Information and Interaction:

On the Specialization of General System Theory

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In the natural sciences, there is a prevalent structure: a structure that involves things that interact with each other through time. In classical physics, there is a theory that describes this structure and that is Newtonian mechanics. In quantum physics, the theory that describes interaction between things is quantum mechanics. The theory that describes interaction between organisms is biology; the theory that describes interaction between prices of competing products in a market is economics (Von Bertalanffy 1969). Therefore, in this thesis, we ask the question:

> If the natural sciences are, abstractly speaking, the study of attaching mathematical models to "systems that involve interaction," how do we describe these systems mathematically and in full generality?

In our search for an answer, we argue for the acceptance of several basic assumptions about the nature of "interacting systems," and that the implied similarities to (*para*-)category theory are indicative that we base our framework for interacting systems in the language of (*para*-)category theory. We see at the end of the thesis that these systems can be reduced to a paracategory equipped with a set of faithful outgoing functors (with generally distinct codomains).

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1 Semantics for Information and Interaction

It had been established by Ludwig von Bertelanffy in the late 1960's that the natural sciences lack a unified semantics for discussing "dynamical systems" (Wallis 2020). Dynamical systems appear everywhere – economics, computer science, biology, and of course physics – however, despite this mathematically meaningful connection (granted that it is a rather loose one), such fields generally are different in overall approach (Von Bertalanffy 1969). Bertalanffy's aim as a systems theorist was to develop a more general approach to systems. What was later dubbed general system theory is just that: an approach. One can utilize general system theory to rapidly develop a new natural science (or at least, that is one goal), but ultimately it lacks the substantial theory one might desire for the purposes of uncovering deeper mathematical reasons why general systems theory is as effective as it is. General system theory remains very abstract – almost to the point of philosophy – but one of its core paradigms is that a system is composed of a *web* of relationships between *things*. The goal of this thesis is to specialize this paradigm to a more mathematically precise set of axioms, specifically in the context where we interpret "relationships" as "interactions" and "things" as "physical objects."

Systems in which agents interact are arguably equally prevalent as *webs of things between which exist relationships*, as it ultimately reduces to the meaning of the words used therein. In this sense, one might see what follows as a complete mathematization of general system theory, but I would like to hold onto the idea that we are specifically looking at systems where there are things that interact. Such systems should be easy to conjure into one's imagination. In biology, cells interact with other cells and animals interact with other animals; in physics, particles interact with other particles, a balls roll down inclined planes. I see it as a (possibly far-fetched) dream that we may one day be able to study such processes that involve "interaction" (which we will henceforth call *interactive systems*), and deduce rules by which all interactive systems obey.

Current scientific language is inadequate in discussing such systems in full generality and this is partly due to the fact that "interaction", as an abstract concept, is not rigidly assigned meaning; instead, an interaction between two agents in a physical or theoretical system is inferred from the context, as one can see as stated above in physics and biology (Von Bertalanffy 1969). For example, we could very well define an interaction as follows:

> Given an interactive system S, and two agents X and Y belonging to S, an *interaction* between X and Y consists of evidence that one of X or Y has influenced the state of the other.

Again, this definition is too imprecise to be called mathematical, but it is a somewhat consistent catch-all for interactions that we may want to study in the natural sciences. Let us consider a couple examples:

i) Let S be the system consisting of two pool balls X and Y on a pool table. Let the states of X and Y be given by their positions and velocities. Suppose we have information that, at time 0 they are moving towards each other (on a course for collision), and at time 1, they are heading away from each other. With this information, we can deduce (using our knowledge of Newtonian mechanics) that there is some intermediary time during which they interacted.

ii) In a slightly sillier example, let S be the system where X is my cousin Alan, and Y is an apple pie I just bought. Let the state of X be given by its position and its apparent level of hunger; let the state of Y be given by its position and the volume of pie still on the tray. If we have information that X has decreased in apparent hunger, Y has decreased in size, and both agents are in close proximity between between times 0 and 1, this is evidence that Alan has consumed some of my pie. *Note however, that this is only evidence that Alan has interacted with my pie – it is not proof of interaction. The same holds for the above example. However, in the natural sciences, we only ever have evidence to work with and proof is a luxury that can only ever be reasonably expected in the context of mathematics. This lends to the perspective that an "interaction" truly does involve some level of statistical, non-Boolean, information.*

Upon closer inspection, however, we can see that this definition is unfortunately slightly circular: in order for us to obtain information pertaining to the states of X and Y, we must have interacted with them in some way. This means that an interaction between X and Y is really only presently defined *in terms* of interactions between X and Z, and Y and Z, with Z being the observer of the interactive system. This raises a number of questions.

How do we resolve this apparent circularity?

How can we formally define an interactive system in such a way that it has the flexibility to realize the observer as an agent that is operating and interacting within?

What can we do to give information and interaction rigorous but morally correct definitions in the very abstract scope of all possible interactive systems? There are many different contexts we can consider, be it quantum, biological, or pie-related – we want information and interaction to have a uniform meaning across all sites of interest.

This motivates a central goal of this paper: to find a consistent meaning, or *semantics*, for information and interaction across all interactive systems. And, of course, we would also like to precisely state what an interactive system *is*. Many disparate fields find some problems isomorphic, but our mathematical toolbox is not yet developed enough to make these relationships precise, to prove them, or to even understand the underlying necessary and sufficient conditions for these isomorphisms (Bertalanffy 37). To delineate such isomorphsims, it seems obligatory to find a context under which all interactive systems can be adequately described. Before this, however, it is necessary to discuss the criteria we will use to define interactive systems and the paradigms we will use to view them.

1.1 Criteria and Paradigms for Interactive Systems

First we will list a collection of rules that interactive systems should follow. We are not yet at the level of precision where we can state these rules as axioms, so the following should be seen as a sketch of what we might reasonably assume an interactive system looks like (though, one that is by no means unique). There are two kinds of rules that I would like to consider. *Criteria* are the rules that are arguably essential for interactive systems, while *Paradigms* are the rules that may confer, if implemented as a mathematical underpinning for interactive systems, a great amount of flexibility, insight, or mathematical development. We will begin with criteria.

Criterium 1. Objects An interactive system S has the data of a collection of objects.

Criterium 2. Interactions For any pair of objects X and Y in an interactive system S, S has the data of the collection of possible interactions between those objects.

Paradigm 1. Objects $\cong$ Observers, Measurements $\hookrightarrow$ Interactions

Loosely speaking, this paradigm states that there are two nearly equivalent ways to view an interactive system. We can view such a system as a collection of objects interacting with each other, and we can also view such a system as a collection of observers measuring each other's states. The paradigm states that, between these two outlooks, every object can be naturally regarded as an observer and every observer can be naturally regarded as an object. On the other hand, the paradigm also supposes that there is a way to regard a measurement as an interaction (and moreover that distinct measurements represent distinct interactions, in some way), though we need not assume that every interaction

This paradigm is central to quantum mechanics, insofar as the human measuring the state of a subatomic particle. The idea captured by this paradigm is that, while the electron, for instance, has changed state as a result of the human having measured it, the human has also changed in state – and, in the latter case, there is a "canonical" way to associate a physical change induced on the human by way of the interaction (including, but not limited to, the physical changes to the state of their brain upon measurement), and their conscious perception that the given measurement has been *made*. **Paradigm 1** extends this association to the electron, imparting it with its own "perspective."

The supposition that the electron is also making some sort of "measurement" during this process is a matter of philosophical consistency, and has been the subject of some debate. For the purposes of describing interactive systems, we do not assume consciousness to have any inherent, special properties. (Taking consciousness to be defined loosely as the process consisting of the statements or *propositions* we are able to make, through time) We instead assume that consciousness is semantically identical to the varying state of any other object in the given interactive system. The merits of this assumption are threefold. First, as previously mentioned, it offers consistency in that consciousness can be represented the same way as any other state of any other object; there is nothing special about (human) consciousness aside from the powerful computational and deductive traits it exhibits. Second, this last part gives us a simplified semantics for consciousness: *consciousness* is isomorphic to the process given by the state of some object. Third, and as we will see later, it grants us the power to model such an abstract process as a functor $F : (\mathbb{R}, \leq) \to \mathscr{C}$ with $(\mathbb{R}, \leq)$ the category of real numbers as a partially ordered set, and $\mathscr{C}$ some subcategory of C*-algebras.

In addition to the above, a semantic implication from measurement to interaction leads us to view some interactions between objects X and Y as transmissions of information about their states to one another, for that is what it means to measure an object. To illustrate, every teenage drama has a line to the effect of, "But Jenny said that Dave said that Carlos said that you have a car." The speaker of such a quote is listing a series of interactions, each person conveying a trace of information about the state of their mind in that they are expressing their perception. By the end of this list of interactions, the speaker has some level of insight into Carlos' mind – they even have insight into the listener's status as the owner of a car. This motivates the following paradigm.

Paradigm 2. Compositionality If X interacts with Y and then Y interacts with Z, this is the same as an interaction between X and Z.

This paradigm immediately implies that interaction (and therefore measurement) is a directed relation. Alice can hear Bob screaming from a mile away, but Bob may not hear Alice. This suggests that category theory may have a fundamental role in modeling the behaviour of interactions. In fact, category theory is very closely related to the natural sciences, paradigmatically. Here is a short summary of each practice, for comparison.

- In category theory, you pick a collection of objects you care about and study them by instead looking at the morphisms between them.
- In a given natural science, you pick a collection of (physical) objects you care about and study them by measuring them and testing how they interact.

Remark. Note that "compositionality of interactions" does not require that *every* interaction from X to Y can be composed with an interaction from Y to Z. We require that, viewed as morphisms, $X \to Y$ *precedes* $Y \to Z$ from the perspective of Y (whatever that means), in order for them to be composed. This is not in line with category theory – in category theory, every such pair of morphisms should be composable. This gives us reason to believe we will actually end up working in *para*category theory where composistion is a partial function, rather than an actual function. For now, however, we will speak in terms of category theory, as paracategory theory shares many theorems common to category theory (Hermida and Mateus 2003).

This motivates a crucial paradigm.

Paradigm 3. **Natural Sciences** $\hookrightarrow$ **Category Theory** The above similarity is more than just a linguistic coincidence.

The similarity given above is even more alluring, given the "category" structure suggested by **Paradigm 2**. Even so, the natural sciences are different from category theory in that a scientist is never given the entire category-in-question to work with. The scientist only has a very small slice of information about a much larger category. **Paradigm 3** therefore suggests that the natural sciences can be viewed as a specific type of category theory: category theory enriched with some extra structure.

1.2 Representation of Information

Here we will begin modeling interactions between agents in an interactive system according to the conditions stated in the preceding section, and we will attempt to do so as constructively as possible. Recall that **Paradigm 1** stipulates that one valid perspective on the notion of an interactive system is that of a collection of observers measuring each other – transmitting information about their states between one another. The first question we must ask is: what, precisely are these observers measuring? If we remain steadfast that all measurements made by all observers should have a common global interpretation, the classical answer would be: all observers are measuring the values of *random variables* at discrete points in time.

Definition 1.2.1. Recall that a *σ-algebra* on a set Ω is a subset $\mathcal{A} \subset \mathcal{P}(\Omega)$ such that $\Omega \in \mathcal{A}$, $a \in \mathcal{A}$ implies that $a^c \in \mathcal{A}$, and for any countable collection of sets $a_i \in \mathcal{A}$, we have $\bigcup a_i \in \mathcal{A}$. A pair $(\Omega, \mathcal{A})$ with $\mathcal{A}$ a σ-algebra on Ω, is called a *measurable space*. A *measurable function* $f : (\Omega, \mathcal{A}) \to (\Psi, \mathcal{B})$ is a function $f : \Omega \to \Psi$ for which $f^{-1}|_{\mathcal{B}} : \mathcal{B} \to \mathcal{A}$ is well-defined.

Before we give the formal definition of a random variable, it is important to recognize the meaning that the probabilist can ascribe to the σ-algebra. If Ω is the set of possible states a process can result in, a σ-algebra $\mathcal{A}$ represents a collection of propositions about the result of that process – in particular, an element $a \in \mathcal{A}$ represents the proposition that the process wound up somewhere in a. This gives meaning to the intersection and union as logical conjunction and disjunction, respectively.

Definition 1.2.2. Recall that a *measure space* is a triple $(\Omega, \mathcal{A}, \mu)$ where Ω is a set, $\mathcal{A}$ is a σ-algebra on Ω, and μ is a *measure* on $\mathcal{A}$ – a function $\mu : \mathcal{A} \to \mathbb{R} \cup \{\infty\}$ such that the following axioms hold:

i) $\mu(a) \geq 0$ for all $a \in \mathcal{A}$.

ii) $\mu(\varnothing) = 0$.

iii) $\mu(\coprod_{i=1}^{\infty} a_i) = \Sigma_{i=1}^{\infty} \mu(a_i)$ for any countable collection of pairwise disjoint sets a_i.

Note that if $(\Omega, \mathcal{A}, \mu)$ is a measure space, a measurable function $f : (\Omega, \mathcal{A}) \to (\Psi, \mathcal{B})$ induces a *pushforward measure* $f_*(\mu)$ on $\mathcal{B}$ given by $f_*(\mu) = \mu \circ f^{-1}|_{\mathcal{B}}$. Viewing a σ-algebra $\mathcal{A}$ on a set Ω as a collection of propositions about the outcome of some process, a measure μ on $\mathcal{A}$ can be thought of as a function assigning "probabilistic weight" to each proposition $a \in \mathcal{A}$. A measure μ for which $\mu(\Omega) = 1$ turns $(\Omega, \mathcal{A})$ into a special type of measure space called a *probability space*. A *random variable* X is a measurable function from a measure space $(\Omega, \mathcal{A})$ to $(\mathbb{R}, \mathcal{B})$ where $\mathcal{B}$ is the *Borel σ-algebra* on $\mathbb{R}$ (the smallest σ-algebra on $\mathbb{R}$ containing the standard topology). Given an element $a \in \mathcal{A}$, the indicator random variable of that subset is defined to be:

$$I_a(x) = \begin{cases} 1 & \text{for } x \in a \\ 0 & \text{for } x \notin a \end{cases}$$

It is convention to assume that the domain of a random variable is a probability space with a chosen probability measure, as for our purposes, the specific probability measure is generally unimportant. A random variable describes which values of $\mathbb{R}$ each element of Ω is to be measured as. As noted above, a random variable induces a pushforward measure on the Borel σ-algebra, assigning a real number to each *Borel set* B, indicating the probability that a measurement will return a value in B. Additionally, the set of random variables on a given measure space form a unital $\mathbb{R}$-algebra, given by point-wise addition and multiplication.

Given a measure space $(\Omega, \mathcal{A}, \mu)$, if an observer makes multiple measurements of a random variable X, they may ask what they can, on average, expect that random variable to be measured as – this is the expected value of X, denoted $E[X]$. In general, $E[X]$ depends on the measure μ, according to the *Lebesgue integral*. Viewing the $\mathbb{R}$-algebra of random variables on $(\Omega, \mathcal{A}, \mu)$, $\mathscr{A}$, as a partial order where $X \leq Y$ when $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$, one can say that $E[X]$ is the unique linear functional that sends I_a to $\mu(a)$ for all $a \in \mathcal{A}$, and preserves infima and suprema of the partial orders. This turns out to be well-defined if we assume $(\Omega, \mathcal{A}, \mu)$ to be a probability space, but making this precise is beyond the scope of this paper.

As noted earlier, a σ-algebra $\mathcal{A}$ can be interpreted as a simple propositional logic with propositions given by elements $a \in \mathcal{A}$, conjunction given by $\cap$, and disjunction given by $\cup$. This structure is called a *Boolean algebra*.

Definition 1.2.3. A Boolean algebra is a six-tuple $(\mathcal{A}, 0, 1, \vee, \wedge, \neg)$ where $\mathcal{A}$ is a set, $0, 1 \in \mathcal{A}$, and $\vee, \wedge : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and $\neg\mathcal{A} \to \mathcal{A}$ are functions satisfying the following for all elements $a, b, c \in \mathcal{A}$:

i) *Associativity.* $a \vee (b \vee c) = (a \vee b) \vee c$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$.

ii) *Commutativity.* $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$.

iii) *Absorption.* $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$.

iv) *Identity.* $a \vee 0 = a$ and $a \wedge 1 = a$.

v) *Distributivity.* $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

vi) *Complements.* $a \vee \neg a = 1$ and $a \wedge \neg a = 0$.

These axioms are all familiar from set theory, and it is evident that a Boolean algebra retains the essential algebraic aspects of set theory and logic. The fact that we are seeing this structure arise should be a sign that we are getting closer to modeling the information an object has about its environment through time. We can see already that a single Boolean algebra gives us an algebraic representation of the propositions an observer is able to conjecture about its environment at a given time. If an observer is moving through time, a slightly more complex structure is required.

Definition 1.2.4. Let $(\Omega, \mathcal{A})$ be a measure space. For a partially ordered set P, a P-*filtration* on $(\Omega, \mathcal{A})$ is a homomorphism of partial orders denoted $F_{(-)} : P \to K(\mathcal{A})$, where the latter is the partial order of sub-σ-algebras of $\mathcal{A}$.

Giving $\mathbb{R}$ a partial order from $\leq$, an $\mathbb{R}$-filtration gives a σ-algebra F_t at each time t – and observe that F_t grows as t increases. An $\mathbb{R}$-filtration is often used in probability theory to represent the information *available* to an observer during a given time. While an increasing amount of information is available for the observer to discuss and predict over time, more structure is needed for one to gain information and answer questions. There is, in fact, a more general algebraic definition for an P-filtration, which we will come back to.

Definition 1.2.5. Let $(\Omega, \mathcal{A})$ be a measure space, and let $\mathscr{A}$ be the $\mathbb{R}$-algebra of random variables. Then a P-filtration on $(\Omega, \mathcal{A})$ is a homomorphism of partial orders denoted $\mathcal{F}_{(-)} : P \to L(\mathscr{A})$, where the latter is the partial order of sub-$\mathbb{R}$-algebras of $\mathscr{A}$.

Lemma 1.2.1. Let $(\Omega, \mathcal{A})$ be a measure space, with $\mathscr{A}$ its $\mathbb{R}$-algebra of random variables. As partially ordered sets ordered by inclusion, $K(\mathcal{A})$ is a sub-partial-order of $L(\mathscr{A})$.

Proof. Let $f : K(\mathcal{A}) \to L(\mathscr{A})$ be the function taking a σ-algebra s and sending it to the $\mathbb{R}$-algebra $f(s)$ of random variables supported on (Ω, s).

First, observe that f and g both preserve order. f preserves order since, if a random variable X is measurable on a σ-algebra s, if $s \subset t$ for some other σ-algebra t, X is also measurable on t by definition.

Now observe that f is an injection. If s is strictly contained in t, with $s, t \in K(\mathcal{A})$, there is a set $x \in t \setminus s$. Then $I_x \in f(t)$, but $I_x \notin f(s)$. □

Proposition 1.2.1. The definition given in **Definition 1.2.5** is a true generalization of the definition given in **Definition 1.2.4**.

Proof. Let $F_{(-)}$ be a P-filtration according to **Definition 1.2.4**. Then let f be the inclusion provided by **Lemma 1.2.1**. Then by post-composing, we can see that $f \circ F_{(-)}$ is a P-filtration according to **Definition 1.2.5**. □

This says that we can indeed think about filtrations as "sequences" (or continua) of $\mathbb{R}$-algebras rather than "sequences" (or continua) of σ-algebras. A real reason why we might prefer **Definition 1.2.5** over **Definition 1.2.4** is that the former is arguably more clear about the kind of information that is being gained by the observer at every step in time. According to the former definition, the observer has a growing collection of random variables to aid his or her study of the system in question. The latter definition is not as explicit. This is one reason why we might prefer to view probability through the lens of algebra rather than measure theory. Another reason is provided by the following proposition.

Proposition 1.2.2. Given a σ-algebra $\mathcal{A}$ on a set Ω, the Boolean algebra $(\mathcal{A}, \varnothing, \Omega, \cup, \cap, (-)^c)$ can be fully recovered from the $\mathbb{R}$-algebra $\mathscr{A}$ of random variables on $(\Omega, \mathcal{A})$.

Proof. Given $\mathscr{A}$, let $\mathcal{B}$ be the set of idempotent random variables. Moreover, for $b_1, b_2 \in \mathcal{B}$, let $b_1 \wedge b_2 = b_1 b_2$, let $b_1 \vee b_2 = b_1 + b_2 - b_1 b_2$, and let $\neg b_1 = 1 - b_1$. We claim that $(\mathcal{A}, \varnothing, \Omega, \cup, \cap, (-)^*)$ is isomorphic to $(\mathcal{B}, 0, 1, \vee, \wedge, \neg)$ (meaning that there is a bijection $f : \mathcal{A} \to \mathcal{B}$ for which $f(\varnothing) = 0$, $f(\Omega) = 1$, and the appropriate functions are preserved). First, this makes $\mathcal{B}$ into a Boolean algebra. Since $\mathbb{R}$ is a commutative ring, $\mathscr{A}$ is also commutative since it uses point-wise multiplication. Then using commutativity of $\mathscr{A}$, we can see that $\vee$, $\wedge$, and $\neg$ are well-defined operations on $\mathcal{B}$:

i)

$$(b_1 b_2)(b_1 b_2) = b_1 b_1 b_2 b_2 = b_1 b_2$$

ii)

$$(b_1 + b_2 - b_1 b_2)(b_1 + b_2 - b_1 b_2)$$

$$= b_1 b_1 + b_2 b_1 - b_1 b_2 b_1 + b_1 b_2 + b_2 b_2 - b_1 b_2 b_2 - b_1 b_1 b_2 - b_2 b_1 b_2 + b_1 b_2 b_1 b_2$$

$$= b_1 + b_2 - b_1 b_2$$

iii)

$$(1 - b_1)(1 - b_1) = 1 - b_1 - b_1 + b_1 b_1 = 1 - b_1$$

Let f be the function taking a set $a \in \mathcal{A}$ to the indicator idempotent $I_a \in \mathcal{B}$.

We can see that for $a_1, a_2 \in \mathcal{A}$, $I_{a_1} \wedge I_{a_2} = I_{a_1} I_{a_2}$ takes the value 1 exactly on the region $a_1 \cap a_2$ (and is zero elsewhere), similarly that $I_{a_1} \vee I_{a_2} = I_{a_1} + I_{a_2} - I_{a_1} I_{a_2}$ takes the value 1 exactly on the set $a_1 \cup a_2$ (and is zero elsewhere), and finally that $\neg I_{a_1} = 1 - I_{a_1}$ takes the value 1 exactly on the region a_1^c (and is zero elsewhere).

f takes $\varnothing$ to the random variable that is constantly 0, and f takes Ω to the random variable that is constantly 1. The fact that f is a bijection comes from the fact that $\mathbb{R}$ is a domain so any idempotent random variable must only take values on 0 or 1. Then, for any idempotent random variable $b \in \mathcal{B}$, its preimage under f is the set $a = b^{-1}(1) \in \mathcal{A}$. This shows that $(\mathcal{A}, \varnothing, \Omega, \cup, \cap, (-)^*)$ is isomorphic to $(\mathcal{B}, 0, 1, \vee, \wedge, \neg)$ as algebraic structures. Hence, we have completely recovered the Boolean algebra structure induced on $\mathcal{A}$ from the algebraic structure of $\mathscr{A}$.

$\square$

In fact, given a σ-algebra $\mathcal{A}$ on some set Ω, one can recover the partial order on $\mathcal{A}$ from the Boolean algebra given by $(\mathcal{A}, \varnothing, \Omega, \cup, \cap, (-)^c)$. We can see this as follows: for all $a, b \in \mathcal{A}$, let $a \leq b$ if $a \wedge b = a$. From set theory we can see $a \leq b$ exactly when $a \subset b$. Hence, we have the following corollary.

Corollary 1.2.1. The partial order on a σ-algebra over a set Ω can also be recovered from the corresponding $\mathbb{R}$-algebra of random variables.

It follows that if $(\Omega, \mathcal{A})$ is *sober* as a topological space (that is, every irreducible element of $\mathcal{A}$ is the closure of precisely one point of Ω), $(\Omega, \mathcal{A})$ is completely determined by its $\mathbb{R}$-algebra of random variables.

This is to say that the $\mathbb{R}$-algebra of random variables for a measure space $(\Omega, \mathcal{A})$ contains much of the information provided by the original measure space – sometimes it contains just as much information. An $\mathbb{R}$-algebra of random variables gives us strictly more information than the Boolean algebra of propositions (unless, of course, the σ-algebra is the power set, in which case they offer equivalent information). We will eventually use this as motivation to move away from classical measure-theoretic probability in favor of a purely algebraic formulation.

This is an important observation because sometimes classical probability theory, outlined above, is not adequate in representing measurement of physical phenomena. In particular, classical probability theory assumes that conjunction is commutative, and that conjunction distributes over disjunction. These axioms do not, in general, hold when it comes to making measurements about quantum events. This is one reason why *quantum probability theory* looks so different from classical probability theory.

In quantum probability theory, a state space Ω is replaced by a *Hilbert space* H of states, and random variables are replaced by *observables*.

Definition 1.2.6. For our purposes, a Hilbert space is a complex vector space H equipped with a positive definite sesquilinear form $\langle \cdot, \cdot \rangle$ and is analytically complete with respect to the induced norm $v \to \sqrt{\langle v, v \rangle}$. An observable on a Hilbert space H is a linear operator $T : H \to H$ that is self-adjoint with respect to the Hermitian adjoint – that is, $T = T^*$. If T is bounded with respect to the induced norm, one can define the *absolute value* of T, denoted $|T|$ to be the unique positive semi-definite operator $|T|$ such that $|T|^2 = T^*T$. The trace of a bounded operator T, if it exists, is given by

$$\text{Tr}(T) = \Sigma_k \langle |T| e_k, e_k \rangle$$

where $(e_k)_k$ is an orthonormal basis for H (and this is well-defined, as this sum, or series in the case that H is infinite dimensional, is independent of choice of basis). For our purposes, we will assume that H is Hilbert space where τ is defined everywhere.

Like classical probability theory, Hilbert spaces and observables are constructed depending on the measurement one intends to make. However, in quantum mechanics, the state of a system is not necessarily a single element of the chosen Hilbert space. Quantum systems can also inhabit *mixed states* – superpositions of more elementary *pure states*. The assumed present state of a quantum system, mixed or pure, is encoded as a positive semi-definite operator ρ. The eigenvectors of an observable T form an orthonormal basis $(e_k)_k$, and each element of this basis represents a different measurement outcome. The probability that an observer measures the observable T in state e_k is given by Tr$(P_{e_k}\rho)$, where P_{e_k} is the self-adjoint idempotent (or projection) onto the subspace spanned by e_k. Viewing P_{e_k} as the proposition that T results in state e_k, we can see that Tr$(P_{e_k}\rho)$ is acting as the expected value of P_{e_k} being true. In fact, given any observable T, the expected value of T given that the system occupies state ρ is Tr$(T\rho)$.

In quantum probability theory, propositions take the form of subspaces of the Hilbert space H – these are propositions that the observer will measure an outcome corresponding to a state somewhere in that subspace. Much like how idempotent random variables are projections from measurable sets – and equivalent to propositions in a suitable Boolean algebra – self-adjoint idempotent observables are projections to subspaces that represent propositions about quantum events. We could very well attempt to construct a Boolean algebra for this collection of propositions given by $(\mathcal{Q}, 0, H, +, \cap, \neg)$, where:

- $\mathcal{Q}$ is the set of subspaces of H.
- 0 is the zero subspace.
- $+$ is vector space addition, the join in the partially ordered set of subspaces of H.
- $\cap$ is intersection, the meet in the partially ordered set of subspaces of H.

But there is no easy choice for $\neg$ here, since the set theoretic complement of a subspace is not a subspace, and a direct complement is not unique. In fact, even if we did resolve this problem, the above could never be a Boolean algebra. As stated before, this logic is non-distributive: it is not the case that for subspaces $U, V, W \subset H$, we have $U \cap (V+W) = (U \cap V) + (U \cap W)$. Moreover, if we wish to realize this purely algebraically (as we did with random variables) we would again see that the product of self-adjoint idempotents represents conjunction, and multiplication fails to commute when we generalize to the space of all operators.

This is evidence against the "realness" of Boolean logic. We therefore must be very careful about the assumptions we are making about the means by which information is stored and transmitted in an interacting object. But this doesn't mean that there is no hope. Quantum probability theory nevertheless sees continuous processes similarly to that which is suggested by **Definition 1.2.5**.

Definition 1.2.7. Given a Hilbert space H, a *quantum process* is an $\mathbb{R}$-action (viewing $\mathbb{R}$ as an additive abelian group) on the space of operators End(H). Generally, one would require that this action be continuous with respect to a chosen operator topology, but this is not relevant for now.

This definition is motivated by the fact that a quantum mechanical state evolving according to the Schrödinger equation can equivalently be realized as a constant state with evolving observables given by an action of $\mathbb{R}$ (Pillet 1970).

Proposition 1.2.3. A quantum process on a Hilbert space H is the same data as a functor $\mathcal{F} : (\mathbb{R}_{\geq 0}, \leq) \to (\text{End}(H), \cong)$ for which the morphism $\mathcal{F}(s \leq t)$ only depends on the difference $(t - s)$, where the latter category is a category with one object, End(H), and whose morphisms are given by $\mathbb{C}$-algebra automorphisms.

Proof. Suppose we have a quantum process on H given by a group homomorphism $\varphi : \mathbb{R} \to \text{Aut}(\text{End}(H))$. Then consider the functor $\mathcal{F} : (\mathbb{R}_{\geq 0}, \leq) \to (\text{End}(H), \cong)$ sending every element of $\mathbb{R}$ to End(H), and every morphism $s \leq t$ to the automorphism $\varphi(t - s)$. This is indeed a functor, as:

- *Identity.* $\mathcal{F}$ sends identity morphisms $t \leq t$ to $\varphi(t - t) = \varphi(0) = 1$, which is the identity morphism on End(H).

- *Correct Domains.* In order for $\mathcal{F}$ to be a functor, we need to make sure that for objects $s, t \in \mathbb{R}$ with $s \leq t$, we have $\mathcal{F}(s \leq t) : \mathcal{F}(s) \to \mathcal{F}(t)$. But this is trivial since the codomain of $\mathcal{F}$ only has one object.

- *Preservation of Composition.* Given morphisms $r \leq s$ and $s \leq t$, their composition $r \leq t$ is sent to

$$\mathcal{F}(r \leq t) = \varphi(t - r)$$

$$= \varphi((s - r) + (t - s))$$

$$= \varphi(s - r)\varphi(t - s)$$

$$= \mathcal{F}(r \leq s) \circ \mathcal{F}(s \leq t)$$

By definition, $\mathcal{F}(s \leq t)$ only depends on the difference $(t - s)$. This shows that a quantum process induces the appropriate functor. On the other hand, if we are given a functor $\mathcal{F} : (\mathbb{R}_{\geq 0}, \leq) \to (\text{End}(H), \cong)$, we would like to see that there is an induced $\mathbb{R}$-action on End(H) (and moreover, that these inducements are inverse correspondences).

Indeed, given $\mathcal{F}$, we can let $r \in \mathbb{R}$ act on $T \in \text{End}(H)$ by $r \cdot T := \mathcal{F}(0 \leq |r|)^{\text{sgn}(r)}T$, where sgn : $\mathbb{R} \to \{-1, 0, 1\}$ is the sign function. To see that this is indeed an $\mathbb{R}$-action, consider $(s + t) \cdot T = \mathcal{F}(0 \leq |s + t|)^{\text{sgn(s+t)}}T$ and without loss of generality the following two cases:

- Case 1: s and t have the same sign. Then $|s + t| = |s| + |t|$ and $0 \leq |s| \leq |s + t|$.

Since $\mathcal{F}$ preserves composition, we have

$$(s+t)\cdot T$$

$$= \mathcal{F}(0 \leq |s|)^{\text{sgn}(s+t)}\mathcal{F}(|s| \leq |s+t|)^{\text{sgn}(s+t)}T$$

$$= \mathcal{F}(0 \leq |s|)^{\text{sgn}(s)}\mathcal{F}(0 \leq |t|)^{\text{sgn}(t)}T$$

where the last equality uses that $\mathcal{F}(x \leq y)$ depends only on the difference $y - x$.

- Case 2: $s = -t$. Then we have $(s+t)\cdot T = 0 \cdot T = \mathcal{F}(0 \leq 0)^0 T = T$.
- Case 3: s is positive, t is negative, and $|t| < |s|$. Then we have $|s+t| + |t| = |s|$. Since $\mathcal{F}$ preserves composition and $0 \leq |s+t| \leq |s|$, we have

  $$\mathcal{F}(0 \leq |s|)$$

  $$= \mathcal{F}(0 \leq |s+t|)\mathcal{F}(|s+t| \leq |s|)$$

  $$= \mathcal{F}(0 \leq |s+t|)\mathcal{F}(0 \leq |t|)$$

  where the last equality uses that $\mathcal{F}(x \leq y)$ depends only on the difference $y - x$. Since each of these morphisms is an isomorphism, we get that

  $$\mathcal{F}(0 \leq |s+t|) = \mathcal{F}(0 \leq |s|)\mathcal{F}(0 \leq |t|)^{-1}$$

- Case 4: If s is negative, t is positive, and $|t| < |s|$, by definition, $\mathcal{F}(0 \leq |s+t|)$ is the inverse of the above, which is what we want.

This shows that the action we've given is associative and therefore a true action on

End(H). □

The preceding proposition shows that a quantum process is a certain special type of functor. The fact that we require $\mathcal{F}$, as a function on morphisms $s \leq t$, to depend only on the difference $(t - s)$, is a result of the symmetry conferred by have a group act on End(H), and a similar argument to the one above can show that this is also the same as a functor $(\mathbb{R}, \leq) \to (\text{End}(H), \cong)$ with a similar dependence constraint on morphisms. This is a symmetry that appears in some quantum mechanical systems, but we do not want to assume that this symmetry exists in general. If we relax this requirement on $\mathcal{F}$, it becomes a functor from $(\mathbb{R}_{\geq 0}, \leq)$ to $(\text{End}(H), \cong)$. If we further relax the codomain, we have a significantly more general definition of quantum process.

Definition 1.2.8. For a partially ordered set P, a generalized P-process on a Hilbert space H is a functor $\mathcal{F} : P \to \mathbb{C}\text{Alg}$, where the latter category is the category of $\mathbb{C}$-algebras.

In fact, we can also similarly relax the definition for P-filtration we gave above, to look like:

Definition 1.2.9. For a partially ordered set P, a generalized P-filtration on a measure space $(\Omega, \mathcal{A})$ is a functor $\mathcal{F} : P \to \mathbb{R}\text{Alg}$, where the latter category is the category of $\mathbb{R}$-algebras.

Something must be going on here – it seems that functors of the form $\mathcal{F} : P \to R\text{Alg}$, for a given ring R, make for a nice unifying formalization for truly *generic* information propagation through time. These more general definitions also relax another unwanted constraint that we saw in **Definition 1.2.5** for the P-filtration and in **Definition 1.2.7** for the quantum process: both definitions describe processes in which no information is "lost" or made "unavailable" over time. The ability to lose information over time is also a key component of the kind of process we wish to model.

Both of the preceding definitions lend to the perspective that the information available to object over time, or rather, the the data representing the interactions it can detect, looks like an algebra of *pseudo-random-variable-observable* things, evolving over time. Both quantum and classical definitions presented above are very similar algebraically, but it is not clear whether we should *think* about elements of these algebras as random variables or as quantum observables.

In fact, there is no need to decide since there is already a theory in place that essentially generalizes both classical and quantum probability theory. This is *free probability theory*. Rather than R-algebras for a well-chosen ring R, we will now consider the merits of working with $*$-*algebras*.

Definition 1.2.10. A $*$-ring is a (potentially non-commutative) ring $\mathcal{R}$ with a map $* : \mathcal{R} \to \mathcal{R}$ that is an involution and an antiautomorphism. A $*$-algebra $\mathscr{S}$ is a $*$-ring with involution that is an associative algebra over a commutative $*$-ring R with involution $'$ such that $(rx)^* = r'x^*$ for $r \in \mathcal{R}$ and $x \in \mathscr{S}$. A morphism of $\mathcal{R}$ $*$-algebras is an $\mathcal{R}$-algebra homomorphism that commutes with the equipped involutions.

Example 1.1. Any algebra over any ring is a $*$-algebra with trivial involutions on both itself and the underlying ring. Hence, it is easy to that $\mathscr{A}$, the algebra of random variables on a measurable space, is a $*$-algebra.

Example 1.2. End(H) for a Hilbert space is a $*$-algebra with the involution given by the Hermitian adjoint †, and the involution on the base ring $\mathbb{C}$ given by complex conjugation.

It worth noting that $\mathbb{C}$ is a good base ring to have in mind. Much of the work we are about to do is often done under the assumption that $\mathcal{R} = \mathbb{C}$, but to avoid potential confusion between $\mathbb{C}$ $*$-algebras and C*-algebras, we will leave the underlying ring arbitrary. C*-algebras are a similar algebraic structure, but are generally assumed to be Banach algebras (which are normed).

Definition 1.2.11. A *free probability space* is an $\mathcal{R}$ $*$-algebra $\mathscr{S}$ equipped with an $\mathcal{R}$-linear functional $\tau : \mathscr{S} \to \mathcal{R}$ that maps 1 to 1. Elements of free probability spaces will be referred to as observables.

As we've noted before, both $\mathbb{R}$-algebras of random variables and $\mathbb{C}$-algebras of observables have their own "expectation" functional. Free probability theory is the study of probability from the algebraic perspective that τ, seen as a generalized expectation functional, encodes the most important aspects of a distribution. In fact, it has been shown that for certain $*$-algebras, the expectation uniquely determines a probability measure on the measure space $(U, \mathcal{B}(U))$ with U a closed interval and $\mathcal{B}(U)$ the σ-algebra of Borel subsets of U (Tao 2010). While τ only gives the expectation for a random variable X, it also gives the expectation for X^n for any n, and therefore $P(X)$ for any polynomial with coefficients in the base ring. This is why the expectation is such a powerful tool. And it is one reason why we call the values $\tau(X), \tau(X^2), \tau(X^3), ...$ the (free) distribution of X.

But, the expectation τ actually encodes even more than this. τ encodes *dependencies* between random variables or observables.

Example 1.3. Consider two fish, Alice and Bob, competing for food (which we will assume is scarce enough for them at this time that it actually incentivizes competition). Let A be the random variable given by the mass of food acquired by Alice and B the random variable given by the mass of food acquired by Bob. If food is scarce and Alice acquires a great surplus of food, we can be reasonably certain that Bob won't be able to get much. This exhibits a "dependency" between the random variables A and B. Or as we would like to view it, this exhibits an *interaction* between A and B. In free probability theory, the distributions $\tau(A), \tau(A^2), ...$ and $\tau(B), \tau(B^2), ...$ do not yield much information about the interactions between A and B. Much more information can be gleaned from their *joint* distribution: the collection of values $\{P(X,Y)\}_{P\in\mathcal{R}\langle X,Y\rangle}$ where P ranges over all elements of the ring $\mathcal{R}\langle X,Y\rangle$ of non-commuting polynomials. Elements $X_1, X_2, ..., X_j$ of a free probability space are called *freely independent* if

$$\tau(\Pi_{k=1}^{j}(P_k(X_{i_k}) - \tau(P_k(X_{i_k})))) = 0$$

whenever P_k are elements of $\mathcal{R}\langle X\rangle$ (and $i_{k_1}, i_{k_2} \in \{1,2,3,..,j\}$ are distinct if $|k_1 - k_2| = 1$). This, at least, is how one can detect the independence of random variables without referencing their classical measure-theoretic distribution.

Then, this brings us back to modeling the information available to and measurements able to be made by an object. We can say that the information available to an object at any given instant is a free probability space $\mathscr{S}$ over some ring $\mathcal{R}$. The elements of $\mathscr{S}$, however, do not have any inborn meaning – they are not observables, and they are not random variables.

Remark. Endowing each with the correct topology, it has been shown that the space spanned by indicator random variables I_a is dense in the space of random variables, while the **Spectral Representation Theorem** for Hilbert spaces states that the space spanned by self-adjoint projections is dense in the space of observables (or at least, its closure contains the space of observables) (Nielsen and Chuang 2010; Pillet 1970). Since we have seen that self-adjoint projections represent proposition, this motivates the following semantics for elements of free probability spaces.

Elements of free probability spaces are "generalized propositions":

limits of linear combinations of self-adjoint idempotents, which represent propositions.

Making this statement precise may perhaps be an interesting avenue of development, but that could very well take another paper.

If we assign a free probability space to each object at each point in time, where each free probability space represents the collection of "generalized propositions" that object can make about its environment, would this provide a sufficient context for the discussion of "transmitting information between objects"? We argue yes – it is, and for a several big reasons.

1. Free probability spaces can serve as a consistent representation for both "random variables" *and* "observables."
2. $*$-algebras can be categorified, and this lends to the possibility that free probability spaces can also be categorified, which calls to mind **Paradigm 3.**
3. Free probability spaces, with their functional τ are encoded with enough information to talk about entropy and mutual information.

The entropy of a random variable or observable is a real number (unless it is infinite) that tells us how unpredictible it is. If a random variable has low entropy, then it is easy to predict (and we have a lot of information about it), while high entropy indicates that it is much more difficult to predict (and we have very little information about it). Entropy can also be computed for multiple random variables simultaneously, giving the "total" uncertainty in all random variables. It has been shown that when $\mathcal{R} = \mathbb{C}$, entropy can be generalized to the context of free probability spaces – that is, entropy of an element may be deduced from its free distribution. However, such "entropies" have not (yet) been demonstrated unique. But since free probability spaces form such an advantageous context, we will fix an entropy function h according to a few rules that it has been demonstrated satisfy (Voiculescu 2001).

1. The domain of h is the set of finite subsets of a free probability space; the codomain is $\mathbb{R}$. Given a free probability space $\mathscr{A}$, we will denote the set of finite subsets of $\mathscr{A}$ as $\mathcal{P}^{<\infty}(\mathscr{A})$.

2. $h(X) := h(\{X\})$ (we omit the curly brackets for convenience) only depends on the free distribution of X. That is, two observables with the same free distribution have the same entropy.

3. $h(X_1, ..., X_n, X_{n+1}, ..., X_{n+m}) \leq h(X_1, ..., X_n) + h(X_1, ..., X_m)$, which equality exactly when $X_1, ..., X_{n+m}$ are freely independent.

4. $h(X)$ is maximized for some X. (If X has maximum entropy, it has maximum uncertainty.)

5. If $h(X)$ has maximum value M, then $h(X_1, ..., X_n)$ is at most nM. If $h(X_1, ..., X_n) = nM$ then $h(X_i) = M$ for all $1 \leq i \leq n$. This follows from (3) and (4).

We would also like to have $0 \leq h(X) \leq 1$ with 0 denoting absolute certainty regarding the value of X and 1 denoting absolute uncertainty. Given a now fixed entropy function h, the mutual information function i can be derived. i is a function $\mathcal{P}^{<\infty}(\mathscr{A}) \times \mathcal{P}^{<\infty}(\mathscr{A}) \to \mathbb{R}$ given by

$$i(X_1, ..., X_n | X_{n+1}, ..., X_{n+m})$$

$$= h(X_1, ..., X_n) + h(X_1, ..., X_m) - h(X_1, ..., X_n, X_{n+1}, ..., X_{n+m})$$

The mutual information between two collections of observables $\{X_i\}_i$ and $\{Y_j\}_j$ measures how much one collection can tell you about the other – a low value indicates low information, while a high value indicates high information. If these collections are freely independent, for example, we would expect that one collection wouldn't indicate *anything* about the other. And in fact, from (3) above, we can see that the mutual information between two freely independent observables is indeed 0.

Equipping a free probability space with an entropy function lays the groundwork for discussing how information can travel between objects and what interaction actually, formally, *is*.

2 Interaction

The primary motivation now for annexing an entropy function to a free probability space is that we can now ask how much information an observer may have about a given collection of observables at any given time. This is how we will detect interaction mathematically. If two objects A and B gain information about each other (or in other words, change states by interacting – and remember, this identification being made here between interaction and measurement is **Paradigm 1**), then:

- The entropy of A's observables that represent B – the collection of generalized propositions about B – should have decreased.
- The entropy of B's observables that represent A – the collection of generalized propositions about A – should have also decreased.

Definition 2.0.1. An *entropic free probability space* or EFP space is a triple $(\mathscr{A}, \tau, h)$, where $(\mathscr{A}, \tau)$ forms a free probability space and h is an "entropy function" which follows the rules in the preceding chapter. An morphism of EFP spaces is a morphism of the underlying $*$-algebras.

This definition for a morphism of EFP spaces may seem ad hoc, but our motivation for this goes back to our notions of generalized P-filtrations and generalized P-processes. These were functors taking values in categories that already had well-defined morphisms – these all happened to be algebra (or $*$-algebra) homomorphisms. In order to discuss how information is pushed forward to future states of an object, we need a similar definition and the one above seems the most appropriate. To this end, a morphism between EFP spaces is one way we can formalize an object's changing available information through time: we should interpret a morphism $f : \mathscr{A} \to \mathscr{B}$ of EFP spaces as:

- $\mathscr{A}$ is collection of observables (information, propositions, things that it can interact with), as seen from the perspective of an object according to some internal time t_1.
- $\mathscr{B}$ is a collection of observables, as seen from the same object according to some internal time t_2 after t_1.
- f is the function that takes the observables of $\mathscr{A}$ to what they are subsequently seen as in $\mathscr{B}$, at internal time t_2.

From this, we will now define an *abstract P-process* to be a functor $\mathcal{F} : P \to$ EFPSp, where P is a partially ordered set and EFPSp is the (now well-defined) category of EFP spaces. Note that we use the phrase "internal time" above to make it clear there is *no universal stream of time*. This perspective on time is conferred by an understanding of general relativity. In general relativity, spacetime is modelled as a *Lorentzian manifold*.

Definition 2.0.2. A Lorentzian manifold is an n-manifold $\mathcal{M}$ equipped with a smooth section $\sigma : \mathcal{M} \to S^2(T^*\mathcal{M})$ to the bundle of symmetric bilinear forms over $\mathcal{M}$, such that:

- $\sigma(p)$ is a non-degenerate form on $T_p(\mathcal{M})$.
- $\sigma(p)$ has signature $[-,+,+,...,+]$ on $T_p(\mathcal{M})$. That is, every basis of $T_p(\mathcal{M})$ that is orthogonal with respect to $\sigma(p)$ can be ordered as $(e_i)_{1\leq i\leq n}$ such that $\sigma(p)(e_1,e_1) < 0$ and $\sigma(p)(e_j,e_j) > 0$ for $j > 1$.

A vector $v \in T_p(\mathcal{M})$ is said to be *timelike* if $\sigma(p)(v,v) < 0$, *spacelike* if $\sigma(p)(v,v) > 0$ and *lightlike* if $\sigma(0)(v,v) = 0$. A vector $v \in T_p(\mathcal{M})$ is called *future-directed* if it is timelike or lightlike. A smooth curve $\gamma : [0,1] \to \mathcal{M}$ is called timelike, spacelike, lightlike, or future-directed, if its derivative at every point $x \in [0,1]$ is timelike, spacelike, lightlike, or future-directed respectively. $\mathcal{M}$ is said to be *causal* if there are no non-constant future-directed loops $\gamma : [0,1] \to \mathcal{M}$.

Note that a causal Lorentzian manifold $\mathcal{M}$ can be partially ordered where, given $x, y \in \mathcal{M}$, we let $x \leq y$ if there is a future-directed path from x to y (García-Parrado and Senovilla 2005). This partial ordering on $\mathcal{M}$ is a partial order of times called the *causal structure* or *causal partial order* of $\mathcal{M}$, and is much more generic than $(\mathbb{R}, \leq)$, which we had been considering earlier. It is therefore important to note that our conception of an abstract P-process can have the above meaning for non-linear partial orders P – two "times" that are incomparable in P may represent "times" that are only separated by a spacelike curve through timespace. Hence, it is useful both for simplicity and for generality's sake to forget notions of space, and for now, encode only time-relatedness into our partial order P. We will generally refer to elements of P as t_i to make this evident, but there is no harm in assuming that $t_i \in P$ may also have spatial qualities.

However, our eventual choice of P is a tricky one, as it ultimately comes down to how we define a physical object. For instance, we can consider the two most obvious choices for P:

- *Case* $P = (\mathbb{R}, \leq)$. A physical object A, under the assumption that $P = \mathbb{R}$ and that objects be represented by abstract P-processes, is given by a functor $\mathcal{F}_A : (\mathbb{R}, \leq) \to \text{EFPSp}$. Each EFP space in the codomain of $\mathcal{F}_A$ represents the collection of observables A has about its environment. Then, A is, by definition, a linear stream of collections of observables, despite the fact that A may actually inhabit a Lorentzian manifold with a *non*-linear causal partial order.
- *Case* $P = (\mathcal{M}, \leq)$ for some Lorentzian manifold $\mathcal{M}$. In this case, a physical object A is defined by a functor $\mathcal{G}_A : (\mathcal{M}, \leq) \to \text{EFPSp}$, and therefore, by the collections of observables A has about its environment at *every possible future*. This may seem more natural, as we are taking into account a more general topological structure of space, however this choice of P suggests that an object may "branch" into multiple pieces as it travels through spacetime.

Giving a mathematical definition for a physical object will be tricky because it isn't obviously "wrong" for an object to "branch" through spacetime. To that end, what is the extent to which a model plane is different from the collection of its components, once I have taken it apart? Physical deconstruction is one way in which we can see this "branching" manifest, unless we stipulate that physical deconstruction is the termination of one process and the initiation of another. This requirement does not play nice with **Paradigm 1**, however. If we wish to see interaction as a morphism between objects in some category, my deconstruction of a model plane should constitute a morphism between the model plane and I – but this cannot happen if all increasingly infinitesimal alterations made to the model plane terminate the model plane as a process.

Remark. Another potential hazard in mathematically defining a physical object as an abstract P-process $\mathcal{F}$ is that the transfer of information pushing observables in $\mathcal{F}(t_1)$ forward to observables in $\mathcal{F}(t_2)$, for $t_1 \leq t_2$, can be regarded as a kind of "interaction" from the observer at time t_1 to time t_2.

In light of this, we will attempt to be as constructive as possible with the definition of an object. For now, we will begin by defining a *simple object* as an abstract $\{\cdot\}$-process, where $\{\cdot\}$ is the singleton partial order. Strictly speaking, a simple object is just an EFP space containing the observables that simple object has about its environment at a single moment in time.

Let us note that simple objects solve the problem noted in *Remark 2* by viewing an abstract P-process as a collection of simple objects $\mathcal{F}(t_1)$ that may be potentially interacting via transfer of information through time. However, note that simple objects are not complex enough for interaction to be defined between them. An interaction between two objects, as stated in **Paradigm 1**, requires that at least one of those objects changes state. In order for an object to change state, we must define more complex objects that experience time according to some non-trivial partial order P. Before this, however, note that a subcategory $\mathcal{O}$ of EFPSp can be interpreted as a collection of simple objects, where morphisms between simple objects represent a transfer of information forward through time (according to some potentially more complex object whose representative abstract P-process contains the simple objects in question).

Note additionally that if we have two simple objects $\mathscr{A}, \mathscr{B} \in \mathcal{O}$, we expect that $\mathscr{A}$ has a subalgebra (or more precisely a sub-$*$-algebra) of observables corresponding to generalized propositions $\mathscr{A}$ could make about $\mathscr{B}$, while we also expect that $\mathscr{B}$ should have a subalgebra of observables corresponding to generalized propositions $\mathscr{B}$ could make about $\mathscr{A}$.

Definition 2.0.3. Given an EFP space $\mathscr{A}$, let $S(\mathscr{A})$ be the set of subalgebras of $\mathscr{A}$. Let $\mathcal{O}$ be a subcategory of EFPSp. Then $\mathcal{O}$ is a *simple interacting system* if $\mathcal{O}$ comes equipped with a family of functions $\{r_{\mathscr{A}} : \mathcal{O} \to S(\mathscr{A})\}_{\mathscr{A}\in\mathcal{O}}$, such that:

- For all $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathcal{O}$, and $f : \mathscr{B} \to \mathscr{C}$, we have $f(r_{\mathscr{B}}(\mathscr{A})) \subset r_{\mathscr{C}}(\mathscr{A})$. That is, morphisms take observables about a simple object to observables about that same simple object.

An *interacting system* is a simple interacting system $(\mathcal{O}, (r_{\mathscr{A}})_{\mathscr{A}\in\mathcal{O}})$ with a set of functors $J = \{\mathcal{F}_i : P_i \to \mathcal{O}\}_{i\in I}$ where P_i is a partially ordered set for each i – that is, each $\mathcal{F}_i$ is an abstract P-process on $\mathcal{O}$ for some P. Elements of J are called the *objects* of the interacting system $(\mathcal{O}, J, (r_{\mathscr{A}})_{\mathscr{A}\in\mathcal{O}})$.

Here, $r_{\mathscr{A}}(\mathscr{B})$ represents the subalgebra of the simple object $\mathscr{A}$ that enumerates the observables $\mathscr{A}$ has about $\mathscr{B}$. Given an interacting system $(\mathcal{O}, J, (r_{\mathscr{A}})_{\mathscr{A}\in\mathcal{O}})$, it then remains to determine how we can detect an interaction between two elements $\mathcal{F}, \mathcal{G} \in J$. This is easy, since each EFP space comes equipped with an entropy function h. But there are two ways we could go about detecting interaction:

1. We could look at $r_{\mathcal{F}(s_1)}(\mathcal{G}(t_1))$, the subalgebra in $\mathcal{F}$ at (internal) time s_1 corresponding to observables about $\mathcal{G}$ at (internal) time t_1, and likewise at $r_{\mathcal{G}(t_1)}(\mathcal{F}(s_1))$. We could pick a finite set of elements $\{x_i\} \subset r_{\mathcal{F}(s_1)}(\mathcal{G}(t_1))$, a finite set of elements $\{y_j\} \subset r_{\mathcal{G}(t_1)}(\mathcal{F}(s_1))$, and given futures $s_2 \geq s_1$ for $\mathcal{F}$ and $t_2 \geq t_1$ for $\mathcal{G}$, we could check to see if $h(\{x_i\}) \geq h(\mathcal{F}(s_1 \leq s_2)(\{x_i\}))$ or $h(\{y_i\}) \geq h(\mathcal{G}(t_1 \leq t_2)(\{y_j\}))$. That is, we could check to see if there is a collection of generalized propositions $\mathcal{F}$

could make about $\mathcal{G}$ that increased in certainty over the (internal) span of time between s_1 and s_2, or whether something similar happened to generalized propositions $\mathcal{G}$ could make about $\mathcal{F}$. If $\mathcal{F}$, for instance, has become more certain regarding some aspect of $\mathcal{G}$, we can reasonably assume there was some sort of interaction between them.

2. On the other hand, we could attempt to look at elements of $\mathcal{O}$ from the perspective of some object $\mathcal{H}$. As an observer, if $\mathcal{H}$ considers an observable $a \in \mathcal{H}(t)$, relating to some other object $\mathcal{I}$, we know that a exists in the subalgebra of $\mathcal{H}(t)$ generated by all subalgebras of observables relating to $\mathcal{I}$ at (internal) time i, $\bigvee_i r_{\mathcal{H}(t)}(\mathcal{I}(i))$. We will denote this algebra $\mathcal{H}_t^{\mathcal{I}}$ – specifically, it is the subalgebra of all observables $\mathcal{H}$ can have about $\mathcal{I}$ at some (internal) time t. Given another object $\mathcal{K}$, $\mathcal{H}$ may want to try and detect if $\mathcal{I}$ and $\mathcal{K}$ have interacted. Then, like above, we can pick a finite set of elements $\{x_i\} \subset \mathcal{H}_t^{\mathcal{I}}$ and a finite set of elements $\{y_j\} \subset \mathcal{H}_t^{\mathcal{K}}$, and we can check to see if their mutual information $i(\{x_i\}|\{y_j\})$ (induced by the entropy function h) has increased as we move forward into $\mathcal{H}_s$, for some $s \geq t$: that is, $i(\{x_i\}|\{y_j\}) \leq i(\mathcal{H}(t \leq s)(\{x_i\})|\mathcal{H}(t \leq s)(\{y_j\}))$. If $\mathcal{H}$ detects that, some time in the future, the subalgebras corresponding to $\mathcal{I}$ and $\mathcal{K}$ are *more* dependent than they were previously, one can also deduce that $\mathcal{I}$ and $\mathcal{K}$ have interacted in some way.

Then, which interpretation of interaction do we use? As suggested by **Paradigm 1**, the most natural thing is to do both – our first interpretation can be seen as a definition of *measurement* between objects, while the second can be seen as a definition of *interaction*. That is, measurement is when an object gains greater certainty about the state of another object, while an interaction according to an object $\mathcal{H}$ is when two objects appear to confer greater mutual information between each other.

Observe that $\bigvee_i r_{\mathcal{H}(t)}(\mathcal{I}(i))$ defines a functor $\varphi^{\mathcal{H}}(I) : P \to$ EFPSp, given by $\varphi^{\mathcal{H}}(\mathcal{I}) : t \mapsto \bigvee_i r_{\mathcal{H}(t)}(\mathcal{I}(i))$, where we assume the domain of $\mathcal{H}$ to be the partially ordered set P. Then, provided all the information encoded in $\mathcal{H}$ – all the information $\mathcal{H}$ could hope to obtain – $\mathcal{H}$ will assume other objects in $(\mathcal{O}, J)$ take the form of functors $P \to \mathcal{O}$. We can define a *primitive interaction according to* $\mathcal{H}$ between objects $\mathcal{I}$ and $\mathcal{K}$ to be as in 2. above: a choice of times $s \leq t \in P$ along with som finite $\{x_i\} \subset \varphi^{\mathcal{H}}(\mathcal{I})(s)$ and $\{y_j\} \subset \varphi^{\mathcal{H}}(\mathcal{K})(s)$ increase in mutual information by time t. We will denote such a primitive interaction according to $\mathcal{H}$ between $\mathcal{I}$ and $\mathcal{K}$ as $(s,t) : \mathcal{I} \to \mathcal{K}$. We can then say that an *interaction according to* $\mathcal{H}$ from $\mathcal{I}$ to $\mathcal{L}$ is a formal composition of primitive interactions $(r, s_1) : \mathcal{I} \to \mathcal{K}_1$, $(s_1, s_2) : \mathcal{K}_1 \to \mathcal{K}_2$,..., $(s_m, t) : \mathcal{K}_m \to \mathcal{L}$, with $r \leq s_1 \leq ... \leq s_m \leq t$.

On the other hand, we define *primitive measurement* between objects $\mathcal{F} : P_1 \to \mathcal{O}$ and $\mathcal{G} : P_2 \to \mathcal{O}$ as in 1. above: a choice of times $s_1 \leq s_2 \in P_1$, $t_1 \leq t_2 \in P_2$, along with some finite $\{x_i\}$ in $\mathcal{F}(s_1)$'s subalgebra of observables corresponding to $\mathcal{G}(t_1)$ that decreases in entropy over the span of time, which to $\mathcal{F}$ appears to be the span s_1 to s_2, and to $\mathcal{G}$ appears to be the span t_1 to t_2. We denote such a simple measurement as $(s_1, s_2, t_1, t_2) : \mathcal{G} \to \mathcal{F}$ (that is to say, $\mathcal{G}$ has transferred information about its state to $\mathcal{F}$). Similarly, a *measurement* between objects $\mathcal{F}$ and $\mathcal{H}$ is defined as a formal composition of primitive measurements $(r_1, r_2, s_1, s_2), (s_3, s_4, s_5, s_6), ..., (s_{4m-1}, s_{4m}, t_1, t_2)$ such that

$$r_1 \leq r_2$$

$$s_1 \leq s_2 \leq s_3 \leq s_4$$

$$\dots$$

$$s_{4m-3} \leq s_{4m-2} \leq s_{4m-1} \leq s_{4m-1}$$

$$t_1 \leq t_2$$

We use formal composition to define measurements as a means of accounting for the fact that some measurements and interactions are the product of "string" of smaller interactions: the Newton's cradle, for instance. It is important to note that a primitive interaction according to $\mathcal{H}$ given by (s, t) *depends* also on the finite sets of elements that increase in mutual information over the time from s to t – the same is true for primitive measurements (a, b, c, d) – however, we choose to omit labelling these sets for convenience.

Recall from above that, given an object $\mathcal{H}$ in an interacting system $(\mathcal{O}, J)$ with $\mathcal{H} : P \to \mathcal{O}$, $\varphi^{\mathcal{H}}(I)$ is an abstract P-process. That is, $\varphi^{\mathcal{H}}$ is a map from J to the class of abstract P-processes PProc. In fact, the image of $\varphi^{\mathcal{H}}$ can be given the structure of a *paracategory* – a category with only partially defined composition – where a morphism from $\varphi^{\mathcal{H}}(\mathcal{I})$ to $\varphi^{\mathcal{H}}(\mathcal{K})$ is given by an interaction according to $\mathcal{H}$ from $\mathcal{I}$ to $\mathcal{K}$. J also has the structure of a paracategory, taking a morphism between two objects to be a measurement.

Specifically, a paracategory $\mathcal{C}$ is all the data of a category without the constraint that all morphisms $s : U \to W$ can be composed with $t : W \to V$. A functor between paracategories is a morphism of the underlying quivers that preserves composition whenever it exists. Note that PProc and J are not genuine categories since a morphsim (s, t) in PProc can only be composed with another morphism (u, v) if $t \leq u$ or $v \leq s$. Likewise with morphisms in J.

However, we now have the ability to state a more sophisticated version of **Paradigm 1**.

Axiom of Faithfulness. The maps $\varphi^{\mathcal{H}} : J \to$ PProc induce *faithful* functors of paracategories for all $\mathcal{H}$ in J.

This paradigm, viewed now as an axiom for interacting systems, adds an important depth to the semantics that we've constructed. Having modified **Paradigm 1** in this way, interacting systems have more structure than they did before: we have a precise method of turning a measurement into an interaction. The fact that we may now express **Paradigm 1** with mathematical precision shows that we indeed have constructed semantics – though perhaps not the only semantics – for studying the natural sciences in full generality.

From here, we would like to now use these tools – particularly an interacting system satisfying our axiom of faithfulness – to then consider how an object $\mathcal{F}$ may approach the *study* of the interacting system it *belongs* to. After all, this brings us back to what natural sciences actually *are*. If we study interacting systems from the perspective of an object $\mathcal{F}$, we are studying an abstraction of the natural sciences. This is an important conclusion. An interacting system allows us to see interactions from the perspective of an observe $\mathcal{F}$, but also from a global perspective – given our abstract definition for an interacting system, one may have the power to study interactions, in general, and derive more general theories than those classically espoused in the natural sciences, though this remains to be seen.

If I continue this study, I would like to try and apply interacting systems in view of this goal.

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