A CATEGORICAL $\mathfrak{sl}_2$ ACTION ON SOME MODULI SPACES OF SHEAVES

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RYAN TAKAHASHI

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Student: Ryan Takahashi

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This dissertation has been accepted and approved in partial fulfillment of the requirements for the Doctor of Philosophy degree in the Department of Mathematics by:

Nicolas Addington	Chair
Ben Elias	Core Member
Alexander Polishchuk	Core Member
Nicholas Proudfoot	Core Member
Julia Widom	Institutional Representative

and

Kate Mondloch	Interim Vice Provost and Dean of the Graduate School

Original approval signatures are on file with the University of Oregon Graduate School.

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DISSERTATION ABSTRACT

Ryan Takahashi

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We study a certain sequence of moduli spaces of stable sheaves on a K3 surface of Picard rank 1 over $\mathbb{C}$. We prove that this sequence can be given the structure of a geometric categorical $\mathfrak{sl}_2$ action, a global version of an action studied by Cautis, Kamnitzer, and Licata. As a corollary, we find that the moduli spaces in this sequence which are birational are also derived equivalent.

CURRICULUM VITAE

NAME OF AUTHOR: Ryan Takahashi

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:

University of Oregon, Eugene, OR
Harvey Mudd College, Claremont, CA

DEGREES AWARDED:

Doctor of Philosophy, Mathematics, 2020, University of Oregon
Bachelor of Science, Mathematics, 2013, Harvey Mudd College

AREAS OF SPECIAL INTEREST:

Algebraic Geometry and Homological Algebra

PROFESSIONAL EXPERIENCE:

Graduate Employee, University of Oregon, 2013-2020

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CHAPTER I

INTRODUCTION

Given a K3 surface $\mathcal{S}$ over $\mathbb{C}$, one can study various moduli spaces of stable sheaves on $\mathcal{S}$ having fixed topological type. This topological information is encoded by fixing the Mukai vector $v(\mathcal{F}) := \mathrm{ch}(\mathcal{F})\sqrt{\mathrm{td}(\mathcal{S})}$, which for a K3 surface takes a particularly simple form:

$$v(\mathcal{F}) = (\mathrm{rk}(\mathcal{F}), c_1(\mathcal{F}), \chi(\mathcal{F}) - \mathrm{rk}(\mathcal{F})) \in H^0(\mathcal{S}, \mathbb{Z}) \oplus H^2(\mathcal{S}, \mathbb{Z}) \oplus H^4(\mathcal{S}, \mathbb{Z}).$$

We denote these moduli spaces $\mathcal{M}_{\mathcal{S}}(v) = \mathcal{M}_{\mathcal{S}}(r, c, s)$. If $r < 0$, we follow Markman in defining $\mathcal{M}_{\mathcal{S}}(r, c, s) = \mathcal{M}_{\mathcal{S}}(-r, c, -s)$; one can also view it as a moduli space of complexes concentrated in degree 1.

Under mild conditions on v (satisfied, in particular, if c generates $\mathrm{Pic}(\mathcal{S})$ as will be the case in our main example), $\mathcal{M}_{\mathcal{S}}(v)$ is a smooth projective hyperkähler variety. When $\langle v, v\rangle = c^2 - 2rs < -2$, the moduli space is empty; otherwise it has dimension $\langle v, v\rangle + 2$.

We assume that $\mathrm{Pic}(\mathcal{S}) = \mathbb{Z} \cdot H$, and for fixed r and s study the sequence of moduli spaces

$$\cdots \qquad \mathcal{M}_{\mathcal{S}}(r-1, H, s-1) \qquad \mathcal{M}_{\mathcal{S}}(r, H, s) \qquad \mathcal{M}_{\mathcal{S}}(r+1, H, s+1) \qquad \cdots$$

The order of the Brauer class obstructing the existence of a universal sheaf on $\mathcal{M}_{\mathcal{S}}(r+n, H, s+n)$ is the gcd over all Mukai vectors w of $(r+n, H, s+n) \cdot w$, or $\gcd(r+n, H^2, s+n)$. We further assume that $\gcd(r-s, H^2) = 1$, so that $\gcd(r+n, H^2, s+n) = 1$ for all n, and so each moduli space in the sequence has a universal sheaf.

From the dimension formula, we see that this sequence of spaces is bounded. The pairs of spaces $\mathcal{M}_{\mathcal{S}}(a, H, b)$ and $\mathcal{M}_{\mathcal{S}}(-b, H, -a)$ which are equidistant from the center admit well-studied birational isomorphisms, called stratified Mukai flops: one notes that for a general sheaf in one moduli space, the spherical twist (or inverse spherical twist) around $\mathcal{O}$ is (quasi-isomorphic to) a sheaf with the other Mukai vector. The indeterminacy locus of this map is precisely the Brill-Noether locus where the cohomology of the sheaves jumps in rank; we study these loci more carefully in Chapter III. The birational geometry of this picture is studied in depth by Markman in [18].

On the other hand, Cautis, Kamnitzer, and Licata study in [5] a sequence of cotangent bundles to Grassmannians, the local model for our sequence of moduli spaces. They construct correspondences between these spaces, and use these correspondences to define Fourier-Mukai functors between the derived categories of the spaces, then show that these functors satisfy certain $\mathfrak{sl}_2$-type relations. They refer to this collection of data as a *geometric categorical* $\mathfrak{sl}_2$ *action*; we review the precise definition in Chapter III. Our primary goal is to show that this same structure which exists in the local model can also be obtained globally on the moduli spaces.

Theorem 1.0.1. *Let $\mathcal{S}$ be a K3 surface over $\mathbb{C}$ with Pic$(\mathcal{S}) = \mathbb{Z} \cdot H$, let r and s be integers with $r - s$ coprime to H^2, and consider the sequence $\mathcal{M}_{\mathcal{S}}(r+n, H, s+n)$, where $n \in \mathbb{Z}$, of moduli spaces of Gieseker-(semi)stable sheaves on $\mathcal{S}$. Then there exist correspondences between these moduli spaces, and line bundles on those correspondences, so that this sequence of spaces and Fourier-Mukai kernels has the structure of a geometric categorical $\mathfrak{sl}_2$ action.*

Combining this theorem with general results of Cautis, Kamnitzer, and Licata on geometric categorical $\mathfrak{sl}_2$ actions, we obtain the following corollary. Halpern-Leistner ([11]) has also announced a proof using alternative methods.

Corollary 1.0.2. *If $\mathcal{S}$ is a K3 surface over $\mathbb{C}$ with Pic$(\mathcal{S}) = \mathbb{Z}{\cdot}H$, then the pairs of moduli spaces $\mathcal{M}_{\mathcal{S}}(a, H, b)$ and $\mathcal{M}_{\mathcal{S}}(-b, H, -a)$, which are birational via spherical twists, are also derived equivalent.*

We are hopeful that Corollary 1.0.2 can be used to obtain the suggested generalization of equation (0.3) in [2], with this derived equivalence taking the place of the Kawamata-Namikawa equivalence.

We conclude the section by giving a brief overview of the paper's organization. In Chapter II we develop some general facts about Grassmannians of coherent sheaves which will be used repeatedly in our later work. In Chapter III, we give a precise definition of geometric categorical $\mathfrak{sl}_2$ action, and construct the action described in Theorem 1.0.1. As in [5], we start by constructing correspondences between our moduli spaces, and define our Fourier-Mukai kernels as line bundles on these correspondences. Finally, in Chapter IV, we verify that our constructions satisfy the various compatibilities required of a geometric categorical $\mathfrak{sl}_2$ action.

CHAPTER II

GRASSMANNIANS OF COHERENT SHEAVES

The correspondences mentioned in Theorem 1.0.1 will be constructed as Grassmannians of coherent sheaves on the moduli spaces. We establish some general facts about these Grassmannians for later use.

Let X be an integral, Cohen-Macaulay scheme, and $\mathcal{F}$ a coherent sheaf of rank r on X. For any $k \leq r$, we consider the Grassmannian of rank k quotients,

$$\mathrm{Gr}(\mathcal{F}, k) \xrightarrow{\pi} X,$$

characterized by the universal property that giving a map $T \to \mathrm{Gr}(\mathcal{F},\mathrm{k})$ is the same as giving a map $T \xrightarrow{f} X$ and a surjection $f^*\mathcal{F} \twoheadrightarrow V^k$ onto a rank k vector bundle. In particular, the Grassmannian carries a tautological exact sequence

$$0 \to S_\pi \to \pi^*\mathcal{F} \to Q_\pi \to 0$$

of sheaves, where Q_π is locally free but S_π need not be. As expected, one has $\mathrm{Gr}(\mathcal{F}, k)|_x = \mathrm{Gr}(\mathcal{F}|_x, k)$ for any point $x \in X$. For a more detailed introduction in the case $k = 1$, see [8, p. 103].

For each i, let

$$X_{\mathcal{F},i} := \{x \in X \mid \dim \mathcal{F}|_x \geq r + i\}$$

be the closed subscheme determined by the $(r + i - 1)$th Fitting ideal. We will simply write X_i when the sheaf is understood.

Theorem 2.0.1. *If $\mathcal{F}$ has homological dimension at most 1 and* $\mathrm{codim}_X(X_i) > ik$ *for all* $i > 0$*, then*

1) *$Gr(\mathcal{F}, k)$ is integral and Cohen-Macaulay. If additionally X is smooth, then $Gr(\mathcal{F}, k)$ is a local complete intersection.*

2) *$L_i\pi^*\mathcal{F} = 0$ for all $i \neq 0$.*

3) *The natural map $\mathcal{O}_X \to R\pi_*\mathcal{O}_{Gr(\mathcal{F},k)}$ is an isomorphism.*

4) *The pushforward of the map $\pi^*\mathcal{F} \to Q_\pi$ is an isomorphism $\mathcal{F} \xrightarrow{\sim} R\pi_*Q_\pi$.*

All the statements are étale-local on X, so after shrinking X if necessary, we may choose a resolution

$$0 \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0 \tag{2.0.1}$$

of $\mathcal{F}$ by vector bundles $\mathcal{E}_0$ and $\mathcal{E}_1$ of ranks s_0 and s_1, respectively. Denote by ρ the structure map $\mathrm{Gr}(\mathcal{E}_0, k) \to X$.

Lemma 2.0.2. *$Gr(\mathcal{F}, k)$ is cut out of $Gr(\mathcal{E}_0, k)$ by a section of the rank ks_1 vector bundle $\rho^*\mathcal{E}_1^\vee \otimes Q_\rho$.*

Proof. Pull the sequence (2.0.1) back along ρ. We get an exact sequence

$$\rho^*\mathcal{E}_1 \to \rho^*\mathcal{E}_0 \to \rho^*\mathcal{F} \to 0,$$

and so $\rho^*\mathcal{E}_0 \twoheadrightarrow Q_\rho$ descends to $\rho^*\mathcal{F} \twoheadrightarrow Q_\rho$ if and only if the composite $\rho^*\mathcal{E}_1 \to Q_\rho$ is zero. □

Proof of 1). To show irreducibility, we note that by Lemma 2.0.2, $\mathrm{Gr}(\mathcal{F}, k)$ is cut out of a space of dimension $\dim X + k(s_0 - k)$ by a section of a vector bundle of rank ks_1, so each irreducible component of $\mathrm{Gr}(\mathcal{F}, k)$ has dimension at least

$$\dim X + k(s_0 - k) - ks_1 = \dim X + k(r - k).$$

However, we see that

$$\dim \pi^{-1}(X_i \setminus X_{i+1}) = \dim(X_i \setminus X_{i+1}) + k(r + i - k).$$

When $i = 0$, the preimage of the open stratum $X_0 \setminus X_1$ has dimension

$$\dim(X_0 \setminus X_1) + k(r - k) = \dim X + k(r - k),$$

the expected dimension. By the codimension condition in Theorem 2.0.1, we find that for $i > 0$, the preimages of the strata have dimension

$$\dim(X_i \setminus X_{i+1}) + k(r + i - k) < \dim X - ik + k(r + i - k) = \dim X + k(r - k).$$

In particular, none of these belong to separate irreducible components.

Now $\mathrm{Gr}(\mathcal{F}, k)$ is cut out of $\mathrm{Gr}(\mathcal{E}_0, k)$ by the right number of equations, and the latter is Cohen-Macaulay (being smooth over the Cohen-Macaulay scheme X), so the former is as well. If X (and thus $\mathrm{Gr}(\mathcal{E}_0, k)$) is smooth, then $\mathrm{Gr}(\mathcal{F}, k)$ is moreover a local complete intersection. In either case, since $\mathrm{Gr}(\mathcal{F}, k)$ is Cohen-Macaulay and generically reduced, it is reduced. □

Proof of 2). Apply π^* to the resolution (2.0.1): the long exact sequence

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow L_2\pi^*\mathcal{F} \longrightarrow 0 \longrightarrow 0 \longrightarrow L_1\pi^*\mathcal{F} \longrightarrow \pi^*\mathcal{E}_1 \longrightarrow \pi^*\mathcal{E}_0 \longrightarrow \pi^*\mathcal{F} \longrightarrow 0.$$

shows that $L_i\pi^*\mathcal{F}$ vanishes for $i \geq 2$ and that $L_1\pi^*\mathcal{F}$ injects into the vector bundle $\pi^*\mathcal{E}_1$. But on the open subscheme $X_0 \setminus X_1$, $\mathcal{F}$ is locally free, so $L_1\pi^*\mathcal{F}$ is supported on $\pi^{-1}(X_1)$. Since the Grassmannian is irreducible by 1), this is a torsion subsheaf of a torsion-free sheaf, hence zero. $\square$

Remark 2.0.3. If we ask only for the weak inequality $\operatorname{codim}_X(X_i) \geq ik$, then both 1) and 2) fail: the Grassmannian is Cohen-Macaulay but may be reducible, and $L_1\pi^*\mathcal{F}$ may be non-zero. Indeed, if $\operatorname{codim}_X(X_i) = ik$ for some i, then the preimage $\pi^{-1}(X_i \setminus X_{i+1})$ is an irreducible component of $\operatorname{Gr}(\mathcal{F}, k)$, and $L_1\pi^*\mathcal{F}$ has rank i on this component.

Lemma 2.0.4. *Parts 3) and 4) of Theorem 2.0.1 hold for $k = 1$, i.e. for $\pi : \mathbb{P}\mathcal{F} \to X$.*

Proof. As in Lemma 2.0.2, let ρ denote the map $\mathbb{P}\mathcal{E}_0 \to X$, and let i be the inclusion $\mathbb{P}\mathcal{F} \hookrightarrow \mathbb{P}\mathcal{E}_0$. The Koszul complex

$$0 \to \rho^*\left(\bigwedge^{s_1} \mathcal{E}_1\right)(-s_1) \to \cdots \to \rho^*\mathcal{E}_1(-1) \to \mathcal{O}_{\mathbb{P}\mathcal{E}_0} \to i_*\mathcal{O}_{\mathbb{P}\mathcal{F}} \to 0$$

is exact: $\mathbb{P}\mathcal{E}_0$ is Cohen-Macaulay since X is, and $\mathbb{P}\mathcal{F}$ has the expected codimension, so the sequence that cuts it out locally is automatically regular, by [19, Theorem 17.4(iii)].

So we have a resolution of $i_*\mathcal{O}_{\mathbb{P}\mathcal{F}}$, and we can compute the pushforward $Rf_*i_*\mathcal{O}_{\mathbb{P}\mathcal{F}} = R\pi_*\mathcal{O}_{\mathbb{P}\mathcal{F}}$ by means of the spectral sequence $E_1^{p,q} = R^qF(A^p) \implies R^{p+q}F(A^\bullet)$ [14, Remark 2.67].

We now show that upon applying $R^i\rho_*$ to the Koszul complex, all but the two rightmost terms vanish. Indeed,

$$R^i\rho_*\left(\rho^*\left(\bigwedge^j \mathcal{E}_1\right)(-j)\right) \cong \bigwedge^j \mathcal{E}_1 \otimes R^i\rho_*\left(\mathcal{O}_{\mathbb{P}\mathcal{E}_0}(-j)\right),$$

by the projection formula. Since $\mathcal{E}_0$ is locally free, we know ([8, p. 103]) that $\rho^{-1}(U_i) \cong \mathbb{P}^{s_0-1}_{U_i}$ for an affine cover $X = \bigcup U_i$. In particular, since $R^i\rho_*\left(\mathcal{O}_{\mathbb{P}\mathcal{E}_0}(-j)\right)$ is the sheafification of

$$U \mapsto H^i(\rho^{-1}(U), \mathcal{O}_{\mathbb{P}\mathcal{E}_0}(-j)|_{\rho^{-1}(U)}),$$

we see that the pushforward vanishes for all i and for all $0 < j < s_0$ (see [13, III.5.1] for cohomology of projective space over an arbitrary noetherian base). In particular, $s_1 < s_0$ (in general, $k \leq r = s_0 - s_1$), proving the claim.

So the spectral sequence is degenerate, and $R\pi_*\mathcal{O}_{\mathbb{P}\mathcal{F}} = R\pi_*\mathcal{O}_{\mathbb{P}\mathcal{E}_0}$. Computing $R^i\pi_*\mathcal{O}_{\mathbb{P}\mathcal{E}_0}$ directly as above shows that $R\pi_*\mathcal{O}_{\mathbb{P}\mathcal{E}_0}$ is a line bundle in degree zero. We note that this automatically means the natural map $\mathcal{O}_X \to R\pi_*\mathcal{O}_{\mathbb{P}\mathcal{E}_0}$ is a line bundle (locally, if a ring map $A \to B$ makes B a free A-module of rank 1, then the ring map is an isomorphism), so this completes the proof of 3).

For part 4), consider the following commutative diagram.

$$\begin{CD}
0 @>>> \rho^*\mathcal{E}_1 @>>> \rho^*\mathcal{E}_0 @>>> \rho^*\mathcal{F} @>>> 0\\
@. @| @VVV @VVV @.\\
\cdots @>>> \rho^*\mathcal{E}_1 @>>> \mathcal{O}_{\mathbb{P}\mathcal{E}_0}(1) @>>> i_*\mathcal{O}_{\mathbb{P}\mathcal{F}}(1) @>>> 0
\end{CD}$$

We observe that the rightmost map factors through $i_*\pi^*\mathcal{F} \to i_*\mathcal{O}_{\mathbb{P}\mathcal{F}}(1)$, the pushforward of the map under consideration. Moreover, the map $\rho^*\mathcal{F} \to i_*\pi^*\mathcal{F}$ pushes forward to an isomorphism $R\rho_*\rho^*\mathcal{F} \to R\pi_*\pi^*\mathcal{F}$ (using the projection formula and what we have proven above), so it suffices to show that the vertical map $\rho^*\mathcal{F} \to i_*\mathcal{O}_{\mathbb{P}\mathcal{F}}(1)$ also becomes an isomorphism when pushed forward. Computing $R\rho_*i_*\mathcal{O}_{\mathbb{P}\mathcal{F}}(1)$ by the same spectral sequence argument as above, we see that applying $R\rho_*$ to the previous diagram gives a morphism

$$\begin{CD}
\mathcal{E}_1 @>>> \mathcal{E}_0 @>>> \mathcal{F}\\
@| @VV{\wr}V @VVV\\
\mathcal{E}_1 @>>> \mathcal{E}_0 @>>> R\pi_*\mathcal{O}_{\mathbb{P}\mathcal{F}}(1)
\end{CD}$$

of distinguished triangles. By the Five Lemma, the right-hand map is also an isomorphism. □

We now complete the proof of Theorem 2.0.1 by induction on k, using the preceding lemma in both the base case and the inductive step.

Proof of 3) and 4). The base case $k = 1$ is proven. For the inductive step, consider the commutative diagram below.

$$\begin{CD}
\mathrm{Flag}(\mathcal{F}, k, k-1) @>{j}>> \mathrm{Gr}(\mathcal{F}, k)\\
@V{h}VV @VV{g}V\\
\mathrm{Gr}(\mathcal{F}, k-1) @>{f}>> X
\end{CD}$$

If we write

$$0 \to S_f \to f^*\mathcal{F} \to Q_f \to 0 \tag{2.0.2}$$

and

$$0 \to S_g \to g^*\mathcal{F} \to Q_g \to 0 \tag{2.0.3}$$

for the tautological sequences on $\mathrm{Gr}(\mathcal{F}, k)$ and $\mathrm{Gr}(\mathcal{F}, k-1)$, respectively, we observe that the partial flag variety can be identified either with $\mathbb{P}(S_f) \xrightarrow{h} \mathrm{Gr}(\mathcal{F}, k-1)$ or with $\mathrm{Gr}(Q_g, k-1) \xrightarrow{j} \mathrm{Gr}(\mathcal{F}, k)$.

We now prove that S_f satisfies the hypotheses of Theorem 2.0.1 (with $k = 1$). In the sequence (2.0.2), we note that $f^*\mathcal{F}$ has homological dimension at most 1 while Q_f is locally free, so S_f also has homological dimension at most 1.

To check the codimension property, note that S_f jumps rank (together with $f^*\mathcal{F}$) on the preimages of the jumping loci for $\mathcal{F}$. Using our previous notation,

$$\mathrm{Gr}(\mathcal{F}, k-1)_{S_f,i} = \mathrm{Gr}(\mathcal{F}, k-1)_{f^*\mathcal{F},i} = f^{-1}\left(X_{\mathcal{F},i}\right),$$

whose dimension is $\dim X_{\mathcal{F},i} + (k-1)(r+i-k+1)$. Since $\mathcal{F}$ satisfies the codimension condition of Theorem 2.0.1, this is at most $\dim X - ik + (k-1)(r+i-k+1)$. The dimension of the Grassmannian is $\dim X + (k-1)(r-k+1)$, and so the codimension in which the rank of S_f jumps by i is at least the difference between these expressions, $ik - i(k-1) = i$, as desired.

Now we can apply Lemma 2.0.4 to the map h to conclude that $Rh_*\mathcal{O} \cong \mathcal{O}$. Since Q_g is a vector bundle, it (trivially) also satisfies the hypotheses of Theorem 2.0.1, so we may assume inductively that $Rj_*\mathcal{O} \cong \mathcal{O}$. Therefore,

$$Rg_*\mathcal{O} \cong Rg_*Rj_*\mathcal{O} \cong Rf_*Rh_*\mathcal{O} \cong Rf_*\mathcal{O} \cong \mathcal{O},$$

proving part 3) of the theorem.

Finally, if we denote the tautological quotient bundle for $\mathrm{Gr}(Q_g, k-1)$ by Q_j, then on the flag variety, we have maps

$$j^*g^*\mathcal{F} \twoheadrightarrow j^*Q_g \twoheadrightarrow Q_j. \tag{2.0.4}$$

We may inductively assume that the second surjection becomes an isomorphism under Rj_*. Now the composition $j^*g^*\mathcal{F} \twoheadrightarrow Q_j$ identifies a rank $k-1$ quotient of a pullback of $\mathcal{F}$, so by the universal property of $\mathrm{Gr}(\mathcal{F}, k-1)$, this map must be the pullback (under h) of $f^*\mathcal{F} \twoheadrightarrow Q_f$. So we have

$$Rg_*Rj_*(j^*g^*\mathcal{F} \twoheadrightarrow Q_j) = Rf_*Rh_*(h^*f^*\mathcal{F} \twoheadrightarrow h^*Q_f) = Rf_*(f^*\mathcal{F} \twoheadrightarrow Q_f),$$

which by induction is an isomorphism. So we see that in (2.0.4), both the second surjection and the composition become isomorphisms when pushed down to X. Thus

$$Rg_*Rj_*(j^*g^*\mathcal{F} \twoheadrightarrow j^*Q_g) = Rg_*(g^*\mathcal{F} \twoheadrightarrow Q_g)$$

is an isomorphism $\mathcal{F} \xrightarrow{\sim} Rg_*Q_g$, as desired. □

Under ideal conditions, we can improve part 1 of the theorem as follows.

Proposition 2.0.5. *If X is smooth, and the strata $X_i \setminus X_{i+1}$ are smooth of exactly the expected codimension $i(k+i)$, then $Gr(\mathcal{F}, k)$ is smooth.*

Proof. The result follows from a general fact: if $f : Z \to Y$ is an equivariant map of smooth G-manifolds, then any map $g : X \to Y$ which is transverse to the G-orbits of Y is transverse to f. In particular, the fiber product $X \times_Y Z$ is smooth for any such map.

In our situation, by passing to an open cover of X, we may assume that the bundles $\mathcal{E}_0$ and $\mathcal{E}_1$ which resolve $\mathcal{F}$ are trivial. So we get a map $g : X \to \mathrm{Hom}(\mathbb{C}^{s_0}, \mathbb{C}^{s_1})$. Let $Z = \{A : \mathbb{C}^{s_0} \to \mathbb{C}^{s_1}, \operatorname{coker} A \twoheadrightarrow \mathbb{C}^k\}$. This Z has a forgetful map to $\mathrm{Hom}(\mathbb{C}^{s_0}, \mathbb{C}^{s_1})$, which is equivariant with respect to the natural $G := GL_{s_0} \times GL_{s_1}$ action on both. Since the fiber product of these maps is $\mathrm{Gr}(\mathcal{F}, k)$, it suffices to argue that g is transverse to the G-orbits of $\mathrm{Hom}(\mathbb{C}^{s_0}, \mathbb{C}^{s_1})$. But the G-orbits are given by the rank stratification of $\mathrm{Hom}(\mathbb{C}^{s_0}, \mathbb{C}^{s_1})$, so their preimages are precisely the strata $X_i \setminus X_{i+1}$. Since these were assumed to be smooth of the expected codimension, g is transverse to the orbits, completing the proof. □

We conclude the section by mentioning a special case that will be useful to us later.

Proposition 2.0.6. *If $k = r$ and X_1 is of expected codimension $k+1$, then $\pi : Gr(\mathcal{F}, r) \to X$ is the blowup along X_1.*

Proof. It is enough to embed $\mathrm{Gr}(\mathcal{F}, r)$ in $\mathbb{P}(\mathcal{I}_{X_1})$; then the blowup and the Grassmannian agree over the dense open subset $X \setminus X_1$, and are both integral and closed.

We know that $\mathrm{Gr}(\mathcal{F}, r)$ is cut out of $\mathrm{Gr}(\mathcal{E}_0, r)$ by a section s of $\mathcal{H}om(\mathcal{E}_1, Q)$, and $\mathrm{Gr}(\mathcal{E}_0, r)$ is in turn cut out of $\mathbb{P}(\bigwedge^r E_0)$ by Plücker relations.

Since X_1 has expected codimension, the Eagon-Northcott complex

$$\cdots \to \bigwedge^{s_1+2} \mathcal{E}_0^* \otimes \mathrm{Sym}^2(\mathcal{E}_1) \otimes \det \mathcal{E}_1 \to \bigwedge^{s_1+1} \mathcal{E}_0^* \otimes \mathcal{E}_1 \otimes \det \mathcal{E}_1 \to \bigwedge^{s_1} \mathcal{E}_0^* \otimes \det \mathcal{E}_1 \to \mathcal{I}_{X_1}$$

associated to the map $\mathcal{E}_0^* \to \mathcal{E}_1^*$ is exact. Twisting this sequence by $\det \mathcal{E}_0 \otimes \det \mathcal{E}_1^*$ gives

$$\cdots \to \bigwedge^{r-2} \mathcal{E}_0 \otimes \mathrm{Sym}^2(\mathcal{E}_1) \to \bigwedge^{r-1} \mathcal{E}_0 \otimes \mathcal{E}_1 \to \bigwedge^{r} \mathcal{E}_0 \to \mathcal{I}_{X_1} \otimes \det \mathcal{E}_0 \otimes \det \mathcal{E}_1^*,$$

and we find that $\mathbb{P}(\mathcal{I}_{X_1}) = \mathbb{P}(\mathcal{I}_{X_1} \otimes \det \mathcal{E}_0 \otimes \det \mathcal{E}_1^*)$ is cut out of $\mathbb{P}(\bigwedge^r \mathcal{E}_0)$ by a section of $\mathcal{H}om(\bigwedge^{r-1} \mathcal{E}_0 \otimes \mathcal{E}_1, \det Q)$, the composition of the map $\bigwedge^{r-1} \mathcal{E}_0 \otimes \mathcal{E}_1 \to \bigwedge^r \mathcal{E}_0$ above with the natural map $\bigwedge^r \mathcal{E}_0 \to \det Q$. So when s vanishes (i.e. the natural map $\mathcal{E}_0 \to Q$ annihilates the image of $\mathcal{E}_1 \to \mathcal{E}_0$), then also $\bigwedge^r \mathcal{E}_0 \to \det Q$ annihilates the image of $\bigwedge^{r-1} \mathcal{E}_0 \otimes \mathcal{E}_1 \to \bigwedge^r \mathcal{E}_0$. $\square$

CHAPTER III

CONSTRUCTION OF THE ACTION

We first recall the definition of geometric categorical $\mathfrak{sl}_2$ action ([5], Def. 2.2.2). We will always assume the base field is $\mathbb{C}$, as well as adjusting notation slightly. Moreover, following Remark 2.6 in [5], we will ignore the $\mathbb{C}^*$ action. With these modifications, we arrive at the following definition. Here and throughout the paper, $D(X)$ denotes the *bounded* derived category of coherent sheaves.

Definition 3.0.1 ([5], Def. 2.2.2)**.** A *geometric categorical* $\mathfrak{sl}_2$ *action* consists of the following data.

(i) A sequence of smooth varieties $\mathcal{M}_{-N}, \mathcal{M}_{-N+1}, \ldots, \mathcal{M}_{N-1}, \mathcal{M}_N$ over $\mathbb{C}$.

(ii) Fourier-Mukai kernels

$$E^{(r)}(\lambda) \in D(\mathcal{M}_{\lambda-r} \times \mathcal{M}_{\lambda+r}) \text{ and } F^{(r)}(\lambda) \in D(\mathcal{M}_{\lambda+r} \times \mathcal{M}_{\lambda-r}).$$

We write $E(\lambda)$ for $E^{(1)}(\lambda)$ and take $E^{(0)}(\lambda) = \mathcal{O}_\Delta$.

(iii) For each $\mathcal{M}_\lambda$, a flat deformation $\widetilde{\mathcal{M}}_\lambda \to \mathbb{A}^1$.

These data are required to satisfy the following conditions.

(i) The Hom space between any two objects of $D(\mathcal{M}_\lambda)$ is finite dimensional.

(ii) $E^{(r)}(\lambda)$ and $F^{(r)}(\lambda)$ are left and right adjoints of each other up to shift. More precisely,

$$E^{(r)}(\lambda)_R = F^{(r)}(\lambda)[r\lambda],$$

and

$$E^{(r)}(\lambda)_L = F^{(r)}(\lambda)[-r\lambda].$$

(iii) At the level of cohomology of complexes we have

$$\mathcal{H}^*(E(\lambda+r) * E^{(r)}(\lambda-1)) \cong E^{(r+1)}(\lambda) \otimes_{\mathbb{C}} H^*(\mathbb{P}^r),$$

where the grading of $H^*(\mathbb{P}^r)$ is centered around 0.

(iv) If $\lambda \leq 0$ then

$$F(\lambda+1) * E(\lambda+1) \cong E(\lambda-1) * F(\lambda-1) \oplus \mathcal{P},$$

where $\mathcal{H}^*(\mathcal{P}) \cong \mathcal{O}_\Delta \otimes_{\mathbb{C}} H^*(\mathbb{P}^{-\lambda-1})$.

Similarly, if $\lambda \geq 0$ then

$$E(\lambda-1) * F(\lambda-1) \cong F(\lambda+1) * E(\lambda+1) \oplus \mathcal{P}',$$

where $\mathcal{H}^*(\mathcal{P}') \cong \mathcal{O}_\Delta \otimes_{\mathbb{C}} H^*(\mathbb{P}^{\lambda-1})$.

(v) We have

$$\mathcal{H}^*(i_{23*}E(\lambda+1) * i_{12*}E(\lambda-1)) \cong E^{(2)}(\lambda)[-1] \oplus E^{(2)}(\lambda)[2],$$

where i_{12} and i_{23} are the closed immersions

$$i_{12} : \mathcal{M}_{\lambda-2} \times \mathcal{M}_\lambda \to \mathcal{M}_{\lambda-2} \times \widetilde{\mathcal{M}}_\lambda$$

$$i_{23} : \mathcal{M}_\lambda \times \mathcal{M}_{\lambda+2} \to \widetilde{\mathcal{M}}_\lambda \times \mathcal{M}_{\lambda+2}.$$

(vi) If $\lambda \leq 0$ and $k \geq 1$ then the image of $\mathrm{supp}(E^{(r)}(\lambda - r))$ under the projection to $\mathcal{M}_\lambda$ is not contained in the image of $\mathrm{supp}(E^{(r+k)}(\lambda - r - k))$ also under the projection to $\mathcal{M}_\lambda$. Similarly, if $\lambda \geq 0$ and $k \geq 1$ then the image of $\mathrm{supp}(E^{(r)}(\lambda + r))$ in $\mathcal{M}_\lambda$ is not contained in the image of $\mathrm{supp}(E^{(r+k)}(\lambda + r + k))$.

(vii) All $E^{(r)}$'s and $F^{(r)}$'s are sheaves (i.e. complexes supported in degree zero).

Between [5, Theorem 2.5] and [6, Theorem 2.8], Cautis, Kamnitzer, and Licata prove the following, which gives our Corollary 1.0.2

Theorem 3.0.2 (Cautis, Kamnitzer, Licata)**.** *Given a geometric categorical* $\mathfrak{sl}_2$ *action, there exists for each* λ *an equivalence of categories* $D(\mathcal{M}_{-\lambda}) \to D(\mathcal{M}_\lambda)$.

3.1. Construction of the correspondences

Fix, once and for all, a K3 surface $\mathcal{S}$ over $\mathbb{C}$ with $\mathrm{Pic}(\mathcal{S}) = \mathbb{Z} \cdot H$. Fix also integers r and s, and consider the sequence of Mukai vectors $(r + n, H, s + n)$ and corresponding moduli spaces. In definition 3.0.1, we set $\mathcal{M}_{r+s+2n} := \mathcal{M}_\mathcal{S}(r + n, H, s + n)$, or in other words, $\mathcal{M}_\mathcal{S}(v) = \mathcal{M}_{\chi(v)}$. Note that χ always has the same parity as $r + s$; the remaining $\mathcal{M}$'s are defined to be empty.

Recall that a general point of $\mathcal{M}_\mathcal{S}(r, c, \chi - r)$ is a sheaf $\mathcal{F}$ with either $h^0(\mathcal{F}) = \chi$ and $h^1(\mathcal{F}) = 0$ (if $\chi \geq 0$), or $h^0(\mathcal{F}) = 0$ and $h^1(\mathcal{F}) = \chi$ (if $\chi \leq 0$). Note that $h^2(\mathcal{F}) = 0$ for any non-trivial $\mathcal{F}$ since $H^2(\mathcal{F})$ is Serre dual to $\mathrm{Hom}(\mathcal{F}, \mathcal{O})$ and $\mathcal{F}$ is stable. We denote by ${}_t\mathcal{M}_\mathcal{S}(v)$ the Brill-Noether locus where $h^0(\mathcal{F})$ and $h^1(\mathcal{F})$ both jump by t. We will often make use of the following computation of Markman:

Lemma 3.1.1 ([18, Corollary 34])**.** *The codimension of* ${}_t\mathcal{M}_\mathcal{S}(r, c, \chi - r)$ *in* $\mathcal{M}_\mathcal{S}(r, c, \chi - r)$ *is* $t(|\chi| + t)$.

For any positive integer k and any χ such that $\mathcal{M}_{\chi-k}$ and $\mathcal{M}_{\chi+k}$ are non-empty, we construct a correspondence between these moduli spaces. Informally, a $\mathbb{C}$-point of our correspondence will be either a pair $([\mathcal{F}] \in \mathcal{M}_{\chi-k}, H^1(\mathcal{F}) \twoheadrightarrow \mathbb{C}^k)$, or a pair $([\mathcal{G}] \in \mathcal{M}_{\chi+k}, \mathbb{C}^k \hookrightarrow H^0(\mathcal{G}))$, which we will argue are the same data.

To see this, note that a subspace of $H^0(\mathcal{G})$ determines a map $\mathcal{O}^k \to \mathcal{G}$, which is injective with stable cokernel $\mathcal{F}$ by [18, Lemma 25], and has $\chi(\mathcal{F}) = \chi - k$, since $\chi(\mathcal{O}) = 2$. Say $\mathcal{G}$ belongs to the tth Brill-Noether stratum of $\mathcal{M}_{\chi+k}$.

$$
\begin{array}{cccc}
 & \mathcal{O}^k \longrightarrow & \mathcal{G} \longrightarrow & \mathcal{F} \\
h^0 & k & \chi+k+t & \chi+t \\
h^1 & 0 & t & k+t \\
h^2 & k & 0 & 0
\end{array}
\tag{3.1.1}
$$

We see that $\mathcal{F}$ belongs to the $(k+t)$th stratum of $\mathcal{M}_{\chi-k}$ and comes with a surjection $H^1(\mathcal{F}) \twoheadrightarrow \mathbb{C}^k$.

The opposite direction is much the same: a quotient $H^1(\mathcal{F}) \twoheadrightarrow \mathbb{C}^k$ is dual to a subspace $\mathbb{C}^k \hookrightarrow \mathrm{Ext}^1(\mathcal{F}, \mathcal{O})$, which gives an extension of $\mathcal{E}$ by $\mathcal{O}^k$, i.e. same short exact sequence as above. We get a sheaf $\mathcal{G}$ with the correct numerics, along with k sections. Stability again comes from [18, Lemma 25].

More formally, let $\mathcal{U}_{\chi-k}$ (resp. $\mathcal{U}_{\chi+k}$) be a universal sheaf on $\mathcal{S} \times \mathcal{M}_{\chi-k}$ (resp. $\mathcal{S} \times \mathcal{M}_{\chi+k}$) and let q (resp. q') be the projection to the moduli space. We can construct the correspondence as a scheme by considering the Grassmannians $\mathrm{Gr}(R^1 q_* \mathcal{U}_{\chi-k}, k) \to \mathcal{M}_{\chi-k}$ and $\mathrm{Gr}(\mathcal{E}xt^2_{q'}(\mathcal{U}_{\chi+k}, \mathcal{O}), k) \to \mathcal{M}_{\chi+k}$, whose $\mathbb{C}$-points are the pairs described above. We point out that the fiber of $R^1 q_* \mathcal{U}_{\chi-k}$ over a point

$[\mathcal{F}]$ really is $H^1(\mathcal{F})$, and similarly for $\mathcal{E}xt^2_{q'}(\mathcal{U}_{\chi+k}, \mathcal{O})$, although the same cannot be said, for example, about $R^0 q_* \mathcal{U}_{\chi-k}$ and $\mathcal{E}xt^1_{q'}(\mathcal{U}_{\chi+k}, \mathcal{O})$ ([13, Theorem III.12.11]).

Proposition 3.1.2. *$Gr(R^1 q_* \mathcal{U}_{\chi-k}, k) \cong Gr(\mathcal{E}\mathrm{xt}^2_{q'}(\mathcal{U}_{\chi+k}, \mathcal{O}), k)$.*

Proof. Temporarily denote $\mathrm{Gr}(R^1 q_* \mathcal{U}_{\chi-k}, k)$ by X and $\mathrm{Gr}(\mathcal{E}xt^2_{q'}(\mathcal{U}_{\chi+k}, \mathcal{O}), k)$ by X'. Let us show that for any χ and k, at least one of $X \xrightarrow{g} \mathcal{M}_{\chi-k}$ and $X' \xrightarrow{h} \mathcal{M}_{\chi+k}$ satisfies the conditions of Theorem 2.0.1. Markman constructs in [18, Eq. (70)] an exact sequence

$$0 \to R^0 q_* \mathcal{U} \to V_0 \to V_1 \to R^1 q_* \mathcal{U} \to 0, \tag{3.1.2}$$

where V_0 and V_1 are locally free. We consider two cases.

If $\chi \le 0$, then certainly $\chi - k \le 0$, so $R^0 q_* \mathcal{U}_{\chi-k}$ is torsion, as a general sheaf of Euler characteristic $\chi - k$ has no sections. But by (3.1.2), $R^0 q_* \mathcal{U}_{\chi-k}$ is a subsheaf of a locally free sheaf, so it vanishes, making (3.1.2) the desired resolution of $R^1 q_* \mathcal{U}_{\chi-k}$. Moreover, the locus ${}_i\mathcal{M}_{\chi-k}$ where the rank of $R^1 q_* \mathcal{U}_{\chi-k}$ jumps by at least i has codimension

$$i(k - \chi + i) \ge i(k + i) > ik,$$

by Lemma 3.1.1. In particular, X is integral and a local complete intersection by Theorem 2.0.1. We point out that in this case X is also normal: if we have the strict inequality $\chi - k < -k$, we can immediately apply Serre's criterion for normality (recall that being Cohen-Macaulay is equivalent to satisfying the Serre condition S_n for all n), as X is smooth away from the preimage of ${}_1\mathcal{M}_{\chi-k}$, whose codimension in X is

$$\begin{aligned} &k(\mathrm{rk}(R^1 q_* \mathcal{U}_{\chi-k}) - k) + \mathrm{codim}_{\mathcal{M}_{\chi-k}}({}_1\mathcal{M}_{\chi-k}) - k(\mathrm{rk}(R^1 q_* \mathcal{U}_{\chi-k}) + 1 - k) \\ =& -k\chi + (k - \chi + 1) - k(1 - \chi) = 1 - \chi > 1 \end{aligned}$$

On the other hand, if $\chi - k = k$, then g is a blowup by Proposition 2.0.6. So away from the preimage of ${}_2\mathcal{M}_{\chi-k}$, X is the blowup of a smooth variety along a smooth center, and is thus smooth. The same calculation as above shows that the codimension of this preimage is 4 (regardless of k), so we again apply the Serre criterion.

If instead $\chi \geq 0$, then the first term vanishes in the dual sequence to (3.1.2):

$$0 \to \mathcal{E}xt^1_q(\mathcal{U}, \mathcal{O}) \to V_1^* \to V_0^* \to \mathcal{E}xt^2_q(\mathcal{U}, \mathcal{O}) \to 0, \tag{3.1.3}$$

yielding a two-step resolution of $\mathcal{E}xt^2_q(\mathcal{U}_{\chi+k}, \mathcal{O})$. By a very similar computation, X' is integral and normal in this case.

From here, our idea is to construct via universal properties morphisms $X \to X'$ and $X' \to X$ inducing the aforementioned bijection on $\mathbb{C}$-points. So in fact both X and X' turn out to be integral. By a form of Zariski's Main Theorem (see, e.g., [20, III.9]), a bijective morphism of integral varieties over $\mathbb{C}$ with normal target is an isomorphism, so this will complete the proof.

On X', we have a tautological quotient map $h^*\,\mathcal{E}xt^2_{q'}(\mathcal{U}_{\chi+k}, \mathcal{O}) \twoheadrightarrow Q_h$, where Q_h is a rank k vector bundle. The idea now is to use Grothendieck duality to produce a map $\mathcal{O} \boxtimes Q_h^* \to (1 \times h)^*\mathcal{U}_{\chi+k}$, the family version of the map $\mathcal{O}^k \to \mathcal{G}$ in (3.1.1).

We note that $L_{-i}h^*R^jq'_*R\mathcal{H}om(\mathcal{U}_{\chi+k}, \mathcal{O})$ vanishes for all $i > 0$ and $j > 2$, and so $h^*\,\mathcal{E}xt^2_{q'}(\mathcal{U}_{\chi+k}, \mathcal{O})$ is the second and last cohomology sheaf of $Lh^*Rq'_*R\mathcal{H}om(\mathcal{U}_{\chi+k}, \mathcal{O})$. So we have a map from

$$Lh^*Rq'_*R\mathcal{H}om(\mathcal{U}_{\chi+k}, \mathcal{O})[2] = Lh^*Rq'_*R\mathcal{H}om(\mathcal{U}_{\chi+k}, q'^!\mathcal{O}) = Lh^*R\mathcal{H}om(Rq'_*\mathcal{U}_{\chi+k}, \mathcal{O})$$

to $h^* \mathcal{E}xt^2_{q'}(\mathcal{U}_{\chi+k}, \mathcal{O})$. We compose this with the tautological quotient map and take derived duals to produce a map $Q_h^* \to Lh^* Rq'_* \mathcal{U}_{\chi+k}$. Next we pull back to $\mathcal{S} \times X'$ and use commutativity of the diagram

$$\begin{array}{ccc} \mathcal{S} \times X' & \xrightarrow{1\times h} & \mathcal{S} \times \mathcal{M}_{\chi+k} \\ \downarrow{\scriptstyle \widetilde{q}'} & & \downarrow{\scriptstyle q'} \\ X' & \xrightarrow{h} & \mathcal{M}_{\chi+k} \end{array}$$

On $\mathcal{S} \times X'$, we obtain a map from $\mathcal{O} \boxtimes Q_h^* \to L\widetilde{q}'^* Lh^* Rq'_* \mathcal{U}_{\chi+k} = L(1 \times h)^* Lq'^* Rq'_* \mathcal{U}_{\chi+k}$. We compose this with the counit $Lq'^* Rq'_* \to \mathrm{id}$ to produce the desired map

$$\mathcal{O} \boxtimes Q_h^* \to (1 \times h)^* \mathcal{U}_{\chi+k} \qquad (3.1.4)$$

(where we have dropped the left derived pullback as $\mathcal{U}_{\chi+k}$ is flat over q').

We have already analyzed this map at each point of X', so we know it is injective, and that the cokernel is a flat family of sheaves. If $\mathcal{M}_{\chi+k} = \mathcal{M}_{\mathcal{S}}(v)$, then the fibers of the cokernel have Mukai vector $v - (k, 0, k)$. So by the universal property of the moduli space, we get a map $g' : X' \to \mathcal{M}_{\chi-k}$ so that

$$0 \to \mathcal{O} \boxtimes Q_h^* \to (1 \times h)^* \mathcal{U}_{\chi+k} \to (1 \times g')^* \mathcal{U}_{\chi-k} \otimes (\mathcal{O} \boxtimes L) \to 0 \qquad (3.1.5)$$

is exact, where L is some line bundle on X'.

Now apply $\widetilde{q}'_*$ to (3.1.5): since $R^2 \widetilde{q}'_* (1 \times h)^* \mathcal{U}_{\chi+k} = h^* R^2 q'_* \mathcal{U}_{\chi+k} = 0$, the connecting morphism $g'^* R^1 q_* \mathcal{U}_{\chi-k} \otimes L \to Q_h^*$ in the long exact sequence is surjective. After tensoring by L^*, we have produced on X' a surjection of $g'^* R^1 q_* \mathcal{U}_{\chi-k}$ onto a rank k bundle, so the universal property gives a map $\phi : X' \to X$. If Q_g is the tautological quotient bundle on X, we have

$$\phi^*(g^* R^1 q_* \mathcal{U}_{\chi-k} \twoheadrightarrow Q_g) = g'^* R^1 q_* \mathcal{U}_{\chi-k} \twoheadrightarrow Q_h^* \otimes L^*.$$

The construction of the inverse map is similar: we have a tautological quotient $g^*R^1q_*\mathcal{U}_{\chi-k} \twoheadrightarrow Q_g$ on X and a diagram

$$\begin{array}{ccc} \mathcal{S}\times X & \xrightarrow{1\times g} & \mathcal{S}\times \mathcal{M}_{\chi-k} \\ \big\downarrow{\scriptstyle \widetilde{q}} & & \big\downarrow{\scriptstyle q} \\ X & \xrightarrow{\quad g \quad} & \mathcal{M}_{\chi-k} \end{array}$$

Now $q^!Rq_*\mathcal{U}_{\chi-k}$ maps to its last cohomology $q^!R^1q_*\mathcal{U}_{\chi-k}[-1] = q^*R^1q_*\mathcal{U}_{\chi-k}[1]$, and we compose this with the unit of adjunction $\mathrm{id} \to q^!Rq_*$. Pulling back along $1\times g$ yields a map $(1\times g)^*\mathcal{U}_{\chi-k} \to \widetilde{q}^*g^*R^1q_*\mathcal{U}_{\chi-k}[1]$. Finally, we pull back the tautological quotient map along $\widetilde{q}$ and compose to get a map $(1 \times g)^*\mathcal{U}_{\chi-k} \to \mathcal{O}\boxtimes Q_g[1]$, corresponding to an extension

$$0 \to \mathcal{O}\boxtimes Q_g \to C \to (1\times g)^*\mathcal{U}_{\chi-k} \to 0.$$

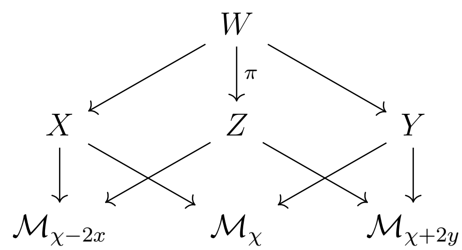
So if $\mathcal{M}_{\chi-k} = \mathcal{M}_{\mathcal{S}}(v')$, then C is a family of sheaves on $\mathcal{S}$ with Mukai vector $v' + (k, 0, k)$, and so we get a map $h' : X \to \mathcal{M}_{\chi+k}$ and the sequence can be written as

$$0 \to \mathcal{O}\boxtimes Q_g \to (1\times h')^*\mathcal{U}_{\chi+k}\otimes(\mathcal{O}\boxtimes L') \to (1\times g)^*\mathcal{U}_{\chi-k} \to 0, \tag{3.1.6}$$

where L' is some line bundle on X. Tensor (3.1.6) by $\mathcal{O} \boxtimes L'^*$ and apply $R\widetilde{q}_*R\mathcal{H}om(-,\mathcal{O})$. On second cohomology we get a map $h'^*\,\mathcal{E}xt^2_q(\mathcal{U}_{\chi+k},\mathcal{O}) \twoheadrightarrow Q_g^*\otimes L'$. By the universal property, this is the pullback of $h^*\,\mathcal{E}xt^2_q(\mathcal{U}_{\chi+k},\mathcal{O}) \twoheadrightarrow Q_h$ along some map $\psi : X \to X'$. □

From now on, we identify the two Grassmannians via this isomorphism. Note that this identification makes the line bundles L and L' in the proof dual, since $Q_g = Q_h^*\otimes L^* = (Q_g^*\otimes L')^*\otimes L^*$, and so the sequences (3.1.5) and (3.1.6) are twists of one another.

Remark 3.1.3. Since there is a unique correspondence between any two moduli spaces, we deliberately avoid developing a cumbersome notation for the correspondences. We will often consider diagrams of the following shape, where X, Y, and Z are the correspondences (necessarily x, y, and $x+y$ steps, respectively), and $W = X \times_{\mathcal{M}_\chi} Y$ is the fiber product.



To see why the map π exists, notice that a point of W consists of a sheaf $[\mathcal{F}]$ in $\mathcal{M}_\chi$, a subspace $\mathbb{C}^x \hookrightarrow H^0(\mathcal{F})$, and a quotient $H^1(\mathcal{F}) \twoheadrightarrow \mathbb{C}^y$. As in the preceding proof, this quotient gives an extension

$$0 \to \mathcal{O}^y \to \mathcal{G} \to \mathcal{F} \to 0$$

for some sheaf $[\mathcal{G}]$ in $\mathcal{M}_{\chi+2y}$, as well as a subspace $\mathbb{C}^y \hookrightarrow H^0(\mathcal{G})$. Moreover, the subspace $\mathbb{C}^x \hookrightarrow H^0(\mathcal{F})$ gives a subspace $\mathbb{C}^{x+y} \hookrightarrow H^0(\mathcal{G})$ which contains this $\mathbb{C}^y$. So points in W can be described as flags $\mathbb{C}^y \hookrightarrow \mathbb{C}^{x+y} \hookrightarrow H^0(\mathcal{G})$, or similarly, as flags $H^1(\mathcal{E}) \twoheadrightarrow \mathbb{C}^{x+y} \twoheadrightarrow \mathbb{C}^x$, where $[\mathcal{E}]$ is in $\mathcal{M}_{\chi-2x}$. In particular, W is a $\mathrm{Gr}(y, x+y) = \mathrm{Gr}(x+y, x)$ bundle over Z.

While the proof of Proposition 3.1.2 shows that these correspondences are normal, we can now see more.

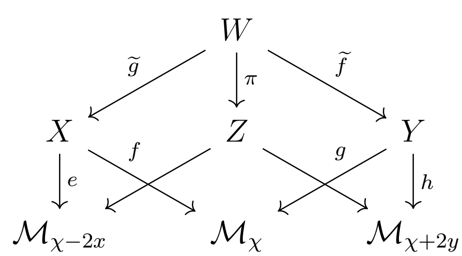
Proposition 3.1.4. *$Gr(R^1 q_* \mathcal{U}_{\chi-k}, k)$ is smooth.*

Proof. We know that the moduli spaces are smooth and that the Brill-Noether strata have expected codimension, by Lemma 3.1.1. So to use Proposition 2.0.5, we only have to argue that the Brill-Noether strata are smooth.

Assume $\chi \geq 0$; the other case is similar. Then the map $\mathrm{Gr}(R^1 q_* \mathcal{U}_{\chi-k}, k) \to \mathcal{M}_{\chi-k}$ is an isomorphism over the stratum ${}_k\mathcal{M}_{\chi-k} \setminus {}_{k+1}\mathcal{M}_{\chi-k}$. On the other hand, the same locus in the Grassmannian is a $\mathrm{Gr}(k, \chi+k)$ bundle over the open (hence smooth) stratum $\mathcal{M}_{\chi+k} \setminus {}_1\mathcal{M}_{\chi+k}$. So each Brill-Noether stratum is smooth because it is a bundle over the dense stratum of a smaller moduli space. □

3.2. Construction of the line bundles

We now construct inductively a line bundle L_Z on the correspondence Z between $\mathcal{M}_{\chi-k}$ and $\mathcal{M}_{\chi+k}$. We use L_Z (or rather, its pushforward to the product) as the kernel $E^{(k)}(\chi)$ in Definition 3.0.1, and $L_Z^* \otimes \omega_Z$ as the kernel $F^{(k)}(\chi)$. We label a diagram as in Remark 3.1.3.

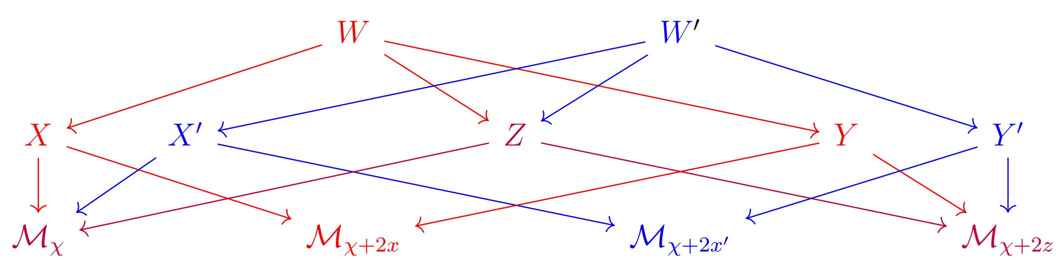


(3.2.7)

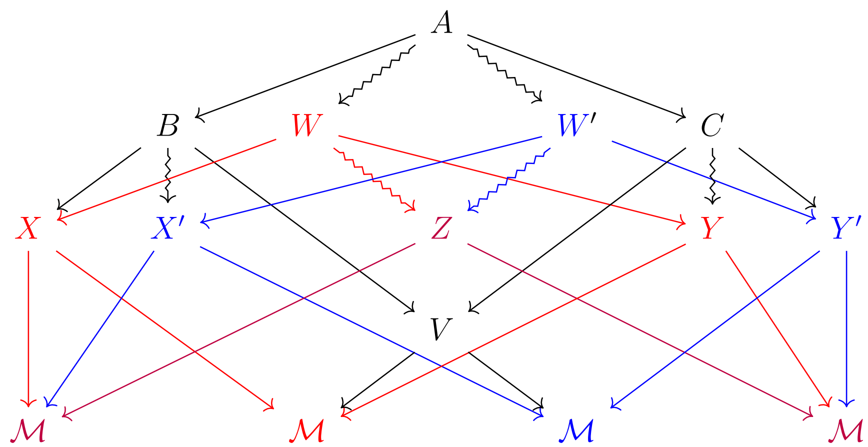
We want (for reasons that will become clear in section 4.2) our line bundles to satisfy the compatibility $\widetilde{g}^* L_X \otimes \widetilde{f}^* L_Y = \pi^* L_Z \otimes \omega_\pi$. Our strategy is to simply choose (in a non-unique way) some line bundles on the one-step correspondences, and show that this compatibility uniquely determines the bundles on the larger correspondences.

So begin by choosing any 1-step correspondence – say, X in diagram (3.2.7), with $x = 1$ – and any line bundle L_X on it. We now choose the line bundle L_Y by requiring that $\widetilde{g}^* L_X \otimes \widetilde{f}^* L_Y \otimes \omega_\pi^*$ be trivial on the fibers of π. Note that there is always such an L_Y : the determinant of the tautological quotient bundle Q_h has degree 1 on the fiber of h, which contains the fiber of π as a linear subspace, so for example we could choose L_Y to be an appropriate tensor power of $\det Q_h$. Moreover, since π is a $\mathbb{P}^1$-bundle, we know that $\widetilde{g}^* L_X \otimes \widetilde{f}^* L_Y \otimes \omega_\pi^*$ is the pullback of a (unique) line bundle on Z, and we define L_Z to be this line bundle.

Now if Z is any correspondence, we construct a line bundle on it in the same fashion, assuming inductively that we have defined line bundles on some X and Y whose fiber product W is a projective space bundle over Z. Since there may be multiple such W's, we show that the line bundle we get is independent of this choice. Consider the diagram



where both the red and blue pictures (with purple belonging to both) are copies of diagram (3.2.7). Here X, Y, X', Y', Z are x, y, x', y', z step correspondences, respectively, where $x + y = x' + y' = z$, although the numbers are unimportant. Having chosen line bundles on X, Y, X', and Y', there is a potential ambiguity as to what line bundle to put on Z, since both W and W' are bundles over it. To show there is no real ambiguity, we reluctantly augment the diagram further.



Here V is the $|x - x'| = |y - y'|$ step correspondence, $B = X \times_{\mathcal{M}} V$, $C = V \times_{\mathcal{M}} Y'$, and $A = B \times_{X'} W' = W \times_Y C$. The squiggly arrows indicate the maps which are Grassmannian bundles.

Let us suppress the pullbacks in the notation; all equalities written are on A. Our two candidates for line bundle on Z are $L_Z = L_X \otimes L_Y \otimes \omega^*_{W/Z}$ and $L'_Z = L_{X'} \otimes L_{Y'} \otimes \omega^*_{W'/Z}$. But by construction, $L_{X'} = L_X \otimes L_V \otimes \omega^*_{B/X'}$ and $L_Y = L_V \otimes L_{Y'} \otimes \omega^*_{C/Y}$. Substituting,

$$L_Z = L_X \otimes L_V \otimes L_{Y'} \otimes \omega^*_{C/Y} \otimes \omega^*_{W/Z}$$

and

$$L'_Z = L_X \otimes L_V \otimes \omega^*_{B/X'} \otimes L_{Y'} \otimes \omega^*_{W'/Z}.$$

Now $\omega_{C/Y}$ pulls back to $\omega_{A/W}$ and $\omega_{B/X'}$ to $\omega_{A/W'}$, so we have

$$L_Z = L'_Z = L_X \otimes L_V \otimes L_{Y'} \otimes \omega^*_{A/Z},$$

and the two line bundles on Z agree as desired.

CHAPTER IV

VERIFICATION OF CONDITIONS

Of the seven conditions in Definition 3.0.1, four are easy to verify and three are more difficult. In section 4.1, we check the easy conditions: (i), (ii), (vi), and (vii). The harder conditions each have their own section, and occupy the remainder of the chapter.

4.1. Straightforward conditions

Lemma 4.1.1 (Condition (i)). *For any χ and any $\mathcal{E}, \mathcal{F} \in D(\mathcal{M}_\chi), \mathrm{Hom}(\mathcal{E}, \mathcal{F})$ is finite dimensional.*

Proof. $\mathcal{M}_\chi$ is proper and smooth, so its bounded derived category is hom-finite.

□

Lemma 4.1.2 (Condition (ii)). *$E^{(r)}(\lambda)$ and $F^{(r)}(\lambda)$ are left and right adjoints up to shift:*

$$E^{(r)}(\lambda)_R = F^{(r)}(\lambda)[r\lambda],$$

and

$$E^{(r)}(\lambda)_L = F^{(r)}(\lambda)[-r\lambda].$$

Proof. Let X be the r-step correspondence, so that we have a diagram

$$\mathcal{M}_{\lambda-r} \xleftarrow{\;f\;} X \xrightarrow{\;g\;} \mathcal{M}_{\lambda+r}$$

Then if L_X is the line bundle constructed in section 3.2, $E^{(r)} = g_* \circ (L_X \otimes -) \circ f^*$, so we have

$$E_R^{(r)} = f_* \circ (L_X^{-1} \otimes -) \circ g^! = f_* \circ (L_X^{-1} \otimes \omega_X \otimes -) \circ g^*[\dim g] = F^{(r)}[\dim g].$$

If $\lambda \geq 0$, then g is a generic $\mathrm{Gr}(r, \lambda + r)$ bundle, so $\dim g = r\lambda$, as desired.

If instead $\lambda < 0$, then g is birational onto the Brill-Noether locus of $\mathcal{M}_{\lambda+r}$ where h^0 jumps to r. If $\lambda + r \geq 0$, then this is ${}_{-\lambda}\mathcal{M}_{\lambda+r}$, which has codimension $(-\lambda)(|\lambda + r| - \lambda) = -r\lambda$, and so $\dim g = r\lambda$. If $\lambda + r < 0$, this is ${}_r\mathcal{M}_{\lambda+r}$, which has codimension $r(|\lambda + r| + r) = r(-\lambda - r + r) = -r\lambda$, and again $\dim g = r\lambda$.

The computation of the left adjoint is similar: we get

$$E_L^{(r)} = f_! \circ (L_X^{-1} \otimes -) \circ g^* = F^{(r)}[\dim f]$$

and check in three cases that $\dim f = -r\lambda$. □

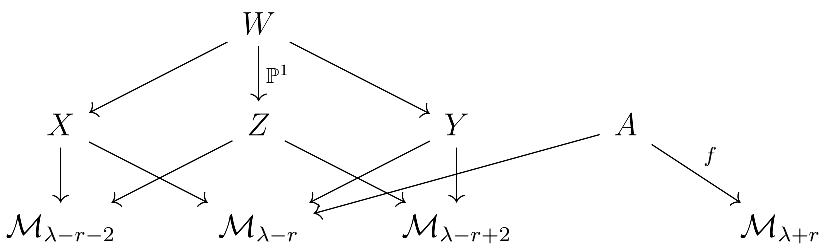
Lemma 4.1.3 (Condition (vi))**.** *If $\lambda \leq 0$ and $k \geq 1$ then the image of $\mathrm{supp}(E^{(r)}(\lambda - r))$ under the projection to $\mathcal{M}_\lambda$ is not contained in the image of $\mathrm{supp}(E^{(r+k)}(\lambda - r - k))$ also under the projection to $\mathcal{M}_\lambda$. Similarly, if $\lambda \geq 0$ and $k \geq 1$ then the image of $\mathrm{supp}(E^{(r)}(\lambda + r))$ in $\mathcal{M}_\lambda$ is not contained in the image of $\mathrm{supp}(E^{(r+k)}(\lambda + r + k))$.*

Proof. In either case, the kernel $E^{(r)}$ is supported over the Brill-Noether locus ${}_r\mathcal{M}_\chi$, which is not contained in ${}_{(r+k)}\mathcal{M}_\chi$. □

Lemma 4.1.4 (Condition (vii))**.** *All $E^{(r)}$'s and $F^{(r)}$'s are sheaves.*

Proof. The kernels are pushforwards of line bundles on the correspondences, so it suffices to argue that the map from a correspondence to the product $\mathcal{M}_{\lambda-r} \times \mathcal{M}_{\lambda+r}$

is finite. Moreover, since both spaces are projective, it is enough to show that the map is quasi-finite. We consider the following diagram; as in the previous section, X, Y, Z, and A are correspondences, and W the fiber product, which is a $\mathbb{P}^1$ bundle over Z.

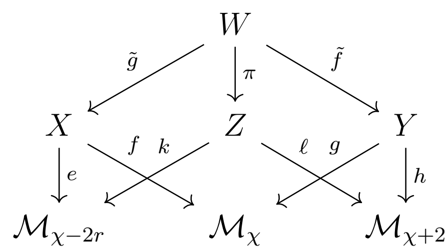


We will show that for any $[\mathcal{F}] \in \mathcal{M}_{\lambda+r}$, the fiber $f^{-1}([\mathcal{F}])$ maps finitely to $\mathcal{M}_{\lambda-r}$, and thus the map from A to $\mathcal{M}_{\lambda-r} \times \mathcal{M}_{\lambda+r}$ has finite fibers. Since this fiber is a Grassmannian $\mathrm{Gr}(r, h^0(\mathcal{F}))$, it is enough to show that the map is non-constant (cf. [13, Ex. II.7.3], or see [16] for a more general fact). Note also that if $h^0(\mathcal{F}) < k+1$, then the fiber itself is finite or empty, so there is nothing to show.

So assume $h^0(\mathcal{F}) \geq k+1$, and fix a flag $\mathbb{C}^{r-1} \subset \mathbb{C}^{r+1} \subset H^0(\mathcal{F})$. We observe that this data also specifies a point of Z : the cokernel of $\mathcal{O}^{r+1} \to \mathcal{F}$ is a sheaf $\mathcal{G}$ in $\mathcal{M}_{\lambda-r-2}$, and $\mathbb{C}^{r+1}/\mathbb{C}^{r-1}$ is a two-dimensional subspace of $H^0(\mathcal{G})$. Thus $\mathbb{P}(\mathbb{C}^{r+1}/\mathbb{C}^{r-1})$ on the one hand gives a varying subspace $\mathbb{C}^k \subset H^0(\mathcal{F})$, i.e. a $\mathbb{P}^1 \subset f^{-1}([\mathcal{F}])$, and on the other hand gives the fiber of W over this point of Z. We will later (Corollary 4.4.3) see that the latter maps finitely to $\mathcal{M}_{\lambda-r}$, so the map from the Grassmannian fiber is non-constant. □

4.2. Condition (iii)

We will now see that our line bundles are constructed precisely to give condition (iii) in Definition 3.0.1. Recall the notation of Section 3.2: we denote the line bundles on the correspondences by L_X, L_Y, and L_Z and work with the diagram below, where π is a $\mathbb{P}^r$ bundle.



To relate $E(\chi + r)E^{(r)}(\chi - 1)$ and $E^{(r+1)}(\chi - r + 1)$, we need the following excess base change result, whose proof we defer until after the proof of Lemma 4.2.3. See also [7, Proposition A.5] and [3, VII, Prop. 2.5].

Recall (see [9, §6.3]) that when f is a regular embedding, the excess normal bundle E of the fiber square

$$\begin{array}{ccc} A & \longrightarrow & C \\ \big\downarrow & & \big\downarrow{\scriptstyle g} \\ B & \xrightarrow{f} & D \end{array}$$

is defined by the exact sequence

$$0 \to N_{A/B} \to g^* N_{C/D} \to E \to 0.$$

If f is arbitrary but D is smooth, as in our case, then the excess normal bundle is by definition the excess normal bundle of the fiber square

$$\begin{array}{ccc} A & \hookrightarrow & B \times C \\ \big\downarrow & & \big\downarrow \\ \Delta & \hookrightarrow & D \times D \end{array}$$

Proposition 4.2.1. *The Fourier Mukai kernel inducing $g^* f_*$ is supported on $W \subset X \times Y$ and has cohomology sheaves $\bigwedge^* E_{\mathcal{M}}^*$, where $E_{\mathcal{M}}$ is the excess normal bundle for the fiber square in diagram* (4.2).

So the kernel $E(\chi+r)E^{(r)}(\chi-1)$ is also supported on W, and has cohomology sheaves $\tilde{g}^* L_X \otimes \tilde{f}^* L_Y \otimes \bigwedge^* E^*_{\mathcal{M}}$, or (by construction of the line bundles)

$$\pi^* L_Z \otimes \omega_\pi \otimes \bigwedge^* E^*_{\mathcal{M}}.$$

We wish to show that the pushforward of this kernel along π has cohomology sheaves $L_Z \otimes_{\mathbb{C}} H^*(\mathbb{P}^r)$, and it suffices to make the following observation.

Proposition 4.2.2. *We have $E_{\mathcal{M}} \cong \Omega_\pi$, the relative cotangent bundle of the $\mathbb{P}^r$ bundle π.*

We first give a helpful description of the relative cotangent bundle. As in Section 3.1, let Q_e, Q_f, etc. denote the tautological quotient bundles.

Lemma 4.2.3. *The relative cotangent bundle is*

$$\Omega_\pi = \mathcal{H}\mathrm{om}(\tilde{g}^* Q_f^*, \tilde{f}^* Q_g).$$

Proof. The argument is similar to the proof of Proposition 3.1.2; we suppress all pullbacks in the notation to help readability. Recall from that proof that we have on $\mathcal{S} \times Z$ a short exact sequence

$$0 \to Q_\ell^* \to \mathcal{U}_{\chi+2} \to \mathcal{U}_{\chi-2r} \otimes L \to 0, \tag{4.2.1}$$

where L is some line bundle. On the other hand, the fiber product W can also be described as $\mathbb{P}Q_\ell$, so it carries a tautological sequence

$$0 \to Q_\pi^* \to Q_\ell^* \to C \to 0 \tag{4.2.2}$$

(where C is defined as the cokernel), and we have $\Omega_\pi = \mathcal{H}om(C, Q_\pi^*)$.

Pulling back everything to $\mathcal{S} \times W$, the composition $Q_\pi^* \hookrightarrow Q_\ell^* \hookrightarrow \mathcal{U}_{\chi+2}$ gives a family of one-dimensional subspaces in H^0 of the fibers of $\mathcal{U}_{\chi+2}$, and so we get a map to Y which identifies Q_π^* with Q_h^*; in particular, the cokernel of this map is given by equation (3.1.5) as $\mathcal{U}_\chi \otimes L'$, where L' is a line bundle such that $Q_h^* = Q_g \otimes L'$. So (still on $\mathcal{S} \times W$) we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q_\pi^* & = & Q_\pi^* & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Q_\ell^* & \longrightarrow & \mathcal{U}_{\chi+2} & \longrightarrow & \mathcal{U}_{\chi-2r} \otimes L & \longrightarrow & 0 \end{array}$$

for which the sequence of cokernels is

$$0 \to C \to \mathcal{U}_\chi \otimes L \to \mathcal{U}_{\chi-2r} \otimes L \to 0.$$

Now we repeat a similar argument: $C \otimes (L')^*$ gives a family of r-dimensional subspaces in H^0 of the fibers of $\mathcal{U}_\chi$, so we get a map to X which identifies C with $Q_f^* \otimes L'$. Putting everything together, we have

$$\Omega_\pi = \mathcal{H}om(C, Q_\pi^*) = \mathcal{H}om(Q_f^* \otimes L', Q_g \otimes L'),$$

as desired.

□

Proof of Proposition 4.2.2. Let $\mathcal{U}$ be a universal sheaf on $\mathcal{S} \times \mathcal{M}_\chi$, and let $q : \mathcal{S} \times \mathcal{M}_\chi \to \mathcal{M}_\chi$ be the projection. Suppose first that $\chi \leq 0$. We again make use of the resolution (3.1.2):

$$0 \to V_0 \to V_1 \to R^1 q_* \mathcal{U} \to 0, \qquad (4.2.3)$$

where V_0 and V_1 are locally free. For readability, we again suppress all pullbacks in the notation throughout this proof, instead stating explicitly where each diagram lives.

On $Y = \mathbb{P}(R^1 q_* \mathcal{U})$ we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_0 & \longrightarrow & V_1 & \longrightarrow & R^1 q_* \mathcal{U} & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & Q_g & & \end{array} \tag{4.2.4}$$

where the pullback is exact by Theorem 2.0.1. From this diagram, we see that $Y = \mathbb{P}(R^1 q_* \mathcal{U})$ is cut out of the projective bundle $P := \mathbb{P}(V_1)$ by the vanishing of a section of $\mathcal{H}om(V_0, Q_g)$ (or rather, $\mathcal{H}om(V_0, Q_P)$, where Q_P is the tautological bundle on P, but note that this restricts to Q_g), and the codimension of Y agrees with the rank of this bundle.

However, when we pull back to W, the codimension and rank no longer agree. Note that on $X = \mathrm{Gr}(\mathcal{E}xt^2_q(\mathcal{U}, \mathcal{O}), r)$ we have a similar diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_1^* & \longrightarrow & V_0^* & \longrightarrow & \mathcal{E}xt^2_q(\mathcal{U}, \mathcal{O}) & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & Q_f & & \end{array} \tag{4.2.5}$$

Thus when we pull back diagram (4.2.4) to W, it no longer remains exact; we get a diagram

$$\begin{array}{ccccccccc} & & & & V_0/Q_f^* & & & & \\ & & & \nearrow & & \searrow & & & \\ 0 \longrightarrow & \ker & \longrightarrow & V_0 & \longrightarrow & V_1 & \longrightarrow & R^1q_*\mathcal{U} & \longrightarrow 0 \\ & \uparrow & \nearrow & & & & \searrow & \downarrow & \\ & Q_f^* & & & & & & Q_g & \end{array} \qquad (4.2.6)$$

We see that the fiber product W is cut out (in $P' := X \times_{\mathcal{M}} P$) by a section of $\mathcal{H}om(V_0/Q_f^*, Q_g)$, not $\mathcal{H}om(V_0, Q_g)$. Combined with Lemma 4.2.3, we have on W the following identifications:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_{W/P'} & \longrightarrow & N_{Y/P} & \longrightarrow & \Omega_\pi & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \mathcal{H}om(V_0/Q_f^*, Q_g) & \longrightarrow & \mathcal{H}om(V_0, Q_g) & \longrightarrow & \mathcal{H}om(Q_f^*, Q_g) & \longrightarrow & 0 \end{array}$$

It remains only to show that this cokernel is also the excess normal bundle $E_{\mathcal{M}}$. On W, we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_{W/X\times Y} & \longrightarrow & N_{W/X\times P} & \longrightarrow & N_{Y/P} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N_{W/X\times Y} & \longrightarrow & N_{P'/X\times P} = N_{\Delta/\mathcal{M}_\chi \times \mathcal{M}_\chi} & \longrightarrow & E_{\mathcal{M}} & \longrightarrow & 0 \end{array}$$

where the identification on the bottom row is because $P \to \mathcal{M}_\chi$ is flat (and thus there is no excess normal bundle on P'). By the Snake Lemma, $N_{Y/P} \to E_{\mathcal{M}}$ is surjective and its kernel is $N_{W/P'}$, as desired.

If instead we have $\chi \geq 0$, we replace (3.1.2) by the dual sequence, Y by X, and make the same argument. □

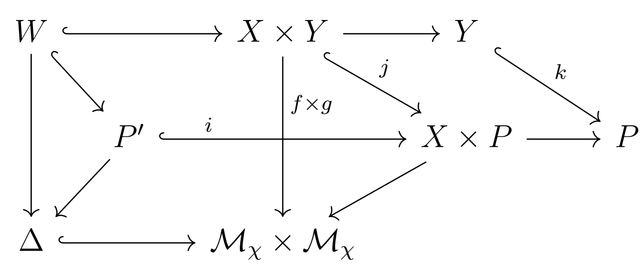
Maintaining the above notation, we now prove Proposition 4.2.1.

Proof of Proposition 4.2.1. Denote by ϕ the map from $P' = X \times_{\mathcal{M}} P$ to P. We first argue that the pullback $\phi^* \mathcal{O}_Y$ has the desired cohomology sheaves $\bigwedge^* E^*_{\mathcal{M}}$. From the proof of Lemma 4.2.3, we know that $\mathcal{O}_Y$ is quasi-isomorphic to the Koszul complex $\text{Kosz}(\mathcal{H}om(V_0, Q_g), s)$ of some section s, that $\mathcal{O}_W$ is quasi-isomorphic to $\text{Kosz}(\mathcal{H}om(V_0/Q^*_f, Q_g), s')$, and that there is a short exact sequence

$$0 \to \mathcal{H}om(V_0/Q^*_f, Q_g) \to \phi^* \mathcal{H}om(V_0, Q_g) \to \mathcal{H}om(Q^*_f, Q_g) \to 0$$

on P'. It follows (see Lemma A.0.1 in the appendix) that $\mathcal{H}^{-i}(\tilde{f}^* \mathcal{O}_Y) = \bigwedge^i \mathcal{H}om(Q^*_f, Q_g) \otimes \mathcal{O}_W$, which we have already shown is $\bigwedge^* E^*_{\mathcal{M}}$.

Thus it suffices to show that $\phi^* \mathcal{O}_Y$ and the kernel inducing $g^* f_*$ have the same cohomology sheaves. Since we are interested only in the cohomology sheaves, we can check this claim after pushing both objects (along the closed embeddings) into $X \times P$. We consider the following diagram.



Note that the fiber squares originating at $X \times_{\mathcal{M}} P$ and at $X \times Y$ are both Tor-independent, the former by [1, Proposition A.1].

Now the kernel inducing $g^* f_*$ is $(f \times g)^* \mathcal{O}_\Delta$ ([14, Ex. 5.12]), and we push this forward along j as discussed. The pullback factors through $X \times P$, so when we base change around the lower square, we find that this pushforward is

$$j_* j^* i_* \mathcal{O}_{P'} = i_* \mathcal{O}_{P'} \otimes j_* \mathcal{O}_{X \times Y}.$$

On the other hand, we push $\phi^* k_* \mathcal{O}_Y$ forward along i, and find after base changing around the right square that we also get

$$i_* i^* j_* \mathcal{O}_{X \times Y} = i_* \mathcal{O}_{P'} \otimes j_* \mathcal{O}_{X \times Y},$$

completing the proof. □

Corollary 4.2.4 (Condition (iii))**.** *At the level of cohomology of complexes we have*

$$\mathcal{H}^*(E(\lambda + r) * E^{(r)}(\lambda - 1)) \cong E^{(r+1)}(\lambda) \otimes_{\mathbb{C}} H^*(\mathbb{P}^r),$$

where the gradings in $H^*(\mathbb{P}^r)$ are centered around 0.

Proof. As discussed, we need to compute the higher direct images along π of a complex on W with $(-j)$th cohomology sheaf $\pi^* L_Z \otimes \omega_\pi \otimes \bigwedge^j E^*_{\mathcal{M}_\chi}$. By Proposition 4.2.2, we see this is simply

$$\pi^* L_Z \otimes \omega_\pi \otimes \bigwedge^j T_\pi = \pi^* L_Z \otimes \Omega_\pi^{r-j}.$$

From the long Euler sequence for $W = \mathbb{P}(Q_\ell^*)$, we get exact sequences

$$0 \to \Omega_\pi^p \to \left(\pi^* \bigwedge^p Q_\ell^*\right)(-p) \to \left(\pi^* \bigwedge^{p-1} Q_\ell^*\right)(-p+1) \to \cdots \to (\pi^* Q_\ell^*)(-1) \to \mathcal{O}_W \to 0$$

showing that

$$R^i \pi_*(\pi^* L_Z \otimes \Omega_\pi^{r-j}) = \begin{cases} L_Z, & \text{if } i = r - j \\ 0, & \text{else.} \end{cases}$$

Thus in the Grothendieck spectral sequence

$$E_2^{i,-j} = R^i\pi_*(\mathcal{H}^{-j}(E(\chi+r)E^{(r)}(\chi-1))) \implies R^{i-j}\pi_*(E(\chi+r)E^{(r)}(\chi-1)),$$

the E_2 term is non-zero if and only if $i = r-j$, so the spectral sequence degenerates immediately, and we find that the non-vanishing pushforwards are

$$R^{r-2j}\pi_*(E(\chi+r)E^{(r)}(\chi-1)) = L_Z.$$

In other words, the cohomology sheaves of the pushforward agree with $L_Z \otimes_{\mathbb{C}} H^*(\mathbb{P}^r) = E^{(r+1)} \otimes_{\mathbb{C}} H^*(\mathbb{P}^r)$, as desired. □

4.3. Condition (iv)

We check condition (iv) in the case that $\lambda = \chi \geq 0$; the other case is similar.

Theorem 4.3.1 (Condition (iv))**.** *For $\lambda \geq 0, E(\lambda-1)F(\lambda-1) = F(\lambda+1)E(\lambda+1) \oplus P$, where $\mathcal{H}^*(P) = \mathcal{O}_\Delta \otimes_{\mathbb{C}} H^*(\mathbb{P}^{\lambda-1})$.*

For brevity, we write simply $EF = FE \oplus P$, where these objects belong to $D(\mathcal{M}_\chi \times \mathcal{M}_\chi)$. Our strategy is to prove the equality on $\mathcal{M}_\chi \times \mathcal{M}_\chi \setminus \Delta({}_2\mathcal{M}_\chi)$, then use the following fact, which we prove in Appendix B. Recall that the homological dimension of a complex $\mathcal{E}$ is the least n for which $\mathcal{E}$ is quasi-isomorphic to a length n complex of vector bundles ([4, Definition 5.2]).

Proposition 4.3.2. *Let Z be a closed subscheme of a Cohen-Macaulay scheme X, and $\mathcal{E}, \mathcal{F} \in D(X)$. If both $\mathcal{E}$ and $\mathcal{F}$ have homological dimension less than or equal to $\operatorname{codim}_X(Z)-2$, then they are isomorphic if and only if their restrictions to $X \setminus Z$ are isomorphic.*

Remark 4.3.3. From the proof we see that in fact it suffices for Z to have a Cohen-Macaulay neighborhood in X.

To prove Theorem 4.3.1, we will apply Proposition 4.3.2 inductively, extending the isomorphism from $\mathcal{M}_\chi \times \mathcal{M}_\chi \setminus \Delta({}_2\mathcal{M}_\chi)$ to $\mathcal{M}_\chi \times \mathcal{M}_\chi \setminus \Delta({}_3\mathcal{M}_\chi)$, and so on. If $m = \dim \mathcal{M}_\chi$, then the codimension of $\Delta({}_t\mathcal{M}_\chi)$ in the product is $m + t(\chi + t)$, so it is enough to show:

Lemma 4.3.4. *For any $t \geq 2$ and any points $p, q \in \mathcal{M}_\chi \setminus {}_{t+1}\mathcal{M}_\chi$, $\operatorname{Ext}^k_{\mathcal{M}_\chi \times \mathcal{M}_\chi}(\mathcal{O}_{(p,q)}, EF)$ vanishes for all k outside of the range*

$$[-t + m - \chi + 1, t + 2m + \chi - 1],$$

and similarly with EF replaced by FE or P.

The lemma will show (cf. [4, Proposition 5.4]) that (some shift of) EF and $FE \oplus P$ have homological dimension at most $2t + m + 2\chi - 2 = m + 2(\chi + t) - 2$. Since we begin our induction at the $t = 2$ stratum, this is less than or equal to $m + t(\chi + t) - 2$, as required by Proposition 4.3.2.

Proof. Since $\mathcal{O}_\Delta$ has projective dimension m and $\mathcal{M}_\chi \times \mathcal{M}_\chi$ has dimension $2m$, one finds (recalling the grading convention of Definition 3.0.1) that $\operatorname{Ext}^k_{\mathcal{M}_\chi \times \mathcal{M}_\chi}(\mathcal{O}_{(p,q)}, P)$ vanishes for k outside of $[m - \chi, 2m + \chi]$, which is sufficient since $t \geq 2$.

Using Lemma 4.1.2 and standard manipulations on Fourier-Mukai kernels (see, e.g., [4, Eq. (3)]), one finds that

$$\operatorname{Ext}^k_{\mathcal{M}_\chi \times \mathcal{M}_\chi}(\mathcal{O}_{(p,q)}, EF) = \operatorname{Ext}^\ell_{\mathcal{M}_{\chi-2}}(F\mathcal{O}_p, F\mathcal{O}_q),$$

where $l = k - m + \chi - 1$. If p and q belong to different Brill-Noether strata of $\mathcal{M}_\chi$, then $F\mathcal{O}_p$ and $F\mathcal{O}_q$ have disjoint supports and these Exts vanish. So assume that both p and q lie in the tth stratum ${}_t\mathcal{M}_\chi \setminus_{t+1} \mathcal{M}_\chi$. We will show that $F\mathcal{O}_p$ and $F\mathcal{O}_q$ are complexes with non-zero cohomology sheaves only in degrees 0 through t. Thus Exts between them vanish for ℓ outside of $[-t, \dim \mathcal{M}_{\chi-2} + t] = [-t, t+m+2\chi-2]$, i.e. k outside of $[-t+m-\chi+1, t+2m+\chi-1]$.

To verify this claim, we argue first that the pullback of $\mathcal{O}_p$ (or $\mathcal{O}_q$) to the correspondence X has non-zero cohomology sheaves only in degrees 0 through t. Since $\mathcal{M}$ is smooth, the Koszul complex for p is exact. The pullback of this complex need not be exact, but by [19, Theorem 16.8], it may have cohomology only up to degree t, the difference between the actual and expected dimension of the fiber X_p. But now as argued in the proof of Lemma 4.1.4, this fiber maps finitely to $\mathcal{M}_{\chi-2}$, so the pushforward is exact, proving the claim.

For the case of FE, the analogous computation shows that

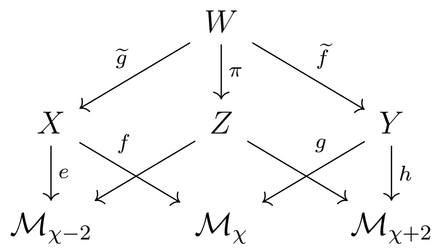
$$\mathrm{Ext}^k_{\mathcal{M}_\chi \times \mathcal{M}_\chi}(\mathcal{O}_{(p,q)}, FE) = \mathrm{Ext}^n_{\mathcal{M}_{\chi+2}}(E\mathcal{O}_p, E\mathcal{O}_q),$$

where $n = k - m - \chi - 1$. If $t = 0$, the fibers of the correspondence Y over p and q are empty, so we are done. If not, the fibers have dimension $t-1$, and $\dim Y - \dim \mathcal{M}_\chi = -\chi - 1$, so the dimension failure is $t + \chi$, and thus the pullbacks of $\mathcal{O}_p$ and $\mathcal{O}_q$ to Y have non-zero cohomology sheaves up to at most this degree. Hence this Ext vanishes for n outside of $[-t-\chi, t+\chi+\dim \mathcal{M}_{\chi+2}] = [-t-\chi, t+m-\chi-2]$, i.e. k outside of $[-t+m+1, t+2m-1]$. Since $\chi \geq 0$, this again falls within the desired range. $\square$

It remains only to prove the base case for the induction, which occupies the remainder of the section.

Proposition 4.3.5. *Theorem 4.3.1 holds on* $\mathcal{M}_\chi \times \mathcal{M}_\chi \setminus \Delta({}_2\mathcal{M}_\chi)$.

To prove this, we begin with the usual diagram



By definition the kernel EF induces the functor $f_*(L_X \otimes e^*e_*(L_X^{-1} \otimes \omega_X \otimes f^*(-)))$, so we have ([14, Ex. 5.12])

$$EF = (f \times f)_*[(e \times e)^*\mathcal{O}_\Delta \otimes (L_X \boxtimes (L_X^{-1} \otimes \omega_X))],$$

and similarly

$$FE = (g \times g)_*[(h \times h)^*\mathcal{O}_\Delta \otimes ((L_Y^{-1} \otimes \omega_Y) \boxtimes L_Y)].$$

In particular, EF and FE are pushforwards of complexes supported on $X \times_{\mathcal{M}_{\chi-2}} X$ and $Y \times_{\mathcal{M}_{\chi+2}} Y$. Since $\chi \geq 0$, the latter is irreducible, but the former may not be. Say $X \times_{\mathcal{M}_{\chi-2}} X = X \cup \overline{U}$, where X is the diagonal copy and U its complement. In terms of points, a $\mathbb{C}$-point of the fiber product is a sheaf $\mathcal{D} \in \mathcal{M}_{\chi-2}$ along with two quotients $H^1(\mathcal{D}) \twoheadrightarrow \mathbb{C}$. The diagonal X is where these quotients coincide, and $\overline{U}$ is the closure of where they differ, i.e. where $h^1(\mathcal{D}) \geq 2$. We remark that $\overline{U}$ always has the expected dimension of the fiber product: it is generically a $\mathbb{P}^1 \times \mathbb{P}^1$ bundle over the locus in $\mathcal{M}_{\chi-2}$ where $h^1(\mathcal{D}) \geq 2$, and X is a $\mathbb{P}^1$ bundle over the same locus. In particular, $U = \overline{U} \setminus X$ is a local complete intersection.

Now $Y \times_{\mathcal{M}_{\chi+2}} Y$ has the expected dimension, and $Y \times Y$ is Cohen-Macaulay, so the Koszul complex for $\mathcal{O}_\Delta$ in $\mathcal{M}_{\chi+2} \times \mathcal{M}_{\chi+2}$ remains exact when pulled back, and thus $(h \times h)^* \mathcal{O}_\Delta = \mathcal{O}_{Y \times_{\mathcal{M}_{\chi+2}} Y}$. Our strategy is to consider the following diagram.

$$\begin{array}{ccccc} & & W \times_Z W & & \\ & \swarrow^{\phi} & & \searrow^{\psi} & \\ X \times_{\mathcal{M}_{\chi-2}} X = X \cup \overline{U} & & & & Y \times_{\mathcal{M}_{\chi+2}} Y \\ & \searrow_{f \times f} & & \swarrow_{g \times g} & \\ & & \mathcal{M}_\chi \times \mathcal{M}_\chi & & \end{array} \tag{4.3.7}$$

Since we are ignoring $\Delta({}_2\mathcal{M}_\chi)$, we may assume ϕ is an isomorphism onto $\overline{U}$, as follows. A point of $W \times_Z W$ can be described as a sheaf $\mathcal{D} \in \mathcal{M}_{\chi-2}$ and a diagram

$$\begin{array}{ccccc} & & & & \mathbb{C} \\ & & & \nearrow & \\ H^1(\mathcal{D}) & \twoheadrightarrow & \mathbb{C}^2 & & \\ & & & \searrow & \\ & & & & \mathbb{C} \end{array}$$

So $W \times_Z W$ maps to $\overline{U}$, and is a bijection both away from the diagonal X and away from the locus ${}_3\overline{U}$ where $h^1(\mathcal{D}) \geq 3$. Since the intersection of these maps to $\Delta({}_2\mathcal{M}_\chi)$, we may assume ϕ is a bijection. From here, one can either verify that $\overline{U}$ is normal (it is smooth along X and Serre's criterion applies away from X) or simply construct the inverse morphisms on $\overline{U} \setminus X$ and $\overline{U} \setminus {}_3\overline{U}$.

As a final preliminary, we record the following computation.

Lemma 4.3.6. *There is a line bundle on $W \times_Z W$ whose pushforward along ϕ is $\left(L_X \boxtimes (L_X^{-1} \otimes \omega_X)\right)\Big|_{\overline{U}}$ and whose pushforward along ψ is $(L_Y^{-1} \otimes \omega_Y) \boxtimes L_Y$.*

Proof. Suppressing pullbacks in the notation, we consider the line bundle $L_X \boxtimes (L_X^{-1} \otimes \omega_X)$ on $W \times_Z W$. The pushforward along ϕ is clear since we can assume ϕ is an isomorphism onto $\overline{U}$. For the ψ direction, we show that the pushforward of the difference

$$\left(L_X \boxtimes (L_X^{-1} \otimes \omega_X)\right) \otimes \left((L_Y^{-1} \otimes \omega_Y) \boxtimes L_Y\right)^{-1}$$

is trivial. By construction of the line bundles, $L_X \otimes L_Y = L_Z \otimes \omega_{W/Z}$, so we can rewrite this as

$$\left(L_X \boxtimes (L_X^{-1} \otimes \omega_X)\right) \otimes \left((L_X^{-1} \otimes L_Z \otimes \omega_{W/Z} \otimes \omega_Y^{-1}) \boxtimes L_X \otimes L_Z^{-1} \otimes \omega_{W/Z}^{-1}\right)$$

$$= (L_Z \otimes \omega_{W/Z} \otimes \omega_Y^{-1}) \boxtimes (L_Z^{-1} \otimes \omega_X \otimes \omega_{W/Z}^{-1}).$$

Since $W \times_Z W$ maps diagonally to Z, this further simplifies to

$$(\omega_W \otimes \omega_Y^{-1}) \boxtimes (\omega_X \otimes \omega_W^{-1}).$$

Now we recall that the excess normal bundle E fits into an exact sequence

$$0 \to T_W \to T_X \oplus T_Y \oplus T_{\mathcal{M}_\chi} \to E \to 0.$$

By Proposition 4.2.2, $E = \Omega_{W/Z}$, so taking determinants yields the identity

$$\omega_X = \omega_Y^{-1} \otimes \omega_W^2 \otimes \omega_Z^{-1}.$$

Thus our difference bundle is

$$(\omega_W \otimes \omega_Y^{-1}) \boxtimes (\omega_Y^{-1} \otimes \omega_W) \otimes \omega_Z = \omega_{W \times_Z W / Y \times_{\mathcal{M}_\chi} Y}.$$

By Grothendieck Duality, the pushforward along ψ is $(\psi_*\mathcal{O})^\vee$. As with ϕ, we see that ψ is an isomorphism away from the diagonal; in particular, ψ is birational. On the other hand, away from ${}_1\mathcal{M}_{\chi+2}$, both $W \times_Z W$ and $Y \times_{\mathcal{M}_{\chi+2}} Y$ are smooth, and so ψ is a rational resolution ([17], Theorem 5.10). Since the intersection of the diagonal with ${}_1\mathcal{M}_{\chi+2}$ maps to $\Delta({}_2\mathcal{M}_\chi)$, it can be ignored, and so we have $\psi_*\mathcal{O} = \mathcal{O}$, completing the proof. □

We are now ready to prove Proposition 4.3.5.

Proof. We consider three cases: $\chi = 0$, $\chi = 1$, and $\chi \geq 2$.

If $\chi = 0$, then $X \times_{\mathcal{M}_{\chi-2}} X$ is also irreducible, i.e. $X \subset \overline{U}$. Moreover, $(e \times e)^*\mathcal{O}_\Delta = \mathcal{O}_{X \times_{\mathcal{M}_{\chi-2}} X}$ by the same reasoning used on the other side. So the line bundle of Lemma 4.3.6 pushes forward to EF via the ϕ direction and to FE via the ψ direction, completing the proof.

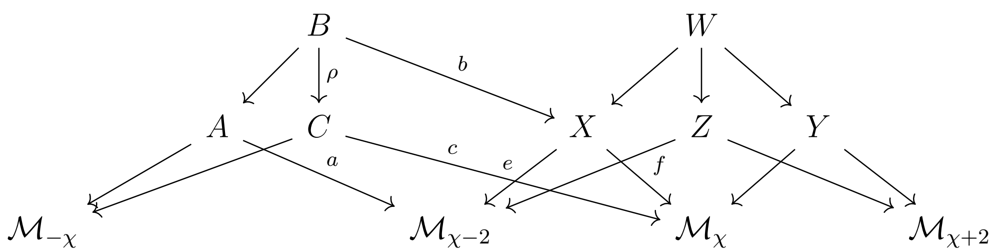
If $\chi = 1$, then $X \times_{\mathcal{M}_{\chi-2}} X$ is reducible, but both components have expected dimension, so we still have $(e \times e)^*\mathcal{O}_\Delta = \mathcal{O}$. We tensor the Mayer-Vietoris sequence

$$0 \to \mathcal{O}_{X \cup \overline{U}} \to \mathcal{O}_X \oplus \mathcal{O}_{\overline{U}} \to \mathcal{O}_{X \cap \overline{U}} \to 0$$

with $L_X \boxtimes (L_X^{-1} \otimes \omega_X)$ and push forward along $f \times f$. The $\mathcal{O}_{X \cup \overline{U}}$ term gives EF and by Lemma 4.3.6, the $\mathcal{O}_{\overline{U}}$ term gives FE, so it is enough to show that the $\mathcal{O}_X$ term gives $\mathcal{O}_{\mathcal{M}_\chi}$ (the diagonal copy of $\mathcal{M}_\chi$ tensored with the cohomology of $\mathbb{P}^0$) and the $O_{X \cap \overline{U}}$ term vanishes. We observe that the restriction of $L_X \boxtimes (L_X^{-1} \otimes \omega_X)$ to (the diagonal copy of) X is simply ω_X. Since $\chi = 1$, Proposition 2.0.6 shows that X is the blowup of $\mathcal{M}_\chi$ along ${}_1\mathcal{M}_\chi$, which has codimension 2; the exceptional divisor is precisely $E = X \cap \overline{U}$. So in our Mayer-Vietoris sequence, we get $\omega_X = \mathcal{O}_X(E)$

in the middle term, and $\omega_X|_E = \mathcal{O}_E(E)$ in the right term, which have the desired pushforwards.

For the case $\chi \geq 2$, the dimension of the diagonal X is now larger than expected, and we must consider an expanded diagram; as usual, A and C are correspondences, and B the fiber product of A and X, which is also a bundle over C.



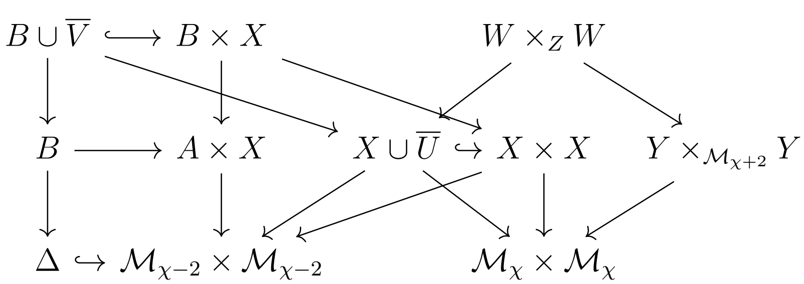
(4.3.8)

We remark that c blows up $\mathcal{M}_\chi$ in ${}_1\mathcal{M}_\chi$ by Proposition 2.0.6. Moreover, if

$$0 \to S_f \to f^* \mathcal{E}xt^2(\mathcal{U}_\chi, \mathcal{O}) \to Q_f \to 0$$

is the tautological sequence on X, then B is identified with $\mathrm{Gr}(S_f, \chi - 1)$ (cf. the proof of Theorem 2.0.1, parts 3 and 4), so b also blows up X in $f^{-1}({}_1\mathcal{M}_\chi)$.

We correspondingly replace diagram (4.3.7) with the following augmentation.



(4.3.9)

Here $B \cup \overline{V}$ is the fiber product, with the decomposition defined analogously to $X \cup \overline{U}$. That is, a $\mathbb{C}$-point is a sheaf $\mathcal{D}$ together with two quotients $H^1(\mathcal{D}) \twoheadrightarrow \mathbb{C}$ and a subspace $\mathbb{C}^{\chi-1} \hookrightarrow H^0(\mathcal{D})$. The diagonal copy of B is where the two quotients agree, and $\overline{V}$ the closure of where they differ. Note that $B \cap \overline{V}$ is the exceptional divisor of the blowup b.

The point of this construction is that $B \cup \overline{V}$ is equidimensional; similar to the $\chi = 1$ case, we can now use the Mayer-Vietoris sequence

$$0 \to \mathcal{O}_{B \cup \overline{V}} \to \mathcal{O}_B \oplus \mathcal{O}_{\overline{V}} \to \mathcal{O}_{B \cap \overline{V}} \to 0.$$

Note first that $(e \times e)^* \mathcal{O}_\Delta$ can be computed by pulling back to $B \times X$ before pushing forward (since $\mathcal{O}_B$ pushes forward to $\mathcal{O}_X$ by Theorem 2.0.1). This in turn can be related to something on $B \cup \overline{V}$ by base-changing around the fiber squares on the left of the diagram. In fact, since $B \cup \overline{V}$ has expected dimension in each component, the upper square is Tor-independent. The lower square is treated in the previous section: the pullback of $\mathcal{O}_\Delta$ has cohomology sheaves $\bigwedge^i T_{B/C}$, by Propositions 4.2.1 and 4.2.2.

As in the $\chi = 1$ case, we note that $L_X \boxtimes (L_X^{-1} \otimes \omega_X)$ restricted to the diagonal is simply ω_X. We now claim that $b^* \omega_X = \omega_{B/C} \otimes \mathcal{O}(B \cap \overline{V})$. First, observe that we can ignore ${}_2\mathcal{M}_\chi$ for this computation as its preimage in B has codimension 4 (shown in the proof of Proposition 3.1.2). So we may assume b and c are smooth (recall that X is smooth by Proposition 3.1.4) blowups over the same locus ${}_1\mathcal{M}_\chi$. Since the codimension k of ${}_1\mathcal{M}_\chi$ in $\mathcal{M}_\chi$ is one more than that of $f^{-1}({}_1\mathcal{M}_\chi)$ in X, the standard blowup formula gives

$$\rho^* \omega_C = \mathcal{O}(k(B \cap \overline{V})) \text{ and } \omega_B = b^* \omega_X \otimes \mathcal{O}((k-1)(B \cap \overline{V})),$$

from which the claim follows.

So when we pull back the line bundle $L_X \boxtimes (L_X^{-1} \otimes \omega_X)$ and tensor with $(e \times e)^* \mathcal{O}_\Delta|_B$, we get a complex whose cohomology sheaves are

$$\bigwedge^i T_{B/C} \otimes \omega_{B/C} \otimes \mathcal{O}(B \cap \overline{V}) = \Omega^j_{B/C} \otimes \mathcal{O}(B \cap \overline{V}).$$

We push these forward to $\mathcal{M}_\chi$ (i.e. $\Delta(\mathcal{M}_\chi \times \mathcal{M}_\chi)$) via C: on C we get $\mathcal{O}_C(E) \otimes H^*(\mathbb{P}^\chi)$ by the computation shown in Corollary 4.2.4, and thus $\mathcal{O}_\Delta \otimes H^*(\mathbb{P}^\chi)$ on $\mathcal{M}_\chi \times \mathcal{M}_\chi$.

If we instead restrict to $B \cap \overline{V}$, the computation is the same except that on C we get shifted copies of $\mathcal{O}_E(E)$, whose pushforward vanishes.

Finally, we turn to the $\mathcal{O}_{\overline{V}}$ term. We first claim that the pushforward of $\mathcal{O}_{\overline{V}}$ is a line bundle on $\overline{U}$. To check this, consider the open sets $\overline{U} \setminus X$ and $\overline{U} \setminus {}_3\overline{U}$, where ${}_3\overline{U}$ again denotes the preimage of ${}_3\mathcal{M}_{\chi-2}$, i.e. the locus where $h^1(\mathcal{D}) \geq 3$. Note that the intersection $X \cap {}_3\overline{U}$ lies over $\Delta({}_2\mathcal{M}_\chi)$, which we are ignoring. Now $\overline{U} \setminus X$ has expected dimension, so on this patch we can directly compute $(e \times e)^* \mathcal{O}_\Delta|_{\overline{U}} = \mathcal{O}_{\overline{U}}$ as in the $\chi = 0$ and $\chi = 1$ cases. On the other hand, the map $\overline{V} \to \overline{U}$ is a $\mathbb{P}^{\chi-1}$ bundle over $\overline{U} \setminus X$, and so $\mathcal{O}_{\overline{V}}$ pushes forward to $\mathcal{O}_{\overline{U}}$ on this patch as well.

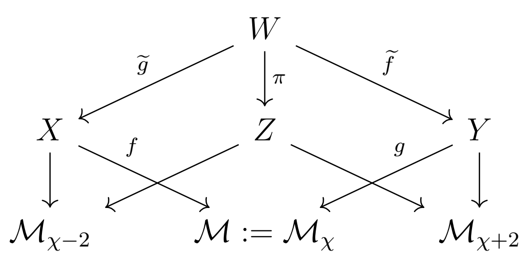
Now we know that the pushforward of $\mathcal{O}_{\overline{V}}$ is a line bundle (in particular, has homological dimension zero) and is isomorphic to $\mathcal{O}_{\overline{U}}$ on the complement of ${}_3\overline{U}$. We hope to extend this isomorphism to the complement of $X \cap {}_3\overline{U}$, i.e. extend it over ${}_3\overline{U} \setminus X$. One finds[1] that ${}_3\overline{U} \setminus X$ has codimension $\chi + 1 \geq 3$, so we apply Proposition 4.3.2 (or rather, Remark 4.3.3 – $\overline{U}$ is not CM, but is a local complete intersection away from X, so ${}_3\overline{U} \setminus X$ has a CM neighborhood $\overline{U} \setminus X$).

[1]Since ${}_3\overline{U}$ is a generic $\mathbb{P}^2 \times \mathbb{P}^2$ bundle over the codimension $3(\chi - 2 + 3)$ locus ${}_3\mathcal{M}_{\chi-2}$, whereas $\overline{U}$ itself is a generic $\mathbb{P}^1 \times \mathbb{P}^1$ bundle over the codimension $2(\chi - 2 + 2)$ locus ${}_2\mathcal{M}_{\chi-2}$

So we have shown that $\mathcal{O}_{\overline{V}} = \bigwedge^0 T_{B/C}$ pushes forward to $\mathcal{O}_{\overline{U}}$ (which as in the previous case gives the kernel FE on $\mathcal{M}_\chi \times \mathcal{M}_\chi$). It now suffices to prove that the higher wedges $\bigwedge^{>0} T_{B/C}$ vanish when restricted to $\overline{V}$ and pushed down. By Lemma 4.2.2, $T_{B/C}$ can be described as $\mathcal{H}om_{\mathcal{O}_B}(Q_e, Q_a^*)$, where Q_e and Q_a are (pullbacks of) tautological bundles for the maps in diagram (4.3.8). In particular, Q_e is pulled back from the X factor of $A \times X$, or the second factor of $X \times X$, and is trivial on the fibers of $\overline{V} \to \overline{U}$. On the other hand, Q_a^* is the tautological subbundle for the Grassmannian $\overline{V} \to \overline{U}$, so the pushforwards of its wedges vanish by Borel-Weil-Bott. We conclude that the pushforward of the whole $\mathcal{O}_{\overline{V}}$ term in our Mayer-Vietoris sequence is simply $\mathcal{O}_{\overline{U}}$, and this completes the proof. □

4.4. Condition (v)

Condition (v) concerns only the composition of one step kernels; we work with the usual diagram, writing $\mathcal{M}$ for $\mathcal{M}_\chi$ as it will be constantly referenced.



(4.4.10)

To obtain the appropriate $\mathbb{A}^1$-deformations of our moduli spaces, we have to take twistor deformations, and therefore work with coherent *analytic* sheaves. In practice, this point can mostly be ignored – see, for example, [21] to verify that our computational techniques are still valid.

If we choose a deformation $\mathcal{M} \hookrightarrow \widetilde{\mathcal{M}}$, then W is the fiber product of X and Y over either $\mathcal{M}$ or $\widetilde{\mathcal{M}}$. Denote the excess normal bundles for these fiber squares by $E_{\mathcal{M}}$ and $E_{\widetilde{\mathcal{M}}}$, and consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_{W/X\times Y} & \longrightarrow & (f\times g)^* N_{\Delta_{\mathcal{M}}/\mathcal{M}\times\mathcal{M}} & \longrightarrow & E_{\mathcal{M}} & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N_{W/X\times Y} & \longrightarrow & (f\times g)^* N_{\Delta_{\widetilde{\mathcal{M}}}/\widetilde{\mathcal{M}}\times\widetilde{\mathcal{M}}} & \longrightarrow & E_{\widetilde{\mathcal{M}}} & \longrightarrow & 0 \end{array}$$

We note that $N_{\Delta_{\mathcal{M}}/\mathcal{M}\times\mathcal{M}} \cong T_{\mathcal{M}}$, and similarly for $\widetilde{\mathcal{M}}$, so applying the Snake Lemma shows that the cokernel of $E_{\mathcal{M}} \to E_{\widetilde{\mathcal{M}}}$ is $(f\times g)^* N_{\mathcal{M}/\widetilde{\mathcal{M}}} \cong \mathcal{O}$. So we have a short exact sequence

$$0 \to E_{\mathcal{M}} \to E_{\widetilde{\mathcal{M}}} \to \mathcal{O} \to 0. \tag{4.4.11}$$

Moreover, we see that the excess map $(f\times g)^* T_{\mathcal{M}} \to E_{\mathcal{M}}$ induces on H^1 a map sending the pullback of the Kodaira-Spencer class – that is, the class of the extension

$$0 \to (f\times g)^* T_{\mathcal{M}} \to (f\times g)^* T_{\widetilde{\mathcal{M}}} \to \mathcal{O} \to 0$$

– to the class of (4.4.11).

Proposition 4.4.1. *The deformation* $\mathcal{M} \hookrightarrow \widetilde{\mathcal{M}}$ *may be chosen so that* (4.4.11) *is a non-split extension.*

By the previous observation, it suffices to show that the induced map on H^1 is non-zero, since by [10, Proposition 25.7] the Kähler class used to construct the twistor line agrees up to a scalar with the Kodaira-Spencer class of the resulting deformation. To prove the map on H^1 is non-zero, we need the following identification.

Proposition 4.4.2. *The restriction of* $(f \times g)^* T_{\mathcal{M}}$ *to a general fiber of* π *is*

$$\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\chi} \oplus \mathcal{O}_{\mathbb{P}^1}^{n} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\chi} \oplus \mathcal{O}_{\mathbb{P}^1}(2),$$

where $n = \dim \mathcal{M} - 2\chi - 2$.

The proof of Proposition 4.4.2 occupies most of the section. Before beginning the setup, we point out the following corollary, which was used in the proof of Lemma 4.3.4. Note that in that proof, we are applying this corollary to the fiber product over $\mathcal{M}_{\chi-2}$, not over $\mathcal{M}_{\chi}$.

Corollary 4.4.3. *In diagram* (4.4.10)*, the restrictions of* f *to a non-empty fiber of* $X \to \mathcal{M}_{\chi-2}$ *or of* g *to a non-empty fiber of* $Y \to \mathcal{M}_{\chi+2}$ *are non-constant.*

Proof. By Proposition 4.4.2, the map on the fibers of π cannot be constant: the pullback of $T_{\mathcal{M}}$ to a general fiber has non-vanishing first cohomology, and therefore the pullback to *any* fiber has non-vanishing first cohomology (and in particular is non-trivial), by upper semicontinuity. But these fibers of π map to the fibers of f and of g, completing the proof. □

We now prepare the proof of Proposition 4.4.2. First, some notation. A point z of Z represents a sheaf $\mathcal{F} \in \mathcal{M}_{\chi+2}$ and a two-dimensional subspace $\mathbb{C}^2 \hookrightarrow H^0(\mathcal{F})$. Suppose z is general, so that $h^1(\mathcal{F}) = 0$. We consider the following diagram, where p and q are the projections, and $i = g \circ \tilde{f} = f \circ \tilde{g}$ is the inclusion of a fiber of π.

$$\begin{array}{ccccc}
\mathcal{S} & \xleftarrow{\;p\;} & \mathcal{S} \times \mathbb{P}^1 & \xrightarrow{\;1 \times i\;} & \mathcal{S} \times \mathcal{M} \\
 & & \big\downarrow q & & \big\downarrow \tilde{q} \\
 & & \mathbb{P}^1 & \xrightarrow{\;i\;} & \mathcal{M}
\end{array}$$

Let $\mathcal{U}$ be a universal sheaf on $\mathcal{S} \times \mathcal{M}$. Then we have $T_{\mathcal{M}} \cong \mathcal{E}xt^1_{\tilde{q}}(\mathcal{U},\mathcal{U})$ ([15, Theorem 10.2.1]), and we wish to compute the pullback along i.

Lemma 4.4.4. *Let $\mathcal{V} = L_0(1 \times i)^*\mathcal{U}$. Then*

$$i^* \mathcal{E}\mathrm{xt}^1_q(\mathcal{U},\mathcal{U}) = \mathcal{E}\mathrm{xt}^1_q(\mathcal{V},\mathcal{V}).$$

Proof. By an analogous statement to [13, Corollary III.12.9], the $\mathcal{E}xt^k_{\tilde{q}}(\mathcal{U},\mathcal{U})$ are vector bundles, so the spectral sequence

$$L_{-j}i^* \mathcal{E}xt^k_{\tilde{q}}(\mathcal{U},\mathcal{U}) \implies \mathcal{H}^{j+k}(Li^* R\tilde{q}_* R\mathcal{H}om(\mathcal{U},\mathcal{U}))$$

is degenerate, yielding

$$L_0 i^* \mathcal{E}xt^1_q(\mathcal{U},\mathcal{U}) = \mathcal{H}^1(Li^* R\tilde{q}_* R\mathcal{H}om(\mathcal{U},\mathcal{U})).$$

By flat base change, this is

$$\mathcal{H}^1(Rq_* L(1 \times i)^* R\mathcal{H}om(\mathcal{U},\mathcal{U})) = \mathcal{H}^1(Rq_* R\mathcal{H}om(L(1 \times i)^*\mathcal{U}, L(1 \times i)^*\mathcal{U})).$$

It remains only to argue that the pullbacks are underived. Vanishing of the higher pullbacks is local, so we may assume $(1 \times i)$ is affine (once we have proven Proposition 4.4.2, we will see that in fact $1 \times i$ was finite). Then $(1 \times i)_*$ is exact, and so we have

$$\begin{aligned}(1\times i)_*\mathcal{H}^k(L(1\times i)^*\mathcal{U}) &= \mathcal{H}^k((1\times i)_*L(1\times i)^*\mathcal{U}) \\ &= \mathcal{H}^k(\mathcal{U}\otimes(1\times i)_*\mathcal{O}) && \text{(by the projection formula)} \\ &= \mathcal{H}^k(\mathcal{U}\otimes\tilde{q}^* i_*\mathcal{O}). && \text{(by flat base change)}\end{aligned}$$

Since $\mathcal{U}$ is flat over $\tilde{q}$, this is 0 for $k \neq 0$, and since $(1\times i)_*$ is faithful, this shows that $L(1\times i)^*\mathcal{U} = L_0(1\times i)^*\mathcal{U} = \mathcal{V}$, completing the proof. □

So we have on $\mathcal{S}\times\mathbb{P}^1$ a sequence

$$0 \to q^*\mathcal{O}(-1) \to p^*\mathcal{F} \to \mathcal{V} \to 0 \tag{4.4.12}$$

whose fibers over a point in $\mathbb{P}^1$ are

$$0 \to \mathcal{O} \to \mathcal{F} \to \mathcal{E} \to 0.$$

For ease of reference, we record the cohomology of these sheaves.

	$\mathcal{O}$	$\mathcal{F}$	$\mathcal{E}$
h^0	1	$\chi+2$	$\chi+1$
h^1	0	0	1
h^2	1	0	0

To prove Proposition 4.4.2, we apply $\mathcal{H}om(\mathcal{V},-)$ to (4.4.12) and use the long exact sequence. We start by identifying the relevant portion.

Lemma 4.4.5.

$$0 \to \mathcal{E}\mathrm{xt}^1_q(\mathcal{V}, p^*\mathcal{F}) \to \mathcal{E}\mathrm{xt}^1_q(\mathcal{V},\mathcal{V}) \to \mathcal{E}\mathrm{xt}^2_q(\mathcal{V}, q^*\mathcal{O}(-1)) \to 0 \tag{4.4.13}$$

is exact.

Proof. To check exactness in the first place, we will show that the previous segment of the long exact sequence is

$$\mathcal{H}om_q(\mathcal{V}, p^*\mathcal{F}) = 0 \to \mathcal{H}om_q(\mathcal{V}, \mathcal{V}) \xrightarrow{\sim} \mathcal{E}xt^1_q(\mathcal{V}, q^*\mathcal{O}(-1)). \tag{4.4.14}$$

The vanishing of $\mathcal{H}om_q(\mathcal{V}, p^*\mathcal{F})$ follows again from [13, Corollary III.12.9], since $\mathcal{H}om(\mathcal{E}, \mathcal{F}) = 0$ by stability. Now the natural map $\mathcal{O} \to \mathcal{H}om_q(\mathcal{V}, \mathcal{V})$ is an isomorphism (on fibers it is $k \cong \mathrm{Hom}(\mathcal{E}, \mathcal{E})$, an isomorphism since $\mathcal{E}$ is stable), so we would like to show that $\mathcal{E}xt^1_q(\mathcal{V}, q^*\mathcal{O}(-1))$ is also $\mathcal{O}$. We have

$$\begin{aligned}
\mathcal{E}xt^1_q(\mathcal{V}, q^*\mathcal{O}(-1)) &= \mathcal{H}^1(Rq_* R\mathcal{H}om(\mathcal{V}, q^*\mathcal{O}(-1))) \\
&= \mathcal{H}^1(R\mathcal{H}om(Rq_*\mathcal{V}[2], \mathcal{O}(-1))) \quad \text{(by Grothendieck duality)} \\
&= \mathcal{E}xt^1(Rq_*\mathcal{V}[2], \mathcal{O}(-1))
\end{aligned}$$

We compute this via the spectral sequence

$$E_2^{i,j} = \mathcal{E}xt^i(R^{-j}q_*\mathcal{V}[2], \mathcal{O}(-1)) \implies \mathcal{E}xt^{i+j}(Rq_*\mathcal{V}[2], \mathcal{O}(-1)]).$$

Since all the fibers $\mathcal{E}$ of $\mathcal{V}$ have the same cohomology, the $R^{-j}q_*\mathcal{V}[2]$ are locally free by [13, Corollary III.12.9], so the left-hand side above vanishes unless $i = 0$. Thus

$$\mathcal{E}xt^1(Rq_*\mathcal{V}[2], \mathcal{O}(-1)) = \mathcal{H}om(R^{-1}q_*\mathcal{V}[2], \mathcal{O}(-1)) = \mathcal{H}om(R^1q_*\mathcal{V}, \mathcal{O}(-1)).$$

Since $H^1(\mathcal{F}) = H^2(\mathcal{F}) = 0$, the long exact sequence obtained by pushing forward (4.4.12) shows

$$R^1 q_* \mathcal{V} = R^2 q_* q^* \mathcal{O}(-1) = R^2 q_* \mathcal{O} \otimes \mathcal{O}(-1).$$

By flat base change, $R^2 q_* \mathcal{O} = \mathcal{O}$, so we have shown that

$$\mathcal{E}xt^1_q(\mathcal{V}, q^*\mathcal{O}(-1)) = \mathcal{H}om(\mathcal{O}(-1), \mathcal{O}(-1)) = \mathcal{O},$$

as desired.

Before continuing to check exactness of (4.4.13), we point out that the strategy above similarly computes its last term to be

$$\mathcal{E}xt^2_q(\mathcal{V}, q^*\mathcal{O}(-1)) = \mathcal{H}om(R^0 q_* \mathcal{V}, \mathcal{O}(-1)).$$

Applying q_* to (4.4.12) yields an exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^{\chi+2} \to R^0 q_* \mathcal{V} \to 0.$$

Now $R^0 q_* \mathcal{V}$ is a vector bundle by [13, Corollary III.12.9]. Since the map from $\mathcal{O}^{\chi+2}$ is surjective, it has no negative summands, and so by rank and degree must be $\mathcal{O}^{\chi} \oplus \mathcal{O}(1)$, yielding

$$\mathcal{E}xt^2_q(\mathcal{V}, q^*\mathcal{O}(-1)) = \mathcal{O}(-1)^{\chi} \oplus \mathcal{O}(-2). \qquad (4.4.15)$$

Returning to the proof at hand, we observe that exactness of (4.4.13) in the last place is similar: the next segment of the long exact sequence is

$$\mathcal{E}xt^2_q(\mathcal{V}, p^*\mathcal{F}) \xrightarrow{\sim} \mathcal{E}xt^2_q(\mathcal{V}, \mathcal{V}) \to \mathcal{E}xt^3_q(\mathcal{V}, q^*\mathcal{O}(-1)) = 0. \qquad (4.4.16)$$

The vanishing is simply by dimension, and since the relative $\mathcal{E}xt$ sheaves are vector bundles by [13, Corollary III.12.9], the isomorphism here is Grothendieck dual to that in (4.4.14). □

Proof of Proposition 4.4.2. By Lemma 4.4.5, (4.4.13) is exact; we now compute its first term via the sequence

$$0 = \mathcal{H}om_q(\mathcal{V}, p^*\mathcal{F}) \to \mathcal{H}om_q(p^*\mathcal{F}, p^*\mathcal{F}) \to \mathcal{H}om_q(q^*\mathcal{O}(-1), p^*\mathcal{F})$$

$$\to \mathcal{E}xt^1_q(\mathcal{V}, p^*\mathcal{F}) \to \mathcal{E}xt^1_q(p^*\mathcal{F}, p^*\mathcal{F}) \to \mathcal{E}xt^1_q(q^*\mathcal{O}(-1), p^*\mathcal{F}) = 0 \qquad (4.4.17)$$

Vanishing of the first term we have already described. We have

$$Rq_* R\mathcal{H}om(q^*\mathcal{O}(-1), p^*\mathcal{F}) = R\mathcal{H}om(\mathcal{O}(-1), Rq_*p^*\mathcal{F}),$$

and by flat base change, $Rq_*p^*\mathcal{F} = R^0q_*p^*\mathcal{F} = \mathcal{O} \otimes H^0(\mathcal{F}) = \mathcal{O}^{\chi+2}$. So we find that $\mathcal{H}om_q(q^*\mathcal{O}(-1), p^*\mathcal{F}) = \mathcal{O}(1)^{\chi+2}$, and $\mathcal{E}xt^1_q(\mathcal{O}(0,1), p^*\mathcal{F}) = 0$.

The remaining two terms of (4.4.17) are trivial bundles: we have

$$\mathcal{E}xt^i_q(p^*\mathcal{F}, p^*\mathcal{F}) = \mathcal{H}^i(Rq_*R\mathcal{H}om(p^*\mathcal{F}, p^*\mathcal{F})) = \mathcal{H}^i(Rq_*p^*R\mathcal{H}om(\mathcal{F}, \mathcal{F})),$$

which by flat base change is

$$\mathcal{H}^i(\mathcal{O}_{\mathbb{P}^1} \otimes R\mathrm{Hom}(\mathcal{F},\mathcal{F})) = \mathcal{O}_{\mathbb{P}^1} \otimes \mathrm{Ext}^i(\mathcal{F},\mathcal{F}).$$

So $\mathcal{H}om_q(p^*\mathcal{F}, p^*\mathcal{F}) = \mathcal{O}$ since $\mathcal{F}$ is stable, and we say $\mathcal{E}xt^1_q(p^*\mathcal{F}, p^*\mathcal{F}) = \mathcal{O}^n$.

With these identifications, (4.4.17) becomes

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\chi+2} \to \mathcal{E}xt^1_q(\mathcal{V}, p^*\mathcal{F}) \to \mathcal{O}^n \to 0. \tag{4.4.18}$$

Now $\mathcal{E}xt^1_q(\mathcal{V}, p^*\mathcal{F})$ is a vector bundle by [13, Corollary III.12.9], and thus so is

$$\ker(\mathcal{E}xt^1_q(\mathcal{V}, p^*\mathcal{F}) \to \mathcal{O}^n) = \mathrm{coker}(\mathcal{O} \to \mathcal{O}(1)^{\chi+2}).$$

So this kernel is $\mathcal{O}(1)^\chi \oplus \mathcal{O}(2)$, and since there are no extensions of $\mathcal{O}^n$ by this bundle, we must have

$$\mathcal{E}xt^1_q(\mathcal{V}, p^*\mathcal{F}) = \mathcal{O}^n \oplus \mathcal{O}(1)^\chi \oplus \mathcal{O}(2).$$

Substituting in (4.4.13), we have an exact sequence

$$0 \to \mathcal{O}^n \oplus \mathcal{O}(1)^\chi \oplus \mathcal{O}(2) \to \mathcal{E}xt^1_q(\mathcal{V}, \mathcal{V}) \to \mathcal{O}(-1)^\chi \oplus \mathcal{O}(-2) \to 0 \tag{4.4.19}$$

We see there are no extensions of the last term by the first, and so $\mathcal{E}xt^1_q(\mathcal{V}, \mathcal{V})$ is their direct sum, completing the proof. □

We are now in a position to prove Proposition 4.4.1.

Proof of Proposition 4.4.1. We claim there is a commutative diagram

$$\begin{array}{ccc}
H^1(W,(f\times g)^*\Omega_{\mathcal{M}}) & \xrightarrow{\ res\ } & H^1(\mathbb{P}^1,(f\times g)^*\Omega_{\mathcal{M}}|_{\mathbb{P}^1}) \\
\big\downarrow \wr & & \big\downarrow \wr \\
H^1(W,(f\times g)^*T_{\mathcal{M}}) & \xrightarrow{\ res^*\ } & H^1(\mathbb{P}^1,(f\times g)^*T_{\mathcal{M}}|_{\mathbb{P}^1}) \\
\big\downarrow \varphi & & \big\downarrow \wr \\
H^1(W,E_{\mathcal{M}}) & \xrightarrow{\ \psi\ } & H^1(\mathbb{P}^1,E_{\mathcal{M}}|_{\mathbb{P}^1}) \\
\big\downarrow \wr & & \big\downarrow \wr \\
H^0(Z,\mathcal{O}_Z) & \xrightarrow{\ \sim\ } & H^0(z,\mathcal{O}_z)
\end{array}$$

where res is the restriction of $(1,1)$-forms and φ is induced by the excess map.

The vertical isomorphisms in the top square simply come from the symplectic form $\Omega_{\mathcal{M}} \xrightarrow{\sim} T_{\mathcal{M}}$ on $\mathcal{M}$. The vertical isomorphisms in the bottom square come from the Leray spectral sequence: by Lemma 4.2.2, we know that $R^0\pi_*E_{\mathcal{M}} = 0$ and $R^1\pi_*E_{\mathcal{M}} = \mathcal{O}_Z$, so the spectral sequence gives $H^1(W,E_{\mathcal{M}}) = H^0(Z,R^1\pi_*E_{\mathcal{M}}) = H^0(Z,\mathcal{O}_Z)$ (and similarly on $\mathbb{P}^1$).

Finally, we consider the middle-right vertical map, induced by the restriction of the excess map. By the identifications in Lemma 4.2.2 and Proposition 4.4.2 this is, for a general fiber, a map

$$\mathcal{O}(-2)\oplus\mathcal{O}(-1)^{\chi}\oplus\mathcal{O}^n\oplus\mathcal{O}(1)^{\chi}\oplus\mathcal{O}(2)\to\mathcal{O}(-2).$$

Moreover, this map of sheaves is surjective since the excess map is, so it must be an isomorphism on the $\mathcal{O}(-2)$ summand, and therefore induces an isomorphism on H^1.

Now res (and thus res^*) is non-zero since, for example, the restriction of an ample class is positive, so the diagram shows that ψ is non-zero for a general fiber.

In other words, there exists a deformation $\widetilde{\mathcal{M}}$ so that the restriction of (4.4.11) to a general fiber is the nonsplit extension

$$0 \to \mathcal{O}(-2) \to \mathcal{O}(-1)^2 \to \mathcal{O} \to 0.$$

This is already enough to see that (4.4.11) is not globally split, but we point out that we have shown more. We have $\mathrm{Ext}^1(\mathcal{O}, E_{\mathcal{M}}) = H^1(E_{\mathcal{M}}) = H^0(R^1\pi_* E_{\mathcal{M}})$ by the Leray spectral sequence, and since $R^1\pi_* E_{\mathcal{M}} = \mathcal{O}_Z$, its sections are constant, so for such a deformation (4.4.11) is nonsplit on *all* fibers. □

For each of our moduli spaces, we choose such a deformation. Then we have

Corollary 4.4.6 (Condition (v))**.** *We have*

$$\mathcal{H}^*(i_{23*}E(\lambda+1) * i_{12*}E(\lambda-1)) \cong E^{(2)}(\lambda)[-1] \oplus E^{(2)}(\lambda)[2],$$

where i_{12} and i_{23} are the closed immersions

$$i_{12} : \mathcal{M}_{\lambda-2} \times \mathcal{M}_{\lambda} \to \mathcal{M}_{\lambda-2} \times \widetilde{\mathcal{M}}_{\lambda}$$

$$i_{23} : \mathcal{M}_{\lambda} \times \mathcal{M}_{\lambda+2} \to \widetilde{\mathcal{M}}_{\lambda} \times \mathcal{M}_{\lambda+2}.$$

Proof. Similar to the proof of Corollary 4.2.4, we apply Proposition 4.2.1 and find that $i_{23*}E(\lambda+1) * i_{12*}E(\lambda-1)$ is the pushforward to the product of a complex on W whose $(-k)$th cohomology sheaf is $\widetilde{g}^* L_X \otimes \widetilde{f}^* L_Y \otimes \bigwedge^k E^*_{\widetilde{\mathcal{M}}}$ on W, or equivalently (by construction of the line bundles)

$$\pi^* L_Z \otimes \omega_\pi \otimes \bigwedge^k E^*_{\widetilde{\mathcal{M}}}. \tag{4.4.20}$$

Once again we will push these forward along π and find that the spectral sequence comparing the cohomology of the pushforward and the pushforward of the cohomology degenerates. Note, however, that we have $E_{\widetilde{\mathcal{M}}}$ and not $E_{\mathcal{M}}$, so this does not simplify as in Corollary 4.2.4.

By Grothendieck duality, applying $R\pi_*$ to (4.4.20) gives $L_Z \otimes \left(R\pi_* \bigwedge^k E_{\widetilde{\mathcal{M}}}\right)^* [-1]$, and we simply compute this for $k = 0, 1, 2$. Since π is a $\mathbb{P}^1$ bundle, $R\pi_*\mathcal{O}_W = \mathcal{O}_Z$. Since $\bigwedge^2 E_{\widetilde{\mathcal{M}}} = \det E_{\widetilde{\mathcal{M}}} = \omega_\pi$ by taking determinants in (4.4.11), its pushforward is $\mathcal{O}_Z[-1]$. Finally, since $\bigwedge^1 E_{\widetilde{\mathcal{M}}} = E_{\widetilde{\mathcal{M}}}$ is $\mathcal{O}_{\mathbb{P}^1}(-1)^2$ on every fiber of π, its pushforward vanishes. So indeed, the only non-zero terms on the E_2 page of the spectral sequence are $R^1\pi_*\mathcal{H}^0 = L_Z$ and $R^0\pi_*\mathcal{H}^{-2} = L_Z$, and thus we get

$$\mathcal{H}^1(i_{23*}E(\lambda+1) * i_{12*}E(\lambda-1)) = \mathcal{H}^{-2}(i_{23*}E(\lambda+1) * i_{12*}E(\lambda-1)) = L_Z = E^{(2)},$$

as desired. $\square$

APPENDIX A

LEMMA ON KOSZUL COMPLEXES

Let

$$0 \to S \to \mathcal{E} \to Q \to 0$$

be a short exact sequence of vector bundles on a Cohen-Macaulay scheme X, and let s be a section of S. Suppose that s cuts out a subscheme Z of the expected codimension, so that the Koszul complex $\operatorname{Kosz}(S, s)$ is quasi-isomorphic to $\mathcal{O}_Z$. By an abuse of notation, we again write s for the corresponding section of $\mathcal{E}$, and consider $\operatorname{Kosz}(\mathcal{E}, s)$. If the sequence of vector bundles is split, it is clear that

$$\operatorname{Kosz}(\mathcal{E}, s) = \operatorname{Kosz}(Q, 0) \otimes \operatorname{Kosz}(S, s) = \bigwedge^{*} Q^{\vee} \otimes \mathcal{O}_Z.$$

We show that even if the sequence is not split, this holds at the level of cohomology sheaves.

Lemma A.0.1. $\mathcal{H}^{-i} \operatorname{Kosz}(\mathcal{E}, s) = \bigwedge^{i} Q^{\vee} \otimes \mathcal{O}_Z$.

Proof. Each term $\bigwedge^{k} \mathcal{E}^{\vee}$ of $\operatorname{Kosz}(\mathcal{E}, s)$ has a filtration

$$0 = F^k_{k+1} \hookrightarrow F^k_k \hookrightarrow \cdots \hookrightarrow F^k_1 \hookrightarrow F^k_0 = \bigwedge^{k} \mathcal{E}^{\vee}$$

with subquotients $F^k_i / F^k_{i+1} \cong \bigwedge^{i} Q^{\vee} \otimes \bigwedge^{k-i} S^{\vee}$ (see for example [13, Ex. II.5.16]). We assemble these into a filtration

$$0 = F_{n+1} \hookrightarrow F_n \hookrightarrow \cdots \hookrightarrow F_1 \hookrightarrow F_0 = \operatorname{Kosz}(\mathcal{E}, s)$$

with $\mathrm{cone}(F_{k+1} \to F_k) = \bigwedge^k Q^\vee \otimes \mathrm{Kosz}(\mathcal{E}, s)[k]$.

We now claim that $\mathcal{H}^{-i}(F_k) = \bigwedge^i Q^\vee \otimes \mathrm{Kosz}(\mathcal{E}, s)$ for all $i \geq k$. We see immediately that this holds for $F_n \cong \bigwedge^n Q^\vee \otimes \mathrm{Kosz}(\mathcal{E}, s)[n]$. Now if the claim holds for F_{k+1}, we consider the long exact sequence of cohomology sheaves coming from the triangle

$$F_{k+1} \to F_k \to \bigwedge^k Q^\vee \otimes \mathrm{Kosz}(\mathcal{E}, s)[k].$$

We see that $\mathcal{H}^{-i}(F_k) \cong \mathcal{H}^{-i}(F_{k+1})$ for all $i \neq k$ (since the cone has cohomology only in degree $-k$). But $\mathcal{H}^{-k}(F_{k+1}) = 0$, so we also have $\mathcal{H}^{-k}(F_k) \cong \bigwedge^k Q^\vee \otimes \mathrm{Kosz}(\mathcal{E}, s)$, proving the claim. In particular, $F_0 = \mathrm{Kosz}(\mathcal{E}, s)$ has the desired cohomology sheaves. □

APPENDIX B

A COHEN-MACAULAY SWINDLE

To prove Proposition 4.3.2, we prove a slightly stronger statement:

Proposition B.0.1. *With the notation of Proposition 4.3.2, if $\mathcal{F}$ has homological dimension less than or equal to* $\operatorname{codim}_X(Z) - 2$, *then*

$$\operatorname{Hom}(\mathcal{E}, \mathcal{F}) = \operatorname{Hom}(\mathcal{E}|_{X \setminus Z}, \mathcal{F}|_{X \setminus Z}).$$

Proof. Replace $\mathcal{E}$ and $\mathcal{F}$ by complexes

$$\cdots \to \mathcal{E}^{-2} \to \mathcal{E}^{-1} \to \mathcal{E}^{0}$$

and

$$\mathcal{F}^{-n} \to \cdots \to \mathcal{F}^{-1} \to \mathcal{F}^{0}$$

of vector bundles, where $n \leq \operatorname{codim}_X(Z) - 2$. Then $R\mathcal{H}om(\mathcal{E}, \mathcal{F})$ can be represented by the complex of vector bundles whose pth term is $\bigoplus_{j-i=p} \mathcal{H}om(\mathcal{E}^i, \mathcal{F}^j)$. In particular, this vanishes for $p < -n$.

We have ([14, Remark 2.67]) a spectral sequence

$$E_1^{p,q} = H^q \left(\bigoplus_{j-i=p} \mathcal{H}om(\mathcal{E}^i, \mathcal{F}^j) \right) \implies \operatorname{Ext}^{p+q}(\mathcal{E}, \mathcal{F}),$$

from which $\operatorname{Hom}(\mathcal{E}, \mathcal{F})$ is computed along the $p + q = 0$ diagonal of the infinity page. Note that this diagonal is determined[1] by the entries $E_1^{p,q}$ where $q \leq n$; the

[1] More technically, one can repeat the proof of [22, Theorem 5.2.12].

entries with $q > n$ that could have maps to this diagonal have $p < -n$ and so are zero.

On the other hand, we can restrict the resolutions of $\mathcal{E}$ and $\mathcal{F}$ to $X \setminus Z$, and by the same argument, find that $\mathrm{Hom}(\mathcal{E}|_{X\setminus Z}, \mathcal{F}|_{X\setminus Z})$ is determined by the entries

$$H^q\left(\bigoplus_{j-i=p} \mathcal{H}om(\mathcal{E}^i|_{X\setminus Z}, \mathcal{F}^j|_{X\setminus Z})\right) = H^q\left(\left(\bigoplus_{j-i=p} \mathcal{H}om(\mathcal{E}^i, \mathcal{F}^i)\right)\Bigg|_{X\setminus Z}\right)$$

where $q \leq n$. Then the following lemma completes the proof. □

Lemma B.0.2. *Let Z and X be as in Proposition 4.3.2. Then for any vector bundle V and any $q \leq \mathrm{codim}_X(Z) - 2$, the natural map $H^q(V) \to H^q(V|_{X\setminus Z})$ is an isomorphism.*

Proof. Let $i : X \setminus Z \hookrightarrow X$ be the inclusion. By [12, Corollary 1.9], there is an exact sequence

$$\cdots \to H^q_Z(V) \to H^q(V) \to H^q(i_* i^* V) \to H^{q+1}_Z(V) \to \cdots$$

But by [12, Theorem 3.8], $\underline{H}^k_Z(V) = 0$ for $k < \mathrm{depth}_Z(V)$, which by the Cohen-Macaulay assumption is $\mathrm{codim}_X(Z)$. In particular, the spectral sequence $H^p(\underline{H}^q_Z(V)) \implies H^{p+q}_Z(V)$ shows that $H^q_Z(V) = H^{q+1}_Z(V) = 0$, and we are done. □

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