ON SOME NOTIONS OF COHOMOLOGY FOR FUSION CATEGORIES

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DISSERTATION ABSTRACT

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In this dissertation, we study two main topics: superfusion categories, and embeddings of symmetric fusion categories into modular fusion categories. Using a construction of Brundan and Ellis, we give a formula relating the fermionic $6j$-symbols of a superfusion category to the $6j$-symbols of the corresponding underlying fusion category, and prove a version of Ocneanu rigidity for superfusion categories. Inspired by the work of Lan, Kong, and Wen on the group of modular extensions of a symmetric fusion category, we also give definitions for the low cohomology groups of a finite supergroup and show these definitions are functorial.

This dissertation includes previously published material.

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CHAPTER I

INTRODUCTION

This dissertation includes previously published material. Chapters III and IV appeared in [Ush18], and Chapters I and II include material from [Ush18].

This dissertation is a contribution to the study of fusion categories, which are abstract objects expressing the idea of quantum symmetries. We study two topics: some generalizations of fusion categories associated with fermionic matter, and embeddings of symmetric fusion categories into modular fusion categories. In both cases some notion of cohomology plays a crucial role.

A *fusion category* is a semisimple $\mathbb{C}$-linear rigid monoidal category with finitely many isomorphism classes of simple objects, finite-dimensional spaces of morphisms between objects, and with simple unit object. Fusion categories play an important role in condensed matter physics. In [TV92, Tur94], Turaev and Viro constructed invariants of 3-manifolds from quantum $6j$-symbols, and showed that these lead to a $(2+1)$-dimensional topological quantum field theory (TQFT). Barrett and Westbury [BW96] showed that these invariants can be constructed from any spherical fusion category. Following this, Kirillov and Balsam [KB10], and Turaev and Virelizier [TV10] proved that the Turaev-Viro-Barrett-Westbury invariants of a spherical fusion category $\mathcal{A}$ are the same as the Reshetikhin-Turaev invariants [RT91] derived from the Drinfeld center $\mathcal{Z}(\mathcal{A})$.

More recently, Douglas, Schommer-Pries and Snyder [DSS13] showed that fusion categories are fully dualizable objects in the 3-category of monoidal categories, and so by the cobordism hypothesis [Lur09] we can associate a fully local 3-dimensional TQFT to any fusion category.

Let **sVec** be the category of superspaces with morphisms the even linear maps between them. A *supercategory* is a category enriched over **sVec**, i.e. the collection of morphisms between objects forms a superspace and composition is an even linear map. A *superfusion category* over $\mathbb{C}$ is a semisimple rigid monoidal supercategory with finitely many isomorphism classes of simple objects, finite-dimensional superspaces of morphisms between objects, and with simple unit object. The tensor product of morphisms satisfies the *super interchange law*

$$(f \otimes g) \circ (h \otimes k) = (-1)^{|g||h|}(f \circ h) \otimes (g \circ k). \tag{1.1}$$

Gaiotto and Kapustin [GK16], following the work of Gu, Wang and Wen [GWW15] described a fermionic analogue of the Turaev-Viro construction whose initial data is a spherical superfusion category, and Bhardwaj, Gaiotto and Kapustin [BGK17] have further studied spin-TQFTs. In comparison to the fusion category case however, not much is known about how to construct TQFTs using superfusion categories.

Fusion categories also have applications to the study of topological phases of matter (sometimes called topological orders [Wen90]). Indeed, Lan, Kong, and Wen [LKW17] have conjectured that (2 + 1)-dimensional symmetry protected topological orders with symmetry a symmetric fusion category $\mathcal{E}$ are classified (up to equivalence) by modular extensions of $\mathcal{E}$ with central charge equal to zero (mod 8) .

In Chapter II, we provide a basic review of monoidal categories, group cohomology, fusion categories, braided fusion categories, algebras and modules in a fusion category, the Deligne tensor product of abelian categories, module categories, and group actions on categories.

In Chapter III, we study superfusion categories and $6j$-symbols. Following [GWW15], a simple object X in a superfusion category is called *bosonic* if $\mathrm{End}(X) \simeq \mathbb{C}^{1|0}$, and *Majorana* if $\mathrm{End}(X) \simeq \mathbb{C}^{1|1}$. A superfusion category is called *bosonic* if all of its simple objects are bosonic. The associator in a fusion category admits a description in terms of *6j-symbols* satisfying a version of the pentagon equation, see i.e. [Tur94], [Wan10]. In a similar way, the associator in a superfusion category can be described in terms of *fermionic 6j-symbols* satisfying the *super pentagon equation* [GWW15]. Using a construction of Brundan and Ellis [BE17], one can describe the *underlying fusion category* of a superfusion category, which is naturally endowed with the structure of a fusion category over sVec (in the sense of [DGNO10, Definition 7.13.1]).

The main result of Chapter III is to give an explicit formula for the $6j$-symbols of the underlying fusion category in terms of the fermionic $6j$-symbols of the superfusion category (Definition 3.30), and show that these $6j$-symbols satisfy the pentagon equation for a monoidal category (Theorem 3.31). If $\mathcal{C}$ is a bosonic pointed superfusion category, i.e. a bosonic superfusion category such that the isomorphism classes of simple objects form a group G, then the fermionic $6j$-symbols in $\mathcal{C}$ are described by a 3-supercocycle [GWW15] $\widetilde{F} : G^3 \to \mathbb{C}^\times$ satisfying

$$\widetilde{F}(g,h,k)\widetilde{F}(g,hk,l)\widetilde{F}(h,k,l) = (-1)^{\omega(g,h)\omega(k,l)}\widetilde{F}(gh,k,l)\widetilde{F}(g,h,kl) \qquad (1.2)$$

where $\omega \in H^2(G, \mathbb{Z}/2\mathbb{Z})$ is a 2-cocycle on G. In this situation, our formula for the $6j$-symbols on the underlying fusion category gives a 3-cocycle on the $\mathbb{Z}/2\mathbb{Z}$-central extension of G determined by ω, whose restriction to G is $\widetilde{F}$. In particular, this implies that every 3-supercocycle on G arises as the restriction of a (genuine) 3-cocycle on a central extension of G by $\mathbb{Z}/2\mathbb{Z}$ (Corollary 3.34).

In Chapter IV we prove a version of Ocneanu rigidity for fusion categories. Ocneanu rigidity is the statement that (i) the number of fusion categories (up to equivalence) is countable, and (ii) the number of fusion categories (up to equivalence) with a given Grothendieck ring is finite. To prove a similar result in the superfusion category setting, we must first decide what the appropriate notion of the Grothendieck ring of a superfusion category should be.

Let $\mathbb{Z}^\pi = \mathbb{Z}[\pi]/(\pi^2 - 1)$, then Brundan and Ellis [BE17] defined the π-*Grothendieck ring* of a superfusion category $\mathcal{C}$ to be the $\mathbb{Z}^\pi$-module $\mathrm{sGr}(\mathcal{C})$ generated by isomorphism classes of objects $[X] \in \mathcal{C}$, subject to the relation that if $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is a short exact sequence with f and g homogeneous morphisms, then $[Y] = [X]\pi^{|f|} + [Z]\pi^{|g|}$. The tensor product on $\mathcal{C}$ induces an associative multiplication on $\mathrm{sGr}(\mathcal{C})$ making $\mathrm{sGr}(\mathcal{C})$ into a $\mathbb{Z}^\pi$-algebra. The main result of Chapter IV is to prove a version of Ocneanu rigidity for superfusion categories and the π-Grothendieck ring (Theorem 4.3).

In Chapter V, we suggest a notion of cohomology for symmetric fusion categories. Let (G, z) be a finite supergroup, i.e. a finite group G together with a central element $z \in Z(G)$ such that $z^2 = 1$, then $\mathrm{Rep}(G, z)$ is the category of finite-dimensional representations of G with braiding given by:

$$c^z_{X,Y}(x \otimes y) = (-1)^{mn} y \otimes x \text{ if } x \in X, y \in Y, zx = (-1)^m x \text{ and } zy = (-1)^n y, \quad (1.3)$$

for irreducible representations X, Y of G. This braiding makes $\mathrm{Rep}(G, z)$ into a symmetric fusion category. A result of Doplicher-Roberts and Deligne [DR89, Del02] says that every symmetric fusion category is of this form.

Our definition of cohomology was inspired by the work of Lan, Kong, and Wen [LKW17] on the group of *modular extensions* of a symmetric fusion category. Let $\mathcal{E}$ be a symmetric fusion category, then a *modular extension* of $\mathcal{E}$ is a modular category $\mathcal{M}$ together with a braided full embedding $\mathcal{E} \hookrightarrow \mathcal{M}$ such that $\mathcal{E}'|_{\mathcal{M}} = \mathcal{E}$. Lan, Kong, and Wen showed that the set $\mathcal{M}_{\text{ext}}(\mathcal{E})$ of equivalence classes of modular extensions of $\mathcal{E}$ admits a group structure making it into a finite abelian group, and that $\mathcal{M}_{\text{ext}}(\text{Rep}(G)) \xrightarrow{\sim} H^3(G, \mathbb{C}^\times)$ [LKW17, Theorem 4.2].

This isomorphism forms part of the following dictionary between the low cohomology groups of G and the category $\text{Rep}(G)$:

$$H^1(G, \mathbb{C}^\times) = \text{Invertible objects in } \text{Rep}(G),$$

$$H^2(G, \mathbb{C}^\times) = \text{Invertible } \text{Rep}(G)\text{-module categories},$$

$$H^3(G, \mathbb{C}^\times) = \text{Modular extensions of } \text{Rep}(G).$$

This connection motivates the following definitions for the cohomology of a finite supergroup (G, z) in terms of the category $\text{Rep}(G, z)$:

$$H^1(G, z) = \text{Invertible objects in } \text{Rep}(G, z),$$

$$H^2(G, z) = \text{Invertible } \text{Rep}(G, z)\text{-module categories},$$

$$H^3(G, z) = \text{Modular extensions of } \text{Rep}(G, z).$$

Since group cohomology is functorial, it is natural to ask whether these definitions are functorial as well. The first main result of Chapter V is to prove that these definitions of first, second, and third cohomology are contravariant functors (Theorems 5.16, 5.33 and 5.51).

Of particular interest is the case of third cohomology: given a supergroup homomorphism $f : (G, z) \to (H, w)$, we construct a homomorphism $\mathcal{M}_{\text{ext}}(\text{Rep}(H, w)) \to \mathcal{M}_{\text{ext}}(\text{Rep}(G, z))$ between the corresponding groups of modular extensions. Given a supergroup (G, z) with $z \neq 1$, there is a canonical homomorphism $i : (\mathbb{Z}/2\mathbb{Z}, -1) \to (G, z)$, and thus an induced homomorphism

$$i^* : \mathcal{M}_{\text{ext}}(\text{Rep}(G, z)) \to \mathcal{M}_{\text{ext}}(\text{sVec}) = \mathbb{Z}/16\mathbb{Z}. \tag{1.4}$$

There are five possible images for i^*. The second main result of Chapter V, which was proven independently of us in [GVR17], is that i^* is surjective if and only if i is split (Theorem 5.59).

CHAPTER II

PRELIMINARIES

Chapter II includes portions of [Ush18].

We begin by providing a basic review of fusion category theory and related topics. The standard reference for fusion categories is [ENO05]. Additional related references include [BJ00], [DGNO10]. Where possible, we use definitions as formulated in [EGNO15].

Abelian Categories

We assume familiarity with abelian categories; a good textbook is [Lan98]. In this section, we recall some necessary definitions from the theory of abelian categories. The prototypical example of an abelian category to keep in mind throughout this section is the category of (left) modules over a unital ring R. Most of the definitions in this section are formulated as in [EGNO15, Chapter 1].

Definition 2.1. Let $\mathcal{C}$ be an abelian category. An object $X \in \mathcal{C}$ is *simple* if there are precisely two subobjects of X, namely 0 and X. An object $X \in \mathcal{C}$ is *semisimple* if it is a direct sum of finitely many simple objects, and $\mathcal{C}$ is *semisimple* if every object of $\mathcal{C}$ is semisimple.

We will be primarily interested in semisimple categories with finitely many simple objects. Recall the classical version of Schur's Lemma: if R is a unital ring, and M, N are simple R-modules, then any morphism $M \to N$ is either zero or an isomorphism. An equivalent statement holds in any abelian category.

Lemma 2.2 (Schur's Lemma)**.** *Let $\mathcal{C}$ be an abelian category, and $X, Y \in \mathcal{C}$ simple objects. Then any morphism $X \to Y$ is either zero, or an isomorphism.*

In particular, if $X \in \mathcal{C}$ is simple, then $\mathrm{Hom}_{\mathcal{C}}(X, X)$ is a division ring.

Definition 2.3. Let $\mathcal{C}$ be an abelian category. An object $X \in \mathcal{C}$ has *finite length* if there exists a filtration

$$0 = X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X \tag{2.1}$$

such that X_i/X_{i-1} is simple for all i. Such a filtration is called a *Jordan-Hölder filtration* of X. If $Y \in \mathcal{C}$ is simple, then we say that this filtration *contains* Y with multiplicity m if X_i/X_{i-1} is isomorphic to Y for m distinct values of i.

Theorem 2.4 (Jordan-Hölder)**.** *Let $\mathcal{C}$ be an abelian category, and suppose $X \in \mathcal{C}$ has finite length. Then any two Jordan-Hölder filtrations of X contain any simple object with the same multiplicity.*

Definition 2.5. Let $\mathcal{C}$ be an abelian category, and suppose $X \in \mathcal{C}$ has finite length. If $Y \in \mathcal{C}$ is simple, then define $[X : Y]$ to be the *multiplicity* of Y in any Jordan-Hölder filtration of X. If $m = [X : Y]$ is non-zero, we say that X *contains* Y with multiplicity m.

Definition 2.6. Let k be a field. We say an abelian category $\mathcal{C}$ is k-linear if, for any $X, Y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is equipped with the structure of a k-vector space such that composition of morphisms is k-linear.

In the language of enriched category theory (see e.g. [Kel05]), a k-linear category is precisely a category enriched over the category of k-vector spaces.

Example 2.7. The category $\mathbf{Vec}_{\mathbb{C}}$ of $\mathbb{C}$-vector spaces is a $\mathbb{C}$-linear abelian category. The category $\mathrm{Vec}_{\mathbb{C}}$ of finite-dimensional $\mathbb{C}$-vector spaces is a semisimple $\mathbb{C}$-linear abelian category (it has one simple object up to isomorphism, namely the one-dimensional vector space $\mathbb{C}$).

In this work we always take $k = \mathbb{C}$, so we omit subscripts and write **Vec** and Vec for the categories of $\mathbb{C}$-vector spaces and finite-dimensional $\mathbb{C}$-vector spaces respectively.

Example 2.8. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$. The category $\mathrm{Rep}_{\mathbb{C}}(\mathfrak{g})$ of finite-dimensional representations of $\mathfrak{g}$ is a semisimple $\mathbb{C}$-linear abelian category.

Monoidal Categories

Monoidal categories are a categorification of the notion of a monoid. In this section, we provide the definition of a monoidal category, monoidal functors, monoidal natural transformations, and rigid monoidal categories. We also provide some examples of monoidal categories. Most of the definitions in this section are formulated as in [EGNO15, Chapter 2]. Another reference for monoidal category theory is [Lan98, VII, XI].

Definition 2.9. A *monoidal category* consists of a category $\mathcal{C}$, together with the data of a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (called the *tensor product* bifunctor), a natural isomorphism $a : (- \otimes -) \otimes \xrightarrow{\sim} - \otimes (- \otimes -)$:

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z), \qquad X, Y, Z \in \mathcal{C} \tag{2.2}$$

called the *associativity isomorphism*, an object $\mathbf{1}_{\mathcal{C}} \in \mathcal{C}$ (called the *unit object*), and natural isomorphisms $\ell_X : \mathbf{1}_{\mathcal{C}} \otimes X \to X$, $r_X : X \otimes \mathbf{1}_{\mathcal{C}} \to X$ (called the left and right *unit isomorphisms*, respectively), subject to the following axioms.

1. The pentagon axiom:

The diagram

$$\begin{tikzcd}
& ((W \otimes X) \otimes Y) \otimes Z \arrow[dl, "a_{W,X,Y} \otimes \mathrm{id}_Z"'] \arrow[dr, "a_{W \otimes X, Y, Z}"] & \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[d, "a_{W, X \otimes Y, Z}"] & & (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "a_{W,X,Y \otimes Z}"] \\
W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "\mathrm{id}_W \otimes a_{X,Y,Z}"] & & W \otimes (X \otimes (Y \otimes Z))
\end{tikzcd} \tag{2.3}$$

is commutative for all W, X, Y, $Z \in \mathcal{C}$.

2. The triangle axiom:

The diagram

$$\begin{tikzcd}
(X \otimes \mathbf{1}_{\mathcal{C}}) \otimes Y \arrow[rr, "a_{X, \mathbf{1}_{\mathcal{C}}, Y}"] \arrow[dr, "r_X \otimes \mathrm{id}_Y"] & & X \otimes (\mathbf{1}_{\mathcal{C}} \otimes Y) \arrow[dl, "\mathrm{id}_X \otimes \ell_Y"] \\
& X \otimes Y &
\end{tikzcd} \tag{2.4}$$

is commutative for all $X, Y \in \mathcal{C}$.

To simplify our notation, we will frequently omit the subscript on the unit object and write $\mathbf{1} := \mathbf{1}_{\mathcal{C}}$.

Example 2.10. The category $\mathbf{Vec}_{\mathbb{C}}$ of $\mathbb{C}$-vector spaces from Example 2.7 is a monoidal category. The tensor product functor is given by the usual tensor product of vector spaces, and the associativity isomorphism $a_{X,Y,Z} : (X \otimes_{\mathbb{C}} Y) \otimes_{\mathbb{C}} Z \xrightarrow{\sim} X \otimes_{\mathbb{C}} (Y \otimes_{\mathbb{C}} Z)$ is given by

$$(x \otimes y) \otimes z \mapsto x \otimes (y \otimes z), x \in X,\ y \in Y,\ z \in Z. \tag{2.5}$$

The unit object is the one-dimensional vector space $\mathbb{C}$, with the obvious left $\ell_X : \mathbb{C} \otimes_{\mathbb{C}} X \xrightarrow{\sim} X$ and right $r_X : X \otimes_{\mathbb{C}} \mathbb{C} \xrightarrow{\sim} X$ unit isomorphisms.

More generally, if R is a commutative ring then the category of R-modules is a monoidal category. The tensor product is given by the tensor product of R-modules, the unit object is R, and the associativity and unit isomorphisms are the obvious ones.

Example 2.11. Let G be a finite group, then the category $\mathbf{Rep}_{\mathbb{C}}(G)$ of representations of G over $\mathbb{C}$ is a monoidal category. The tensor product is given by the tensor product of representations, the unit object is the trivial representation, and the associativity and unit isomorphisms are the obvious ones.

We denote by $\mathrm{Rep}_{\mathbb{C}}(G) \subset \mathbf{Rep}_{\mathbb{C}}(G)$ the full monoidal subcategory of finite-dimensional representations of G over $\mathbb{C}$. We will omit the subscript and refer to this category simply as $\mathrm{Rep}(G)$.

Example 2.12. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$, then the category $\mathbf{Rep}_{\mathbb{C}}(\mathfrak{g})$ of representations of $\mathfrak{g}$ over $\mathbb{C}$ is a monoidal category. The tensor product of representations V and W in $\mathbf{Rep}_{\mathbb{C}}(\mathfrak{g})$ is defined to be $V \otimes_{\mathbb{C}} W$, with $\mathfrak{g}$-action given by the familiar Leibniz rule.

Monoidal Functors and Natural Transformations.

Definition 2.13. Let $\mathcal{C}$ and $\mathcal{D}$ be two monoidal categories. A *monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$, together with a natural isomorphism

$$J_{X,Y} : F(X) \otimes F(Y) \xrightarrow{\sim} F(X \otimes Y) \tag{2.6}$$

such that $F(\mathbf{1}_\mathcal{C})$ is isomorphic to $\mathbf{1}_\mathcal{D}$, and that the diagram (called the *monoidal structure axiom*):

$$\begin{array}{ccc}
(F(X)\otimes F(Y))\otimes F(Z) & \xrightarrow{a'_{F(X),F(Y),F(Z)}} & F(X)\otimes(F(Y)\otimes F(Z)) \\
\big\downarrow{\scriptstyle J_{X,Y}\otimes \mathrm{id}_{F(Z)}} & & \big\downarrow{\scriptstyle \mathrm{id}_{F(X)}\otimes J_{Y,Z}} \\
F(X\otimes Y)\otimes F(Z) & & F(X)\otimes F(Y\otimes Z) \\
\big\downarrow{\scriptstyle J_{X\otimes Y,Z}} & & \big\downarrow{\scriptstyle J_{X,Y\otimes Z}} \\
F((X\otimes Y)\otimes Z) & \xrightarrow{F(a_{X,Y,Z})} & F(X\otimes(Y\otimes Z)),
\end{array} \tag{2.7}$$

is commutative for all $X, Y, Z \in \mathcal{C}$, where a (respectively a') denotes the associator in $\mathcal{C}$ (respectively $\mathcal{D}$). We often refer to the monoidal functor (F, J) simply as F.

Example 2.14. Suppose G is a group, and $H \leq G$ a subgroup. The restriction functor $\mathrm{res}^G_H : \mathrm{Rep}(G) \to \mathrm{Rep}(H)$ is a monoidal functor.

Definition 2.15. A monoidal functor $F : \mathcal{C} \to \mathcal{D}$ is a *monoidal equivalence* if it is an equivalence in the normal sense. In this case, we say that $\mathcal{C}$ and $\mathcal{D}$ are *monoidally equivalent.*

Recall that a functor is an equivalence (in the normal sense) if and only if it is full, faithful, and essentially surjective, see e.g. [Lan98, IV.4 Theorem 1].

Definition 2.16. Let $\mathcal{C}$ and $\mathcal{D}$ be two monoidal categories, and let (F, J) and (F', J') be two monoidal functors $\mathcal{C} \to \mathcal{D}$. A *natural transformation* of monoidal functors $\eta : (F, J) \to (F', J')$ is a natural transformation $\eta : F \to F'$ such that $\eta_{\mathbf{1}_\mathcal{C}} : F(\mathbf{1}_\mathcal{C}) \to F'(\mathbf{1}_\mathcal{C})$ is an isomorphism, and the diagram

$$\begin{array}{ccc}
F(X)\otimes F(Y) & \xrightarrow{J_{X,Y}} & F(X\otimes Y) \\
\big\downarrow{\scriptstyle \eta_X\otimes\eta_Y} & & \big\downarrow{\scriptstyle \eta_{X\otimes Y}} \\
F'(X)\otimes F'(Y) & \xrightarrow{J'_{X,Y}} & F'(X\otimes Y)
\end{array} \tag{2.8}$$

is commutative for all $X, Y \in \mathcal{C}$.

Rigid Monoidal Categories.

Definition 2.17. Let $\mathcal{C}$ be a monoidal category. An object X^* is a *left dual* of X if there exists morphisms $\mathrm{ev}_X : X^* \otimes X \to \mathbf{1}_{\mathcal{C}}$ and $\mathrm{coev}_X : \mathbf{1}_{\mathcal{C}} \to X \otimes X^*$, called the *evaluation* and *coevaluation* morphisms respectively, such that the compositions

$$X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_X} (X \otimes X^*) \otimes X \xrightarrow{a_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X, \tag{2.9}$$

and

$$X^* \xrightarrow{\mathrm{id}_{X^*} \otimes \mathrm{coev}_X} X^* \otimes (X \otimes X^*) \xrightarrow{a^{-1}_{X^*,X,X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_{X^*}} X^* \tag{2.10}$$

are given by the identity morphism.

If $X \in \mathcal{C}$ has a left dual, then it is unique up to a unique isomorphism.

Example 2.18. If $V \in \mathrm{Vec}$ is a finite-dimensional vector space, then $V^* = \mathrm{Hom}_{\mathrm{Vec}}(V, \mathbb{C})$ is a left dual of V, with evaluation map defined on pure tensors by $f \otimes v \mapsto f(v)$, and coevaluation map defined by $z \mapsto \sum_{i=1}^n z v_i \otimes \widehat{v}_i$, where $v_1, \ldots, v_n$ is any basis of V, and $\widehat{v}_1, \ldots, \widehat{v}_n$ is the corresponding dual basis of V^*.

There is a similar notion of a *right dual* of an object, see e.g. [EGNO15, §2.10] for details. In all examples we care about, every object has isomorphic left and right duals, so we do not include the definition here.

Definition 2.19. Let $\mathcal{C}$ be a monoidal category. An object $X \in \mathcal{C}$ is *rigid* if it has left and right duals. We say that $\mathcal{C}$ is *rigid* if every object of $\mathcal{C}$ is rigid.

Group Cohomology

The language of group cohomology is often useful for working with monoidal categories. Let G be a group, and A a G-module, that is, an abelian group with a G-action. In this section, we provide a definition of the group cohomology $H^*(G, A)$

of G with values in A in terms of the cohomology of the *standard complex* of G with values in A. We follow the discussion in [EGNO15, §1.7]; other references include [Bro82] and [Wei95, Chapter 6].

Definition 2.20. Let G be a group, and A a G-module. The *standard complex* of G with values in A is the chain complex (C, d) with terms $C^n(G, A) := \mathrm{Fun}(G^n, A)$, and differential $d_n : C^{n-1} \to C^n$ given by

$$\begin{aligned} d_n(f)(g_1, \dots, g_n) = g_1 \cdot f(g_2, \dots, g_n) - f(g_1 g_2, \dots, g_n) + \dots \\ + (-1)^{n-1} f(g_1, \dots, g_{n-1} g_n) + (-1)^n f(g_1, \dots, g_{n-1}). \end{aligned} \tag{2.11}$$

We call an element of $C^n(G, A)$ an *n-cochain*, $f \in \ker(d_{n+1})$ an *n-cocycle*, and the *group cohomology of G with values in A* is defined to be

$$H^*(G, A) = H^*((C, d)). \tag{2.12}$$

For the convenience of the reader, we unpack this definition to write down the equation that an n-cochain must satisfy to be an n-cocycle for $n = 0, 1, 2, 3$.

Observe that a 0-cochain is a function (not necessarily a group homomorphism) $f : \mathbf{1} \to A$ from the trivial group to A, which is completely determined by $f(1) \in A$. Under this identification, Definition 2.20 implies the following.

Definition 2.21. A 0-*cocycle* on G with values in A is an element $a \in A$ satisfying the equation

$$0 = g \cdot a - a, \quad \text{for all } g \in G. \tag{2.13}$$

Thus 0-cocycles are precisely the G-invariant elements of A, so $H^0(G, A) = A^G$.

Definition 2.22. A 1-*cocycle* on G with values in A is a function $f : G \to A$ satisfying the equation

$$0 = g \cdot f(h) - f(gh) + f(g), \quad \text{for all } g, h \in G. \tag{2.14}$$

Definition 2.23. Let G be a group, and A a G-module. We say that A is a *trivial* G-module if $g \cdot a = a$ for all $g \in G$, $a \in A$.

If A is a trivial G-module, then a 1-cocycle on G with values in A is precisely a group homomorphism $G \to A$, and $H^1(G, A) = \mathrm{Hom}(G, A)$ in this case.

Definition 2.24. A 2-*cocycle* on G with values in A is a function $f : G \times G \to A$ satisfying the equation:

$$0 = g \cdot f(h,k) - f(gh,k) + f(g,hk) - f(g,h), \quad \text{for all } g, h, k \in G, \tag{2.15}$$

or, written multiplicatively:

$$f(g,h)f(gh,k) = g \cdot f(h,k)f(g,hk), \quad \text{for all } g, h, k \in G. \tag{2.16}$$

We refer to Eqs. (2.15) and (2.16) as the 2-*cocycle condition*. If A is a trivial G-module then there is an important connection between the second cohomology group and central extensions, which we describe now.

Definition 2.25. A *central extension* of a group G by an abelian group A is a short exact sequence of groups

$$1 \to A \to E \to G \to 1 \tag{2.17}$$

such that $A \subseteq Z(E)$. We say that two central extensions E, E' of G by A are *equivalent* if there exists an isomorphism $\phi : E \to E'$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}_A} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathrm{id}_G} & & \\ 0 & \longrightarrow & A & \longrightarrow & E' & \longrightarrow & G & \longrightarrow & 0 \end{array} \tag{2.18}$$

commutes.

The following theorem relates central extensions of G by A to the second cohomology group $H^2(G, A)$.

Theorem 2.26. *Let G be a group, and A a trivial G-module. The second cohomology group $H^2(G, A)$ is in bijection with equivalence classes of central extensions of G by A.*

We will not prove this theorem, however we will briefly describe the constructions involved. Given a central extension $1 \to A \to E \xrightarrow{p} G \to 1$, choose a set-theoretic section $s : G \to E$ of p. Then $s(\rho(gh)\rho(h)^{-1}\rho(g)^{-1}) = 1$, so by exactness there exists $\alpha(g, h) \in A$ such that

$$\rho(gh) = \alpha(g, h)\rho(g)\rho(h), \text{ for } g, h \in G. \tag{2.19}$$

associativity of G implies that $\alpha : G \times G \to A$ satisfies the 2-cocycle condition, so α represents a class $[\alpha] \in H^2(G, A)$. It can be shown that this class only depends on the equivalence class of the central extension; in particular, it does not depend on the choice of section.

For the reverse construction, let $[\alpha] \in H^2(G, A)$, and choose a cocycle representative $\alpha : G \times G \to A$. Define the group E to be the set $G \times A$ with multiplication given by

$$(g_1, a_1) \cdot (g_2, a_2) = (g_1g_2, a_1a_2\alpha(g_1g_2)), \ g_1, g_2 \in G, \ a_1, a_2 \in A. \tag{2.20}$$

Associativity of this multiplication is equivalent to α satisfying the 2-cocycle condition (2.16). This makes E a central extension of G by A, and it can be shown that the equivalence class of this central extension is independent of the chosen cocycle representative.

Definition 2.27. A 3-*cocycle* on G with values in A is a function $f : G \times G \times G \to A$ satisfying the equation

$$0 = g \cdot f(h,k,l) - f(gh,k,l) + f(g,hk,l) - f(g,h,kl) + f(g,h,k), \quad \text{for all } g,h,k,l \in G, \tag{2.21}$$

or, written multiplicatively:

$$f(g,h,kl)f(gh,k,l) = g \cdot f(h,k,l)f(g,hk,l)f(g,h,k), \quad \text{for all } g,h,k,l \in G. \tag{2.22}$$

As in the $n = 2$ case, we refer to Eqs. (2.21) and (2.22) as the 3-*cocycle condition*.

Example 2.28. Let G be a group. Let Vec_G be the category whose objects are finite-dimensional G-graded $\mathbb{C}$-vector spaces, and whose morphisms are linear maps preserving the G-grading.

We equip Vec_G with a monoidal structure as follows. The tensor product of G-graded vector spaces V and W is defined to be $V \otimes_{\mathbb{C}} W$ with G-grading given by

$$(V \otimes_{\mathbb{C}} W)_g = \oplus_{\substack{h,k\in G \\ hk=g}} V_h \otimes W_k, \;\; g \in G, \tag{2.23}$$

with the obvious associativity and unit isomorphisms.

We can obtain other monoidal structures on Vec_G by twisting the standard associator by a 3-cocycle. Indeed, let $\omega : G \times G \times G \to \mathbb{C}^\times$ be a 3-cocycle, and for $g \in G$ let $\mathbb{C}_g$ denote the one-dimensional G-graded vector space concentrated in degree g. We can define a new associator on Vec_G by the formula

$$\begin{aligned} a^\omega_{\mathbb{C}_g,\mathbb{C}_h,\mathbb{C}_k} : (\mathbb{C}_g \otimes_{\mathbb{C}} \mathbb{C}_h) \otimes_{\mathbb{C}} \mathbb{C}_k &\to \mathbb{C}_g \otimes_{\mathbb{C}} (\mathbb{C}_h \otimes_{\mathbb{C}} \mathbb{C}_k) \\ a^\omega_{\mathbb{C}_g,\mathbb{C}_h,\mathbb{C}_k} &= \omega(g,h,k) a_{\mathbb{C}_g,\mathbb{C}_h,\mathbb{C}_k}, \end{aligned} \tag{2.24}$$

for $g,h,k \in G$, and then extending linearly to all objects of Vec_G. That a^ω satisfies the pentagon equation (2.3) is equivalent to ω satisfying the 3-cocycle condition (2.22). We denote by Vec_G^ω the monoidal category obtained in this way.

Remark 2.29. The category Vec_G^ω depends (up to monoidal equivalence) only on the class $[\omega] \in H^3(G, \mathbb{C}^\times)$ [EGNO15, Proposition 2.6.1].

Fusion Categories

The following definition is from [ENO05, §2].

Definition 2.30. A *fusion category* over $\mathbb{C}$ is a semisimple rigid $\mathbb{C}$-linear monoidal category $\mathcal{C}$ with finitely many isomorphism classes of simple objects, finite-dimensional spaces of morphisms between objects, and with simple unit object.

Example 2.31. Let G be a finite group, then $\mathrm{Rep}(G)$, the category of finite-dimensional representations of G, is a fusion category.

Example 2.32. Let G be a finite group, then the category Vec_G from Example 2.28 is a fusion category. The simple objects in Vec_G are the one-dimensional spaces $\mathbb{C}_g$ for $g \in G$. If $\omega : G \times G \times G \to \mathbb{C}^\times$ is a 3-cocycle on G, then Vec_G^ω is also a fusion category.

Recall (Definition 2.19) that rigidity means we have evaluation $\mathrm{ev}_X : X^* \otimes X \to \mathbf{1}_\mathcal{C}$ and coevaluation $\mathrm{coev}_X : \mathbf{1}_\mathcal{C} \to X \otimes X^*$ morphisms for each object $X \in \mathcal{C}$.

Definition 2.33. Let $\mathcal{C}$ be a fusion category. We say an object $X \in \mathcal{C}$ is *invertible* if the evaluation ev_X and coevaluation coev_X morphisms are isomorphisms.

Equivalently, an object $X \in \mathcal{C}$ is invertible if there exists $Y \in \mathcal{C}$ such that $X \otimes Y \xrightarrow{\sim} Y \otimes X \xrightarrow{\sim} \mathbf{1}$.

Example 2.34. If G is a finite group, then invertible objects in $\mathrm{Rep}(G)$ are precisely the 1-dimensional representations of G.

Definition 2.35. We say that a fusion category $\mathcal{C}$ is *pointed* if all of its simple objects are invertible.

Let G be a finite group, and $\omega : G \times G \times G \to \mathbb{C}^\times$ a 3-cocycle on G with values in $\mathbb{C}^\times$, then the category Vec_G^ω considered in Example 2.28 is pointed. In fact, every pointed fusion category is monoidally equivalent to a category of this form [ENO05, §8].

Definition 2.36. A *fusion subcategory* of a fusion category $\mathcal{C}$ is a full abelian subcategory $\mathcal{D} \subset \mathcal{C}$ closed under subquotients and tensor products.

It follows from [EGNO15, Corollary 4.11.4] that a fusion subcategory $\mathcal{D} \subset \mathcal{C}$ is itself a fusion category. The invertible objects of $\mathcal{C}$ form a fusion subcategory of $\mathcal{C}$, which we denote by $\mathrm{Inv}(\mathcal{C})$.

Grothendieck Rings and the Frobenius-Perron Dimension. We being by recalling the notion of the Grothendieck ring of a monoidal abelian category.

Definition 2.37. Let $\mathcal{C}$ be an abelian category. The *Grothendieck group* $\mathrm{Gr}(\mathcal{C})$ of $\mathcal{C}$ is the abelian group generated by the symbols $[X]$ for $X \in \mathcal{C}$, such that if

$$0 \to X \to Y \to Z \to 0 \tag{2.25}$$

is an exact sequence in $\mathcal{C}$, then we have the relation

$$[Y] - [X] - [Z] = 0 \tag{2.26}$$

in $\mathrm{Gr}(\mathcal{C})$.

If $\mathcal{C}$ is a *monoidal* abelian category, then the tensor product on $\mathcal{C}$ induces an associative multiplication on $\mathrm{Gr}(\mathcal{C})$ given by the formula:

$$[X \otimes Y] := [X] \cdot [Y], \ \ X, Y \in \mathcal{C}. \tag{2.27}$$

In this situation, we call $\mathrm{Gr}(\mathcal{C})$ the *Grothendieck ring* of $\mathcal{C}$.

We can say even more in the situation where $\mathcal{C}$ is a fusion category. Let X_i, $i \in I$ be a complete set of representatives of the isomorphism classes of simple objects in $\mathcal{C}$. Then $\mathrm{Gr}(\mathcal{C})$ is a free abelian group with basis $[X_i]$, $i \in I$. For $X \in \mathcal{C}$, we have the decomposition:

$$[X] = \sum_{i \in I} [X : X_i][X_i]. \tag{2.28}$$

The following notion was first developed in the fusion category context in [ENO05].

Definition 2.38. Let $\mathcal{C}$ be a fusion category, and take $X \in \mathcal{C}$. The *Frobenius-Perron dimension* $\mathrm{FPdim}(X)$ of X is the largest positive real eigenvalue of the matrix of (left) multiplication by $[X]$ on $\mathrm{Gr}(\mathcal{C})$.

That this definition is well-defined follows from the Frobenius-Perron theorem [Per07, Fro12]. We recall a simplified version of that theorem here.

Theorem 2.39 (Frobenius-Perron)**.** *Let A be a square matrix with non-negative real entries. Then A has a positive real eigenvalue λ_A such that if $\mu \in \mathbb{C}$ is any other eigenvalue of A, then $|\mu| < \lambda_A$.*

The Frobenius-Perron dimension extends to a homomorphism $\mathrm{Gr}(\mathcal{C}) \to \mathbb{R}$ with the property that $\mathrm{FPdim}([X]) \geq 0$ for all $X \in \mathcal{C}$. It turns out that this property characterizes the Frobenius-Perron dimension.

Proposition 2.40 ([EGNO15, Proposition 3.3.6])**.** *Let $\mathcal{C}$ be a fusion category. The Frobenius-Perron dimension determines a homomorphism* $\mathrm{FPdim} : \mathrm{Gr}(\mathcal{C}) \to \mathbb{R}$ *such that* $\mathrm{FPdim}([X]) \geq 0$ *for all $X \in \mathcal{C}$. Moreover, it is the only homomorphism* $\mathrm{Gr}(\mathcal{C}) \to \mathbb{R}$ *that takes non-negative values on $[X]$ for all $X \in \mathcal{C}$.*

The following characterization of invertible objects in a fusion category is often useful.

Lemma 2.41. *Let $\mathcal{C}$ be a fusion category. An object $X \in \mathcal{C}$ is invertible if and only if* $\mathrm{FPdim}(X) = 1$.

In [ENO05], the notion of the Frobenius-Perron dimension of a fusion category was also developed.

Definition 2.42. Let $\mathcal{C}$ be a fusion category, and X_i, $i \in I$ a complete set of representatives of the isomorphism classes of simple objects in $\mathcal{C}$. The *Frobenius-Perron dimension* $\mathrm{FPdim}(\mathcal{C})$ of $\mathcal{C}$ is defined to be

$$\mathrm{FPdim}(\mathcal{C}) := \sum_{i \in I} \mathrm{FPdim}(X_i)^2. \tag{2.29}$$

Example 2.43. Consider the category $\mathcal{C} = \mathrm{Rep}(S_3)$ of finite-dimensional representations of S_3 over $\mathbb{C}$. Let $\mathbf{1}$ be the trivial representation, χ the sign representation, and S the standard representation. Then $\mathrm{Gr}(\mathrm{Rep}(S_3))$ has basis $[\mathbf{1}], [\chi], [S]$, with multiplication given by:

$$[\chi] \cdot [S] = [S], \qquad [\chi] \cdot [\chi] = [\mathbf{1}], \qquad [S] \cdot [S] = [\mathbf{1}] + [\chi] + [S]. \tag{2.30}$$

A straightforward computation shows that $\mathrm{FPdim}(\mathbf{1}) = \mathrm{FPdim}(\chi) = 1$ and $\mathrm{FPdim}(S) = 2$. Thus

$$\mathrm{FPdim}(\mathrm{Rep}(S_3)) = \mathrm{FPdim}(\mathbf{1}_{\mathrm{triv}})^2 + \mathrm{FPdim}(\chi)^2 + \mathrm{FPdim}(S)^2 = 6 = |S_3|. \tag{2.31}$$

Remark 2.44. It follows from Proposition 2.40 that if G is a finite group, then $\mathrm{FPdim}(X) = \dim_{\mathbb{C}}(X)$ for all $X \in \mathrm{Rep}(G)$, so $\mathrm{FPdim}(\mathrm{Rep}(G)) = |G|$.

Observe that the category $\mathrm{Rep}(G)$ has the property that the Frobenius-Perron dimensions of all objects are integers.

Definition 2.45. A fusion category $\mathcal{C}$ is called *integral* if $\mathrm{FPdim}(X) \in \mathbb{Z}$ for all $X \in \mathcal{C}$.

In [ENO05, Theorem 8.33] it was proved that a fusion category is integral if and only if it is category of representations of a finite-dimensional quasi-Hopf algebra. Not every fusion category is integral however, as the following important example shows.

Example 2.46 (see [DGNO10])**.** An *Ising fusion category* $\mathcal{I}$ has three isomorphism classes of simple objects: the unit object $\mathbf{1}$, an invertible object π, and a non-invertible object X, satisfying the multiplication rules:

$$[\pi] \cdot [\pi] = [\mathbf{1}], \qquad [\pi] \cdot [X] = [X] \cdot [\pi] = [X], \qquad [X] \cdot [X] = [\mathbf{1}] + [\pi] \tag{2.32}$$

in $\mathrm{Gr}(\mathcal{I})$. It is straightforward to check that:

$$\mathrm{FPdim}(\mathbf{1}) = 1, \qquad \mathrm{FPdim}(\pi) = 1, \qquad \mathrm{FPdim}(X) = \sqrt{2}. \tag{2.33}$$

so $\mathrm{FPdim}(\mathcal{I}) = 4$. In fact, every non-pointed fusion category with Frobenius-Perron dimension 4 is an Ising fusion category.

Braided Fusion Categories

Recall that monoidal categories categorify the notion of a monoid. Braided monoidal categories, introduced in [JS93], categorify the notion of a commutative monoid. Most of the definitions in this section are formulated as in [EGNO15, Chapter 8].

Definition 2.47. A *braiding* on a monoidal category $\mathcal{C}$ is a natural isomorphism

$$c_{X,Y} : X \otimes Y \to Y \otimes X \text{ for } X, Y \in \mathcal{C}, \tag{2.34}$$

such that the diagrams

$$\begin{tikzcd}[column sep=small]
 & X \otimes (Y \otimes Z) \arrow[r, "c_{X,Y\otimes Z}"] & (Y \otimes Z) \otimes X \arrow[dr, "a_{Y,Z,X}"] & \\
(X \otimes Y) \otimes Z \arrow[ur, "a_{X,Y,Z}"] \arrow[dr, "c_{X,Y}\otimes \mathrm{id}_Z"'] & & & Y \otimes (Z \otimes X), \\
 & (Y \otimes X) \otimes Z \arrow[r, "a_{Y,X,Z}"] & Y \otimes (X \otimes Z) \arrow[ur, "\mathrm{id}_Y \otimes c_{X,Z}"'] &
\end{tikzcd} \tag{2.35}$$

and

$$\begin{tikzcd}[column sep=small]
 & (X \otimes Y) \otimes Z \arrow[r, "c_{X\otimes Y,Z}"] & Z \otimes (X \otimes Y) \arrow[dr, "a^{-1}_{Z,X,Y}"] & \\
X \otimes (Y \otimes Z) \arrow[ur, "a^{-1}_{X,Y,Z}"] \arrow[dr, "\mathrm{id}_X\otimes c_{Y,Z}"'] & & & (Z \otimes X) \otimes Y, \\
 & X \otimes (Z \otimes Y) \arrow[r, "a^{-1}_{X,Z,Y}"] & (X \otimes Z) \otimes Y \arrow[ur, "c_{X,Z} \otimes \mathrm{id}_Y"'] &
\end{tikzcd} \tag{2.36}$$

commute for all $X, Y, Z \in \mathcal{C}$.

Definition 2.48. A *braided* monoidal category is a monoidal category together with a braiding. A *braided fusion category* is a fusion category equipped with a braiding.

Example 2.49. Let G be a group, then $\mathrm{Rep}(G)$ admits a braiding given by transposition of factors:

$$\begin{aligned} c_{V,W} : V \otimes W &\to W \otimes V \\ v \otimes w &\mapsto w \otimes v, \end{aligned} \tag{2.37}$$

for $v \in V$, $w \in W$ and $V, W \in \mathrm{Rep}(G)$.

Example 2.50. Let R be a commutative ring, then the category of R-modules is a braided monoidal category, with braiding given by transposition of factors. If G is an abelian group, then Vec_G (see Example 2.28) is braided, with braiding given by transposition of factors.

Example 2.51. The Ising fusion categories considered in Example 2.46 can be given a braiding, see [DGNO10, Appendix B].

Braided Monoidal Functors.

Definition 2.52. Let $\mathcal{C}$ and $\mathcal{D}$ be braided monoidal categories with braidings c and c' respectively. A monoidal functor $(F, J) : \mathcal{C} \to \mathcal{D}$ is called *braided* if the following diagram commutes:

$$\begin{tikzcd} F(X) \otimes F(Y) \arrow[r, "c'_{F(X),F(Y)}"] \arrow[d, "J_{X,Y}"] & F(Y) \otimes F(X) \arrow[d, "J_{Y,X}"] \\ F(X \otimes Y) \arrow[r, "F(c_{X,Y})"] & F(Y \otimes X) \end{tikzcd} \tag{2.38}$$

for all $X, Y \in \mathcal{C}$.

Definition 2.53. A braided monoidal functor $F : \mathcal{C} \to \mathcal{D}$ is a *braided monoidal equivalence* if it is an equivalence in the normal sense. In this case, we say that $\mathcal{C}$ and $\mathcal{D}$ are *braided equivalent*.

Centralizers. The following definition was introduced by Müger in [Müg03].

Definition 2.54. Let $\mathcal{D}$ be a fusion subcategory of a braided fusion category $\mathcal{C}$. The *centralizer* $\mathcal{D}'|_{\mathcal{C}}$ of $\mathcal{D}$ in $\mathcal{C}$ is defined to be

$$\mathcal{D}'|_{\mathcal{C}} = \{X \in \mathcal{C} \ : \ c_{X,Y} \circ c_{Y,X} = \mathrm{id}_{X \otimes Y} \text{ for all } Y \in \mathcal{D}\} \tag{2.39}$$

i.e. $\mathcal{D}'|_{\mathcal{C}}$ is the full subcategory of objects in $\mathcal{C}$ that centralize every object in $\mathcal{D}$.

We will occasionally write $\mathcal{D}'$ instead of $\mathcal{D}'|_{\mathcal{C}}$.

Definition 2.55. A braided fusion category $\mathcal{C}$ is called *symmetric* if $\mathcal{C}' = \mathcal{C}$.

Equivalently, $\mathcal{C}$ is symmetric if and only if $c_{X,Y} \circ c_{Y,X} = \mathrm{id}_{X \otimes Y}$ for all $X, Y \in \mathcal{C}$. The braided categories discussed in Examples 2.49 and 2.50 are symmetric. Observe that a fusion subcategory $\mathcal{D} \subseteq \mathcal{C}$ of a braided fusion category is symmetric if and only if $\mathcal{D} \subseteq \mathcal{D}'|_{\mathcal{C}}$.

The following notion was defined by Müger in [Müg03].

Definition 2.56. A braided fusion category $\mathcal{C}$ is called *non-degenerate* if $\mathcal{C}' = \mathrm{Vec}$.

Example 2.57. Any Ising braided fusion category (see Example 2.51) is non-degenerate [DGNO10, Corollary B.12].

We call a symmetric braided fusion category a *symmetric fusion category*, and a non-degenerate braided fusion category a *non-degenerate fusion category.* Symmetric fusion categories are those braided fusion categories whose centralizer is as large as possible, while non-degenerate fusion categories are the braided fusion categories whose centralizer is as small as possible.

As we will later see, there is a complete classification of symmetric fusion categories in terms of supergroups. Non-degenerate fusion categories (and the closely related notion of a modular fusion category) have been classified in low ranks (see e.g. [RSW09]), though much less is known than in the symmetric case.

Drinfeld Center. In this section we describe the Drinfeld center of a monoidal category. The Drinfeld center of a fusion category is an example of a non-degenerate braided fusion category, and will play an important role in Chapter V. This construction is due to Drinfeld (unpublished), and was given in [JS91, Maj91].

Definition 2.58 (see e.g. [EGNO15, Definition 7.13.1])**.** Let $\mathcal{C}$ be a monoidal category. The *Drinfeld center* of $\mathcal{C}$ is the category $\mathcal{Z}(\mathcal{C})$ whose objects are pairs (Z, γ) where $Z \in \mathcal{C}$ and

$$\gamma_X : X \otimes Z \xrightarrow{\sim} Z \otimes X, \;\; X \in \mathcal{C} \tag{2.40}$$

is a natural isomorphism (sometimes called a *half-braiding*), such that the diagram

$$
\begin{tikzcd}
 & X\otimes(Z\otimes Y) \arrow[r, "a^{-1}_{X,Z,Y}"] & (X\otimes Z)\otimes Y \arrow[dr, "\gamma_X\otimes \mathrm{id}_Y"] & \\
X\otimes(Y\otimes Z) \arrow[ur, "\mathrm{id}_X\otimes\gamma_Y"] \arrow[dr, "a^{-1}_{X,Y,Z}"'] & & & (Z\otimes X)\otimes Y, \\
 & (X\otimes Y)\otimes Z \arrow[r, "\gamma_{X\otimes Y}"] & Z\otimes(X\otimes Y) \arrow[ur, "a^{-1}_{Z,X,Y}"'] &
\end{tikzcd}
\tag{2.41}
$$

commutes for all $X, Y \in \mathcal{C}$.

A morphism $(Z,\gamma) \to (Z',\gamma')$ in $\mathcal{Z}(\mathcal{C})$ is a morphism $f : Z \to Z'$ in $\mathcal{C}$ such that the diagram

$$
\begin{tikzcd}
X\otimes Z \arrow[r, "\gamma_X"] \arrow[d, "\mathrm{id}_X\otimes f"] & Z\otimes X \arrow[d, "f\otimes \mathrm{id}_X"] \\
X\otimes Z' \arrow[r, "\gamma'_X"] & Z'\otimes X,
\end{tikzcd}
\tag{2.42}
$$

commutes for all $X \in \mathcal{C}$.

The Drinfeld center $\mathcal{Z}(\mathcal{C})$ has a monoidal structure given as follows. If $(Z,\gamma), (Z',\gamma') \in \mathcal{Z}(\mathcal{C})$, then

$$(Z,\gamma)\otimes(Z',\gamma') := (Z\otimes Z', \widetilde{\gamma}), \tag{2.43}$$

where $\widetilde{\gamma}_X : X\otimes(Z\otimes Z') \to (Z\otimes Z')\otimes X$ is defined by the following diagram

$$
\begin{tikzcd}
X\otimes(Z\otimes Z') \arrow[r, "a^{-1}_{X,Z,Z'}"] \arrow[d, "\widetilde{\gamma}_X"] & (X\otimes Z)\otimes Z' \arrow[r, "\gamma_X\otimes \mathrm{id}_{Z'}"] & (Z\otimes X)\otimes Z' \arrow[d, "a_{Z,X,Z'}"] \\
(Z\otimes Z')\otimes X & Z\otimes(Z'\otimes X) \arrow[l, "a^{-1}_{Z,Z',X}"] & Z\otimes(X\otimes Z'). \arrow[l, "\mathrm{id}_Z\otimes\gamma'_X"]
\end{tikzcd}
\tag{2.44}
$$

The unit object of $\mathcal{Z}(\mathcal{C})$ is $(\mathbf{1}_{\mathcal{C}}, r^{-1}\ell)$ where r and ℓ are the right and left unit constraints in $\mathcal{C}$ respectively.

The Drinfeld center $\mathcal{Z}(\mathcal{C})$ comes equipped with a braiding:

$$c_{(Z,\gamma),(Z',\gamma')} := \gamma'_Z. \tag{2.45}$$

Theorem 2.59 ([EGNO15, Theorem 7.16.6, Corollary 8.20.13, Theorem 9.3.2])**.** *Let $\mathcal{C}$ be a fusion category. Then*

(i) The center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$ is a fusion category,

(ii) $\mathrm{FPdim}(\mathcal{Z}(\mathcal{C})) = \mathrm{FPdim}(\mathcal{C})^2$*, and*

(iii) $\mathcal{Z}(\mathcal{C})$ is non-degenerate.

Algebras and Modules in a Fusion Category

The following definitions make sense in any monoidal category, but we restrict our attention to fusion categories here.

Definition 2.60. Let $\mathcal{C}$ be a fusion category. An *algebra* in $\mathcal{C}$ (sometimes called an $\mathcal{C}$-algebra) is a triple (A, m, η) with A an object in $\mathcal{C}$, multiplication morphism $m : A \otimes A \to A$, and unit morphism $\eta : \mathbf{1} \to A$, satisfying the following conditions

$$m \circ (\mathrm{id}_A \otimes m) \circ a_{A,A,A} = m \circ (m \otimes \mathrm{id}_A), \tag{2.46}$$

$$m \circ (\eta \otimes \mathrm{id}_A) = \mathrm{id}_A = m \circ (\mathrm{id}_A \otimes \eta). \tag{2.47}$$

If $\mathcal{C}$ is a braided fusion category, then the algebra A is called *commutative* if

$$m = m \circ c_{A,A}. \tag{2.48}$$

Example 2.61. A **Vec**-algebra is precisely an associative $\mathbb{C}$-algebra with unit. A commutative **Vec**-algebra is precisely an associative commutative $\mathbb{C}$-algebra with unit.

Example 2.62. If V is a finite dimensional vector space, then $A = V \otimes V^*$ has a natural algebra structure given by:

$$\begin{aligned} m : V \otimes V^* \otimes V \otimes V^* &\to V \otimes V^* \\ v \otimes f \otimes w \otimes g &\mapsto f(w)\,(v \otimes g)\,, \end{aligned} \tag{2.49}$$

for $v, w \in V$ and $f, g \in V^*$. Let $v_1, \ldots, v_n$ be a basis for V, and $\widehat{v}_1, \ldots, \widehat{v}_n$ the corresponding dual basis for V^*, then the unit element in this algebra is given by

$$\eta = \sum_{i=1}^{n} v_i \otimes \widehat{v}_i. \tag{2.50}$$

This makes $V \otimes V^*$ into a Vec-algebra.

Observe that the previous example involved the evaluation and coevaluation maps from Example 2.18. This suggests the following generalization of the previous example.

Example 2.63. Let $\mathcal{C}$ be a fusion category, and $X \in \mathcal{C}$. Define $A := X \otimes X^*$, then A is a $\mathcal{C}$-algebra with multiplication morphism

$$m : A \otimes A = X \otimes X^* \otimes X \otimes X^* \xrightarrow{\mathrm{id} \otimes \mathrm{ev}_X \otimes \mathrm{id}} X \otimes X^* \tag{2.51}$$

and unit morphism $\eta = \mathrm{coev}_X : \mathbf{1}_{\mathcal{C}} \to X \otimes X^*$.

Definition 2.64. Let $\mathcal{C}$ be a fusion category, and A a $\mathcal{C}$-algebra. A *right A-module* is a pair (M, μ), with $M \in \mathcal{C}$ and $\mu : M \otimes A \to M$ (called the *right action morphism*) such that

$$\mu \circ (\mu \otimes \mathrm{id}_M) = \mu \circ (\mathrm{id}_M \otimes m) \circ a_{M,A,A}, \tag{2.52}$$

$$\mu \circ (\mathrm{id}_M \otimes \eta) = \mathrm{id}_M. \tag{2.53}$$

Definition 2.65. Let $\mathcal{C}$ be a fusion category, and A a $\mathcal{C}$-algebra. A *left A-module* is a pair (M, μ), with $M \in \mathcal{C}$ and $\mu : A \otimes M \to M$ (called the *left action morphism*) such that

$$\mu \circ (m \otimes \mathrm{id}_M) = \mu \circ (\mathrm{id}_A \otimes \mu) \circ a_{A,A,M}, \tag{2.54}$$

$$\mu \circ (\eta \otimes \mathrm{id}_M) = \mathrm{id}_M. \tag{2.55}$$

Definition 2.66. Let $\mathcal{C}$ be a fusion category, and A a $\mathcal{C}$-algebra. We define $\mathcal{C}_A$ to be the category of *right A-modules in $\mathcal{C}$*, with morphisms being the A-module homomorphisms between them.

Example 2.67. Let $\mathcal{C}$ be a fusion category, A a $\mathcal{C}$-algebra, and $X \in \mathcal{C}$. We can construct the *free A-module* $X \otimes A$, which has right action morphism

$$\mu : (X \otimes A) \otimes A \xrightarrow{a_{X,A,A}} X \otimes (A \otimes A) \xrightarrow{\mathrm{id} \otimes m} X \otimes A. \tag{2.56}$$

The association $X \mapsto X \otimes A$ defines a functor $\mathcal{C} \to \mathcal{C}_A$ left adjoint to the forgetful functor $\mathcal{C}_A \to \mathcal{C}$ [EGNO15, Lemma 7.8.12].

Definition 2.68. Let $\mathcal{C}$ be a fusion category, and suppose A and B are $\mathcal{C}$-algebras. An (A, B)-*bimodule* in $\mathcal{C}$ is a triple (M, p, q) where M is an object in $\mathcal{C}$, $p : A \otimes M \to M$ and $q : M \otimes B \to M$ are morphisms in $\mathcal{C}$, such that (M, p) is a left A-module, (M, q) is a right B-module, and

$$p \circ (\mathrm{id}_A \otimes q) \circ a_{A,M,B} = q \circ (p \otimes \mathrm{id}_B) \tag{2.57}$$

as morphisms $(A \otimes M) \otimes B \to M$.

Example 2.69. If A is an algebra in $\mathcal{C}$ with multiplication map $m : A \otimes A \to A$, then the multiplication map $m : A \otimes A \to A$ endows A with the structure of an (A, A)-bimodule in $\mathcal{C}$.

Definition 2.70. Let $\mathcal{C}$ be a fusion category, and suppose A and B are $\mathcal{C}$-algebras. We denote by ${}_A\mathcal{C}_B$ the category of (A, B)-bimodules in $\mathcal{C}$, with morphisms being the (A, B)-bimodule homomorphisms between them.

Connected Étale Algebras. If R is a commutative ring, then the category of R-modules admits a tensor product. Similarly, if A is a commutative $\mathcal{C}$-algebra, then the category $\mathcal{C}_A$ of right A-modules can be equipped with a tensor product. There is no guarantee, however, that $\mathcal{C}_A$ will be a fusion category. In this section, we recall the definition of a connected étale algebra, a class of algebras whose category of modules is guaranteed to be a fusion category.

Definition 2.71. An algebra $A \in \mathcal{C}$ is *separable* if the multiplication morphism $m : A \otimes A \to A$ splits as a morphism of A-bimodules. If $\mathcal{C}$ is braided, then we say that an algebra $A \in \mathcal{C}$ is *étale* if it is both commutative and separable. We say an étale algebra is *connected* if $\dim \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A) = 1$.

Suppose A is a connected étale algebra in a braided fusion category $\mathcal{C}$. Given a right A-module (X, μ), the braiding on $\mathcal{C}$ allows us to define left A-module structures on X by

$$\mu^+ : A \otimes X \xrightarrow{c_{A,X}} X \otimes A \xrightarrow{\mu} X \tag{2.58}$$

$$\mu^- : A \otimes X \xrightarrow{c_{X,A}^{-1}} X \otimes A \xrightarrow{\mu} X \tag{2.59}$$

making $(X, \mu^{\pm}, \mu)$ into an A-bimodule. This defines full embeddings

$$F_{\pm} : \mathcal{C}_A \to {}_A\mathcal{C}_A. \tag{2.60}$$

The category ${}_A\mathcal{C}_A$ of A-bimodules has a tensor product, and so we obtain a tensor product on $\mathcal{C}_A$.

Theorem 2.72 ([DMNO13, §3.3 and Lemma 3.11])**.** *Let $\mathcal{C}$ be a braided fusion category, and $A \in \mathcal{C}$ a connected étale algebra. Then $\mathcal{C}_A$ is a fusion category, and*

$$\mathrm{FPdim}(\mathcal{C}_A)\mathrm{FPdim}(A) = \mathrm{FPdim}(\mathcal{C}). \tag{2.61}$$

We recall the following characterization of connected étale algebras from [DMNO13, §3].

Definition 2.73. Let $\mathcal{C}$ be a braided fusion category, and $\mathcal{A}$ a fusion category. Suppose $F : \mathcal{C} \to \mathcal{A}$ is a tensor functor, then the structure of a *central functor* on F is a braided tensor functor $F' : \mathcal{C} \to \mathcal{Z}(\mathcal{A})$ whose composition with the forgetful functor $\mathcal{Z}(\mathcal{A}) \to \mathcal{A}$ equals F.

In particular, any braided tensor functor $\mathcal{C} \to \mathcal{A}$ has the structure of a central functor. Moreover, the following lemma shows that every central functor gives rise to a connected étale algebra.

Lemma 2.74 ([DMNO13, Lemma 3.5])**.** *Let $\mathcal{C}$ be a braided fusion category, $\mathcal{A}$ a fusion category, and $F : \mathcal{C} \to \mathcal{A}$ a central functor. Let $I : \mathcal{A} \to \mathcal{C}$ be the right adjoint functor of F. Then $A = I(\mathbf{1}_{\mathcal{A}})$ has a canonical structure of a connected étale algebra.*

If $A \in \mathcal{C}$ is a connected étale algebra, then the *free module* functor $- \otimes A : \mathcal{C} \to \mathcal{C}_A$ (see Example 2.67) admits the structure of a central functor. The right adjoint functor is the forgetful functor $I : \mathcal{C}_A \to \mathcal{C}$, and $I(\mathbf{1}_{\mathcal{C}_A}) \xrightarrow{\sim} A$ as $\mathcal{C}$-algebras [DMNO13, Lemma 3.9]. Thus there is an equivalence between connected étale algebras and central functors.

Local Modules. The following notion is due to Pareigis [Par95].

Definition 2.75. Let $\mathcal{C}$ be a braided fusion category, and A a $\mathcal{C}$-algebra. Let (M, μ) be a right A-module. We say that M is a *local* A-module if

$$\mu \circ c_{A,M} \circ c_{M,A} = \mu. \tag{2.62}$$

Remark 2.76. Pareigis [Par95] refers to local modules as *dyslectic* modules.

If M is a local A-module, then the embeddings $F_{\pm} : \mathcal{C}_A \to {}_A\mathcal{C}_A$ defined using Eqs. (2.58) and (2.59) coincide.

Definition 2.77. Let $\mathcal{C}$ be a braided fusion category, and A a connected étale algebra in $\mathcal{C}$. We denote by $\mathcal{C}_A^{\mathrm{loc}}$ the category of *local A-modules in $\mathcal{C}$*.

The tensor product of A-modules in $\mathcal{C}$ preserves $\mathcal{C}_A^{\mathrm{loc}}$, and the braiding on $\mathcal{C}$ induces a braiding on $\mathcal{C}_A^{\mathrm{loc}}$ [Par95, KO02], so $\mathcal{C}_A^{\mathrm{loc}}$ is a braided fusion category.

Lemma 2.78 ([DMNO13, Corollary 3.32])**.** *Let A be a connected étale algebra in a non-degenerate fusion category $\mathcal{C}$. Then $\mathcal{C}_A^{\mathrm{loc}}$ is a non-degenerate fusion category, and*

$$\mathrm{FPdim}(\mathcal{C}_A^{\mathrm{loc}})\mathrm{FPdim}(A)^2 = \mathrm{FPdim}(\mathcal{C}). \tag{2.63}$$

Deligne Tensor Product of Abelian Categories

Definition 2.79. Let $\mathcal{C}$ be a $\mathbb{C}$-linear abelian category. We say $\mathcal{C}$ is *locally finite* if:

(i) for any $X, Y \in \mathcal{C}$, $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{C}}(X, Y) < \infty$, and

(ii) every object in $\mathcal{C}$ has finite length.

Fusion categories are locally finite. The following notion is due to Deligne [Del90].

Definition 2.80. Let $\mathcal{C}$ and $\mathcal{D}$ be two locally finite $\mathbb{C}$-linear abelian categories. The *Deligne tensor product* $\mathcal{C} \boxtimes \mathcal{D}$ is an abelian $\mathbb{C}$-linear category, together with a bifunctor:

$$\begin{gathered} \boxtimes : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \boxtimes \mathcal{D} \\ (X, Y) \mapsto X \boxtimes Y \end{gathered} \tag{2.64}$$

which is right exact in both variables, and such that if $F : \mathcal{C} \times \mathcal{D} \to \mathcal{A}$ is a right exact in both variables bifunctor, then there exists a unique right exact functor $\overline{F} : \mathcal{C} \boxtimes \mathcal{D} \to \mathcal{A}$ such that $\overline{F} \circ \boxtimes = F$.

The Deligne tensor product $\mathcal{C} \boxtimes \mathcal{D}$ as defined above always exists, and is a locally finite abelian category. If $\mathcal{C}$ and $\mathcal{D}$ are fusion categories, then $\mathcal{C} \boxtimes \mathcal{D}$ can be given a monoidal structure such that:

$$(X_1 \boxtimes Y_1) \otimes (X_2 \boxtimes Y_2) = (X_1 \otimes X_2) \boxtimes (Y_1 \otimes Y_2),\ \ X_1, X_2 \in \mathcal{C},\ \ Y_1, Y_2 \in \mathcal{D}, \tag{2.65}$$

This makes $\mathcal{C} \boxtimes \mathcal{D}$ into a fusion category [EGNO15, Corollary 4.6.2].

Module Categories

Definition 2.81. Let $\mathcal{C}$ be a monoidal category. A *left module category* over $\mathcal{C}$ is a category $\mathcal{M}$ equipped with an *action bifunctor* $\odot : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$, a natural isomorphism:

$$m_{X,Y,M} : (X \otimes Y) \odot M \xrightarrow{\sim} X \odot (Y \odot M),\ \ X, Y \in \mathcal{C},\ \ M \in \mathcal{M} \tag{2.66}$$

called the *module associativity constraint*, and a *unit isomorphism* $\ell_M : \mathbf{1}_{\mathcal{C}} \odot M \xrightarrow{\sim} M$, subject to the following axioms.

1. The pentagon axiom:

   The diagram

$$\begin{tikzcd}
 & ((X \otimes Y) \otimes Z) \odot M \arrow[dl, "a_{X,Y,Z} \odot \mathrm{id}_M"'] \arrow[dr, "m_{X \otimes Y, Z, M}"] & \\
(X \otimes (Y \otimes Z)) \odot M \arrow[d, "m_{X, Y \otimes Z, M}"] & & (X \otimes Y) \odot (Z \odot M) \arrow[d, "m_{X,Y,Z \odot M}"] \\
X \odot ((Y \otimes Z) \odot M) \arrow[rr, "\mathrm{id}_X \odot m_{Y,Z,M}"] & & X \odot (Y \odot (Z \odot M))
\end{tikzcd} \tag{2.67}$$

   is commutative for all $X, Y, Z \in \mathcal{C}$ and $M \in \mathcal{M}$.

2. The triangle axiom:

The diagram

$$\begin{tikzcd}
(X \otimes \mathbf{1}_{\mathcal{C}}) \odot M \arrow[rr, "m_{X,\mathbf{1}_{\mathcal{C}},M}"] \arrow[dr, "r_X \otimes \mathrm{id}_M"'] & & X \odot (\mathbf{1}_{\mathcal{C}} \odot M) \arrow[dl, "\mathrm{id}_X \otimes \ell_M"] \\
& X \odot M &
\end{tikzcd} \tag{2.68}$$

is commutative for all $X \in \mathcal{C}$ and $M \in \mathcal{M}$.

Remark 2.82. A *right* $\mathcal{C}$*-module category* is a left $\mathcal{C}^{\mathrm{rev}}$-module category, where $\mathcal{C}^{\mathrm{rev}}$ is the category $\mathcal{C}$ with reversed tensor product.

Definition 2.83. Let $\mathcal{M}$ and $\mathcal{N}$ be two module categories over $\mathcal{C}$. A $\mathcal{C}$*-module functor* from $\mathcal{M}$ to $\mathcal{N}$ consists of a functor $F : \mathcal{M} \to \mathcal{N}$ and a natural isomorphism:

$$s_{X,M} : F(X \odot M) \to X \odot F(M),\ X \in \mathcal{C}, M \in \mathcal{M}, \tag{2.69}$$

such that the diagrams

$$\begin{tikzcd}
& F((X \otimes Y) \odot M) \arrow[dl, "F(m_{X,Y,M})"'] \arrow[dr, "s_{X \otimes Y,M}"] & \\
F(X \odot (Y \odot M)) \arrow[d, "s_{X,Y \odot M}"] & & (X \otimes Y) \odot F(M) \arrow[d, "n_{X,Y,F(M)}"] \\
X \odot F(Y \odot M) \arrow[rr, "\mathrm{id}_X \odot s_{Y,M}"] & & X \odot (Y \odot F(M))
\end{tikzcd} \tag{2.70}$$

and

$$\begin{tikzcd}
F(\mathbf{1}_{\mathcal{C}} \odot M) \arrow[rr, "s_{\mathbf{1}_{\mathcal{C}},M}"] \arrow[dr, "F(\ell_M)"'] & & \mathbf{1}_{\mathcal{C}} \odot F(M) \arrow[dl, "\ell_{F(M)}"] \\
& F(M) &
\end{tikzcd} \tag{2.71}$$

commute for all $X, Y \in \mathcal{C}$ and $M \in \mathcal{M}$.

We will always assume that our module categories are semisimple locally finite abelian categories over $\mathbb{C}$, and that all module functors are $\mathbb{C}$-linear, unless otherwise stated.

Definition 2.84. Let $\mathcal{C}$, $\mathcal{D}$ be two fusion categories. A $(\mathcal{C},\mathcal{D})$-*bimodule category* is a category $\mathcal{M}$ that has left $\mathcal{C}$-module and right $\mathcal{D}$-module category structures with associativity constraints $m_{X,Y,M} : (X \otimes Y) \odot M \xrightarrow{\sim} X \odot (Y \odot M)$ and $n_{M,W,Z} : M \odot (W \otimes Z) \xrightarrow{\sim} (M \odot W) \odot Z$ respectively, together with a collection of natural isomorphisms $b_{X,M,Z} : (X \odot M) \odot Z \xrightarrow{\sim} X \odot (M \odot Z)$ such that the diagrams:

$$\begin{tikzcd}
& ((X \otimes Y) \odot M) \odot Z \arrow[dl, "m_{X,Y,M} \otimes \mathrm{id}_Z"'] \arrow[dr, "b_{X\otimes Y,M,Z}"] & \\
(X \odot (Y \odot M)) \odot Z \arrow[d, "b_{X,Y\odot M,Z}"] & & (X \otimes Y) \odot (M \odot Z) \arrow[d, "m_{X,Y,M\odot Z}"] \\
X \odot ((Y \odot M) \odot Z) \arrow[rr, "\mathrm{id}_X \otimes b_{Y,M,Z}"] & & X \odot (Y \odot (M \odot Z))
\end{tikzcd} \tag{2.72}$$

and

$$\begin{tikzcd}
& X \odot (M \odot (W \otimes Z)) \arrow[dl, "\mathrm{id}_X \otimes n_{M,W,Z}"'] & \\
X \odot ((M \odot W) \odot Z) & & (X \odot M) \odot (W \otimes Z) \arrow[ul, "b_{X,M,W\otimes Z}"'] \arrow[d, "n_{X\odot M,W,Z}"] \\
(X \odot (M \odot W)) \odot Z \arrow[u, "b_{X,M\odot W,Z}"] & & ((X \odot M) \odot W) \odot Z \arrow[ll, "b_{X,M,W} \otimes \mathrm{id}_Z"]
\end{tikzcd} \tag{2.73}$$

commute for all $X, Y \in \mathcal{C}$, $W, Z \in \mathcal{D}$, and $M \in \mathcal{M}$.

Remark 2.85. Equivalently, a $(\mathcal{C},\mathcal{D})$-bimodule is a module category over $\mathcal{C} \boxtimes \mathcal{D}^{\mathrm{rev}}$.

The tensor product of module categories was described in [ENO10].

Definition 2.86. Let $\mathcal{M}$ be a right $\mathcal{C}$-module category, $\mathcal{N}$ be a left $\mathcal{C}$-module category, and $\mathcal{A}$ a semisimple abelian category. Suppose $F : \mathcal{M} \times \mathcal{N} \to \mathcal{A}$ is a bifunctor additive in every argument. We say that F is $\mathcal{C}$-*balanced* if there is a natural family of isomorphisms:

$$b_{M,X,N} : F(M \odot X, N) \xrightarrow{\sim} F(M, X \odot N), \tag{2.74}$$

such that the diagram

$$\begin{tikzcd}
F(M \odot (X \otimes Y), N) \arrow[rr, "F(m_{M,X,Y}, \mathrm{id}_N)"] \arrow[d, "b_{M,X\otimes Y,N}"] & & F((M \odot X) \odot Y, N) \arrow[d, "b_{M\odot X,Y,N}"] \\
F(M, (X \otimes Y) \odot N) \arrow[dr, "F(\mathrm{id}_M, n_{X,Y,N})"] & & F(M \odot X, Y \odot N) \arrow[dl, "b_{M,X,Y\odot N}"] \\
& F(M, X \odot (Y \odot N)) &
\end{tikzcd} \tag{2.75}$$

commutes for all $M \in \mathcal{M}$, $N \in \mathcal{N}$, and $X, Y \in \mathcal{C}$.

Definition 2.87 ([ENO10, Definition 3.3])**.** A *tensor product* of a right $\mathcal{C}$-module category $\mathcal{M}$ and a left $\mathcal{C}$-module category $\mathcal{N}$ is an abelian category $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$, together with a $\mathcal{C}$-balanced functor:

$$B_{M,N} : \mathcal{M} \times \mathcal{N} \to \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \tag{2.76}$$

such that if $F : \mathcal{M} \times \mathcal{N} \to \mathcal{A}$ is a $\mathcal{C}$-balanced functor with $\mathcal{A}$ an abelian category, then there exists a unique additive functor $\overline{F} : \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \to \mathcal{A}$ such that $\overline{F} \circ B = F$.

In [ENO10, §3.2] (see also [Gre10]), it is shown that the tensor product of module categories exists.

Remark 2.88. If $\mathcal{M}$ is a $(\mathcal{A}, \mathcal{C})$-bimodule category, and $\mathcal{N}$ is a $(\mathcal{C}, \mathcal{D})$-bimodule category, then $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ is a $(\mathcal{A}, \mathcal{D})$-bimodule category. It is universal for $\mathcal{C}$-balanced $(\mathcal{A}, \mathcal{D})$-bimodule functors $\mathcal{M} \times \mathcal{N} \to \mathcal{A}$.

Group Actions on Categories

Definition 2.89 (see e.g. [EGNO15, §2.7])**.** Let G be a finite group. We denote by $\underline{G}$ the monoidal category whose objects are the elements of G, the only morphisms are the identity homomorphism, and the tensor product is given by multiplication in G.

Definition 2.90. Let $\mathcal{C}$ be a fusion category. Let $\underline{\mathrm{Aut}}_{\otimes}(\mathcal{C})$ denote the monoidal category of $\mathbb{C}$-linear monoidal autoequivalences of $\mathcal{C}$. An *action* of G on a fusion category $\mathcal{C}$ is a monoidal functor $\varrho : \underline{G} \to \underline{\mathrm{Aut}}_{\otimes}(\mathcal{C})$.

Definition 2.91. Let $\mathcal{B}$ be a braided fusion category. Let $\underline{\mathrm{Aut}}^{\mathrm{br}}_{\otimes}(\mathcal{B})$ denote the monoidal category of $\mathbb{C}$-linear braided autoequivalences of $\mathcal{B}$. A *braided action* of G on a braided fusion category $\mathcal{B}$ is a monoidal functor $\varrho : \underline{G} \to \underline{\mathrm{Aut}}^{\mathrm{br}}_{\otimes}(\mathcal{B})$.

Given an action of G on $\mathcal{C}$, let $T_g = \varrho(g)$ for $g \in G$, then the monoidal structure on ϱ gives an isomorphism $\gamma_{g,h} : T_g \circ T_h \xrightarrow{\sim} T_{gh}$ for $g, h \in G$.

Definition 2.92. A *G-equivariant object* in $\mathcal{C}$ is a pair (X, u), consisting of an object $X \in \mathcal{C}$ and a family of isomorphisms $u = \{u_g : T_g(X) \xrightarrow{\sim} \mid g \in G\}$, such that the diagram

$$\begin{tikzcd} T_g(T_h(X)) \arrow[r, "T_g(u_h)"] \arrow[d, "\gamma_{g,h}(X)"] & T_g(X) \arrow[d, "u_g"] \\ T_{gh}(X) \arrow[r, "u_{gh}"] & X \end{tikzcd} \tag{2.77}$$

commutes for all $g, h \in G$.

Definition 2.93. A *G-equivariant morphism* $(X, u) \to (Y, v)$ is a morphism $X \to Y$ in $\mathcal{C}$ such that that the diagram

$$\begin{tikzcd} T_g(X) \arrow[r, "T_g(f)"] \arrow[d, "u_g"] & T_g(Y) \arrow[d, "v_g"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{2.78}$$

commutes for all $g \in G$.

Definition 2.94. If $\mathcal{C}$ is a fusion category with a G-action $\varrho : \underline{G} \to \underline{\mathrm{Aut}}_{\otimes}(\mathcal{C})$, then we can form the category $\mathcal{C}^G$ of *G-equivariant objects* in $\mathcal{C}$, with morphisms being the G-equivariant maps. There is a monoidal structure on $\mathcal{C}^G$: the tensor product of (X, u) with (Y, v) is $X \otimes Y$ with G-equivariant structure given by the composition

$$T_g(X \otimes Y) \xrightarrow{J^g_{X,Y}} T_g(X) \otimes T_g(Y) \xrightarrow{u_g \otimes v_g} X \otimes Y \tag{2.79}$$

where $J^g_{X,Y}$ is the monoidal structure on T_g. This makes $\mathcal{C}^G$ into a fusion category.

Example 2.95. Any group G has a unique action on Vec, the trivial action. A G-equivariant object in Vec is therefore a vector space V, together with a collection of automorphisms $u_g : V \xrightarrow{\sim} V$ for $g \in G$ satisfying Eq. (2.77). In particular, this endows V with a G-action, and so $\mathrm{Vec}^G \xrightarrow{\sim} \mathrm{Rep}(G)$.

Observe that if G acts on a fusion category $\mathcal{C}$, then we have an embedding $\mathrm{Rep}(G) \xrightarrow{\sim} \mathrm{Vec}^G \hookrightarrow \mathcal{C}^G$. The process of taking $\mathcal{C}$ to the category $\mathcal{C}^G$ is known as *equivariantization.*

Definition 2.96. Let $\mathcal{C}$ be a fusion category, and G be a group. A *grading* of $\mathcal{C}$ by G is a decomposition:

$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g, \tag{2.80}$$

where $\mathcal{C}_g \subset \mathcal{C}$ are abelian subcategories, such that the tensor product maps $\mathcal{C}_g \times \mathcal{C}_h$ to $\mathcal{C}_{gh}$. The subcategory $\mathcal{C}_1$ is monoidal, and we call it the *trivial component* of the grading. We say the grading is *faithful* if $\mathcal{C}_g \neq 0$ for all $g \in G$.

The following notion is due to Turaev [Tur00].

Definition 2.97. A *braided G-crossed fusion category* is a fusion category $\mathcal{C}$ equipped with the following structures:

(i) a (not necessarily faithful) grading $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$,

(ii) an action $g \mapsto T_g$ of G on $\mathcal{C}$ such that $T_g(\mathcal{C}_h) \subset \mathcal{C}_{ghg^{-1}}$, and

(iii) a natural collection of isomorphisms, called the *G-braiding*:

$$c_{X,Y} : X \otimes Y \xrightarrow{\sim} T_g(Y) \otimes X, \;\; X \in \mathcal{C}_g, \;\; g \in G \text{ and } Y \in \mathcal{C}. \tag{2.81}$$

satisfying some compatibility conditions which we omit here, see e.g. [EGNO15, Definition 8.24.1(a)-(c)].

Remark 2.98. The trivial component $\mathcal{C}_1$ of a braided G-crossed fusion category is a braided fusion category with a braided G-action.

Let $\mathcal{B}$ be a braided fusion category containing $\mathrm{Rep}(G)$ as a symmetric fusion subcategory. Let $A = \mathrm{Fun}(G) \in \mathrm{Rep}(G) \subset \mathcal{B}$, and consider the category $\mathcal{B}_A$ of right A-modules in $\mathcal{B}$, then $\mathcal{B}_A$ is called the *de-equivariantization* of $\mathcal{B}$. The following theorem is due to [Kir01] and [Müg04a], though we use the statement of [EGNO15, Theorem 8.24.3].

Theorem 2.99. *The equivariantization and de-equivariantization constructions establish a bijection between the set of equivalences classes of braided G-crossed fusion categories and the set of equivalence classes of braided fusion categories containing* $\mathrm{Rep}(G)$ *as a symmetric fusion subcategory.*

Given a braided fusion category $\mathcal{B}$ containing $\mathrm{Rep}(G)$, the de-equivariantization $\mathcal{B}_A$ is a braided G-crossed fusion category such that $(\mathcal{B}_A)^G \xrightarrow{\sim} \mathcal{B}$. We have the following useful description of the trivial component of $\mathcal{B}_A$.

Proposition 2.100 ([Müg04b])**.** *Let $\mathcal{B}$ be a braided fusion category containing* $\mathrm{Rep}(G)$ *as a symmetric fusion subcategory. Then $(\mathcal{B}_A)_1$ is the full subcategory of local A-modules in $\mathcal{B}$, i.e. $(\mathcal{B}_A)_1 = \mathcal{B}_A^{\mathrm{loc}}$.*

The following property of de-equivariantization will be useful.

Proposition 2.101 ([DGNO10, Proposition 4.56(ii)]). *Let $\mathcal{B}$ be a braided fusion category containing* $\mathrm{Rep}(G)$ *as a symmetric fusion subcategory. Then $\mathcal{B}$ is non-degenerate if and only if $(\mathcal{B}_A)_1$ is non-degenerate and the grading on $\mathcal{B}_A$ is faithful.*

CHAPTER III

FERMIONIC $6J$-SYMBOLS IN SUPERFUSION CATEGORIES

Chapter III appeared in [Ush18].

In this chapter, we recall the definitions of the $6j$-symbols of a fusion category, and the fermionic $6j$-symbols of a superfusion category. Using a construction of Brundan and Ellis [BE17], one can describe the underlying fusion category of a superfusion category. The main goal of this chapter is to derive an explicit formula for the $6j$-symbols of the underlying fusion category in terms of the fermionic $6j$-symbols of the original superfusion category. Using our formula, we also investigate the special case where our superfusion category is pointed.

$6j$-symbols in Fusion Categories

We begin by describing how the associator $a : (- \otimes -) \otimes - \xrightarrow{\sim} - \otimes (- \otimes -)$ in a fusion category can be described in terms of *$6j$-symbols*, closely following the discussion in [Wan10, Chapter 4], see also [Tur94, Chapter VI].

We begin by introducing some notation. Let $\mathcal{A}$ be a fusion category, and X_i, $i \in I$ a complete set of representatives of the isomorphism classes of simple objects in $\mathcal{A}$. The monoidal structure on $\mathcal{A}$ determines the *fusion rules* of $\mathcal{A}$:

$$X_i \otimes X_j \simeq \bigoplus_{m \in I} N_m^{ij} X_m, \tag{3.1}$$

where

$$N_m^{ij} = [X_i \otimes X_j : X_m] = \dim \operatorname{Hom}_{\mathcal{A}}(X_m, X_i \otimes X_j) = \dim \operatorname{Hom}_{\mathcal{A}}(X_i \otimes X_j, X_m). \tag{3.2}$$

In other words, N_m^{ij} is the multiplicity (see Definition 2.3) of X_m in $X_i \otimes X_j$.

The notion of admissibility will be useful.

Definition 3.1 (see [Wan10, Definition 4.1])**.** Let $\mathcal{A}$ be a fusion category with isomorphism classes of simple objects indexed by a set I. We say a quadruple

$(i, j, m, \alpha) \in I^3 \times \mathbb{Z}_{\geq 0}$ is *admissible* if $1 \leq \alpha \leq N_m^{ij}$. A decuple $(i, j, m, k, n, t, \alpha, \beta, \eta, \varphi) \in I^6 \times \mathbb{Z}_{\geq 0}^4$ is *admissible* if each of the quadruples (i, j, m, α), (m, k, n, β), (j, k, t, η) and (i, t, n, φ) are admissible.

Unpacking this definition, that a quadruple $(i, j, m, \alpha) \in I^3 \times \mathbb{Z}_{\geq 0}$ is admissible means that the simple object X_m occurs in the tensor product $X_i \otimes X_j$ with multiplicity at least α. Put another way, if X_m occurs in $X_i \otimes X_j$, then the set of admissible triples of the form $\{(i, j, m, \alpha) \mid 1 \leq \alpha \leq N_m^{ij}\}$ label the occurrences of X_m in $X_i \otimes X_j$.

Remark 3.2. A fusion category is called *multiplicity-free* if $N_m^{ij} \in \{0, 1\}$ for all $i, j, m \in I$ [Wan10, Definition 4.5]. In the multiplicity-free case, an admissible decuple is completely described by the sextuple (i, j, m, k, n, t), in which case this definition recovers [Wan10, Definition 4.7].

Example 3.3. Let A be a finite abelian group, then the category $\mathrm{Rep}(A)$ of finite-dimensional representations of A is a multiplicity-free category. In fact, any pointed fusion category is multiplicity-free.

For each $i, j, m \in I$, choose basis vectors $\mathbf{e}_m^{ij}(\alpha)$ $(1 \leq \alpha \leq N_m^{ij})$ for the space $\mathrm{Hom}_{\mathcal{A}}(X_i \otimes X_j, X_m)$. Given this choice, an admissible quadruple (i, j, m, α) corresponds to the basic vector $\mathbf{e}_m^{ij}(\alpha)$. We wish to describe the associator

$$a(X_i, X_j, X_k) : (X_i \otimes X_j) \otimes X_k \to X_i \otimes (X_j \otimes X_k) \tag{3.3}$$

in terms of our chosen basis. Indeed, fixing admissible quadruples (i, j, m, α) and (m, k, n, β), we can form the composition

$$(X_i \otimes X_j) \otimes X_k \xrightarrow{\mathbf{e}_m^{ij}(\alpha) \otimes \mathrm{id}_{X_k}} X_m \otimes X_k \xrightarrow{\mathbf{e}_n^{mk}(\beta)} X_n. \tag{3.4}$$

We will represent this composition graphically by Fig. 1.

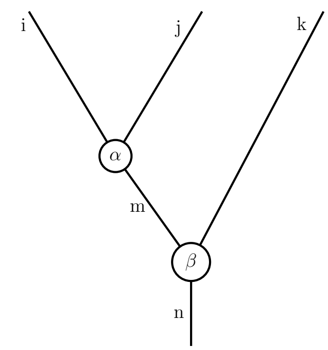


Figure 1. Graphical representation of the composition (3.4).

Let $t \in I$. If (j, k, t, η) and (i, t, n, φ) are admissible, then we have the composition

$$(X_i \otimes X_j) \otimes X_k \xrightarrow{a(X_i,X_j,X_k)} X_i \otimes (X_j \otimes X_k) \xrightarrow{\mathrm{id}_{X_i} \otimes \mathbf{e}_t^{jk}(\eta)} X_i \otimes X_t \xrightarrow{\mathbf{e}_n^{it}(\varphi)} X_n. \quad (3.5)$$

We will represent this composition graphically by Fig. 2.

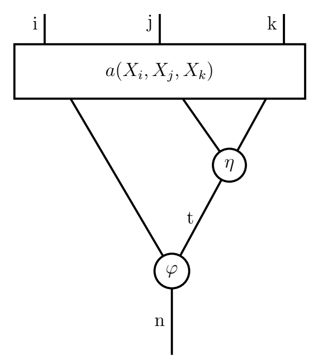
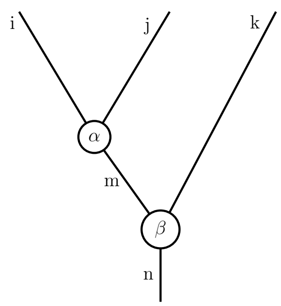


Figure 2. Graphical representation of the composition (3.5).

Fix $i, j, k, n \in I$. Taking the direct sum of the above compositions over all $t \in I$ such that (j, k, t, η) and (i, t, n, φ) are admissible gives an isomorphism [Tur94, Lemma 1.1.1, Lemma 1.1.2]

$$\bigoplus_{t\in I} \mathrm{Hom}_{\mathcal{A}}(X_j \otimes X_k, X_t) \otimes \mathrm{Hom}_{\mathcal{A}}(X_i \otimes X_t, X_n) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}((X_i \otimes X_j) \otimes X_k, X_n)$$
$$\mathbf{e}_t^{jk}(\eta) \otimes \mathbf{e}_n^{it}(\varphi) \mapsto \mathbf{e}_n^{it}(\varphi) \circ (\mathrm{id}_{X_i} \otimes \mathbf{e}_t^{jk}(\eta)) \circ a(X_i, X_j, X_k) \tag{3.6}$$

Expressing (3.4) in terms of this basis determines a constant $F^{ijm,\alpha\beta}_{knt,\eta\varphi} \in \mathbb{C}$ for each admissible decuple $(i, j, m, k, n, t, \alpha, \beta, \eta, \varphi)$ in $\mathcal{A}$, defined by the graphical equation in Fig. 3.



$$= \sum_{t\in I} \sum_{\eta=1}^{N_t^{jk}} \sum_{\varphi=1}^{N_n^{it}} F^{ijm,\alpha\beta}_{knt,\eta\varphi}$$

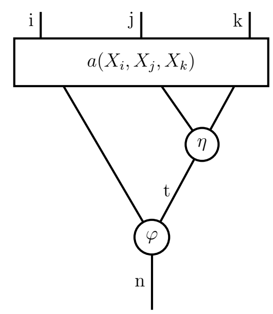


Figure 3. Graphical definition of $6j$-symbols.

This describes the associator in $\mathcal{A}$ as a collection of matrices

$$\begin{aligned} F^{ijm}_{knt} &: \mathrm{Hom}_{\mathcal{A}}(X_i \otimes X_j, X_m) \otimes \mathrm{Hom}_{\mathcal{A}}(X_m \otimes X_k, X_n) \\ &\to \mathrm{Hom}_{\mathcal{A}}(X_j \otimes X_k, X_t) \otimes \mathrm{Hom}_{\mathcal{A}}(X_i \otimes X_t, X_n) \end{aligned} \tag{3.7}$$

whose entries are the constants defined above. The matrices F^{ijm}_{knt} are called *6j-symbols*, as they depend on six indices. If $(i, j, m, k, n, t, \alpha, \beta, \eta, \varphi)$ is not admissible, then by convention we set $F^{ijm,\alpha\beta}_{knt,\eta\varphi} = 0$. The pentagon axiom in $\mathcal{A}$ is then equivalent to the following equation in terms of $6j$-symbols.

Lemma 3.4 (Pentagon equation)**.** *Let $\mathcal{A}$ be a fusion category with simple objects indexed by a set I. For each $i, j, m \in I$, let $N^{ij}_m = \dim \mathrm{Hom}_{\mathcal{A}}(X_i \otimes X_j, X_m)$, and choose basis vectors $\mathbf{e}^{ij}_m(\alpha)$ ($1 \le \alpha \le N^{ij}_m$) for* $\mathrm{Hom}_{\mathcal{A}}(X_i \otimes X_j, X_m)$*. Given these* *choices, if* $i, j, k, l, m, n, t, p, q, s \in I$ *and* $\alpha, \beta, \eta, \chi, \gamma, \delta, \phi \in \mathbb{Z}_{\geq 0}$*, then*

$$\sum_{t \in I} \sum_{\eta=1}^{N_t^{jk}} \sum_{\varphi=1}^{N_n^{it}} \sum_{\kappa=1}^{N_s^{tl}} F_{knt,\eta\varphi}^{ijm,\alpha\beta} F_{lps,\kappa\gamma}^{itn,\varphi\chi} F_{lsq,\delta\phi}^{jkt,\eta\kappa} = \sum_{\epsilon=1}^{N_p^{mq}} F_{lpq,\delta\epsilon}^{mkn,\beta\chi} F_{qps,\delta\gamma}^{ijm,\alpha\epsilon} \tag{3.8}$$

Example 3.5 (see [EGNO15, Example 2.3.8])**.** Let G be a finite group, and $\omega : G \times G \times G \to \mathbb{C}^\times$ a 3-cocycle on G (see Example 2.28). Recall that the fusion category Vec_G^ω has pairwise non-isomorphic simple objects $\mathbb{C}_g$, $g \in G$, satisfying $\mathbb{C}_g \otimes \mathbb{C}_h \xrightarrow{\sim} \mathbb{C}_{gh}$. The admissible quadruples in this category are of the form $(g, h, gh, 1)$ for all $g, h \in G$. Thus given $g, h, k \in G$, we can write

$$F(g, h, k) := F_k^{gh} \in \mathbb{C}^\times \tag{3.9}$$

for the corresponding $6j$-symbol unambiguously. The pentagon equation (3.8) then reduces to

$$F(g, h, k)F(g, hk, l)F(h, k, l) = F(gh, k, l)F(g, h, kl) \quad g, h, k, l \in G \tag{3.10}$$

i.e. F is a 3-cocycle on G with values in $\mathbb{C}^\times$ (see Eq. (2.22)).

6*j*-symbols in Superfusion Categories

Superfusion Categories. In this section, we will recall the definition of a superfusion category using the language of [BE17], and describe the associator in a superfusion category in terms of so-called *fermionic 6j-symbols*, following [GK16]. We begin by describing some basics of super linear algebra.

Definition 3.6. A *superspace* is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-vector space V. The parity of a homogeneous element $v \in V$ will be denoted by $|v|$.

Definition 3.7. Let <u>**sVec**</u> be the category whose objects are superspaces, and whose morphisms are even linear maps, i.e. linear maps preserving the grading.

We can make $\underline{\mathbf{sVec}}$ into a monoidal category by defining the tensor product of superspaces V and W to be the space $V \otimes W$ with grading

$$\begin{aligned}(V \otimes W)_0 &:= (V_0 \otimes W_0) \oplus (V_1 \otimes W_1)\\ (V \otimes W)_1 &:= (V_1 \otimes W_0) \oplus (V_0 \otimes W_1),\end{aligned} \tag{3.11}$$

with the tensor product of morphisms defined in the obvious way.

The braiding

$$c_{V,W}(v \otimes w) = (-1)^{|v||w|} w \otimes v$$

defined on homogeneous $v \in V$ and $w \in W$ makes $\underline{\mathbf{sVec}}$ into a symmetric monoidal category. We denote by sVec $\subset$ $\underline{\mathbf{sVec}}$ the full monoidal subcategory of finite-dimensional superspaces.

Definition 3.8 (see [BE17, Definition 1.1] and [Kel05, Section 1.2] for details)**.** A *supercategory* is a $\underline{\mathbf{sVec}}$-enriched category. A *superfunctor* between supercategories is a $\underline{\mathbf{sVec}}$-enriched functor. A *supernatural transformation* $\beta : F \Rightarrow G$ between superfunctors $F, G : \mathcal{A} \to \mathcal{B}$ is a collection of morphisms $\beta_X : F(X) \to G(X)$ satisfying a supernaturality condition, see [BE17, Definition 1.1.(iii)] for details. We say that a supernatural transformation is *even* if all its component maps are even.

In particular, if $\mathcal{A}$ is a supercategory, then $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is a superspace for all X, $Y \in \mathcal{A}$, and composition

$$\mathrm{Hom}_{\mathcal{A}}(Z, Y) \otimes \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}}(X, Z) \tag{3.12}$$

is an *even* linear map for all $X, Y, Z \in \mathcal{A}$.

Definition 3.9. A superfunctor $F : \mathcal{A} \to \mathcal{B}$ is a *superequivalence* if there is a superfunctor $G : \mathcal{B} \to \mathcal{A}$ such that $F \circ G$ and $G \circ F$ are isomorphic to $\mathrm{id}_{\mathcal{B}}$ (respectively $\mathrm{id}_{\mathcal{A}}$) by even supernatural transformations. As observed in [BE17, Definition 1.1(iv)], a superfunctor $F : \mathcal{A} \to \mathcal{B}$ is a superequivalence if and only if it is full, faithful, and evenly dense, that is, every object of $\mathcal{B}$ is evenly isomorphic to an object in the image of F.

Given supercategories $\mathcal{A}$ and $\mathcal{B}$, we can form their tensor product $\mathcal{A} \boxtimes \mathcal{B}$. Objects of $\mathcal{A} \boxtimes \mathcal{B}$ are pairs (X, Y) with $X \in \mathcal{A}$ and $Y \in \mathcal{B}$. Morphisms in $\mathcal{A} \boxtimes \mathcal{B}$ are given by $\mathrm{Hom}_{\mathcal{A} \boxtimes \mathcal{B}}((X, Y), (W, Z)) := \mathrm{Hom}_{\mathcal{A}}(X, W) \otimes \mathrm{Hom}_{\mathcal{B}}(Y, Z)$, with composition in $\mathcal{A} \boxtimes \mathcal{B}$ defined using the braiding in $\underline{\mathbf{sVec}}$, see [BE17] for details.

Definition 3.10 ([BE17, Definition 1.4])**.** A *monoidal supercategory* is a supercategory $\mathcal{D}$, together with a tensor product superfunctor $- \otimes - : \mathcal{D} \boxtimes \mathcal{D} \to \mathcal{D}$, a unit object $\mathbf{1}_{\mathcal{D}}$, and even supernatural isomorphisms $a : (- \otimes -) \otimes - \xrightarrow{\sim} - \otimes (- \otimes -)$, $l : \mathbf{1}_{\mathcal{D}} \otimes - \xrightarrow{\sim} -$ and $r : - \otimes \mathbf{1}_{\mathcal{D}} \xrightarrow{\sim} -$ satisfying axioms analogous to the ones of a monoidal category. A *monoidal superfunctor* between monoidal supercategories $\mathcal{D}$ and $\mathcal{E}$ is a superfunctor $F : \mathcal{D} \to \mathcal{E}$ such that $F(\mathbf{1}_{\mathcal{D}})$ is evenly isomorphic to $\mathbf{1}_{\mathcal{E}}$, together with even coherence maps $J : F(-) \otimes F(-) \to F(- \otimes -)$ satisfying the usual axioms. A monoidal superfunctor $F : \mathcal{D} \to \mathcal{E}$ is said to be a *monoidal superequivalence* if it is a superequivalence of supercategories.

An important feature of monoidal supercategories is the *super interchange law*

$$(f \otimes g) \circ (h \otimes k) = (-1)^{|g||h|} (f \circ h) \otimes (g \circ k) \tag{3.13}$$

describing the composition of tensor products of morphisms. We recall the following definitions from [GWW15, Appendix C], which should be compared to the definition of a (non-super) fusion category (see Definition 2.30).

Definition 3.11. A *superfusion category* over $\mathbb{C}$ is a semisimple rigid monoidal supercategory $\mathcal{C}$ with finitely many isomorphism classes of simple objects, finite dimensional superspaces of morphisms, and with simple unit object.

A simple object $X \in \mathcal{C}$ is *bosonic* if $\mathrm{End}_{\mathcal{C}}(X) \simeq \mathbb{C}^{1|0}$, and *Majorana* if $\mathrm{End}_{\mathcal{C}}(X) \simeq \mathbb{C}^{1|1}$. A superfusion category is called *bosonic* if all its simple objects are bosonic.

We say two superfusion categories are *superequivalent* if there is a monoidal superequivalence between them, and we say that two superfusion categories are *equivalent* if there is a monoidal superfunctor between them which is an equivalence of abstract categories.

We will later see examples of superfusion categories that are equivalent but not superequivalent, and so care must be taken to distinguish the two notions.

Remark 3.12. The superalgebra version of Wedderburn's theorem says that there are two families of finite-dimensional simple $\mathbb{C}$-superalgebras, namely $M_{n,m}$ and Q_n, see e.g. [Kle05, Theorem 12.2.9] for details. Consequently if V is a finite dimensional simple supermodule over a $\mathbb{C}$-superalgebra A, then $\mathrm{End}_A(V)$ is isomorphic to either $\mathbb{C}^{1|0}$ or $\mathbb{C}^{1|1}$, in which case V is said to be of type M or of type Q, respectively. In the literature, one also finds the terminology type M or type Q used in place of the language bosonic or Majorana adopted here.

Given a superfusion category $\mathcal{C}$, the unit object $\mathbf{1}_{\mathcal{C}}$ is always bosonic. Indeed, since $\mathbf{1}_{\mathcal{C}} \otimes \mathbf{1}_{\mathcal{C}} \simeq \mathbf{1}_{\mathcal{C}}$, the tensor product functor induces an even embedding

$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}}, \mathbf{1}_{\mathcal{C}}) \otimes \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}}, \mathbf{1}_{\mathcal{C}}) \to \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}} \otimes \mathbf{1}_{\mathcal{C}}, \mathbf{1}_{\mathcal{C}} \otimes \mathbf{1}_{\mathcal{C}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}}, \mathbf{1}_{\mathcal{C}}) \quad (3.14)$$

which implies $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}}, \mathbf{1}_{\mathcal{C}}) \simeq \mathbb{C}^{1|0}$.

Remark 3.13. Let $\mathcal{C}$ be a superfusion category. The hypothesis that $\mathcal{C}$ is rigid means that for each $X \in \mathcal{C}$ we have a left dual $X^* \in \mathcal{C}$ and a right dual ${}^*X \in \mathcal{C}$, together with even morphisms $\mathrm{ev}_X : X^* \otimes X \to \mathbf{1}_{\mathcal{C}}$, $\mathrm{coev}_X : \mathbf{1}_{\mathcal{C}} \to X \otimes X^*$,

$\mathrm{ev}'_X : X \otimes^* X \to \mathbf{1}_{\mathcal{C}}$, and $\mathrm{coev}'_X : \mathbf{1}_{\mathcal{C}} \to^* X \otimes X$ satisfying the usual equations, see [EGNO15, Section 2.10] for details.

Fermionic 6j-symbols. In this section we develop a notion of 6j-symbols for superfusion categories that parallels the notion of 6j-symbols for fusion categories. Proceeding as before, let $\mathcal{C}$ be a superfusion category, and X_i, $i \in I$ a complete set of representatives of the isomorphism classes of simple objects in $\mathcal{C}$. The monoidal structure on $\mathcal{C}$ determines the *superfusion rules*

$$X_i \otimes X_j \simeq \bigoplus_{m \in I} N_m^{ij} X_m \tag{3.15}$$

where

$$N_m^{ij} = \dim \mathrm{Hom}_{\mathcal{C}}(X_i \otimes X_j, X_m) = \dim \mathrm{Hom}_{\mathcal{C}}(X_m, X_i \otimes X_j) \in \mathbb{Z}_{\geq 0} \tag{3.16}$$

i.e. N_m^{ij} is the (ordinary vector space) dimension of the superspace $\mathrm{Hom}_{\mathcal{C}}(X_i \otimes X_j, X_m)$. With this notation, our notion of admissible quadruple and decuple remain the same as in Definition 3.1. As in the fusion category case, for each $i, j, m \in I$ we choose *homogeneous* basis vectors $\mathbf{e}_m^{ij}(\alpha)$ ($1 \leq \alpha \leq N_m^{ij}$) for the superspace $\mathrm{Hom}_{\mathcal{C}}(X_i \otimes X_j, X_m)$. Let $s_m^{ij}(\alpha) = |\mathbf{e}_m^{ij}(\alpha)|$ denote the parity of the corresponding basis vector.

Definition 3.14. We say that an admissible decuple $(i, j, m, k, n, t, \alpha, \beta, \eta, \varphi)$ is *parity admissible* if

$$s_m^{ij}(\alpha) + s_n^{mk}(\beta) = s_t^{jk}(\eta) + s_n^{it}(\varphi). \tag{3.17}$$

In exactly the same way as in the fusion category case, we have constants $\widetilde{F}^{ijm,\alpha\beta}_{knt,\eta\varphi} \in \mathbb{C}$ for each admissible decuple $(i, j, m, k, n, t, \alpha, \beta, \eta, \varphi)$ in $\mathcal{C}$, defined by the graphical equation in Fig. 4.

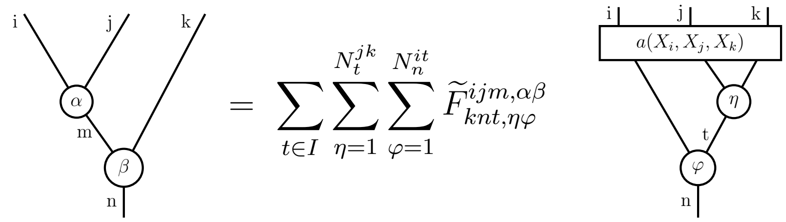


Figure 4. Graphical definition of fermionic $6j$-symbols.

Remark 3.15. We recover the parity admissibility condition (3.17) by comparing the parity of both sides of the equation in Fig. 4. In particular, the constant $\widetilde{F}^{ijm,\alpha\beta}_{knt,\eta\varphi}$ is non-zero only for parity admissible decuples $(i, j, m, k, n, t, \alpha, \beta, \eta, \varphi)$.

This describes the associativity constraint in $\mathcal{C}$ as a collection of matrices

$$\begin{aligned} \widetilde{F}^{ijm}_{knt} &: \mathrm{Hom}_{\mathcal{C}}(X_i \otimes X_j, X_m) \otimes \mathrm{Hom}_{\mathcal{C}}(X_m \otimes X_k, X_n) \\ &\to \mathrm{Hom}_{\mathcal{C}}(X_j \otimes X_k, X_t) \otimes \mathrm{Hom}_{\mathcal{C}}(X_i \otimes X_t, X_n) \end{aligned} \tag{3.18}$$

whose entries are the constants defined above.

Definition 3.16. In the situation above, the matrices $\widetilde{F}^{ijm}_{knt}$ are called *fermionic 6j-symbols*. If $(i, j, m, k, n, t, \alpha, \beta, \eta, \varphi)$ is not (parity) admissible, then by convention we set $\widetilde{F}^{ijm,\alpha\beta}_{knt,\eta\varphi} = 0$.

The super pentagon axiom in $\mathcal{C}$ is equivalent to the following equation in terms of fermionic $6j$-symbols, called the *fermionic pentagon identity* in [GWW15].

Lemma 3.17 (Super pentagon equation)**.** *Let $\mathcal{C}$ be a superfusion category, and X_i, $i \in I$ a complete set of representatives of the isomorphism classes of simple objects in $\mathcal{C}$. For each $i, j, m \in I$, let $N^{ij}_m = \dim \mathrm{Hom}_{\mathcal{C}}(X_i \otimes X_j, X_m)$, and choose homogeneous basis vectors $\boldsymbol{e}^{ij}_m(\alpha)$ $(1 \leq \alpha \leq N^{ij}_m)$ for $\mathrm{Hom}_{\mathcal{C}}(X_i \otimes X_j, X_m)$. Let $s^{ij}_m(\alpha) = |\boldsymbol{e}^{ij}_m(\alpha)|$. Given these choices, if $i, j, k, l, m, n, t, p, q, s \in I$ and*

$\alpha, \beta, \eta, \chi, \gamma, \delta, \phi \in \mathbb{Z}_{\geq 0}$, *then*

$$\sum_{t \in I} \sum_{\eta=1}^{N_t^{jk}} \sum_{\varphi=1}^{N_n^{it}} \sum_{\kappa=1}^{N_s^{tl}} \widetilde{F}^{ijm,\alpha\beta}_{knt,\eta\varphi} \widetilde{F}^{itn,\varphi\chi}_{lps,\kappa\gamma} \widetilde{F}^{jkt,\eta\kappa}_{lsq,\delta\phi} = (-1)^{s_m^{ij}(\alpha) s_q^{kl}(\delta)} \sum_{\epsilon=1}^{N_p^{mq}} \widetilde{F}^{mkn,\beta\chi}_{lpq,\delta\epsilon} \widetilde{F}^{ijm,\alpha\epsilon}_{qps,\delta\gamma}. \quad (3.19)$$

Let G be a finite group, and $\omega : G \times G \times G \to \mathbb{C}^\times$ a 3-cocycle on G with values in A. We saw in Example 3.5 that the pentagon equation for Vec_G^ω said that the $6j$-symbols for Vec_G^ω satisfy the 3-cocycle condition. It turns out an analogous result holds in the superfusion case.

Definition 3.18 (compare to Definition 2.35)**.** We say a superfusion category $\mathcal{C}$ is *pointed* if any simple object $X \in \mathcal{C}$ is invertible, that is, there exists $Y \in \mathcal{C}$ such that $X \otimes Y \xrightarrow{\sim} Y \otimes X \xrightarrow{\sim} \mathbf{1}_{\mathcal{C}}$.

Example 3.19. Let $\mathcal{C}$ be a bosonic pointed superfusion category, and let G be the (finite) group of isomorphism classes of simple objects in $\mathcal{C}$. Let X_g, $g \in G$ be a complete set of representatives of the isomorphism classes of simple objects in $\mathcal{C}$. Then:

$$X_g \otimes X_h \xrightarrow{\sim} X_{gh}, \quad \text{for all } g, h \in G, \quad (3.20)$$

so admissible quadruples in $\mathcal{C}$ are of the form $(g, h, gh, 1)$ for all $g, h \in G$. Let $\omega(g, h)$ denote the parity of the one-dimensional superspace $\mathrm{Hom}_{\mathcal{C}}(X_g \otimes X_h, X_{gh})$, then the parity admissibility condition (3.17) implies:

$$\omega(g, h) + \omega(gh, k) = \omega(h, k) + \omega(g, hk),$$

for all $g, h, k \in G$, so ω is a 2-cocycle on G with values in $\mathbb{Z}/2\mathbb{Z}$. The super pentagon equation (3.19) implies:

$$\widetilde{F}(g, h, k)\widetilde{F}(g, hk, l)\widetilde{F}(h, k, l) = (-1)^{\omega(g,h)\omega(k,l)} \widetilde{F}(gh, k, l)\widetilde{F}(g, h, kl),$$

for all $g, h, k, l \in G$, so following [GWW15] we say $\widetilde{F}$ is a *3-supercocycle* on G.

$6j$-symbols of the Underlying Fusion Category.

Definition 3.20. Let $\mathcal{C}$ be a superfusion category, together with an object π and an odd isomorphism $\zeta : \pi \xrightarrow{\sim} \mathbf{1}_{\mathcal{C}}$. In this situation, we say that $(\mathcal{C}, \pi, \zeta)$ is a Π-*complete superfusion category.*

In this section, we describe Brundan and Ellis' construction [BE17] of the Π-*envelope* of a superfusion category, and the *underlying fusion category* of a Π-complete superfusion category, which is a fusion category over sVec in the sense of [DGNO10]. Given a superfusion category $\mathcal{C}$, we use these constructions to define the underlying fusion category of $\mathcal{C}$, and give a formula relating the $6j$-symbols of the underlying fusion category to the fermionic $6j$-symbols of $\mathcal{C}$.

Suppose $(\mathcal{C}, \pi, \zeta)$ is a Π-complete superfusion category, then every object in $\mathcal{C}$ is the target of an odd isomorphism. It turns out that every superfusion category is equivalent to a Π-complete superfusion category, by the following construction described in [BE17].

Definition 3.21 (see [BE17, Definition 1.16])**.** Let $\mathcal{C}$ be a superfusion category. The Π-*envelope* of $\mathcal{C}$ is the rigid monoidal supercategory $\mathcal{C}_\pi$ with objects of the form $X^{\underline{a}}$, where $X \in \mathcal{C}$ and $\underline{a} \in \mathbb{Z}/2\mathbb{Z}$, and morphisms defined by

$$\mathrm{Hom}_{\mathcal{C}_\pi}(X^{\underline{a}}, Y^{\underline{b}})_{\underline{c}} := \mathrm{Hom}_{\mathcal{C}}(X, Y)_{\underline{a}+\underline{b}+\underline{c}}$$

If $f : X \to Y$ is a homogeneous morphism in $\mathcal{C}$ with parity $|f|$, then let $f^{\underline{b}}_{\underline{a}}$ denote the corresponding morphism $X^{\underline{a}} \to Y^{\underline{b}}$ which has parity $\underline{a} + \underline{b} + |f|$ in $\mathcal{C}_\pi$. The composition in $\mathcal{C}_\pi$ is induced by the composition in $\mathcal{C}$, and the tensor product of objects and morphisms is defined by

$$\begin{aligned} X^{\underline{a}} \otimes Y^{\underline{b}} &:= (X \otimes Y)^{\underline{a}+\underline{b}} \\ f^{\underline{b}}_{\underline{a}} \otimes g^{\underline{d}}_{\underline{c}} &:= (-1)^{(\underline{c}+\underline{d}+|g|)\underline{a}+\underline{d}|f|}(f \otimes g)^{\underline{b}+\underline{d}}_{\underline{a}+\underline{c}} \end{aligned} \tag{3.21}$$

The unit object of $\mathcal{C}_\pi$ is $\mathbf{1}_{\mathcal{C}}^{\underline{0}}$, and the maps a, l, and r in $\mathcal{C}$ extend to $\mathcal{C}_\pi$ in the obvious way. The left dual of an object $X^{\underline{a}} \in \mathcal{C}_\pi$ is given by $(X^*)^{\underline{a}}$, where evaluation and coevaluation morphisms are given by

$$\mathrm{ev}_{X^{\underline{a}}} := (\mathrm{ev}_X)_{\underline{0}}^{0} : (X^*)^{\underline{a}} \otimes X^{\underline{a}} \to \mathbf{1}^{\underline{0}}$$

and

$$\mathrm{coev}_{X^{\underline{a}}} := (\mathrm{coev}_X)_{\underline{0}}^{0} : \mathbf{1}^{\underline{0}} \to X^{\underline{a}} \otimes (X^*)^{\underline{a}}$$

Similarly, the right dual of $X^{\underline{a}} \in \mathcal{C}_\pi$ is $({}^*X)^{\underline{a}} \in \mathcal{C}_\pi$, where $\mathrm{ev}'_{X^{\underline{a}}} := (\mathrm{ev}'_X)_{\underline{0}}^{0}$ and $\mathrm{coev}'_{X^{\underline{a}}} := (\mathrm{coev}'_X)_{\underline{0}}^{0}$.

The functor $J : \mathcal{C} \to \mathcal{C}_\pi$ sending $X \mapsto X^{\underline{0}}$ and $f \mapsto (f)_{\underline{0}}^{0}$ is full, faithful, and essentially surjective, so $\mathcal{C}$ and $\mathcal{C}_\pi$ are equivalent as superfusion categories. However J need not be a superequivalence in general, indeed, in [BE17, Lemma 4.1] it is shown that J is a superequivalence if and only if $\mathcal{C}$ is Π-complete.

Definition 3.22. The *superadditive envelope* $\mathcal{C}_\pi^+$ of a superfusion category $\mathcal{C}$ is the superfusion category obtained by taking the additive envelope of the Π-envelope of $\mathcal{C}$.

In $\mathcal{C}_\pi^+$ we have the odd isomorphism $\zeta := (\mathrm{id}_{\mathbf{1}})_{\underline{1}}^{0} : \mathbf{1}_{\mathcal{C}}^{\underline{1}} \to \mathbf{1}_{\mathcal{C}}^{\underline{0}}$, so $(\mathcal{C}_\pi^+, \mathbf{1}^{\underline{1}}, \zeta)$ is a Π-complete superfusion category.

Definition 3.23 ([DGNO10, Definition 4.16])**.** A fusion category over sVec is a fusion category $\mathcal{A}$ equipped with a braided functor sVec $\to \mathcal{Z}(\mathcal{A})$. Equivalently, this is an object (π, β) in the Drinfeld center $\mathcal{Z}(\mathcal{A})$ together with an isomorphism $\xi : \pi \otimes \pi \xrightarrow{\sim} \mathbf{1}$ such that

$$\begin{aligned} &(\xi^{-1} \otimes \mathrm{id}_X) \circ l_X^{-1} \circ r_X \circ (\mathrm{id}_X \otimes \xi) \\ &= a(\pi, \pi, X)^{-1} \circ (\mathrm{id}_\pi \otimes \beta_X) \circ a(\pi, X, \pi) \circ (\beta_X \otimes \mathrm{id}_\pi) \circ a(X, \pi, \pi)^{-1} \end{aligned} \tag{3.22}$$

for all $X \in \mathcal{A}$, and

$$\beta_\pi = -\mathrm{id}_{\pi\otimes\pi} \in \mathrm{Hom}_{\mathcal{A}}(\pi \otimes \pi, \pi \otimes \pi). \tag{3.23}$$

In this situation we say $(\mathcal{A}, \pi, \beta, \xi)$ is a *fusion category over* sVec.

In the language of [BE17], the quadruple $(\mathcal{A}, \pi, \beta, \xi)$ is an example of a *monoidal* Π-*category*.

Definition 3.24. Let $(\mathcal{L}, \pi, \zeta)$ be a Π-complete superfusion category. The *underlying fusion category* $\underline{\mathcal{L}}$ of $\mathcal{L}$ is the fusion category with the same objects as $\mathcal{L}$, but only the even morphisms.

That $(\mathcal{L}, \pi, \zeta)$ is Π-complete allows us to endow $\underline{\mathcal{L}}$ with the structure of a fusion category over sVec. Indeed, define the even supernatural transformation $\beta : - \otimes \pi \xrightarrow{\sim} \pi \otimes -$ by letting β_X be the composition

$$X \otimes \pi \xrightarrow{\mathrm{id}_X \otimes \zeta} X \otimes \mathbf{1} \xrightarrow{r_X} X \xrightarrow{l_X^{-1}} \mathbf{1} \otimes X \xrightarrow{\zeta^{-1} \otimes \mathrm{id}_X} \pi \otimes X \tag{3.24}$$

for $X \in \underline{\mathcal{L}}$. It is straightforward to check that β is an even supernatural transformation, and that (π, β) is an object of the Drinfeld center $\mathcal{Z}(\underline{\mathcal{L}})$ of $\underline{\mathcal{L}}$. Let $\xi = l_1 \circ (\zeta \otimes \zeta) : \pi \otimes \pi \xrightarrow{\sim} \mathbf{1}$, then ξ is even and thus may be viewed as an isomorphism $\pi \otimes \pi \xrightarrow{\sim} \mathbf{1}$ in $\underline{\mathcal{L}}$. The following is a special case of [BE17, Lemma 3.2].

Lemma 3.25. $(\underline{\mathcal{L}}, \pi, \beta, \xi)$ *is a fusion category over* sVec*.*

Thus to every Π-complete superfusion category there is a corresponding fusion category over sVec. In [BE17, §5], the inverse construction is given, which takes $(\mathcal{A}, \pi, \beta, \xi)$ a fusion category over sVec to its *associated superfusion category* $\widehat{\mathcal{A}}$, which is a Π-complete superfusion category. The category $\widehat{\mathcal{A}}$ has the same objects as $\mathcal{A}$, with morphisms defined by $\mathrm{Hom}_{\widehat{\mathcal{A}}}(X, Y)_0 := \mathrm{Hom}_{\mathcal{A}}(X, Y)$ and

$\mathrm{Hom}_{\widehat{\mathcal{A}}}(X,Y)_1 := \mathrm{Hom}_{\mathcal{A}}(X, \pi \otimes Y)$, with the tensor product of objects being identical to that in A. We will not describe the composition or tensor product of morphisms in this category here.

The following follows from [BE17, Lemma 5.4], and will be crucial when we investigate Ocneanu rigidity for superfusion categories.

Lemma 3.26. *Every* Π-*complete superfusion category is the associated superfusion category of a fusion category over* sVec.

Definition 3.27. Let $\mathcal{C}$ be a superfusion category, and let $\underline{\mathcal{C}_\pi^+}$ be the underlying fusion category of the superadditive envelope of $\mathcal{C}$ (see Definitions 3.21 and 3.22). We call $\underline{\mathcal{C}_\pi^+}$ the *underlying fusion category* of $\mathcal{C}$.

Our goal is to give an explicit formula for the $6j$-symbols of the underlying fusion category $\underline{\mathcal{C}_\pi^+}$ in terms of the fermionic $6j$-symbols of $\mathcal{C}$. Recall that for $X, Y \in \mathcal{C}$ and $\underline{a}, \underline{b} \in \mathbb{Z}/2\mathbb{Z}$, we have

$$\mathrm{Hom}_{\underline{\mathcal{C}_\pi^+}}(X^{\underline{a}}, Y^{\underline{b}}) = \mathrm{Hom}_{\mathcal{C}}(X,Y)_{\underline{a}+\underline{b}} \tag{3.25}$$

If $f : X \to Y$ is a homogeneous morphism in $\mathcal{C}$ and $\underline{a} + \underline{b} = |f|$, then we denote by $f_{\underline{a}}^{\underline{b}}$ the corresponding morphism $X^{\underline{a}} \to Y^{\underline{b}}$ in $\underline{\mathcal{C}_\pi^+}$. The tensor product of objects and morphisms in $\underline{\mathcal{C}_\pi^+}$ is defined by

$$\begin{aligned} X^{\underline{a}} \otimes Y^{\underline{b}} &:= (X \otimes Y)^{\underline{a}+\underline{b}} \\ f_{\underline{a}}^{\underline{b}} \otimes g_{\underline{c}}^{\underline{d}} &:= (-1)^{\underline{d}|f|}(f \otimes g)_{\underline{a}+\underline{c}}^{\underline{b}+\underline{d}} \end{aligned} \tag{3.26}$$

From Lemma 3.25 we get that $(\underline{\mathcal{C}_\pi^+}, \mathbf{1}^{\underline{1}}, \beta, \xi)$ is a fusion category over sVec, where

$$\beta_{X^{\underline{a}}} = (-1)^{\underline{a}} \cdot (l_X^{-1} \circ r_X)_{\underline{a}+1}^{\underline{a}+1} : X^{\underline{a}} \otimes \mathbf{1}^{\underline{1}} \xrightarrow{\sim} \mathbf{1}^{\underline{1}} \otimes X^{\underline{a}},\ X^{\underline{a}} \in \underline{\mathcal{C}_\pi^+} \tag{3.27}$$

and

$$\xi = (l_1)_{\underline{0}}^{\underline{0}} : \mathbf{1}^{\underline{1}} \otimes \mathbf{1}^{\underline{1}} \xrightarrow{\sim} \mathbf{1}^{\underline{0}} \tag{3.28}$$

Let X_i, $i \in I$ be a complete set of representatives of isomorphism classes[1] of simple objects in a superfusion category $\mathcal{C}$. Define

$$J = \{(i, \underline{a}) \in I \times \mathbb{Z}/2\mathbb{Z} \text{ such that } \underline{a} = 0 \text{ if } X_i \text{ is Majorana}\} \tag{3.29}$$

We denote the element $(i, \underline{a}) \in J$ by $i^{\underline{a}}$. The isomorphism classes of simple objects in $\underline{\mathcal{C}_\pi^+}$ are labeled by J. Indeed, suppose X_i is bosonic, then we have a pair of non-isomorphic simple objects $X_i^{\underline{0}}$ and $X_i^{\underline{1}}$ in $\underline{\mathcal{C}_\pi^+}$ corresponding to the labels $i^{\underline{0}}$ and $i^{\underline{1}}$ respectively. If X_i is Majorana, then $X_i^{\underline{0}}$ and $X_i^{\underline{1}}$ are isomorphic in $\underline{\mathcal{C}_\pi^+}$, so we choose $X_i^{\underline{0}}$ as our representative simple object, and label it by $i^{\underline{0}}$.

Remark 3.28. If $\mathcal{C}$ is a bosonic superfusion category, then the underlying fusion category $\underline{\mathcal{C}_\pi^+}$ has twice as many simple objects (up to isomorphism) as $\mathcal{C}$, labeled by elements of $J = I \times \mathbb{Z}/2\mathbb{Z}$.

Example 3.29. Let $\mathcal{C}$ be a bosonic pointed superfusion category, as in Example 3.19. The underlying fusion category $\underline{\mathcal{C}_\pi^+}$ is pointed, so let G_ω denote the (finite) group of isomorphism classes of simple objects in $\underline{\mathcal{C}_\pi^+}$. As a set, we have $G_\omega = \mathbb{Z}/2\mathbb{Z} \times G$, though we would like to describe the group structure on G_ω. The isomorphisms $\mathbf{e}(g,h) : X_g \otimes X_h \xrightarrow{\sim} X_{gh}$ in $\mathcal{C}$ induce isomorphisms in $\underline{\mathcal{C}_\pi^+}$

$$\mathbf{e}(g^{\underline{a}}, h^{\underline{b}}) = (\mathbf{e}(g,h))_{\underline{a}+\underline{b}}^{\underline{a}+\underline{b}+\omega(g,h)} : X_g^{\underline{a}} \otimes X_h^{\underline{b}} \xrightarrow{\sim} X_{gh}^{\underline{a}+\underline{b}+\omega(g,h)}$$

for all $g^{\underline{a}}, h^{\underline{b}} \in G_\omega$, and so the group structure on G_ω is given by

$$g^{\underline{a}} \cdot h^{\underline{b}} := (gh)^{\underline{a}+\underline{b}+\omega(g,h)}.$$

[1] We say that two objects in a superfusion category lie in the same isomorphism class if there is a (not necessarily even) isomorphism between them.

Comparing this to Eq. (2.20), we see that G_ω is the central extension of G by $\mathbb{Z}/2\mathbb{Z}$ determined by the 2-cocycle ω.

Let $\mathcal{C}$ be a superfusion category, and let J label the simple objects in $\underline{\mathcal{C}_\pi^+}$, as described in Eq. (3.29). Let $i^{\underline{a}}, j^{\underline{b}}, m^{\underline{c}} \in J$, and suppose that (i,j,m,α) is an admissible quadruple in $\mathcal{C}$. If $\underline{c} = \underline{a} + \underline{b} + s_m^{ij}(\alpha)$ then $\mathbf{e}_m^{ij} : X_i \otimes X_j \to X_m$ induces a morphism

$$X_i^{\underline{a}} \otimes X_j^{\underline{b}} \to X_m^{\underline{c}}$$

in $\underline{\mathcal{C}_\pi^+}$, in which case $(i^{\underline{a}}, j^{\underline{b}}, m^{\underline{c}}, \alpha)$ is an admissible quadruple in $\underline{\mathcal{C}_\pi^+}$. This implies that every admissible quadruple in $\underline{\mathcal{C}_\pi^+}$ can be written unambiguously in the form

$$(i^{\underline{a}}, j^{\underline{b}}, m, \alpha)$$

where $i^{\underline{a}}, j^{\underline{b}}, m^{\underline{a}+\underline{b}+s_m^{ij}(\alpha)} \in J$ and (i,j,m,α) is an admissible quadruple in $\mathcal{C}$. In the same way, every admissible decuple in $\underline{\mathcal{C}_\pi^+}$ can be written unambiguously as

$$(i^{\underline{a}}, j^{\underline{b}}, m, k^{\underline{c}}, n, t, \alpha, \beta, \eta, \varphi)$$

where $i^{\underline{a}}$, $j^{\underline{b}}$, $m^{\underline{a}+\underline{b}+s_m^{ij}(\alpha)}$, $t^{\underline{b}+\underline{c}+s_t^{jk}(\eta)}$, $n^{\underline{a}+\underline{b}+\underline{c}+s_m^{ij}(\alpha)+s_n^{mk}(\beta)} \in J$, and $(i,j,m,k,n,t,\alpha,\beta,\eta,\varphi)$ is a parity admissible decuple in $\mathcal{C}$.

Definition 3.30. Let $\mathcal{C}$ be a superfusion category, and $\underline{\mathcal{C}_\pi^+}$ its underlying fusion category. If $(i^{\underline{a}}, j^{\underline{b}}, m, k^{\underline{c}}, n, t, \alpha, \beta, \eta, \varphi)$ is an admissible decuple in $\underline{\mathcal{C}_\pi^+}$, let

$$F_{k^{\underline{c}}nt,\eta\varphi}^{i^{\underline{a}}j^{\underline{b}}m,\alpha\beta} := (-1)^{\underline{c}s_m^{ij}(\alpha)} \widetilde{F}_{knt,\eta\varphi}^{ijm,\alpha\beta}.$$

If $(i^{\underline{a}}, j^{\underline{b}}, m, k^{\underline{c}}, n, t, \alpha, \beta, \eta, \varphi)$ is not admissible, then let $F_{k^{\underline{c}}nt,\eta\varphi}^{i^{\underline{a}}j^{\underline{b}}m,\alpha\beta} = 0$.

We claim that the symbols defined above are in fact the $6j$-symbols of $\underline{\mathcal{C}_\pi^+}$. Indeed, they satisfy the following version of the pentagon equation.

Theorem 3.31 (Pentagon equation)**.** *Let $\mathcal{C}$ be a superfusion category with simple objects indexed by a set I, and $\underline{\mathcal{C}_\pi^+}$ the underlying fusion category. For each $i, j, m \in$*

I, let $N_m^{ij} = \dim \mathrm{Hom}_{\mathcal{C}}(X_i \otimes X_j, X_m)$, and choose a homogeneous basis $\boldsymbol{e}_m^{ij}(\alpha)$ $(1 \leq \alpha \leq N_m^{ij})$ for $\mathrm{Hom}_{\mathcal{C}}(X_i \otimes X_j, X_m)$. Let $s_m^{ij}(\alpha) = |e_m^{ij}(\alpha)|$. Given these choices, we have

$$\sum_{t\in I}\sum_{\eta=1}^{N_t^{jk}}\sum_{\varphi=1}^{N_n^{it}}\sum_{\kappa=1}^{N_s^{tl}} F^{i^{\underline{a}}j^{\underline{b}}m,\alpha\beta}_{k^{\underline{c}}nt,\eta\varphi} F^{i^{\underline{a}}t^{\underline{b}+\underline{c}+s_t^{jk}(\eta)}n,\varphi\chi}_{l^{\underline{d}}ps,\kappa\gamma} F^{j^{\underline{b}}k^{\underline{c}}t,\eta\kappa}_{l^{\underline{d}}sq,\delta\phi}$$
$$= \sum_{\epsilon=1}^{N_p^{mq}} F^{m^{\underline{a}+\underline{b}+s_m^{ij}(\alpha)}k^{\underline{c}}n,\beta\chi}_{l^{\underline{d}}pq,\delta\epsilon} F^{i^{\underline{a}}j^{\underline{b}}m,\alpha\epsilon}_{q^{\underline{c}+\underline{d}+s_q^{kl}(\delta)}ps,\delta\gamma} \tag{3.30}$$

for all $i,j,k,l,m,n,t,p,q,s \in I$, $\underline{a},\underline{b},\underline{c} \in \mathbb{Z}/2\mathbb{Z}$, and $\alpha,\beta,\eta,\chi,\delta,\phi \in \mathbb{Z}_{\geq 0}$.

Proof. By combining Definition 3.30 with the super pentagon equation (3.19), we have the equality

$$\sum_{t\in I}\sum_{\eta=1}^{N_t^{jk}}\sum_{\varphi=1}^{N_n^{it}}\sum_{\kappa=1}^{N_s^{tl}} (-1)^{\underline{c}s_m^{ij}(\alpha)+\underline{d}s_n^{it}(\varphi)+\underline{d}s_t^{jk}(\eta)} F^{ijm,\alpha\beta}_{knt,\eta\varphi} F^{itn,\varphi\chi}_{lps,\kappa\gamma} F^{jkt,\eta\kappa}_{lsq,\delta\phi}$$
$$= (-1)^{s_m^{ij}(\alpha)s_q^{kl}(\delta)} \sum_{\epsilon=1}^{N_p^{mq}} (-1)^{\underline{d}s_n^{mk}(\beta)+(\underline{c}+\underline{d}+s_q^{kl}(\delta))s_m^{ij}(\alpha)} F^{mkn,\beta\chi}_{lpq,\delta\epsilon} F^{ijm,\alpha\epsilon}_{qps,\delta\gamma}$$

and thus it suffices to show that

$$\underline{c}s_m^{ij}(\alpha) + \underline{d}s_n^{it}(\varphi) + \underline{d}s_t^{jk}(\eta) = s_m^{ij}(\alpha)s_q^{kl}(\delta) + \underline{d}s_n^{mk}(\beta) + (\underline{c} + \underline{d} + s_q^{kl}(\delta))s_m^{ij}(\alpha)$$

for all admissible decuples $(i^{\underline{a}}, j^{\underline{b}}, m, k^{\underline{c}}, n, t, \alpha, \beta, \eta, \varphi)$ in $\underline{\mathcal{C}_\pi^+}$. This immediately reduces to showing that

$$\underline{d}s_n^{it}(\varphi) + \underline{d}s_t^{jk}(\eta) = \underline{d}s_n^{mk}(\beta) + \underline{d}s_m^{ij}(\alpha)$$

which holds by the parity compatibility condition (3.17). □

Remark 3.32. Our definition of the $6j$-symbols in $\underline{\mathcal{C}_\pi^+}$ can be recovered directly from the construction of $\underline{\mathcal{C}_\pi^+}$, in which case Theorem 3.31 can be viewed as a corollary of the pentagon axiom in $\underline{\mathcal{C}_\pi^+}$. Indeed, for each admissible quadruple

$(i^{\underline{a}}, j^{\underline{b}}, m, \alpha)$ in $\underline{\mathcal{C}_\pi^+}$, let

$$\mathbf{e}_m^{i^{\underline{a}} j^{\underline{b}}}(\alpha) := \left(\mathbf{e}_m^{ij}(\alpha)\right)_{\underline{a}+\underline{b}}^{\underline{a}+\underline{b}+s_m^{ij}(\alpha)} : X_i^{\underline{a}} \otimes X_j^{\underline{b}} \to X_m^{\underline{a}+\underline{b}+s_m^{ij}(\alpha)} \tag{3.31}$$

For ease of notation, set $\underline{d} = \underline{a} + \underline{b} + s_m^{ij}(\alpha)$ and $\underline{e} = \underline{a} + \underline{b} + \underline{c}s_m^{ij}(\alpha) + s_n^{mk}(\beta)$, then (3.4) is given by

$$(X_i^{\underline{a}} \otimes X_j^{\underline{b}}) \otimes X_k^{\underline{c}} \xrightarrow{\mathbf{e}_m^{i^{\underline{a}} j^{\underline{b}}}(\alpha) \otimes \mathrm{id}_{X_k^{\underline{c}}}} X_m^{\underline{d}} \otimes X_k^{\underline{c}} \xrightarrow{\mathbf{e}_n^{m^{\underline{d}} k^{\underline{c}}}(\beta)} X_n^{\underline{e}} \tag{3.32}$$

where we have

$$\mathbf{e}_m^{i^{\underline{a}} j^{\underline{b}}}(\alpha) \otimes \mathrm{id}_{X_k^{\underline{c}}} = (-1)^{\underline{c}s_m^{ij}(\alpha)} \left(\mathbf{e}_m^{ij}(\alpha) \otimes \mathrm{id}_{X_k}\right)_{\underline{a}+\underline{b}+\underline{c}}^{\underline{c}+\underline{d}} \tag{3.33}$$

by definition of the tensor product on $\underline{\mathcal{C}_\pi^+}$. Next, fix an admissible quadruple $(j^{\underline{b}}, k^{\underline{c}}, t, \eta)$. The composition (3.5) is given by

$$(X_i^{\underline{a}} \otimes X_j^{\underline{b}}) \otimes X_k^{\underline{c}} \xrightarrow{a(X_i^{\underline{a}}, X_j^{\underline{b}}, X_k^{\underline{c}})} X_i^{\underline{a}} \otimes (X_j^{\underline{b}} \otimes X_k^{\underline{c}}) \xrightarrow{\mathrm{id}_{X_i^{\underline{a}}} \otimes \mathbf{e}_t^{j^{\underline{b}} k^{\underline{c}}}(\eta)} X_i^{\underline{a}} \otimes X_t^{\underline{f}} \xrightarrow{\mathbf{e}_n^{i^{\underline{a}} t^{\underline{f}}}(\varphi)} X_n^{\underline{e}} \tag{3.34}$$

where $\underline{f} = \underline{b} + \underline{c} + s_t^{jk}(\eta)$. We compute

$$\mathrm{id}_{X_i^{\underline{a}}} \otimes \mathbf{e}_t^{j^{\underline{b}} k^{\underline{c}}}(\eta) = \left(\mathrm{id}_{X_i} \otimes \mathbf{e}_t^{jk}(\eta)\right)_{\underline{a}+\underline{b}+\underline{c}}^{\underline{a}+\underline{f}} \tag{3.35}$$

and so the compositions (3.32) and (3.34) in $\underline{\mathcal{C}_\pi^+}$ are induced by the corresponding compositions (3.4) and (3.5) in $\mathcal{C}$ up to a factor of $(-1)^{\underline{c}s_m^{ij}(\alpha)}$, as expected.

Example 3.33. Let $\mathcal{C}$ be a bosonic pointed superfusion category, as in Examples 3.19 and 3.29. For all $g^{\underline{a}}, h^{\underline{b}}, k^{\underline{c}} \in G_\omega$ we can unambiguously write $F(g^{\underline{a}}, h^{\underline{b}}, k^{\underline{c}}) \in \mathbb{C}^\times$ for the corresponding $6j$-symbol in $\underline{\mathcal{C}_\pi^+}$. With this notation, Definition 3.30 implies

$$F(g^{\underline{a}}, h^{\underline{b}}, k^{\underline{c}}) = (-1)^{\underline{c}\omega(g,h)} \widetilde{F}(g, h, k)$$

for all $g^{\underline{a}}, h^{\underline{b}}, k^{\underline{c}} \in G_\omega$. The pentagon equation (3.30) implies that F is a 3-cocycle on the central extension G_ω with values in $\mathbb{C}^\times$.

Viewing G as the subset of G_ω consisting of elements of the form $g^{\underline{0}}$, we have the following corollary.

Corollary 3.34. *Let $\widetilde{F} : G^3 \to \mathbb{C}^\times$ be a 3-supercocycle on G with 2-cocycle ω. Then there exists a 3-cocycle $F : G_\omega^3 \to \mathbb{C}^\times$ on G_ω such that*

$$F|_{G^3} = \widetilde{F}$$

In other words, every 3-supercocycle on G arises as the restriction of a 3-cocycle on a $\mathbb{Z}/2\mathbb{Z}$-central extension of G.

CHAPTER IV

OCNEANU RIGIDITY FOR SUPERFUSION CATEGORIES

Chapter IV appeared in [Ush18].

The following result, known as Ocneanu rigidity, was originally proved by Ocneanu, Blanchard, and Wassermann (unpublished) in certain cases. The first published proof was in [ENO05, §7].

Theorem 4.1 (Ocneanu rigidity)**.** *(i) The number of fusion categories (up to equivalence) is countable, and*

(ii) The number of fusion categories (up to equivalence) with a given Grothendieck ring is finite.

The goal of this chapter is to prove a version of Ocneanu rigidity for superfusion categories. To do this, we must first decide what the Grothendieck ring of a superfusion category should be. Brundan and Ellis [BE17] suggested the following definition.

Definition 4.2 (compare with Definition 2.37)**.** Let $\mathbb{Z}^\pi := \mathbb{Z}[\pi]/(\pi^2 - 1)$. The *$\pi$-Grothendieck ring* $\mathrm{sGr}(\mathcal{C})$ is the $\mathbb{Z}^\pi$-module generated by isomorphism classes of objects $[X]$ in $\mathcal{C}$ subject to the relation that if

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0 \tag{4.1}$$

is a short exact sequence with f and g homogeneous morphisms, then

$$[Y] = [X]\pi^{|f|} + [Z]\pi^{|g|}. \tag{4.2}$$

The tensor product on $\mathcal{C}$ then induces an associative multiplication on $\mathrm{sGr}(\mathcal{C})$, making $\mathrm{sGr}(\mathcal{C})$ into a $\mathbb{Z}^\pi$-algebra.

With this definition, we prove the following version of Ocneanu rigidity for superfusion categories, and give some examples of these π-Grothendieck rings.

Theorem 4.3 (Ocneanu rigidity for superfusion categories)**.** *(i) The number of superfusion categories (up to superequivalence) is countable, and*

(ii) The number of superfusion categories (up to superequivalence) with a given π-Grothendieck ring is finite.

Superforms

Let $\mathcal{D}$ be a Π-complete superfusion category.

Definition 4.4. A *superform* of $\mathcal{D}$ is a superfusion category $\mathcal{C}$ such that $\mathcal{C} \simeq \mathcal{D}$ are equivalent (but not necessarily superequivalent) superfusion categories.

Our goal is to prove the following.

Proposition 4.5. *A Π-complete superfusion category $\mathcal{D}$ has only finitely many superforms, up to superequivalence of superfusion categories.*

To show this, the following notion will be useful.

Definition 4.6. Let $\mathcal{C}$ and $\mathcal{D}$ be superfusion categories, and $F : \mathcal{C} \to \mathcal{D}$ a tensor superfunctor. Its *even essential image*, denoted $F(\mathcal{C})$, is the full subcategory of $\mathcal{D}$ consisting of objects evenly isomorphic to $F(X)$ for some $X \in \mathcal{C}$.

Recall that a tensor superfunctor $F : \mathcal{C} \to \mathcal{D}$ is a superfunctor such that $F(\mathbf{1}_{\mathcal{C}})$ is evenly isomorphic to $\mathbf{1}_{\mathcal{D}}$, together with an even natural isomorphism $c_{X,Y} : F(X) \otimes F(Y) \xrightarrow{\sim} F(X \otimes Y)$ satisfying the usual diagram (see e.g. [EGNO15, §2.4]).

Lemma 4.7. *Given a tensor superfunctor $F : \mathcal{C} \to \mathcal{D}$, its even essential image $F(\mathcal{C})$ is a full tensor subcategory of $\mathcal{D}$.*

Proof. Indeed given $Y, Y' \in F(\mathcal{C})$, there exists $X, X' \in \mathcal{C}$ such that $F(X) \xrightarrow{\sim} Y$ and $F(X') \xrightarrow{\sim} Y'$ are evenly isomorphic. Then $F(X \otimes X') \xrightarrow{\sim} F(X) \otimes F(X') \xrightarrow{\sim} Y \otimes Y'$ is an even isomorphism, whence $Y \otimes Y' \in F(\mathcal{C})$. □

It turns out that if $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of superfusion categories, then $\mathcal{C}$ is determined (up to superequivalence) by $F(\mathcal{C})$. More precisely, we have the following theorem.

Theorem 4.8. *If $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{A} \to \mathcal{D}$ are equivalences of superfusion categories with $G(\mathcal{A}) = F(\mathcal{C})$, then $\mathcal{A}$ and $\mathcal{C}$ are superequivalent superfusion categories.*

Proof. If $X \in \mathcal{A}$, then $G(X) \in G(\mathcal{A}) = F(\mathcal{C})$, so there exists $X_{\mathcal{C}} \in \mathcal{C}$ such that $F(X_{\mathcal{C}}) \xrightarrow{\sim} G(X)$ are evenly isomorphic. For each $X \in \mathcal{A}$, we pick such a $X_{\mathcal{C}} \in \mathcal{C}$ together with an even isomorphism $q_X : F(X_{\mathcal{C}}) \xrightarrow{\sim} G(X)$. We define a superfunctor $K : \mathcal{A} \to \mathcal{C}$ as follows. On objects, let $K(X) = X_{\mathcal{C}}$. On morphisms, if $f \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$ then let $K(f) = F^{-1}(q_Y^{-1} \circ G(f) \circ q_X)$, i.e. $K(f)$ is the image of f under the even isomorphism

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(X, Y) &\xrightarrow{G} \mathrm{Hom}_{\mathcal{D}}(G(X), G(Y)) \\ &\xrightarrow{(q_Y^{-1})_* \circ (q_X)^*} \mathrm{Hom}_{\mathcal{C}}(F(X_{\mathcal{C}}), F(Y_{\mathcal{C}})) \\ &\xrightarrow{F^{-1}} \mathrm{Hom}_{\mathcal{C}}(X_{\mathcal{C}}, Y_{\mathcal{C}}) \end{aligned} \tag{4.3}$$

Functorality of F and G implies that K is a superfunctor, and it is immediate that K is full and faithful. As we saw in Definition 3.9, to show that K is a superequivalence it remains to prove that $K(\mathcal{A}) = \mathcal{C}$. Let $Y \in \mathcal{C}$, then $F(Y) \in F(\mathcal{C}) = G(\mathcal{A})$ so there exists $X \in \mathcal{A}$ together with an even isomorphism $G(X) \xrightarrow{\sim} F(Y)$, so $F(X_c) \xrightarrow{\sim} F(Y)$ are evenly isomorphic. This implies that

$K(X) = X_c \xrightarrow{\sim} Y$ are evenly isomorphic, i.e. $Y \in K(\mathcal{A})$. Thus K is a superequivalence.

It remains to endow K with the structure of a monoidal superfunctor. To do this, we must define even coherence maps $J_{X,Y} : K(X) \otimes K(Y) \to K(X \otimes Y)$ satisfying the usual axioms. Let c and d denote the coherence maps for F and G respectively. Let $\varphi_{X,Y} : F(X_{\mathcal{C}} \otimes Y_{\mathcal{C}}) \xrightarrow{\sim} F((X \otimes Y)_{\mathcal{C}})$ be the composition

$$\begin{aligned} F(X_{\mathcal{C}} \otimes Y_{\mathcal{C}}) \xrightarrow{c^{-1}_{X_{\mathcal{C}},Y_{\mathcal{C}}}} F(X_{\mathcal{C}}) \otimes F(Y_{\mathcal{C}}) \xrightarrow{q_x \otimes q_y} G(X) \otimes G(Y) \\ \xrightarrow{d_{X,Y}} G(X \otimes Y) \xrightarrow{q^{-1}_{X \otimes Y}} F((X \otimes Y)_{\mathcal{C}}) \end{aligned} \tag{4.4}$$

With this notation, let $J_{X,Y} := F^{-1}(\varphi_{X,Y})$. It is straightforward to check that (K, J) satisfies the axioms for a monoidal superfunctor, and so K is a superequivalence of superfusion categories. □

We are now ready to prove the above proposition.

Proof of Proposition 4.5. Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of superfusion categories, where $\mathcal{D}$ is Π-complete. Let Y_i, $i \in I$ be a complete set of representatives of simple objects of $\mathcal{D}$. Since F is an equivalence, for each $i \in I$, there exists an object $X_i \in \mathcal{C}$ such that $F(X_i) \xrightarrow{\sim} Y_i$. Since $\mathcal{D}$ is Π-complete, for each $i \in I$, there exists $Y_i' \in \mathcal{D}$ such that $Y_i \xrightarrow{\sim} Y_i'$ are oddly isomorphic. Fix $i \in I$. If Y_i is Majorana, then $\mathrm{Hom}_{\mathcal{D}}(F(X_i), Y_i) \simeq \mathbb{C}^{1|1}$, so $Y_i \in F(\mathcal{C})$ and $Y_i' \in F(\mathcal{C})$. If Y_i is bosonic, then the space $\mathrm{Hom}_{\mathcal{D}}(F(X_i), Y_i)$ is one-dimensional, either even or odd. So $Y_i \in F(\mathcal{C})$ or $Y_i' \in F(\mathcal{C})$ (or possibly both). Since the subcategory $F(\mathcal{D})$ is determined by the choice of Y_i or Y_i' (or both) for all $i \in I$ such that Y_i is bosonic, and there are finitely many such choices, there are finitely many possibilities for $F(\mathcal{C})$. By Theorem 4.8, we are done. □

We are now ready to prove the main result of this chapter.

Proof of Theorem 4.3. For (i), observe that Ocneanu rigidity [ENO05, Theorem 2.28, Theorem 2.31] implies there are countably many fusion categories over sVec. Since every Π-complete superfusion category is the associated superfusion category of a fusion category over sVec, there are countably many Π-complete superfusion categories. Every superfusion category is equivalent to a Π-complete superfusion category, so Proposition 4.5 implies (i).

For (ii), fix a superfusion category $\mathcal{C}$, and suppose that $\mathcal{D}$ is a superfusion category with $\mathrm{sGr}(\mathcal{C}) \simeq \mathrm{sGr}(\mathcal{D})$. We will show that there are finitely many possibilities for $\mathcal{D}$, up to superequivalence. Since $\mathrm{sGr}(\mathcal{C}) \simeq \mathrm{sGr}(\mathcal{D})$, the underlying fusion categories $\underline{\mathcal{C}_\pi^+}$ and $\underline{\mathcal{D}_\pi^+}$ have isomorphic Grothendieck rings. By Ocneanu rigidity [ENO05, Theorem 2.28], there are finitely many fusion categories with a given Grothendieck ring, and moreover each of these fusion categories $\mathcal{A}$ admits only finitely many tensor functors $\mathrm{sVec} \to \mathcal{Z}(\mathcal{A})$ [ENO05, Theorem 2.31], hence there are finitely many fusion categories over sVec with Grothendieck ring isomorphic to $\mathrm{Gr}(\underline{\mathcal{C}_\pi^+})$. Since every Π-complete fusion category is the associated superfusion category of a fusion category over sVec, there are finitely many possibilities for $\mathcal{D}_\pi^+$ up to superequivalence, so by Proposition 4.5 there are finitely many possibilities for $\mathcal{D}$ up to superequivalence. □

Examples of π-Grothendieck Rings

In this section, we compute the π-Grothendieck ring of some superfusion categories.

Example 4.9. Let $\mathcal{C} = \mathrm{SVec}$ denote the monoidal supercategory of finite dimensional superspaces, together with all linear maps between them. Let $\mathbb{C}^{p|q} = \mathbb{C}^p \oplus \mathbb{C}^q$ denote the superspace with even part $\mathbb{C}^p$ and odd part $\mathbb{C}^q$, then $[\mathbb{C}^{p|q}] = (p + q\pi)[\mathbb{C}^{1|0}]$ in sGr(SVec), where we used that $[\mathbb{C}^{0|1}] = \pi[\mathbb{C}^{0|1}]$. Thus sGr(SVec) is a free $\mathbb{Z}^\pi$-module, generated by $[\mathbb{C}^{1|0}]$. Moreover, the tensor product on SVec gives

$$[\mathbb{C}^{p|q}][\mathbb{C}^{p'|q'}] = [\mathbb{C}^{pp'+qq'|pq'+qp'}], \tag{4.5}$$

and so sGr(SVec) is free as a $\mathbb{Z}^\pi$-algebra.

Example 4.10. Let $\mathcal{I}$ be an Ising braided category (see Examples 2.46 and 2.51, and [DGNO10, Appendix B]). Recall that $\mathcal{I}$ has three isomorphism classes of simple objects: the unit object $\mathbf{1}$, an invertible object π, and a non-invertible object X, satisfying the fusion rules:

$$\pi \otimes \pi \simeq \mathbf{1}, \quad \pi \otimes X \simeq X \simeq X \otimes \pi, \quad X \otimes X \simeq \mathbf{1} \oplus \pi. \tag{4.6}$$

The fusion subcategory $\mathcal{I}_{ad} \subset \mathcal{I}$ generated by $\mathbf{1}$ and π is braided equivalent to sVec [DGNO10, Lemma B.11], and thus $\mathcal{I}$ is a fusion category over sVec. Let us consider the associated superfusion category $\widehat{\mathcal{I}}$.

The isomorphism $\pi \otimes \pi \simeq \mathbf{1}$ in $\mathcal{I}$ induces an odd isomorphism $\pi \xrightarrow{\sim} \mathbf{1}$ in $\widehat{\mathcal{I}}$. Similarly, the isomorphism $\pi \otimes X \simeq X$ in $\mathcal{I}$ induces an odd isomorphism $X \xrightarrow{\sim} X$ in $\widehat{\mathcal{I}}$. Thus $\widehat{\mathcal{I}}$ has a bosonic simple object $\mathbf{1} \xrightarrow{\sim} \pi$, and a Majorana simple object X. From the fusion rules, we get the relations

$$[X] = \pi[X], \quad [X]^2 = (1+\pi)[1]$$

in $\mathrm{sGr}(\widehat{\mathcal{I}})$.

Example 4.11 (see [EGNO15, §8.18.2])**.** Generalizing the previous example, take $k \equiv 2 \bmod 4$, and let $C_k(q)$ denote the braided fusion category of integrable $\widehat{\mathfrak{sl}_2}$ modules at level k. This category has simple objects V_i, $i = 0, \ldots, k$ with unit object $V_0 = \mathbf{1}$ and fusion rule given by the truncated Clebsch-Gordan rule:

$$V_i \otimes V_j \simeq \bigoplus_{l=\max(i+j-k,0)}^{\min(i,j)} V_{i+j-2l} \tag{4.7}$$

The fusion subcategory $D_k(q) \subset C_k(q)$ generated by $\mathbf{1}$ and $\pi := V_k$ is braided equivalent to sVec, and so $C_k(q)$ is a fusion category over sVec. Let $\mathcal{C}_k := \widehat{C_k(q)}$ denote the associated superfusion category.

Since $\pi \otimes V_i \simeq V_{k-i}$ in $C_k(q)$ for all $i = 0, \ldots, k$, we have $V_i \xrightarrow{\sim} V_{k-i}$ in $\mathcal{C}_k$. Thus $C_k(q)$ has $k/2$ bosonic simple objects $V_0, V_1, \ldots, V_{k/2-1}$, and a single Majorana simple object $V_{k/2}$.

CHAPTER V

COHOMOLOGY OF SYMMETRIC FUSION CATEGORIES

In this chapter we describe a notion of cohomology for symmetric fusion categories. Recall that a braided fusion category $\mathcal{C}$ with braiding $c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ is *symmetric* if:

$$c_{X,Y} \circ c_{Y,X} = \mathrm{id}_{Y \otimes X}, \tag{5.1}$$

for all $X, Y \in \mathcal{C}$. We saw in Example 2.49) that $\mathrm{Rep}(G)$, the category of finite-dimensional representations of G over $\mathbb{C}$ with braiding given by transposition of factors, is a symmetric fusion category.

Example 5.1. Let sVec be the fusion category of finite-dimensional superspaces (compare with Definition 3.7). There is a braiding on sVec defined on homogeneous vectors by:

$$c_{V,W}(v \otimes w) = (-1)^{|v||w|} w \otimes v, \ v \in V, \ w \in W. \tag{5.2}$$

This braiding makes sVec into a symmetric fusion category.

Example 5.1 is an instance of the following general construction.

Example 5.2. Let G be a finite group and let $z \in G$ be a central element such that $z^2 = 1$. Then there is a braiding on $\mathrm{Rep}(G)$ making it into a symmetric fusion category, defined by the following formula.

$$c^z_{X,Y}(x \otimes y) = (-1)^{mn} y \otimes x \text{ if } x \in X, y \in Y, zx = (-1)^m x \text{ and } zy = (-1)^n y, \tag{5.3}$$

for irreducible representations X, Y of G.

Let $\mathrm{Rep}(G, z)$ denote the category $\mathrm{Rep}(G)$ equipped with the braiding of Example 5.2.

Definition 5.3. A *supergroup* is a pair (G, z) with G a group, and $z \in Z(G)$ a central element such that $z^2 = 1$. We say a supergroup is *non-trivial* if $z \neq 1$, and *trivial* otherwise.

Let (G, z) be a finite supergroup, then by Example 5.2 we have a symmetric fusion category $\mathrm{Rep}(G, z)$. The following result of Doplicher-Roberts and Deligne states that every symmetric fusion category arises this way (up to equivalence).

Theorem 5.4 ([DR89, Del02])**.** *Any symmetric fusion category $\mathcal{C}$ is braided equivalent to a category of the form* $\mathrm{Rep}(G, z)$ *with* (G, z) *a finite supergroup.*

Given the connection between symmetric fusion categories and finite supergroups, we proceed by describing some basic properties of supergroups. Let (G, z) be a non-trivial supergroup, then we have the following exact sequence of groups:

$$1 \to \langle z \rangle \hookrightarrow G \to G/\langle z \rangle \to 1. \tag{5.4}$$

In particular, G is a $\mathbb{Z}/2\mathbb{Z}$-central extension of the quotient $G/\langle z \rangle$, and so by Theorem 2.26 determines a cohomology class $[G, z] \in H^2(G, \langle z \rangle, \mathbb{Z}/2\mathbb{Z})$.

Definition 5.5. We say a non-trivial supergroup (G, z) is *split* if $[G, z] = 0$ in $H^2(G/\langle z \rangle, \mathbb{Z}/2\mathbb{Z})$, and *non-split* otherwise.

In particular, a supergroup (G, z) is split if and only if it can be written as a product $G = G/\langle z \rangle \times \langle z \rangle$.

Example 5.6. Let $z \in SL_2(\mathbb{F}_5)$ denote the non-trivial central element of $SL_2(\mathbb{F}_5)$, then $(SL_2(\mathbb{F}_5), z)$ is a non-trivial supergroup. In this situation, the central extension of Eq. (5.4) gives the following:

$$1 \to \langle z \rangle \to SL_2(\mathbb{F}_5) \to A_5 \to 1. \tag{5.5}$$

This central extension corresponds to the non-trivial element of $H^2(A_5, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}/2\mathbb{Z}$. In other words, $(SL_2(\mathbb{F}_5), z)$ is a non-split supergroup.

Definition 5.7. A *supergroup homomorphism* $(G, z) \to (H, w)$ is a group homomorphism $f : G \to H$ such that $f(z) = w$.

Remark 5.8. Any non-trivial supergroup is determined by a pair (G, α), where G is a group and $[\alpha] \in H^2(G, \mathbb{Z}/2\mathbb{Z})$. Indeed, this data describes a central extension $1 \to \mathbb{Z}/2\mathbb{Z} \hookrightarrow \widetilde{G} \to G \to 1$ of G by $\mathbb{Z}/2\mathbb{Z}$, and so $\widetilde{G}$ is a non-trivial supergroup in the sense of Definition 5.3. From this viewpoint, a supergroup homomorphism $f : (G, \alpha) \to (H, \beta)$ is a group homomorphism $G \to H$ such that $f^*(\beta) = \alpha$.

Remark 5.9. Due to Theorem 5.4, we will restrict our attention to finite supergroups; whenever we refer to a supergroup from now on, we always mean a *finite* supergroup.

Definition 5.10. Let sGrp be the category whose objects are finite supergroups, and whose morphisms are the supergroup homomorphisms between them. We call sGrp the *category of finite supergroups*.

We proceed by describing the functorial nature of the association $(G, z) \mapsto \mathrm{Rep}(G, z)$. Suppose $f : G \to H$ is a group homomorphism. Then f^* induces a braided monoidal functor $f^* : \mathrm{Rep}(H) \to \mathrm{Rep}(G)$. Concretely, given $V \in \mathrm{Rep}(H)$, let $f^*(V) = V$ as a vector space, where the G-action is given by $g \cdot v := f(g) \cdot v$ for all $g \in G$ and $v \in f^*(V)$. On morphisms, if $A : V \to V'$ is an H-linear map, then A is G-linear, considered as a map $f^*(V) \to f^*(V')$.

Suppose now that $f : (G, z) \to (H, w)$ is a supergroup homomorphism. Since $\mathrm{Rep}(G)$ and $\mathrm{Rep}(G, z)$ (similarly, $\mathrm{Rep}(H)$ and $\mathrm{Rep}(H, w)$) have the same monoidal structure, we have a monoidal functor $f^* : \mathrm{Rep}(H, w) \to \mathrm{Rep}(G, z)$. It is natural to ask whether f^* is braided.

Lemma 5.11. *Suppose $f : (G, z) \to (H, w)$ is a supergroup homomorphism. Then $f^* : \mathrm{Rep}(H, w) \to \mathrm{Rep}(G, z)$ is a braided monoidal functor.*

Proof. Showing that f^* is braided (see Definition 2.52) reduces to showing that:

$$c^z_{f^*(U),f^*(V)}(u \otimes v) = f^*(c^w_{U,V})(u \otimes v) \text{ for all } u \in U,\ v \in V, \tag{5.6}$$

for irreducible representations U and V of H. This follows immediately from the requirement that $f(z) = w$. □

The following lemma is obvious.

Lemma 5.12. *Let $f_1 : (G, z) \to (H, w)$ and $f_2 : (H, w) \to (K, x)$ be supergroup homomorphisms. Then $(f_2 \circ f_1)^* = f_1^* \circ f_2^*$ as functors $\mathrm{Rep}(K, x) \to \mathrm{Rep}(G, z)$.*

Lemmas 5.11 and 5.12 together say that the association $(G, z) \mapsto \mathrm{Rep}(G, z)$ forms part of a contravariant functor between sGrp and the category of braided fusion categories.

The First Cohomology Group of a Supergroup

Definition 5.13. Let $\mathcal{C}$ be a fusion category. We denote by $\underline{\mathrm{Inv}(\mathcal{C})}$ the (finite) group of isomorphism classes of invertible objects in $\mathcal{C}$.

If G is a finite group, recall from Example 2.34 that invertible objects of $\mathrm{Rep}(G)$ are precisely the one-dimensional representations of G. This gives an isomorphism:

$$\underline{\mathrm{Inv}(\mathrm{Rep}(G))} \xrightarrow{\sim} \mathrm{Hom}(G, \mathbb{C}^\times). \tag{5.7}$$

Recall that $H^1(G, \mathbb{C}^\times) = \mathrm{Hom}(G, \mathbb{C}^\times)$ for the trivial G-action on $\mathbb{C}^\times$. Thus we have an isomorphism:

$$H^1(G, \mathbb{C}^\times) \xrightarrow{\sim} \underline{\mathrm{Inv}(\mathrm{Rep}(G))}. \tag{5.8}$$

This isomorphism motivates the following definition.

Definition 5.14. Suppose $\mathcal{E}$ is a symmetric fusion category. We define the *first cohomology* of $\mathcal{E}$ to be:

$$H^1_{\mathrm{sym}}(\mathcal{E}) := \underline{\mathrm{Inv}(\mathcal{E})}, \tag{5.9}$$

the group of invertible objects of $\mathcal{E}$. Given a finite supergroup (G, z), we define the *first cohomology* of (G, z) to be:

$$H^1(G, z) := H^1_{\mathrm{sym}}(\mathrm{Rep}(G, z)) = \underline{\mathrm{Inv}(\mathrm{Rep}(G, z))}. \tag{5.10}$$

We will show that $H^1 : \mathrm{sGrp} \to \mathrm{Ab}$ is a contravariant functor. In particular, given a supergroup homomorphism $f : (G, z) \to (H, w)$, we construct an induced map $H^1(f) : H^1(H, w) \to H^1(G, z)$ on first cohomology.

Lemma 5.15. *Let $f : (G, z) \to (H, w)$ be a supergroup homomorphism. If $U \in \mathrm{Rep}(H, w)$ is invertible, then so is $f^*(U) \in \mathrm{Rep}(G, z)$. In particular, restricting f^* to invertible objects determines a homomorphism:*

$$H^1(f) : \underline{\mathrm{Inv}(\mathrm{Rep}(H, w))} \to \underline{\mathrm{Inv}(\mathrm{Rep}(G, z))}. \tag{5.11}$$

Proof. If $U \in \mathrm{Rep}(H, w)$ is invertible, then $1 = \mathrm{FPdim}(U) = \mathrm{FPdim}(f^*(U))$ implies $f^*(U)$ is invertible, by Lemma 2.41. Restricting f^* to invertible thus determines a map $H^1(f) : \underline{\mathrm{Inv}(\mathrm{Rep}(H, w))} \to \underline{\mathrm{Inv}(\mathrm{Rep}(G, z))}$. That f^* is monoidal implies $H^1(f)$ is a group homomorphism. $\square$

Thus we have the following (obvious) result.

Theorem 5.16. $H^1 : \mathrm{sGrp} \to \mathrm{Ab}$ *is a contravariant functor.*

Recall that any finite group G can be thought as a trivial supergroup of the form $(G,1)$. In this case:

$$H^1(G,1) = \underline{\mathrm{Inv}(\mathrm{Rep}(G))} = \mathrm{Hom}(G,\mathbb{C}^\times) = H^1(G,\mathbb{C}^\times). \tag{5.12}$$

Thus in the case of trivial supergroups, our definition of first cohomology agrees with ordinary first group cohomology.

Example 5.17. If (G,z) is a non-trivial split supergroup, then $\mathrm{Rep}(G,z) \xrightarrow{\sim} \mathrm{Rep}(G/\langle z\rangle) \boxtimes \mathrm{sVec}$, so writing $\widetilde{G} = G/\langle z\rangle$, we have

$$\underline{\mathrm{Inv}(\mathrm{Rep}(G,z))} = \underline{\mathrm{Inv}(\mathrm{Rep}(\widetilde{G}))} \times \underline{\mathrm{Inv}(\mathrm{sVec})} = H^1(\widetilde{G},\mathbb{C}^\times) \times \mathbb{Z}/2\mathbb{Z}. \tag{5.13}$$

The Second Cohomology Group of a Supergroup

If R is a commutative ring, then every left R-module is automatically an R-bimodule. Similarly, if $\mathcal{B}$ is a braided fusion category, then every left $\mathcal{B}$-module category is automatically a $\mathcal{B}$-bimodule category, and so it makes sense to take the tensor product of $\mathcal{B}$-module categories (see Definition 2.87).

Definition 5.18 ([ENO10, Definition 4.1])**.** Let $\mathcal{B}$ be a braided fusion category. We say that a $\mathcal{B}$-module category $\mathcal{M}$ is *invertible* if there exists a $\mathcal{B}$-module category $\mathcal{N}$ such that:

$$\mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N} \xrightarrow{\sim} \mathcal{N} \boxtimes_{\mathcal{B}} \mathcal{M} \xrightarrow{\sim} \mathcal{B} \tag{5.14}$$

as $\mathcal{B}$-module categories.

Definition 5.19 ([ENO10, §4.4])**.** Let $\mathcal{B}$ be a braided fusion category. Let the *Picard group* $\mathrm{Pic}(\mathcal{B})$ of $\mathcal{B}$ be the group of invertible $\mathcal{B}$-module categories.

In [Gre10], Greenough proved that the Picard group of $\mathrm{Rep}(G)$ is isomorphic to $H^2(G,\mathbb{C}^\times)$. We proceed by describing this isomorphism in detail.

Definition 5.20. Let G be a group. A *projective representation* of G over $\mathbb{C}$ is a $\mathbb{C}$-vector space V, together with a group homomorphism $\rho : G \to \mathrm{PGL}(V)$.

Recall that $\mathrm{PGL}(V)$ is the quotient of $\mathrm{GL}(V)$ by the subgroup of non-zero scalar matrices. Given a projective representation $\rho : G \to \mathrm{PGL}(V)$, choose lifts $L(g) \in \mathrm{GL}(V)$ for $g \in G$. These lifts then satisfy:

$$L(gh) = \alpha(g,h)L(g)L(h), \;\; g,h \in G, \tag{5.15}$$

for scalars $\alpha(g,h) \in \mathbb{C}^\times$. A straightforward computation shows that $\alpha : G \times G \to \mathbb{C}^\times$ is a 2-cocycle on G with values in $\mathbb{C}^\times$. In this situation, we say that V is a *projective representation of G with 2-cocycle α.*

Definition 5.21. Suppose G is a finite group, and let (H, α) be a pair with $H \leq G$ a subgroup and $[\alpha] \in H^2(H, \mathbb{C}^\times)$. Let $\mathcal{M}(H, \alpha)$ denote the category of projective representations of H with 2-cocycle α.

The category $\mathcal{M}(H, \alpha)$ can be given the structure of a $\mathrm{Rep}(G)$-module category (see Definition 2.81). Given a projective representation V of H with 2-cocycle α, and an (ordinary) representation W of G, define $W \odot V := \mathrm{Res}^G_H W \otimes V$ with diagonal H-action. Then $\mathrm{Res}^G_H W \otimes V$ is a projective representation of H with 2-cocycle α. This makes $\mathcal{M}(H, \alpha)$ into semisimple indecomposable $\mathrm{Rep}(G)$-module category, and every semisimple indecomposable $\mathrm{Rep}(G)$-module category is of this form [Ost03, Theorem 3.2] [EGNO15, Corollary 7.12.20].

The following is [Gre10, Corollary 8.11].

Theorem 5.22. *Let G be a finite group. The* $\mathrm{Rep}(G)$*-module categories of the form $\mathcal{M}(G, \alpha)$ with $[\alpha] \in H^2(G, \mathbb{C}^\times)$ are invertible, and the map*

$$\begin{gathered} H^2(G, \mathbb{C}^\times) \to \mathrm{Pic}(\mathrm{Rep}(G)) \\ [\alpha] \mapsto \mathcal{M}(G, \alpha), \end{gathered} \tag{5.16}$$

is an isomorphism.

Theorem 5.22 suggests the following definition.

Definition 5.23. Suppose $\mathcal{E}$ is a symmetric fusion category. We define the *second cohomology* of $\mathcal{E}$ to be the Picard group of $\mathcal{E}$:

$$H^2_{\mathrm{sym}}(\mathcal{E}) := \mathrm{Pic}(\mathcal{E}). \tag{5.17}$$

Given a finite supergroup (G, z), we define the *second cohomology* of (G, z) to be:

$$H^2(G, z) := H^2_{\mathrm{sym}}(\mathrm{Rep}(G, z)) = \mathrm{Pic}(\mathrm{Rep}(G, z)). \tag{5.18}$$

As was the case for first cohomology, we will show that $H^2 : \mathrm{sGrp} \to \mathrm{Ab}$ is a contravariant functor. To define the induced map on second cohomology, we first describe how a supergroup homomorphism $f : (G, z) \to (H, w)$ endows $\mathrm{Rep}(G, z)$ with the structure of a $(\mathrm{Rep}(G, z), \mathrm{Rep}(H, w))$-bimodule category.

Lemma 5.24. *Let $f : (G, z) \to (H, w)$ be a supergroup homomorphism, and let $f^* : \mathrm{Rep}(H, w) \to \mathrm{Rep}(G, z)$ be the braided monoidal functor induced by f. Define a right action of $\mathrm{Rep}(H, w)$ on $\mathrm{Rep}(G, z)$ by the formula*

$$X \odot V := X \otimes f^*(V), \;\; X \in \mathrm{Rep}(G, z), \;\; V \in \mathrm{Rep}(H, w)$$

Then $\mathrm{Rep}(G, z)$ is a $(\mathrm{Rep}(G, z), \mathrm{Rep}(H, w))$-bimodule category.

Proof. The tensor product structure on the contravariant functor $f^* : \mathrm{Rep}(H, w) \to \mathrm{Rep}(G, z)$ ensures that the formula given above endows $\mathrm{Rep}(G, z)$ with a right $\mathrm{Rep}(H, w)$-module category structure. The left and right actions are compatible via the associativity isomorphism of $\mathrm{Rep}(G, z)$. □

We will denote by $\mathrm{Rep}(G, z)_f$ the category $\mathrm{Rep}(G, z)$ with the $(\mathrm{Rep}(G, z), \mathrm{Rep}(H, w))$-bimodule category structure induced by f, as described in Lemma 5.24. We suggest the following definition for the induced map on second cohomology.

Definition 5.25. Suppose $f : (G,z) \to (H,w)$ is a supergroup homomorphism. Define $H^2(f) : \mathrm{Pic}(\mathrm{Rep}(H,w)) \to \mathrm{Pic}(\mathrm{Rep}(G,z))$ by the formula:

$$H^2(f)(\mathcal{M}) := \mathrm{Rep}(G,z)_f \boxtimes_{\mathrm{Rep}(H,w)} \mathcal{M},\ \mathcal{M} \in \mathrm{Pic}(\mathrm{Rep}(H,w)). \tag{5.19}$$

Remark 5.26. This definition is similar to the notion of *Picard induction* described in [MN18, §2.5]. Given a braided fusion category $\mathcal{B}$ and a fusion subcategory $\mathcal{D} \subset \mathcal{B}$, the Picard induction homomorphism $\mathrm{Pic}(\mathcal{D}) \to \mathrm{Pic}(\mathcal{B})$ is defined by $\mathcal{M} \mapsto \mathcal{B} \boxtimes_{\mathcal{D}} \mathcal{M}$.

Our immediate goal is to prove the following proposition.

Proposition 5.27. *Let $f : (G,z) \to (H,w)$ be a supergroup homomorphism. Then:*

(i) If $\mathcal{M} \in \mathrm{Pic}(\mathrm{Rep}(H,w))$, then $H^2(f)(\mathcal{M}) \in \mathrm{Pic}(\mathrm{Rep}(G,z))$, and

(ii) $H^2(f) : \mathrm{Pic}(\mathrm{Rep}(H,w)) \to \mathrm{Pic}(\mathrm{Rep}(G,z))$ is a group homomorphism.

The following lemma will be useful in the proof of Proposition 5.27.

Lemma 5.28. *Let $f : (G,z) \to (H,w)$ be a supergroup homomorphism, and let $\mathcal{M}, \mathcal{N} \in \mathrm{Pic}(\mathrm{Rep}(H,w))$. Then we have an equivalence:*

$$H^2(f)(\mathcal{M}) \boxtimes_{\mathrm{Rep}(G,z)} H^2(f)(\mathcal{N}) \xrightarrow{\sim} H^2(f)(\mathcal{M} \boxtimes_{\mathrm{Rep}(H,w)} \mathcal{N}) \tag{5.20}$$

of $\mathrm{Rep}(G,z)$-module categories.

Proof. We have the following chain of $\operatorname{Rep}(G,z)$-module equivalences:

$$\begin{aligned}
&H^2(f)(\mathcal{M} \boxtimes_{\operatorname{Rep}(H,w)} \mathcal{N}) \\
&\qquad = \operatorname{Rep}(G,z)_f \boxtimes_{\operatorname{Rep}(H,w)} (\mathcal{M} \boxtimes_{\operatorname{Rep}(H,w)} \mathcal{N}) \\
&\qquad \xrightarrow{\sim} (\operatorname{Rep}(G,z)_f \boxtimes_{\operatorname{Rep}(H,w)} \mathcal{M}) \boxtimes_{\operatorname{Rep}(H,w)} \mathcal{N} \\
&\qquad \xrightarrow{\sim} (\operatorname{Rep}(G,z)_f \boxtimes_{\operatorname{Rep}(H,w)} \mathcal{M}) \boxtimes_{\operatorname{Rep}(G,z)} (\operatorname{Rep}(G,z)_f \boxtimes_{\operatorname{Rep}(H,w)} \mathcal{N}) \\
&\qquad = H^2(f)(\mathcal{M}) \boxtimes_{\operatorname{Rep}(G,z)} H^2(f)(\mathcal{N}).
\end{aligned} \tag{5.21}$$

□

Proof of Proposition 5.27. For both (i) and (ii) we will need that $H^2(\operatorname{Rep}(H,w)) = \operatorname{Rep}(G,z)_f \boxtimes_{\operatorname{Rep}(H,w)} \operatorname{Rep}(H,w) \xrightarrow{\sim} \operatorname{Rep}(G,z)$ as $\operatorname{Rep}(G,z)$-module categories.

(i) Suppose $\mathcal{M}$ is an invertible $\operatorname{Rep}(H,w)$-module category, then there exists a $\operatorname{Rep}(H,w)$-module category $\mathcal{N}$ such that $\mathcal{M} \boxtimes_{\operatorname{Rep}(H,w)} \mathcal{N} \xrightarrow{\sim} \mathcal{N} \boxtimes_{\operatorname{Rep}(H,w)} \mathcal{M} \xrightarrow{\sim} \operatorname{Rep}(H,w)$. By Lemma 5.28, we have:

$$f^2(\mathcal{M}) \boxtimes_{\operatorname{Rep}(G,z)} f^2(\mathcal{N}) \xrightarrow{\sim} f^2(\mathcal{M} \boxtimes_{\operatorname{Rep}(G,z)} \mathcal{N}) \xrightarrow{\sim} f^2(\operatorname{Rep}(H,w)) \xrightarrow{\sim} \operatorname{Rep}(G,z), \tag{5.22}$$

as $\operatorname{Rep}(G,z)$-module categories. A similar computation shows that $f^2(\mathcal{N}) \boxtimes_{\operatorname{Rep}(G,z)} f^2(\mathcal{M}) \xrightarrow{\sim} \operatorname{Rep}(G,z)$ as $\operatorname{Rep}(G,z)$-module categories, so $f^2(\mathcal{M})$ is invertible.

(ii) This follows immediately from Lemma 5.28.

□

Lemma 5.29. *Suppose $f : (G,z) \to (H,w)$ and $g : (H,w) \to (K,x)$ are supergroup homomorphisms. Then there is an equivalence*

$$\operatorname{Rep}(G,z)_f \boxtimes_{\operatorname{Rep}(H,w)} \operatorname{Rep}(H,w)_g \xrightarrow{\sim} \operatorname{Rep}(G,z)_{gf} \tag{5.23}$$

of $(\operatorname{Rep}(G,z), \operatorname{Rep}(K,x))$*-bimodule categories.*

Proof. Let the functor $F : \mathrm{Rep}(G,z)_f \times \mathrm{Rep}(H,w)_g \to \mathrm{Rep}(G,z)_{gf}$ be given by $F(V,W) = V \otimes f^*(W)$, with the obvious map on morphisms. We proceed as follows.

(i) F is a left $\mathrm{Rep}(G,z)$-module functor.

Let $s_{X,(V,W)} : F(X \odot (V,W)) \to X \odot F(V,W)$ be given by $s_{X,(V,W)} = a_{X,V,f^*(W)}$ for $X, V \in \mathrm{Rep}(G,z)$ and $W \in \mathrm{Rep}(H,w)$. Then s satisfies Eq. (2.67) by the pentagon axiom, so F is a left $\mathrm{Rep}(G,z)$-module functor.

(ii) F is a right $\mathrm{Rep}(K,x)$-module functor.

Let $t_{(V,W),X} : F((V,W) \odot X) \to F(V,W) \odot X$ be given by $t_{(V,W),X} = a^{-1}_{V,f^*(W),(gf)^*(X)}$ for $V \in \mathrm{Rep}(G,z)$, $W \in \mathrm{Rep}(H,w)$, and $X \in \mathrm{Rep}(K,x)$. Then t satisfies Eq. (2.67) by the pentagon axiom, so F is a right $\mathrm{Rep}(K,x)$-module functor.

(iii) F is $\mathrm{Rep}(H,w)$-balanced.

Let $b_{V,X,W} : (V \otimes f^*(X)) \otimes f^*(W) \to V \otimes f^*(X \otimes W)$ be given by $b_{V,X,W} = a_{V,f^*(W),f^*(W)}$. The pentagon axiom implies that b satisfies Eq. (2.75), so F is a $\mathrm{Rep}(H,w)$-balanced functor.

Let $B : \mathrm{Rep}(G,z)_f \times \mathrm{Rep}(H,w)_g \to \mathrm{Rep}(G,z)_f \boxtimes_{\mathrm{Rep}(H,w)} \mathrm{Rep}(H,w)_g$ be the canonical $\mathrm{Rep}(H,w)$-balanced functor from Definition 2.87. By (i), (ii), and (iii), we get a $(\mathrm{Rep}(G,z), \mathrm{Rep}(K,x))$-bimodule functor:

$$\overline{F} : \mathrm{Rep}(G,z)_f \boxtimes_{\mathrm{Rep}(H,w)} \mathrm{Rep}(H,w)_g \to \mathrm{Rep}(G,z)_{gf}. \tag{5.24}$$

such that $\overline{F}B = F$. Define $G : \mathrm{Rep}(G,z)_{fg} \to \mathrm{Rep}(G,z)_f \boxtimes \mathrm{Rep}(H,w)_g$ by $G(X) = B(X, \mathbf{1})$, then G is a $(\mathrm{Rep}(G,z), \mathrm{Rep}(K,x))$-bimodule functor. We have natural isomorphisms:

$$\begin{aligned}\overline{F}G(X) &= \overline{F}B(X, \mathbf{1}) \\ &= F(X, \mathbf{1}) \\ &= X \otimes f^*(\mathbf{1}) \\ &\xrightarrow{\sim} X,\end{aligned} \tag{5.25}$$

for $X \in \mathrm{Rep}(G, z)$, and so $\overline{F}G \xrightarrow{\sim} \mathrm{id}$. On the other hand:

$$\begin{aligned}G\overline{F}(B(V, W)) &= GF(V, W) \\ &= G(V \otimes f^*(W)) \\ &= B(V \otimes f^*(W), \mathbf{1}) \\ &= B(V \odot W, \mathbf{1}) \\ &\xrightarrow{\sim} B(V, W \odot \mathbf{1}) \\ &\xrightarrow{\sim} B(V, W)\end{aligned} \tag{5.26}$$

for $V \in \mathrm{Rep}(G, z)$, $W \in \mathrm{Rep}(H, w)$, and so $G\overline{F} \xrightarrow{\sim} \mathrm{id}$. Thus $(\overline{F}, G)$ forms an equivalence of $(\mathrm{Rep}(G, z), \mathrm{Rep}(K, x))$-bimodule categories. $\square$

Remark 5.30. In the case where $g = \mathrm{id}_H : (H, w) \to (H, w)$, the above lemma produces the familiar equivalence:

$$\mathrm{Rep}(G, z)_f \boxtimes_{\mathrm{Rep}(H,w)} \mathrm{Rep}(H, w) \xrightarrow{\sim} \mathrm{Rep}(G, z)_f \tag{5.27}$$

of $(\mathrm{Rep}(G, z), \mathrm{Rep}(H, w))$-bimodule categories.

Lemma 5.31. *Suppose $f : (G, z) \to (H, w)$ and $g : (H, w) \to (K, x)$ are supergroup homomorphisms. Then*

$$H^2(g \circ f) = H^2(f) \circ H^2(g), \tag{5.28}$$

as homomorphisms $\mathrm{Pic}(\mathrm{Rep}(K, x)) \to \mathrm{Pic}(\mathrm{Rep}(G, z))$.

Proof. Let $\mathcal{M} \in \mathrm{Pic}(\mathrm{Rep}(K, x))$, then by Lemma 5.29:

$$\begin{aligned}(H^2(f) \circ H^2(g))(\mathcal{M}) &= \mathrm{Rep}(G, z)_f \boxtimes_{\mathrm{Rep}(H,w)} (\mathrm{Rep}(H, w)_g \boxtimes_{\mathrm{Rep}(K,x)} \mathcal{M}) \\ &\xrightarrow{\sim} (\mathrm{Rep}(G, z)_f \boxtimes_{\mathrm{Rep}(H,w)} \mathrm{Rep}(H, w)_g) \boxtimes_{\mathrm{Rep}(K,x)} \mathcal{M} \\ &\xrightarrow{\sim} \mathrm{Rep}(G, z)_{gf} \boxtimes_{\mathrm{Rep}(K,x)} \mathcal{M} \\ &= H^2(g \circ f)(\mathcal{M}),\end{aligned} \tag{5.29}$$

as required. $\square$

Lemma 5.32. *Let* $\mathrm{id} = \mathrm{id}_{(G,z)}$ *be the identity homomorphism. Then* $H^2(\mathrm{id}) = \mathrm{id}_{H^2(G,z)}$.

Proof. Let $\mathcal{M} \in \mathrm{Pic}(\mathrm{Rep}(G, z))$, then $H^2(\mathrm{id})(\mathcal{M}) = \mathrm{Rep}(G, z) \boxtimes_{\mathrm{Rep}(G,z)} \mathcal{M} \xrightarrow{\sim} \mathcal{M}$. $\square$

Combining Proposition 5.27 and Lemmas 5.31 and 5.32, we get the following.

Theorem 5.33. $H^2 : \mathrm{sGrp} \to \mathrm{Ab}$ *is a contravariant functor.*

Modular Extensions and the Third Cohomology Group

Our goal in this section is to define the third cohomology group of a finite supergroup (G, z) as the group of *modular extensions* of $\mathrm{Rep}(G, z)$. The notion of (minimal) modular extension is to due Müger [Müg03], though we follow the discussion in [LKW17].

Definition 5.34. Let $\mathcal{E}$ be a symmetric fusion category. A *modular extension* of $\mathcal{E}$ is a non-degenerate braided fusion category $\mathcal{M}$, together with a braided full embedding $\iota : \mathcal{E} \hookrightarrow \mathcal{M}$, such that $\mathcal{E}'|_{\mathcal{M}} = \mathcal{E}$.

Remark 5.35. Extensions as in the previous definition are typically called *non-degenerate extensions*. We will abuse terminology and always refer to them as modular extensions.

Theorem 5.36 ([Müg03],[DGNO10])**.** *Let $\mathcal{C}$ be a non-degenerate braided fusion category, and $\mathcal{D} \subset \mathcal{C}$ a fusion subcategory. Then*

$$\mathrm{FPdim}(\mathcal{D})\mathrm{FPdim}(\mathcal{D}') = \mathrm{FPdim}(\mathcal{C}). \tag{5.30}$$

Corollary 5.37. *Let $\mathcal{E}$ be a symmetric fusion category, $\mathcal{M}$ a non-degenerate braided fusion category, and $\iota : \mathcal{E} \hookrightarrow \mathcal{M}$ a braided full embedding. Then $(\mathcal{M}, \iota)$ is a modular extension if and only if* $\mathrm{FPdim}(\mathcal{M}) = \mathrm{FPdim}(\mathcal{E})^2$.

In particular, this implies that all non-degenerate extensions of $\mathcal{E}$ have the same Frobenius-Perron dimension, and so $\mathcal{E}$ has finitely many non-degenerate extensions.

Example 5.38. Let $\mathcal{E}$ be a symmetric fusion category, and let $\mathcal{Z}(\mathcal{E})$ be the Drinfeld center of $\mathcal{E}$ (see Definition 2.58). Let $\iota_0 : \mathcal{E} \hookrightarrow \mathcal{Z}(\mathcal{E})$ denote the canonical braided full embedding $X \mapsto (X, c_{X,-})$. By Theorem 2.59 we know that $\mathcal{Z}(\mathcal{E})$ is a non-degenerate braided fusion category, and $\mathrm{FPdim}(\mathcal{Z}(\mathcal{E})) = \mathrm{FPdim}(\mathcal{E})^2$, so $(\mathcal{Z}(\mathcal{E}), \iota_0)$ is a modular extension of $\mathcal{E}$.

In particular, the set of modular extensions of a fixed symmetric fusion category $\mathcal{E}$ is non-empty.

Definition 5.39 ([LKW17, Definition 4.9])**.** We say that two modular extensions $(\mathcal{M}, \iota_{\mathcal{M}})$, $(\mathcal{N}, \iota_{\mathcal{N}})$ of $\mathcal{E}$ are *equivalent* if there is a braided equivalence $f : \mathcal{M} \to \mathcal{N}$ such that $f \circ \iota_M \simeq \iota_N$.

Group of Modular Extensions. Let $\mathcal{M}_{\mathrm{ext}}(\mathcal{E})$ denote the set of equivalence classes of modular extensions of a symmetric fusion category $\mathcal{E}$, then we have seen that $\mathcal{M}_{\mathrm{ext}}(\mathcal{E})$ is non-empty and finite. In [LKW17], Lan, Kong, and

Wen constructed a multiplication on $\mathcal{M}_{\text{ext}}(\mathcal{E})$, making it into a finite abelian group. We proceed by describing this group structure.

Recall that the tensor product functor

$$\otimes : \mathcal{E} \boxtimes \mathcal{E} \to \mathcal{E} \tag{5.31}$$

on a symmetric fusion category $\mathcal{E}$ is braided [JS93, Proposition 5.4]. Let $I : \mathcal{E} \to \mathcal{E} \boxtimes \mathcal{E}$ denote the right adjoint functor to $\otimes$, then

$$L_{\mathcal{E}} := I(\mathbf{1}_{\mathcal{E}}) = \oplus_{X \in \mathcal{O}(\mathcal{E})} X \boxtimes X^* \tag{5.32}$$

is a connected étale algebra in $\mathcal{E} \boxtimes \mathcal{E}$ by Lemma 2.74. Observe that $L_{\mathcal{E}} \cap (\mathcal{E} \boxtimes \mathbf{1}_{\mathcal{E}}) = \mathbf{1}_{\mathcal{E}} \boxtimes \mathbf{1}_{\mathcal{E}}$, so by [LKW17, Proposition 3.4] we obtain a braided full embedding:

$$\begin{aligned} \iota : \mathcal{E} &\hookrightarrow (\mathcal{E} \boxtimes \mathcal{E})_{L_{\mathcal{E}}} = (\mathcal{E} \boxtimes \mathcal{E})^{\text{loc}}_{L_{\mathcal{E}}} \\ X &\mapsto (X \boxtimes \mathbf{1}_{\mathcal{E}}) \otimes L_{\mathcal{E}}, \end{aligned} \tag{5.33}$$

where $(X \boxtimes \mathbf{1}_{\mathcal{E}}) \otimes L_{\mathcal{E}}$ is the free $L_{\mathcal{E}}$-module described in Example 2.67. Theorem 2.72 implies $\text{FPdim}((\mathcal{E} \boxtimes \mathcal{E})^{\text{loc}}_{L_{\mathcal{E}}}) = \text{FPdim}(\mathcal{E})$, so ι is a braided equivalence. We therefore have a braided embedding:

$$\iota_{\mathcal{M}} \boxtimes_{\mathcal{E}} \iota_{\mathcal{N}} : \mathcal{E} = (\mathcal{E} \boxtimes \mathcal{E})^{\text{loc}}_{L_{\mathcal{E}}} \xrightarrow{\iota_{\mathcal{M}} \boxtimes \iota_{\mathcal{N}}} (\mathcal{M} \boxtimes \mathcal{N})^{\text{loc}}_{L_{\mathcal{E}}}. \tag{5.34}$$

That $\mathcal{M} \boxtimes \mathcal{N}$ is non-degenerate implies $(\mathcal{M} \boxtimes \mathcal{N})^{\text{loc}}_{L_{\mathcal{E}}}$ is non-degenerate [DMNO13, Corollary 3.30]. We therefore define:

$$\mathcal{M} \boxtimes_{\mathcal{E}}^{(\iota_{\mathcal{M}}, \iota_{\mathcal{N}})} \mathcal{N} := ((\mathcal{M} \boxtimes \mathcal{N})^{\text{loc}}_{L_{\mathcal{E}}}, \iota_{\mathcal{M}} \boxtimes_{\mathcal{E}} \iota_{\mathcal{N}}). \tag{5.35}$$

Theorem 5.40 ([LKW17, Theorem 4.20])**.** *The multiplication $\boxtimes_{\mathcal{E}}^{(-,-)} : \mathcal{M}_{\text{ext}}(\mathcal{E}) \times \mathcal{M}_{\text{ext}}(\mathcal{E}) \to \mathcal{M}_{\text{ext}}(\mathcal{E})$ defined above makes $\mathcal{M}_{\text{ext}}(\mathcal{E})$ into a finite abelian group with identity element $(\mathcal{Z}(\mathcal{E}), \iota_0)$.*

The following example will motivate our definition of the third cohomology of a supergroup.

Example 5.41 ([LKW17, §4.3])**.** Let G be a finite group. Suppose $(\mathcal{M}, \iota_{\mathcal{M}})$ is a modular extension of $\mathrm{Rep}(G)$. Let $A = \mathrm{Fun}(G) \in \mathrm{Rep}(G) \subset \mathcal{M}$, then by Theorem 2.99 the de-equivariantization $\mathcal{M}_A$ is a braided G-crossed fusion category. In particular, $\mathcal{M}_A$ admits a G-grading $\mathcal{M}_A = \bigoplus_{g\in G}(\mathcal{M}_A)_g$. Since $\mathcal{M}$ is non-degenerate, the trivial component $(\mathcal{M}_A)_1$ is non-degenerate, and the G-grading on $\mathcal{M}_A$ is faithful by Proposition 2.101. Thus $\mathrm{FPdim}((\mathcal{M}_A)_g) = 1$ for all $g \in G$ [ENO05, Proposition 8.20], which implies that $\mathcal{M}_A$ is pointed with underlying group of simple objects G. Thus $\mathcal{M}_A \xrightarrow{\sim} \mathrm{Vec}_G^{\omega_{(\mathcal{M},\iota_{\mathcal{M}})}}$ for some $[\omega_{(\mathcal{M},\iota_{\mathcal{M}})}] \in H^3(G, \mathbb{C}^\times)$, and so we get an isomorphism [LKW17, Theorem 4.22]:

$$\begin{aligned} \mathcal{M}_{\mathrm{ext}}(\mathrm{Rep}(G)) &\xrightarrow{\sim} H^3(G, \mathbb{C}^\times) \\ (\mathcal{M}, \iota_{\mathcal{M}}) &\mapsto [\omega_{(M,\iota_{\mathcal{M}})}]. \end{aligned} \tag{5.36}$$

The Third Cohomology Group of a Supergroup.

Definition 5.42. Suppose $\mathcal{E}$ is a symmetric fusion category. We define the *third cohomology* of $\mathcal{E}$ to be

$$H^3_{\mathrm{sym}}(\mathcal{E}) := \mathcal{M}_{\mathrm{ext}}(\mathcal{E}), \tag{5.37}$$

the group of modular extensions of $\mathcal{E}$. Given a finite supergroup (G, z), we define the *third cohomology* of (G, z) to be:

$$H^3(G, z) := H^3_{\mathrm{sym}}(\mathrm{Rep}(G, z)) = \mathcal{M}_{\mathrm{ext}}(\mathrm{Rep}(G, z)). \tag{5.38}$$

We will show that $H^3 : \mathrm{sGrp} \to \mathrm{Ab}$ is a contravariant functor. To define the induced map on third cohomology, we must first describe some useful properties of the category $\mathrm{Rep}(G, z)$. To simplify our notation, we will sometimes write $\mathcal{E}_G$ for the category $\mathrm{Rep}(G, z)$. Recall from Example 5.2 that if X, Y are simple objects of $\mathrm{Rep}(G, z)$, then the braiding on $\mathrm{Rep}(G, z)$ is given by the formula:

$$c^z_{X,Y}(x \otimes y) = (-1)^{mn} y \otimes x,\ \ x \in X, y \in Y,\ \ z \cdot x = (-1)^m x,\ \ z \cdot y = (-1)^n y, \tag{5.39}$$

with m, n equal to zero or one. Write $|x| = m$ and $|y| = n$, then $c^z_{X,Y}(x \otimes y) = (-1)^{|x||y|} y \otimes x$. The following two lemmas are clear.

Lemma 5.43. *Let (G, z) be a supergroup, and X a simple object of* $\mathrm{Rep}(G, z)$*. Then $|g \cdot x| = |x|$ for all $g \in G$ and $x \in X$.*

Lemma 5.44. *Let (G, z) and (H, w) be supergroups, X a simple object of* $\mathrm{Rep}(G, z)$*, Y a simple object of* $\mathrm{Rep}(H, w)$*. Then $X \otimes_{\mathbb{C}} Y$ is a simple object of* $\mathrm{Rep}(G \times H, (z, w))$*, and $|x \otimes y| = |x| + |y|$ for all $x \in X$ and $y \in Y$.*

Proposition 5.45. *Let (G, a) and (H, b) be supergroups. Then there is a braided equivalence*

$$\widetilde{K} : \mathrm{Rep}(G, z) \boxtimes \mathrm{Rep}(H, w) \to \mathrm{Rep}(G \times H, (z, w)) \tag{5.40}$$

sending $V \boxtimes W$ to $V \otimes_{\mathbb{C}} W$.

Proof. Given $V \in \mathcal{E}_{G,z}$ and $W \in \mathcal{E}_{H,w}$, let $\widetilde{K}(V \boxtimes W) = V \otimes_{\mathbb{C}} W$ with the obvious $G \times H$-action. It is well-known that $\widetilde{K}$ is an equivalence of abstract categories (every irreducible representation of $G \times H$ is of the form $V \otimes_{\mathbb{C}} W$ with V and W irreducible representations of G and H respectively). We need only show that $\widetilde{K}$ is a braided monoidal functor.

We define a monoidal structure on $\widehat{K}$ by the formula:

$$\begin{aligned} J_{X_1 \boxtimes X_2, Y_1 \boxtimes Y_2} : \widetilde{K}(X_1 \boxtimes X_2) \otimes \widetilde{K}(Y_1 \boxtimes Y_2) &\to \widetilde{K}((X_1 \otimes Y_1) \boxtimes (X_2 \otimes Y_2)) \\ (x_1 \otimes x_2) \otimes (y_1 \otimes y_2) &\mapsto (-1)^{|x_2||y_1|} (x_1 \otimes y_1) \otimes (x_2 \otimes y_2). \end{aligned} \tag{5.41}$$

That J is $G \times H$-linear follows from a straightforward computation and Lemma 5.43. That J satisfies the monoidal structure axiom (Definition 2.13) reduces to showing that

$$|x_2||y_1| + |x_2 \otimes y_2||z_1| \equiv |y_2||z_1| + |x_2||y_1 \otimes z_1| \pmod 2, \tag{5.42}$$

which follows from Lemma 5.44. That $\widetilde{K}$ is braided reduces to showing that

$$|x_2||y_1|+|x_1||y_1|+|x_2||y_2| \equiv |y_2||x_1|+|x_1 \otimes x_2||y_1 \otimes y_2| \pmod 2, \tag{5.43}$$

which also follows from Lemma 5.44. $\square$

Suppose $f : (G,z) \to (H,w)$ be a supergroup homomorphism. We are now ready to describe the induced map $H^3(f) : \mathcal{M}_{\mathrm{ext}}(\mathrm{Rep}(H,w)) \to \mathcal{M}_{\mathrm{ext}}(\mathrm{Rep}(G,z))$. Let R_f be right adjoint to the braided functor given by the composition:

$$\mathrm{Rep}(H,w) \boxtimes \mathrm{Rep}(G,z) \xrightarrow{f^* \boxtimes \mathrm{id}} \mathrm{Rep}(G,z) \boxtimes \mathrm{Rep}(G,z) \xrightarrow{\otimes} \mathrm{Rep}(G,z). \tag{5.44}$$

Let $A_f := R_f(\mathbf{1}_{\mathcal{E}_{G,z}})$, then A_f is a connected étale algebra in $\mathrm{Rep}(H,w) \boxtimes \mathrm{Rep}(G,z)$ (Lemma 2.74) and $\mathrm{FPdim}(A_f) = |H|$ ([EGNO15, Lemma 6.2.4]).

Suppose $(\mathcal{M}, \iota_{\mathcal{M}}) \in \mathcal{M}_{\mathrm{ext}}(\mathrm{Rep}(H,w))$. For $X \in \mathrm{Rep}(G,z)$, we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{M}}(\mathbf{1} \boxtimes X, A_f) &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Rep}(G,z)}(f^*(\mathbf{1}) \otimes X, \mathbf{1}) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Rep}(G,z)}(f^*(\mathbf{1}), X^*) \end{aligned} \tag{5.45}$$

Since $f^*(\mathbf{1}) = \mathbf{1}$, we get that $\mathbf{1} \boxtimes X$ is a summand of A_f if and only if $X = \mathbf{1}$, and so $(\mathbf{1} \boxtimes \mathrm{Rep}(G,z)) \cap A_f = \mathbf{1} \boxtimes \mathbf{1}$. Thus the free module functor

$$\begin{aligned} \mathrm{Rep}(G,z) &\to (\mathrm{Rep}(H,w) \boxtimes \mathrm{Rep}(G,z))^{\mathrm{loc}}_{A_f} = (\mathrm{Rep}(H,w) \boxtimes \mathrm{Rep}(G,z))_{A_f} \\ X &\mapsto (\mathbf{1} \boxtimes X) \otimes A_f \end{aligned} \tag{5.46}$$

is a braided full embedding [LKW17, Proposition 3.4]. Moreover,

$$\begin{aligned} \mathrm{FPdim}((\mathrm{Rep}(H,w) \boxtimes \mathrm{Rep}(G,z))_{A_f}) &= |G| \\ &= \mathrm{FPdim}(\mathrm{Rep}(G,z)), \end{aligned} \tag{5.47}$$

so $(\mathbf{1} \boxtimes -) \otimes A_f$ is a braided equivalence $\mathrm{Rep}(G,z) \xrightarrow{\sim} (\mathrm{Rep}(H,w) \boxtimes \mathrm{Rep}(G,z))^{\mathrm{loc}}_{A_f}$, and so we obtain a braided full embedding

$$\begin{aligned} f^*(\iota_{\mathcal{M}}) : \mathrm{Rep}(G,z) &\xrightarrow{(\mathbf{1} \boxtimes -) \otimes A_f} (\mathrm{Rep}(H,w) \boxtimes \mathrm{Rep}(G,z))^{\mathrm{loc}}_{A_f} \\ &\xrightarrow{\iota_{\mathcal{M}} \boxtimes \iota_G} (\mathcal{M} \boxtimes \mathcal{Z}(\mathrm{Rep}(G,z)))^{\mathrm{loc}}_{A_f}, \end{aligned} \tag{5.48}$$

where $\iota_G : \mathrm{Rep}(G,z) \hookrightarrow \mathcal{Z}(\mathrm{Rep}(G,z))$ is the canonical embedding.

Theorem 5.46. *Let $f : (G,z) \to (H,w)$ be a supergroup homomorphism, and $\mathcal{M}$ a modular extension of* $\mathrm{Rep}(H,w)$*. Then $f^*(\mathcal{M},\iota_{\mathcal{M}}) := (\mathcal{M} \boxtimes \mathcal{Z}(\mathrm{Rep}(G,z)))^{\mathrm{loc}}_{A_f}, f^*(\iota_{\mathcal{M}}))$ is a modular extension of* $\mathrm{Rep}(G,z)$*.*

Proof. By Lemma 2.78, we have

$$\mathrm{FPdim}((\mathcal{M} \boxtimes \mathcal{Z}(\mathrm{Rep}(G,z))))^{\mathrm{loc}}_{A_f} = |G|^2 = \mathrm{FPdim}(\mathrm{Rep}(G,z))^2, \tag{5.49}$$

so $(\mathcal{M}\boxtimes\mathcal{Z}(\mathrm{Rep}(G,z)))^{\mathrm{loc}}_{A_f}$ is a modular extension of $\mathrm{Rep}(G,z)$ by Corollary 5.37. □

Thus every supergroup homomorphism induces a map between the corresponding groups of modular extensions:

$$\begin{aligned} H^3(f) : \mathcal{M}_{\mathrm{ext}}(\mathrm{Rep}(H,w)) &\to \mathcal{M}_{\mathrm{ext}}(\mathrm{Rep}(G,z)) \\ (\mathcal{M},\iota_{\mathcal{M}}) &\mapsto f^*(\mathcal{M},\iota_{\mathcal{M}}). \end{aligned} \tag{5.50}$$

Theorem 5.47. *Suppose $f : (G,z) \to (H,w)$ is a supergroup homomorphism. Then*

$$H^3(f) : \mathcal{M}_{\mathrm{ext}}(\mathrm{Rep}(H,w)) \to \mathcal{M}_{\mathrm{ext}}(\mathrm{Rep}(G,z)) \tag{5.51}$$

is a group homomorphism.

Proof. Let $(\mathcal{M},\iota_{\mathcal{M}}), (\mathcal{N},\iota_{\mathcal{N}}) \in \mathcal{M}_{\mathrm{ext}}(\mathcal{E}_H)$, then

$$H^3(f)(\mathcal{M}) \boxtimes_{\mathcal{E}_G} H^3(f)(\mathcal{N}) = ((\mathcal{M} \boxtimes Z(\mathcal{E}_G))^{\mathrm{loc}}_{A_f} \boxtimes (\mathcal{N} \boxtimes \mathcal{Z}(\mathcal{E}_G))^{\mathrm{loc}}_{A_f})^{\mathrm{loc}}_{L_{\mathcal{E}_G}} \tag{5.52}$$

Let L_1 be the braided functor given by the composition:

$$\mathcal{E}_H \boxtimes \mathcal{E}_G \boxtimes \mathcal{E}_H \boxtimes \mathcal{E}_G \xrightarrow{f^*\boxtimes\mathrm{id}\boxtimes f^*\boxtimes\mathrm{id}} \mathcal{E}_G \boxtimes \mathcal{E}_G \boxtimes \mathcal{E}_G \boxtimes \mathcal{E}_G \xrightarrow{\otimes\circ(\otimes\boxtimes\otimes)} \mathcal{E}_G, \tag{5.53}$$

and let L_2 be the braided functor given by the composition:

$$\mathcal{E}_H \boxtimes \mathcal{E}_G \boxtimes \mathcal{E}_H \boxtimes \mathcal{E}_G \xrightarrow{\mathrm{id}\boxtimes\tau_{G,H}\boxtimes\mathrm{id}} \mathcal{E}_H \boxtimes \mathcal{E}_H \boxtimes \mathcal{E}_G \boxtimes \mathcal{E}_G \xrightarrow{\otimes\boxtimes\otimes} \mathcal{E}_H \boxtimes \mathcal{E}_G \xrightarrow{\otimes\circ(f^*\boxtimes\mathrm{id})} \mathcal{E}_G. \tag{5.54}$$

Then $L_1 \xrightarrow{\sim} L_2$. Let R_1 and R_2 be right adjoint to L_1 and L_2 respectively, then

$$\begin{aligned}((\mathcal{M} \boxtimes Z(\mathcal{E}_G))^{\mathrm{loc}}_{A_f} \boxtimes (\mathcal{N} \boxtimes \mathcal{Z}(\mathcal{E}_G))^{\mathrm{loc}}_{A_f})^{\mathrm{loc}}_{L_{\mathcal{E}_G}} &\xrightarrow{\sim} (\mathcal{M} \boxtimes Z(\mathcal{E}_G) \boxtimes \mathcal{N} \boxtimes \mathcal{Z}(\mathcal{E}_G))^{\mathrm{loc}}_{R_1(\mathbf{1})} \\ &\xrightarrow{\sim} (\mathcal{M} \boxtimes \mathcal{N} \boxtimes \mathcal{Z}(\mathcal{E}_G) \boxtimes \mathcal{Z}(\mathcal{E}_G))^{\mathrm{loc}}_{R_2(\mathbf{1})} \\ &\xrightarrow{\sim} ((\mathcal{M} \boxtimes \mathcal{N})^{\mathrm{loc}}_{\mathcal{E}_H} \boxtimes (\mathcal{Z}(\mathcal{E}_G) \boxtimes \mathcal{Z}(\mathcal{E}_G))^{\mathrm{loc}}_{\mathcal{E}_G})^{\mathrm{loc}}_{A_f}\end{aligned} \tag{5.55}$$

But $(\mathcal{Z}(\mathcal{E}_G) \boxtimes \mathcal{Z}(\mathcal{E}_G))^{\mathrm{loc}}_{\mathcal{E}_G} \xrightarrow{\sim} \mathcal{Z}(\mathcal{E}_G)$, and so we have

$$H^3(f)(\mathcal{M}) \boxtimes_{\mathcal{E}_G} H^3(f)(\mathcal{N}) \xrightarrow{\sim} ((\mathcal{M} \boxtimes \mathcal{N})^{\mathrm{loc}}_{\mathcal{E}_H} \boxtimes \mathcal{Z}(\mathcal{E}_G))^{\mathrm{loc}}_{A_f} = H^3(f)(\mathcal{M} \boxtimes_{\mathcal{E}_H} \mathcal{N}). \tag{5.56}$$

□

Remark 5.48. We outline an alternative proof of Theorem 5.47. Let $\alpha : G \to H \times G$ be given by $\alpha(x) = (f(x), x)$, then α allows us to view G as a subgroup of $H \times G$. Let $\alpha^* : \mathrm{Rep}(H \times G, (w, z)) \to \mathrm{Rep}(G, z)$ be the corresponding restriction functor, and let R_α be right adjoint to α^*. Then $A := R_\alpha(\mathbf{1})$ is a connected étale algebra in $\mathcal{E}_{H\times G}$, so by [LKW17, Proposition 5.7] the map

$$\begin{gathered}\widehat{\alpha} : \mathcal{M}_{\mathrm{ext}}(\mathrm{Rep}(H \times G, (w, z))) \to \mathcal{M}_{\mathrm{ext}}(\mathrm{Rep}(H, w)) \\ \mathcal{M} \mapsto \mathcal{M}^{\mathrm{loc}}_A\end{gathered} \tag{5.57}$$

is a group homomorphism. Observe that the diagram

$$\begin{tikzcd} \mathcal{E}_H \boxtimes \mathcal{E}_G \arrow[r, "\sim"] \arrow[d, "f^* \boxtimes \mathrm{id}"] & \mathcal{E}_{H\times G} \arrow[d, "\alpha^*"] \\ \mathcal{E}_G \boxtimes \mathcal{E}_G \arrow[r, "\otimes"] & \mathcal{E}_G \end{tikzcd} \tag{5.58}$$

is commutative, so A is the image of A_f under the equivalence $\mathcal{E}_H \boxtimes \mathcal{E}_G \xrightarrow{\sim} \mathcal{E}_{H\times G}$. Thus we get:

$$H^3(f)(\mathcal{M}) = (\mathcal{M} \boxtimes \mathcal{Z}(\mathcal{E}_G))^{\mathrm{loc}}_{A_f} \xrightarrow{\sim} (\mathcal{M} \boxtimes \mathcal{Z}(\mathcal{E}_G))^{\mathrm{loc}}_{A} = \widehat{\alpha}(\mathcal{M} \boxtimes \mathcal{Z}(\mathcal{E}_G)). \tag{5.59}$$

That $\widehat{\alpha}$ is a group homomorphism then implies (after a straightforward computation) that $H^3(f)$ is as well.

Lemma 5.49. *Let (G, z) be a supergroup, then $H^3(\mathrm{id}) : \mathcal{M}_{\mathrm{ext}}(G, z) \to \mathcal{M}_{\mathrm{ext}}(G, z)$ is the identity homomorphism.*

Proof. Suppose $(\mathcal{M}, \iota_{\mathcal{M}}) \in \mathcal{M}_{\mathrm{ext}}(\mathrm{Rep}(G, z))$, then $A_{\mathrm{id}} = L_{\mathcal{E}_G} \in \mathcal{E}_G \boxtimes \mathcal{E}_G$, and so $H^3(\mathrm{id})(\mathcal{M}, \iota_{\mathcal{M}}) = \mathcal{M} \boxtimes_{\mathcal{E}_G} \mathcal{Z}(\mathcal{E}_G) \xrightarrow{\sim} \mathcal{M}$. $\square$

Lemma 5.50. *Let $f : (G, z) \to (H, w)$ and $g : (H, w) \to (K, y)$ be supergroup homomorphisms, then*

$$H^3(g \circ f) = H^3(f) \circ H^3(g). \tag{5.60}$$

Proof. Let $(\mathcal{M}, \iota_{\mathcal{M}}) \in \mathcal{M}_{\mathrm{ext}}(\mathrm{Rep}(K, y))$. Define braided functors $F_f := \otimes \circ (f^* \otimes \mathrm{id})$, $F_g := \otimes \circ (g^* \otimes \mathrm{id})$, and $F_{gf} = \otimes \circ ((g \circ f)^* \otimes \mathrm{id})$. Let L_1 be the braided functor given by the composition

$$\mathcal{E}_K \boxtimes \mathcal{E}_H \boxtimes \mathcal{E}_G \xrightarrow{\mathrm{id} \boxtimes F_f} \mathcal{E}_K \boxtimes \mathcal{E}_G \xrightarrow{F_{gf}} \mathcal{E}_G, \tag{5.61}$$

and let L_2 be the braided functor given by the composition

$$\mathcal{E}_K \boxtimes \mathcal{E}_H \boxtimes \mathcal{E}_G \xrightarrow{F_g \boxtimes \mathrm{id}} \mathcal{E}_H \boxtimes \mathcal{E}_G \xrightarrow{F_f} \mathcal{E}_G. \tag{5.62}$$

Observe that $L_1 \xrightarrow{\sim} L_2$. Let R_1 and R_2 be right adjoint to L_1 and L_2 respectively, then

$$\begin{aligned} H^3(f) \circ H^3(g)(\mathcal{M}, \iota_{\mathcal{M}}) &= ((\mathcal{M} \boxtimes \mathcal{Z}(\mathcal{E}_H))^{\mathrm{loc}}_{A_g} \boxtimes \mathcal{Z}(\mathcal{E}_G))^{\mathrm{loc}}_{A_f} \\ &\xrightarrow{\sim} (\mathcal{M} \boxtimes \mathcal{Z}(\mathcal{E}_H) \boxtimes \mathcal{Z}(\mathcal{E}_G))^{\mathrm{loc}}_{R_2(\mathbf{1})} \\ &\xrightarrow{\sim} (\mathcal{M} \boxtimes \mathcal{Z}(\mathcal{E}_H) \boxtimes \mathcal{Z}(\mathcal{E}_G))^{\mathrm{loc}}_{R_1(\mathbf{1})} \\ &\xrightarrow{\sim} (\mathcal{M} \boxtimes (\mathcal{Z}(\mathcal{E}_H) \boxtimes \mathcal{Z}(\mathcal{E}_G))^{\mathrm{loc}}_{A_f})^{\mathrm{loc}}_{A_{gf}} \\ &\xrightarrow{\sim} (\mathcal{M} \boxtimes \mathcal{Z}(\mathcal{E}_G))^{\mathrm{loc}}_{A_{gf}} \\ &= H^3(g \circ f)(\mathcal{M}, \iota_{\mathcal{M}}). \end{aligned} \tag{5.63}$$

$\square$

The following is immediate from Theorem 5.47 and Lemmas 5.49 and 5.50.

Theorem 5.51. $H^3 : \mathrm{sGrp} \to \mathrm{Ab}$ *is a contravariant functor.*

Connections and Applications. Suppose (G, z) is a supergroup, and $H \leq G$ a subgroup with $z \in H$, then $A = \mathrm{Fun}(G/H)$ is a connected étale algebra in $\mathrm{Rep}(G, z)$. If $\mathcal{M}$ is a modular extension of $\mathrm{Rep}(G, z)$, then $\mathcal{M}_A$ is a modular extension of $\mathrm{Rep}(H, z)$, so by [LKW17, Proposition 5.7] we have the group homomorphism

$$\begin{aligned} \mathcal{M}_{\mathrm{ext}}(\mathrm{Rep}(G, z)) &\to \mathcal{M}_{\mathrm{ext}}(\mathrm{Rep}(H, z)) \\ (\mathcal{M}, \iota_{\mathcal{M}}) &\mapsto (\mathcal{M}_A^{\mathrm{loc}}, \iota_{\mathcal{M}}). \end{aligned} \tag{5.64}$$

On the other hand, the inclusion homomorphism $i : (H, z) \to (G, z)$ induces a (potentially different) homomorphism $H^3(i) : \mathcal{M}_{\mathrm{ext}}(\mathrm{Rep}(G, z)) \to \mathcal{M}_{\mathrm{ext}}(\mathrm{Rep}(H, z))$. The following proposition says that these homomorphisms are in fact the same.

Proposition 5.52. *Suppose* (G, z) *is a supergroup, and* $H \leq G$ *a subgroup with* $z \in H$*. Let* $i : (H, z) \to (G, z)$ *be the inclusion homomorphism, and* $A := \mathrm{Fun}(G/H) \in \mathrm{Rep}(G, z)$*. Then*

$$H^3(i)(\mathcal{M}, \iota_{\mathcal{M}}) = (\mathcal{M}_A^{\mathrm{loc}}, \iota_{\mathcal{M}}) \tag{5.65}$$

for all $(\mathcal{M}, \iota_{\mathcal{M}}) \in \mathcal{M}_{\mathrm{ext}}(\mathrm{Rep}(G, z))$*.*

Proof. Since i is the inclusion of a subgroup, we have $i^* = \mathrm{Res}_H^G$. The right adjoint functor to i^* is $R = \mathrm{Hom}_{\mathbb{C}H}(\mathbb{C}G, -)$, so $R(\mathbf{1}) = \mathrm{Fun}(G/H) = A$. Let S be right adjoint to the braided functor $\mathcal{E}_G \boxtimes \mathcal{E}_H \xrightarrow{i^* \boxtimes \mathrm{id}} \mathcal{E}_H \boxtimes \mathcal{E}_H$, then $S(\mathbf{1}) = A \boxtimes \mathbf{1}$. In particular, we get that

$$(\mathcal{M} \boxtimes \mathcal{Z}(\mathcal{E}_H))_{A_i}^{\mathrm{loc}} \xrightarrow{\sim} (\mathcal{M}_A^{\mathrm{loc}} \boxtimes \mathcal{Z}(\mathcal{E}_H))_{L_{\mathcal{E}_H}}^{\mathrm{loc}} \xrightarrow{\sim} \mathcal{M}_A^{\mathrm{loc}}. \tag{5.66}$$

since $\mathcal{Z}(\mathcal{E}_H)$ is the identity element in $\mathcal{M}_{\mathrm{ext}}(\mathrm{Rep}(H, z))$. $\square$

Let (G, z) be a non-trivial supergroup, then there is a canonical homomorphism $i : (\mathbb{Z}/2\mathbb{Z}, -1) \to (G, z)$, so we obtain a group homomorphism

$$H^3(i) : \mathcal{M}_{\text{ext}}(\text{Rep}(G, z)) \to \mathcal{M}_{\text{ext}}(\text{sVec}) = \mathbb{Z}/16\mathbb{Z}. \tag{5.67}$$

We focus our attention on $H^3(i)$. To simplify our notation, we write $\widehat{G} = G/\langle z \rangle$. Observe that $\text{Rep}(G, z)$ contains $\text{Rep}(\widehat{G})$ as the fusion subcategory of representations on which z acts trivially. Let $A = \text{Fun}(\widehat{G}) \in \text{Rep}(\widehat{G}) \subset \text{Rep}(G, z)$. Given a modular extension $\mathcal{M} \in \mathcal{M}_{\text{ext}}(\text{Rep}(G, z))$ we can form the de-equivariantization $\mathcal{M}_A$. By Proposition 2.100 the trivial component of $\mathcal{M}_A$ is given by $\mathcal{M}_A^{\text{loc}}$. The following properties of $\mathcal{M}_A$ follow from Theorem 2.99, Proposition 5.52, and [DGNO10, Proposition 4.30].

Lemma 5.53. *Let $\mathcal{M} \in \mathcal{M}_{\text{ext}}(\text{Rep}(G, z))$. Then*

(i) *$\mathcal{M}_A$ is a faithfully graded braided $\widehat{G}$-crossed fusion category,*

(ii) *the trivial component of the grading $(\mathcal{M}_A)_1 = \mathcal{M}_A^{\text{loc}}$ is equivalent to $H^3(i)(\mathcal{M})$,*

(iii) $\text{sVec} \subset \mathcal{M}_A^{\text{loc}}$ *is a $\widehat{G}$-stable fusion subcategory, and*

(iv) $\text{sVec}^{\widehat{G}} \xrightarrow{\sim} \text{Rep}(G, z)$.

A classification of faithfully graded braided G-crossed fusion categories was given by Etingof, Nikshych, and Ostrik in [ENO10]. We recall that classification now.

Theorem 5.54 ([ENO10, Theorem 7.12])**.** *Let $\mathcal{B}$ be a braided fusion category. Equivalence classes of braided G-crossed categories $\mathcal{C}$ having a faithful G-grading with trivial component $\mathcal{B}$ are parametrized by triples (c, M, α), where:*

(i) *$c : G \to \text{Pic}(\mathcal{B})$ is a group homomorphism,*

(ii) M belongs to a certain torsor over $H^2(G, \mathrm{Inv}(\mathcal{B}))$, and

(iii) α belongs to a certain torsor over $H^3(G, \mathbb{C}^\times)$,

subject to the requirement that certain obstructions:

(i) $O_3(c) \in H^3(G, \mathrm{Inv}(\mathcal{B}))$, and

(ii) $O_4(c, M) \in H^4(G, \mathbb{C}^\times)$,

vanish.

Remark 5.55. Let $\mathcal{B}$ be a braided fusion category. Denote by $\mathrm{Aut}_\otimes^{\mathrm{br}}(\mathcal{B})$ the group of isomorphism classes of braided autoequivalences of $\mathcal{B}$. If $\mathcal{B}$ is non-degenerate, then by [ENO10, Theorem 5.2] we have:

$$\mathrm{Pic}(\mathcal{B}) \xrightarrow{\sim} \mathrm{Aut}_\otimes^{\mathrm{br}}(\mathcal{B}). \tag{5.68}$$

Given a homomorphism $c : G \to \mathrm{Pic}(\mathcal{B}) \xrightarrow{\sim} \mathrm{Aut}_\otimes^{\mathrm{br}}(\mathcal{B})$, the first obstruction $O_3(c) \in H^3(G, \mathrm{Inv}(\mathcal{B}))$ determines whether c can be lifted to a braided action $\underline{c} : \underline{G} \to \underline{\mathrm{Aut}_\otimes^{\mathrm{br}}}(\mathcal{B})$ as in Definition 2.91.

Let $\mathcal{B}$ be a modular extension of sVec equipped with a braided $\widehat{G}$-action such that $\mathrm{sVec} \subset \mathcal{B}$ is $\widehat{G}$-stable. Restricting this action to sVec gives a braided action $\varrho : \underline{\widehat{G}} \to \underline{\mathrm{Aut}_\otimes^{\mathrm{br}}}(\mathrm{sVec})$, and so we have natural isomorphisms $\gamma_{g,h} : \mathrm{id}_{\mathrm{sVec}} \xrightarrow{\sim} \mathrm{id}_{\mathrm{sVec}}$ for $g, h \in \widehat{G}$. Let $\delta \in \mathrm{sVec}$ be the non-trivial simple object, then $(\gamma_{g,h})_\delta = \alpha(g,h)\mathrm{id}_\delta$ for some $\alpha(g,h) \in \mathbb{C}^\times$. Since $\delta \otimes \delta \xrightarrow{\sim} \mathbf{1}_{\mathrm{sVec}}$, we have $\alpha(g,h)^2 = 1$ so $\alpha(g,h) \in \mathbb{Z}/2\mathbb{Z}$. The monoidal structure axiom for ϱ says that α is a 2-cocycle on $\widehat{G}$ with values in $\mathbb{Z}/2\mathbb{Z}$, so we obtain a cohomology class $[\varrho] \in H^2(\widehat{G}, \mathbb{Z}/2\mathbb{Z})$.

Definition 5.56. Let $\varrho : \underline{\widehat{G}} \to \underline{\mathrm{Aut}_\otimes^{\mathrm{br}}}(\mathrm{sVec})$ be a braided action of $\widehat{G}$ on sVec. The class $[\varrho] \in H^2(\widehat{G}, \mathbb{Z}/2\mathbb{Z})$ described above is called the *cohomology class of ϱ*.

Recall from Theorem 2.26 that $[\varrho] \in H^2(\widehat{G}, \mathbb{Z}/2\mathbb{Z})$ determines a $\mathbb{Z}/2\mathbb{Z}$-central extension $\widetilde{G}$ of $\widehat{G}$, equivalently, a non-trivial supergroup $(\widetilde{G}, z)$. On the other hand, $\mathrm{sVec}^{\widehat{G}}$ is a symmetric fusion category [DGNO10, Corollary 4.31], so by Theorem 5.4 is equivalent to the category of finite-dimensional representations of some supergroup. The following proposition says that this supergroup is $(\widetilde{G}, z)$.

Proposition 5.57. *Let $\varrho : \underline{\widehat{G}} \to \underline{\mathrm{Aut}}^{\mathrm{br}}_{\otimes}(\mathrm{sVec})$ be a braided action of a finite group $\widehat{G}$ on* sVec*. Then there is a braided equivalence*

$$\mathrm{sVec}^{\widehat{G}} \xrightarrow{\sim} \mathrm{Rep}(\widetilde{G}, z) \tag{5.69}$$

where $\widetilde{G}$ is the $\mathbb{Z}/2\mathbb{Z}$-central extension of $\widehat{G}$ determined by the $[\varrho] \in H^2(\widehat{G}, \mathbb{Z}/2\mathbb{Z})$ of ϱ.

Proof. Let $(V, u) \in \mathrm{sVec}^{\widetilde{G}}$. Write $V = V_0 \oplus V_1$, then the $\widehat{G}$-equivariance condition implies we have equations:

$$u^0_g u^0_h = u^0_{gh}, \text{ and} \tag{5.70}$$

$$u^1_g u^1_h = \alpha(g, h) u^1_{gh}, \tag{5.71}$$

where $u^0_g : V_0 \to V_0$ and $u^1_g : V_1 \to V_1$ denote the even and odd components of u_g respectively. Observe that Eq. (5.70) describes a $\widehat{G}$-action on V_0, and Eq. (5.71) describes a projective $\widehat{G}$-action on V_1 with 2-cocycle α. That V_1 is a projective representation of $\widehat{G}$ with 2-cocycle α is equivalent to saying that V_1 is a genuine representation of the central extension $\widetilde{G}$ where $z \in \widetilde{G}$ acts by -1. We can also view V_0 as a $\widetilde{G}$-representation via the homomorphism $\widetilde{G} \to \widehat{G}$, and so $V \in \mathrm{Rep}(\widetilde{G}, z)$. With this $\widetilde{G}$-action, a $\widehat{G}$-equivariant morphism $f : (V, u) \to (W, v)$ gives a $\widetilde{G}$-linear map $V \to W$ in $\mathrm{Rep}(\widetilde{G}, z)$, and so we have a (faithful) functor

$$\mathcal{G} : \mathrm{sVec}^{\widehat{G}} \to \mathrm{Rep}(\widetilde{G}, z). \tag{5.72}$$

A straightforward computation shows that a $\widetilde{G}$-linear morphism $f : V \to W$ is automatically $\widehat{G}$-equivariant, so $\mathcal{G}$ is full. Comparing the $\widetilde{G}$-actions shows that $\mathcal{G}$ is a monoidal functor. Since $z \in \widetilde{G}$ acts by 1 on V_0 and by -1 on V_1, $\mathcal{G}$ is moreover braided. Thus $\mathcal{G}$ is a fully faithful braided monoidal functor. But $\mathrm{FPdim}(\mathrm{sVec}^{\widehat{G}}) = 2|\widehat{G}| = |\widetilde{G}| = \mathrm{FPdim}(\mathrm{Rep}(\widetilde{G}, z))$, so $\mathcal{F}$ is a braided equivalence [EO04, Proposition 2.19]. □

Remark 5.58. The inverse equivalence $\mathcal{F} : \mathrm{Rep}(\widetilde{G}, z) \to \mathrm{sVec}^{\widehat{G}}$ can be constructed explicitly. A representation $V \in \mathrm{Rep}(\widetilde{G}, z)$ inherits a $\mathbb{Z}/2\mathbb{Z}$-grading with $V_0 = \{v \in V \mid zv = v\}$ and $V_1 = \{v \in V \mid zv = -v\}$, making V into a superspace.

If $\varphi : \widetilde{G} \to \mathrm{GL}(V)$ is the $\widetilde{G}$-action on V, then $u_g = \varphi(1, g)$ is a $\widehat{G}$-equivariant structure on V, so define $\mathcal{F}(V) = (V, u) \in \mathrm{sVec}^{\widehat{G}}$.

This proposition implies the following characterization for when i^* is surjective, which was proven independently of us in [GVR17].

Theorem 5.59. *Let (G, z) be a finite non-trivial supergroup. The map $i : (\mathbb{Z}/2\mathbb{Z}, z) \to (G, z)$ splits if and only if $i^* : \mathcal{M}_{\mathrm{ext}}(\mathrm{Rep}(G, z)) \to \mathbb{Z}/16\mathbb{Z}$ is surjective.*

Proof. If i splits, then i^* is surjective by functorality of $H^3 : \mathrm{sGrp} \to \mathrm{Ab}$. For the reverse direction, suppose i^* is surjective. Then there exists a modular extension $\mathcal{M} \in \mathcal{M}_{\mathrm{ext}}(\mathrm{Rep}(G, z))$ with $i^*(\mathcal{M}) \xrightarrow{\sim} \mathcal{I}$ braided equivalent to an Ising fusion category. Since any tensor autoequivalence of an Ising category is isomorphic to the identity functor [DGNO10, Remark B.6 (i)], the restricted $\widehat{G}$-action on $\mathrm{sVec} \subset \mathcal{I}$ corresponds to the trivial element of $H^2(G/\langle z \rangle, \mathbb{Z}/2\mathbb{Z})$, so by Proposition 5.57 we have $\mathrm{Rep}(G, z) \xrightarrow{\sim} \mathrm{sVec}^{\widehat{G}} \xrightarrow{\sim} \mathrm{Rep}(\mathbb{Z}/2\mathbb{Z} \times \widehat{G}, (-1, 1)) \xrightarrow{\sim} \mathrm{sVec} \boxtimes \mathrm{Rep}(\widehat{G})$. Thus G admits a character $\chi : G \to \{1, -1\}$ with $\chi(z) = -1$, which is precisely a splitting of i. □

Proposition 5.57 and Lemma 5.53 imply the following characterization of when the $\widehat{G}$-equivariantization of a faithfully graded braided $\widehat{G}$-crossed fusion category is a modular extension of $\mathrm{Rep}(G, z)$.

Corollary 5.60. *Let (G, z) be a supergroup, and $\widehat{G} = G/\langle z \rangle$. Let $\mathcal{M}$ be a faithfully graded braided $\widehat{G}$-crossed fusion category such that:*

(i) *the trivial component $\mathcal{M}_1$ is a modular extension of* sVec,

(ii) sVec $\subset \mathcal{M}_1$ *is $\widehat{G}$-stable, and*

(iii) *G is the $\mathbb{Z}/2\mathbb{Z}$-central extension of $\widehat{G}$ corresponding to the cohomology class of the restricted action $\varrho : \widehat{G} \to \underline{\mathrm{Aut}}^{\mathrm{br}}_{\otimes}(\mathrm{sVec})$.*

Then $\mathcal{M}^{\widehat{G}}$ is a modular extension of $\mathrm{Rep}(G, z)$, *and every modular extension of* $\mathrm{Rep}(G, z)$ *arises in this way.*

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