THE IDENTIFICATION PROBLEM IN THE TWO VARIABLE, LINEAR CASE

BY

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CHAPTER I

GENERAL CONCEPTS

Introduction

The application of statistical techniques and mathematical tools to economics has encountered various problems that required some new developments in mathematical statistics. Among these new developments are those which are somewhat loosely referred to as identification problems. Their nature can be understood by contrast with the more usual type of statistical problem found in the natural sciences. A typical problem of the physical world utilizes a theory of the effect of a set of variables in determining a single variable: the effect of pressure and temperature on the volume of a gas, the effect of temperature, food, etc., on biological growth, and the effect of latitude and altitude on rainfall are familiar examples. In these problems some assumptions regarding the effect of the determining variables on a single dependent variable is made, and this assumption is tested by statistical investigation. The assumption involved is a specification in the R. A. Fisher sense [3] --some postulation about the form of a single probability distribution with a rigid separation of determining or causal variables from the determined or dependent variables. Usually errors of measurement are assumed negligible in the determining variables.

Economic theories usually cannot be specified in such a manner. There are usually two or more probability distributions involved, the causal and dependent variables may not be separable, and errors of observation are likely to occur in all variables. The differences account for the need of a development of new statistical methods known as identification methods.

Consider for example the Keynesian theory of employment. In its simplified form it can be written

$$Y = I + C$$

$$S = f(Y)$$

$$I = S$$

$$I = g(i)$$

$$i = h(M)$$

where Y = income, I = investment, C = consumption, S = savings, i = rate of interest, and M = quantity of money. None of these variables can be accurately measured and no one equation can be considered alone. The five equations must be studied together since in the absence of an experimental or laboratory method observed data must come from the real world where many forces interact.

The study of the price and quantity of a commodity, for example, is not only the study of the function of one in terms of the other but also of the relations between the prices and quantities of this commodity and how these are simultaneously affected by the actions of the prices of other commodities. Thus, too, in a model set up by C. Clark [1], consumption is recognized as being related to income and previous income (to consider the most important influences as interpreted by Mr. Clark), while imports are a function of income, producers goods are a function of income and of previous durable goods, etc., until a complete system of seven linear equations incorporating seven variable quantities is constructed which, presumably, will explain the major activities of the U. S. business cycle.

These equations may be set up by using economic ideas and theories--by applying principles of economics in combination with legal requirements, institutions affecting behavior, and technological knowledge. But, in addition, combined with theory we may utilize systematically collected statistical data to help construct the equations.

These simultaneous equations merely translate into a mathematical language the fact that many things act on each other at the same instant of time, and they describe the method by which they interact. Along with the varying quantities or variables which occur in the equations there are certain parameters or constant quantities--constant for the particular system under consideration. The solution of these simultaneous equations--the values for the variables which will at the same time satisfy all the conditions as stated by the equations--is in terms of these constants. Also these constants tell us how the variables are related to each other. Having decided to construct a system of some n equations in n variables, the values to be used for the constants must be determined in some way. Usually, data on past values of the variables are collected, and, by some method such as least squares, the values of the constants which are in keeping with the past actions of the variables are found. These constants may be adjusted to take care of expected future developments, but essentially they are found by use of past values of the variables concerned.

The whole plan of procedure discussed above assumes that the variables which occur in the system of equations (sometimes called structural equations) can be measured directly and accurately. To go back to Clark's within-the-system variables (consumption, imports, producers' goods, new construction, change in inventories, sales, and gross national income) the assumption is that it is possible to get the true value for each of these, and that each is directly measurable. If this is the case then the classical procedures allow us to construct an econometric model. However, if the variables are not directly observable, if we cannot measure them directly and accurately, then a whole field of statistical inference about the variables which are not directly observable in terms of some which are opens up. In this situation, each of the unobservable variables may be expressed as a function of an observable variable and of random errors of measurement.

The problem of statistical inference with respect to a system of equations, then, is the study of what conclusions (inferences) can be drawn about the true unobservable variables and about the structural relationships or equations in the system from the observed values. In the remainder of this chapter, something of the history of this problem and the general lines of progress toward solution to date will be given. In the second chapter, the simple regression problem of two observed variables will be considered when the distributions are normal in character. In the third chapter the regression problem in one non-normal case will be studied--the case where two variables are subject to correlated errors, and where a linear regression exists between nonobservable variables, one of which is distributed rectangularly.

History

Pigou [11] was among the first to discuss a problem of identification. Henry Schultz [14] considered several aspects of the problem in his detailed studies of demand. But a formulation of the problem was not forthcoming until recent years. In the past decade much work has been done by J. Marschak [9], O. Reiersøl [12], Hurwicz [5], T. Koopmans [7], Rubin [13], A. Wald [17], J. Neyman [10], E. Scott [15], among others, so that by this time a fairly systematic formulation of the identification problem in general has been given both from the point of view of mathematical analysis and application to economics. Also the well known work by

Frisch [4] has done much in the examination of the more general ideas of regression among two or more variables.

The Fundamental Ideas and Terms

Before proceeding with a formulation of what is meant by "the identification problem" we will define several terms. We will consider certain variables

$$\eta_1, \eta_2, \eta_3, \ldots, \quad u_1, u_2, u_3, \ldots,$$

and

$$y_1, y_2, y_3, \ldots .$$

The η_i will represent true variables which we will assume to be non-observable, directly. The u_i will represent errors of measurement or errors of observation or disturbances. Sometimes these two types of variables are grouped together and referred to as latent variables as they both have the character of being nonobservable. Associated with these latent variables we have certain observable variables, represented by the y_i. Relationships or equations which connect the observable variables with the nonobservable variables and the errors of observation are called structural relationships. Typical examples of such structural relationships are $\eta_1 = \alpha + \beta\eta_2$ and $y_i = \eta_i - u_i$.

Along with these structural relationships between the observable and the nonobservable variables we must assume (specify) particular probability distribution functions of the nonobservable variables and of the errors of observation; that is, we have density functions $P(u_i)$ and $P(\eta_i)$ or a $P(u_i, \eta_i)$. These distributions together with functional relations of the variables form a structure.

We have given probability distributions of η_i and u_i. Since y_i are determined by the η's and u's, we may derive a corresponding probability distribution for the observable variables y_i, $P(y_i)$. Hence $P(y_i)$ is uniquely determined by the structure. Since y_i are observable variables we can apply $P(y_i)$ to the data.

Different structural relationships and different probability distribution functions of the latent variables will comprise a set of different structures. Any set of structures may be called a model. According to Koopmans and Reiersøl, to specify a model is to describe or define a set of structures which contains the particular structure which generates the distribution of the observed variables. Any two or more structures which generate the same distribution of observed variables are said to be undistinguishable or equivalent. Specification in the R. A. Fisher sense is the simpler concept of determining or deciding upon the mathematical form of the distribution of the hypothetical population from which a sample is to be regarded as drawn. It is the choosing a single mathematical form to represent the distribution of a population, rather than choosing a set of forms, one of which represents the population.

The basic problem of identification is whether the probability distribution of the y_i (observed values) determines uniquely the probability distribution of the latent variables. Can the probability distribution of the y_i be generated by only one structure in the model? This will depend on the model and also, usually, on the given structure. But, if this is so, then we say that the model identifies the structure, or that the structure is identifiable.

In some cases the structure as a whole may be unidentifiable, but it still may be possible to uniquely determine some of its characteristics. A <u>structural parameter</u> is some quantity which is inherent in the particular structure, a quantity which has one value for a particular structure and different values for other structures (in other words these are the constants in the relating equations and probability distributions). These structural parameters characterize the structure. We say that a model identifies a parameter in a structure if that parameter has the same value in all equivalent structures in the model. If all the parameters of a structure are identifiable, then the structure itself is identifiable (provided that the structure has been defined except for the values of the parameters).

An example given by Koopmans and Reiersøl [6], [12], the proof of which and examples of which are given in the discussion which follows, shows that under some changes in the conditions of the structure a parameter becomes identifiable. Once a parameter has been identified we can devise some statistic which may be used to estimate the parameter, then make some observations and from this sample compute the value of the statistic; that is, get the estimated value of the parameter. We have done this for two simple examples.

As yet there are few established methods of estimating structural parameters. Wald [16] has given some results and confidence intervals for parameters in certain special cases where the u_i are uncorrelated. Scott has given a technique for obtaining a consistent estimate when the u_i are correlated but she does not give any measurement of the precision of her estimate. Both of these cases assume the structural relationships are of a particular linear type. The problem of finding sampling distributions of estimates in fairly general cases is largely unsolved.

Classical Regression Analysis

It might be well to contrast the above concepts with the classical regression problem. In the usual regression analysis we think of having certain values or observations for one variable, say $x_1, x_2, x_3, \ldots, x_n$ and for any one of these x_i we then want to find a corresponding value of another variable quantity, say y_i. Due to inaccuracies of measurements, however, we cannot obtain the true value of y_i. In many measurements of a y_i we would obtain a variety of values for y_i. Thus we would get a scatter of observed y_i's around the true value of y_i, corresponding to a given x_i.

The true y_i is made up of an observed part which we can write as y_i' and an error of observation u_{1i}; in other words

$$y_i = y_i' - u_{1i}$$

We assume in this concept of regression that we have an exact measurement of the x_1's; that is, the value that we observe for an x_1 is the true value. However, it is more realistic to think of having an error of measurement in the x_1's also. This is the assumption made on each of the variables which we consider in our study of the problem of identification. Thus the true x_1 is made up of an observed part, call it x_1', and an error of observation u_{21}, or

$$x_1 = x_1' - u_{21}$$

In the classical regression theory the variance of u_{21}, then, is zero.

In the usual analysis there is assumed to be an exact relationship between the x and the true value of y, the true value of y or the mean value of all the possible observed values of y for any given x, and the x is measured exactly. In the concept described in the section of this chapter on "The Fundamental Ideas and Terms," both the x and y are subject to errors of observation, and we assume that there is an exact relationship between the true values of x and the true values of y.

CHAPTER II

LINEAR REGRESSION WITH NORMAL "ERRORS"

Suppose we have a structure S involving a regression between the unobservable variables η_1 and η_2, that is

(1) $\eta_1 = \alpha + \beta\eta_2$, α, β are constants,

and the observed variables y_1 and y_2 contain errors u_1 and u_2.

Specifically, take

$$(2) \quad \begin{aligned} y_1 &= \eta_1 + u_1 \\ y_2 &= \eta_2 + u_2 \end{aligned}$$

Assume $E(u_1) = 0$ and $E(u_2) = 0$; that is, the expected value or true mean of u_1 and of u_2 is zero. The values of the standard deviations σ_{u_1} and σ_{u_2}, as well as the covariance of u_1 and u_2 are unspecified.

Suppose, in addition, that η_1 and η_2 are independent of u_1 and u_2 and that the distribution of u_1 and u_2 and that the distribution of u_1 and u_2 is normal. Now to complete the structure we must assume also a distribution for η_1 and η_2 (or for η_1, since η_2 is in exact linear relationship with η_1 and knowing the distribution of η_1 we also have the distribution of η_2).

The identification of α and β depends, in this particular example on the distribution of η_1. We will consider in this chapter these two possibilities: (a) the case in which the distribution of η_1 is normal and (b) the case in which the distribution of η_1 is not normal.

<u>Theorem I:</u> If, in the structure S, the distribution of η_1 is normal then there are many values of β which will give the same joint probability distribution of y_1 and y_2; i.e., β is not identifiable.

<u>Theorem II:</u> If, in the structure S, the distribution of η_1 is not normal then there is a unique value of β for any probability distribution of y_1 and y_2; thus, β is identifiable.

Proof of Theorem I: If η_1 and η_2, u_1 and u_2 are normally distributed, then since

$$y_1 = \eta_1 + u_1$$

$$y_2 = \eta_2 + u_2$$

the joint probability distribution of y_1 and y_2 $P(y_1, y_2)$ is also normal, and it is completely characterized by the following five <u>parameters:</u> $E(y_1)$, $E(y_2)$, $\sigma^2(y_1)$, $\sigma^2(y_2)$ and covariance of y_1y_2. (Covariance of y_1y_2 can be indicated by $\rho\,\sigma_{y1}, \sigma_{y2}$.) These parameters are expressible in terms of the parameters of the distributions of u_1 and u_2 and η_1, as follows:

$$E(y_1) = E(\eta_1)$$

$$E(y_2) = \alpha + \beta E(\eta_1)$$

$$\sigma^2(y_1) = \sigma^2(u_1 + \eta_1) = \sigma^2(u_1) + \sigma^2(\eta_1)$$

$$\sigma^2(y_2) = \sigma^2(u_2 + \eta_2) = \sigma^2(u_2) + \sigma^2(\eta_2)$$

$$= \sigma^2(u_2) + \beta^2 \sigma^2(\eta_1)$$

$$\begin{aligned} \operatorname{Cov}(y_1 y_2) &= \operatorname{cov} u_1 u_2 + \operatorname{cov} \eta_1 \eta_2 \\ &= \operatorname{cov} u_1 u_2 + \operatorname{cov} \eta_1(\alpha + \beta \eta_1) \\ &= \operatorname{cov} u_1 u_2 + \beta \sigma^2(\eta_1) . \end{aligned}$$

Suppose we choose any value for β. It can be seen that these equations can still be satisfied by an appropriate choice of $E(\eta_1)$, α, $\sigma^2(u_1)$, $\sigma^2(\eta_1)$, $\sigma^2(u_2)$ and cov u_1u_2. Almost any values may be chosen for these six parameters subject only to the restriction that

$$D = \begin{vmatrix} \sigma^2(u_1) & \operatorname{cov}(u_1 u_2) \\ \operatorname{cov}(u_1 u_2) & \sigma^2(u_2) \end{vmatrix}$$

~~where D~~ has to be positive. (Schwartz' inequality for integrals.)

Proof of Theorem II: This is the case where the joint probability distribution of y_1 and y_2 is not normal.

Let $\Phi_{y_1 y_2}(t_1, t_2)$ be the characteristic function of $P(y_1, y_2)$:

$$\Phi_{y_1,y_2}(t_1,t_2) = E(e^{y_1it_1 + y_2it_2}) =$$

$$= \int_{y_1}\int_{y_2} e^{y_1it_1 + y_2it_2}\, P(y_1y_2)dy_1dy_2$$

$$= E\left[e^{(u_1 + \eta_1)it_1 + (u_2 + \eta_2)it_2}\right]$$

$$= E\left[e^{u_1it_1 + u_2it_2} \cdot e^{\eta_1it_1 + \eta_2it_2}\right]$$

$$= E(e^{u_1it_1 + u_2it_2}) \cdot E(e^{\eta_1it_1 + \eta_2it_2})$$

since u_1, u_2 are independent of η_1, η_2.

Taking the log of the characteristic function we have

$$\log \Phi_{y_1,y_2}(t_1,t_2) = \log\left[E(e^{u_1it_1 + u_2it_2})\right] + \log\left[E(e^{\eta_1it_1 + \eta_2it_2})\right].$$

However u_1 and u_2 are normally distributed, so $\log\left[E(e^{u_1it_1 + u_2it_2})\right]$ will be made up of terms of second degree or less. But the distribution of y_1 and y_2 is not normal, and so $\log \Phi_{y_1,y_2}(t_1,t_2)$ is a polynomial of degree higher than two. We may write

$$\log \Phi_{y_1,y_2}(t_1,t_2) = \frac{\sum_r \sum_s K_{rs}t_1^r t_2^s}{r!\, s!}$$

Also,

$$\log\left[E(e^{\eta_1it_1 + \eta_2it_2})\right] = \log\left[E(e^{\eta_1it_1 + \alpha it_2 + \beta\eta_1t_2})\right]$$

$$= \log e^{\alpha it_2} \cdot E\left[e^{i\eta_1(t_1 + \beta t_2)}\right]$$

$$= \alpha it_2 + \log E\left[e^{i\eta_1(t_1 + \beta t_2)}\right]$$

The second term on the right hand side is a function of $(t_1 + \rho t_2)$; η_1 is not normally distributed, therefore this term is also a polynomial of degree greater than two, and we can write

$$\log[E(e^{\eta_1 it_1 + \eta_2 it_2})] = \alpha it_2 + \sum_n \frac{K_n(t_1 + \beta t_2)^n}{n!}$$

Hence

$$\sum_r \sum_s \frac{K_{rs} t_1^r t_2^s}{r!\, s!} = at_1^2 + bt_1t_2 + ct_2^2 + dt_1 + et_2 + f$$

$$+ \alpha it_2 + \sum_n \frac{K_n(t_1 + \beta t_2)^n}{n!}$$

$$= at_1^2 + bt_1t_2 + ct_2^2 + dt_1 + et_2 + f$$

$$+ \alpha it_2 + \sum_n \sum_{j=0}^{n} \frac{K_n t_1^j (\beta t_2)^{n-j}\, n!}{n!\, j!\, (n-j)!}$$

Since only the left hand member and the last term of the right hand member have terms of degree higher than two, we can equate coefficients of terms with like power of t_1 and t_2.

Consider the two expressions

$$\frac{K_{rs} t_1^r t_2^s}{r!\, s!} \quad \text{and} \quad \frac{K_n \beta^{n-j} t_1^j t_2^{n-j}}{j!\, (n-j)!}.$$

If these are two particular terms of like degree, then we can identify the subscripts.

Let $j = r$

$n - j = s$.

We then have

$$K_{rs} = K_{s-r}\,\beta^{s}.$$

If this is true for all r and s, it must be true also for r + 1 and s - 1.

Hence

$$K_{r+1,s-1} = K_{r+s}\,\beta^{s-1}$$

or

$$K_{rs} = \frac{K_{r+1,s-1}}{\beta^{s-1}} \cdot \beta^{s}$$

$$= K_{s+1,s-1} \cdot \beta$$

Thus, we have a unique value for β. It is identifiable.

If we assume, in a particular case, that a structure such as the one above exists, we can attempt to estimate β by taking a sample from the population of observable variables y_1 and y_2 and then calculate sample values for K_{rs} and $K_{r+1,s-1}$. The ratio $K_{rs} / K_{r-1,s-1}$ gives us an estimate of β. This estimate β_* is a consistent estimate of β ; that is, as the sample size $n \rightarrow \infty$ $Pr\{|\beta_* - \beta| < \epsilon\} \rightarrow 1$, and this may be shown as follows:

Given $\lim_{n \rightarrow \infty} Pr\{|K_{rs} - \mathcal{K}_{rs}| < \epsilon\} \rightarrow 1$. That is, there exists an N such that for any given ϵ and δ

$$Pr\{|K_{rs} - \mathcal{K}_{rs}| < \epsilon\} > 1 - \delta$$

$$\lim_{n \rightarrow \infty} Pr\{|K_{r+1,s-1} - \mathcal{K}_{r+s,s-1}| < \epsilon\} > 1 - \delta$$

Therefore,

$$-\epsilon < \left| K_{rs} - \mathcal{X}_{rs} \right| < \epsilon$$

$$-\epsilon < \left| K_{r+1,s-1} - \mathcal{X}_{r+1,s-1} \right| < \epsilon$$

Consider

$$\left| \frac{K_{rs}}{K_{r+1,s-1}} - \frac{\mathcal{X}_{rs}}{\mathcal{X}_{r+1,s-1}} \right|$$

$$\left| \frac{K_{rs}}{K_{r+1,s-1}} - \frac{\mathcal{X}_{rs}}{\mathcal{X}_{r+1,s-1}} \right|$$

$$= \left| \frac{K_{rs}\mathcal{X}_{r+1,s-1} - \mathcal{X}_{r+1,s-1}\mathcal{X}_{rs} - \mathcal{X}_{rs}K_{r+1,s-1} + \mathcal{X}_{r+1,s-1}\mathcal{X}_{rs}}{K_{r+1,s-1}\mathcal{X}_{r+1,s-1}} \right|$$

$$\leq \frac{\left|\mathcal{X}_{r+1,s-1}\right| \left|(K_{rs} - \mathcal{X}_{rs})\right| - \left|\mathcal{X}_{rs}\right| \left|(K_{r+1,s-1} - \mathcal{X}_{r+1,s-1})\right|}{\left|K_{r+1,s-1}\mathcal{X}_{r+1,s-1}\right|}$$

$$\leq \frac{\left|\mathcal{X}_{r+1,s-1}\right| \left|\epsilon\right| - \left|\mathcal{X}_{rs}\right| \left|\epsilon\right|}{\left|K_{r+1,s-1}\mathcal{X}_{r+1,s-1}\right|} \rightarrow 0$$

In identifying β we make use of those cumulants with value different from zero. In fact only in the case where they are not zero is β identifiable. Therefore, $\mathcal{X}_{r+1,s-1}$ will not be zero in the linear non-normal case, and $K_{r+1,s-1}$ can be made different from zero with a large sample; hence the denominator $K_{r+1,s-1}\mathcal{X}_{r+1,s-1}$ will be different from zero.

CHAPTER III

THE RECTANGULAR, NORMAL CASE

In this chapter we will work several computational problems, specifying a particular structure, to illustrate the general example, Case II, about which we proved a theorem in the preceding chapter.

Assume S to be known to be the following (using the same notation as in Chapter II):

(1) $\eta_1 = 10 + 2\eta_2$ (Here $\alpha = 10$, $\beta = 2$.)

(2) $y_1 = \eta_1 + u_1$

$y_2 = \eta_2 + u_2$

(3) Distribution of η_2 is rectangular, with

$$P\{\eta_2 \in d\eta_2\} = \frac{1}{4}\, d\eta_2$$

and with a range for this distribution of

$$-2 < \eta_2 < 2$$

(4) Joint distribution of u_1 and u_2 is normal with

$$\sigma(u_1) = 1$$

$$\sigma(u_2) = 1$$

$$\operatorname{cov}(u_1 u_2) = \rho\, \sigma(u_1)\sigma(u_2) = \frac{1}{2}$$

$$\bar{u}_1 = 0$$

$$\bar{u}_2 = 0$$

Or, we can write

$$p(u_1,u_2) = \frac{1}{2\pi(.75)^{\frac{1}{2}}} e^{\frac{1}{2(.75)}(u_1^2 - u_1u_2 + u_2^2)}$$

Suppose, now, that we take a sample of ten random values of η_2 and a sample of ten random values each for u_1 and for u_2. To obtain ten random values of η_2 we will use a table of random numbers [2],[8]. In this table the numbers range from zero to .999 so we must transform each of the ten numbers which we select in such a manner as will give us mean zero and range of -2 to -2 for the population of η_2. To do this we multiply each of the ten numbers selected by 4 and subtract 2 from that product.

Table I below shows the ten values chosen and the corresponding η_1 and η_2.

Table I. Ten Random Values For η_1 and η_2

Value Drawn from R.N. Table	η_2	η_1 ($\eta_1 = 10 + 2\eta_2$)
.537	.148	10.296
.634	.536	11.072
.353	-.588	8.824
.634	.536	11.072
.983	1.932	13.864
.026	-1.896	6.208
.851	1.604	13.208
.646	.584	11.128
.586	.344	10.688
.349	-.604	8.792

To obtain ten random numbers for u_1 and u_2 we select ten pairs of values from a table of normal random numbers [2],[8]. In the table used, however, the correlation between the two values in each pair is zero. Our population of u_1u_2 specified by S has $\rho = \frac{1}{2}$. We must transform these ten pairs of values in such a way as to make $\rho = \frac{1}{2}$ and keep $\sigma(u_1) = \sigma(u_2) = 1$.

To obtain the desired results we will proceed as follows: Let $V_i (i = 1 \cdots 10)$ and W_i $(i = 1 \cdots 10)$ be the pairs of values drawn from the normal random number table.

Let u_1 and u_2 be a linear combination of V and W.

$$u_{1i} = A_1V_i + A_2W_i$$
$$u_{2i} = B_1V_i + B_2W_i$$

and impose the conditions

(1) $\quad \sigma^2(u_1) = A_1^2 + A_2^2 = 1$

(2) $\quad \sigma^2(u_2) = B_1^2 + B_2^2 = 1$

(3) $\quad \text{cov}(u_1u_2) = A_1B_1 + A_2B_2 = \frac{1}{2}$

Substitute values from (1) - (2) into (3):

$$\sqrt{1 - A_2^2}\,\sqrt{1 - B_2^2} + A_2B_2 = \frac{1}{2}$$

or

$$A_2^2 + B_2^2 - A_2B_2 = \frac{3}{4}$$

If we select a value of A_2 and of B_2--say, $A_2 = \sqrt{\frac{3}{4}}$ and $B_2 = 0$, then $A_1 = \frac{1}{2}$ and $B_1 = 1$.

This is one set of values for A_1, A_2, B_1, B_2 which will satisfy the condition that $\rho = \frac{1}{2}$ and $\sigma(u_1) = \sigma(u_2) = 1$. Our two transforming equations, then, will be

$$u_{1i} = V_i$$

$$u_{2i} = \frac{1}{2} V_i + \sqrt{\frac{3}{4}}\, W_i$$

The ten values obtained for u_1 and u_2 are given in Table 2 below.

Table 2. Ten Pairs of Normal Random Numbers

V	W	$\frac{1}{2} V$	$\sqrt{\frac{3}{4}}\, W$	u_1	u_2
.835	.230	.418	.199	.835	.617
.584	.089	.292	.077	.584	.369
-.366	-.880	-.183	-.762	-.366	-.945
.255	.888	.178	.769	.255	.947
.525	.282	.262	.244	.525	.506
-.547	.297	-.274	.257	-.547	-.017
1.145	.261	.572	.226	1.145	.798
-1.726	.107	-.863	.093	-1.726	-.770
-.645	.185	-.372	.160	-.645	-.212
-.406	.550	-.203	.476	-.406	.273

Having ten ϕ values for η_1, η_2, u_1, u_2 we can calculate ten values for y_1 and for y_2 by using the relations

$$y_1 = \eta_1 + u_1$$

$$y_2 = \eta_2 + u_2.$$

The values obtained are recorded in Table 3, below.

Next, we will want to compute some $k_{r+1,s-1}$ and k_{rs} to obtain an estimate of β (the true value of which is 2) from y_1 and y_2. The constants k_{rs} and $k_{r+1,s-1}$ are cumulants of the joint distribution of y_1, y_2 and it is possible to express these cumulants in terms of moments and the derivation of formulas relating the cumulants of a

Table 3. Values of y_1 and y_2

y_1	y_2
11.131	.765
11.656	.905
8.458	-1.533
11.327	1.483
14.389	2.438
5.661	-1.879
14.353	2.402
9.402	-.186
9.843	.132
8.386	-.331

joint distribution of two variables in terms of the moments of such a distribution is given in the Appendix of this paper. The moments about zero are easily calculated from combinations of powers of the values of y_1 and y_2 and the cumulants then can be calculated from these moments by using the following relations:

Let moments be written:

$$m_{rs}\ (y_1 y_2) = \frac{y_{1i}^{r}\ y_{2i}^{s}}{n}$$

Let k_{rs} indicate the cumulants

$$k_{10} = m_{10}$$

$$k_{01} = m_{01}$$

$$k_{20} = m_{20} - m_{10}^2$$

$$k_{11} = m_{11} - m_{10}m_{01}$$

$$k_{02} = m_{02} - m_{01}^2$$

$$k_{30} = m_{30} - 3m_{10}m_{20} + 2m_{10}^3$$

$$k_{21} = m_{21} - 2m_{10}m_{11} - m_{20}m_{01} + 2m_{10}^2 m_{01}$$

$$k_{12} = m_{12} - m_{10}m_{02} - 2m_{11}m_{01} + 2m_{10}m_{01}$$

$$k_{03} = m_{03} - 3m_{01}m_{02} + 2m_{01}^3$$

$$k_{40} = m_{40} - 4m_{10}m_{30} - 3m_{20}^2 + 12m_{10}^2 m_{20} - 6m_{10}^4$$

$$k_{31} = m_{31} - 3m_{10}m_{21} - m_{01}m_{30} - 3m_{20}m_{11} + 6m_{10}m_{01}m_{20} + 6m_{11}m_{10}^2 - 6m_{10}^3 m_{01}$$

$$k_{22} = m_{22} - 2m_{10}m_{12} - 2m_{01}m_{21} - m_{20}m_{02} + 2m_{01}^2 m_{20} + 8m_{11}m_{10}m_{10} - 6m_{10}^2 m_{01}^2 - 2m_{11}^2 + 2m_{10}^2 m_{02}$$

$$k_{13} = m_{13} - m_{10}m_{03} - 3m_{01}m_{12} - 3m_{11}m_{02} + 6m_{01}^2 m_{11} + 6m_{10}m_{02}m_{01} - 6m_{10}m_{01}^3$$

$$k_{04} = m_{04} - 4m_{01}m_{03} - 3m_{02}^2 + 12m_{01}^2 m_{02} - 6m_{01}^4$$

Calculations are given below.

	y_1	y_1^2	y_1^3	y_2	y_2^2	y_3^3
1	11.131	123.899	1,379.122	.765	.5852	.4477
2	11.656	135.862	1,583.611	.905	.8190	.7412
3	8.458	71.538	605.066	1.533	2.3501	3.6027
4	11.327	128.301	1,453.265	1.483	2.1993	3.2615
5	14.389	207.043	2,979.146	2.438	5.9438	14.4911
6	5.661	32.047	181.418	-1.879	3.5306	-6.6341
7	14.353	206.009	2,956.842	2.402	5.7696	13.8586
8	9.402	88.398	831.114	-.186	.0346	-.0064
9	9.843	96.885	953.636	.132	.0174	.0023
10	8.386	70.325	589.745	-.331	.1096	-.0363

$\Sigma y_1 = 104.606$		$m_{10} = 10.461$	
$\Sigma y_1^2 = 1,160.307$		$m_{20} = 116.031$	
$\Sigma y_1^3 = 15,012.986$		$m_{30} = 1,551.297$	
$\Sigma y_1^4 = 163,686.315$		$m_{40} = 16,368.632$	
$\Sigma y_2 = 7.282$		$m_{01} = .728$	
$\Sigma y_2^2 = 21.559$		$m_{02} = 2.156$	
$\Sigma y_2^3 = 29.723$		$m_{03} = 2.973$	
$\Sigma y_2^4 = 92.469$		$m_{04} = 9.247$	
$\Sigma y_1 y_2 = 104.622$		$m_{11} = 10.462$	
$\Sigma y_1 y_2^2 = 260.588$		$m_{12} = 26.059$	
$\Sigma y_1^2 y_2 = 1,650.133$		$m_{21} = 165.013$	
$\Sigma y_1^2 y_2^2 = 3,178.379$		$m_{22} = 317.838$	
$\Sigma y_1^3 y_2 = 10,371.650$		$m_{31} = 1,037.265$	
$\Sigma y_1 y_2^3 = 460.368$		$m_{13} = 46.037$	

When we substitute the above values for m_{ij} in the expressions for k_{rs} we obtain

$k_{40} = -23.466$

$k_{31} = -0.97$

estimate of $\beta = 4.102$.

Other values for the fourth order cumulants are:

$k_{22} = -1.531; \quad k_{13} = -1.217; \quad k_{04} = -1.120.$

Therefore, the other estimates of β are:

(1) using k_{31} and k_{22} , β = 4.533,

(2) using k_{22} and k_{13} , β = 1.258

(3) using k_{13} and k_{04} , β = 1.084.

The parameter β is uniquely determined by the two cumulants where the cumulants are of the population of y_1y_2. Ours was a sample of only 10.

Next, let us derive the joint probability distribution of y_1y_2 and then write its characteristic function, and take the log of the characteristic function. If we then expand the log of the characteristic function of the probability distribution of y_1y_2, we can directly obtain the true values of the cumulants which identify β.

We will recall that in our special case, $P(\eta_2)$ was a rectangular distribution such that $P\{\eta_2 \epsilon\, d\eta_2 = \frac{1}{4} d\eta_2$ and that

$$P(u_1u_2) = \frac{1}{2\pi(.75)^{\frac{1}{2}}}\, e^{-\frac{1}{2(.75)}(u_1^2 - u_1u_2 + u_2^2)}$$

The η_i are independent of the u_{1i} and u_{2i}, so we may write

$$P(u_1, u_2, \eta_2) = \frac{1}{4} \cdot \frac{1}{2\pi(.75)^{\frac{1}{2}}}\, e^{-\frac{1}{2(.75)}(u_1^2 - u_1u_2 + u_2^2)}.$$

Next we will make the transformation to y_1y_2 space, using our structural relationships.

$$y_1 = 10 + 2\eta_2 + u_1$$

$$y_2 = u_2 + \eta_2$$

and one other arbitrary one, as

$$y_3 = \eta_2$$

Solved for u_1, u_2 and η_2, we have

$$u_1 = y_1 - 10 - 2y_3$$

$$u_2 = y_2 - y_3$$

$$\eta_2 = y_3$$

The Jacobian of the transformation is

$$J = \begin{vmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

Hence we may write the joint distribution of $y_1 y_2$ and y_3

$$P(y_1, y_2, y_3) = \frac{1}{4(2\pi)(.75)^{\frac{1}{2}}} \exp -\frac{1}{2(.75)}(y_1 - 10 - 2\eta_2)^2 - (y_1 - 10 - 2\eta_2)\cdot(y_2 - \eta_2) + (y_2 - \eta_2)^2$$

The characteristic function $\phi_{y_1 y_2}(t_1 t_2) = \iiint e^{it_1 y_1 + it_2 y_2} P(y_1 y_2 y_3)\, dy_1\, dy_2\, dy_3$

or

$$\phi_{y_1 y_2}(t_1 t_2) = \frac{1}{4(2\pi)(.75)^{\frac{1}{2}}} \int_{-2}^{+2} d\eta_2 \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dy_2\, e^{it_1 y_1 + it_2 y_2}$$

$$\cdot \exp\left\{-\frac{1}{2(.75)}(y_1 - 10 - 2\eta_2)^2 - (y_1 - 10 - 2\eta_2)\cdot(y_2 - \eta_2) + (y_2 - \eta_2)^2\right.$$

Now let

$$z_1 = \frac{(y_1 - 10 - 2\eta_2)}{\sqrt{.75}} - \frac{y_2 - \eta_2}{2\sqrt{.75}}$$

$$z_2 = y_2 - \eta_2$$

Solving for y_1 and y_2, we have

$$y_1 = \sqrt{.75}\, z_1 + 10 + 2\eta_2 + \frac{z_2}{2}$$

$$y_2 = z_2 + \eta_2$$

$$J = \begin{vmatrix} \sqrt{.75} & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = \sqrt{.75}$$

Therefore,

$$\phi_{y_1 y_2}(t_1 t_2) = \frac{1}{4(2\pi)} \int_{-2}^{+2} d\eta_2 \int_{-\infty}^{+\infty} dz_1 \int_{-\infty}^{+\infty} dz_2\, e^{it_1(\sqrt{.75}z_1 + 10 - 2\eta_2) + it_2(z_2 + \eta_2)}$$

$$\cdot \exp\left\{ -\frac{1}{2(.75)^2} \left(\sqrt{.75}\, z_1 + \frac{z_2}{2} \right)^2 - \left(\sqrt{.75} + \frac{z_2}{2} \right)(z_2) + z_2^2 \right\}$$

The second exponent reduces to

$$-\frac{1}{2}(z_1^2 + z_2^2)$$

or

$$\phi_{y_1 y_2}(t_1, t_2) = \frac{1}{4\sqrt{2\pi}} \int_{-2}^{+2} d \int_{-\infty}^{+\infty} dz_2\, e^{it_1(\frac{z_2}{2} + 10 + 2\eta_2) + it_2(z_2 + \eta_2)}$$

$$\cdot e^{-\frac{1}{2}z_2^2} \cdot \frac{1}{\sqrt{2}} \int_{-\infty}^{+\infty} e^{it\sqrt{.75}z_1 - \frac{1}{2}z_1^2} dz_1$$

but

$$\frac{1}{\sqrt{2}} \int_{-\infty}^{+\infty} \exp\left\{it_1\sqrt{.75}z_1 + \frac{1}{2}z_1^2\, dz_1\right\} =$$

$$e^{-t_1^2(.75/2)} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z_1 - it_1\sqrt{.75})^2} dz_1 =$$

$$e^{-.75/2\, t_1^2}$$

and

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}z_2^2 + \frac{it_1 z_2}{2} + it_2 z_2} dz_2 =$$

$$\frac{1}{\sqrt{2\pi}} = e^{-\frac{1}{2}(t_1/2 + t_2)^2} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(t_2 - it_1/2 - it_2)^2} dz_2 =$$

$$e^{-\frac{1}{2}(t_1/2 + t_2)^2}$$

Therefore,

$$\phi_{y_1y_2}(t_1,t_2) = \frac{1}{4}\int_{-2}^{-2} d\eta_2\, e^{it_1(10 + 2\eta_2) + it_2\eta_2}\,\cdot$$

$$e^{-.75/2\, t_1^2 - \frac{1}{2}(t_1/2 + t_2)^2}$$

$$= \frac{1}{4}\exp\left\{10it_1 - .75/2\, t_1^2 - \frac{1}{8}(t_1/2 + t_2)^2\right\}\int_{-2}^{-2} e^{(2it_1 + it_2)\eta_2}\, d\eta_2$$

$$= \frac{1}{4}\exp\left\{10it_1 - .75/2\, t_1^2 - \frac{1}{2}(t_1/2 + t_2)^2\right.\cdot \left.\frac{e^{(2it_1 + it_2)\eta_2}}{2it_1 + it_2}\right|_{-2}^{-2}$$

$$= \frac{\exp\left\{10it_1 - .75/2\, t_1^2 - \frac{1}{8}(t_1/2 + t_2)^2\right\}}{4(2it_1 + it_2)}\,\cdot$$

$$\left[e^{2(2it_1 + it_2)} - e^{-2(2it_1 + it_2)}\right]$$

$$= \frac{\exp\left\{10it_1 - .75t_1^2/2 - \frac{1}{8}(t_1/2 + t_2)^2\right\}}{2(2t_1 + t_2)}\cdot \sin 2(2t_1 + t_2)$$

Having found $\phi_{y_1y_2}(t_1,t_2)$, the next step is to find $\log \phi_{y_1,y_2}(t_1,t_2)$ and then to expand this last in a power series:

$$\log \phi_{y_1,y_2}(t_1,t_2) = 10it_1 - .75/2\, t_1 - \frac{1}{8}(t_1/2 + t_2)^2 -$$

$$\log\left\{\frac{\sin 2(2t_1 + t_2)}{2(2t_1 + t_2)}\right\}$$

We must find the expansion for $\log\left\{\dfrac{\sin 2(2t_1 + t_2)}{2(2t_1 + t_2)}\right\}$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdot \cdot \cdot \cdot \cdot$$

$$\sin 2x = 2x - \frac{8x^3}{3!} - \frac{32x^5}{5!} - \cdot \cdot \cdot \cdot \cdot$$

so we may write in terms of the expressions for sin 2x

$$\log\left\{\frac{\sin 2x}{2x}\right\} = \log\left\{1 - \frac{4x^2}{3!} - \frac{16x^4}{5!} - \cdot \cdot \cdot \cdot\right\}$$

which may be rewritten as

$$\log\left\{\frac{\sin 2x}{2x}\right\} = \log\left\{1 - \left(\frac{4x^2}{3!} + \frac{16x^4}{5!} - \cdot \cdot \cdot \cdot\right)\right\}$$

In other words, the right hand expression is of the form $\log(1 + v)$, and

$$\log(1 - v) = -v - \frac{v^2}{2} - \frac{v^3}{3} - \frac{v^4}{4} - \cdot \cdot \cdot \cdot$$

Therefore,

$$\log \frac{\sin 2x}{2x} = -\frac{4x^2}{3!} + \frac{16x^4}{5!} - \cdot \cdot \cdot \cdot - \frac{16x^4}{2(3!)^2} - \cdot \cdot \cdot \cdot, \text{ etc.}$$

$$= -\frac{4x^2}{6} + \frac{16x^4}{120} - \frac{8x^4}{36} - \cdot \cdot \cdot \cdot$$

$$= \frac{4x^2}{6} - \frac{4}{45}x^4 - \cdot \cdot \cdot \cdot$$

For x we will now substitute the value $(2t_1 + t_2)$, and obtain

$$\log\left\{\frac{\sin 2(2t_1 + t_2)}{2(2t_1 + t_2)}\right\} = -\frac{4(2t_1 + t_2)^2}{6} - \frac{4}{25}(2t_1 + t_2)^4 - \cdot \cdot \cdot \cdot$$

$$= -\frac{4}{6}(4t_1^2 + 4t_1t_2 + t_2^2)$$

$$= -\frac{4}{45}(16t_1^4 + 32t_1^3t_2 + 24t_1^2t_2^2 + 8t_1t_2^3 + t_2^4) - \cdot \cdot \cdot$$

Hence, we will write

$$\text{Log } \phi_{y_1y_2}(t_1,t_2) = 10it_1 - \frac{.75}{2}t_1^2 - \frac{t_1^2}{8} - \frac{t_1t_2}{2} - \frac{t_2^2}{2}$$

$$- \frac{16}{6}t_1^2 - \frac{16}{6}t_1t_2 - \frac{4}{6}t_2^2$$

$$- \frac{64}{45}t_1^4 - \frac{128}{45}t_1^3t_2 - \frac{96}{45}t_1^2t_2^2$$

$$- \frac{32}{45}t_1t_2^3 - \frac{4}{45}t_2^4 - \cdot \cdot \cdot \cdot$$

and finally, collecting terms,

$$\text{Log } \phi_{y_1y_2}(t_1,t_2) = 10it_2 - \frac{19}{6}t_1^2 - \frac{19}{6}t_1t_2 - \frac{5}{6}t_2^2 - \frac{64}{45}t_1^4$$

$$- \frac{128}{45}t_1^3t_2 - \frac{96}{45}t_1^2t_2^2 - \frac{32}{45}t_1t_2^3 - \frac{4}{45}t_2^4 - \cdot \cdot \cdot \cdot$$

The cumulants of our joint probability distribution of y_1y_2 are, by definition, the coefficients of the expansions multiplied by the appropriate factorials; i.e., K_{rs} is the coefficient of the $t_1^rt_2^s$ term multiplied by r!s!.

To find the value of β in our structure we need a pair of cumulants $K_{r+1,s-1}$ and $K_{r,s}$. Suppose we select r = 3, s = 1, then we will use the relation

$$K_{40} = \beta \, K_{31}$$

or

$$\frac{64}{45} 4! = \beta \left(\frac{128}{45} \cdot 3! \, 1! \right)$$

$$\beta = 2.$$

Extension to General Rectangular-Normal Case

In this section we are going to carry out the procedure of finding the expressions for the cumulants for the generalized case in which the probability distribution of η_2 is rectangular and in which the errors of measurement are normally distributed with $E(u_1) = E(u_2) = 0$, but with $\sigma(u_1)$ and $\sigma(u_2)$ as well as the correlation unspecified. The structure to be considered, then, is

$$\eta_1 = b + c\,\eta_2$$

$$y_1 = \eta_1 + u_1$$

$$y_2 = \eta_2 + u_2$$

$$y_3 = \eta_2$$

$$p(u_1, u_2) = \frac{1}{2\pi(1 - {}^2)^{\frac{1}{2}} \sigma(u_1)\sigma(u_2)} \, e^{-\frac{1}{2(1-{}^2)}\left\{\frac{u_1^2}{\sigma_1^2} + \frac{2\, u_1 u_2}{\sigma_1 \sigma_2} + \frac{u_2^2}{\sigma_2^2}\right\}}$$

and the distribution of η_2 is such that $P(\eta_2 \epsilon \, d\eta_2) = \frac{1}{2A} d\eta_2$ and the range of the distribution is from $-A$ to A. Again, as in the preceding section, we want to find the probability distribution of y_1 and y_2, then its characteristic function and finally the logarithm of the characteristic function.

$$p(u_1, u_2, \eta_2) = \frac{1}{2\Lambda} \cdot \frac{1}{2\pi\sqrt{1-\rho^2}\,\sigma(u_1)\sigma(u_2)} \cdot$$

$$\exp\left\{-\frac{1}{2(1-\rho^2)}\;\frac{u_1^2}{\sigma(u_1^2)} - \frac{2\rho u_1 u_2}{\sigma(u_1)\sigma(u_2)} + \frac{u_2^2}{\sigma(u_2^2)}\right\}$$

and since

$$u_1 = y_1 - b - c\eta_2$$

$$u_2 = y_2 - y_3$$

$$\eta_2 = y_3$$

$$J = \begin{vmatrix} 1 & 0 & -c \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = 1,$$

therefore,

$$p(y_1 y_2 y_3) = \frac{1}{2\Lambda \cdot 2\pi\sqrt{1-\rho^2}\,\sigma_1\sigma_2} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(y_1 - b - c\eta_2)^2}{\sigma_1^2}\right.\right.$$

$$\left.\left. - \frac{2\rho(y_1 - b - c\eta_2)(y_2 - \eta_2)}{\sigma_1\sigma_2} + \frac{(y_2 - \eta_2)}{\sigma_2^2}\right]\right\}$$

and

$$\phi(t_1 t_2) = \frac{1}{2A\, 2\pi\sqrt{1-\rho^2}\,\sigma_1\sigma_2} \int_{-A}^{+A} d\eta_2 \int_{-\infty}^{+\infty} dy_1 \int_{-\infty}^{\infty} dy_2 \cdot e^{it_1y_1 + it_2y_2} \cdot$$

$$\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(y_1 - b - c\eta_2)^2}{\sigma_1^2} - \frac{2\rho(y_1 - b - c\eta_2)(y_2 - \eta_2)}{\sigma_1\sigma_2}\right.\right.$$

$$\left.\left. + \frac{y_2 - \eta_2}{\sigma_2^2}\right]\right\}$$

Make the transformation

$$z_1 = \frac{(y_1 - b - c\eta_2)}{\sigma_1\sqrt{1-\rho^2}} - \frac{(y_2 - \eta_2)}{\sigma_2\sqrt{1-\rho^2}} \; ; \quad z_2 = \frac{y_2 - \eta_2}{\sigma_2^2}$$

$$z_1 = \frac{1}{\sqrt{1-\rho^2}}\left(\frac{y_1 - b - c\eta_2}{\sigma_1} - \frac{\sigma_1\rho z_2}{\sigma_1}\right)$$

$$\sigma_1\rho z_2 - \sigma_1\sqrt{1-\rho^2}\,z_1 + b + c\eta_2 = y_1$$

$$\sigma_2 z_2 + \eta_2 = y_2$$

$$J = \begin{vmatrix} \sigma_1\sqrt{1-\quad^2} & \sigma_1\rho \\ 0 & \sigma_2 \end{vmatrix} = \sigma_1\sigma_2\sqrt{1-\rho^2}$$

$$\phi(t_1t_2) = \frac{1}{2A\,2\pi}\int_{-A}^{A} d\eta_2 \int_{-\infty}^{-\infty} dz_1 \int_{-\infty}^{-\infty} dz_2 \;\exp\Big[it_1(\sigma_1\rho z_2 + \sigma_1\sqrt{1-\rho^2}z_1$$

$$+ b + c\eta_2 + it_2(\sigma_2 z_2 + \eta_2)\Big]\cdot \exp\Big[-\frac{1}{2(1-\rho^2)}$$

$$(z_1\sqrt{1-\rho^2} + \rho z_2)^2 - 2\rho(z_1\sqrt{1-\rho^2} + \rho z_2)(z_2) + z_2^2\Big\}\Big]$$

$$\phi(t_1t_2) = \frac{1}{2A\,2\pi}\int_{-A}^{A} d\eta_2 \int_{-\infty}^{\infty} dz_1 \int_{-\infty}^{\infty} dz_2 \;\exp\Big[it_1(\sigma_1\rho z_2 + \sigma_1\sqrt{1-\rho^2}z_1$$

$$+ b + c\eta_2 + it_2(\sigma_2 z_2 - \eta_2)\Big]\cdot e^{-\frac{1}{2}(z_1^2 + z_2^2)}$$

$$\phi(t_1t_2) = \frac{1}{2A\,2\pi}\int_{-A}^{A} d\eta_2 \int_{-\infty}^{-\infty} dz_1 \int_{-\infty}^{-\infty} dz_2 \; e^{-\frac{1}{2}z_1^2 + it_1\sigma_1\sqrt{1-\rho^2}z_1}\cdot$$

$$e^{-\frac{1}{2}z_2^2 + it_1\sigma_1\rho z_2^2 + it_2\sigma^2 z_2}\cdot e^{it_1 b + it_1 c\eta_2 + it_2\eta_2}$$

But

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dz_1 e^{-\frac{1}{2}(z_1^2 - 2it_1\sigma_1\sqrt{1-\rho^2}z_1)}$$

$$= e^{\frac{i^2 t_1^2\sigma_1^2(1-\rho^2)}{2}}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\infty} e^{-\frac{1}{2}(z_1 - it_1\sigma_1\sqrt{1-\rho^2})^2}dz_1$$

$$= e^{-\frac{1}{2}t_1^2\sigma_1^2(1-\rho^2)}$$

and

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz_2 \, e^{-\frac{1}{2}\left[z_2^2 - 2(it_1 \sigma_1 \rho + it_2 \sigma_2) z_2\right.}$$

$$= e^{\frac{-(t_1 \sigma_1 \rho + t_2 \sigma_2)^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[z_2 - (it_1 \sigma_1 \rho + it_2 \sigma_2)\right]^2} dz_2$$

$$= e^{-\frac{1}{2}(t_1 \sigma_1 \rho + t_2 \sigma_2)^2}$$

and

$$\int_{-A}^{A} d\eta_2 e^{ibt_1 + it_1 c \eta_2 + it_2 \eta_2} =$$

$$e^{ibt_1} \int_{-A}^{A} e^{it_1 c \eta_2 + it_2 \eta_2} d\eta_2 =$$

$$\frac{e^{ibt_1} \cdot \left(e^{it_1 c \eta_2 + it_2 \eta_2}\right)\Big|_{-A}^{A}}{it_1 c + it_2} =$$

$$\frac{e^{ibt_1}\left[e^{A(it_1 c + it_2)} - e^{-A(it_1 c + it_2)}\right]}{it_1 c + it_2}$$

Therefore,

$$\phi(t_1t_2) = \tfrac{1}{2}e^{-\frac{1}{2}t_1^2\sigma_1^2(1-\rho^2) - \frac{1}{2}(t_1\sigma_1\rho + t_2\sigma_2)^2 + ibt_1} \cdot$$

$$\frac{e^{A(it_1c + it_2)} - e^{-A(it_1c + it_2)}}{A(ict_1 + it_2)}$$

but

$$\frac{e^{A(it_1c + it_2)} - e^{A(it_1c + it_2)}}{A(ict_1 + it_2)} = \frac{2\sin(At_1c + At_2)}{At_1c + At_2}$$

$$\left(\text{since } e^{i\theta} = \cos\theta - i\sin\theta;\ \frac{e^{i\theta} - e^{-i\theta}}{i\theta} = \frac{2\sin\theta}{\theta}.\right)$$

$$\phi(t_1t_2) = e^{-\frac{1}{2}t_1^2\sigma_1^2(1-\rho^2) - \frac{1}{2}(t_1\sigma_1 + t_2\sigma_2)^2 + ibt_1} \cdot$$

$$\frac{\sin(At_1c + At_2)}{At_1c + At_2}$$

and

$$\log\phi(t_1t_2) = ibt_1 - \tfrac{1}{2}t_1^2\sigma_1^2(1-\rho^2) - \tfrac{1}{2}(t_1\sigma_1\rho + t_2\sigma_2)^2$$

$$\log\left[\frac{\sin(At_1c + At_2)}{At_1c + At_2}\right]$$

Now we must find the expansion for

$$\log\left[\frac{\sin At_1c + At_2}{At_1c + At_2}\right]$$

To find $\log\left[\frac{\sin A(ct_1 + t_2)}{A(ct_1 + t_2)}\right]$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin Ax = Ax - \frac{(Ax)^3}{3!} + \frac{(Ax)^5}{5!} - \frac{(Ax)^7}{7!} + \cdots$$

$$\frac{\sin Ax}{Ax} = 1 - \frac{(Ax)^2}{3!} + \frac{(Ax)^4}{5!} - \frac{(Ax)^6}{7!} + \cdots$$

$$\log\left[\frac{\sin Ax}{Ax}\right] = \log 1 - \left[\frac{(Ax)^2}{3!} - \frac{(Ax)^4}{5!} + \frac{(Ax)^6}{7!} - \cdots\right]$$

where we can let $v = \frac{(Ax)^2}{3!} - \frac{(Ax)^4}{5!} + \frac{(Ax)^6}{7!} - \cdots$

$\log \frac{\sin Ax}{Ax} = \log(1 - v)$. Log $(1-v)$ can be expanded:

$$\log(1 - v) = -v - \frac{v^2}{2} - \frac{v^3}{3} - \frac{v^4}{4} - \cdots$$

$$\log \frac{\sin Ax}{Ax} = -\frac{(Ax)^2}{3!} + \frac{(Ax)^4}{5!} - \frac{(Ax)^6}{7!} + \cdots - \frac{(Ax)^4}{2(3!)^2} + \frac{2(Ax)^6}{3!5!} - $$

$$= -\frac{(Ax)^2}{3!} - \left(\frac{1}{5!} - \frac{1}{3!}\right)(Ax)^4 - \left(\frac{1}{7!} - \frac{2}{3!5!}\right)(Ax)^6 - \cdots$$

Now, $\frac{Ax^2}{3!} = A^2(ct_1 + t_2)^2 = \frac{A^2c^2t_1^2 + 2A^2ct_1t_2 + At_2^2}{6}$

and

$$\left[\frac{1}{5!} - \frac{1}{2(3!)}(Ax)^4\right] = \frac{2A^4}{360}(ct_1 + t_2)^4$$

$$= \frac{2}{360}(A^4c^4t_1^4 + 4A^4c^3t_1^3t_2 + 6A^4c^2t_1^2t_2^2 +$$

$$4A^4c\,t_1t_2^3 + A^4t_2^4)$$

therefore,

$$\log\left[\frac{\sin(Act_1 + At_2)}{Act_1 + At_2}\right] = -\frac{Ac^2t_1^2}{6} - \frac{Act_1t_2}{3} - \frac{At_2^2}{6} - \frac{2A^4c^4t_1^4}{360} -$$

$$\frac{2A^4c^3t_1^3t_2}{90} - \frac{2A^4c^2t_1^2t_2^2}{60} - \frac{2A^4ct_1t_2^3}{90} -$$

$$\frac{2A^4t_2^4}{360} - \cdots$$

Hence we obtain

$$\log \phi(t_1t_2) = -\tfrac{1}{2}t_1^2\sigma_1^2(1-\rho^2) - \tfrac{1}{2}(t_1\sigma_1\rho + \sigma_2 t_2)^2 + ibt_1 - \frac{Ac^2t_1^2}{6}$$

$$-\frac{Act_1t_2}{3} - \frac{At_2^2}{6} - \frac{1}{180}A^4c^4t_1^4 - \frac{2}{90}A^4c^3t_1^3t_2$$

$$-\frac{1}{30}A^4c^2t_1^2t_2^2 - \frac{1}{45}A^4c\,t_1t_2^3 - \frac{1}{180}A^4t_2^4 - \cdots$$

Here, as in the particular case in the preceding section we have only to multiply each coefficient to get the cumulants of the distribution of y_1 and y_2.

Hence the Cumulants for this distribution are:

$$k_{10} = b \qquad\qquad k_{01} = 0$$

$$k_{20} = \sigma_1^2(1-\rho^2) + \sigma_1^2\rho^2 + \frac{Ac^2}{3}$$

$$k_{11} = \tfrac{1}{2}\sigma_1\sigma_2\rho + \frac{Ac}{3}$$

$$k_{02} = \sigma_2^2 + \frac{A}{3}$$

$$k_{30} = 0; \quad k_{12} = 0; \quad k_{21} = 0; \quad k_{03} = 0;$$

$$k_{40} = \frac{-2}{15}A^4c^4; \qquad k_{13} = k_{22} = k_{31} = k_{04} = \frac{-2A^4c^4}{15};$$

etc.

It can be seen upon examination of these equations relating the parameters to the cumulants that we have in

$$k_{20} = \sigma_1^2(1-\rho^2) + \sigma_1^2\rho^2 + \frac{Ac^2}{3}$$

$$k_{11} = \tfrac{1}{2}\sigma_1\sigma_2\rho + \frac{Ac}{3}$$

$$k_{02} = \sigma_2^2 + \frac{A}{3}$$

essentially three equations in four unknowns since from the cumulants of order higher than two we obtain a unique value for c. Therefore, we cannot identify the whole structure or even any of the parameters other than band c. But if we knew any other fact about the various correlation coefficients or if we knew A, we could identify the whole structure.

AN EXAMPLE USING ACTUAL DATA

In any application of the theory discussed in the previous sections to the real world we would have to specify a structure and then gather data for the observable variables. In the problem which follows we will specify the same type of structure that we discussed in Chapter II (Theorem II); that is;

$$\eta_1 = a + b\,\eta_2$$

and the probability distribution of η_2 is assumed rectangular. Also,

$$y_1 = \eta_1 + u_1$$
$$y_2 = \eta_2 + u_2$$

The probability distribution of u_1 and u_2 is assumed normal.

The problem which we will study is the relatioship between per capita income and per capita auto registrations, by states, for the year 1947. Our observed values of per capita income will be indicated by y_1 and observed per capita auto registrations by y_2. In finding β we have the expected increase in per capita income needed for an increase of one auto registration. Calculations are given in the following pages.

State	Per Capita Income, 1947	Auto Reg. 1947	Per Capita Auto Reg. 1947
Conn.	1618	505,277	.2956
Maine	1151	180,516	.2131
Mass.	1421	889,530	.2061
N. H.	1156	114,608	.2332
R. I.	1491	178,608	.2504
Vt.	1142	92,599	.2578
Del.	1622	62,711	.2353
Md.	1454	445,041	.2444
N. J.	1540	1,028,389	.2472
N. Y.	1747	2,481,478	.1841
Penn.	1333	1,985,275	.2005
W. Va.	1042	270,961	.1425
Ala.	844	359,403	.1269
Ark.	751	241,120	.1237
Fla.	1140	553,019	.2915
Ga.	925	499,600	.1599
Ky.	847	428,423	.1506
La.	910	357,288	.1511
Miss.	681	239,092	.1095
N. C.	896	615,574	.1724
S. C.	789	356,416	.1876
Tenn	911	476,056	.1633
Va.	1083	564,676	.2109
Ariz.	1135	142,945	.2863
N. M.	1048	113,052	.2126
Okla.	959	465,686	.1993
Texas	1164	1,585,645	.2472
Ill.	1602	1,748,482	.2214
Ind.	1274	950,072	.2772
Iowa	1133	677,829	.2670
Mich.	1425	1,597,137	.3039
Minn.	1215	724,207	.2594
Mo.	1190	829,157	.219.
Ohio	1426	1,965,307	.2845
Wis.	1340	792,891	.2527
Colo.	1447	326,970	.2911
Idaho	1306	140,000	.2667
Kan.	1285	534,097	.2966
Mont.	1677	131,468	.2350
Neb.	1233	369,975	.2812
N. D.	1654	152,208	.2371
S. D.	1367	167,991	.2613
Utah	1208	148,517	.2699
Wyo.	1458	73,357	.2926
Calif.	1657	2,992,060	.4332
Nev.	1860	44,447	.4033
Ore.	1284	405,015	.3717
Wash.	1419	576,655	.3321

Σy_1 = 60,260		m_{10} = 1,255.417	
Σy_1^2 = 79,452,276		m_{20} = 1,655,255.750	
Σy_1^3 = 109,269,705,002		m_{30} = 2,276,452.187	
Σy_1^4 = 155,767,604,956,152		m_{40} = 3,245,158,416,000.	
Σy_2 = 11.5600		m_{01} = .2408	
Σy_2^2 = 3.0026		m_{02} = .0626	
Σy_2^3 = .8845		m_{03} = .0184	
Σy_2^4 = .2469		m_{04} = .00514	
Σy_1y_2 = 15,108.5370		m_{11} = 312.678	
$\Sigma y_1y_2^2$ = 4,055.3791		m_{12} = 84.487	
$\Sigma y_1^2y_2$ = 20,555,045.0033		m_{21} = 428,230.104	
$\Sigma y_1^2y_2^2$ = 5,668,571.4107		m_{22} = 118,095.238	
$\Sigma y_1^3y_2$ = 28,966,277,040.9660		m_{31} = 603,464,104.903	
$\Sigma y_1y_2^3$ = 1,160.1931		m_{13} = 24.170	

Using the relationships derived in the Appendix,

$$k_{40} = -4,434,430,000$$

$$k_{31} = -10,165,700$$

and therefore an estimate of β is $\beta^* =$ 437.90.

Since $\alpha = k_{10} = m'_{10} = 1,255.417$, our estimated relationship between η_1 and η_2 is

$$\eta_1 = 1,255.417 + 437.900\ \eta_2.$$

Or if we wanted the the relationship for η_2 given η_1, the equation is

$$\eta_2 = .2408 + .00228\ \eta_1 .$$

If we were to use the classical regression procedure, which is to assume that in finding the regression of y_2 on y_1 the measurements for y_1 are true values f0r per capita income, we would obtain the following relationship:

$$y_2 = .04377 + .000156 y_1 .$$

And similarly the regression of y_1 on y_2 is

$$y_1 = 598.7865 + 2,726.492 y_2 .$$

APPENDIX

Cumulants of a Distribution Expressed in Terms of Moments about Zero

To derive the expressions for the cumulants of a distribution in terms of the moments about zero of that distribution we first write the characteristic function for any distribution in terms of its moments:

$$\phi(t_1t_2) = 1 + i(m_{10}t_1 + m_{01}t_2) + \frac{i^2}{2!}(m_{10}^2t_1^2 + 2m_{11}t_1t_2 + m_{01}^2t_2^2) + \cdots$$

$$\log \phi(t_1t_2) = \log\left\{1 + \left[i(m_{10}t_1 + m_{01}t_2) + \frac{i^2}{2!}(m_{10}^2t_1^2 + 2m_{11}t_1t_2 + m_{01}^2t_2^2) \cdots\right]\right\}$$

Let

$$v = i(m_{10}t_1 + m_{01}t_2) + \frac{i^2}{2}(m_{10}^2t_1^2 + 2m_{11}t_1t_2 + m_{01}^2t_2^2) \cdots$$

Then we can write

$$\log \phi(t_1t_2) = \log(1 + v) = v + \frac{v^2}{2} + \frac{v^3}{3} + \frac{v^4}{4} \cdots$$

$$\log(1+v) = \Big[i(m_{10}t_1 + m_{01}t_2) + \frac{i^2}{2}(m_{20}t_1^2 + 2m_{11}t_1t_2 + m_{02}t_2^2)$$

$$+ \frac{i^3}{3!}(m_{30}t_1^3 + 3m_{21}t_1^2t_2 + 3m_{12}t_1t_2^2 + m_{03}t_2^3)$$

$$+ \frac{i^4}{4!}(m_{40}t_1^4 + 4m_{31}t_1^3t_2 + 6m_{22}t_1^2t_2^2 + 4m_{13}t_1t_2^3 + m_{04}t_2^4)$$

$$+ \cdot \cdot \cdot \Big]$$

$$- \frac{1}{2}\Big[i^2 (m_{10}t_1 + m_{01}t_2)^2$$

$$+ \frac{2i^3}{2!}(m_{20}t_1^2 + 2m_{11}t_1t_2 + m_{02}t_2^2)(m_{10}t_1 + m_{01}t_2)$$

$$+ \frac{2i^4}{3!}(m_{30}t_1^3 + 3m_{21}t_1^2t_2 + 3m_{12}t_1t_2^2 + m_{03}t_2^3)(m_{10}t_1 + m_{01}t_2)$$

$$+ \frac{i^4}{4!}(m_{20}t_1^2 + 2m_{11}t_1t_2 + m_{02}t_2^2)^2 + \cdot \cdot \cdot \Big]$$

$$+ \frac{1}{3}\Big[i^3 (m_{10}t_1 + m_{01}t_2)^3$$

$$+ \frac{3i^4}{2}(m_{10}t_1 + m_{01}t_2)^2 (m_{20}t_1^2 + 2m_{11}t_1t_2 + m_{02}t_2^2)$$

$$+ \cdot \cdot \cdot \Big]$$

$$- \frac{1}{4}\Big[i^4 (m_{10}t_1 + m_{01}t_2)^4 + \cdot \cdot \cdot \Big]$$

$$\begin{aligned}
\log(1+v) = & \Big[i m_{10} t_1 + i m_{01} t_2 + \frac{i^2}{2} m_{20} t_1^2 + i^2 m_{11} t_1 t_2 + \frac{i^2}{2} m_{02} t_2^2 \\
& + \frac{i^3}{6} m_{30} t_1^3 + \frac{i^3}{2} m_{21} t_1^2 t_2 + \frac{i^3}{2} m_{12} t_1 t_2^2 + \frac{i^3}{6} m_{03} t_2^3 \\
& + \frac{i^4}{24} m_{40} t_1^4 + \frac{i^4}{6} m_{31} t_1^3 t_2 + \frac{i^4}{4} m_{22} t_1^2 t_2^2 + \frac{i^4}{6} m_{13} t_1 t_2^3 + \frac{i^4}{24} m_{04} t_2^4 \\
& + \cdot \cdot \cdot \Big] \\
& - \frac{1}{2} \Big[i^2 m_{10}^2 t_1^2 + 2 i^2 m_{10} m_{01} t_1 t_2 + i^2 m_{01}^2 t_2^2 \\
& + i^3 m_{10} m_{20} t_1^3 + 2 i^3 m_{10} m_{11} t_1^2 t_2 + i^3 m_{10} m_{02} t_1 t_2^2 \\
& + i^3 m_{20} m_{01} t_1^2 t_2 + 2 i^3 m_{11} m_{01} t_1 t_2^2 + i^3 m_{01} m_{02} t_2^3 \\
& + \frac{i^4}{3} m_{10} m_{30} t_1^4 + i^4 m_{10} m_{21} t_1^3 t_2 + i^4 m_{10} m_{12} t_1^2 t_2^2 + \frac{i^4}{3} m_{10} m_{03} t_1 t_2^3 \\
& + \frac{i^4}{3} m_{01} m_{30} t_1^3 t_2 + i^4 m_{01} m_{21} t_1^2 t_2^2 + i^4 m_{01} m_{12} t_1 t_2^3 + \frac{i^4}{3} m_{01} m_{03} t_2^4 \\
& + \frac{i^4}{4} m_{20}^2 t_1^4 + i^4 m_{20} m_{11} t_1^3 t_2 + \frac{i^4}{2} m_{20} m_{02} t_1^2 t_2^2 \\
& + i^4 m_{11}^2 t_1^2 t_2^2 + i^4 m_{11} m_{02} t_1 t_2^3 + \frac{i^4}{4} m_{02}^2 t_2^4 \\
& + \cdot \cdot \cdot \Big] \\
& + \frac{1}{3} \Big[i^3 m_{10}^3 t_1^3 + 3 i^3 m_{10}^2 m_{01} t_1^2 t_2 + 3 i^3 m_{10} m_{01}^2 t_1 t_2^2 + i^3 m_{01}^3 t_2^3 \\
& + \frac{3}{2} i^4 m_{10}^2 m_{20} t_1^4 + 3 i^4 (m_{10} m_{01} m_{20} + m_{11} m_{10}^2) t_1^3 t_2 \\
& + \frac{3}{2} i^4 (m_{01}^2 m_{20} + 4 m_{11} m_{10} m_{01}) t_1^2 t_2^2 + 3 i^4 (m_{01}^2 m_{11} + m_{10} m_{02} m_{01}) t_1 t_2^3 \\
& + \frac{3}{2} i^4 m_{01} m_{02} t_2^4 + \cdot \cdot \cdot \Big] \\
& - \frac{1}{4} \Big[i^4 m_{10}^4 t_1^4 + 4 i^4 m_{10}^3 m_{01} t_1^3 t_2 + 6 i^4 m_{10}^2 m_{01}^2 t_1^2 t_2^2 + 4 i^4 m_{10} m_{01}^3 t_1 t_2^3 \\
& + i^4 m_{01}^4 t_2^4 + \cdot \cdot \cdot
\end{aligned}$$

$$
\begin{aligned}
\log(1+v) = {} & i m_{10}t_1 + i m_{01}t_2 + \left(\frac{i^2}{2}m_{20} - \frac{i^2}{2}m_{10}^2\right)t_1^2 + \left(i^2 m_{11} - i^2 m_{10}m_{01}\right)t_1t_2 \\
& + \left(\frac{i^2}{2}m_{02} - \frac{i^2}{2}m_{01}^2\right)t_2^2 + \left(\frac{i^3}{6}m_{30} - \frac{i^3}{2}m_{10}m_{20} + \frac{i^3}{3}m_{10}^3\right)t_1^3 \\
& + \left(\frac{i^3}{2}m_{21} - i^3 m_{10}m_{11} - \frac{i^3}{2}m_{20}m_{01} + i^3 m_{10}m_{01}\right)t_1^2t_2 \\
& + \left(\frac{i^3}{2}m_{12} - \frac{i^3}{2}m_{10}m_{02} - i^3 m_{11}m_{01} + i^3 m_{10}m_{01}\right)t_1t_2^2 \\
& + \left(\frac{i^3}{6}m_{03} - \frac{i^3}{2}m_{01}m_{02} + \frac{i^3}{3}m_{01}^3\right)t_2^3 \\
& + \left(\frac{i^4}{24}m_{40} - \frac{i^4}{6}m_{10}m_{30} - \frac{i^4}{8}m_{20}^2 + \frac{i^4}{2}m_{10}^2m_{20} - \frac{i^4}{4}m_{10}^4\right)t_1^4 \\
& + \left(\frac{i^4}{6}m_{31} - \frac{i^4}{2}m_{10}m_{21} - \frac{i^4}{6}m_{01}m_{30} - \frac{i^4}{2}m_{20}m_{11} + i^4 m_{10}m_{01}m_{20}\right. \\
& \qquad \left. + i^4 m_{11}m_{10}^2 - i^4 m_{10}^3 m_{01}\right)t_1^3t_2 \\
& + \left(\frac{i^4}{4}m_{22} - \frac{i^4}{2}m_{10}m_{12} - \frac{i^4}{2}m_{01}m_{21} - \frac{i^4}{4}m_{20}m_{02} + \frac{i^4}{2}m_{01}^2m_{20}\right. \\
& \qquad \left. + 2i^4 m_{11}m_{10}m_{01} - \frac{i^4}{2}m_{11}^2 + \frac{i^4}{4}m_{10}^2m_{02} - \frac{3}{4}i^4 m_{10}^2m_{01}^2\right) \\
& + \left(\frac{i^4}{6}m_{13} - \frac{i^4}{6}m_{10}m_{03} - \frac{i^4}{2}m_{01}m_{12} - \frac{i^4}{2}m_{11}m_{02} + i^4 m_{01}^2m_{11}\right. \\
& \qquad \left. + i^4 m_{10}m_{02}m_{01} - i^4 m_{10}m_{01}^3\right)t_1t_2^3 \\
& + \left(\frac{i^4}{24}m_{04} - \frac{i^4}{6}m_{01}m_{03} - \frac{i^4}{8}m_{02}^2 + \frac{i^4}{2}m_{01}^2m_{02} - \frac{i^4}{4}m_{01}^4\right)t_2^4 \quad +
\end{aligned}
$$

The coefficients multiplied by the appropriate factorials are the cumulants of the distribution in terms of the moments about zero. Hence:

$$K_{10} = m_{10}$$

$$K_{01} = m_{01}$$

$$K_{20} = m_{20} - m_{10}^2$$

$$K_{11} = m_{11} - m_{10}m_{01}$$

$$K_{02} = m_{02} - m_{01}^2$$

$$K_{30} = m_{30} - 3m_{10}m_{20} + 2m_{10}^3$$

$$K_{21} = m_{21} - 2m_{10}m_{11} - m_{20}m_{01} + 2m_{10}^2m_{01}$$

$$K_{12} = m_{12} - m_{10}m_{02} - 2m_{11}m_{01} + 2m_{10}m_{01}^2$$

$$K_{03} = m_{03} - 3m_{01}m_{02} + 2m_{01}^3$$

$$K_{40} = m_{40} - 4m_{10}m_{30} - 3m_{20}^2 + 12\,m_{10}^2m_{20} - 6m_{10}^4$$

$$K_{31} = m_{31} - 3m_{10}m_{21} - m_{01}m_{30} - 3m_{20}m_{11} + 6m_{10}m_{01}m_{20} + 6m_{11}m_{10}^2 - 6m_{10}^3m_{01}$$

$$K_{22} = m_{22} - 2m_{10}m_{12} - 2m_{01}m_{21} - m_{20}m_{02} + 2m_{01}^2m_{20} + 8m_{11}m_{10}m_{01} - 6m_{10}^2m_{01}^2 - 2m_{11}^2 + 2m_{10}^2m_{02}$$

$$K_{13} = m_{13} - m_{10}m_{03} - 3m_{01}m_{12} - 3m_{11}m_{02} + 6m_{01}^2m_{11} + 6m_{10}m_{02}m_{01} - 6m_{10}m_{01}^3$$

$$K_{04} = m_{04} - 4m_{01}m_{03} - 3m_{02}^2 + 12m_{01}^2m_{02} - 6m_{01}^4$$

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