

REGULARITY OF FOURTH AND SECOND ORDER NONLINEAR ELLIPTIC
EQUATIONS

by

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DISSERTATION ABSTRACT

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Title: Regularity of Fourth and Second Order Nonlinear Elliptic Equations

In this thesis, we prove regularity theory for nonlinear fourth order and second order elliptic equations. First we show that for a certain class of fourth order equations in the double divergence form, where the nonlinearity is in the Hessian, solutions that are $C^{2,\alpha}$ enjoy interior estimates on all derivatives. Next, we consider the fourth order Lagrangian Hamiltonian stationary equation for all phases in dimension two and show that solutions, which are $C^{1,1}$ will be smooth and we also derive a $C^{2,\alpha}$ estimate for it. We also prove explicit $C^{2,\alpha}$ interior estimates for viscosity solutions of fully nonlinear, uniformly elliptic second order equations, which are close to linear equations and we compute an explicit bound for the closeness.

This dissertation includes previously published co-authored material.

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CHAPTER I

INTRODUCTION

In this thesis, we discuss the regularity of second and fourth order nonlinear partial differential equations that arise naturally in differential geometry. In particular, we study the fourth order Lagrangian Hamiltonian stationary equation and solutions of fourth order nonlinear elliptic equations in the double divergence form; for example, critical points of convex or concave functionals defined on Hessian spaces. We also study fully nonlinear second order equations which are neither convex nor concave but are close to linear equations.

Chapter two introduces functionals of the form (2.40) defined on the space of matrices. The Euler-Lagrange form of such functionals are fourth order nonlinear equations of the double divergence form that may not necessarily be factored into second order operators. In section (2.2) we prove estimates for the bounded mean oscillations of solutions of constant coefficient fourth order elliptic equations in the double divergence form and in section (2.3) we prove Schauder type estimates for solutions of linear fourth order elliptic equations in the double divergence form. Combining the above two results we prove our main result in section (2.4), i.e, we prove smoothness and derive interior estimates for solutions of a certain class of Euler Lagrange equations that arise from functionals of the form (2.40). In section (2.5) we show some important applications of our result. Much of this chapter draws from the paper [1].

Chapter three introduces the second order special Lagrangian equation (3.2) and the fourth order Lagrangian Hamiltonian stationary equation (3.3). In section (3.2), we prove interior Schauder estimates for solutions of the non homogeneous special Lagrangian equation where the phase is a C^α function and lies in the

sub-critical region. Using this result along with regularity results for the special Lagrangian equation with a supercritical phase, we prove our main results in section (3.3) and section (3.4), i.e., we prove that $C^{1,1}$ solutions of the Hamiltonian stationary equation, for all phases in dimension two, are smooth with uniform interior $C^{2,\alpha}$ estimates. Much of this chapter draws from the paper [2].

Chapter four introduces the theory of a priori estimates for viscosity solutions of second order equations with convexity and uniform ellipticity and how the structure of F plays a key role in deriving higher order estimates for fully nonlinear equations of the form (4.2 and 4.1). We consider viscosity solutions of fully nonlinear, uniformly elliptic equations that are neither convex nor concave but are close to linear equations and in section (4.2), we derive an explicit formula to compute the distance between (4.1) and the Laplace equation and also construct a quadratic polynomial that separates from u by a distance of $r^{2+\alpha}$ on the ball of radius r . Using this result along with results involving $W^{2,p}$ estimates for concave equations [3] we prove our main theorems in section (4.3) and section (4.4), i.e, we derive $C^{2,\alpha}$ interior estimates for (4.2 and 4.1) along with computing an explicit bound for the closeness. Much of this chapter draws from the paper [4].

CHAPTER II

REGULARITY BOOTSTRAPPING: 4TH ORDER EQUATIONS

2.1. Background and Introduction

In this chapter, we develop Schauder and bootstrapping theory for solutions to fourth order non linear elliptic equations of the following double divergence form

$$\int_{\Omega} a^{ij,kl}(D^2u)u_{ij}\eta_{kl}dx = 0, \quad \forall \eta \in C_0^\infty(\Omega) \quad (2.1)$$

in $B_1 = B_1(0)$. For the Schauder theory, we require the standard Legendre-Hadamard ellipticity condition,

$$a^{ij,kl}(D^2u(x))\xi_{ij}\xi_{kl} \geq \Lambda|\xi_{rs}|^2 \quad (2.2)$$

while in order to bootstrap, we will require the following condition:

$$b^{ij,kl}(D^2u(x)) = a^{ij,kl}(D^2u(x)) + \frac{\partial a^{pq,kl}}{\partial u_{ij}}(D^2u(x))u_{pq}(x) \quad (2.3)$$

satisfies

$$b^{ij,kl}(D^2u(x))\xi_{ij}\xi_{kl} \geq \Lambda_1 \|\xi\|^2. \quad (2.4)$$

Our main result is the following: Suppose that conditions (2.1) and (2.4) are met on some open set $U \subseteq S^{n \times n}$ (space of symmetric matrices). If u is a $C^{2,\alpha}$ solution with $D^2u(B_1) \subset U$, then u is smooth on the interior of the domain B_1 .

One example of such an equation is the Hamiltonian Stationary Lagrangian equation, which governs Lagrangian surfaces that minimize the area functional

$$\int_{\Omega} \sqrt{\det(I + (D^2u)^T D^2u)} dx \tag{2.5}$$

among potential functions u . (cf. [5], [6, Proposition 2.2]). The minimizer satisfies a fourth order equation, that, when smooth, can be factored into a Laplace type operator on a nonlinear quantity. Recently in [7], the authors have shown that a C^2 solution is smooth. The results in [7] are the combination of an initial regularity boost, followed by applications of the second order Schauder theory as in [3].

More generally, for a functional F on the space of matrices, one may consider a functional of the form

$$\int_M F(D^2u) dx.$$

The Euler-Lagrange equation will generically be of the following double-divergence type:

$$\frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{\partial F}{\partial u_{ij}}(D^2u) \right) = 0. \tag{2.6}$$

(2.6) need not factor into second order operators, so it may be genuinely a fourth order double-divergence elliptic type equation. It should be noted that in general, (2.6) need not take the form of (2.1). It does when $F(D^2u)$ can be written as a function of $D^2u^T D^2u$ (as for example (3.3)). Our results in this chapter apply to a class of Euler-Lagrange equations arising from such functionals. In particular, we will show that if F is a convex function of D^2u and a function of $D^2u^T D^2u$ (such as 3.3 when $|D^2u| \leq 1$) then $C^{2,\alpha}$ solutions will be smooth.

The Schauder theory for second order divergence and non-divergence type elliptic equations is by now well-developed, see [8], [9] and [3]. For higher order

non-divergence equations, Schauder theory is available, see [10]. However, for higher order equations in divergence form, much less is known. One expects the results to be different. For second order equations, solutions to divergence type equations with C^α coefficients are known to be $C^{1,\alpha}$, [8, Theorem 3.13], whereas for non-divergence equations, solutions will be $C^{2,\alpha}$ [9, Chapter 6]. Recently, in [11], the authors have obtained general Schauder theory results for parabolic equations (of order $2m$) in divergence form, where the time coefficients are allowed to be merely measurable. Their proof (like ours) is in the spirit of Campanato techniques, but requires smooth initial conditions. Our result is aimed at showing that weak solutions are in fact smooth. Classical Schauder theory for general systems has been developed, [12, Chapter 5,6]. However, it is non-trivial to apply the general classical results to obtain the result we are after. Even so, it is useful to focus on a specific class of fourth order double-divergence operators, and offer random access to the non-linear Schauder theory for these cases. Regularity for fourth order equations remains an important developing area of geometric analysis.

Our proof goes as follows: We start with a $C^{2,\alpha}$ solution of (2.1) whose coefficient matrix is a smooth function of the Hessian of u . We first prove that $u \in W^{3,2}$ by taking a difference quotient of (2.1) and give a $W^{3,2}$ estimate of u in terms of its $C^{2,\alpha}$ norm. Again by taking a difference quotient and using the fact that now $u \in W^{3,2}$, we prove that $u \in C^{3,\alpha}$.

Next, we make a more general proposition where we prove a $W^{3,2}$ estimate for $u \in W^{2,2}$ satisfying a uniformly elliptic equation of the form

$$\int (c^{ij,kl} u_{ik} + h^{jl}) \eta_{jl} dx = 0$$

in $B_1(0)$, where $c^{ij,kl}, h^{kl} \in W^{1,2}(B_1)$ and η is a test function in B_1 . Using the fact that $u \in W^{3,2}$, we prove that $u \in C^{3,\alpha}$ and also derive a $C^{3,\alpha}$ estimate of u in terms of its $W^{3,2}$ norm. Finally, using difference quotients and dominated convergence, we achieve all higher orders of regularity.

Definition 2.1.1. *We say an equation of the form (2.1) is **regular on** $U \subseteq S^{n \times n}$ when the coefficients of the equation satisfy the following conditions on U :*

1. *The coefficients $a^{ij,kl}$ depend smoothly on D^2u .*
2. *The coefficients $a^{ij,kl}$ satisfy (3.16).*
3. *Either b^{ijkl} or $-b^{ijkl}$ (given by (2.3)) satisfy (2.4).*

The following is our main result.

Theorem 2.1.2. *Suppose that $u \in C^{2,\alpha}(B_1)$ satisfies the following fourth order equation*

$$\int_{B_1(0)} a^{ij,kl}(D^2u(x))u_{ij}(x)\eta_{kl}(x)dx = 0$$

$$\forall \eta \in C_0^\infty(B_1(0))$$

If $a^{ij,kl}$ is regular on an open set containing $D^2u(B_1(0))$, then u is smooth on $B_r(0)$ for $r < 1$.

To prove this, we will need the following two Schauder type estimates.

Proposition 2.1.3. *Suppose $u \in W^{2,\infty}(B_1)$ satisfies the following*

$$\int_{B_1(0)} [c^{ij,kl}(x)u_{ij}(x) + f^{kl}(x)] \eta_{kl}(x)dx = 0$$

$$\forall \eta \in C_0^\infty(B_1(0))$$
(2.7)

where $c^{ij,kl}, f^{kl} \in W^{1,2}(B_1)$, and $c^{ij,kl}$ satisfies (3.16). Then $u \in W^{3,2}(B_{1/2})$ and

$$\|D^3u\|_{L^2(B_{1/2})} \leq C(\|u\|_{W^{2,\infty}(B_1)}, \|f^{kl}\|_{W^{1,2}(B_1)}, \|c^{ij,kl}\|_{W^{1,2}}, \Lambda_1).$$

Proposition 2.1.4. *Suppose $u \in C^{2,\alpha}(B_1)$ satisfies (2.7) in B_1 where $c^{ij,kl}, f^{kl} \in C^{1,\alpha}(B_1)$ and $c^{ij,kl}$ satisfies (3.16). Then we have $u \in C^{3,\alpha}(B_{1/2})$ with*

$$\|D^3u\|_{C^{0,\alpha}(B_{1/4})} \leq C(1 + \|D^3u\|_{L^2(B_{3/4})})$$

and $C = C(|c^{ij,kl}|_{C^\alpha(B_1)}, |f^{kl}|_{C^\alpha(B_1)}, \Lambda_1, \alpha)$ is a positive constant.

We note that the above estimates are appropriately scaling invariant: Thus we can use these to obtain interior estimates for a solution in the interior of any sized domain.

2.2. Preliminaries

We begin by considering a constant coefficient double divergence equation.

Theorem 2.2.1. *Suppose $w \in H^2(B_r)$ satisfies the constant coefficient equation*

$$\int c_0^{ik,jl} w_{ik} \eta_{jl} dx = 0 \tag{2.8}$$

$$\forall \eta \in C_0^\infty(B_r(0)).$$

Then for any $0 < \rho \leq r$ there holds

$$\int_{B_\rho} |D^2 w|^2 \leq C_1 (\rho/r)^n \|D^2 w\|_{L^2(B_r)}^2$$

$$\int_{B_\rho} |D^2 w - (D^2 w)_\rho|^2 \leq C_2 (\rho/r)^{n+2} \int_{B_r} |D^2 w - (D^2 w)_r|^2.$$

Here $(D^2 w)_\rho$ is the average value of $D^2 w$ on a ball of radius ρ .

Proof. By dilation we may consider $r = 1$. We restrict our consideration to the range $\rho \in (0, a]$ noting that the statement is trivial for $\rho \in [a, 1]$ where a is some constant in $(0, 1/2)$.

First, we note that w is smooth [13, Theorem 33.10]. Recall [14, Lemma 2, Section 4, applied to elliptic case] : For an elliptic 4th order L_0

$$L_0 u = 0 \text{ on } B_R$$

$$\implies \|Du\|_{L^\infty(B_{R/4})} \leq C_3(\Lambda, n) \|u\|_{L^2(B_R)}.$$

We may apply this to the second derivatives of w to conclude that

$$\|D^3 w\|_{L^\infty(B_a)}^2 \leq C_4(\Lambda, n) \int_{B_1} \|D^2 w\|^2. \quad (2.9)$$

For small enough $a < 1$. Now

$$\begin{aligned}
\int_{B_\rho} |D^2 w|^2 &\leq C_5(n)\rho^n \|D^2 w\|_{L^\infty(B_a)}^2 \\
&= C_5\rho^n \inf_{x \in B_a} \sup_{y \in B_a} |D^2 w(x) + D^2 w(y) - D^2 w(x)|^2 \\
&\leq C_5\rho^n \inf_{x \in B_a} \left[D^2 w(x) + 2a \|D^3 w\|_{L^\infty(B_a)} \right]^2 \\
&\leq 2C_5\rho^n \left[\inf_{x \in B_a} \|D^2 w(x)\|^2 + 4a^2 \|D^3 w\|_{L^\infty(B_a)}^2 \right] \\
&\leq 2C_5\rho^n \left[\frac{1}{|B_a|} \|D^2 w\|_{L^2(B_a)}^2 + 4a^2 C_4 \|D^2 w\|_{L^2(B_a)}^2 \right] \\
&\leq C_6(a, n)\rho^n \|D^2 w\|_{L^2(B_1)}^2.
\end{aligned}$$

Similarly

$$\begin{aligned}
\int_{B_\rho} |D^2 w - (D^2 w)_\rho|^2 &\leq \int_{B_\rho} |D^2 w - D^2 w(0)|^2 \\
&\leq \int_{S^{n-1}} \int_0^\rho r^2 |D^3 w|^2 r^{n-1} dr d\phi \\
&= C_7\rho^{n+2} \|D^3 w\|_{L^\infty(B_a)}^2. \tag{2.10}
\end{aligned}$$

Next, observe that (2.8) is purely fourth order, so the equation still holds when a second order polynomial is added to the solution. In particular, we may choose

$$D^2 \bar{w} = D^2 w - (D^2 w)_1$$

for \bar{w} also satisfying the equation. Then

$$D^3 \bar{w} = D^3 w$$

so by the Poincare inequality we have

$$\begin{aligned} \|D^3 w\|_{L^\infty(B_a)}^2 &= \|D^3 \bar{w}\|_{L^\infty(B_a)}^2 \\ &\leq C_4 \int_{B_1} \|D^2 \bar{w}\|^2 = C_4 \int_{B_1} \|D^2 w - (D^2 w)_1\|^2. \end{aligned} \quad (2.11)$$

We conclude from (2.11) and (2.10)

$$\int_{B_\rho} |D^2 w - (D^2 w)_\rho|^2 \leq C_7 \rho^{n+2} C_4 \int_{B_1} \|D^2 w - (D^2 w)_1\|^2.$$

□

Next, we have a corollary to the above theorem.

Corollary 2.2.2. *Suppose w is as in the Theorem 2.2.1. Then for any $u \in H^2(B_r)$, and for any $0 < \rho \leq r$, there holds*

$$\int_{B_\rho} |D^2 u|^2 \leq 4C_1(\rho/r)^n \|D^2 u\|_{L^2(B_r)}^2 + (2 + 8C_1) \|D^2(u - w)\|_{L^2(B_r)}^2. \quad (2.12)$$

and

$$\begin{aligned} \int_{B_\rho} |D^2 u - (D^2 u)_\rho|^2 &\leq 4C_2(\rho/r)^{n+2} \int_{B_r} |D^2 u - (D^2 u)_r|^2 \\ &\quad + (8 + 16C_2) \int_{B_r} |D^2(u - w)|^2 \end{aligned} \quad (2.13)$$

Proof. Let $v = u - w$. Then (2.12) follows from direct computation:

$$\begin{aligned}
\int_{B_\rho} |D^2 u|^2 &\leq 2 \int_{B_\rho} |D^2 w|^2 + 2 \int_{B_\rho} |D^2 v|^2. \\
&\leq 2C_1(\rho/r)^n \|D^2 w\|_{L^2(B_r)}^2 + 2 \int_{B_r} |D^2 v|^2 \\
&\leq 4C_1(\rho/r)^n \left[\|D^2 v\|_{L^2(B_r)}^2 + \|D^2 u\|_{L^2(B_r)}^2 \right] + 2 \int_{B_r} |D^2 v|^2 \\
&= 4C_1(\rho/r)^n \|D^2 u\|_{L^2(B_r)}^2 + 2[1 + 2C_1(\rho/r)^n] \|D^2 v\|_{L^2(B_r)}^2.
\end{aligned}$$

Similarly

$$\begin{aligned}
\int_{B_\rho} |D^2 u - (D^2 u)_\rho|^2 &\leq 2 \int_{B_\rho} |D^2 w - (D^2 w)_\rho|^2 + 2 \int_{B_\rho} |D^2 v - (D^2 v)_\rho|^2 \\
&\leq 2 \int_{B_\rho} |D^2 w - (D^2 w)_\rho|^2 + 8 \int_{B_\rho} |D^2 v|^2 \\
&\leq 2C_2(\rho/r)^{n+2} \int_{B_r} |D^2 w - (D^2 w)_r|^2 + 8 \int_{B_\rho} |D^2 v|^2 \\
&\leq 2C_2(\rho/r)^{n+2} \left\{ \begin{array}{l} 2 \int_{B_r} |D^2 u - (D^2 u)_r|^2 \\ + 2 \int_{B_r} |D^2 v - (D^2 v)_r|^2 \end{array} \right\} + 8 \int_{B_r} |D^2 v|^2 \\
&\leq 4C_2(\rho/r)^{n+2} \int_{B_r} |D^2 u - (D^2 u)_r|^2 \\
&\quad + (8 + 16C_2(\rho/r)^{n+2}) \int_{B_r} |D^2 v|^2.
\end{aligned}$$

The statement follows, noting that $\rho/r \leq 1$. □

We will be using the following Lemma frequently, so we state it here for the reader's convenience.

Lemma 2.2.3. [8, Lemma 3.4]. *Let ϕ be a nonnegative and nondecreasing function on $[0, R]$. Suppose that*

$$\phi(\rho) \leq A \left[\left(\frac{\rho}{r} \right)^\alpha + \varepsilon \right] \phi(r) + Br^\beta$$

for any $0 < \rho \leq r \leq R$, with A, B, α, β nonnegative constants and $\beta < \alpha$. Then for any $\gamma \in (\beta, \alpha)$, there exists a constant $\varepsilon_0 = \varepsilon_0(A, \alpha, \beta, \gamma)$ such that if $\varepsilon < \varepsilon_0$ we have for all $0 < \rho \leq r \leq R$

$$\phi(\rho) \leq c \left[\left(\frac{\rho}{r} \right)^\gamma \phi(r) + Br^\beta \right]$$

where c is a positive constant depending on A, α, β, γ . In particular, we have for any $0 < r \leq R$

$$\phi(r) \leq c \left[\frac{\phi(R)}{R^\gamma} r^\gamma + Br^\beta \right].$$

2.3. Proofs of the propositions

We begin by proving Proposition 2.1.3.

Proof. By approximation, (2.7) holds for $\eta \in W_0^{2,2}$. We are assuming that $u \in W^{2,\infty}$, so (2.7) must hold for the test function

$$\eta = -[\tau^4 u^{h_p}]^{-h_p}$$

where $\tau \in C_c^\infty$ is a cutoff function in B_1 that is 1 on $B_{1/2}$, and the superscript h_p refers to taking difference quotient in the e_p direction. We choose h small enough after having fixed τ , so that η is well defined. We have

$$\int_{B_1} (c^{ij,kl} u_{ij} + f^{kl}) [\tau^4 u^{h_p}]_{kl}^{-h_p} dx = 0$$

For h small we can integrate by parts with respect to the difference quotient to get

$$\int_{B_1} (c^{ij,kl}u_{ij} + f^{kl})^{h_p} [\tau^4 u^{h_p}]_{kl} dx = 0.$$

Using the product rule for difference quotients we get

$$\int_{B_1} [(c^{ij,kl}(x))^{h_p} u_{ij}(x) + c^{ij,kl}(x + he_p) u_{ij}^{h_p} + (f^{kl})^{h_p}] [\tau^4 u^{h_p}]_{kl} dx = 0$$

Letting $v = u^{h_p}$, differentiating the second factor gives:

$$\begin{aligned} \int_{B_1} & [(c^{ij,kl}(x))^{h_p} u_{ij}(x) + c^{ij,kl}(x + he_p) v_{ij}(x) + (f^{kl})^{h_p}(x)] \\ & \times \left[\begin{array}{c} \tau^4 v_{kl} + 4\tau^3 \tau_k v_l + 4\tau^3 \tau_l v_k \\ + 4v(\tau^3 \tau_{kl} + 3\tau^2 \tau_k \tau_l) \end{array} \right] (x) dx = 0 \end{aligned}$$

from which

$$\begin{aligned} \int_{B_1} \tau^4 c^{ij,kl}(x + he_p) v_{ij} v_{kl} dx &= - \int_{B_1} [(c^{ij,kl}(x))^{h_p} u_{ij}(x) + \\ & c^{ij,kl}(x + he_p) v_{ij}(x) + (f^{kl})^{h_p}(x)] \\ & \times \left[\begin{array}{c} 4\tau^3 \tau_k v_l + 4\tau^3 \tau_l v_k \\ + 4v(\tau^3 \tau_{kl} + 3\tau^2 \tau_k \tau_l) \end{array} \right] dx \quad (2.14) \\ & - \int_{B_1} [(c^{ij,kl}(x))^{h_p} u_{ij}(x) + (f^{kl})^{h_p}(x)] \tau^4 v_{kl} dx. \end{aligned}$$

First we bound the terms on the right side of (2.14). Starting at the top:

$$\begin{aligned}
& \int_{B_1} [(c^{ij,kl}(x))^{h_p} u_{ij}(x) + (f^{kl})^{h_p}(x)] \times \left[\begin{array}{l} 4\tau^3 \tau_k v_l + 4\tau^3 \tau_l v_k \\ + 4v(\tau^3 \tau_{kl} + 3\tau^2 \tau_k \tau_l) \end{array} \right] dx \\
& \leq \left[\|u\|_{W^{2,\infty}(B_1)}^2 + 1 \right] \int_{B_1} \left(|(c^{ij,kl}(x))^{h_p}|^2 + |(f^{kl})^{h_p}(x)|^2 \right) dx \\
& \quad + C_8(\tau, D\tau, D^2\tau) \int_{B_1} (|Dv|^2 + |v|^2) dx.
\end{aligned} \tag{2.15}$$

Next, by Young's inequality we have:

$$\begin{aligned}
& \int_{B_1} c^{ij,kl}(x + he_p) v_{ij}(x) \times \\
& [4\tau^3 \tau_j v_l + 4\tau^3 \tau_l v_j + 4v(\tau^3 \tau_{jl} + 3\tau^2 \tau_j \tau_l)] dx \\
& \leq \frac{C_9(\tau, D\tau, D^2\tau, c^{ij,kl})}{\varepsilon} \int_{B_1} (|Dv|^2 + v^2) dx + \varepsilon \int_{B_1} \tau^4 |D^2v|^2 dx
\end{aligned} \tag{2.16}$$

and also

$$\begin{aligned}
& \int_{B_1} [(c^{ij,kl}(x))^{h_p} u_{ij}(x) + (f^{kl})^{h_p}(x)] \tau^4 v_{kl} dx \\
& \leq \varepsilon \int_{B_1} \tau^4 \|D^2v\|^2 dx \\
& \quad + \frac{C_{10}}{\varepsilon} (\|u\|_{W^{2,\infty}(B_1)}^2, |\tau|_{L^\infty(B_1)}) \int_{B_1} [|(c^{ij,kl})^{h_p}|^2 + |(f^{kl})^{h_p}|^2] dx
\end{aligned} \tag{2.17}$$

Now by uniform ellipticity (3.16), the left hand side of (2.14) is bounded below by

$$\Lambda \int_{B_1} \tau^4 \|D^2v\|^2 dx \leq \int_{B_1} \tau^4 c^{ij,kl}(x + he_p) v_{ik}(x) v_{kl}(x) dx \tag{2.18}$$

Combining all (2.14), (2.15), (2.17), (2.16) and (2.18) and choosing ε appropriately, we get

$$\begin{aligned}
& \frac{\Lambda}{2} \int_{B_1} \tau^4 \|D^2 v\|^2 dx \\
& \leq C_{11} (\|\tau\|_{W^{2,\infty}(B_1)}, \|u\|_{W^{2,\infty}(B_1)}) \left(\int_{B_1} |(f^{kl})^{h_p}|^2 + |c^{ij,kl}|^2 + |(c^{ij,kl})^{h_p}|^2 \right) \\
& \leq C_{12} (\|\tau\|_{W^{2,\infty}(B_1)}, \|u\|_{W^{2,\infty}(B_1)}, \|f^{kl}\|_{W^{1,2}(B_1)}^2, \|c^{ij,kl}\|_{W^{1,2}(B_1)}^2, \Lambda).
\end{aligned}$$

Now this estimate is uniform in h and direction e_p so we conclude that the difference quotients of u are uniformly bounded in $W^{2,2}(B_{1/2})$. Hence $u \in W^{3,2}(B_{1/2})$ and

$$\begin{aligned}
& \|D^3 f\|_{L^2(B_{1/2})} \\
& \leq \frac{2C_{12}}{\Lambda} (\|\tau\|_{W^{2,\infty}(B_1)}, \|u\|_{W^{2,\infty}(B_1)}, \|f^{kl}\|_{W^{1,2}(B_1)}^2, \|c^{ij,kl}\|_{W^{1,2}(B_1)}^2, \Lambda).
\end{aligned}$$

□

We now prove Proposition 2.1.4

Proof. We begin by taking a difference quotient of the equation

$$\int (c^{ij,kl} u_{ij} + f^{kl}) \eta_{kl} dx = 0$$

along the direction h_m . This gives

$$\int [(c^{ij,kl}(x))^{h_m} u_{ij}(x) + c^{ij,kl}(x + h e_m) u_{ij}^{h_m}(x) + (f^{kl})^{h_m}] \eta_{kl}(x) dx = 0$$

which gives us the following PDE in $u_{ij}^{h_m}$:

$$\int c^{ij,kl}(x + he_m) u_{ij}^{h_m}(x) \eta_{kl}(x) dx = \int q(x) \eta_{kl}(x) dx$$

where

$$q(x) = -(f^{kl})^{h_m}(x) - (c^{ij,kl}(x))^{h_m} u_{ij}(x)$$

Note that $q \in C^\alpha(B_1)$ and $c^{ij,kl}(x + he_m)$ is still an elliptic term for all x in B_1 . For compactness of notation we denote

$$g = u^{h_m} \tag{2.19}$$

and replace $c^{ij,kl}(x + he_m)$ with $c^{ij,kl}$, as the difference is immaterial. Our equation reduces to

$$\int c^{ij,kl} g_{ij} \eta_{kl} dx = \int q \eta_{kl} dx \tag{2.20}$$

Using integration by parts we have

$$\begin{aligned} \int c^{ij,kl} g_{ij} \eta_{kl} dx &= - \int q_l \eta_k dx \\ &= - \int (q - q(0))_l \eta_k dx \\ &= \int (q - q(0)) \eta_{kl} dx \end{aligned}$$

Now for each fixed $r < 1$ we write $g = v + w$ where w satisfies the following constant coefficient PDE on $B_r \subseteq B_1$:

$$\int_{B_1(0)} c^{ij,kl}(0) w_{ij} \eta_{kl} dx = 0 \quad (2.21)$$

$$\forall \eta \in C_0^\infty(B_r(0))$$

$$w = g \text{ on } \partial B_r$$

$$\nabla w = \nabla g \text{ on } \partial B_r.$$

By the Lax Milgram Theorem the above PDE with the given boundary condition has a unique solution in the space H_0^2 . By combining (2.20) and (2.21) we conclude

$$\int_{B_r} c^{ij,kl}(0) v_{ij} \eta_{kl} dx = \int_{B_r} (c^{ij,kl}(0) - c^{ij,kl}(x)) g_{ij} \eta_{kl} dx + \int_{B_r} q \eta_{kl} dx \quad (2.22)$$

Now w is smooth (again see [13, Theorem 33.10]), and $g = u^{h_m}$ is $C^{2,\alpha}$, so $v = g - w$ is $C^{2,\alpha}$ and can be well approximated by smooth test functions in $H_0^2(B_r)$. It follows that v can be used as a test function in (2.22): On the left hand side we have by (3.16)

$$\left[\int_{B_r} c^{ij,kl}(0) v_{ij} v_{kl} dx \right]^2 \geq \left[\Lambda \int_{B_r} |D^2 v|^2 dx \right]^2.$$

Defining

$$\zeta(r) = \sup\{ |c^{ij,kl}(x) - c^{ij,kl}(y)| : x, y \in B_r \} \quad (2.23)$$

and using the Cauchy-Schwarz inequality we get

$$\left[\int_{B_r} (c^{ij,kl}(0) - c^{ij,kl}(x)) g_{ij} v_{kl} dx \right]^2 \leq \zeta^2(r) \int_{B_r} |D^2 g|^2 dx \int_{B_r} |D^2 v|^2 dx.$$

Using Holder's inequality

$$\left[\int_{B_r} |(q(x) - q(0))v_{kl}(x)| dx \right]^2 \leq \int_{B_r} |q(x) - q(0)|^2 dx \int_{B_r} |D^2v|^2 dx$$

This gives us

$$\begin{aligned} \Lambda^2 \left[\int_{B_r} |D^2v|^2 dx \right]^2 &\leq \zeta^2(r) \int_{B_r} |D^2g|^2 dx \int_{B_r} |D^2v|^2 dx + \\ &\int_{B_r} |q(x) - q(0)|^2 dx \int_{B_r} |D^2v|^2 dx \end{aligned}$$

which implies

$$\Lambda^2 \int_{B_r} |D^2v|^2 dx \leq \zeta^2(r) \int_{B_r} |D^2g|^2 dx + \int_{B_r} |q(x) - q(0)|^2 dx. \quad (2.24)$$

Using corollary 2.2.2 for any $0 < \rho \leq r$ we get

$$\int_{B_\rho} |D^2g|^2 dx \leq 4C_1(\rho/r)^n \|D^2g\|_{L^2(B_r)}^2 + (2 + 8C_1) \|D^2v\|_{L^2(B_r)}^2 \quad (2.25)$$

Now combing (2.25) and (2.24) we get

$$\begin{aligned} \int_{B_\rho} |D^2g|^2 dx &\leq 4C_1(\rho/r)^n \|D^2g\|_{L^2(B_r)}^2 \\ &+ \frac{(2 + 8C_1)}{\Lambda^2} \left[\zeta^2(r) \int_{B_r} |D^2g|^2 dx + \int_{B_r} |q(x) - q(0)|^2 dx \right] \\ &= \left[\frac{(2 + 8C_1) \zeta^2(r)}{\Lambda^2} + 4C_1(\rho/r)^n \right] \int_{B_r} |D^2g|^2 dx \\ &+ \frac{(2 + 8C_1)}{\Lambda^2} \int_{B_r} |q(x) - q(0)|^2 dx. \end{aligned} \quad (2.26)$$

Also from Corollary 2.2.2

$$\begin{aligned}
\int_{B_\rho} |D^2g - (D^2g)_\rho|^2 dx &\leq 4C_2(\rho/r)^{n+2} \int_{B_r} |D^2g - (D^2g)_r|^2 dx \\
&\quad + (8 + 16C_2) \int_{B_r} |D^2v|^2 dx \\
&\leq 4C_2(\rho/r)^{n+2} \int_{B_r} |D^2g - (D^2g)_\rho|^2 dx \\
&\quad + \frac{(8 + 16C_2)}{\Lambda^2} \left[\zeta^2(r) \int_{B_r} |D^2g|^2 dx + \int_{B_r} |q(x) - q(0)|^2 dx \right].
\end{aligned}$$

Because $c^{ij,kl} \in C^{1,\alpha}$ we have from (2.23) that

$$\zeta(r)^2 \leq C_{13}r^{2\alpha} \tag{2.27}$$

Again q is a C^α function which implies

$$|q(x) - q(0)| \leq \|q\|_{C^\alpha(B_1)} |x - 0|^\alpha$$

and

$$\int_{B_r} |q - q(0)|^2 dx \leq C_{14} \|q\|_{C^\alpha(B_1)}^2 r^{n+2\alpha}$$

So we have

$$\begin{aligned}
&\int_{B_\rho} |D^2g - (D^2g)_\rho|^2 \tag{2.28} \\
&\leq 4C_2(\rho/r)^{n+2} \int_{B_r} |D^2g - (D^2g)_\rho|^2 \\
&\quad + \frac{(8 + 16C_2)}{\Lambda^2} C_{13}r^{2\alpha} \int_{B_r} |D^2g|^2 \\
&\quad + \frac{(8 + 16C_2)}{\Lambda^2} C_{14} \|q\|_{C^\alpha(B_1)}^2 r^{n+2\alpha}.
\end{aligned}$$

For $r < r_0 < 1/4$ to be determined, we have (2.26)

$$\int_{B_\rho} |D^2 g|^2 \leq C_{15} \left\{ [(\rho/r)^n + r^{2\alpha}] \int_{B_r} |D^2 g|^2 + r_0^{2\alpha+2\delta} r^{n-2\delta} \right\}.$$

Where δ is some positive number. Now we apply [8, Lemma 3.4]. In particular, take

$$\phi(\rho) = \int_{B_\rho} |D^2 g|^2$$

$$A = C_{15}$$

$$B = r_0^{2\alpha+2\delta}$$

$$\alpha = n$$

$$\beta = n - 2\delta$$

$$\gamma = n - \delta.$$

There exists $\varepsilon_0(A, \alpha, \beta, \gamma)$ such that if

$$r_0^{2\alpha} \leq \varepsilon_0 \tag{2.29}$$

we have

$$\phi(\rho) \leq C_{15} \left\{ [(\rho/r)^n + \varepsilon_0] \phi(r) + r_0^{2\alpha+2\delta} r^{n-2\delta} \right\}$$

and the conclusion of [8, Lemma 3.4] says that for $\rho < r_0$

$$\begin{aligned} \phi(\rho) &\leq C_{16} \left\{ [(\rho/r)^\gamma] \phi(r) + r_0^{2\alpha+2\delta} \rho^{n-2\delta} \right\} \\ &\leq C_{16} \frac{1}{r_0^{n-\delta}} \rho^{n-\delta} \|D^2 g\|_{L^2(B_{r_0})} + r_0^{2\alpha+2\delta} \rho^{n-2\delta} \\ &\leq C_{17} \rho^{n-\delta} \end{aligned}$$

This C_{17} depends on r_0 which is chosen by (2.29) and $\|D^2g\|_{L^2(B_{3/4})}$. So there is a positive uniform radius upon which this holds for points well in the interior. In particular, we choose $r_0 \in (0, 1/4)$ so that the estimate can be applied uniformly at points centered in $B_{1/2}(0)$ whose balls remain in $B_{3/4}(0)$. Turning back to (2.28), we now have,

$$\begin{aligned} \int_{B_\rho} |D^2g - (D^2g)_\rho|^2 &\leq 4C_2(\rho/r)^{n+2} \int_{B_r} |D^2g - (D^2g)_\rho|^2 + C_{18}r^{2\alpha}\rho^{n-\delta} \\ &\quad + C_{19}\|q\|_{C^\alpha(B_1)}r^{n+2\alpha} \\ &\leq 4C_2(\rho/r)^{n+2} \int_{B_r} |D^2g - (D^2g)_\rho|^2 + C_{20}r^{n+2\alpha-\delta} \end{aligned}$$

Again we apply [8, Lemma 3.4]: This time, take

$$\begin{aligned} \phi(\rho) &= \int_{B_\rho} |D^2g - (D^2g)_\rho|^2 \\ A &= 4C_2 \\ B &= C_{20} \\ \alpha &= n + 2 \\ \beta &= n + 2\alpha - \delta \\ \gamma &= n + 2\alpha \end{aligned}$$

and conclude that for any $r < r_0$

$$\begin{aligned} \int_{B_r} |D^2g - (D^2g)_\rho|^2 &\leq C_{21} \left\{ \frac{1}{r_0^{n+2\alpha}} \int_{B_{r_0}} |D^2g - (D^2g)_{r_0}|^2 r^{n+2\alpha} + C_{20}r^{n+2\alpha-\delta} \right\} \\ &\leq C_{22}r^{n+2\alpha-\delta} \end{aligned}$$

with C_{22} depending on r_0 , $\|D^2g\|_{L^2(B_{3/4})}$, $\|q\|_{C^\alpha(B_1)}$ etc. It follows by [8, Theorem 3.1] that $D^2g \in C^{(2\alpha-\delta)/2}(B_{1/4})$, in particular, must be bounded locally:

$$\|D^2g\|_{L^\infty(B_{1/4})} \leq C_{23} \left\{ 1 + \|D^2g\|_{L^2(B_{1/2})} \right\}. \quad (2.30)$$

This allows us to bound

$$\int_{B_r} |D^2g|^2 \leq C_{24}r^n$$

which we can plug back in to (2.28):

$$\begin{aligned} \int_{B_\rho} |D^2g - (D^2g)_\rho|^2 &\leq 4C_2(\rho/r)^{n+2} \int_{B_r} |D^2g - (D^2g)_\rho|^2 + C_{25}r^{2\alpha}C_{24}r^n \\ &\quad + C_{19} \|q\|_{C^\alpha(B_1)} r^{n+2\alpha} \\ &\leq C_{26}r^{n+2\alpha} \end{aligned}$$

This is precisely the hypothesis in [8, Theorem 3.1]. We conclude that

$$\|D^2g\|_{C^\alpha(B_{1/4})} \leq C_{27} \left\{ \sqrt{C_{26}} + \|D^2g\|_{L^2(B_{1/2})} \right\}.$$

Recalling (2.19) we see that u must enjoy uniform $C^{3,\alpha}$ estimates on the interior, and the result follows. \square

2.4. Proof of the Main Theorem

The propositions in the previous section allow us to prove the following Corollary, from which the Main Theorem will follow.

Corollary 2.4.1. *Suppose $u \in C^{N,\alpha}(B_1)$, $N \geq 2$, and satisfies the following regular (recall (2.3)) fourth order equation*

$$\int_{\Omega} a^{ij,kl}(D^2u)u_{ij}\eta_{kl}dx = 0, \quad \forall \eta \in C_0^\infty(\Omega).$$

Then

$$\|u\|_{C^{N+1,\alpha}(B_r)} \leq C(n, b, \|u\|_{W^{N,\infty}(B_1)}).$$

In particular

$$u \in C^{N,\alpha}(B_1) \implies u \in C^{N+1,\alpha}(B_r)$$

Case 1 $N = 2$. The function $u \in C^{2,\alpha}(B_1)$ and hence also in $W^{2,\infty}(B_1)$. By approximation (2.1) holds for $\eta \in W_0^{2,\infty}$, in particular, for

$$\eta = -[\tau^4 u^{h_m}]^{-h_m}$$

where $\tau \in C_c^\infty(B_1)$ is a cut off function in B_1 that is 1 on $B_{1/2}$, and superscript h_m refers to the difference quotient. As before, we have chosen h small enough (depending on τ) so that η is well defined . We have

$$\int_{\Omega} a^{ij,kl}(D^2u)u_{ij} [\tau^4 u^{h_m}]_{kl} dx = 0.$$

Integrating by parts as before with respect to the difference quotient, we get

$$\int_{B_1} [a^{ij,kl}(D^2u)u_{ij}]^{h_m} [\tau^4 u^{h_m}]_{kl} dx = 0$$

Let $v = u^{h_m}$. Observe that the first difference quotient can be expressed as

$$\begin{aligned}
[a^{ij,kl}(D^2u)u_{ij}]^{h_m}(x) &= a^{ij,kl}(D^2u(x + he_m)) \frac{u_{ij}(x + he_m) - u_{ij}(x)}{h} \\
&+ \frac{1}{h} [a^{ij,kl}(D^2u(x + he_m)) - a^{ij,kl}(D^2u(x))] u_{ij}(x) \\
&= a^{ij,kl}(D^2u(x + he_m)) v_{ij}(x) \\
&+ \left[\int_0^1 \frac{\partial a^{ij,kl}}{\partial u_{pq}}(tD^2u(x + he_m) + (1-t)D^2u(x)) dt \right] v_{pq}(x) u_{ij}(x).
\end{aligned} \tag{2.31}$$

We get

$$\int_{B_1} \tilde{b}^{ij,kl} v_{ij} [\tau^4 v]_{kl} dx = 0 \tag{2.32}$$

where

$$\begin{aligned}
\tilde{b}^{ij,kl}(x) &= a^{ij,kl}(D^2u(x + he_m)) + \left[\int_0^1 \frac{\partial a^{pq,kl}}{\partial u_{ij}}(tD^2u(x + he_m) + \right. \\
&\quad \left. (1-t)D^2u(x)) dt \right] u_{pq}(x).
\end{aligned} \tag{2.33}$$

Expanding derivatives of the second factor in (2.32) and collecting terms gives us

$$\int_{B_1} \tilde{b}^{ij,kl} v_{ij} \tau^4 v_{kl} dx \leq \int_{B_1} \left| \tilde{b}^{ij,kl} \right| |v_{ij}| \tau^2 C_{28}(\tau, D\tau, D^2\tau) (1 + |v| + |Dv|) dx$$

Now for h small, $\tilde{b}^{ij,kl}$ very closely approximates $b^{ij,kl}$, so we may assume h is small.

Applying (2.4) and Young's inequality

$$\int_{B_1} \tau^4 \Lambda_1 |D^2v|^2 \leq C_{28} \sup \tilde{b}^{ij,kl} \int_{B_1} \left(\varepsilon \tau^4 |D^2v|^2 + C_{32} \frac{1}{\varepsilon} (1 + |v| + |Dv|)^2 \right) dx.$$

That is

$$\int_{B_{1/2}} |D^2v|^2 \leq C_{29} \int_{B_1} (1 + |v| + |Dv|)^2 dx.$$

Now this estimate is uniform in h (for h small enough) and direction e_m , so we conclude that the derivatives are in $W^{2,2}(B_{1/2})$. This also shows that

$$\|D^3u\|_{L^2(B_{1/2})} \leq C_{30} \left(\|Du\|_{L^2(B_1)}, \|D^2u\|_{L^2(B_1)} \right).$$

Remark : We only used uniform continuity of D^2u to allow us to take the limit, but we did require the precise modulus of continuity.

For the next step, we are not quite able to use Proposition 2.1.4 because the coefficients $a^{ij,kl}$ are only known to be $W^{1,2}$. So we proceed by hand.

We begin by taking a single difference quotient

$$\int_{B_1} [a^{ij,kl}(D^2u)u_{ij}]^{h_m} \eta_{kl} dx = 0$$

and arriving at the equation in the same fashion as to (2.32) above (this time letting $g = u^{h_m}$) we have

$$\int_{B_1} \tilde{b}^{ij,kl} g_{ij}(x) \eta_{kl} dx = 0.$$

Inspecting (2.33) we see that $\tilde{b}^{ij,kl}$ is C^α :

$$\left\| \tilde{b}^{ij,kl}(x) - \tilde{b}^{ij,kl}(y) \right\| \leq C_{31} |x - y|^\alpha$$

where C_{31} depends on $\|D^2u\|_{C^\alpha}$ and on bounds of $Da^{ij,kl}$ and $D^2a^{ij,kl}$. As in the proof of Proposition 2.1.4, for a fixed $r < 1$ we let w solve the boundary value

problem

$$\int_{B_r} \tilde{b}^{ij,kl}(0) w_{ij} \eta_{kl} dx = 0, \forall \eta \in C_0^\infty(B_r)$$

$$w = g \text{ on } \partial B_r$$

$$\nabla w = \nabla g \text{ on } \partial B_r$$

Let $v = g - w$. Note that

$$\int_{B_r} \tilde{b}^{ij,kl}(0) v_{ij} \eta_{kl} dx = \int_{B_r} \left(\tilde{b}^{ij,kl}(0) - \tilde{b}^{ij,kl}(x) \right) g_{ij} \eta_{kl} dx.$$

Now v vanishes to second order on the boundary, and we may use v as a test function. We get

$$\int_{B_r} \tilde{b}^{ij,kl}(0) v_{ij} v_{kl} dx = \int_{B_r} \left(\tilde{b}^{ij,kl}(0) - \tilde{b}^{ij,kl}(x) \right) g_{ij} v_{kl} dx.$$

As before,

$$\left(\Lambda \int_{B_r} |D^2 v|^2 dx \right)^2 \leq \left[\sup_{x \in B_r} \left| \tilde{b}^{ij,kl}(0) - \tilde{b}^{ij,kl}(x) \right| \right]^2 \int_{B_r} |D^2 g|^2 dx \int_{B_r} |D^2 v|^2 dx.$$

Defining

$$\begin{aligned} \zeta(r) &= \sup \left\{ \left| \tilde{b}^{ij,kl}(x) - \tilde{b}^{ij,kl}(y) \right| \mid x, y \in B_r \right\} \\ &\leq 4^\alpha C_{31} r^{2\alpha} \end{aligned} \tag{2.34}$$

then

$$\int_{B_r} \left(\tilde{b}^{ij,kl}(0) - \tilde{b}^{ij,kl}(x) \right) g_{ij} v_{kl} dx)^2 \leq \zeta^2(r) \int_{B_r} |D^2 g|^2 \int_{B_r} |D^2 v|^2.$$

So now we have :

$$\int_{B_r} |D^2 v|^2 \leq \frac{\zeta^2(r)}{\Lambda^2} \int_{B_r} |D^2 g|^2.$$

Using Corollary 2.2.2, for any $0 < \rho \leq r$ we get

$$\begin{aligned} \int_{B_\rho} |D^2 g - (D^2 g)_\rho|^2 &\leq 4C_2(\rho/r)^{n+2} \int_{B_r} |D^2 g - (D^2 g)_r|^2 \\ &\quad + (8 + 16C_2) \int_{B_r} |D^2 v|^2 \\ &\leq 4C_2(\rho/r)^{n+2} \int_{B_r} |D^2 g - (D^2 g)_r|^2 \\ &\quad + \frac{(8 + 16C_2)\zeta^2(r)}{\Lambda^2} \|D^2 g\|_{L^2(B_r)}^2. \end{aligned} \tag{2.35}$$

Also by Corollary 2.2.2

$$\begin{aligned} \int_{B_\rho} |D^2 g|^2 &\leq 4C_1(\rho/r)^n \|D^2 g\|_{L^2(B_r)}^2 + (2 + 8C_1) \|D^2 v\|_{L^2(B_r)}^2 \\ &\leq 4C_1(\rho/r)^n \|D^2 g\|_{L^2(B_r)}^2 + (2 + 8C_1) \frac{\zeta^2(r)}{\Lambda^2} \|D^2 g\|_{L^2(B_r)}^2. \end{aligned}$$

This implies

$$\int_{B_\rho} |D^2 g|^2 \leq (4C_1(\rho/r)^n + (2 + 8C_1) 4^{2\alpha} C_{31}^2 r^{2\alpha}) \|D^2 g\|_{L^2(B_r)}^2.$$

Now we can apply [8, Lemma 3.4] again, this time with

$$\begin{aligned}
\phi(\rho) &= \int_{B_\rho} |D^2g|^2 \\
A &= 4C_1 \\
\alpha &= n \\
B, \beta &= 0 \\
\gamma &= n - 2\delta \\
\varepsilon &= (2 + 8C_1) 4^{2\alpha} C_{31}^2 r^{2\alpha}.
\end{aligned}$$

There exists a constant $\varepsilon_0(A, \alpha, \gamma)$ such that by choosing

$$r_0^{2\alpha} \leq \frac{\varepsilon_0}{(2 + 8C_1) 4^{2\alpha} C_{31}^2} < \frac{1}{4}$$

we may conclude that for $0 < r \leq r_0$

$$\int_{B_r} |D^2g|^2 \leq C_{32} r^{n-2\delta} \frac{\int_{B_{r_0}} |D^2g|^2}{r_0^{n-2\delta}}. \quad (2.36)$$

Next, for small $\rho < r < r_0$ we have combining (2.35) (2.34) and (2.36)

$$\begin{aligned}
\int_{B_\rho} |D^2g - (D^2g)_\rho|^2 &\leq 4C_2(\rho/r)^{n+2} \int_{B_r} |D^2g - (D^2g)_r|^2 \\
&+ \frac{(8 + 16C_2) 4^\alpha \int_{B_{r_0}} |D^2g|^2}{\Lambda^2 r_0^{n-2\delta}} C_{31} C_{32} r^{n-2\delta} r^{2\alpha} \\
&\leq C_{33} r^{n+2\alpha-\delta}
\end{aligned} \quad (2.37)$$

with C_{33} depending on $\|D^2g\|_{L^2(B_{3/4})}, r_0, \varepsilon_0$. Again, we apply [8, Theorem 3.1] to $D^2g \in C^{(2\alpha-\delta)/2}(B_{1/4})$. From here, the argument is identical to the argument

following (2.30). We conclude that

$$\|D^2g\|_{C^\alpha(B_{1/4})} \leq C_{34} \left\{ 1 + \|D^2g\|_{L^2(B_{3/4})} \right\}.$$

Substituting $g = u^{h_m}$ we see that u must enjoy uniform $C^{3,\alpha}$ estimates on the interior, and the result follows.

Case 2 $N = 3$. We may take a difference quotient of (2.1) directly.

$$\int_{\Omega} [a^{ij,kl}(D^2u)u_{ij}]^{h_m} \eta_{kl} dx = 0, \quad \forall \eta \in C_0^\infty(\Omega).$$

(To be more clear we are using a slightly offset test function $\eta(x + he_m)$ and then using a change of variables, subtracting, and dividing by h .)

We get

$$\int_{B_1} \left[a^{ij,kl}(D^2u(x + he_m))u_{ij}^{h_m}(x) + \frac{\partial a^{ij,kl}}{\partial u_{pq}}(M^*(x))u_{pq}^{h_m}(x)u_{ij}(x) \right] \eta_{kl} = 0.$$

where $M^*(x) = t^*D^2u(x + h_m) + (1 - t^*)D^2u(x)$ and $t^* \in [0, 1]$. Now we are assuming that $u \in C^{3,\alpha}(B_1)$, so the first and second derivatives of the difference quotient will converge to the second and third derivatives, uniformly. We can then apply dominated convergence, passing the limit as $h \rightarrow 0$ inside the integral and recalling $u_m = v$ as before, we obtain

$$\int_{B_1} \left[a^{ij,kl}(D^2u(x))v_{ij}(x) + \frac{\partial a^{pq,kl}}{\partial u_{ij}}(D^2u(x))v_{ij}(x)u_{pq}(x) \right] \eta_{kl} = 0$$

that is

$$\int_{B_1} b^{ij,kl}(D^2u(x))v_{ij}(x)\eta_{kl}(x) = 0, \quad \forall \eta \in C_0^\infty(\Omega). \quad (2.38)$$

It follows that $v \in C^{2,\alpha}$ satisfies a fourth order double divergence equation, with coefficients in $C^{1,\alpha}$. First, we apply Proposition 2.1.3 :

$$\|D^3 v\|_{L^2(B_{1/2})} \leq C_{35} (\|v\|_{W^{2,\infty}(B_1)}) (1 + \|b^{ij,kl}\|_{W^{1,2}(B_1)}).$$

In particular, $u \in W^{4,2}(B_{1/2})$. Next, we apply 2.1.4

$$\begin{aligned} \|D^3 v\|_{C^{0,\alpha}(B_{1/4})} &\leq C(1 + \|D^3 v\|_{L^2(B_{1/2})}) \leq C(\|u\|_{W^{2,\infty}(B_1)}, \|b^{ij,kl}\|_{W^{1,2}(B_1)}) \\ &\leq C_{36}(n, b, \|u\|_{C^{3,\alpha}(B_1)}). \end{aligned}$$

We conclude that $u \in C^{4,\alpha}(B_r)$ for any $r < 1$.

Case 3 $N \geq 4$. Let $v = D^\alpha u$ for some multindex α with $|\alpha| = N - 2$. Observe that taking the first difference quotient and then taking a limit yields (2.38), when $u \in C^{3,\alpha}$. Now if $u \in C^{4,\alpha}$ we may take a difference quotient and limit of (2.38) to obtain

$$\int_{B_1} \left[b^{ij,kl}(D^2 u(x)) u_{ijm_1 m_2}(x) + \frac{\partial b^{ij,kl}}{\partial u_{pq}}(D^2 u(x)) u_{pqm_2} u_{ij} \right] \eta_{kl}(x) = 0, \quad \forall \eta \in C_0^\infty(\Omega).$$

and if $u \in C^{N,\alpha}$, then $v \in C^{2,\alpha}$, so we may take $N - 2$ difference quotients to obtain

$$\int_{B_1} [b^{ij,kl}(D^2 u(x)) v_{ij}(x) + f^{kl}(x)] \eta_{kl}(x) = 0, \quad \forall \eta \in C_0^\infty(\Omega). \quad (2.39)$$

where

$$f^{kl} = D^\beta (b^{ij,kl}(D^2 u(x)) u_{ij}) - b^{ij,kl}(D^2 u(x)) D^\alpha u_{ij}$$

where $|\beta| = |\alpha| - 1$. One can check by applying the chain rule repeatedly that f^{kl} is $C^{1,\alpha}$. So we may apply Proposition 2.1.3 to (2.39) and obtain that

$$\|D^3v\|_{L^2(B_{1/2})} \leq C_{37}(\|v\|_{W^{2,\infty}(B_1)})(1 + \|b^{ij,kl}\|_{W^{1,2}(B_1)})$$

that is

$$\|u\|_{W^{N+1,2}(B_r)} \leq C_{38}(n, b, \|u\|_{W^{N,\infty}(B_1)}).$$

Now apply Proposition 2.1.4:

$$\|D^3v\|_{C^{0,\alpha}(B_{1/4})} \leq C_{39}(1 + \|D^3v\|_{L^2(B_{3/4})})$$

that is

$$\|u\|_{C^{N+1,\alpha}(B_r)} \leq C_{40}(n, b, \|u\|_{W^{N,\infty}(B_1)}).$$

The Main Theorem follows.

2.5. Critical Points of Convex Functions of the Hessian

Suppose that $F(D^2u)$ is either a convex or a concave function of D^2u , and we have found a critical point of

$$\int_{\Omega} F(D^2u) dx \tag{2.40}$$

for some $\Omega \subset \mathbb{R}^n$, where we are restricting to compactly supported variations, so that the Euler-Lagrange equation is (2.6). If we suppose that F also has the additional structure condition,

$$\frac{\partial F(D^2u)}{\partial u_{ij}} = a^{pq,ij}(D^2u)u_{pq} \tag{2.41}$$

for a some $a^{ij,kl}$ satisfying (3.16), then we can derive smoothness from $C^{2,\alpha}$, as follows.

Corollary 2.5.1. *Suppose $u \in C^{2,\alpha}(B_1)$ is critical point of (2.40), where F is a smooth function satisfying (2.41) with $a^{ij,kl}$ satisfying (3.16) and F is uniformly convex or uniformly concave on $U \subseteq S^{n \times n}$ where U is the range of $D^2u(B_1)$ in the Hessian space.*

Then $u \in C^\infty(B_r)$, for all $r < 1$.

Proof. If u is a critical point of (2.40), then it satisfies the weak (2.1), for $a^{ij,kl}$ in (2.41). To apply the main Theorem, all we need to show is that

$$b^{ij,kl}(D^2u(x)) = a^{ij,kl}(D^2u(x)) + \frac{\partial a^{pq,kl}}{\partial u_{ij}}(D^2u(x))u_{pq}(x)$$

satisfies (3.16). From (2.41):

$$\frac{\partial}{\partial u_{kl}} \left(\frac{\partial F(D^2u)}{\partial u_{ij}} \right) = a^{kl,ij}(D^2u) + \frac{\partial a^{pq,ij}(D^2u)}{\partial u_{kl}}u_{pq}. \quad (2.42)$$

So

$$b^{ij,kl}(D^2u(x))\xi_{ij}\xi_{kl} = \frac{\partial}{\partial u_{kl}} \left(\frac{\partial F(D^2u)}{\partial u_{ij}} \right) \xi_{ij}\xi_{kl} \geq \Lambda |\xi|^2$$

for some $\Lambda > 0$, because F is convex. If F is concave, u is still a critical point of $-F$ and the same argument holds. \square

We mention one special case.

Lemma 2.5.2. *Suppose $F(D^2u) = f(w)$ where $w = (D^2u)^T(D^2u)$. Then*

$$\frac{\partial F(D^2u)}{\partial u_{ij}} = a^{ij,kl}(D^2u)u_{kl} \quad (2.43)$$

Proof. Let

$$w_{kl} = u_{ka} \delta^{ab} u_{bl}.$$

Then

$$\begin{aligned} \frac{\partial F(D^2 u)}{\partial u_{ij}} &= \frac{\partial f(w)}{\partial w_{kl}} \frac{\partial w_{kl}}{\partial u_{ij}} \\ &= \frac{\partial f(w)}{\partial w_{kl}} (\delta_{ka,ij} \delta^{ab} u_{bl} + u_{ka} \delta^{ab} \delta_{bl,ij}) \\ &= \frac{\partial f(w)}{\partial w_{kl}} (\delta_{ki} u_{jl} + u_{ki} \delta_{lj}) \\ &= \frac{\partial f(w)}{\partial w_{il}} \delta_{jm} u_{ml} + \frac{\partial f(w)}{\partial w_{kj}} u_{km} \delta_{im} \\ &= \frac{\partial f(w)}{\partial w_{il}} \delta_{jk} u_{kl} + \frac{\partial f(w)}{\partial w_{kj}} u_{kl} \delta_{il}. \end{aligned}$$

This shows (2.43) for

$$a^{ij,kl} = \frac{\partial f(w)}{\partial w_{il}} \delta_{jk} + \frac{\partial f(w)}{\partial w_{kj}} \delta_{il}.$$

□

CHAPTER III

THE HAMILTONIAN STATIONARY EQUATION

3.1. Background and Introduction

In this chapter, we study the regularity of the Lagrangian Hamiltonian stationary equation, which is a fourth order nonlinear PDE. Consider the function $u : B_1 \rightarrow \mathbb{R}$ where B_1 is the unit ball in \mathbb{R}^2 . The gradient graph of u , given by $\{(x, Du(x)) | x \in B_1\}$ is a Lagrangian submanifold of the complex Euclidean space. The function θ is called the Lagrangian phase for the gradient graph and is defined by

$$\theta = F(D^2u) = \text{Im} \log \det(I + iD^2u)$$

or equivalently,

$$\theta = \sum_i \arctan(\lambda_i) \tag{3.1}$$

where λ_i represents the eigenvalues of the Hessian.

The non homogeneous special Lagrangian equation is given by the following second order nonlinear equation

$$F(D^2u) = f(x). \tag{3.2}$$

The Hamiltonian stationary equation is given by the following fourth order nonlinear PDE

$$\Delta_g \theta = 0 \tag{3.3}$$

where Δ_g is the Laplace-Beltrami operator, given by:

$$\Delta_g = \sum_{i,j=1}^2 \frac{\partial_i(\sqrt{\det g} g^{ij} \partial_j)}{\sqrt{\det g}}$$

and g is the induced Riemannian metric from the Euclidean metric on \mathbb{R}^4 , which can be written as

$$g = I + (D^2u)^2.$$

Recently, in [7], the authors proved that in any dimension, a $C^{1,1}$ solution of the Hamiltonian stationary equation will be smooth with uniform estimates of all orders if the phase $\theta \geq \delta + (n - 2)\pi/2$, or, if the bound on the Hessian is small. In the two dimensional case, using the result in [7], we get uniform estimates for u when $|\theta| \geq \delta > 0$ (by symmetry). In this chapter, we consider the Hamiltonian stationary equation for all phases in dimension two without imposing a smallness condition on the Hessian or on the range of θ , and we derive uniform estimates for u , in terms of the $C^{1,1}$ bound which we denote by Λ . We write $\|u\|_{C^{1,1}(B_1)} = \|Du\|_{C^{0,1}(B_1)} = \Lambda$. Our main results are the following:

Theorem 3.1.1 (Main Theorem 1). *Suppose that $u \in C^{1,1}(B_1)$ and satisfies (3.3) on $B_1 \subset \mathbb{R}^2$ where $\theta \in W^{1,2}(B_1)$. Then u is a smooth function with interior Hölder estimates of all orders, based on the $C^{1,1}$ bound of u .*

Theorem 3.1.2 (Main Theorem 2). *Suppose that $u \in C^{1,1}(B_1)$ and satisfies (3.2) on $B_1 \subset \mathbb{R}^2$. If $f \in C^\alpha(B_1)$, then there exists $R = R(2, \Lambda, \alpha) < 1$ such that $u \in C^{2,\alpha}(B_R)$ and satisfies the following estimate*

$$|D^2u|_{C^\alpha(B_R)} \leq C_1(\|u\|_{L^\infty(B_1)}, \Lambda, |f|_{C^\alpha(B_1)}). \quad (3.4)$$

Our proof goes as follows: we denote

$$\theta(x) = F(D^2u(x)) = f(x). \tag{3.5}$$

We start by applying the De Giorgi-Nash theorem to the uniformly elliptic Hamiltonian stationary equation (3.3) on B_1 to prove that $\theta \in C^\alpha(B_{1/2})$. Next we consider the non-homogeneous special Lagrangian equation (3.2) where $\theta \in C^\alpha(B_{1/2})$. Using a rotation of [15] we rotate the gradient graph so that the new phase $\bar{\theta}$ of the rotated gradient graph satisfies $|\bar{\theta}| \geq \delta > 0$. Now we apply the result in [3] to the new potential \bar{u} of the rotated graph to obtain a $C^{2,\alpha}$ interior estimate for it. On rotating back the rotated gradient graph to our original gradient graph, we see that our potential u turns out to be $C^{2,\alpha}$ as well. A computation involving change of co-ordinates gives us the corresponding $C^{2,\alpha}$ estimate, shown in (3.4). Once we have a $C^{2,\alpha}$ solution of (3.3), smoothness follows by [7, Corollary 5.1].

In two dimensions, solutions to the second order special Lagrangian equation

$$F(D^2u) = C$$

enjoy full regularity estimates in terms of the potential u [16]. For higher dimensions, such estimates fail [17] for $\theta = C$ with $|C| < (n - 2)\pi/2$.

3.2. Proof of the Lemma

We first prove the second Theorem, followed by the proof of the first Theorem. We prove Theorem 3.1..2 using the following lemma. Recalling (3.5) we state the following lemma:

Lemma 3.2.1. *Suppose that $u \in C^{1,1}(B_1)$ satisfies (3.2) on $B_1 \subset \mathbb{R}^2$. Suppose*

$$0 \leq \theta(0) < (\pi/2 - \arctan \Lambda)/4. \quad (3.6)$$

If $\theta \in C^{\bar{\alpha}}(B_1)$, then there exists $0 < \alpha < \bar{\alpha}$ and C_0 such that

$$|D^2u(x) - D^2u(0)| \leq C_0(\|u\|_{L^\infty(B_1)}, \Lambda, |\theta|_{C^\alpha(B_1)}) * |x|^\alpha.$$

Proof. Consider the gradient graph $\{(x, Du(x)) | x \in B_1\}$ where u has the following Hessian bound

$$-\Lambda I_n \leq D^2u \leq \Lambda I_n$$

a.e. where it exists.

Define δ as

$$\delta = (\pi/2 - \arctan \Lambda)/2 > 0. \quad (3.7)$$

Since by (4.17) we have $0 \leq \theta(0) < \delta/2$, there exists $R'(\delta, |\theta|_{C^{\bar{\alpha}}}) > 0$ such that

$$|\theta(x) - \theta(0)| < \delta/2$$

for all $x \in B_{R'} \subseteq B_1$. This implies for every x in $B_{R'}$ for which D^2u exists, we have

$$\delta > \theta > \theta(0) - \delta/2.$$

So now we rotate the gradient graph $\{(x, Du(x)) | x \in B_{R'}\}$ downward by an angle of δ .

Let the new rotated co-ordinate system be denoted by (\bar{x}, \bar{y}) where

$$\bar{x} = \cos(\delta)x + \sin(\delta)Du(x) \quad (3.8)$$

$$\bar{y} = -\sin(\delta)x + \cos(\delta)Du(x). \quad (3.9)$$

On differentiating \bar{x} (3.8) with respect to x we see that

$$\frac{d\bar{x}}{dx} = \cos(\delta)I_n + \sin(\delta)D^2u(x) \leq \cos(\delta)I_n + \Lambda \sin(\delta)I_n$$

Thus

$$\cos(\delta)I_n - \Lambda \sin(\delta)I_n \leq \frac{d\bar{x}}{dx} \leq \cos(\delta)I_n + \Lambda \sin(\delta)I_n.$$

To obtain Lipschitz constants so that

$$\frac{1}{L_2}I_n \leq \frac{d\bar{x}}{dx} \leq L_1I_n \quad (3.10)$$

let

$$L_1 = \cos(\delta) + \Lambda \sin(\delta)$$

$$L_2 = \max\left\{\left|\frac{1}{\cos(\delta)I_n + D^2u(x) \sin(\delta)}\right| \mid x \in B_{R'}\right\}.$$

To find the value of L_2 , we see that in $B_{R'}$ we have the following:

let $\min\{\theta_1, \theta_2\} \geq -A$ where $A = \arctan \Lambda$.

$$\begin{aligned}
\cos(\delta)I_n + \sin(\delta)D^2u(x) &\geq \cos(\delta) - \sin(\delta)\tan(A) \\
&= \cos(\delta)(1 - \tan(\delta)\tan(A)) \\
&= \cos(\delta)\frac{\tan(\delta) + \tan(A)}{\tan(\delta + A)} \\
&= \cos(\delta)\frac{\tan(\delta) + \tan(A)}{\tan(\frac{\pi/2 - A}{2} + A)} \\
&= \cos(\delta)\frac{\tan(\delta) + \tan(A)}{\tan(\pi/2 - \delta)}.
\end{aligned}$$

This shows that

$$\frac{1}{L_2} = \cos(\delta)\frac{\tan(\delta) + \tan(A)}{\tan(\pi/2 - \delta)}.$$

Clearly $1/L_2$ is positive.

Now, by [7, Prop 4.1] we see that there exists a function \bar{u} such that

$$\bar{y} = D_{\bar{x}}\bar{u}(\bar{x})$$

where

$$\bar{u}(x) = u(x) + \sin \delta \cos \delta \frac{|Du(x)|^2 - |x|^2}{2} - \sin^2(\delta)Du(x) \cdot x \quad (3.11)$$

defines \bar{u} implicitly in terms of \bar{x} (since \bar{x} is invertible). Here \bar{x} refers to the rotation map (3.8).

Note that

$$\bar{\theta}(\bar{x}) - \bar{\theta}(\bar{y}) = \theta(x) - \theta(y)$$

which implies that $\bar{\theta}$ is also a $C^{\bar{\alpha}}$ function

$$\frac{|\bar{\theta}(\bar{x}_1) - \bar{\theta}(\bar{x}_2)|}{|\bar{x}_1 - \bar{x}_2|^\alpha} = \frac{|\theta(x_1) - \theta(x_2)|}{|x_1 - x_2|^\alpha} * \frac{|x_1 - x_2|^{\bar{\alpha}}}{|\bar{x}_1 - \bar{x}_2|^{\bar{\alpha}}}$$

thus,

$$|\bar{\theta}|_{C^{\bar{\alpha}}(B_{r_0})} \leq L_2^{\bar{\alpha}} |\theta|_{C^{\bar{\alpha}}(B_{R'})}.$$

Let $\Omega = \bar{x}(B_{R'})$. Note that $B_{r_0} \subset \Omega$ where $r_0 = R'/2L_2$. So our new gradient graph is $\{(\bar{x}, D_{\bar{x}}\bar{u}(\bar{x})) | \bar{x} \in \Omega\}$. The function \bar{u} satisfies the equation

$$F(D_{\bar{x}}^2\bar{u}) = \bar{\theta}(\bar{x})$$

in B_{r_0} where $\bar{\theta} \in C^{\bar{\alpha}}(B_{r_0})$. Observe that on B_{r_0} we have

$$\bar{\theta} = \theta - 2\delta < \delta - 2\delta = -\delta < 0$$

as $\theta < \delta$ on $B_{R'}$.

Claim 3.2.2. : *If $|\bar{\theta}| > \delta$, then $F(D^2\bar{u}) = \bar{\theta}$ is a solution to a uniformly elliptic concave equation.*

Proof. The proof follows from [18, lemma 2.2] and also from [7, pg 24].

□

Now using [3, Corollary 1.3] we get interior Schauder estimates for \bar{u} :

$$|D^2\bar{u}(\bar{x}) - D^2\bar{u}(0)| \leq C(\|\bar{u}\|_{L^\infty(B_{r_0/2})} + |\bar{\theta}|_{C^\alpha(B_{r_0/2})}) \quad (3.12)$$

for all \bar{x} in $B_{r_0/2}$ where $C = C(\Lambda, \alpha)$. This is our $C^{2,\alpha}$ estimate for \bar{u} .

Next, in order to show the same Schauder type inequality as (3.12) for u in place of \bar{u} , we establish relations between the following pairs:

- (i) oscillations of the Hessian of D^2u and $D^2\bar{u}$
- (ii) oscillations of θ and $\bar{\theta}$

(iii) the supremum norms of u and \bar{u} .

We rotate back to our original gradient graph by rotating up by an angle of δ and consider again the domain $B_{R'}(0)$. This gives us the following relations:

$$\begin{aligned}x &= \cos(\delta)\bar{x} - \sin(\delta)D_{\bar{x}}\bar{u}(\bar{x}) \\y &= \sin(\delta)\bar{x} + \cos(\delta)D_{\bar{x}}\bar{u}(\bar{x}).\end{aligned}\tag{3.13}$$

This gives us:

$$\begin{aligned}\frac{dx}{d\bar{x}} &= \cos(\delta)I_n - \sin(\delta)D_{\bar{x}}^2\bar{u}(\bar{x}) \\D_{\bar{x}}y &= \sin(\delta)I_n + \cos(\delta)D_{\bar{x}}^2\bar{u}(\bar{x}).\end{aligned}$$

So we have

$$D_x^2u(x) = D_{\bar{x}}y \frac{d\bar{x}}{dx} = [\sin(\delta)I_n + \cos(\delta)D_{\bar{x}}^2\bar{u}(\bar{x})][\cos(\delta)I_n - \sin(\delta)D_{\bar{x}}^2\bar{u}(\bar{x})]^{-1}.$$

The above expression is well defined everywhere because $D_{\bar{x}}^2\bar{u}(\bar{x}) < \cot(\delta)I_n$ for all $\bar{x} \in B_{r_0}$.

Note that we have $\cos(\delta)I_n - D_{\bar{x}}^2\bar{u}(\bar{x}) \sin(\delta) \geq \frac{1}{L_1}$, since

$$\frac{dx}{d\bar{x}} = \cos(\delta)I_n - \sin(\delta)D_{\bar{x}}^2\bar{u}(\bar{x}) = \left(\frac{d\bar{x}}{dx}\right)^{-1} \geq \frac{1}{L_1}I_n$$

by (4.8).

Next,

$$\begin{aligned}
D_x^2 u(x) - D_x^2 u(0) &= [\sin(\delta)I_n + \cos(\delta)D_{\bar{x}}^2 \bar{u}(\bar{x})][\cos(\delta)I_n - \sin(\delta)D_{\bar{x}}^2 \bar{u}(\bar{x})]^{-1} \\
&\quad - [\sin(\delta)I_n + \cos(\delta)D_{\bar{x}}^2 \bar{u}(0)][\cos(\delta)I_n - \sin(\delta)D_{\bar{x}}^2 \bar{u}(0)]^{-1}. \quad (3.14)
\end{aligned}$$

For simplification of notation we write

$$\begin{aligned}
D_{\bar{x}}^2 \bar{u}(\bar{x}) &= A \\
D_{\bar{x}}^2 \bar{u}(0) &= B \\
\cos(\delta) &= c, \sin(\delta) = s.
\end{aligned}$$

Noting that $[sI_n + cA]$ and $[cI_n - sA]^{-1}$ commute with each other we can write (3.14) as the following equation

$$\begin{aligned}
D_x^2 u(x) - D_x^2 u(0) &= \\
&= [cI_n - sB]^{-1}[cI_n - sB][sI_n + cA][cI_n - sA]^{-1} - \\
&= [cI_n - sB]^{-1}[sI_n + cB][cI_n - sA][cI_n - sA]^{-1}.
\end{aligned}$$

Again we see that

$$[cI_n - sB][sI_n + cA] - [sI_n + cB][cI_n - sA] = A - B.$$

This means

$$D_x^2 u(x) - D_x^2 u(0) = [cI_n - sB]^{-1}[A - B][cI_n - sA]^{-1}.$$

We have already shown that

$$|cI_n - sA| \geq \frac{1}{L_1}$$

which implies

$$|cI_n - sA|^{-1} \leq L_1.$$

Thus we get

$$\begin{aligned} |D_x^2 u(x) - D_x^2 u(0)| &\leq L_1^2 |D_{\bar{x}}^2 \bar{u}(\bar{x}) - D_{\bar{x}}^2 \bar{u}(0)| \\ &\leq CL_1^2 (|\bar{u}|_{L^\infty(B_{r_0/2})} + |\bar{\theta}|_{C^\alpha(B_{r_0/2})}) |\bar{x}|^\alpha \\ &\leq CL_1^{2+\alpha} (|\bar{u}|_{L^\infty(B_{r_0/2})} + |\bar{\theta}|_{C^\alpha(B_{r_0/2})}) |x|^\alpha \end{aligned} \quad (3.15)$$

where L_1 is the Lipschitz constant of the co-ordinate change map. This implies

$$\frac{1}{L_1^{\alpha+2}} |D_x^2 u(x)|_{C^\alpha(B_R)} \leq |D_{\bar{x}}^2 u(\bar{x})|_{C^\alpha(B_{r_0/2})}. \quad (3.16)$$

Recall from (3.11) that

$$\bar{u}(x) = u(x) + v(x).$$

This shows

$$\begin{aligned} \|\bar{u}(\bar{x})\|_{L^\infty(B_{r_0/2})} &= \|\bar{u}(x)\|_{L^\infty(\bar{x}^{-1}(B_{r_0/2}))} \leq \|\bar{u}(x)\|_{L^\infty(B_{R'})} \\ &\leq \|u(x)\|_{L^\infty(B_{R'})} + \|v\|_{L^\infty(B_{R'})}. \end{aligned} \quad (3.17)$$

Note that

$$\|v\|_{L^\infty(B_R)} \leq R \|Du\|_{L^\infty(B_R)} + \frac{1}{2} [R^2 + \|Du\|_{L^\infty(B_R)}^2] \quad (3.18)$$

and combining (3.16), (3.17), (3.18) with (3.15) we get

$$|D_x^2 u(x) - D_x^2 u(0)| \leq CL_1^{\alpha+2} \left\{ \begin{array}{l} \|u\|_{L^\infty(B_{R'})} + R\|Du\|_{L^\infty(B_R)} + \\ \frac{1}{2}[R^2 + \|Du\|_{L^\infty(B_R)}^2] + L_2^\alpha r_0 |\theta|_{C^\alpha(B_{R'})} \end{array} \right\} |x|^\alpha.$$

This proves the Lemma. □

3.3. Proof of Main Theorem 2

Proof. First note that the lemma provides a bound for the Hölder norm of the Hessian on any interior ball, so by a rescaling of the form

$$u_\rho(x) = \frac{u(\rho x)}{\rho^2}$$

for values of $\rho > 0$ and translation of any point to the origin. Consider the gradient graph $\{(x, Du(x)) | x \in B_1\}$ where u satisfies

$$F(D^2 u) = \theta$$

on B_1 and $\theta \in C^{\bar{\alpha}}(B_1)$. Then there exists a ball of radius r inside B_1 on which $osc\theta < \delta/4$ where δ is as defined in (4.34).

Now this means that either we have $\theta(x) < \delta/2$ in which case, by the above lemma we see that $u \in C^{2,\alpha}(B_r)$ satisfying the given estimates; or we have $\theta(x) > \delta/4$ in which case $u \in C^{2,\alpha}(B_r)$ with uniform estimates, by claim (3.2.2) and [3, Corollary 1.3]. □

3.4. Proof of Main Theorem 1

Proof. Since $u \in C^{1,1}(B_1)$ and $\theta \in W^{1,2}(B_1)$ satisfies the uniformly elliptic equation

$$\Delta_g \theta = 0,$$

by the De Giorgi-Nash Theorem we have that $\theta \in C^\alpha(B_{1/2})$. This means that u satisfies

$$F(D^2 u) = \theta.$$

By Theorem 3.1..2 we see that $u \in C^{2,\alpha}(B_r)$ where $r < 1/2$. Smoothness follows by [7, Corollary 5.1]. □

CHAPTER IV

ALMOST LINEAR ELLIPTIC EQUATIONS

4.1. Background and Introduction

In this chapter, we derive an a priori interior $C^{2,\alpha}$ estimate for viscosity solutions of the non-linear, uniformly elliptic equation

$$F(D^2u) = f(x), \tag{4.1}$$

under the assumption that $f(x) \in C^\alpha$ and F is almost linear.

For viscosity solutions of second order, fully non-linear equations of the form

$$F(D^2u) = 0 \tag{4.2}$$

where F is concave and uniform elliptic, the theory of a priori estimates is well developed after the work of [19] and [20]. For general F , regularity theory for solutions to fully nonlinear uniformly elliptic equations of the form (4.2) include interior $C^{1,\alpha}$ estimates [3] and partial regularity results [21]. The structure of F plays a key role in deriving higher order estimates for fully non-linear elliptic equations of the forms (4.1) and (4.2). In [22], the authors have produced counterexamples to Evans-Krylov type estimates for general fully nonlinear equations. The classical approach to regularity of equations of the above form, is to differentiate the equation with respect to a direction i , and then u_i solves a linearized equation which is treated like a linear equation with bounded measurable coefficients. In [23], Savin proved interior $C^{2,\alpha}$ estimates for flat viscosity solutions

of (4.2) where F is smooth. He showed that viscosity solutions of (4.2) that are sufficiently close to a quadratic polynomial are, in fact, classical solutions.

Here, we consider a space of uniformly elliptic, non-linear equations of the form (4.2) and (4.1) where we assume that F is uniformly differentiable and there exists a universal constant ε_0 , such the value of DF at any two points in S_n is always ε_0 close to each other. This means that there exists a linear operator in the neighborhood of $DF(0)$ such that F is ε_0 close to it in the sense of (4.7). We formally define this property of F in definition 4.1.1. Cordes [24] had proved interior $C^{1,\alpha}$ estimates for uniformly elliptic linear equations of the form $a_{ij}(x)u_{ij}(x) = g(x)$ where the coefficients a_{ij} are close to the Laplace equation. Nirenberg [25, Chapter 6] proved that the same estimate [24] holds under the assumption that there exists constant $d > 0$ such that in the intersection S of the domain of x with every closed sphere of radius d the coefficients $a_{ij}(x)$ of the linear equation have small oscillation for all values of the arguments. Note that unlike the equation in Cordes, the equation we consider is a fully nonlinear equation of the form (4.1) where F is close to a linear equation in the sense of (4.7). It is important to note that Cordes's result cannot be applied to a linearization of equation (4.1) to obtain the result we are after since f is merely a C^α here. We also provide explicit values of how close the operator F should be to the Laplace equation and prove explicit $C^{2,\alpha}$ estimates for viscosity solutions of (4.2). In Theorem 4.1.3, we prove interior $C^{2,\alpha}$ for solutions to (4.1) using the $C^{2,\alpha}$ estimates we derive for solutions of (4.2) in Theorem 4.1.2 along with [3]'s result on $W^{2,p}$ estimates for concave equations.

This chapter is divided into the following sections. In this section we state definitions and our main results. In section 2, we prove Theorem 4.1.2 and in section 3, we prove Theorem 4.1.3.

Definitions and notations

We first define a few terms that we will be using to state the properties of the operator F .

Definition 4.1.1. *We define the uniformly elliptic, non-linear operator F to be almost linear with constant ε_0 if*

$$|DF(M) - DF(N)| \leq \varepsilon_0 \tag{4.3}$$

for all $M, N \in S_n$ where S_n is the space of all real symmetric $n \times n$ matrices. We define ε_0 to be the **closeness constant** of F . Through out this chapter we make the assumption $F(0) = 0$.

Theorem 4.1.2 (Main Theorem 1). *Given λ, Λ there exist universal constants $0 < \bar{\alpha} < 1$ and $\varepsilon_0 > 0$ such that if F is almost linear with constant ε_0 and $u \in C(B_1)$ is viscosity solution of (4.2) on B_1 , then $u \in C^{2,\bar{\alpha}}(B_{1/2})$ and satisfies the following estimate*

$$\|u\|_{C^{2,\bar{\alpha}}(B_{1/2})} \leq C_1 \|u\|_{L^\infty(B_1)} \tag{4.4}$$

where

$$C_1 = 2^{4+\bar{\alpha}} C(n) \left(\frac{5}{4} + \frac{n}{\lambda^{1/2}} + \frac{2n^2 e}{\lambda} + 32n^2 e \frac{\varepsilon_0}{\lambda^2} \right) \tag{4.5}$$

and $C(n)$ is a dimensional constant.

Theorem 4.1.3 (Main Theorem 2). *Given λ, Λ , and $0 < \bar{\alpha} < 1$ there exists a universal constant $\varepsilon_0 > 0$ such that if F is almost linear with constant ε_0 and $u \in C(B_1)$ is viscosity solution of (4.1) on B_1 with $f \in C^\alpha(B_1)$, then $u \in C^{2,\alpha}(B_{1/2})$ and the following estimate holds*

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C_2 \tag{4.6}$$

where $C_2 = C_2(n, \lambda, \Lambda, C_1)$, $0 < \alpha < \bar{\alpha}$ and $C_1, \bar{\alpha}$ are as defined in the previous Theorem.

The methods involved in our proof include comparing equation (4.2) to the Laplace equation with boundary data equal to a mollification of u . We use the Krylov-Safanov theorem [26] along with harmonic estimates to construct a quadratic polynomial that would be separated from u by a distance of $r^{2+\alpha}$ on the ball of radius r . This is used in the construction of an iterative sequence of quadratic polynomials that leads to our desired estimate in the first theorem. We used results involving $W^{2,p}$ estimates for concave equations [3] to prove the second theorem.

4.2. Proof of Main Theorem 1

Observe that condition (4.3) of definition (1.1) implies that F satisfies the property that there exists a linear operator L_0 such that

$$|F(N) - L_0(N)| \leq \varepsilon_0 |N| \tag{4.7}$$

for all $N \in S_n$. On fixing the matrix M to be the zero matrix in equation (4.3) and using the fundamental theorem of calculus along with the fact that $F(0) = 0$, one can easily see that the linear operator can be defined as

$$L_0(N) = DF(0)N. \tag{4.8}$$

For the following lemma and proposition we assume the operator L_0 to be the Laplacian. The proof of Theorem 4.1.2 follows as a corollary to proposition 4.2.3.

Lemma 4.2.1. *Given $\bar{\alpha}, \lambda, \Lambda$ there exist universal constants $\varepsilon_0 > 0$ and $r_0 > 0$, such that if the uniformly elliptic operator F satisfies*

$$|F(N) - \text{tr}(N)| < \varepsilon_0 |N| \quad (4.9)$$

for all $N \in S_n$, then for any viscosity solution $u \in C(B_2)$ of (4.2) in $B_2(0)$, we can find a polynomial P of degree 2 satisfying

$$\begin{aligned} F(D^2P) &= 0 \\ \sup_{B_{r_0}} |u - P| &\leq r_0^{2+\bar{\alpha}} \|u\|_{L^\infty(B_1)} \\ \|P\|_{L^\infty(B_1)} &\leq C_0 \|u\|_{L^\infty(B_1)}. \end{aligned} \quad (4.10)$$

We compute the explicit values of the universal constants to be

- (i) $r_0 = [\frac{1}{4}(\frac{1}{4n^2e})^3]^{\frac{1}{1-\bar{\alpha}}}$
- (ii) $C_0 = \frac{5}{4} + n + 2n^2e + n^2e\frac{\varepsilon_0}{\lambda}$
- (iii) $\varepsilon_0 = \min\{\frac{\lambda r_0^{\bar{\alpha}}}{64n^2e}, \frac{\alpha_0 r_0^{2+\bar{\alpha}}}{2(3+\alpha_0)K_2} [\frac{3r_0^{2+\bar{\alpha}}}{2(3+\alpha_0)K_1}]^{\frac{3}{\alpha_0}}\}$ where K_1, α_0, K_2 are defined in (4.12) and (4.20) respectively.

Proof. Let's denote $\|u\|_{L^\infty(B_1)} = M$. We consider a function h that satisfies the following boundary value problem:

$$\begin{aligned} \Delta h &= 0 \text{ in } B_1 \\ h &= u^\gamma \text{ on } \partial B_1. \end{aligned} \quad (4.11)$$

Here u^γ refers to a mollification of u for any $\gamma > 0$. It is defined by

$$u^\gamma = \eta_\gamma * u$$

where

$$\eta_\gamma(x) = \frac{1}{\gamma^n} \eta\left(\frac{x}{\gamma}\right)$$

and $\eta \in C^\infty(\mathbb{R}^n)$ is defined by

$$\eta(x) = \begin{cases} C \exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

with the constant $C > 0$ being chosen such that $\int_{\mathbb{R}^n} \eta dx = 1$. Note that since u is defined on all of \mathbb{R}^n , the mollifier sequence u^γ is well defined everywhere. From the Krylov-Safanov theorem (stated below), we get the following estimate

$$\|u\|_{C^{\alpha_0}(B_1)} \leq K_1 \|u\|_{L^\infty(B_2)}. \quad (4.12)$$

The Krylov-Safanov theorem states the following:

Theorem 4.2.2. *[26, Theorem 1] [Krylov-Safanov] Let $u \in C^0$ be a viscosity solution of $S(\frac{\lambda}{n}, \Lambda, 0) = 0$ in B_1 . Then u is Hölder continuous and*

$$\|u\|_{C^{\alpha_0}(B_{1/2})} \leq C\left(\frac{\lambda}{n}, \Lambda\right) \|u\|_{L^\infty(B_1)}$$

with (small) $\alpha_0 = \alpha_0(\frac{\lambda}{n}, \Lambda) > 0$.

This implies that u^γ converges to u uniformly on \bar{B}_1 and satisfies the following estimate:

$$\|u^\gamma - u\|_{L^\infty(B_1)} \leq K_1 \gamma^{\alpha_0} M. \quad (4.13)$$

Since h is harmonic and thus analytic there exists a polynomial $P_0(x)$ of degree two

$$P_0(x) = h(0) + x \cdot Dh(0) + \frac{1}{2} x \cdot D^2 h(0) x$$

such that for all $|x| < 1/2$, we get

$$|h(x) - P_0(x)| \leq |R_3(x)|$$

where R_3 is the remainder term of order 3 in the Taylor series expansion of h . So we get

$$|h(x) - P_0(x)| \leq \left(\frac{|x|}{1/4}\right)^3 n^2 e^3 M.$$

We can choose $r_0 \in (0, 1)$ such that the following holds

$$\sup_{B_{r_0}} |h(x) - P_0(x)| \leq \frac{1}{4} M r_0^{2+\bar{\alpha}}$$

and we compute the value of r_0 to be

$$r_0 = \left[\frac{1}{4} \left(\frac{1}{4n^2 e}\right)^3\right]^{\frac{1}{1-\bar{\alpha}}}. \quad (4.14)$$

Next, using the fact that F satisfies (4.9) and $\Delta P_0 = 0$, we define a new quadratic polynomial P such that

$$F(D^2 P) = 0.$$

We define $P(x) = P_0(x) + \frac{|x|^2}{2\lambda}c|D^2h(0)|$ where $|c| < \varepsilon_0$. Using harmonic estimates we see that

$$|D^2h(0)| \leq \frac{2n^2eM}{(1/4)^2} = 32n^2eM$$

on B_{r_0} . We observe that

$$\sup_{B_{r_0}} |h - P| < \sup_{B_{r_0}} |h - P_0| + M \frac{|r_0^2|}{2\lambda} \varepsilon_0 32n^2e. \quad (4.15)$$

We want the RHS of the above expression to be less than $\frac{1}{2}Mr_0^{2+\bar{\alpha}}$ and for that we need

$$M \frac{|r_0^2|}{2\lambda} \varepsilon_0 32n^2e \leq M \frac{1}{4} r_0^{2+\bar{\alpha}}.$$

This is possible when

$$\varepsilon_0 \leq \frac{\lambda r_0^{\bar{\alpha}}}{64n^2e} = \frac{\lambda}{64n^2e} \left[\frac{1}{4} \left(\frac{1}{4n^2e} \right)^3 \right]^{\frac{\bar{\alpha}}{1-\bar{\alpha}}}. \quad (4.16)$$

Again using harmonic estimates, we get the following estimate for P :

$$\|P\|_{L^\infty(B_1)} \leq C_0M$$

where

$$C_0 = \frac{5}{4} + n + 2n^2e + n^2e \frac{\varepsilon_0}{\lambda}. \quad (4.17)$$

Next, by definition of F , we know that

$$|F(N) - \text{tr}(N)| \leq \varepsilon_0|N| \quad \text{when } N \in S_n.$$

This gives us the following:

$$\begin{aligned} |F(D^2h)| &= |F(D^2h) - \Delta(h) + \Delta(h)| = |F(D^2h) - \Delta(h)| \\ &\leq \varepsilon_0 |D^2h| \leq \varepsilon_0 K \end{aligned} \quad (4.18)$$

where $\|D^2h\|_{L^\infty(\bar{B}_1)} = K$.

Now, we compute the value of K . From [9, Chapter 6] we see that

$$K = \|D^2h\|_{L^\infty(\bar{B}_1)} \leq \bar{K} \|u_\gamma\|_{C^3(\bar{B}_1)}$$

where \bar{K} is a dimensional constant. We compute the value of $\|u_\gamma\|_{C^3(\bar{B}_1)}$.

Let p be a multi-index such that $|p| = 3$. For any $x \in \bar{B}_1$ we observe the following:

$$\begin{aligned} |D^p(u_\gamma(x))| &= |D^p(\eta_\gamma) * u(x)| = \left| \int_{\mathbb{R}^n} D^p \eta_\gamma(x-y) u(y) dy \right| \\ &\leq \sup_{y \in \text{supp}(\eta_\gamma)} |u(y)| \int_{\mathbb{R}^n} |D^p \eta_\gamma(x-y)| dy \\ &\leq M \int_{\mathbb{R}^n} \left| \frac{1}{\gamma^{n+3}} D^p \eta\left(\frac{x-y}{\gamma}\right) \right| dy. \end{aligned}$$

We do a change of variable $z = \frac{x-y}{\gamma}$ to reduce the above expression to

$$\leq M \frac{1}{\gamma^3} \int_{\mathbb{R}^n} \left| \frac{1}{\gamma^n} D^p \eta(z) \gamma^n \right| dz = M \frac{1}{\gamma^3} \int_{\mathbb{R}^n} |D^p \eta(z)| dz$$

This shows that

$$K \leq \bar{K} M \frac{1}{\gamma^3} \int_{\mathbb{R}^n} |D^p \eta(z)| dz. \quad (4.19)$$

Let's define

$$K_2 = \bar{K} \int_{\mathbb{R}^n} |D^p \eta(z)| dz, \quad |p| = 3. \quad (4.20)$$

For the sake of simplifying notation, we do not substitute the value of K_2 . So for the rest of the proof the constant K_2 refers to the value obtained in (4.20). Using uniform ellipticity we see that the following inequalities holds on B_1 :

$$\begin{aligned} F(D^2h + D^2(\frac{K\varepsilon_0}{2\lambda}(1 - |x|^2))) &\leq 0. \\ F(D^2h - D^2(\frac{K\varepsilon_0}{2\lambda}(1 - |x|^2))) &\geq 0. \end{aligned}$$

Using comparison principles [9, theorem 17.1] and (4.13) we see that for all $x \in B_1$ we have:

$$|u(x) - h(x)| \leq K_1M\gamma^{\alpha_0} + \varepsilon_0K_2M\frac{1}{\gamma^3}. \quad (4.21)$$

On combining (4.21), (4.15) we see that:

$$\begin{aligned} \sup_{B_{r_0}} |u - P| &< \sup_{B_{r_0}} |u - h| + \sup_{B_{r_0}} |h - P| \\ &< K_1M\gamma^{\alpha_0} + \varepsilon_0K_2M\frac{1}{\gamma^3} + M\frac{1}{2}r_0^{2+\bar{\alpha}}. \end{aligned} \quad (4.22)$$

Our goal is to find a ε_0 for which the RHS of the above inequality will be equal to $Mr_0^{2+\bar{\alpha}}$. That's possible when

$$K_1M\gamma^{\alpha_0} + \varepsilon_0K_2M\frac{1}{\gamma^3} \leq \frac{1}{2}Mr_0^{2+\bar{\alpha}}$$

for some choice of γ . We choose γ small that maximizes the following function of γ

$$\frac{\gamma^3}{2K_2}r_0^{2+\bar{\alpha}} - \frac{K_1}{K_2}\gamma^{3+\alpha_0}.$$

We see that the above function of γ reaches a maximum when

$$\gamma = \left[\frac{3r_0^{2+\bar{\alpha}}}{2K_1(3+\alpha_0)} \right]^{\frac{1}{\alpha_0}}. \quad (4.23)$$

Using this value of γ we compute the value of ε_0 to be

$$\varepsilon_0 \leq \frac{\alpha_0 r_0^{2+\bar{\alpha}}}{2(3+\alpha_0)K_2} \left[\frac{3r_0^{2+\bar{\alpha}}}{2(3+\alpha_0)K_1} \right]^{\frac{3}{\alpha_0}} \quad (4.24)$$

where K_1 , α_0 and K_2 are defined in (4.12) and (4.20) respectively.

From (4.16) and (4.24) we see that

$$\varepsilon_0 = \min \left\{ \frac{\lambda r_0^{\bar{\alpha}}}{64n^2 e}, \frac{\alpha_0 r_0^{2+\bar{\alpha}}}{2(3+\alpha_0)K_2} \left[\frac{3r_0^{2+\bar{\alpha}}}{2(3+\alpha_0)K_1} \right]^{\frac{3}{\alpha_0}} \right\} \quad (4.25)$$

□

We now make a proposition similar to the statement of Theorem 1.2 and we assume that the linear operator L_0 in (4.8) is the Laplacian. Throughout this proof the constants C_0 and r_0 will refer to the constants obtained in (4.17) and (4.14) respectively.

Proposition 4.2.3. *Given λ, Λ there exist universal constants $0 < \bar{\alpha} < 1$ and $\varepsilon_0 > 0$ such that if F is almost linear with constant ε_0 and satisfies condition (4.9), then any viscosity solution $u \in C(B_1)$ of (4.2) will be in $C^{2,\bar{\alpha}}(B_{1/2})$ and satisfy the following estimate*

$$\|u\|_{C^{2,\bar{\alpha}}(B_{1/2})} \leq C_1 \|u\|_{L^\infty(B_1)}$$

where C_1 is as stated in 4.5.

Proof. We first prove that the $C^{2,\bar{\alpha}}$ estimate holds at the origin. As before, we denote $\|u\|_{L^\infty(B_1)} = M$.

We prove that there exists a polynomial P of degree 2 such that

$$|u(x) - P(x)| \leq MC'_0 |x|^{2+\bar{\alpha}} \quad \forall x \in B_1 \quad (4.26)$$

$$F(D^2P) = 0$$

$$\|P\|_{L^\infty(B_1)} \leq MC'_0$$

where $C'_0 = C_0(1 + \frac{3}{1-r_0^{2+\bar{\alpha}}})$. In order to prove the existence of such a polynomial P , we need the following claim.

Claim 4.2.4. *There exists a sequence of polynomials $\{P_k\}_{k=1}^\infty$ of degree 2 such that*

$$F(D^2P_k) = 0 \quad (4.27)$$

$$\|u - P_k\|_{L^\infty(B_{r_0^k})} \leq Mr_0^{k(2+\bar{\alpha})} \quad (4.28)$$

where F and u are as defined in Theorem 4.1.2.

We first prove the claim.

Proof. : Let $P_0 = 0$. Then (4.28) holds good for the $k = 0$ case. We assume that (4.28) holds for $k \leq i$ and we prove it for $k = i + 1$.

Consider

$$v_i(x) = \frac{u(r_0^i x) - P_i(r_0^i x)}{r_0^{2i}}$$

for all $x \in B_1$. Define

$$F_i(N) = F(N + D^2P_i)$$

for all $N \in S_n$. Since $F(D^2P_i) = 0$ we see that $F_i(D^2v_i) = 0$. Since

$$\|u - P_i\|_{L^\infty(B_{r_0^i})} \leq Mr_0^{i(2+\bar{\alpha})},$$

we observe that

$$\|v_i\|_{L^\infty(B_1)} \leq \frac{Mr_0^{i(2+\bar{\alpha})}}{r_0^{2i}} = Mr_0^{i\bar{\alpha}}.$$

Note that the operator F_i satisfies the same properties as the operator F :

$$|DF_i(M) - DF_i(N)| = |DF(M + D^2P_i) - DF(N + D^2P_i)| \leq \varepsilon_0$$

and F_i also has the same ellipticity constants as F . We apply the above corollary to the equation $F_i(D^2v_i) = 0$. This gives us the existence of a quadratic polynomial \bar{P}_i such that

$$F_i(D^2\bar{P}_i) = 0 \tag{4.29}$$

$$\begin{aligned} \|v_i - \bar{P}_i\|_{L^\infty(B_{r_0})} &\leq Mr_0^{i\bar{\alpha}}r_0^{(2+\bar{\alpha})} \\ \|\bar{P}_i\|_{L^\infty(B_1)} &\leq C_0Mr_0^{i\bar{\alpha}}. \end{aligned}$$

Next, we define

$$P_{i+1} = P_i + r_0^{2i}\bar{P}_i(r_0^{-i}x). \tag{4.30}$$

From (4.29) we see that

$$F(D^2P_{i+1}) = F_i(D^2\bar{P}_i) = 0$$

and on substituting the value of v_i in the second inequality of (4.29) we see that

$$\left\| \frac{u(r_0^i x) - P_i(r_0^i x)}{r_0^{2i}} - \bar{P}_i \right\|_{L^\infty(B_{r_0})} \leq Mr_0^{i\bar{\alpha}}r_0^{(2+\bar{\alpha})}r_0^{2i}$$

which reduces to

$$\|u - P_{i+1}\|_{L^\infty(B_{r_0^{i+1}})} \leq Mr_0^{(i+1)(2+\bar{\alpha})}.$$

This completes the inductive construction of the quadratic polynomial sequence.

Hence the claim 4.2.4. □

Using the above claim, we return to proving Theorem 4.1.2.

We show that this sequence $\{P_k\}_{k=1}^{\infty}$ is convergent and $\lim_{k \rightarrow \infty} P_k = P$ is the required polynomial in (4.26).

From (4.30) we see that

$$P_{i+1} - P_i = r_0^{2i} a_i + r_0^i b_i \cdot x + x^T c_i \cdot x.$$

From the third inequality of (4.29) we observe that the terms of the series $\sum_{i=1}^{\infty} (P_{i+1} - P_i)$ is bounded by a convergent geometric series

$$|P_{i+1} - P_i| \leq MC_0 r_0^{i(2+\bar{\alpha})}.$$

Hence the series $\sum_{i=1}^{\infty} (P_{i+1} - P_i)$ being telescopic converges and we define

$$P = \lim_{i \rightarrow \infty} P_i = \sum_{i=1}^{\infty} (P_{i+1} - P_i).$$

Note that $F(D^2 P) = 0$ as $F(D^2 P_i) = 0$ for all i .

For all $x \in B_{r_0^i}$ we have

$$\begin{aligned} |P(x) - P_i(x)| &\leq \\ \sum_{j=i}^{\infty} |P_{j+1} - P_j| &\leq MC_0 \sum_{j=i}^{\infty} (r_0^{2j} r_0^{j\bar{\alpha}} + r_0^j r_0^{j\bar{\alpha}} r_0^j + r_0^j r_0^{j\bar{\alpha}} r_0^j) \\ &\leq \frac{3MC_0}{1 - r_0^{2+\bar{\alpha}}} r_0^{i(2+\bar{\alpha})}. \end{aligned}$$

This shows that

$$\|P\|_{L^\infty(B_1)} = \|P - P_0\|_{L^\infty(B_1)} \leq \frac{3MC_0}{1 - r_0^{2+\bar{\alpha}}}$$

in B_1 . Note that P_i is a uniformly bounded sequence of quadratic polynomials converging uniformly, so the limit P should be a quadratic polynomial as well.

If we fix $x \in B_1$, we can choose an integer i such that

$$r_0^{i+1} < |x| \leq r_0^i.$$

Then we have the estimate

$$\begin{aligned} |u(x) - P(x)| &\leq |u(x) - P_i(x)| + |P_i(x) - P(x)| \\ &\leq MC_0 r_0^{i(2+\bar{\alpha})} + \frac{3MC_0}{1 - r_0^{2+\bar{\alpha}}} r_0^{i(2+\bar{\alpha})} \\ &\leq MC'_0 |x|^{2+\bar{\alpha}} \end{aligned}$$

where

$$C'_0 = C_0 \left(1 + \frac{3}{1 - r_0^{2+\bar{\alpha}}}\right).$$

This completes the proof of (4.26).

Next, consider any point x_0 in $B_{1/2}$. Let $v(x) = 4u(x/2 + x_0)$ where $x \in B_1$. Note that $B_{1/2}(x_0) \subset B_1$ and hence $F(D^2v) = 0$ in B_1 . Now we repeat the same argument as before for v in order to obtain the same estimate as (4.26) on the ball $B_{1/2}(x_0)$. By [3, Remark 3, page 74] it follows that $u \in C^{2,\alpha}(\bar{B}_{1/2}(0))$ with bounds given by

$$\|D^2u\|_{C^{\bar{\alpha}}(\bar{B}_{1/2}(0))} \leq C_1 \|u\|_{L^\infty(B_1)}.$$

where

$$C_1 = 2^{2+\bar{\alpha}}C(n)4C_0(n, \bar{\alpha}, \lambda, \Lambda) \quad (4.31)$$

and $C(n)$ is a dimensional constant. This proves the estimate in (4.4). \square

Proof. of Theorem 4.1.2:

From definition (1.1) we know that there exists a linear operator L_0 such that

$$|F(N) - L_0(N)| \leq \varepsilon_0|N|$$

where $L_0(N) = DF(0)N$. By the following transformation we show that we can assume the linear operator L_0 to be the Laplacian and hence the desired result follows from the above proposition.

Let $W = DF(0)$. Since F is elliptic, W^{-1} exists with eigenvalues in $[\frac{1}{\Lambda}, \frac{1}{\Lambda}]$. We define

$$A = (W^T)^{-1} \quad (4.32)$$

$$\tilde{F}(N) = F(AN).$$

Note that the eigenvalues of A^T lie in $[\frac{1}{\Lambda}, \frac{1}{\lambda}]$. Now we make the following observation

$$\begin{aligned}
\frac{\partial \tilde{F}}{\partial n_{ij}} &= \frac{\partial F}{\partial a_{pq}} \frac{\partial (AN)_{pq}}{\partial n_{ij}} \\
&= \sum_{p,q} W_{pq} \frac{\partial (\sum_s A_{ps} N_{sq})}{\partial n_{ij}} \\
&= \sum_{p,q} W_{pq} \sum_s A_{ps} \frac{\partial (N_{sq})}{\partial n_{ij}} \\
&= \sum_{p,q} W_{pj} \sum_s A_{ps} \delta^{si} \delta^{qj} \\
&= \sum_p W_{pj} A_{pi} \\
&= \sum_p W_{jp}^T A_{pi} \\
&= (W^T A)_{ji} \\
&= \left(W^T (W^T)^{-1} \right)_{ji} \\
&= \delta_{ij}.
\end{aligned}$$

It follows that $D\tilde{F}(0) = I$. Note that \tilde{F} has ellipticity constants in $[\frac{\lambda}{\Lambda}, \frac{\Lambda}{\lambda}]$ since

$$\begin{aligned}
&\tilde{F}(N + P) - \tilde{F}(P) = F(AN + AP) - F(AP) \\
\implies \frac{\lambda}{\Lambda} |P| &\leq \lambda |AP| \leq \tilde{F}(N + P) - \tilde{F}(P) \leq \Lambda |AP| \leq \frac{\Lambda}{\lambda} |P|.
\end{aligned}$$

Finally, note that

$$\begin{aligned}
\left| D\tilde{F}(M) - D\tilde{F}(N) \right| &= \left| A^T \cdot DF(AM) - A^T \cdot DF(AN) \right| \\
&= \left| A^T \cdot (DF(AM) - DF(AN)) \right| \\
&\leq \varepsilon_0 |A^T|.
\end{aligned}$$

Therefore, \tilde{F} is almost linear with constant $\varepsilon_0 |A^T|$ and satisfies the corresponding condition in (4.9). Hence the proof. □

For the rest of the chapter, the constant C_1 will refer to the fixed constant defined in (4.31).

4.3. Proof of Main Theorem 2

Note that without loss of generality we can write $f(0) = 0$ since we can write (4.1) as $G(D^2u) = g(x)$ where $G(D^2u) = F(D^2u) - f(0)$ and $g(x) = f(x) - f(0)$. Now, we prove Theorem 4.1.3. In order to do so we will require the following two Lemmas. Lemma 4.3.1 below is a modified version of [3, Lemma 7.9], which we derive on substituting $\varepsilon = 0$ in (7.18) of [3, Lemma 7.9].

Lemma 4.3.1. *Let u be a viscosity solution of (4.1) in $B_{4/7}$ such that $\|u\|_{L^\infty(B_{4/7})} \leq 1$ and $f \in L^n(B_{4/7})$. Assume that $F(D^2w) = 0$ has $C^{1,1}$ interior estimates (with constant C_1). Then there exists a function $h \in C^2(\bar{B}_{3/7})$ such that h*

satisfies $\|h\|_{C^{1,1}(\bar{B}_{3/7})} \leq c(n)C_1$ (for a constant $c(n)$ depending only on n) and

$$\begin{aligned} \|u - h\|_{L^\infty(B_{3/7})} &\leq C_3 \|f\|_{L^n(B_{4/7})} \\ F(D^2h) &= 0 \text{ in } B_{1/2} \\ h &= u \text{ on } \partial B_{1/2}. \end{aligned} \tag{4.33}$$

Here C_3 is a positive constant depending on n, λ, Λ, C_1 .

Note: We say that $F(D^2w) = 0$ has $C^{1,1}$ interior estimates (with constant C_1) if for any $w_0 \in C(\partial B)$ there exists a solution $w \in C^2(B_1) \cap C(\bar{B}_1)$ of

$$\begin{aligned} F(D^2w) &= 0 \quad \text{in } B_1 \\ w &= w_0 \quad \text{on } \partial B_1 \end{aligned}$$

such that $\|w\|_{C^{1,1}(\bar{B}_{1/2})} \leq C_1 \|w_0\|_{L^\infty(\partial B_1)}$.

Proof. [3, lemma 7.9]. □

Remark 4.3.2. Recall that the constant C_1 was defined in Theorem 4.1.2 to prove interior $C^{2,\bar{\alpha}}$ estimates (4.4) for $F(D^2u) = 0$. Since $u \in S(\frac{\lambda}{n}, \Lambda, 0)$, using maximum principles and Theorem 4.1.2 we see that the same constant C_1 works as the interior $C^{1,1}$ estimates constant for $F(D^2w) = 0$ in the above lemma.

Lemma 4.3.3. There exists $\delta > 0$ depending on $n, \lambda, \Lambda, C_1, \bar{\alpha}, \alpha$ such that if u is a viscosity solution of (4.1) in B_1 with

$$\|u\|_{L^\infty(B_1)} \leq 1$$

and

$$\left(\frac{1}{|B_r|} \int_{B_r} |f|^n\right)^{1/n} \leq \delta r^\alpha \quad \forall r \leq 1 \quad (4.34)$$

then there exists a polynomial P of degree 2 such that

$$\begin{aligned} \|u - P\|_{L^\infty(B_r)} &\leq C_4 r^{2+\alpha} \quad \forall r \leq 1, \\ |DP(0)| + \|D^2P\| &\leq C_4 \end{aligned} \quad (4.35)$$

for some constant $C_4 > 0$ depending only on $n, \lambda, \Lambda, C_1, \bar{\alpha}, \alpha$. Here $\bar{\alpha}$ and α are fixed constants from Theorem 4.1.2 and Theorem 4.1.3 respectively.

Proof. The proof follows from the following claim.

Claim 4.3.4. *There exists $0 < \mu < 1$ and a sequence and $\{P_k\}$ depending on $n, \lambda, \Lambda, C_1, \bar{\alpha}, \alpha$ such that*

$$\begin{aligned} P_k(x) &= a_k + b_k \cdot x + \frac{1}{2} x^t c_k \cdot x \\ F(D^2 P_k) &= 0 \\ \|u - P_k\|_{L^\infty(B_{\mu^k})} &\leq \mu^{k(2+\alpha)} \\ |a_k - a_{k-1}| + \mu^{k-1} |b_k - b_{k-1}| + \mu^{2(k-1)} |c_k - c_{k-1}| &\leq C_1 \mu^{(k-1)(2+\alpha)}. \end{aligned} \quad (4.36)$$

Here F and u are as defined in Lemma 4.3.3.

We first prove the claim.

Proof. Let $P_0 = 0$. For $k = 0$, we see that (4.36) holds for any $\mu > 0$. We assume that (4.36) holds for $k \leq i$ and we prove that it holds good for $k = i + 1$.

We choose μ small enough such that

$$2C_1 \mu^{\bar{\alpha}} \leq \mu^\alpha < 3/7. \quad (4.37)$$

We define

$$\begin{aligned} v_i(x) &= \frac{(u - P_i)(\mu^i x)}{\mu^{i(2+\alpha)}} \\ F_i(N) &= \frac{F(\mu^{i\alpha} N + c_i)}{\mu^{i\alpha}} \\ f_i(x) &= \frac{f(\mu^i x)}{\mu^{i\alpha}} \end{aligned}$$

where $P_i(x) = a_i + b_i \cdot x + \frac{1}{2}x^T \cdot c_i x$. This shows that

$$F_i(D^2 v_i(x)) = f_i(x). \quad (4.38)$$

Now we choose δ small enough such that

$$C_3 \delta \leq C_1 \mu^{2+\bar{\alpha}} \quad (4.39)$$

where C_3 is the constant appearing in the first inequality of (4.33) in Lemma 4.3.1.

We consider the equation (4.38). Observe that

$$\|f_i\|_{L^n(B_1)} = \mu^{-i\alpha} \mu^{-i} \|f\|_{L^n(B_{\mu^i})} \leq 2\mu^{-i\alpha} \mu^{-i} \delta \mu^{i\alpha} \mu^i = \delta.$$

Note that F_i satisfies the same properties as F . Since $\|v_i\|_{L^\infty(B_1)} \leq 1$, by applying Lemma 4.3.1 to (4.38), we see that there exists $h \in C^2(\bar{B}_{3/7})$ such that

$$\|v_i - h\|_{L^\infty(B_{3/7})} \leq C_3 \delta \quad (4.40)$$

and h solves the following boundary value problem:

$$\begin{aligned} F_i(D^2h) &= 0 \text{ in } B_{1/2} \\ h &= v_i \text{ on } \partial B_{1/2}. \end{aligned}$$

Then from the definition of F_i above, it follows that

$$F(\mu^{i\alpha} D^2h + c_i) = 0 \text{ in } B_{1/2}. \quad (4.41)$$

Note that F_i has the same ellipticity constants as F and DF_i satisfies the condition (4.3) with the same constant ε_0 . So now since h satisfies the above equation, by Theorem 4.1.2 and maximum principles we see that

$$\|h\|_{C^{2,\bar{\alpha}}(B_{1/4})} \leq C_1 \|v_i\|_{L^\infty(\partial B_{1/2})} \leq C_1, \quad (4.42)$$

where the last inequality follows by using $|v_i| \leq 1$. Since h is $C^{2,\bar{\alpha}}$, there exists a polynomial \bar{P} given by

$$\bar{P}(x) = h(0) + Dh(0) \cdot x + \frac{1}{2} x^t D^2h(0) \cdot x$$

such that

$$\|h - \bar{P}\|_{L^\infty(B_\mu)} \leq C_1 \mu^{2+\bar{\alpha}}. \quad (4.43)$$

From (4.40) and (4.43) we have

$$\begin{aligned}
\|v_i - \bar{P}\|_{L^\infty(B_\mu)} &\leq \|v_i - h\|_{L^\infty(B_\mu)} + \|h - \bar{P}\|_{L^\infty(B_\mu)} \\
&\leq C_3\delta + C_1\mu^{2+\bar{\alpha}} \\
&\leq 2C_1\mu^{2+\bar{\alpha}} \\
&\leq \mu^{2+\alpha} \tag{4.44}
\end{aligned}$$

where the last two inequalities follow from (4.39) and (4.37).

Rescaling back (4.44) we see that

$$|u(x) - P_i(x) - \mu^{i(2+\alpha)}\bar{P}(\mu^{-i}x)| \leq \mu^{(2+\alpha)(i+1)} \tag{4.45}$$

for all $x \in B_{\mu^{i+1}}$.

We define

$$P_{i+1}(x) = P_i(x) + \mu^{i(2+\alpha)}\bar{P}(\mu^{-i}x) \tag{4.46}$$

and we have

$$c_{i+1} = c_i + \mu^{i\alpha}D^2h(0).$$

From (4.45) we see that

$$\|u - P_{i+1}\|_{L^\infty(B_{\mu^{i+1}})} \leq \mu^{(i+1)(2+\alpha)}$$

and from (4.41) we get

$$F(c_{i+1}) = 0.$$

Next, from (4.42) and (4.46) we see that

$$\begin{aligned}
& |a_{i+1} - a_i| + \mu^i |b_{i+1} - b_i| + \mu^{2i} |c_{i+1} - c_i| \\
&= \mu^{i(2+\alpha)} (|h(0)| + |Dh(0)| + |D^2h(0)|) \\
&\leq \mu^{i(2+\alpha)} C_1.
\end{aligned}$$

This proves claim 4.3.4. □

Now we return to proving the lemma. By the same argument used in the proof of Theorem 4.1.2 we can show that the sequence $\{P_i\}$ is uniformly convergent in B_1 . Let's define $P = \lim_{i \rightarrow \infty} P_i = \sum_{i=1}^{\infty} (P_{i+1} - P_i)$. We see that for $i \geq 0$

$$\begin{aligned}
\|u - P\|_{L^\infty(B_{\mu^i})} &\leq \|u - P_i\|_{L^\infty(B_{\mu^i})} + \sum_{j=i}^{\infty} \|P_{j+1} - P_j\|_{L^\infty(B_{\mu^j})} \\
&\leq \mu^{i(2+\alpha)} + \sum_{j=i}^{\infty} [|a_{j+1} - a_j| + \mu^j |b_{j+1} - b_j| + \frac{1}{2} \mu^{2j} |c_{j+1} - c_j|] \\
&\leq \mu^{i(2+\alpha)} + C_1 \sum_{j=i}^{\infty} [\mu^{(j)(2+\alpha)} + \mu^j \mu^{j(1+\alpha)} + \mu^{2j} \mu^{j\alpha}] \\
&\leq C'_4 \mu^{i(2+\alpha)}
\end{aligned}$$

where $C'_4 = 1 + \frac{3C_1}{1-\mu^{2+\alpha}}$. Clearly we have

$$|DP(0)| + \|D^2P\| \leq C''_4$$

for some constant $C''_4 > 0$. By defining $C_4 = \max\{C'_4, C''_4\}$ we see that (4.35) holds good. This proves the lemma. □

Proof. of Theorem 4.1.3: We first prove the estimate (4.6) at the origin. We show that there exists a polynomial of degree 2 and $r_1 < 1$ such that

$$\begin{aligned} \|u - P\|_{L^\infty(B_r)} &\leq C'_2 r^{2+\alpha} \quad \forall r \leq r_1 \\ |DP(0)| + \|D^2P\| &\leq C'_2 \end{aligned} \tag{4.47}$$

where $C'_2 = C'_2(\|u\|_{L^\infty(B_1)}, |f|_{C^\alpha(B_1)}, n, \lambda, \Lambda, \bar{\alpha}, \alpha, C_1)$, $0 < \alpha < \bar{\alpha}$ and $\bar{\alpha}$ is the Hölder power appearing in (4.4):

$$\|u\|_{C^{2,\bar{\alpha}}(B_{1/2})} \leq C_1 \|u\|_{L^\infty(B_1)}.$$

Note that the C^α function $f(x)$ satisfies the following

$$\left(\frac{1}{|B_1|} \int_{B_1} |f|^n\right)^{1/n} \leq |f|_{C^\alpha(B_1)}.$$

The proof follows directly from Lemma 4.3.3, if we do the following rescaling for all $x \in B_1$. Consider the following function

$$\tilde{u}(x) = \frac{r_1^{-2}u(r_1x)}{\delta^{-1}|f|_{C^\alpha(B_1)} + r_1^{-2}\|u\|_{L^\infty(B_1)}} = \frac{r_1^{-2}u(r_1x)}{T}$$

where r_1 is chosen such that

$$r_1^\alpha |f|_{C^\alpha(B_1)} < \delta T \tag{4.48}$$

and δ is as defined in (4.39). Observe that $\|\tilde{u}\|_{L^\infty(B_1)} \leq 1$. Now we consider the operator

$$F_T(N) = \frac{1}{T}F(TN)$$

defined for all $N \in S_n$.

Note that F_T satisfies the following properties:

- (i) F_T has the same ellipticity constants λ and Λ as F .
- (ii) DF_T satisfies condition (4.3) with the same constant ε_0 .

We see that \tilde{u} satisfies the equation

$$F_T(D^2\tilde{u}(x)) = \frac{1}{T}F(TD^2\tilde{u}(x)) = \frac{1}{T}F(D^2u(r_1x)) = \frac{f(r_1x)}{T} = f_T(x),$$

where for $r \leq 1$ we have

$$\left(\frac{1}{|B_r|} \int_{B_r} |f_T|^n\right)^{1/n} \leq \left(\frac{r_1^\alpha |f|_{C^\alpha(B_{rr_1})}}{T}\right) r^\alpha < \delta r^\alpha.$$

Therefore, the equation

$$F_T(D^2\tilde{u}(x)) = f_T(x)$$

satisfies all the conditions of Lemma 4.3.3 and hence the function \tilde{u} satisfies the estimates (4.35).

Now by rescaling back (4.35) we get the desired estimate (4.47).

Next, consider any point x_0 in $B_{1/2}$. Let $v(x) = 4u(x/2 + x_0)$ where $x \in B_1$. Note that $B_{1/2}(x_0) \subset B_1$ and hence $F(D^2v) = 0$ in B_1 . Now we repeat the same argument as before for v in order to obtain the same estimate as (4.47) on the ball $B_{1/2}(x_0)$. By [3, Remark 3, page 74] it follows that $u \in C^{2,\alpha}(\bar{B}_{1/2}(0))$ with bounds given by

$$\|D^2u\|_{C^\alpha(\bar{B}_{1/2}(0))} \leq C_2$$

where $0 < C_2 = C_2(n, \Lambda, \lambda, C_1)$. This proves the estimate in (4.6). Hence the proof. □

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