STABILITY WITHIN T^2-SYMMETRIC EXPANDING SPACETIMES

by

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DISSERTATION ABSTRACT

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We prove a nonpolarized analogue of the asymptotic characterization of T^2-symmetric Einstein flow solutions completed recently by LeFloch and Smulevici. We impose a far weaker condition, but obtain identical rates of decay for the normalized energy and associated quantities. We describe numerical simulations which indicate that there is a locally attractive set for T^2-symmetric solutions not subject to this weakened condition. This local attractor is distinct from the local attractor in our main theorem, thereby indicating that the polarized asymptotics are unstable.

This dissertation includes unpublished coauthored material.

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For Jim,

my friend, mentor, and collaborator.

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CHAPTER I

INTRODUCTION

This dissertation is concerned with the stability properties of a special class of expanding vacuum spacetimes. There exist broad conjectures about the expanding direction behavior of such spacetimes (see Section 1.2), but currently little is known about some of the most elementary examples. In the special case that the spacetime has spatial topology T^3, admits two spacelike Killing vector fields, and satisfies a further technical condition (that the spacetime is *polarised*, cf. Section 1.1) results of [LS16] show that there is a local attractor of the Einstein Flow in the expanding direction. It is natural to ask whether the condition that the spacetime be polarised is necessary. Do spacetimes on T^3 with two spacelike Killing vector fields necessarily become approximately polarised? Do they then flow to the polarised attractor?

We partially resolve these questions by analytic and numerical means. Our main theorem states that solutions which are not polarised will have the expanding direction asymptotics of polarised solutions if they satisfy a certain weaker condition: that one of the two conserved quantities of the flow vanishes. See Section 1.1 for the precise statement of this condition; we will call such solutions B_0 or $B = 0$ *solutions*. The conserved quantity B vanishes for all polarised solutions, but setting $B = 0$ defines a much larger set. In particular, the set of $B = 0$ solutions is of codimension one in the space of all solutions in these coordinates while polarised solutions occupy a set of infinite codimension.

Our main theorem states that the condition $B = 0$ suffices to ensure that a solution has polarised asymptotics. Solutions with $B = 0$ flow toward polarised solutions, and thus flow toward the polarised attractor. In the latter portion of the paper, we present numerical evidence that the condition $B = 0$ is necessary for the solution to have polarised asymptotics and flow toward the polarised attractor. There appears to be an attractor for solutions satisfying $B \neq 0$, which shares some formal properties with the $B = 0$ attractor. However, such solutions flow away from the $B = 0$ set, and so the $B = 0$ asymptotics appear to be unstable within the set of vacuum spacetimes with spatial topology T^3 and two spacelike Killing vector fields.

Previous to this work, numerical simulations conducted by Berger [Ber15a, Ber15b] indicated that all solutions, without regard to the polarisation condition, flowed toward the polarised attractor. This work relied on a technique for solving the constraints which was not general. Our contribution to that numerical work has been to solve the constraints in a random way. This was necessary to demonstrate that earlier simulations had implicitly satisfied $B = 0$, and to discover that solutions which did not satisfy this condition have distinct behavior.

Our stability proof follows the technique of [LS16]. There are, however, considerable new difficulties introduced by assuming the much weaker condition that $B = 0$. We overcome these issues by using a new foliation to make use of a correction to the energy that differs from those previously considered. All chapters of this dissertation are joint work with Jim Isenberg and Beverly K Berger.

1.1. The Spacetimes under Consideration, and Their Asymptotics

The class of solutions under consideration here are the T^2-symmetric spacetimes, which have spatial topology T^3 and two spacelike Killing vector fields. It was shown in [BCIM97] that these conditions suffice to determine a unique, global, areal foliation of the spacetime; all such Einstein Flows have a metric of the form

$$g = e^{\widehat{l}-V+4\tau}\left(-d\tau^2 + e^{2(\rho-\tau)}d\theta^2\right) + e^V\left[dx + Qdy + (G+QH)d\theta\right]^2 + e^{-V+2\tau}\left[dy + Hd\theta\right]^2$$

where ∂_x and ∂_y are the Killing vector fields. The area of the $\{\partial_x, \partial_y\}$ orbit is $e^{2\tau}$, so the singularity occurs as $\tau \to -\infty$ and the spacetime expands as $\tau \to \infty$. Relative to the coordinates t, P, α, λ used in [Rin15], our quantities are given by

$$\begin{aligned} V :=& P + \log t \\ \rho :=& -\frac{1}{2}\log\alpha \\ \tau :=& \log t \\ \widehat{l} :=& P + \frac{1}{2}\lambda - \frac{3}{2}\log t. \end{aligned}$$

See the Appendix for a complete concordance of notations between the cited papers and this dissertation.

There are two quantities associated to the Killing vector fields which, as a consequence of the equations, are constant: K_x and K_y [Ger71, Ger72]. Any constant nondegenerate linear transformation of ∂_x and ∂_y yields an isometry of the spacetime, and without loss of generality we may perform such a transformation to arrange for one of K_x or K_y to vanish identically. We denote the nonvanishing constant simply by K. Then in these coordinates the vacuum Einstein Field Equations reduce to

$$\widehat{l}_\theta = V_\theta V_\tau + e^{2(V-\tau)} Q_\theta Q_\tau$$

and

$$\begin{aligned} V_{\tau\tau} - e^{2(\tau-\rho)} V_{\theta\theta} =& -\rho_\tau V_\tau - e^{2(\tau-\rho)}\rho_\theta V_\theta + e^{2(V-\tau)}\left(Q_\tau^2 - e^{2(\tau-\rho)} Q_\theta^2\right) \quad (1.1) \\ Q_{\tau\tau} - e^{2(\tau-\rho)} Q_{\theta\theta} =& -\rho_\tau Q_\tau - e^{2(\tau-\rho)}\rho_\theta Q_\theta - 2\left(Q_\tau V_\tau - e^{2(\tau-\rho)} Q_\theta V_\theta\right) + 2Q_\tau \quad (1.2) \\ \widehat{l}_\tau + \rho_\tau + 2 =& \frac{1}{2}\left[V_\tau^2 + e^{2(\tau-\rho)} V_\theta^2 + e^{2(V-\tau)}\left(Q_\tau^2 + e^{2(\tau-\rho)} Q_\theta^2\right)\right] \\ \rho_\tau =& \frac{1}{2} K^2 e^{\widehat{l}}. \quad (1.3) \end{aligned}$$

It is of interest to note that equation (1.3) actually appears as a consequence of the constraint equations. We, however, take it to be the evolution equation for the quantity

ρ. The remaining variables G and H satisfy evolution equations which may be integrated after $(V, Q, \widehat{l}, \rho)$ are determined.

The condition $Q \equiv 0$ is often imposed when studying these solutions in the collapsing direction. Such solutions are called *polarised*. A simple class of solutions are those which are spatially homogeneous (l, V, Q are independent of θ) and satisfy $\rho \equiv 0$. Such solutions are called *Kasner* and form an important class of anisotropic examples. Let us note that, in our coordinates, polarised Kasner solutions take the form

$$V = a\tau + b, \quad \widehat{l} = \frac{1}{2}a^2\tau + c$$

for some constants $a, b, c \in \mathbb{R}$. Solutions which satisfy only the condition $K = 0$ (in this case one can perform a change of coordinates to ensure that $\rho \equiv 0$) are called *Gowdy*. These, too, have been studied extensively in the direction of the singularity. In the expanding direction, the dynamics of Gowdy solutions are known [Rin17] and appear to be very different than those of non-Gowdy solutions. Non-Gowdy solutions such that $\widehat{l}, V, Q$ are independent of θ are called *pseudo-homogeneous* or *PH* [Rin15]. The future asymptotics of polarised PH solutions are known to be of the form

$$|V - (a\tau + b)| \to 0, \quad \left|\widehat{l} - \left(\frac{1}{2}a^2\tau + c\right)\right| \to 0, \quad a \in (-2, 2).$$

That is, PH solutions have asymptotics of the same form as a Kasner solution, but the value of V_τ at $\tau = \infty$ is not entirely free. The major accomplishment of [Rin15] is to show that a solution is PH if and only if $\int_{S^1} e^\rho \, d\theta$ is bounded.

Since the future behavior of Gowdy and PH solutions is understood, we will only be concerned with non-Gowdy, non-PH solutions; that is, solutions with $K \neq 0$ and $\int_{S^1} e^\rho \, d\theta$

unbounded as $\tau \to \infty$. In this case, note that we can shift $\widehat{l}$ by a constant

$$l := \widehat{l} + \log(K^2/2)$$

so that

$$\widehat{l}_\theta = l_\theta, \quad \widehat{l}_\tau = l_\tau$$

and

$$\begin{aligned} l_\tau + \rho_\tau + 2 =& \frac{1}{2}\left[V_\tau^2 + e^{2(\tau-\rho)}V_\theta^2 + e^{2(V-\tau)}\left(Q_\tau^2 + e^{2(\tau-\rho)}Q_\theta^2\right)\right] && (1.4)\\ \rho_\tau =& e^l && (1.5)\\ l_\theta =& V_\theta V_\tau + e^{2(V-\tau)}Q_\theta Q_\tau && (1.6)\end{aligned}$$

Thus the unknowns will be (V, Q, l, ρ) and the evolution equations will be (1.1), (1.2), (1.4) and (1.5) subject to the constraint (1.6). It is sometimes useful to write equations (1.1) and (1.2) in the following form:

$$\begin{aligned} \partial_\tau\left(e^\rho V_\tau\right) =& \partial_\theta\left(e^{2\tau-\rho}V_\theta\right) + e^{2(V-\tau)+\rho}\left(Q_\tau^2 - e^{2(\tau-\rho)}Q_\theta^2\right) && (1.7)\\ \partial_\tau\left(e^{\rho+2(V-\tau)}Q_\tau\right) =& \partial_\theta\left(e^{-\rho+2V}Q_\theta\right). && (1.8)\end{aligned}$$

As shown in [Rin15], the system has two conserved quantities:

$$\begin{aligned} A &:= \int_{S^1} e^\rho\left(V_\tau - e^{2(V-\tau)}Q_\tau Q\right)\, d\theta\\ B &:= \int_{S^1} e^{\rho+2(V-\tau)}Q_\tau\, d\theta. \end{aligned}$$

Definition. *Let B_0 be the class of non-Gowdy (that is, $K \neq 0$), non-pseudo-homogeneous (that is, not all of l, V, Q are θ-independent) solutions for which $B = 0$.*

We refer to elements of this set with the adjective B_0. Pseudo-homogeneous solutions satisfying $B = 0$ were shown in [Rin15] to have slightly different asymptotics from general PH solutions.

1.2. Conjectures Regarding the Limiting Behavior of Expanding Spacetimes

Given the variety of subsets of T^2-symmetric spacetimes which have been previously considered, it is natural to ask which solutions are stable under the inclusion into a larger class. We are interested in, for example, the stability in the expanding direction of polarized T^2-symmetric spacetimes within the class of all T^2-symmetric spacetimes. Efforts to understand the stability properties of this restricted class are concentrated on a conjecture due to Anderson, Fischer and Moncrief on the geometrization of expanding Einstein Flows.

Before describing this conjecture, note that when the curvature is negative there exist examples (see, eg. [AM04]) of future stable behavior. The case considered in this dissertation seems to be very different. The Kasner metrics on T^3 are unstable within the class of Gowdy metrics, which appear to be unstable within the class of all T^2-symmetric metrics. Given the prevalence of instabilities, symmetric solutions may not appear to be useful tools in the understanding of the behavior of generic solutions on compact topologies. Let us state one generic conjecture in the case that Σ has closed spatial topology. The definition makes use of the notion of proper time distance between Cauchy surfaces. See [Rin13] for a precise definition of this concept.

Conjecture ([And01])**.** *Let (M, g) be a vacuum spacetime satisfying the following conditions:*

1. *(M, g) has a compact spacelike Cauchy hypersurface Σ,*

2. *the Yamabe invariant $\sigma(\Sigma) \leq 0$,*

3. *M is foliated by CMC hypersurfaces exhausting the interval $[H_0, 0)$, and*

4. (M, g) *is future causally geodesically complete.*

Furthermore, let the Cauchy surface of constant mean curvature τ *be called* Σ_τ*, and let the induced metric on* Σ_τ *be called* $g(\tau)$*. Let* $\widehat{t}(\Sigma_\tau)$ *be the proper time distance from* Σ_τ *to the fixed surface* Σ*. Define*

$$h(\tau) := \left(\widehat{t}(\Sigma_\tau)\right)^{-2} g(\tau).$$

Then Σ *admits a decomposition* $\Sigma = H \cup S$*, where the union is along 2-tori, such that*

1. h *converges on* H *as* $\tau \to 0$ *to a complete hyperbolic metric of finite positive volume,*

2. S *is Siefert fibered and* $\lim_{\tau \to 0} \mathrm{vol}_h S = 0$*.*

See [FM01] for a similar formulation. As a statement about geometry, the conjecture says that the Einstein Flow, properly normalized, on CMC surfaces which are not Yamabe positive, is geometrizing. As a statement about cosmology, note that the hyperbolic metric which is the limit of h on H is isotropic. So the conjecture states that the probability that a randomly chosen observer in Σ observes Σ to be locally isotropic increases to 1 as $\tau \to 0$.

The space of examples where this rescaling procedure has been carried out in full is very small; our work provides an example of stable expanding behavior in the case that Σ consists of a single Seifert fibered component: $\Sigma = T^3$.

1.3. An Outline of This Dissertation

The focus of the current work is to prove a result analogous to the local stability theorem of [LS16]. In the large, the technique of proof is straightforward and replicates the technique of that work. That is, we make an ansatz that the means of some of the metric components and their derivatives are close to their asymptotic values, linearize the system around these values thereby introducing error terms, and then use a bootstrap argument to show that these estimates are improved by the flow as long as the initial data are close enough to their asymptotic values. As a corollary of the bootstrap, we obtain strong enough control of the error terms to show that these are negligible as $\tau \to \infty$, which finishes the proof.

There is considerable work in determining precisely the right quantities to linearize. In particular, the right side of equation (1.4) is natural to use as an energy for V and Q. We compute the derivative of this energy in Chapter II and also make some introductory definitions. This energy, however, involves time derivatives, which is not desirable. One thus modifies the energy by a term schematically of the form

$$\int_{S^1} f_\tau \left(\int_{S^1} f \, d\phi - f \right) d\theta$$

which, after differentiating, produces terms of the form appearing in the energy, but where the time and space components have opposite sign. It is this corrected energy that we actually use in the stability proof, and so it is necessary to bound the error terms produced by adjusting the energy by this correction. We define the correction, compute its derivatives, and compute bounds on the associated error terms in Chapter III. In Chapter IV we compute the evolution of the energy with the addition of the correction, and derive a number of bounds on error terms which will appear in the bootstrap proof, which is contained in Chapter VI. Before proceeding with the bootstrap, however, we determine in Chapter V which quantities we will linearize and derive a bound on the distance from a solution to the linearized solution. In Chapter VII, we use the bootstrap argument to derive bounds on the metric components. Chapter VIII consists of numerical evidence that the $B = 0$ attractor is not an attractor for solutions with $B \neq 0$. Finally, the Appendix includes a concordance of notations between this document and many of the previous works concerned with T^2-symmetric Einstein flows.

CHAPTER II

PRELIMINARY COMPUTATIONS

This chapter is joint work with Jim Isenberg and Beverly K Berger.

Before proceeding with the proof of the main theorem, we define the energy under consideration and calculate its evolution. First, however, it will be useful to have a convenient notation for the mean of a function in the θ-direction.

Definition (S^1-mean)**.** *For $f\colon S^1 \to \mathbb{R}$, let*

$$\langle f \rangle := \int_{S^1} f(\theta)\, d\theta.$$

Note that in [LS16], the authors choose to use the volume form $e^\rho\, d\theta$ for their mean. Our choice is almost identical to that used in [Rin15], but we normalize so that $\int_{S^1} d\theta = 1$. Either choice would suffice.

Define the following energy

$$\begin{aligned} J :=& \frac{1}{2}\left[V_\tau^2 + e^{2(\tau-\rho)}V_\theta^2 + e^{2(V-\tau)}\left(Q_\tau^2 + e^{2(\tau-\rho)}Q_\theta^2\right)\right] \\ E :=& \int_{S^1} e^{\rho-2\tau} J\, d\theta \end{aligned}$$

and the S^1-volume

$$\Pi := \langle e^\rho \rangle = \int_{S^1} e^\rho\, d\theta.$$

Note that equation (1.4) now reads $l_\tau + \rho_\tau + 2 = J$. We will use the terms V-energy and Q-energy loosely to refer to $V_\tau^2 + e^{2(\tau-\rho)}V_\theta^2$ and $e^{2(V-\tau)}\left(Q_\tau^2 + e^{2(\tau-\rho)}Q_\theta^2\right)$, respectively.

One may compute using the evolution equations for V and Q that

$$\partial_\tau \left(e^\rho J\right) = 2e^\rho J - \rho_\tau e^\rho J - e^\rho V_\tau^2 - e^{2V-\rho} Q_\theta^2 + \partial_\theta \left(e^{2\tau-\rho} V_\theta V_\tau + e^{2V-\rho} Q_\theta Q_\tau\right)$$

so the energy E evolves by

$$E_\tau = \int_{S^1} -\rho_\tau e^{\rho-2\tau} J - e^{\rho-2\tau} V_\tau^2 - e^{2(V-\tau)-\rho} Q_\theta^2 \, d\theta.$$

The terms $-e^{\rho-2\tau} V_\tau^2 - e^{2(V-\tau)-\rho} Q_\theta^2$ appearing here are undesirable for proving energy inequalities. This necessitates the modification of E by a term which trades V_τ^2 for V_θ^2. This is the main topic of Chapter III.

CHAPTER III

CORRECTIONS AND THEIR BOUNDS

This chapter is joint work with Jim Isenberg and Beverly K Berger.

Define the correction

$$\Lambda := \frac{1}{2} e^{-2\tau} \int_{S^1} V_\tau \left(V - \langle V \rangle - 1 \right) e^\rho \, d\theta. \tag{3.1}$$

Corrections to the energy of essentially this form were used previously in the Gowdy case [Rin04] and in the existing results on T^2-symmetric spacetimes [Rin15, LS16]. Our definition differs only slightly from those previously used. Differentiating (3.1) and using integration by parts yields the two components of the V-energy, but with opposite sign. This allows us to replace time derivatives by space derivatives, which may be bounded. At the same time, the correction has better decay properties than the energy, and so we are able to draw conclusions about the energy in the expanding direction.

To trade V_τ^2 for V_θ^2 and Q_τ^2 for Q_θ^2, it would be more natural to consider the corrections

$$\frac{1}{2} e^{-2\tau} \int_{S^1} V_\tau \left(V - \langle V \rangle \right) e^\rho \, d\theta, \quad \text{and} \quad \frac{1}{2} e^{-2\tau} \int_{S^1} e^{2(V-\tau)} Q_\tau \left(Q - \langle Q \rangle \right) e^\rho \, d\theta$$

separately as other authors have done. Then, by differentiating the Q-correction one would hope to obtain terms of the form $Q_\tau^2 - e^{2(\tau-\rho)} Q_\theta^2$, perhaps with a leading factor. Our definition exploits the fact that (1.7) contains exactly the expression that we would like to obtain from the Q-correction.

Lemma 1. *Consider a non-Gowdy T^2-symmetric Einstein flow. The correction defined in* (3.1) *evolves by*

$$\begin{aligned}\partial_\tau \Lambda \ &= -2\Lambda + \frac{1}{2}e^{-2\tau}\int_{S^1} -e^{2\tau}e^{-\rho}V_\theta^2\, d\theta + \frac{1}{2}e^{-2\tau}\int_{S^1} V_\tau^2 e^\rho\, d\theta \\ &+ \frac{1}{2}e^{-2\tau}\int_{S^1} e^{2(V-\tau)+\rho}\left(Q_\tau^2 - e^{2(\tau-\rho)}Q_\theta^2\right)(V - \langle V\rangle - 1)\, d\theta - \langle V_\tau\rangle \left\langle \frac{1}{2}e^{\rho-2\tau}V_\tau\right\rangle.\end{aligned}$$

Proof. We compute straightforwardly using equations (1.7), (1.8) and integration by parts. From the definition of Λ we have

$$\begin{aligned}\partial_\tau \Lambda =& -2\Lambda + \frac{1}{2}e^{-2\tau}\int_{S^1} (e^\rho V_\tau)_\tau (V - \langle V\rangle - 1)\, d\theta + \frac{1}{2}e^{-2\tau}\int_{S^1} V_\tau \partial_\tau (V - \langle V\rangle - 1) e^\rho\, d\theta \\ =& -2\Lambda \\ &+ \frac{1}{2}e^{-2\tau}\int_{S^1}\left[e^{2\tau}\left(e^{-\rho}V_\theta\right)_\theta + e^{2(V-\tau)+\rho}\left(Q_\tau^2 - e^{2(\tau-\rho)}Q_\theta^2\right)\right](V - \langle V\rangle - 1)\, d\theta \\ &+ \frac{1}{2}e^{-2\tau}\int_{S^1} V_\tau \partial_\tau (V - \langle V\rangle - 1) e^\rho\, d\theta \\ =& -2\Lambda + \frac{1}{2}e^{-2\tau}\int_{S^1} -e^{2\tau}e^{-\rho}V_\theta^2\, d\theta + \frac{1}{2}e^{-2\tau}\int_{S^1} V_\tau^2 e^\rho\, d\theta \\ &+ \frac{1}{2}e^{-2\tau}\int_{S^1} e^{2(V-\tau)+\rho}\left(Q_\tau^2 - e^{2(\tau-\rho)}Q_\theta^2\right)(V - \langle V\rangle - 1)\, d\theta \\ &- \langle V_\tau\rangle \left\langle \frac{1}{2}e^{\rho-2\tau}V_\tau\right\rangle\end{aligned}$$

which completes the proof. $\square$

We will modify the energy E by Λ to obtain only spatial terms. It will then be desirable to know that Λ has better decay than E. To that end, note that

$$\|V - \langle V\rangle\|_{C^0} \lesssim \int_{S^1} |V_\theta|\, d\theta \le \left(\int_{S^1} V_\theta^2 e^{-\rho}\, d\theta\right)^{1/2} \Pi^{1/2} \le (\Pi E)^{1/2}. \tag{3.2}$$

As is standard (cf. [RS14]), we use the notation $f \lesssim h$ to mean that there is a constant $C \ge 0$ which is independent of the solution under consideration, such that $f \le Ch$. If a constant depends on the solution or the time at which we specify initial data, we will explicitly insert it.

One finds the following bound using Hölder's Inequality.

Lemma 2 ([Rin15], Lemma 72). *Consider a non-Gowdy T^2-symmetric Einstein flow. Then*

$$\left|\Lambda + \left\langle \frac{1}{2} e^{\rho - 2\tau} V_\tau \right\rangle \right| = \left| \frac{1}{2} e^{-2\tau} \int_{S^1} V_\tau \left(V - \langle V \rangle\right) e^{\rho} \, d\theta \right| \lesssim e^{-\tau} \Pi E$$

For the following bound on the Q correction, cf. [Rin15], Lemma 73 where the author assumes a uniform bound on Π which we don't assume here. The proof is essentially the same.

Lemma 3. *For any a non-Gowdy T^2-symmetric Einstein flow,*

$$\left| e^{-2\tau} \int_{S^1} e^{2(V-\tau)} Q_\tau \left(Q - \langle Q \rangle\right) e^{\rho} \, d\theta \right| \lesssim e^{-\tau} e^{2(\Pi E)^{1/2}} \Pi E$$

Proof. Note that we have already bounded $\|V - \langle V \rangle\|_{C_0}$ in equation (3.2), and so we may commute out factors of e^V to obtain

$$\begin{aligned}
\left\| e^{V} \left(Q - \langle Q \rangle\right) \right\|_{C^0} = & \left\| e^{V - \langle V \rangle + \langle V \rangle} \left(Q - \langle Q \rangle\right) \right\|_{C^0} \\
= & e^{\|V - \langle V \rangle\|_{C^0}} e^{\langle V \rangle} \left\| Q - \langle Q \rangle \right\|_{C^0} \\
\leq & e^{2\|V - \langle V \rangle\|_{C^0}} \left(\int_{S^1} e^{2V} Q_\theta^2 e^{-\rho} \, d\theta \right)^{1/2} \Pi^{1/2} \\
\leq & e^{2\|V - \langle V \rangle\|_{C^0}} e^{\tau} E^{1/2} \Pi^{1/2}
\end{aligned}$$

via Hölder's inequality. So we may compute, using the bound on $\|V - \langle V\rangle\|_{C_0}$, Hölder's inequality, and the definition of E

$$\begin{aligned}\left|e^{-2\tau}\int_{S^1} e^{2(V-\tau)}Q_\tau\left(Q-\langle Q\rangle\right)e^\rho\,d\theta\right| \lesssim & e^{-4\tau}\left\|e^V\left(Q-\langle Q\rangle\right)\right\|_{C^0}\left|\int_{S^1} e^V Q_\tau e^\rho\,d\theta\right| \\ \lesssim & e^{2\|V-\langle V\rangle\|_{C^0}}e^{-3\tau}E^{1/2}\Pi^{1/2}\left|\int_{S^1} e^V Q_\tau e^\rho\,d\theta\right| \\ \leq & e^{2\|V-\langle V\rangle\|_{C^0}}e^{-\tau}E\Pi \\ \lesssim & e^{-\tau}e^{2(\Pi E)^{1/2}}\Pi E. \qquad \square\end{aligned}$$

We only need the Q correction for the following identity, which follows directly from the definitions of the conserved quantities A, B:

$$\left\langle e^{\rho-2\tau}V_\tau\right\rangle = e^{-2\tau}\left(A + B\langle Q\rangle + \int_{S^1} e^{2(V-\tau)}Q_\tau\left(Q-\langle Q\rangle\right)e^\rho\,d\theta\right).$$

For B_0 solutions, however, we use the bound on the Q correction to obtain the following bound

$$\left|\left\langle e^{\rho-2\tau}V_\tau\right\rangle\right| - e^{-2\tau}|A| \lesssim e^{-\tau}e^{2(\Pi E)^{1/2}}\Pi E \tag{3.3}$$

which yields the desired estimate on the correction.

Proposition 1. *Fow any a non-Gowdy, B_0 T^2-symmetric Einstein flow,*

$$|\Lambda| - e^{-2\tau}|A| \lesssim e^{-\tau}\left(1 + e^{2(\Pi E)^{1/2}}\right)\Pi E. \tag{3.4}$$

We write these last two inequalities in this way merely to show that the bound occurs with no arbitrary constant in front of $e^{-2\tau}|A|$.

The correction Λ introduces significant new error terms after differentiation. However, these terms have good bounds, and the modified energy $E + \Lambda$ has significantly better properties upon comparison to E alone. The evolution of this modified energy is the focus of the next chapter.

CHAPTER IV

EVOLUTION OF THE CORRECTED ENERGY

This chapter is joint work with Jim Isenberg and Beverly K Berger.

One would like to show that, up to error terms, Π and E satisfy an ODE. While this is true asymptotically, it is more useful to compute with an energy which has been modified by the correction.

One computes that

$$\begin{aligned}
(E+\Lambda)_\tau =& \int_{S^1} -e^{\rho-2\tau}\rho_\tau J - e^{\rho-2\tau}V_\tau^2 - e^{2(V-\tau)-\rho}Q_\theta^2 \, d\theta - 2\Lambda \\
&+ \frac{1}{2}e^{-2\tau}\int_{S^1} -e^{2\tau}e^{-\rho}V_\theta^2 \, d\theta + \frac{1}{2}e^{-2\tau}\int_{S^1} V_\tau^2 e^\rho \, d\theta \\
&+ \frac{1}{2}e^{-2\tau}\int_{S^1} e^{2(V-\tau)+\rho}\left(Q_\tau^2 - e^{2(\tau-\rho)}Q_\theta^2\right)\left(V - \langle V\rangle - 1\right) \, d\theta \\
&- \langle V_\tau \rangle \left\langle \frac{1}{2}e^{\rho-2\tau}V_\tau \right\rangle \\
=& - \left(1 + \frac{\Pi_\tau}{\Pi}\right)(E+\Lambda) + \left(\frac{\Pi_\tau}{\Pi}E - \int_{S^1} e^{\rho-2\tau}\rho_\tau J \, d\theta\right) - \left(1 - \frac{\Pi_\tau}{\Pi}\right)\Lambda \\
&+ \frac{1}{2}e^{-2\tau}\int_{S^1} e^{2(V-\tau)+\rho}\left(Q_\tau^2 - e^{2(\tau-\rho)}Q_\theta^2\right)\left(V - \langle V\rangle\right) \, d\theta - \langle V_\tau\rangle \left\langle \frac{1}{2}e^{\rho-2\tau}V_\tau\right\rangle.
\end{aligned}$$

The leading term on the right leads us to the ansatz that $\Pi(E+\Lambda)$ (and so ΠE) should decay like $e^{-\tau}$. Accordingly, define the corrected, normalized energy $H := \Pi(E+\Lambda)$. One computes that

$$\begin{aligned}
\partial_\tau\left(e^\tau H\right) =& e^\tau H + e^\tau \Pi_\tau (E+\Lambda) + e^\tau \Pi (E+\Lambda)_\tau \\
=& e^\tau \Pi \left((E+\Lambda)\left(1 + \frac{\Pi_\tau}{\Pi}\right) + (E+\Lambda)_\tau\right) \\
=& e^\tau \Pi \left[\left(\frac{\Pi_\tau}{\Pi}E - \int_{S^1} e^{\rho-2\tau}\rho_\tau J \, d\theta\right) - \left(1 - \frac{\Pi_\tau}{\Pi}\right)\Lambda \right. \\
&\left. + \frac{1}{2}e^{-2\tau}\int_{S^1} e^{2(V-\tau)+\rho}\left(Q_\tau^2 - e^{2(\tau-\rho)}Q_\theta^2\right)\left(V - \langle V\rangle\right) d\theta - \langle V_\tau\rangle\left\langle \frac{1}{2}e^{\rho-2\tau}V_\tau\right\rangle\right]
\end{aligned} \tag{4.1}$$

The ansatz in the local stability proof will be that $e^\tau H$ is of constant order. The proof is via a bootstrap argument, where we will want to bound all of the terms of $\partial_\tau(e^\tau H)$ in terms of Π, E, H and τ. The following Proposition deals with each of these error terms.

Proposition 2. *Consider the evolution of a B_0 T^2-symmetric solution with initial data given at time $\tau = s_0$. The following estimates hold.*

$$\left|\frac{\Pi_\tau}{\Pi}E - \int_{S^1} e^{\rho-2\tau}\rho_\tau J\,d\theta\right| \lesssim E\int_{S^1} e^{\rho-\tau}\rho_\tau J\,d\theta \tag{4.2}$$

$$\left|\left(1-\frac{\Pi_\tau}{\Pi}\right)\Lambda\right| \lesssim |A|e^{-2\tau}\left(1+\frac{\Pi_\tau}{\Pi}\right) + e^{-\tau}\left(1+e^{2(\Pi E)^{1/2}}\right)(\Pi+\Pi_\tau)\,E \tag{4.3}$$

$$\left|\langle V_\tau\rangle\left\langle \frac{1}{2}e^{\rho-2\tau}V_\tau\right\rangle\right| \lesssim C_{\rho(s_0)}e^{-\tau}\left(|A| + e^\tau e^{2(\Pi E)^{1/2}}\Pi E\right)E^{1/2} \tag{4.4}$$

and

$$\left|e^{-2\tau}\int_{S^1} e^{2(V-\tau)+\rho}\left(Q_\tau^2 - e^{2(\tau-\rho)}Q_\theta^2\right)(V-\langle V\rangle)\,d\theta\right| \lesssim \Pi^{1/2}E^{3/2}. \tag{4.5}$$

Proof. For (4.2), using Young's inequality, we note that

$$\begin{aligned}
|l_\theta| \le& |V_\tau V_\theta| + |e^{V-\tau}Q_\tau e^{V-\tau}Q_\theta| \\
=& |e^{(\rho-\tau)/2}V_\tau e^{-(\rho-\tau)/2}V_\theta| + |e^{V-\tau}e^{(\rho-\tau)/2}Q_\tau e^{V-\tau}e^{-(\rho-\tau)/2}Q_\theta| \\
\le& \frac{1}{2}\left[e^{\rho-\tau}V_\tau^2 + e^{\tau-\rho}V_\theta^2 + e^{2(V-\tau)}e^{\rho-\tau}Q_\tau^2 + e^{2(V-\tau)}e^{\tau-\rho}Q_\theta^2\right] \\
=& e^{\rho-\tau}J.
\end{aligned} \tag{4.6}$$

Thus we may use the Poincaré inequality to compute that

$$
\begin{aligned}
\left|\frac{\Pi_\tau}{\Pi}E - \int_{S^1} e^{\rho-2\tau}\rho_\tau J\, d\theta\right| =& \Pi^{-1}\left|\Pi_\tau E - \Pi\int_{S^1} e^{\rho-2\tau}\rho_\tau J\, d\theta\right| \\
=& \Pi^{-1}\left|\int_{S^1}\int_{S^1} e^{\rho(\phi)}e^{\rho(\theta)-2\tau}J(\theta)\left(\rho_\tau(\phi)-\rho_\tau(\theta)\right)\, d\phi d\theta\right| \\
\leq& \Pi^{-1}\left|\int_{S^1}\int_{S^1} e^{\rho(\phi)}e^{\rho(\theta)-2\tau}J(\theta)\sup_{a,b\in S^1}|\rho_\tau(a)-\rho_\tau(b)|\, d\phi d\theta\right| \\
=& \Pi^{-1}\Pi\int_{S^1} e^{\rho(\theta)-2\tau}J(\theta)\, d\theta \sup_{a,b\in S^1}|\rho_\tau(a)-\rho_\tau(b)| \\
\lesssim& E\int_{S^1}\rho_\tau|l_\theta|\, d\theta \\
\leq& E\int_{S^1}\rho_\tau e^{\rho-\tau}J\, d\theta.
\end{aligned}
$$

Inequality (4.3) follows directly from inequality (3.4). To prove (4.5), we first commute out the V-mean.

$$
\begin{aligned}
&\left|e^{-2\tau}\int_{S^1} e^{2(V-\tau)+\rho}\left(Q_\tau^2 - e^{2(\tau-\rho)}Q_\theta^2\right)(V-\langle V\rangle)\, d\theta\right| \\
\leq& e^{-2\tau}\|V-\langle V\rangle\|_{C^0}\int_{S^1} e^{2(V-\tau)+\rho}\left|Q_\tau^2 - e^{2(\tau-\rho)}Q_\theta^2\right|\, d\theta \\
\leq& e^{-2\tau}(\Pi E)^{1/2}\int_{S^1} e^{\rho}e^{2(V-\tau)}\left(Q_\tau^2 + e^{2(\tau-\rho)}Q_\theta^2\right)\, d\theta \\
\lesssim& (\Pi E)^{1/2}\int_{S^1} e^{\rho-2\tau}J\, d\theta \\
=& \Pi^{1/2}E^{3/2}.
\end{aligned}
$$

Lastly, for (4.4) recall that ρ is increasing and compute that

$$
\begin{aligned}
|\langle V_\tau\rangle| \leq& \left(\int_{S^1} V_\tau^2 e^{\rho}\, d\theta\right)^{1/2}\left(\int_{S^1} e^{-\rho}\, d\theta\right)^{1/2} \\
\leq& C_{\rho(s_0)}e^{\tau}\left(\int_{S^1} V_\tau^2 e^{\rho-2\tau}\, d\theta\right)^{1/2} \\
\lesssim& C_{\rho(s_0)}e^{\tau}E^{1/2}
\end{aligned}
$$

and use (3.3). This completes the proof. □

Now that we have an energy satisfying a good differential equation with good bounds on the error, we must proceed to the linearization. This is the topic of the following chapter.

CHAPTER V

LINEARIZATION

This chapter is joint work with Jim Isenberg and Beverly K Berger.

In [LS16], the authors present an argument that certain asymptotic rates of Π, E should be preferred, based on the assumption that $e^\tau H$ should be of constant order. In this section we briefly summarize that argument as it appears in our context.

Definition. *Let* $Y := \left\langle e^{l+\rho+2\tau} \right\rangle$.

Note that we have defined Y so that $Y_\tau = \langle e^{l+\rho+2\tau}(l_\tau + \rho_\tau + 2)\rangle = \langle e^{l+\rho+2\tau} J\rangle$. We want to form a system of ordinary differential equations from the means, however. So we distribute the integral over the product, introducing the error term Ω. One computes

$$\Pi_\tau = e^{-2\tau} Y \tag{5.1}$$

$$Y_\tau = e^{2\tau} E Y \Pi^{-1} + \Omega \tag{5.2}$$

where

$$\Omega := \left\langle e^{l+\rho+2\tau} J \right\rangle - e^{2\tau} E Y \Pi^{-1}$$

is an error term satisfying

$$|\Omega| \lesssim e^{4\tau} E \left\langle e^{l+\rho-\tau} J \right\rangle = e^\tau E Y_\tau.$$

Note that our quantity E contains the terms Q_θ and Q_τ, and so is not identical to the energy in [LS16]. Nonetheless, the quantities Π, Y, E satisfy similar relations to the relations LeFloch and Smulevici's quantities do. Normalizing, we compute that

$$
\begin{aligned}
\partial_\tau\left(e^{-\tau}H^{-1/2}\Pi\right) =& e^{-\tau}H^{-1/2}\Pi_\tau - e^{-\tau}H^{-1/2}\Pi - \frac{1}{2}e^{-\tau}H^{-1/2}\Pi\frac{H_\tau}{H} \\
=& \left(e^{-3\tau}H^{-1/2}Y\right) + \left(e^{-\tau}H^{-1/2}\Pi\right)\left(-1 - \frac{1}{2}\frac{H_\tau}{H}\right) \\
\partial_\tau\left(e^{-3\tau}H^{-1/2}Y\right) =& e^{-3\tau}H^{-1/2}Y_\tau - 3e^{-3\tau}H^{-1/2}Y - \frac{1}{2}e^{-3\tau}H^{-1/2}Y\frac{H_\tau}{H} \\
=& e^{-3\tau}H^{-1/2}\left(e^{2\tau}EY\Pi^{-1} + \Omega\right) + \left(e^{-3\tau}H^{-1/2}Y\right)\left(-3 - \frac{1}{2}\frac{H_\tau}{H}\right) \\
=& \frac{\left(e^{-3\tau}H^{-1/2}Y\right)}{\Pi^2}e^{2\tau}\Pi E + \left(e^{-3\tau}H^{-1/2}Y\right)\left(-3 - \frac{1}{2}\frac{H_\tau}{H}\right) + e^{-3\tau}H^{-1/2}\Omega
\end{aligned}
$$

$$
\begin{aligned}
=& \frac{\left(e^{-3\tau}H^{-1/2}Y\right)}{\left(e^{-\tau}H^{-1/2}\Pi\right)^2}\frac{\Pi E}{H} + \left(e^{-3\tau}H^{-1/2}Y\right)\left(-3 - \frac{1}{2}\frac{H_\tau}{H}\right) + e^{-3\tau}H^{-1/2}\Omega \\
=& \frac{\left(e^{-3\tau}H^{-1/2}Y\right)}{\left(e^{-\tau}H^{-1/2}\Pi\right)^2} + \left(e^{-3\tau}H^{-1/2}Y\right)\left(-3 - \frac{1}{2}\frac{H_\tau}{H}\right) + e^{-3\tau}H^{-1/2}\Omega \\
&+ \frac{\left(e^{-3\tau}H^{-1/2}Y\right)}{\left(e^{-\tau}H^{-1/2}\Pi\right)^2}\left(\frac{\Pi E}{H} - 1\right).
\end{aligned}
$$

We insert our ansätze that $\frac{H_\tau}{H} \to -1$, $e^{-3\tau}H^{-1/2}\Omega \to 0$, and $\left(\frac{\Pi E}{H} - 1\right) \to 0$, to obtain the ODE

$$
\begin{aligned}
\partial_\tau\left(e^{-\tau}H^{-1/2}\Pi\right) =& \left(e^{-3\tau}H^{-1/2}Y\right) + \left(e^{-\tau}H^{-1/2}\Pi\right)\left(-\frac{1}{2}\right) \\
\partial_\tau\left(e^{-3\tau}H^{-1/2}Y\right) =& \frac{\left(e^{-3\tau}H^{-1/2}Y\right)}{\left(e^{-\tau}H^{-1/2}\Pi\right)^2} + \left(e^{-3\tau}H^{-1/2}Y\right)\left(-\frac{5}{2}\right)
\end{aligned}
$$

which has a fixed point at

$$
\frac{\Pi}{e^\tau\sqrt{H}} = \frac{2}{\sqrt{10}}, \quad \frac{Y}{e^{3\tau}\sqrt{H}} = \frac{1}{\sqrt{10}}.
$$

So we conjecture that the quantities

$$
c := \frac{\Pi}{e^\tau\sqrt{H}} - \frac{2}{\sqrt{10}}, \quad d := \frac{Y}{e^{3\tau}\sqrt{H}} - \frac{1}{\sqrt{10}}
$$

decay and compute the evolution of these quantities using (5.1) and (5.2). We find that

$$\partial_\tau \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} -1/2 & 1 \\ -5/2 & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} - \frac{1}{2}\partial_\tau \log\left(e^\tau H\right) \begin{pmatrix} c \\ d \end{pmatrix}$$
$$\underbrace{-\frac{1}{2}\partial_\tau \log\left(e^\tau H\right) \begin{pmatrix} \frac{2}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{f(d,c)}{\left(c+\frac{2}{\sqrt{10}}\right)^2} + \left(\frac{\Pi E}{H} - 1\right)\frac{d+\frac{1}{\sqrt{10}}}{\left(c+\frac{2}{\sqrt{10}}\right)^2} + \frac{\Omega}{e^{3\tau}H^{1/2}} \end{pmatrix}}_{=:\widetilde{\Omega}}$$

where $f(c,d) = \frac{1}{4}c\left(c\left(10c - 10d + 3\sqrt{10}\right) - 4\sqrt{10}d\right)$ has vanishing linear part. In the end, the following estimate is obtained (cf. [LS16], Proposition 5.1).

Proposition 3. *Consider the evolution of a B_0 T^2-symmetric solution. Provided the corrected energy H is positive one has for $s \geq s_0$*

$$\left|\begin{pmatrix} c \\ d \end{pmatrix}\right|(s) \lesssim e^{(s_0-s)/4}\left(\frac{e^{s_0}H(s_0)}{e^s H(s)}\right)^{1/2}\left|\begin{pmatrix} c \\ d \end{pmatrix}\right|(s_0) + \int_{s_0}^{s} e^{(\tau-s)/4}\left(\frac{e^\tau H(\tau)}{e^s H(s)}\right)^{1/2}|\omega(\tau)|\, d\tau,$$

where

$$|\omega| \lesssim \left|\widetilde{\Omega}\right|.$$

Quickly note a bound on one of the terms appearing in $\widetilde{\Omega}$.

Lemma 4. *Consider the evolution of a B_0 T^2-symmetric solution. The following estimate holds.*

$$\left|e^{-3\tau}H^{-1/2}\Omega\right| \lesssim e^{-2\tau}|H|^{-1/2}EY_\tau.$$

The proof of this lemma proceeds in the same manner as the proof of inequality (4.2). The remaining three terms in $\widetilde{\Omega}$ will be estimated directly. In the next section, we will perform a bootstrap argument to bound these errors, provided the initial data is sufficiently close to the asymptotic behavior.

CHAPTER VI

THE BOOTSTRAP LEMMA

This chapter is joint work with Jim Isenberg and Beverly K Berger.

The technique of proof follows [LS16]. The idea is to impose some smallness assumptions on the means of the energy, the S^1 volume, and their derivatives. We will then use a bootstrap argument to show that these assumptions are improved. The reason for obtaining the estimates of Lemma 2 is to bound the evolution of the corrected energy H. Let us discuss how that proof will go. We have computed $\partial_\tau (e^\tau H)$ in equation (4.1). Note that we may bound the right side of that equation by an expression of the form

$$|\partial_\tau (e^\tau H)| \lesssim e^\tau \Pi E F + \widetilde{F} = e^\tau H \frac{\Pi E}{H} F + \widetilde{F}$$

where, using the results of Lemma 2 we can write

$$F := \frac{1}{E}\left[E \int_{S^1} e^{\rho-\tau} \rho_\tau J \, d\theta + e^{-\tau}\left(1 + e^{2(\Pi E)^{1/2}}\right)(\Pi + \Pi_\tau) E + \Pi^{1/2} E^{3/2} + e^{2(\Pi E)^{1/2}} \Pi E^{3/2}\right]$$

$$= \int_{S^1} e^{\rho-\tau} \rho_\tau J \, d\theta + e^{-\tau}\left(1 + e^{2(\Pi E)^{1/2}}\right)(\Pi + \Pi_\tau) + (\Pi E)^{1/2} + e^{2(\Pi E)^{1/2}} \Pi E^{1/2} \tag{6.1}$$

and

$$\widetilde{F} := e^\tau \Pi \left(|A| e^{-2\tau}\left(1 + \frac{\Pi_\tau}{\Pi}\right) + e^{-\tau}|A| E^{1/2}\right) = |A|\Pi\left(e^{-\tau}\left(1 + \frac{\Pi_\tau}{\Pi}\right) + E^{1/2}\right). \tag{6.2}$$

Note that F and $\widetilde{F}$ are nonnegative. We will then be concerned with the quantities

$$\int_{s_0}^\infty F(\tau)\, d\tau, \quad \text{and} \quad \int_{s_0}^\infty \widetilde{F}(\tau)\, d\tau$$

which will bound the evolution of $e^\tau H$ in the bootstrap proof.

Lemma 5. *There exist $M, s_0, \epsilon > 0$, functions U_ϵ, L_ϵ and an open set of B_0 Einstein Flows satisfying $|A| < 1$,*

$$\begin{aligned} |c| \leq& \epsilon \\ |d| \leq& \epsilon \\ \left| \frac{\Pi E}{H} - 1 \right| \leq& 1, \end{aligned}$$

and

$$M \leq L_\epsilon \leq e^{s_0} H(s_0) \leq U_\epsilon \leq \epsilon \tag{6.3}$$

at time $\tau = s_0$ such that, for all $\tau \in [s_0, \infty)$, the following weaker estimates hold:

$$|c| \leq \epsilon^{1/4} \tag{6.4}$$

$$|d| \leq \epsilon^{1/4}$$

$$\left| \frac{\Pi E}{H} - 1 \right| < 3 \tag{6.5}$$

$$\frac{1}{2} L_\epsilon \leq e^\tau H(\tau) \leq 2U_\epsilon \tag{6.6}$$

The rest of this chapter consists of a proof of this lemma. The technique of proof will be a straightforward "open closed" argument:

1. Suppose estimates (6.4) to (6.6) are satisfied for $\tau \in [s_0, s)$.

2. We will improve each of the five estimates (6.4) to (6.6) at $\tau = s$ by choosing ϵ small enough and s_0, M large enough.

3. In the course of these estimates, we will define the functions

$$U_\epsilon = e^{s_0} H(s_0) + C\epsilon^{1/2} e^{-\frac{s_0}{4}}$$
$$L_\epsilon = e^{s_0} H(s_0) - C\epsilon^{1/2} e^{-\frac{s_0}{4}}$$

which involve ϵ and s_0. We will then need to verify for this U_ϵ, L_ϵ that it is in fact possible to pick a nonempty open set of initial data satisfying (6.3) by choosing s_0 large depending on ϵ.

6.1. Auxiliary Bootstrap Estimates

From the bootstrap assumptions, we have that

$$\frac{1}{2} M e^{-\tau} \leq \frac{1}{2} L_\epsilon e^{-\tau} \leq H \leq 2U_\epsilon e^{-\tau} \leq 2\epsilon e^{-\tau},$$

and

$$\left| \frac{\Pi}{e^\tau \sqrt{H}} - \frac{2}{\sqrt{10}} \right| = |c| < \epsilon^{1/4}, \quad \left| \frac{Y}{e^{3\tau} \sqrt{H}} - \frac{1}{\sqrt{10}} \right| = |d| < \epsilon^{1/4}$$

so

$$\Pi \lesssim \left(\frac{2}{\sqrt{10}} + \epsilon^{1/4} \right) e^\tau H^{1/2} \leq \left(\frac{2}{\sqrt{10}} + \epsilon^{1/4} \right) \epsilon^{1/2} e^{\tau/2} \lesssim \epsilon^{1/2} e^{\tau/2},$$
$$e^{2\tau} \Pi_\tau = Y \lesssim \left(\frac{1}{\sqrt{10}} + \epsilon^{1/4} \right) e^{3\tau} H^{1/2} \lesssim \left(\frac{1}{\sqrt{10}} + \epsilon^{1/4} \right) \epsilon^{1/2} e^{5\tau/2} \lesssim \epsilon^{1/2} e^{5\tau/2}.$$

Note that (6.5) implies that $\Pi E \lesssim 4H$ on this interval, which implies that

$$\Pi E \lesssim \epsilon e^{-\tau} \lesssim \epsilon, \quad \text{and} \quad 1 + e^{2(\Pi E)^{1/2}} \lesssim 1 + e^{\epsilon^{1/2}} \leq 3$$

for sufficiently small ϵ. The bound on Π and the fact that $\Pi, Y > 0$ together imply that, for $a < -1/2$,

$$\int_{s_0}^{\infty} e^{a\tau}\Pi_\tau \, d\tau = \lim_{s\to\infty} \left(e^{as}\Pi(s)\right) - e^{as_0}\Pi(s_0) - a\int_{s_0}^{\infty} e^{a\tau}\Pi \, d\tau \lesssim e^{\left(a+\frac{1}{2}\right)s_0}\epsilon^{1/2},$$

and

$$\begin{aligned}\int_{s_0}^{\infty} e^{(a-2)\tau}Y_\tau \, d\tau = &\lim_{s\to\infty} \left(e^{(a-2)s}Y(s)\right) - e^{(a-2)s_0}Y(s_0) - (a-2)\int_{s_0}^{\infty} e^{(a-2)\tau}Y \, d\tau \\ \lesssim & e^{\left(a+\frac{1}{2}\right)s_0}\epsilon^{1/2}.\end{aligned}$$

6.2. Bound on Λ

To improve inequality (6.5), first Note also that $\left|\frac{\Pi E}{H} - 1\right| = \frac{\Pi}{H}|\Lambda|$. Then we may use (3.4) to obtain, for some constant C independent of the solution,

$$\begin{aligned}\frac{\Pi}{H}|\Lambda| \leq & \frac{\Pi}{H}\left[e^{-2\tau}|A| + Ce^{-\tau}\left(1 + e^{2(\Pi E)^{1/2}}\right)\Pi E\right] \\ = & e^{-2\tau}|A|\frac{\Pi}{H} + Ce^{-\tau}\left(1 + e^{2(\Pi E)^{1/2}}\right)\frac{\Pi^2 E}{H}.\end{aligned}$$

For the latter term, note that

$$Ce^{-\tau}\left(1 + e^{2(\Pi E)^{1/2}}\right)\frac{\Pi^2 E}{H} \lesssim e^{-\tau}\frac{\Pi}{H}(\Pi E) \lesssim e^{-2\tau}\frac{\epsilon^{1/2}e^{\tau/2}}{\frac{1}{2}Me^{-\tau}}\epsilon \lesssim \epsilon^{1/2}e^{-\tau/2}.$$

while for the first summand we estimate in a similar manner

$$e^{-2\tau}|A|\frac{\Pi}{H} \lesssim e^{-2\tau}\frac{\epsilon^{1/2}e^{\tau/2}}{\frac{1}{2}Me^{-\tau}} \lesssim \epsilon^{1/2}e^{-\tau/2}.$$

Thus in total we obtain

$$\left|\frac{\Pi E}{H} - 1\right| = \frac{\Pi}{H}|\Lambda| \lesssim \epsilon^{1/2}e^{-\tau/2} \tag{6.7}$$

which is less than 3 for small ϵ and large s_0.

6.3. An Upper and Lower Bound on H

For the energy H we have the following estimate:

$$\begin{aligned}
|\partial_\tau (e^\tau H)| =& e^\tau \Pi \left| \left(\frac{\Pi_\tau}{\Pi} E - \int_{S^1} e^{\rho - 2\tau} \rho_\tau J \, d\theta \right) - \left(1 - \frac{\Pi_\tau}{\Pi} \right) \Lambda \right. \\
& \left. + \frac{1}{2} e^{-2\tau} \int_{S^1} e^{2(V-\tau)+\rho} \left(Q_\tau^2 - e^{2(\tau-\rho)} Q_\theta^2 \right) (V - \langle V \rangle) \, d\theta - \langle V_\tau \rangle \left\langle \frac{1}{2} e^{\rho - 2\tau} V_\tau \right\rangle \right| \\
\lesssim & e^\tau \Pi E F + \widetilde{F} \\
=& e^\tau H \frac{\Pi E}{H} F + \widetilde{F} \\
\lesssim & e^\tau H (1 + C\epsilon^{1/2}) F + \widetilde{F}.
\end{aligned}$$

The quantities F and $\widetilde{F}$ are the nonnegative quantities defined in equations (6.1) and (6.2). We then integrate to obtain the following lower bound

$$e^{s_0} H(s_0) - \int_{s_0}^{s} \widetilde{F} \, d\tau - (1 + C\epsilon^{1/2}) \int_{s_0}^{s} e^\tau H(\tau) F \, d\tau \lesssim e^s H(s)$$

and upper bound

$$e^s H(s) \lesssim e^{s_0} H(s_0) + \int_{s_0}^{s} \widetilde{F} \, d\tau + (1 + C\epsilon^{1/2}) \int_{s_0}^{s} e^\tau H(\tau) F \, d\tau$$

and use the integral version of Grönwall's inequality (note that the quantity $e^{s_0} H(s_0) + \int_{s_0}^{s} \widetilde{F} \, d\tau$ is nondecreasing in s) to obtain

$$\left(e^{s_0} H(s_0) - \int_{s_0}^{s} \widetilde{F} \, d\tau \right) \exp \left[-(1 + C\epsilon^{1/2}) \int_{s_0}^{s} F \, d\tau \right] \lesssim e^s H(s), \tag{6.8}$$

and

$$e^s H(s) \lesssim \left(e^{s_0} H(s_0) + \int_{s_0}^{s} \widetilde{F}\, d\tau\right) \exp\left[(1 + C\epsilon^{1/2}) \int_{s_0}^{s} F\, d\tau\right]. \tag{6.9}$$

What we want, then, is for $\int_{s_0}^{s} \widetilde{F}\, d\tau$ to be bounded in terms of s_0 and ϵ, and for $\int_{s_0}^{s} F\, d\tau \to 0$ as $\epsilon \to 0$. Recall that we have assumed $|A| < 1$ and note that

$$\begin{aligned}\widetilde{F} =& |A| \left(e^{-\tau} (\Pi + \Pi_\tau) + \Pi E^{1/2}\right)\\ \lesssim& \left(e^{-\tau} \left(\epsilon^{1/2} e^{\tau/2} + \epsilon^{1/2} e^{\tau/2}\right) + \epsilon^{3/4} e^{-\tau/4}\right)\\ \lesssim& \epsilon^{1/2} e^{-\tau/4}.\end{aligned}$$

so

$$\left|\int_{s_0}^{\infty} \widetilde{F}\, d\tau\right| \lesssim \epsilon^{1/2} e^{-\frac{s_0}{4}}. \tag{6.10}$$

So let $U_\epsilon := e^{s_0} H(s_0) + C\epsilon^{1/2} e^{-\frac{s_0}{4}}$ (C is the constant associated to the $\lesssim$ in inequality (6.10), and does not depend on the solution) and take $s_0 > 4 \log\left(\frac{4C}{\sqrt{\epsilon}}\right)$ and $|H(s_0)| < \frac{\epsilon}{2} e^{-s_0}$. Similarly, let $L_\epsilon := e^{s_0} H(s_0) - C\epsilon^{1/2} e^{-\frac{s_0}{4}}$. Then it is obvious that

$$e^{s_0} H(s_0) - C\epsilon^{1/2} e^{-\frac{s_0}{4}} \leq e^{s_0} H(s_0) \leq e^{s_0} H(s_0) + C\epsilon^{1/2} e^{-\frac{s_0}{4}}$$

and by the choice of s_0, we have

$$\begin{aligned} 0 <& e^{s_0} H(s_0) - \frac{\epsilon}{4} < e^{s_0} H(s_0) - C\epsilon^{1/2} e^{-\frac{s_0}{4}}\\ & e^{s_0} H(s_0) + C\epsilon^{1/2} e^{-\frac{s_0}{4}} < \frac{\epsilon}{2} + \frac{\epsilon}{4} < \epsilon.\end{aligned}$$

It is clear that there is an open set of data satisfying these inequalities.

Inequalities (6.8) and (6.9) become

$$L_\epsilon \exp\left[-(1+C\epsilon^{1/2})\int_{s_0}^{s} F\,d\tau\right] \lesssim e^s H(s) \lesssim U_\epsilon \exp\left[(1+C\epsilon^{1/2})\int_{s_0}^{s} F\,d\tau\right].$$

Now we turn to the bound on F.

$$\begin{aligned}
F &= \int_{S^1} e^{\rho-\tau}\rho_\tau J\,d\theta + e^{-\tau}\left(1+e^{2(\Pi E)^{1/2}}\right)(\Pi+\Pi_\tau) + (\Pi E)^{1/2} + e^{2(\Pi E)^{1/2}}\Pi E^{1/2} \\
&\lesssim e^{-3\tau}\int_{S^1} e^{\rho+l+2\tau} J\,d\theta + e^{-\tau}(\Pi+\Pi_\tau) + (\Pi E)^{1/2} + \Pi E^{1/2} \\
&\lesssim e^{-3\tau}Y_\tau + \epsilon^{1/2}e^{-\tau/2} + \epsilon^{1/2}e^{-\tau/2} + \epsilon^{1/2}e^{-\tau/2} + \epsilon^{3/4}e^{-\tau/4} \\
&\lesssim e^{-3\tau}Y_\tau + \epsilon^{1/2}e^{-\tau/4}.
\end{aligned}$$

We have previously bounded the integral of the latter term in time by $C\epsilon^{1/2}e^{-\frac{s_0}{4}}$, so it remains to compute

$$\int_{s_0}^{\infty} e^{-3\tau}Y_\tau\,d\tau \lesssim e^{-\frac{s_0}{2}}\epsilon^{1/2}.$$

So

$$\int_{s_0}^{\infty} F\,d\tau \lesssim \epsilon^{1/2}e^{-\frac{s_0}{4}}.$$

Thus, in total for H, we have

$$L_\epsilon\frac{2}{3} < L_\epsilon \exp\left[-(1+C\epsilon^{1/2})\int_{s_0}^{s} F\,d\tau\right] \lesssim e^s H(s) \lesssim U_\epsilon \exp\left[(1+C\epsilon^{1/2})\int_{s_0}^{s} F\,d\tau\right] \le U_\epsilon\frac{3}{2}$$

for sufficiently small ϵ.

6.4. Bounds on Π, Y

At this stage, we must improve the bounds on c and d. Let us determine what the smallness assumptions of Lemma 5 imply for the error term of the ODE system of Chapter V. Recall the conclusion of Proposition 3: if $H > 0$, then

$$\left|\begin{pmatrix} c \\ d \end{pmatrix}\right|(s) \lesssim e^{(s_0-s)/4}\left(\frac{e^{s_0}H(s_0)}{e^s H(s)}\right)^{1/2}\left|\begin{pmatrix} c \\ d \end{pmatrix}\right|(s_0) + \int_{s_0}^{s} e^{(\tau-s)/4}\left(\frac{e^{\tau}H(\tau)}{e^s H(s)}\right)^{1/2}|\omega(\tau)|\,d\tau, \tag{6.11}$$

where

$$\begin{aligned}|\omega| \lesssim \left|\widetilde{\Omega}\right| = \Bigg| &-\frac{1}{2}\partial_\tau \log\left(e^\tau H\right)\begin{pmatrix} \frac{2}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{pmatrix} \\ &+ \begin{pmatrix} 0 \\ \frac{f(d,c)}{\left(c+\frac{2}{\sqrt{10}}\right)^2} + \left(\frac{\Pi E}{H}-1\right)\frac{d+\frac{1}{\sqrt{10}}}{\left(c+\frac{2}{\sqrt{10}}\right)^2} + \frac{\Omega}{e^{3\tau}H^{1/2}} \end{pmatrix}\Bigg|\end{aligned}$$

and

$$\left|e^{-3\tau}H^{-1/2}\Omega\right| \lesssim e^{-2\tau}|H|^{-1/2}EY_\tau.$$

To begin with, note that $e^\tau H(\tau)$ has both upper and lower bounds, and so both terms of the form $\left(\frac{e^\tau H(\tau)}{e^s H(s)}\right)^{1/2}$ can be bounded above in terms of L_ϵ and U_ϵ:

$$\begin{aligned}\left|\begin{pmatrix} c \\ d \end{pmatrix}\right|(s) &\lesssim e^{(s_0-s)/4}\left|\begin{pmatrix} c \\ d \end{pmatrix}\right|(s_0) + \int_{s_0}^{s} e^{(\tau-s)/4}|\omega(\tau)|\,d\tau \\ &\lesssim \epsilon + \int_{s_0}^{s} e^{(\tau-s)/4}|\omega(\tau)|\,d\tau.\end{aligned} \tag{6.12}$$

The contribution to the right side of (6.11) from the error term $e^{-3\tau}H^{-1/2}\Omega$ is

$$
\begin{aligned}
\int_{s_0}^{s} e^{(\tau-s)/4}\left|e^{-3\tau}H^{-1/2}\Omega\right| d\tau \lesssim & e^{-s/4}\int_{s_0}^{s} e^{-7\tau/4}\left|H^{-1/2}\Pi^{-1}\right|\Pi E Y_\tau\, d\tau \\
= & e^{-s/4}\int_{s_0}^{s} e^{-11\tau/4}\left|\frac{\Pi E}{H}\frac{1}{c+\frac{2}{\sqrt{10}}}\right| Y_\tau\, d\tau \\
\lesssim & e^{-s/4}\int_{s_0}^{s} e^{-11\tau/4}\left|\frac{\Pi E}{H}Y_\tau\right| d\tau \\
\lesssim & e^{-s/4}\int_{s_0}^{s} e^{-3\tau/4-2\tau}Y_\tau\, d\tau \\
\lesssim & e^{-s/2}\epsilon^{1/2}
\end{aligned}
$$

where we have used the fact that $\frac{e^{\tau}H^{-1/2}}{c+\frac{2}{\sqrt{10}}} = \Pi^{-1}$ and the bootstrap assumptions.

The contribution from $\left(\frac{\Pi E}{H}-1\right)\frac{d+\frac{1}{\sqrt{10}}}{\left(c+\frac{2}{\sqrt{10}}\right)^2}$ is

$$
\begin{aligned}
\int_{s_0}^{s} e^{(\tau-s)/4}\left|\left(\frac{\Pi E}{H}-1\right)\frac{d+\frac{1}{\sqrt{10}}}{\left(c+\frac{2}{\sqrt{10}}\right)^2}\right| d\tau \lesssim & e^{-s/4}\int_{s_0}^{s} e^{\tau/4}\left|\left(\frac{\Pi E}{H}-1\right)\right| d\tau \\
\lesssim & e^{-s/4}\int_{s_0}^{s} e^{\tau/4}\epsilon^{1/2}e^{-\tau/2}\, d\tau \\
\lesssim & \epsilon^{1/2}e^{-s/4}\int_{s_0}^{s} e^{-\tau/4}\, d\tau \\
\lesssim & \epsilon^{1/2}e^{-s/2}
\end{aligned}
$$

where we have used inequality (6.7).

Turning to $\frac{f(d,c)}{\left(c+\frac{2}{\sqrt{10}}\right)^2}$, we recall that f has vanishing linear part, so

$$
\int_{s_0}^{s} e^{(\tau-s)/4}\left|\frac{f(d,c)}{\left(c+\frac{2}{\sqrt{10}}\right)^2}\right| d\tau \lesssim \epsilon^{1/2}\left(4-4e^{\frac{s0-s}{4}}\right) \lesssim \epsilon^{1/2}e^{-s/4}
$$

To bound $\partial_\tau \log(e^\tau H)$, note that $e^\tau H$ is uniformly bounded away from 0, and use the estimates on F and $\widetilde{F}$ obtained above to compute

$$\begin{aligned}
|\partial_\tau \log(e^\tau H)| =& \frac{1}{e^\tau H} |\partial_\tau (e^\tau H)| \\
\lesssim& \frac{1}{e^\tau H} \left(e^\tau H \frac{\Pi E}{H} F + \widetilde{F} \right) \\
=& \frac{\Pi E}{H} F + \frac{1}{e^\tau H} \widetilde{F} \\
\lesssim& F + \widetilde{F} \\
\lesssim& \epsilon^{1/2} e^{-\tau/4} + e^{-3\tau} Y_\tau .
\end{aligned}$$

So the contribution to (6.11) is

$$\int_{s_0}^{s} e^{(\tau - s)/4} \left| C_{s_0} \left(\epsilon^{1/2} e^{-\tau/4} + e^{-3\tau} Y_\tau \right) \right| d\tau \lesssim \epsilon^{1/2} e^{-s/2}.$$

Combining these estimates, we have from inequality (6.12) that

$$\begin{aligned}
\left| \begin{pmatrix} c \\ d \end{pmatrix} \right| (s) \lesssim& e^{(s_0 - s)/4} \left| \begin{pmatrix} c \\ d \end{pmatrix} \right| (s_0) + \int_{s_0}^{s} e^{(\tau - s)/4} |\omega(\tau)| \, d\tau \\
\lesssim& C_{s_0} \epsilon e^{-s/4} + \epsilon^{1/2} e^{-s/4} \\
\lesssim& \epsilon^{1/2} e^{-s/4} \\
\lesssim& \epsilon^{1/4} .
\end{aligned} \tag{6.13}$$

Thus we have improved all of the bootstrap inequalities, and the proof is complete.

CHAPTER VII

ASYMPTOTIC BEHAVIOR

This chapter is joint work with Jim Isenberg and Beverly K Berger.

We are now in a position to present the B_0 version of the main result of [LS16]. The primary difference is that the fine grained asymptotics of V and its mean are lost. Forthcoming work will describe the behavior of V and Q, and the dependence of that behavior on the conserved quantity B. Given our estimates above, the proof of the theorem is nearly identical to the polarized case.

Theorem 1. *Let $C_1 > 0$ and $\tau_{min} > 0$ be fixed. There exists an ϵ_0 such that if $0 \le \epsilon \le \epsilon_0$ and $s_0 \ge \tau_{min}$, for any B_0 initial data set satisfying the smallness conditions of Lemma 5, the associated solution satisfies for $\tau \in [s_0, \infty)$*

$$\left| e^\tau H - C_\infty^2 \right| \lesssim e^{-\tau/4} \tag{7.1}$$

$$\left| \Pi - \frac{2}{\sqrt{10}} C_\infty e^{\tau/2} \right| \lesssim e^{\tau/4} \tag{7.2}$$

$$\left| Y - \frac{1}{\sqrt{10}} C_\infty e^{5\tau/2} \right| \lesssim e^{9\tau/4} \tag{7.3}$$

$$\left| E - \frac{\sqrt{10}}{2} C_\infty e^{-3\tau/2} \right| \lesssim e^{-7\tau/4} \tag{7.4}$$

$$\left| \langle l \rangle - l \right| \lesssim e^{-\tau/2} \tag{7.5}$$

$$\left| e^l - 1 \right| \lesssim e^{-\tau/4} \tag{7.6}$$

$$\left| \Pi^{-1} e^\rho - e^{\rho_\infty} \right| \lesssim e^{-\tau/2} \tag{7.7}$$

$$\left| H - \frac{4}{\sqrt{10}} C_\infty e^{\tau/2} e^{\rho_\infty} \right| \lesssim e^{\tau/4} \tag{7.8}$$

for some $C_\infty > 0$ and $\rho_\infty \colon S^1 \to \mathbb{R}$.

Proof. The proof proceeds as in [LS16]. We may, for example, notice that (6.13) implies that $|c|, |d| \lesssim e^{-\tau/4}$. Similarly, we may apply the bounds we have found on $F, \widetilde{F}$ to (6.8) and (6.9) to find that

$$\left|e^{\tau}H - C_{\infty}^{2}\right| \lesssim e^{-\tau/4}$$

for some $C_{\infty} > 0$, giving (7.1). Combining this with the bound on c, d and rearranging yields (7.2) and (7.3).

Recall that $H = \Pi(E + \Lambda)$ and

$$|\Lambda| \lesssim e^{-2\tau}|A| + e^{-\tau}\left(1 + e^{2(\Pi E)^{1/2}}\right)\Pi E \lesssim e^{-2\tau}.$$

Then combine (7.1) and (7.2) to obtain (7.4). The estimate (7.5) follows from (4.6) and (7.4).

Once we know that l converges pointwise to a constant, to estimate e^{l} let us note that

$$e^{-2\tau}Y = \Pi\langle e^{l}\rangle + \left(\int_{S^1} e^{\rho+l}\,d\theta - \Pi\langle e^{l}\rangle\right).$$

We may then estimate the error term as usual, and combine the estimates on Y, Π to arrive at (7.6).

For (7.7), we begin by noting that inequality (4.6) implies that

$$|\rho_{\tau\theta}| = \left|e^{l}l_{\theta}\right| \leq e^{-3\tau}\left|e^{\rho+l+2\tau}J\right| = e^{-3\tau}Y_{\tau}$$

which is integrable over $S^1 \times [\tau_{\min}, \infty)$. Thus ρ converges to some function ρ_{∞} as $\tau \to \infty$, and we may apply the Poincaré inequality to obtain (7.7).

Finally, (7.8) follows directly from the definition of c and bounds (7.2) and (7.7).

$\square$

CHAPTER VIII

NUMERICAL EVIDENCE AND CONJECTURES

This chapter is joint work with Jim Isenberg and Beverly K Berger.

The full Einstein flow is a large, quasilinear system of partial differential equations about which it is difficult even to make conjectures. This remains true even in the simplified T^2-symmetric case considered in this work. It has been crucial to this work to base our conjectures on evidence garnered from numerical simulations of T^2-symmetric Einstein flows.

Our code is a reimplementation of code previously developed by Berger to simulate T^2-symmetric spacetimes in the contracting direction, and then later in the expanding direction. We reimplemented this code in OCaml, and made a number of modifications to improve the accuracy and speed. Most importantly, we developed code to produce solutions of the T^2-symmetric constraint equation via a random process, which allowed us to probe the behavior of generic T^2-symmetric Einstein flows.

8.1. Numerical Methodology

The code consists of two parts which work only for non-Gowdy T^2-symmetric solutions in the coordinates used here. However, the codes work equally well in in the polarized, B_0, and $B \neq 0$ cases. With some trivial modifications, the code can be made to evolve Gowdy solutions, although we do not present any such simulations here. The two basically independent parts of the code are

1. code to randomly generate initial data which solves the constraint equations and
2. code to evolve an initial data set according to equations (1.1), (1.2), (1.4), and (1.5).

We note, however, that it is preferable to formulate both the constraints and the evolution equations in the form used in [BIW01].

Solving the Constraints Numerically

This code more or less accomplishes a division, but via a spectral decomposition of the functions. The key is to note that the constraint equation in the coordinates of [BIW01] is

$$\pi_P P_\theta + \pi_Q Q_\theta + \pi_\lambda \lambda_\theta = 0. \tag{8.1}$$

This can be thought of as a linear equation if one chooses the free data correctly. We make the ansatz the functions which constitute initial data, $P, \pi_P, Q, \pi_Q, \lambda$ and π_λ, are approximated by Fourier series of some finite order k. In this case, the constraint (8.1) becomes a system of quadratic algebraic equations in the Fourier coefficients. One then chooses some coefficients to be free data, which we choose according to a uniform distribution on the box $[-C, C]^m$. One could in principle use another distribution; in our tests, the exponential terms in the evolution equations lead to numerical overflow for initial data which are too large. Thus, we experimentally determined a value of C which generates data that is numerically stable on the timescales we can simulate.

If we choose the coefficients correctly, after they are determined what was an underdetermined quadratic system of equations becomes a linear system with precisely one solution. We then call BLAS/LAPACK to solve the system, and recover the Fourier coefficients.

This method of solving the constraints introduces two limitations to our simulations. First, the probability distribution we use to generate the free data could be chosen differently. We have already mentioned that this limitation is essentially a matter of numerical stability. Second, our solutions have finite Fourier order. For linear equations, this is a strong restriction; for a recent illustration of the limitation, see [Rin17] where the behavior of solutions with finite order Fourier expansions differs qualitatively from generic solutions. Since the flow is quasilinear in the case under consideration and we solve the system by finite differences, the solution has infinite support in Fourier space for all times after the initial time. Thus we do not think this is a strong restriction in our case.

Evolving the Solution

Given a solution $P, \pi_P, Q, \pi_Q, \lambda, \pi_\lambda$ satisfying (8.1), we implement a finite difference scheme to evolve the solution forward in time. We represent S^1 by a finite number of points $\{\theta_i\}_{i=1}^n$. In practice, we have most often chosen $n \in \{2^8, 2^9, \cdots, 2^{12}\}$. All of the plots in Figures 1 through 3 were generated with $n = 2^{10}$. Our numerical scheme is a reimplementation of a code in FORTRAN written by Berger. This code uses finite differences which are centered in space and 4th order accurate. In time, the integrator uses the Iterated Crank Nicolson scheme with exactly 3 iterations [Teu00]. We observe that the scheme has good convergence properties; as n increases, the errors decrease at exactly the expected rate.

To obtain confidence that our simulations depict behavior which is generic for the class under consideration, we simulated on the order of 20 randomly chosen initial constraints solutions in each of the following classes: polarized, B_0, and generic T^2-symmetric. The qualitative behavior depicted in Figures 1 through 3 is observed to be the same for all simulations in that class. That is, for example, all B_0 solutions have energies E_V and E_Q which converge to constants.

8.2. Prototypical Simulations

For a choice of initial data and spatial resolution, the code produces a set of points $\{(\theta_i, \tau_i)\}$ and the functions $V, V_\tau, Q, Q_\tau, l, \rho$ at each of these points. From these functions of two variables, one can generate various scalar functions of time which reveal qualities of interest of the flow. Below, we will be interested in the following four quantities:

$$\mathscr{L} := \int_{S^1} l\, e^{\rho-\tau/2} d\theta, \quad \mathscr{R} := \int_{S^1} \rho\, e^{\rho-\tau/2} d\theta,$$
$$E_V := \int_{S^1} \left[V_\tau^2 + e^{2(\tau-\rho)} V_\theta^2\right] e^{\rho-\tau/2} d\theta, \quad E_Q := \int_{S^1} e^{2(V-\tau)} \left(Q_\tau^2 + e^{2(\tau-\rho)} Q_\theta^2\right) e^{\rho-\tau/2} d\theta,$$
$$\mathscr{V} := \int_{S^1} V_\tau\, e^{\rho-\tau/2} d\theta$$

First, note that the volume form $e^{\rho-\tau/2}d\theta$ is used to smooth out the graphs (the integrals will generally oscillate without this normalization).

In Figures 1 through 3, we plot three numerical simulations; one each of polarised, B_0, and fully generic (subject to the restrictions inherent in our numerical method).

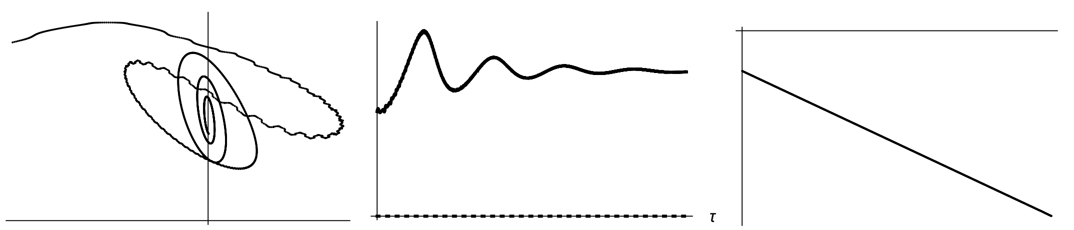

FIGURE 1 A typical polarised solution.

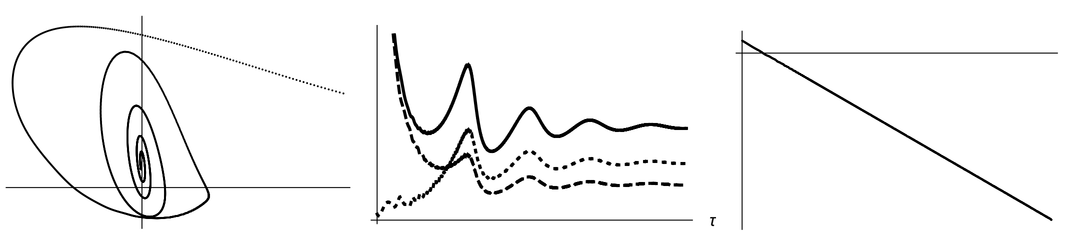

FIGURE 2 A typical B_0 solution.

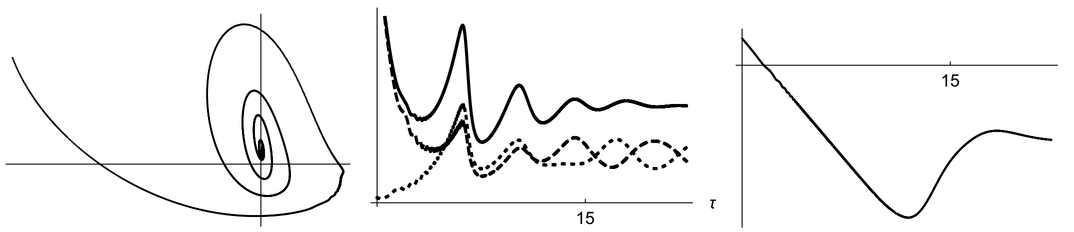

FIGURE 3 A typical generic solution.

On the left is a plot of $\mathscr{L}_\tau$ on the horizontal axis and $\mathscr{R}_\tau$ on the vertical axis. One can see the rotational sink which is the basis for the local stability proof. It turns out that Π and Y satisfy better algebraic identities, and these are the quantities used in the proof. Note that, once the asymptotic regime is reached, all three types of non-Gowdy solutions demonstrate similar behavior.

In the center plot, we have graphed E_V, E_Q and $E_V + E_Q$ against τ on the horizontal axis. In the polarized case, $E_Q = 0$. Note in the polarized and B_0 cases, all three quantities achieve a limit as $\tau \to \infty$. In the generic case, however, the total energy converges (to the same constant that it converges to in the other two cases) but E_V and E_Q do not. These quantities "slosh" energy as $\tau \to \infty$, and the amplitude of the sloshing does not decay. The period of the sloshing is exactly the period of the rotational sink depicted in the left plot.

This sloshing turns out to lead to an instability in the asymptotics of V. In the plot on the right is the expression

$$\log |\mathscr{V}| = \log \left| \int_{S^1} V_\tau \, e^{\rho - \tau/2} d\theta \right|$$

plotted against τ on the horizontal axis. While $\mathscr{V} \to 0$ exponentially for B_0 solutions (and so for polarised solutions as well), $\mathscr{V}$ converges to a nonzero constant generically. This feature has not been observed by other authors, and formed the largest impediment to our proof of an analogue of Theorem 1 in the generic T^2-symmetric case.

8.3. A Conjecture Based on the Simulations

In [LS16], the authors are able to determine the first order behavior of the energy and Π, but also the first order behavior of V and the rate of its decay to the mean value. We have so far been unable to derive estimates for V and Q in the B_0 case. From the numerics, however, it is clear that there is are constants a, C_V such that

$$\left| \int_{S^1} V \, d\theta - V \right| = O(e^{-\tau/2}), \quad |V - C_V \tau - a| = O(e^{-\tau/2})$$

and that

$$C_V = \begin{cases} 0 & \text{if } B = 0 \\ -\frac{1}{2} & \text{if } B \neq 0 \end{cases} .$$

Other conjectures will be the subject of forthcoming work.

APPENDIX

CONCORDANCE OF NOTATIONS BETWEEN [BCIM97], [BIW01], [Rin15], AND [LS16]

This chapter is joint work with Jim Isenberg and Beverly K Berger.

The Einstein flows under consideration in the this work have been studied extensively, including many important special subsets of solutions. Unfortunately, authors have used many different coordinates for exactly the same set of spacetimes, and this document adds yet another set of coordinates. As an aid to the reader who wishes to read the cited works together, we provide in this appendix a concordance of notations used in the most frequently cited of these works.

To the best of our knowledge, all of the works in the table rely on the foliation and equations derived in [BCIM97]. This dissertation, [BCIM97], [BIW01] and [Rin15] use coordinates for T^2-symmetric Einstein flows which are completely general. The analysis in [LS16] only applies to polarized T^2-symmetric Einstein flows, and so relies on the assumption that some metric components vanish identically. In [Rin04], future asymptotics of Gowdy solutions are derived. The notation used there is exactly the notation of [Rin15] if one imposes the conditions $\alpha \equiv 1, K = 0$ so we omit it from the table.

In the table below, each column uses the notation internal to the document named in the first row. All of the expressions in a given row are equal. For example, the function called P in [Rin15] has the expression $2U - \log R$ in [LS16]. Since [LS16] only deals with polarized flows, the expressions in this column will only be equal to those in other documents if the polarization condition is imposed.

[BCIM97]	[BIW01]	[Rin15]	[LS16]	this document
$\log t$	$-\tau$	$\log t$	$\log R$	τ
$2U$	$P-\tau$	$P+\log t$	$2U$	V
$\alpha^{-1/2}$	$2\pi_\lambda$	$\alpha^{-1/2}$	a^{-1}	e^ρ
$\frac{K^2}{2}t^{-2}\alpha e^{2\nu}$	$\frac{K^2}{2}e^{P+\frac{1}{2}\lambda+3\tau/2}$	$\frac{K^2}{2}t^{-3/2}e^{P+\frac{1}{2}\lambda}$	$\frac{K^2}{2}R^{-2}e^{2\eta}$	e^l
$2tU_t$	$1-\dfrac{\pi_P}{2\pi_\lambda}$	tP_t+1	$2RU_R$	V_τ
$t^{-1}\int_{S^1}\alpha^{-1/2}e^{4U}Q_t\,d\theta$	$-\int_{S^1}\pi_Q\,d\theta$	$\int_{S^1}\alpha^{-1/2}e^{2P}tQ_t\,d\theta$	0	$\int_{S^1}e^{\rho+2(V-\tau)}Q_\tau\,d\theta =: B$

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