EQUIVARIANT KHOVANOV HOMOTOPY TYPE AND PERIODIC LINKS

by

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DISSERTATION ABSTRACT

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In this dissertation, we give two equivalent definitions for a group G acting on a strictly-unitary-lax-2-functor $D : \underline{2^n} \to \mathscr{B}$ from the cube category to the Burnside category. We then show that the natural $\mathbb{Z}/p\mathbb{Z}$ action on a p-periodic link L induces such an action on Lipshitz and Sarkar's Khovanov functor $F_{Kh}(L) : \underline{2^n} \to \mathscr{B}$ which makes the Khovanov homotopy type $\mathcal{X}(L)$ into an equivariant knot invariant. That is, if a link L' is equivariantly isotopic to L, then $\mathcal{X}(L')$ is Borel homotopy equivalent to $\mathcal{X}(L)$.

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To my family.

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CHAPTER I

INTRODUCTION

In 1985, Jones described a new polynomial knot invariant satisfying a skein relation [Jon85]. Two years later, Kauffman gave a state model definition for the Jones polynomial by defining what is now called the Kauffman Bracket [Kau87]. The Jones Polynomial is celebrated not only because it is relatively good at distinguishing knots, but also because it was used by Kauffman [Kau87], Murasugi [Mur87], Thistlethwaite [Thi87] [Thi88], and Menasco and Thistlethwaite [MT93] to prove the Tait Conjectures from 1898 [Tai98]. In 2000, based on an idea of Crane and Frenkle [CF94], Khovanov categorified the Jones polynomial by assigning a bigraded abelian group to an oriented link [Kho00]. This is a refinement of the work of Jones in the sense that the graded Euler characteristic of Khovanov Homology is the unnormalized Jones polynomial. As a further refinement of the Jones polynomial and Khovanov homology, Lipshitz and Sarkar [LS14] constructed the Khovanov stable homotopy type $\mathcal{X}_{Kh}(L)$ by using the notion of flow categories described by Cohen, Jones, and Segal in [CJS95]. More precisely, for each oriented link diagram L, Lipshitz and Sarkar constructed a family of suspension spectra $\mathcal{X}(L) = \bigvee_j \mathcal{X}^j_{Kh}(L)$ such that

(1) The reduced cohomology of $\mathcal{X}^j_{Kh}(L)$ is the same as the Khovanov homology $Kh^{*,j}(L)$:

$$\tilde{H}^i\left(\mathcal{X}^j_{Kh}(L)\right) = Kh^{i,j}(L)$$

(2) The homotopy type of $\mathcal{X}^j_{Kh}(L)$ is determined by the isotopy class of the link L.

Shortly afterwards, a similar spectrum invariant was described by Hu, Kriz, and Kriz utilizing different techniques [HKK12]. In [LLS15a], Lawson, Lipshitz, and Sarkar gave an equivalent construction of the Lipshitz-Sarkar Khovanov homotopy type by defining a strictly-unitary-lax-2-functor $F_{Kh}(L) : \underline{2^n} \to \mathscr{B}$ from the cube category to the Burnside category, and by using this reformulation they showed that the Lisphitz-Sarkar and Hu-Kriz-Kriz invariants were homotopy equivalent.

In 1961, Fox first suggested studying classes of knots with various forms of symmetries. One such class of links is periodic links, which are links that possess a diagram with a rotation symmetry [Fox61]. In 1988, Murasugi showed that there is a relationship between the Jones polynomials of a periodic link and its quotient link, creating an obstruction for when links can be periodic [Mur88]. In 2007, Chbili defined a G-equivariant Khovanov homology when G is a cyclic group of odd order [Chb07]. In 2015, Politarczyk defined another equivariant version of Khovanov homology for periodic links that is an analogue to Borel equivariant cohomology [Pol15]. In 2018, Borodicz, Politarczyk, and Silvero extended Politarczyk's work by utilizing equivariant cubical flow categories to define an equivariant Khovanov homotopy type [BP17]. They also related the Borel equivariant homology of the homotopy type to Politarczyk's equivariant Khovanov homology. In 2018, Stoffregen and Zhang also constructed a Khovanov homotopy type for periodic links by constructing an equivariant version of Lawson, Lipshitz, and Sarkar's Khovanov functor [SZ18]. It is currently unknown if these two constructions result in equivalent equivariant homotopy types.

In the following chapters, we will give a third construction of an equivariant Khovanov homotopy type $\mathcal{X}_{Kh}(L)$ for periodic links. More precisely, we will prove the following

Theorem 1.1. *For a p-periodic link L, the natural action of $\mathbb{Z}/p\mathbb{Z}$ on L induces a $\mathbb{Z}/p\mathbb{Z}$ action on $\mathcal{X}_{Kh}(L)$ which makes $\mathcal{X}_{Kh}(L)$ a naive $\mathbb{Z}/p\mathbb{Z}$-spectrum.*

Theorem 1.2. *If L and L' are equivariantly isotopic p-periodic links, then $\mathcal{X}_{Kh}(L')$ is Borel homotopy equivalent to $\mathcal{X}_{Kh}(L)$.*

By Borel homotopy equivalent, we mean that we can find collection of naive $\mathbb{Z}/p\mathbb{Z}$-spectra X_i and Y_{i-1} such that we get a composition of roofs

$$\begin{tikzcd}
 & X_1 \arrow[dl] \arrow[dr] & & X_2 \arrow[dl] \arrow[dr] & & {} \arrow[dr] & & X_i \arrow[dl] \arrow[dr] & \\
\mathcal{X}_{Kh}(L) & & Y_1 & & {} & \cdots & Y_{i-1} & & \mathcal{X}_{Kh}(L')
\end{tikzcd}$$

where the downward maps are equivariant and induce homotopy equivalences but the inverse maps $Y_j \to X_i$ need not be equivariant.

To prove theorems 1.1 and 1.2 we will do the following: we will give two equivalent definitions for what we mean for a group G to act on a strictly-unitary-lax-2-functor (chapter 3); we will then define how $\mathbb{Z}/p\mathbb{Z}$ acts on the Khovanov functor $F_{Kh}(L) : \underline{2^n} \to \mathscr{B}$ and how this group action can be extended to an action on $\mathcal{X}_{Kh}(L)$ (chapter 4); and finally we will show that $\mathcal{X}_{kh}(L)$ is an equivariant knot invariant (chapter 5).

CHAPTER II

BACKGROUND

In this chapter we will recall definitions and set notation so that we can describe both the Khovanov functor $F_{Kh} : \underline{2^n} \to \mathscr{B}$ and the construction of the Khovanov homotopy type $\mathcal{X}_{Kh}(L)$. We begin with our main object of study, periodic links.

Definition 2.1. *A link L of m components is a piecewise linear embedding of m disjoint copies of S^1 in S^3. A knot is a link with only one component.*

It is often easier to represent a knot by using a knot diagram, which is the projection of the knot onto a plane with small breaks to indicate where one strand crosses over another strand. The convention is that the projection of the over-strand remains intact while the projection of the under-strand is broken. Figure 1 contains a few examples of knot diagrams.

Slight changes to the embedding of the circles that make up a link L do not affect how "knotted" or "linked" the components of L are in S^3, and so links are only considered up to isotopy.

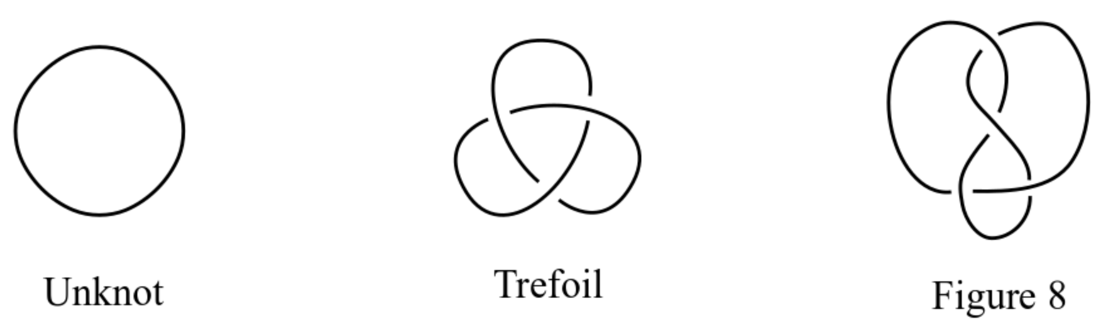


Figure 1. Diagrams for the Unknot, Trefoil, and Figure-8 knot.

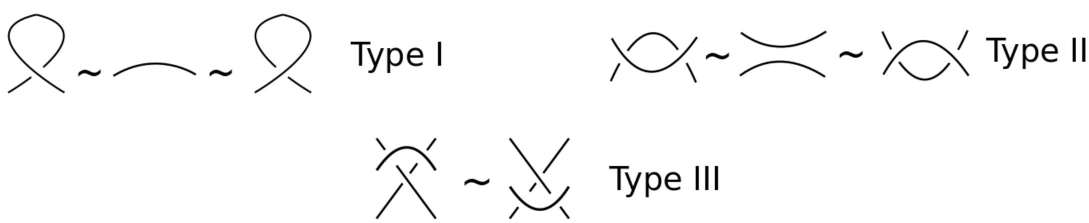
Definition 2.2. *Let $f, g : \amalg_{i=1}^{m} S^1 \to S^3$ be two piecewise linear embeddings of $\amalg_{i=1}^{m} S^1$ in S^3. An isotopy from f to g is a piecewise linear continuous map $H : \amalg_{i=1}^{m} S^1 \times [0,1] \to S^3$ such that $H(-,0) = f$, $H(-,1) = g$, and $H(-,t)$ is a piecewise linear embedding of $\amalg_{i=1}^{m} S^1$ in S^3 for all $t \in [0,1]$.*

Using this definition, we can now describe an equivalence relation for links.

Definition 2.3. *Two links L_1 and L_2 are equivalent if there exists an isotopy between them.*

Explicitly describing the isotopy between two links can often be quite difficult, and so it is often helpful to use the following theorem of Reidemeister to determine when two knot diagrams represent the same equivalence class of links.

Theorem 2.4. *[Rei74] Two links L_1 and L_2 are equivalent if and only if a diagram D_1 representing L_1 can be transformed into a diagram D_2 representing L_2 by a sequence of the following three types of moves*



The main focus of this paper will be a specific class of links called periodic links.

Definition 2.5. *A link L is called p-periodic if it possesses a knot diagram with a $\frac{2\pi}{p}$ rotational symmetry about a point not in the image of L.*

In figure 2, rotating each knot diagram about the marked center point shows that the Hopf Link is 2-periodic and that the Trefoil is 3-periodic.

Figure 2. A 2-periodic diagram for the Hopf Link and a 3-periodic diagram for the Trefoil

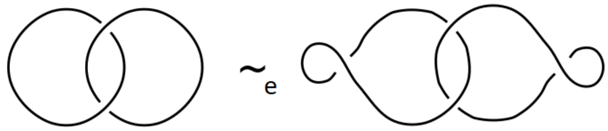
Two p-periodic links L_1 and L_2 are equivariantly isotopic if there exists an isotopy between L_1 and L_2 that respects the $\frac{2\pi}{p}$ rotation symmetry of the two links. As with the non-equivariant case, it is often easier to think of this equivariant isotopy in terms of diagrams, so we will now define the concept of equivariant Reidemeister moves.

Definition 2.6. *Given a p-periodic knot diagram D for a link L, an equivariant Reidemeister move of type I (resp. II or III) is the result of performing a regular Reidemeister move of type I (resp. II or III) and the $p-1$ images of that move under the rotational action on D.*

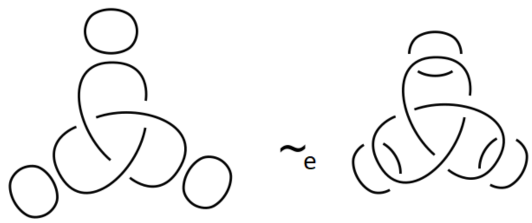
Examples of the first and second equivariant Reidemeister moves can be seen in figure 3. We now give a proof of the following proposition, which can be thought of as an equivariant verison of Reidemeister's theorem.

Proposition 2.7 ([BPS18] - Proposition 2.7)**.** *Let L_1 and L_2 be two p-periodic links and let D_1 and D_2 be two p-periodic diagrams representing L_1 and L_2, respectively. Every equivariant isotopy from L_1 to L_2 can be realized by a sequence of equivariant Reidemeister moves from D_1 to D_2.*

Proof. Quotienting D_1 and D_2 by the rotation action will result in two isotopic diagrams D_1^q and D_2^q representing the quotients of the links L_1 and L_2. Since D_1^q



Equivariant Reidemeister Move I



Equivariant Reidemeister Move II

Figure 3. Examples of equivariant Reidemeister moves.

and D_2^q are isotopic, there is a sequence of regular Reidemeister moves transforming one into the other. This sequence of moves between D_1^q and D_2^q lifts to a sequence of equivariant Reidemeister moves from D_1 to D_2. □

Another important fact about periodic links is that $\mathbb{Z}/p\mathbb{Z}$ acts naturally on any p-periodic link L. More precisely, $i \in \mathbb{Z}/p\mathbb{Z}$ acts on L by rotating the link $\frac{2i\pi}{p}$ radians. This is the natural action that will be used to induce an action on $\mathcal{X}_{Kh}(L)$.

2.1 The Cube Category

The objects of the n-dimensional cube category $\underline{2^n}$ are elements of the product $\{0,1\}^n$. There is a partial ordering on $\mathrm{Ob}\big(\underline{2^n}\big)$ with $(u_1, \ldots, u_n) \geq (v_1, \ldots, v_n)$ whenever $u_i \geq v_i$ for all $1 \leq i \leq n$. This partial ordering also induces a grading on $\mathrm{Ob}\big(\underline{2^n}\big)$ given by the L^1-norm

$$|u| = \sum_{i=1}^{n} u_i.$$

It will occasionally be useful to know the difference in grading between u and v, and so we will sometimes write $u >_k v$ when $u \geq k$ and $|u| - |v| = k$. Additionally, we will occasionally write $u \mathrel{-\!\!\bullet} v$ when $u >_1 v$ in order to emphasize that u and

v are joined by a single edge in the cube. The partial ordering on the objects also induces the morphism structure in $\underline{2}^n$ with there being a unique morphism $\varphi_{u,v}$ between u and v whenever $u \geq v$ and no morphism otherwise. (i.e $\mathrm{Hom}_{\underline{2}^n}(u,v) = \{\varphi_{u,v}\}$ when $u \geq v$ and $\mathrm{Hom}_{\underline{2}^n}(u,v) = \emptyset$ otherwise.) We will view $\underline{2}^n$ as a strict 2-category that contains no non-identity 2-morphisms. It will be helpful later to have the following sign function

Definition 2.8. *For $u = \{u_1, u_2, \ldots\} >_1 v = \{v_1, v_2, \ldots\}$, let k be the unique element of $\{1, 2, \ldots, n\}$ such that $u_k > v_k$. Define*

$$sgn_{u,v} = \sum_{i=1}^{k-1} u_i \pmod{2}.$$

2.2 The Thickened Cube Category

The objects of the thickened cube category $\widehat{\underline{2}^n}$ are composable pairs of morphisms $u \xrightarrow{\varphi_{u,v}} v \xrightarrow{\varphi_{v,w}} w$ for any $u, v, w \in \mathrm{Ob}(\underline{2}^n)$. A morphism between $u \xrightarrow{\varphi_{u,v}} v \xrightarrow{\varphi_{v,w}} w$ and $u' \xrightarrow{\varphi_{u',v'}} v' \xrightarrow{\varphi_{v',w'}} w'$ is a commutative diagram of the form

$$\begin{tikzcd}
u \arrow[r, "\varphi_{u,v}"] \arrow[d, "\varphi_{u,u'}"'] & v \arrow[r, "\varphi_{v,w}"] & w \arrow[d, "\varphi_{w,w'}"'] \\
u' \arrow[r, "\varphi_{u',v'}"'] & v' \arrow[u, "\varphi_{v',v}"'] \arrow[r, "\varphi_{v',w'}"'] & w'.
\end{tikzcd}$$

Note the upward direction of the middle vertical map. We will occasionally refer to a morphism by the triple $(\varphi_{u,u'}, \varphi_{v',v}, \varphi_{w,w'})$ of its vertical maps. The composition of two morphisms $(\varphi_{u,u'}, \varphi_{v',v}, \varphi_{w,w'})$ and $(\varphi_{u',u''}, \varphi_{v'',v'}, \varphi_{w',w''})$ is just formed by vertically stacking the two commutative diagrams, or more succinctly $(\varphi_{u',u''}, \varphi_{v'',v'}, \varphi_{w',w''}) \circ (\varphi_{u,u'}, \varphi_{v',v}, \varphi_{w,w'}) = (\varphi_{u',u''} \circ \varphi_{u,u'}, \varphi_{v',v} \circ \varphi_{v'',v'}, \varphi_{w,w'} \circ \varphi_{w',w''})$. This category is the result of a general thickening process applied to the cube category. A similar process can be applied to any small category.

2.3 The Burnside Category

The objects of the Burnside category, $\mathscr{B}$, are finite sets. A morphism between X and Y in $\mathscr{B}$ is a triple (A, s, t) where A is a finite set, $s : A \to X$ is a set map, and $t : A \to Y$ is a set map. (s and t are often called the source map and target map respectively.) The triple (A, s, t) is often called a correspondence (or span) between X and Y and is usually depicted as

$$\begin{tikzcd} & A \arrow[dl, "s"'] \arrow[dr, "t"] & \\ X & & Y \end{tikzcd}.$$

Given two correspondences $(A, s_A, t_A) \in \mathrm{Hom}_{\mathscr{B}}(X, Y)$ and $(B, s_B, t_B) \in \mathrm{Hom}_{\mathscr{B}}(Y, Z)$ the composition of these two morphisms is given by (C, s_C, t_C) where

$$C = A \times_Y B = \{(a, b) \in A \times B \mid t_A(a) = s_B(b)\}$$
$$s_C(a, b) = s_A(a) \qquad t_C(a, b) = t_B(b).$$

Diagrammatically this looks like

$$\begin{tikzcd} & & A \times_y B \arrow[dl] \arrow[dr] & & \\ & A \arrow[dl, "s_A"'] \arrow[dr, "t_A"] & & B \arrow[dl, "s_B"'] \arrow[dr, "t_B"] & \\ X & & Y & & Z. \end{tikzcd}$$

Additionally, given two correspondences (A, s_A, t_A) and (B, s_B, t_B) from X to Y, a morphism of correspondences is a bijection between A and B such that following triangles commute

$$\begin{tikzcd} & A \arrow[dl] \arrow[d, "\cong"] \arrow[dr] & \\ & B \arrow[dl] \arrow[dr] & \\ X & & Y. \end{tikzcd}$$

The Burnside category is a weak 2-category with the morphisms of correspondences acting as the 2-morphisms. See, for instance, [LLS15b, Section 3] for more details.

In order to define the Khovanov functor in section 2.5, it will help to define a functor $\mathcal{A} : \mathscr{B} \to Ab$ from $\mathscr{B}$ to the category of abelian groups.

Definition 2.9. *For $X \in Ob(\mathscr{B})$, define $\mathcal{A}(X) = \mathbb{Z}\langle X \rangle$, the free abelian group with basis X. For a correspondence $(A, s, t) \in Hom_{\mathscr{B}}(X, Y)$ define the map $\mathcal{A}(A) : \mathcal{A}(X) \to \mathcal{A}(Y)$ on each of the basis elements $x \in X$ as*

$$\mathcal{A}(A)(x) = \sum_{y \in Y} |s^{-1}(x) \cap t^{-1}(y)| y.$$

If we consider the objects of $\mathscr{B}$ to be isomorphism classes of correspondences instead of just correspondences, then we can view $\mathscr{B}$ as a regular category. In this case, $\mathcal{A}$ identifies $\mathscr{B}$ as a full subcategory of Ab whose objects are finitely generated free abelian groups and whose morphisms are matrices with non-negative entries.

2.4 Functors from the Cube to the Burnside Category

We will now give two equivalent definitions for a functor $D : \underline{2^n} \to \mathscr{B}$ from the cube category to the Burnside category. As mentioned in sections 2.1 and 2.3 we can view $\underline{2^n}$ as a strict 2-category and $\mathscr{B}$ as a weak 2-category. This means that when we refer to a functor from the cube category to the Burnside category, we really mean a strictly-unitary-lax-2-functor. A more detailed explanation for these types of functors can be found in [Béa67], where they are referred to as strictly unitary homomorphisms.

Definition 2.10. *A strictly-unitary-lax-2-functor D from the cube category $\underline{2^n}$ to the Burnside category $\mathscr{B}$ consists of*

(i) A finite set $D(u) \in \mathscr{B}$ for every $u \in Ob\,(\underline{2^n})$

(ii) A finite correspondence $D(\varphi_{u,v}) \in Hom_{\mathscr{B}}\big(D(u), D(v)\big)$ *for every* $u \geq v$

(iii) A 2-isomorphism $D_{u,v,w} : D(\varphi_{v,w}) \times_{D(v)} D(\varphi_{u,v}) \to D(\varphi_{u,w})$ *for all* $u > v > w$

such that for all $u > v > w > z$ *the following diagram commutes*

$$\begin{tikzcd}
D(\varphi_{w,z}) \times_{D(w)} D(\varphi_{v,w}) \times_{D(v)} D(\varphi_{u,v}) \arrow[r, "Id \times D_{u,v,w}"] \arrow[d, "D_{v,w,z} \times Id"'] & D(\varphi_{w,z}) \times_{D(w)} D(\varphi_{u,w}) \arrow[d, "D_{u,w,z}"] \\
D(\varphi_{v,z}) \times_{D(v)} D(\varphi_{u,v}) \arrow[r, "D_{u,v,z}"'] & D(\varphi_{u,z})
\end{tikzcd}$$

.

It should be noted that $D(\varphi_{w,z}) \times_{D(w)} D(\varphi_{v,w}) \times_{D(v)} D(\varphi_{u,v})$ denotes $D(\varphi_{w,z}) \times_{D(w)} \big(D(\varphi_{v,w}) \times_{D(v)} D(\varphi_{u,v})\big)$ when going across the top of the diagram, and $\big(D(\varphi_{w,z}) \times_{D(w)} D(\varphi_{v,w})\big) \times_{D(v)} D(\varphi_{u,v})$ when going down the left side. These are not the same but they are canonically identified.

Notice that condition (ii) in the above definition requires a finite correspondence be given for every morphism in $\underline{2^n}$. In some sense this is the complete set of data for the morphisms, and so we will often use the phrase "complete definition" when we want to specifically refer to definition 2.10.

The second definition is composed of a similar set of data. However, instead of giving a correspondence $D(\varphi_{u,v})$ for every morphism $\varphi_{u,v}$ in $\underline{2^n}$, this definition only gives a correspondence for each edge of the cube. With this in mind (and to contrast the above definition), we will often refer to definition 2.11 as the "edge definition" for D.

Definition 2.11. *A strictly-unitary-lax-2-functor* D *from the cube category* $\underline{2^n}$ *to the Burnside category* $\mathscr{B}$ *consists of*

(e.i) A finite set $D(u) \in \mathscr{B}$ *for every* $u \in \underline{2^n}$

(e.ii) *A finite correspondence* $D(\varphi_{u,v}) \in Hom_{\mathscr{B}}\big(D(u), D(v)\big)$ *for every edge* $u \to v$

(e.iii) *An isomorphism* $D_{u,v,v',w} : D(\varphi_{v,w}) \times_{D(v)} D(\varphi_{u,v}) \to D(\varphi_{v',w}) \times_{D(v')} D(\varphi_{u,v'})$ *for every face*

$$\begin{tikzcd}[row sep=small, column sep=small] & v \arrow[dr] & \\ u \arrow[ur] \arrow[dr] & & w \\ & v' \arrow[ur] & \end{tikzcd}.$$

Such that the following two conditions are satisfied

$(C1)$ *For every face*

$$\begin{tikzcd}[row sep=small, column sep=small] & v \arrow[dr] & \\ u \arrow[ur] \arrow[dr] & & w \\ & v' \arrow[ur] & \end{tikzcd}, \quad D_{u,v',v,w} = D^{-1}_{u,v,v',w}$$

$(C2)$ *For every three dimensional face*

$$\begin{tikzcd}[row sep=small, column sep=small] & v \arrow[r] \arrow[dr] & w'' \arrow[dr] & \\ u \arrow[ur] \arrow[r] \arrow[dr] & v' \arrow[ur] \arrow[dr] & w' \arrow[r] & z \\ & v'' \arrow[ur] \arrow[r] & w \arrow[ur] & \end{tikzcd}$$

the following square commutes

$$\begin{tikzcd}[column sep=small] D(\varphi_{w'',z}) \times_{D(w'')} D(\varphi_{v,w''}) \times_{D(v)} D(\varphi_{u,v}) \arrow[d, "Id \times D_{u,v,v',w''}"'] \arrow[r, "D_{v,w'',w',z} \times Id"] & D(\varphi_{w',z}) \times_{D(w')} D(\varphi_{v,w'}) \times_{D(v)} D(\varphi_{u,v}) \arrow[d, "Id \times D_{u,v,v'',w'}"] \\ D(\varphi_{w'',z}) \times_{D(w'')} D(\varphi_{v',w''}) \times_{D(v')} D(\varphi_{u,v'}) \arrow[d, "D_{v',w'',w,z} \times Id"'] & D(\varphi_{w',z}) \times_{D(w')} D(\varphi_{v'',w'}) \times_{D(v'')} D(\varphi_{u,v''}) \arrow[d, "D_{v'',w',w,z} \times Id"] \\ D(\varphi_{w,z}) \times_{D(w)} D(\varphi_{v',w}) \times_{D(v')} D(\varphi_{u,v'}) \arrow[r, "Id \times D_{u,v',v'',w}"'] & D(\varphi_{w,z}) \times_{D(w)} D(\varphi_{v'',w}) \times_{D(v'')} D(\varphi_{u,v''}) \end{tikzcd}$$

Given the data from the complete definition, it is easy to produce the required data for the edge definition by simply setting $D_{u,v,v',w} = D^{-1}_{u,v',w} \circ D_{u,v,w}$. Showing that the data from the edge definition is sufficient to produce the data for the total definition requires selecting maximal chains between any two vertices. A maximal chain $\mathfrak{m}^{u,v}$ between vertices $u >_k v$ is a choice of edges

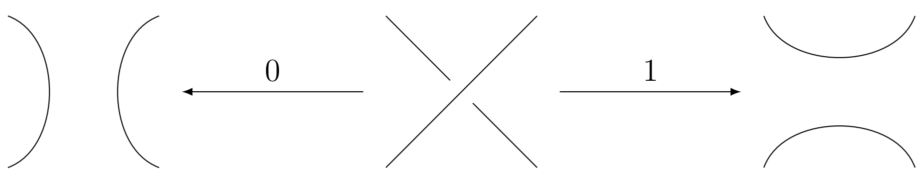
$$u = z_0^{u,v} \mathrel{-\!\!\bullet} \dots \mathrel{-\!\!\bullet} z_i^{u,v} \mathrel{-\!\!\bullet} \dots\mathrel{.} \mathrel{-\!\!\bullet} z_k^{u,v} = v.$$

connecting u and v. By using these maximal chains and the face isomorphisms from $(e.iii)$, it is possible to recover the complete definition of D. For a more detailed explanation, see [LLS15a, Prop. 4.3].

2.5 The Khovanov Functor

Following section 4 of [LLS15a], we now define the Khovanov functor $F_{Kh}(L) : \underline{2^n} \to \mathscr{B}$ which is a specific strictly-unitary-lax-2-functor from the cube category to the Burnside category that captures the information from the Khovanov chain complex. To do this, we will first define a functor $F_{Kh,Ab} : (\underline{2^n})^{op} \to Ab$ and then refine it to produce $F_{Kh}(L)$.

For an oriented link diagram, L, with n crossings and a fixed ordering on the crossings, we can construct a cube of resolutions for L by replacing each crossing with one of the following two resolutions



which are referred to as the the 0-resolution of 1-resolution respectively. Given a vertex $u = \{u_1, \dots u_n\} \in Ob(\underline{2^n})$, performing the u_i-resolution on the ith crossing of the link results in a collection of embedded circles in S^2, denoted L_u. Additionally, for an edge $u \mathrel{-\!\!\bullet} v$, L_u can be changed into L_v by either merging two circles together or splitting apart a circle into two circles. Let $V = \mathbb{Z}\langle x_+, x_- \rangle$ be a free rank-2 $\mathbb{Z}$ module with the following multiplication and comultiplication

$$
\begin{aligned}
m(x_+ \otimes x_+) &= x_+ & \qquad \Delta(x_+) &= x_+ \otimes x_- + x_- \otimes x_+ \\
m(x_+ \otimes x_-) &= x_- & \qquad \Delta(x_-) &= x_- \otimes x_- \\
m(x_- \otimes x_+) &= x_- & & \\
m(x_+ \otimes x_+) &= 0 & &
\end{aligned}
$$

We can define a function $F_{Kh,Ab} : (\underline{2^n})^{op} \to Ab$ as follows. For $u \in \underline{2^n}$, $F_{Kh,Ab}(u) = \bigotimes_{S \in \pi_0(L_u)} V$. For the morphism $\varphi_{u,v}$ corresponding to the edge $u \to v$, there are two cases. When L_u is obtained from L_v by merging two circles, $F_{Kh,Ab}(\varphi^{op}_{u,v})$ applies the multiplication map to the corresponding factors of $F_{Kh,Ab}(u)$ and the identity map to the remaining factors. In the other case, L_u is obtained from L_v by splitting apart a single circle, and so $F_{K_h,Ab}(\varphi^{op}_{u,v})$ applies the comultiplication map to corresponding factor of $F_{Kh,Ab}(u)$ and the identity map to the remaining factors. The total complex of the cube, which can be formed by multiplying the edge map $u \to v$ by $(-1)^{sgn_{u,v}}$ and summing over the vertices of each grading, is the Khovanov complex.

To refine $F_{Kh,Ab} : (\underline{2^n})^{op} \to Ab$ into a strictly-unitary-lax-2-functor $F_{Kh} : \underline{2^n} \to \mathscr{B}$, it suffices to describe a set of data for the vertices, edges, and faces of the cube that satisfy condition $(C1)$ and $(C2)$ from definition 2.13. For each u, $F_{Kh}(u) = \{x : \pi_o(L_u) \to \{x_+, x_-\}\}$ is the preferred basis of the Khovanov generators. For the edge morphism $u \to v$ and for each $y \in F_{Kh}(v)$, notice that $F_{Kh,Ab}(\varphi^{op}_{u,v})(y) = \sum_{x \in F_{Kh}(u)} \epsilon_{x,y} x$ where $\epsilon_{x,y}$ is a matrix whose entries are 0 and 1. This means, we can define $F_{Kh}(\varphi_{u,v}) = \{(y, x) \in F_{Kh}(v) \times F_{Kh}(u) | \epsilon_{x,y} = 1\}$.

To define the isomorphism $F_{Kh\; u,v,v',w} : F_{Kh}(\varphi_{v,w}) \times_{F_{Kh}(v)} F_{Kh}(\varphi_{u,v}) \to F_{Kh}(\varphi_{v',w}) \times_{F_{Kh}(v')} F_{Kh}(\varphi_{u,v'})$ for the face $\begin{tikzcd}[row sep=tiny, column sep=tiny] & v \arrow[dr] & \\ u \arrow[ur] \arrow[dr] & & w \\ & v' \arrow[ur] & \end{tikzcd}$, we first note that since $F_{Kh,Ab}$ is a commutative diagram there is a 2-isomorphism between $F_{Kh}(\varphi_{v,w}) \times_{F_{Kh}(v)}$

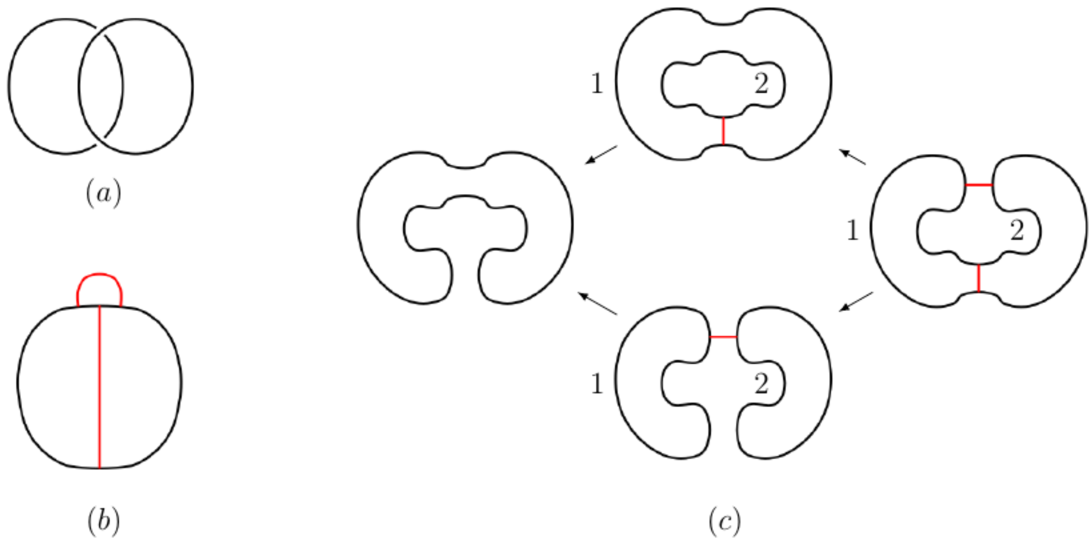


Figure 4. (a) An example of a link L that would result in a ladybug configuration. (b) An isotopy of the 00-resultion of L which gives the ladybug configuration its name. (c) The resolution of the link from (a) with the right arc pairs 1 and 2 labeled.

$F_{Kh}(\varphi_{u,v})$ and $F_{Kh}(\varphi_{v',w}) \times_{F_{Kh}(v')} F_{Kh}(\varphi_{u,v'})$. Namely, for $x \in F_{Kh}(u)$ and $z \in F_{Kh}(w)$, the cardinalities of

$$A_{x,z} := s^{-1}(z) \cap t^{-1}(x) \subseteq F_{Kh}(\varphi_{v,w}) \times_{F_{Kh}(v)} F_{Kh}(\varphi_{u,v}) \qquad \text{and}$$
$$A'_{x,z} := s^{-1}(z) \cap t^{-1}(x) \subseteq F_{Kh}(\varphi_{v',w}) \times_{F_{Kh}(v')} F_{Kh}(\varphi_{u,v'})$$

are the (x, z) entries in the matrix $F_{Kh,Ab}(\varphi^{op}_{u,v}) \circ F_{Kh,Ab}(\varphi^{op}_{v,w})$ and $F_{Kh,Ab}(\varphi^{op}_{u,v'}) \circ F_{Kh,Ab}(\varphi^{op}_{v',w})$ respectively, and these two matrices are the same. In most cases the cardinalities of the above sets are either zero or one. In either case, there is a unique isomorphism $F_{Kh\ u,v,v',w}|_{A_{x,z}} : A_{x,z} \to A'_{x,z}$. The only exceptional case is when a circle C_w in L_w splits to form two circles C^1_v and C^2_v in L_v and two circles $C^1_{v'}$ and $C^2_{v'}$ in $L_{v'}$; these two circles merge back to a single circle C_u in L_u; x labels C_u by x_-; and z labels C_w by x_+.

In this case, we can define the isomorphism by using the ladybug configuration (See figure 4). To use the ladybug configuration, we first draw the circle C_w from L_w. We then draw an arc a_v where we would need to pinch C_w together to form the two circles C_{v_1} and C_{v_2} in L_v. We then draw a second arc $a_{v'}$ where we would need to pinch the C_w circle together to form the two circles $C_{v'_1}$ and $C_{v'_2}$ in L'_v. Using a_v and $a_{v'}$ we can define the *right pair* of arcs in $(C_w, \partial a_v \cup \partial a_{v'})$ as the arcs you get by walking along a_v and $a_{v'}$ and then turning right. We then can choose to label one of the right pair arcs as 1 and the other as 2. (This choice of labeling will not affect $F_{Kh\ u,v,v',w}$). Each right pair arc is contained entirely within one of the two circles in both L_v and $L_{v'}$. This means we can label the two circle of L_v as C_v^1 and C_v^2 based on which right pair arc they contain. Similarly, we can label the two circles of $L_{v'}$ as $C_{v'}^1$ and $C_{v'}^2$. With these identifications we can define the two elements of $A_{x,z}$ as

$$\alpha = ((C_w \to x_+), (C_v^1, C_v^2) \to (x_-, x_+), (C_u \to x_-)),$$

$$\beta = ((C_w \to x_+), (C_v^1, C_v^2) \to (x_+, x_-), (C_u \to x_-))$$

and the two elements of $A'_{x,z}$ as

$$\alpha' = ((C_w \to x_+), (C_{v'}^1, C_{v'}^2) \to (x_-, x_+), (C_u \to x_-)),$$

$$\beta' = ((C_w \to x_+), (C_{v'}^1, C_{v'}^2) \to (x_+, x_-), (C_u \to x_-)).$$

With these two two-element sets identified $F_{Kh\ u,v,v',w}$ can be defined by mapping $\alpha \mapsto \alpha'$ and $\beta \mapsto \beta'$. From the definition of $F_{Kh\ u,v,v',w}$ it is clear that $F^{-1}_{Kh\ u,v',v,w}$ $= F_{Kh\ u,v,v',w}$ so condition $(C1)$ is satisfied. Checking condition $(C2)$ requires fixing a three-dimensional face
$$\begin{tikzcd}
 & v \arrow[r] \arrow[dr] & w'' \arrow[dr] & \\
u \arrow[ur] \arrow[r] \arrow[dr] & v' \arrow[ur] \arrow[dr] & w' \arrow[r] & z \\
 & v'' \arrow[ur] \arrow[r] & w \arrow[ur] &
\end{tikzcd}$$
and fixing Khovanov generators $x \in F_{Kh}(u)$ and $z \in F_{Kh}(z)$. There are six correspondences coming from the six

paths through the cube which correspond to three bijections F_{Kh,u,v^*,w^*} and three bijections $F_{Kh\ v^*,w*,z}$. One needs to check that these bijections agree (taking into account the ladybug formation.) The proof that these bijections do in fact agree follows from lemmas 5.14 and 5.17 in [LS14].

2.6 Homotopy Colimits

In order to define the Khovanov homotopy type, we will end up needing to take a homotopy colimit, so for completeness sake, we include Vogt's definition [Vog73] as described by Lawson, Lipshitz, and Sarkar in [LLS15a, Section 2.9].

Definition 2.12. *Let $\mathscr{C}$ be a small category and $D : \mathscr{C} \to Top_\bullet$ be a $\mathscr{C}$-diagram in $Top_\bullet$ (or Top). Let*

$$\mathscr{C}_n(A,B) = \{(f_n, \dots, f_1) \in (Mor(\mathscr{C}))^n | f_n \circ \cdots \circ f_1 : A \to B \text{ is defined in } \mathscr{C}\} \quad n > 0$$
$$\mathscr{C}_0(A,A) = \{(id_A)\} \qquad\qquad \mathscr{C}_0(A,B) = \emptyset \text{ for } A \neq B.$$

The homotopy colimit of D, hocolim(D), is

$$\left(\coprod_{A,B \in \mathscr{C}} \coprod_{n=0}^{\infty} \mathscr{C}_n(A,B) \times I^n \times D(A) \right) \cup \{*\} / \sim$$

where I is the unit interval and $\{*\}$ an extra point, where $\sim$ is given as follows:

$$(t_n, f_n, \dots, t_1, f_1; x) \sim \begin{cases} (t_n, f_n, \dots, t_2, f_2; x) & f_1 = id \\ (t_n, f_n, \dots, f_{i+1}, t_i t_{i-1}, f_{i-1}, \dots, t_1, f_1; x) & f_i = id, 1 < i \\ (t_n, f_n, \dots, t_{i+1}, f_{i+1} \circ f_i, t_{i-1}, , \dots, t_1, f_1; x) & t_i = 1, i < n \\ (t_{n-1}, f_{n-1}, \dots, t_1, f_1; x) & t_n = 1 \\ (t_n, f_n, \dots, f_{i+1}; D((f_i \circ f_1)(x)) & t_i = 0 \\ * & x = \text{base point} \end{cases}$$

with $\{*\}$ as base point for a diagram D in $Top_\bullet$. The unbased version is obtained by deleting $\{*\}$ and the last relation.

2.7 The Khovanov Homotopy Type

We now have all the required background needed to define the Khovanov homotopy type $\mathcal{X}_{Kh}(L)$ given a link diagram L and the Khovanov functor $F_{Kh}(L) : \underline{2^n} \to \mathscr{B}$. The construction from [LLS15a, Section 4] begins by defining a family of functors $\widehat{F^k_{Kh}} : \underline{\widehat{2^n}} \to Top_\bullet$ for eack $k \in \mathbb{N}$. For each $u \xrightarrow{\varphi_{u,v}} v \xrightarrow{\varphi_{v,w}} w \in \mathrm{Ob}\big(\underline{\widehat{2^n}}\big)$ we define

$$\widehat{F^k_{Kh}}\big(u \xrightarrow{\varphi_{u,v}} v \xrightarrow{\varphi_{v,w}} w\big) = \bigvee_{a \in F_{Kh}(\varphi_{u,v})} \prod_{\substack{b \in F_{Kh}(\varphi_{v,w}) \\ s(b)=t(a)}} S^k.$$

To define the image of a morphism $(\varphi_{u,u'}, \varphi_{v',v}, \varphi_{w,w'})$ between $u \xrightarrow{\varphi_{u,v}} v \xrightarrow{\varphi_{v,w}} w$ and $u' \xrightarrow{\varphi_{u',v'}} v' \xrightarrow{\varphi_{v',w'}} w'$ we need to define a map

$$\bigvee_{a \in F_{Kh}(\varphi_{u,v})} \prod_{\substack{b \in F_{Kh}(\varphi_{v,w}) \\ s(b)=t(a)}} S^k \longrightarrow \bigvee_{a' \in F_{Kh}(\varphi_{u',v'})} \prod_{\substack{b' \in F_{Kh}(\varphi_{v',w'}) \\ s(b')=t(a')}} S^k.$$

To do this we note that it suffices to construct the map on each piece of the wedge sum, so fix an $a \in F_{Kh}(\varphi_{u,v})$. The maps $F_{Kh\, u,u',v}$ and $F_{Kh\, u,v',v}$ induce a bijection

$$F_{Kh}(\varphi_{u,v}) \cong F_{Kh}(\varphi_{v',v}) \times_{F_{Kh}(v)} F_{Kh}(\varphi_{u',v'}) \times_{F_{Kh}(u')} F_{Kh}(\varphi_{u,u'}).$$

This means each a can be identified with a triple (y, a', x) in the fiber product above. With this identification, we can define $\widehat{F^k_{Kh}}$ to send the wedge summand corresponding to a to the wedge summand corresponding to a'. Similarly, the maps $F_{Kh\, v',w,w'}$ and $F_{Kh\, u,v',v}$ give an isomorphism

$$F_{Kh}(\varphi_{v',w'}) = F_{Kh}(\varphi_{w,w'}) \times_{F_{Kh}(w)} F_{Kh}(\varphi_{v,w}) \times_{F_{Kh}(v)} F_{Kh}(\varphi_{v',v})$$

so for each $b \in F_{Kh}(\varphi_{v',w'})$ we get $b' = (z, \bar{b}, \bar{y})$. If we let Δ_b represent the diagonal map then, we can consider the sequence of maps

$$\prod_{\substack{b \in F_{Kh}(\varphi_{v,w}) \\ s(b)=t(a)}} S^k \xrightarrow{\prod_b \Delta_b} \prod_{\substack{b \in F_{Kh}(\varphi_{v,w}) \\ s(b)=t(a)}} \prod_{\substack{b'=(z,\bar{b},\bar{y}) \in F_{Kh}(\varphi_{v',w}) \\ \bar{b}=b \\ \bar{y}=y}} S^k \cong \prod_{\substack{b' \in F_{Kh}(\varphi_{v',w}) \\ b'=(z,b,y) \\ s(b)=t(a)}} S^k$$

We now note that $\{b' \in F_{Kh}(\varphi_{u',v'}) | b' = (z, b, y)\}$ is a subset of $\{b' \in F_{Kh}(\varphi_{u',v'}) | t(a') = s(b')\}$ since $s(b') = s(y) = t(a')$. This means we can extend the above map to a map

$$\prod_{\substack{b \in F_{Kh}(\varphi_{v,w}) \\ s(b)=t(a)}} S^k \xrightarrow{\prod_b \Delta_b} \prod_{\substack{b \in F_{Kh}(\varphi_{v,w}) \\ s(b)=t(a)}} \prod_{\substack{b'=(z,\bar{b},\bar{y}) \in F_{Kh}(\varphi_{v',w}) \\ \bar{b}=b \\ \bar{y}=y}} S^k \cong \prod_{\substack{b' \in F_{Kh}(\varphi_{v',w}) \\ b'=(z,b,y) \\ s(b)=t(a)}} S^k \to \prod_{\substack{b' \in F_{Kh}(\varphi_{v',w'}) \\ s(b')=t(a')}} S^k$$

by mapping to the base point in the remaining factors. This is our desired map. Applying this map to every part of the wedge sum gives the definition of $\widehat{F^k_{Kh}}((\varphi_{u,u'}, \varphi_{v',v}, \varphi_{w,w'}))$.

We now note that there are natural transformations $S^n \wedge \widehat{F^k_{Kh}}(L) \to \widehat{F^{k+n}_{Kh}}(L)$ between our family of thickened Khovanov functors $\widehat{F^k_{Kh}}(L)$ given by

$$S^n \wedge \left(\bigvee_{a \in F_{Kh}(\varphi_{u,v})} \prod_{\substack{b \in F_{Kh}(\varphi_{v,w}) \\ s(b)=t(a)}} S^k \right) \cong \bigvee_{a \in F_{Kh}(\varphi_{u,v})} S^n \wedge \prod_{\substack{b \in \varphi_{v,w} \\ s(b)=t(a)}} S^k \to \bigvee_{a \in F_{Kh}(\varphi_{u,v})} \prod_{\substack{b \in \varphi_{v,w} \\ s(b)=t(a)}} S^{n+k}$$

The first part of this map is just given by commuting the smash product with the wedge sum. The second part of this map is given by applying the following map to each

$$\sigma^n : S^n \wedge \prod_i X_i \to \prod_i S^n \wedge X_i$$

where we view $S^n \wedge X$ as $[0,1]^n \times X / (\partial[0,1]^n \times X \cup [0,1]^n \times \{*\})$ and where $\sigma^n(y, x_1, \ldots, x_n) = ((y, x_1) \ldots (y, x_n))$. If all of the X_i are $(k-1)$-connected this σ^n induces isomorphisms on π_i for $0 \leq i \leq 2k-2$. With these natural transformations, we can view all of the $\widehat{F^k_{Kh}}$'s as being a diagram of spectra $\widehat{F_{Kh}} : \underline{\widehat{2^n}} \to \mathscr{S}$.

Finally, we let $\underline{\widehat{2^n}}_+$ be the category obtained from $\underline{\widehat{2^n}}$ by adding a new object $*$ and new morphisms $((u \to v \to w) \to *)$ from each vertex of $\underline{\widehat{2^n}}$ with $w \neq \vec{0}$. Similarly, we will define $\widehat{F^k_{Kh}}^+ : \underline{\widehat{2^n}}_+ \to \mathscr{S}$ by setting $\widehat{F^k_{Kh}}^+|_{\underline{\widehat{2^n}}} = \widehat{F^k_{Kh}}$ and $\widehat{F^k_{Kh}}^+(*) = \{*\}$. Taking the homotopy colimit of $\widehat{F_{Kh}}^+$ and formally desuspending by the number of negative crossings n_- in L produces $\mathcal{X}_{Kh}(L)$. That is $\mathcal{X}_{Kh}(L) = \Sigma^{-n_-} hocolim(\widehat{F_{Kh}}^+)$.

We should mention that in definition 2.15 we described the homotopy colimit for a functor $D : \mathscr{C} \to \mathrm{Top}_\bullet$ and not a functor to the category of spectra. To resolve this discrepancy, we note that the homotopy colimits of the the $\widehat{F^k_{Kh}}^+$'s together with the structure maps σ^n form a classical spectrum $(hocolim(\widehat{F^k_{Kh}}^+), \sigma^n)$ and that $(hocolim(\widehat{F_{Kh}}^+))_k = hocolim(\widehat{F^k_{Kh}}^+)$, and so $(hocolim(\widehat{F^k_{Kh}}^+), \sigma^n) = \mathcal{X}_{Kh}(L)$.

We should also mention that for the remainder of our discussion, we will suppress the formal grading shift as it is clear in the proofs of invariance that any grading shifts will agree with the above.

CHAPTER III

GROUP ACTIONS ON CATEGORIES AND FUNCTORS

The main goal of this chapter is to give two equivalent definitions for what it means for a group G to act on a strictly-unitary-lax two functor $D : \underline{2^n} \to \mathscr{B}$ from the cube category to the Burnside category. We begin by giving a definition for what it means for a group G to act on a category C.

Definition 3.1. *For a group G, the group action of G on a category C is the following collection of data:*

(1) an autoequivalence $\mathcal{G}_g : C \to C$ for each $g \in G$

(2) an isomorphism of functors $\eta_{g,h} : \mathcal{G}_h\mathcal{G}_g \cong \mathcal{G}_{hg}$ for each pair $g, h \in G$

such that for all $g, h, i \in G$ the following diagram of functors is commutative:

$$\begin{tikzcd}
\mathcal{G}_i\mathcal{G}_h\mathcal{G}_g \arrow[r] \arrow[d] & \mathcal{G}_i\mathcal{G}_{hg} \arrow[d] \\
\mathcal{G}_{ih}\mathcal{G}_g \arrow[r] & \mathcal{G}_{ihg}.
\end{tikzcd}$$

Given $g \in G$, $A \in Ob(C)$, and f a morphism in C, we will usually write gA to mean $\mathcal{G}_g(A)$ and gf to mean $\mathcal{G}_g(F)$. We now define what it means for G to act on a functor between two 1-categories.

Definition 3.2. *Let $\mathcal{F} : C \to C'$ be a functor between categories C and C' and let G be a group that acts on C. A group action of G on $\mathcal{F}$ is a collection of maps $R_{g,A} : \mathcal{F}(A) \to \mathcal{F}(gA)$ such that for all $A, B \in Ob(C)$, $f \in Hom(A, B)$, and for all $g, h \in G$, the following hold:*

(1) For the identity element $e \in G$, $R_{e,A}$ is the identity morphism

(2) $R_{h,gA} \circ R_{g,A} = R_{hg,A}$

(3) The following diagram commutes

$$\begin{tikzcd} \mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "R_{g,A}"'] & \mathcal{F}(B) \arrow[d, "R_{g,B}"] \\ \mathcal{F}(gA) \arrow[r, "(\mathcal{F} \circ \mathcal{G}_g)(f)"'] & \mathcal{F}(gB). \end{tikzcd}$$

Unfortunately, the above definition does not immediately apply to a strictly-unitary-lax-2-functor $D : \underline{2^n} \to \mathscr{B}$ because such functors only preserve compositions of morphisms up to an isomorphism. Instead, we describe the following larger set of data to explain precisely how G acts on D.

Definition 3.3. *Let $D : \underline{2^n} \to \mathscr{B}$ be a strictly-unitary-lax-2-functor from the cube category to the Burnside category described by the "complete set of data" (definition 2.10) and let G be a group that acts on $\underline{2^n}$. A group action of G on D is a collection of maps*

$$R_{g,u} : D(u) \to D(gu) \text{ and } S_{g,\varphi_{u,v}} : D(\varphi_{u,v}) \to D(\varphi_{gu,gv})$$

for any $u, v, w \in Ob(\underline{2^n})$ with $u >_k v >_l w$ and for any $g, h \in G$, such that

(c.i) For the identity element $e \in G$, $R_{e,u} : D(u) \to D(eu)$ and $S_{e,\varphi_{u,v}} : D(\varphi_{u,v}) \to D(\varphi_{eu,ev})$ are the identity maps

(c.ii) $R_{h,gu} \circ R_{g,u} = R_{hg,u}$

(c.iii) $S_{h,\varphi_{u,v}} \circ S_{g,\varphi_{u,v}} = S_{hg,\varphi_{u,v}}$

(c.iv) $R_{g,u} \circ s = s \circ S_{g,\varphi_{u,v}}$ *and* $R_{g,u} \circ t = t \circ S_{g,\varphi_{u,v}}$

(c.v) $S_{g,\varphi_{u,w}} \circ D_{u,v,w} = D_{gu,gv,gw} \circ (S_{g,\varphi_{v,w}} \times S_{g,\varphi_{v,w}})$.

We have chosen to label the conditions ($c.i$) through ($c.v$) in order to emphasize that this definition is for a group action on D where D is described by the complete set of data. The following definition is a similar collection data but for D described by the edge set of data. To distinguish between the two definitions, we will include the superscript ^{e} on the maps and we will label the conditions ($e.i$) - ($e.v$).

Definition 3.4. *Let* $D : \underline{2^n} \to \mathscr{B}$ *be a strictly-unitary-lax-2-functor from the cube category to the Burnside category described by the edge set of data (definition* 2.11*) and let* G *be a group that acts on* $\underline{2^n}$*. A group action of* G *on* D *is a collection of maps*

$$R^e_{g,u} : D(u) \to D(gu) \quad \text{and} \quad S^e_{g,\varphi_{u,v}} : D(\varphi_{u,v}) \to D(\varphi_{gu,gv})$$

for any $u, v, v', w \in Ob(\underline{2^n})$ *with* $\begin{tikzcd}[row sep=tiny, column sep=small] & v \arrow[dr] & \\ u \arrow[ur] \arrow[dr] & & w \\ & v' \arrow[ur] & \end{tikzcd}$ *and for any* $g, h \in G$*, such that*

(e.i) *For the identity element* $e \in G$*,* $R_{e,u} : D(u) \to D(eu)$ *and* $S_{e,\varphi_{u,v}} : D(\varphi_{u,v}) \to D(\varphi_{eu,ev})$ *are the identity maps*

(e.ii) $R^e_{h,gu} \circ R^e_{g,u} = R^e_{hg,u}$

(e.iii) $S^e_{h,\varphi_{u,v}} \circ S^e_{g,\varphi_{u,v}} = S^e_{hg,\varphi_{u,v}}$

(e.iv) $R^e_{g,u} \circ s = s \circ S^e_{g,\varphi_{u,v}}$ *and* $R^e_{g,u} \circ t = t \circ S^e_{g,\varphi_{u,v}}$

(e.v) $(S^e_{g,\varphi_{v',w}} \times_{R^e_{g,v'}} S^e_{g,\varphi_{u,v'}}) \circ D_{u,v,v',w} = D_{gu,gv,gv',gw} \circ \left(S^e_{g,\varphi_{v,w}} \times_{R^e_{g,v}} S^e_{g,\varphi_{u,v}}\right)$

An advantage of the second definition is that it contains much less data since the maps $S^e_{g,\varphi_{u,v}}$ need only be defined for the edges of the cube. Additionally, we only need to check that certain conditions hold for each face of the cube. We will now show the promised equivalence between definition 3.3 and definition 3.4.

Proposition 3.5. *The complete set of data for the action of G on $D : \underline{2^n} \to \mathscr{B}$ can be used to construct the edge set of data for the group action in such a way that $R_{g,u} = R^e_{g,u}$ for all $u \in \underline{2^n}$ and that $S_{g,\varphi_{u,v}} = S^e_{g,\varphi_{u,v}}$ for all edges $u \mathrel{-\!\!\bullet} v$. Similarly, the edge set of data for the group action can be used to construct the complete set of data in such a way that $R^e_{g,u} = R_{g,u}$ for all $u \in \underline{2^n}$ and that $S^e_{g,\varphi_{u,v}} = S_{g,\varphi_{u,v}}$ for all edges $u \mathrel{-\!\!\bullet} v$.*

Proof. We begin by showing that the complete data definition of the G-action produces the edge set of data. To do this, we need to define $R^e_{g,u}$ and $S^e_{g,\varphi_{u,v}}$ and show that conditions $(e.i) - (e.v)$ hold. We require that $R^e_{g,u} = R_{g,u}$ for all vertices and that $S^e_{g,\varphi_{u,v}} = S_{g,\varphi_{u,v}}$ for all edges, which immediately implies that conditions $(e.i) - (e.iv)$ follow directly from $(c.i) - (c.iv)$. All that remains to be shown is that condition $(e.v)$ holds. To prove this, we consider any face $\begin{tikzcd}[row sep=tiny, column sep=tiny] & v \arrow[dr, -{Circle}] & \\ u \arrow[ur, -{Circle}] \arrow[dr, -{Circle}] & & w \\ & v' \arrow[ur, -{Circle}] & \end{tikzcd}$ and note that in the following diagram, the two center squares commute because of condition $(c.v)$.

$$\begin{tikzcd}
D(\varphi_{v,w}) \times_{D(v)} D(\varphi_{u,v}) \arrow[r, "D_{u,v,w}"] \arrow[d, "S^e_{g,\varphi_{v,w}} \times S^e_{g,\varphi_{u,v}}"] \arrow[rr, bend left, "D_{u,v,v',w}"] & D(\varphi_{u,w}) \arrow[r, "D^{-1}_{u,v',w}"] \arrow[d, "S^e_{g,\varphi_{u,w}}"] & D(\varphi_{v',w}) \times_{D(v')} D(\varphi_{u,v'}) \arrow[d, "S^e_{g,\varphi_{v',w}} \times S^e_{g,\varphi_{u,v'}}"] \\
D(\varphi_{gv,gw}) \times_{D(gv)} D(\varphi_{gu,gv}) \arrow[r, "D_{gu,gv,gw}"'] \arrow[rr, bend right, "D_{gu,gv,gv',gw}"'] & D(\varphi_{gu,gw}) \arrow[r, "D^{-1}_{gu,gv',gw}"'] & D(\varphi_{gv',gw}) \times_{D(gv')} D(\varphi_{gu,gv'})
\end{tikzcd}$$

Since $D_{u,v,v',w} = D^{-1}_{u,v',w} \circ D_{u,v,w}$ the outer square of the diagram commutes and so condition $(e.v)$ holds, and so definition 3.3 can be used to construct the data for definition 3.4.

To show the other direction, we need to show that the collections of maps for the edge definition of the G-action give us a collection of maps for the complete G-action. We must again define $R_{g,u} = R^e_{g,u}$. This trivially satisfies the first half of condition $(c.i)$ and condition $(c.ii)$. In order to define $S_{g,\varphi_{u,v}}$ we will use the idea of maximal chains. For each pair of vertices $u >_k v$, choose any sequence of k edges

$$u = z_0^{u,v} \multimap ... \multimap z_i^{u,v} \multimap .... \multimap z_k^{u,v} = v.$$

to be the maximal chain $\mathfrak{m}^{u,v}$ from u to v. Given these choices of maximal chains we can define $D(\varphi_{u,v}) = D(\varphi_{z_k^{u,v}, z_{k-1}^{u,v}}) \times_{D(z_{k-1}^{u,v})} D(\varphi_{z_{k-1}^{u,v}, z_{k-2}^{u,v}}) \times ... \times_{D(z_1^{u,v})} D(\varphi_{z_1^{u,v}, z_0^{u,v}})$. Given $(x_k, ..., x_1) \in D(\varphi_{u,v})$ we can now apply our $S^e_{g,*}$ maps to each element in the tuple $(x_k, ..., x_1)$ since each element comes from an edge in the cube. This gives us a map from $D(\varphi_{u,v})$ to $D(\varphi_{gz_k^{u,v}, gz_{k-1}^{u,v}}) \times_{D(gz_{k-1}^{u,v})} D(\varphi_{gz_{k-1}^{u,v}, gz_{k-2}^{u,v}}) \times ... \times_{D(gz_1^{u,v})} D(\varphi_{gz_1^{u,v}, gz_0^{u,v}})$ which sends

$$(x_k, x_{k-1}, \ ... \ , x_2, x_1) \longmapsto \left(S^e_{g,\varphi_{z_k^{u,v}, z_{k-1}^{u,v}}}(x_k), \ ... \ , S^e_{g,\varphi_{z_1^{u,v}, z_0^{u,v}}}(x_1) \right).$$

If $k = 1$ then then $S_{g,\varphi_{u,v}} = S^e_{g,\varphi_{u,v}}$ for the edge $u \multimap v$ as required. However, if $k > 1$, then $\left(S^e_g, \varphi_{z_k^{u,v}, z_{k-1}^{u,v}}(x_k), \dots, S^e_g, \varphi_{z_1^{u,v}, z_0^{u,v}}(x_1) \right)$ may not be in $D(\varphi_{gu,gv})$ since the action of G may not send $\mathfrak{m}^{u,v}$ to $\mathfrak{m}^{gu,gv}$. More explicitly, $gu = gz_0^{u,v} \multimap \dots \multimap gz_i^{u,v} \multimap \dots \multimap gz_k^{u,v} = gv$ need not equal $gu = z_0^{gu,gv} \multimap ... \multimap z_i^{gu,gv} \multimap .... \multimap z_k^{gu,gv} = gv$. However, since both of these chains start at gu and gv, we can apply a series of face isomorphisms $D_{*,*,*,*}$ that are included in the data of our functor to get a map from

$$D(\varphi_{gz_k^{u,v}, gz_{k-1}^{u,v}}) \times_{D(gz_{k-1}^{u,v})} D(\varphi_{gz_{k-1}^{u,v}, gz_{k-2}^{u,v}}) \times ... \times D(\varphi_{gz_1^{u,v}, gz_0^{u,v}})$$

to

$$D(\varphi_{z_k^{gu,gv},z_{k-1}^{gu,gv}}) \times_{D(z_{k-1}^{gu,gv})} D(\varphi_{z_{k-1}^{gu,gv},z_{k-2}^{gu,gv}}) \times ... \times D(\varphi_{z_1^{gu,gv},z_0^{gu,gv}}) = D(\varphi_{gu,gv})$$

as required. This means, we can define $S_{g,\varphi_{u,v}}$ as the composition of the product of our $S^e_{g,*}$ maps and some number of face isomorphisms. It follows immediately that the second part of condition $(c.i)$ holds. To check that condition $(c.iii)$ holds, we first note that by our description the map $S_{h,\varphi_{u,v}} \circ S_{g,\varphi_{u,v}}$ is made up of four parts: a product of $S^e_{g,*}$ maps, a composition of face isomorphisms $D^{g_i}_{*,*,*,*} \circ \cdots \circ D^{g_1}_{*,*,*,*}$, a product of $S^e_{h,*}$ maps, and finally a second composition of face isomorphisms $D^{h_i}_{*,*,*,*} \circ \cdots \circ D^{h_1}_{*,*,*,*}$. By making use of condition $(e.v)$ we can commute the product of $S^e_{h,*}$ maps past the first set of face isomorphisms, and by using condition $(e.iii)$, we can compose the product of $S^e_{h,*}$ maps with the product of the $S^e_{g,*}$ maps to give us

$$\begin{aligned}
&\left(S_{h,\varphi_{u,v}} \circ S_{g,\varphi_{u,v}}\right)(x) \\
&= D^{h_{j'}}_{*,*,*,*} \circ \cdots \circ D^{h_1}_{*,*,*,*} \circ (S_{h,*}, \ldots S_{h,*}) \circ D^{g_j}_{*,*,*,*} \circ \cdots \circ D^{g_1}_{*,*,*,*} \circ \left(S^e_{g,*}(x_k), \ldots S^e_{g,*}(x_1)\right) \\
&= D^{h_{j'}}_{*,*,*,*} \circ \cdots \circ D^{h_1}_{*,*,*,*} \circ D^{g_j}_{h*,h*,h*,h*} \circ \cdots \circ D^{g_1}_{h*,h*,h*,h*} \circ \left((S^e_{h,*} \circ S^e_{g,*})(x_k), \ldots (S^e_{h,*} \circ S^e_{g,*})(x_1)\right) \\
&= D^{h_{j'}}_{*,*,*,*} \circ \cdots \circ D^{h_1}_{*,*,*,*} \circ D^{g_j}_{h*,h*,h*,h*} \circ \cdots \circ D^{g_1}_{h*,h*,h*,h*} \circ \left((S^e_{hg,*}(x_k), \ldots (S^e_{hg,*}(x_1)\right) \\
&= S_{hg,\varphi_{u,v}}(x)
\end{aligned}$$

as required. It is easy to check that condition $(c.iv)$ follows directly from condition $(e.iv)$. Finally, for condition $(c.v)$, we know that the map $D_{u,v,w}$ is defined to be a composition of face isomorpisms between $\mathfrak{m}^{u,v} \cup \mathfrak{m}^{v,w}$ and $\mathfrak{m}^{u,w}$ and that these isomorphisms commute with the group action in the sense of $(e.v)$. Hence, the proof of $(c.v)$ is analogous to our proof of $(c.iii)$. Thus, the complete set of data for the group action can be constructed from the edge set of data.

□

CHAPTER IV

A $\mathbb{Z}/p\mathbb{Z}$-ACTION ON THE KHOVANOV HOMOTOPY TYPE

The goal of this chapter is to prove theorem 1.1., *i.e.* the natural group action of $\mathbb{Z}/p\mathbb{Z}$ on a p-periodic link L induces an action on $\mathcal{X}_{Kh}(L)$ that makes $\mathcal{X}_{Kh}(L)$ a naive G-spectrum.

We begin by explaining how the action of $\mathbb{Z}/p\mathbb{Z}$ on L induces an action on $\underline{2^n}$. Recall from section 2.5 that if we fix an ordering of the n crossings of a knot diagram for L, then given a vertex $u = \{u_1, \ldots, u_n\} \in Ob(\underline{2^n})$ performing the u_i-resolution on the ith crossing of L forms a collection of embedded circles L_u. Since $g \in \mathbb{Z}/p\mathbb{Z}$ acts on L by rotating the knot diagram $\frac{2g\pi}{p}$ radians about a central axis, we can define gL_u to be the image of L_u under this rotation. It is clear that gL_u is identical to some other resolution $L_{u'}$ in Khovanov's cube of resolution where $u' = \{u'_1, \ldots u'_n\}$ is another vertex in $Ob(\underline{2^n})$, and so it makes sense to define $gu = u'$. Clearly $|u| = |gu|$ since the rotation action does not change any of the resolutions in L_u. Further, for any vertices $u >_k v$, we know that $gu >_k gv$, and since the Hom sets in $\underline{2^n}$ contain either 0 or 1 elements the only way to satisfy the condition in definition 3.1 is for g to map $\varphi_{u,v}$ to $\varphi_{gu,gv}$.

A more succinct way to state the above is that the action of $\mathbb{Z}/p\mathbb{Z}$ of L induces a permutation of the n crossings of L, and so $\mathbb{Z}/p\mathbb{Z}$ acts on $\underline{2^n}$ by permuting the n-coordinates of $\{0,1\}^n$ in the same manner.

We can extend the induced action of $\mathbb{Z}/p\mathbb{Z}$ on $\underline{2^n}$ to an action on $\widehat{\underline{2^n}}$, by defining g to act on objects by sending $(u \to v \to w) \mapsto (gu \to gv \to gw)$ and to act on morphisms by sending $(\varphi_{u,u'}, \varphi_{v',v}, \varphi_{w,w'}) \mapsto (\varphi_{gu,gu'}, \varphi_{gv',gv}, \varphi_{gw,gw'})$.

Since we now have a description for the induced group action on $\underline{2^n}$ and $\widehat{\underline{2^n}}$, we can now describe how $\mathbb{Z}/p\mathbb{Z}$ acts on the Khovanov functor F_{Kh} and the thickened Khovanov functor $\widehat{F_{Kh}}$.

Proposition 4.1. *Let L be a p-periodic link and let $F_{Kh}(L) : \underline{2^n} \to \mathscr{B}$ be the associated Khovanov functor. The natural action of $\mathbb{Z}/p\mathbb{Z}$ on L induces an action $\underline{2^n}$ which induces an action on $F_{Kh}(L)$.*

Proof. We will prove this proposition by describing a collection of maps that satisfy the edge definition for a group action on a strictly-unitary-lax-2-functor (definition 3.4). That is, we will define maps $R^e_{g,u} : F_{Kh}(u) \to F_{Kh}(gu)$ and $S^e_{g,\varphi_{u,v}} : F_{Kh}(\varphi_{u,v}) \to F_{Kh}(\varphi_{gu,gv})$ that meet conditions $(e.i) - (e.v)$.

For any vertex u, $F_{Kh}(u) = \{x : \pi_0(L_u) \to \{x_+, x_-\}\}$ is the set of labellings of the circles in L_u by the preferred Khovanov generators. Since L_{gu} is just a rotation of L_u, there is an induced map $g_* : \pi_0(L_u) \to \pi_0(L_{gu})$. This means that for any $x \in F_{Kh}(u)$, we can define $R_{g,u}(x) = x \circ (g_*)^{-1}$. It is clear that this definition of $R_{g,u}$ satisfies the first half of condition $(e.i)$ and condition $(e.ii)$.

For the edge morphism $u \multimap v$, $F_{Kh}(\varphi_{u,v})$ is defined by $F_{Kh}(\varphi_{u,v}) = \{(y,x) \in F_{Kh}(v) \times F_{Kh}(u) | \epsilon_{x,y} = 1\}$, so we define $S_{g,\varphi_{u,v}} : F_{Kh}(\varphi_{u,v}) \to F_{Kh}(\varphi_{gu,gv})$ by setting $S^e_{g,\varphi_{u,v}}((y,x)) = (R^e_{g,v}(y), R^e_{g,u}(x))$. It is clear that this satisfied the second half of condition $(e.i)$. We can check directly that condition $(e.iii)$ is satisfied since

$$\begin{aligned}
(S_{h,\varphi_{gu,gv}} \circ S_{g,\varphi_{u,v}})((y,x)) &= (S_{h,\varphi_{gu,gv}})(R_{g,v}(y), R_{g,u}(x)) \\
&= \Big((R_{h,gv} \circ R_{g,v})(y), (R_{h,gu} \circ R_{g,u})(x)\Big) \\
&= (R_{hg,v}(y), R_{hg,u}(x)) \\
&= (S_{hg,\varphi_{u,v}}(y,x)).
\end{aligned}$$

Again, checking directly, we get that

$$(s \circ S_{g,\varphi_{u,v}})((y,x)) = s(R_{g,v}(y), R_{g,u}(x)) = R_{g,u}(x) = R_{g,u}(s(y,x)) = (R_{g,u} \circ s)((y,x))$$

and that

$$(t \circ S_{g,\varphi_{u,v}})((y,x)) = t(R_{g,v}(y), R_{g,u}(x)) = R_{g,v}(y) = R_{g,v}(t(y,x)) = (R_{g,u} \circ t)((y,x))$$

and so condition $(e.iv)$ is satisfied.

All that remains for us to check is that our definitions satisfy condition $(e.v)$. That is, we need to check that for any face $\begin{smallmatrix} & v & \\ u & & w \\ & v' & \end{smallmatrix}$ the following diagram commutes

$$\begin{tikzcd}
F_{Kh}(\varphi_{v,w}) \times_{F_{Kh}(v)} F_{Kh}(\varphi_{u,v}) \arrow[r, "F_{u,v,v',w}"] \arrow[d, "S^e_{g,\varphi_{v,w}} \times_{R^e_{g,v}} S^e_{g,\varphi_{u,v}}"] & F_{Kh}(\varphi_{v',w}) \times_{F_{Kh}(v')} F_{Kh}(\varphi_{u,v'}) \arrow[d, "S^e_{g,\varphi_{v',w}} \times_{R^e_{g,v'}} S^e_{g,\varphi_{u,v'}}"] \\
F_{Kh}(\varphi_{gv,gw}) \times_{F_{Kh}(gv)} F_{Kh}(\varphi_{gu,gv}) \arrow[r, "F_{gu,gv,gv',gw}"'] & F_{Kh}(\varphi_{gv',gw}) \times_{F_{Kh}(gv')} F_{Kh}(\varphi_{gu,gv'})
\end{tikzcd}.$$

In section 2.6, we noted that for $x \in F_{Kh}(u)$ and $z \in F_{Kh}(w)$ we can define the sets

$$A_{x,z} := s^{-1}(x) \cap t^{-1}(z) \subseteq F_{Kh}(\varphi_{v,w}) \times_{F_{Kh}(v)} F_{Kh}(\varphi_{u,v})$$
$$A'_{x,z} := s^{-1}(x) \cap t^{-1}(z) \subseteq F_{Kh}(\varphi_{v',w}) \times_{F_{Kh}(v')} F_{Kh}(\varphi_{u,v'})$$

Applying the vertical maps in the diagram above to these two sets sends the sets $A_{x,z}$ and $A'_{x,z}$ to

$$A_{R_{g,u}(x),R_{g,w}(z)} := s^{-1}(R_{g,u}(x)) \cap t^{-1}(R_{g,w}(z)) \subseteq F_{Kh}(\varphi_{gv,gw}) \times_{F_{Kh}(gv)} F_{Kh}(\varphi_{gu,gv})$$

and

$$A'_{R_{g,u}(x),R_{g,w}(z)} := s^{-1}(R_{g,u}(x)) \cap t^{-1}(R_{g,w}(z)) \subseteq F_{Kh}(\phi_{gv',gw}) \times_{F_{Kh}(gv')} F_{Kh}(\phi_{gu,gv'})$$

respectively. We know the cardinality of these four sets will always be the same and will always be 0,1, or 2. When the cardinality of the sets is either 0 or 1 the above square obviously commutes, so we only need to consider the case when the cardinality is 2. In this case, we used the ladybug configuration which involved drawing arcs a_v and $a_{v'}$, which allowed us to define a right pair of arcs in the circle C_w. Labeling one of the right pair arcs as 1 and the other as 2 allowed us to label the two circles in L_v and $L_{v'}$ and define the elements of $A_{x,z}$ as

$$\alpha = ((C_w \to x_+), (C_v^1, C_v^2) \to (x_-, x_+), (C_u \to x_-))$$
$$\beta = ((C_w \to x_+), (C_v^1, C_v^2) \to (x_+, x_-), (C_u \to x_-))$$

and the two elements of $A'_{x,z}$ as

$$\alpha' = ((C_w \to x_+), (C_{v'}^1, C_{v'}^2) \to (x_-, x_+), (C_u \to x_-))$$
$$\beta' = ((C_w \to x_+), (C_{v'}^1, C_{v'}^2) \to (x_+, x_-), (C_u \to x_-)).$$

This allowed us to define $F_{u,v,v',w}$ as the map sending $\alpha \mapsto \alpha'$ and $\beta \mapsto \beta'$. Using the ladybug configuration for L_{gw}, we can also define the elements of $A_{R_{g,u}(x),R_{g,w}(z)}$ as

$$\alpha_g = ((C_{gw} \to x_+), (C_{gv}^1, C_{gv}^2) \to (x_-, x_+), (C_{gu} \to x_-))$$
$$\beta_g = ((C_{gw} \to x_+), (C_{gv}^1, C_{gv}^2) \to (x_+, x_-), (C_{gu} \to x_-))$$

and the two elements of $A'_{R_{g,u}(x),R_{g,w}(z)}$ as

$$\alpha'_g = ((C_{gw} \to x_+), (C_{gv'}^1, C_{gv'}^2) \to (x_-, x_+), (C_{gu} \to x_-))$$
$$\beta'_g = ((C_{gw} \to x_+), (C_{gv'}^1, C_{gv'}^2) \to (x_+, x_-), (C_{gu} \to x_-)).$$

$F_{Kh\ gu,gv,gv',gw}$ is defined to be the map that sends $\alpha_g \mapsto \alpha'_g$ and $\beta_g \mapsto \beta'_g$. If we let ga_v and $ga_{v'}$ be the image of the arcs a_v and $a_{v'}$ under the rotation action of g, it is clear that $ga_v = a_{gv}$, $ga_{v'} = a_{gv'}$ and that the image of the right pair of arcs in L_w is the same as the right pair of arcs in L_{gw}. However, the image of the *right pair* arc labeled 1 in L_w may not coincide with the *right pair* arc labeled 1 in L_{gw} since we independently chose these labellings when defining $F_{Kh\; u,v,v',w}$ and $F_{Kh\; gu,gv,gv',gw}$. If the image of the *right pair* arc labeled 1 in L_w coincides with the *right pair* arc labeled 1 in L_{gw}, then the vertical maps in the diagram send $\alpha \mapsto \alpha_g$, $\beta \mapsto \beta_g$, $\alpha' \mapsto \alpha'_g$, and $\beta' \mapsto \beta'_g$. If the image of the *right pair* arc labeled 1 in L_w coincides with the *right pair* arc labeled 2 in L_{gw}, then the vertical maps in the diagram send $\alpha \mapsto \beta_g$, $\beta \mapsto \alpha_g$, $\alpha' \mapsto \beta'_g$, and $\beta' \mapsto \alpha'_g$. In either case, the diagram commutes. $\square$

The proof of the previous proposition used the edge definition of a group acting on a strictly-unitary-lax-2-functor to show that there is an induced action of $\mathbb{Z}/p\mathbb{Z}$ on $F_{Kh}(L) : \underline{2^n} \to \mathscr{B}$. By proposition 3.5, we know that the we can use the maps $R^e_{g,u} : F_{Kh}(u) \to F_{Kh}(gu)$ and $S^e_{g,\varphi_{u,v}} : F_{Kh}(\varphi_{u,v}) \to F_{Kh}(\varphi_{gu,gv})$ defined in the previous proof to construct maps $R_{g,u} : F_{Kh}(u) \to F_{Kh}(gu)$ and $S_{g,\varphi_{u,v}} : F_{Kh}(\varphi_{u,v}) \to F_{Kh}(\varphi_{gu,gv})$ that satisfy conditions $(c.i)$ through $(c.v)$ from the complete definition of g acting on $F_{Kh}(L) : \underline{2^n} \to \mathscr{B}$. We will now use these complete definition maps to prove the following proposition.

Proposition 4.2. *Let L be an p-periodic link and let $F_{Kh}(L) : \underline{2^n} \to \mathscr{B}$ be the Khovanov functor. The natural action of $\mathbb{Z}/p\mathbb{Z}$ on L induces an action on the thickened Khovanov functor $\widehat{F^k_{Kh}} : \underline{\widehat{2^n}} \to Top_\bullet$.*

Proof. As in definition 3.2, we need to construct a map

$$R_{g,u\to v\to w} : \widehat{F^k_{Kh}}(u \to v \to w) \to \widehat{F^k_{Kh}}(gu \to gv \to gw)$$

that satisfies the three conditions in definition 3.2. This means we need a map

$$R_{g,u\to v\to w}: \bigvee_{a\in F_{Kh}(\varphi_{u,v})} \prod_{\substack{b\in F_{Kh}(\varphi_{v,w})\\ s(b)=t(a)}} S^k \longrightarrow \bigvee_{S_{g,\varphi_{u,v}}(a)\in F_{Kh}(\varphi_{gu,gv})} \prod_{\substack{S_{g,\varphi_{v,w}}(b)\in F_{Kh}(\varphi_{gv,gw})\\ s(S_{g,\varphi_{v,w}}(b))=t(S_{g,\varphi_{u,v}}(a))}} S^k$$

We will define this map on each component of the wedge sum, and then extend this across the wedge sum by mapping the component corresponding to $a \in F_{Kh}(\varphi_{u,v})$ to the wedge component corresponding to $S_{g,\varphi_{u,v}}(a) \in F_{Kh}(\varphi_{gu,gv})$. So fix an $a \in F_{Kh}(\varphi_{u,v})$, label the elements of $\{b \in F_{Kh}(\varphi_{v,w}); s(b) = t(a)\}$ arbitrarily by $b_1, \ldots, b_\ell$, and let p_{b_i} be a point in the S^k component of the product corresponding to b_i. Then we can define

$$R_{g,u\to v\to w}\big((p_{b_1},\ \ldots\ ,\ p_{b_\ell})\big) = (p_{S_{g,\varphi_{v,w}}(b_1)},\ \ldots\ , p_{S_{g,\varphi_{v,w}}(b_\ell)}).$$

The facts that $R_{e,u\to v\to w}$ is the identity morphism and that $R_{h,gu\to gv\to gw} \circ R_{g,u\to v\to w} = R_{hg,u\to v\to w}$ follow directly from conditions $(c.i)$ and $(c.ii)$ of the group action on $F_{Kh}: \underline{2^n} \to \mathscr{B}$.

All that remains is to verify the diagram in condition (3) of definition 3.2 commutes. Recall that in section 2.7, we defined the map $\widehat{F^k_{Kh}}(\varphi_{u,u'}, \varphi_{v',v}, \varphi_{w,w'})$ from $\widehat{F^k_{Kh}}(u \to v \to w)$ to $\widehat{F^k_{Kh}}(u' \to v' \to w')$ on each component of the wedge sum and then extended it to the whole wedge sum. Since $R_{g,u\to v\to w}$ is similarly defined, it suffices to check the commutative diagram in definition 3.2 component-wise as well. So fix an $a \in \widehat{F^k_{Kh}}(\varphi_{u,v})$. Also recall that in $\widehat{\underline{2^n}}$ the morphism

$$\begin{tikzcd}
u \arrow[r, "\varphi_{u,v}"] \arrow[d, "\varphi_{u,u'}"'] & v \arrow[r, "\varphi_{v,w}"] & w \arrow[d, "\varphi_{w,w'}"] \\
u' \arrow[r, "\varphi_{u',v'}"'] & v' \arrow[u, "\varphi_{v',v}"'] \arrow[r, "\varphi_{v',w'}"'] & w'
\end{tikzcd}$$

gives us two isomorphisms

$$F_{Kh}(\varphi_{u,v}) \cong F_{Kh}(\varphi_{v',v}) \times_{F_{Kh}(v')} F_{Kh}(\varphi_{u',v'}) \times_{F_{Kh}(u')} F_{Kh}(\varphi_{u,u'})$$

and

$$F_{Kh}(\varphi_{v',w'}) \cong F_{Kh}(\varphi_{w,w'}) \times_{F_{Kh}(w)} F_{Kh}(\varphi_{v,w}) \times_{F_{Kh}(v)} F_{Kh}(\varphi_{v',v}).$$

So for each $a \in F_{Kh}(\varphi_{u,v})$ there is a triple (y, a', x) in the composition, and similarly for each $b' \in F_{Kh}(\varphi_{v',w'})$ there is a corresponding triple $(z, \bar{b}, \bar{y})$. The induced group action on $\underline{\widehat{2^n}}$ also gives us the following isomorphisms

$$F_{Kh}(\varphi_{gu,gv}) \cong F_{Kh}(\varphi_{gv',gv}) \times_{F_{Kh}(gv')} F_{Kh}(\varphi_{gu',gv'}) \times_{F_{Kh}(gu')} F_{Kh}(\varphi_{gu,gu'})$$

and

$$F_{Kh}(\varphi_{gv',gw'}) \cong F_{Kh}(\varphi_{gw,gw'}) \times_{F_{Kh}(gw)} F_{Kh}(\varphi_{gv,gw}) \times_{F_{Kh}(gv)} F_{Kh}(\varphi_{gv',gv}).$$

By applying condition $(c.v)$ from definition 3.3, we see that for $S_{g,\varphi_{u,v}}(a) \in F_{Kh}(\varphi_{gu,gv})$ we have $S_{g,\varphi_{u,v}}(a) = \big(S_{g,\varphi_{v',v}}(y), S_{g,\varphi_{u,v'}}(a'), S_{g,\varphi_{u,u'}}(x)\big)$, and that for $S_{g,\varphi_{v',w}}(b') \in F_{Kh}(\varphi_{gv',gw})$ we have $S_{g,\varphi_{v',w}}(b') = \big(S_{g,\varphi_{w,w'}}(z), S_{g,\varphi_{u,v'}}(\bar{b}), S_{g,\varphi_{u,u'}}(\bar{y})\big)$. With these isomorphisms in mind, we let $m_i = |\{b' = (z, \bar{b}, \bar{y}) \in F_{Kh}(\varphi_{v',w'}) \mid \bar{b} = b_i\ \bar{y} = y\}|$. Since a is fixed, we know that y in the above isomorphisms is also fixed. This means each triple $(z, \bar{b}, \bar{y})$ is uniquely determined by b_i and a $z \in F_{Kh}(\varphi_{w,w'})$. So we will label the these triples as $(b_i, z_{i,1}),\ \cdots,\ (b_i, z_{i,m_i})$, so under the diagonal map $\Pi_b\Delta_b$, we will have that $p_{b_i} \mapsto (p_{b_i,z_{i,1}}, p_{b_i,z_{i,2}},\ \cdots,\ p_{b_i,z_{i,m_i}})$.

We need to show that the following diagram commutes, but first for clarity we will describe the maps involved in the diagram. The two horizontal maps are given by the group action. More specifically, the top horizontal map is given by $R_{g,u\to v\to w}$ and the bottom horizontal map $R_{g,u'\to v'\to w'}$. The first pair of vertical maps is just a diagonal map applied to each element in $F_{Kh}(\varphi_{u,v})$ and $F_{Kh}(\varphi_{gu,gv})$. The second pair of vertical maps is just a relabeling of the elements in the product under the bijections $b' = (z, b_i, \bar{y})$ and $S_{g,\varphi_{v',w}}(b') = \big(S_{g,\varphi_{w,w'}}(z), S_{g,\varphi_{u,v'}}(\bar{b}), S_{g,\varphi_{u,u'}}(\bar{y})\big)$ described above. For the the last pair of vertical maps, recall from section 2.7 that $\{b' \in F_{Kh}(\varphi_{u',v'}) | b' = (z, b, y)\}$

is a subset of $\{b' \in F_{Kh}(\varphi_{u',v'}) | t(a') = s(b')\}$ and that $\{S_{g,\varphi_{v,w}}(b') \in F_{Kh}(\varphi_{gu',gv'}) | S_{g,\varphi_{v,w}}(b') = (S_{g,\varphi_{w,w'}}(z), S_{g,\varphi_{v,w}}(b), S_{g,\varphi_{v',v}}(y))\}$ is a subset of $\{S_{g,\varphi_{v,w}}(b') \in F_{Kh}(\varphi_{gu',gv'}) | t(S_{g,\varphi_{u,v}}(a')) = s(S_{g,\varphi_{v,w}}(b'))\}$. So these final verical maps are just extensions of the previous vertical maps that map to the base point in the remaining factors.

$$
\begin{tikzcd}
\displaystyle\prod_{\substack{b \in F_{Kh}(\varphi_{v,w}) \\ s(b)=t(a)}} S^k \arrow[r, "R_{g,u\to v\to w}"] \arrow[d] & \displaystyle\prod_{\substack{S_{g,\varphi_{v,w}}(b) \in F_{Kh}(\varphi_{gv,gw}) \\ s(S_{g,\varphi_{v,w}}(b)) = t(S_{g,\varphi_{u,v}}(a))}} S^k \arrow[d] \\
\displaystyle\prod_{\substack{b \in F_{Kh}(\varphi_{v,w}) \\ s(b)=t(a)}} \prod_{\substack{b'=(z,\bar{b},\bar{y}) \in F_{Kh}(\varphi_{v',w'}) \\ \bar{b}=b \\ \bar{y}=y}} S^k \arrow[d, "\cong"'] & \displaystyle\prod_{\substack{S_{g,\varphi_{v,w}}(b) \in F_{Kh}(\varphi_{gv,gw}) \\ s(S_{g,\varphi_{v,w}}(b))=t(S_{g,\varphi_{u,v}}(a))}} \prod_{\substack{b'^*=(z^*,\bar{b^*},\bar{y^*}) \in F_{Kh}(\varphi_{gv',gw'}) \\ \bar{b^*}=S_{g,\varphi_{v,w}}(b) \\ \bar{y^*}=S_{g,\varphi_{v',v}}(y)}} S^k \arrow[d, "\cong"] \\
\displaystyle\prod_{\substack{b' \in F_{Kh}(\varphi_{v',w'}) \\ b'=(z,b,y) \\ s(b)=t(a)}} S^k \arrow[d] & \displaystyle\prod_{\substack{S_{g,\varphi_{v',w'}}(b') \in F_{Kh}(\varphi_{gv',gw'}) \\ S_{g,\varphi_{v',w'}}(b') = (S_{g,\varphi_{w,w'}}(z),\, S_{g,\varphi_{v,w}}(b),\, S_{g,\varphi_{v',v}}(y)) \\ s(S_{g,\varphi_{v,w}}(b)) = t(S_{g,\varphi_{u,v}}(a))}} S^k \arrow[d] \\
\displaystyle\prod_{\substack{b' \in F_{Kh}(\varphi_{v',w'}) \\ s(b')=t(a')}} S^k \arrow[r, "R_{g,u'\to v'\to w'}"'] & \displaystyle\prod_{\substack{S_{g,\varphi_{v',w'}}(b') \in F_{Kh}(\varphi_{gv',gw'}) \\ s(S_{g,\varphi_{gv',gw'}}(b'))=t(S_{g,\varphi_{gu',gv'}}(a'))}} S^k
\end{tikzcd}
$$

Going down and then right we get that

$$\begin{aligned}
&\big(p_{b_1}, \ \dots \ , p_{b_l}\big)\\
&\mapsto \big(p_{b_1,z_{1,1}}, \dots, \ p_{b_1,z_{1,m_1}}, p_{b_2,z_{2,1}}, \dots, p_{b_l,z_{l,m_l}}\big)\\
&\mapsto \big(p_{b'_{1,1}}, \ \dots \ , \ p_{b_{1,m_1}}, p_{b_{2,1}}, \dots, p_{b'_{\ell,m_\ell}}\big)\\
&\mapsto \big(p_{b'_{1,1}}, \ \dots \ , \ p_{b_{1,m_1}}, p_{b_{2,1}}, \dots, p_{b'_{\ell,m_\ell}}, *, \dots, *\big)\\
&\mapsto \big(p_{S_{g,\varphi_{v',w'}}(b'_{1.1})}, \ \dots \ , \ p_{S_{g,\varphi_{v',w'}}(b'_{1,m_\ell})}, p_{S_{g,\varphi_{v',w'}}(b'_{2,1})}, \dots, p_{S_{g,\varphi_{v',w'}}(b'_{\ell,m_\ell})}, *, \dots, *\big)
\end{aligned}$$

where $*$ denotes the base point.

Going right and then down we get

$$\begin{aligned}
&\big(p_{b_1}, \ \dots \ , \ p_{b_l}\big)\\
&\mapsto \big(p_{S_{g,\varphi_{v,w}}(b_1)}, \ \dots \ , \ p_{S_{g,\varphi_{v,w}}(b_l)}\big)\\
&\mapsto \big(p_{S_{g,\varphi_{v',w'}}(b_1),S_{g,\varphi_{w,w'}}(z_{1,1})}, \dots \ , \ p_{S_{g,\varphi_{v',w'}}(b_1),S_{g,\varphi_{w,w'}}(z_{1,m_1})},\\
&\qquad\qquad p_{S_{g,\varphi_{v,w}}(b_2),S_{g,\varphi_{w,w'}}(z_{2,1})}, \dots, p_{S_{g,\varphi_{v,w}}(b_\ell),S_{g,\varphi_{w,w'}}(z_{\ell,m_\ell})}\\
&\mapsto \big(p_{S_{g,\varphi_{v',w'}}(b'_{1.1})}, \ \dots \ , \ p_{S_{g,\varphi_{v',w'}}(b'_{1,m_\ell})}, p_{S_{g,\varphi_{v',w'}}(b'_{2,1})}, \dots, p_{S_{g,\varphi_{v',w'}}(b'_{\ell,m_\ell})}\big)\\
&\mapsto \big(p_{S_{g,\varphi_{v',w'}}(b'_{1.1})}, \ \dots \ , \ p_{S_{g,\varphi_{v',w'}}(b'_{1,m_\ell})}, p_{S_{g,\varphi_{v',w'}}(b'_{2,1})}, \dots, p_{S_{g,\varphi_{v',w'}}(b'_{\ell,m_\ell})}, *, \dots, *\big).
\end{aligned}$$

Thus, the diagram commutes and so condition (3) is satisfied. $\square$

Since we know that $\mathbb{Z}/p\mathbb{Z}$ acts on all of the $\widehat{F^k_{Kh}}$'s, we will now check that the action commutes with the natural transformations $S^n \wedge \widehat{F^k_{Kh}}(L) \to \widehat{F^{k+n}_{Kh}}(L)$. That is, we will check that the following diagram commutes:

$$
\begin{tikzcd}
S^n \wedge \left(\bigvee\limits_{a \in F_{Kh}(\varphi_{u,v})} \prod\limits_{\substack{b \in F_{Kh}(\varphi_{v,w}) \\ s(b)=t(a)}} S^k \right) \arrow[d] \arrow[dr, "id \wedge R_{g,u\to v\to w}"] & \\
\bigvee\limits_{a \in F_{Kh}(\varphi_{u,v})} \prod\limits_{\substack{b \in F_{Kh}(\varphi_{v,w}) \\ s(b)=t(a)}} S^{n+k} \arrow[dr, "R_{g,u\to v\to w}"'] & S^n \wedge \left(\bigvee\limits_{S_{g,\varphi_{u,v}}(a) \in F_{Kh}(\varphi_{gu,gv})} \prod\limits_{\substack{S_{g,\varphi_{v,w}}(b) \in F_{Kh}(\varphi_{gv,gw}) \\ s(S_{g,\varphi_{v,w}}(b))=t(S_{g,\varphi_{u,v}}(a))}} S^k \right) \arrow[d] \\
& \bigvee\limits_{S_{g,\varphi_{u,v}}(a) \in F_{Kh}(\varphi_{gu,gv})} \prod\limits_{\substack{S_{g,\varphi_{u,v}}(b) \in \varphi_{gv,gw} \\ s(S_{g,\varphi_{u,v}}(b))=t(S_{g,\varphi_{u,v}}(a))}} S^{n+k}.
\end{tikzcd}
$$

Recall that the suspension maps involve commuting the smash product past the wedge sum and then applying the following map to each summand

$$\sigma^n : S^n \wedge \prod_i X_i \to \prod_i S^n \wedge X_i$$

where we view $S^n \wedge X$ as $[0,1]^n \times X/(\partial[0,1]^n \times X \cup [0,1]^n \times \{*\})$ and where $\sigma^n(y, x_1, \ldots, x_n) = ((y, x_1) \ldots (y, x_n))$.

The action of $\mathbb{Z}/p\mathbb{Z}$ on $\widehat{F^k_{Kh}}$ and $\widehat{F^{k+n}_{Kh}}$ permutes the parts of the wedge sum and then permutes the S^k's in each of the products. Since we commute the smash product past the wedge sum the first permutation of the parts of the wedge sum, will not affect the natural transformation. Similarly, permuting the S^k's just corresponds to permuting the x_i's in the description of $S^n \wedge \prod_i X_i$ which clearly commutes with applying the map σ^n.

We now know G acts on $\widehat{F_{Kh}}(L) : \underline{\widehat{2^n}} \to \mathscr{B}$, which can be extended by to an action on $\widehat{F_{Kh}}^+(L)$ by having G fix the added point $*$. We can now prove theorem 1.1.

Theorem 1.1 *For a p-periodic link L, the natural action of $\mathbb{Z}/p\mathbb{Z}$ on L induces a $\mathbb{Z}/p\mathbb{Z}$ action on $\mathcal{X}_{Kh}(L)$, which makes $\mathcal{X}_{Kh}(L)$ a naive $\mathbb{Z}/p\mathbb{Z}$-spectrum.*

Proof. Let $Z = u \to v \to w$, $Z' = u' \to v' \to w'$, $gZ = gu \to gv \to gw$, and $gZ' = gu' \to gv' \to gw'$ be objects in $\underline{\widehat{2^n}}$. Since $\mathcal{X}_{Kh}(L) = hocolim(\widehat{F_{Kh}}^+(L))$, we know that the kth space in the spectrum is $hocolim(\widehat{F^k_{Kh}}^+(L))$, which is defined as

$$\left(\coprod_{Z,Z' \in \underline{\widehat{2^n}}} \coprod_{n=0}^{\infty} \mathscr{C}_n(Z,Z') \times I^n \times \widehat{F^k_{Kh}}^+(Z) \right) \cup \{*\}/\sim.$$

For $g \in \mathbb{Z}/p\mathbb{Z}$, g acts on $hocolim(\widehat{F^k_{Kh}}(L))$ by sending the above collection of cells to

$$\left(\coprod_{gZ,gZ' \in \underline{\widehat{2^n}}} \coprod_{n=0}^{\infty} \mathscr{C}_n(gZ,gZ') \times I^n \times R_{g,Z}\big(\widehat{F^k_{Kh}}^+(Z)\big) \right) \cup \{*\}/\sim.$$

In more detail, G acts on the homotopy colimit by sending a chain of composible morphisms in $\underline{\widehat{2^n}}$ to its image under the action of G on $\underline{\widehat{2^n}}$ (that is it sends an element of $\mathscr{C}_n(Z,Z')$ to an element of $\mathscr{C}_n(gZ,gZ')$), by sending I^n to I^n by the identity map, and by sending elements of $\widehat{F^k_{Kh}}^+(Z)$ to the elements of $\widehat{F^k_{Kh}}^+(gZ)$ by using the map $R_{g,Z}$ given by the action of G on $\widehat{F^k_{Kh}}^+(L) : \underline{\widehat{2^n}} \to \mathscr{B}$. The relations $\sim$ only involve the composition of morphisms in $\underline{\widehat{2^n}}$ which we know commutes with the action of G. Similarly, the conditions $(c.i)$ - $(c.v)$ ensure that this action satisfies the conditions for G acting on $hocolim(\widehat{F^k_{Kh}}^+(L))$ as a topological space.

Furthermore, the suspension map between $hocolim(\widehat{F^k_{Kh}}^+(L))$ and $hocolim(\widehat{F^{k+1}_{Kh}}^+(L))$ is given by natural transformation $S^1 \wedge \widehat{F^k_{Kh}}(L) \to \widehat{F^{k+1}_{Kh}}(L)$, and we know that this commutes with the action of g. Thus, g acts on each space in $\mathcal{X}_{Kh}(L)$ and the action of g commutes with the suspension in $\mathcal{X}_{Kh}(L)$, and so $\mathcal{X}_{Kh}(L)$ is a naive $\mathbb{Z}/p\mathbb{Z}$-spectrum as desired. $\square$

CHAPTER V

PROOF OF INVARIANCE

The goal of this chapter is to prove theorem 1.2. In order to do that, we need to show that if L and L' are two equivariantly isotopic p-periodic links, then $\mathcal{X}_{Kh}(L) = hocolim(\widehat{F_{Kh}}^{+}(L))$ is Borel stable homotopy equivalent to $\mathcal{X}_{Kh}(L') = hocolim(\widehat{F_{Kh}}^{+}(L'))$. This is equivalent to showing that the Khovanov homotopy type is invariant under the three equivariant Reidemeister moves (def 2.6). To do this, we need the following definition of an insular subfunctor, which is a special case of [LLS17, Def 3.25].

Definition 5.1. *Given a strictly-unitary-lax-2-functor $D : \underline{2^n} \to \mathscr{B}$ an insular subfunctor E of D is a collection of subsets $E(u) \subset D(u)$ for each $u \in Ob(\underline{2^n})$ such that for all $u > v$*

$$s^{-1}(E(u)) \cap t^{-1}(D(v) \setminus E(v)) = \emptyset \subset D(\varphi_{u,v}).$$

We can extend E to a strictly-unitary-lax-2-functor by defining $E(\varphi_{u,v}) \subset D(\varphi_{u,v}) = s^{-1}(E(u)) \cap t^{-1}(E(v))$ and letting $E_{u,v,w} : E(\varphi_{v,w}) \times_{E(v)} E(\varphi_{u,v}) \to E(\varphi_{u,w})$ be the map induced by $D_{u,v,w} : D(\varphi_{v,w}) \times_{D(v)} D(\varphi_{u,v}) \to D(\varphi_{u,w})$.

Given an insular subfunctor E of D, we can define the corresponding quotient functor $(D/E) : \underline{2^n} \to \mathscr{B}$ be setting $(D/E)(u) = D(u) \setminus E(u)$, $(D/E)(\varphi_{u,v}) = s^{-1}(D/E(u)) \cap t^{-1}(D/E)(v)$ and letting $(D/E)_{u,v,w}$ be the map induced by $D_{u,v,w}$. This is again a special case of the corresponding quotient functor described in [LLS17].

Our reason for introducing insular subfunctors and their quotient functors is the following important lemma.

Lemma 5.2. *Given an insular subfunctor E of a strictly-unitary-lax-2-functor D there exists a cofiber sequence*

$$hocolim(\widehat{E}^{+}) \hookrightarrow hocolim(\widehat{D}^{+}) \longrightarrow hocolim(\widehat{D/E}^{+}).$$

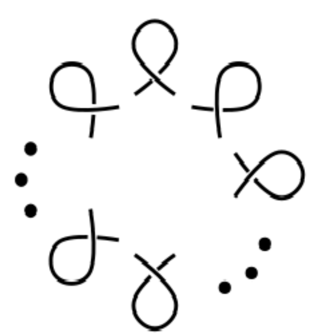
In particular, if the inclusion map (resp. quotient map) in the sequence above corresponds to an acyclic subcomplex of $Tot(\mathcal{A}(D))$, then the quotient map (resp. inclusion map) is a stable homotopy equivalence.

Proof. This follows directly from [LLS17] □

In order to prove that $\mathcal{X}_{Kh}(L)$ is Borel homotopy equivalent to $\mathcal{X}_{Kh}(L')$, it suffices to find insular subfunctors of $F_{Kh}(L)$ that are closed under the induced $\mathbb{Z}/p\mathbb{Z}$-action, that have corresponding chain complexes which are acyclic, and whose quotient functors are isomorphic to the Khovanov functor of L. Then we can apply the previous lemma to get the desired Borel homotopy equivalence.

Proposition 5.3. *If L and L' are two equivariantly isotopic p-periodic links that differ by a equivariant Reidemeister move of type I, then $\mathcal{X}_{Kh}(L)$ is Borel homotopy equivalent to $\mathcal{X}_{Kh}(L')$.*

Proof. Let $KC(L)$ and $KC(L')$ be the respective Khovanov chain complexes for L and L'. We know that L' differs from L by an equivariant Reidemeister move of type I, which is the same as performing p copies of a regular Reidemeister I move. We can depict these crossings as

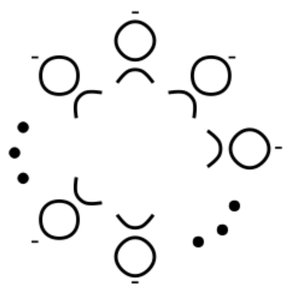


.

Figure 5. (a) The 1-resolution of a Reidemeister I move, (b) the 0-resolution of a Reidemeister I move, (c) a 0-resolution indicating both x_+ and x_- generators, (d) a resolution indicating all the generators for both the 1-resolution and the 0-resolution.

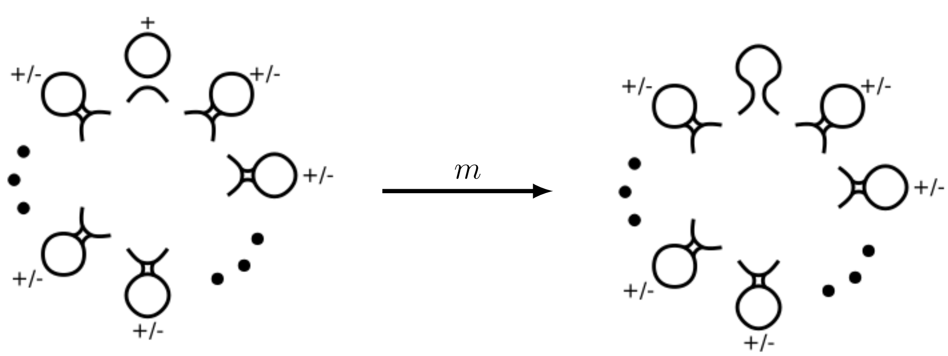
We will label these crossings 1 to p. Now in $KC(L')$ these crossing are resolved as either the 1-resolution or the 0-resolution (respectively (a) and (b) in figure 5). Additionally, the 0-resolution contains a unique circle that can be labeled as $+$ or $-$ to specify a part of the complex that only contains one of the Khovanov generators x_+ or x_- for that circle. Sometimes we want to allow for both generators in which case we will label the circle $+/-$ like in figure 5 (c). Similarly, when we want to refer to part of the complex containing the generators for any of the above resolutions, we will use the notation in figure 5 (d).

We will let C_1 be the subcomplex of $KC(L')$ containing all the generators for vertices where at least one of the crossings involved in the p Reidemeister moves of type I is the 1-resolution or at least one of the circles in the p 0-resolutions is labeled with an x_+. It is clear that C_1 is closed under the natural group action. The quotient $KC(L')/C_1$, which is depicted below, is the quotient complex that contains all the generators for the vertices where all the p Reidemeister I moves are the 0-resolution with the circle labeled x_-.



It is clear that $KC(L')/C_1$ is isomorphic to $KC(L)$ and that $KC(L')/C_1$ corresponds to an insular subfunctor $F_{KC(L')/C_1}$ of $F_{Kh}(L')$. If we can show that C_1 is an acyclic subcomplex, then we will be able to apply lemma 5.2 to get that $hocolim(\widehat{F_{KC(L')/C_1}}^{+}) \to hocolim(\widehat{F_{Kh}}^{+}(L'))$ is a homotopy equivalence and that $\mathcal{X}_{Kh}(L)$ is Borel homotopy equivalent to $\mathcal{X}_{Kh}(L')$ as required. To show this fact, we will describe how C_1 is built out of a series of acyclic subcomplexes (similar to the ones described by Bar-Natan [BN02]).

Let $C_{1.1}$ be the subcomplex of $KC(L')$ depicted below. That is, the subcomplex of all the generators for vertices where the first Reidemeister I move is the 0-resolution with the an x_+ labeled circle or the first Reidemeister I move is the 1-resolution.

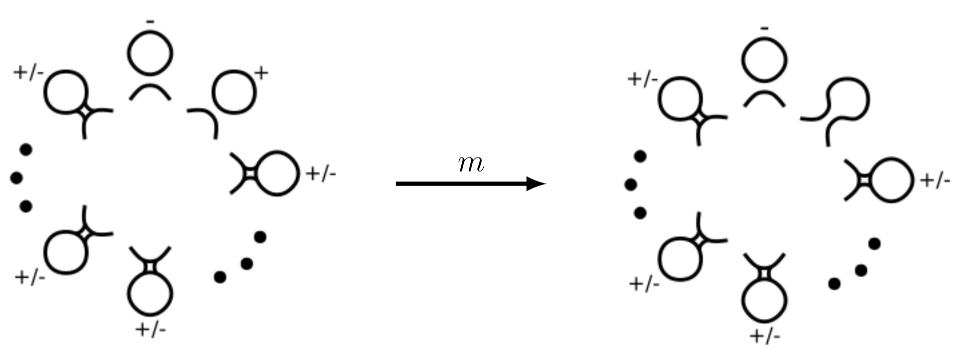


$C_{1.1}$

Note that the two types of vertices in $C_{1.1}$ are connected by an edge where the x_+ circle in one of 0-resolutions merges to form a 1-resolution. Each of these merge maps is an isomorphism, which means $C_{1.1}$ is an acyclic subcomplex. In a similar fashion, let $C_{1.2}$ be the subcomplex of $KC(L')$ containing all generators for the vertices where

- the first Reidemeister I move is the 0-resolution with an x_- marked circle, and the second Reidemeister I move is the 0-resolution with an x_+ marked circle, or

- the first Reidemeister I move is the 0-resolution with an x_- marked circle, and the second Reidemeister I move is the 1-resolution.

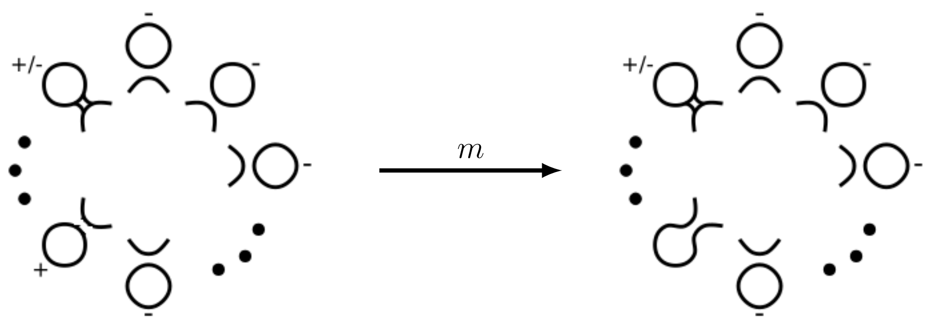


$C_{1.2}$

Again, the merge maps between the 0-resolutions and 1-resolutions in the second Reidemeister I move are all isomorphisms, so $C_{1.2}$ is an acyclic subcomplex. Continuing in this manner, let $C_{1.i}$ be the subcomplex containing all the generators for vertices where

- the first $i-1$ Reidemeister I moves are the 0-resolution with an x_- marked circle and the ith Reidemeister I move is the 0-resolution with an x_+ marked circle, or

- the first $i-1$ Reidemeister I moves are the 0-resolution with an x_- marked circle and the ith Reidemeister I move is the 1-resolution.



$C_{1.i}$

Again $C_{1.i}$ is acyclic. By construction, $C_1 = C_{1.1} \oplus \cdots \oplus C_{1.n}$, and since each of the $C_{1.i}$'s is acyclic, it follows that C_1 is an acyclic subcomplex. □

In the previous proposition, we described in detail a subcomplex corresponding to an insular subfunctor that was closed under the group action and was made up of acyclic subcomplexes corresponding to each of the p copies of the Reidemeister I move. We will use a similar proof technique for equivariant Reidemeister move of type II.

Proposition 5.4. *If L and L' are two equivariantly isotopic p-periodic links that differ by an equivariant Reidemeister move of type II, then $\mathcal{X}_{Kh}(L)$ is Borel homotopy equivalent to $\mathcal{X}_{Kh}(L')$.*

Proof. We again let $KC(L)$ and $KC(L')$ be the respective Khovanov chain complexes for L and L', and we note that L' differs from L by p copies of the normal Reidemeister move of type II. Each of these p type II moves in L' introduces two crossings. A depiction of two of the p type II moves can be seen in figure 6 (a) below. We will refer to the left move as the first of the p Reidemeister

II moves, and the right one as the ith Reidemeister II move. Each of the two crossings can be resolved in two ways, and so each Reidemeister II move represents four vertices in the cube of resolutions. We can label these four vertices as the 00-resolution, 10-resolution, 11-resolution, and the 01-resolution. (See figure 6 (b)). Note that the 01-resolution of each of the p-copies of the Reidemeister II move contains a central circle.

Let C_1 be the subcomplex of $KC(L')$ consisting of all the generators corresponding to vertices where

- one of the p copies of the Reidemeister II move is the 01-resolution with the central circle labeled x_+, or
- one of the p copies of the Reidemeister II move is the 11-resolution.

It is clear this is an equivariant subcomplex. To see that C_1 is acyclic, we will let $C_{1.1}$ be the subcomplex containing all the generators for vertices where

- the first Reidemeister II move is the 01-resolution with the central circle labeled x_+, or
- the first Reidemeister II move is the 11-resolution.

(See figure 6 (c)). The merge maps connecting the vertices in $C_{1.1}$ are isomorphisms, which means $C_{1.1}$ is acyclic. Similar to our method in the previous proof, we let $C_{1.i}$ be the subcomplex containing all generators corresponding to vertices where

- the first $i-1$ Reidemeister II moves are the 01-resolution with the central circle labeled x_- and the ith Reidemeister II move is the 01-resolution with the central circle labeled x_+, or

- the first $i-1$ Reidemeister II moves are the 01-resolution with the central circle labeled x_- and the ith Reidemeister II move is the 11-resolution.

See figure 6 (d). Again, the merge maps between these vertices are isomorphisms, and so $C_{1.i}$ is acyclic. Since $C_1 = C_{1.1} \oplus \cdots \oplus C_{1.n}$, we see that C_1 is an acyclic subcomplex that is closed under the group action.

Letting $C_2 = KC(L')/C_1$ we see that C_2 is the complex pictured in figure 7 (a). We now let C_3 be the subcomplex of C_2 consisting of all the generators for vertices where

- at least one of the p copies of the Reidemeister 11 moves is the 00-resolution, or
- at least one of the p copies of the Reidemeister II moves is the 01-resolution with the central circle labeled x_-.

It is clear from the description that C_3 is closed under the group action.

To show that C_3 is acyclic, we will let $C_{3.1}$ be the subcomplex of C_2 that contains all the generators for vertices where

- the first copy of the p type II Reidemeister moves is the 00-resolution, or
- the first copy of the p type II Reidemeister moves is the 01-resolution with the central circle labeled x_-

See figure 7 (b). Since C_2 only contains generators for vertices where the 01-resolutions have an x_- labeled central circle, the splitting maps between the vertices in $C_{3.1}$ are isomorphisms, which means $C_{3.1}$ is acyclic. Defining $C_{3.i}$ in the same manner described above, it is clear that $C_{3.i}$ is acyclic and that $C_3 = C_{3.1} \oplus \cdots \oplus C_{3.n}$ (See figure 7 (c)).

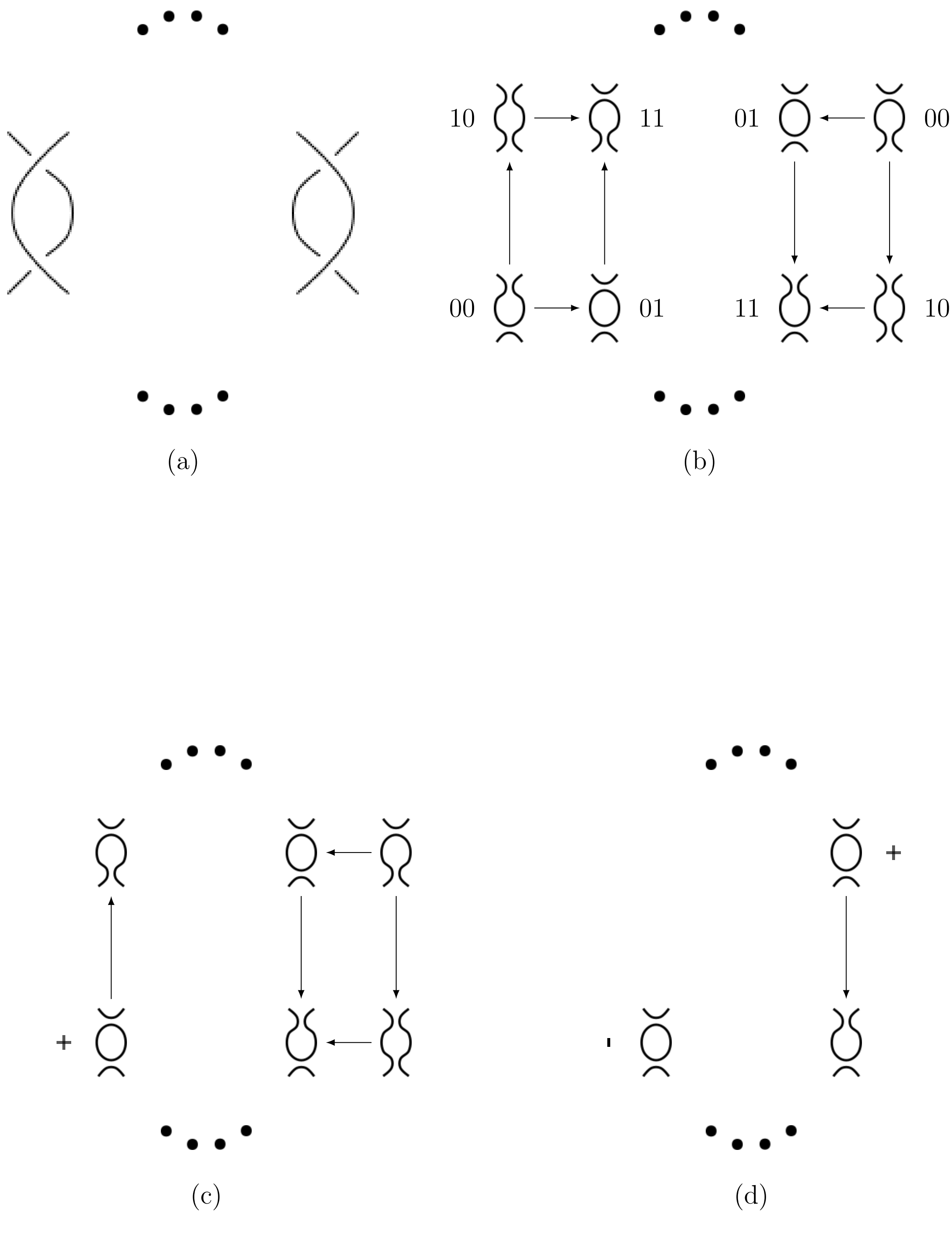


Figure 6. (a) a depiction of two of the p Reidemeister II moves that make up the equivariant Reidemeister II move, (b) the resolutions of the crossings in the two Reidemeister II moves, (c) the subcomplex $C_{1.1}$, (d) the subcomplex $C_{1.i}$

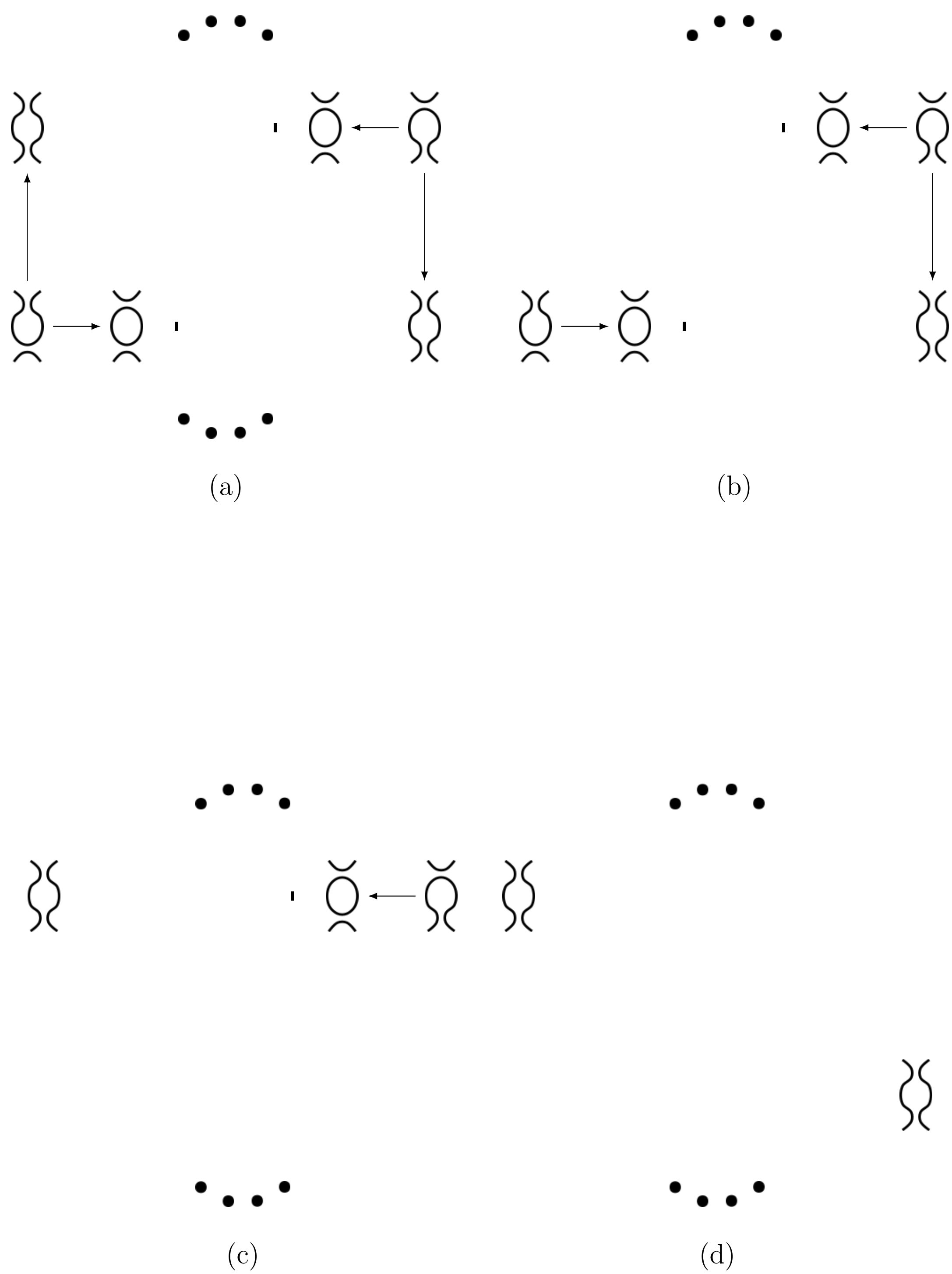


Figure 7. (a) the quotient complex C_2, (b) the subcomplex $C_{3.1}$, (c) the subcomplex $C_{3.i}$, (d) the quotient complex C_2/C_3.

Now, we note that C_2 corresponds to an insular subfuctor F_{C_2} of $F_{Kh}(L')$, so applying lemma 5.2 we get that the inclusion map $hocolim(\widehat{F_{C_2}}^+) \hookrightarrow hocolim(\widehat{F_{Kh}}^+(L'))$ is a homotopy equivalence. Additionally, we note that C_3 corresponds to an insular subfunctor F_{C_3} functor of F_{C_2}, which means we can apply lemma 5.2 a second time to get that the quotient map $hocolim(\widehat{F_{C_2}}^+) \to hocolim(\widehat{F_{C_2/C_3}}^+)$ is a homotopy equivalence. Since C_2/C_3 is isomorphic to $KC(L)$ (see figure 7 (d)), it follows that $hocolim(\widehat{F_{C_2/C_3}}^+)$ is homotopy equivalent to $hocolim(\widehat{F_{Kh}}^+(L))$, and so it follows that $\mathcal{X}_{Kh}(L')$ is Borel homotopy equivalent to $\mathcal{X}_{Kh}(L)$. □

Lipshitz and Sarkar note that the above two proofs depend upon the following facts.

Remark 5.5. [LS14] Let u and v be vertices in a (partial) cube of resolution such that there is an arrow from v to u, and one of the following holds.

(1) The arrow from v to u merges a circle U of the (partial) resolution at v. Let S be the set of all generators that correspond to u; and let T be the set of all generators corresponding to v with the circle U labeled by x_+.

(2) The arrow from v to u splits off a circle U in the (partial) resolution at U. Let S be the set of all generators that correspond to u with the circle U labeled by x_-; and let T be the set of all generators that correspond to v.

Let C be the chain complex generated by S and T; it is an acyclic complex, and therefore we can delete it without changing the homology. If, in addition, C is a subcomplex or a quotient complex of the original chain complex, then in deleting C, we do not introduce any new boundary maps.

In addition, we also note that if we want to delete an equivariant subcomplex, we can take the subcomplex where at least one of p copies of the non-equivariant Reidemeister moves contains one of the deletions described above. We can then show that this equivariant subcomplex (or the corresponding equivariant quotient complex) is acyclic by finding further acyclic subcomplexes. With this in mind we can now proceed to the equivariant Reidemeister move of type III, and note that since we have already shown that $\mathcal{X}(L)$ is invariant under equivariant Reidemeister moves of type I and type II it suffices to check the following braid-like version of the Reidemeister move of type III.

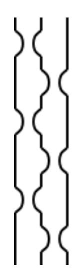
For more infomation about this braid-like Reidemeister move, see [Bal10, Section 7.3].

Proposition 5.6. *If L and L' are two equivariantly isotopic p-periodic links that differ by the braid-like version of the equivariant Reidemeister move of type III, then $\mathcal{X}_{Kh}(L)$ is Borel homotopy equivalent to $\mathcal{X}_{Kh}(L')$.*

Proof. We again let $KC(L)$ and $KC(L')$ correspond to the Khovanov complex of L and L' respectively. This means that L' differs from L by a series of p-copies of the braid version of a Reidemeister move of type III. We know from the previous remarks that it suffices to find acyclic equivariant subcomplexes corresponding to insular subfunctors whose inclusion/quotient result in a complex isomorphic to $KC(L)$. We also know that these equivariant subcomplexes can be built up from smaller complexes, each focusing on a single Reidemeister III type move. It suffices to use the complexes described by Lipshitz and Sarkar in [LS14, Proposition 6.4].

Let C_1 be the subcomplex of $KC(L')$ where at least one of the p Reidemeister type III braid-like moves contains a generator for one of the vertices depicted in figure 8 with all of the + marked circles corresponding to the x_+ generator. Note that C_1 is equivariant by construction, and it can be checked directly that the quotient complex $C_2 = KC(L')/C_1$ corresponds to an insular subfunctor F_{C_2}. This means that if C_1 is acyclic, then the inclusion map $hocolim(\widehat{F_{C_2}}^+) \to hocolim(\widehat{F_{Kh}}^+(L'))$ will induce a homotopy equivalence by lemma 5.2.

Let C_3 be the subcomplex of C_2 where at least one of the p Reidemeister type III braid-like moves contains a generator corresponding to one of the vertices depicted in figure 9. Again, the $-$ marked circle corresponds only to the x_- generator. By construction C_3 is equivariant and it can be checked that C_3 corresponds to an insular subfunctor F_{C_3}. If we can show that C_3 is acyclic, then it will again follow by lemma 5.2 that the quotient map $hocolim(\widehat{F_{C_2}}^+) \to hocolim(\widehat{F_{C_2/C_3}}^+)$ is a homotopy equivalence. The subcomplex C_2/C_3 results in p copies of the following 111000 resolution



which is clearly isotopic to $KC(L)$. So if we can show that C_1 and C_3 are acyclic then it will follow that $hocolim(\widehat{F_{C_2/C_3}}^+)$ will be homotopy equivalent to $hocomlim\widehat{F_{Kh}}^+(L)$ and that $\mathcal{X}_{Kh}(L)$ is Borel homotopy equivalent $\mathcal{X}_{Kh}(L')$.

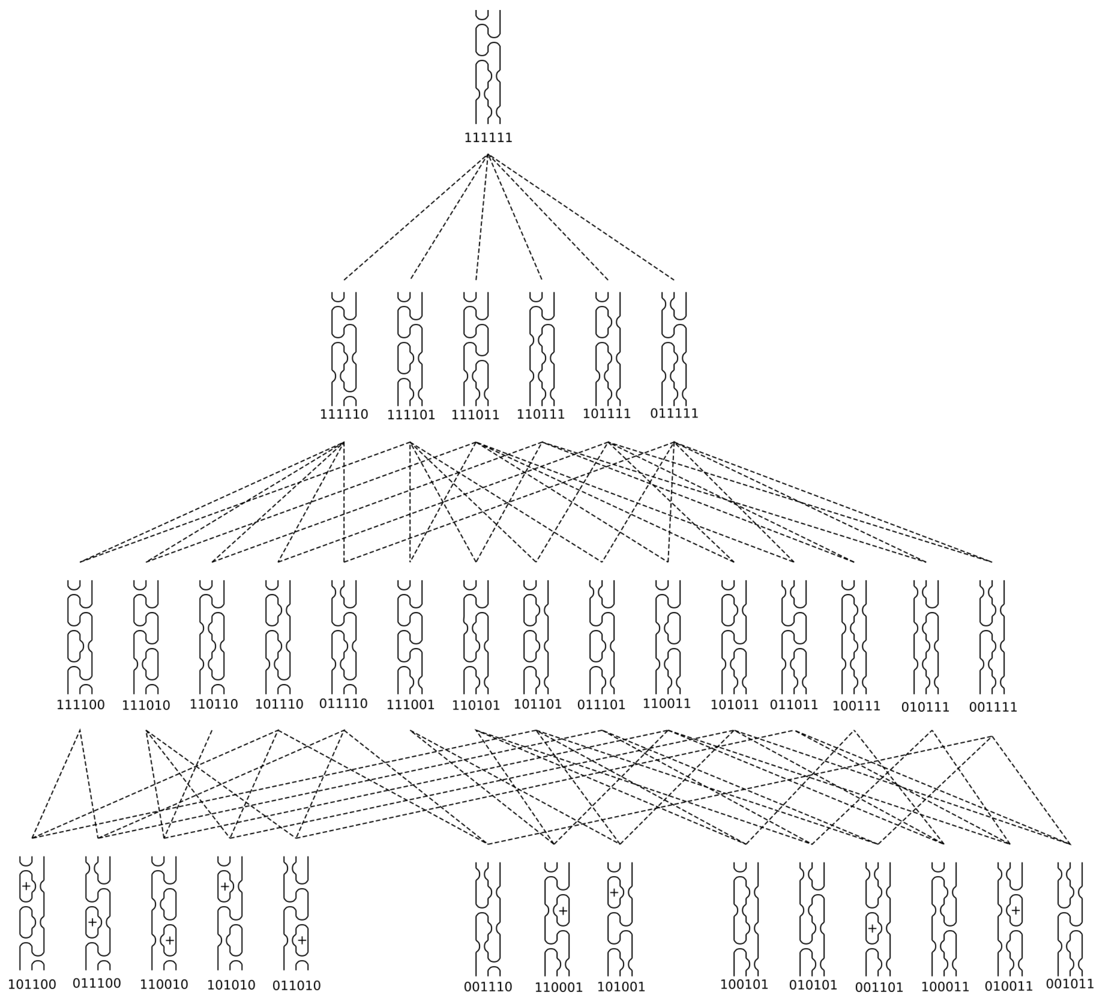


Figure 8. A subcomplex of C_1 for one of the p copies of the Reidemeister III move. The symbol $+$ indicates the corresponding circle is labeled x_+.

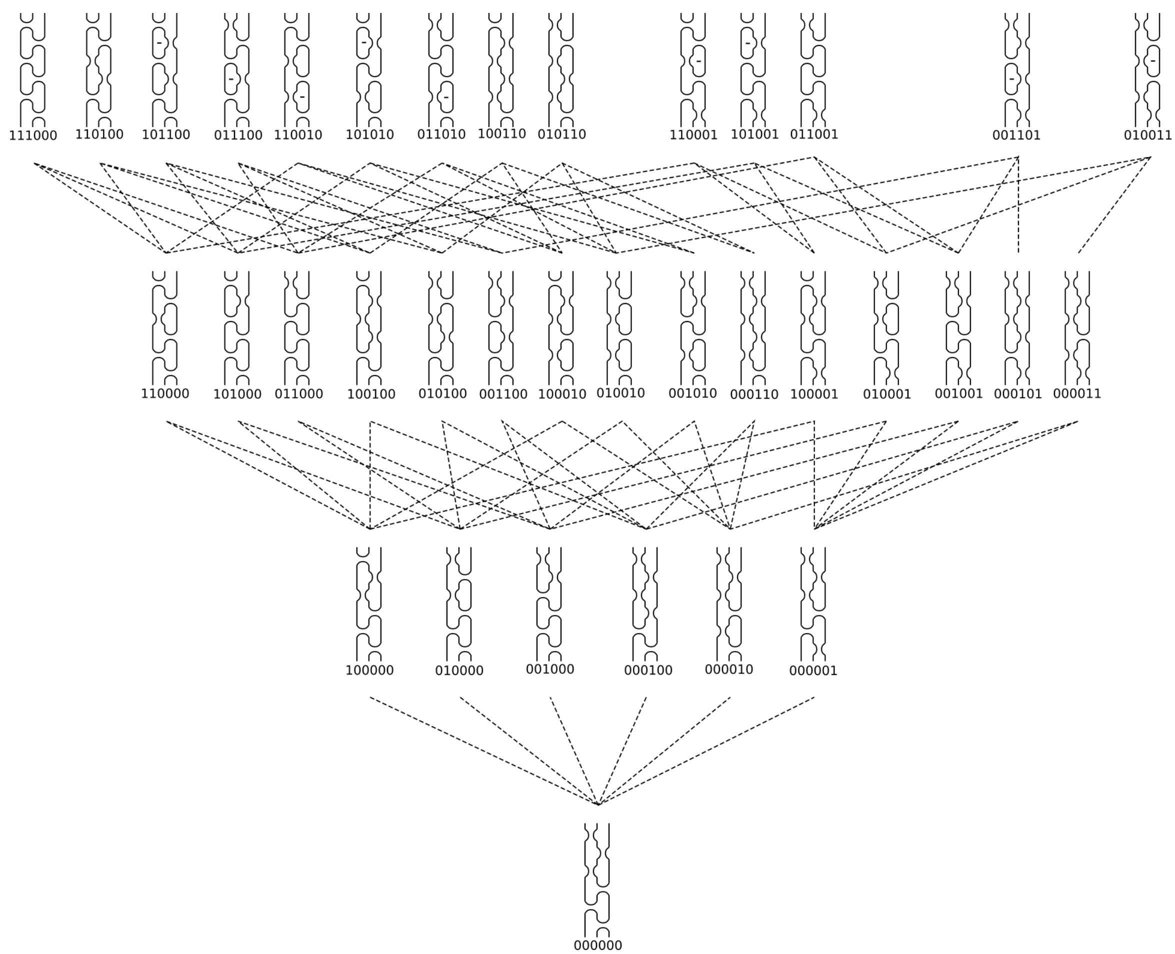


Figure 9. A subcomplex of C_3 for one of the p copies of the Reidemeister III move. The symbol $-$ indicates the corresponding circle is labeled x_-.

To see why these complexes are acyclic, we note the following list of deletions described by Lipshitz and Sarkar.

Top half 1*1111, 1*1110, 1*1101, 1*1011, 110*11, 0111*1, *01111, 1*1100, 1*1010, 1*1001, 0111*01, 110*01, *01110, 01101*, *01011, 110*10, 010*11, 1*0011, 01*101, *01101, 1001*1, 10*101.

Bottom half 0000*0, 1000*0, 0100*0, 0010*0, 00*100, 0*0001, 00001*, 0110*0, 1100*0, 1*1000, 10101*0, 01*100, 1*0001, *01001, 00*101, 01*001, 01001*, 010*10, 0*0110, *01100, *01010, 100*10, 1*0100, 10*100.

Here 1*1111 means to cancel along the edge from the 101111 resolution to the 111111 resolution. It can be checked directly that each of these cancellations corresponds to taking either a subcomplex or a quotient complex. This means during all of the cancellations no additional maps are introduced.

We can build C_1 and C_3 out of subcomplexes $C_{1,i}$ and $C_{3,i}$ as we did in propositions 5.3 and 5.4 above. That is, we can apply the deletions described by Lipshitz and Sarkar to each of the p copies of the Reidemeister III moves in turn. Each of the $C_{1,i}$'s and the $C_{3,i}$'s will be acyclic by construction and so C_1 and C_3 are acyclic as required. □

We can now prove our second main theorem.

Theorem 1.2 *For a p-periodic link L, the natural action of $\mathbb{Z}/p\mathbb{Z}$ on L induces a $\mathbb{Z}/p\mathbb{Z}$ action on $\mathcal{X}_{Kh}(L)$ which makes $\mathcal{X}_{Kh}(L)$ a naive $\mathbb{Z}/p\mathbb{Z}$-spectrum. Furthermore, if a link L' is equivariantly isotopic to L, then $\mathcal{X}_{Kh}(L')$ is Borel homotopy equivalent to $\mathcal{X}_{Kh}(L)$.*

Proof. If L is equivariantly isotopic to L', we know that L can be transformed into L' by a series of equivariant Reidemeister moves. Applying propositions 5.3, 5.4, and 5.6 in the same order as the series of equivariant Reidemeister moves gives us a roof of morphisms from $\mathcal{X}_{Kh}(L)$ to $\mathcal{X}_{Kh}(L')$. $\square$

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