ON SYMMETRIES OF KNOTS AND THEIR SURGERIES

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DISSERTATION ABSTRACT

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Title: On Symmetries of Knots and Their Surgeries

We investigate relationships between some knot invariants and symmetries of knots. In the first chapter, we recall the definitions of knots, the symmetries we will investigate, and some classical knot invariants including the signature, the genus, and the Alexander polynomial.

In the second chapter we investigate the relation between the knot Floer homology of a periodic knot and the knot Floer homology of its quotient knot. Specifically, we prove a rank inequality between them using a spectral sequence of Hendricks, Lipshitz, and Sarkar. We further conjecture a filtration on this inequality for which we provide evidence and consequences including a signature inequality for alternating periodic knots.

In the third chapter we define Dehn surgery, and discuss covering maps between Dehn surgeries on the same knot. We classify such covers for torus knots, and conjecture some strong restrictions on when such a covering can occur for hyperbolic knots. We check this conjecture for knots with 8 or fewer crossings.

In the final chapter we prove that the quotient of a definite knot is definite.

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CHAPTER I

INTRODUCTION

1.1. Background

What is a knot?

A *knot* is a closed loop in 3-dimensional space, and two knots are equivalent if one can be deformed into the other without cutting and rejoining the loop (or passing the loop through itself). More precisely, a knot is a smooth embedding of S^1 into S^3, and two knots are equivalent if there is a smooth isotopy between them. Any knot which can be deformed into the leftmost knot shown in Figure 1 is called an *unknot*. When drawing a knot, we indicate that a strand crosses behind another strand by leaving a gap. See Figure 1 for some knot diagrams with such crossings.

Our main interest throughout this document is symmetries of knots, mostly rotational symmetries. We call a knot with an n-fold rotational symmetry *n-periodic*. For example, the trefoil in Figure 1 is 3-periodic as can be easily seen from the center diagram. More precisely, a knot K is n-periodic if there exists a

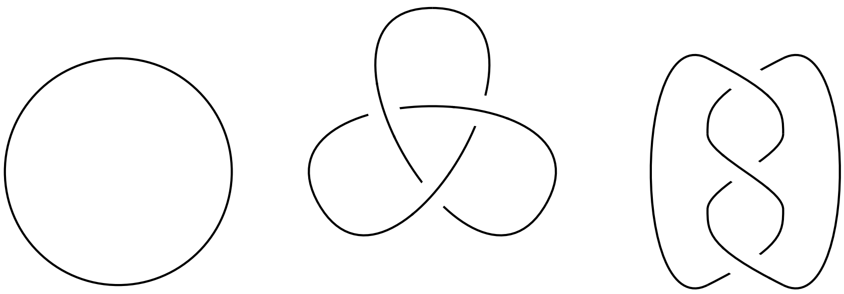

FIGURE 1. An unknot (left), the left-handed half-hitch, or left-handed trefoil (center), and another diagram of the left-handed trefoil (right).

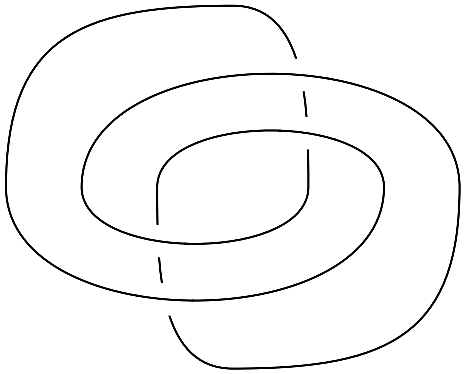

FIGURE 2. A 2-periodic diagram for the left-handed trefoil.

$\mathbb{Z}/n\mathbb{Z}$ action on S^3 which preserves K set-wise and has fixed set an unknot disjoint from K. We will refer to this fixed set as the *axis* of rotation.

Sometimes periodic actions on knots are obvious, but it may be the case that a given diagram for a knot does not display all or even any of the rotational symmetries of the knot. However, there does always exist some diagram which displays any given symmetry. The trefoil in Figure 1, for example, is also 2-periodic, as can be seen by the diagram in Figure 2.

Given a periodic knot, we are also interested in considering the *quotient knot*. This is the knot you get by cutting an n-periodic diagram into n equivalent pieces, discarding all but one of them, and then connecting the loose ends without crossing them. The quotient of the 3-periodic action on the trefoil is shown in Figure 3. More precisely, the quotient knot is the image of K in the quotient of S^3 under the $\mathbb{Z}/n\mathbb{Z}$ rotation. Note that the quotient of S^3 by a finite order rotation is again S^3.

The broad questions we are interested in studying include:

1. How can you tell if a knot is periodic?

2. If you know a knot is periodic, what can you say about its quotient knot?

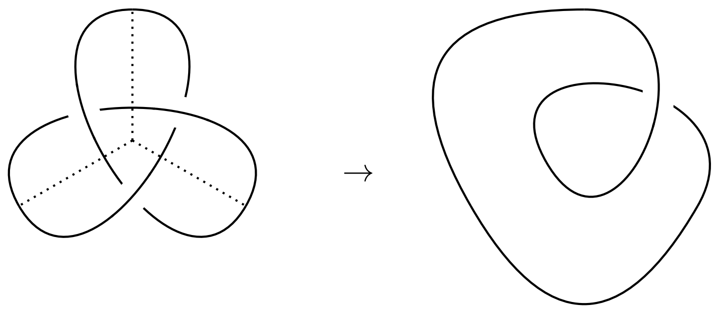

FIGURE 3. The quotient of the trefoil under its 3-periodic action is the unknot.

3. How do various properties and invariants of periodic knots interact with the periodic action?

To approach these questions, we will use several knot invariants. An *invariant* of a knot is any other mathematical object which we can assign to a knot and does not change under deformation (isotopy) of the knot. These are useful for many reasons, for example as a method of proving that two knots which look different actually are not isotopic. The main knot invariants we will use are the signature and the genus (which are integers), the Alexander polynomial, knot Floer homology, and Dehn surgery, the first three of which will be defined briefly in the following section.

Knot Invariants

We first consider the notion of a Seifert surface for a knot K. A *Seifert surface* is a 2-dimensional surface S which has a single edge (boundary) equal to K, and which is orientable. That is, it has two distinct sides, as opposed to a surface like a Möbius strip which has only one side. See Figure 4 for a Seifert surface for the trefoil.

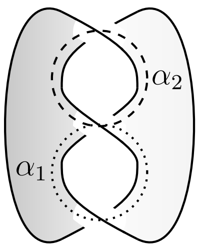

FIGURE 4. An orientable surface with boundary the trefoil. One side is shaded light gray, and the other side is shaded dark gray. At each of the three points where the boundary crosses over itself, there is a half-twisted band connecting the left and right regions of the surface. The curves α_1 and α_2 form a basis for H_1 of this surface.

From a Seifert surface we can define the genus, signature, and Alexander polynomial of a knot. The *genus* of a knot K is the minimal possible genus of a Seifert surface for K. The *genus* of a surface S is half of the rank of $H_1(S;\mathbb{Z})$, which can be thought of as half the number of holes in the surface.

To define the signature and Alexander polynomial, we need additionally the Seifert matrix. Let n be twice the genus of the Seifert surface S, and let $\{\alpha_i\}$ be any basis for $H_1(S;\mathbb{Z})$. For example, we could choose the α_i to each wrap once around one of the n holes in the surface S. Then a *Seifert matrix* for S is an n by n matrix where the (i,j)th entry counts the linking number of α_i pushed slightly to one side of S with α_j pushed slightly to the other side of S. Here the linking number just counts (with sign) the number of times one curve wraps around the other. Note that this depends on the choice of α_i as well as the choice of S and so the Seifert matrix itself is not a knot invariant.

Looking back at Figure 4, n is 2, and we can choose α_1 and α_2 as shown. Then the Seifert matrix is $M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Indeed, α_1 and α_2 each link with themselves after being pushed off the surface, and pushing α_1 and α_2 off in one way links them, and in the other way does not.

The *signature* of a knot is defined to be the signature of the symmetric bilinear form given by $M + M^T$, where M^T is the transpose of M. That is, the number of positive eigenvalues minus the number of negative eigenvalues. It turns out that the signature does not depend on which Seifert surface we had, so it is a knot invariant, which we denote $\sigma(K)$. In our example, $M + M^T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, which has two positive eigenvalues 3 and 1, so the signature of the left-handed trefoil is 2. The signature of the unknot turns out to be 0, which can be used to verify that the trefoil and unknot are not isotopic.

The *Alexander polynomial* of K is the polynomial $\Delta_K(t)$ defined to be the determinant of $M - tM^T$. In our example, $M - tM^T = \begin{pmatrix} 1-t & -t \\ 1 & 1-t \end{pmatrix}$, which has determinant $t^2 - t + 1$. After a normalization, this again turns out to be a knot invariant. For more details about these definitions, see [Lic97].

We will defer the definition of Dehn surgery until chapter III, and omit entirely any definition of knot Floer homology since it is somewhat more complicated to define. However, we note that knot Floer homology has the structure of a bigraded abelian group, and that it can easily recover both the Alexander polynomial and the genus.

1.2. Summary of Results

In Chapter II we discuss the relationship between the knot Floer homology of a periodic knot and the knot Floer homology of its quotient. In particular we prove an inequality relating the coefficients of these polynomials, and conjecture a stronger version. As a consequence, we get a relationship between the Alexander polynomials and the signatures of these knots.

In Chapter III we consider Dehn surgeries. In particular, we consider two Dehn surgeries on the same knot and ask whether there is a covering map between them. This can be considered as a hidden type of symmetry of the knot.

In Chapter IV we consider knots for which the absolute value of the signature is twice the genus. Such a knot is called *definite*, and we prove that the quotient knot of a periodic definite knot is again definite.

CHAPTER II

RANK INEQUALITIES ON KNOT FLOER HOMOLOGY OF PERIODIC KNOTS

2.1. Introduction

Periodic knots have been studied extensively, and although hyperbolic geometry and other tools can often determine a knot's periods and quotients in specific cases, many relations between periodic knots and knot invariants are unknown. Useful tools for these questions come from Murasugi, who proved in [Mur71] that the Alexander polynomial of the quotient knot divides the Alexander polynomial of the periodic knot, and Edmonds, who proved in [Edm84] an inequality involving the genus of the periodic knot and the genus of the quotient.

A potential newer tool to study these questions is the knot invariant called knot Floer homology, developed by Ozsváth and Szabó [OS04] and independently Rasmussen [Ras03]. Knot Floer homology is a bigraded abelian group $\widehat{HFK}_i(K,a)$, which is defined using techniques from symplectic geometry. This invariant categorifies the Alexander polynomial in the sense that the Alexander polynomial is the Euler characteristic of $\widehat{HFK}$ [OS08]. Since the Alexander polynomial is useful for studying periodic knots, it is natural to expect that $\widehat{HFK}$ is as well.

Some work has already been done in the direction of understanding the relationship between periodic knots and knot Floer homology. Hendricks [Hen15], with refinement by Hendricks, Lipshitz, and Sarkar [HLS16], developed a spectral sequence from $\widehat{HFK}(\widetilde{K})$ to $\widehat{HFK}(K)$ for 2-periodic knots $\widetilde{K}$, using a localization theorem of Seidel and Smith [SS10].

This chapter concerns Theorem 2.1.1, a corollary of the spectral sequence [HLS16, Theorem 1.16], and Conjecture 2.1.2, a refinement of this rank inequality filtered by homological grading. Theorem 2.1.1 and Conjecture 2.1.2 each give new information about the Alexander polynomials of periodic knots.

Theorem 2.1.1. *Let $\widetilde{K}$ be a 2-periodic knot in S^3 with quotient knot K. Let λ be the linking number of the axis with K. Then there is a rank inequality*

$$\sum_i \operatorname{rank}\left(\widehat{HFK}_i(\widetilde{K}, 2a + \frac{\lambda-1}{2}) \oplus \widehat{HFK}_i(\widetilde{K}, 2a + \frac{\lambda+1}{2})\right) \geq \sum_i \operatorname{rank} \widehat{HFK}_i(K, a)$$

for all $a \in \mathbb{Z}$.

The following conjecture proposes a Maslov grading filtered version of the rank inequality in Theorem 2.1.1.

Conjecture 2.1.2. *Let $\widetilde{K}$ be a 2-periodic knot in S^3 with quotient knot K and axis A, and let λ be $lk(K, A)$. Then for all $a, q \in \mathbb{Z}$,*

$$\sum_{i \geq q} \operatorname{rank}\left(\widehat{HFK}_i(\widetilde{K}, \widetilde{a}) \oplus \widehat{HFK}_i(\widetilde{K}, \widetilde{a}+1)\right) \geq \sum_{2i \geq q+1} \operatorname{rank} \widehat{HFK}_i(K, a)$$

and

$$\sum_{i \leq q} \operatorname{rank}\left(\widehat{HFK}_i(\widetilde{K}, -\widetilde{a}) \oplus \widehat{HFK}_i(\widetilde{K}, -\widetilde{a}-1)\right) \geq \sum_{2i \leq q-1} \operatorname{rank} \widehat{HFK}_i(K, -a),$$

where $\widetilde{a} = 2a + \dfrac{\lambda-1}{2}$.

Note that the second inequality in this conjecture would follow from the first by taking the mirrors of K and $\widetilde{K}$.

Organization

In Section 2.2 we lay out the motivation for Theorem 2.1.1 and Conjecture 2.1.2, and prove the corresponding statements in Morse homology. In Section 2.3 we prove Theorem 2.1.1, and state some additional theorems on knot Floer homology which will be useful in Section 2.4. In Section 2.4 we prove applications of Theorem 2.1.1 and Conjecture 2.1.2 to the Alexander polynomial. Finally, in Section 2.5 we provide computational and theoretical evidence for Conjecture 2.1.2, and explain where the proof in Section 2.2 breaks down when applied to knot Floer homology.

2.2. Motivation from Morse Homology

Floer homology theories are modeled on Morse homology, and Theorem 2.1.1 and Conjecture 2.1.2 are Floer-theoretic analogs of rank inequalities in Morse homology. Specifically, Theorem 2.1.1 is an analog of the following classical result of Smith theory.

Theorem 2.2.1. *Let X be finite-dimensional G-CW complex for a finite order p-group G, with fixed set F. Then*

$$\sum_{i\in\mathbb{Z}} \operatorname{rank} H_i(X;\mathbb{F}_p) \geq \sum_{i\in\mathbb{Z}} \operatorname{rank} H_i(F;\mathbb{F}_p).$$

A first attempt at refining this statement might be to restrict the inequality to each homological grading. However, this is immediately false. Consider the case that $X = S^2$, and $G = \mathbb{F}_2$ acts by reflection so that $F = S^1$. Then $H_1(S^2;\mathbb{F}_2) = 0$, but $H_1(S^1;\mathbb{F}_2) \neq 0$.

However, with more care two refinements to this inequality have been shown. One is our model for Conjecture 2.1.2 and comes from the following result of Floyd. Another was proved more recently in [May87]. We have also included a modern proof of Floyd's theorem here in the hope that it may be adapted to the knot Floer homology case. See Section 2.5.3 for further discussion.

Theorem 2.2.2. *[Flo52, Theorem 4.4] Let X be a locally compact finite dimensional Hausdorff space. Let τ be a periodic map on X of prime period p, and let F be the fixed set of τ. Then for all $n \in \mathbb{Z}$*

$$\sum_{i \geq n} \operatorname{rank} H_i(X; \mathbb{F}_p) \geq \sum_{i \geq n} \operatorname{rank} H_i(F; \mathbb{F}_p).$$

Floyd's original proof of this fact uses certain long exact sequences in homology. However, in the case where X is a finite dimensional $\mathbb{Z}/p$-CW complex, we can reprove this statement using a spectral sequence similar to (2.3.1). We will restrict to the case $p = 2$ for simplicity. The key step in the proof which does not immediately generalize to the knot Floer homology case is the following lemma.

Lemma 2.2.3. *Let $C_*(X)$ be the complex of cellular chains on X. Then the subspace of $C_*(X)$ generated by fixed cells is a subcomplex, $C_*^{fix}(X)$.*

Proof. By definition of G-CW complex, if a cell has a fixed point then the entire cell is fixed, and by continuity of the group action if a cell is fixed then so is its boundary. □

We will apply this lemma in the context of the following bicomplex of cellular chains on X.

$$\dots \xleftarrow{1+\tau} C_*(X) \xleftarrow{1+\tau} C_*(X) \xleftarrow{1+\tau} \dots$$

Consider the spectral sequence ${}^hE^r_{p,q}$ coming from taking the horizontal $(1+\tau)$ differentials first. Note that this specrtral sequence converges since it is bounded both above and below in the q grading.

Lemma 2.2.4. *This spectral sequence converges to*

$$H_q(F) \otimes \mathbb{F}_2[u, u^{-1}] \cong \bigoplus_{p+q=i} {}^hE^\infty_{p,q},$$

where u is in p-grading 1 and q-grading 0, and the isomorphism respects only the $(p+q)$-grading.

Proof. The E^1 page is $C_i(F) \otimes \mathbb{F}_2[u, u^{-1}]$, since any basis for the image of $1+\tau$ can be extended to the kernel of $1+\tau$ by adding exactly the cells which generate $C_i(F)$. Then the differential on the E^2 page is precisely the differential in $C_i(F)$, and all further differentials are 0. Indeed, a non-zero differential on a subsequent page would include a non-zero map from a fixed cell to a non-fixed cell, contradicting Lemma 2.2.3. □

On the other hand we also have a spectral sequence ${}^vE^r_{p,q}$ from taking the vertical differentials first. This spectral sequence has

$$H_q(X) \otimes \mathbb{F}_2[u, u^{-1}] \cong \bigoplus_{p+q=i} {}^vE^1_{p,q},$$

where again u is in q-grading 0 and p-grading 1. However, this spectral sequence must converge to the same homology as ${}^hE^r_{p,q}$ since X is finite-dimensional and hence has a bounded cellular chain complex. Hence we get a spectral sequence from

$H_i(X) \otimes \mathbb{F}_2[u, u^{-1}]$ to $H_i(F) \otimes \mathbb{F}_2[u, u^{-1}]$. This implies the classical Smith inequality

$$|H_*(X; \mathbb{F}_2)| \geq |H_*(F; \mathbb{F}_2)|$$

where $|H_*(X; \mathbb{F}_2)|$ is the total dimension of $H_*(X; \mathbb{F}_2)$.

We would like to refine this result to be filtered by the vertical grading in the spectral sequence. To do so, we will need the following definitions and proposition, which apply more generally to any bicomplex of $\mathbb{F}_2$-vector spaces. In this setting we will refer to the horizontal differential as ∂_h and the vertical differential as ∂_v.

Definition 2.2.5. A *square* is any bicomplex of $\mathbb{F}_2$-vector spaces consisting of four non-zero generators a, b, c, and d with $\partial_h(b) = a, \partial_h(d) = c, \partial_v(a) = c$, and $\partial_v(b) = d$. That is, any bicomplex of the form as shown in the left part of Figure 5.

Similarly, a *staircase* is any bicomplex of $\mathbb{F}_2$-vector spaces as shown in the right part of Figure 5. That is, a collection of generators $\{a_i, b_i | 0 \leq i \leq n\}$ with $\partial_h(b_i) = a_i$ and $\partial_v(b_i) = a_{i+1}$, where a_0 or b_n may be 0, but all other a_i and b_i are non-zero.

The *length* of a staircase is the number of isomorphisms $\partial_h(b_i) = a_i$ and $\partial_v(b_i) = a_{i+1}$ in the diagram, so that a staircase of length 0 is a single generator, and a staircase of length 1 is a single isomorphism between generators.

This terminology allows us to break apart bicomplexes into understandable pieces. The following Proposition is also proved in [Ste18].

Proposition 2.2.6. *[Kho07] Vertically bounded bicomplexes of $\mathbb{F}_2$-vector spaces which are finite dimensional in each bigrading decompose as direct sums of staircases and squares.*

To prove this proposition, we first give the following lemmas and definition.

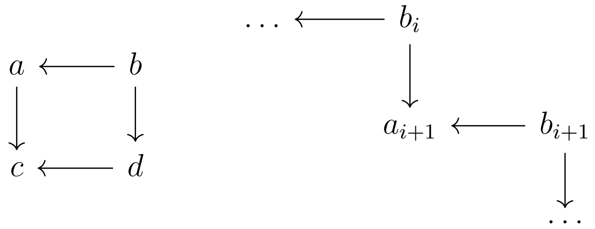

FIGURE 5. Square (left) and staircase (right) bicomplexes.

Lemma 2.2.7. *Every square subcomplex of a bicomplex of* $\mathbb{F}_2$*-vector spaces is a direct summand.*

Proof. Any bicomplex C of $\mathbb{F}_2$-vector spaces is a module over $\mathbb{F}_2[x, y]/(x^2, y^2)$, which is a Frobenius algebra. Any square is a rank 1 free module, and hence projective. However, projective modules over a Frobenius algebra are injective as well, and hence summands. $\square$

Definition 2.2.8. Let $S = \{a_i, b_i | 0 \leq i \leq n\}$ and $S' = \{a'_i, b'_i | 0 \leq i \leq m\}$ be a pair of disjoint staircase summands of a bicomplex C such that the bigrading of a_0 is the same as the bigrading of a'_0, the length of S is less than or equal to the length of S', and either $a_0, a'_0 \neq 0$ or $a_0 = a'_0 = 0$. That is, S and S' occupy the same diagonal, S is not longer than S', and they begin in the same bigrading. Then let *the sum of* S *and* S', $S + S'$, be the staircase $\{a_i + a'_i, b_i + b'_i | 0 \leq i \leq n\} \cup \{a'_i, b'_i | n < i \leq m\}$ if $b_n \neq 0$, or $\{a_i + a'_i, b_i + b'_i | 0 \leq i \leq n\}$ if $b_n = 0$.

Lemma 2.2.9. *In the notation above,* $S + S'$ *is a summand of* C*. Furthermore, if* $b_n = 0$ *then* $S \oplus S' = (S + S') \oplus S'$*, and if* $b_n \neq 0$*, then* $S \oplus S' = (S + S') \oplus S$*. In particular, we can replace one of* S *or* S' *with* $S + S'$ *in a staircase decomposition of* C*. See Figure 6 for an example.*

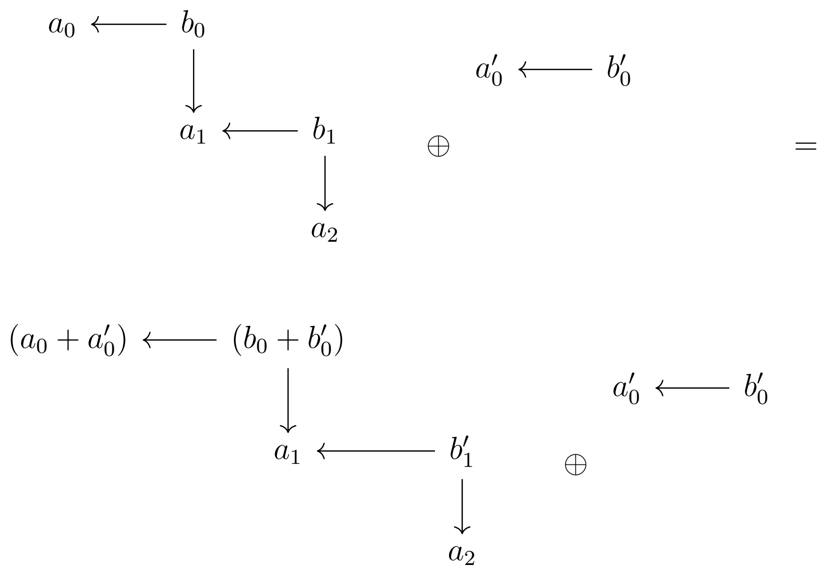

FIGURE 6. Replacing a staircase with the sum of two staircases in a direct sum.

Proof. It is clear that if $b_n = 0$, then $S \cup S' = S' \cup (S + S')$ and if $b_n \neq 0$, then $S \cup S' = S \cup (S + S')$, and so to check that $S + S'$ is a summand of C it is enough to check that it is a summand of $S \cup S'$. The only place where this might fail is at a_n or b_n where S ends.

First, if $b_n = 0$, then S ends in a vertical differential, and indeed the final element $a_{n-1} + a'_{n-1}$ of $S + S'$ is not in the image of ∂_h since a_{n-1} is not but a'_{n-1} is.

Second, if $b_n \neq 0$ then S ends in a horizontal differential, and applying the vertical differential to the element $b_n + b'_n$ of $S + S'$ is exactly b'_{n+1} so that $S + S'$ is a staircase, as desired. □

Proof of Proposition 2.2.6. Let C be a vertically bounded bicomplex of $\mathbb{F}_2$ vector spaces which is finite dimension in each bigrading and with horizontal differential ∂_h and vertical differential ∂_v. By Lemma 2.2.7, any square subcomplex of C is a summand, so we can quotient these out to get a new bicomplex without any square subcomplexes. We therefore assume there are no square subcomplexes, and in particular, no compositions of horizontal and vertical isomorphisms

$$\begin{array}{ccc} y & \xleftarrow[\partial_h]{} & x \\ \downarrow{\scriptstyle \partial_v} & & \\ z & & \end{array} \qquad \text{or} \qquad \begin{array}{ccc} & & x \\ & & \downarrow{\scriptstyle \partial_v} \\ z & \xleftarrow[\partial_h]{} & y \end{array},$$

since either of these would necessarily complete to a square by commutativity of the bicomplex. That is, $\partial_h \circ \partial_v = \partial_v \circ \partial_h = 0$.

We now claim that there is a choice of basis which splits C into a direct sum of staircases. We will prove this claim by induction on the number of non-trivial vertical degrees.

For the base case, we have a single horizontal chain complex. We first choose a basis for the image of ∂_h, then extend it to a basis for the kernel of ∂_h. Now we choose preimages of the kernel basis elements, and use these elements to extend the basis to the entire complex. By construction this decomposes our complex into trivial staircases (basis elements which are in the kernel but not the image of ∂_h), and length 1 staircases (isomorphisms between the 1-dimensional subspaces spanned by basis elements given by ∂_h). Furthermore, each of these is a summand.

Now consider a bicomplex C' with bounded vertical degrees, which by the inductive assumption has a basis which decomposes it into staircase summands. We will add a horizontal chain complex C_{top} in a new top vertical degree to get a complex $C = C_{top} \to C'$, and we will construct a staircase summand in C which begins in an arbitrary grading of C_{top}.

We consider two cases. To begin, suppose (C_{top}, ∂_h) is not exact, and choose a basis for C_{top} splitting it into staircase summands as in the base case, and choose a basis element $a \in C_{top}$ which is in the kernel but not the image of ∂_h. We will construct a staircase summand containing a. Any staircase containing a must start at a since $\partial_h(a) = 0$, so it remains to consider $\partial_v(a)$.

In this direction, write $\partial_v(a) = b_1 + b_2 + \cdots + b_n$ for some basis elements b_i in C'. Each b_i is contained in a unique staircase summand in C' by the inductive assumption, and since $\partial_h \circ \partial_v = 0$, $\partial_h(b_i) = 0$ so that these staircases all start in the same bigrading. Now by Lemma 2.2.9 and induction we can find a change of basis for C' decomposing it into new staircase summands (of C') so that $b_1+b_2+\cdots+b_n = b_1'$ is a basis element and hence contained in one of the staircases.

We then have a staircase in C, but it may be the case that for some other basis elements c in C_{top} and $\{b_i'\}$ in C', $\partial_v(c) = b_1' + b_2' + \cdots + b_m'$. That is, it is not obvious in this basis that our staircase is a summand. To fix this, we will change the basis of C_{top} by replacing c with $c + a$, and repeat as necessary until for any basis element c in C_{top} the image under ∂_v is a disjoint sum of basis elements from $\partial_v(a)$. Since $\partial_v \circ \partial_h = 0$, none of these basis elements are in the image of ∂_h, and so we then have a staircase summand in C.

Alternatively, suppose that (C_{top}, ∂_h) is exact. Then choose a basis element a in C_{top} which is not in the image of ∂_h, and apply basis changes as above. It then remains to consider $\partial_h(a)$. By construction this is a basis element, and so our staircase will be a summand unless $\partial_h(c) = \partial_h(a)$ for some other basis element c in C_{top}. In this case replace c with $c + a$ in the basis, and since C_{top} is exact, $c + a$ is in the image of ∂_h and hence in the kernel of ∂_v since $\partial_v \circ \partial_h = 0$. In particular, this change will preserve the condition that $\partial_v(c)$ is a disjoint set of basis elements from $\partial_v(a)$.

Now for any complex C_{top} we have constructed a staircase summand for C, and hence by induction we can decompose C into staircase summands since C is finite dimensional in each bigrading. □

We now return to the bicomplex of cellular chains on X, and give a final lemma before completing the proof of Theorem 2.2.2.

Lemma 2.2.10. *There exists a decomposition of the bicomplex*

$$\dots \xleftarrow{1+\tau} C_*(X) \xleftarrow{1+\tau} C_*(X) \xleftarrow{1+\tau} \dots$$

as in Proposition 2.2.6 such that each staircase with $a_0 = 0$ is length 1.

Proof. Start with the decomposition into summands given from Proposition 2.2.6. Consider a summand S consisting of a single staircase of length greater than 1, and for which $a_0 = 0$. That is, S begins with a vertical isomorphism $d(b_0) = b_1 + \tau b_1$. Then observe that $b_0 + \tau b_0 = 0$, and hence b_0 is fixed by τ. Now we can write $b_0 = \alpha + \beta + \tau\beta$ where $\alpha \in C_*^{\text{fix}}(X)$ and β is in the subspace spanned by generators which are not fixed by τ.

Since $C_*^{\text{fix}}(X)$ is a subcomplex by Lemma 2.2.3, $d(\alpha) = 0$. This implies that $d(\beta + \tau\beta) = d(b_0) = b_1 + \tau b_1$, and hence that $\beta + \tau\beta \xrightarrow{d} b_1 + \tau b_1$ was not part of a square summand. In particular then, we have

$$\begin{array}{ccc} \beta + \tau\beta & \xleftarrow[1+\tau]{} & \beta \\ \downarrow{\scriptstyle d} & & \\ b_1 + \tau b_1 & & \end{array}$$

is contained in S, which is a contradiction with Proposition 2.2.6. □

Proof of theorem 2.2.2 in the case $p = 2$ and X is an $\mathbb{Z}/p$-CW complex. Combining Proposition 2.2.6 and Lemma 2.2.10, we see that all generators of ${}^{v}E^{\infty}_{p,q}$ are represented by staircases in the bicomplex with $a_0 = 0$ and $b_n = 0$. That is, staircases which end with a horizontal arrow on the top, and a vertical arrow on the bottom.

Now for any generator of $H_*(F)$, consider the staircase that represents it in the bicomplex. The corresponding generator on ${}^{v}E^{1}_{p,q}$ will be in a higher (or equal if the staircase has length 0) vertical grading than the generator in ${}^{h}E^{1}_{p,q}$. This gives the desired inequality since the vertical grading on ${}^{v}E^{1}_{p,q}$ gives the grading on $H_*(X)$, and the vertical grading on ${}^{h}E^{1}_{p,q}$ gives the grading on $H_*(F)$. □

2.3. Knot Floer Homology Background

In this section we will prove Theorem 2.1.1, and recall some other useful theorems on knot Floer homology. Throughout the rest of the chapter, let $\widetilde{K}$ be a 2-periodic knot with axis $\widetilde{A}$, and let K be the quotient knot with axis A. Let λ be the linking number of K with A. We now deduce Theorem 2.1.1 using [HLS16, Theorem 1.16].

Theorem 2.1.1. There is a rank inequality

$$\sum_i \operatorname{rank}\left(\widehat{HFK}_i(\widetilde{K}, 2a + \frac{\lambda - 1}{2}) \oplus \widehat{HFK}_i(\widetilde{K}, 2a + \frac{\lambda + 1}{2})\right) \geq \sum_i \operatorname{rank} \widehat{HFK}_i(K, a)$$

for all $a \in \mathbb{Z}$.

Proof. Let V and W be 2-dimensional vector spaces with gradings as shown in figure 7.

$V:$

$\mathrm{gr}(V)$	0	1
-1	$\mathbb{F}_2$	0
0	0	$\mathbb{F}_2$

$W:$

$\mathrm{gr}(W)$	-1	0
0	$\mathbb{F}_2$	$\mathbb{F}_2$

FIGURE 7. The 2-dimensional vector spaces V and W. Each columns is a Maslov grading, and each row is an Alexander grading.

Then [HLS16, Theorem 1.16] provides a spectral sequence

$$\widehat{HFK}_*(\widetilde{K}) \otimes V \otimes W \otimes \mathbb{F}_2[\theta, \theta^{-1}] \Rightarrow \widehat{HFK}_*(K) \otimes W \otimes \mathbb{F}_2[\theta, \theta^{-1}] \tag{2.3.1}$$

which splits along Alexander gradings, taking the grading $2a + \dfrac{\lambda - 1}{2}$ on the E^1 page to the grading a on the E^∞ page, and collapsing elements in gradings of the other parity on the E^1 page to 0 on the E^∞ page.

In particular, looking at the grading $\widetilde{a} = 2a + \dfrac{\lambda - 1}{2}$ on the E^1 page, there are exactly two gradings ($\widetilde{a}$ and $\widetilde{a} + 1$) in $\widehat{HFK}(\widetilde{K})$ which contribute to that $\widetilde{a}$ grading in the tensor product. Furthermore, these two gradings do not contribute to any other gradings in the tensor product. Hence the spectral sequence (2.3.1) gives the result. □

The following theorems of Ozsváth and Szabó characterize knot Floer homology for alternating knots and L-space knots respectively in such a way that they can be recovered from the Alexander polynomial. These will be useful in obtaining applications of Conjecture 2.1.2.

Theorem 2.3.2. *[OS03, Theorem 1.3] Let $K \subset S^3$ be an alternating knot, and write its (symmetrized) Alexander polynomial as*

$$\Delta_K(t) = a_0 + \sum_{s>0} a_s(t^s + t^{-s}).$$

Then $\widehat{HFK}(S^3, K, s)$ *is supported entirely in Maslov grading* $s + \sigma(K)/2$*, and*

$$\widehat{HFK}(S^3, K, s) \cong \mathbb{Z}^{|a_s|}.$$

Theorem 2.3.3. *[OS05, Theorem 1.2] Let* $K \subset S^3$ *be an L-space knot. Then there is an increasing sequence of integers*

$$n_{-k} < \cdots < n_k$$

with $n_i = -n_{-i}$*, and for* $-k \leq i \leq k$*,* $\widehat{HFK}(K, a) = 0$ *unless* $a = n_i$ *for some* i*. In this case* $\widehat{HFK}(K, a) \cong \mathbb{Z}$ *and is supported entirely in dimension* δ_i*, where*

$$\delta_i = \begin{cases} 0 & \text{if } i = k \\ \delta_{i+1} - 2(n_{i+1} - n_i) + 1 & \text{if } k - i \text{ is odd} \\ \delta_{i+1} - 1 & \text{if } k - i > 0 \text{ is even.} \end{cases}$$

2.4. Consequences of a Filtered Rank Inequality

The goal of this section is to prove some interesting consequences of Theorem 2.1.1 and Conjecture 2.1.2. Specifically, we will prove some restrictions on the Alexander polynomials of certain periodic knots. To begin, we restate the conjecture.

Conjecture 2.1.2. Let $\widetilde{K} \subset S^3$ be 2-periodic with quotient knot K. Then for all $a, q \in \mathbb{Z}$,

$$\sum_{i \geq q} \operatorname{rank} \left(\widehat{HFK}_i(\widetilde{K}, \widetilde{a}) \oplus \widehat{HFK}_i(\widetilde{K}, \widetilde{a} + 1) \right) \geq \sum_{2i \geq q+1} \operatorname{rank} \widehat{HFK}_i(K, a)$$

and

$$\sum_{i\le q} \operatorname{rank}\left(\widehat{HFK}_i(\widetilde{K}, -\widetilde{a}) \oplus \widehat{HFK}_i(\widetilde{K}, -\widetilde{a}-1)\right) \ge \sum_{2i\le q-1} \operatorname{rank} \widehat{HFK}_i(K, -a),$$

where $\widetilde{a} = 2a + \dfrac{\lambda - 1}{2}$.

Theorem 2.1.1 and this conjecture both have some nice consequences for the Alexander polynomials of 2-periodic alternating and L-space knots.

Theorem 2.4.1. *Let $\widetilde{K}$ be a 2-periodic alternating knot in S^3 with alternating quotient K and having linking number λ with the axis. Notate the Alexander polynomials of $\widetilde{K}$ and K as*

$$\Delta_{\widetilde{K}}(t) = \widetilde{a}_0 + \sum_{\widetilde{s}>0} \widetilde{a}_{\widetilde{s}}(t^{\widetilde{s}} + t^{-\widetilde{s}}), \text{ and } \Delta_{K(t)} = a_0 + \sum_{s>0} a_s(t^s + t^{-s}),$$

as in Theorem 2.3.2. Then for each s,

$$|\widetilde{a}_{2s+\frac{\lambda-1}{2}} - \widetilde{a}_{2s+\frac{\lambda+1}{2}}| \ge a_s,$$

and in particular the number of terms in $\Delta_{\widetilde{K}}$ is at least the number of terms in Δ_K. Additionally, if Conjecture 2.1.2 holds then

$$|2\sigma(K) - \sigma(\widetilde{K})| \le \lambda + 1.$$

Proof. The statement follows directly from applying the two inequalities in Conjecture 2.1.2 to Theorem 2.3.2. In particular since the inequality is split into

Alexander gradings, we can consider Δ_K one term at a time. Then the inequality $|\widetilde{a}_{2a+\frac{\lambda-1}{2}} - \widetilde{a}_{2a+\frac{\lambda+1}{2}}| \geq a_s$ comes from the total rank inequality in Theorem 2.1.1, noting that signs on the coefficients of $\widetilde{K}$ alternate. The grading refinement immediately gives

$$\widetilde{s} + \frac{\sigma(\widetilde{K})}{2} \geq 2s + \sigma(K) - 1,$$

for each grading $\widetilde{s}$ getting sent to s. However, we know that the gradings $2s + \frac{\lambda-1}{2}$ and $2s + \frac{\lambda+1}{2}$ get sent to s, so this simplifies to

$$2\sigma(K) + \lambda + 1 \geq \sigma(\widetilde{K}).$$

Finally, by considering the mirror of K we also get that

$$\sigma(\widetilde{K}) \geq 2\sigma(K) - \lambda - 1,$$

as desired. □

Example 2.4.2. Consider the knot 10_{122} which is 2-periodic over 4_1 with $\lambda = 1$. 10_{122} has signature 0 and Alexander polynomial

$$-2t^{-3} + 11t^2 - 24t + 31 - 24t^{-1} + 11t^{-2} - 2t^{-3},$$

whereas 4_1 also has signature 0, but Alexander polynomial

$$-t + 3 - t^{-1}.$$

Looking back at Theorem 2.3.2, we have Alexander gradings given by the exponents in Δ_K so that $s \in \{-1, 0, 1\}$ with $a_s \in \{1, 3, 1\}$ respectively. Since $\lambda = 1$

these will lift to give $\widetilde{s} = 2s + 0$, and indeed the first inequality is then $2 + 11 \geq 1$, $24 + 31 \geq 3$, and $24 + 11 \geq 1$. The signature inequality is also satisfied since both signatures are 0.

Remark 2.4.3. The fact that the number of terms in $\Delta_{\widetilde{K}}$ is at least the number of terms in Δ_K also follows from a theorem of Murasugi that all terms in the Alexander polynomial of an alternating knot are nonzero [Mur58, Theorem 1.1] and that Δ_K divides $\Delta_{\widetilde{K}}$.

Theorem 2.4.4. *Let $\widetilde{K}$ be a 2-periodic L-space knot in S^3 with L-space quotient K. Then there are at least as many terms in $\Delta_{\widetilde{K}}$ as in Δ_K. Furthermore let n be the width of Δ_K, again normalize the Alexander polynomial as in Theorem 2.3.2, and suppose that Conjecture 2.1.2 holds. Then there is at most 1 term in $\Delta_{\widetilde{K}}$ with exponent larger than*

$$2n + \frac{\lambda + 1}{2},$$

and in particular there are at most $4n + \lambda + 4$ terms in $\Delta_{\widetilde{K}}$ total.

Proof. As we will see, all statements follow from Theorem 2.3.3, the characterization of $\widehat{HFK}(K)$ in terms of Δ_K.

The inequality between the number of terms in $\Delta_{\widetilde{K}}$ and Δ_K is clear from Theorem 2.1.1.

For the other claims, observe that the largest δ_i in Theorem 2.3.3 is zero (so that on the maximal Maslov grading Conjecture 2.1.2 will be trivially satisfied). The other conclusions will follow by considering the minimal Maslov grading. Observe that the smallest δ_i is negative the width of the Alexander polynomial, $n_{-k} - n_k$, as follows. Since the Alexander polynomial is symmetric each gap $n_{i+1} - n_i$ has a mirrored gap $n_{-i} - n_{-i-1}$, and exactly one of these contributes $2(n_{i+1} - n_i) + 1$, while the other contributes -1. Summing these gives that indeed the minimal δ_i is $n_{-k} - n_k$.

This gives the stated bound on the number of terms in $\Delta_{\widetilde{K}}$ of degree larger than $2n + (\lambda + 1)/2$ since otherwise the δ_i for $\widetilde{K}$ corresponding to the minimal δ_i for K would be less than $n_{-k} - n_k$ contradicting that the degree of $\Delta_{\widetilde{K}}$ is larger than the degree of Δ_K.

Finally, the bound on the number of terms in $\Delta_{\widetilde{K}}$ follows from symmetry. Specifically there is also at most one term in $\Delta_{\widetilde{K}}$ with exponent less than $-2n - (\lambda + 1)/2$, and hence there are at most $4n + \lambda + 4$ terms total. □

This theorem can be somewhat improved by further assuming the L-space conjecture of Boyer, Gordon and Watson.

Conjecture 2.4.5. *[BGW13, Conjecture 1] Let M be a closed, connected, irreducible, orientable 3-manifold. Then M is not an L-space if and only if $\pi_1(M)$ is left-orderable.*

In particular, assuming this conjecture allows us to drop the assumption that K is an L-space knot in Theorem 2.4.4.

Proposition 2.4.6. *Let $\widetilde{K}$ be a p-periodic knot with quotient K. If Conjecture 2.4.5 holds and $\widetilde{K}$ is an L-space knot, then K is an L-space knot.*

Proof. Since $\widetilde{K}$ is an L-space knot, all sufficiently large surgeries on $\widetilde{K}$ are L-spaces. In particular, by taking any large surgery with surgery coefficient a multiple of p, we get an L-space surgery $\widetilde{Y} = S^3_{pn}(\widetilde{K})$ with a surgery curve that is equivariant with respect to the periodic action. This then induces a surgery on the quotient knot $Y = S^3_n(K)$. Furthermore, $\widetilde{Y}$ is a p-fold branched cover of Y with branch set the union of the core of the surgery and the axis of the original periodic action. We can also assume that $\widetilde{Y}$ and Y are irreducible, since there are only finitely many reducible surgeries on a given knot.

Now we claim that if $\pi_1(Y)$ is left-orderable, then so is $\pi_1(\widetilde{Y})$. This follows directly from [BRW05, Theorem 1.1(1)] if the induced map $\pi_1(\widetilde{Y}) \to \pi_1(Y)$ is non-trivial. Suppose that the map is trivial. Then we can lift the map $\widetilde{Y} \to Y$ to the universal cover $\overline{Y}$ of Y. If $\overline{Y}$ is not S^3, then $H_3(\overline{Y}) = 0$, and so the map $\widetilde{Y} \to Y$ has degree 0, contradicting it being a p-fold branched cover. On the other hand, if $\overline{Y}$ is S^3, then $\pi_1(Y)$ is finite and hence not left-orderable.

Now Conjecture 2.4.5 implies that if $\widetilde{Y}$ is an L-space then so is Y. □

2.5. Evidence for the Main Conjecture

There is strong evidence for Conjecture 2.1.2, both theoretically and computationally.

Computational Evidence

To check Conjecture 2.1.2, we generated pseudorandom knots and verified the conjecture for each one as follows.

First we construct a tangle K on 5 strands by choosing 18 random operations from the set $\{c_i, o_i, u_i\}$. Here c_i refers to a cup cap pair connecting the ith strand to the $i+1$th strand, o_i refers to the ith strand crossing over the $i+1$th strand, and u_i refers to crossing the ith strand under the $i+1$th strand.

Next, we check that each K we construct has closure a knot, and that the tangle for $\widetilde{K}$ constructed by repeating the operations for K also has closure a knot. If either condition fails, then we choose 18 new random operations.

Once we have a 2-periodic knot described by a tangle, we use Ozsváth and Szabó's knot Floer homology calculator [OS] based on [OS18] to compute $\widehat{HFK}(K)$ and $\widehat{HFK}(\widetilde{K})$, and verify the conjecture for this pseudorandom 2-periodic knot.

While verifying the conjecture for each knot, we also tabulated the Alexander polynomial and the total rank of the knot Floer homology for each periodic knot. The total rank of $\widehat{HFK}(\widetilde{K})$ ranged from 1 to 907253 with an average of about 7761.52. These data confirm that we have verified the conjecture for over 500 distinct knots.

The Case of Torus Knots

It does not seem easy to check many special cases of Theorem 2.1.1 or Conjecture 2.1.2. For torus knots, specific examples may be computed by Theorem 2.3.3, which we have done for many torus knots.

Proposition 2.5.1. *Conjecture 2.1.2 is true for $\widetilde{K} = T(2p, q)$ and $K = T(p, q)$ for all $p, q < 60$.*

Proof. Since torus knots have an explicit formula for their Alexander polynomials, and are L-space knots, we used a computer to directly compute $\widehat{HFK}$ using Theorem 2.3.3. □

On the other hand, computations for any infinite family involve understanding all terms in some cyclotomic polynomials. Nonetheless, we can check the main conjecture in this case if we restrict to only the maximal Alexander gradings, and we can verify the results of Theorem 2.4.4 for torus knots even without assuming the conclusion of Conjecture 2.1.2.

Proposition 2.5.2. *The first inequality in Conjecture 2.1.2 is true for the maximal Alexander gradings on the 2-periodic torus knots $T(2p, q) \to T(p, q)$.*

Proof. Since torus knots have L-space surgeries, we can use Theorem 2.3.3 to compute $\widehat{HFK}$. Recall that

$$\Delta_{T(p,q)}(t) = \frac{(t^{pq} - 1)(t - 1)}{(t^p - 1)(t^q - 1)}$$

has degree $(p - 1)(q - 1)$, and that in this case the linking number between the axis and knot is $\lambda = q$. By Theorem 2.3.3, the maximum Alexander grading for $T(p, q)$ is $(p - 1)(q - 1)/2$, half the width of $\Delta_{T(p,q)}$, which lifts to the Alexander grading

$$(p - 1)(q - 1) + \frac{q - 1}{2} = \frac{2pq - 2p - q + 1}{2} = \frac{(2p - 1)(q - 1)}{2}.$$

Conveniently, this is the maximum Alexander grading for $\Delta_{T(2p,q)}$. And indeed, these Alexander polynomials are monic, and both the δ_is from Theorem 2.3.3 are 0, giving the desired result. □

Remark 2.5.3. The above proposition is also true, with essentially the same proof, for the mirror knots, or equivalently for the minimum Alexander grading in the second inequality in Conjecture 2.1.2.

Proposition 2.5.4. *The conclusions of Theorem 2.4.4 hold for torus knots, without assuming Conjecture 2.1.2.*

Proof. This follows immediately by checking the degrees of the Alexander polynomials for torus knots. As in the previous proposition, we see that there are no terms in $\Delta_{T(2p,q)}$ larger than $2\cdot\text{width}(\Delta_{T(p,q)}) + (q + 1)/2$. □

Adapting the Morse Homology Proof

Finally, we would like to point out where we got stuck in adapting the proof of Theorem 2.2.2 to prove Conjecture 2.1.2. In fact, most of the proof works similarly.

Proposition 2.5.5. *If the spectral sequence (2.3.1) does not contain any staircases beginning with a vertical differential on the top left and ending with a horizontal differential on the bottom right, then Conjecture 2.1.2 holds.*

Proof. This condition is a slightly weaker replacement of Lemma 2.2.3. From there, the proof follows identically to that of Theorem 2.2.2. The factor of 2 in the grading shift comes from the identification of the E^∞ page with $\widehat{HFK}_*(K) \otimes W \otimes \mathbb{F}_2[\theta, \theta^{-1}]$ as in [HLS16]. The shift by 1 in the grading comes from the extra V vector space in the spectral sequence. $\square$

CHAPTER III

COVERING MAPS BETWEEN SURGERIES ON THE SAME KNOT

3.1. Introduction

Dehn Surgery

Consider a knot $K \subset S^3$ (although this construction will work just as well in any 3-manifold). Then *Dehn surgery* is a method for constructing a new 3-manifold by cutting out a small tube around K, and gluing it back in with a twist. Specifically, a tubular neighborhood N of K is a solid torus, and cutting N out of S^3 leaves us with two pieces: $S^3 - N$, and $N \cong S^1 \times D^2$, each of which has a torus boundary. By choosing any homeomorphism from the torus $S^1 \times S^1$ to itself, we can construct a new 3-manifold by gluing these two pieces together with that identification on the boundary. Thus Dehn surgery takes a knot K and a homeomorphism $f : S^1 \times S^1 \to S^1 \times S^1$, and produces a 3-manifold $S^3_f(K)$. Furthermore, the homeomorphism type of this manifold is invariant under isotopy of the knot K and the homeomorphism f.

To describe this gluing process more concretely, we need to define the meridian and longitude of K, both of which are curves on the boundary of N. The *meridian* is a curve which has linking number 1 with K, and bounds a disk in N. The *longitude* is a curve which is isotopic to K in S^3, and which has linking number 0 with K. Now we will use p/q to refer to the curve on the boundary of N which wraps p times around the meridian and q times around the longitude. That is, p/q refers to the curve which is p times the meridian plus q times the longitude in $H_1(\partial N; \mathbb{Z})$. It turns out that the manifold $S^3_f(K)$ depends only on the image of the meridian under f. So to specify $S^3_f(K)$, it is enough to specify the p/q curve on the boundary of N which is the image of the meridian on $S^3 - N$, and we use the notation $S^3_{p/q}(K)$ for $S^3_f(K)$. Note that p and q must be relatively prime for this to give a homeomorphism of $S^1 \times S^1$, so p/q is well defined as a rational number.

In addition to being a knot invariant, Dehn surgery is an important method for constructing 3-manifolds. In fact, every compact orientable 3-manifold can be constructed by iterating this construction, see [Wal60] or [Lic63]. Extensive work has been done to understand this construction, but many elementary questions remain unresolved. For example, let M be a closed oriented 3-manifold, K a knot in M, and γ, γ' surgery slopes along K (or just rational numbers in the case $M = S^3$). Denote by $M_\gamma(K)$ Dehn surgery on K in M along γ. One may ask when $M_\gamma(K)$ is homeomorphic to $M_{\gamma'}(K)$. In particular, the following conjecture regarding the uniqueness of Dehn surgery along knots has been around since at least 1991 [Gor91, Conjecture 6.1].

Conjecture 3.1.1 (Cosmetic Surgery Conjecture)**.** *If $M - K$ is not a solid torus and there exists an orientation preserving homeomorphism between $M_\gamma(K)$ and $M_{\gamma'}(K)$ then there exists a self-homeomorphism of $M - K$ taking γ to γ'.*

Many partial results have been shown. For example, in 1990, Mathieu showed [Mat90] that the orientation preserving requirement is necessary by constructing an orientation reversing counterexample. See also [BHW99]. In 2015 Ni and Wu [NW15] proved that if surgery on γ and γ' provide a counterexample to the conjecture for a knot in S^3, then $\gamma = -\gamma'$. More recently Jeon proved [Jeo16] in 2016 that the conjecture is true for all but finitely many surgeries on each knot in a fairly general class of hyperbolic knots.

As a generalization of the cosmetic surgery question Lidman and Manolescu [LM16, Question 1.15] asked when $M_\gamma(K)$ is a covering space of $M_{\gamma'}(K)$. Restricting to knots in S^3, we use the homological framing to write γ as $p/q \in \mathbb{Q}$ with $\gcd(p, q) = 1$. With this notation, a naive generalization of Conjecture 3.1.1 for knots in S^3 might be

Conjecture 3.1.2 (Virtual Cosmetic Surgery Conjecture)**.** *If $K \subset S^3$ is not the unknot, $p'/q' \neq p/q, p/q \neq \infty$, and there exists a covering map of degree d from $S^3_\gamma(K)$ to $S^3_{\gamma'}(K)$, then there exists a degree d self-covering map of $S^3 - K$ taking the p/q curve to the p'/q' curve.*

Remark 3.1.3. The $p/q \neq \infty$ condition is necessary since there exist lens space surgeries on hyperbolic knots. We are going to consider the case $d > 1$, and we consider the more general case of unoriented manifolds.

Main Results

We prove that Conjecture 3.1.2 is false for torus knots $T(r, s)$ in S^3, see Examples 3.6.5 and 3.6.6, and we will classify counterexamples. In order to do so, we prove a structure theorem for covers between Seifert fiber spaces (see Proposition 3.4.4), which reduces the question to classifying all covers between orbifolds with base space S^2 and 3 or fewer cone points. These are called *small* Seifert fiber spaces, see section 3.6.

Theorem 3.1.4. *Let $S^2(a, b, c) \to S^2(a', b', c')$ be a degree $n > 1$ cover of 2-orbifolds over S^2 with cone points of orders a, b, c, and a', b', c' respectively. Then*

1. *If* $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}<1$, *then* $(a,b,c),(a',b',c'),n$ *are one of the following up to reordering of* (a,b,c) *and* (a',b',c'), *for some* $x,y\in\mathbb{Z}$.

(a,b,c)	(a',b',c')	n	(a,b,c)	(a',b',c')	n
(x,x,y)	$(2,x,2y)$	2	$(x,4x,4x)$	$(2,3,4x)$	6
$(2,x,2x)$	$(2,3,2x)$	3	$(3,3,7)$	$(2,3,7)$	8
(x,x,x)	$(3,3,x)$	3	$(2,7,7)$	$(2,3,7)$	9
$(3,x,3x)$	$(2,3,3x)$	4	$(3,8,8)$	$(2,3,8)$	10
$(x,2x,2x)$	$(2,4,2x)$	4	$(4,8,8)$	$(2,3,8)$	12
(x,x,x)	$(2,3,2x)$	6	$(9,9,9)$	$(2,3,9)$	12
$(4,4,5)$	$(2,4,5)$	6			

2. *If* $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$, *then* $(a,b,c),(a',b',c'),n$ *are one of the following up to reordering of* (a,b,c) *and* (a',b',c'), *where* $n=x^2+xy+y^2$ *and* $m=x^2+y^2$ *for some* $x,y\in\mathbb{Z}$.

(a,b,c)	(a',b',c')	n
$(2,3,6)$	$(2,3,6)$	n
$(2,4,4)$	$(2,4,4)$	m
$(3,3,3)$	$(3,3,3)$	n
$(3,3,3)$	$(2,3,6)$	$2n$

3. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} > 1$*, then* $(a,b,c),(a',b',c'),n$ *are one of the following up to reordering of* (a,b,c) *and* (a',b',c')*, for some* $x,y \in \mathbb{Z}$*.*

(a,b,c)	(a',b',c')	n	*conditions*	(a,b,c)	(a',b',c')	n
$(1,x,y)$	$(1,nx,ny)$	n		$(2,3,3)$	$(2,3,4)$	2
$(1,d,d)$	$(2,2,x)$	$2x/d$	$d\|x$	$(2,2,3)$	$(2,3,4)$	4
$(2,2,d)$	$(2,2,x)$	x/d	$d\|x$	$(2,3,3)$	$(2,3,5)$	5
$(1,d,d)$	$(2,3,3)$	$12/d$	$d \in \{1,2,3\}$	$(2,2,5)$	$(2,3,5)$	6
$(1,d,d)$	$(2,3,4)$	$24/d$	$d \in \{1,2,3,4\}$	$(2,2,3)$	$(2,3,5)$	10
$(1,d,d)$	$(2,3,5)$	$60/d$	$d \in \{1,2,3,5\}$			

Furthermore, we construct all of the above covers.

Remark 3.1.5. It is interesting to note that many Seifert fibered surgeries on other knots are also known to be small, for example alternating hyperbolic knots [IM16], and hence the covers between Seifert fibered surgeries on such knots are also understood through Theorem 3.1.4.

The covers in Theorem 3.1.4 give counter examples to Conjecture 3.1.2 for torus knots, but we provide a nice structure theorem in the cases where these exceptional covers do not occur.

Theorem 3.1.6. *Let* $r,s > 2$*,* $(r,s) \neq (3,4),(3,5),(4,5),(3,7)$*, or* $(3,8)$*. Then* $S^3_{p/q}(T(r,s))$ *covers* $S^3_{p'/q'}(T(r,s))$ *if and only if all of the following hold.*

1. $|rsq-p| = |rsq'-p'|$

2. $p|p'$

3. $gcd(p/p', rsq-p) = gcd(p/p', rs) = 1$

4. $p/p' \equiv 1 \mod rs$

If these are satisfied, then the degree of the cover is p'/p.

One might hope that in this case Conjecture 3.1.2 is satisfied, but in fact even covers over a fixed base orbifold can give counterexamples. See Example 3.6.6.

In the case of hyperbolic knots, Mostow rigidity implies that there are no non-trivial self covers of the knot complements. In this case Conjecture 3.1.2 would reduce to the cosmetic surgery conjecture on hyperbolic knots for trivial covers, and the following conjecture.

Conjecture 3.1.7 (Hyperbolic Virtual Cosmetic Surgery Conjecture)**.** *If* $p/q \neq p'/q' \in \mathbb{Q}$*, then* $S^3_{p/q}(K)$ *does not non-trivially cover* $S^3_{p'/q'}(K)$ *for any hyperbolic knot* K*.*

An argument pointed out by a referee of [Boy18] shows that the following proposition, which is precisely stated later as Corollary 3.7.3, is a consequence of [FKP08, Theorem 1.1].

Proposition 3.1.8. *Conjecture 3.1.7 is true for all but at most* 32 p'/q' *slopes on each hyperbolic knot* $K \subset S^3$*.*

Focusing on low crossing number knots, some computations in SnapPy [CDGW] along with known information about exceptional surgeries on twist knots and pretzel knots give the following.

Proposition 3.1.9. *Conjecture 3.1.7 is true for all hyperbolic knots with* 8 *or fewer crossings.*

Outline of the Chapter

The organization of the chapter is as follows. In section 3.2 we provide some background. In sections 3.3 through 3.6 we discuss the case of torus knots, proving Theorem 3.1.4 in section 3.5 and Theorem 3.1.6 in section 3.6. In section 3.7 we discuss the case of hyperbolic knots, culminating in the proofs of Propositions 3.1.8 and 3.1.9.

3.2. Background

All 3-manifolds are assumed compact, connected and orientable, although not oriented. For convience throughout, we will only work with non-trivial positive torus knots $T(r, s)$ with $r, s > 0$.

We will use the notation $S^2(\alpha_1, \ldots, \alpha_n)$ to mean the orbifold with underlying surface S^2, and n cone points points with $\mathbb{Z}/\alpha_i\mathbb{Z}$ isotropy subgroups. In the 1970s, Moser classified Dehn surgeries on torus knots:

Theorem 3.2.1. *[Mos71, Theorem 1] Let K be the (r, s) torus knot, and M be $S^3_{p/q}(K)$. Then*

(1) If $|rsq - p| > 1$ then M is a Seifert fiber space with base orbifold $S^2(r, s, |rsq - p|)$, and the orientation preserving homeomorphism type is determined by p.

(2) If $|rsq - p| = 1$ then M is the lens space $L(p, qs^2)$.

(3) If $rsq - p = 0$ then M is $L(r, s)\#L(s, r)$.

Note that $L(-m, n)$ is understood to mean $L(m, -n)$ when $m > 0$, and that since $p/q = -p/(-q)$ give the same surgery, it can always be arranged that $rsq - p \geq 0$. Note that we are only considering manifolds up to orientation reversing homeomorphism.

Let M be an oriented Seifert fiber space with base orbifold $S^2(\alpha_1, \dots, \alpha_n)$ and Seifert invariants $b, \{(\alpha_i, \beta_i)\}$. For convenience we will not require the normalization $0 < \beta_i < \alpha_i$. We will use the standard notation

$$\{b; (o_1, 0); (\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)\}.$$

Throughout, we will omit the $(o_1, 0)$ term, which indicates that the base orbifold is S^2 and that M is orientable, since this will be true for all of our Seifert fiber spaces. For more information see [JN83]. It will be useful to recall some facts about orbifold covers and Seifert fiber spaces. We use Thurston's definition of a covering map of orbifolds, see [Thu, Chapter 13].

Definition 3.2.2. The *orbifold Euler characteristic* of a compact 2-dimensional orbifold Σ with underlying manifold S, r corner reflectors of orders $\{n_i\}$ and s cone points of orders $\{m_j\}$ is

$$\chi(\Sigma) := \chi(S) - \frac{1}{2}\sum_{i=1}^{r}\left(1 - \frac{1}{n_i}\right) - \sum_{j=1}^{s}\left(1 - \frac{1}{m_j}\right).$$

Note that by the Riemann-Hurwitz formula, $\chi(\Sigma)$ is multiplicative under finite covers. In the case at hand, suppose $S^2(a, b, c) \to S^2(a', b', c')$ is a covering space of degree d. Then

$$\chi(S^2) - \left(1 - \frac{1}{a}\right) - \left(1 - \frac{1}{b}\right) - \left(1 - \frac{1}{c}\right) = d\left(\chi(S^2) - \left(1 - \frac{1}{a'}\right) - \left(1 - \frac{1}{b'}\right) - \left(1 - \frac{1}{c'}\right)\right).$$

More succinctly,

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 = d\left(\frac{1}{a'} + \frac{1}{b'} + \frac{1}{c'} - 1\right). \tag{3.2.3}$$

Additionally, looking at the preimages of the orbifold points a', b', and c', there is an obvious condition on d which we will now describe.

For any partition $\lambda_a = \{a_1, \dots a_n\}$ of d where $a_i | a$, let λ^a refer to the set $\{a/a_1, \dots a/a_n\}$. Now observe that given a cover $S^2(a,b,c) \to S^2(a',b',c')$ of degree d, there exist partitions $\lambda_{a'}, \lambda_{b'}$ and $\lambda_{c'}$ of d by divisors of a', b', and c' respectively so that the union $\lambda^{a'} \cup \lambda^{b'} \cup \lambda^{c'}$ consists entirely of 1s except for a single a, b, and c. We will refer to this as the *partition condition* for orbifold covers.

Definition 3.2.4. A *Seifert neighborhood* of a fiber γ in a Seifert fiber space is a fiber preserving and orientation preserving homeomorphism from a neighborhood of γ to $I \times D^2/ \sim$ where $(0,z) \sim (1, e^{2\pi i q/p} z)$ for some pair of relatively prime integers p and q, and the fibers are cycles of vertical fibers $I \times *$. Once such a homeomorphism is fixed we will refer to such a neighborhood as $N_{\frac{q}{p}}(\gamma)$.

By definition a Seifert neighborhood exists for every fiber, and p is the *index* of the fiber. A fiber is *regular* if $p = 1$ and *singular* otherwise.

Definition 3.2.5. A *Seifert cover* is a covering map of Seifert fiber spaces which takes fibers to fibers.

Definition 3.2.6. Given a Seifert covering $f : \widetilde{M} \to M$, a *pre-regular* fiber $\gamma \subset \widetilde{M}$ is a Seifert fiber of $\widetilde{M}$ such that $f(\gamma)$ is a regular fiber of M. A *pre-singular* fiber γ is one such that $f(\gamma)$ is a singular fiber of M.

The following is a restatement of an observation in [Mos71], which will be needed to discuss realizations of Seifert fiber spaces as surgeries on specific torus knots. We assume throughout that $r, s > 0$.

Lemma 3.2.7. *Fix a torus knot $T(r,s)$. If p/q surgery on $T(r,s)$ is a small Seifert fiber space, then the b and (α_i, β_i) Seifert invariants are determined by r, s, p, and q.*

Proof. See [Mos71] or [GL14, Section2.5]. □

3.3. Lens Spaces and Connect Sums of Lens Spaces

In this section we will resolve Conjecture 3.1.2 in the case when the base space is a lens space or a connect sum of lens spaces. That is, we consider case (2) in Theorem 3.2.1.

Lemma 3.3.1. *Let M and M' be obtained from Dehn surgery on a torus knot K which is not the unknot. Then if either M or M' is of type (2) in Moser's classification, then there is no covering map $f : M \to M'$.*

Proof. On a non-trivial torus knot $T(p, q)$ there is a unique reducible surgery $S^3_{pq/1}(T(p, q))$ by Theorem 3.2.1. Indeed, all other surgeries are Seifert fiber spaces over S^2 (and are not $S^2 \times S^1$, since $T(p, q)$ is non-trivial), and hence are irreducible. However, by the sphere theorem any cover of a reducible 3-manifold is reducible, since π_2 is preserved by covers. □

Lemma 3.3.2. *If $L(p, q)$ and $L(p', q')$ are lens spaces obtained from surgeries on the same torus knot, then $L(p, q)$ covers $L(p', q')$ if and only if p divides p'.*

Proof. The lens space $L(p', x)$ has a unique cover for each divisor d of p', and that cover is $L(p'/d, x)$, so the only if direction is clear. On the other hand, looking at which lens spaces are possible as surgeries on the same torus knot, we get from (2) in Theorem 3.2.1 that $\gcd(r, p') = 1$ and that $q'rs \equiv 1 \bmod p'$, after choosing p', q' so that $rsq' + p' \geq 0$. Hence we can write $q's^2$ as $sr^{-1} \bmod p'$.

Now suppose that $L(p', x)$ and $L(p'/d, y)$ occur as (p', q') and (p, q) surgery respectively on the same torus knot, so that $x = q's^2$ and $y = qs^2$. Then $qrs \equiv \pm 1 \bmod p$ so that $x \equiv \pm sr^{-1} \bmod p$ (and the same for y), giving $x \equiv \pm y \bmod p$.

Then by the classification of (unoriented) lens spaces $L(p'/d, y) \cong L(p'/d, x)$, and so $L(p'/d, y)$ covers $L(p', x)$. □

Since the only covers of lens spaces are lens spaces, this finishes the case where the base 3-manifold is a lens space.

3.4. Covers of Seifert Fiber Spaces

Throughout this section let M be an orientable Seifert fiber space with the underlying surface of the base orbifold S^2, i.e. $M \cong \{b; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}$. Let $f : \widetilde{M} \to M$ be a covering map. Then there is an induced Seifert fiber structure on $\widetilde{M}$ where the fibers are the preimages of the fibers in M; see for example [JN83, lemma 8.1]. In particular, there is a choice of Seifert fiber structure on $\widetilde{M}$ so that f is a Seifert cover. Note however, that $\widetilde{M}$ may have other Seifert fiber structures for which f is not even homotopic to a Seifert cover. Similar results to those in this section are observed in [Hua02, Section 2].

Definition 3.4.1. A *fiberwise cover* is a Seifert cover $f : \widetilde{M} \to M$ for which the preimage of each fiber of M is a single fiber of $\widetilde{M}$.

We will observe below that fiberwise covers induce an isomorphism between the base orbifolds.

Definition 3.4.2. A *pullback cover* is a Seifert cover $f : \widetilde{M} \to M$ which induces a covering map $f_* : \widetilde{\Sigma} \to \Sigma$ of base orbifolds with $\deg(f) = \deg(f_*)$.

Remark 3.4.3. The term *pullback* is justified by the following proposition, which implies the universal property, and hence uniqueness, of such covers.

Proposition 3.4.4. *Given a cover of Seifert fiber spaces $f : \widetilde{M} \to M$, f factors as a composition of a fiberwise cover $f_2 : \widetilde{M} \to \overline{M}$ and a pullback cover $f_1 : \overline{M} \to M$. In particular, f induces a covering map of base orbifolds $\widetilde{\Sigma} \to \Sigma$. This is notated as*

$$\begin{array}{ccccc}
S^1 & \xrightarrow{deg(f_2)} & S^1 & \xrightarrow{id} & S^1 \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{M} & \xrightarrow{f_2} & \overline{M} & \xrightarrow{f_1} & M \\
\downarrow & & \downarrow & & \downarrow \rho \\
\widetilde{\Sigma} & \xrightarrow{id} & \widetilde{\Sigma} & \xrightarrow{deg(f_1)} & \Sigma,
\end{array}$$

where $\overline{M}$ is the pullback of the bottom right square, the columns are Seifert fibrations and the bottom row are the base orbifolds. The top left S^1 is a pre-regular fiber of $\widetilde{M}$.

To prove this proposition, we use the following lemma describing the local behavior.

Lemma 3.4.5. *Given a Seifert cover $f : \widetilde{N} \to N$ of Seifert neighborhoods, the covering map is equivalent (as covering spaces) to one whose deck transformation groups acts as rotation on both coordinates of $\partial \widetilde{N}$. Furthermore, f is determined (up to covering space isomorphism) by this action on the boundary.*

Proof. The map f is a covering map with cyclic deck transformation group G since N is homotopy equivalent to a circle. Pick a generator g of G. The generator g acts on $\widetilde{N}$ taking fibers to fibers and has finite order, so it decomposes into an action g_1 on the central fiber, S^1, and an action g_2 on D^2, a disk transverse to each fiber. By classification of 1-manifolds g_1 is conjugate to a rotation, and by [vK19], g_2 is conjugate to a rotation, so up to isomorphism of covering spaces, g rotates $\widetilde{N}$ on both coordinates. □

We are now ready to prove Proposition 3.4.4.

Proof of Proposition 3.4.4. First, given a Seifert cover, we describe the induced cover on base orbifolds. Consider a Seifert neighborhood $N_{p'/q'}$ of a fiber γ in M. Each connected component of $f^{-1}(\gamma)$ is a Seifert neighborhood by construction of the Seifert structure on $\widetilde{M}$. It is also clear that if γ is a regular fiber, then so is each connected preimage of γ since in a regular Seifert neighborhood every fiber generates π_1. Now quotienting by the S^1 action induces homeomorphisms $D^2 \to D^2$ so that f induces a cover between base orbifolds near smooth points. If γ is instead a singular fiber with nearby fibers homotopic to k times γ, then a connected component $\widetilde{\gamma}$ of $f^{-1}(\gamma)$ will have nearby fibers homotopic to k/d times $\widetilde{\gamma}$, where d is the degree of the cover $\widetilde{\gamma} \to \gamma$, by Lemma 3.4.5. Indeed, the fibers near γ generate $k\mathbb{Z} \subset \mathbb{Z} = \pi_1(\gamma)$, so the fibers near $\widetilde{\gamma}$ must generate $k/d\mathbb{Z} \subset \mathbb{Z} = \pi_1(\widetilde{\gamma})$. Thus we have an induced map of base orbifolds $D^2(k/d) \to D^2(k)$ by the obvious quotient, so that f induces a cover on base orbifolds near singular fibers as well.

Now, let $\underline{f} : \widetilde{\Sigma} \to \Sigma$ be the induced cover of base orbifolds, let $\rho : M \to \Sigma$ be the projection, and define

$$\overline{M} := \{(m, \widetilde{s}) | m \in M, \widetilde{s} \in \widetilde{\Sigma}, \rho(m) = \underline{f}(\widetilde{s})\}.$$

Now it is easy to check that the projection $f_1 : \overline{M} \to M$ given by $f_1(m, \widetilde{s}) = m$ is a cover of the same degree as $(\underline{f})$, and that lifting the Seifert fiber structure on M to $\overline{M}$ makes $\overline{M}$ a pullback cover of M. Similarly, the map $f_2 : \widetilde{M} \to \overline{M}$ given by $f_2(\widetilde{m}) = (f(\widetilde{m}), \rho(\widetilde{m}))$ is a fiberwise cover since by construction it induces the identity map on base orbifolds.

□

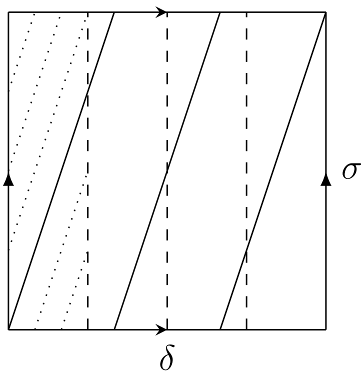

FIGURE 8. The degree 4 fiberwise quotient of $N_{1/3}(\gamma)$ is $N_{4/3}(f(\gamma))$. δ is a fiber in $N_{1/3}$, σ and the dashed lines are sections of $\partial N_{1/3}$ with the same image in $N_{4/3}$, the solid diagonal line is a meridian of $N_{1/3}$, and the dotted line is its image in $N_{4/3}$.

It will also be useful to describe explicitly the effect of fiberwise and pullback covers on the standard Seifert fiber form, which is stated in the following two corollaries.

Corollary 3.4.6. *Let* $f : \widetilde{M} \to M$ *be a fiberwise cover with* $\widetilde{M} = \{b; (\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)\}$. *Then* $M = \{d_f b; (\alpha_1, d_f \beta_1), \ldots, (\alpha_k, d_f \beta_k)\}$, *where* d_f *is the degree of* f.

Proof. Begin by rewriting $\widetilde{M}$ as $\{0; (\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k), (\alpha_{k+1}, b)\}$ with $\alpha_{k+1} = 1$. Then applying Proposition 3.4.4 to a neighborhood of each listed fiber gives the result. See Figure 8.

□

Corollary 3.4.7. *Let* $f : \widetilde{M} \to M$ *be a pullback of base orbifolds with*

$$M = \{b; (\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)\}.$$

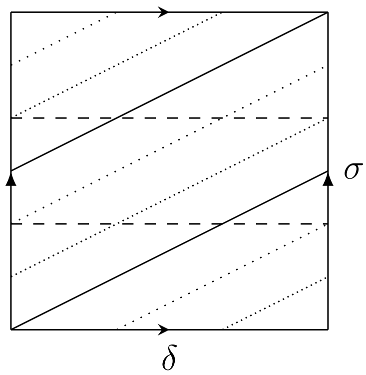

FIGURE 9. The pullback of $N_{2/3}(\gamma)$ along $f_* : D^2 \to D^2(3)$ is $N_{2/1}(f^{-1}(\gamma))$. δ and the dashed lines are fibers of $N_{2/1}$ with image the same fiber in $N_{2/3}$, and σ is a section of $\partial N_{2/1}$. The diagonal lines are meridians of $\partial N_{2/1}$ with the same image in $N_{2/3}$.

Then

$$\widetilde{M} = \{db; \left(\frac{\alpha_1}{\lambda_1(\alpha_1)}, \beta_1\right), \ldots, \left(\frac{\alpha_1}{\lambda_{r_1}(\alpha_1)}, \beta_1\right), \ldots, \left(\frac{\alpha_k}{\lambda_k(\alpha_k)}, \beta_k\right), \ldots, \left(\frac{\alpha_k}{\lambda_{r_k}(\alpha_k)}, \beta_k\right)\}$$

where d is the degree of f, $\lambda(\alpha_i)$ is the partition of d by divisors of α_i coming from the cover of base orbifolds, $\lambda_j(\alpha_i)$ is the jth part of the partition $\lambda(\alpha_i)$ (in any order), and r_i is the length of $\lambda(\alpha_i)$.

Proof. The α Seifert invariants are determined by the cone points from the orbifold cover, which are determined from the partitions as stated. See for example [EKS84, section 1]. The β Seifert invariants are left unchanged by Proposition 3.4.4. Writing the b from M as a $(1, b)$ fiber, this then lifts to d-many $(1, b)$ fibers in $\widetilde{M}$ by Proposition 3.4.4, which can then be reconsolidated into db. See figure 9. □

3.5. Orbifold Covers

In this section we will classify all orbifold covers of the form $S^2(a,b,c) \to S^2(a',b',c')$. Taking $a' = r$ and $b' = s$, Moser's classification along with Proposition 3.4.4 will allow us to classify coverings between surgeries on $T(r,s)$.

Since the orbifold Euler characteristic (or just orbifold characteristic, χ_{orb}) is multiplicative under covers, we can further decompose the problem into the cases $\chi_{orb} < 0$, $\chi_{orb} = 0$, and $\chi_{orb} > 0$. These correspond to the three cases in Theorem 3.1.4.

Covers of Negative Orbifold Characteristic

Proposition 3.5.1. *The only non-trivial covers of orbifolds $S^2(a,b,c) \to S^2(a',b',c')$ with negative orbifold characteristic are*

(a,b,c)	(a',b',c')	*degree*	(a,b,c)	(a',b',c')	*degree*
(x,x,y)	$(2,x,2y)$	2	$(4,4,5)$	$(2,4,5)$	6
$(2,x,2x)$	$(2,3,2x)$	3	$(3,3,7)$	$(2,3,7)$	8
(x,x,x)	$(3,3,x)$	3	$(2,7,7)$	$(2,3,7)$	9
$(3,x,3x)$	$(2,3,3x)$	4	$(3,8,8)$	$(2,3,8)$	10
$(x,2x,2x)$	$(2,4,2x)$	4	$(4,8,8)$	$(2,3,8)$	12
(x,x,x)	$(2,3,2x)$	6	$(9,9,9)$	$(2,3,9)$	12
$(x,4x,4x)$	$(2,3,4x)$	6			

where $x,y \in \mathbb{Z}$ are large enough that $\chi_{orb} < 0$.

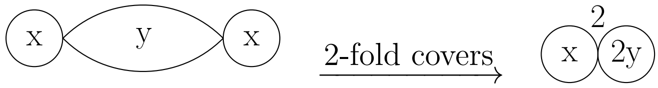

FIGURE 10. $S^2(x,x,y)$ 2-fold covers $S^2(2,x,2y)$

Observe that since Seifert fiber spaces over these orbifolds have a unique base orbifold [JN83, Theorem 5.2], the only possible torus knots these covers can occur on are $T(2,x), T(4,5), T(3,7)$ and $T(3,8)$.

Proof. To begin with, multiplicativity of the orbifold characteristic gives

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-1=n\left(\frac{1}{a'}+\frac{1}{b'}+\frac{1}{c'}-1\right)$$

where n is the degree of the cover. By assumption, $\chi_{orb}<0$, so both $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-1$ and $\frac{1}{a'}+\frac{1}{b'}+\frac{1}{c'}-1$ are between 0 and -1. We first consider the case $n\geq 7$. In this case $\frac{6}{7}<\frac{1}{a'}+\frac{1}{b'}+\frac{1}{c'}<1$, and so there are finitely many potential triples (a',b',c'). For each of these triples, the partition condition on covers gives a finite list of triples (a,b,c) and degrees n for which we might have a cover $S^2(a,b,c)\to S^2(a',b',c')$.

Now we associate to each degree n orbifold cover $S^2(a,b,c)\to S^2(a',b',c')$ a cover of $S^1\vee S^1$ also of degree n in the following way. Split the base S^2 into three regions with a wedge of two circles such that each region contains one orbifold point. Then the original cover gives a gluing of some covers of the resulting disk orbifolds onto a cover of $S^1\vee S^1$. This is shown for $S^2(x,x,y)\to S^2(2,x,2y)$ in figure 10. Now since the problem is reduced to covers of degree less than 7, plus some finite number of potential exceptions, we can use a brute force search to obtain the stated list of covers.

□

(a,b,c)	(a',b',c')	degree
$(2,3,6)$	$(2,3,6)$	n
$(2,4,4)$	$(2,4,4)$	m
$(3,3,3)$	$(3,3,3)$	n
$(3,3,3)$	$(2,3,6)$	$2n$

TABLE 1. Covers of zero orbifold charactersitic. $n = x^2 + xy + y^2$ and $m = x^2 + y^2$ for x, y not both 0.

Covers of Zero Orbifold Characteristic

Proposition 3.5.2. *The only covers* $S^2(a,b,c) \to S^2(a',b',c')$ *with* $\chi_{orb} = 0$ *are given in Table 1.*

Note that these covers only occur on $T(2,3)$ since the base orbifold of Seifert fiber spaces with these base orbifolds is unique [JN83, Theorem 5.2].

Proof. First, recall that the only triples (a,b,c) with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$ are $(2,4,4)$, $(3,3,3)$, and $(2,3,6)$. Unlike the other cases, the multiplicativity of the orbifold characteristic tells us nothing about the degree of any potential covers. In particular, each of these orbifolds has many self-covers. The key observation to classify these covers is the connection to lattices. $S^2(2,3,6)$ and $S^2(3,3,3)$ are the fundamental domains of the hexagonal lattice for the p6 and p3 wallpaper groups respectively, and $S^2(2,4,4)$ is the fundamental domain of the square lattice for the p4 wallpaper group. We can then identify covers of these orbifolds with sublattices, keeping track of the symmetries of the sublattice.

Consider the hexagonal lattice for the $S^2(3,3,3)$ orbifold. That is, a hexagonal lattice with a $\mathbb{Z}/3$ symmetry at each vertex. Any self cover would give a hexagonal sublattice with the same symmetries, and we can identify these sublattices (along with a chosen shortest length vector) with vectors in the original lattice in the following way. Overlay the lattice on $\mathbb{C}$ with 1 corresponding to a shortest length vector. To get a hexagonal sublattice from a vector, multiply each vector in the lattice by the chosen complex number to generate a new lattice, which will induce identical symmetries. See also [CS99, section 2.2].

Additionally, neither $S^2(2,3,6)$ or $S^2(2,4,4)$ can cover $S^2(3,3,3)$ since they both have either corner reflectors or a cone point of order 2, neither of which can cover a cone point of order 3. Now the index of the sublattice (and hence the degree of the cover) will be given by the square of the norm of the chosen vector and hence degrees of these self covers are given by outputs of the quadratic form $x^2 + xy + y^2$. See also [CM80, Table 4].

Next consider the hexagonal lattice for $S^2(2,3,6)$. Precisely the same argument will classify self covers. However in this case, for any hexagonal sublattice (where the vertices have a $\mathbb{Z}/6$ rotation action), there is an additional cover corresponding to the same sublattice given by the two fold cover $S^2(3,3,3) \to S^2(2,3,6)$ with partitions $2 = \frac{3}{3} + \frac{3}{3} = \frac{2}{1} = \frac{6}{3}$. That is, corresponding to each hexagonal sublattice, we can forget a 2-fold symmetry and recover $S^2(3,3,3)$. Again, see also [CM80, Figure 4]. Hence we have $S^2(3,3,3)$ covers $S^2(2,3,6)$ with degree $2(x^2 + xy + y^2)$.

Finally, for $S^2(2,4,4)$ we have a square lattice, and as above, we consider square sublattices with the same symmetries. These have indices $x^2 + y^2$. We also note that these sublattices correspond additionally to covers of $S^2(2,4,4)$ by $S^2(2,2,2,2)$ or by T^2 by forgetting additional symmetries. $\square$

Remark 3.5.3. It is helpful to observe that a priori the degrees of the covers $S^2(3,3,3) \to S^2(2,3,6)$ are of the form $2nn'$ for $n = x^2 + xy + z^2$ and $n' = z^2 + wz + w^2$. However, nn' is again of this form, since compositions of self covers of $S^2(3,3,3)$ must again be self covers of $S^2(3,3,3)$.

Remark 3.5.4. In all of these cases, covers of a specified degree are not necessarily unique. For example $49 = 7^2 + 7 \cdot 0 + 0^2 = 5^2 + 5 \cdot 3 + 3^2$, and hence there are two inequivalent self covers of $S^2(2,3,6)$ of degree 49.

Covers of Positive Orbifold Characteristic

Proposition 3.5.5. *The only non-trivial covers of orbifolds* $S^2(a,b,c) \to S^2(a',b',c')$ *with positive orbifold characteristic are the following.*

(a,b,c)	(a',b',c')	*degree*	*conditions*	(a,b,c)	(a',b',c')	*degree*
$(1,x,y)$	$(1,nx,ny)$	n		$(2,3,3)$	$(2,3,4)$	2
$(1,d,d)$	$(2,2,x)$	$2x/d$	$d \mid x$	$(2,2,4)$	$(2,3,4)$	3
$(1,d,d)$	$(2,3,3)$	$12/d$	$d \in \{1,2,3\}$	$(2,2,3)$	$(2,3,4)$	4
$(2,2,2)$	$(2,3,3)$	3		$(2,2,2)$	$(2,3,4)$	6
$(1,d,d)$	$(2,3,4)$	$24/d$	$d \in \{1,2,3,4\}$	$(2,3,3)$	$(2,3,5)$	5
$(1,d,d)$	$(2,3,5)$	$60/d$	$d \in \{1,2,3,5\}$	$(2,2,5)$	$(2,3,5)$	6
$(2,2,d)$	$(2,2,x)$	x/d	$d \mid x$	$(2,2,3)$	$(2,3,5)$	10
				$(2,2,2)$	$(2,3,5)$	15

Here n,x,y *are any positive integers. Note that since Seifert fiber spaces over these orbifolds (i.e. lens spaces) do not necessarily have unique base orbifolds, these covers may (and in fact do) occur on* $T(3,4), T(2,x)$ *and* $T(3,5)$ *in addition to* $T(2,3)$*.*

Proof. Using only multiplicativity of orbifold characteristic and the classification of elliptic 2-orbifolds (see for example [Thu, section 13.3]), the potential covers are

1. $S^2(x,y) \to S^2(nx,ny)$

2. $S^2(d,d) \to S^2(2,2,x)$ with $d|x$,
3. $S^2(d,d) \to S^2(2,3,3)$ where $d|12$,
4. $S^2(d,d) \to S^2(2,3,4)$ where $d|24$,
5. $S^2(d,d) \to S^2(2,3,5)$ where $d|60$,
6. $S^2(2,2,d) \to S^2(2,2,x)$ with $d|x$,
7. $S^2(2,2,2) \to S^2(2,3,3)$,
8. $S^2(2,2,3) \to S^2(2,3,3)$,
9. $S^2(2,2,2) \to S^2(2,3,4)$,
10. $S^2(2,2,3) \to S^2(2,3,4)$,
11. $S^2(2,2,4) \to S^2(2,3,4)$,
12. $S^2(2,2,3) \to S^2(2,3,5)$,
13. $S^2(2,3,3) \to S^2(2,3,4)$,
14. $S^2(2,2,2) \to S^2(2,3,5)$,
15. $S^2(2,3,3) \to S^2(2,3,5)$,
16. $S^2(2,2,5) \to S^2(2,3,5)$.

Not all of these satisfy the partition condition, so applying that restriction as well gives

1. $S^2(x,y) \to S^2(nx,ny)$
2. $S^2(d,d) \to S^2(2,2,x)$ with $d|x$,

3. $S^2(d,d) \to S^2(2,3,3)$, $d \in \{1,2,3\}$,

4. $S^2(d,d) \to S^2(2,3,4)$, $d \in \{1,2,3,4\}$,

5. $S^2(d,d) \to S^2(2,3,5)$, $d \in \{1,2,3,5\}$,

6. $S^2(2,2,d) \to S^2(2,2,x)$, $d|x$,

7. $S^2(2,2,d) \to S^2(2,3,4)$, $d \in \{2,3,4\}$

8. $S^2(2,3,3) \to S^2(2,3,4)$,

9. $S^2(2,2,2) \to S^2(2,3,3)$

10. $S^2(2,2,2) \to S^2(2,3,5)$,

11. $S^2(2,2,3) \to S^2(2,3,5)$,

12. $S^2(2,2,5) \to S^2(2,3,5)$,

13. $S^2(2,3,3) \to S^2(2,3,5)$.

In fact these are all orbifold covers, which can be shown in the same way as for the negative orbifold case. This is shown for some cases in figures 11 and 12. The cases $S^2(d,d) \to S^2(2,3,5)$ for $d \in \{1,2,3,5\}$ and $S^2(d,d) \to S^2(2,3,4)$ for $d \in \{1,2,3\}$ are specifically omitted since they are compositions of other covers on the list. $S^2(x,y) \to S^2(nx,ny)$ corresponds to an n-fold cover of a single circle. $S^2(2,2,d) \to S^2(2,2,x)$ is similar to $S^2(x,x,y) \to S^2(2,x,2y)$ from figure 10. As a final remark we note that there is not necessarily a unique covering space, or even a unique partition for each entry. For example with respect to the cover $S^2(2,2) \to S^2(2,2,4)$, we have

$$4 = \frac{2}{1} + \frac{2}{1} = \frac{2}{1} + \frac{2}{1} = \frac{4}{2} + \frac{4}{2},$$

but also

$$4 = \frac{2}{1} + \frac{2}{2} + \frac{2}{2} = \frac{2}{1} + \frac{2}{1} = \frac{4}{1}.$$

□

Proof of Theorem 3.1.4. This is now a direct consequence of Propositions 3.5.1, 3.5.2, and 3.5.5. □

3.6. Realization of Orbifold Covers

Now that we have a complete list of possible base orbifold covers, we aim to understand when these covers are realized by Seifert covers of surgeries on a torus knot. By Proposition 3.4.4 we can split this problem into two parts. First, given a Seifert fiber space $M = S^3_{p/q}(K)$ with base orbifold Σ and $\widetilde{\Sigma} \to \Sigma$ a non-trivial cover of orbifolds, when is the pullback of M along this cover also realized by surgery on K? We discuss this in Section 3.6.1. Second, given a fixed base orbifold Σ, which coverings of Seifert fiber spaces occur over Σ as surgery on the same torus knot? We discuss this in Section 3.6.2. Finally, composing a fiberwise cover and a pullback cover may be realized even if the intermediate cover is not. An example is given in Section 3.6.3.

Realization of Pullbacks of Orbifold Covers

Lemma 3.6.1. *Pullbacks along the following coverings of base 2-orbifolds do not occur for surgeries on any torus knot.*

1. $S^2(d,d) \to S^2(2,s,2)$ *where* $d|s$ *and* s *is odd,*

2. $S^2(d,d) \to S^2(2,3,3)$ *where* $d \in \{1,2,3\}$*,*

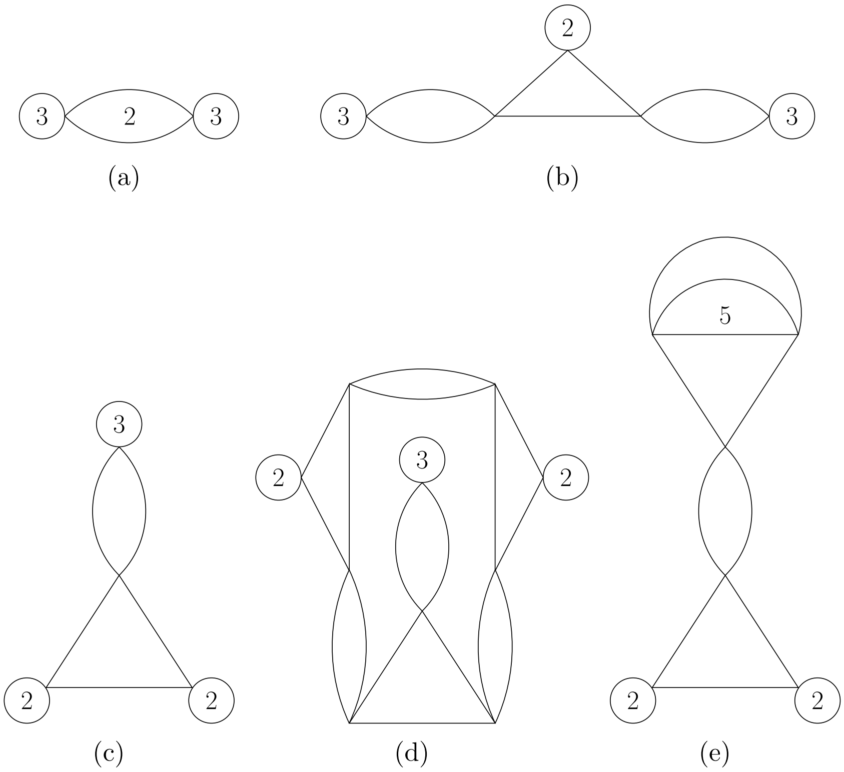

FIGURE 11. Some orbifold covers from Proposition 3.5.5.

(a): $S^2(2,3,3) \to S^2(2,3,4)$

(b): $S^2(2,3,3) \to S^2(2,3,5)$

(c): $S^2(2,2,3) \to S^2(2,3,4)$

(d): $S^2(2,2,3) \to S^2(2,3,5)$

(e): $S^2(2,2,5) \to S^2(2,3,5)$

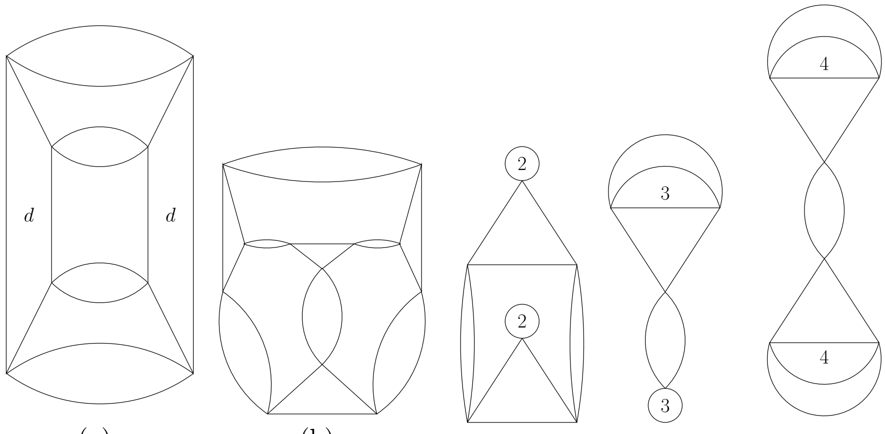

FIGURE 12. More orbifold covers from Proposition 3.5.5. Figure (A) is drawn for $x/d = 4$. In general, d would be labeling an x/d-gon.

(a): $S^2(1, d, d) \to S^2(2, 2, x)$, $x = 4d$

(b): $S^2(1, d, d) \to S^2(2, 3, 3)$, $d = 1$

(c): $S^2(1, 2, 2) \to S^2(2, 3, 3)$

(d): $S^2(1, 3, 3) \to S^2(2, 3, 3)$

(e): $S^2(1, 4, 4) \to S^2(2, 3, 4)$

3. $S^2(d,d) \to S^2(2,3,4)$ *where* $d \in \{1,2,3,4\}$,

4. $S^2(d,d) \to S^2(2,3,5)$ *where* $d \in \{1,2,3,5\}$.

Proof. We first consider (1). By Moser's classification $S^2(2,s,2)$ can only occur as a base orbifold from surgery on the torus knot $T(2,s)$. We will check that $S^2(d,d)$ never occurs from surgery on this knot. Since Seifert fiber spaces over $S^2(d,d)$ are lens spaces, Moser's classification implies $|2sq - p| = 1$ in the cover. In particular $p \equiv \pm 1 \bmod 2s$. Computing p (the order of H_1) from the Seifert invariants however, gives

$$p = \pm|H_1(\{b; (d,\beta_1), (d,\beta_2)\})| = d^2 b + d\beta_1 + d\beta_2 \equiv 0 \bmod d.$$

Hence $p \not\equiv \pm 1 \bmod 2s$ unless (potentially) $d = 1$. In this case we would have the space

$$\{b; (2,\beta_1), (s,\beta_2), (2,\beta_3)\}$$

lifting to

$$\{2sb; (1, s\beta_1), (1, 2\beta_2), (1, s\beta_3)\} = L(s(2b + \beta_1 + \beta_3) + 2\beta_2, 1).$$

In particular then, we would have $p = s(2b + \beta_1 + \beta_3) + 2\beta_2 \not\equiv \pm 1 \bmod 2s$ since it is even. Cases (2)-(4) are similar with the same kind of modular arithmetic obstructions. □

Remark 3.6.2. While pullbacks along these covers do not occur from surgeries on a torus knot, more general covers which induce these covers of base orbifolds may.

In contrast to the case of Lemma 3.6.1, in other cases pullbacks along covers of base orbifolds are often realized as surgeries.

Example 3.6.3. Given a surgery with one of the base orbifolds listed below, the pullback along the listed cover is often also a surgery on that torus knot.

1. $S^2(2, s, s) \to S^2(2, s, 4)$ on T(2,s),
2. $S^2(2, 2, 3) \to S^2(2, 3, 4)$ on T(2,3),
3. $S^2(2, 3, 3) \to S^2(2, 3, 5)$ on T(2,3),
4. $S^2(2, 2, 3) \to S^2(2, 3, 5)$ on T(2,3),
5. $S^2(2, 2, 5) \to S^2(2, 3, 5)$ on T(2,5).

First consider (1). Then we have as a base space

$$\{b; (2, 1), (s, \beta_2), (4, \beta_3)\},$$

where $\beta_3 \in \{1, 3\}$. This lifts along the degree 2 cover (1) with corresponding partitions $2 = \frac{4}{2} = \frac{2}{1} = \frac{s}{s} + \frac{s}{s}$ to give

$$\{2b; (1, 1), (s, \beta_2), (s, \beta_2), (2, \beta_3)\} = \{2b + 1; (s, \beta_2), (s, \beta_2), (2, \beta_3)\}.$$

In particular,

$$p = \pm|H_1(\{2b + 1; (s, \beta_2), (s, \beta_2), (2, \beta_3)\})| = 2s^2(2b + 1) + s^2\beta_3 + 4s\beta_2 \equiv s \bmod 2s.$$

By Moser's classification this base orbifold is realized whenever $|2sq-p| = s$. In fact for any choice of $p \equiv s \bmod 2s$, there is a choice of q so that $|2sq - p| = s$. Since p determines b, β_2, and β_3 by Lemma 3.2.7, this space

$$\{2b + 1; (s, \beta_2), (s, \beta_2), (2, \beta_3)\}$$

is realized as surgery on T(2,s) as long as p and q are relatively prime. It is easy to check that this often happens. The other cases (2)-(5) are similar.

Realization of Covers over a Fixed Orbifold

In this case the only possible covers are fiberwise covers, which are determined by Corollary 3.4.6. Since the b and β invariants are determined by p (see Lemma 3.2.7), it is enough to compute p (the order of H_1) in the cover, and see if surgery with that p can produce the base orbifold in question. We provide an example:

Consider the Seifert fiber space obtained by $-2/3$ surgery on T(2,5). This has base orbifold $S^2(2,5,32)$ with H_1 of order 2. The standard Seifert form is therefore

$$\{-2; (2,1), (5,3), (32,29)\}.$$

Taking this as a degree d fiberwise cover gives

$$\{-2d; (2,d), (5,3d), (32,29d)\}$$

which has H_1 of order $2d$, which will be p/q surgery on T(2,5) precisely when $|10q-p| = 32$ and $2d = |p|$. Additionally, the value of q is then determined by $|10q-p| = 32$, and must be relatively prime to p. For example $p = -12, q = 2$ is a solution, but not a valid surgery, whereas $p = -22, q = 1$ is.

Remark 3.6.4. This example agrees with [LM16, Theorem 1.12], since although $2/3 < 1$, $\lceil 3/2 \rceil > \lfloor 1/22 \rfloor$ so this (regular) cover is consistent with their theorem.

Realization of Compositions of Covers

We describe the general method for checking if one Seifert fiber space $\widetilde{M}$ covers another Seifert fiber space M, according to Proposition 3.4.4.

1. First check if there exists a cover between the base orbifolds. Note that M comes with a specified base orbifold, but if $\widetilde{M}$ is a lens space, then we must check all $S^2(d,d)$ which cover the base orbifold of M. For small Seifert fiber spaces this is classified in section 3.5.

2. Next compute the pullback of the proposed base manifold M along the cover of base orbifolds from (1), as described in section 3.6.1

3. Finally check if the proposed cover $\widetilde{M}$ covers this pullback as described in section 3.6.2.

Proof of Theorem 3.1.6. By Lemma 3.3.2, we can reduce to the case that at least one of the two surgeries is not a lens space. Theorem 3.1.4 classifies covers of base orbifolds in this case. All such non-trivial covers could only occur on the listed exceptional torus knots, so the remaining covers are fiberwise covers. It remains to check that if $\widetilde{M} \to M$ is a degree d fiberwise cover, then $d \cdot |H_1(\widetilde{M})| = |H_1(M)|$. Suppose

$$\widetilde{M} = \{b; (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)\}.$$

Then according to [Mos71], $|H_1(\widetilde{M})| = |\alpha_1\alpha_2\alpha_3 b + \alpha_1\alpha_2\beta_3 + \alpha_1\beta_2\alpha_3 + \beta_1\alpha_2\alpha_3|$, and by Corollary 3.4.6

$$M = \{db; (\alpha_1, d\beta_1), (\alpha_2, d\beta_2), (\alpha_3, d\beta_3)\}.$$

This gives that $|H_1(M)| = |\alpha_1\alpha_2\alpha_3 db + \alpha_1\alpha_2 d\beta_3 + \alpha_1 d\beta_2\alpha_3 + d\beta_1\alpha_2\alpha_3| = d \cdot |H_1(\widetilde{M})|$ as desired. Conversely, since $|H_1(M)|$ and the base orbifold determine M, as long as $|rsq - p| = |rsq' - p'|$, we can try to take an appropriate degree fiberwise cover of $S^3_{p'/q'}(T(r,s))$ to get $S^3_{p/q}(T(r,s))$. This cover will exist if and only if $p'|p$ and p/p' is relatively prime to the indices of the singular fibers, r, s, and $|rsq' - p'|$.

□

We conclude with a pair of examples.

Example 3.6.5. Let $\widetilde{M}$ be $(5,1)$ surgery on $T(2,3)$ and let M be $(45,7)$ surgery on $T(2,3)$. Then by Moser's classification M is given by

$$\{1; (2,1), (3,1), (3,2)\}$$

with base orbifold $S^2(2,3,3)$. Since $\widetilde{M}$ is a lens space, we should check pullbacks along $S^2 \to S^2(2,3,3)$, $S^2(2,2) \to S^2(2,3,3)$, and $S^2(3,3) \to S^2(2,3,3)$. We will first pull back along $S^2(3,3) \to S^2(2,3,3)$, which will turn out to be sufficient. The partitions for this degree 4 cover are $4 = \frac{2}{1} + \frac{2}{1} = \frac{3}{1} + \frac{3}{3} = \frac{3}{1} + \frac{3}{3}$ as computed from figure 12. This gives the Seifert fiber space

$$\{4; (1,1), (1,1), (1,1), (3,1), (1,2), (3,2)\} = \{9; (3,1), (3,2)\} = L(90, -29).$$

This is 19-fold covered by $L(5,-29) = L(5,1)$, which by Moser's classification is $\widetilde{M}$. In fact no cover $\widetilde{M} \to M$ could come from a cover of the complement of $T(2,3)$, since such a cover would necessarily be fiber preserving on the knot complement. Alternatively, since the complement of $T(2,3)$ is also Seifert fibered (with Seifert invariants $(2,1),(3,\pm 1)$, depending on orientation), it is also possible to compute all self covers directly.

Example 3.6.6. Let $\widetilde{M}$ be 105/4 surgery on $T(4,7)$ and let M be 21/1 surgery on $T(4,7)$. Then by Theorem 3.1.6 $\widetilde{M}$ is a 5-fold cover of M, both of which have base orbifold $S^2(4,7,7)$. However, this cover does not restrict to a self cover of the $T(4,7)$ complement, as can be seen from the Seifert invariants.

$$\widetilde{M} = \{-1; (4,1),(7,5),(7,4)\}, \quad M = \{-1; (4,1),(7,5),(7,1)\}.$$

The degree 5 cover between them sends the $(7,5)$ fiber to the $(7,1)$ fiber, whereas in a self cover of the knot complement, the $(7,5)$ fiber must be preserved.

3.7. Hyperbolic Knots

In this section we will first use a theorem of Futer, Kalfagianni, and Purcell to prove Proposition 3.1.8, and then we will use computations of the hyperbolic volume and identification of exceptional surgeries to prove Proposition 3.1.9.

First we will give a necessary definition. For more background information see [Rat06]. We will use the homological framing for knots in S^3, so that the longitude refers to the framing curve having linking number 0 with the knot. Using the standard identification of the boundary of a horoball neighborhood of the cusp with a torus quotient of $\mathbb{C}$, we can define complex lengths for the longitude and meridian. These are only determined up to scaling the horoball, so we use the following.

Definition 3.7.1. The *cusp shape* $s \in \mathbb{C}$ of a hyperbolic knot is $s = l/m$, where l is the complex length of the longitude, and m is the complex length of the meridian.

This is independent of the choice of horoball since the longitude and meridian scale together.

Our first goal is to prove Proposition 3.1.8, here restated as Corollary 3.7.3, which is a corollary of the following theorem of Futer, Kalfagianni, and Purcell.

Theorem 3.7.2. *[FKP08, Theorem 1.1] Let K be a hyperbolic knot in S^3, and let l be the length of a surgery slope p/q on the knot complement which is greater than 2π. Then*

$$Vol(K_{p/q}) \geq \left(1 - \left(\frac{2\pi}{l_{p/q}}\right)^2\right)^{3/2} \cdot Vol(S^3 - K).$$

Corollary 3.7.3. *Let $K \subset S^3$ be a hyperbolic knot, and $p/q \in \mathbb{Q}$. Then there are at most 32 $p'/q' \in \mathbb{Q}$ such that $K_{p'/q'}$ is non-trivially covered by $K_{p/q}$.*

Remark 3.7.4. A somewhat similar theorem of Hodgson and Kerckhoff [HK05, Theorem 5.9, Corollary 6.7] gives a similar result, but with a bound of 60 surgeries.

Proof of Corollary 3.7.3. We will use Theorem 3.7.2 to bound from above the surgery length of hyperbolic surgeries which could contradict the conjecture. Let $\mathrm{Vol}(K_{p/q})$ be the hyperbolic volume of $K_{p/q}$, and let $K_{p/q} \to K_{p'/q'}$ be a degree n cover. Since hyperbolic volume is multiplicative under covers (see for example [Rat06, Theorem 11.6.3]),

$$\mathrm{Vol}(K_{p/q}) = n\mathrm{Vol}(K_{p'/q'}).$$

Furthermore a theorem of Thurston [Thu, Theorem 6.5.6] gives the inequality $\mathrm{Vol}(S^3 - K) > \mathrm{Vol}(K_{p/q}), \mathrm{Vol}(K_{p'/q'})$. Hence by non-triviality of the cover,

$$\mathrm{Vol}(K_{p'/q'}) < \mathrm{Vol}(S^3 - K)/2. \tag{3.7.5}$$

Now we can solve for $l_{p'/q'}$ in Theorem 3.7.2 to get

$$l_{p'/q'} < \frac{2\pi}{\sqrt{1-(1/2)^{2/3}}} = 10.328942\ldots$$

We claim there are at most 32 p'/q' for which this is satisfied. Let p'/q' and r/s be slopes such that the above equation is satisfied, and let area(T) be the area of the cusp torus T for K. Then as in the proof of [Ago00, Theorem 8.1],

$$|p's - rq'| < \frac{(10.33)^2}{\text{area}(T)}.$$

Furthermore, area$(T) \geq 2\sqrt{3}$ (see for example [CM01], note that equality holds if and only if K is the knot 4_1). Combining these results then gives

$$|p's - rq'| < 30.84.$$

But by [Ago00, Lemma 8.2], there are at most $P(k) + 1$ slopes with intersection number at most k where $P(k)$ is the smallest prime larger than k, so there are at most 32 p'/q' such that $K_{p'/q'}$ is non-trivially covered by $K_{p/q}$.

□

The rest of this section is devoted to checking that none of the 32 potential exceptions for low crossing number knots give rise to counterexamples. We proceed by using the computer program SnapPy [CDGW] to check the hyperbolic surgeries. First, SnapPy will compute the cusp shape $s \in \mathbb{C}$ of a hyperbolic knot. From this it is easiest to compute the normalized surgery length, so we normalize the cusp to have area 1, and to have positive real meridian. Computing this normalized meridian m and longitude l in terms of the cusp shape s given by SnapPy gives

$$m = \frac{1}{\sqrt{|\mathrm{Im}(s)|}}, \quad l = sm.$$

The following lemma will then let us bound which p/q may give rise to the 32 potentially exceptional surgeries.

Lemma 3.7.6. *Let $k \in \mathbb{R}_{>0}$, $a = \dfrac{|k \cdot Re(l)|}{|m \cdot Im(l)|} + \dfrac{k}{m}$ and $b = \dfrac{k}{|Im(l)|}$, and suppose either $|p| > a$ or $|q| > b$. Then (p, q) surgery on K has surgery curve of normalized length greater than k.*

Proof. The normalized surgery length is $|pm + ql|$, and since m is real, $|q \cdot \mathrm{Im}(l)| \leq |pm + ql|$. In particular, as long as $|q| > \dfrac{k}{|\mathrm{Im}(l)|}$ then $|pm + ql| > k$. Now suppose $|q| \leq \dfrac{k}{|\mathrm{Im}(l)|}$, but that $|p| > \dfrac{|k \cdot \mathrm{Re}(l)|}{|m \cdot \mathrm{Im}(l)|} + k$. Then

$$|pm + ql| \geq |\mathrm{Re}(pm + ql)| = |\mathrm{Re}(pm) + \mathrm{Re}(ql)| = |pm + \mathrm{Re}(ql)|.$$

But $|\mathrm{Re}(ql)|$ is at most $\dfrac{k \cdot |\mathrm{Re}(l)|}{|\mathrm{Im}(l)|}$, so as long as $|pm|$ is at least $\dfrac{|k \cdot \mathrm{Re}(l)|}{|\mathrm{Im}(l)|} + k$ then $|pm + ql| > k$, or equivalently as long as $|p| \geq \dfrac{|k \cdot \mathrm{Re}(l)|}{|m \cdot \mathrm{Im}(l)|} + \dfrac{k}{m}$, then $|pm + ql| > k$, as desired. □

Now we can use Lemma 3.7.6 and SnapPy to finish the case of hyperbolic surgeries on knots with 8 or fewer crossings.

Proposition 3.7.7. *Let K be a hyperbolic knot with 8 or fewer crossings. Then there is no pair of hyperbolic surgeries $S^3_{p/q}(K)$ and $S^3_{p'/q'}(K)$ with a non-trivial covering between them.*

Proof. Using Corollary 3.7.3, it would be enough to check that among the shortest 32 surgery lengths all have hyperbolic volume greater than $\mathrm{Vol}(S^3 - K)/2$. The volumes are checked with SnapPy using Lemma 3.7.6 to ensure that we check at least the 32 shortest curves.

For all of them except $S^3_{\pm 5/1}(4_1)$ and $S^3_{1/1}(6_1)$, the volume of the surgered manifold is more than half the volume of the knot complement. Hence by Equation 3.7.5 they cannot be covered by other surgeries on the same knot. For the remaining two hyperbolic surgeries, we have

$$\mathrm{Vol}(S^3_{5/1}(4_1)) = 0.9813688\ldots \text{ and } \mathrm{Vol}(S^3_{1/1}(6_1)) = 1.3985088\ldots$$

whereas

$$\mathrm{Vol}(4_1) = 2.0298832\ldots \text{ and } \mathrm{Vol}(6_1) = 3.1639632\ldots$$

For these two surgeries the volume is more than a third the volume of the knot complement. Hence it is enough to check that these two manifolds have no two fold covers. But

$$|H_1(S^3_{\pm 5/1}(4_1))| = 5, \text{ and } |H_1(S^3_{1/1}(6_1))| = 1$$

are both odd, so there are no maps from $H_1 \to \mathbb{Z}/2\mathbb{Z} = S_2$, so there are no two fold covers. □

This leaves the case of exceptional (non-hyperbolic) surgeries on knots with 8 or fewer crossings to which we devote the rest of this section. We first consider alternating knots for which exceptional surgeries are classified in [IM16, Corollary 1.2]. In particular, among alternating hyperbolic knots, only twist knots have more than one exceptional surgery. The Regina software [BBP$^+$16] was used to identify the Seifert fibered and toroidal exceptional surgeries, and the zero-surgeries. The case of the toroidal ±4-surgery is also worked out in [Ter13, Section 2], and is

Knot	+1-surgery	+2-surgery	+3-surgery
4_1	$\{;(2,1),(3,1),(7,1)\}$	$\{;(2,1),(4,1),(5,1)\}$	$\{;(3,1),(3,1),(4,1)\}$
5_2	$\{;(2,1),(3,1),(11,2)\}$	$\{;(2,1),(4,1),(7,2)\}$	$\{;(3,1),(3,1),(5,2)\}$
$m6_1$	$\{;(2,1),(3,1),(13,2)\}$	$\{;(2,1),(4,1),(9,2)\}$	$\{;(3,1),(3,1),(7,2)\}$
$m7_2$	$\{;(2,1),(3,1),(17,3)\}$	$\{;(2,1),(4,1),(11,3)\}$	$\{;(3,1),(3,1),(8,3)\}$
$m8_1$	$\{;(2,1),(3,1),(19,3)\}$	$\{;(2,1),(4,1),(13,3)\}$	$\{;(3,1),(3,1),(10,3)\}$

TABLE 2. The exceptional Seifert fiber surgeries on hyperbolic twist knots with 8 or fewer crossings. The m refers to the mirror of the knot, and for 4_1 there are the additional $-1, -2, -3$-surgeries since it is amphichiral. For each of the listed surgeries, the b Seifert invariant is -1 and so is omitted.

the union of a twisted interval bundle over the Klein bottle and a torus knot complement. Table 2 gives the Seifert fibered surgeries, and Table 3 gives the toroidal surgeries. For convenience we use the mirrors of $6_1, 7_2$, and 8_1, and since 4_1 is amphichiral we only list its non-negative surgeries.

Covers of Seifert fiber spaces are Seifert fiber spaces, and the multiplicativity of orbifold Euler characteristic gives an obstruction to covers between the surgeries in Table 2. We now consider the toroidal surgeries in Table 3.

Lemma 3.7.8. *Let M and N be 3-manifolds. If dim $H_1(M;\mathbb{R}) >$ dim $H_1(N;\mathbb{R})$ then N cannot cover M.*

Proof. Suppose $f : N \to M$ is a covering map. Then the transfer homomorphism composed with the induced map f_* on homology induces multiplication by $\deg(f)$ on $H_1(M;\mathbb{R})$, which is an isomorphism. This implies that the transfer homomorphism is injective and hence that dim $H_1(M;\mathbb{R}) \leq$ dim $H_1(N;\mathbb{R})$. □

Twist knot	0-surgery	+4-surgery
4_1	$[A:(1,1)]/\begin{pmatrix}0 & 1\\ 1 & -2\end{pmatrix}$	$(S^3 - T(2,3)) \cup K_I$
5_2	$[A:(2,1)]/\begin{pmatrix}0 & 1\\ 1 & -1\end{pmatrix}$	$(S^3 - T(2,3)) \cup K_I$
$m6_1$	$[A:(2,1)]/\begin{pmatrix}0 & 1\\ 1 & -2\end{pmatrix}$	$(S^3 - T(2,5)) \cup K_I$
$m7_2$	$[A:(3,2)]/\begin{pmatrix}0 & 1\\ 1 & -1\end{pmatrix}$	$(S^3 - T(2,5)) \cup K_I$
$m8_1$	$[A:(3,1)]/\begin{pmatrix}0 & 1\\ 1 & -2\end{pmatrix}$	$(S^3 - T(2,7)) \cup K_I$

TABLE 3. The exceptional toroidal surgeries on hyperbolic twist knots with 8 or fewer crossings. K_I refers to the nontrivial interval bundle over the Klein bottle coming from the mapping cylinder of the orientation cover. $[A : (x,y)]$ refers to the Seifert fiber space with base surface the annulus and a single exceptional fiber (x,y). Quotienting by a matrix refers to gluing the two torus boundary components together via that element of the mapping class group. The framing is given by choosing the fiber and a section. As in Table 2 the m refers to the mirror of the knot, and we omit the -4-surgery on 4_1.

By Lemma 3.7.8, 0-surgery on a knot can never be covered by any non-zero surgery on a knot. It remains to check that 4-surgery is not covered by 0-surgery for twist knots. To do so, we consider the geometric decomposition surface of [AFW15, Section 1.9]. This is similar to the geometric torus decomposition, except that it additionally allows Klein bottles coming from K_I components, as we have in Table 3. Observe that for 4-surgery on a twist knot we have a single Klein bottle as the geometric decomposition surface, since torus knot complements admit an $\widetilde{SL_2(\mathbb{R})}$ geometry (see for example [Tsa13]). Now by [AFW15, Theorem 1.9.3] this geometric decomposition surface lifts to the geometric decomposition surface of any finite cover. In particular, if 0-surgery on a twist knot covered 4-surgery on a twist knot, then it would have a (non-empty) geometric decomposition surface cutting it into pieces which each cover the respective torus knot complement.

However, the geometric decomposition surface for the twist knot 0-surgeries has at most one torus, since the obvious torus cuts it into a single Seifert fiber space $[A : (x, y)]$. However, by multiplicativity of the orbifold characteristic, $[A : (1, 1)]$ does not cover $D^2(2, 3) = S^3 - T(2, 3)$ (and similarly for the other twist knots we consider). Hence 0-surgery cannot cover 4-surgery on these twist knots. In particular,

Proposition 3.7.9. *Conjecture 3.1.7 is true for alternating knots with 8 or fewer crossings.*

The final case is that of the non-alternating hyperbolic knots of 8 or fewer crossings, the knots 8_{20} and 8_{21}.

SnapPy [CDGW] verifies that all surgeries on the knot 8_{21}, and all surgeries except the 0, 1, and 2 surgery on the knot 8_{20} are hyperbolic. In fact, the volumes of surgeries on the knot 8_{21} and of hyperbolic surgeries on the knot 8_{20} are all large enough to obstruct any non-trivial covers, as in Corollary 3.7.3.

The knot 8_{20} is also the pretzel knot $P(-3,3,2)$, and [Mei14, Theorem 1.1], or Wu [Wu11, Theorem 1.1] can be checked to see that the only toroidal surgery on 8_{20} is the 0-surgery. Hence the Seifert fiber space surgeries on $P(-3,3,2)$ are the $+1$ and $+2$ surgeries, which are identified by Regina as

$$\{-1; (3,1), (4,1), (5,2)\} \text{ and } \{-1; (2,1), (4,1), (9,2)\}$$

respectively. These base orbifolds have orbifold characteristic $-13/60$ and $-5/36$ respectively, so there is no cover between these spaces. This concludes the case of hyperbolic knots with 8 or fewer crossings.

Proposition 3.7.10. *Let K be a hyperbolic knot with 8 or fewer crossings. Then $S^3_{p/q}(K)$ does not non-trivially cover $S^3_{p'/q'}(K)$. In particular, Conjecture 3.1.7 is true for these knots.*

This also completes the proof of Proposition 3.1.9.

CHAPTER IV

QUOTIENTS OF DEFINITE KNOTS ARE DEFINITE

4.1. Introduction

Let K be a knot in S^3 with signature $\sigma(K)$ and genus $g(K)$. Then K is *definite* if $|\sigma(K)| = 2g(K)$. This is a relatively small class of knots, but this condition has a nice geometric interpretation. Specifically, a knot is definite if and only if it has a Seifert surface with definite linking form.

A knot $K \subset S^3$ is *periodic* if it is fixed by a finite cyclic group acting on S^3 with fixed set an unknot disjoint from K. In this case we refer to the image of K in $S^3/(\mathbb{Z}/p)$ as the *quotient knot*.

The goal of this chapter is to investigate periodic definite knots, and in particular apply a result of Edmonds [Edm84, Theorem 4] to prove the following theorem.

Theorem 4.1.1. *The quotient of a periodic definite knot is definite.*

4.2. Background

Definition 4.2.1. A quadratic form $\langle -, - \rangle$ is *positive (resp. negative) definite* if $\langle x, x \rangle > 0$ (resp. < 0) for all $x \neq 0$.

We will also use the equivalent charaterization that a matrix is positive (resp. negative) definite if and only if all of its eigenvalues are positive (resp. negative).

Definition 4.2.2. A Seifert surface S for K is *positive (resp. negative) definite* if the (symmetrized) linking form $\text{lk}(-, -)$ on $H_1(S)$ as defined in [GL78, Section 2] is positive (resp. negative) definite. That is, the symmetrized Seifert matrix for S is definite.

Definition 4.2.3. A knot is *definite* if it has a definite Seifert surface.

Lemma 4.2.4. *Let $K \subset S^3$ be a knot. Then the following are equivalent.*

1. *K is definite.*

2. *Every minimal genus Seifert surface for K is definite.*

3. *$|\sigma(K)| = 2g(K)$, where $g(K)$ is the genus of K.*

Proof. (2) implies (1) is obvious, and we will show that (1) implies (3) and (3) implies (2).

To see that (1) implies (3), suppose K is definite with definite Seifert surface S and corresponding symmetrized Seifert matrix $M \in M_n(\mathbb{Z})$. Since M is definite, $\sigma(M) = \pm n = \sigma(K)$. In particular, M is a minimal dimensional symmetrized Seifert matrix and so S is a minimal genus Seifert surface. Hence $|\sigma(K)| = 2g(K)$.

On the other hand, suppose $|\sigma(K)| = 2g(K)$. Then taking any minimal genus Seifert surface S with symmetrized Seifert matrix $M \in M_n(\mathbb{Z})$, we see that $|\sigma(K)| = |\sigma(M)| \leq \dim(M) = 2g(K)$, and hence $|\sigma(M)| = n$ so M is definite. □

The following proposition gives a strong restriction on the Alexander polynomial of definite knots.

Proposition 4.2.5. *Let $K \subset S^3$ be a definite knot. Then $|\Delta_K(t)| = |\sigma(K)| = 2g(K)$, where $|\Delta_K(t)|$ is the width of the Alexander polynomial.*

Proof. Let S be a definite Seifert surface for K with Seifert matrix $M \in M_n(\mathbb{Z})$, and recall that $\Delta_K(t) = \det(M^T - tM)$. Since M is definite $\det(M^{-1}) \neq 0$, so multiplying both sides by $\det(M^{-1})$ makes it clear that the first and last terms of $\Delta_K(t)$ will be $\det(M)t^n$ and $\det(M)$ respectively. Hence the width of the Alexander polynomial is $n = |\sigma(M)| = |\sigma(K)|$. The second equality is proved in Lemma 4.2.4. □

4.3. Periodic Definite Knots

Theorem 4.1.1. The quotient knot of a periodic definite knot is definite.

The proof of this theorem relies on the following theorem of Edmonds.

Theorem 4.3.1. *[Edm84, Theorem 4] Let $\widetilde{K}$ be a periodic knot. Then there exists a minimal genus Seifert surface $\widetilde{S}$ for $\widetilde{K}$ which is preserved by the periodic action. Furthermore, the image of $\widetilde{S}$ in the quotient is a Seifert surface for the quotient knot K.*

We will also need the following lemma.

Lemma 4.3.2. *If the preimage of a Seifert surface S under a $\mathbb{Z}/p$ rotation action in S^3 is a positive (resp. negative) definite Seifert surface $\widetilde{S}$, then S is positive (resp. negative) definite.*

Proof. Consider a curve $C \subset S$ which is homologically non-trivial. Let $\widetilde{C}$ be the (possibly disconnected) preimage of C in $\widetilde{S}$. Note that since C is homologically non-trivial, so is $\widetilde{C}$. Now suppose $\widetilde{S}$ is positive definite so that $\mathrm{lk}(\widetilde{C}, \widetilde{C}) > 0$. We claim that $\mathrm{lk}(C, C) > 0$, so that S is also positive definite. The linking number $\mathrm{lk}(C, C)$ is the sum of (signed) intersection points between C and the Seifert surface Σ for a positive push-off of C. Let $\widetilde{\Sigma}$ be the preimage of Σ which is an equivariant Seifert surface for a positive push-off of $\widetilde{C}$. Then each intersection point between C

and Σ lifts to p intersection points (with the same sign) between $\widetilde{C}$ and $\widetilde{\Sigma}$. Hence $\text{lk}(\widetilde{C}, \widetilde{C}) = p \cdot \text{lk}(C, C)$, and so $\text{lk}(C, C) > 0$. □

Proof of Theorem 4.1.1. By Theorem 4.3.1 any periodic knot $\widetilde{K}$ has an equivariant minimal genus Seifert surface $\widetilde{S}$ with quotient S. By Lemma 4.2.4, $\widetilde{S}$ is definite, and so by Lemma 4.3.2 S is as well. □

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