The Formation of Communication Networks in Cooperative Games*

by

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Abstract

I survey the literature on network formation in situations where the possible gains from cooperation of coalitions of agents are modeled by a coalitional game. I discuss the models that appear in the literature and their predictions on the networks that will be formed according to various equilibrium concepts, as well as the eventual payoffs to the players in equilibrium networks.

*Most of what I cover in this chapter is also covered in Slikker and van den Nouweland (2001a). I am grateful to Marco Slikker, Matthew Jackson, and Myrna Wooders for commenting on an earlier version of this chapter. Their comments have greatly improved it.

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1 Introduction

Network structures play an important role in many economic situations. The types of networks that I consider in this chapter are those that connect many individuals who each must establish and maintain their own links. I refer to such networks as communication networks; they describe the bilateral channels through which individuals can communicate and thereby coordinate their actions. The worth that coalitions of individuals can obtain by coordinating their actions is modeled by a coalitional game, which specifies for each coalition S of individuals a worth $v(S)$. Suppose, for example, that there are three individuals, one seller who has one indivisible unit of a good for sale, and two potential buyers. Suppose the value of the good is 0 to the seller (s), 1 to the first buyer (b_1), and 2 to the second buyer (b_2). This situation can be modeled as a coalitional game with player set $\{s, b_1, b_2\}$ and $v(\{s\}) = v(\{b_1\}) = v(\{b_2\}) = v(\{b_1, b_2\}) = 0$, $v(\{s, b_1\}) = 1$, and $v(\{s, b_2\}) = v(\{s, b_1, b_2\}) = 2$. If only the two buyers are linked (the only link formed is b_1b_2), then the seller cannot communicate with any of the buyers, so that no worth can be generated. If the seller is linked to the first buyer and the first buyer, in turn, is linked to the second buyer (the two links sb_1 and b_1b_2 have been formed), then all three can communicate and coordinate their actions (the seller and the second buyer do so through the first buyer) and a worth of 2 can be generated by selling the good to the second buyer.

Note that in the approach based on coalitional games as explained above, the worth that players in a network can obtain primarily depends on which players are connected with one another (directly or indirectly) and not on how exactly these players are connected. Hence, issues such as the deterioration of information as it has to travel along longer paths are not taken into account. When we want to take these types of issues into account, we end up in the realm of value functions on networks, which are covered in Jackson (2004). Also, the approach based on coalitional games precludes externalities between different groups of interconnected players; the worth generated by a group of interconnected players does not depend on whether or not players not in the group are connected to each other.

I illustrated above how the worth that the players can obtain depends on the network. The discussion there concentrated on the worth that can be obtained by the players as a group if they cooperate. This does not address the issue of how this worth will then be divided among the players. In general, this division will depend on the positions that the players take in the network. For example, if in the situation described above the two links sb_1 and b_1b_2 have been formed and the worth 2 is generated by selling the good to the second buyer, then the first buyer might get some share of the profits of the sale because his cooperation is needed for the seller and the second buyer to make a deal as they cannot communicate directly (link sb_2 has not been formed). If, on the other hand, link sb_2 has been formed, then the worth 2 can still be generated by selling the good to the second buyer, but now the seller and this second buyer do not need the assistance of the first buyer to complete the transaction and it seems reasonable that the seller and the second buyer will share the worth 2 between themselves in some way. The specific way in which the worth generated by the players in a network is shared among them is specified by an allocation rule. In most of the models that I describe in this chapter, it is assumed that such an allocation rule is exogenously given and the players decide which links to form given this allocation rule.[1]

A network is a collection of bilateral links between players who must establish and maintain their own links. Forming or not forming certain links is a strategic decision for each player forming them, as such links have an influence on the position that this player will take in the network and thereby have an influence on the payoff that the player expects to obtain, as specified by the allocation rule. Because the formation of links is a strategic decision, it is appropriate to model network formation as a non-cooperative game. Perhaps the simplest possible way to model network formation is by means of a strategic-form game in which players simultaneously announce which links they want to form and in which a link between two players is formed if and only if both these players want to form it.[2] In this model, every network can emerge in a Nash equilibrium when the underlying coalitional game and the allocation rule are such that every player wants to have as many links as possible. This is because it takes the consent of two players to form a link, so that a single player cannot form any new links through unilateral deviation. This motivates the study of refinements of Nash equilibrium for network-formation models.

In this chapter, I discuss the various (non-cooperative) models of network formation in coalitional games that have been studied in the literature. The questions answered for these models are which networks can be formed according to Nash equilibria or refinements thereof and what the payoffs are to the players in such networks. The models illustrate how differences in the circumstances under which the players are

[1]The exception is the model in Section 7, in which players bargain over link formation and payoff division simultaneously.

[2]This model is described in Section 4.

forming links and differences in ways in which jointly generated profits are distributed between them influence the networks they form.

I shall proceed according to the following outline.

1. *Introduction*
2. *Definitions*
3. *Network-Formation Game in Extensive Form*
4. *Network-Formation Game in Strategic Form*
5. *Comparison of the Network-Formation Models in Extensive and Strategic Forms*
6. *Network Formation with Costs for Establishing Links*
7. *Simultaneous Bargaining over Network Formation and Payoff Division*
8. *Related Literature*
9. *References*

2 Definitions

All the models that I cover in this chapter start from a coalitional game (N, v), in which $N = \{1, 2, ..., n\}$ is the set of players and $v : 2^N \to I\!R$ is the characteristic function that assigns to each coalition $S \subseteq N$ of players a worth $v(S)$. It is always assumed that $v(\emptyset) = 0$. The interpretation of the worth $v(S)$ of a coalition S is that this is the payoff that the members of S can obtain for themselves if they coordinate their actions, independently of the actions taken by the players who are not included in the coalition. However, in order to coordinate their actions, players have to be able to communicate.

Communication takes place through bilateral channels, which I refer to as (communication) links. The link between players i and j is denoted by ij and the set of all possible communication links between the players in N is denoted by $g^N = \{ij \mid i, j \in N, i \neq j\}$. In the models that I survey, it takes the consent of both players to form the link between them and an existing link ij can be used costlessly by both players i and j. A network is a pair (N, g), where N is the set of players and $g \subseteq g^N$ is a set of links. When this does not lead to confusion, a network (N, g) is identified with its set of links and is simply denoted g. The network with all possible links, g^N, is referred to as the complete network. In a network g, two players can communicate (and, hence, coordinate their actions) if and only if there exists a path of communication between them. A path in g between players i and j is a sequence of players $i = i_1, i_2, ..., i_t = j$ with $t \geq 2$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, 2, ..., t-1\}$. If there exists such a path with $t = 2$, then $ij \in g$ and players i and j can communicate directly. Players can communicate indirectly through other players if there exists a longer path between them. Players who are not connected by a path are in different components of the network, while players who are connected by a path are in the same component. The component containing player i is denoted $C_i(g) = \{j \in N \mid j = i \text{ or } j \text{ and } i \text{ are connected by a path in } g\}$, and the set of all components of network (N, g) is denoted by $\pi(N, g) = \{C_i(g) \mid i \in N\}$. Network (N, g) is connected if it contains only one component, i.e., if $C_i(g) = N$ for each $i \in N$, so that all players can communicate with each other.

A coalitional game and a network comprise a communication situation. Formally, a communication situation is a triple (N, v, g), consisting of a player set N, a characteristic function v, and a set of links g. I am interested in the formation of links given a coalitional game (N, v) and I denote the set of all communication situations with player set N and characteristic function v by CS_v^N.

For a communication situation (N, v, g), the network-restricted game (N, v^g) incorporates both the possible gains from cooperation for coalitions of players as modeled by the coalitional game (N, v) and the restrictions on communication reflected by communication network g. The players in a coalition $T \subseteq N$ have available the links in $g(T) = \{ij \in g \mid i \in T \text{ and } j \in T\}$ and this induces a partition $\pi(T, g)$ of this coalition into the components of the network $(T, g(T))$. The players in T can only coordinate their actions within these components. This motivates the definition of the characteristic function v^g as $v^g(T) = \sum_{C \in \pi(T,g)} v(C)$ for each $T \subseteq N$. Note that this definition implicitly assumes that there are no externalities between different components of a network; the worth of the players in a network is simply the sum of the worths of its components. Jackson and Wolinsky (1996) refer to this property as component additivity of the value function that assigns a value to every network. This property of the network-restricted game derived from a coalitional game is in accordance with the interpretation of the coalitional game that it assigns to each coalition of players the worth that they can obtain *independently of the other players*. In addition, the definition of the characteristic function v^g is such that the worth of a coalition of players who form a component of a network does not depend on exactly which links exist between these players, but only on whether or not these players are connected at all. This reflects the interpretation of the links as (costless) communication channels which allow players to cooperate.

Using the network-restricted game (N, v^g), Myerson (1977) defined an allocation rule for communication situations. An allocation rule on a class CS of communication situations is a function γ that assigns to each communication situation $(N, v, g) \in CS$ a payoff vector $\gamma(N, v, g) \in I\!R^N$. If allocation rule γ is used, then a player $i \in N$ expects to get a payoff $\gamma_i(N, v, g)$ from being in communication situation (N, v, g). The Myerson value μ is the Shapley value Φ (cf. Shapley (1953)) of the network-restricted game;

$$\mu_i(N, v, g) = \Phi_i(N, v^g) = \sum_{T \subset N : i \notin T} \frac{|T|!(|N| - 1 - |T|)!}{|N|!} \left(v^g(T \cup i) - v^g(T)\right).^3$$

The Myerson value is by far the most widely used allocation rule for communication situations in the literature. One of the reasons for this popularity is undoubtedly the firm axiomatic grounding of this allocation rule. Myerson (1977) showed that for any coalitional game (N, v) the Myerson value is the unique allocation rule on CS_v^N that satisfies the two properties component efficiency and fairness. Component efficiency of an allocation rule means that the players in a component distribute the value of this component, which they can obtain for themselves irrespective of the actions taken by the players not in the component, among themselves. Fairness reflects the idea that two players should gain or lose equally from forming the link between them. An allocation rule is fair if the payoffs of two players i and j in- or decrease by the same amount whenever the link connecting them is severed.[4]

Several other allocation rules for communication situations appear in the literature. The main one is the position value (cf. Borm et al. (1992)), which focuses on the importance of the communication links rather than the role of the players. Finding an axiomatic characterization of this allocation rule for a fixed coalitional game and variable networks has proven rather elusive over the years, but Slikker (2003) recently succeeded in finding such an axiomatic characterization. He uses two axioms in his characterization of the position value, component efficiency and balanced total threats. Balanced total threats is reminiscent of the balanced contributions axiom that Myerson (1980) used to axiomatize the Myerson value and that can replace fairness in a characterization of the Myerson value. The axiomatization of the position value may well lead to new applications of this allocation rule in the future.

I will cover both extensive-form games of network formation and strategic-form ones. The network-formation game in extensive form is most easily described casu-

[3]Here, $|T|$ denotes the number of elements in a set T. Also, following popular tradition, I omit the set brackets { and } when I don't think such an omission will cause the reader any confusion.

[4]Jackson (2003) refers to this property as equal bargaining power.

ally and I will do exactly that in Section 3. A game in strategic form is a tuple $(N, (S_i)_{i\in N}, (f_i)_{i\in N})$ consisting of a player set N, a strategy set S_i for each player $i \in N$, and a payoff function $f_i : \prod_{i\in N} S_i \to I\!R$ for each player $i \in N$. In such a game, every player $i \in N$ chooses one of his strategies $s_i \in S_i$ and then each player i gets a payoff $f_i(s)$, where $s = (s_i)_{i\in N}$ denotes the strategy profile chosen. A strategy profile is a Nash equilibrium (cf. Nash (1950)) if no player can increase his payoff by unilaterally deviating to a different strategy. In formula, $s \in \prod_{i\in N} S_i$ is a Nash equilibrium if for each player $i \in N$ and each strategy $t_i \in S_i$ it holds that $f_i(s_i) \geq f_i(t_i, s_{-i})$, where $s_{-i} = (s_j)_{j\in N\setminus i}$ denotes the (fixed) strategies of the players other than i and (t_i, s_{-i}) denotes the strategy profile in which player i plays his strategy t_i and every player $j \in N\setminus i$ plays his strategy s_j.

I will use several refinements of Nash equilibria. A strategy s_i of a player i is undominated if there is no other strategy that gives this player at least as high a payoff and sometimes a higher payoff, for all possible strategy choices of the other players. Hence, $s_i \in S_i$ is undominated if there is no other strategy $t_i \in S_i$ such that $f_i(s_i, s_{-i}) \leq f_i(t_i, s_{-i})$ for all $s_{-i} \in \prod_{j\in N\setminus i} S_j$, with the inequality being strict for at least one s_{-i}. An undominated Nash equilibrium is a Nash equilibrium in which each player plays an undominated strategy.

A strong Nash equilibrium (cf. Aumann (1959) and Bernheim et al. (1987)) is a strategy profile that is stable against deviations not only by single players, but by any coalition of players. Hence, $s \in \prod_{i\in N} S_i$ is a strong Nash equilibrium if for each coalition $T \subseteq N$ it holds that there is no strategy tuple $t_T = (t_i)_{i\in T} \in \prod_{i\in T} S_i$ such that $f_i(t_T, s_{N\setminus T}) \geq f_i(s)$ for all $i \in T$ and $f_i(t_T, s_{N\setminus T}) > f_i(s)$ for at least one $i \in T$. This definition of strong equilibrium is actually slightly different from that put forward in Aumann (1959) and Bernheim et al. (1987) in that it allows a coalition to deviate to a strategy profile that strictly increases the payoffs of some of its members without decreasing those of the other members, whereas the original definition allows only deviations that strictly increase the payoffs of all members of a deviating coalition. Both interpretations of strong Nash equilibrium are prominent in the literature, and in most games the two definitions lead to the same sets of strong Nash equilibria, but the one that I use here is slightly more appealing in the context of network-formation games (see, eg. Jackson and van den Nouweland (2001)).

Coalition-proof Nash equilibria (cf. Bernheim et al. (1987)) are similar to strong Nash equilibria in that they require a strategy profile to be stable against deviations by all coalitions of players. However, in a coalition-proof Nash equilibrium, the deviations are restricted to be stable themselves against further deviations by subcoalitions. To provide the formal definition of coalition-proof Nash equilibrium, which is inductive, I need some additional notation. Let $\Gamma = (N; (S_i)_{i\in N}; (f_i)_{i\in N})$ be a game in strategic form, $T \subset N$ a coalition, and $s^*_{N\setminus T} \in \prod_{i\in N\setminus T} S_i$. The strategic-form game $\Gamma(s^*_{N\setminus T}) = (T; (S_i)_{i\in T}; (f^*_i)_{i\in T})$ induced on the players of T by the strategies $s^*_{N\setminus T}$ has payoff functions $f^*_i : \prod_{i\in T} S_i \to I\!R$ given by $f^*_i(s_T) = f_i(s_T, s^*_{N\setminus T})$ for all $s_T \in \prod_{i\in T} S_i$, for each $i \in T$. In a 1-player game $(\{i\}; S_i; f_i)$, a strategy $s^*_i \in S_i$ is a coalition-proof Nash equilibrium if s^*_i maximizes f_i over S_i. Let $\Gamma = (N, (S_i)_{i\in N}, (f_i)_{i\in N})$ be a game with $|N| > 1$ players and suppose that coalition-proof Nash equilibria have been defined for games with fewer than $|N|$ players. A strategy profile $s^* \in \prod_{i\in N} S_i$ is called self enforcing if for all $T \subset N$, it holds that s^*_T is a coalition-proof Nash equilibrium of the game $\Gamma(s^*_{N\setminus T})$. A strategy profile s^* is a coalition-proof Nash equilibrium of Γ if s^* is self enforcing and there is no other self-enforcing strategy profile $s \in S_N$ such that $f_i(s) > f_i(s^*)$ for all $i \in N$.

The last refinement of Nash equilibria that I use is defined only for strategic-form games that are potential games. A game $(N; (S_i)_{i\in N}; (f_i)_{i\in N})$ is a potential game (cf. Monderer and Shapley (1996)) if there exists a potential function $P : \prod_{i\in N} S_i \to I\!R$ such that for every strategy profile $s \in \prod_{i\in N} S_i$, every $i \in N$, and every $t_i \in S_i$, it holds that $f_i(s_i, s_{-i}) - f_i(t_i, s_{-i}) = P(s_i, s_{-i}) - P(t_i, s_{-i})$. The function P provides information on changes in payoffs for every player when he changes his strategy choice unilaterally. For a potential game, the potential maximizer selects the strategy profiles that maximize a potential function P. Monderer and Shapley (1996) prove that the potential maximizer is a well-defined refinement of Nash equilibria. Ui (2001) shows that Nash equilibria that maximize a potential function are generically robust, and Garratt and Qin (2001) provide justification for using the potential maximizer in network-formation games.

3 Network-Formation Game in Extensive Form

Network-formation games in extensive form were introduced by Aumann and Myerson (1988). The network-formation process in these games is sequential. Pairs of players get opportunities to form links according to some exogenous rule of order on the links. Links are formed one at a time and players observe which pairs of players form links or decline to form links as the game progresses. Moreover, links cannot be broken once they have been formed. After each pair of players has had an opportunity to form a link, in the order determined by the exogenous rule of order, every pair of payers that has declined to form a link is given another opportunity to do so, in an order determined by the same rule of order. This processes is repeated as long as new links are formed in each round but it stops when, after the latest link has been formed, all pairs of players who have not formed a link have had one final opportunity to change their minds and form it, but declined. The payoffs to the players are those found by applying the Myerson value μ to the network that is formed in combination with an underlying coalitional game. The network-formation game in extensive form as described above with underlying coalitional game (N, v), exogenous allocation rule μ (the Myerson value), and exogenous rule of order σ on all possible links between the players in N, is denoted by $\Delta^{nf}(N, v, \mu, \sigma)$.

For the extensive-form games of network formation $\Delta^{nf}(N, v, \mu, \sigma)$, no general results have been obtained to the best of my knowledge. 'Results' for these games are limited to a series of examples that illustrate which networks are supported by subgame-perfect Nash equilibria of the network-formation games for various underlying coalitional games. A subgame-perfect Nash equilibrium is a profile of strategies, one for each player, that satisfies the requirement that at each point in the game every player's strategy gives him the highest possible payoff in the remainder of the game, given the strategies of the other players.[5]

The following example is taken from Aumann and Myerson (1988) and shows that for a superadditive coalitional game, the extensive-form network-formation process might not lead to the formation of a complete network or even a connected network. A coalitional game (N, v) is superadditive if any two disjoint coalitions of players T and R (so, $T \cap R = \emptyset$) can only benefit from joining forces, i.e., $v(T \cup R) \geq v(T) + v(R)$.

Example 1 *Consider player set* $N = \{1, 2, 3\}$ *and assume that the economic possibilities of the players are captured in the superadditive coalitional game* (N, v)*, where* $v(T) = 0$ *if* $|T| \leq 1$*,* $v(T) = 60$ *if* $|T| = 2$*, and* $v(N) = 72$*. For this underlying game, the payoffs to the players as determined by the Myerson value for the various possible networks are as follows. In the empty network, every player receives zero;*

$$\mu_i(N, v, \emptyset) = 0 \text{ for each } i \in N.$$

In a network with one link, the two linked players equally divide the value of a 2-player

[5] Subgame-perfect Nash equilibria always exist for these games because they are finite and have perfect information.

coalition and the isolated player gets zero;

$$\mu_k(N, v, \{ij\}) = \begin{cases} 0 & \text{if } k \notin \{i, j\}; \\ 30 & \text{if } k \in \{i, j\}. \end{cases}$$

In a network with two links, the payoffs are

$$\mu_r(N, v, \{ij, jk\}) = \begin{cases} 44 & \text{if } r = j; \\ 14 & \text{if } r \in \{i, k\}. \end{cases}$$

Finally, in the complete network each player receives the same payoff and

$$\mu_i(N, v, g^N) = 24 \text{ for each } i \in N.$$

To find which networks are supported by subgame-perfect Nash equilibria of the game $\Delta^{nf}(N, v, \mu, \sigma)$, first note that each player receives a positive payoff if he forms any links at all, whereas all players receive zero if no links are formed. It follows that at least one link will be formed in a subgame-perfect Nash equilibrium. So, suppose that exactly one link has been formed, say link ij. If no additional links are formed, players i and j will each receive a payoff of 30. Note that if one of them, say i, forms a link with the remaining player k, he would increase his payoff to 44. However, in the network $(N, \{ij, ik\})$ players j and k receive only 14 and they can increase their payoffs to 24 by forming link jk. Hence, none of the players i and j will form a link with player k, as this will cause the other player to do so as well and both their payoffs will be only 24, rather than the 30 they each get in the network $(N, \{ij\})$. This shows that in a subgame-perfect Nash equilibrium exactly one link will be formed. The order σ on the links is not important in the sense that for any order σ and for any one of the three links, there is a subgame-perfect Nash equilibrium that results in the formation of this particular link, as is discussed in detail in Slikker (2000).[6]

As for superadditive coalitional games the extensive-form network-formation process might not lead to the formation of a complete network, some research has focussed on strengthening the requirement of superadditivity to convexity. A coalitional game (N, v) is convex if a player's contribution to a coalition increases (weakly) with the size of the coalition he joins, i.e., for any player i and any two coalitions $T, R \subseteq N \backslash i$ with $T \subseteq R$, it holds that $v(T \cup i) - v(T) \leq v(R \cup i) - v(R)$. Convexity of a coalitional

[6]For some games, the order σ might influence which networks are supported by subgame-perfect Nash equilibria. Slikker and van den Nouweland (2001a) demonstrate this in example 6.2.

game seems to provide strong incentives for players to cooperate in the largest possible coalition. It is a long-standing open conjecture that for a convex coalitional games, subgame-perfect Nash equilibria of the extensive-form game of network formation support the complete network. This conjecture was addressed in van den Nouweland (1993) and later in Slikker and Norde (2000). Van den Nouweland (1993) shows that the conjecture holds true if a second conjecture holds. This second conjecture is that for any convex game and any network that is not the complete network, there exist two players who have not formed a link and whose Myerson values in the network are weakly smaller than their respective Myerson values in the complete network. However, this second conjecture was disproved by an example due to R. Holzman (private communication), which can be found as example 6.3 in Slikker and van den Nouweland (2001a).

Slikker and Norde (2000) study the extensive-form network-formation games $\Delta^{nf}(N, v, \mu, \sigma)$ for underlying games that are convex and symmetric. A coalitional game (N, v) is symmetric if the worth of every coalition of players depends only on how many members it has, i.e., $v(T) = v(R)$ for all coalitions $T, R \subseteq N$ with $|T| = |R|$. The additional requirement of symmetry makes it easier to analyze a game because it reduces its complexity considerably. Slikker and Norde (2000) are able to show that for convex symmetric coalitional games with at least two and at most five players the complete network is supported by a subgame-perfect Nash equilibrium of the extensive-form games of network formation $\Delta^{nf}(N, v, \mu, \sigma)$, for any order σ. Moreover, they are able to show that for strictly convex[7] symmetric games (N, v) with at least two and at most five players it holds that any network (N, g) that is supported by a subgame-perfect Nash equilibrium is payoff equivalent to the complete network, i.e., $\mu(N, v, g) = \mu(N, v, g^N)$. This result cannot be extended to games with more than five players. Slikker and Norde (2000) study a 6-player strictly convex symmetric game (N, v) and show that for this game there exist networks that are not payoff equivalent to the complete network but that are supported by subgame-perfect Nash equilibria of network-formation games $\Delta^{nf}(N, v, \mu, \sigma)$. However, it is still an open question whether for (strictly) convex (symmetric) games with more than five players the complete network is supported by a subgame-perfect Nash equilibrium of the extensive-form network-formation games.

[7] A game (N, v) is strictly convex if the convexity inequalities all hold with strict inequality, i.e., for any player i and any two coalitions $T, R \subseteq N \backslash i$ with $T \subset R$, it holds that $v(T \cup i) - v(T) < v(R \cup i) - v(R)$.

Other coalitional games for which the extensive-form network-formation game has been studied are weighted majority games. A weighted majority game is a game that results from a situation in which each player $i \in N$ has a number of votes, w_i, and in which a coalition of players needs a total of q votes (the quota) to obtain the surplus, which is normalized to equal 1. The tuple $(N, q, (w_i)_{i \in N})$ is a weighted majority situation. To avoid the existence of two disjoint coalitions of players that can obtain the surplus, it is assumed that $\frac{1}{2}\sum_{i \in N} w_i < q$. The characteristic function v of the weighted majority game (N, v) associated with weighted majority situation $(N, q, (w_i)_{i \in N})$ is defined by

$$v(T) = \begin{cases} 1 & \text{if } \sum_{i \in T} w_i \geq q; \\ 0 & \text{otherwise} \end{cases}$$

for all $T \subseteq N$.

The following example considers a weighted majority game with one powerful player and several small ones, a so-called apex game. This example is due to Aumann and Myerson (1988).

Example 2 *Consider the weighted majority situation $(N, q, (w_i)_{i \in N})$ with player set $N = \{1, 2, 3, 4, 5\}$, quota $q = 4$, and numbers of votes $w_1 = 3$ and $w_2 = w_3 = w_4 = w_5 = 1$. Player 1 has more votes than each of the other players, but still needs at least one of the players with few votes to obtain the surplus. Also, the players with few votes can obtain the surplus without player 1 if all four of them cooperate. The characteristic function of the associated weighted majority game (N, v) is*

$$v(T) = \begin{cases} 1 & \text{if } T = \{2, 3, 4, 5\} \text{ or if } 1 \in T \text{ and } |T| \geq 2; \\ 0 & \text{otherwise.} \end{cases}$$

Note that players 2 through 5 are symmetric in this game. Consequently, it suffices to only discuss the payoffs for some possible networks and those for other networks can be found using this symmetry. If only a link between player 1 and player 2 is formed, then these two players depend on each other to obtain the surplus. This is expressed in the Myerson value, which gives each of them $\frac{1}{2}$. Of course, all isolated players receive 0. Player 1 can increase his payoff by linking with more players with few votes, as this will decrease player 1's dependence on these other players. In a network in which the three links between players 1, 2, and 3 are formed, player 1 gets $\frac{2}{3}$, while players 2 and 3 get $\frac{1}{6}$ each. In a network in which the 6 links between players 1, 2, 3, and 4 are formed, player 1 gets $\frac{3}{4}$ while players 2, 3, and 4 get $\frac{1}{12}$ each. When all the players are *included, player 1's payoff decreases again as now he is no longer essential in obtaining the surplus. In the complete network, player 1 gets $\frac{3}{5}$ and the other players get only $\frac{1}{10}$ each. Players 2, 3, 4, and 5 get more if they do not include player 1; in the network in which the 6 links between players 2, 3, 4, and 5 are formed, these players get $\frac{1}{4}$ each. However, any one of these players has the highest payoff in a network where he alone is linked with player 1. But players 2, 3, 4, and 5 prefer to form the 6 links amongst themselves, excluding player 1, to linking up with player 1 and at least one other player (with few votes).*

It can be shown that only networks in which all links are formed within components are supported by subgame-perfect Nash equilibria, so I restrict attention to those. I identify the formation of all links between the players in a coalition T with the formation of coalition T.

I use backward induction to find the coalitions whose formation is supported by subgame-perfect Nash equilibria. Suppose coalition $\{1, 2, 3, 4\}$ has been formed. Then players 2 and 5 can increase their payoffs (from $\frac{1}{12}$ and 0, respectively, to $\frac{1}{10}$) by forming a link, which eventually will result in the formation of coalition N. Hence, once a coalition with player 1 and three players with few votes has been formed, the remaining links will also be formed. Now, suppose that coalition $\{1, 2, 3\}$ has been formed. It was just shown that if any one of players 1, 2, or 3 forms an additional link with players 4 or 5, this will eventually result in the formation of coalition N. This would decrease the payoff of player 1 from $\frac{2}{3}$ to $\frac{3}{5}$ and that of players 2 and 3 form $\frac{1}{6}$ to $\frac{1}{10}$. Hence, no additional links will be formed once a coalition with player 1 and two players with few votes has been formed. This implies that once coalition $\{1, 2\}$ has been formed, player 1 can permanently improve his payoff from $\frac{1}{2}$ to $\frac{2}{3}$ by forming a link with player 3. As this will also improve player 3's payoff, a coalition with player 1 and one player with few votes cannot be sustained in a subgame-perfect Nash equilibrium.

Alternatively, suppose players 2, 3, 4, and 5 have formed a coalition. If one of them forms a link with player 1, eventually the complete network will be formed, decreasing his payoff from $\frac{1}{4}$ to $\frac{1}{10}$. Hence, no additional links will be formed once coalition $\{2, 3, 4, 5\}$ has been formed.

The analysis above shows that only two types of networks can possibly be supported in subgame-perfect Nash equilibria, coalitions like $\{1, 2, 3\}$ with player 1 and two players with few votes, and coalition $\{2, 3, 4, 5\}$. Note that players 2 through 5 each have a higher payoff in coalition $\{2, 3, 4, 5\}$ than in either one of the coalitions that include player 1 and two players with few votes. This implies that players 2 through 5 will *refuse to form links with player 1 and form all links with each other. Hence, coalition* $\{2,3,4,5\}$ *will be formed in a subgame-perfect Nash equilibrium.*

Aumann and Myerson (1988) mention that in apex games with a large player who has more than one and less than q votes and several small players with one vote each, in general the complete network on a minimal winning coalition of small players will be formed in subgame-perfect Nash equilibria..

To the best of my knowledge, no general results for other types of weighted majority games have been reported in the literature. Aumann and Myerson (1988) do, however, report on two weighted majority situations with two large players each in which both the coalitions with the large players and the coalitions with one large player and all small ones are supported by subgame-perfect Nash equilibria.

The following example is due to Feinberg (1998). In response to a question posed in Aumann and Myerson (1988), this example provides a weighted majority game and a network that is not internally complete with the property that no additional links will be formed in a subgame-perfect Nash equilibrium.

Example 3 *Consider the 8-player weighted majority situation in which the 8 players have 5, 1, 2, 2, 2, 2, 4, and 1 votes, respectively, and the quota is 12 votes. The characteristic function of the associated weighted majority game* (N,v) *assigns a worth of 1 to every coalition* $T \subseteq N = \{1,2,3,4,5,6,7,8\}$ *if and only if* $\sum_{i \in T} w_i \geq 12$ *and a worth of 0 otherwise. Feinberg (1998) shows that in the network-formation games* $\Delta^{nf}(N,v,\mu,\sigma)$ *it holds that, once the network* (N,g) *with*

$$g = g^N \backslash \{37,47,57,67,18,48,58,68\}$$

has been formed, no further links will be formed by the players. In this network, the payoffs according to the Myerson value are

$$\mu(N,v,g) = \left(\frac{123}{420}, \frac{27}{420}, \frac{42}{420}, \frac{38}{420}, \frac{38}{420}, \frac{38}{420}, \frac{91}{420}, \frac{23}{420}\right).$$

The payoffs according to the Myerson value in the complete network are

$$\mu(N,v,g^N) = \left(\frac{122}{420}, \frac{22}{420}, \frac{41}{420}, \frac{41}{420}, \frac{41}{420}, \frac{41}{420}, \frac{90}{420}, \frac{22}{420}\right).$$

This shows that the complete network is preferred to network (N,g) *by players 4, 5, and 6, whereas all other players prefer network* (N,g) *to the complete network. However, in network* (N,g)*, players 4, 5, and 6 have already formed all links except those with* *players 7 and 8. Hence, players 7 and 8 can prevent players 4, 5, and 6 from forming additional links. This shows that once network (N, g) is formed, no additional links will be formed. It is still unknown whether there exists a subgame-perfect Nash equilibrium of $\Delta^{nf}(N, v, \mu, \sigma)$ that results in the formation of network (N, g) if the formation process is started from the empty network.*

4 Network-Formation Game in Strategic Form

The simplest game of network formation in coalitional games is the strategic-form game that was mentioned briefly in Myerson (1991) (p. 448) and studied more extensively in Dutta et al. (1998). In this game, the players each independently indicate the set of other players with whom they would like to form bilateral relations. A link is then formed between two players if both of them indicate they would like to form a relation with each other. The payoffs to the players are those found by applying an exogenous allocation rule to the network that is formed in combination with the underlying coalitional game.

To describe the network-formation game in strategic form formally, let (N, v) be a coalitional game and let γ be an allocation rule for communication situations. The strategy set of player $i \in N$ is $S_i = \{T \mid T \subseteq N \backslash i\}$, where a particular strategy $s_i \in S_i$ represents the set of players with whom player i would like to form links. If the players play a strategy tuple $s = (s_i)_{i \in N} \in \prod_{i \in N} S_i$, then a link is formed between two players i and j if and only if $j \in s_i$ and $i \in s_j$. Denote the set of all links that are formed according to this rule by $g(s)$. The network-formation game in strategic form $\Gamma^{nf}(N, v, \gamma)$ is described by the tuple $(N; (S_i)_{i \in N}; (f_i^{\gamma})_{i \in N})$, where $f_i^{\gamma}(s) = \gamma_i(N, v, g(s))$ for each $s \in S$.

A game $\Gamma^{nf}(N, v, \gamma)$ will in general have many Nash equilibria. The reason for this is that if a player i indicates that he does not want to form a link with another player j, then it does not matter for the network formed (or the payoffs) whether or not player j wants to form a link with player i, as this link will not be formed in either case. This reasoning underlies theorem 1. To state this theorem formally, I need the three logically independent properties of allocation rules for communication situations that were used in Dutta et al. (1998). The class of allocation rules for communication situations satisfying these three properties is reasonably large and it contains, for example, the Myerson value on the class of communication situations with a superadditive underlying game.

Component Efficiency An allocation rule γ on a class CS of communication situations satisfies component efficiency if for every communication situation $(N, v, g) \in CS$ and every component $C \in \pi(N, g)$ it holds that $\sum_{i \in C} \gamma_i(N, v, g) = v(C)$.

Component efficiency of an allocation rule means that the players in a component distribute the value of this component among themselves.

Weak Link Symmetry An allocation rule γ on a class CS of communication situations satisfies weak link symmetry if for every communication situation $(N, v, g) \in CS$ and every link ij it holds that if $\gamma_i(N, v, g \cup ij) > \gamma_i(N, v, g)$ then $\gamma_j(N, v, g \cup ij) > \gamma_j(N, v, g)$.[8]

Weak link symmetry is a form of fairness where the formation of a new link between two players cannot strictly benefit just one of them.

Improvement Property An allocation rule γ on a class CS of communication situations satisfies the improvement property if for every communication situation $(N, v, g) \in CS$ and every link ij it holds that if there exists a $k \in N \backslash \{i, j\}$ such that $\gamma_k(N, v, g \cup ij) > \gamma_k(N, v, g)$, then $\gamma_i(N, v, g \cup ij) > \gamma_i(N, v, g)$ or $\gamma_j(N, v, g \cup ij) > \gamma_j(N, v, g)$.

This property stipulates that the formation of a new link cannot benefit some player who is not involved in the link without also benefitting at least one of the two players forming it.

Dutta et al. (1998) showed that any allocation rule satisfying the three aforementioned properties necessarily satisfies a fourth property, link monotonicity, if the underlying coalitional game is superadditive. It states that forming an additional link can never harm a player if the allocation rule is link monotonic.

Link Monotonicity An allocation rule γ on a class CS of communication situations satisfies link monotonicity if for every communication situation $(N, v, g) \in CS$ and every link ij it holds that $\gamma_i(N, v, g \cup ij) \geq \gamma_i(N, v, g)$.

[8] I omit details on domains on which a rule γ is defined, as the domains that are considered are always closed with respect to the operations that appear in the properties.

Even though link monotonicity plays an important and prominent role in many of the results that are obtained for network-formation games $\Gamma^{nf}(N, v, \gamma)$, Slikker and van den Nouweland (2001a) demonstrate in examples that replacing weak link symmetry by link monotonicity, even in a context where component efficiency and the improvement property hold, will not guarantee the validity of the statements in the results that are described below. Hence, link monotonicity should be viewed as an intermediate result only, albeit one that is interesting enough to warrant highlighting it as a separate property.

If an allocation rule satisfies link monotonicity, then a player never has an incentive to change his strategy to prevent the formation of one or more of his links.[9] This implies that any network g can be supported by a Nash equilibrium of the network-formation game, namely the strategy profile in which each agent i chooses his strategy $s_i = \{j \in N \mid ij \in g\}$, indicating the will to form exactly the links in g in which he is involved. In this strategy profile, no single player can induce the formation of an additional link, as that would require a change in strategy by two players. A single player could prevent the formation of one or more of his links, but has no incentive to do so under link monotonicity. This shows the validity of the following theorem.

Theorem 1 *(Dutta et al. (1998)) Let (N, v) be a superadditive coalitional game and let γ be an allocation rule on CS_v^N that satisfies component efficiency, weak link symmetry, and the improvement property. Then any network g can be supported by a Nash equilibrium of the network-formation game $\Gamma^{nf}(N, v, \gamma)$.*

A driving force behind this theorem is that it takes two players to form a link, while the Nash equilibrium concept allows only single-player deviations. Hence, if two players each do not indicate that they want to form a link with each other, then none of them can unilaterally cause the link to be formed, even if its formation would benefit the players. There are two ways around this. One is to consider undominated Nash equilibria, and the other is to look at equilibrium refinements that allow for deviations by multiple players.

In an undominated Nash equilibrium, if the formation of a link with another player is beneficial to him, a player should indicate that he wants to form this link, even if the other player does not do so and therefore the link will not be formed. The reason is that if the first player is not certain whether the other player will want to form the link

[9] Note that this shows that link monotonicity implies that the strategy $s_i = N \backslash i$ is a weakly dominant strategy for each player $i \in N$.

or not, then he does not want to block the possibility of the link being formed. For a superadditive coalitional game, the restriction to undominated strategies narrows down the set of equilibria considerably. While there may be multiple undominated Nash equilibria, they all result in the formation of a network in which the payoffs are equal to those in the complete network, $\gamma(N, v, g^N)$. The complete network g^N itself is also supported by an undominated Nash equilibrium, the strategy profile $\bar{s}$ defined by $\bar{s}_i = N \backslash i$ for each $i \in N$.

Theorem 2 *(Dutta et al. (1998)) Let (N, v) be a superadditive coalitional game and let γ be an allocation rule on CS_v^N that satisfies component efficiency, weak link symmetry, and the improvement property. Then $\overline{s}$ is an undominated Nash equilibrium of the network-formation game $\Gamma^{nf}(N, v, \gamma)$. Moreover, if s is an undominated Nash equilibrium of $\Gamma^{nf}(N, v, \gamma)$, then $\gamma(N, v, g(s)) = \gamma(N, v, g^N)$.*

Considering equilibrium refinements that allow for deviations by multiple players, the most obvious such refinement is strong Nash equilibrium.. However, strong Nash equilibria might not exist in the game $\Gamma^{nf}(N, v, \gamma)$, not even under fairly strong requirements on the underlying coalitional game.[10] This motivates the consideration of coalition-proof Nash equilibria, which also allow for deviations by multiple players, but where these deviations are restricted to be immune to further allowed deviations themselves. It turns out that for a superadditive coalitional game (N, v), there are no existence problems for coalition-proof Nash equilibria in the network-formation game $\Gamma^{nf}(N, v, \gamma)$, as the complete network g^N is always supported by a coalition-proof Nash equilibrium. Moreover, even though there might be multiple networks supported by coalition-proof Nash equilibria, the payoffs are the same in all of these networks. This shows that, even though undominated Nash equilibria and coalition-proof Nash equilibria may lead to very different payoffs to the players for strategic-form games in general, for the strategic-form games of network formation coalition-proof Nash equilibria lead to similar outcomes as undominated Nash equilibria.

Theorem 3 *(Dutta et al.(1998)) Let (N, v) be a superadditive coalitional game and let γ be an allocation rule on CS_v^N that satisfies component efficiency, weak link symmetry, and the improvement property. Then the strategy profile $\bar{s}$ is a coalition-proof Nash*

[10]Slikker and van den Nouweland (2001a) show that convexity of the underlying game does not guarantee existence of strong Nash equilibria (example 7.4) and that balancedness is not necessary for existence (example 7.5).

equilibrium of the network-formation game $\Gamma^{nf}(N, v, \gamma)$. Moreover, if s is a coalition-proof Nash equilibrium of $\Gamma^{nf}(N, v, \gamma)$, then $\gamma(N, v, g(s)) = \gamma(N, v, g^N)$.

A different approach was taken by Qin (1996). He showed that for any coalitional game (N, v) and any external allocation rule γ that is component efficient, the strategic-form network-formation game $\Gamma^{nf}(N, v, \gamma)$ is a potential game if and only if $\gamma = \mu$, i.e., if the exogenous allocation rule used is the Myerson value. The essence of the proof that μ is the only component-efficient allocation rule for which $\Gamma^{nf}(N, v, \gamma)$ is a potential game, consists of using a potential of $\Gamma^{nf}(N, v, \gamma)$ to show that γ must satisfy fairness. In his proof of the other implication, that $\Gamma^{nf}(N, v, \mu)$ is a potential game, Qin (1996) used a sort of cyclicity property for potential games that was shown by Monderer and Shapley (1996) to characterize games that admit a potential. Slikker and van den Nouweland (2001a) provide a proof of this implication that uses a representation theorem by Ui (2000), who shows that there is a relation between the existence of potential functions for games in strategic form and Shapley values of coalitional games.

Because the network-formation games $\Gamma^{nf}(N, v, \mu)$ are potential games, for these games the potential-maximizing strategy profiles provide an equilibrium refinement. The following theorem shows that the application of this refinement leads to results similar to those obtained for other refinements.

Theorem 4 *(Qin (1996)) Let (N, v) be a superadditive coalitional game and let P be a potential function for the network-formation game $\Gamma^{nf}(N, v, \mu)$. Then P assumes its maximum value at $\bar{s}$ and if s is a strategy profile in which P is maximal, then $\mu(N, v, g(s)) = \mu(N, v, g^N)$.*

The three theorems above show that undominated Nash equilibrium, coalition-proof Nash equilibrium, and potential-maximizing strategies (when appropriate) all support the formation of the complete network or a network that is payoff equivalent to the complete network when the underlying coalitional game is superadditive. Hence, all three equilibrium concepts provide the same unique prediction on the ultimate payoffs of the players, namely those associated with the complete network.[11] However, the

[11] I point out that pairwise stability (see the chapter by Jackson (2004) in this volume) does not provide the same unique predictions. While it is true that the complete network is pairwise stable if the underlying coalitional game is superadditive, there may be networks that are not payoff equivalent to the complete network that are pairwise stable. The reason for this is that the pairwise stability concept only considers the addition of one link at a time and this can make a smaller network pairwise stable if the addition of more than one link is necessary to increase the payoffs to the players.

theorems above do not necessarily imply that these three types of equilibria support the same networks. They leave open the possibility that a non-complete network that is payoff equivalent to the complete network is supported by one equilibrium concept but not by the other two. Little is known about exactly which networks are supported by the various equilibrium concepts. Exceptions are Garratt and Qin (2001) and Garratt et al. (2002), who shed some light on which networks are supported by potential-maximizing strategy profiles of network-formation games $\Gamma^{nf}(N, v, \mu)$, specifically for 3-player coalitional games.

Slikker at al. (2000a) analyze hypergraph-formation games in strategic form. The hypergraph-formation games that they study are straightforward extensions of the strategic-form network-formation games $\Gamma^{nf}(N, v, \mu)$ to situations in which payers can form multilateral relationships rather than just bilateral ones. They derive results similar to theorems 1, 2, and 3 in this more general setting. In addition, they prove that hypergraph-formation games are weighted potential games (cf. Monderer and Shapley (1996)) if and only if a weighted Myerson value is used as exogenous allocation rule. Using this, they are able to derive a result similar to theorem 4 for weighted potentials.

5 Comparison of the Network-Formation Models in Extensive and Strategic Forms

The differences between the games of network formation in extensive form and the games of network formation in strategic form are illustrated by considering the 3-person game (N, v) with player set $N = \{1, 2, 3\}$ in which the worth of every 1-player coalition equals 0, that of every 2-player coalition equals 60, and that of the 3-player grand coalition equals 72. This is the same game as in example 1. That example illustrated that the prediction of the network-formation game in extensive form is that a network with 1 link will be formed. The reason that no additional links are formed is the following. Suppose that link ij has been formed. Then it seems beneficial for player i to form a link with the third player, k, to increase the payoff of player i from 30 to 44 and that of player k from 0 to 14. However, this would cause a drop in player j's payoff from 30 to 14 and now players j and k have an incentive to form the third link and increase each of their payoffs from 14 to 24. Note that the payoff of player i falls to 24 as a result off this. Hence, player i will not form the link with player k because of player j's threat to retaliate by forming a link with player k as well. Note that executing this threat is in player j's best interest. Also, if all 3 links are formed, both players i and j are worse off than then when only link ij is formed. Hence, the network $(N, \{ij\})$ is sustained by a pair of mutual threats of the kind "If you form a link with k, then so will I." Note, however, that such mutual threats can only be effective if the negotiation process is public and player j can observe whether or not player i forms a link with player k. These threats lose bite if bilateral negotiations are conducted secretly, or when negotiations over different links are carried out simultaneously rather than sequentially and links cannot be broken once they have been formed. In such situations, if player k starts negotiations with players i and j separately after link ij has been formed, players i and j are basically playing the following game.

	l	nl
l	24,24	44,14
nl	14,44	30,30

Here, l and nl denote the strategies of forming a link with k and not forming a link with k, respectively. In this game, it is a dominant strategy for both players i and j to form a link with k. Note that this game describes a prisoners' dilemma situation. It shows that in the network-formation game in strategic form both players i and j will form a link with player k, and the complete network will be formed, simply because players i and j cannot sign a binding agreement to abstain from forming a link with k.

6 Network Formation with Costs for Establishing Links

The network-formation games in extensive form and strategic form that were the subject of the previous two sections do not allow for the inclusion of costs for forming links. In these models, the worth of a connected coalition of players is the same, whether they are connected by a minimum number of links, by all possible links, or something in between. Slikker and van den Nouweland (2000) investigate the effects of introducing costs for forming links into the two aforementioned games of link formation. To isolate the effect that such costs have on the networks that are formed in equilibrium, these costs are taken to be as simple as possible, namely constant across links. A cost-extended communication situation is a tuple (N, v, g, c), in which N is a set of players, (N, v) a coalitional game, (N, g) a communication network, and $c \geq 0$ the cost for establishing a communication link.

When links are costly, the worth that a coalition of players can obtain does no longer just depend on whether they are connected or not, but also on how many links they have formed. As there are costs associated with forming additional links, players have incentives to form only as many links as necessary to connect them to others. However, forming an additional link will put a player in a more central position in a network and might increase his payoff. Therefore, a player needs to carefully balance the cost of an additional link against its benefits. The inclusion of costs necessitates extending the notion of an allocation rule. Slikker and van den Nouweland (2000) extend the Myerson value and use this allocation rule in the link-formation games. To obtain the worth of a coalition of players, we need to subtract the costs of the links formed by these players from the benefits that they can obtain in the presence of these links. Hence, the worth of a coalition $T \subseteq N$ of players in cost-extended communication situation (N, v, g, c) is

$$v^{g,c}(T) = \sum_{C \in \pi(T,g)} v(C) - c\,|g(T)|.$$

The cost-extended Myerson value $\nu(N, v, g, c)$ is the Shapley value of the associated cost-extended network-restricted game $(N, v^{g,c})$, i.e.,

$$\nu(N, v, g, c) = \Phi(N, v^{g,c}).$$

Note that $\nu(N, v, g, c) = \mu(N, v, g)$ whenever $c = 0$, so that the cost-extended Myerson value is indeed an extension of the Myerson value. The cost-extended Myerson value can be interpreted in two methodologically very different ways. One is as a solution to the bargaining problem in which the players bargain over the benefits and the costs of forming links simultaneously. This interpretation stems from the fact that the definition of the cost-extended Myerson value is based on the game $(N, v^{g,c})$, which includes both the benefits and the costs of forming links. Another interpretation is that players first form links and pay the costs for forming them and then, when a network has been formed and the costs are sunk, bargain over the division of the benefits. This interpretation stems form the fact that the cost-extended network-restricted game $(N, v^{g,c})$ can be written in terms of the network-restricted game (N, v^g) as $v^{g,c} = v^g - c\sum_{ij \in g} u_{i,j}$,[12] which implies that

$$\nu_i(N, v, g, c) = \mu_i(N, v, g) - \frac{1}{2}\sum_{ij \in g} c$$

[12] $u_{i,j}$ denotes the characteristic function of the unanimity game on coalition $\{i, j\}$ and is defined by $u_{i,j}(T) = 1$ if $\{i, j\} \subseteq T$ and $u_{i,j}(T) = 0$ otherwise, for all $T \subseteq N$.

for each player i. This shows that benefits are allocated according to the Myerson value, whereas the cost of each link is simply split between the players who form it.

Focusing on the networks rather than the players, a reward function $r^{v,c}$ can be defined that assigns to each network g the net worth that the players can obtain in this network;

$$r^{v,c}(g) = \sum_{C \in \pi(N,g)} v(C) - c\,|g|.$$

Such a reward function is a special case of a value function as discussed in Jackson (2004). Slikker and van den Nouweland (2001a) prove that the cost-extended Myerson value is a special case of the Myerson value that is defined for such value functions in Jackson and Wolinsky (1996).

The trade-off between the costs and benefits of an additional link can be illustrated by considering a 3-player symmetric coalitional game in which the worth of every 1-player coalition equals $v_1 = 0$, that of every 2-player coalition equals $v_2 \geq 0$, and that of the grand coalition consisting of all 3 players equals $v_3 \geq 0$. Then in a network with two links, a player who is involved in only one link has a cost-extended Myerson value $\nu_i(N, v, \{ij, jk\}, c) = \frac{1}{3}v_3 - \frac{1}{6}v_2 - \frac{1}{2}c$. If this player were to form a link with the other player who is involved in only one link, then the complete network would result and each player's cost-extended Myerson value would become $\nu_i(N, v, \{ij, ik, jk\}, c) = \frac{1}{3}v_3 - c$. Hence, player i would prefer to form this third link if and only if $c < \frac{1}{3}v_2$, in which case the extra benefit of being more central in the network ($\frac{1}{6}v_2$) outweighs the extra cost of the link ($\frac{1}{2}c$).

Using the cost-extended Myerson value, the effect of costs for forming links can be studied in both the extensive-form network-formation games $\Delta^{nf}(N, v, c, \nu, \sigma)$ and the strategic-form network-formation games $\Gamma^{nf}(N, v, c, \nu)$. Slikker and van den Nouweland (2000, 2001a, 2002) provide overviews that identify which networks are supported by various equilibrium refinements of these games as the costs for forming links change. They do so for all 3-player games where the underlying coalitional games are symmetric and non-negative. Non-negativity basically means that cooperation is beneficial in the sense that the worth of any multi-player coalition is at least that of the sum of the individual worths of its members. I will not reproduce these overviews here. Overviews for subgame-perfect Nash equilibria of the extensive-form games of network formation and those for undominated Nash equilibria and coalition-proof Nash equilibria of the strategic-form games of network formation were first published in Slikker and van den Nouweland (2000), while those for Nash equilibria are published in Slikker and van den Nouweland (2001a). Slikker and van den Nouweland (2002) extend earlier results to cost-extended communication situations and show that for any coalitional game (N, v), any non-negative cost c, and any component-efficient external allocation rule γ, the strategic-form network-formation game $\Gamma^{nf}(N, v, c, \gamma)$ is a potential game if and only if $\gamma = \nu$, i.e., if the exogenous allocation rule used is the cost-extended Myerson value. Hence, for the potential games $\Gamma^{nf}(N, v, c, \nu)$, potential maximizing strategy profiles can be used as an equilibrium refinement and Slikker and van den Nouweland (2002) provide overviews of networks supported by potential-maximizing strategy profiles.

The most striking results obtained are discussed below.[13]

It turns out that for 3-player symmetric games (N, v) the pattern of equilibrium networks as a function of changing costs for forming links depends on whether or not the underlying game is superadditive and/or convex. This holds for both the extensive-form games of network-formation and the strategic-form games of network formation and also for all of the equilibrium concepts studied. It is well-known that a convex game is superadditive, while the reverse implication is not true in general. Hence, with respect to the network-formation games with costs for forming links, three types of underlying games need to be considered. They are non-superadditive games, superadditive games that are not convex, and convex games.

In strategic-form network-formation games, undominated Nash equilibrium, coalition-proof Nash equilibrium, and potential-maximizing strategy profiles provide the most useful predictions. Surprisingly, it turns out that for these games the predictions of coalition-proof Nash equilibrium refine those of undominated Nash equilibrium at the network level.[14] Also, the patterns of networks supported by potential-maximizing strategies are almost exactly the same as those for coalition-proof Nash equilibrium.[15] Therefore, I concentrate on networks supported by coalition-proof Nash equilibria. These are as follows. For all underlying coalitional games, the complete network is supported for very low costs of link formation[16] and the empty network (in which

[13]I remind the reader that the results discussed cover 3-player coalitional games that are symmetric, zero-normalized, and non-negative.

[14]Note that coalition-proof Nash equilibrium is not a refinement of undominated Nash equilibrium on the strategy level, not even for the class of strategic-form network-formation games.

[15]The only difference appears for nonsuperadditive games, where the level of the cost at which the transition from the complete network to networks with 1 link occurs is higher for the potential-maximizing strategy profiles than for coalition-proof Nash equilibria.

[16]If the game is non-superadditive, then the cut-off for 'very low' costs might be negative, in which case the complete network is not supported for any non-negative cost c.

there are no links and, hence, every player is isolated) is supported for very high costs. It is for intermediate cost levels that the predictions depend on the structure of the underlying coalitional game.[17] If that game is non-superadditive, then for intermediate cost levels the formation of exactly one link is supported. If the underlying game is convex, networks with two links are supported for intermediate cost levels. Finally, if the coalitional game is superadditive but not convex, then networks with two links are supported for lower costs, while for higher costs the formation of only one link is supported.

In extensive-form network-formation games, subgame-perfect Nash equilibrium is the most appropriate solution concept. The networks supported by subgame-perfect Nash equilibria are as follows. If the underlying coalitional game is non-superadditive, then for low costs only networks with one link are supported, while for high costs only the empty network results. Networks with more than one link are not supported for any level of the costs if the game is not superadditive. For convex coalitional games, the complete network is supported for very low costs, networks with two links are supported for intermediate costs, and the empty network is supported for very high costs. The most interesting case turns out to be that of superadditive coalitional games that are not convex. For such games, the number of links whose formation is supported by coalition-proof Nash equilibria varies with the costs of forming links in a non-monotonic way. For very low costs, networks with 3 links are supported.[18] For somewhat higher costs, networks with one link are supported. Then, if the costs increase from there, networks with two links are supported (so an increase in the costs results in the formation of *more* links). If the costs keep increasing, the number of links formed in equilibrium decreases again. First networks with 1 link are supported, while for very high costs the empty network (with 0 links) is supported.

The following example illustrates that an increase in the cost for establishing a communication link can result in more communication between the players in subgame-perfect Nash equilibria of the extensive-form game of network formation.

Example 4 *Let (N, v) be the 3-player symmetric game (N, v) in which the worth of every 1-player coalition equals 0, that of every 2-player coalition equals 60, and that of the 3-player grand coalition equals 72. Note that this game is superadditive but not*

[17]The levels of costs that are considered intermediate vary with the structure of the coalitional game.

[18]For some games in this class the cut-off for 'very low' costs might be negative, in which case the complete network is not supported for any non-negative cost c.

convex. It is the same game as was studied in example 1, where it was shown that if the cost for forming links is 0, only networks with one link are supported by subgame-perfect Nash equilibria. For low costs of forming links (to be precise, for $c < 20$*), the discussion in example 1 is still valid, as is its conclusion that exactly one link will be formed.*

But the analysis changes if the costs are larger. I demonstrate this for $c = 22$*. With these costs, the cost-extended Myerson values are as follows. An isolated player (one who is does not form any links) gets 0. In a network with 1 link, the two players who have formed this link each get 19 (which is one half of the revenue of 60 minus the cost of the link). In a network with 2 links, the central player (who has formed 2 links) gets 22, whereas the other two players each get 3. In the complete network, each player gets 2 (which is one third of the revenue of 72 minus the cots of the three links).*

The higher costs change the incentives of the players. In a network with 2 links, the two players who have not formed a link with each other have no incentive to do so, because that would reduce their payoffs from 3 to 2. Then, in a network with 1 link, a player who is involved in this link has an incentive to also form a link with the isolated player. Doing this will increase his payoff (from 19 to 22) and, unlike for lower levels of the costs, there is no threat of the third link being formed later on. Clearly, at least one link will be formed so that some players get a strictly positive payoff. It follows that networks with 2 links are supported by subgame-perfect Nash equilibria if the costs are equal to 22. Note that this means that the increase of the cost, from say 19 to 22, results in the formation of 2 links rather than 1.

The equilibrium concept for network-formation games in strategic form that is most similar in spirit to subgame perfection is undominated Nash equilibrium. However, there is a multiplicity of networks resulting from undominated Nash equilibria in the strategic-form network-formation games and coalition-proof Nash equilibrium provides a further refinement of these predictions. Therefore, a comparison of the cost-network patterns for subgame-perfect Nash equilibria in the network-formation games in extensive form with those for coalition-proof Nash equilibria in the network-formation games in strategic form is appropriate. The predictions according to subgame-perfect Nash equilibrium in the games in extensive form and those according to coalition-proof Nash equilibrium in the games in strategic form are remarkably similar. For convex games, the predictions in the extensive-form games and those in the strategic-form games are the same for all levels of the costs of link formation. For non-superadditive games, the predictions in both network-formation games are almost the same. The only difference is that the level of the costs that marks the transition from the complete network to a network with one link is possibly positive in the strategic-form game, whereas it is always negative (and therefore does not show up in the overview) in the extensive-form games.[19] The predictions of both types of network-formation games are most dissimilar if the underlying coalitional game is superadditive but not convex. In the extensive-form game a network with one link is supported if the costs are fairly low, but not very low, whereas in the strategic-form game for the same level of costs the complete network is supported. For all other levels of the cost the predictions of both games are the same. The difference between the predictions of both network-formation games is a result of the validity of mutual threats in the network-formation game in extensive form, as discussed in section 5, which is applicable to all games that are superadditive but not convex. These mutual threats also drive the remarkable result that higher costs may result in the formation of more links in the extensive-form network-formation game. Mutual threats will only be credible for lower costs, as for higher costs a player who executes such a threat will permanently decrease his payoff.

For games with more than 3 players, it is no longer true that the pattern of networks formed in equilibrium depends only on whether a game is superadditive and/or convex. Slikker and van den Nouweland (2000) illustrate this with two examples of symmetric 4-player games that are superadditive but not convex and for which the patterns of equilibrium networks as a function of the cost of forming links are different. However, the most interesting result that is obtained for symmetric 3-player games, namely that in the network-formation games in extensive form it is possible that the number of links formed in subgame-perfect Nash equilibria increases as the cost for establishing links increases, is still valid for games with more than 3 players. Slikker and van den Nouweland (2000) illustrate this in an example of a 4-player game and also for n-player games with n odd. In contrast, Slikker and van den Nouweland (2002) extend the result that in strategic-form network-formation games the number of links formed in potential-maximizing strategy profiles decreases as the costs for forming links increase. They prove this for coalitional games with an arbitrary number of players that are not necessarily symmetric.

[19]This level of the cost is higher and always positive if potential-maximizing strategies are used in the strategic-form games of network formation.

7 Simultaneous Bargaining over Network Formation and Payoff Division

In the three previous sections, an exogenous allocation rule was used to determine the payoffs to the players in various networks. This can be interpreted as network formation and bargaining occurring in two sequential stages; the first stage is the network-formation stage and in the second stage the players bargain over payoffs, given the network formed in the first stage. The second stage is collapsed into an exogenous allocation rule that provides the predicted outcome of the process of bargaining over payoffs. In contrast, Slikker and van den Nouweland (2001b)[20] study a model of network formation in which players bargain over the formation of links and the division of the payoffs simultaneously. In such a model, the use of an exogenous allocation rule is no longer justified. The link and claim game provides an integrated approach to network formation and payoff division. Like in the games of network formation in extensive and strategic forms of sections 3 and 4, it is built around a coalitional game (N, v) describing the possibilities of cooperating coalitions of players. To keep notations as simple as possible, it is assumed the game is zero-normalized, i.e., $v(i) = 0$ for each player $i \in N$. The link and claim game $\Gamma^{lc}(N, v)$ is a strategic-form game $(N; (S_i)_{i \in N}; (f_i)_{i \in N})$ with strategies and payoff functions as described below. The strategy set of player i is

$$S_i = \{c^i \in A^N \mid c^i_i = P\},$$

where $A := I\!R_+ \cup \{P\}$, $I\!R_+ = [0, \infty)$, and P stands for Pass. A strategy for player i specifies a $c^i_j \in I\!R_+ \cup \{P\}$ for any player $j \in N$. The interpretation of $c^i_j = P$ is that player i is not willing to form a link with player j, and $c^i_j \in I\!R_+$ means that player i is willing to form a link with player j provided that he gets the amount of his claim c^i_j for forming it. As player i cannot form a link with himself, it is assumed that $c^i_i = P$ for all $c^i \in S_i$. Suppose the players play strategy profile $c = (c^i)_{i \in N} \in \prod_{i \in N} S_i$. The resulting payoffs to the players depend on the network that is formed. According to strategy profile c, the set $l(c)$ of links that the players are willing to form is

$$l(c) = \{ij \mid c^j_i, c^i_j \in I\!R_+\}$$

as the consent of both players is needed to form the link between them. However, the claims of the players for forming these links might add up to more than is available.

[20] Most results in this section are taken from Slikker and van den Nouweland (2001b). However, they also appear in chapter 9 in Slikker and van den Nouweland (2001a), where they are presented more extensively and where the proofs are clearer, in my opinion.

Hence, it needs to be determined which of the links in $l(c)$ carry feasible claims. Network $(N, l(c))$ partitions the player set into components. For every such component, the links in $l(c)$ between its members can only be formed if the total of the claims for these links does not exceed the worth of the coalition. Hence, the set $g(c)$ of links that are formed equals

$$g(c) = \{ij \in l(c) \mid \sum_{km \in l(c): k,m \in C_i(l(c))} (c^k_m + c^m_k) \leq v(C_i(l(c)))\},$$

where $C_i(l(c))$ denotes the component of network $(N, l(c))$ that contains player i. This construction of $g(c)$ implies that if the players in a component of $(N, l(c))$ collectively claim too much, they all end up being isolated.[21] The payoffs to the players can be found by adding their claims for the links that are actually formed;

$$f_i(c) = \sum_{j: ij \in g(c)} c^i_j.^{22}$$

Note that this gives an isolated player his stand-alone payoff of zero.

The link and claim game $\Gamma^{lc}(N, v)$ is illustrated in the following example.

Example 5 *Let (N, v) be the 3-person coalitional game with $N = \{1, 2, 3\}$ and characteristic function v with $v(T) = 0$ if $|T| = 1$, $v(T) = 30$ if $|T| = 2$, and $v(T) = 72$ if $T = N$. Consider the strategy profile*

$$c = (c^1, c^2, c^3) = ((P, 10, 10), (10, P, 10), (P, 10, P))$$

in the link and claim game $\Gamma^{lc}(N, v)$. The link between players 1 and 3 is not in $l(c)$, because, while player 1 would like to form this link ($c^1_3 = 10 \in I\!R_+$), player 3 does not ($c^3_1 = P$). Link 12 is in $l(c)$, because both players 1 and 2 want to form it. Proceeding in this way, it is found that $l(c) = \{12, 23\}$. The network $(N, l(c))$ has one component, $\pi(N, l(c)) = \{\{1, 2, 3\}\}$. The total of the payoffs claimed for forming the links in $l(c)$ equals $c^1_2 + c^2_1 + c^2_3 + c^3_2 = 40 \leq 72 = v(N)$. As these claims are feasible for coalition N, all links in $l(c)$ are formed and $g(c) = \{12, 23\}$. The corresponding payoffs to the players are $f_1(c) = c^1_2 = 10$, $f_2(c) = c^2_1 + c^2_3 = 20$, and $f_3(c) = c^3_2 = 10$.

[21] See Slikker and van den Nouweland (2001a) for a discussion of alternative approaches.

[22] Note that this definition leaves open the possibility that the players claim less than what is available. This, however, will never happen in a Nash equilibrium and so it does not really matter whether the remainder is burned (as is the case for the expression that I provide here) or, for example, divided evenly among the players.

The profile $\hat{c} = ((P, 20, 20), (20, P, P), (P, 20, P))$ *is an example of a strategy profile in which the players claim too much for all the links in* $l(\hat{c})$ *to be formed. It holds that* $l(\hat{c}) = \{12\}$ *and* $\hat{c}_2^1 + \hat{c}_1^2 = 40 > 30 = v(1, 2)$. *Hence,* $g(\hat{c}) = \emptyset$ *and* $f_i(\hat{c}) = 0$ *for every* $i \in N$.

For link and claim games the question is not only which networks are supported by (refinements of) Nash equilibria, but also which payoff vectors are supported. For any coalitional game (N, v) it holds that a strategy profile in $\Gamma^{lc}(N, v)$ is not a Nash equilibrium if it results in the formation of a network containing a cycle and if at least one player claims a positive amount on one of the links in the cycle. A cycle is a path $i_1, i_2, ..., i_{t+1}$ in which $i_1, i_2, ..., i_t$ are all different players and $i_{t+1} = i_1$.

Theorem 5 *(Slikker and van den Nouweland (2001b)) Let* (N, v) *be a zero-normalized coalitional game. For every Nash equilibrium* c *in the link and claim game* $\Gamma^{lc}(N, v)$ *it holds that all claims on links in cycles in* $g(c)$ *are equal to zero.*

I illustrate this theorem in the following example.

Example 6 *Consider the 3-player game* (N, v) *in example 5. The complete network is the only possible network for the 3 players that contains a cycle. This network can only be formed if the players play a strategy profile* c *with* $c_j^i \in I\!R_+$ *for each* $i, j \in N$, $i \neq j$, *and* $c_2^1 + c_3^1 + c_1^2 + c_3^2 + c_1^3 + c_2^3 \leq 72$. *If* c *is a Nash equilibrium of* $\Gamma^{lc}(N, v)$, *then no player can gain from unilaterally deviating to a strategy in which he simply raises one of his claims. Hence,* $c_2^1 + c_3^1 + c_1^2 + c_3^2 + c_1^3 + c_2^3 = 72$ *has to hold. Without loss of generality, assume that* $c_2^1 > 0$, *so that player 1 gets a positive amount for forming link* 12. *Player 2 can increase his payoff by refusing to form link* 12 *and claiming the amount* c_2^1 *for himself by playing strategy* $\hat{c}^2 = (P, P, c_1^2 + c_3^2 + c_2^1)$. *Then network* $(N, \{13, 23\})$ *will be formed, and the players can still obtain* $v(N) = 72$. *Hence,* $f_2(c_1, \hat{c}_2, c_3) = c_1^2 + c_3^2 + c_2^1 > c_1^2 + c_3^2 = f_2(c)$. *This demonstrates that player 2 has a profitable deviation from* c, *so that* c *is not a Nash equilibrium.*

Theorem 5 implies that for a zero-normalized coalitional game (N, v) with at least three players and a positive value for the grand coalition ($v(N) > 0$), Nash equilibria do not support the formation of the complete network. Therefore, attention is shifted to connected networks. As is argued in example 6, if a connected network is formed in a Nash equilibrium, then the payoffs must be efficient for the underlying coalitional game, i.e. sum up to $v(N)$. Slikker and van den Nouweland (2001b) prove that in general a plethora of efficient payoff vectors are supported by Nash equilibria of the link and claim game $\Gamma^{lc}(N,v)$.[23] One of their main results is that all payoffs in the core are supported by Nash equilibria. The core of a coalitional game (N,v) consists of all efficient payoff vectors such that the members of each coalition $T \subseteq N$ collectively get at least the worth $v(T)$ that they can obtain independently of the players who are not included in the coalition; $core(N,v) = \{x \in I\!R^N \mid \sum_{i\in T} x_i \geq v(T)$ for all $T \subseteq N$ and $\sum_{i \in N} x_i = v(N)\}$.

Theorem 6 *(Slikker and van den Nouweland (2001b)) For any zero-normalized coalitional game (N,v) it holds that for every payoff vector $x \in core(N,v)$ there exists a Nash equilibrium c of the link and claim game $\Gamma^{lc}(N,v)$ such that $f(c) = x$.*

As Nash equilibrium itself supports very many payoff vectors, focus is shifted to strong Nash equilibria in an attempt to generate clearer predictions. An example shows that strong Nash equilibria of the link and claim game may support payoff vectors that are not efficient for the underlying coalitional game.

Example 7 *Let (N,v) be the 4-person coalitional game with player set $N = \{1,2,3,4\}$ and characteristic function v with $v(T) = 0$ if $|T| = 1$, $v(T) = 2$ if $|T| = 2$ or $|T| = 3$, and $v(T) = 3$ if $T = N$. The strategy profile c with $c^1 = (P,1,P,P)$, $c^2 = (1,P,P,P)$, $c^3 = (P,P,P,1)$, and $c^4 = (P,P,1,P)$ is a strong Nash equilibrium of $\Gamma^{lc}(N,v)$. It results in the formation of the network $(N,\{12,34\})$ and payoff vector $f(c) = (1,1,1,1)$. This payoff vector is not efficient for the game (N,v) because the sum of the payoffs equals 4 whereas the worth of the grand coalition equals only 3.*

The result in the previous example stems from the fact there exists a partition of the player set into coalitions such that the sum of the worths of these coalitions is larger than the worth of the grand coalition. Slikker and van den Nouweland (2001b) show that if the game (N,v) is such that a partition with this property does not exist, then every strong Nash equilibrium of the link and claim game $\Gamma^{lc}(N,v)$ results in a payoff vector that is not only efficient, but even in the core of the coalitional game (N,v). This, of course, implies that for such a coalitional game the set of strong Nash equilibria of $\Gamma^{lc}(N,v)$ is empty if the core of (N,v) is empty. The following example illustrates that not all payoff vectors in the core are necessarily supported by strong Nash equilibria.

[23] A Nash equilibrium c supports payoff vector x if $f(c) = x$.

Example 8 *Consider the coalitional game* (N, v) *with player set* $N = \{1, 2, 3\}$ *and* $v(T) = 0$ *if* $|T| = 1$, $v(1, 2) = 120$, $v(1, 3) = 60$, $v(2, 3) = 80$, *and* $v(N) = 180$. *Payoff vector* $(60, 60, 60)$ *is in the core of this game. I will show that there is no strong Nash equilibrium that results in payoff vector* $(60, 60, 60)$ *by showing that every Nash equilibrium that supports this payoff vector cannot be strong. Suppose that c is a Nash equilibrium of* $\Gamma^{lc}(N, v)$ *such that* $f(c) = (60, 60, 60)$. *It follows from theorem 5 that* $(N, g(c))$ *must have two links.*

Suppose that $g(c) = \{12, 23\}$. *Together with* $f(c) = (60, 60, 60)$, *this implies that* $c^2 = (c_1^2, P, 60 - c_1^2)$ *for some* $0 \leq c_1^2 \leq 60$, $c^1 = (P, 60, c_3^1)$, *and* $c^3 = (c_1^3, 60, P)$, *where either* $c_3^1 = P$ *or* $c_1^3 = P$ *(or both). Because* $f_2(c) = 60$, *either* $c_1^2 > 0$ *or* $c_3^2 = 60 - c_1^2 > 0$ *(or both). Without loss of generality, assume that* $c_3^2 > 0$. *Strategy profile c is not a strong Nash equilibrium because players 1 and 3 can increase their payoffs by deviating to strategies* $(\hat{c}^1, \hat{c}^3)$ *defined by* $\hat{c}^1 = (P, 60, \frac{c_3^2}{2})$ *and* $\hat{c}^3 = (60 + \frac{c_3^2}{2}, P, P)$. *The strategy profile* $(\hat{c}^1, c^2, \hat{c}^3)$ *results in the formation of links 12 and 13 and payoff vector* $(60 + \frac{c_3^2}{2}, c_1^2, 60 + \frac{c_3^2}{2})$, *which means that players 1 and 3 both improved their payoffs through the deviation.*

It can be demonstrated in a similar manner that c is not a strong Nash equilibrium if $g(c) = \{12, 13\}$ *or* $g(c) = \{13, 23\}$. *This shows that there is no strong Nash equilibrium of* $\Gamma^{lc}(N, v)$ *that supports payoff vector* $(60, 60, 60)$.

The reasoning above exploits the feature of the payoff vector $(60, 60, 60)$ *that all its elements are positive, so that any strategy profile c that supports this payoff vector leads to a middleman who gets a positive payoff. This idea can be extended and leads to the conclusion that none of the payoff vectors in which all coordinates are positive can be supported by a strong Nash equilibrium. For a payoff vector in the core like* $(70, 110, 0)$, *in which one of the players has a payoff of zero, a similar reasoning as before cannot be applied. Indeed, the payoff vector* $(70, 110, 0)$ *is supported by the strong Nash equilibrium c in which* $c^1 = (P, P, 70)$, $c^2 = (P, P, 110)$, *and* $c^3 = (0, 0, P)$.

The results in example 8 hold in general. The following theorem identifies the payoff vectors that are supported by strong Nash equilibria of the link and claim game. The first part identifies a class of games for which the set of payoff vectors supported by strong Nash equilibria coincides with the core of the underlying coalitional game, while the second part describes a class of games for which the payoff vectors supported by strong Nash equilibria are all the payoff vectors in the core in which at least one of the players receives a zero payoff.

Theorem 7 *(Slikker and van den Nouweland (2001b)) Let* (N, v) *be a zero-normalized coalitional game with the property that* $v(N) \geq \sum_{k=1}^{t} v(B_k)$ *for all partitions* $\{B_1, \ldots, B_t\}$ *of* N*, and let* $\Gamma^{lc}(N, v)$ *be the corresponding link and claim game.*

(i) *If there exists a partition* $\{B_1, \ldots, B_t\}$ *of* N *such that* $|B_k| = 2$ *for all* $k \in \{1, \ldots, t\}$ *and* $v(N) = \sum_{k=1}^{t} v(B_k)$*, then*

$$\{f(c) \mid c \text{ is a strong Nash equilibrium of } \Gamma^{lc}(N, v)\} = core(N, v).$$

(ii) *If* $v(N) > \sum_{k=1}^{t} v(B_k)$ *for all partitions* $\{B_1, \ldots, B_t\}$ *of* N *in which* $|B_k| = 2$ *for each* $k \in \{1, \ldots, t\}$[24]*, then*

$$\{f(c) \mid c \text{ is a strong Nash equilibrium of } \Gamma^{lc}(N, v)\} = \{x \in core(N, v) \mid \text{there exists a player } i \in N \text{ such that } x_i = 0\}.$$

Theorem 7 shows that, while strong Nash equilibria of the link and claim games often exist, the strong Nash equilibrium concept seems quite restrictive as it results in at least one of the players receiving a payoff of zero for a large class of coalitional games. One of the players receiving a zero payoff might even be the central player in a star. The reason that the payoff of a player in such a central position is kept low in a strong Nash equilibrium is that other players can avoid having to communicate via him by forming new links between themselves. However, such deviations are not necessarily stable against further deviations. This motivates the consideration of coalition-proof Nash equilibria. The following example demonstrates how the requirement that deviations are self-enforcing prevents the players from making certain deviations, so that some strategies that are not stable against arbitrary deviations become sustainable.

Example 9 *Consider the coalitional game* (N, v) *in example 8 and payoff vector* $x = (60, 60, 60)$*, which is in the core of* (N, v) *but not supported by a strong Nash equilibrium of the associated link and claim game* $\Gamma^{lc}(N, v)$*. Consider strategy profile* c *defined by* $c^1 = (P, 60, P)$*,* $c^2 = (0, P, 60)$*, and* $c^3 = (P, 60, P)$*, for which* $g(c) = \{12, 23\}$ *and* $f(c) = x$*. It is shown below that* c *is a coalition-proof Nash equilibrium of* $\Gamma^{lc}(N, v)$*, which shows that* x *is supported by a coalition-proof Nash equilibrium.*

It is easily seen that c *is a Nash equilibrium as no player can unilaterally deviate to a strategy that gives him a higher payoff. Further, there are no deviations by coalition*

[24] Note that this condition is trivially satisfied for a game (N, v) with an odd number of players.

N that increase the payoffs of all players as $x_1 + x_2 + x_3 = 180 = v(N)$. Hence, to prove that c is a coalition-proof Nash equilibrium, it suffices to show that there are no profitable deviations by 2-player coalitions that are stable against further deviations by members of the deviating coalition.

Start by considering a deviation by coalition $\{1, 2\}$. When c is played, players 1 and 2 together receive 120, which is the worth of coalition $\{1, 2\}$. Hence, to improve their payoffs, players 1 and 2 need to deviate to a strategy profile that results in the formation of at least one link with player 3. Because player 3 is still playing strategy $(P, 60, P)$, this will have to be link 23 and player 3 will still receive 60 after the deviation by coalition $\{1, 2\}$. But then players 1 and 2 together cannot obtain more than $v(N) - 60 = 120 = x_1 + x_2$ after the deviation, so they cannot both improve their payoffs. Similar arguments show that there are no profitable deviations by coalition $\{2, 3\}$.

It remains to consider deviations by coalition $\{1, 3\}$. Because $x_1 + x_3 = 120 > v(1, 3)$, any profitable deviation by players 1 and 3 has to result in the formation of a connected network. To improve their payoffs, players 1 and 3 have to break a link with player 2 on which player 2 has a positive claim. Hence, link 23 will be broken. This is represented by the strategies

$$\begin{array}{ll} \hat{c}^1 = (P, \hat{c}^1_2, \hat{c}^1_3) & \text{with } \hat{c}^1_2 + \hat{c}^1_3 = 60 + 60\alpha,\ 0 < \alpha < 1, \\ \hat{c}^3 = (60 + 60\beta, P, P) & \text{with } 0 < \beta \leq 1 - \alpha. \end{array}$$

However, player 1 can deviate from strategy profile $(\hat{c}^1, c^2, \hat{c}^3)$ by playing $\tilde{c}^1 = (P, 120, P)$, which induces the formation of network $(N, \{12\})$ and improves his payoff from $60+60\alpha$ to $120 = v(1, 2) - c^2_1$. Because $\tilde{c}^1$ is a coalition-proof Nash equilibrium in the reduced game that emerges when the strategies of players 2 and 3 are fixed to c^2 and $\hat{c}^3$, respectively, it follows that deviation $(\hat{c}^1, \hat{c}^3)$ is not self-enforcing. We conclude that c is coalition-proof Nash equilibrium of $\Gamma^{lc}(N, v)$.

Slikker and van den Nouweland (2001a) provide a description of all coalition-proof Nash equilibria of the link and claim game $\Gamma^{lc}(N, v)$ and corresponding payoffs for coalitional games (N, v) with three players. A remarkable result is that for some coalitional games with a non-empty core, some efficient payoff vectors that are not in the core of the game are nevertheless supported by coalition-proof Nash equilibria. For the game (N, v) of examples 8 and 9, the efficient payoff vector $(100, 10, 70)$ is supported by the coalition-proof Nash equilibrium $c = (c^1, c^2, c^3)$ defined by $c^1 = (P, P, 100)$, $c^2 = (P, P, 10)$, and $c^3 = (70, 0, P)$[25], but it is not in the core of (N, v) (players 1 and 2

[25] See lemma 9.2 in Slikker and van den Nouweland (2001a).

get less than $v(1,2)$). The results in Slikker and van den Nouweland (2001a) also imply that there exist coalition-proof Nash equilibria of the link and claim game $\Gamma^{lc}(N,v)$ that result in efficient payoff vectors even if the core of the game (N,v) is empty, as long as $v(T) \leq v(N)$ for at least one 2-player coalition T.

Slikker (2000) shows that all the results obtained in Slikker and van den Nouweland (2001a) on coalition-proof Nash equilibria still hold for something which he calls adjusted coalition-proof Nash equilibria. Adjusted coalition-proof Nash equilibrium is similar to coalition-proof Nash equilibrium, but it limits the size of deviating coalitions to be less than or equal to two. This is an especially appealing limitation in a setting of network formation, where each link is formed by two players and where a single player can break links.

8 Related Literature

In this chapter I have discussed the literature on the formation of networks in coalitional games. A coalitional game describes the possible gains from cooperation for all coalitions of players. The question of which coalitions will be formed by the players in a coalitional game is still largely unresolved. Models of network formation in coalitional games approach this question by adding structure to the game and thereby making it possible to consider bilateral cooperation that facilitates cooperation by larger coalitions of players. A network is a collection of bilateral relations between the players and as such networks can be viewed as a generalization of a coalition structures in the sense that a coalition of players can be identified with a network in which all the members of the coalition have formed bilateral links with each other. Hence, a coalition structure is a collection of complete networks. This is formalized in Slikker and van den Nouweland (2001a), who show that the value for games with coalition structures introduced in Aumann and Drèze (1974) coincides with the value for games with networks introduced in Myerson (1977) for the networks that model the coalition structures as described above. Networks are more general than coalition structures because they allow for non-transitivity of bilateral relations. This opens up the possibility to model communication between players who have no direct relation, but who do have indirect relations via other players who act as intermediates. Just like for coalition structures, players who have no direct or indirect relations cannot effectively communicate and hence not cooperate. For players who can communicate, the worth that they can obtain by cooperating is modeled by a coalitional game. In this approach, the value of a network primarily depends on which players it connects with one another (directly or indirectly) and not on how exactly it connects these players (the only exception being a model in which links are costly, see section 6). Hence, issues such as the deterioration of information as it has to travel along longer paths are not taken into account. When we want to take these types of issues into account, we end up with models in which the worth of players in networks is given by a value function rather than derived from a coalitional game. The literature on networks with value functions is covered in Jackson (2004), Chapter 2 in this volume. Also, the approach based on coalitional games precludes externalities between different groups of interconnected players. Currarini (2002) looks at the formation of networks in situations where such externalities can exist.[26]

There is a lively literature on the tensions between stability of networks and their efficiency in the sense of overall payoff maximization. Jackson (2004) includes a discussion of this literature. Mostly, the models that I discuss illustrate this tension, which was shown in Jackson and Wolinsky (1996) to be quite pervasive even in settings of more general value functions than those based on coalitional games.

In this chapter, I concentrated on the basic models of network formation in which players form bilateral relations that are deterministic and symmetric and in which the possibilities of coalitions of player are given by a coalitional game with transferable utility. I believe that most of the issues that arise when studying network formation in coalitional games, arise for this basic model. I refer the reader to Myerson (1980), van den Nouweland et al. (1992), and Slikker et al. (2000a) for extensions to situations in which players can form multilateral relations, to Calvo et al. (1999) for extensions to situations in which bilateral relations are not deterministic, to Slikker and van den Nouweland (2001a) and Casas-Méndez and Prada-Sánchez (2003) for situations in which utilities are not transferable, and to Slikker et al. (2000b) for situations in which players form relations that are not symmetric. For a much more elaborate treatment of the subject of network formation in coalitional games than I could provide in this survey, I refer the reader to Slikker and van den Nouweland (2001a), where the reader will also find most of the proofs that I have omitted in this chapter.

[26] All of these paper look at static models of network formation. For a survey of the research on learning in networks I refer the reader to Goyal (2004), Chapter 5 in this volume.

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