FAITHFUL TROPICALIZATION OF HYPERTORIC VARIETIES

by

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DISSERTATION ABSTRACT

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The hypertoric variety $\mathfrak{M}_{\mathcal{A}}$ defined by an arrangement $\mathcal{A}$ of affine hyperplanes admits a natural tropicalization, induced by its embedding in a Lawrence toric variety. In this thesis, we explicitly describe the polyhedral structure of this tropicalization and calculate the fibers of the tropicalization map. Using a recent result of Gubler, Rabinoff, and Werner, we prove that there is a continuous section of the tropicalization map.

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For Pa

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CHAPTER I

INTRODUCTION

In this dissertation, we study the tropicalization of the hypertoric variety $\mathfrak{M}_{\mathcal{A}}$ defined by an arrangement $\mathcal{A}$ of affine hyperplanes. Hypertoric varieties were first studied by Bielawski and Dancer [BD00]. They are "hyperkähler analogues" of toric varieties, and examples of conical symplectic resolutions. The relationship between the variety $\mathfrak{M}_{\mathcal{A}}$ and the arrangement $\mathcal{A}$ is analogous to that between a semiprojective toric variety and its polyhedron. See, e.g., [Pro08] for an overview of this relationship. The hypertoric variety $\mathfrak{M}_{\mathcal{A}}$ is not, in general, a toric variety. However, it is naturally defined as a closed subvariety of a toric variety, the Lawrence toric variety $\mathfrak{B}_{\mathcal{A}}$. The Lawrence embedding allows us to define a tropicalization of $\mathfrak{M}_{\mathcal{A}}$.

Given a closed embedding of a variety X in a toric variety, there is a corresponding tropicalization $\mathrm{Trop}(X)$, which is the continuous image of the Berkovich space X^{an} under the tropicalization map. The tropicalization may be endowed with the structure of a finite polyhedral complex. A single variety X may yield many distinct tropicalizations, each given by a different choice of embedding into a toric variety. When we speak of *the* tropicalization of X, it is always with respect to a chosen embedding.

By a result of Foster, Gross, and Payne [FGP14, Pay09], if X has at least one embedding into a toric variety, then the inverse system of all such embeddings induces an inverse system of tropicalizations, and the limit of this system in the category of topological spaces is X^{an}. This raises the question of how well a particular tropicalization approximates the geometry of the analytic space. To this end, a tropicalization is said to be **faithful** if the tropicalization map $X^{\mathrm{an}} \to \mathrm{Trop}(X)$

admits a continuous section, realizing $\mathrm{Trop}(X)$ as (homeomorphic to) a closed subset of X^{an}.

If X is embedded in a torus, then $\mathrm{Trop}(X)$ is the support of a finite polyhedral complex, which is balanced when the polyhedra are weighted by tropical multiplicity. Gubler, Rabinoff, and Werner have proved that such a tropicalization is faithful if all tropical multiplicities are equal to one [GRW16]. Moreover, in this case, the section of tropicalization is uniquely defined. This generalizes work of Baker, Payne, and Rabinoff, who obtained the first results on faithful tropicalization in the case where X is a curve [BPR16].

In the more general situation where X is embedded in a toric variety, $\mathrm{Trop}(X)$ is the union of the tropicalizations $\mathrm{Trop}(X \cap O)$ as O ranges over all torus orbits. In this case, tropical multiplicity one is no longer sufficient to imply faithfulness: it is possible that the continuous sections defined on each of the strata $\mathrm{Trop}(X \cap O)$ do not glue to a continuous section on the entire tropicalization [GRW15, Example 8.11]. However, Gubler, Rabinoff, and Werner [GRW15, Theorem 8.14] have recently proved that if X is embedded in a toric variety with dense torus T, then the resulting tropicalization is faithful, with uniquely defined continuous section, if certain conditions on the embedding and the polyhedral structure of the resulting tropicalization are satisfied. We state a simplified version of this result as Theorem 5.1. While the first results on faithful tropicalizations [CHW14, DP16], required careful study of Berkovich spaces and their skeleta, that analysis is now absorbed into the proof of this theorem, so that faithfulness may be checked by exclusively working "downstairs," with the tropicalization.

In this thesis, we apply this this theorem of Gubler, Rabinoff, and Werner to prove that an arbitrary hypertoric variety $\mathfrak{M}_{\mathcal{A}}$ is faithfully tropicalized by its embedding in the Lawrence toric variety $\mathfrak{M}_{\mathcal{A}}$. We thus obtain many new examples, in every even dimension, of varieties which are faithfully tropicalized by a "natural" embedding into a toric variety. These examples include the cotangent bundle of projective space and the cotangent bundle of a product of projective spaces, as well as many singular varieties. To our knowledge, this is the first application of Gubler, Rabinoff, and Werner's theorem to a class of tropicalizations for which faithfulness was previously unknown.

Furthermore, we shall see that, in all but the most trivial case, the hypertoric variety $\mathfrak{M}_{\mathcal{A}}$ in its Lawrence embedding does not meet all torus orbits in the expected dimension (Corollary 2.12). This is in contrast to several other known examples of "nice" tropicalizations, including the moduli space $\overline{M}_{0,n}$ of stable rational curves [GM10, Tev07], some alternate compactifications of $M_{0,n}$ [CHMR16], and the space of logarithmic stable maps to a projective toric variety [Ran15]. By [GRW15, Corollary 8.15], a variety which meets all torus orbits in the expected dimension, or not at all, is faithfully tropicalized if it has multiplicity one everywhere. Since this result is not available to us, we must find a polyhedral structure on $\mathrm{Trop}(\mathfrak{M}_{\mathcal{A}})$ to work with. Our first main result describes such a polyhedral structure in terms of the combinatorics of the defining arrangement $\mathcal{A}$.

Theorem 4.1. *The tropicalization* $\mathrm{Trop}(\mathfrak{M}_{\mathcal{A}})$ *of the hypertoric variety is the union of cones* $C_{\mathcal{F}}^{(F,\mathcal{R})}$ *indexed by a flat* F *of* $\mathcal{M}$*, a face* $\mathcal{R}$ *of the localization* $\mathcal{A}_F$*, and a flag of flats* $\mathcal{F}$ *in the restriction* $\mathcal{M}^F$*. These cones satisfy*

$$\dim C_{\mathcal{F}}^{(F,\mathcal{R})} = d - \operatorname{codim} \mathcal{R} + \ell(\mathcal{F}).$$

This gives $\mathrm{Trop}(\mathfrak{M}_{\mathcal{A}})$ *the combinatorial structure of a finite polyhedral complex, under the closure relation*

$$C_{\mathcal{F}'}^{(F',\mathcal{R}')} \subseteq \overline{C_{\mathcal{F}}^{(F,\mathcal{R})}} \tag{1.1}$$

if and only if the following conditions hold:

- $F \subseteq F'$;
- $\mathcal{R}' \subseteq \overline{\mathcal{R}}$;
- F' *is a flat in* $\mathcal{F}$, *and* $\mathrm{trunc}_{F'}(\mathcal{F})$ *is a refinement of* $\mathcal{F}'$.

Moreover, this gives each stratum $\mathrm{Trop}(\mathfrak{M}_{\mathcal{A}}) \cap \widetilde{N}_{\mathbb{R}}(\sigma_{F,\mathcal{R}})$ *the structure of a polyhedral fan, which is balanced when all cones are given weight one.*

Equipped with Theorem 4.1, we can describe the interplay between the fan of the toric variety $\mathfrak{B}_{\mathcal{A}}$ and the fan $\mathrm{Trop}(\mathfrak{M}_{\mathcal{A}} \cap \widetilde{T})$, where $\widetilde{T}$ is the dense torus of $\mathfrak{B}_{\mathcal{A}}$. The cones of each of these two fans are described in terms of the combinatorics of the arrangement $\mathcal{A}$. By examining these combinatorics, we see that the conditions of Theorem 5.1 are satisfied, proving the tropicalization is faithful.

Theorem 5.4. *There is a unique continuous section of the tropicalization map* $\mathfrak{M}_{\mathcal{A}}^{\mathrm{an}} \to \mathrm{Trop}(\mathfrak{M}_{\mathcal{A}})$.

The rest of the dissertation is outlined as follows. In Chapter II, we recall basic facts about toric geometry and hyperplane arrangements, and we define the Lawrence toric variety and hypertoric variety associated to an arrangement. We also prove some technical lemmas. Chapter III serves as a brief overview of non-Archimedean analytic spaces and tropicalization. We describe the tropicalization of a linear space (Example 3.1), which we will later use to define the polyhedral structure on $\mathrm{Trop}(\mathfrak{M}_{\mathcal{A}})$. In

Chapter IV, we prove Theorem 4.1. We calculate the fibers of tropicalization, and show that they are affinoid subdomains of $\mathfrak{M}_{\mathcal{A}}^{\mathrm{an}}$ containing a unique Shilov boundary point. Finally, in Chapter V we state Theorem 5.1, due to Gubler, Rabinoff, and Werner, and use it to prove Theorem 5.4.

CHAPTER II

TORIC VARIETIES AND HYPERTORIC VARIETIES

In this chapter, we briefly review the theories of toric varieties and hyperplane arrangements, and we set notation and definitions we will use in the sequel. We define the Lawrence toric variety and the hypertoric variety associated to an arrangement. We note that the Lawrence toric variety is defined here in terms of its fan, whereas typically in the literature it is defined as a GIT quotient. The equivalence of our approach with the standard definition follows from Lemmas 2.6, 2.7, and 2.8. For further background reading on these topics, the interested reader is referred to [Ful93] and [CLS11] on toric varieties; [Oxl11] on matroids; [BD00] and [HS02] on Lawrence toric varieties and hypertoric varieties; and [PW07] and [Pro08] on the relationship between these varieties and the associated hyperplane arrangements.

Throughout the remainder of this dissertation, we fix a lattice $M \cong \mathbb{Z}^d$ and an algebraically closed field K, complete with respect to a non-Archimedean valuation $\nu\colon K \to \mathbb{R} \cup \{\infty\}$, which may be trivial. The dual lattice to M is $N = \operatorname{Hom}(M, \mathbb{Z})$, and we set $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} = \operatorname{Hom}(M, \mathbb{R})$. Let $T = \operatorname{Spec} K[M]$ be the split K-torus with character lattice M and cocharacter lattice N.

2.1. Toric varieties

Let Σ be a (pointed) rational polyhedral fan in $N_{\mathbb{R}}$. Each cone $\sigma \in \Sigma$ defines an **affine toric variety** $Y_\sigma = \operatorname{Spec} K[\sigma^\vee \cap M]$ with dense torus T. For $\tau \prec \sigma$ in Σ, Y_τ is naturally an open subvariety of Y_σ. Gluing along these identifications, we obtain the T-**toric variety** $Y_\Sigma = \bigcup_{\sigma \in \Sigma} Y_\sigma$ defined by Σ.

The action of T partitions Y_Σ into torus orbits. These orbits are in bijection with cones in Σ, with $\sigma \in \Sigma$ corresponding to the orbit $O(\sigma) = \operatorname{Spec} K[\sigma^\perp \cap M]$. The orbit $O(\sigma)$ is a torus of dimension equal to $\operatorname{codim} \sigma$, with character lattice $M(\sigma) = \sigma^\perp \cap M$ and cocharacter lattice $N(\sigma) = N/(\langle\sigma\rangle \cap N)$, where $\langle\sigma\rangle = \mathbb{R}\sigma$ is the linear span of σ in $N_\mathbb{R}$. We have the set-theoretic decomposition $Y_\Sigma = \bigsqcup_{\sigma\in\Sigma} O(\sigma)$.

We set $M_\mathbb{R}(\sigma) = M(\sigma) \otimes_\mathbb{Z} \mathbb{R}$ and $N_\mathbb{R}(\sigma) = N(\sigma) \otimes_\mathbb{Z} \mathbb{R}$. If $\tau \prec \sigma$, then we have the projection $N_\mathbb{R}(\tau) \to N_\mathbb{R}(\sigma)$, which by abuse of notation we denote π_σ. For a cone $\tau \in \Sigma$, the orbit closure $\overline{O(\tau)}$ is a toric variety with dense torus $O(\tau)$. Its fan is the set of cones $\operatorname{star}(\tau) = \{\pi_\tau(\sigma) \mid \sigma \succ \tau\}$ in $N_\mathbb{R}(\tau)$.

A homomorphism of tori $\phi\colon T \to T'$ is uniquely determined by the corresponding homomorphism $\phi^*\colon M' \to M$ of character lattices, or equivalently by the dual homomorphism $\phi_*\colon N \to N'$ of cocharacter lattices. Note that ϕ is injective if and only if ϕ^* is surjective (if and only if ϕ_* is injective). Dually, ϕ is surjective if and only if ϕ^* is injective (if and only if ϕ_* is surjective). We say that an injective or surjective morphism of tori is **split** if the corresponding map of character lattices (or, equivalently, of cocharacter lattices) is split.

If Σ and Σ' are fans in $N_\mathbb{R}$ and $N'_\mathbb{R}$, respectively, then a homomorphism $\phi\colon T \to T'$ extends to an equivariant morphism of toric varieties $Y_\Sigma \to Y_{\Sigma'}$ if and only if for each cone $\sigma \in \Sigma$ there exists $\sigma' \in \Sigma'$ such that $\phi_*(\sigma) \subseteq \sigma'$.

Following [Gro15], we define a **linear subvariety** L of the torus T to be a subvariety in some choice of torus coordinates. That is, there exists an isomorphism $M \cong \mathbb{Z}^d$, inducing $K[M] \cong K[\mathbb{Z}^d] = K[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$, such that the ideal of L is generated by linear forms in the x_i.

Lemma 2.1. *If $\phi\colon T \to T'$ is a split surjection of tori, and $L \subseteq T'$ is a linear subvariety, then $\phi^{-1}(L)$ is a linear subvariety of T with $\operatorname{codim}_T \phi^{-1}(L) = \operatorname{codim}_{T'} L$.*

Proof. Let $\{x_1, \dots, x_d\}$ be an integral basis of M'. Then $\{\phi^*(x_1), \dots, \phi^*(x_d)\}$ is linearly independent in M because ϕ^* is injective, and each $\phi^*(x_i)$ is primitive because x_i is primitive and ϕ^* is split. Therefore, $\{\phi^*(x_1), \dots, \phi^*(x_d)\}$ may be extended to an integral basis of M.

If the ideal of L is generated by linear forms in the variables x_i, then the ideal of $\phi^{-1}(L)$ is generated by linear forms in the variables $\phi^*(x_i)$. This shows that $\phi^{-1}(L)$ is a linear subvariety of T. Moreover, by injectivity of ϕ^*, the ideal of $\phi^{-1}(L)$ is generated by the same number of independent linear forms as is the ideal of L, proving that $\operatorname{codim}_T \phi^{-1}(L) = \operatorname{codim}_{T'} L$. □

2.2. Hyperplane arrangements and matroids

Given a finite set E, a tuple $\mathbf{a} = (a_e) \in N^E$ of nonzero primitive elements, and $r = (r_e) \in \mathbb{Z}^E$, we define the corresponding **arrangement** $\mathcal{A} = \mathcal{A}(\mathbf{a}, r)$ to be the multiset of affine integral hyperplanes

$$H_e = \{m \in M_{\mathbb{R}} \mid \langle m, a_e \rangle + r_e = 0\} \qquad (e \in E)$$

in $M_{\mathbb{R}}$. If $\mathbf{a}$ generates the lattice N, then $\mathbf{a}$ is a **primitive spanning configuration**. Each hyperplane H_e is cooriented by the integral normal vector a_e, with "positive" and "negative" closed halfspaces,

$$H_e^+ = \{u \in M_{\mathbb{R}} \mid \langle u, a_e \rangle + r_e \leq 0\}$$

and

$$H_e^- = \{u \in M_{\mathbb{R}} \mid \langle u, a_e \rangle + r_e \leq 0\},$$

respectively.

The arrangement $\mathcal{A}$ is **simple** if the intersection of any k hyperplanes is either empty or has codimension k, and $\mathcal{A}$ is **unimodular** if every collection of d linearly independent normal vectors $\{a_{e_1}, \ldots, a_{e_d}\}$ is an integral basis of N. An arrangement which is both simple and unimodular is **smooth**.

If $r = 0$, so that each hyperplane H_e is a linear subspace of $M_{\mathbb{R}}$, then we call the arrangement **central**. Given $\mathcal{A} = \mathcal{A}(\mathbf{a}, r)$, we let $\mathcal{A}_0 = \mathcal{A}(\mathbf{a}, 0)$ be the **centralization** of $\mathcal{A}$. We denote by $(H_e)_0$ the translation of H_e to the origin.

For each relation $\sum_{e \in E} c_e a_e$ in N satisfied by the configuration $\mathbf{a}$, we have the corresponding linear form

$$\sum_{e \in E} c_e x_e \in K[x_e \mid e \in E].$$

We let $L = L(\mathbf{a})$ be the d-dimensional linear subspace of $\mathbb{A}^E$ defined by the vanishing of these linear forms. The dependencies among points in the configuration $\mathbf{a}$ are encoded in the underlying **matroid** $\mathcal{M} = \mathcal{M}(\mathbf{a})$ on E. A matroid on E is a combinatorial structure, defined by declaring a collection of subsets of E to be **independent** (the subsets which are not independent are called **dependent**). The collection of independent subsets must satisfy certain axioms inspired by the linear algebraic notion of linear independence.

The **rank function** of $\mathcal{M}$ defines the rank $\operatorname{rk} S$ of a subset $S \subseteq E$ to be the dimension of the subspace of $N_{\mathbb{R}}$ spanned by $\{a_e \mid e \in S\}$. Equivalently, the rank of S is equal to the codimension of the intersection $\bigcap_{e \in S} (H_e)_0$ of all central hyperplanes indexed by S. Observe that the rank of $\mathcal{M}$, defined to be $\operatorname{rk} E$, is equal to d if and only if $\mathbf{a}$ is a primitive spanning configuration.

A subset $F \subseteq E$ is a **flat** of $\mathcal{M}$ if it is maximal for its rank; that is, if $S \supseteq F$, then either $S = F$ or $\operatorname{rk} S > \operatorname{rk} F$. A **flag** of flats in $\mathcal{M}$ is a chain

$$\mathcal{F} = (\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{k-1} \subsetneq F_k = E)$$

where each F_i is a flat. The **length** of such a flag, denoted $\ell(\mathcal{F})$, is the number k of nonempty flats in $\mathcal{F}$. Since the rank must increase at each step, a maximal flag of flats will have length equal to the rank of $\mathcal{M}$. By inserting flats, any flag may be refined into a maximal flag.

It is clear from the definition that the collection of all flats is uniquely determined by the rank function. It is a basic result of matroid theory that the reverse is true, and that the rank function and the lattice of flats each individually determines the collection of all independent sets. Thus, a matroid may be "cryptomorphically" defined in terms of either its rank function or its flats, with each of these structures being subject to appropriate axioms. See [Oxl11] for these axioms and other equivalent characterizations of matroids.

Given a flat F of $\mathcal{M}$, we define the **restriction** of the central arrangement $\mathcal{A}_0$ to F, denoted $\mathcal{A}_0^F$, to be the arrangement of hyperplanes $\{(H_e)_0 \cap H_F \mid e \notin F\}$ in the vector space $H_F = \bigcap_{e \in F} (H_e)_0$. We let $L^F \subseteq \mathbb{A}^{E \smallsetminus F}$ denote the corresponding linear subspace, obtained from L by setting $x_e = 0$ for all $e \in F$. The underlying matroid of $\mathcal{A}_0^F$, denoted $\mathcal{M}^F$, is the matroid on $E \smallsetminus F$ obtained from $\mathcal{M}$ by deleting F. The flats of $\mathcal{M}^F$ are precisely the sets $F' \smallsetminus F$ for $F' \supseteq F$ a flat of $\mathcal{M}$; we therefore identify flats of $\mathcal{M}^F$ with flats of $\mathcal{M}$ which contain F, and flags of flats in $\mathcal{M}^F$ with flags in $\mathcal{M}$ which begin at F.

As a dual construction, we define the **localization** of the arrangement $\mathcal{A}$ at any subset $S \subseteq E$ to be the arrangement of hyperplanes $\{H_e \mid e \in S\}$ in the vector space

$M_{\mathbb{R}}$. The centralization of $\mathcal{A}_S$ is $(\mathcal{A}_0)_S$, and we write $\mathcal{M}_S$ for its underlying matroid. The ground set of $\mathcal{M}_S$ is S, and its flats are precisely those flats of $\mathcal{M}$ which are contained in S.

Remark 2.2. There are notable differences in our definitions of restriction and localization.

(1) While it is possible to define the restriction to a non-flat S, the resulting matroid will contain loops. In this document, we shall only need to restrict to flats, and so we will limit our attention to that case. On the other hand, in order to combinatorially describe the fan of the Lawrence toric variety in Section 2.3, it will be necessary to localize $\mathcal{A}$ at every subset $S \subseteq E$.

(2) There is no canonical way to lift the construction of the restriction $\mathcal{A}_0^F$ to the non-central arrangement $\mathcal{A}$, which is why we defined restriction on the centralization. By contrast, the localization of $\mathcal{A}_0$ at S uniquely determines the localization of $\mathcal{A}$ at S.

Remark 2.3. We have defined localization so that $\mathcal{A}_S$ is an arrangement in $M_{\mathbb{R}}$. As a result, the normal vectors of the hyperplanes in $\mathcal{A}_S$ do not necessarily span N, even if the original arrangement $\mathcal{A}$ was defined by a primitive spanning configuration. Alternatively, one may wish to define $\mathcal{A}_S$ to be an arrangement in $M_{\mathbb{R}}/H_S$; the normal vectors of this arrangement will then be spanning if the normal vectors of $\mathcal{A}$ are spanning (see, e.g., [PW07, §2]). However, it will be convenient for our purposes to have all localizations living in the same vector space $M_{\mathbb{R}}$.

An arrangement $\mathcal{A}$ assigns a sign vector $\operatorname{sgn}_{\mathcal{A}}(m) \in \{+, 0, -\}^E$ to each $m \in M_{\mathbb{R}}$, via

$$\operatorname{sgn}_{\mathcal{A}}(m)_e = \begin{cases} + & \text{if } m \in H_e^+ \smallsetminus H_e, \\ 0 & \text{if } m \in H_e, \\ - & \text{if } m \in H_e^- \smallsetminus H_e. \end{cases}$$

A nonempty fiber of $\operatorname{sgn}_{\mathcal{A}} \colon M_{\mathbb{R}} \to \{+, 0, -\}^E$ is called a **face** of the arrangement $\mathcal{A}$. A **vertex** of $\mathcal{A}$ is a face consisting of a single point. Each face $\mathcal{R}$ defines sets $E^+(\mathcal{R}) = \{e \in E \mid \mathcal{R} \subseteq H_e^+\}$ and $E^-(\mathcal{R}) = \{e \in E \mid \mathcal{R} \subseteq H_e^-\}$. Note that $E = E^+(\mathcal{R}) \cup E^-(\mathcal{R})$. We set $E^0(\mathcal{R}) = E^+(\mathcal{R}) \cap E^-(\mathcal{R}) = \{e \in E \mid \mathcal{R} \subseteq H_e\}$.

Notice that the closure of a face is the intersection of all halfspaces which contain it:

$$\overline{\mathcal{R}} = \Big(\bigcap_{e \in E^+(\mathcal{R})} H_e^+ \Big) \cap \Big(\bigcap_{e \in E^-(\mathcal{R})} H_e^- \Big). \tag{2.1}$$

It follows that the codimension of $\mathcal{R}$ in $M_{\mathbb{R}}$ is the codimension of the intersection of all hyperplanes containing it:

$$\operatorname{codim} \mathcal{R} = \operatorname{codim} \bigcap_{e \in E^0(\mathcal{R})} H_e = \operatorname{codim} \bigcap_{e \in E^0(\mathcal{R})} (H_e)_0 = \operatorname{rk} \mathcal{M}_{E^0(\mathcal{R})}. \tag{2.2}$$

The above discussion of faces applies to localizations of $\mathcal{A}$ as well. If $S \subseteq E$ is any subset, then a face $\mathcal{R}$ of $\mathcal{A}_S$ determines sets $S^+(\mathcal{R})$, $S^0(\mathcal{R})$, and $S^-(\mathcal{R})$, with $S = S^+(\mathcal{R}) \cup S^-(\mathcal{R})$ and $S^0(\mathcal{R}) = S^+(\mathcal{R}) \cap S^-(\mathcal{R})$. Furthermore, $\overline{\mathcal{R}}$ and $\operatorname{codim} \mathcal{R}$ are computed as in (2.1) and (2.2), respectively, with E replaced by S.

Lemma 2.4. *Let $S \subseteq E$. If $\mathcal{R}$ is a face of $\mathcal{A}$ and $\mathcal{R}'$ is a face of the localization $\mathcal{A}_S$, then $\mathcal{R} \subseteq \overline{\mathcal{R}'}$ if and only if $E^+(\mathcal{R}) \supseteq S^+(\mathcal{R}')$ and $E^-(\mathcal{R}) \supseteq S^-(\mathcal{R}')$.*

Proof. Suppose $\mathcal{R} \subseteq \overline{\mathcal{R}'}$. Since the halfspaces H_e^+ and H_e^- are closed, any halfspace which contains $\mathcal{R}'$ also contains $\overline{\mathcal{R}'}$, hence contains $\mathcal{R}$. That is, $E^+(\mathcal{R}) \supseteq S^+(\mathcal{R}')$ and $E^-(\mathcal{R}) \supseteq S^-(\mathcal{R}')$

Conversely, suppose $E^+(\mathcal{R}) \supseteq S^+(\mathcal{R}')$ and $E^-(\mathcal{R}) \supseteq S^-(\mathcal{R}')$. Then for each $e \in S^+(\mathcal{R}')$, we also have $e \in E^+(\mathcal{R})$ and therefore $\mathcal{R} \subseteq H_e^+$. Similarly, $\mathcal{R} \subseteq H_e^-$ for each $e \in S^-(\mathcal{R}')$. By the formula (2.1) applied to the face $\mathcal{R}'$, we conclude that $\mathcal{R} \subseteq \overline{\mathcal{R}'}$. $\square$

2.3. The Lawrence toric variety of an arrangement

We now describe how an arrangement $\mathcal{A} = \mathcal{A}(\mathbf{a}, r)$ in $M_{\mathbb{R}}$, where $\mathbf{a}$ is a primitive spanning configuration, gives rise to a toric variety $\mathfrak{B}_{\mathcal{A}}$, called the Lawrence toric variety. While we continue to fix the torus $T = \operatorname{Spec} K[M]$, the variety $\mathfrak{B}_{\mathcal{A}}$ is not a T-toric variety; rather, it is a $\widetilde{T}$-toric variety, where $\widetilde{T}$ is a (split) extension of $\mathbb{G}_m^E$ by T.

The configuration $\mathbf{a}$ defines a homorphism $\mathbb{Z}^E \to N$, where the generator δ_e is mapped to a_e. This map is surjective because $\mathbf{a}$ is spanning, and its kernel Λ is a lattice because $\mathbf{a}$ is primitive.

Let $\overline{\Delta}\colon \mathbb{Z}^E \to \mathbb{Z}^E \oplus \mathbb{Z}^E$ denote the antidiagonal embedding: If we denote by δ_e^+ the generators of the first copy of $\mathbb{Z}^E$ and by δ_e^- the generators of the second copy of $\mathbb{Z}^E$, then $\overline{\Delta}(\delta_e) = \delta_e^+ - \delta_e^-$. The composition of $\overline{\Delta}$ with the $\iota\colon \Lambda \to \mathbb{Z}^E$ gives an inclusion of Λ into $\mathbb{Z}^E \oplus \mathbb{Z}^E$. Let $\widetilde{N}$ denote the quotient, so that we have the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda & \xrightarrow{\ \iota\ } & \mathbb{Z}^E & \xrightarrow{\ \mathbf{a}\ } & N & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle \overline{\Delta}} & & \Big\downarrow & & \\ 0 & \longrightarrow & \Lambda & \longrightarrow & \mathbb{Z}^E \oplus \mathbb{Z}^E & \longrightarrow & \widetilde{N} & \longrightarrow & 0 \end{array} \tag{2.3}$$

The images ρ_e^+ and ρ_e^- in $\widetilde{N}$ of the generators δ_e^+ and δ_e^-, respectively, provide a natural spanning set of $\widetilde{N}$. Note that $\widetilde{N}$ is a lattice of rank $|E|+d$, and we have the short exact sequence

$$0 \longrightarrow N \longrightarrow \widetilde{N} \longrightarrow \mathbb{Z}^E \longrightarrow 0 \tag{2.4}$$

where $N \to \widetilde{N}$ is the vertical map from (2.3) and $\widetilde{N} \to \mathbb{Z}^E$ is defined by $\rho_e^{\pm} \mapsto \delta_e$. In particular, for each relation $\sum_{e\in E} c_e a_e = 0$ in N, we have

$$\sum_{e\in E} c_e(\rho_e^+ - \rho_e^-) = 0$$

in $\widetilde{N}$, and these are all relations among the generators $\rho_e^{\pm}$.

For each pair $(S, \mathcal{R})$, where $S \subseteq E$ and $\mathcal{R}$ is a face of the localization $\mathcal{A}_S$, define $\sigma_{S,\mathcal{R}}$ to be the cone in $\widetilde{N}_{\mathbb{R}}$ with rays generated by the integral vectors

$$\{\rho_e^+ \mid e \in S^+(\mathcal{R})\} \cup \{\rho_f^- \mid f \in S^-(\mathcal{R})\}.$$

Let $\Sigma_{\mathcal{A}} = \{\sigma_{S,\mathcal{R}} \mid S \subseteq E \text{ and } \mathcal{R} \text{ is a face of } \mathcal{A}_S\}$ be the collection of all such cones. Lemmas 2.6, 2.7, and 2.8 establish that $\Sigma_{\mathcal{A}}$ is a fan in $\widetilde{N}_{\mathbb{R}}$, called the **Lawrence fan**, whose maximal cones are precisely the $(|E|+d)$-dimensional cones $\sigma_{E,\xi}$ indexed by vertices ξ of $\mathcal{A}$. Let $\widetilde{M}$ be the dual lattice to $\widetilde{N}$, and let $\widetilde{T} = \operatorname{Spec} K[\widetilde{M}]$ be the **Lawrence torus**. Then the $\widetilde{T}$-toric variety $\mathfrak{B}_{\mathcal{A}} = Y_{\Sigma_{\mathcal{A}}}$ is called the **Lawrence toric variety** associated to $\mathcal{A}$.

Remark 2.5. In the literature, $\mathfrak{B}_{\mathcal{A}}$ is usually defined as the GIT quotient of the cotangent bundle $T^*\mathbb{A}^E \cong \mathbb{A}^E \times \mathbb{A}^E$ by $G = \operatorname{Spec} K[\Lambda^\vee]$ with respect to the character $\alpha = \iota^*(r) \in \Lambda^\vee$. Hausel and Sturmfels [HS02, Proposition 4.3] identify the fan of this

GIT quotient as the collection of cones $\sigma_{E,\xi}$ together with all of their faces. Hence, this fan is $\Sigma_{\mathcal{A}}$.

However, the maximal cones in [HS02] are indexed not by vertices of $\mathcal{A}$, but by bases of the dual matroid of $\mathcal{M}$. Given such a basis, the intersection of all hyperplanes H_e indexed by the dual basis of $\mathcal{M}$ yields a vertex of $\mathcal{A}$. Every vertex arises in this way, but unless $\mathcal{A}$ is simple, one vertex may arise from multiple bases. (In the extreme example where $\mathcal{A}$ is central, there may be many bases, but each produces the single vertex of $\mathcal{A}$.) Therefore, it is more natural to index the maximal cones by vertices of the arrangement, as we do here.

To our knowledge, no description of the non-maximal cones of $\Sigma_{\mathcal{A}}$ appears in the literature.

Lemma 2.6. *The cone $\sigma_{S,\mathcal{R}}$ has dimension $|S| + \operatorname{codim} \mathcal{R}$.*

Proof. It suffices to prove this in the case $S = E$.

The dimension of $\sigma_{E,\mathcal{R}}$ is equal to the dimension of its real span $\langle \sigma_{E,\mathcal{R}} \rangle \subseteq \widetilde{N}_{\mathbb{R}}$. Define

$$V_1 = \mathbb{R}\langle \rho_i^+, \rho_j^- \mid i \in E^+(\mathcal{R}) \smallsetminus E^0(\mathcal{R}), j \in E^-(\mathcal{R}) \smallsetminus E^0(\mathcal{R}) \rangle$$

and

$$V_2 = \mathbb{R}\langle \rho_i^+, \rho_j^- \mid i, j \in E^0(\mathcal{R}) \rangle.$$

Then we clearly have $\langle \sigma_{E,\mathcal{R}} \rangle = V_1 + V_2$.

Since every relation among the elements $\rho_e^{\pm}$ is of the form

$$\sum_{e \in E} c_e(\rho_e^+ - \rho_e^-) = 0,$$

any nontrivial linear dependence among the generators of $\sigma_{E,\mathcal{R}}$ must occur among the vectors $\{\rho_i^+, \rho_j^- \mid i,j \in E^0(\mathcal{R})\}$ generating V_2. It follows that $\dim V_1 = |E \smallsetminus E^0(\mathcal{R})|$ and $\langle \sigma_{E,\mathcal{R}} \rangle = V_1 \oplus V_2$.

Choose any basis $\mathcal{B} \subseteq E^0(\mathcal{R})$ of the matroid $\mathcal{M}_{E^0(\mathcal{R})}$. We claim that

$$\{\rho_i^+, \rho_j^- \mid i \in \mathcal{B}, j \in E^0(\mathcal{R})\} \tag{2.5}$$

is a basis for V_2. Indeed, any nontrivial linear dependence among these generators must occur among the subset $\{\rho_i^+, \rho_j^- \mid i,j \in \mathcal{B}\}$, but any such dependence must be trivial because $\mathcal{B}$ is independent. Thus, the set in (2.5) is linearly independent. On the other hand, for any $i \in E^0(\mathcal{R}) \smallsetminus \mathcal{B}$, there is a unique expression $a_i = \sum_{b\in\mathcal{B}} c_b a_b$, where $c_b \in \mathbb{Z}$. This implies

$$\rho_i^+ = \rho_i^- + \sum_{b\in\mathcal{B}} c_b(\rho_b^+ - \rho_b^-),$$

and therefore the set in (2.5) spans V_2. By (2.2), we have $\operatorname{codim} \mathcal{R} = \operatorname{rk} \mathcal{M}_{E^0(\mathcal{R})}$. Therefore,

$$\dim V_2 = |\mathcal{B}| + |E^0(\mathcal{R})| = \operatorname{rk} \mathcal{M}_{E^0(\mathcal{R})} + |E^0(\mathcal{R})| = \operatorname{codim} \mathcal{R} + |E^0(\mathcal{R})|,$$

and hence

$$\dim \sigma_{E,\mathcal{R}} = \dim V_1 + \dim V_2 = \operatorname{codim} \mathcal{R} + |E|.$$

□

Lemma 2.7. *Every face of $\sigma_{S,\mathcal{R}}$ is of the form $\sigma_{S',\mathcal{R}'}$, and we have the face relations*

$$\sigma_{S',\mathcal{R}'} \prec \sigma_{S,\mathcal{R}} \quad \textit{if and only if} \quad S \supseteq S' \textit{ and } \mathcal{R} \subseteq \overline{\mathcal{R}'}.$$

Proof. Again, we may assume that $S = E$ and $\mathcal{R}$ is a face of $\mathcal{A}$.

Suppose $\tau \prec \sigma_{E,\mathcal{R}}$ is a face. Then $\tau = u^{\perp} \cap \sigma_{E,\mathcal{R}}$ for some $u \in \sigma^{\vee}_{E,\mathcal{R}} \subseteq \widetilde{M}_{\mathbb{R}}$. We also know that τ is generated by a subset of the rays of $\sigma_{E,\mathcal{R}}$. That is, τ is the cone generated by

$$\{\rho^{+}_{i} \mid i \in I\} \cup \{\rho^{-}_{j} \mid j \in J\}$$

for some subsets $I \subseteq E^{+}(\mathcal{R})$ and $J \subseteq E^{-}(\mathcal{R})$. Set $S' = I \cup J$.

Note that the dual $\overline{\Delta}^{*}$ of the antidiagonal map of (2.3) maps $\widetilde{M}_{\mathbb{R}}$ onto $M_{\mathbb{R}}$. Fix any point $p \in \mathcal{R}$. Set

$$m = \overline{\Delta}^{*}(\epsilon u) + p \in M_{\mathbb{R}},$$

where $\epsilon > 0$ will be fixed shortly. Observe that

$$\langle m, a_e\rangle + r_e = \epsilon\langle u, \overline{\Delta}(a_e)\rangle + \langle p, a_e\rangle + r_e = \epsilon(\langle u, \rho^{+}_{e}\rangle - \langle u, \rho^{-}_{e}\rangle) + (\langle p, a_e\rangle + r_e)$$

for any $e \in E$. Since the sign of $\langle p, a_e\rangle + r_e$ for each e is determined by $\mathcal{R}$, and E is a finite set, it is possible to choose $\epsilon > 0$ sufficiently small so that

$$\operatorname{sgn}_{\mathcal{A}_{S'}}(m)_e = \begin{cases} + & \text{if } e \in I \smallsetminus (I \cap J), \\ 0 & \text{if } e \in I \cap J, \\ - & \text{if } e \in I \smallsetminus (I \cap J). \end{cases}$$

That is, m lies in a face $\mathcal{R}'$ of $\mathcal{A}_{S'}$ such that $S'^{+}(\mathcal{R}') = I$ and $S'^{-}(\mathcal{R}') = J$. This shows that $\tau = \sigma_{S',\mathcal{R}'}$. Since $I \subseteq E^{+}(\mathcal{R})$ and $J \subseteq E^{-}(\mathcal{R})$, Lemma 2.4 implies that $\mathcal{R} \subseteq \overline{\mathcal{R}'}$.

Conversely, let $S' \subseteq E$ and let $\mathcal{R}'$ be a face of $\mathcal{A}_{S'}$ with $\mathcal{R} \subseteq \overline{\mathcal{R}'}$. We shall show that $\sigma_{S',\mathcal{R}'}$ is a face of $\sigma_{E,\mathcal{R}}$. By Lemma 2.4, we have $E^+(\mathcal{R}) \supseteq S'^+(\mathcal{R}')$ and $E^-(\mathcal{R}) \supseteq S'^-(\mathcal{R}')$.

Fix some $p \in \mathcal{R}$ and $m \in \mathcal{R}'$, and for each $k \in E \setminus S'$, choose positive real numbers c_k and d_k such that $c_k - d_k = \langle m - p, a_k \rangle$. Define a functional on $\mathbb{Z}^E \oplus \mathbb{Z}^E$ by

$$u = \sum_{i \in S'^-(\mathcal{R}')} \langle m - p, a_i \rangle x_i^+ - \sum_{j \in S'^+(\mathcal{R}') \setminus S'^0(\mathcal{R}')} \langle m - p, a_j \rangle x_j^- + \sum_{k \in E \setminus S'} (c_k x_k^+ - d_k x_k^-),$$

where $\{x_e^\pm\}$ is the dual basis to $\{\delta_e^\pm\}$. A priori, $u \in \mathbb{R}\langle x_e^\pm \rangle$. However, we have defined u so that

$$\langle \overline{\Delta}^*(u), \delta_e \rangle = \langle u, \overline{\Delta}(\delta_e) \rangle = \langle u, \delta_e^+ - \delta_e^- \rangle$$

is equal to $\langle m - p, a_e \rangle$ for all $e \in E$. Since $\{a_e, e \in E\}$ spans $N_\mathbb{R}$, it follows that $\overline{\Delta}^*(u) = m - p \in M_\mathbb{R}$, and therefore $u \in \widetilde{M}_\mathbb{R}$. By construction, we have $u \in \sigma_{E,\mathcal{R}}^\vee$ and $u^\perp \cap \sigma_{E,\mathcal{R}} = \sigma_{S',\mathcal{R}'}$, proving that $\sigma_{S',\mathcal{R}'} \prec \sigma_{E,\mathcal{R}}$. □

Lemma 2.8. *If ξ and ζ are two vertices of $\mathcal{A}$, then $\sigma_{E,\xi} \cap \sigma_{E,\zeta}$ is a cone of the form $\sigma_{S,\mathcal{R}}$. Moreover, every cone $\sigma_{S,\mathcal{R}}$ is the face of $\sigma_{E,\xi}$ for some vertex ξ.*

Proof. We say that a hyperplane H_e **separates** the vertices ξ and ζ if ξ and ζ lie in opposite halfspaces of H_e and neither lies on H_e. Let $S \subseteq E$ be the set of all e such that H_e does not separate ξ and ζ. Then there is a unique face $\mathcal{R}$ of $\mathcal{A}_S$ such that $\mathcal{R}$ contains every point in the interior of the line segment connecting ξ and ζ, and $\sigma_{E,\xi} \cap \sigma_{E,\zeta} = \sigma_{S,\mathcal{R}}$.

Conversely, if $\sigma_{S,\mathcal{R}}$ is any cone in $\Sigma_\mathcal{A}$, then $\mathcal{R}$ is a union of faces of $\mathcal{A}$. Since $\mathbf{a}$ is spanning, each face of $\mathcal{A}$ contains at least one vertex in its closure, so $\mathcal{R}$ has this property as well. Let ξ be any vertex of $\mathcal{A}$ contained in $\overline{\mathcal{R}}$. Therefore $\sigma_{S,\mathcal{R}} \prec \sigma_{E,\xi}$ by Lemma 2.7. □

2.4. The hypertoric variety of a an arrangement

Consider the surjection $\widetilde{N} \to \mathbb{Z}^E$ in (2.4). Tensoring with $\mathbb{R}$, we obtain a linear map $\Phi_*\colon \widetilde{N}_{\mathbb{R}} \to \mathbb{R}^E$ with $\Phi_*(\sigma_{S,\mathcal{R}}) = \mathbb{R}^S_{\geq 0}$. We thus obtain a surjective map of toric varieties $\Phi\colon \mathfrak{B}_{\mathcal{A}} \to \mathbb{A}^E$. We define the **hypertoric variety** of the arrangement $\mathcal{A}$, denoted $\mathfrak{M}_{\mathcal{A}}$, to be the preimage of the linear space L under the map Φ. It is irreducible of dimension $2d$.

Remark 2.9. Since $\mathbb{Z}^E \oplus \mathbb{Z}^E$ surjects onto $\widetilde{N}$ (cf. (2.3)), the map Φ lifts to a surjection $T^*(\mathbb{A}^E) \cong \mathbb{A}^E \times \mathbb{A}^E \to \mathbb{A}^E$ (this is the moment map for the hamiltonian action of $\mathbb{G}_m^E$ on $\mathbb{A}^E$). If we work over $K = \mathbb{C}$, then the complex points of $\mathbb{A}^E/L$ can be identified with the dual Lie algebra of the torus $G = \operatorname{Spec} K[\Lambda^\vee]$, and the composition of Φ with the projection $\mathbb{A}^E \to \mathbb{A}^E/L$ is then the moment map μ for the hamiltonian action of G on $T^*\mathbb{A}^E$. This endows $\mathfrak{M}_{\mathcal{A}} = \mu^{-1}(0) /\!\!/_\alpha G$ with a canonical Poisson structure. Arbo and Proudfoot have proved that this Poisson structure makes $\mathfrak{M}_{\mathcal{A}}$ a symplectic variety in the sense of Beauville [AP16, Proposition 4.14]. The torus T acts on $\mathfrak{M}_{\mathcal{A}}$ via its antidiagonal embedding in the Lawrence torus $\widetilde{T}$. This action is hamiltonian with moment map $\Phi|_{\mathfrak{M}_{\mathcal{A}}}$.

By [BD00, Theorems 3.2 & 3.3], the hypertoric variety $\mathfrak{M}_{\mathcal{A}}$ has at worst orbifold singularities if and only if the arrangement $\mathcal{A}$ is simple, and is smooth if and only if $\mathcal{A}$ is smooth. The variety $\mathfrak{M}_{\mathcal{A}_0}$ is affine, and if $\mathcal{A}$ is simple then $\mathfrak{M}_{\mathcal{A}} \to \mathfrak{M}_{\mathcal{A}_0}$ is an orbifold resolution of singularities. By [HP04, Lemma 2.2], $\mathfrak{M}_{\mathcal{A}}$ is independent of the coorientations of the hyperplanes in $\mathcal{A}$.

Example 2.10. If $\mathcal{A}$ is the arrangement of coordinate hyperplanes in $M_{\mathbb{R}}$, then the associated hypertoric variety is $\mathfrak{M}_{\mathcal{A}} \cong T^*\mathbb{A}^d \cong \mathbb{A}^{2d}$. The polytope of $\mathbb{P}^d$ is a d-simplex in $M_{\mathbb{R}}$ cut out by $d+1$ affine hyperplanes. The hypertoric variety associated to the arrangement consisting of these hyperplanes is isomorphic to $T^*\mathbb{P}^d$. The same procedure realizes the cotangent bundle of a product of projective spaces as a hypertoric variety. In general, if Y is a projective toric variety with polyhedron P, then the arrangement of hyperplanes cutting out P defines a hypertoric variety which contains T^*Y as a dense open subset [BD00, Theorem 7.1].

Since $\Phi_*(\sigma_{S,\mathcal{R}}) = \mathbb{R}^S_{\geq 0}$, the restriction of Φ to $O(\sigma_{S,\mathcal{R}})$ is a surjection onto the torus orbit $\mathbb{G}_m^{E\smallsetminus S}$ in $\mathbb{A}^E$. This surjection is split because $\widetilde{T} \to \mathbb{G}_m^E$ is split. Moreover,

$$\Phi^{-1}(\mathbb{G}_m^{E\smallsetminus S}) = \bigsqcup_{\mathcal{R}} O(\sigma_{S,\mathcal{R}}),$$

where the (set-theoretic) disjoint union is taken over all faces $\mathcal{R}$ of $\mathcal{A}_S$.

Proposition 2.11. *Let $F \subseteq E$ be a subset and let $\mathcal{R}$ be a face of $\mathcal{A}_F$. The intersection $\mathfrak{M}_{\mathcal{A}} \cap O(\sigma_{F,\mathcal{R}})$ is nonempty if and only if F is a flat of $\mathcal{M}$, in which case it is a linear subvariety of $O(\sigma_{F,\mathcal{R}})$ of dimension $2d - \operatorname{rk} F - \operatorname{codim} \mathcal{R}$. In particular, $\mathfrak{M}_{\mathcal{A}} \cap O(\sigma_{F,\mathcal{R}})$ is irreducible when it is nonempty.*

Proof. Because $\mathfrak{M}_{\mathcal{A}} \cap O(\sigma_{F,\mathcal{R}})$ is the preimage of $L \cap \mathbb{G}_m^{E\smallsetminus F}$ under the surjection $\Phi|_{O(\sigma_{F,\mathcal{R}})} \colon O(\sigma_{F,\mathcal{R}}) \to \mathbb{G}_m^{E\smallsetminus F}$, we have $\mathfrak{M}_{\mathcal{A}} \cap O(\sigma_{F,\mathcal{R}}) \neq \emptyset$ if and only if $L \cap \mathbb{G}_m^{E\smallsetminus F} \neq \emptyset$, and this occurs if and only if F is a flat of $\mathcal{M}$.

Suppose then that F is a flat of $\mathcal{M}$. Since $L \cap \mathbb{G}_m^{E\smallsetminus F}$ is a linear subvariety of $\mathbb{G}_m^{E\smallsetminus F}$ and $\Phi|_{O(\sigma_{F,\mathcal{R}})}$ is a split surjection, we may apply Lemma 2.1 to conclude that $\mathfrak{M}_{\mathcal{A}} \cap O(\sigma_{F,\mathcal{R}})$ is a linear subvariety of $O(\sigma_{F,\mathcal{R}})$ of codimension equal to the codimension of $L \cap \mathbb{G}_m^{E\smallsetminus F}$ in $\mathbb{G}_m^{E\smallsetminus F}$.

Since $\dim(L \cap \mathbb{G}_m^{E \smallsetminus F}) = \dim L^F = \operatorname{rk} \mathcal{M}^F = d - \operatorname{rk} F$, we have

$$\operatorname{codim}_{O(\sigma_{F,\mathcal{R}})}(\mathfrak{M}_{\mathcal{A}} \cap O(\sigma_{F,\mathcal{R}})) = |E \smallsetminus F| - d + \operatorname{rk} F.$$

By Lemma 2.6,

$$\begin{aligned}
\dim(\mathfrak{M}_{\mathcal{A}} \cap O(\sigma_{F,\mathcal{R}})) &= \dim O(\sigma_{F,\mathcal{R}}) - (|E \smallsetminus F| - d + \operatorname{rk} F) \\
&= (d - \operatorname{codim} \mathcal{R} + |E \smallsetminus F|) - (|E \smallsetminus F| - d + \operatorname{rk} F) \\
&= 2d - \operatorname{rk} F - \operatorname{codim} \mathcal{R}.
\end{aligned}$$

□

In general, if X is a subvariety of a toric variety, then the expected dimension of the intersection of X with a torus orbit $O(\sigma)$ is $\dim X - \dim \sigma$. If the dimension of the intersection is equal to the expected dimension, then we say that X intersects the torus orbit **properly**. Proposition 2.11 implies that a hypertoric variety does not, in general, intersect all Lawrence torus orbits properly.

Corollary 2.12. *The hypertoric variety $\mathfrak{M}_{\mathcal{A}}$ does not meet each torus orbit of $\mathfrak{B}_{\mathcal{A}}$ properly unless $\mathfrak{M}_{\mathcal{A}} \cong \mathbb{A}^{2d}$.*

Proof. Let F be a flat of $\mathcal{M}$, so that $\mathfrak{M}_{\mathcal{A}} \cap O(\sigma_{F,\mathcal{R}})$ is nonempty. By Lemma 2.6, the expected dimension of this intersection is

$$\dim \mathfrak{M}_{\mathcal{A}} - \dim \sigma_{F,\mathcal{R}} = 2d - |F| - \operatorname{codim} \mathcal{R}.$$

By Proposition 2.11, $\dim(\mathfrak{M}_{\mathcal{A}} \cap O(\sigma_{F,\mathcal{R}}))$ agrees with the expected dimension if and only if $|F| = \operatorname{rk} F$. This will hold for every flat of $\mathcal{M}$ if and only if $|E| = d$ and $\mathcal{M}$ is the uniform matroid of rank d. In this case, the primitive spanning configuration $\mathbf{a}$ will be an integral basis for N, and therefore $\mathfrak{M}_{\mathcal{A}} \cong \mathbb{A}^{2d}$. □

CHAPTER III

ANALYTIFICATION AND TROPICALIZATION

This chapter contains a brief overview of Berkovich's theory of non-Archimedean analytic spaces, and the definition of the (Kajiwara-Payne) tropicalization of subvarieties of a toric variety. We include Example 3.1, describing the tropicalization of a linear space, which will be of use to us in describing the tropicalization of the hypertoric variety $\mathfrak{M}_{\mathcal{A}}$ in Chapter IV. For further reading, see [Ber90] for foundations of Berkovich spaces, [CHW14, Section 5] for a practical discussion of affinoid algebras, [MS15] for a general treatment of tropicalizations, and [Gub13] for a comprehensive treatment of tropicalization from the perspective of non-Archimedean geometry.

Write $\mathbb{T} = \mathbb{R} \cup \{\infty\}$, which we shall consider as a monoid under addition and as a topological space homeomorphic to $(0, 1]$. Recall that our ground field K is equipped with a non-Archimedean valuation $\nu\colon K \to \mathbb{T}$. Let $|\cdot| = \exp(-\nu(\,\cdot\,))$ be the associated norm on K.

3.1. Affinoid algebras and analytic spaces

For $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$, we have the **weighted Gauss norm** $||\cdot||_r$ on the polynomial ring $K[x_1, \ldots, x_n]$, defined by

$$\left|\left| \sum_{u \in \mathbb{N}^n} c_u x^u \right|\right|_r = \max_{u \in \mathbb{N}} |c_u| r^u$$

(where $c_u = 0$ for all but finitely many u, and $r^u = r_1^{u_1} \cdots r_n^{u_n}$). The completion of $K[x_1, \ldots, x_n]$ with respect to $||\cdot||_r$ is the **generalized Tate algebra**

$$K\langle r_1^{-1}x_1, \ldots, r_n^{-1}x_n\rangle = \left\{ \sum_{u \in \mathbb{N}^n} c_u x^u \;\middle|\; |c_u| r^u \to 0 \text{ as } |u| \to \infty \right\}$$

which can be thought of as the ring of convergent power series on the polydisc of radius r in $\mathbb{A}^n$. It is a Banach algebra, equipped with the norm $||\cdot||_r$.

A K**-affinoid algebra** is a Banach algebra $(\mathscr{A}, ||\cdot||)$, where $\mathcal{A}$ is isomorphic to a quotient $K\langle r_1^{-1}x_1, \ldots, r_n^{-1}x_n\rangle / I$ and $||\cdot||$ is equivalent to the quotient norm. The **Berkovich spectrum** $\mathscr{M}(\mathscr{A})$ of a K-affinoid algebra $\mathscr{A}$ is the set of bounded multiplicative seminorms γ on $\mathscr{A}$, equipped with the coarsest topology such that $\mathrm{ev}_a \colon \mathscr{M}(\mathscr{A}) \to \mathbb{R}_{\geq 0}$, $\gamma \mapsto \gamma(a)$ is continuous for every $a \in \mathscr{A}$.

Similar to the construction of the generalized Tate algebra, we may complete the Laurent polynomial ring $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ with respect to $||\cdot||_r$ to obtain

$$K\langle r_1^{-1}x_1, r_1 x_1^{-1} \ldots, r_n^{-1}x_n, r_n x_n^{-1}\rangle = \left\{ \sum_{u \in \mathbb{Z}^n} c_u x^u \;\middle|\; |c_u| r^u \to 0 \text{ as } |u| \to \infty \right\},$$

which is also a K-affinoid algebra.

Given a K-affinoid algebra $\mathscr{A}$, with norm $||\cdot||$, and $s \in \mathbb{R}_{>0}$, define

$$\mathscr{A}\langle s^{-1}x, sy\rangle = \left\{ \sum_{i,j \geq 0} c_{ij} x^i y^j \;\middle|\; c_{ij} \in \mathscr{A},\ ||c_{ij}|| s^{i-j} \to 0 \text{ as } i+j \to \infty \right\}.$$

For any element $f \in \mathscr{A}$, we can then define the affinoid algebra

$$\mathscr{A}\langle s^{-1}f, sf^{-1}\rangle = \mathscr{A}\langle s^{-1}x, sy\rangle / (x - f, fy - 1).$$

Iterating this construction, given $s_1, \ldots, s_n \in \mathbb{R}_{>0}$ and $f_1, \ldots, f_n \in \mathscr{A}$, we may define the affinoid algebra $\mathscr{A}\langle s_1^{-1} f_1, s_1 f_1^{-1}, \ldots, s_n^{-1} f_n, s_n f_n^{-1}\rangle$. Its Berkovich spectrum $\mathscr{M}(\mathscr{A}\langle s_1^{-1} f_1, s_1 f_1^{-1}, \ldots, s_n^{-1} f_n, s_n f_n^{-1}\rangle)$ includes into $\mathscr{M}(\mathscr{A})$ as the set of all seminorms γ such that $\gamma(f_i) = s_i$ for $i = 1, \ldots, n$.

A **Berkovich K-analytic space** is, roughly speaking, a topological space equipped with a sheaf of analytic functions which is locally isomorphic to the Berkovich spectrum of a K-affinoid algebra, where the ring of analytic functions on $\mathscr{M}(\mathscr{A})$ is $\mathscr{A}$. For details, see [Ber90, Chapters 2 & 3].

Given a K-variety X, there is an analytification functor which associates to X a K-analytic space X^{an}. As a topological space, X^{an} can be described without reference to Berkovich spectra or affinoid algebras. If $X = \operatorname{Spec} A$ is affine, then X^{an} is the set of ring valuations $A \to \mathbb{T}$ extending the valuation ν on K (or, equivalently, as the set of multiplicative seminorms on A extending $|\cdot|$). We give X^{an} the coarsest topology such for every $a \in A$, the evaluation map $\mathrm{ev}_a \colon X^{\mathrm{an}} \to \mathbb{T}$, $\mathrm{val} \mapsto \mathrm{val}(a)$ is continuous. For general X, we may take a cover of X by affine open subschemes $\{U_i\}$, and the analytifications U_i^{an} glue to form X^{an}. (Equivalently, X^{an} can be expressed as a set of equivalence classes of L-valued points, as L varies over all valued field extensions of K [Gub13, Remark 2.2].)

The functor $X \to X^{\mathrm{an}}$ possesses many nice properties [Ber90, Sections 3.4 & 3.5]. For instance, X^{an} is compact if and only if X is proper, and the topological dimension of X^{an} is equal to the algebraic dimension of X. If $\varphi \colon X \to Y$ is a morphism of K-varieties, then many properties (e.g. smooth, étale, flat, finite) of φ are inherited by φ^{an}. Of particular use to us is that φ^{an} is a closed (resp. open) immersion if and only if φ is.

3.2. Tropicalization

Let Σ be a fan in $N_{\mathbb{R}}$, as in Section 2.1. For a cone $\sigma \in \Sigma$, we define $\overline{N}_{\mathbb{R}}^{\sigma}$ to be the set of monoid homomorphisms $\mathrm{Hom}(\sigma^{\vee} \cap M, \mathbb{T})$. We give $\overline{N}_{\mathbb{R}}^{\sigma}$ the topology of pointwise convergence. If $\tau \prec \sigma$, then $\overline{N}_{\mathbb{R}}^{\tau}$ is naturally identified with the open subset of $\overline{N}_{\mathbb{R}}^{\sigma}$ consisting of maps which are finite on $\tau^{\perp} \cap \sigma^{\vee} \cap M$ (in particular, $N_{\mathbb{R}} = \overline{N}_{\mathbb{R}}^{\{0\}}$ is an open subset of each $\overline{N}_{\mathbb{R}}^{\sigma}$). Gluing along these identifications, we obtain $\overline{N}_{\mathbb{R}}^{\Sigma}$, a partial compactification of $N_{\mathbb{R}}$. This mirrors the construction of the toric variety Y_{Σ}: we have a decomposition $\overline{N}_{\mathbb{R}}^{\Sigma} = \bigcup_{\sigma \in \Sigma} \overline{N}_{\mathbb{R}}^{\sigma}$ analogous to the decomposition of Y_{Σ} into affine toric varieties, and a (set-theoretic) decomposition $\overline{N}_{\mathbb{R}}^{\Sigma} = \bigsqcup_{\sigma \in \Sigma} N(\sigma)$ analogous to the decomposition of Y_{Σ} into torus orbits. These two constructions are related by the process of tropicalization.

The **tropicalization map** on the torus $T = \operatorname{Spec} K[M]$ is the continuous surjection

$$\mathrm{trop}\colon T^{\mathrm{an}} \to N_{\mathbb{R}}$$

which takes a valuation $\mathrm{val}\colon K[M] \to \mathbb{T}$ to its restriction $\mathrm{val}|_M \colon M \to \mathbb{R}$. More generally, for a cone $\sigma \in \Sigma$, we have a **tropicalization map**

$$\mathrm{trop}\colon Y_{\sigma}^{\mathrm{an}} \to \overline{N}_{\mathbb{R}}^{\sigma} = \mathrm{Hom}(\sigma^{\vee} \cap M, \mathbb{T})$$

which similarly maps a valuation $\mathrm{val}\colon K[\sigma^{\vee} \cap M] \to \mathbb{T}$ to its restriction to $\sigma^{\vee} \cap M$. These maps glue to give a tropicalization map $\mathrm{trop}\colon Y_{\Sigma}^{\mathrm{an}} \to \overline{N}_{\mathbb{R}}^{\Sigma}$. This map is a continuous and proper surjection, which has the property that its restriction to each torus orbit is the usual tropicalization $O(\sigma)^{\mathrm{an}} \to N_{\mathbb{R}}(\sigma)$ for a torus.

Given a closed subvariety $X \subseteq Y_{\Sigma}$, the analytification X^{an} is a closed subspace of Y_{Σ}^{an}. The **tropicalization** $\mathrm{Trop}(X) \subseteq \overline{N}_{\mathbb{R}}^{\Sigma}$ of X defined by its embedding in Y_{Σ}

is the image of X^{an} under trop. If X is a subvariety of the torus T, then $\operatorname{Trop}(X)$ may be given the structure of a finite polyhedral complex, which is a (not necessarily pointed) fan if X is defined over a subfield of K having trivial valuation. Moreover, this polyhedral complex is of pure dimension $\dim X$ and is equipped with a positive integer-valued weight function, the tropical multiplicity, with respect to which the complex is balanced [MS15, Theorem 3.3.5].

We refer the interested reader to [OP13, Section 2] for a discussion of tropical multiplicities. The basic idea is as follows. Each $w \in N_{\mathbb{R}}$ defines a scheme $\mathcal{T}^w$ over the valuation ring of K with generic fiber T. We think of this as a degeneration of T. The closure of $X \subseteq T$ in $\mathcal{T}$ may or may not intersect the special fiber. This intersection is a scheme over the residue field, called the **initial degeneration** $\operatorname{in}_w X$ of X at w. Part of the so-called Fundamental Theorem of Tropical Geometry [MS15, Theorem 3.2.5] states that $w \in \operatorname{Trop}(X)$ if and only if $\operatorname{in}_w X$ is nonempty. In this case, the **tropical multiplicity** of $\operatorname{Trop}(X)$ at w is the multiplicity of $\operatorname{in}_w X$, i.e., its number of irreducible components, counted with multiplicity.

In the general setting where X is a subvariety of a toric variety, $\operatorname{Trop}(X)$ may be computed orbit-by-orbit: $\operatorname{Trop}(X) \cap N_{\mathbb{R}}(\sigma) = \operatorname{Trop}(X \cap O(\sigma))$. The multiplicity of $\operatorname{Trop}(X)$ at $w \in \operatorname{Trop}(X) \cap N_{\mathbb{R}}(\sigma)$ is equal to the tropical multiplicity of $\operatorname{Trop}(X \cap O(\sigma))$ at w. Thus, $\operatorname{Trop}(X)$ is a partial compactification of the balanced finite polyhedral complex $\operatorname{Trop}(X \cap T)$ by lower-dimensional balanced finite polyhedral complexes. If $X = \overline{X \cap T}$ (in particular, if X is irreducible and $X \cap T$ is nonempty), then $\operatorname{Trop}(X)$ is the closure of $\operatorname{Trop}(X \cap T)$ in $\overline{N}_{\mathbb{R}}^{\sigma}$ [MS15, Corollary 6.2.16].

Tropicalization is functorial with respect to toric morphisms. Let $f\colon Y_\Sigma \to Y_{\Sigma'}$ be such a map. For $\sigma \in \Sigma$, there exists $\sigma' \in \Sigma'$ such that $f_*(\sigma) \subseteq \sigma'$. For such a σ', the restriction of f^* gives a map $M(\sigma') \to M(\sigma)$, inducing $\overline{N}_{\mathbb{R}}^{\sigma} \to \overline{N}_{\mathbb{R}}^{\sigma'}$. These maps glue to give a map $\overline{N}_{\mathbb{R}}^{\Sigma} \to \overline{N}_{\mathbb{R}}^{\Sigma'}$, denoted $\operatorname{Trop}(f)$. See [Pay09] for details. By [MS15, Corollary 6.2.15], if $X \subseteq Y_{\Sigma}$, then $\operatorname{Trop}(f)(\operatorname{Trop}(X)) = \operatorname{Trop}(f(X))$.

Example 3.1. Of particular importance to us will be the tropicalization of a linear space. As in Section 2.2, let $\mathcal{A}$ be an arrangement, with associated linear space $L \subseteq \mathbb{A}^E = \operatorname{Spec} K[x_e \mid e \in E]$ and underlying matroid $\mathcal{M}$.

The torus orbits of $\mathbb{A}^E$ are indexed by subsets $S \subseteq E$, where S corresponds to the torus $\mathbb{G}_m^{E \smallsetminus S} \subseteq \mathbb{A}^E$ defined by $x_e = 0$ if and only if $e \in S$. The intersection $L \cap \mathbb{G}_m^{E \smallsetminus S}$ of L with one of these orbits is nonempty if and only if S is a flat of $\mathcal{M}$, in which case it is the intersection of the restriction $L^S \subseteq \mathbb{A}^{E \smallsetminus S}$ with the torus $\mathbb{G}_m^{E \smallsetminus S}$.

Given a flat F of $\mathcal{M}$, we define $\delta_F = \sum_{e \in F} \delta_e \in \mathbb{R}^E \subseteq \mathbb{T}^E$, where $\delta_e \in \mathbb{R}^E$ is the basis vector corresponding to $e \in E$. For a flag of flats

$$\mathcal{F} = (\emptyset = F_0 \subset F_1 \subset \cdots \subset F_{k-1} \subset F_k = E),$$

we have the cone

$$\beta_{\mathcal{F}} = \mathbb{R}_{\geq 0}\langle \delta_{F_1}, \cdots, \delta_{F_{k-1}}, \pm\delta_E \rangle \subseteq \mathbb{R}^E,$$

where $\dim \beta_{\mathcal{F}} = \ell(\mathcal{F}) = k$. We have $\beta_{\mathcal{F}} \prec \beta_{\mathcal{F}'}$ if and only if $\mathcal{F}'$ is a refinement of $\mathcal{F}$. The collection of cones $\beta_{\mathcal{F}}$ defines a fan in $\mathbb{R}^E$, called the **Bergman fan** of $\mathcal{M}$ (with the *fine* fan structure of [AK06]), with support equal to $\operatorname{Trop}(L \cap \mathbb{G}_m)$. As discussed in [DP16, Section 2], every initial degeneration of L is also a linear space. Therefore, the Bergman fan of $\mathcal{M}$ is a pure polyhedral fan of dimension $\operatorname{rk} \mathcal{M}$, which is balanced when each cone is assigned weight one.

Note that every cone $\beta_{\mathcal{F}}$ in the Bergman fan contains the diagonal copy of $\mathbb{R}$, the span of δ_E. Many authors take the quotient by this lineality space, which is equal to the tropicalization of the projectivization of L. We shall not adopt this convention.

If F is a flat, then $\mathrm{Trop}(L \cap \mathbb{G}_m^{E \setminus F})$ is the support of the Bergman fan of the restriction $\mathcal{M}^F$. Its cones, denoted $\beta_{\mathcal{F}}^{(F)}$, are in correspondence with flags $\mathcal{F}$ of flats in $\mathcal{M}^F$ (such a flag is identified with a flag of flats in $\mathcal{M}$ beginning at F). The full tropicalization $\mathrm{Trop}(L) \subseteq \mathbb{T}^E$, together with the fan structures on each of its strata, is the **extended Bergman fan** of $\mathcal{A}$. It is equal to the closure of

$$\mathrm{Trop}(L \cap \mathbb{G}_m^E) = \mathrm{Trop}(L) \cap \mathbb{R}^E$$

in $\mathbb{T}^E$.

CHAPTER IV

THE TROPICALIZATION OF A HYPERTORIC VARIETY

In this chapter, we describe the structure of the tropicalization of a hypertoric variety induced by its canonical embedding in the Lawrence toric variety. The main result is Theorem 4.1, which describes a polyhedral structure on the tropicalization. We also calculate the fibers of the tropicalization map. The main tool to obtain these results is the moment map Φ, which by functoriality of tropicalization gives a map from the tropicalization of the hypertoric variety to the Bergman fan of the underlying matroid of the arrangement.

Fix an arrangement $\mathcal{A} = \mathcal{A}(\mathbf{a}, r)$ in $M_{\mathbb{R}}$, where $\mathbf{a}$ is a primitive spanning configuration. As in Section 2.2, let $L \subseteq \mathbb{A}^E$ be the associated linear space and $\mathcal{M}$ the underlying matroid.

4.1. Description of the tropicalization

Let $\Phi\colon \mathfrak{B}_{\mathcal{A}} \to \mathbb{A}^E$ be the moment map from Section 2.4. By definition of $\mathfrak{M}_{\mathcal{A}}$, Φ maps $\mathfrak{M}_{\mathcal{A}}$ onto L, and by functoriality of tropicalization, $\mathrm{Trop}(\Phi)$ gives a surjection $\mathrm{Trop}(\mathfrak{M}_{\mathcal{A}}) \to \mathrm{Trop}(L)$. Given a flat F of $\mathcal{M}$, the stratum $\mathrm{Trop}(L \cap \mathbb{G}_m^{E \smallsetminus F})$ of $\mathrm{Trop}(L)$ is the Bergman fan of the restriction $\mathcal{M}^F$. As noted in example 3.1, this fan has cones $\beta_{\mathcal{F}}^{(F)}$ indexed by flags $\mathcal{F}$ of flats in $\mathcal{M}^F$. Given such a flag $\mathcal{F}$, let $C_{\mathcal{F}}^{(F,\mathcal{R})}$ be the preimage of $\beta_{\mathcal{F}}^{(F)}$ under the surjection $\mathrm{Trop}(\mathfrak{M}_{\mathcal{A}} \cap O(\sigma_{F,\mathcal{R}})) \to \mathrm{Trop}(L \cap \mathbb{G}_m^{E \smallsetminus F})$. Since every Bergman fan is balanced when every maximal cone is given weight one, each stratum $\mathrm{Trop}(\mathfrak{M}_{\mathcal{A}} \cap O(\sigma_{F,\mathcal{R}})) = \mathrm{Trop}(\mathfrak{M}_{\mathcal{A}}) \cap \widetilde{N}_{\mathbb{R}}(\sigma_{F,\mathcal{R}})$ inherits this structure of a balanced polyhedral fan with cones $C_{\mathcal{F}}^{(F,\mathcal{R})}$ and all weights equal to one. Our main theorem describes how these fans are pieced together.

Theorem 4.1. *The tropicalization* $\mathrm{Trop}(\mathfrak{M}_\mathcal{A})$ *of the hypertoric variety is the union of cones* $C_\mathcal{F}^{(F,\mathcal{R})}$ *indexed by a flat* F *of* $\mathcal{M}$*, a face* $\mathcal{R}$ *of the localization* $\mathcal{A}_F$*, and a flag of flats* $\mathcal{F}$ *in the restriction* $\mathcal{M}^F$*. These cones satisfy*

$$\dim C_\mathcal{F}^{(F,\mathcal{R})} = d - \operatorname{codim} \mathcal{R} + \ell(\mathcal{F}).$$

This gives $\mathrm{Trop}(\mathfrak{M}_\mathcal{A})$ *the combinatorial structure of a finite polyhedral complex, under the closure relation*

$$C_{\mathcal{F}'}^{(F',\mathcal{R}')} \subseteq \overline{C_\mathcal{F}^{(F,\mathcal{R})}} \tag{4.1}$$

if and only if the following conditions hold:

- $F \subseteq F'$*;*
- $\mathcal{R}' \subseteq \overline{\mathcal{R}}$*;*
- F' *is a flat in* $\mathcal{F}$*, and* $\mathrm{trunc}_{F'}(\mathcal{F})$ *is a refinement of* $\mathcal{F}'$*.*

Moreover, this gives each stratum $\mathrm{Trop}(\mathfrak{M}_\mathcal{A}) \cap \widetilde{N}_\mathbb{R}(\sigma_{F,\mathcal{R}})$ *the structure of a polyhedral fan, which is balanced when all cones are given weight one.*

Given a flat F and a face $\mathcal{R}$ of $\mathcal{A}_F$, there are two fans which live in $\mathrm{Trop}(O(\sigma_{F,\mathcal{R}})) = \widetilde{N}_\mathbb{R}(\sigma_{F,\mathcal{R}})$: the fan $\mathrm{Trop}(\mathfrak{M}_\mathcal{A}) \cap \widetilde{N}_\mathbb{R}(\sigma_{F,\mathcal{R}})$ and the fan of the orbit closure $\overline{O(\sigma_{F,\mathcal{R}})}$. The former fan has cones $C_\mathcal{F}^{(F,\mathcal{R})}$ indexed by flags of flats in $\mathcal{M}^F$, while the latter consists of the projections of the cones $\sigma_{S,\mathcal{R}'}$ with $\sigma_{S,\mathcal{R}'} \succ \sigma_{F,\mathcal{R}}$ (by Lemma 2.7, this is equivalent to $S \supseteq F$ and $\mathcal{R}' \subseteq \overline{\mathcal{R}}$). The following lemma relates these two fans.

Lemma 4.2. *Let F be a flat of $\mathcal{M}$ and $\mathcal{R}$ a face of $\mathcal{A}_F$. Given a set $S \subseteq E$ which contains F and a face $\mathcal{R}'$ of $\mathcal{A}_S$ contained in $\overline{\mathcal{R}}$, the intersection*

$$C_{\mathcal{F}}^{(F,\mathcal{R})} \cap \operatorname{relint}(\pi_{\sigma_{F,\mathcal{R}}}(\sigma_{S,\mathcal{R}'})),$$

for $\mathcal{F}$ a flag of flats in $\mathcal{M}^F$, is nonempty if and only if S is a flat of $\mathcal{M}$ which appears in the flag $\mathcal{F}$. In this case,

$$\overline{C_{\mathcal{F}}^{(F,\mathcal{R})}} \cap \widetilde{N}_{\mathbb{R}}(\sigma_{S,\mathcal{R}'}) = C_{\operatorname{trunc}_S(\mathcal{F})}^{(S,\mathcal{R}')}.$$

Proof. First, because $\Phi(\sigma_{F,\mathcal{R}}) = \mathbb{R}^F_{\geq 0}$, we have that

$$\operatorname{Trop}(\Phi)|_{\widetilde{N}_{\mathbb{R}}(\sigma_{F,\mathcal{R}})} \colon \widetilde{N}_{\mathbb{R}}(\sigma_{F,\mathcal{R}}) = \operatorname{Trop}(O(\sigma_{F,\mathcal{R}})) \to \mathbb{R}^{E \smallsetminus F} = \mathbb{R}^E/\mathbb{R}^F$$

is given by $[v] \mapsto [\Phi(v)]$. (Here we are identifying $\mathbb{R}^E/\mathbb{R}^F$, the vector space spanned by the cocharacter lattice of $O(\mathbb{R}^F_{\geq 0}) = \mathbb{G}_m^{E \smallsetminus F} \subseteq \mathbb{A}^E$ with $\mathbb{R}^{E \smallsetminus F}$ primarily for notational convenience.) In other words, the square

$$\begin{array}{ccc} \widetilde{N}_{\mathbb{R}} & \xrightarrow{\operatorname{Trop}(\Phi)} & \mathbb{R}^E \\ {\scriptstyle \pi_{\sigma_{F,\mathcal{R}}}}\downarrow & & \downarrow \\ \widetilde{N}_{\mathbb{R}}(\sigma_{F,\mathcal{R}}) & \xrightarrow{\operatorname{Trop}(\Phi)} & \mathbb{R}^{E \smallsetminus F} \end{array} \tag{4.2}$$

commutes.

Now, suppose $v \in C_{\mathcal{F}}^{(F,\mathcal{R})} \cap \operatorname{relint}(\pi_{\sigma_{F,\mathcal{R}}}(\sigma_{S,\mathcal{R}'})) \subseteq \widetilde{N}_{\mathbb{R}}(\sigma_{F,\mathcal{R}})$. Every vector in $\sigma_{S,\mathcal{R}'}$ can be written as a linear combination, with non-negative coefficients, of the generators ρ_e^+, ρ_f^- for $e \in S^+(\mathcal{R}')$, $f \in S^-(\mathcal{R}')$. Then v is the image of such a vector under $\pi_{\sigma_{F,\mathcal{R}}}$, which kills all generators of $\sigma_{S,\mathcal{R}'}$ indexed by elements of F (since

$F^{\pm}(\mathcal{R}) \subseteq S^{\pm}(\mathcal{R}')$ by Lemma 2.4). In order for v to be in the relative interior of $\pi_{\sigma_{F,\mathcal{R}}}(\sigma_{S,\mathcal{R}'})$, therefore, it must be that coefficient of ρ_e^+ (resp. ρ_f^-) is positive for $e \in S^+(\mathcal{R}') \smallsetminus F^+(\mathcal{R})$ (resp. $f \in S^-(\mathcal{R}') \smallsetminus F^-(\mathcal{R})$). Since the square (4.2) commutes, it follows that $\mathrm{Trop}(\Phi)(v) \in \mathbb{R}^{E \smallsetminus F}$ will lie in $\mathbb{R}_{>0}^{S \smallsetminus F} \cap \beta_{\mathcal{F}}^{(F)}$. By the definition of $\beta_{\mathcal{F}}^{(F)}$ (cf. Example 3.1), this intersection is nonempty if and only if S is a flat in the flag $\mathcal{F}$.

Conversely, suppose S is a flat in $\mathcal{F}$. For $e \in S \smallsetminus F$, define

$$v_e = \begin{cases} \frac{1}{2}(\rho_e^+ + \rho_e^-) & \text{if } e \in S^0(\mathcal{R}'), \\ \rho_e^+ & \text{if } e \in S^+(\mathcal{R}) \smallsetminus S^0(\mathcal{R}'), \\ \rho_e^- & \text{if } e \in S^-(\mathcal{R}) \smallsetminus S^0(\mathcal{R}'), \end{cases}$$

and let $v = \sum_{e \in S \smallsetminus F} v_e$. Then, by design, we have $v \in \mathrm{relint}(\pi_{\sigma_{F,\mathcal{R}}}(\sigma_{S,\mathcal{R}'}))$ and $\mathrm{Trop}(\Phi)(v) = \delta_{S \smallsetminus F} \in \beta_{\mathcal{F}}$. , it follows that $v \in \mathrm{Trop}(\Phi)^{-1}(\beta_{\mathcal{F}}) = C_{\mathcal{F}}^{(F,\mathcal{R})}$, and therefore $C_{\mathcal{F}}^{(F,\mathcal{R})} \cap \mathrm{relint}(\pi_{\sigma_{F,\mathcal{R}}}(\sigma_{S,\mathcal{R}'})) \neq \emptyset$.

For the final part of the lemma, we assume that S is a flat in the flag $\mathcal{F}$. We have

$$\overline{C_{\mathcal{F}}^{(F,\mathcal{R})}} \cap \widetilde{N}_{\mathbb{R}}(\sigma_{S,\mathcal{R}'})) = \pi_{\sigma_{S,\mathcal{R}'}}(C_{\mathcal{F}}^{(F,\mathcal{R})})$$

by [OR13, Lemma 3.9], so we need only prove that this projection coincides with $C_{\mathrm{trunc}_S(\mathcal{F})}^{(S,\mathcal{R}')}$. The square

$$\begin{array}{ccc} \widetilde{N}_{\mathbb{R}}(\sigma_{F,\mathcal{R}}) & \xrightarrow{\mathrm{Trop}(\Phi)} & \mathbb{R}^{E \smallsetminus F} \\ {\scriptstyle \pi_{\sigma_{S,\mathcal{R}'}}} \downarrow & & \downarrow \\ \widetilde{N}_{\mathbb{R}}(\sigma_{S,\mathcal{R}'}) & \xrightarrow{\mathrm{Trop}(\Phi)} & \mathbb{R}^{E \smallsetminus S} \end{array}$$

commutes for the same reason that that (4.2) commutes. Thus, we see that $\mathrm{Trop}(\Phi)$ maps $\pi_{\sigma_{S,\mathcal{R}'}}(C_{\mathcal{F}}^{(F,\mathcal{R})})$ onto $\beta_{\mathrm{trunc}_S(\mathcal{F})}^{(S)}$, which shows that $\pi_{\sigma_{S,\mathcal{R}'}}(C_{\mathcal{F}}^{(F,\mathcal{R})}) \subseteq C_{\mathrm{trunc}_S(\mathcal{F})}^{(S,\mathcal{R}')}$.

On the other hand, if $w \in C^{(S,\mathcal{R}')}_{\mathrm{trunc}_S(\mathcal{F})}$ and v is a preimage of w under $\pi_{\sigma_{S,\mathcal{R}'}}$, then $\mathrm{Trop}(\Phi)(v)$ need not lie in $\beta^{(F)}_{\mathcal{F}}$, so that v need not be in $C^{(F,\mathcal{R})}_{\mathcal{F}}$. However, we can choose $\eta \in \mathbb{R}^{S \smallsetminus F}$ so that $\mathrm{Trop}(\Phi)(v) + \eta \in \beta^{(F)}_{\mathcal{F}}$. Since $\eta \in \mathbb{R}^{S \smallsetminus F}$, there exists a preimage $v' \in \widetilde{N}_{\mathbb{R}}(\sigma_{F,\mathcal{R}})$ of η under $\mathrm{Trop}(\Phi)$, such that v' is expressed as a sum of generators ρ^+_e and ρ^-_f with $e, f \in S \smallsetminus F$. Then $\pi_{\sigma_{S,\mathcal{R}'}}(v') = 0$, and therefore $v + v' \in C^{(F,\mathcal{R})}_{\mathcal{F}}$ projects to w. This shows the reverse inclusion $C^{(S,\mathcal{R}')}_{\mathrm{trunc}_S(\mathcal{F})} \subseteq \pi_{\sigma_{S,\mathcal{R}'}}(C^{(F,\mathcal{R})}_{\mathcal{F}})$. □

We are now ready to prove the main theorem.

Proof of Theorem 4.1. Suppose that $C^{(F',\mathcal{R}')}_{\mathcal{F}'} \subseteq \overline{C^{(F,\mathcal{R})}_{\mathcal{F}}}$. Then necessarily the intersection $\overline{C^{(F,\mathcal{R})}_{\mathcal{F}}} \cap \widetilde{N}_{\mathbb{R}}(\sigma_{F',\mathcal{R}'})$ is nonempty. By Lemma 4.2, this implies $F' \supseteq F$ is a flat in $\mathcal{F}$ and $\mathcal{R}' \subseteq \overline{\mathcal{R}}$. In this case,

$$\overline{C^{(F,\mathcal{R})}_{\mathcal{F}}} \cap \widetilde{N}_{\mathbb{R}}(\sigma_{F',\mathcal{R}'}) = C^{(F',\mathcal{R}')}_{\mathrm{trunc}_F \mathcal{F}}$$

will contain $C^{(F',\mathcal{R}')}_{\mathcal{F}'}$ as a face if and only if $\mathrm{trunc}_{F'}(\mathcal{F})$ is a refinement of $\mathcal{F}'$.

Let $C^{(F,\mathcal{R})}_{\mathcal{F}}$ be a cone in $\mathrm{Trop}(\mathfrak{M}_{\mathcal{A}}) \cap \widetilde{N}_{\mathbb{R}}(\sigma_{F,\mathcal{R}})$. Since

$$\mathrm{Trop}(\Phi)\colon N_{\mathbb{R}}(\sigma_F, \mathcal{R}) \to \mathbb{R}^{E \smallsetminus F}$$

is a linear surjection of relative dimension $d - \operatorname{codim} \mathcal{R}$, and $C^{(F,\mathcal{R})}_{\mathcal{F}}$ is defined to be the preimage of $\beta^{(F)}_{\mathcal{F}}$, it follows that

$$\dim C^{(F,\mathcal{R})}_{\mathcal{F}} = d - \operatorname{codim} \mathcal{R} + \dim \beta^{(F)}_{\mathcal{F}} = d - \operatorname{codim} \mathcal{R} + \ell(\mathcal{F}).$$

□

Remark 4.3. By Theorem 4.1, a cone $C_{\mathcal{F}}^{(F,\mathcal{R})}$ is inclusion-maximal in the stratum $\mathrm{Trop}(\mathfrak{M}_{\mathcal{A}}) \cap \widetilde{N}_{\mathbb{R}}(\sigma_{F,\mathcal{R}})$ if and only if the flag $\mathcal{F}$ is maximal. A maximal flag in $\mathcal{M}^F$ has length $\operatorname{rk} \mathcal{M}^F = d - \operatorname{rk} F$, so that

$$\dim C_{\mathcal{F}}^{(F,\mathcal{R})} = d - \operatorname{codim} \mathcal{R} + \ell(\mathcal{F}) = 2d - \operatorname{rk} F - \operatorname{codim} \mathcal{R}.$$

Thus $\dim C_{\mathcal{F}}^{(F,\mathcal{R})}$ agrees with $\dim(\mathfrak{M}_{\mathcal{A}} \cap O(\sigma_{F,\mathcal{R}})) = \dim(\mathrm{Trop}(\mathfrak{M}_{\mathcal{A}}) \cap \widetilde{N}_{\mathbb{R}}(\sigma_{F,\mathcal{R}}))$ by Proposition 2.11. This shows that the inclusion-maximal cones in each stratum of $\mathrm{Trop}(\mathfrak{M}_{\mathcal{A}})$ are precisely the dimension-maximal cones, which should be expected because tropicalizations are always pure-dimensional.

4.2. Fibers of tropicalization

In this section, we calculate the fiber of the tropicalization map $\mathfrak{M}_{\mathcal{A}} \to \mathrm{Trop}(\mathfrak{M}_{\mathcal{A}})$ over a point $\theta \in \mathrm{Trop}(\mathfrak{M}_{\mathcal{A}})$, following the approach of [CHW14]. Suppose that $\theta \in \mathrm{Trop}(\mathfrak{M}_{\mathcal{A}}) \cap \widetilde{N}_{\mathbb{R}}(\sigma_{F,\mathcal{R}})$ for some flat F of $\mathcal{M}$ and face $\mathcal{R}$ of the localization $\mathcal{A}_F$. We shall write $\mathrm{trop}^{-1}(\theta) \subseteq O(\sigma_{F,\mathcal{R}})^{\mathrm{an}}$ to denote the fiber over θ of the map $\mathrm{trop}\colon O(\sigma_{F,\mathcal{R}})^{\mathrm{an}} \to \widetilde{N}_{\mathbb{R}}(\sigma_{F,\mathcal{R}})$, and $\mathrm{trop}_{\mathfrak{M}_{\mathcal{A}}}^{-1}(\theta) = \mathrm{trop}^{-1}(\theta) \cap \mathfrak{M}_{\mathcal{A}}^{\mathrm{an}}$ for the fiber over θ of $\mathfrak{M}_{\mathcal{A}}^{\mathrm{an}} \to \mathrm{Trop}(\mathfrak{M}_{\mathcal{A}})$. Let $\eta = (\eta_e) \in \mathbb{R}^{E \smallsetminus S}$ be the image of θ under $\mathrm{Trop}(\Phi)$.

We defined the moment map $\Phi\colon \mathfrak{B}_{\mathcal{A}} \to \mathbb{A}^E$ as an extension to $\mathfrak{B}_{\mathcal{A}}$ of a split surjection of tori $\widetilde{T} \to \mathbb{G}_m^E$. This surjection is given by the map $\widetilde{N} \to \mathbb{Z}^E$ from (2.4) or equivalently by an injection of characters $\Phi^*\colon \mathbb{Z}^E \to \widetilde{M}$. The image under Φ^* of a generator $x_e \in \mathbb{Z}^E$ is the primitive diagonal element $x_e^+ + x_e^- \in \widetilde{M} \subseteq \mathbb{Z}^E \oplus \mathbb{Z}^E$. Inspired by the standard notation for the homogeneous coordinate ring of $\mathfrak{B}_{\mathcal{A}}$, we denote by $z_e w_e$ the monomial in $K[\widetilde{M}]$ corresponding to $x_e^+ + x_e^-$.

The diagonal element $x_e^+ + x_e^-$ is in $\widetilde{M}(\sigma_{F,\mathcal{R}})$ if and only if $e \in E \smallsetminus F$, and the restriction of Φ to $O(\sigma_{F,\mathcal{R}})$ is given by restricting Φ^* to obtain $\mathbb{Z}^{E \smallsetminus F} \to \widetilde{M}(\sigma_{F,\mathcal{R}})$. We may extend the set $\{x_e^+ + x_e^- \mid e \in E \smallsetminus F\}$ to an integral basis of $\widetilde{M}(\sigma_{F,\mathcal{R}})$. By Lemma 2.6 we must add $d - \operatorname{codim} \mathcal{R} = \dim \mathcal{R}$ primitive elements u_i. We write y_i for the monomial corresponding to u_i, so that

$$K[O(\sigma_{F,\mathcal{R}})] = K[\widetilde{M}(\sigma_{F,\mathcal{R}})] \cong K[(z_e w_e)^{\pm 1}, y_i^{\pm 1} \mid e \in E \smallsetminus F, i = 1, \ldots, \dim \mathcal{R}].$$

An element of $\widetilde{N}_{\mathbb{R}}(\sigma_{F,\mathcal{R}}) = \operatorname{Hom}(\widetilde{M}(\sigma_{F,\mathcal{R}}), \mathbb{R})$ is uniquely determined by its values on the integral basis $\{x_e^+ + x_e^-, u_i \mid e \in E \smallsetminus F, i = 1, \ldots, \dim \mathcal{R}\}$ of $\widetilde{M}(\sigma_{F,\mathcal{R}})$. In particular, a valuation $\operatorname{val} \in O(\sigma_{F,\mathcal{R}})^{\mathrm{an}}$ lies in $\operatorname{trop}^{-1}(\theta)$ if and only if $\operatorname{val}(z_e w_e) = \langle x_e^+ + x_e^-, \theta \rangle = \eta_e$ for all $e \in E \smallsetminus F$ and $\operatorname{val}(y_i) = \langle u_i, \theta \rangle$ for all $i = 1, \ldots, \dim \mathcal{R}$.

Let $\mathcal{B}$ be a basis of $\mathcal{M}^F$ which has maximal η-weight; that is, $\sum_{e \in \mathcal{B}} \eta_e$ is maximized at $\mathcal{B}$, and consider the subring

$$A = K[(z_e w_e)^{\pm 1}, (y_i)^{\pm 1} \mid e \in \mathcal{B}, i = 1, \ldots, \dim \mathcal{R}]$$

of $K[O(\sigma_{F,\mathcal{R}})]$. Since $\mathcal{B}$ is a basis of $\mathcal{M}^F$, for each $f \in E \smallsetminus (F \cup \mathcal{B})$, there is a unique element

$$p_f = \sum_{e \in \mathcal{B}} c_e (z_e w_e) \in A$$

such that $p_f - z_f w_f \in K[O(\sigma_{F,\mathcal{R}})]$ lies in the ideal of $\mathfrak{M}_{\mathcal{A}} \cap O(\sigma_{F,\mathcal{R}})$. Since these relations generate the ideal, this shows that A is isomorphic to the coordinate ring $K[\mathfrak{M}_{\mathcal{A}} \cap O(\sigma_{F,\mathcal{R}})]$. Furthermore, because the basis $\mathcal{B}$ is η-maximal, any valuation val on A with $\operatorname{val}(z_e w_e) = \eta_e$ for every $e \in \mathcal{B}$ must necessarily satisfy $\operatorname{val}(p_f) = \eta_f$ for all $f \in E \smallsetminus (F \cup \mathcal{B})$ as well.

We can therefore identify $\mathrm{trop}^{-1}_{\mathfrak{M}_{\mathcal{A}}}(\theta)$ with the set of valuations on A such that $\mathrm{val}(z_e w_e) = \eta_e$ for all $e \in \mathcal{B}$ and $\mathrm{val}(y_i) = \langle u_i, \theta \rangle$ for $i = 1, \ldots, \dim \mathcal{R}$. This is a satisfying description of the fiber; however, it is useful to identify this fiber with the Berkovich spectrum of a particular K-affinoid algebra.

Set $r_e = \exp(-\eta_e)$ and $s_i = \exp(-\langle u_i, \theta \rangle)$. Define the affinoid algebras

$$\mathscr{A} = K\langle r_e^{-1}(z_e w_e), r_e(z_e w_e)^{-1}, s_i^{-1} y_i, s_i y_i^{-1} \mid e \in \mathcal{B}, i = 1, \ldots, \dim \mathcal{R} \rangle.$$

and

$$\mathscr{B} = \mathscr{A}\langle r_f^{-1} p_f, r_f p_f^{-1} \mid f \in E \smallsetminus (F \cup \mathcal{B}) \rangle.$$

Then by construction, each seminorm $\gamma \in \mathscr{M}(\mathscr{B})$ restricts to a seminorm on A with $\gamma(z_e w_e) = r_e$ and $\gamma(y_i) = s_i$ for all $e \in E \smallsetminus F$ and $i = 1, \ldots, \dim \mathcal{R}$. (Equivalently, $-\log \gamma(-)$ is a valuation on A with $-\log \gamma(z_e w_e) = \eta_e$ and $-\log \gamma(y_i) = \langle u_i, \theta \rangle$.) In fact, this is a bijective correspondence, and any such seminorm on A extends uniquely to $\mathscr{B}$. We will not prove this statement, but we refer the reader to the proof of [CHW14, Proposition 5.6], which can be adapted to prove the following.

Proposition 4.4. *The fiber* $\mathrm{trop}^{-1}_{\mathfrak{M}_{\mathcal{A}}}(\theta)$ *is* $\mathscr{M}(\mathscr{B}) \subseteq \mathfrak{M}^{\mathrm{an}}_{\mathcal{A}}$.

It turns out that $\mathscr{M}(\mathscr{B})$ has a unique **Shilov boundary point**: a seminorm $\gamma \in \mathscr{M}(\mathscr{B})$ such that $\mathrm{ev}_b \colon \mathscr{M}(\mathscr{B}) \to \mathbb{R}_{\geq 0}$ is maximized at γ for every $b \in \mathscr{B}$. Again, we shall not prove this, but we refer to [CHW14, Theorem 5.8 & Corollary 5.9] for an outline.

The significance of this result is that the section $\mathrm{Trop}(\mathfrak{M}_{\mathcal{A}}) \to \mathfrak{M}^{\mathrm{an}}_{\mathcal{A}}$ which we will construct in Chapter V will map each $\theta \in \mathrm{Trop}(\mathfrak{M}_{\mathcal{A}})$ to the unique Shilov boundary point of $\mathrm{trop}^{-1}_{\mathfrak{M}_{\mathcal{A}}}(\theta)$.

CHAPTER V

FAITHFUL TROPICALIZATION

In this chapter, we prove that each hypertoric variety is faithfully tropicalized by its Lawrence embedding. We do so by using Theorem 4.1 to show that the conditions of a theorem of Gubler, Rabinoff, and Werner are satisfied.

5.1. The theorem of Gubler-Rabinoff-Werner

Let X be a suvariety of a torus. Gubler, Rabinoff, and Werner have shown that there exists a unique continuous section to the tropicalization map $X^{\mathrm{an}} \to \mathrm{Trop}(X)$ if $\mathrm{Trop}(X)$ has tropical multiplicity one at every point [GRW16, Theorem 10.6].

In the general case, where X is a subvariety of a toric variety Y_Σ which is not necessarily a torus, we can apply the above result on each torus orbit: If $\mathrm{Trop}(X)$ has multiplicity one at every point, then there is a unique section of tropicalization which is continuous on $\mathrm{Trop}(X \cap O(\sigma)) = \mathrm{Trop}(X) \cap N_{\mathbb{R}}(\sigma)$ for each $\sigma \in \Sigma$. However, this section may fail to be continuous on all of $\mathrm{Trop}(X)$. An example of an irreducible hypersurface in $\mathbb{A}^3$ for which this section is not continuous is given in [GRW15, Example 4.9].

In [GRW15], Gubler, Rabinoff, and Werner provide the following sufficient criteria for continuity of this section.

Theorem 5.1 ([GRW15, Proposition 8.8 & Theorem 8.14])**.** *Let Σ be a pointed rational fan in $N_{\mathbb{R}}$, and let $X \subseteq Y_\Sigma$ be a subvariety. Suppose that*

(1) $X \cap T$ is dense in X;

(2) for all $\sigma \in \Sigma$, the subscheme $X \cap O(\sigma)$ is either empty or equidimensional of dimension d_σ;

(3) $\mathrm{Trop}(X)$ *has tropical multiplicity one at every point;*

(4) $\mathrm{Trop}(X) \cap N_{\mathbb{R}}$ *can be covered by finitely many d_0-dimensional polyhedra with the following property: If the recession cone of P meets the relative interior of σ, then $\pi_\sigma(P) = \overline{P} \cap N_{\mathbb{R}}(\sigma)$ has dimension d_σ.*

Then there is a unique continuous section of the tropicalization map $X^{\mathrm{an}} \to \mathrm{Trop}(X)$.

Remark 5.2. It follows from [GRW15, Proposition 8.3] that for each point in the tropicalization of multiplicity one, the fiber of the tropicalization map over that point contains a unique Shilov boundary point. The section produced by Theorem 5.1 maps each point of $\mathrm{Trop}(X)$ to the unique Shilov boundary point in its fiber.

Although the proof of Theorem 5.1 requires careful study of the analytification X^{an}, the criteria (1)–(4) can be checked purely by inspecting $\mathrm{Trop}(X)$ (and X itself). We remark that while Theorem 5.1 is a powerful tool, it does not trivialize the problem of finding a faithful tropicalization of X. Indeed, it does not give any indication as to how to find a faithful tropicalization of X, nor does it imply that one must even exist. Rather, Theorem 5.1 transforms a difficult problem in non-Archimedean geometry—that of verifying that a particular tropicalization is faithful—into a difficult combinatorics problem. For example, as outlined in [GRW15, Example 8.16], Theorem 5.1 can be used to prove that the Plücker embedding yields a faithful tropicalization of $\mathrm{Gr}(2, n)$; however, many ingredients of the original proof in [CHW14] remain necessary to establish conditions (2)–(4).

Remark 5.3. Faithful tropicalization is easy to verify in one situation. If X is irreducible and intersects each torus orbit properly or not at all, then by [GRW15, Theorem 8.15] the resulting tropicalization is faithful if $\mathrm{Trop}(X)$ has multiplicity one everywhere. By Corollary 2.12, a hypertoric variety $\mathfrak{M}_{\mathcal{A}}$ fails to possess this nice property outside of the trivial case $\mathfrak{M}_{\mathcal{A}} = \mathbb{A}^{2d}$. (The Grassmannian $\mathrm{Gr}(2, n)$ also does not intersect torus orbits properly.)

5.2. Faithful tropicalization of hypertoric varieties

Let $\mathfrak{M}_{\mathcal{A}}$ be any hypertoric variety. As in section 4.1, let $\mathrm{Trop}(\mathfrak{M}_{\mathcal{A}})$ denote the tropicalization induced by the Lawrence embedding $\mathfrak{M}_{\mathcal{A}} \subseteq \mathfrak{B}_{\mathcal{A}}$. We now use Theorem 5.1 to prove that this is a faithful tropicalization.

Theorem 5.4. *There is a unique continuous section of the tropicalization map* $\mathfrak{M}_{\mathcal{A}}^{\mathrm{an}} \to \mathrm{Trop}(\mathfrak{M}_{\mathcal{A}})$.

Proof. We shall show that the four conditions of Theorem 5.1 hold. The intersection $\mathfrak{M}_{\mathcal{A}} \cap \widetilde{T}$ is nonempty, and therefore dense in $\mathfrak{M}_{\mathcal{A}}$ because $\mathfrak{M}_{\mathcal{A}}$ is irreducible. By Proposition 2.11, the intersection of $\mathfrak{M}_{\mathcal{A}}$ with any torus orbit in $\mathfrak{B}_{\mathcal{A}}$ is either empty or it is a linear space. In particular, each of these intersections (when nonempty) is equidimensional and the tropical multiplicity of $\mathrm{Trop}(\mathfrak{M}_{\mathcal{A}})$ is one at every point.

It remains to show that (4) holds. We equip $\mathrm{Trop}(\mathfrak{M}_{\mathcal{A}})$ with the polyhedral structure described in Theorem 4.1. Then $\mathrm{Trop}(\mathfrak{M}_{\mathcal{A}}) \cap \widetilde{N}_{\mathbb{R}}$ is covered by the $2d$-dimensional cones $C_{\mathcal{F}}^{(\emptyset, \widetilde{M}_{\mathbb{R}})}$, where $\mathcal{F}$ is a maximal flag of flats in $\mathcal{M}$. For convenience, we will write $C_{\mathcal{F}}$ instead of $C_{\mathcal{F}}^{(\emptyset, \widetilde{M}_{\mathbb{R}})}$. By Lemma 4.2, the cone $C_{\mathcal{F}}$ meets the relative interior of a Lawrence cone $\sigma_{F,\mathcal{R}}$ if and only if F is a flat of $\mathcal{M}$ which appears in the flag $\mathcal{F}$. In this case, Lemma 4.2 gives

$$\pi_{\sigma_{F,\mathcal{R}}}(C_{\mathcal{F}}) = \overline{C_{\mathcal{F}}} \cap \widetilde{N}_{\mathbb{R}}(\sigma_{F,\mathcal{R}}) = C_{\mathrm{trunc}_F(\mathcal{F})}^{(F,\mathcal{R})}.$$

Now, $\text{trunc}_F(\mathcal{F})$ is a maximal flag of flats in $\mathcal{M}^F$, and therefore $C^{(F,\mathcal{R})}_{\text{trunc}_F(\mathcal{F})}$ is an inclusion-maximal cone of the fan $\text{Trop}(\mathfrak{M}_\mathcal{A}) \cap \widetilde{N}_\mathbb{R}(\sigma_{F,\mathcal{R}})$. It follows (cf. Remark 4.3) that $C^{(F,\mathcal{R})}_{\text{trunc}_F(\mathcal{F})}$ has dimension equal to $\dim(\mathfrak{M}_\mathcal{A} \cap O(\sigma_{F,\mathcal{R}}))$. We may therefore apply Theorem 5.1 to conclude that there is a unique continuous section of tropicalization defined on all of $\text{Trop}(\mathfrak{M}_\mathcal{A})$. □

Remark 5.5. We conclude by noting that there is a more general notion of hypertoric variety than we have discussed in this dissertation. Arbo and Proudfoot [AP16] have recently shown how to construct a hypertoric variety from a zonotopal tiling $\mathcal{T}$. Such a hypertoric variety is also embedded in a (generalized) Lawrence toric variety, and agrees with the variety constructed in Section 2.4 in the case where $\mathcal{T}$ is a regular tiling and hence normal to some affine arrangement. We suspect that Theorem 5.4 remains true in this more general setting.

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