z \in W$, therefore $t_x t_y = 0$. If $b = c$, then for any $z \in W$ such that $\gamma_{x,y,z^{-1}} \neq 0$, we must have $y \sim_L z$ and $z^{-1} \sim_L x^{-1}$. The last condition implies $z \sim_R x$ by Proposition 3.2.3, so $z \in \Gamma_a^{-1} \cap \Gamma_d$. It follows that $t_x t_y \in J_{\Gamma_a^{-1} \cap \Gamma_d}$. □

By the definition of Γ_s for $s \in S$, the proposition means that $t_x t_y = 0$ if the last letter of x is not the same as the first letter of y. If the letters are equal, the proposition implies that for any basis element t_z occurring in the product $t_x t_y$ (when the product is nonzero), z must start with the same letter as x and end with the same letter as y. The second case here further splits into two subcases, depending on whether the last dihedral segment of x and the first dihedral segment of y involve the same set of letters. The product $t_x t_y$ is easy to describe if they do not.

Proposition 5.4.3. *Let $x, y \in C$. Suppose x ends with the letter that y starts with, and suppose that the last dihedral segment of x and the first dihedral segment of y involve different sets of letters. Then $t_x t_y = t_{x \circ y}$.*

Proof. Let $x_1, \cdots, x_p$ and $y_1, \cdots, y_q$ be the dihedral segments of x and y, respectively. By the assumptions, $x_1, \cdots, x_k, y_1, \cdots, y_l$ are exactly the dihedral segments of the glued product $x \circ y$, therefore Theorem F implies

$$t_x t_y = t_{x_1} \cdots t_{x_p} t_{y_1} \cdots t_{y_q} = t_{x_1 \circ \cdots \circ x_p \circ y_1 \circ \cdots \circ y_q} = t_{x \circ y}.$$ □

It remains to compute products of the form $t_x t_y$ in J_C where x ends in the letter y starts with and the last dihedral segment x_p of x contains the same set of letters as the first dihedral segment y_1 of y. In this case, Theorem F implies that

$$t_x t_y = t_{x_1} \cdots t_{x_{p-1}} (t_{x_p} t_{y_1}) t_{y_2} \cdots t_{y_q}$$

where $x_1, \cdots, x_p, y_1, \cdots, y_q$ are as in the proof of Proposition 5.4.3 and $t_{x_p} t_{y_1}$ can be computed using Proposition 5.3.1. We may compute $t_{x_p} t_{y_1}$ and distribute the product so as to write $t_x t_y$ as a linear combinations of products of dihedral elements. If such a product has two consecutive factors corresponding to elements in the same dihedral group, we can again use Proposition 5.3.1 to compute the product of these two factors first and then distribute the product to obtain a new linear combination. Repeat such processes until we have $t_x t_y$ as a linear combination of products where no consecutive factors correspond to elements of the same dihedral group. This means the factors appear as the dihedral segment of an element in C, so we may apply Theorem F to each of the products and rewrite $t_x t_y$ as a linear combination of other basis elements, finishing the computation. This algorithm is illustrated in the following example, and the Sage ([Dev16]) code implementing the algorithm is available at [Xu].

Example 5.4.4 (Product of arbitrary elements)**.** Suppose $S = \{1, 2, 3\}$ and $m(1, 2) = 4, m(1, 3) = 5, m(2, 3) = 6$.

(a) Let $x = 123, y = 323213$. Then by Theorem F and Proposition 5.3.1,

$$\begin{aligned} t_x t_y &= t_{12} t_{23} t_{3232} t_{21} t_{13} \\ &= t_{12} (t_{232} + t_{23232}) t_{21} t_{13} \\ &= t_{12} t_{232} t_{21} t_{13} + t_{12} t_{23232} t_{21} t_{13}. \end{aligned}$$

Applying Theorem F again to the last expression, we have

$$t_x t_y = t_{123213} + t_{12323213}.$$

(b) Let $x = 123, y = 3213$. Repeated use of Theorem F and Proposition 5.3.1 yields

$$\begin{aligned}
t_x t_y &= t_{12}t_{23}t_{32}t_{21}t_{13} \\
&= t_{12}(t_2 + t_{232})t_{21}t_{13} \\
&= (t_{12}t_2)t_{21}t_{13} + t_{12}t_{232}t_{21}t_{13} \\
&= (t_{12}t_{21})t_{13} + t_{12}t_{232}t_{21}t_{13} \\
&= (t_1 + t_{121})t_{13} + t_{12}t_{232}t_{21}t_{13} \\
&= t_1t_{13} + t_{121}t_{13} + t_{12}t_{232}t_{21}t_{13} \\
&= t_{13} + t_{1213} + t_{123213}.
\end{aligned}$$

Example 5.4.5. Consider the algebra J_1 arising from the Coxeter system (W, S) given by the following diagram. Let $x = 121$, $y = 12321$, and let y_n denote the glued

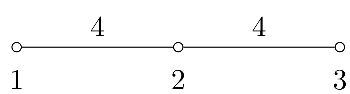

Figure 5.4.1: The Coxeter diagram of (W, S).

product $y \circ y \circ \cdots \circ y$ of n copies of y for each $n \in \mathbb{Z}_{\geq 1}$. It is easy to see that $\Gamma_1 \cap \Gamma_1^{-1}$ consists exactly of 1, x and all y_n where $n \geq 1$ so that J_1 has basis elements t_1, t_x and t_n $(n \geq 1)$ where we set $t_n := t_{y_n}$ for all $n \geq 1$. One efficient way to see this is to draw the subgraph D_1 of the subregular graph of the Coxeter system (see Section 2.4) and recall that elements of $\Gamma_1 \cap \Gamma_1^{-1}$ are in a bijection with the walks on D_1 which start at the top vertex and end at one of the diamond-shaped vertices.

Let us describe the products of all pairs of basis element in J_1. First, we have $t_1 t_w = t_w = t_w t_1$ for each basis element $t_w \in J_1$, as t_1 is the identity. For products involving t_x but not t_1, propositions 5.4.3 and 5.3.1 imply that $t_x t_x = t_{121}t_{121} = t_1$,

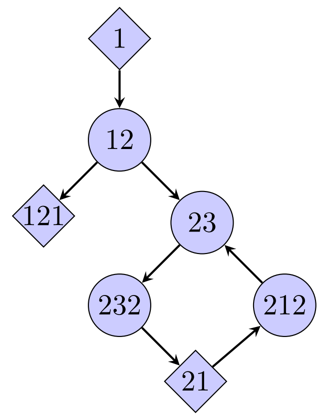

Figure 5.4.2: The graph D_1 for (W, S).

while

$$t_x t_n = t_{121} t_{12321\cdots} = t_{121} t_{12} t_{2321 \circ y_{n-1}} = t_{12} t_{2321 \circ y_{n-1}} = t_n \tag{5.4.4}$$

and similarly $t_n t_x = t_n$ for all $n \geq 1$ (where we set $y_0 = 1$). Finally, to describe products of the form $t_m t_n$ where $m, n \geq 1$, set $t_0 = t_1 + t_x$. Using computations similar to those in Equation (5.4.4), we can easily check that $t_1 t_n = t_{n-1} + t_{n+1}$ for all $n \geq 1$, then show by induction on m that

$$t_m t_n = t_{|m-n|} + t_{m+n} \tag{5.4.5}$$

for all $m, n \geq 1$. We have now described all pairwise products of basis elements in J_1.

CHAPTER VI

J_C AND THE COXETER DIAGRAM

Let (W, S) be an arbitrary Coxeter system, and let J_C be its subregular J-ring. We study the relationship between J_C and the Coxeter diagram of (W, S) in this section.

6.1. Simply-laced Coxeter Systems

Let us recall more graph-theoretic terminology. Let $G = (V, E)$ be an undirected graph. Recall from Section sec:subregular cell that a walk on on G is a sequence $P = (v_1, \cdots, v_k)$ of vertices in G such that $\{v_i, v_{i+1}\}$ is an edge for all $1 \leq i \leq k-1$. We define a *spur* on G to be a walk of the form (v, v', v) where $\{v, v'\}$ forms an edge. Given any walk containing a spur, i.e., a walk of the form $P_1 = (\cdots, u, v, v', v, u', \cdots)$, we may remove the spur to form a new walk $P_2 = (\cdots, u, v, u', \cdots)$; conversely, we can add a spur (v, v', v) to a walk of the form P_2 to obtain the walk P_1.

Recall that a *groupoid* may be viewed as a generalization of a group, in that it is defined to be a pair $(\mathcal{F}, \circ)$, where $\mathcal{F}$ is set and $\circ$ is a partially-defined binary operation on $\mathcal{F}$ that satisfy certain axioms (see [CdSW99]). More precisely, for any topological space X and a chosen subset A of X, the *fundamental groupoid of X based on A* is defined to be $\Pi(X, A) := (\mathcal{P}, \circ)$, where $\mathcal{P}$ are the homotopy equivalence classes of paths on X that connect points in A and $\circ$ is concatenation of paths. Given an undirected graph $G = (V, E)$, we may view G as embedded in a topological surface and hence as a topological space with the subspace topology induced from the surface. We define the *fundamental groupoid of G* to be $\Pi(G) := \Pi(G, V) = (\mathcal{P}, \circ)$, where $\mathcal{P}$ stands for paths (in the topological sense) on G.

Note that paths on G are just walks, and concatenation of paths correspond to concatenation of walks. More precisely, for any two walks $P = (v_1, \cdots, v_{k-1}, v_k)$ and $Q = (u_1, u_2 \cdots u_l)$ on G, we define their *concatenation* to be the walk $P \circ Q = (v_1, \cdots, v_{k-1}, v_k, u_2, \cdots, u_l)$ if $v_k = u_1$; otherwise we leave $P \circ Q$ undefined. Also note that two walks are homotopy equivalent if and only if they can be obtained from each other by a sequence of removals or additions of spurs, and each homotopy equivalence class of walks contains a unique walk with no spurs. We use $[P]$ to denote the class of a walk P. For each path $P = (v_1, v_2, \cdots, v_k)$, we also define its *inverse* to be the walk $P^{-1} := (v_k, \cdots, v_2, v_1)$.

Remark 6.1.1. Note that concatenations of walks are compatible with glued products of words defined in Definition 5.2.2 in that if x and y are reduced words of subregular elements for which their glued product $x \circ y$ is defined, then $P(x \circ y) = P(x) \circ P(y)$, where the map $w \mapsto P(w)$ is the natural map described Section 2.4. Also note that the notations introduced here are compatible with the notation $P \circ C^n \circ P^{-1}$ used in the proof of Theorem 2.4.6, in that the walk denoted by $P \circ C^n \circ P^{-1}$ is exactly the concatenation of the walk P, n copies of the walk C, and the inverse of P in that proof.

For each vertex s in G, we define the *fundamental group of G based at s* to be $\Pi_s(G) = (\mathcal{P}_s, \circ)$, where $\mathcal{P}_s$ are now equivalence classes of walks on G that start and end with s, and $\circ$ is concatenation as before. Note that $\Pi_s(G)$ is actually a group, so it makes sense to talk about the its group algebra $\mathbb{Z}\Pi_s(G)$ over $\mathbb{Z}$. We may define a counterpart of $\mathbb{Z}\Pi_s(G)$ for $\Pi(G)$ by mimicking the construction of a group algebra.

Definition 6.1.2. Let $\Pi(G) = (\mathcal{P}, \circ)$ be the fundamental groupoid of a graph G. We define the *groupoid algebra* of $\Pi(F)$ over $\mathbb{Z}$ to be the free abelian group

$\mathbb{Z}\mathcal{P} = \oplus_{[P]\in\mathcal{P}}\mathbb{Z}[P]$ equipped with an $\mathbb{Z}$-bilinear multiplication $\cdot$ defined by

$$[P]\cdot[Q] = \begin{cases} [P\circ Q] & \text{if } P\circ Q \text{ is defined in } G,\\ 0 & \text{if } P\circ Q \text{ is not defined.}\end{cases}$$

Note that $\mathbb{Z}\Pi(G)$ is clearly associative.

Remark 6.1.3. Recall the well-known notion of a *quiver* and its *path algebra* (see, for example, [Bri12]), and let Q be the quiver that has G as its underlying undirected graph and has an arrow e_{ab} pointing from the vertex a to the vertex b for each edge $\{a, b\}$ in G. Then $\mathbb{Z}\Pi(G)$ may be viewed as the quotient of the path algebra of Q over $\mathbb{Z}$ modulo the relations $e_{ab}e_{ba} = e_a$ for all edges (a, b) in Q, where e_a refers to the trivial path that starts and terminates at a.

Proposition 6.1.4. *Let $G = (V, E)$ where V is finite. Let P_s be the constant walk (s) for all $s \in V$. Then the groupoid algebra $\mathbb{Z}\Pi(G)$ has the structure of a based ring with basis $\{[P]\}_{[P]\in\mathcal{P}}$, with unit 1=$\sum_{s\in V}[P_s]$ (so the distinguished index set simply corresponds to V), and with its anti-involution induced by the map $[P]\mapsto[P^{-1}]$.*

For each $s \in V$, the group algebra $\mathbb{Z}\Pi_s(G)$ has the structure of a unital based ring with basis $\{[P]\}_{[P]\in\mathcal{P}_s}$, with unit $1 = [P_s]$ (so the distinguished index set is simply $\{s\}$), and with its anti-involution induced by the map $[P]\mapsto[P^{-1}]$.

Proof. All the claims are easy to check using definitions. $\square$

Now, suppose (W, S) is a simply-laced Coxeter system, and let G be its Coxeter diagram. Recall that this means $m(s, t) = 3$ for $s, t \in S$ whenever $\{s, t\}$ is an edge in G while $m(s, t) = 2$ otherwise. Let us consider the map $C \to \Pi(G)$ which sends each element $x = s_1\cdots s_k \in C$ to the homotopy equivalence class $[P(x)]$ where $P(x)$ is the walk $(s_1, s_2, \cdots, s_k)$ as in Section 2.4. We claim this is a bijection.

To see this, note that we must have $m(s_i, s_{i+1}) = 3$ for each $1 \leq i \leq k-1$, so $s_{i+2} \neq s_i$ for all $1 \leq i \leq k-2$ by Proposition 2.4.5, therefore P_x contains no spurs. This means $P(x)$ is exactly the unique representative with no spurs in its class. Conversely, given class of walks in $\Pi(G)$, we may take its unique representative $(s_1, \cdots, s_k)$ with no spurs and consider the word $s_1 \cdots s_k$. By Proposition 2.4.5, $s_1 \cdots s_k$ is the reduced word of an element in C. This gives a two-sided inverse to the map $x \mapsto [P(x)]$.

Since C and $\mathcal{P}$ index the basis elements of J_C and $\mathbb{Z}\Pi(G)$, respectively, the bijection $x \mapsto [P(x)]$ induces a unique $\mathbb{Z}$-module isomorphism $\Phi : J_C \to \mathbb{Z}\Pi(G)$ defined by

$$\Phi(t_x) = [P(x)], \qquad \forall x \in C. \tag{6.1.1}$$

We are now ready to prove Theorem A, which is restated below.

Theorem A. *Let (W, S) be an any simply-laced Coxeter system, and let G be its Coxeter diagram. Let $\Pi(G)$ be the fundamental groupoid of G, let $\Pi_s(G)$ be the fundamental group of G based at s for any $s \in S$, let $\mathbb{Z}\Pi(G)$ be the groupoid algebra of $\Pi(G)$, and let $\mathbb{Z}\Pi_s(G)$ be the group algebra of $\Pi_s(G)$. Then $J_C \cong \mathbb{Z}\Pi(G)$ as based rings, and $J_s \cong \mathbb{Z}\Pi_s(G)$ as based rings for all $s \in S$.*

Proof. We show that the $\mathbb{Z}$-module isomorphism $\Phi : J_C \to \mathbb{Z}\Pi(G)$ defined by Equation 6.1.1 is an algebra isomorphism. This would imply $J_s \cong \mathbb{Z}\Pi_s(G)$ for all $s \in S$, since Φ clearly restricts to a $\mathbb{Z}$-module map from J_s to $\mathbb{Z}\Pi_s(G)$. The fact that Φ and the restrictions are actually isomorphisms of based rings is clear once we compare the based ring structure of $J_C, \mathbb{Z}\Pi(G), J_s$ and $\mathbb{Z}\Pi_s(G)$ described in Corollary 4.3.5 and Proposition 6.1.4.

To show Φ is an algebra homomorphism, we need to show

$$[P(x)] \cdot [P(y)] = \Phi(t_x t_y) \tag{6.1.2}$$

for all $x, y \in C$. Let $s_k \cdots s_1$ and $u_1 \cdots u_l$ be the reduced word of x and y, respectively. If $s_1 \neq u_1$, then Equation (6.1.2) holds since both sides are zero by Definition 6.1.2 and Corollary 5.4.2. If $s_1 = u_1$, let $q \leq \min(k, l)$ be the largest integer such that $s_i = u_i$ for all $1 \leq i \leq q$. Then

$$\begin{aligned}
[P_x] \cdot [P_y] &= [(s_k, \cdots, s_{q+1}, s_q, \cdots, s_1) \circ (s_1, \cdots, s_q, u_{q+1}, \cdots, u_l)] \\
&= [(s_k, \cdots, s_{q+1}, s_q, \cdots, s_2, s_1, s_2, \cdots, s_q, u_{q+1}, \cdots, u_l)] \\
&= [(s_k, \cdots, s_{q+1}, s_q, u_{q+1}, \cdots, u_l)],
\end{aligned}$$

where the last equality holds by successive removal of spurs of the form (s_{i+1}, s_i, s_{i+1}). On the other hand, since $m(s_i, s_{i+1}) = 3$ for each $1 \leq i \leq q$, Proposition 5.3.1 implies

$$t_{s_{i+1}s_i} t_{s_i s_{i+1}} = t_{s_{i+1}}, \quad t_{s_i} t_{s_i s_{i+1}} = t_{s_i s_{i+1}}, \tag{6.1.3}$$

therefore by Theorem F,

$$\begin{aligned}
t_x t_y &= (t_{s_k\cdots s_{q+1}s_q} t_{s_q s_{q-1}} \cdots t_{s_3 s_2} t_{s_2 s_1})(t_{s_1 s_2} t_{s_2 s_3} \cdots t_{s_{q-1} s_q} t_{s_q u_{q+1}\cdots u_l}) \\
&= (t_{s_k\cdots s_{q+1}s_q} t_{s_q s_{q-1}} \cdots t_{s_3 s_2}) t_{s_2} (t_{s_2 s_3} \cdots t_{s_{q-1} s_q} t_{s_q u_{q+1}\cdots u_l}) \\
&= (t_{s_k\cdots s_{q+1}s_q} t_{s_q s_{q-1}} \cdots t_{s_3 s_2})(t_{s_2 s_3} \cdots t_{s_{q-1} s_q} t_{s_q u_{q+1}\cdots u_l}) \\
&= \cdots \\
&= t_{s_k\cdots s_{q+1}s_q} t_{s_q u_{q+1}\cdots u_l} \\
&= t_{s_k\cdots s_{q+1} s_q u_{q+1}\cdots s_l'}.
\end{aligned}$$

Here, the last equality follows from Proposition 5.4.3, and the "$\cdots$" signify repeated use of the equations in (6.1.3) to "remove" the products of the form $(t_{s_{i+1}} t_{s_i})(t_{s_i} t_{s_{i+1}})$ where $2 \le i \le q-1$. By the definition of Φ, we then have

$$\Phi(t_x t_y) = [(s_k, \cdots, s_{q+1}, s_q, u_{q+1}, \cdots, u_l)].$$

It follows that $[P(x)] \cdot [P(y)] = \Phi(t_x t_y)$, and we are done. $\square$

6.2. Oddly-connected Coxeter Systems

Define a Coxeter system (W, S) to be *oddly-connected* if for every pair of vertices s, t in its Coxeter diagram G, there is a walk in G of the form $(s = v_1, v_2, \cdots, v_k = t)$ where the edge weight $m(v_i, v_{i+1})$ is odd for all $1 \le i \le k-1$. In this subsection, we discuss how the odd-weight edges affect the structure of the algebras J_C and J_s $(s \in S)$.

We need some relatively heavy notation.

Definition 6.2.1. For any $s, t \in S$ such that $M = m(s,t)$ is odd:

(a) We define

$$z(st) = sts \cdots t$$

to be the alternating word of length $M - 1$ that starts with s. Note that it necessarily ends with t now that M is odd.

(b) We define maps $\lambda_s^t, \rho_t^s : J_C \to J_C$ by

$$\lambda_s^t(t_x) = t_{z(ts)} t_x,$$

$$\rho_s^t(t_x) = t_x t_{z(st)},$$

and define the map $\phi_s^t : J_C \mapsto J_C$ by

$$\phi_s^t(t_x) = \rho_s^t \circ \lambda_s^t(t_x)$$

for all $x \in C$.

Remark 6.2.2. The notation above is set up in the following way. The letters λ and ρ indicate a map is multiplying its input by an element on the left and right, respectively. The subscripts and superscripts are to provide mnemonics for what the maps do on the reduced words indexing the basis elements of J_C: by Corollary 5.4.2, λ_s^t maps $J_{\Gamma_s^{-1}}$ to $J_{\Gamma_t^{-1}}$ and vanishes on $J_{\Gamma_h^{-1}}$ for any $h \in S \setminus \{s\}$. Similarly, ρ_s^t maps J_{Γ_s} to J_{Γ_t} and vanishes on J_{Γ_h} for any $h \in S \setminus \{s\}$.

Proposition 6.2.3. *Let s, t be as in Definition 6.2.1. Then*

(a) $\rho_s^t \circ \lambda_s^t = \lambda_s^t \circ \rho_s^t$.

(b) $\rho_t^s \circ \rho_s^t(t_x) = t_x$ *for any* $x \in \Gamma_s$, $\lambda_t^s \circ \lambda_s^t(t_x) = t_x$ *for any* $x \in \Gamma_s^{-1}$.

(c) $\rho_s^t(t_x)\lambda_s^t(t_y) = t_x t_y$ *for any* $x \in \Gamma_s, y \in \Gamma_s^{-1}$.

(d) The restriction of ϕ_s^t *on* J_s *gives an isomorphism of based rings from* J_s *to* J_t.

Proof. Part (a) holds since both sides of the equation sends t_x to $t_{z(ts)}t_x t_{z(st)}$. Parts (b) and (c) are consequences of the truncated Clebsch-Gordan rule. By the rule,

$$t_{z(st)}t_{z(ts)} = t_s,$$

therefore $\rho_t^s \circ \rho_s^t(t_x) = t_x t_s = t_x$ for any $x \in \Gamma_s$ and $\lambda_t^s \circ \lambda_s^t(t_x) = t_s t_x$ for any $x \in \Gamma_s^{-1}$; this proves (b). Meanwhile, $\rho_s^t(t_x)\lambda_s^t(t_y) = t_x t_{z(st)} t_{z(ts)} t_y = t_x t_s t_y = t_x t_y$ for any $x \in \Gamma_s, y \in \Gamma_s^{-1}$; this proves (c).

For part (d), the fact that ϕ_s^t maps J_s to J_t follows from Remark 6.2.2. To see that is a (unit-preserving) algebra homomorphism, note that

$$\phi_s^t(t_s) = t_{z(ts)}t_s t_{z(st)} = t_{z(ts)}t_{z(st)} = t_t,$$

and for all $t_x, t_y \in J_s$,

$$\phi_s^t(t_x)\phi_s^t(t_y) = (\rho_s^t(\lambda_s^t(t_x)) \cdot (\lambda_s^t(\rho_s^t(t_y)) = \lambda_s^t(t_x) \cdot \rho_s^t(t_y) = \phi_s^t(t_x t_y)$$

by parts (a) and (c). We can similarly check ϕ_t^s is an algebra homomorphism from J_t to J_s. Finally, using calculations similar to those used for part (b), it is easy to check that ϕ_s^t and ϕ_t^s are mutual inverses , therefore ϕ_s^t is an algebra isomorphism.

It remains to check that the restriction is an isomorphism of based rings. In light of Proposition 4.3.5, this means checking that $\phi_s^t(t_{x^{-1}}) = (\phi_s^t(t_x))^*$ for each $t_x \in J_s$, where * is the linear map sending t_x to $t_{x^{-1}}$ for each $t_x \in J_s$. This holds because

$$\phi_s^t(t_{x^{-1}}) = t_{z(ts)}t_{x^{-1}}t_{z(st)} = (t_{z(st)^{-1}}t_xt_{z(ts)^{-1}})^* = (t_{z(ts)}t_xt_{z(st)})^* = (\phi_s^t(t_x))^*,$$

where the second equality follows from the definition of * and the fact that $t_x \mapsto t_{x^{-1}}$ defines an anti-homomorphism in J (see Corollary 4.1.1). $\square$

Now we upgrade the definitions and propositions from a single edge to a walk.

Definition 6.2.4. For any walk $P = (u_1, \cdots, u_l)$ in G where $m(u_k, u_{k+1})$ is odd for all $1 \leq k \leq l-1$, we define maps λ_P, ρ_P by

$$\lambda_P = \lambda_{u_{l-1}}^{u_l} \circ \cdots \lambda_{u_2}^{u_3} \circ \lambda_{u_1}^{u_2},$$

$$\rho_P = \rho_{u_{l-1}}^{u_l} \circ \cdots \rho_{u_2}^{u_3} \circ \rho_{u_1}^{u_2},$$

and define the map $\phi_P : J_C \to J_C$ by

$$\phi_P = \lambda_P \circ \rho_P.$$

Proposition 6.2.5. *Let $P = (u_1, \cdots, u_l)$ be as in Definition 6.2.4 Then*

(a) $\phi_P = \phi_{u_{l-1}}^{u_l} \circ \cdots \circ \phi_{u_2}^{u_3} \circ \phi_{u_1}^{u_2}$.

(b) $\rho_{P^{-1}} \circ \rho_P(t_x) = t_x$ *for any* $x \in \Gamma_{u_1}$, $\lambda_{P^{-1}} \circ \lambda_P(t_x) = t_x$ *for any* $x \in \Gamma_{u_1}^{-1}$.

(c) $\rho_P(t_x)\lambda_P(t_y) = t_xt_y$ *for any* $x \in \Gamma_{u_1}, y \in \Gamma_{u_l}^{-1}$.

(d) The restriction of ϕ_P gives an isomorphism of based rings from J_{u_1} to J_{u_l}.

Proof. Part (1) holds since each left multiplication $\lambda_{u_k}^{u_{k+1}}$ commutes with all right multiplications $\rho_{u_{k'}}^{u_{k'}+1}$. Part (2)-(4) can be proved by writing out each of the maps as a composition of $(l-1)$ appropriate maps corresponding to the $(l-1)$ edges of P and then repeatedly applying their counterparts in Proposition 6.2.5 on the composition components. In particular, part (4) holds since ϕ_P is a composition of isomorphisms of based rings is clearly another isomorphism of based rings. $\square$

We are almost ready to prove Theorem B. Let us restate it here.

Theorem B. *Let (W, S) be an oddly-connected Coxeter system. Then*

(a) $J_s \cong J_t$ *as based rings for all* $s, t \in S$.

(b) $J_C \cong \mathrm{Mat}_{S\times S}(J_s)$ *as based rings for all* $s \in S$. *In particular,* J_C *is Morita equivalent to* J_s *for all* $s \in S$.

Here, for each fixed $s \in S$, the algebra $\mathrm{Mat}_{S\times S}(J_s)$ is the matrix algebra of matrices with rows and columns indexed by S and with entries from J_s. For any $a, b \in S$ and $f \in J_s$, let $E_{a,b}(f)$ be the matrix in $\mathrm{Mat}_{S\times S}(J_s)$ with f at the a-row, b-column and zeros elsewhere. We explain how $\mathrm{Mat}_{S\times S}(J_s)$ is a based ring below.

Proposition 6.2.6. *The ring* $\mathrm{Mat}_{S\times S}(J_s)$ *is a based ring with basis* $\{E_{a,b}(t_x) : a, b \in S, x \in \Gamma_s \cap \Gamma_s^{-1}\}$, *with unit element* $1 = \sum_{s\in S} E_{s,s}(t_s)$, *and with its anti-involution induced by* $E_{a,b}(t_x)^* = E_{b,a}(t_{x^{-1}})$.

Proof. Note that for any $a, b, c, d \in S$ and $f, g \in J_s$,

$$E_{a,b}(f)E_{c,d}(g) = \delta_{b,c}E_{a,d}(fg). \tag{6.2.4}$$

The fact that $\mathrm{Mat}_{S\times S}(J_s)$ is a unital $\mathbb{Z}_+$-ring with $1 = \sum_{s\in S} E_{s,s}(t_s)$ is then straightforward to check. Next, note that

$$(E_{a,b}(f)E_{c,d}(g))^* = 0 = (E_{c,d}(g))^*(E_{a,b}(f))^*$$

when $b \neq c$. When $b = c$,

$$(E_{a,b}(t_x)E_{c,d}(t_y))^* = (E_{a,d}((t_x t_y)))^* = E_{d,a}(t_{y^{-1}}t_{x^{-1}}) = (E_{c,d}(t_y))^*(E_{a,b}(t_x))^*$$

where, like in the proof of Proposition 6.2.3, the second equalities follows from the fact that the map $t_x \mapsto t_{x^{-1}}$ induces an anti-homomorphism of J. The last two displayed equations imply that * induces an anti-involution of $\mathrm{Mat}_{S\times S}(J_s)$. Finally, note that $E_{u,u}(t_s)$ appears in $E_{a,b}(t_x)E_{c,d}(t_y) = \delta_{b,c}E_{a,d}(t_x t_y)$ for some $u \in S$ if and only if $b = c, a = d = u$ and $x = y^{-1}$ (for t_s appears in $t_x t_y$ if and only if $x = y^{-1}$; see Corollary 4.3.5). This proves that Equation (4.3.1) from Definition 4.3.2 holds, and we have completed all the necessary verifications. $\square$

Proof of Theorem B. Part (1) follows from the last part of Proposition 6.2.5, since there is a walk $(s = u_1, u_2, \cdots, u_l = t)$ in G that contains only odd-weight edges now that (W, S) is oddly-connected.

To prove (2), fix $s \in S$. For each $t \in S$, fix a walk $P_{st} = (s = u_1, \cdots, u_l = t)$ and define $P_{ts} = P_{st}^{-1}$. Write λ_{st} for $\lambda_{P_{st}}$, and define $\rho_{st}, \lambda_{ts}, \rho_{ts}$ similarly. Consider the unique $\mathbb{Z}$-module map

$$\Psi : J_C \to \mathrm{Mat}_{S\times S}(J_s)$$

defined as follows: for any $t_x \in J_C$, say $x \in \Gamma_a^{-1} \cap \Gamma_b$ for $a, b \in S$, let

$$\Psi(t_x) = E_{a,b}(\lambda_{as} \circ \rho_{bs}(t_x)).$$

We first show below that Ψ is an algebra isomorphism.

Let $t_x, t_y \in J_C$. Suppose $x \in \Gamma_a^{-1} \cap \Gamma_b$ and $y \in \Gamma_c^{-1} \cap \Gamma_d$ for $a, b, c, d \in S$. If

$b \neq c$, then

$$\Psi(t_x)\Psi(t_y) = 0 = \Psi(t_x t_y),$$

where the first equality follows from Equation (6.2.4) and the second equality holds since $t_x t_y = 0$ by Corollary 5.4.2. If $b = c$, then

$$\begin{aligned}
\Psi(t_x)\Psi(t_y) &= E_{a,b}(\lambda_{as} \circ \rho_{bs}(t_x)) \cdot E_{c,d}(\lambda_{cs} \circ \rho_{ds}(t_y)) \\
&= E_{a,d}([\lambda_{as} \circ \rho_{bs}(t_x)] \cdot [\lambda_{bs} \circ \rho_{ds}(t_y)]) \\
&= E_{a,d}((\lambda_{as} \circ \rho_{ds})[\rho_{bs}(t_x) \cdot \lambda_{bs}(t_y)]) \\
&= E_{a,d}((\lambda_{as} \circ \rho_{ds})[t_x t_y]) \\
&= \Psi(t_x t_y),
\end{aligned}$$

where the second last equality holds by part (3) of Proposition 6.2.5. It follows that Ψ is an algebra homomorphism. Next, consider the map

$$\Psi' : \mathrm{Mat}_{S\times S}(J_s) \to J_C$$

defined by

$$\Psi'(E_{a,b}(f)) = \lambda_{sa} \circ \rho_{sb}(f)$$

for all $a, b \in S$ and $f \in J_s$. Using Part (2) of Proposition 6.2.5, it is easy to check that Ψ and Ψ' are mutual inverses as maps of sets. It follows that Ψ is an algebra isomorphism. Finally, it is easy to compare Proposition 4.3.5 with Proposition 6.2.6 and check that Ψ is an isomorphism of based rings by direct computation. □

Remark 6.2.7. The conclusions of the theorem fail in general when (W, S) is not oddly-connected. As a counter-example, consider based rings J_1 and J_2 arising from the Coxeter system in Example 5.4.5. By the truncated Clebsch-Gordan rule,

$$t_{212}t_{212} = t_2 = t_{232}t_{232},$$

therefore J_2 contains at least two basis elements with multiplicative order 2. However, it is evident from Example 5.4.5 that t_{121} is the only basis element of order 2 in J_1. This implies that J_1 and J_2 are not isomorphic as based rings. Moreover, Equation (6.2.4) implies that for any $s \in S$, the basis elements of $\mathrm{Mat}_{S\times S}(J_s)$ of order 2 must be of the form $E_{u,u}(t_x)$ where $u \in S$ and t_x is a basis element of order 2 in J_s, so $\mathrm{Mat}_{S\times S}(J_1)$ and $\mathrm{Mat}_{S\times S}(J_2)$ have different numbers of basis elements of order 2 as well. It follows that Part (2) of the theorem also fails.

Remark 6.2.8. The isomorphism between J_s and J_t can be easily lifted to a tensor equivalence between their categorifications $\mathcal{J}_s$ and $\mathcal{J}_t$, the subcategories of the category $\mathcal{J}$ mentioned in the introduction that correspond to $\Gamma_s \cap \Gamma_s^{-1}$ and $\Gamma_t \cap \Gamma_t^{-1}$.

Let us end the section by revisiting an earlier example.

Example 6.2.9. Let (W, S) be the Coxeter system from Example 2.4.8, whose Coxeter diagram is shown in Figure 2.4.2. Clearly, (W, S) is oddly-connected, hence $J_3 \cong J_2 \cong J_1$ and $J_C \cong \mathrm{Mat}_{3\times 3}(J_1)$ by Theorem B. Let us study J_1.

Recall that elements of $\Gamma_1 \cap \Gamma_1^{-1}$ correspond to walks on the graph D_1 shown in Figure 2.4.4 that start with the top vertex and end with either the bottom-left or bottom-right vertex. Observe that all such walks can be obtained by concatenating the walks corresponding to the elements $x = 1231, y = 1321, z = 12321, w = 13231$. This means that any reduced word in $\Gamma_1 \cap \Gamma_1^{-1}$ can be written as glued products of x, y, z, w, which implies that t_x, t_y, t_z, t_w generate J_1 by Theorem F and Proposition 5.4.3. Computing the products of these elements reveals that J_1 can be described as the algebra generated by t_x, t_y, t_z, t_w subject to the following six relations:

$$t_x t_y = 1 + t_z,\ t_y t_x = 1 + t_w,\ t_x t_w = t_x = t_z t_x,\ t_y t_z = t_y = t_w t_y,\ t_w^2 = 1 = t_z^2.$$

The first two of the relations show that $t_z = t_x t_y - 1, t_w = t_y t_x - 1$, whence the other four relations can be expressed in terms of only t_x and t_y. Easy calculations then show that J_1 can be presented as the algebra generated by t_x, t_y subject to only the following two relations:

$$t_x t_y t_x = 2t_x,\ t_y t_x t_y = 2t_y.$$

Finally, via the change of variables $X := t_x/2, Y := t_y$, we see that

$$J_1 = \langle X, Y \rangle / \langle XYX = X, YXY = Y \rangle.$$

A simple presentation like this is helpful for studying representations of J_1 and hence J_2, J_3 and J_C.

6.3. Fusion J_s

In this subsection, we describe all fusion rings arising in the form J_s from a Coxeter system (W, S). Recall that for J_s to be a fusion ring, Theorem 4.3.7 places a strong restriction on what the Coxeter diagram of (W, S) could be; this will in turn force the structure of J_s to be rather restricted:

Theorem C. *Let (W, S) be a Coxeter system, and let $s \in S$. Suppose J_s is a fusion ring for some $s \in S$. Then there exists a dihedral Coxeter system (W', S') such that $J_t \cong J_{s'}$ as based rings for all $t \in S$ and for both $s' \in S'$.*

Proof. Let G be the Coxeter diagram of (W, S), and suppose J_s is a fusion ring for some $s \in S$. Then by Theorem 4.3.7, either G is a tree and (W, S) is simply-laced, or G is a tree and there exists a unique pair $a, b \in S$ such that $m(a, b) > 3$.

In the first case where (W, S) is simply-laced, A implies that J_t is isomorphic to group algebra of the fundamental group $\Pi_t(G)$ for all $t \in S$, and the group is trivial since G is a tree. This means J_t is isomorphic to a ring of the form $J_{s'}$ associated with the dihedral system (W', S') with $S' = \{s', s''\}$ and $m(s', s'') = 3$.

In the second case, let $m(a, b) = M$ and let $t \in S$. By the description of G, there must be a walk P in G from t to either a or b such all the edges in the walk have weight 3, so Part (d) of Proposition 6.2.5 implies that J_t is isomorphic to either J_a or J_b as based rings. Without loss of generality, suppose $J_s \cong J_a$. By the description of the set $\{P(w) : w \text{ is the reduced word of a subregular element}\}$ in the proof of Theorem 2.4.6, $\Gamma_a \cap \Gamma_a^{-1}$ contains exactly the elements $a, aba, \cdots, ab \cdots a$ where the reduced words alternate in a, b and contains less than M letters. This means that J_t is isomorphic as a based ring to the fusion ring $J_{s'}$ associated with the dihedral system (W', S') with $S' = \{s', s''\}$ where $m(s', s'') = M$. $\square$

Remark 6.3.1. Recall from 5.3 that any algebra of the form J_s arising from a dihedral Coxeter system is isomorphic to the odd part Ver_M^{odd} of a Verlinde algebra, where $M \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$. Thus, the theorem means that any fusion ring of the form J_s arising from any Coxeter system (W, S) is isomorphic to Ver_M^{odd} for some M as well. Moreover, the proof of the theorem reveals that M can be described simply as the largest edge weight in the Coxeter diagram of (W, S).

CHAPTER VII

FREE FUSION RINGS

We focus on certain Coxeter systems (W, S) whose Coxeter diagrams involve edges of weight ∞ in this section. We show that for suitable choices of $s \in S$, J_s is isomorphic to a *free fusion ring*.

7.1. Background

Free fusion rings are defined as follows.

Definition 7.1.1 ([Rau12]). A *fusion set* is a set A equipped with an involution $^{-}\colon A \to A$ and a *fusion* map $\diamond : A \times A \to A \cup \emptyset$. Given any fusion set $(A, ^{-}, \diamond)$, we extend the operations $^{-}$ and $\diamond$ to the free monoid $\langle A \rangle$ as follows:

$$\overline{a_1 \cdots a_k} = \bar{a}_k \cdots \bar{a}_1,$$

$$(a_1 \cdots a_k) \diamond (b_1 \cdots b_l) = a_1 \cdots a_{k-1}(a_k \diamond b_1) b_2 \cdots b_l,$$

where the right side of the last equation is taken to be $\emptyset$ whenever $k = 0, l = 0$ or $a_k \diamond b_1 = \emptyset$. We then define the *free fusion ring* associated with the fusion set $(A, ^{-}, \diamond)$ to be the free abelian group $R = \mathbb{Z}\langle A \rangle$ on $\langle A \rangle$, with multiplication $\cdot : R \times R \to R$ given by

$$v \cdot w = \sum_{v = xz, w = \bar{z}y} xy + x \diamond y \tag{7.1.1}$$

for all $v, w \in \langle A \rangle$, where xy means the juxtaposition of x and z.

It is well known that $\cdot$ is associative (see [Rau12]). It is also easy to check that R is a unital based ring with its basis given by the free monoid $\langle A \rangle$, with unit given by the empty word, and with its anti-involution $^* : \langle A \rangle \to \langle A \rangle$ given by the map $^{-}$.

Free fusion rings were introduced in [Rau12] to capture the tensor rules in certain semisimple tensor categories arising from the theory of operator algebras. More specifically, the categories are categories of representations of *compact quantum groups*, and their Grothendieck rings fit the axiomatization of free fusion rings in Definition 7.1.1. In [Fre14], A. Freslon classified all free fusion rings arising as the Grothendieck rings of compact quantum groups in terms of their underlying fusion sets. Further, while a free fusion ring may appear as the Grothendieck ring of multiple non-isomorphic compact quantum groups, Freslon described a canonical way to associate a *partition quantum group*—a special type of compact quantum group—to any free fusion ring arising from a compact quantum group. These special quantum groups correspond via a type of Schur-Weyl duality to *categories of non-crossing partitions*, which can in turn be used to study the representations of the quantum groups.

All the free fusion rings appearing as J_s in our examples fit in the classification of [Fre14]. In each of our examples, we will identify the associated partition quantum group $\mathbb{F}$. The fact that J_s is connected to $\mathbb{F}$ is intriguing, and it would be interesting to see how the categorification of J_s arising from Soergel bimodules connects to the representations of $\mathbb{F}$ on the categorical level.

7.2. Example 1: O_N^+

One of the simplest fusion sets is the singleton set $A = \{a\}$ with identity as its involution and with fusion map $a \diamond a = \emptyset$. The associated free fusion ring is $R =$

$\oplus_{n\in\mathbb{Z}_{\geq 0}}\mathbb{Z}a^n$, where

$$a^k \cdot a^l = a^{k+l} + a^{k+l-2} + \cdots + a^{|k-l|}$$

by Equation 7.1.1. The partition quantum group associated to R is the *free orthogonal quantum group* O_N^+, and its corresponding category of partitions is that of all noncrossing *pairings*; see [Ban96] and [BS09].

Let (W,S) be the infinite dihedral system with $S=\{1,2\}$ and $W=I_2(\infty)$, the infinite dihedral group. We claim that J_1 is isomorphic to R as based rings. To see this, recall from the discussion following Definition 5.3.3 that J_s is the $\mathbb{Z}$-span of basis elements t_{1_n}, where n is odd and $1_n = 121\cdots 1$ alternates in $1,2$ and has length n. For $m=2k+1$ and $n=2l+1$ for some $k,l\geq 1$, the truncated Clebsch-Gordan rule implies that

$$t_{1_m}\cdot t_{1_n} = t_{1_{2k+1}}t_{1_{2l+1}} = t_{1_{2(k+l)+1}} + t_{1_{2(k+l-1)+1}} + \cdots + t_{1_{2|k-l|+1}}.$$

It follows that $R\cong J_1$ as based rings via the unique $\mathbb{Z}$-module map with $a^k\mapsto t_{1_{2k+1}}$ for all $k\in\mathbb{Z}_{\geq 0}$. Similarly, $R\cong J_2$ as based rings.

7.3. Example 2: U_N^+

In this subsection we consider the free fusion ring R arising from the fusion set $A=\{a,b\}$ with $\bar{a}=b$ and $a\diamond a = a\diamond b = b\diamond a = b\diamond a = \emptyset$. The partition quantum group associated to R is the *free unitary quantum group* U_N^+. In the language of [Fre14], this quantum group corresponds to the category of $\mathcal{A}$-*colored* noncrossing partitions where $\mathcal{A}$ is a *color set* containing two colors *inverse* to each other.

Consider the Coxeter system (W,S) given by the Coxeter diagram G below.

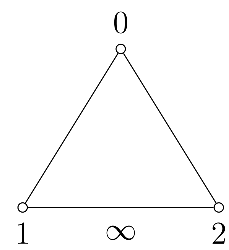

Figure 7.3.1: The Coxeter diagram of (W, S).

Theorem D. *We have a isomorphism $R \cong J_0$ of based rings.*

Our strategy to prove Theorem D is to describe a bijection between the free monoid $\langle A \rangle$ and the set $\Gamma_0 \cap \Gamma_0^{-1}$, use it to define a $\mathbb{Z}$-module isomorphism from R to J_0, then show that it is an isomorphism of based rings. To establish the bijection, recall the definition of the walk $P(x)$ for each $x \in C$ from Section 2.4 , then encode each $x \in \Gamma_s \cap \Gamma_s^{-1}$ by a word in $\langle A \rangle$ in the following way: travel along the walk $P(x)$, write down an "a" every time an edge in the walk goes from 1 to 2, a "b" every time an edge goes from 2 to 1, and write nothing down otherwise. Call the resulting word w_x. For example, the element $x = 012120120$ corresponds to the word $w_x = abaa$. Note that w_x records all parts of $P(x)$ that travel along the edge $\{1, 2\}$, but "ignores" the parts that involve the edges containing 0.

We claim that the map $\varphi : \Gamma_0 \cap \Gamma_0^{-1} \to \langle A \rangle, x \mapsto w_x$ gives our desired bijection. To see that φ is injective, note that by Proposition 2.4.5, the elements of $\Gamma_0 \cap \Gamma_0^{-1}$ correspond to walks on G that start and end with 0 but contain no spurs involving 0. The latter condition means that the parts of the walk $P(x)$ that are "ignored" in w_x, i.e., the parts involving the edges $\{0, 1\}$ or $\{0, 2\}$, can be recovered from w_x. More precisely, given any word $w = w_x$ for some $x \in \Gamma_0 \cap \Gamma_0^{-1}$, we may read the letters of w from left to right and write down P_x using the following principles:

(a) The empty word $w = \emptyset$ corresponds to the element $0 \in \Gamma_0 \cap \Gamma_0^{-1}$, for $P(x)$ involves the edge $\{1,2\}$ for any other element of the intersection.

(b) The only way w can start with a is for $P(x)$ to start with 0, immediately travel to 1, then travel from 1 to 2, so $P(x)$ must start with (0,1,2) if w start with a. Similarly, $P(x)$ starts with $(0,2,1)$ if w starts with b.

(c) If the last letter we have read from w is an "a", the last vertex we have recovered in the sequence for $P(x)$ must be 2.

    (a) If this "a" is the last letter of w, $P(x)$ must involve no more traversals of the edge $\{1,2\}$ and hence immediately return to 2 from 0, so adding one more 0 to the current sequence returns $P(x)$.

    (b) If the "a" is followed by another "a", the next traversal of $\{1,2\}$ in $P(x)$ after the sequence already written down must be from 1 to 2 again. This forces $P(x)$ to travel to 0 next, and to avoid a spur it must go on to 1, then to 2, so we add (0,1,2) to the sequence for $P(x)$. If the "a" is followed by a "b", $P(x)$ must next immediately travel to 1 and we add 1 to the sequence, for otherwise $P(x)$ would have to travel along the cycle $2 \to 0 \to 1 \to 2$ as we just described and the "a" would be followed by another "a".

(d) If the last letter we have read from w is an "b", the last vertex we have recovered in the sequence for $P(x)$ must be 1. The method to recover more of $P(x)$ from the rest of w is similar to the one described in (3).

To illustrate the recovery of w_x from x, suppose we know $abaa = w_x$ for some $x \in \Gamma_0 \cap \Gamma_0^{-1}$, we would get $P_x = (0,1,2,1,2,0,1,2,0)$ by successively writing down

$(0,1,2),(1),(2),(0,1,2)$ and (0), so $x = 012120120$. Indeed, note that we may run the process for any word w in $\langle A \rangle$ to get an element in $\Gamma_0 \cap \Gamma_0^{-1}$. This gives us a map $\phi : \langle A \rangle \to \Gamma_0 \cap \Gamma_0^{-1}$ that is a mutual inverse to φ and ϕ, so both φ and ϕ are bijective.

We can now prove Theorem D. We present an inductive proof that can be easily adapted to prove Theorem E later.

Proof of Theorem D. Let $\Phi : R \to J_0$ be the $\mathbb{Z}$-module homomorphism defined by

$$\Phi(w) = t_{\phi(w)}.$$

Since ϕ is a bijection, this is an isomorphism of $\mathbb{Z}$-modules. We will show that Φ is an algebra isomorphism by showing that

$$\Phi(v)\Phi(w) = \Phi(v \cdot w) \tag{7.3.2}$$

for all $v, w \in \langle A \rangle$. Note that this is true if v or w is empty, since then $t_v = t_0$ or $t_w = t_0$, which is the identity of J_0 by Corollary 5.1.4.

Now, assume neither v nor w is empty. We prove Equation (7.3.2) by induction on the *length* $l(v)$ of v, i.e., on the number of letters in v. For the base case, suppose $l(v) = 1$ so that $v = a$ or $v = b$. If $v = a$, then $\phi(a) = 0120$. There are two cases:

(a) Case 1: w starts with a.

Then $\phi(w)$ has the form $\phi(w) = 012\cdots$, so

$$\Phi(v)\Phi(w) = t_{0120}t_{012\cdots} = t_{0120 \circ 012\cdots} = t_{012012\cdots} = t_{\phi(aw)}$$

by Proposition 5.4.3. Meanwhile, since $\bar{a} \neq a$ and $a \diamond a = \emptyset$ in A,

$$v \cdot w = aw$$

in R, therefore $\Phi(v \cdot w) = t_{\phi(aw)}$ as well. Equation (7.3.2) follows.

(b) Case 2: w starts with b.

In this case, suppose the longest alternating subword $bab\cdots$ appearing in the beginning of w has length k, and and write $w = bw'$. Then $\phi(w)$ takes the form $\phi(w) = 0212\cdots$, its first dihedral segment is 02 and the second is $(2,1)_{k+1}$, hence $\phi(w) = 02 \circ (2,1)_{k+1} \circ x$ where x is the glued product of all the remaining dihedral segments. Direct computation using Theorem F and propositions 5.4.3 and 5.3.1 then yields

$$\begin{aligned} \Phi(v)\Phi(w) &= t_{01}[t_{(1,2)_{k+2}} + t_{(1,2)_k}]t_x \\ &= t_{01\circ(1,2)_{k+2}\circ x} + t_{01\circ(1,2)_k\circ x} \\ &= t_{\phi(w)} + t_{\phi(w')}. \end{aligned}$$

Meanwhile, since $\bar{a} = b$ and $a \circ b = \emptyset$ in A,

$$v \cdot w = a \cdot bab\cdots = abab\cdots + ab\cdots = w + w'$$

in R, therefore $\Phi(v \cdot w) = t_{\phi(w)} + t_{\phi(w')}$ as well. Equation (7.3.2) follows.

The proof for the case $l(v) = 1$ and $v = b$ is similar.

For the inductive step of our proof, assume Equation (7.3.2) holds whenever v is nonempty and $l(v) < L$ for some $L \in \mathbb{N}$, and suppose $l(v) = L$. Let $\alpha \in A$ be the first letter of v, and write $v = \alpha v'$. Then $l(v') < L$, and by (7.1.1),

$$a \cdot v' = v + \sum_{u \in U} u$$

where U is a subset of $\langle A \rangle$ where all words have length smaller than L. Using the inductive hypothesis on α, v', u and the $\mathbb{Z}$-linearity of Φ, we have

$$\begin{aligned} \Phi(v)\Phi(w) &= \Phi\left(\alpha \cdot v' - \sum_{u \in U} u\right)\Phi(w) \\ &= \Phi(\alpha)\Phi(v')\Phi(w) - \sum_{u \in U} \Phi(u)\Phi(w) \\ &= \Phi(\alpha)\Phi(v' \cdot w) - \Phi\left(\sum_{u \in U} u \cdot w\right) \end{aligned}$$

Here, the element $v' \cdot w$ may be a linear combination of multiple words in R, but applying the inductive hypothesis on α still yields

$$\Phi(\alpha)\Phi(v' \cdot w) = \Phi(\alpha \cdot (v' \cdot w))$$

by the $\mathbb{Z}$-linearity of Φ and $\cdot$. Consequently,

$$\begin{aligned} \Phi(v)\Phi(w) &= \Phi(\alpha \cdot (v' \cdot w)) - \Phi\left(\sum_{u \in U} u \cdot w\right) \\ &= \Phi\left((\alpha \cdot v') \cdot w - \sum_{u \in U} u \cdot w\right) \\ &= \Phi\left(\left[(\alpha \cdot v') - \sum_{u \in U} u\right] \cdot w\right) \\ &= \Phi(v \cdot w). \end{aligned}$$

by the associativity of $\cdot$ and the $\mathbb{Z}$-linearity of Φ and $\cdot$. This completes the proof that Φ is an algebra isomorphism.

The fact that Φ is in addition an isomorphism of based rings is straightforward to check. In particular, observe that $\phi(\bar{w}) = \phi(w)^{-1}$ so that $\Phi(\bar{w}) = t_{\phi(\bar{w})} = t_{\phi(w)^{-1}} = (\Phi(w))^*$, therefore Φ is compatible with the respective involutions in R and J_0. We omit the details of the other necessary verifications. $\square$

7.4. Example 3: $Z_N^+(\{e\}, n-1)$

In this subsection, we consider an infinite family of fusion rings $\{R_n : n \in \mathbb{Z}_{\geq 2}\}$, where each R_n arises from the fusion set

$$A_n = \{e_{ij} : i, j \in [n]\}$$

with $\bar{e}_{ij} = e_{ji}$ for all $i, j \in [n]$ and

$$e_{ij} \diamond e_{kl} = \begin{cases} e_{il} & \text{if} \quad j = k \\ \emptyset & \text{if} \quad j \neq k \end{cases}$$

for all $i, j, k, l \in [n]$. We may think of the fusion set as the usual matrix units for $n \times n$ matrices and think of the fusion map as an analog of matrix multiplication, with the fusion product being $\emptyset$ whenever the matrix product is 0. In the notation of [Fre14], the partition quantum group corresponding to R_n is denoted by $Z_N^+(\{e\}, n-1)$, which equals the *amalgamated free product* of $(n-1)$ copies of $\tilde{H}_N^+$ amalgamated along S_N^+, where S_N^+ stands for the *free symmetric group*, H_N^+ stands for the *free hyperoctohedral group*, and $\tilde{H}_N^+$ stands for the *free complexification* of H_N^+. In particular, $R_2 = \tilde{H}_N^+$.

For $n \in \mathbb{Z}_{\geq 2}$, let (W_n, S_n) be the Coxeter system where $S_n = \{0, 1, 2, \cdots, n\}$, $m(0, i) = \infty$ for all $i \in [n]$, $m(i, i+1) = 3$ for all $i \in [n-1]$, and $m(i, j) = 2$ otherwise. The Coxeter diagrams G_n of (W_n, S_n) are shown in Figure 7.4.2, where a blue edge stands for an edge of weight ∞ and a black edge stands for an edge of weight 3.

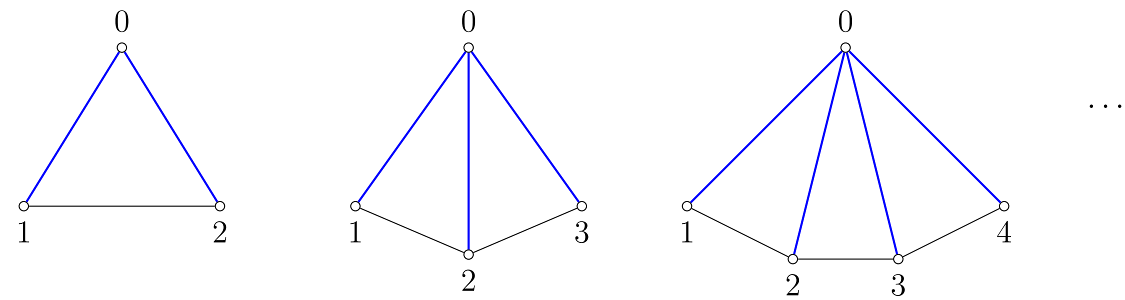

Figure 7.4.2: The Coxeter diagrams of (W_n, S_n).

Let $J_0^{(n)}$ denote the subring J_0 of the subregular J-ring of (W_n, S_n).

Theorem E. *For each $n \in \mathbb{Z}_{\geq 2}$, $R_n \cong J_0^{(n)}$ as based rings.*

For each $n \geq 2$, the strategy to prove the isomophism $R_n \cong J_0^{(n)}$ is similar to the one used for Theorem D; that is, we will first describe a bijection $\phi : \langle A_n \rangle \to \Gamma_0 \cap \Gamma_0^{-1}$, then show that the $\mathbb{Z}$-module map $\Phi : R_n \to J_0^0$ given by $\Phi(w) = t_{\phi(w)}$ is an isomorphism of based rings.

To describe ϕ, note that for $i, j \in [n]$, there is a unique shortest walk P_{ij} from i to j on the "bottom part" of G_n, i.e., on the subgraph of G_n induced by the vertex subset $[n]$. Define $\phi(e_{ij})$ to be element in $\Gamma_0 \cap \Gamma_0^{-1}$ corresponding to the walk on G that starts from 0, travels to i, travels to j along the path P_{ij}, then returns to 0. For example, when $n = 4$, $\phi(e_{24}) = 02340, \phi(e_{43}) = 0430, \phi(e_{44}) = 040$. Next, for any word w in $\langle A_n \rangle$, define $\phi(w)$ to be the glued product of the ϕ-images of its letters. For example, $\phi(e_{24}e_{43}e_{44}e_{44}) = 023404304040$.

It is clear that ϕ is a bijection, with inverse φ given as follows: for any $x \in \Gamma_0 \cap \Gamma_0^{-1}$, write x as the glued product of subwords that start and end with 0 but do not contain 0 otherwise; each such subword must be of the form $\phi(e_{ij})$. We define $\varphi(x)$ to be the concatenation of these letters. For example, $\varphi(0230404) = e_{23}e_{44}e_{44}$ since $02304040 = (0230) \circ (040) \circ (040)$.

Before we prove Theorem 7.4, let us record one useful lemma:

Lemma 7.4.1. *Let $x_{ij} = i \cdots j$ be the element in C corresponding to the walk P_{ij} for all $i, j \in [n]$. Then $t_{x_{ij}} t_{x_{jk}} = t_{x_{ik}}$ for all $i, j, k \in [n]$.*

Proof. This follows by carefully considering the possible relationships between i, j, k and repeatedly using Proposition 5.3.1 to compute $t_{x_{ij}} t_{x_{jk}}$ in each case. Alternatively, notice that the simple reflection 0 is not involved in x_{ij} for any $i, j \in [n]$, hence the computation of $t_{x_{ij}} t_{x_{jk}}$ can be done in the subregular J-ring of the Coxeter system with the "bottom part" of G_n as its diagram. This system is simply-laced, so the result follows immediately from Theorem A. □

Proof of Theorem E. Let $n \geq 2$, and let ϕ and Φ be as above. As in the proof of Theorem 7.3, we show that Φ is an algebra isomorphism by checking that

$$\Phi(v)\Phi(w) = \Phi(v \cdot w) \tag{7.4.3}$$

for all $v, w \in \langle A_n \rangle$. Once again, we may assume that both v and w are non-empty again use induction on the length $l(v)$ of v. The inductive step of the proof will be identical with the one for Theorem D. For the base case where $l(v) = 1$, suppose $v = e_{ij}$ for some $i, j \in [n]$. There are two cases.

(a) Case 1: w starts with a letter $e_{j'k}$ where $j' \neq j$.

Then $\phi(v)$ and $\phi(w)$ take the form $\phi(v) = \cdots j0, \phi(w) = 0j' \cdots$, so

$$\Phi(v)\Phi(w) = t_{\cdots j0}t_{0j'\cdots} = t_{\cdots j0\circ 0j'\cdots} = t_{\phi(e_{ij})\circ\phi(w)} = t_{\phi(e_{ij}w)}$$

by Proposition 5.4.3. Meanwhile, since $\bar{e}_{ij} \neq e_{j'k}$ and $e_{ij} \diamond e_{j'k} = \emptyset$ in A_n,

$$v \cdot w = e_{ij}w$$

in R, therefore $\Phi(v \cdot w) = t_{\phi(e_{ij}w)}$ as well. Equation (7.4.3) follows.

(b) Case 2: w starts with e_{jk} for some $k \in [n]$.

Write $w = e_{jk}w'$. We need to carefully consider four subcases, according to how they affect the dihedral segments of $\phi(v)$ and $\phi(w)$.

(a) $i = j = k$. Then $v = e_{jj}$, $\phi(v) = 0j0 = (0, j)_3$, and w starts with $e_{jj}\cdots$, hence $\phi(w)$ starts with $0j0\cdots$. Suppose the first dihedral segment of $\phi(w)$ is $(0, j)_L$, and write $\phi(w) = (0, j)_L \circ x$. Then Theorem F and propositions 5.4.3 and 5.3.1 yield

$$\begin{aligned}
\Phi(v)\Phi(w) &= t_{(0,j)_3}t_{(0,j)_L}t_x \\
&= t_{(0,j)_{L+2}\circ x} + t_{(0,j)_L\circ x} + t_{(0,j)_{L-2}\circ x} \\
&= t_{\phi(e_{jj}w)} + t_{\phi(w)} + t_{\phi(w')},
\end{aligned}$$

while

$$v \cdot w = e_{jj} \cdot e_{jj}w' = e_{jj}e_{jj}w' + e_{jj}w' + w' = e_{jj}w + w + w'$$

since $\bar{e}_{jj} = e_{jj}$ and $e_{jj} \diamond e_{jj} = e_{jj}$. It follows that Equation (7.4.3) holds.

(b) $i = j$, but $j \neq k$. In this case, $v = e_{jj}, \phi(v) = (0, j)_3$ as in (a), while $\phi(w) = 0j \circ x$ for some reduced word x which starts with j but not $j0$. We have

$$\Phi(v)\Phi(w) = t_{0j0}t_{j0}t_x = t_{0j0j\circ x} + t_{0j\circ x} = t_{\phi(e_{jj}w)} + t_{\phi(w)},$$

while

$$v \cdot w = e_{jj} \cdot e_{jk}w' = e_{jj}e_{jk}w' + e_{jk}w' = e_{jj}w + w$$

since $\bar{e}_{jj} \neq e_{jk}$ and $e_{jj} \diamond e_{jk} = e_{jk}$. This implies Equation (7.4.3).

(c) $i \neq j$, but $j = k$. In this case, $v = e_{ij}$ and $\phi(v) = y \circ j0$ for some reduced word y which ends in j but not $0j$, and $\phi(w)$ can be written as $\phi(w) = (0, j)_L \circ x$ as in (a). We have

$$\begin{aligned} \Phi(v)\Phi(w) &= t_y t_{j0} t_{(0,j)_L} t_x \\ &= t_{y\circ(j,0)_{L+1}\circ x} + t_{y\circ(j,0)_{L-1}\circ x} \\ &= t_{\phi(e_{ij}w)} + t_{\phi(e_{ij}w')}, \end{aligned}$$

while

$$v \cdot w = e_{ij} \cdot e_{jj}w' = e_{ij}w + e_{ij}w'$$

since $\bar{e}_{ij} \neq e_{jj}$ and $e_{ij} \diamond e_{jj} = e_{ij}$. This implies Equation (7.4.3).

(d) $i \neq j$, and $j \neq k$. In this case, $\phi(v) = 0i \circ x_{ij} \circ j0$ (recall the definition of x_{ij} from Lemma 7.4.1), and $\phi(w) = 0j \circ x_{jk} \circ x$ for some x which starts with $k0$. We have

$$\begin{aligned}
\Phi(v)\Phi(w) &= t_{0i}t_{x_{ij}}t_{j0}t_{0j}t_{x_{jk}}t_x \\
&= t_{0i}t_{x_{ij}}t_{j0j}t_{x_{jk}}t_x + t_{0i}t_{x_{ij}}t_jt_{x_{jk}}t_x \\
&= t_{0i\circ x_{ij}\circ j0j\circ x_{jk}\circ x} + t_{0i}t_{x_{ij}}t_{x_{jk}}t_x \\
&= t_{\phi(e_{ij}w)} + t_{0i}t_{x_{ik}}t_x,
\end{aligned}$$

where the fact $t_{x_{ij}}t_{x_{jk}} = t_{x_{ik}}$ comes from Lemma 7.4.1. Now, if $i \neq k$, $t_{0i}t_{x_{ik}}t_x = t_{0i\circ x_{ik}\circ x} = t_{\phi(e_{ik}w')}$, so

$$\Phi(v)\Phi(w) = t_{\phi(e_{ij}w)} + t_{\phi(e_{ik}w')}.$$

If $i = k$, note that $t_{0i}t_{ik}t_x = t_{0k}t_kt_x = t_{0k}t_x$. Suppose the first dihedral segment of x is $(k,0)_{L'}$ for some $L' \geq 2$, and write $x = (k,0)_{L'} \circ x'$. Then $t_{0k}t_x = t_{0k}t_{(k,0)_{L'+1}}t_{x'} = t_{(0,k)_{L'+1}\circ x'} + t_{(0,k)_{L'-1}\circ x'} = t_{\phi(e_{kk}w')+\phi(w')}$, so

$$\Phi(v)\Phi(w) = t_{\phi(e_{ij}w)} + t_{\phi(e_{ik}w')} + t_{\phi(w')}.$$

In either case, Equation (7.4.3) holds again, because

$$v \cdot w = e_{ij} \cdot e_{jk}w' = e_{ij}e_{jk}w' + e_{ik}w' + \delta_{ik}e_{w'} = e_{ij}w + e_{ik}w' + \delta_{ik}e_{w'}$$

now that $\bar{e}_{ij} = e_{ji}$ and $e_{ij} \diamond e_{jk} = e_{ik}$.

We have now proved Φ is an algebra isomorphism. As in Theorem D, the fact that Φ is in addition an isomorphism of based rings is again easy to check, and we omit the details. $\square$

7.5. Comparison with J_s

In this section, we discuss a feature shared by free fusion rings and rings of the form J_s that arise from Coxeter systems, namely, that the products of basis elements in both types of rings are controlled "locally and inductively". Let us explain this below.

First we consider a ring of the form J_s. Let $x, y \in \Gamma_s \cap \Gamma_s^{-1}$, and let their respective dihedral segments be $x_1, x_2, \cdots, x_p$ and $y_1, y_2, \cdots, y_q$ as in the discussion following Proposition 5.4.3. By the algorithm for computing $t_x t_y$ from that discussion, to compute $t_x t_y$ we should first compute $t_{x_p} t_{y_1}$, a "local" product in that it only depends on the last dihedral segment of x and the first dihedral segment of y. Further, write $x' = x_1 \circ \cdots \circ x_{p-1}$ and $y' = y_2 \circ \cdots \circ y_q$, suppose $t_{x_p} t_{y_1} = \sum_{z \in Z} t_z$ for some set Z, and let s' be the letter x_p starts with, then Z includes s' as a summand if and only if $y_1 = x_p^{-1}$ by Proposition 5.3.1, and the algorithm mentioned above implies that

$$t_x t_y = \begin{cases} \sum_{z \in Z} t_{x' \circ z \circ y'} & \text{if} \quad y_1 \neq x_p^{-1} \\ \sum_{z \in Z \setminus \{s'\}} t_{x' \circ z \circ y'} + t_{x'} t_{y'} & \text{if} \quad y_1 = x_p^{-1}, \end{cases} \tag{7.5.4}$$

where the term $t_{x'} t_{y'}$ in the second case appears since $t_{s'}$ appears in $t_{x_p} t_{y_1}$ and $t_{x'} t_{s'} t_{y'} = t_{x'} t_{y'}$. This formula is illustrated in all the computations in Example 5.4.4. Note that the second case means that the computation of $t_x t_y$ now reduces to that of $t_{x'} t_{y'}$, which would have been done "by induction".

Now let R be a free fusion ring given by a fusion set $\langle A, ^-, \diamond \rangle$. Let $v, w \in \langle A \rangle$, and write $v = x_1 \cdots x_p$ and $w = y_1 \cdots y_q$ for $x_1, \cdots, x_p, y_1, \cdots, y_q \in A$. Write

$v' = x_1 \cdots x_{p-1}$ and $w' = y_1 \cdots y_{q-1}$, and suppose $x_p \cdot y_1 = \sum_{z \in Z} z$. Then

$$Z = \begin{cases} \{x_p y_1\} & \text{if} \quad y_1 \neq \bar{x}_p \\ \{x_p y_1, x_p \diamond y_1\} & \text{if} \quad y_1 = \bar{x}_p \end{cases} \tag{7.5.5}$$

by Equation 7.1.1, and Equation 7.1.1 is easily seen to be equivalent to the following.

$$v \cdot w = \begin{cases} \sum_{z \in Z} v' z w' = v'(x_p y_1) w' = vw & \text{if} \quad y_1 \neq \bar{x}_p \\ \sum_{z \in Z} v' z w' + v' \cdot w' = vw + v'(x_p \diamond y_1) w' + v' \cdot w' & \text{if} \quad y_1 = \bar{x}_p. \end{cases} \tag{7.5.6}$$

The equation may be interpreted as saying that the computation of $v \cdot w$ begins with the "local" computation $x_p \cdot y_1$ involving only the last letter of v and first letter of w, and that the local computation is either enough to finish the computation or reduces it to the computation of $v' \cdot w'$, which would have been done "by induction". Further, whether the inductive result is needed depends on whether x_p and y_1 are dual to each other, just like in Equation 7.5.4 where the duality is provided by the inverse map. In these regards, Equation 7.5.4 and Equation 7.5.6 are very similar.

The main difference between the two types of computations discussed above is that in the free fusion ring setting, the set Z obtained from the "local" product $x_p \cdot y_1$ contains their juxtaposition and at most 1 more element, while the set Z obtained from $t_{x_p} t_{y_1}$ in J_s may contain more summands. It would be interesting to know whether or how one may modify the axiomatization of free fusion rings to account for this difference and create a notion that would encompass more rings of the form J_s.

We focus on a specific series of Coxeter systems for the rest of the section. For $n \geq 4$, let (W_n, S) be the Coxeter system with $S = \{s, t, u\}$, $m(s,t) = 4, m(s,u) = 2$

and $m(t,u) = n$. Its Coxeter diagram is shown in Figure 7.5.3. We focus on the ring J_s associated with (W_m, S), which we denote by $J_s^{(n)}$, for each $n \geq 4$. Note that when $n \geq 4$, we recover the Coxeter system and the ring J_1 discussed in Example 5.4.5.

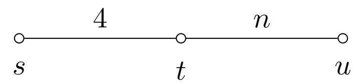

Figure 7.5.3: The Coxeter diagram of (W_n, S).

Let $e = s$ and denote the word $st \circ (t,u)_k \circ ts$ by $[k]$ for each odd integers $k \geq 1$, where the notation $(t,u)_k$ denotes the alternating word $tut\cdots$ involving k letters as in Definition 2.4.3. For example, $[1] = sts$ and $[5] = stututs$. For each $n \geq 4$, let A_n be the set of all odd integers k with $3 \leq k < n$. Then as in Example 5.4.5, it is easy to draw the graph D_1 of (W_n, S) and examine its cycles to conclude that the elements of $\Gamma_1 \cap \Gamma_1^{-1}$ in W_n are exactly $e, [1]$ and the words in $\langle A_n \rangle$, where a word $k_1 k_2 \cdots k_p$ stands for the word $[k_1] \circ [k_2] \circ \cdots \circ [k_p]$ in W_n. Here, t_e is the identity of $J_s^{(n)}$ by Corollary 4.2.4, $t_1^2 = t_e$ by Proposition 5.3.1, and it is easy to verify that $t_1 t_w = t_w = t_w t_1$ for any $w \in \langle A_n \rangle$ by using Proposition 5.3.1 and noting that the first and last dihedral segments of w are st and ts, respectively.

It remains to discuss products of the form $t_v t_w$ in $J_1^{(n)}$ for $v, w \in \langle A_n \rangle$. To do so, note that by using Proposition 5.3.1 and the results of Section 5.4, it is straightforward to check the following facts.

(a) If both v and w contain only one letter, say $v = k$ and $w = l$ for $k, l \in A_n$, then

$$t_v t_w = t_k t_l = t_{kl} + \sum_{z \in T(k,l)} t_z + \delta_{k,l} t_e,$$

where $T(k,l)$ contains all the numbers of the form $\phi(d)$ in Proposition 5.3.1

(with n in place of M), i.e., the $z's$ are the numbers produced with the truncated Clebsch-Gordan rule. For example, when $n = 8$, $t_5t_7 = t_{57} + t_3$ while $t_5t_5 = t_{55} + t_5 + t_3 + t_1 + t_e$. We write $t_kt_l = \sum_{z\in Z(k,l)} t_z$.

(b) If v contains only one letter but w contains at least two letter, say with $v = k$ and $w = lw'$ for $k, l \in A_n$ and $w' \in \langle A_n\rangle$, then

$$t_vt_w = \begin{cases} \sum_{z\in Z(k,l)} t_{zw'} & \text{if} \quad k \neq l \\ t_{vw} + \sum_{z\in T(k,l)} t_{zw'} & \text{if} \quad k = l \end{cases},$$

where $Z(k,l)$ and $T(k,l)$ are as in (a) and zw' is defined to be w'. For example, when $n = 8$, $t_5t_{735} = t_{5735} + t_{335}$ while $t_5t_{535} = t_{5535} + t_{535} + t_{335} + t_{35}$. Similarly, if v contains at least two letters and w contains only one letter, say $v = v'k$ and $w = l$, then

$$t_vt_w = \begin{cases} \sum_{z\in Z(k,l)} t_{v'z} & \text{if} \quad k \neq l \\ t_{vw} + \sum_{z\in T(k,l)} t_{v'z} & \text{if} \quad k = l \end{cases}.$$

(c) If both v and w contain at least letters, suppose $v = v'k$ and $w = lw'$ for $k, l \in A_n, v', w' \in \langle A_n\rangle$. Then

$$t_vt_w = t_vt_w = \begin{cases} \sum_{z\in Z(k,l)} t_{v'zw'} & \text{if} \quad k \neq l \\ t_{vw} + \sum_{z\in T(k,l)\setminus\{1\}} t_{v'zw'} + t_{v'}t_{w'} - t_{v'w'} & \text{if} \quad k \neq l \end{cases}.$$

For example, when $n = 8$, $t_{35}t_{759} = t_{35759} + t_{3359}$ while $t_{35}t_{559} = t_{35559} + t_{3559} + t_{3359} + t_3t_{59} - t_{359}$. Note that by (a), (b), (c) and an easy induction, $t_{v'}t_{w'} - t_{v'w'}$, hence t_vt_w, is always a nonnegative linear combination of basis elements in $J_s^{(n)}$.

While it is worth noting that the displayed equations in parts (b) and (c) differ from Equation 7.5.4 in that the letters in A_n do not represent dihedral segments of elements in $\Gamma_s \cap \Gamma_s^{-1}$, the equations in (c) again illustrate how $t_v t_w$ depend "locally" on $t_k t_l$ and "inductively" on $t_{v'} t_{w'}$, and the equations in (b) are of the same spirit. Together, the facts of (a), (b) and (c) means that once all products of the form $t_k t_l$ are known for all $k, l \in A_n$, we can quickly describe the product $t_v t_w$ for arbitrary $v, w \in \langle A_n \rangle$ in terms of word manipulations. For example, when $n = 6$, by four direct computations, we have

$$t_3 t_3 = t_{33} + t_5 + t_3 + t_1 + t_e, \quad t_5 t_5 = t_{55} + t_1 + t_e,$$

$$t_3 t_5 = t_3 + t_{35}, \quad t_5 t_3 = t_5 + t_{53}.$$

From these equalities, we may compute a generic product such as $t_{353} t_{335}$ as follows.

$$\begin{aligned} t_{353} t_{335} &= t_{353335} + t_{35535} + t_{35335} + t_{35} t_{35} - t_{3535} \\ &= t_{353335} + t_{35535} + t_{35335} + t_{355} + t_{3535} - t_{3535} \\ &= t_{353335} + t_{35535} + t_{35335} + t_{355} \end{aligned}$$

Example 7.5.1. When $n = 4$, $A_n = \{3\}$. For each $p \geq 1$, we may denote the word $33 \cdots 3$ containing p copies of 3 by 3_p, and denote t_{3_p} by y_p. Using the facts of (a), (b) and (c), it is then easy to check that

$$y_p y_q = y_{|p-q|} + y_{p+q},$$

where y_0 is defined to be $t_e + t_1$. This recovers Equation 5.4.5 of Example 5.4.5.

Example 7.5.2. When $n = 5$, we still have $A_n = \{3\}$. Define y_p as in the previous example. Using the facts of (a), (b) and (c), it is easy to check that

$$y_p y_q = y_{|p-q|} + y_{p+q} + \sum_{z \in Z(p,q)} y_z,$$

where y_0 is again defined as $t_e + t_1$ and $Z(p,q)$ contains the $\min(p,q)$ numbers $|p-q|+1, |p-q|+3, \cdots, p+q-3, p+q-1$ of the same parity.

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