

the violinist, has now verified the leading mathematical theories of instrument vibration. In fact, the game style, fast swells and oscillations are found in the patterns with all violin tunings and can confirm my research" and we will now proceed, and therefore when instrument tuning is corrected this further work today's conditions have been resolved by themselves. It will apparently however, work whether violin's are tuned exactly the same pitch or just a certain degree. It is an interesting question how often we will find a small difference in

THE MATHEMATICS OF THE VIOLIN

By Nina Hyatt
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Instrument tuning is a very interesting field of study. When playing a violin, one finds that the sound is not always consistent. This is due to the various "partial" pitches that are present in the tuning system. These partials are called "overtones" and are produced by the instrument when it is being played. In this paper I will discuss the mathematics behind these overtones.

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For centuries, men have admired the mellow, resonant tones of Stradivari violins. In fact, the late 17th and early 18th centuries are known as the "Golden Age" of violins because of the skills of luthiers¹ such as Amati, Stradivari, and Guarneri, whose instruments possess a resonant tone quality which today's craftsmen have been unable to reproduce. It is apparent, however, that earlier models of the violin did not meet with as great a success. It is on rare occasion that one may find a modern luthier attempting to imitate the style of a pre-17th century instrument, and perhaps as rare is the occasion when we hear one played. Indeed, if we give credence to the words of men who were witness to the quality of the violin's precursors¹, perhaps there is little reason to revive interest in the early forms:

--Now the musicians
Hover with nimble sticks o'er squeaking crowds
Tickling the dried guts of a mewing cat.

The fiddler's croud now squeaks aloud,
His fidlinge strings begin to trole;
He loves a wake and a wedding cake,³
A bride-house and a brave May-pole.

¹ For this and other italicized technical terms, see Glossary.

² William Sandys, F.S.A. and Simon Andrew Forster, The History of the Violin (London: John Russel Smith, 1864) 31, as quoted from Marston's "What You Will."

³ Sandys 31, as quoted from "Cupid's Banishment," 1617.

I' th' head of all this warlike rabble
 Crowdero march'd expert and able,
 A squeaking engine he applied
 Unto his neck on north-east side,
 His warped ear hung o'er the strings,
 Which was but souse to chitterlings.
 His grizly beard was long and thick,
 With which he strung his fiddle-stick;
 For he to horse-tail scorn'd to owe.⁴
 For what on his own chin did grow.

Scientists and luthiers alike have been attempting to discover the secrets of Stradivarius and his contemporaries since the masters themselves were alive. What is it, then, that makes the "Golden Age" violins so special, and why have modern luthiers been unable to produce instruments of comparable quality? Is there, in fact, a violin model which is theoretically superior to today's model (i.e. the model which originated with the Golden Age masters), and if so, why did the rapid evolution of the violin species come to a rather abrupt halt during the 17th century? We can not, of course, expect to answer these questions fully; however, considering that the development of various models seems to have been the result of trial and error on the part of the luthiers, a mathematical approach may provide some new insight.

The violin, as a vibrational system, poses some unique, and rather complicated mathematical problems which most of its researchers either steer clear of, or treat in a very generalized manner. Because of the many connected vibrating bodies (for example, the strings, the bridge, the top and

⁴ Sandys 32, as quoted from Hudibras.



back plates, the soundpost, the bass bar, the tailpiece, and the ribs) which form the violin, and consequently have a role in its sound production, we must, in this initial work, narrow our considerations to those characteristics which seem to be most influential in sound production. We shall reserve consideration of additional characteristics for later works, in which our mathematical description of the instrument may be expanded upon to include these additional variables. Thus, we consider this work to be a foundation for further study, and therefore hope to provide the reader with some understanding of how the violin works, and how mathematics can be applied to describe both the instrument and the tone which it produces.

In order to determine which characteristics we should examine in this work, we would do well to consider the violin's history and development, both in order to clearly define the term "violin", and then to shed light on which of its characteristics developed for practical or decorative purposes, and which developed because of their contribution to the instrument's tone. We may then use our chosen characteristics in a mathematical discussion of the violin as a system of vibrations which produces sound. We will also be able to consider specific instruments from our historical discussion in light of this mathematical analysis, and discuss the possibility of finding a theoretically "superior" model through mathematical



comparison of various models and the frequencies at which they vibrate best.

HISTORY

The exact origins of the violin are somewhat clouded. Most of the records of the violin's precursors are to be found in artists' renditions (in drawings, paintings, or literature) of the instruments, because few of the actual instruments still exist. A great number of sources are available today, in which the authors have researched these older instruments. Though individual artists may not have been dogmatic about their portrayal of the instrument (in many instances artists omit small but essential characteristics), there are enough representations for historians to get a fairly accurate idea of the nature of some of the violin's early forms. The features of the violin's inner structure, including the bass bar, soundpost, and endblocks are, however, left unmentioned in most cases, as they are not observed by the common viewer. Thus, since few of the early instruments are still in existence, we are often left to our own hypotheses as to whether or not they existed in a particular model. Historians differ in their views as to which of the numerous stringed instruments were the actual ancestors of the violin, and which were merely similar in appearance, but non-related in origin. For our purposes, however, "lineage" is not as important a

consideration as physical characteristics. Therefore, we will rely on these authors principally for their physical descriptions of early models of stringed instruments, and the approximate time period during which they were made. We may then select specific instruments (and measurements), or specific characteristics of a class of instruments for closer examination and comparison to today's violin model. We will pay special attention to the size, shape, and tunings of these instruments because these characteristics are important factors in determining the nature of the mathematical equations which we will use in later sections. Size and shape determine the boundary conditions which limit our equations, and tunings provide us with frequency ranges of instruments which we can compare to the frequencies in our mathematical solutions (see Appendix C for note/frequency tables⁵). All physical aspects, however influence the vibrations of the instrument to some extent, and are therefore important to a mathematical analysis of the instrument; thus we will also consider other characteristics such as the bridge, soundpost, bass bar, tailpiece, and sound holes in various instruments. We will first examine the modern violin in order to provide a basis for comparison of earlier instruments.

⁵ Tables from Robert C. Weast, ed., CRC Handbook of Chemistry and Physics, 63rd ed. (Boca Raton: CRC Press, Inc., 1982-83) E-45.



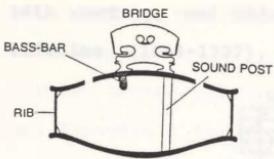


Figure 1

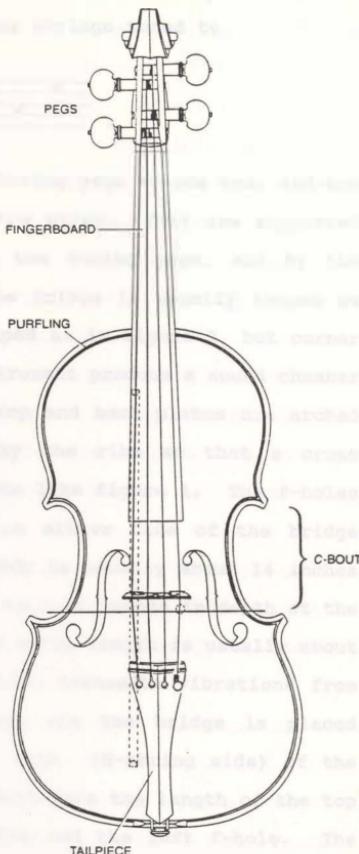
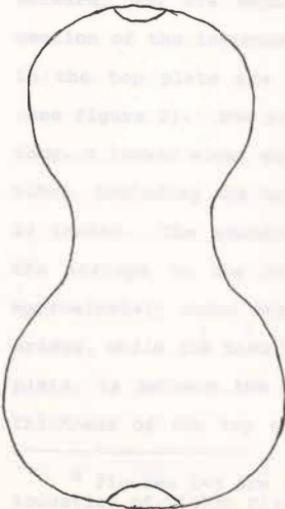
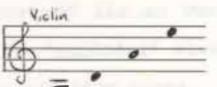


Figure 2



The violin, which first appeared under that name in the 16th century, and which Stradivarius perfected during his lifetime (1644-1737), has four strings tuned to



The strings are wound around tuning pegs at one end, and are attached to the endpiece at the other. They are supported by the nut at the end near the tuning pegs, and by the bridge near the tailpiece. The bridge is usually shaped as in figure 1. The body is shaped as in figure 2, but corner and end blocks inside the instrument produce a sound chamber shaped like figure 3.⁶ The top and back plates are arched outward, and are separated by the ribs so that a cross section of the instrument looks like figure 1. The f-holes in the top plate are found on either side of the bridge (see figure 2). The violin body is usually about 14 inches long, 9 inches wide, and 1.18 to 1.26 inches in depth at the ribs. Including the neck, the total length is usually about 23 inches. The soundpost, which transmits vibrations from the strings to the back plate via the bridge is placed approximately under the right foot (E-string side) of the bridge, while the bass bar, which runs the length of the top plate, is between the left foot and the left f-hole. The thickness of the top and back plates decreases from center

⁶ Figures 1-3 are from Carleen Maley Hutchins, "The Acoustics of Violin Plates," Scientific American October 1981: 174.



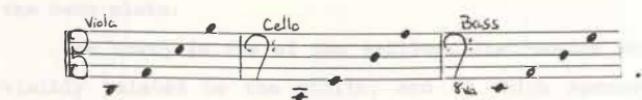
to outer edge, but exact specifications vary, depending on the maker's preference. Appendix A contains an example of a thicknessing diagram which was kindly provided by M. le luthier Arnaud Camurat of Aix en Provence, France. Also, see Appendix E for an "exploded" view⁷ of a violin showing the placement of the interior parts. The violin is made of light woods, which promote elasticity, but are also resistant enough to withstand the pressure exerted by the tightened strings. Fir, pine, or spruce is usually used for the top plate, the bass bar, and the sound post. Maple is used for the back plate, the ribs, and the neck. According to M. Camurat, the back plate thus amplifies and projects the vibrations received from the top plate through the ribs and soundpost.⁸ Ebony is used for the fingerboard, the nut, the tuning pegs, and the tailpiece, because it is very hard and resistant, and is not worn down by the rubbing of the strings, as other woods would be. Its dark color also prevents dirt from the player's fingers or the rubbing of the strings from being too visible. The bridge is usually made of spotted maple.

The violin family, today, includes instruments from four different size categories, of which the violin is the treble, or smallest instrument, the viola is the alto, the cello (violoncello) is the tenor or baritone, and the bass

⁷ from Carleen Maley Hutchins, "The Physics of Violins," Scientific American November 1962: 80

⁸ Arnaud Camurat, letter to the author, 22 January 1988.

violin is the bass. The viola, cello, and bass are tuned, respectively, as follows:



The viola is played under the chin like the violin, and is only about two inches longer than the violin. It is also interesting to note that the bass violin is tuned in fourths, instead of fifths like the violin, viola and cello. The bass also has the sloping shoulders of the viol family.

This model of the violin has remained unchanged since the time of Stradivari except in a few minor details such as the lengthening and angling of the neck and fingerboard, the heightening of the bridge, and the reinforcement of the bass bar and soundpost, which were brought about when the tuning "A" was raised to a frequency of 440 cycles per second (during the 19th century). These changes produced instruments with brighter tones, which are better able to balance the brass and percussion in orchestral playing, and which carry more easily in large auditoriums.

In order to keep from digressing too far, we will direct most of our attention toward those instruments which produce sound in the same manner as the modern violin--that is, those which are bowed, and which have a neck, against which the fingers of the left hand hold the strings in order to produce pitches between (or above) the open strings. The instruments must also have one or more sound holes, some

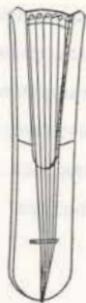


type of bridge, and preferably, a sound post, or some sort of structure which transmits vibrations from the bridge to the back plate.

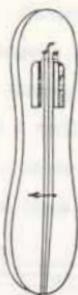
The crwth is one of the earliest instruments which is visibly related to the violin, and of which accounts are consistent and detailed enough to be of some use. This instrument first appeared during the Middle Ages, and remained in use until about 1800 (see timeline in Appendix D for relative lifespans of early instruments). The crwth was also known, depending on locale and time period, as the rotta, crot, cruit, crud, crudh, crowd, crowde, chorus, or crouth. In its earliest forms, the crwth had from three to six strings (usually five), with no neck, and was held on the lap and plucked. It was usually solid with no resonant chamber. Its shape and size varied widely. By the 13th century (though possibly earlier according to some sources), soundholes and a neck and fingerboard had been added, and the instrument was played with a bow. The size, shape, and placement of the soundholes, however, remained inconsistent.

But the latest model of the crwth, which was predominant from the 13th to the 18th centuries, is the most interesting as relates to our discussion. It possesses all of the "defining characteristics" of the violin (i.e. those which are most influential in the sound production of the violin) except, perhaps, the bass bar, but its body is of a very different shape (see figures A and B for illustrations

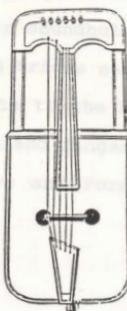
Figure A



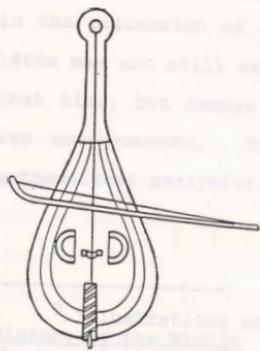
German
Rotta: 5th and
7th Century.



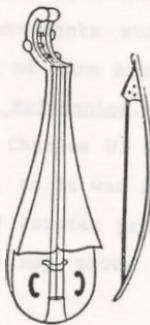
French
Crouth (Rotta):
11th Century.



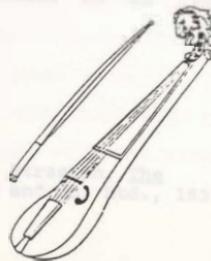
Welsh Crwth
18th Century.



Lyra: 13th Century



Rebe: 16th
Century, from Kastner's
"Danse des Morts"
(1852).



Italian Giga: 15th Century



of the crwth and other pre-17th-century instruments⁹). The body of the instrument is shaped like a rectangular box with rounded lower corners. It has two circular soundholes, one on each side of the bridge. It has a tailpiece, a neck and fingerboard, and six strings which, according to Sandys and Forster,¹⁰ are "stated to have been thus tuned



Two of the strings extend to the left of the fingerboard, and were, thus, probably used as drones, and not fingered.

A few of these instruments still exist today, for example one in the Museum of Fine Arts in Boston, which is pictured in Encyclopaedia Britannica.¹¹ A crwth which was in the possession of one Charles W. G. Wynne, Esq. in the 1860s may not still exist, as it was in a poor condition at that time, but Sandys and Forster provide us with some of its measurements. These may prove useful in our later mathematical analysis:

⁹ Illustrations are from E. van der Straeten, The History of the Violin (London: Cassell and Co. Ltd., 1933) Chapters 1, 2, 4, and 5.

¹⁰ Sandys 34.

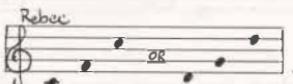
¹¹ "Crwth," Encyclopaedia Britannica, 1980 ed.



length=22in., width=9.5in., maximum depth=2in.,
length of fingerboard=10.25in.¹²

The instrument had no bridge, tail piece, or strings at the time, but six tuning pegs were still intact.

The rebec (originally called the rubebe) developed in Europe in about the 11th century from the similar Arab rebab, which was introduced in Spain with Muslim culture. Many historians also believe the medieval lyra to have had some influence on its development. It had a shallow body shaped like a pear half, but unlike the skin rebab, the rebec had a wooden belly and a fingerboard. A tailpiece, tuning pegs, and a bridge were also features of the rebec, as well as sound holes, which were, however, inconsistent in size and shape. The instrument was played with a bow, and had three strings. According to Straeten, the pitches of the strings varied around the following two tunings:



He states that "the rule was to pull up the first string as high as it would bear without breaking and then tune the two others from it in accordance with the given intervals."¹³ Use of the rebec, in its various forms, continued into the early 18th century, but diminished greatly with the advent of the 16th century viols. Because of its harsh, dry tone,

12 Sandys 35.

13 Straeten 8.

the rebec was used primarily for dance music, especially at fairs or in taverns. It was rarely used by minstrels of a higher order, such as those at court.

By the late 15th century, the rebec had undergone some minor modifications and improvements. Its newer form became known as the gigue (French) or giga (Italian) from the German "geige", which began in the 12th century as their name for the rebec, was applied indiscriminantly to all bowed instruments in the 16th and 17th centuries, and from the 18th century onward, was used exclusively for the violin. This modified rebec was constructed in three different sizes, and usually had three strings tuned in fifths, though the Italian models often had four. The pitches of the strings were:



The gigue had a more developed scroll than the rebec, and the French and Italian forms were usually more slender in outline. Like the rebec, the gigue had no frets, but possessed a bridge, a fingerboard, a tailpiece, and soundholes. The kit, pochette, and Taschengeige were all small versions of the rebec, which lasted into the 18th century.

The name lira (or lyra) was first applied to a Byzantine instrument much like the rebec in the 9th century.

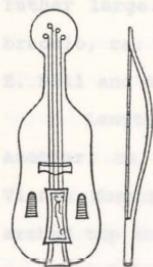


This instrument was pear-shaped, and had a bridge, a tailpiece, a rude fingerboard, and two semi-circular sound holes. It usually had from three to five melody strings, and one or two *bourdons*. The lira had no scroll, however, its head being flat with tuning pegs set from front to back. The evolution of this instrument is unclear, and it seems the name lira was often used interchangeably during the 11th and 12th centuries with the word fiddle. By the 15th century, however, an instrument had evolved which claimed the name as its own. This instrument came in two sizes: the treble, or *lira da braccio*, and the bass, or *lira da gamba* (also called the *lirone*). Like the medieval fiddle (and also the early lira), the *lira da braccio* had no scroll. Its tuning pegs ran front to back in a flat peg disc. Also like its older namesake, the *lira da braccio* usually had from three to five melody strings and two *bourdons*. Other than these details, however, the *lira da braccio* was much like today's violin model. It had a bridge, a tailpiece, a fingerboard, and two sound holes, which were usually f- or c- shaped. It was one of the first instruments to possess the shallow ribs of the violin rather than the rounded pear-shaped body. It was also wider in proportion to its length compared to many of the earlier instruments, and possessed the middle *bouts* which characterize the violin's shape, though they were somewhat shallower on the lira. Unlike the violin, the lira (especially in later models) had an incurvation at the



lower and on the side of the bow. Low bow introduced to facilitate bowing which was slow development, as it was

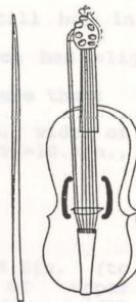
rather large. Stradivari's first dimensions of a lira da braccio, or fiddle, which he had in the possession of Cesare de' Medici and was at Siena in 1600; now in the collection of the Accademia Nazionale dei Lincei.



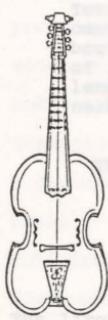
Reinmar's
Fiddle with Bow:
13th Century.



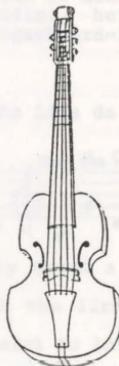
Italian Fidel:
1500.



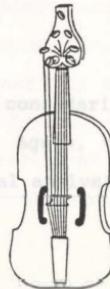
Lyra da
Braccio and Bow:
1499.



Pareia's
Viol: 1482.
Note holes and pegs
for six strings.



Viol of 1542



Lyra da Braccio
1505.

14 Stradivari 31.

15 Stradivari 32.

16 Stradivari 33.



lower end of the body, which may have been introduced to facilitate the chin's grip on the instrument, as it was rather large. Straeten gives the dimensions of a lira da braccio, ca. 1540, which was in the possession of Messrs. W. E. Hill and Sons of London in 1933:

Length of body=15.48in., Width=9.4in.¹⁴

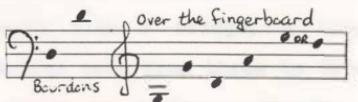
Another, ca. 1580, which was (and may still be) in the Vienna Museum of old instruments, and which had slightly arched top and back plates, he found to measure thus:

Total length=28in., length of body=19in., width of upper bouts=9in., width of lower bouts=10.6in., and height of ribs=1.6in.¹⁵

A lirone from the Heyer collection measured:

Total length=36.8in., length of body=23.6in. (to center of incurvation), width of upper bouts=12.6in., width of middle bouts=9.2in., width of lower bouts=16in., height of ribs=2.32in., length of fingerboard=12.4in., length of neck=6.4in.¹⁶

The tuning of the lira da braccio was usually



The lirone was probably tuned a third lower, considering a size ratio of 5:4 with the lira da braccio. Again, these measurements can be useful to later mathematical analysis as

¹⁴ Straeten 31.

¹⁵ Straeten 32.

¹⁶ Straeten 32.



boundary conditions, and tunings can be useful for comparing frequencies of theoretical models to those produced by actual models.

Also known as the fiedel (German) or vielle (French), the medieval fiddle first appeared in 10th century Europe, and may have been derived from the lira of that time. Such an origin would account for the similarity of the two instruments which led to the interchangeable use of the names "fiddle" (or "fiedel" or "vielle") and "lira" for several centuries. The word fiddle is still often used as a generic term for any bowed instrument with a neck. Instruments which seem to have fallen under the title "fiddle" most often are those with a relatively long neck in proportion to the body, some sort of incurvations on the sides, though usually not as developed as those of the liras and viols, and some sort of sound holes (not usually f- or c-shaped--shapes vary between instruments, and seem to play a decorative as well as functional role). Fiddles are also usually portrayed with a bridge, a fingerboard, and a tailpiece. Strings were from three to five in number, and were usually tuned in fifths. Straeten mentions three different tunings¹⁷ which were used for five-stringed fiddles, and one for the four-stringed "guitar-fiddle" (so named because its shape was like that of the ancient Egyptian guitar):

¹⁷ Straeten 13, 15.



These are obviously tunings for the larger, or bass version of the fiddle. It is interesting to note, however, that the strings are of only four different pitch classes--namely, C, G, D, and A--the same pitches to which we tune viola and cello strings today. The guitar-fiddle, which usually had four strings, lasted longest, continuing to be in use until the middle of the 15th century.

The viol developed during the 15th century, and shared many characteristics with the guitar-fiddle. The bridge, fingerboard, and tailpiece remained much the same on the viol as they had been on the fiddle and the lira. The middle bouts, however, were deepened and given outer corners, and the shoulders were more sloping, somewhat reminiscent of the pear shape of the rebec, though the back plate was flat. It was also on the viol that the sound holes first took the "f" shape that they have today, and the scroll first appeared in today's form.

The viol, like the lira, was divided into two classes: the viols da gamba, and the viols da braccio. The early viols, however, had treble, tenor, and bass in the "da gamba" class, and treble and alto in the "da braccio" class, whereas the lira had only one of each class. Instruments of the viol family were usually fretted, unlike the violin, and had five or six strings (usually six), which were all played over the fingerboard. The viols were much prized for



the clarity of tone which the gut frets provided, giving each note the quality of an open string. Straeten gives a number of different tunings for viols. Older bass viols were tuned



The older treble instruments were tuned to the same pitches an octave higher. The Perfect Viols, a class of viols which came into existence during the second half of the 15th century, were constructed in four sizes, and tuned thus:



Straeten also gives the tuning and measurements of a viol da gamba made by Hans Vohar (probably around 1550) which was owned by a Mr. Arnold Dolmetsch:



Total length=34.4in., length of body=19.2in., width of upper bouts=9.0in., width of lower bouts=10.6in., and height of ribs=3.4in.¹⁸

¹⁸ Straeten 26.



The treble and alto viols da braccio were superseded in the late 16th century (except for a smaller version of the treble instrument used in France until the 18th century) by the viols da gamba, but later re-emerged with modifications as the violin and viola. The viols da gamba had deeper ribs, and produced a more powerful tone in the register to which they were tuned. During the 17th century, however, the viols da gamba were, in turn, superseded in popularity by the violin and viola. Establishment of orchestras in the early 17th century favored the violin family because of their stronger tone, and only the bass version of the viol lasted into the 18th century. The viols were revived at the end of the 19th century, however, to play music that had been written for them in the past.

There have been, of course, many instruments made over the centuries which we have not considered. Even instruments which, by name, fall under one of our seven categories may have had very different characteristics than those which we described. Variations in body shape were especially frequent, along with changes in size, shape, and placement and number of f-holes. The instruments which we have chosen to discuss are those which appear most often in both art and literature of their era. These seem to have been the most common and the most long-lived, and thus probably had the best tone quality of those made. The amount of information on these models (especially dimensions) also allows us to treat them mathematically,



whereas we are provided with only the shape (from works of art) or the name (from literature) of many lesser used models.

THE VIOLIN AS A VIBRATIONAL SYSTEM

The violin is made up of a number of parts which, when linked together, form a very complex vibrational system. The vibrations begin at the point of contact between the bow and the string (or the finger and the string if the instrument is being plucked), but the string alone is not the producer of the tone we hear. Each element of the vibrational system influences the vibrations so that what we hear is a combination of the sound waves which were most enhanced (or least damped) during their "journey" through the violin. The frequency at which the string is vibrating is most prominent, but it inspires sympathetic vibrations in other parts of the instrument which add overtones to the sound which we hear. The loudness of the sound (i.e. the amplitude of the waves) is also greatly increased by the linkage of the violin's parts. A bowed string attached to a rigid support at each end, for example, produces a much weaker tone than the assembled violin because its surface area is too small to have an appreciable affect on the air around it. As Hutchins put it, "Trying to make music with an unamplified string would be like trying to fan oneself



with a toothpick."¹⁹ Let us, then, follow the path of a vibration through the violin. We will first ignore the effects of the coupling in order to simplify the system, and then we will discuss complications which coupling introduces. For this initial analysis, therefore, we will consider vibrations in one direction as they travel from string to ear.

The bow influences the string through a combination of sticking and sliding friction, first displacing the string, and then releasing it when the force of the string's tension overcomes the force of the friction between the string and the bow. Thus the effect of bowing on the string is much like the effect of extremely rapid *pizzicato* (it is, of course, much more rapid than the human finger could reproduce). Thus, when we consider the mathematics of this system in this initial work, we will, for simplicity's sake, consider the behavior of a string plucked only once. It is important to note, however, that bowing not only provides a constant stimulus to the string, but also propagates the vibration of the entire system through time. The rapid stimulus from the bow lessens the decay of the vibration between stimuli, which is caused by the friction and resistance of various parts of the violin, and which occurs to the greatest extent when the string is plucked only once. Thus, because of the effects of coupling, which we will discuss later, bowing allows the system to vibrate at more

¹⁹ Hutchins, "The Physics of Violins," 83.



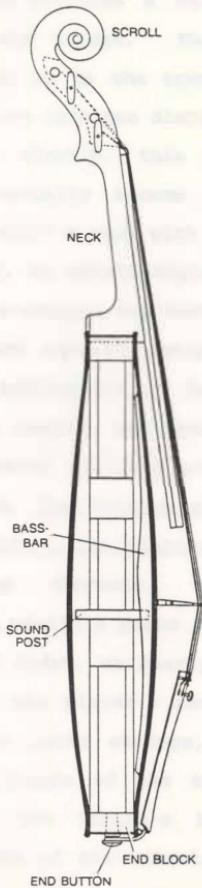
frequencies than does pizzicato, and thus produces a fuller tone (i.e. a tone with more overtones).

The string is wound around a tuning peg at one end but, for all vibrational considerations, it ends where it crosses the nut, as its tension causes it to behave as if fixed at the point of contact. At the other end, the string crosses over the bridge, and is attached at the tailpiece (see figures 2 and 4), which is attached to the end of the violin's body. The open strings of the violin have frequencies of approximately 196, 294, 440, and 659 cycles per second. The viola, which is of similar construction, but is slightly larger in dimension, is tuned to frequencies of approximately 131, 196, 294, and 440 cycles per second. The fingers, as they are placed on the string, raise the frequency of vibration by making the string, and thus its wavelength shorter.

The vibrations of the string cause three types of motion in the bridge. The twisting motion, and the vertical flexing of the upper bridge are negligible in their additions to the sound we hear because they do not reach the resonant chamber (the legs of the bridge are held by friction to the top plate, and slide very little), and are therefore not amplified. These two components may, however, redirect certain vibrations, and thus prevent some frequencies from being amplified. But they are, in any case, only minor components of the bridge's motion in comparison with the third type, which is rocking. The main



Figure 4



component of the string's motion is from side to side. This component causes the bridge's feet to move, alternately, downward, and thus produces a rocking motion around the central axis of the bridge. The main function of the bridge, thus, seems to be the translation of the string's horizontal vibrations into two distinct vertical components. As we will find shortly, this translation allows the vibrations to eventually become three-dimensional sound waves, which can "fill" a room with sound.

At this point, we should digress momentarily from our vibrational path to examine two features of the violin which are unknown to most concert goers and music lovers, but which are highly influential in determining the nature of the violin's tone, namely, the bass bar and the soundpost. Cremer discusses these features and their effects at some length in his work, The Physics of the Violin. Hyacinth Abele, in The Violin: Its History and Construction, also provides a very thorough, though non-mathematical description of the violin's parts and their functions. The bass bar is placed under, or nearly under the left foot of the bridge (from the player's perspective--i.e. the foot closest to the two lower strings, "G" and "D"). It is connected to the inside of the top plate, and runs the entire length of the violin's body, thus damping the lengthwise component of the vibrations. This damping, in effect, reduces the number of overtones which the top plate can produce because it limits certain modes of vibration.

Because of its comparative thickness and rigidity, the bass bar also strengthens the lateral vibrations, which would normally diminish toward the ends of the body, by providing a stimulus which has equal (or nearly equal) intensity along the entire length of the plate. The soundpost, which is placed under the right foot (closest to the "A" and "E" strings) of the bridge provides a direct stimulus to the back plate. The soundpost itself is too short and thick to vibrate in the frequencies at which the rest of the violin vibrates, thus it has no direct influence on the configuration of the waves which it transmits. It acts, for our purposes, as a rigid structure which causes the back plate to vibrate in synchrony with the right foot of the bridge and part of the top plate. Thus the bass bar and soundpost, along with the f-holes, divide the plates into two regions which vibrate differently. The first region, which includes the left half of the top plate, and the portion around the outside of the right f-hole (see figure 2), receives its principal stimulus from the left foot of the bridge. The second region, which receives its principal stimulus from the right foot of the bridge, includes a small area between the longitudinal center of the plate and the right f-hole, which Cremer calls the "island", and the back plate. The f-holes here serve not only to allow more flexibility in the top plate, but also to help separate the two out of phase vibrational regions. Thus, the periodic motion of the feet caused by the rocking of the bridge



transmits the vibrations to the two plates. It is interesting to note, here, that when one foot is at its highest point the other foot is at its lowest point.

The top and back plates are also connected around the instrument's outer edges by the ribs. In our work, we will use the ribs as a frame of reference, considering them to be fixed while the other parts of the instrument and the surrounding air vibrate with respect to the ribs.

Besides moving the air outside the violin, the non-synchronous vibration of the plates causes air pressure changes inside the resonant cavity. These changes in pressure cause periodic disturbances in the air which exit through the soundholes. These disturbances are more commonly known as sound waves.

THE MATHEMATICS OF THE VIOLIN'S MAJOR VIBRATING BODIES

Having outlined the path of the vibrations from the string through the bridge to the plates, and then to the air, we can see that the vibrations caused by the bowing or plucking of a violin string can be divided into three main categories. The vibrations begin at the string in one dimension, the bridge translates that motion into the two-dimensional vibration of the plates, and the out of phase motion of the plates causes the three-dimensional vibration of the air. Neglecting air friction, we can thus describe



these three types of motion using the wave equation in one, two, and three dimensions, that is

$$\rho \frac{\partial^2}{\partial t^2} \Phi = \nabla \cdot [\tau \nabla \Phi] ,$$

where Φ is dependent on one, two, or three variables, and describes the displacement or change in pressure of each point at time t , ρ = mass density of the vibrating medium, and τ = tension or pressure at a given point. The solutions to this equation are derived in different ways, depending on the shape of the vibrating body. For example, in two dimensions, different coordinate systems are used for circles and rectangles because points on circles are more easily described in terms of radius (r) and angle of elevation (θ), while rectangles are most easily considered in terms of width and length (x and y).

In order to begin with a relatively uncomplicated system, let us consider a violin shaped somewhat like the crwth, which we discussed in an earlier section. Our model is, thus, a rectangular box with plates and strings of constant thickness and density. Let us also assume that plate and string tensions are constant. This model will allow us to do some fundamental calculations which may be helpful in later research when added variables and more intricate shapes increase the complexity of the problem. Under these conditions, our general equation takes the form

$$\rho \frac{\partial^2}{\partial t^2} \Phi = \tau \nabla^2 \Phi .$$



Thus, to describe the string, plates, and air respectively, we get

$$\frac{\rho}{\tau} \frac{\delta^2}{\delta t^2} \Phi(x, t) = \frac{\delta^2}{\delta x^2} \Phi(x, t),$$

$$\frac{\rho}{\tau} \frac{\delta^2}{\delta t^2} \Phi(x, y, t) = \frac{\delta^2}{\delta x^2} \Phi(x, y, t) + \frac{\delta^2}{\delta y^2} \Phi(x, y, t),$$

$$\frac{\rho}{\tau} \frac{\delta^2}{\delta t^2} \Phi(x, y, z, t) = \frac{\delta^2}{\delta x^2} \Phi(x, y, z, t) + \frac{\delta^2}{\delta y^2} \Phi(x, y, z, t) + \frac{\delta^2}{\delta z^2} \Phi(x, y, z, t).$$

Note that rho and tau take on different values in each dimensional case, depending on the qualities of the vibrating medium. To simplify our calculations slightly, we let $a = \sqrt{\frac{\tau}{\rho}}$.

It is important to note, however, that there are minor vibrational components which are not included in these equations, but which we have chosen to neglect for the time being for the sake of simplicity and practicality. The vertical vibrations of the string are one example of such an omission. They are a small component of the string's motion, and their influence on the plates is much diminished due to the bridge's flexibility, but their inclusion would make our calculations substantially more complicated.

We will, in this work, show the general solutions to the wave equation in one, two, and three dimensions. Particular solutions depend on initial conditions and specific properties (shape, size, density, tension, etc.) of the vibrating medium. We will discuss some particular solutions in a later section.



The solution to the wave equation in one dimension describes the principal motion of the vibrating string. We want to find all functions $\Psi(x)$ and $T(t)$ such that

$$\Phi(x,t) = \Psi(x) T(t)$$

and

$$\frac{1}{a^2} \frac{\delta^2}{\delta t^2} \Phi(x,t) = \frac{\delta^2}{\delta x^2} \Phi(x,t) .$$

Using *, we find

$$\frac{\delta^2}{\delta t^2} \Phi(x,t) = \Psi(x) T''(t) ,$$

$$\frac{\delta^2}{\delta x^2} \Phi(x,t) = T(t) \Psi''(x) .$$

And substituting into **, we get

$$\frac{1}{a^2} \Psi(x) T''(t) = T(t) \Psi''(x) ,$$

$$\frac{1}{a^2} T'' \frac{(t)}{T(t)} = \frac{\Psi''(x)}{\Psi(x)} .$$

Since the left side now depends only on t , and the right side depends only on x , and since t and x are independant variables, we have that

$$\frac{1}{a^2} T'' \frac{(t)}{T(t)} = \frac{\Psi''(x)}{\Psi(x)} = -\lambda ,$$

where $-\lambda$ is a constant.

Using

$$\frac{1}{a^2} \frac{T''(t)}{T(t)} = -\lambda ,$$

we get

$$T''(t) = -\lambda a^2 T(t) .$$

In other words, using the differential operator $D^* = \frac{d^2}{dt^2}$,

$$D^2 T = -\lambda a^2 T$$



$$(D^2 + \lambda a^2)T = 0$$

$$(D + ai\sqrt{\lambda})_s(D - ai\sqrt{\lambda})T = 0$$

Now letting

$$(D - ai\sqrt{\lambda})T = u$$

we get

$$(D + ai\sqrt{\lambda})u = 0$$

$$e^{ai\sqrt{\lambda}t}Du + e^{ai\sqrt{\lambda}t}ai\sqrt{\lambda}u = 0$$

$$(e^{ai\sqrt{\lambda}t}u)' = 0$$

$$e^{ai\sqrt{\lambda}t}u = c_1$$

$$u = c_1 e^{-ai\sqrt{\lambda}t}$$

where c_1 is a constant.

Substituting back for u , we get

$$(D - ai\sqrt{\lambda})T = c_1 e^{-ai\sqrt{\lambda}t}$$

$$e^{-ai\sqrt{\lambda}t}DT - e^{-ai\sqrt{\lambda}t}ai\sqrt{\lambda}T = (e^{-ai\sqrt{\lambda}t}T)' = c_1 e^{-2ai\sqrt{\lambda}t}$$

$$e^{-ai\sqrt{\lambda}t}T = \int c_1 e^{-2ai\sqrt{\lambda}t} = \frac{c_1}{-2ai\sqrt{\lambda}} e^{-2ai\sqrt{\lambda}t} + c_2$$

$$T = \frac{c_1}{-2ai\sqrt{\lambda}} e^{-ai\sqrt{\lambda}t} + c_2 e^{ai\sqrt{\lambda}t}$$

But $\frac{c_1}{-2ai\sqrt{\lambda}}$ only exists for non-zero values of λ , thus

$$T(t) = C_1 e^{-ai\sqrt{\lambda}t} + C_2 e^{ai\sqrt{\lambda}t}$$

where C_1 and C_2 are constants and λ is not equal to zero.

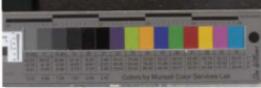
For $\lambda = 0$, we get

$$T''(t) = 0$$

from our original equation. Integrating twice gives us the general solution

$$T(t) = B_1 + B_2 t$$

where B_1 and B_2 are constants.



Thus, using the definition $e^{ix} = \cos(x) + i\sin(x)$, and $e^{-ix} = \cos(x) - i\sin(x)$, our solution takes the form

$$T(t) = \begin{cases} K_1 \cos(a\sqrt{\lambda}t) + K_2 \sin(a\sqrt{\lambda}t) & \text{for } \lambda > 0 \\ B_1 + B_2 t & \text{for } \lambda = 0 \\ C_2 e^{a\sqrt{\mu}t} + C_3 e^{-a\sqrt{\mu}t} & \text{for } \lambda < 0 \end{cases}$$

Now using

$$\frac{\Psi''(x)}{\Psi(x)} = -\lambda$$

and following the same procedure, we get

$$\Psi(x) = \begin{cases} K_3 \cos(\sqrt{\lambda}x) + K_4 \sin(\sqrt{\lambda}x) & \text{for } \lambda > 0 \\ B_3 + B_4 x & \text{for } \lambda = 0 \\ C_3 e^{\sqrt{\mu}x} + C_4 e^{-\sqrt{\mu}x} & \text{for } \lambda < 0 \end{cases}$$

Using boundary conditions, we can restrict our family of solutions to those which apply to our specific situation. Since the string is fixed at both ends,

$$\Phi(0, t) = \Phi(l_s, t) = 0$$

$$\Phi(0, t) = 0 \rightarrow \Psi(0) T(0) = 0$$

$$\Phi(l_s, t) = 0 \rightarrow \Psi(l_s) T(t) = 0 ,$$

where l_s is the length of the string.

Assuming neither Ψ or T are identically zero, we can restate our boundary conditions

$$\Psi(0) = \Psi(l_s) = 0 .$$

Then using $\Psi(x)$ from above, we get

$$0 = \Psi(0) = \begin{cases} K_3 & \text{for } \lambda > 0 \\ B_3 & \text{for } \lambda = 0 \\ C_3 + C_4 & \text{for } \lambda < 0 \end{cases}$$

$$0 = \Psi(l_s) = \begin{cases} K_4 \sin(\sqrt{\lambda}l_s) & \text{for } \lambda > 0 \\ B_4 l_s & \text{for } \lambda = 0 \\ C_3 e^{\sqrt{\mu}l_s} - C_4 e^{-\sqrt{\mu}l_s} & \text{for } \lambda < 0 \end{cases}$$



The solution to the wave equation in two dimensions describes the potential energy of a membrane plate.

We can eliminate $\lambda=0$ from consideration because

$$B_4 l_s = 0 \rightarrow B_4 = 0 \rightarrow \Psi(x) = 0.$$

We also eliminate the solution for negative values of λ because

$$0 = C_3(e^{\sqrt{|\lambda|}l_s} - e^{-\sqrt{|\lambda|}l_s}) \rightarrow C_3 = 0 \rightarrow \Psi(x) = 0,$$

and so we are left with positive values for λ , and

$$0 = K_4 \sin(\sqrt{\lambda} l_s)$$

$$K_4 = 0 \rightarrow \Psi(x) \equiv 0 \text{ therefore } K_4 \neq 0$$

$$0 = \sin(\sqrt{\lambda} l_s)$$

$$\sqrt{\lambda} l_s = n\pi \quad n = 0, 1, 2, \dots$$

$$\text{Thus } \lambda = \left(\frac{n\pi}{l_s}\right)^2$$

$$\Psi(x) = K_4 \sin\left(\frac{n\pi}{l_s}x\right),$$

and since we have positive λ , we get

$$T(t) = K_1 \cos(a\sqrt{\lambda}t) + K_2 \sin(a\sqrt{\lambda}t).$$

Therefore, we have solutions for Φ which look like

$$\Phi(x, t) = \sum_{n=0}^{\infty} K_n \sin\left(\frac{n\pi}{l_s}x\right) [K_1 \cos(a\frac{n\pi}{l_s}t) + K_2 \sin(a\frac{n\pi}{l_s}t)],$$

and given a specific initial displacement and velocity for "initial conditions", we can easily solve for the constants K_1 , K_2 , and K_4 .



The solution to the wave equation in two dimensions describes the principal motion of a vibrating plate. We want to find $\Theta(x, y)$ and $T(t)$ such that

$$\star \quad \Phi(x, y, t) = \Theta(x, y) T(t)$$

and

$$** \quad \frac{1}{a^2} \frac{\delta^2}{\delta t^2} \Phi(x, y, t) = \frac{\delta^2}{\delta x^2} \Phi(x, y, t) + \frac{\delta^2}{\delta y^2} \Phi(x, y, t) .$$

Using *, we find

$$\begin{aligned} \frac{\delta^2}{\delta t^2} \Phi(x, y, t) &= \Theta(x, y) T''(t) \\ \frac{\delta^2}{\delta x^2} \Phi(x, y, t) &= T(t) \frac{\delta^2}{\delta x^2} \Theta(x, y) \\ \frac{\delta^2}{\delta y^2} \Phi(x, y, t) &= T(t) \frac{\delta^2}{\delta y^2} \Theta(x, y) . \end{aligned}$$

Thus, substituting into **, we get

$$\frac{1}{a^2} \Theta(x, y) T''(t) = T(t) \frac{\delta^2}{\delta x^2} \Theta(x, y) + T(t) \frac{\delta^2}{\delta y^2} \Theta(x, y)$$

$$\frac{1}{a^2} \frac{T''(t)}{T(t)} = \frac{\frac{\delta^2}{\delta x^2} \Theta(x, y) + \frac{\delta^2}{\delta y^2} \Theta(x, y)}{\Theta(x, y)} .$$

Since the left side now depends only on t , and the right side depends only on x and y , and since t , x , and y are independant variables, we have that

$$\frac{1}{a^2} \frac{T''(t)}{T(t)} = \frac{\frac{\delta^2}{\delta x^2} \Theta(x, y) + \frac{\delta^2}{\delta y^2} \Theta(x, y)}{\Theta(x, y)} = -\lambda ,$$

where $-\lambda$ is constant.

Using the left side, we get, as in the case of the vibrating string

$$T(t) = \begin{cases} K_1 \cos(a\sqrt{\lambda}t) + K_2 \sin(a\sqrt{\lambda}t) & \text{for } \lambda > 0 \\ B_1 + B_2 t & \text{for } \lambda = 0 \\ C_1 e^{a\sqrt{\lambda}t} + C_2 e^{-a\sqrt{\lambda}t} & \text{for } \lambda < 0 \end{cases} .$$



Using

$$\frac{\delta^2}{\delta x^2} \Theta(x,y) + \frac{\delta^2}{\delta y^2} \Theta(x,y) = -\lambda \Theta(x,y),$$

and letting

$$\Theta(x,y) = \Psi(x) \Gamma(y)$$

$$\frac{\delta^2}{\delta x^2} \Theta(x,y) = \Psi''(x) \Gamma(y)$$

$$\frac{\delta^2}{\delta y^2} \Theta(x,y) = \Gamma''(y) \Psi(x),$$

we get

$$\Psi''(x) \Gamma(y) + \Gamma''(y) \Psi(x) = -\lambda \Psi(x) \Gamma(y)$$

$$\frac{\Psi''(x)}{\Psi(x)} + \frac{\Gamma''(y)}{\Gamma(y)} = -\lambda$$

$$\frac{\Psi''(x)}{\Psi(x)} = -\lambda - \frac{\Gamma''(y)}{\Gamma(y)} = -d,$$

where d is a constant.

We then have two independant equations in x and y, and can solve as we did in the string problem. Thus, we get

$$\Psi(x) = \begin{cases} K_1 \cos(\sqrt{d}x) + K_2 \sin(\sqrt{d}x) & \text{for } d > 0 \\ B_3 + B_4 x & \text{for } d = 0 \\ C_1 e^{\sqrt{|d|}x} + C_2 e^{-\sqrt{|d|}x} & \text{for } d < 0 \end{cases}$$

$$\Gamma(y) = \begin{cases} K_3 \cos(\sqrt{\lambda-d}y) + K_4 \sin(\sqrt{\lambda-d}y) & \text{for } (\lambda-d) > 0 \\ B_5 + B_6 y & \text{for } (\lambda-d) = 0 \\ C_5 e^{\sqrt{\lambda-d}y} + C_6 e^{-\sqrt{\lambda-d}y} & \text{for } (\lambda-d) < 0 \end{cases}$$

Various shapes of plate will give us very different boundary conditions, thus narrowing the family of solutions is much more complicated in two dimensions than in one



dimension. Such variations also make the problem a more interesting one, however, and since shape and size of the body of stringed instruments seem to be the characteristics which varied most before the 17th century, we will deal with this two dimensional problem in greater depth in a later section. For now, then, let us simply note that the family of solutions for Φ in two dimensions depends, as it did in one dimension, on the boundary conditions, which determine the nature of Ψ , Γ , and T .

The solution to the wave equation in three dimensions describes the air pressure at a point (x, y, z) at time t . We want to find $F(x, y, z)$ and $T(t)$ such that

$$* \quad \Phi(x, y, z, t) = F(x, y, z) T(t)$$

and

$$** \quad \frac{1}{a^2} \frac{\delta^2}{\delta t^2} \Phi(x, y, z, t) = \frac{\delta^2}{\delta x^2} \Phi(x, y, z, t) + \frac{\delta^2}{\delta y^2} \Phi(x, y, z, t) + \frac{\delta^2}{\delta z^2} \Phi(x, y, z, t) .$$

From *, we get

$$\frac{\delta^2}{\delta t^2} \Phi(x, y, z, t) = F(x, y, z) T''(t) \quad \frac{\delta^2}{\delta y^2} \Phi(x, y, z, t) = \frac{\delta^2}{\delta y^2} F(x, y, z) T(t)$$

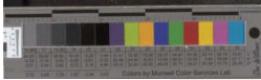
$$\frac{\delta^2}{\delta x^2} \Phi(x, y, z, t) = \frac{\delta^2}{\delta x^2} F(x, y, z) T(t) \quad \frac{\delta^2}{\delta z^2} \Phi(x, y, z, t) = \frac{\delta^2}{\delta z^2} F(x, y, z) T(t) *$$

And we can write, from **,

$$\frac{1}{a^2} F(x, y, z) T''(t) = T(t) \left[\frac{\delta^2}{\delta x^2} F(x, y, z) + \frac{\delta^2}{\delta y^2} F(x, y, z) + \frac{\delta^2}{\delta z^2} F(x, y, z) \right]$$

$$\frac{1}{a^2} \frac{T''(t)}{T(t)} = \frac{\frac{\delta^2}{\delta x^2} F(x, y, z) + \frac{\delta^2}{\delta y^2} F(x, y, z) + \frac{\delta^2}{\delta z^2} F(x, y, z)}{F(x, y, z)} *$$

As in the one- and two- dimensional problems, each side of the equation depends on different independant variables,



and we can thus set each side equal to a constant $-\lambda$.

Using the equation in t , we get the same solution as before:

$$T(t) = \begin{cases} K_1 \cos(a\sqrt{\lambda}t) + K_2 \sin(a\sqrt{\lambda}t) & \text{for } \lambda > 0 \\ B_1 + B_2 t & \text{for } \lambda = 0 \\ C_1 e^{a\sqrt{\lambda}t} + C_2 e^{-a\sqrt{\lambda}t} & \text{for } \lambda < 0 \end{cases}$$

Using the equation in x, y , and z , and letting

$$F(x,y,z) = \Theta(x,y) Z(z) ,$$

we can solve for Θ and Z as we did in the two-dimensional case for Θ and T . Thus, we find

$$\frac{\delta^2}{\delta x^2} F(x,y,z) = \frac{\delta^2}{\delta x^2} \Theta(x,y) Z(z)$$

$$\frac{\delta^2}{\delta y^2} F(x,y,z) = \frac{\delta^2}{\delta y^2} \Theta(x,y) Z(z)$$

$$\frac{\delta^2}{\delta z^2} F(x,y,z) = \Theta(x,y) Z''(z) ,$$

and so

$$-\lambda \Theta(x,y) Z(z) = \frac{\delta^2}{\delta x^2} \Theta(x,y) Z(z) + \frac{\delta^2}{\delta y^2} \Theta(x,y) Z(z) + \Theta(x,y) Z''(z)$$

$$-\lambda - \frac{Z''(z)}{Z(z)} = \frac{\frac{\delta^2}{\delta x^2} \Theta(x,y) + \frac{\delta^2}{\delta y^2} \Theta(x,y)}{\Theta(x,y)} = -p ,$$

where $-p$ is a constant.

From the equation in z , we get

$$\frac{Z''(z)}{Z(z)} = -(\lambda - p)$$

$$Z(z) = \begin{cases} K_3 \cos(\sqrt{\lambda - p}z) + K_4 \sin(\sqrt{\lambda - p}z) & \text{for } (\lambda - p) > 0 \\ B_3 + B_4 z & \text{for } (\lambda - p) = 0 \\ C_3 e^{\sqrt{\lambda - p}z} + C_4 e^{-\sqrt{\lambda - p}z} & \text{for } (\lambda - p) < 0 \end{cases}$$

and from the equation in x and y , letting

$$\Theta(x,y) = \Psi(x) \Gamma(y) ,$$

we get



$$\frac{\delta^2}{\delta x^2} \Theta(x, y) = \Psi''(x) \Gamma(y) \quad \frac{\delta^2}{\delta y^2} \Theta(x, y) = \Psi'(x) \Gamma'(y)$$

$$\Psi''(x) \Gamma(y) + \Psi(x) \Gamma''(y) = -p \Psi(x) \Gamma(y)$$

$$\frac{\Psi''(x)}{\Psi(x)} = -p - \frac{\Gamma''(y)}{\Gamma(y)} = -d$$

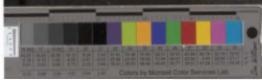
$$\Psi(x) = \begin{cases} K_s \cos(\sqrt{d}x) + K_b \sin(\sqrt{d}x) & \text{for } d > 0 \\ B_s + B_b x & \text{for } d = 0 \\ C_s e^{\sqrt{|d|}x} + C_b e^{-\sqrt{|d|}x} & \text{for } d < 0 \end{cases}$$

$$\Gamma(y) = \begin{cases} K_s \cos(\sqrt{p-d}y) + K_b \sin(\sqrt{p-d}y) & \text{for } (p-d) > 0 \\ B_s + B_b y & \text{for } (p-d) = 0 \\ C_s e^{\sqrt{p-d}y} + C_b e^{-\sqrt{p-d}y} & \text{for } (p-d) < 0 \end{cases}$$

where $-d$ is a constant.

Again, as in the two-dimensional problem, we need specific boundary conditions to narrow the family of solutions to a more manageable size, thus we need to know the exact shape and size of the instrument before we can describe the nature of the vibrations its parts produce.

Before we apply our two dimensional solutions to a specific example, let us first discuss the problems involved with describing the vibrations when one-, two-, and three-dimensional systems are linked together, as they are in the violin. As we have seen, describing the motion of each of the separate components is a fairly straightforward process. When the vibrational systems are linked, however, each exerts an influence on those to which it is connected. Though the string excites the movement of the plates initially, once the plates are moving, their vibrations also



affect the motion of the string via the bridge. A similar relationship exists between the plates and the air. Though the plates give the air its initial stimulus, the motion (or pressure changes) of the air in the sound chamber affects the vibration of the plates (and therefore also that of the strings). This linkage makes the solution of the wave equation very difficult, if not impossible, because we must solve the one-, two-, and three-dimensional equations simultaneously in order to find a solution which accounts for the influence of each of the three vibrating media on the others.

We can discuss the relationship between the string and each plate in our rectangular example with the equations

$$\frac{1}{a_s^2} \frac{\delta^2}{\delta t^2} \Phi_1(x_s, t) - \nabla^2 \Phi_1(x_s, t) = B(x_s, t)$$

$$\frac{1}{a_p^2} \frac{\delta^2}{\delta t^2} \Phi_2(x_p, y_p, t) - \nabla^2 \Phi_2(x_p, y_p, t) = 0$$

$$\Phi_2(x_{po}, y_{po}, t) = \Phi_1(x_{po}, t) ,$$

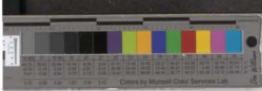
where Φ_1 describes string displacement, Φ_2 describes plate displacement, and (x_{po}, y_{po}) and (x_{so}) are the points at which the plate and string (respectively) touch the bridge, and $B(x, t)$ describes bow movement. We can, similarly, discuss the coupling between the plate and the air using the equations

$$\frac{1}{a_p^2} \frac{\delta^2}{\delta t^2} \Phi_2(x_p, y_p, t) - \nabla^2 \Phi_2(x_p, y_p, t) = A(x_p, y_p, t)$$

$$\frac{1}{a_A^2} \frac{\delta^2}{\delta t^2} \Phi_3(x_A, y_A, z_A, t) - \nabla^2 \Phi_3(x_A, y_A, z_A, t) = 0$$

$$\Phi_3(x, y, H, t) = \Phi_{1 top}(x, y, t)$$

$$\Phi_3(x, y, 0, t) = \Phi_{2 back}(x, y, t) ,$$



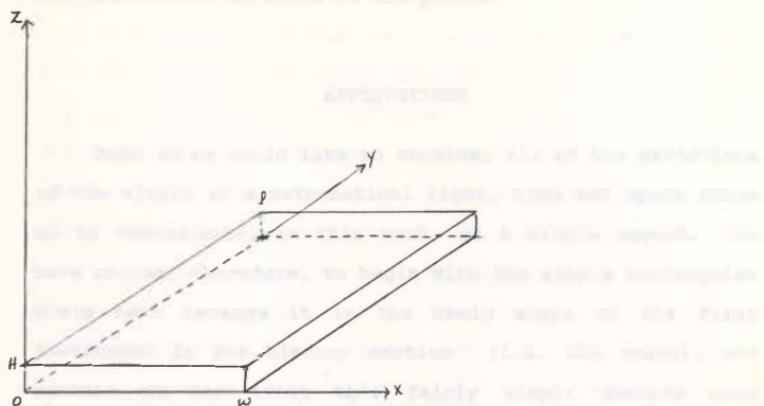


Figure 5

and keep the same conditions. we have chosen the plane as our space, take a coordinate system in the plane such that each other variable or vector to change the time quality of the situation. It will be different shaped and there will be different dimensions and different situations with other times, and different properties. The solution, which path does the car take also also depends whether the situation on the plane is a long distance or a short one, which time of the starting and the end, the position will enable to receive because the longer the distance between the two points in the plane in the same time, the greater the speed, the smaller will the time.

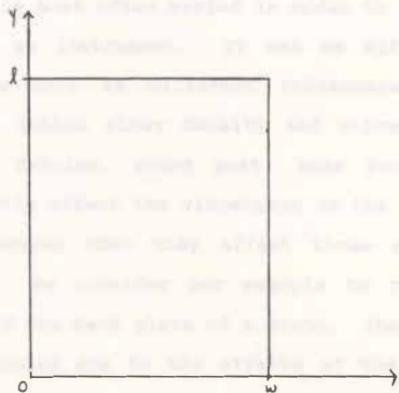


Figure 6

where H is the height of the ribs (see figure 5), and $A(x_p, y_p, t)$ describes the air pressure changes caused by the non-synchronous movement of the plates.

APPLICATIONS

Much as we would like to consider all of the variations of the violin in a mathematical light, time and space force us to concentrate, in this work, on a single aspect. We have chosen, therefore, to begin with the simple rectangular plate both because it is the basic shape of the first instrument in our history section (i.e. the crwth), and because we can treat this fairly simple example more thoroughly in the allotted time than we could a plate with more complex boundary conditions. We have chosen the plate as our aspect under consideration because it is the characteristic which is most often varied in order to change the tone quality of an instrument. It can be different shapes and sizes as well as different thicknesses and different materials (which alter density and vibrational capabilities). The f-holes, sound post, bass bar, and bridge also all directly affect the vibrations of the plates in a more obvious manner than they affect those of the string and the air. We consider our example to roughly describe the motion of the back plate of a crwth. The front plate is more complicated due to the effects of the sound holes and the bass bar. We have chosen to examine the



effects of changing the position of the soundpost (the point of stimulus) and changing the length and width of the plate.

First, recalling that we have decided to consider the ribs as fixed, we can establish the boundary conditions of our rectangular plate. In order to establish a frame of reference for our discussion, we must situate our plate in a coordinate system. Let us label the lower left corner (from the player's perspective) as the origin. We then get a system which looks like figure 6. Let us call the length of the plate (i.e. the "body length" of the instrument) "l", and the width "w". Note that both these measurements are positive numbers. Since the plate is attached to the ribs at the edges, and therefore does not vibrate with respect to the ribs at the edges, we must have

$$\begin{aligned}\Phi(0,y,t) &= 0 & \Phi(w,y,t) &= 0 \\ \Phi(x,0,t) &= 0 & \Phi(x,l,t) &= 0\end{aligned}$$

Using, from our previous calculations, that

$$\Phi(x,y,t) = \Psi(x) \Gamma(y) T(t)$$

$$\Psi(x) = \begin{cases} K_1 \cos(\sqrt{d}x) + K_2 \sin(\sqrt{d}x) & \text{for } d > 0 \\ B_1 + B_2 x & \text{for } d = 0 \\ C_1 e^{\sqrt{d}x} + C_2 e^{-\sqrt{d}x} & \text{for } d < 0 \end{cases}$$

$$T(t) = \begin{cases} K_1 \cos(a\sqrt{\lambda}t) + K_2 \sin(a\sqrt{\lambda}t) & \text{for } \lambda > 0 \\ B_1 + B_2 t & \text{for } \lambda = 0 \\ C_1 e^{a\sqrt{\lambda}t} + C_2 e^{-a\sqrt{\lambda}t} & \text{for } \lambda < 0 \end{cases}$$

$$\Gamma(y) = \begin{cases} K_5 \cos(\sqrt{\lambda-d}y) + K_6 \sin(\sqrt{\lambda-d}y) & \text{for } (\lambda-d) > 0 \\ B_5 + B_6 y & \text{for } (\lambda-d) = 0 \\ C_5 e^{\sqrt{\lambda-d}y} + C_6 e^{-\sqrt{\lambda-d}y} & \text{for } (\lambda-d) < 0 \end{cases},$$

and noting that neither Ψ , Γ , nor T can be identically zero (otherwise, we have $\Phi(x, y, z, t) = 0$, which implies that no vibration is occurring), we have

$$\Phi(0, y, t) = 0 \rightarrow \Psi(0) \Gamma(y) T(t) = 0 \rightarrow \Psi(0) = 0$$

$$\Phi(w, y, t) = 0 \rightarrow \Psi(w) \Gamma(y) T(t) = 0 \rightarrow \Psi(w) = 0$$

$$\Phi(x, 0, t) = 0 \rightarrow \Psi(x) \Gamma(0) T(t) = 0 \rightarrow \Gamma(0) = 0$$

$$\Phi(x, l, t) = 0 \rightarrow \Psi(x) \Gamma(l) T(t) = 0 \rightarrow \Gamma(l) = 0$$

Thus, we get

$$0 = \Psi(0) = \begin{cases} K_3 & \text{for } d > 0 \\ B_3 & \text{for } d = 0 \\ C_3 + C_4 & \text{for } d < 0 \end{cases}$$

$$0 = \Gamma(0) = \begin{cases} K_5 & \text{for } (\lambda-d) > 0 \\ B_5 & \text{for } (\lambda-d) = 0 \\ C_5 + C_6 & \text{for } (\lambda-d) < 0 \end{cases},$$

and we can write

$$\Psi(x) = \begin{cases} K_4 \sin(\sqrt{d}x) & \text{for } d > 0 \\ B_4 x & \text{for } d = 0 \\ C_4 (e^{\sqrt{d}x} - e^{-\sqrt{d}x}) & \text{for } d < 0 \end{cases}$$

$$\Gamma(y) = \begin{cases} K_5 \sin(\sqrt{\lambda-d}y) & \text{for } (\lambda-d) > 0 \\ B_5 y & \text{for } (\lambda-d) = 0 \\ C_5 (e^{\sqrt{\lambda-d}y} - e^{-\sqrt{\lambda-d}y}) & \text{for } (\lambda-d) < 0 \end{cases},$$

Then using the other two boundary conditions, we have



$$0 = \Psi(w) = \begin{cases} K_4 \sin(\sqrt{d}w) & \text{for } d > 0 \\ B_4 w & \text{for } d = 0 \\ C_3(e^{\sqrt{|d|}w} - e^{-\sqrt{|d|}w}) & \text{for } d < 0 \end{cases}$$

$$0 = \Gamma(l) = \begin{cases} K_6 \sin(\sqrt{\lambda-d}l) & \text{for } (\lambda-d) > 0 \\ B_6 l & \text{for } (\lambda-d) = 0 \\ C_5(e^{\sqrt{\lambda-d}l} - e^{-\sqrt{\lambda-d}l}) & \text{for } (\lambda-d) < 0 \end{cases}$$

But

$$B_4 w = 0 \rightarrow B_4 = 0 \rightarrow \Psi = 0$$

$$C_3(e^{\sqrt{d}w} - e^{-\sqrt{d}w}) = 0 \rightarrow C_3 = 0 \rightarrow \Psi = 0$$

$$B_6 l = 0 \rightarrow B_6 = 0 \rightarrow \Gamma = 0$$

$$C_5(e^{\sqrt{\lambda-d}l} - e^{-\sqrt{\lambda-d}l}) = 0 \rightarrow C_5 = 0 \rightarrow \Gamma = 0.$$

Therefore, we must reject the solutions for $d=0$, d negative, $(\lambda-d)=0$, and $(\lambda-d)$ negative, and we are left with

$$\Psi(w) = 0 = K_4 \sin(\sqrt{d}w) \quad d > 0$$

$$\Gamma(l) = 0 = K_6 \sin(\sqrt{\lambda-d}l) \quad (\lambda-d) > 0.$$

Since K_4 and K_6 can not be zero (otherwise $\Phi = 0$), we have

$$\sin(\sqrt{d}w) = 0 \rightarrow d = \left(\frac{n\pi}{w}\right)^2 \quad n = 0, 1, 2, \dots$$

$$\sin(\sqrt{\lambda-d}l) = 0 \rightarrow \lambda - d = \left(\frac{m\pi}{l}\right)^2 \rightarrow \lambda = \left(\frac{m\pi}{l}\right)^2 + \left(\frac{n\pi}{w}\right)^2 \quad m = 0, 1, 2, \dots$$

And since λ is positive, we get

$$T(t) = K_1 \cos(a \sqrt{\left(\frac{m\pi}{l}\right)^2 + \left(\frac{n\pi}{w}\right)^2} t) + K_2 \sin(a \sqrt{\left(\frac{m\pi}{l}\right)^2 + \left(\frac{n\pi}{w}\right)^2} t).$$

Thus our solution for Φ takes the form

$$\Phi(x, y, t) = \sum_{m,n} [A_{mn} \cos(a \sqrt{\left(\frac{m\pi}{l}\right)^2 + \left(\frac{n\pi}{w}\right)^2} t) + B_{mn} \sin(a \sqrt{\left(\frac{m\pi}{l}\right)^2 + \left(\frac{n\pi}{w}\right)^2} t)] \sin\left(\frac{n\pi}{w}y\right) \sin\left(\frac{m\pi}{l}x\right).$$



According to Fourier Analysis, given initial position and initial velocity

$$\Phi(x,y,0) = f(x,y) \quad \frac{\delta}{\delta t} \Phi(x,y,0) = g(x,y) ,$$

we find

$$A_{nm} = \frac{4}{wl} \iint_R f(x,y) \sin\left(\frac{n\pi}{w}\right)x \sin\left(\frac{m\pi}{l}\right)y \, dx \, dy$$

$$B_{nm} a\left(\left(\frac{m\pi}{l}\right)^2 + \left(\frac{n\pi}{w}\right)^2\right) = \frac{4}{wl} \iint_R g(x,y) \sin\left(\frac{n\pi}{w}\right)x \sin\left(\frac{m\pi}{l}\right)y \, dx \, dy .$$

In our problem, initial velocity is zero, so B_{nm} is also zero. To find A_{nm} , we must first find $f(x,y)$, and define our limits of integration. But $f(x,y)$ depends on the position of the soundpost, thus we have chosen two examples, one with the soundpost in the center of the plate, and one with the soundpost halfway between the center and the edge on the longitudinal axis of the plate. We find that each $f(x,y)$ is divided into eight regions. Integrating over the eight regions separately, and adding the results gives us A_{nm} for each example. The integral table in Appendix B may be helpful to those readers who wish to verify these calculations.

If we had a perfectly rigid bridge, "h" would be equal to the amplitude of the string's vibration. The bridge is not perfectly rigid, however, so we label the initial displacement at the point of the soundpost as "h". Note that negative numbers refer to direction with respect to the ribs, and not to magnitude, therefore "h" is a



negative number because the soundpost pushes the plate away from the center of the ribs.

With the soundpost at the center of the plate, we find

Region 1 (A and B)

$$f(y) = \frac{-2h}{l} y \quad \begin{aligned} 0 \leq y \leq \frac{l}{w} x & \text{ and } 0 \leq x \leq \frac{w}{2} \\ 0 \leq y \leq -\frac{l}{w} x + l & \text{ and } \frac{w}{2} < x \leq w \end{aligned}$$

Region 2 (A and B)

$$f(x) = \frac{-2h}{w} x \quad \begin{aligned} 0 \leq x \leq \frac{w}{l} y & \text{ and } 0 \leq y \leq \frac{l}{2} \\ 0 \leq x \leq -\frac{w}{l} y + w & \text{ and } \frac{l}{2} < y \leq l \end{aligned}$$

Region 3 (A and B)

$$f(y) = \frac{2h}{l} y - 2h \quad \begin{aligned} -\frac{l}{w} x + l \leq y \leq l & \text{ and } 0 \leq x \leq \frac{w}{2} \\ \frac{l}{w} x \leq y \leq l & \text{ and } \frac{w}{2} < x \leq w \end{aligned}$$

Region 4 (A and B)

$$f(x) = \frac{2h}{w} x - 2h \quad \begin{aligned} -\frac{w}{l} y + w \leq x \leq w & \text{ and } 0 \leq y \leq \frac{l}{2} \\ \frac{w}{l} y \leq x \leq w & \text{ and } \frac{l}{2} < y \leq l \end{aligned}$$

$$A_{mn} = 0 \quad \text{for } m \neq n$$

$$A_{nn} = \frac{8h}{n^2 \pi^2} \left[\frac{1}{2} - \frac{1}{2}(-1)^n - 1 + (-1)^n \right] \quad \text{for } m = n$$

Positioning the soundpost at $(\frac{w}{2}, \frac{l}{4})$, we get



Region 1 (A and B)

$$f(y) = \frac{-4h}{l} y \quad \begin{aligned} 0 \leq y \leq \frac{l}{2w} x \text{ and } 0 \leq x \leq \frac{w}{2} \\ 0 \leq y \leq \frac{-l}{2w} x + \frac{l}{2} \text{ and } \frac{w}{2} < x \leq w \end{aligned}$$

Region 2 (A and B)

$$f(x) = \frac{-2h}{w} x \quad \begin{aligned} 0 \leq x \leq \frac{2w}{l} y \text{ and } 0 \leq y \leq \frac{l}{4} \\ 0 \leq x \leq \frac{-2w}{3l} y + \frac{2w}{3} \text{ and } \frac{l}{4} < y \leq l \end{aligned}$$

Region 3 (A and B)

$$f(y) = \frac{4h}{3l} y - \frac{4h}{3} \quad \begin{aligned} \frac{-3l}{2w} x + l \leq y \leq l \text{ and } 0 \leq x \leq \frac{w}{2} \\ \frac{3l}{2w} x - \frac{l}{2} \leq y \leq l \text{ and } \frac{w}{2} < x \leq w \end{aligned}$$

Region 4 (A and B)

$$f(x) = \frac{2h}{w} x - 2h \quad \begin{aligned} \frac{-2w}{l} y + w \leq x \leq w \text{ and } 0 \leq y \leq \frac{l}{4} \\ \frac{2w}{3l} y + \frac{w}{3} \leq x \leq w \text{ and } \frac{l}{4} < y \leq l \end{aligned}$$

$$A_{nm} = \frac{256mh}{\pi^3 (4n^2 - m^2) (9m^2 - 4n^2)} [\sin \frac{\pi}{4}(m-2n) - \sin \frac{\pi}{4}(m+2n)] \quad \text{for } m \neq 2n \text{ and } 3m \neq 2n$$

$$A_{nm} = \frac{h}{n^2 \pi^2} [-1 + (-1)^n] \quad \text{for } m = 2n$$

$$A_{nm} = \frac{2h}{m^2 \pi^2} \left[1 - (-1)^{\frac{m}{2}} \right] \quad \text{for } 3m = 2n.$$

To exemplify the usefulness of these calculations, let us first consider a plate with the dimensions of the crwth given in our history section--that is, we will set



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$w=9\frac{1}{2}'' = 19\frac{1}{2}''$ and $l=22''$. Thus doing, we can find the frequencies (W_{nm}) at which the violin will vibrate, and their corresponding amplitude (A_{nm}) for each of our examples (see Table 1).

Table 1

Soundpost at $(w/2, l/2)$	W_{nm}	Soundpost at $(w/2, l/4)$
$A_{11}=-0.811h$	$W_{11}=0.360a$	$A_{11}=-0.778h$
$A_{12}=0$	$W_{12}=0.437a$	$A_{12}=-0.203h$
$A_{13}=0$	$W_{13}=0.541a$	$A_{13}=-0.091h$
$A_{14}=0$	$W_{14}=0.660a$	$A_{14}=-0.039h$
$A_{21}=0$	$W_{21}=0.677a$	$A_{21}=0$
$A_{22}=0$	$W_{22}=0.720a$	$A_{22}=0$
$A_{23}=0$	$W_{23}=0.788a$	$A_{23}=0$
$A_{24}=0$	$W_{24}=0.874a$	$A_{24}=0$
$A_{31}=0$	$W_{31}=1.002a$	$A_{31}=-0.012h$
$A_{32}=0$	$W_{32}=1.032a$	$A_{32}=0.027h$
$A_{33}=-0.090h$	$W_{33}=1.081a$	$A_{33}=-0.029h$
$A_{34}=0$	$W_{34}=1.145a$	$A_{34}=-0.031h$
$A_{41}=0$	$W_{41}=1.330a$	$A_{41}=0$
$A_{42}=0$	$W_{42}=1.353a$	$A_{42}=0$
$A_{43}=0$	$W_{43}=1.390a$	$A_{43}=0$
$A_{44}=0$	$W_{44}=1.441a$	$A_{44}=0$

For an optimal violin model, we would want the strongest frequencies (i.e. the frequencies with the greatest amplitude) to fall within the pitch range of the instrument. For the crwth in our history section, for example, we would want the frequencies to range between 196 and 784 because the instrument was tuned to play in this range. We would probably find that the most pleasing combinations of overtones are those in which the natural vibrational modes of the plate enhance tones which form intervals of thirds, fourths, fifths, and octaves with the

fundamental. Though each individual has his or her own idea of what is pleasing, composers have used these intervals for centuries as consonant or "stable" intervals. In the case of the plucked string, we would expect the fundamental to be the lowest natural frequency of the plate because its only nodes would be at the edges of the plate, and it would have a longer wavelength than any of the other vibrational components of the plate. It seems that an optimum plate, therefore, would have its fundamental at the pitch of the lowest string, and would have strong overtones at the pitches of the other strings. For our crwth, this would mean we want the frequencies 196.00, 261.63, 293.66, 392.00, 523.25, and 587.33 to be some of the frequencies with the highest amplitude, or " A_{nm} " value. From our calculations, it seems that $(\frac{w}{2}, \frac{1}{4})$ would be the better of the two placements for the soundpost because it would stimulate strong vibration of the plate at more frequencies than the model with the soundpost at $(\frac{w}{2}, \frac{1}{2})$. We would expect the latter model, therefore, to produce a "fuller," or richer tone.

Our calculations, of course, represent only a minor part of what would be needed in order to describe the entire violin mathematically. There are still a great number of aspects of just our simple rectangular back plate which we have left unexplored, however, and which would also provide some interesting possibilities for study. Among these are:

- 1.) the effects of variation of materials (how different



materials affect "a", which affects the frequencies of tones produced), 2.) the effects of size variation on wave frequency (we know that larger sizes produce lower frequencies, but one might try to find a ratio which relates size to frequency), 3.) the role of the ribs in linking the top and back plates, and 4.) the effects of varying the curvature of the plates (arching of the plates makes the body deeper, thus giving the sound chamber a greater volume). In further research, one may want to explore such factors as the following: 1.) the effects of the flexible bridge on vibrations between the string and the plates, 2.) the possibility of finding a method of solution for the coupled system of equations describing the vibrational behavior of the strings, plates and air, 3.) the effects of changing the ratio between the volume of the sound chamber and the area of the f-holes, 4.) the possibility of finding the relationship between the volume of the sound chamber and the frequency range of the instrument, 5.) the effects of variations on shape and size of the instrument, and 6.) the effects of varnish on the wood's properties (especially tension, or resistance, and density). In order to find a theoretically optimum model, however, we would have to choose characteristics of a specific instrument and compare them to other models. Since there are so many possible variables on the violin, it is impossible to test all models and find an ultimately superior model for the violin. But assuming that it is possible to describe the violin



mathematically, one might use calculations similar to those in this work to compare a theoretical model to an existing model and judge the overtone combinations that different parts of the theoretical instrument would produce to be desireable or undesireable based on this comparison.

The ultimate goal which one could hope to achieve through a complete mathematical description of the violin would be the ability to choose the frequencies and their corresponding amplitudes which one desires in the instrument's tone, and work the equations "backwards" to find the measurements and composition required for a model to produce such a tone. In this way, scientists and luthiers could work together to create instruments which sound like those of the masters.

Because of the complexity of the mathematics involved, it is not likely that scientists will ever be able to describe the violin completely enough, using mathematics, to accurately simulate its vibrational behavior. It is possible, however, that one might use math to study the effects, or approximate effects, which small changes in a violin's structure have on the instrument's tonal configuration. In this way, a luthier could conceivably test parts of a violin before actually making them, to determine whither a model would produce elements of the desired tone quality. This mathematical testing would save time and materials which may otherwise be spent trying to modify an undesireable instrument. But we will not forsake

all hope of attaining a complete mathematical model--after all, man had enough scientific ingenuity and persistence to reach the moon. Perhaps we will yet be able to produce instruments of a quality which equals, or even surpasses that of the Golden Age masters.

guitar - a stringed instrument consisting of a wooden sound board with a neck having six tuning pegs at one end, and a fingerboard with six strings at the other; the strings are plucked with the fingers and bowed with a bow made of hair, horsehair, or wire.

guitar player - person who plays the guitar with one hand and plucks the strings with the other hand while holding the instrument.

guitarist - the artist who plays the guitar.

guitarist - the person who plays the guitar.



GLOSSARY

bass bar - a bar of fir, pine, or spruce which is glued to, and runs the length of the top plate of the violin, passing under the foot of the bridge nearest the lowest string

bourdon - a drone string, i.e. a string that is not fingered

bout - the lengthwise shape of the violin is divided into three sections called upper, middle, and lower bouts. The shoulders of the instrument form the upper bouts, the inward "c" curves form the middle bouts, and the lower bouts comprise the end of the instrument nearest the player.

corner blocks - thick blocks of wood which fill the corners made by the violin's middle bouts inside the sounding chamber

crowd - the middle English spelling of crwth

da braccio - (as in "lira da braccio") Italian for "of the arm"

da gamba - (as in "lira da gamba") Italian for "of the legs"

endblocks - thick blocks of wood at the inside ends of the violin

fingerboard - an ebony piece which forms the top part of the violin's neck, and extends over the body to within about two inches of the bridge (the strings are held against this piece when notes are fingered)

luthier - violin maker

nut - a small raised ebony piece on the end of the fingerboard near the scroll which holds the strings above the fingerboard

overtones - vibrations which are of a higher frequency than the fundamental, and which are inspired by vibration at the fundamental frequency

pizzicato - plucking of the string with the finger

ribs - the wood which separates the top and back plates around the edges of the violin (the top and back plates are glued to the ribs)

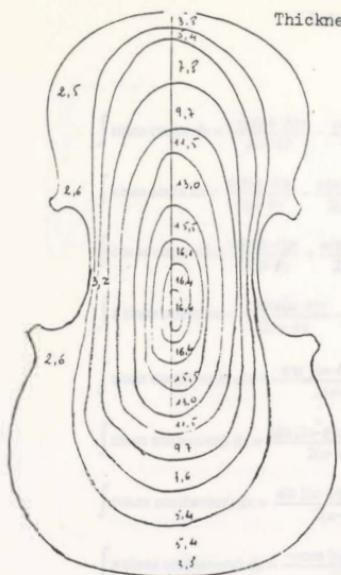
soundpost - a short rod of fir, pine, or spruce which is held between the plates of the violin by tension, and is placed under the foot of the bridge opposite the bass bar

tailpiece - an ebony piece to which the strings are attached at the end of the violin closest to the player

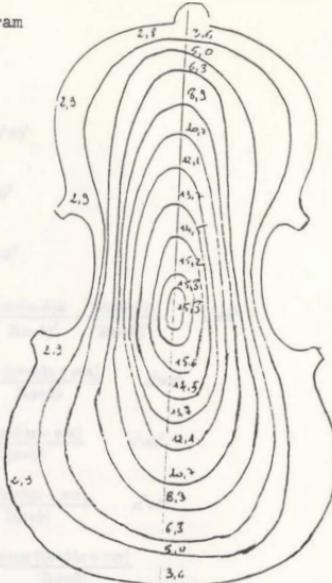


APPENDIX A

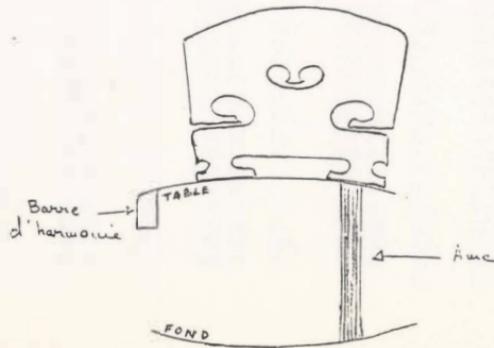
Thicknessing Diagram



TABLE



FEB. 2



APPENDIX E

Integral Table

$$\int \sin ax \cos bx dx = \frac{-\cos(a-b)x}{2(a-b)} - \frac{\cos(a+b)x}{2(a+b)} \quad a^2 \neq b^2$$

$$\int \sin ax \sin bx dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)} \quad a^2 \neq b^2$$

$$\int \cos ax \cos bx dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)} \quad a^2 \neq b^2$$

$$\int x \sin ax \cos bx dx = \frac{-x \cos(a-b)x}{2(a-b)} - \frac{x \cos(a+b)x}{2(a+b)} + \frac{\sin(a-b)x}{2(a-b)^2} + \frac{\sin(a+b)x}{2(a+b)^2} \quad a^2 \neq b^2$$

$$\int \sin ax \cos(bx+m\pi) dx = \frac{-\cos[(a-b)x - m\pi]}{2(a-b)} - \frac{\cos[(a+b)x + m\pi]}{2(a+b)} \quad a^2 \neq b^2$$

$$\int \sin ax \sin(bx+m\pi) dx = \frac{\sin[(a-b)x - m\pi]}{2(a-b)} - \frac{\sin[(a+b)x + m\pi]}{2(a+b)} \quad a^2 \neq b^2$$

$$\int \cos ax \cos(bx+m\pi) dx = \frac{\sin[(a-b)x - m\pi]}{2(a-b)} + \frac{\sin[(a+b)x + m\pi]}{2(a+b)} \quad a^2 \neq b^2$$

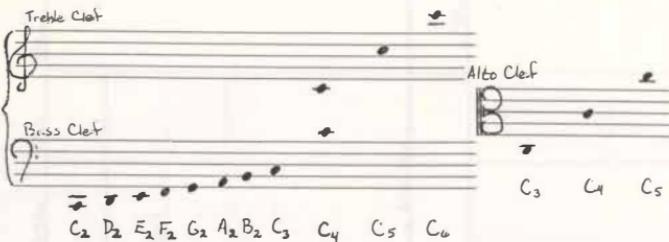
$$\int x \sin ax \cos(bx+m\pi) dx = \frac{-x \cos[(a-b)x - m\pi]}{2(a-b)} - \frac{x \cos[(a+b)x + m\pi]}{2(a+b)}$$

$$+ \frac{\sin[(a-b)x - m\pi]}{2(a-b)} + \frac{\sin[(a+b)x + m\pi]}{2(a+b)} \quad a^2 \neq b^2$$



APPENDIX C

Note Frequencies

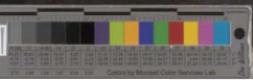


EQUAL TEMPERED CHROMATIC SCALE

A₄ = 440

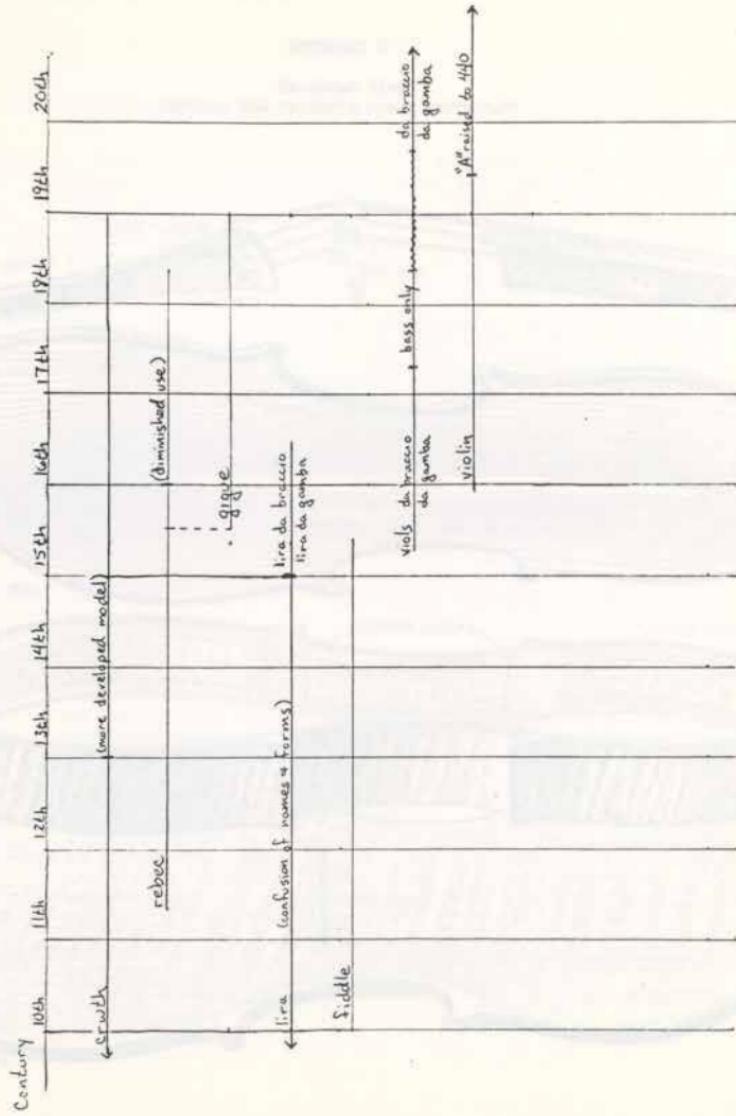
American Standard pitch. Adopted by the American Standards Association in 1936

Note	Frequency	Note	Frequency	Note	Frequency	Note	Frequency
C ₀	16.35	C ₁	65.41	C ₂	261.63	C ₃	1046.50
C# ₀	17.32	C# ₁	69.30	C# ₂	277.18	C# ₃	1108.73
D ₀	18.35	D ₁	73.42	D ₂	293.66	D ₃	1174.66
D# ₀	19.45	D# ₁	77.78	D# ₂	311.13	D# ₃	1244.51
E ₀	20.60	E ₁	82.41	E ₂	329.63	E ₃	1318.51
F ₀	21.83	F ₁	87.31	F ₂	349.23	F ₃	1396.91
F# ₀	23.12	F# ₁	92.50	F# ₂	369.99	F# ₃	1479.98
G ₀	24.50	G ₁	98.00	G ₂	392.00	G ₃	1567.98
G# ₀	25.96	G# ₁	103.83	G# ₂	415.30	G# ₃	1661.22
A ₀	27.50	A ₁	110.00	A ₂	440.00	A ₃	1760.00
A# ₀	29.14	A# ₁	116.54	A# ₂	466.16	A# ₃	1864.66
B ₀	30.87	B ₁	123.47	B ₂	493.88	B ₃	1975.53
C ₁	32.70	C ₂	130.81	C ₃	523.25	C ₄	2093.00
C# ₁	34.65	C# ₂	138.59	C# ₃	554.37	C# ₄	2217.46
D ₁	36.71	D ₂	146.83	D ₃	587.33	D ₄	2349.32
D# ₁	38.89	D# ₂	155.56	D# ₃	622.25	D# ₄	2489.02
E ₁	41.20	E ₂	164.81	E ₃	659.26	E ₄	2637.02
F ₁	43.65	F ₂	174.61	F ₃	698.46	F ₄	2793.83
F# ₁	46.25	F# ₂	185.00	F# ₃	739.99	F# ₄	2959.96
G ₁	49.00	G ₂	196.00	G ₃	783.99	G ₄	3135.96
G# ₁	51.91	G# ₂	207.65	G# ₃	830.61	G# ₄	3322.44
A ₁	55.00	A ₂	220.00	A ₃	880.00	A ₄	3520.00
A# ₁	58.27	A# ₂	233.08	A# ₃	932.33	A# ₄	3729.31
B ₁	61.74	B ₂	246.94	B ₃	987.77	B ₄	3951.07
						C ₅	4186.01



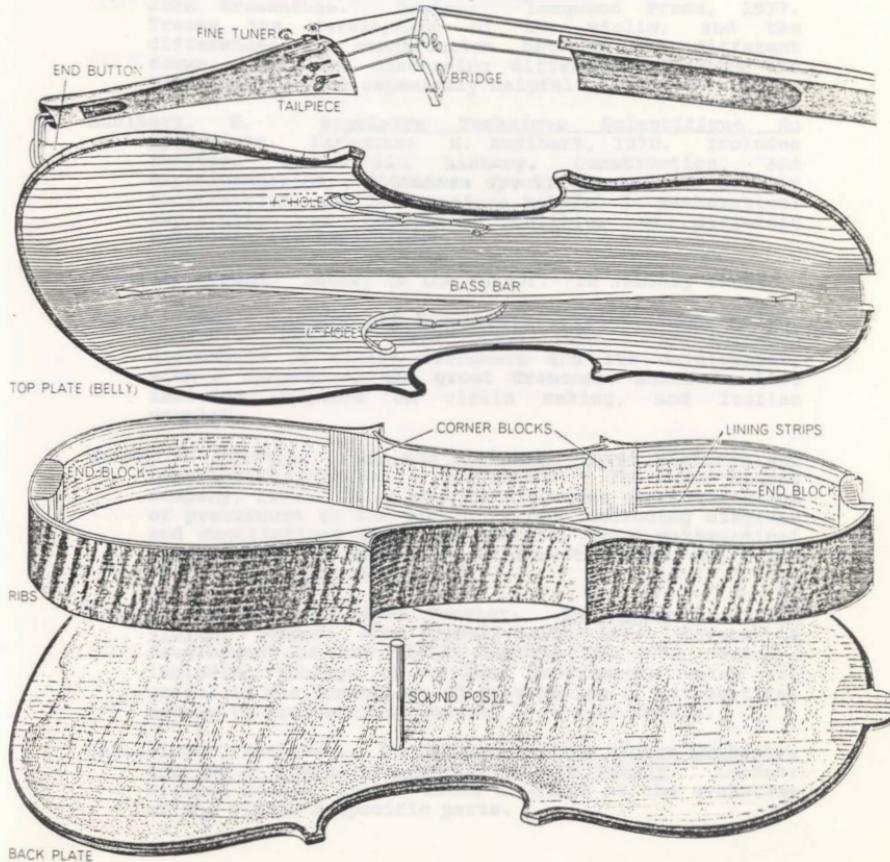
APPENDIX D

Timeline



APPENDIX E

Exploded View
showing the violin's inner structure



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Violin History and Construction

Abele, Hyacinth. The Violin: Its History and Construction Illustrated and Described From Many Sources. Trans. John Broadhouse. Boston: Longwood Press, 1977. Traces the development of the violin, and the differences in construction between its different forms. Sections discussing different structure and tone quality were especially helpful to me.

Audibert, H. Breviaire Technique Scientifique du violoniste. Narbonne: H. Audibert, 1970. Includes chapters on violin history, construction, and experimentation. Discusses specific materials used in construction, and is therefore helpful in mathematical explanation of resonance qualities of different materials and instruments.

Camurat, Arnaud. Letter to the author. 22 January 1988.

Farga, Franz. Violins and Violinists. Trans. Egon Larsen. London: Camelot Press Ltd., 1950. An in-depth discussion of violin structure and its development, with a chapter on the great Cremonese makers. Also includes chapters on violin making, and Italian varnish.

Gates, W. Francis. Pipe and Strings. Three Historic and Descriptive Sketches. Cincinnati: The John Church Company, 1895. Third section discusses characteristics of precursors to the violin family, including diagrams and descriptions which are helpful to a mathematical analysis of the merits of different shapes and sizes for the resonant chamber.

Sandys, William and Simon Forster. The History of the Violin. London: John Russell Smith, 1864. Covers the development of the violin through the 17th century, including tunings of various instruments which are useful in the comparison of frequencies of mathematical models to frequencies produced by actual models.

Schlesinger, Kathleen. Instruments of the Modern Orchestra and Early Precursors of the Violin Family. London: William Reever, 1969. Looks in depth at the evolution of the violin's specific parts.



van der Straeten, E. The History of the Violin: Its Ancestors and Collateral Instruments, from earliest times to the present day. 2 vols. London: Cassell and Co. Ltd., 1933. Covers violin history through the 17th century, providing numerous diagrams, tunings, and measurements which are useful specifics in the mathematical analysis of various instruments.

Specific Measurements and Characteristics

Hargrave, Roger and Michael Tollmann. "G.B. Guadagnini 1776." The Strad 95 (1985): 760. Gives description and specific measurements of Guadagnini violins. Will be useful for mathematical analysis.

Merrill, Albert Herve. Master Violins and How to Make Them: A Complete and Scientific Course in Tone Production Based on the Laws of Sound and Proven by 38 Years of Successful Experience in Violin Making and Retuning. Portland: T.G. Robison, 1930. Instructions on how to make a "master violin". No mathematics, but gives specific measurements. Easy reading.

Nelson, Sheila M. The Violin and Viola. New York: W.W. Norton and Co. Inc., 1972. Two relevant chapters--one on violin structure and makers, and one on string acoustics. Includes actual measurements of violins. Will be useful for further mathematical analysis.

"Royal Approval." The Strad 96 (1986): 656-63. A description of the 1818 premier prix-winning violin made by Nicolas Lupot. Includes specific measurements.

Vigdorchic, Isaac. "Micro Chip." The Strad 96 (1986): 944-49. An interview. Isaac Vigdorchik explains, among other things, his theory of chromatic plate tunings. The article deals with specific measurements and alterations, and is therefore conducive to mathematical application.

General Background on Acoustics and Resonance

Berg, Richard E. and David G. Stork. The Physics of Sound. Englewood Cliffs: Prentice-Hall, Inc., 1982. Several pertinent chapters on waves, sound, and overtones. Also includes a chapter on string instruments.



Culver, Charles. Musical Acoustics. 3rd ed. New York: The Blakiston Company, 1951. A non-technical view of the principles of acoustics. Includes chapters on sound, resonance, pitch, strings, and stringed instruments. Good for a basic understanding of acoustics.

Dale, E. Brock. "Demonstration of the Effect of Altering the Cavity Resonance of a Violin." American Journal of Physics 47 (1979): 201. Discusses the effects of CO₂ and He gases on violin cavity resonance. May relate to differences in resonance produced by wood aging.

Gough, C.E. "The Resonant Response of a Violin G-String and the Excitation of the Wolf-Note." Acustica 44 (1980): 113-23. Confirms, and more technically discusses Schelleng's explanation of the wolf-note phenomenon.

Hunter, Arthur. "Violins by Computer." The Strad 95 (1984): 485. Briefly discusses the merits and problems of cross-ply construction of violins.

Hutchins, Carleen Maley. "The Acoustics of Violin Plates." Scientific American Oct. 1981: 170-186. Discusses vibrational properties of violin top and back before they are joined. Author cites Savart's study on eigenmodes, and other similar studies in a discussion of how physical properties of the plates affect their vibrational frequencies. A good general description of phenomena to which mathematics may be applied. Especially useful because of its discussion on eigenmodes.

---. "A History of Violin Research." Journal of the Acoustical Society of America 73 (1983): 1421-40. Excellent source on researchers of the violin from a structural perspective. Includes a section on wood choice, and an excellent bibliography.

---. "The Physics of Violins." Scientific American Nov. 1962: 78-93. Discusses violin parts and their functions in sound production (without specific measurements), and the general relationship between volume (varying size and thickness) and resonance quality. Includes sections on "The Basic Violin," "The Resonant Box," "Loudness Curves," "Experimental Instruments," "Tap Tones," and "New Violins." Good for a basic understanding of violin anatomy and physiology.

John, Leonard. "Fibre Option." The Strad 95 (1984): 256-7. The author discusses his experiment of making two violins from graphite. Provides an interesting comparison to resonance properties of wood violins.

Johnson, Derek. "Clear Finish." The Strad 95 (1985): 897. The author describes the experience which led him to conclude that general construction, arching and thickness, and use of a straight-grained softwood similar in density to spruce for the belly are the most important factors of construction which determine a good finished violin. Helpful in narrowing my topic.

Lawergren, B. "Harmonics of S Motion on Bowed Strings." Journal of the Acoustical Society of America 73 (1983): 2174-79. A short article on complex string motion. Helped with an understanding of the workings of the violin as a whole before I concentrated on the back plate.

Lee, A.R. and M.P. Rafferty. "Longitudinal Vibrations in Violin Strings." Journal of the Acoustical Society of America 73 (1983): 1361-65. Discusses the longitudinal vibrations detected in bowed strings by the rocking of the bridge. Important for its discussion on how string motion affects bridge motion. Also mentions that different bowing style or other factors may affect string vibration, thus altering instrument tone. Such factors must be taken into account when instruments are altered and then tested for different tone quality.

Olson, Harry. Musical Engineering. New York: McGraw-Hill Book Co., Inc., 1952. A technical source on physical forces involved in the production of sound using musical instruments. Good for understanding how physics explains acoustical phenomena mathematically.

Richardson, E.G. The Acoustics of Orchestral Instruments and of the Organ. London: Edward Arnold and Co., 1929. Contains a chapter on stringed instruments, including discussion of the functions of specific violin parts, and non-technical mathematical equations.

Rokos, Kurt. "Timbre Changes." The Strad 96 (1985): 256. A method of elimination wolf note, and the theory behind it.

Schelling, John C. "The Physics of the Bowed String." Scientific American Jan. 1974: 87-95. In-depth discussion on the motion of bowed strings and the effects of variations in string tension, bow pressure, and bow distance from the bridge. May be especially useful in consideration of the effects of bridge placement and violin length on tone quality in various pitch ranges.



Waller, Mary Desiree. Chladni Figures: A Study in Symmetry. London: C. Bell and Sons Ltd., 1961. The author has repeated and extended Chladni's observations on vibrating plates. The book is a collection of pictures, data, and notes on these experiments. It will be helpful to an understanding of other sources, many of which refer to Chladni figures. Not violin specific.

Technical Mathematical Sources

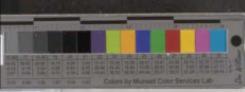
Berg, Paul and James L. McGregor. Elementary Partial Differential Equations. San Francisco: Holden-Day, Inc., 1966. Includes technical mathematical discussion of the vibrating string, including methods of solution of the wave equation. Very helpful for an understanding of the mathematics involved in my topic.

Butkov, Eugene. Mathematical Physics. Menlo Park, CA: Addison-Wesley Publishing Company, 1968. Includes pertinent sections on vibrating membranes, propagation of initial conditions, and special functions which can be applied to vibrating media of various shapes. This work is very technical, and would be helpful in further research of my topic.

Coates, Kevin. Geometry, Proportion and the Art of Lutherie. Oxford: Clarendon Press, 1985. Extensive drawings, observations, and mathematical analysis of the violin and its precursors. Presents mathematical solutions to the design of 16th-18th century stringed instruments. An excellent source which relates closely to my topic on all points.

Crandall, Irving B. Vibrating Systems and Sound. New York: D. Van Nostrand Co., 1926. An excellent technical mathematical source on how vibrations are transmitted. Chapters on simple vibrating systems, resonators and filters, propagation of sound, radiation and transmission problems, the acoustics of closed spaces, absorption, and reflection and transmission are relevant.

Cremer, Lothar. The Physics of the Violin. Trans. John S. Allen. Cambridge: MIT Press, 1984. An excellent specific and mathematical discussion on the workings of the violin. Draws heavily on Helmholtz's studies. Includes in depth mathematical analysis of features such as the sound post, the bass bar, the sound holes, and the bridge. Closely related to my mathematical analysis on all counts.



Faanhof, Al. "Parabolic Arching of Violin Plates." The Strad 93 (1982): 342-46. Excellent source on the arching of violin plates, including specific mathematics and measurements. Helpful to mathematical analysis of the violin.

Helmholtz, Hermann L.F. Sensations of Tone as a Physiological Basis for the Theory of Music. New York: Longmans, Green, and Co., 1895. An excellent primary source. Many other authors refer to Helmholtz's works. Discusses in depth: vibrations, harmony, and relationship of musical tones. Contains a great deal of helpful mathematical discussion in appendices.

Italian Physical Society. Frontiers in Physical Acoustics. New York: North Holland Physics Publishing, 1986. A good up-to-date view of acoustics. The section by J.E. Ffowcs Williams on "Physical Origins of Acoustical Noise" is of particular interest--especially its section on three-dimensional acoustic source distributions.

Leipp, Emile. The Violin. Toronto: Univ. of Toronto Press, 1969. Contains four sections: History, Aesthetics, The Workshop, and Acoustics. The last two are mathematical, and are especially relevant. Includes discussion of wood choice (acoustical qualities of various woods) and preparation, and varnish.

Lindsay, R. Bruce, ed. Physical Acoustics. Stroudsburg: Dowden, Hutchinson, and Ross, Inc., 1974. A collection of physical acoustics-related magazine articles. All involve technical mathematics. Helpful in gaining an understanding of the mathematics involved in other technical sources.

Main, Iain G. Vibrations and Waves in Physics. Cambridge: Cambridge Univ. Press, 1978. An explanation of physical principles which can be applied to the violin. Chapters 8-10 are especially relevant. Includes technical mathematics.

Morse, Philip M. Vibration and Sound. New York: Mc Graw Hill Book Co., Inc., 1948. Sections on mathematics, forced oscillations, and coupled oscillations is useful. Also includes technical chapters on "The Flexible String" and "Membranes and Plates," which are useful for their mathematics.



Pippard, A.B. The Physics of Vibration. 2 vols. Cambridge: Cambridge Univ. Press, 1978. Technical mathematical source--not violin specific. More pertinent chapters include "Applications of Complex Variables to Linear Systems," "Fourier Series and Integral," and "Waves and Resonators."

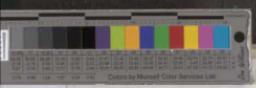
Skudrzyk, Eugen. The Foundations of Acoustics. New York: Springer-Verlag, 1971. Extremely technical in mathematics, but a good mathematical view of acoustics. Chapter 22 on "The Wave Equation in Spheroidal Coordinates and Its Solutions" is of special interest.

Strutt, John William Lord Rayleigh. The Theory of Sound. 2 vols. New York: Macmillan and Co., 1894 and 1896. A good primary source, cited by several prominent authors in the field. Highly mathematical. Includes technical discussion on vibrating systems which was helpful to my understanding of the mathematics involved in violin acoustics. Volume 2 includes chapters on aerial vibrations, secondary waves, and resonators.

Tolstov, Georgi P. Fourier Series. Trans. Richard A. Silverman. New York: Dover Publications, Inc., 1962. Very useful for an understanding of Fourier Analysis.

Tull, Alan. "The Acoustically Tuned Bridge." 2 parts. The Strad 93 (1982): 326-28 and 488-91. Specific and mathematical analysis of the acoustics of the violin bridge, including materials, dimensions, mass, rigidity, and action. Helpful for an understanding of the complete violin system before concentrating on specifics of the resonating cavity. Part 2 explores the functional parameters of the violin bridge. Tull's question on the bridge is similar to mine on the violin shape--"Since the form has remained unchanged for so long, can there be good reason for trying to change it?"

Wood, Alexander. The Physics of Music. London: Methuen and Co., Ltd., 1944. A mathematical approach to the nature of sound including wave equations. Chapters on intensity and loudness, pitch, musical quality, and vibrations of strings help to understand the effects of different adjustments on the tone quality and loudness in various ranges of the violin.



OTHER RELATED WORKS

Davidson, Peter. The Violin: Its Construction, Theoretically and Practically Treated. 4th ed. London: F. Pitman, 1881. Cited in De Fidelicus Bibliographia, by Edward Heron Allen, p.6. Heron-Alen says it contains "an excellent translation of, and commentary upon the articles descriptive of the Savart violin..." If the title of the book is at all descriptive of its content, it should be an excellent source.

Heron-Allen, Edward. Violin Making as it Was and Is. London: Ward, Lock and Co., 1884. I got the reference from Heron-Allen's De Fidelicus Bibliographia, p. 13. The book gives minute details of violin construction. Part I is on violin history, and Part II deals with the theoretical principles which govern the structure of the violin.

Maugin, J.C. Manuel du Luthier. Paris: Roret, 1834. I found the reference in Heron-Allen's De Fidelicus Bibliographia, p. 14. According to Heron-Allen, this work "aims at being a complete guide to the fiddle-maker's art."

Peterlongo, Paolo. The Violin: Its Physical and Acoustic Principles. Revised English ed. Elek: 1980. The reference, which I found in a book review in The Strad 91 (1980): 193, does not include the name of the publishers. In his book, Peterlongo discusses physical principles and functioning of bowed stringed instruments, with explanations on why new violins sound more shrill than the great Cremonese instruments. I feel that this source would contribute to an understanding of the acoustical system of the violin.

Richelme, Marius. Etudes et Observations sur la Lutherie Ancienne et Moderne. Marseille: F. Canquois, 1868. This work was cited in Heron-Allen's De Fidelicus Bibliographia, p. 21. According to Heron-Allen, "Richelme was a practical maker who devoted much study to the scientific principles of his art. These studies he embodied in this little work..." If this book, indeed, contains any scientific analysis of the violin, it would be helpful to my research.



Savart, Felix. Memoire sur la Construction des Instruments a Cordes et a Archet. Leipzig: F. Kistner, 1844. I found this reference in Heron-Allen's De Fidelicus Bibliographia, p. 22, which states, "This is perhaps the most scientific work extant on the theoretical and scientific principles which govern construction of, and tone-production on, the violin, besides being a complete description for all practical and scientific purposes of the celebrated invention known as the Savart Trapezoid Violin, or box-fiddle." Sounds like an excellent source!

Vigdorchick, Isaac. The Acoustical Systems of Stradivarius and Other Cremona Makers. This book was cited in "Micro Chip." The Strad 96 (1986): 944-49 by Dudley Reed. He did not include publication information. According to Reed, this book summarizes early 20th century Russian acoustical research on the violin, and includes extensive discussion of plate thicknesses, which would be helpful to my mathematical analysis.

