ESSAYS ON AGENT HETEROGENEITY AND ADAPTIVE LEARNING

by

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DISSERTATION ABSTRACT

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Agent heterogeneity has been a widely discussed topic in the recent decade. However, most of the models that emerged from the literature draw their conclusions from a rational expectations equilibrium. These models impose strong assumptions on what agents know and how much they understand the models operate from one period to the next. Adaptive learning offers a straightforward response to this criticism by assuming agents are econometric learners. My dissertation aims to investigate the implications of combining these two features – agent heterogeneity and adaptive learning – together to see how models behave differently from the traditional models. My research relaxes these rational expectation assumptions in several widely-studied macroeconomic models.

In the first chapter of the dissertation, I traduce a novel concept of local rationality in a real business cycle model and with heterogeneous agents. The heterogeneity is introduced through ex-ante identical idiosyncratic income shocks. To understand how heterogeneity plays a role in the result, I implement a series of experiments that include different versions of the model with representative agents and heterogeneous agents. Both rational expectation results and locally rational expectation results are obtained. Both chapters find novel results that aggregate variables behave differently under adaptive learning primarily due to wealth-rich agents' learning behaviors. The simulations show that the rational expectations

equilibrium can be approximated with adaptive learning in these otherwise hard-to-solve models.

The last chapter focuses on a different type of heterogeneity with adaptive learning agents – expectational heterogeneity. The agents observe different signals to forecast relevant variables about the future. I show analytically that multiple sunspots can be used by agents in the model simultaneously, and these equilibria near an indeterminate steady state can still be E-stable. The analysis in the model holds for both the linear and the nonlinear versions of the model.

Overall, my dissertation makes contributions in the intersection fields of agent-heterogeneity and adaptive learning. The interaction could either be used as a computational method to approximate the rational expectations equilibrium (REE) or introduces extra friction in the model to have different aggregate responses given aggregate shocks.

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To Arthur and Jonathan

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CHAPTER I
INTRODUCTION

Economic agents making forward-looking decisions is one of the major differences between economics and natural sciences. In most modern economic theory, the outcomes partially depend on what people expect to happen. The current standard technique for modeling expectations is to assume rational expectations (RE). Intuitively, Rational expectations define an equilibrium where reality and belief align with each other. Formally, RE is defined as the conditional expectation of the relevant model-specific variables. While RE provides an intuitive solution to most macroeconomic models, the approach presumes that decision-makers have extensive knowledge about the economy, including the models' structure and the variables that go into agents' decision rules. These RE assumptions become even more stringent when the relevant variables take the form of a distribution.

My research aims to relax these rational expectation assumptions in several widely-studied macroeconomic models. I show that an alternative expectation assumption, a.k.a adaptive learning, provides a deeper understanding of the rational expectation assumptions imposed in several macroeconomic dynamic models. It shows that adaptive learning can explain a range of empirical observations. All chapters of my dissertation share the common ground in that I introduce some bounded rationality to account for empirical evidence that full rationality fails. My research is motivated by the fact that two individuals in the economy hardly hold identical beliefs on the future outlook. Also, expectations are formed with a certain level of irrationality because people do not have the necessary information or understanding of how the economy operates. However, the most recent macroeconomics development leaves out the discussions of expectations due to increasing model complexity, albeit the relevant role expectations play in the actual economy. In terms of applications, each chapter of the dissertation focus on the following topics: the effects of monetary policies and economy-wide shocks on consumption and income inequality in the United States, moment matching in RBC models, stability of sunspot equilibrium with multiple sunspots, and zero-lower bound on the interest rate and liquidity trap.

In the first chapter, a new behavioral concept, local rationality, is developed within a simple heterogeneous-agent model with incomplete markets. To make savings decisions, agents must forecast the shadow price of asset holdings. In the absence of aggregate uncertainty, locally-rational agents predict shadow prices rationally and make optimal state-contingent decisions. These agents then use estimated econometric models to extend their rational shadow-price forecasts to accommodate aggregate uncertainty. This chapter finds novel results that aggregate variables behave differently under bounded rationality, primarily due to wealth-rich agents learning behaviors. I introduce the local rationality concept in a real business cycle model (RBC) to account for the second-moment fluctuations observed in the data.

The second chapter introduces local rationality to a New Keynesian economy with incomplete markets and sticky nominal prices. Households are heterogeneous and face idiosyncratic wage risks. Both aggregate productivity shocks and monetary policy shocks are incorporated into the model. Both households and intermediate goods producers are assumed to be locally rational because they make optimal state-contingent decisions in the absence of aggregate uncertainties. Agents use estimated econometric models to forecast their shadow prices to accommodate aggregate uncertainties. In a calibrated model that attempts to capture features of US income inequality, I implement multiple monetary and fiscal experiments. I show that the aggregate responses to policies differ from their counterparts in a similar model with entirely rational agents. I found that the interaction of agent heterogeneity and adaptive learning can make aggregate shocks induce distributional effects. The effects are driven mainly by the top-rich agents in the economy.

In the last chapter, I investigate the implication of introducing multiple finite-state Markov extrinsic sunspot processes in a general univariate forward-looking model. In this model, each agent either does not observe any sunspots or observes only one of the sunspots. I show adaptively stable restricted perception Markov stationary sunspot equilibria (RP-SSEs) can exist near an indeterminate steady state for both the linear and nonlinear cases. I present the analytical conditions for the existence and E-stability. I also show that the model would prefer a sunspot equilibrium to a steady state equilibrium under model selection dynamics.

CHAPTER II
INTERACTIONS OF ADAPTIVE LEARNING AND HETEROGENEITY IN A REAL BUSINESS CYCLE MODEL

II.1 Introduction

Contemporary micro-founded macroeconomics models are identified partially by the notion of rationality. A central aspect is that expectations can influence the time path of the economy. Rationality is widely used in economic theories to ensure internal consistency within the model. The equilibrium notion based on rationality is a two-sided relationship where agents form expectations that lead to dynamics that match their expectations. Rationality comes in two essential parts: i) knowing the probability distribution of the endogenous and exogenous variables and can form optimal forecasts, and ii) given these forecasts, agents make optimal choices to maximize their objectives. The criticism is that the sophistication required of agents by rationality is substantial.

Literature on bounded rationality and adaptive learning has developed to respond to the criticism of rational agents' knowledge of the model structure. Instead of knowing the data generating process, agents act like econometricians and estimate forecasting models to form expectations. In turn, the boundedly rational expectations feedback into the dynamic system and generate new data for the agents to update their models. However, there hasn't been much exploration on the criticism that comes with agents making the optimal decision by solving infinite-horizon programming problems. The pioneering work of Evans and McGough (2020) first introduced shadow-price learning as a behavior primitive as a response to the criticism on the optimal decisions. Informally, agents are assumed to act as if they solve a two-period optimization problem in each period and forecast "shadow prices" to trade-off between choices today and the impact tomorrow. Shadow price learning turns a complex dynamic control optimization problem into a forecasting problem closely linked to bounded rationality and adaptive learning.

The agents' sophistication level becomes even higher when the model deviates from the representative-agent setting to a heterogeneous-agent one. In a representative-agent model, all agents would act identically and hence know everyone makes the same decisions in each period. However, in a model with heterogeneous agents, rational expectations assume agents know the state variables' whole distribution and how it translates to prices. As a more realistic modeling strategy than the traditional presentative-agent models, heterogeneous-agent models have attracted a lot of attention in the literature. For example, Philip Bergmann (2020) finds that energy price shocks decrease inequalities for both income and wealth in a real business cycle (RBC) model with heterogeneous agents. New results have been found through the lens of the HANK model. Kaplan et al. (2018) find that the indirect effects of an unexpected decrease in interest rates operating through a general equilibrium increase in labor demand outweigh the direct effects of inter-temporal substitution. McKay et al. (2016) find the power of forward guidance smaller in a HANK model than in the standard model. Bhandari et al. find that the Ramsey planner's optimal policy responses differ from the representative agent economy in magnitudes and directions. However, findings are drawn from a rational expectation equilibrium and impose strong assumptions on agents' knowledge about the economy's structure and the law of motions of some large-dimension states.

This chapter intends to relax this strong assumption on the agent's ability to solve the dynamic optimization problem by introducing *local rationality* in a dynamic model. Local rationality assumes that agents can make fully optimal decisions under idiosyncratic shocks in the absence of aggregate uncertainty. However, agents do not know the equilibrium mapping from the economy's aggregate states to the distribution of state variables and market-clearing prices. The rationality is local in the sense that, to account for the aggregate shocks, agents use an econometric model as guidance for deviating from the fully optimal decisions in the absence of aggregate uncertainty. The aggregate states and individual household's beliefs determine the direction and magnitude of the deviations. In this environment, agents form beliefs based on idiosyncratic shocks paths and react differently to aggregate shocks.

Under local rationality, the economy is self-referential: the shifts in individual beliefs determine the new distribution of beliefs, combined with labor productivity and asset holdings distributions, affect current market-clearing conditions. The market-clearing conditions, in turn, reinforce the individual beliefs. A prominent unique feature of our environment is that the interaction of learning and idiosyncratic shock dynamics plays a significant role in expectation formation. Mainly, learning introduces a parameter that governs the speed of agents' updating of their econometric models. When the speed is low, households spend a relatively long time updating the forecast rule. During the same periods, the agents might have experienced a wide range of idiosyncratic shocks. Some agents end up with a high position in asset holdings from the interaction, but their beliefs are also impacted by their personal experience when they held less wealth. These asset-wealthy agents react to aggregate shocks as if they were poorer than they are. The opposite cases could also arise. Namely, asset-poor agents might respond to aggregate shocks as if they were richer than they are.

Under the shadow price learning with heterogeneous agents in the RBC model, we found novel interactions between the learning mechanism and the distribution of individual variables. When the learning rule is simple and only uses aggregate capital and aggregate shocks as the regressors, the constant learning gain plays an important role in determining the aggregate variable behaviors. Specifically, asset-rich agents use data from the periods when they hold low levels of assets and are more sensitive to aggregate shocks. Their behaviors are close to what it is like when they are asset-poor. When we allow the agents to have a more complicated learning rule that includes the individual variables, the agents can learn the "correct" beliefs that match their asset holdings and idiosyncratic shocks. As a result, the impulse response functions under the simple learning for the aggregate variables are different from the rational expectations. The difference disappears when the learning rule is extended to include the individual variables. We also show that heterogeneity in agents asset holding is necessary for the adaptive learning introduces the difference in the impulse response functions. As a counterexample, the representative-agent version of the model doesn't exhibit the same properties as the heterogeneous-agent model with local rationality.

II.2 Literature Review

There has been a wide range of papers that intends to explore the possibility of explaining business cycle fluctuations with shifts in expectations. Early work by Benhabib and Farmer (1994, 1996, 2000) and Farmer and Guo (1994, 1995) introduced the possibility of multiple equilibrium and sunspots into the study of standard equilibrium business cycles under rational expectations. Eusepi and Preston (2011) further extended the framework to include learning dynamics that create changes in expectations and generate business cycles that better match the data's comovements.

Another reason to deviate from rational expectations is that the equilibrium solutions often impose relatively strong assumptions on agents' knowledge about the economy's structure in a model.

This paper belongs to two broad literature work. One intends to reconcile the predictions of real business cycle theory with observed data. The other is to investigate the effects of movements in income-and-wealth distribution. The former includes Hansen (1985), Rogerson (1988), Christiano and Eichenbaum (1992), Benhabib and Farmer (1994), Andolfatto (1996), Schmitt-Grohe (2000), and Eusepi and Preston (2011). These papers introduce a range of frictions that range from the indeterminacy of rational expectations equilibrium to long forecasting horizons for future prices. The latter literature includes Krusell and Smith (1998), Castaneda et al. (2003)

II.3 Environment

The following section considers a standard heterogeneous environment in Aiyagari's (1993) style, including endogenous labor choice and aggregate shocks similar to Krusell and Smith (1998). We assume a unit mass of workers who make choices to maximize their present discounted value of lifetime utility evaluated over stochastic streams of consumption and leisure.

$$\hat{E}_0^j \sum_{t=0}^{\infty} \beta^t U(c_t^j, l_t^j), \tag{II.1}$$

subject to the flow budget constraint and borrowing constraint

$$c_t^j + a_{t+1}^j = (1 + r_t)a_t^j + w_t \epsilon_t^j (1 - l_t^j), \tag{II.2}$$

where c^j, l^j, a^j, and ϵ^j denote household j's consumption, labor choice, asset holding in the form of capital claims, and individual labor productivity. $\hat{E}^j$ denotes the subjective expectation held by household j that might or might not be rational. As is the standard assumption, different households have different efficiency units of labor per hour worked. In return to supplying labor, households receive a wage that can be separated into two components: an aggregate component w_t, which is the same across all workers, and an idiosyncratic component ϵ_t^j which will be independent and identically distributed across all workers.

We assume that ϵ_t^j is a finite-state Markov process with the same transition matrix for all households $\Pi(\epsilon_t^j, \epsilon_{t-1}^j)$. Furthermore, we will assume agents cannot directly ensure against this idiosyncratic risk but can buy and sell claims to capital up to an exogenously given borrowing constraint $\underline{a}$. The worker's problem is then to, taking the stochastic process of r_t and w_t as given, choose streams of consumption and labor to maximize (II.1), subject to the borrowing limit $a_{t+1}^j \geq \underline{a}$ and time allocation $0 \leq l_t^j \leq 1$. Household optimality then yields standard first-order conditions

$$U_c(c_t^j, l_t^j) \geq \beta \hat{E}_t^j \left[\lambda_{t+1}^j\right], \tag{II.3}$$

$$U_l(c_t^j, l_t^j) = w_t \epsilon_t U_c(c_t^j, l_t^j), \tag{II.4}$$

$$\lambda_t^j = (1 + r_t) U_c(c_t^j, l_t^j) \tag{II.5}$$

where equation (II.3) and (II.4) are euler equation and labor-leisure choice. Equation (II.5) defines shadow price, λ_t^i. The shadow price is the marginal utility of saving from the last period. The shadow price has a very clear economic meaning that the agents are aware of instead of just a mathematical number. The euler equation can hold with inequality only if $a_{t+1}^j = 0$. Standard methods can show that, given a stochastic process for r_t and w_t, an allocation $\{c_t^j, l_t^j\}$ solves the household's problem if and only if it satisfies (II.2),(II.3), and (II.4). The production technology is standard.

There is a representative firm which produces output under perfect competition. The firm rents capital at rental rate r_t and hires effective labor from the household at wage w_t, thus the firm solves

$$\max_{K_t, N_t} \theta_t f(K_t, N_t) - w_t N_t - (r_t + \delta) K_t, \tag{II.6}$$

where θ_t is a stochastic variable that affects total factor productivity. The firm chooses capital and labor inputs, K_t and N_t, to maximize profits, taking factor prices w_t and r_t as given. Capital wear-off rate is δ. The optimal condition on behalf of the firm then yields the first-order conditions

$$w_t = \theta_t f_N(K_t, N_t) \tag{II.7}$$

$$r_t = \theta_t f_K(K_t, N_t) - \delta, \tag{II.8}$$

which equate factor prices with their real marginal productions. r^t is the rental rate of capital, and w_t is the real wage. The environment introduced here includes the standard representative-agent growth model or Aiyagari (1994) model, when idiosyncratic randomness or aggregate randomness is shut down.

II.4 Rational Expectations Equilibrium (REE)

In the absence of aggregate uncertainty, agents are assumed to make fully optimal decisions under idiosyncratic shocks. To account for aggregate shocks, they use an econometric model as guidance for deviating from the fully optimal decisions. We leave out the time subscripts for notional convenience and use an apostrophe to indicate the next period. Naturally, the definition of local rationality consists of two parts: agents' behavior without aggregate uncertainty and adaptive learning behaviors under aggregate uncertainty.

II.5 Stationary Recursive Equilibrium

Before introducing our bounded rationality assumptions, we first define the stationary rational expectations equilibrium absent aggregate shocks by setting $\theta_t = 1$ for all t. This definition will be used as a benchmark for later comparisons.

Definition 1

A *Stationary Recursive Equilibrium* consists of a distribution measure $\bar{\mu}$ over (a, ϵ), policy rules $\bar{c}(a, \epsilon)$, $\bar{l}(a, \epsilon)$,$\bar{\lambda}(a, \epsilon)$ and $\bar{a}(a, \epsilon)$, prices $\bar{w}$ and $\bar{r}$, and aggregate capital and labor supply $\bar{K}$ and $\bar{N}$ such that

1. The policy rules $\bar{c}(a, \epsilon)$, $\bar{l}(a, \epsilon)$,$\bar{\lambda}(a, \epsilon)$ and $\bar{a}(a, \epsilon)$ solve recursive versions of (II.2)-(II.5) for all (a, ϵ)

$$\begin{aligned}
\bar{c}(a, \epsilon) + \bar{a}(a, \epsilon) &= (1 + \bar{r})a + \bar{w}\epsilon\bar{l}(a, \epsilon) \\
U_l(\bar{c}(a, \epsilon), \bar{l}(a, \epsilon)) &= \bar{w}\epsilon U_c(\bar{c}(a, \epsilon), \bar{l}(a, \epsilon)) \\
\bar{\lambda}(a, \epsilon) &= (1 + \bar{r})U_c(\bar{c}(a, \epsilon), \bar{l}(a, \epsilon)) \\
U_c(\bar{c}(a, \epsilon), \bar{l}(a, \epsilon)) &\geq \beta\mathbb{E}\left[\bar{\lambda}(\bar{a}(a, \epsilon), \epsilon')\right]
\end{aligned}$$

   where $\mathbb{E}$ is taken over ϵ'

2. Firm optimally conditions hold $\bar{w} = f_N(\bar{K}, \bar{N})$ and $\bar{r} = f_K(\bar{K}, \bar{N}) - \delta$.
3. The labor market clears $\bar{N} = \int \epsilon(1 - \bar{l}(a, \epsilon))d\bar{\mu}(a, \epsilon)$
4. The asset market clears $\bar{K} = \int a d\bar{\mu}(a, \epsilon)$

5. $\bar{\mu}$ is stationary under the policy rules and Π: for any Borel set $\mathcal{B}$

$$\bar{\mu}(\mathcal{B}, \epsilon') = \sum_{\epsilon} \Pi(\epsilon', \epsilon)\bar{\mu}\left(\{a : \bar{a}(a, \epsilon) \in \mathcal{B}\}, \epsilon\right).$$

II.5.1 Stochastic Recursive Equilibrium - REE

Now consider the rational expectations equilibrium with the aggregate shocks. The stochastic recursive equilibrium adds aggregate shocks θ_t to the stationary version. Agents' decisions on asset holding, consumption, and labor supply also depend on prices w_t and r_t. The prices, in turn, are implied by the market clearing conditions.

Definition 2

A *Stochastic Recursive Equilibrium* consists of prices prices w_t and r_t, a distribution measure μ_t as a function of prices over asset holding and idiosyncratic labor productivity (a_t, ϵ_t), policy rules $c_t(a_t, \epsilon_t; w_t, r_t)$, $l_t(a_t, \epsilon_t; w_t, r_t)$, and $a_{t+1}(a_t, \epsilon_t; w_t, r_t)$, , and aggregate capital and labor supply $K_t(w_t, r_t)$ and $N_t(w_t, r_t)$ such that

1. The policy rules $c_t(a, \epsilon)$, $l_t(a, \epsilon)$, and $a_t(a, \epsilon)$ solve recursive versions of (II.2)-(II.5) for all (a_t, ϵ_t)
2. Firm optimally conditions hold $w_t = f_N(K_t, N_t)$ and $r_t = f_K(K_t, N_t) - \delta$.
3. The labor market clears $N_t(w_t, r_t) = \int \epsilon_t(1 - l_t(a_t, \epsilon_t; w_t, r_t))d\bar{\mu}(a_t, \epsilon_t; w_t, r_t)$
4. The asset market clears $K_t(w_t, r_t) = \int a_t d\mu_t(a_t, \epsilon_t; w_t, r_t)$
5. The distribution $\mu_t(w_t, r_t)$ evolves under the policy rules and the transition probability Π.

To compare the results to the bounded rational equilibrium, we define the induced shadow price from the $\lambda_t(a_t, \epsilon_t; w_t, r_t) = (1 + r_t)U_c(c_t(a_t, \epsilon_t; w_t, r_t), l_t((a_t, \epsilon_t; w_t, r_t)))$. Let I denote a mapping that gives the current information set of the agents, i.e., $x = I(\Omega)$. We keep this arbitrary for now, but we will allow for specific functions in future sections[1]. Let Λ^{RE} be the indeuced ergodic distribution of λ_t and X_{t-1} from the REE. We define $\Psi_{RE} = \mathbb{E}(\Lambda^{RE})(xx')^{-1}\mathbb{E}(\Lambda^{RE})[x \log(\lambda/\bar{\lambda})]$ where $\bar{\lambda}$ is the stationary steady state from the stationary recursive equilibrium.

II.6 Local Rationality

The difficulty faced in solving a Stochastic Recursive Equilibrium lies in the fact that policy rules and the law of motion depend on μ, a high dimensional object. The literature has used multiple approaches to approximate these equilibria. There are two different approaches. The first type uses projection methods based on Krusell and Smith (1998) to summarize the distribution with a finite set of moments. The exact procedure can vary but generally faces the problem that each additional moment adds an extra dimension to the state space. Thus, the curse of dimensionality quickly appears. The second approach, first introduced by Reiter (2009), instead linearizes policy rules around the Stationary Recursive Equilibrium. Our bounded rationality equilibrium will borrow from both of these works of literature and representative agent learning literature.

The behavior of rational agents has two interesting limits. The first natural limit is when the size of the aggregate shocks approaches zero. It's clear that in the absence of aggregate shocks, *Stationary Recursive Equilibrium* is a special case of a *Locally*

[1] For example, I could give log deviations of aggregate capital and θ from their stationary recursive equilibrium values $\bar{K}$ and θ.

Rational Recursive Equilibrium. Without the aggregate shocks, the model reduces to a Bewley/Aiyagari model.

With small aggregate shocks, a locally rational equilibrium's behavior inherits properties from the stationary recursive equilibrium, such as the wealth distribution and level of precautionary savings. This structure allows us to isolate how agents learn in the presence of aggregate shocks. In the other direction, we can take the limit of when the size of idiosyncratic shocks $\epsilon_{i,t}$ approaches zero and the initial distribution μ being a point mass with all agents having the same initial wealth and beliefs. In this limit, the distribution of agents will remain a point mass throughout time, and we recover behavior similar to the Euler equation learning of Evans and McGough (2020).

II.6.1 Locally Rational Agents

One hallmark of the rational expectations equilibrium is that agents know the current distribution of agents and its law of motion and its effect on prices. All of this is incorporated into the agents' decision-making process through the expectation term. This section embraces the bounded rationality assumption and instead assumes agents don't have access to the entire state variable's distribution. They form expectations by learning from their experience. A novel aspect of our approach is that we assume agents know how to behave optimally in the absence of aggregate risk and only learn how aggregate shocks should affect their decisions.

In doing so, we adjust the decision problem of the agent as follows. Agents' information set is x, and a vector summarizing their beliefs, ψ. They use their information and beliefs ψ from expectations over the future marginal value of savings, which we denote by $\mathbb{E}^{\psi}[\lambda']$. Given current prices, r and w, we posit that the agent's decisions rules then solve the following equation system.

$$c(a,\epsilon;x,\psi,r,w) + a'(a,\epsilon;x,\psi,r,w) = (1+r)a + w\epsilon l(a,\epsilon;x,\psi,r,w) \tag{II.9}$$

$$U_l(c(a,\epsilon;x,\psi,r,w), l(a,\epsilon;x,\psi,r,w) = w\epsilon U_c(c(a,\epsilon;x,\psi,r,w), l(a,\epsilon;x,\psi,r,w)) \tag{II.10}$$

$$\lambda(a,\epsilon;x,\psi,r,w) = (1+r)U_c(c(a,\epsilon;x,\psi,r,w), l(a,\epsilon;x,\psi,r,w)) \tag{II.11}$$

$$U_c(c(a,\epsilon;x,\psi,r,w), l(a,\epsilon;x,\psi,r,w)) \geq \beta \mathbb{E}^{\psi}\left[\lambda' | a'(a,\epsilon;x,\psi,r,w),\epsilon,x\right] \tag{II.12}$$

with equality only if $a(a,\epsilon;x,\psi,r,w) = \underline{a}$. Note that equations (II.9)-(II.12) are *behavorial primitives*: they are imposed assumptions on the behavior the the households. In order to determine an agent's choices, we need to specify how the expectation $\mathbb{E}^{\psi}[\lambda'|a,\epsilon,x]$ is formed. Our *local rationality* assumption is that agents form expectations *relative* to how they would rationally behave in the the stationary recursive equilibrium. Specifically we assume $\mathbb{E}^{\psi}[\lambda'|a(a,\epsilon;x,\psi,r,w),\epsilon',x] = \bar{\lambda}(a'(a,\epsilon;x,\psi,r,w),\epsilon')\exp(\psi' x)$. Taking expectations over ϵ' we then recover

$$\mathbb{E}^{\psi}\left[\lambda'|a(a,\epsilon;x,\psi,r,w),\epsilon,x\right] = \left(\sum_{\epsilon'} \Pi(\epsilon,\epsilon')\bar{\lambda}(a'(a,\epsilon;x,\psi,r,w),\epsilon')\right)\exp(\psi' x). \tag{II.13}$$

Consistent with the forecasting rule, we assume that households update ψ by regressing log deviations of $\lambda(a,\epsilon;x,\psi,r,w)$ from $\bar{\lambda}(a,\epsilon)$ on the previous periods information set x_{-}. Beliefs are then updated each period via a recursive constant gain learning rule as follows. We let R_{-} represent the previous period's estimate for the covariance matrix of x. The covariance matrix of x is updated via

$$R(x_{-},R_{-}) = R_{-} + \gamma({x_{-}}' x_{-} - R_{-}). \tag{II.14}$$

While beliefs ψ are updated according to

$$\psi'(a,\epsilon;x,x_{-},\psi,r,w) = \psi + \gamma R(x_{-},R_{-})^{-1} x_{-}\left(\log\left(\frac{\lambda(a,\epsilon;x,\psi,r,w)}{\bar{\lambda}(a,\epsilon)}\right) - \psi' x_{-}\right). \tag{II.15}$$

The current state of the economy is $\Omega = (\mu, \theta, x_{-}, R_{-})$ where μ is the joint distribution over (a, ϵ, ψ). We are now able to define locally rational recursive dynamics.

Definition 3

A *locally rational recursive dynamics* consists of policy rules $c(a, \epsilon; x, \psi, r, w)$, $l(a, \epsilon; x, \psi, r, w)$, $a'(a, \epsilon; x, \psi, r, w)$,and $\lambda(a, \epsilon; x, \psi, r, w)$; evolution of beliefs $R(x_{-}, R_{-})$ and $\psi'(a, \epsilon; x, x_{-}, \psi, r, w)$; pricing functions $r(\Omega)$ and $w(\Omega)$; aggregate firm choices $N(\Omega)$ and $K(\Omega)$; a function specifying the information set $x = I(\Omega)$; and a law of motion for the aggregate distribution $H(\Omega)$ such that

1. Given prices r and w, $c(a, \epsilon; x, \psi, r, w)$, $n(a, \epsilon; x, \psi, r, w)$, $a'(a, \epsilon; x, \psi, r, w)$, and $\lambda(a, \epsilon; x, \psi, r, w)$ solve (II.9)-(II.13)
2. Firms behave optimally: $w(\Omega) = \theta(\Omega) f_N(K(\Omega), N(\Omega))$ and $r(\Omega) = \theta(\Omega) f_K(K(\Omega), N(\Omega)) - \delta$.
3. The labor market clears $N(\Omega) = \int \epsilon(1 - l(a, \epsilon; x, \psi, r(\Omega), w(\Omega)))d\mu(a, \epsilon, \psi)$
4. The asset market clears $K(\mu, \theta) = \int a d\mu(a, \epsilon, \psi)$
5. Beliefs evolve according to constant gain learning: $R(x_{-}, R_{-})$ and $\psi'(a, \epsilon; x, x_{-}, \psi, r, w)$ satisfy (II.14) and (II.15)
6. The law of motion H is consistent with $a'(a, \epsilon; x, \psi, r(\Omega), w(\Omega))$, Π,$\psi'(a, \epsilon; x, x_{-}, \psi, r(\Omega), w(\Omega))$, $I(\Omega)$, and $R(x_{-}, R_{-})$.

II.6.2 Restricted Perception Equilibrium - LREE

We are ready to define the locally rational expectations equilibrium as a restricted perceptions equilibrium based on the definition of locally rational recursive dynamics. The folk theorem of the learning literature states that the long-run beliefs will converge to a restricted perceptions equilibrium (RPE) if the equilibrium is E-stable. Intuitively, RPE is characterized by self-confirming beliefs. Hold beliefs fixed at $\psi_t = \bar{\psi}$ for all t and all agents. Feed these fixed beliefs ψ_t into the locally rational recursive dynamics without the belief evolving part in the fifth bullet point. Let $\Lambda(\bar{\psi})$ be the induced ergodic distribution of the shadow price and information set (λ_{t+1} and X_t) from the dynamics. We can construct the linear projection of $log(\lambda/\bar{\lambda})$ on X under measure $\Lambda(\bar{\psi})$. Effectively, the locally rational recursive dynamics defines a T-map for the agents beliefs with the form $\bar{\psi}' = T(\bar{\psi})$. Here $T(\bar{\psi}) = \mathbb{E}_{\Lambda(\bar{\psi})}[XX']^{-1}\mathbb{E}_{\Lambda(\bar{\psi})}[X \log(\lambda/\bar{\lambda})]$ and defines the the coefficients of the learning model from the ergodic distribution of the dynamics. We have the following definition

Definition 4

A locally rational expectations equilibrium is a locally rational recursive dynamics with the ergodic distributions of the beliefs ψ^* such that $\psi^* = \mathbb{E}_{\Lambda(\psi^*)}[XX']^{-1}\mathbb{E}_{\Lambda(\psi^*)}[X \log(\lambda/\bar{\lambda})]$.

This framework approximately nests the rational expectations equilibrium. Let Λ^{RE} be the induced ergodic distribution of λ_t and X_{t-1} from the rational expectations equilibrium.

II.7 Calibration and Simulation

The utility function is given by the following form $U(c,l) = \frac{c^{1-\sigma}}{1-\sigma} - \chi\frac{(1-l)^{1-\gamma}}{1-\gamma}$ and the production function takes the standard Cobb-Douglas form. $f(K,N) = K^{\alpha}N^{1-\alpha}$. The calibration of the model is done through moment matching in the stationary stochastic equilibrium (SSE) without the aggregate shocks. There are three sets of parameters that need to be set: (i) parameters related to household preferences, productions, and aggregate shocks; (ii) initial conditions; and (iii) stochastic processes for idiosyncratic shocks. The details of calibrations are summarized in Table (1). I explain the parameters in the following three subsections.

Variable	Moment	RA	HA
σ	CES parameter	2.0000	2.0000
γ	Frisch elasticity	2.0000	2.0000
β	Capital-Output Ratio	0.9612	0.9277
χ	Leisure Ratio = 0.33	55.206	54.738
$\underline{a}$	Borrowing constraint	0.0000	0.0000
α	65% output $\rightarrow$ labor	0.3500	0.3500
ρ_θ	Krueger et al. (2009)	0.8150	0.8150
σ_θ	Krueger et al. (2009)	0.0140	0.0140
ρ_{z^p}	Permanent ρ	-	0.9923
σ_{z^p}	Permanent idio.	-	0.1960
σ_{z^t}	I.I.D idio.	-	0.2300

TABLE 1 Benchmark Yearly Calibrations

II.7.1 Preferences, Productions and Aggregate Shocks

The settings of parameters match standard representative agent calibrations such as Schmitt-Grohe and Uribe (2004) and Siu (2004). I set the CES parameters for household consumption and labor at $\sigma = 2$, $\gamma = 2$, and the discount factor β is set so that the aggregate capital to GDP ratio is 10.26 for the yearly calibration. The firms operate a decreasing return to scale technology so that the labor income accounts for 65% of the total output. We calibrate χ to target $1 - \bar{l}$ to be 0.333 in the steady state, which implies spending approximately 33.3% of their time allocation working. We assume that the information available to the agents is given by.

$$I(\Omega) = \left(1, \log\left(\int a d\mu(a, \epsilon, \psi)\right) - \log\left(\bar{K}\right), \log(\theta)\right), \tag{II.16}$$

Agents respond to the log deviation of capital and productivity from their steady-state values. Finally, agents cannot borrow which means $\underline{a} = 0$.

II.7.2 Initial Conditions and Learning

For the representative agent model, the initial conditions for each variable are at the steady-state level. For the heterogeneous agent model, the initial conditions are drawn from the stationary distribution of the stationary recursive equilibrium. In addition to the standard calibrations, our simulations also require specifying initial conditions, including the joint distribution (μ) of assets, productivity, and beliefs. Also, the covariance matrix R needs initialization. The joint distribution of assets and productivity are taken to be the joint distribution of assets and productivity from the stationary recursive equilibrium,$\bar{\mu}$. The initial covariance matrix R_0, which is shared across all agents, is derived from data generated by the rational expectations equilibrium.

II.7.3 Stochastic Processes

I calibrate the aggregate productivity process following Krueger, who estimated a process for disposable earnings after taxes and transfers. They estimated an annual persistence of innovations to be $\rho_\theta = 0.815$ with a standard deviation of $\sigma_\theta = 0.014$. We assume the idiosyncratic log productivity process is the sum of an AR(1) and i.i.d. component. The calibration follows the practices in Krueger (2005), who estimated a process for disposable earnings after taxes and transfers. They estimated an annual persistence of innovations to be 0.992 with a standard deviation of 0.098. The standard deviation of the transitory component they estimated to be 0.23. The combination of the permanent shocks and transitory shocks can help the model account for the wealth and income inequality of the model. The idiosyncratic productivity process is approximated using the Rouwenhorst method with 7 grid points for the permanent component and 3 grid points for the transitory component.

II.7.4 Numeric Method

To compute the stationary recursive equilibrium, we approximate the agents' income process using a finite-state Markov chain. We discretize the AR(1) component of productivity with seven grid points using Rouwenhorst's method in Kopecky and Suen (2010) and the transitory component with three grid points Gauss-Hermite quadrature. The agent's decision rules are approximated along the asset dimension with 100 grid points, non-linearly spaced. We solve for the agents' optimal decision rules conditional on prices via the endogenous grid method of Caroll. The stationary distribution is solved by approximating the distribution with a histogram of $10,00$ data points. We construct a transition matrix with the approximated policy rules and then solve the transition matrix's stationary distribution. Finally, β and χ are chosen to target the capital-to-output ratio and aggregate labor supply through a non-linear solver. To approximate the Recursive Competitive Equilibrium, we follow Boppart et al. (2018) to linearize policy rules around the Stationary Recursive Equilibrium by constructing impulse response functions. Details, as well as our tests verifying the linearity assumption, are provided in Appendix A. We apply algorithm 1 to simulate the locally rational equilibrium conditional on initial beliefs. The simulation requires solving the temporary equilibrium each period conditional on the distribution of beliefs and aggregate states. We apply a variant of the endogenous grid method to approximate each agents' policies conditional on prices quickly.

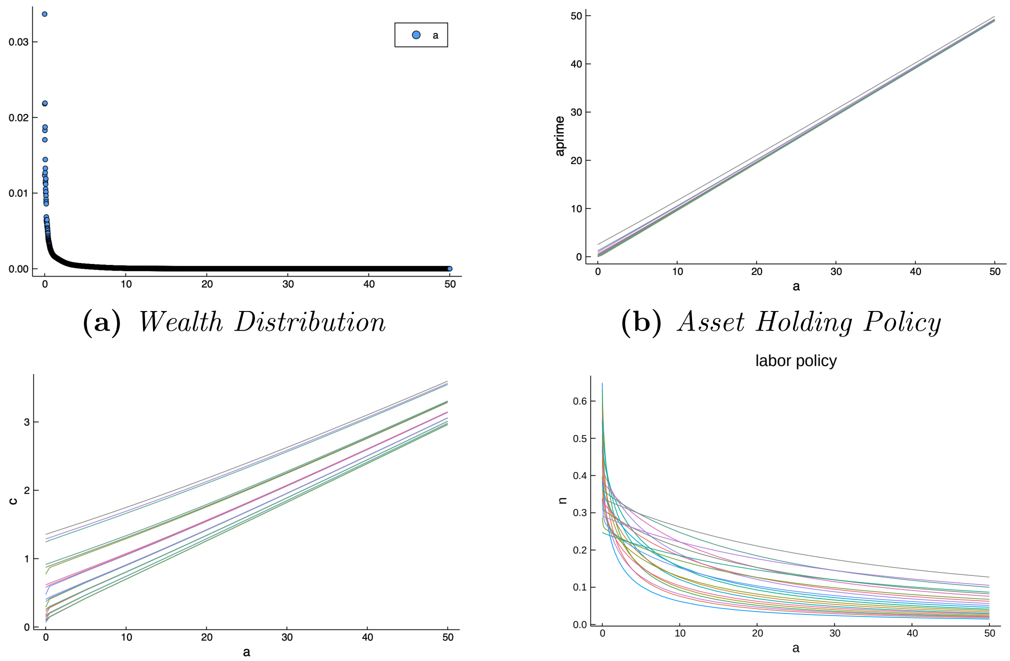
II.7.5 Calibrated Distributions and Policy Rules

Figure (1) shows the asset distribution and the policies rules for the asset holding, consumption, and labor supply from the stationary stochastic equilibrium. In general, most agents hold an asset level less than 5, and there is a mass probability for agents to be on the borrowing constraint. The different colors in figure (1b)-(1d) stand for varying levels of idiosyncratic income shocks. In general, when the idiosyncratic income shock is higher for an agent, they tend to consume more and also have a higher asset holding for the future period. From the policies functions, we can see that the asset holding policy function varies very little across different income shock levels, whereas the income shocks have a relatively large impact on the consumption levels for the agents. The labor supply function is a little different from the other two policy functions in the sense that agents behave differently with a high or a low asset holding. Given a low productivity shock, an agent would work more when they are asset-poor and work less when they are asset-rich. This behavior change is shown through the crossovers of the line plot in figure(1d).

II.7.6 Simulation Algorithm

A fairly simple algorithm can be constructed to simulate the dynamics with locally rational agents., shown as follows. Given the parameters set by the calibration set by the previous section, we can first solve for the stationary distribution for the shadow price $\bar{\lambda}(a, \epsilon)$, the algorithm basically finds the fixed point of prices and distributions that satisfies the inter-temporal conditions. There are several

features of this algorithm to consider. The simulation without aggregate risk only requires solving a single Bellman equation to determine the shadow price $\bar{\lambda}(a, \epsilon)$. This compares favorably to Giusto (2014), which requires repeatedly solving the



(a) *Wealth Distribution* **(b)** *Asset Holding Policy* **(c)** *Consumption Policy* **(d)** *Labor Policy*

FIGURE 1. *Stationary Stochastic Equilibrium*

Note: The first graph shows that the wealth distribution from the stationary equilibrium is heavily skewed. Most of the wealth is held by a small number of rich agents. The rest of the three figures show the policy functions for an agent in the stationary equilibrium. These policy functions are the optimal choices in the absence of aggregate risks.

Bellman equation after each period, and value functions, which take aggregate and idiosyncratic states as function inputs. The most computationally intensive part of this process is step 4 which requires solving a non-linear equation in r , w, and $\{a_i', l_i, c_i, \lambda_i\}$. Intuitively, this can be achieved by determining the choices for each agent that solve (II.9)-(II.12) for a given (r, w). The process speeds up by noting that differences in individual agents' decisions depend only on $\eta_i \exp(\psi_i' x)$; thus, it is possible to pre-compute those decisions, further speeding up the algorithms.

Intialize: For current parameterization solve for steady state $\bar{\lambda}(a, \epsilon)$. Let Ω be the current aggregate state, assume the $\mu \in \Omega_t$ is populated by a finite number M of agents indexed by i **for** $t \in 1$ **to** T **do**

Compute $x = I(\Omega_t)$ Find for r, w and $\{a'_i, l_i, c_i, \lambda_i\}$ such the policies $a'_i, l_i, c_i, \lambda_i$ solve equations (II.9)-(II.12) for each $a_i, \psi_i, \epsilon_i \in \mu(\Omega_t)$ and the market clearing conditions

$$
\begin{aligned}
N &= \frac{1}{M}\sum_{i=1}^{M} \epsilon_i (1 - l_i) \\
K &= \frac{1}{M}\sum_{i=1}^{M} a_i \\
r &= \theta(\Omega_t) f_K(N, K) - \delta \\
w &= \theta(\Omega_t) f_N(N, K)
\end{aligned}
$$

hold Update beliefs ψ'_i and R according to (II.14) and (II.15) Draw new aggregate shock θ' Draw new productivity ϵ'_i for each agent i and construct μ' from $\{a'_i, \epsilon'_i, \psi'_i\}$ Update $\Omega_{t+1} = (\mu', \theta', x, R)$

end

Algorithm 1: Simulation of Economy Locally Rational Agents

II.8 Model Results

This section presents the computational results from the models and equilibrium concepts. We first show that the restricted perception equilibrium exists by simulation and then draw the connection between the RPE beliefs and REE beliefs. We also compare the impulse response functions (IRF) from RPE and REE and show that learning gain can play a role in how the aggregate variables respond to the shocks. We provide detailed explanations for the IRFs and argue that the results are coming from the interactions of learning mechanisms and the heterogeneity across the agents.

II.8.1 Existence of RPE

By simulations, we show that the locally rational expectations equilibrium exists. We use the calibration from the previous section and set the constant learning gain to be 0.005, which is a relatively small gain. For the benchmark model, we let the information mapping function to be

$$x_t = I(\Omega_t) = (1, \log(K_t/\bar{K}), \log(\theta_t))$$

where $\bar{K}$ is the steady state of aggregate capital from the stationary recursive equilibrium. For the benchmark model, the learning gain is 0.01. To compute the locally rational expectations equilibrium, we follow the definition and execute the following two steps. First, we simulate the locally rational expectations dynamics with initial beliefs $(0, 0, 0)$ for ψ_{i0} for all agents. The simulation runs for $100,000$ agents for $30,000$ periods of time. We take the average belief over all agents from the last $50,000$ periods as the potential candidate for the restricted perception equilibrium (RPE) beliefs. To check that this average belief is indeed from the RPE, we fix the beliefs at this average level and simulate the locally rational expectations dynamics again for $10,000$ periods[2] with $100,000$ agents. The induced belief distribution is given in Figure (2). The vertical red lines in Figure (2) represent the average beliefs (loading on the constant term, aggregate capital, and aggregate shock) computed from the last $5,000$ of the initial $30,000$ periods of simulation. The histogram shows the ergodic distribution of the induced beliefs from the simulation holding the beliefs

[2]We have simulated the model for more than 1 million periods with various different calibrations, the per-period temporary equilibrium can always be solved. Eventually, the temporary equilibrium can be represented by the function that is a mapping from aggregate labor supply to itself. This mapping always crosses the 45 degree line and hence has a solution.

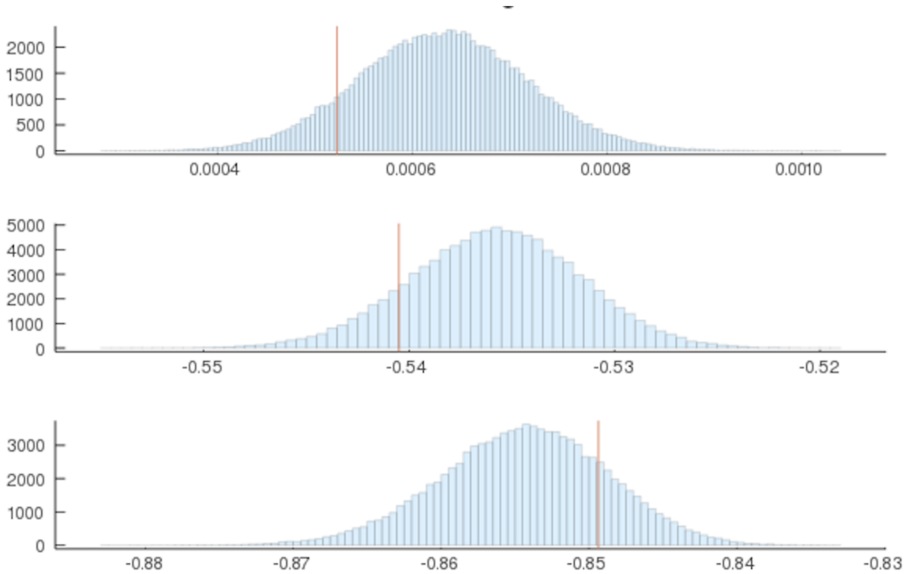


FIGURE 2. *Induced Belief Distribution*

Note: The blue histograms are the induced belief distributions for the three estimates of the learning rule. They correspond to the belief loading on the constant, aggregate capital, and aggregate income shock. The red vertical line shows where the initial beliefs are located.

fixed at the red line level. The simulation shows that the RPE beliefs fall into the distribution of the ergodic beliefs distribution and hence are self-referential.

II.8.2 Learnability of REE

There are two well-documented features of this economy that allow this forecast to be simplified further. First, the economy is well approximated linear policy rules around the stationary recursive equilibrium. Second, approximate aggregation holds in general. We first define the estimates for the forecasting rule under rational expectations where agents have full knowledge of how their expected future marginal value of savings $\mathbb{E}\left[\lambda(a', \epsilon', H(\mu, \theta), \theta')|a', \epsilon', \mu, \theta\right]$ Given a rational expectations equilibrium, it is then possible to construct functions $\psi^{RE}(a, \epsilon)$ such that

$\psi^{RE}(a,e) = \text{cov}(x, x|a, \epsilon)^{-1}\text{cov}(x, \log(\lambda/\bar{\lambda})|a, \epsilon)$. The REE beliefs are the aggregate beliefs over the distribution of agent's asset holding and individual income shocks. The computational results for REE beliefs are represented by the blue dashed line in Figure (3) We show that not only do the RPE beliefs exist but also that they

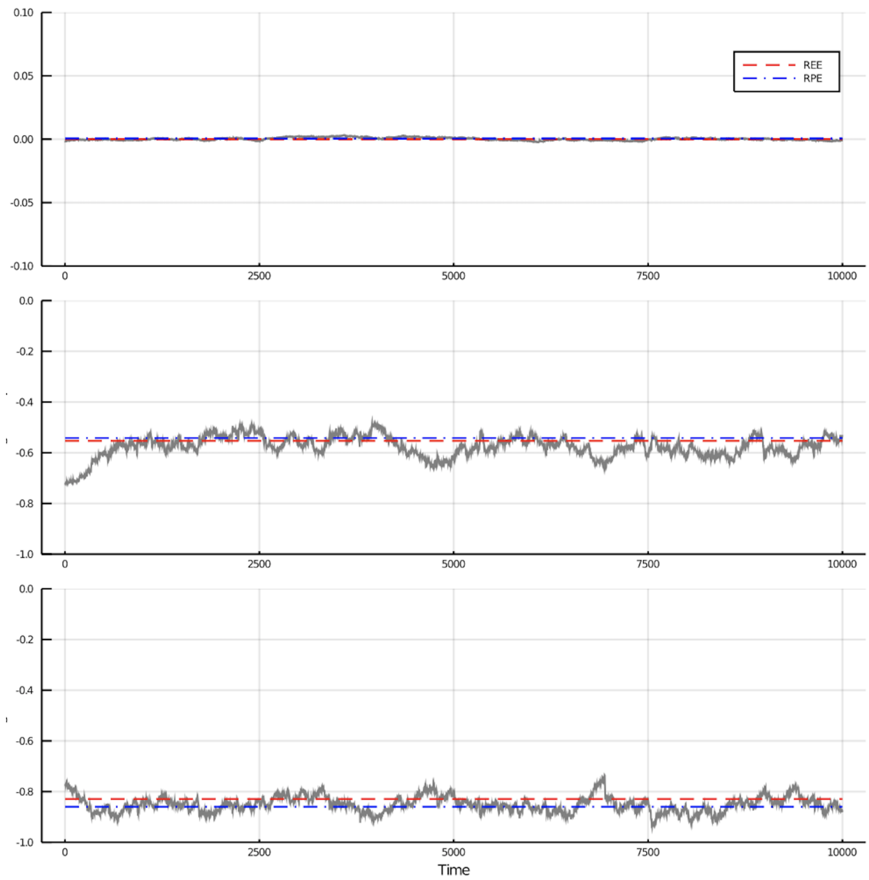


FIGURE 3. *RPE and REE Beliefs*

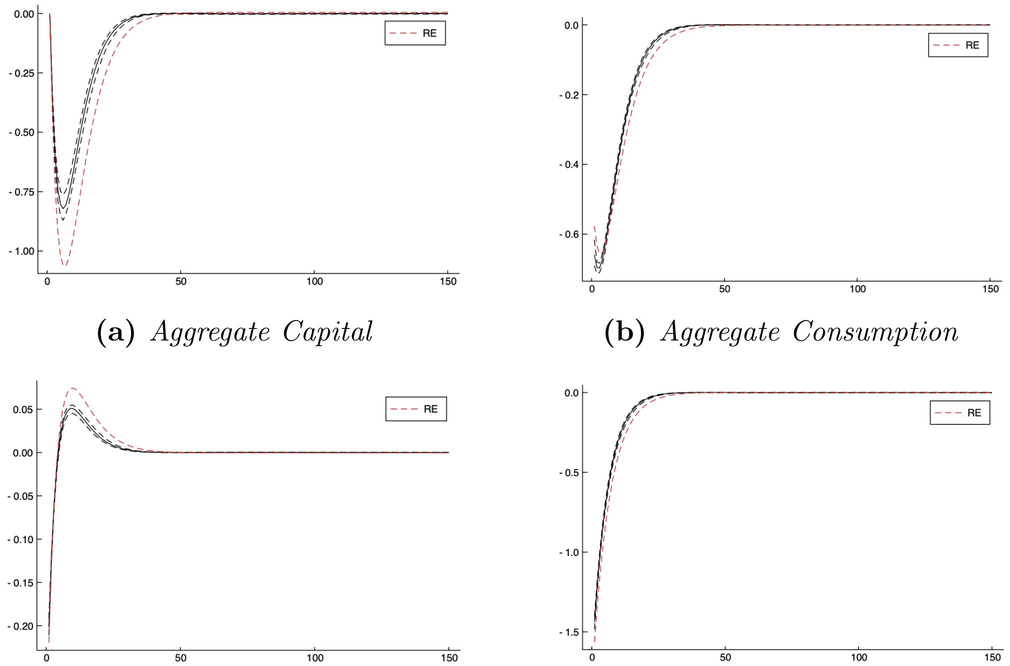
Note: The three graphs correspond to the time paths of the belief loading on the constant, aggregate capital, and aggregate income shock. They show that the average RPE beliefs converge to the rational expectations equilibrium beliefs for each belief loading.

are similar to the rational expectations equilibrium beliefs. Figure (3) shows the learning dynamics from the LREE compared to the REE counterpart. The three subplots in the figure represent each estimate for the coefficient in the learning rule. In this case, the belief components are for the constant term, aggregate capital deviation, and aggregate income shock deviation. These results computationally show that the aggregate rational expectations equilibrium beliefs in a heterogeneous-agent environment could be learned by the agents if they are locally rational. However, the convergence of aggregate beliefs to the REE levels does not imply the models behave identically under the RPE and REE. Details are presented in the next section.

II.8.3 Impulse Response Functions

To compute the impulse response functions (IRFs) from the locally rational expectations equilibrium, we draw distributions of the asset holdings, individual shocks, and beliefs from the ergodic distribution of the restricted perception equilibrium. Give the model a negative aggregate income shock and compute the impulse responses of the aggregate variables. We repeat the process 500 times and use the median time paths of the aggregate variables as the IRFs. We also compute the top 2.5 percentile and bottom 2.5 percentile as a proxy of the 95 percentile of the IRFs. Figure (4) displays the corresponding impulse responses of changes in the aggregate asset and aggregate consumption implied by one standard deviation of exogenous innovation of aggregate income shock ϵ_θ.

Figure (4) shows when the learning gain is 0.001, which is set to be relatively low, there is a clear difference between the impulse responses in the LREE and REE. Specifically, aggregate capital stock falls less than their rational expectations counterparts, with aggregate capital falling 25% less in response to a one-standard-deviation fall in productivity. Meanwhile, consumption falls by more than the rational expectations counterpart. However, this discrepancy disappears when the learning gain is relatively high. The average path of all three variables lies almost exactly in



(a) *Aggregate Capital* **(b)** *Aggregate Consumption*

(c) *Interest Rate* **(d)** *Aggregate Output*

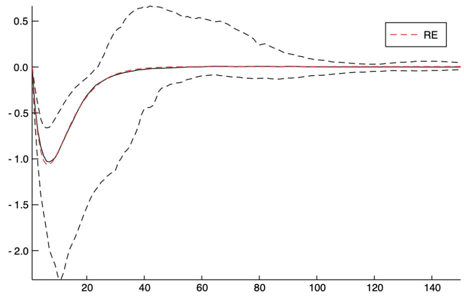
FIGURE 4. *Impulse Response Functions with Learning Gain of* 0.001

Note: This figure is simulated with a heterogeneous-agent model. The impulse response functions from the LREE (black dashed lines) with low learning gain compared to REE (red dashed lines). The LREE cuts consumption more than the REE counterpart under a negative productivity shock. The three black dash-lines are 97.5%, 50%, and 0.5% responses from the ergodic distributions of the LREE.

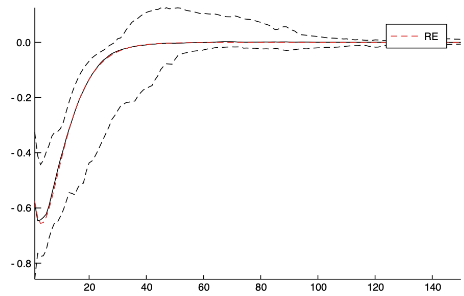
line with the rational expectations impulse response functions. Figure (5) shows the IRFs from LREE with a learning gain of 0.1.

Higher gain implies that agents place greater weight on more recent experiences when forecasting future values, which means a higher variance of beliefs in the ergodic distribution. The intuition for the difference is that under lower gain, agents essentially use the long periods of time[3] to estimate the forecasting model. Meanwhile, a large proportion of the asset is held by rich agents in the economy because the wealth

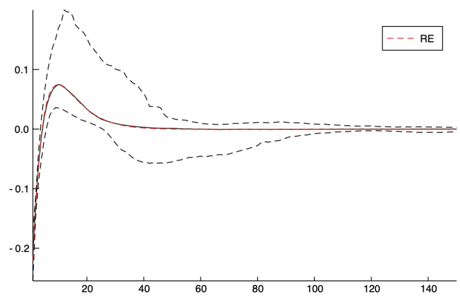
[3]If γ_{gain} be the learning gain, agents use $\frac{1}{\gamma_{\text{gain}}}$ to estimate the forecasting rule.



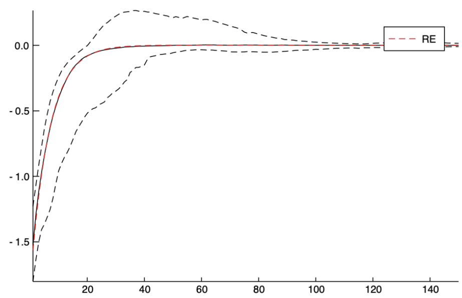
(a) *Aggregate Capital*



(b) *Aggregate Consumption*



(c) *Interest Rate*



(d) *Aggregate Output*

FIGURE 5. *Impulse Response Functions with Learning Gain of* 0.1

Note: This figure is simulated with a heterogeneous-agent model. The impulse response functions from the LREE (black dashed lines) with high learning gain compared to REE (red dashed lines). The aggregate responses are comparable. The three black dash-lines are 97.5%, 50%, and 0.5% responses from the ergodic distributions of the LREE.

distribution is heavily skewed according to Figure (7a). As a result, these rich agents who are currently holding a high level of assets also used their historical data from when they were poor. In some way, these agents behave more like the asset-poor type than the asset-rich type in the rational expectations equilibrium. Whereas with a high learning gain, the agents can quickly adjust to the beliefs commensurate to their asset-holding level.

To understand why the IRFs from the LREE are different in the specific direction compared to the REE, we need to dive into how agents' asset holding affects their beliefs. Specifically, in a low gain setting, the agents who are currently poor but who also still have beliefs consistent with being rich behave in a manner that is consistent with some notion of decreased risk aversion. We first need to understand the implication of having low and high beliefs in the environment we have set up. We show in Figure (6) how beliefs are associated with asset holding levels in a rational expectations equilibrium. The blue line represents how the belief loading on capital in the learning rule in the REE model for the agents who have the lowest idiosyncratic ($s = 1$) shock, whereas the golden line to the right stands for the case when the agents have the highest idiosyncratic shock ($s = 21$).

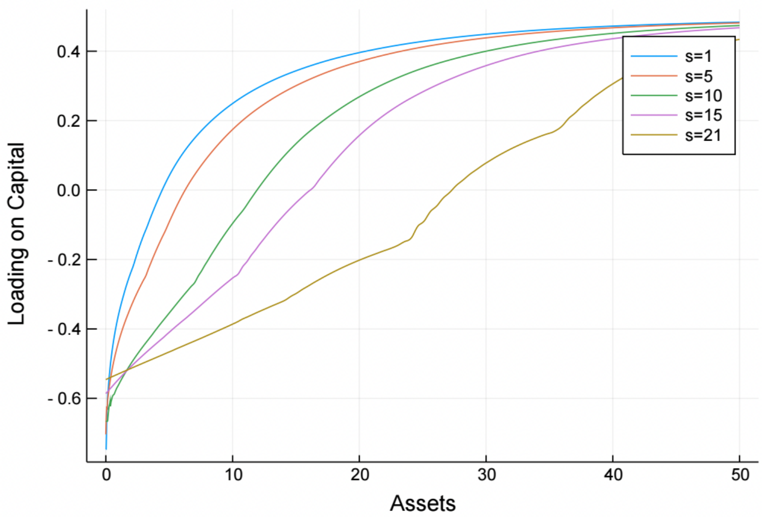


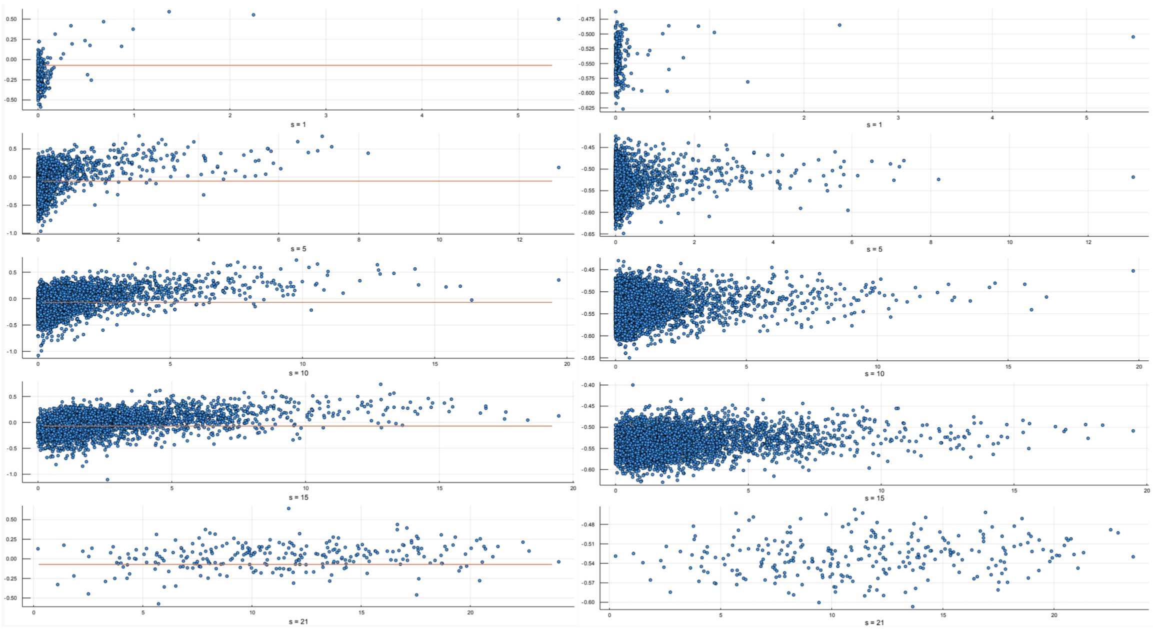
FIGURE 6. *Belief Loading on Aggregate Capital in REE*

Note: Each "s" stands for an individual productivity shock level. The figure shows the loading on aggregate capital from a rational expectations equilibrium. It shows that the REE beliefs should be a function of both idiosyncratic shocks and individual asset holding. The functional form might be highly nonlinear as $s = 21$ shows.

We see that, in general, there is a positive association of loading on the aggregate capital in the learning rule to the asset level. The loading on the aggregate capital shows how the agents believe a negative aggregate shock affects their future marginal utility from consumption. When the loading on capital becomes higher, the agent thinks a negative aggregate shock will have a smaller impact on their future, and hence they would cut their consumption less today. The extreme case is that when the agents are very wealthy, and the loading on capital becomes positive, they would increase their consumption today under a negative aggregate income shock. There will be more labor supply under a negative aggregate shock, and hence the marginal return for capital increases according to the Cobb-Douglas production function. In this situation, for an asset-rich agent, the increase in the capital returns cancels out. It even outweighs the negative impact of a lower labor income from the negative aggregate shock. As a result, the very wealthy agents with a positive loading on the capital increase consumption when there is a negative productivity shock.

If the learning gain is low, the asset-rich agents use a long period of historical data to estimate the beliefs for the learning rule. As a result, their beliefs on aggregate capital are underestimated. Consequently, the association of belief loading on aggregate capital to the asset level will become less strong in a low gain setting. Figure (7) shows the scatter plot of a large number of agents drawn from the ergodic distribution with locally rational expectations. We can see that the association between asset and belief is much weaker under the low gain setting than under the high-gain setting. An important takeaway here is that the simple learning rule that only uses aggregate capital and aggregate shock as the regressors omit two variables important to forecasting the shadow price: idiosyncratic shocks and individual asset holding. We will extend the learning rule to include these two variables in later sections to show the corresponding results.

Now we can analyze impulse response functions in Figure (4) and Figure (5) are different from each other. Specifically, with a low learning gain as in Figure (4), rich agents use extended historical data to forecast their future shadow price and remember what it was like to have a low level of asset holding from the past. As a result, these agents underestimate the belief loading on the aggregate capital. Facing an aggregate shock, these rich agents tend to cut their consumption more than what the REE would justify. As a result, we see that consumption has a more considerable fall in Figure (4b) than the impulse responses under REE. Simultaneously, these rich agents with underestimated beliefs also tend to save more than the REE would justify, which is shown in Figure (4a).



(a) *Learning Gain* = 0.1 **(b)** *Learning Gain* = 0.001

FIGURE 7. *Belief Estimates on Aggregate Capital*

Note: The scatter plots of asset holding and belief loading on aggregate capital. The left panel is simulated with a high learning gain, whereas the right panel is simulated with a small learning gain. The high learning gain presents a positive association of asset holding and belief loading.

This observation shows that locally rational agents have a certain level of habit persistence behaviors. However, this habit persistence only exists in a heterogeneous-agent model. Recall that the idiosyncratic income shocks introduce heterogeneity to the population. Consequently, the asset holdings can present a distribution over a wide range. Intuitively, the agents who currently enjoy high-income levels might have gone through a low-income phase and vise versa. Whereas in the representative-agent model, all agents always have the same levels of income. The variation in income only comes from the aggregate shocks, which only cause the wealth to fluctuate slightly away from the steady-state level. As a result, the representative agents can learn the corresponding beliefs of the rational expectations equilibrium over time because they are never too far away from the steady-state wealth level. To illustrate the insight, we simulate the impulse response functions from the LREE with the same level of low learning gain. It shows that the LREE impulse responses match the REE ones.

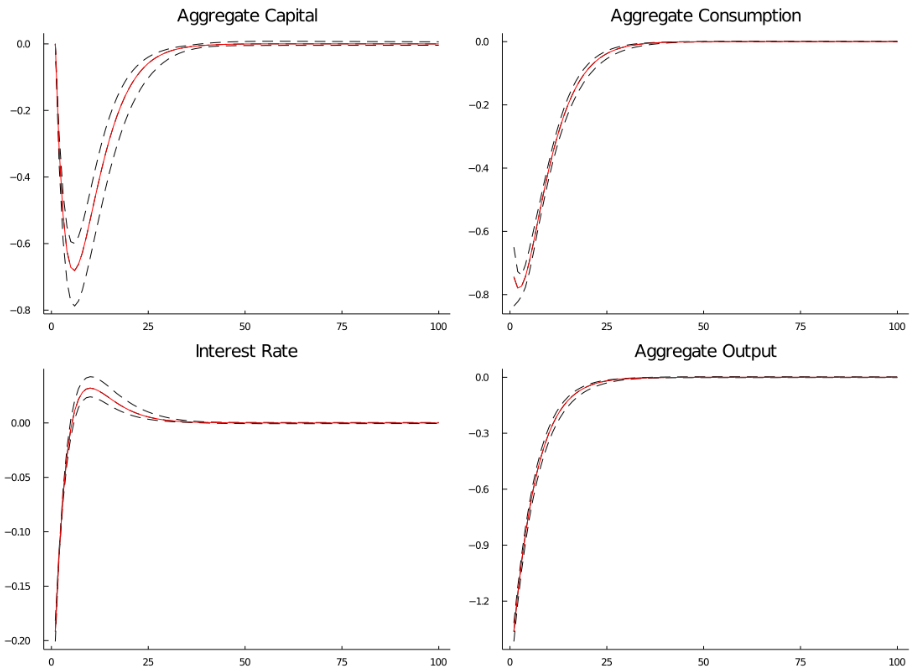


FIGURE 8. *IRFs from Representative Agent*

Note: This figure is simulated with a representative-agent model. The impulse response functions from the LREE (black dashed lines) with low learning gain compared to REE (red dashed lines). The three black dash-lines are 97.5%, 50%, and 0.5% responses from the ergodic distributions of the LREE.

II.8.4 Extended Forecasting Function

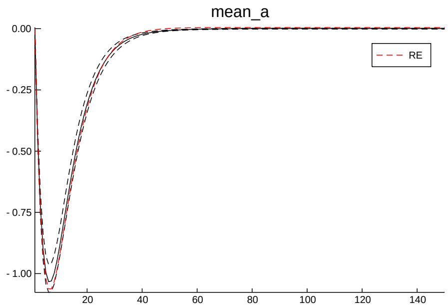
We can also endow agents with a more complicated forecasting rule. To do this, we amend the information function to be

$$I(a,\epsilon,\Omega) = \Bigg(1, \log\left(\int a d\mu(a,\epsilon,\psi)/\bar{K}\right), \log(\theta), \log\left(\int a d\mu(a,\epsilon,\psi)/\bar{K}\right)(a-\bar{a}),$$
$$\log(\theta)(a-\bar{a}), \log\left(\int a d\mu(a,\epsilon,\psi)/\bar{K}\right)\epsilon, \log(\theta)\epsilon\Bigg).$$

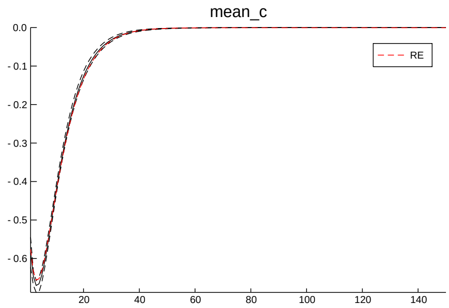
Now when learning and forecasting their future marginal utility of saving, agents take into account how their individual states (asset holding and idiosyncratic shocks) interact with aggregate variables. Specifically, this extended learning rule can approximate the relationship between the belief loadings and individual state variables represented in Figure (6).

This extended forecasting rule captures the monotonic dependence of the beliefs of the agents on individual states. Comparing impulse responses in Figure (9) to Figure (4), we see that including just a set of interaction terms generates impulse response that almost precisely line up with those of the rational expectations equilibrium. This simulation from the extended learning rule justifies using REE in a heterogeneous-agent environment. The agents do not necessarily need to know the whole distribution of the state variable and the economy's structure to make optimal decisions. Figure (9) shows that when individual variables are introduced in the learning rule, the agents can effectively learn the positive associations between the belief estimates and the asset. Agents can still learn the correct beliefs quickly even when the constant learning gain is as low as 0.001.

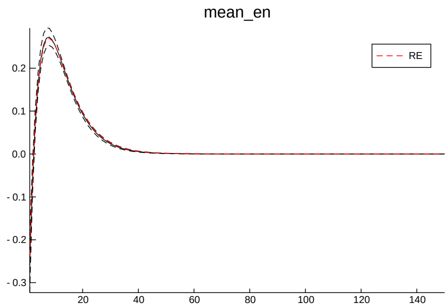
An interesting observation is that the borrowing constraint plays a very small role in the LREE results here. It is true that when the borrowing constraint is



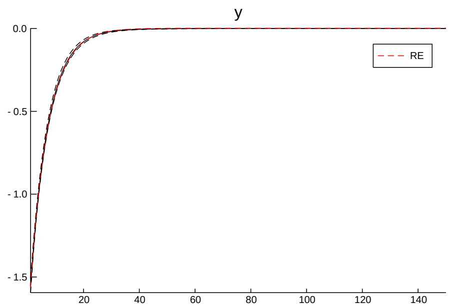
(a) *Aggregate Capital*



(b) *Aggregate Consumption*



(c) *Aggregate Labor Hours*



(d) *Aggregate Output*

FIGURE 9. *IRFs from Expanded Learning Rule with Gain of* 0.001
Note: This figure is simulated with a heterogeneous-agent model. The learning rule includes both aggregate variable and individual variables. The impulse response functions from the LREE (black dashed lines) with low learning gain compared to REE (red dashed lines). The three black dash-lines are 97.5%, 50%, and 0.5% responses from the ergodic distributions of the LREE.

relaxed, there will be fewer agents who are on the constraint. However, the main result is driven by the rich agents who underestimate their belief loadings due to a low learning gain. This is very different from most heterogeneous-agent papers that find that the model behaves differently due to the borrowing constraint. In the LREE setting, the deviation to the REE is derived from the interaction of learning and heterogeneity.

II.8.5 Conclusion

This chapter extends the analysis of adaptive learning and shadow-price learning to a heterogeneous-agent environment. A novel concept of local rationality is introduced. Local rationality assumes that agents can make fully optimal decisions under idiosyncratic shocks in the absence of aggregate uncertainty. However, agents do not know the equilibrium mapping from the economy's aggregate states to the distribution of state variables and market-clearing prices. We found novel interactions between the learning mechanism and the distribution of individual variables under adaptive learning in the heterogeneous-agent model. Suppose the agents use a simple learning rule that contains only the aggregate variables. In that case, the locally rational agents can behave differently from the rational agents because they can't adjust quickly to the beliefs that correspond to their asset level. The two exceptions are when the learning gain is high and the learning rule is extended to include the individual variables. This chapter contributes to two works of literature. First, the LREE environment can be used as an efficient computational method for approximating the impulse response functions under rational expectations. To achieve this approximation, the modeler needs to include idiosyncratic variables in the learning rule. This finding also justifies using rational expectations in the recent development of heterogeneous-agent models where most results are drawn from the REE assumption. Although the assumption is based on the agents' understanding of the vastly complicated model and its dynamics, they can learn it with relatively simple information such as aggregate shocks and individual shocks. In addition to the computational contribution, future modelers could also investigate the behavioral aspect of shadow-price learning. Future work could be done in re-examining how the heterogeneous agent New Keynesian (HANK) model would still hold the results when the model deviates from REE to LREE.

CHAPTER III
LOCALLY RATIONAL HETEROGENEOUS AGENT NEW KEYNESIAN MODEL

III.1 Introduction

This chapter is a natural extension of the previous one. Here, I introduce local rationality to a New Keynesian (LRHANK) economy with incomplete markets and sticky nominal prices. Households are heterogeneous and face idiosyncratic wage risks. Both aggregate productivity shocks and monetary policy shocks are incorporated into the model. Both households and intermediate-good producers are assumed to be locally rational because they make optimal state-contingent decisions in the absence of aggregate uncertainties. Agents use estimated econometric models to forecast their shadow prices to accommodate aggregate uncertainties. For simplicity, the model is set to have a zero-inflation trend. In a calibrated model that captures income inequality, I implement a monetary experiment to see how monetary policy plays a role in household wealth inequality. I show that the aggregate responses to policies differ from their counterparts in a similar model with entirely rational agents, which further confirms that adaptive learning introduces behavior implications in heterogeneous-agent models.

I explore the implication of local rationality introduced in the previous chapter to a Heterogeneous Agent New Keynesian (HANK) model. As a more realistic model than the representative-agent counterpart, the HANK model can be used to match the distributions of wealth and marginal propensities to consume. New results have been found through the lens of the HANK model. Kaplan et al. (2018) find that the indirect effects of an unexpected decrease in interest rates operating through a general equilibrium increase in labor demand outweigh the direct effects of inter-temporal substitution. McKay et al. (2016) find the power of forward guidance smaller in a HANK model than in the standard model. Bhandari et al. find that the Ramsey planner's optimal policy responses differ from the representative agent economy in magnitudes and directions. However, all of the results are drawn from a rational expectation equilibrium. REE imposes strong assumptions on agents' knowledge about the economic structure and the law of motions of some large-dimension states. In this paper, agents are instead assumed to be locally rational and make optimal state-contingent decisions in the absence of aggregate monetary policy uncertainty. To accommodate aggregate uncertainties, these agents use estimated econometric models to extend their rational shadow-price forecasts. The local rationality concept is identical to the previous chapter. The only difference here is the environment that includes price-stickiness.

In a calibrated model, I show that the aggregate responses to policies in a restricted perception equilibrium differ from their counterparts in a similar model with rational agents. Specifically, I show that wealth inequality's response to a monetary shock or a productivity shock differs in an LRHANK model than a HANK model. In a HANK model with rational expectations, the wealth distribution does not react much to aggregate shocks. In contrast, we get a lot of movements in the wealth inequality from the LRHANK model. I further show that the movement in the wealth distribution under LRHANK is mainly coming from the top wealthy agents.

III.2 Literature Review

Although the traditional monetary policy tools are not well-suited to achieve distributional goals, it is still important for policymakers to understand and monitor the effects on different groups within society. In general, monetary policy affects inequality, and rising inequality affects the effectiveness of the policies. There is a wide range of papers that connect monetary policy to inequality. The related literature is trying to understand the connection from the empirical perspective. Cobian et al. (2017) studied the effects of monetary policy shocks on consumption and income inequality in the United States using data from the Consumer Expenditure Survey. They found that monetary policy shocks account for a non-trivial component of the historical variations in inequality. Specifically, a contractionary monetary policy shock systematically increases inequality. Other research by Ostry et al. (2019) over recent decades supports these findings. Their study finds that an unanticipated 100 basis-point decline in the interest rate lowers the Gini measure of inequality by 1.25 percent in the short term and by 2.25 in the medium term. These results are in line with the general finding that contractionary monetary policy makes wealth inequality worse. Amaral (2017) discussed a wide range of channels that monetary policy might have a distributional effect but commented that the link between monetary policy and inequality is still inconclusive. A small body of works exists to establish the connection between monetary policy and income/wealth inequality by reviewing the theoretical channels. Dolado et al. (2018) focus exclusively on the earnings heterogeneity channel through the asymmetric nature of searching-and-matching frictions.

However, all of the theoretical models are drawn from the assumption of rational expectations equilibrium. This paper serves as the first one to understand how adaptive learning can introduce the distributional effects from the aggregate shocks. I found that a contractionary monetary policy can exacerbate wealth inequality when the agents are locally rational with a simple learning rule that only includes aggregate variables and a relatively low learning gain. This distributional effect disappears when the agents are entirely rational, which matches the results from Amaral (2017). The productivity shock also produces movement in the wealth distribution, but the direction is more mixed than a monetary shock. I further present the computational results that the top-wealthy agents primarily drive the distributional effects from the monetary and productivity shocks. In this learning environment, these top-wealthy agents behave more like an asset-poor type in their rational expectations counterpart.

III.3 Baseline HANK Model

I consider a benchmark new Keynesian economy with heterogeneous agents, incomplete markets, and nominal rigidities. There are four sectors: households, final-good producers, intermediate-good producers, and the government. Price stickiness is introduced in the style of Rotemberg (1982). The model is purposefully set to be a simple one to give insight into what local rationality brings in a HANK environment. Both aggregate productivity shocks and aggregate monetary shocks are considered.

III.3.1 Households

A unit mass of households makes choices to maximize their present discounted value of lifetime utility evaluated over stochastic streams of the final consumption good $\{c_t(\omega)\}_t$ and labor $\{n_t(\omega)\}_t$. Individual ω's preferences are ordered by

$$E_0^{\omega} \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t(\omega)^{1-\sigma} - 1}{1-\sigma} - \frac{n_t(\omega)^{1+\chi}}{1+\chi} \right) \tag{III.1}$$

where $\sigma,\ \chi,\ \gamma > 0$. Household ω supplies $z_t(\omega)n_t(\omega)$ units of labor at time t in the labor market in return of common wage w_t per unit of labor. Here $z_t(\omega)$ is an idiosyncratic productivity shock that affects household ω. Households trade one-period riskless bonds b_t at time t up to a borrowing constraint $\underline{b}$ with each other and with the government. The real price of one unit of the riskless bond is normalized to 1. The bond purchased at time t has a nominal rate return of $1+i_t$ at time $t+1$. Let Π_t be the inflation rate at time t and $d_t(\omega)$ be the dividend received by household ω from the intermediate-good producers measured in units of the final good. Finally, the government takes a lump-sum tax T_t from the household in each period. The household's problem is to, taking idiosyncratic productivity shock $z_t(\omega)$, lump-sum tax T_t, dividend $d_t(\omega)$, inflation rate Π_t,wage w_t, the nominal interest rate i_t and initial bond holding b_{-1}, as given, choose streams of the final good $\{c_t(\omega)\}$, labor supply $\{n_t(\omega)\}$, and bond holdings $\{b_t(\omega)\}$ to maximize (III.1) subject to a period budget constraint and a borrowing constraint

$$c_t(\omega) + b_t(\omega) = z_t(\omega)n_t(\omega)w_t + \left(\frac{1+i_{t-1}}{1+\Pi_t}\right) b_{t-1}(\omega) + d_t(\omega) - T_t \tag{III.2}$$

$$b_t(\omega) > \underline{b} \tag{III.3}$$

Household ω's utility maximization problem yields the following first-order conditions

$$c_t^{-\sigma}(\omega) = \beta(1+i_t)E_t^{\omega}\lambda_{t+1}(\omega) \tag{III.4}$$

$$\lambda_t(\omega) = \frac{c_t^{-\sigma}(\omega)}{1+\Pi_t} \tag{III.5}$$

$$n_t^{\chi}(\omega) = c_t^{-\sigma}(\omega)z_t(\omega)w_t \tag{III.6}$$

where Eq.(III.4)-(III.5) are the intertemporal Euler equation and Eq.(III.6) is the standard intra-temporal labor leisure trade-off equation. It will show that it is convenient to write the Euler equation into (III.4)-(III.5) for the introduction of local rationality. Furthermore, aggregate consumption C_t is defined as the integration of each agent's consumption.

$$C_t = \int_0^1 c_t(\omega)d\omega \tag{III.7}$$

III.3.2 Final Good Producers

A final good Y_t is produced by competitive firms that use a continuum of intermediate goods $\{y_t(\omega)\}_{\omega\in(0,1)}$ in a production function

$$Y_t = \left(\int_0^1 y_t(\omega)^{\frac{\nu-1}{\nu}} d\omega\right)^{\frac{\nu}{\nu-1}} \tag{III.8}$$

with $\nu > 1$ denoting the elasticity of substitution of different intermediate goods used in the production function. The final-good producer solves the following profit maximization problem, taking the final-good prices P_t and intermediate-good prices $\{p_t(\omega)\}_\omega$ as given

$$\max_{\{y_t(\omega)\}_{\omega\in[0,1]}} P_t \left(\int_0^1 y_t(\omega)^{\frac{\nu-1}{\nu}} d\omega\right)^{\frac{\nu}{\nu-1}} - \int_0^1 p_t(\omega)y_t(\omega)d\omega \tag{III.9}$$

The profit maximization problem in (III.9) yields a demand function for intermediate goods

$$y_t(\omega) = \left(\frac{p_t(\omega)}{P_t}\right)^{-\nu} Y_t \tag{III.10}$$

The shadow price of producing one extra unit of final good implies the nominal price P_t which is written as follows

$$P_t = \left(\int_0^1 p_t(\omega)^{1-\nu} \right)^{\frac{1}{1-\nu}} \tag{III.11}$$

III.3.3 Intermediate Good Producers

There is a unit mass of intermediate-good producers. Intermediate goods are produced by these firms and sold in monopolistically competitive markets. Firm ω uses the following technology to produce intermediate good $y_t(\omega)$ at time t

$$y_t(\omega) = \theta_t \ell_t^{\alpha}(\omega) \tag{III.12}$$

where θ_t is an economy-wise aggregate productivity shock that hits every intermediate firm, and $\ell_t(\omega)$ is the amount of effective labor hired by firm ω. The logarithm of θ_t follows an AR(1) process as follows

$$\log(\theta_t) = \rho_\theta \log(\theta_{t-1}) + \xi_t^\theta \tag{III.13}$$

$$\xi_t^\theta \sim \text{Normal}(0, \sigma_\theta^2) \tag{III.14}$$

Cost minimization implies that the intermediate good producer ω's marginal cost for producing one extra unit is as follows.

$$\mathcal{M}_t(p_t(\omega)) = \frac{w_t}{\alpha\theta_t} \left(\frac{p_t(\omega) Y_t}{\theta_t P_t} \right)^{\frac{1-\alpha}{\alpha}} \tag{III.15}$$

These monopolistic firms face downward-sloping demand curves specified by Eq.(III.10) and choose prices $p_t(\omega)$ while bearing quadratic Rotemberg (1982) price adjustment costs measured in units of the final consumption good written as follows.

$$\mathcal{C}(p_t(\omega), p_{t-1}(\omega)) = -\frac{\phi}{2}\left(\frac{p_t(\omega)}{p_{t-1}(\omega)} - 1\right)^2 \tag{III.16}$$

Firm ω chooses prices $\{p_t(\omega)\}$ to maximize profit, taking aggregate consumption $\{C_t\}$, aggregate final goods production $\{Y_t\}$, aggregate price $\{P_t\}$, wage $\{w_t\}$, aggregate productivity shock $\{\theta_t\}$ as given. Each firm is assumed to value profit streams with a stochastic discount factor driven by aggregate consumption. Intermediate firm ω's profit maximization problem is given as follows. The derivation is shown in Appendix I.2.

$$\max_{\{p_t(\omega)\}_{t=0}^{t=\infty}} E_0 \sum_t \beta^t \frac{C_t^{-\sigma}}{C_0^{-\sigma}} \left\{ \left(\frac{p_t(\omega)}{P_t}\right)^{-\nu} Y_t \left[\frac{p_t(\omega)}{P_t} - \mathcal{M}_t(p_t(\omega))\right] + \mathcal{C}(p_t(\omega), p_{t-1}(\omega)) \right\} \tag{III.17}$$

Assume all of the intermediate firms fully believe that the economy is in a symmetric equilibrium. The first-order condition for intermediate firms' price-setting problem can be written as follows. The derivation is provided in Appendix I.1.

$$C_t^{-\sigma}\left((\nu-1)Y_t + \left(\frac{1-\alpha-\alpha\nu}{\alpha^2}\right)\left(\frac{Y_t}{\theta_t}\right)^{\frac{1}{\alpha}} w_t + \phi\Pi_t(1+\Pi_t)\right) = \phi\beta E_t\Lambda_{t+1} \tag{III.18}$$

$$\Lambda_t = C_t^{-\sigma}\Pi_t(1+\Pi_t) \tag{III.19}$$

where Λ_t is the shadow price for the intermediate-good producers. This shadow price is not just a numeric number but has an economic meaning. Specifically, the shadow price is the marginal revenue for the firm if they increase the price by one unit in the next period. The optimality condition states that the firm will choose a price such that the marginal cost is equal to the marginal revenue. For simplicity, we assume the firms know the aggregate consumption when they make choices about the current price. Eq.(III.18)-(III.19) state that if the intermediate producers expect higher inflation from t to $t+1$, they will choose higher P_t to smooth the price adjustment cost. This smoothing behavior comes from the adjustment cost being a quadratic function in the price change. Finally, the dividends from the intermediate firm are uniformly distributed across the households so that.

$$d_t(\omega) = D_t = Y_t - w_t L_t - \frac{\phi}{2}\Pi_t^2 \tag{III.20}$$

There are important implications for different dividends schedules. For the benchmark model, I only consider the simple uniform dividend schedule for now.

III.3.4 Government

According to a Taylor rule, the monetary authority sets the nominal interest rate on bonds i_t up to an aggregate policy shock.

$$i_t = \bar{i} + \phi_\Pi \Pi_t + \varepsilon_t \tag{III.21}$$

where $\bar{i}$ is the nominal interest rate target, and ϕ is the feedback parameter for inflation. Note that in this benchmark model, the inflation target is 0. ε_t is the aggregate policy shock that follows a stochastic process and follows an AR(1) process as follows

$$\varepsilon_t = \rho_\varepsilon \varepsilon_{t-1} + \xi_t^\varepsilon \tag{III.22}$$

$$\xi_t^\varepsilon \sim \text{Normal}(0, \sigma_\varepsilon^2) \tag{III.23}$$

The government borrows by selling bonds B_t to the households, and the outstanding debt is financed through a lump-sum tax. The government budget constraint is written as

$$\frac{1+i_{t-1}}{1+\Pi_t}B_{t-1} = B_t + T_t \tag{III.24}$$

Assume that the total bond supply or government debt B_t is constant and equals $\bar{B}$ in each period, and the government chooses T_t to balance the budget constraint.

III.3.5 Market Clearing Conditions

In a symmetric equilibrium, $p_t(\omega) = P_t$, $y_t(\omega) = Y_t$ for all $\omega \in [0, 1]$. Also the labor demand from each intermediate firm is the same and equals to the aggregate labor demand, denoted as L_t, i.e. $\ell_t(\omega) = L_t$ for all $\omega \in [0, 1]$. Market clearing conditions are

$$L_t = \int_0^1 z_t(\omega)n_t(\omega)d\omega \tag{III.25}$$

$$C_t = Y_t - \frac{\phi}{2}\Pi_t^2 \tag{III.26}$$

$$B_t = \int_0^1 b_t(\omega)d\omega \tag{III.27}$$

where Eq.(III.25), Eq.(III.26), and Eq.(III.27) are the market clearing conditions for labor market, final good market and bond market respectively.

III.4 Equilibria

III.4.1 Representative Agent

This subsection introduces the equilibria of the model with representative agents. I define the rational expectations equilibrium and the bounded rational equilibrium here.

Definition 5

Given an initial condition, a constant sequence of $\{B_t\}$ satisfying $B_t = \bar{B}$, and sequences of aggregate shocks, a *rational expectations equilibrium* is a stochastic sequence of prices and inflation $\{i_t, w_t, \Pi_t\}$, household allocations $\{b_t, n_t, c_t, \lambda_t\}$, aggregates $\{Y_t, L_t, \Lambda_t, D_t, T_t\}$ such that

1. Given prices and inflation $\{i_{t-1}, i_t, w_t, \Pi_t\}$, transfers $\{D_t, T_t\}$, household allocations $\{b_t, n_t, c_t, \lambda_t\}$ solve the household's problem

$$\begin{aligned}
c_t + b_t &= n_t w_t + \left(\frac{1 + i_{t-1}}{1 + \Pi_t}\right) b_{t-1} + D_t - T_t \\
c_t^{-\sigma} &= \beta(1 + i_t) E_t \lambda_{t+1} \\
\lambda_t &= \frac{c_t^{-\sigma}}{1 + \Pi_t} \\
n_t^{\chi} &= c_t^{-\sigma} w_t
\end{aligned}$$

2. Given prices $\{i_t, w_t\}$ and the consumption $\{c_t\}$, the intermediate firms choose $\{\Pi_t, Y_t, L_t, D_t\}$ maximize their profit

$$c_t^{-\sigma}\left((\nu-1)Y_t + \left(\frac{1-\alpha-\alpha\nu}{\alpha^2}\right)\left(\frac{Y_t}{\theta_t}\right)^{\frac{1}{\alpha}} w_t + \phi\Pi_t(1+\Pi_t)\right) = \phi\beta E_t\Lambda_{t+1}$$

$$\Lambda_t = c_t^{-\sigma}\Pi_t(1+\Pi_t)$$

$$D_t = Y_t - w_t L_t - \frac{\phi}{2}\Pi_t^2$$

$$Y_t = \theta_t L_t^{\alpha}$$

3. Given prices and inflation $\{i_{t-1}, \Pi_t\}$, government chooses $\{i_t, T_t\}$

$$\frac{1+i_{t-1}}{1+\Pi_t}B_{t-1} = B_t + T_t$$

$$i_t = \bar{i} + \phi_\Pi \Pi_t$$

4. All markets clear

$$L_t = n_t$$

$$c_t = Y_t - \frac{\phi}{2}\Pi_t^2$$

$$b_t = \bar{B}$$

The adaptive learning approach typically assumes agents have a correctly specified forecasting model with unknown parameters. Before defining the locally rational expectations equilibrium (LREE) for the representative agent model, it is useful to get the forecasting model for both the households and the intermediate firms. Note that expectations enter the temporary equilibrium through both Eq.(III.4) and Eq.(III.18). The state variables are i_{t-1}, θ_t and ε_t. Consider an approximation of the rational expectations equilibrium around the steady state by log-linearization. Note that the rational expectations equilibrium implies that the linearized solutions for λ_{t+1} are as follows.

$$\log\left(\frac{\lambda_{t+1}}{\bar{\lambda}}\right) = \bar{\psi}_0^{\lambda} + \bar{\psi}_1^{\lambda} \cdot i_t + \bar{\psi}_2^{\lambda} \cdot \log(\theta_{t+1}) + \bar{\psi}_3^{\lambda} \cdot \varepsilon_{t+1} \tag{III.28}$$

The law of motion for the state variables are as follows.

$$i_t = \bar{\psi}_0^i + \bar{\psi}_1^i \cdot i_{t-1} + \bar{\psi}_2^i \cdot \log(\theta_t) + \bar{\psi}_3^i \cdot \varepsilon_t \tag{III.29}$$

$$\log(\theta_{t+1}) = \rho_\theta \log(\theta_t) + \xi_{t+1}^{\theta} \tag{III.30}$$

$$\varepsilon_{t+1} = \rho_\varepsilon \varepsilon_t + \xi_{t+1}^{\varepsilon} \tag{III.31}$$

These four equations above imply.

$$\log\left(\frac{\lambda_{t+1}}{\bar{\lambda}}\right) = \bar{\psi}_0 + \bar{\psi}_1 i_{t-1} + \bar{\psi}_2 \log(\theta_t) + \bar{\psi}_3 \varepsilon_t + \bar{\psi}_{\xi 1} \xi_{t+1}^{\theta} + \bar{\psi}_{\xi 2} \xi_{t+1}^{\varepsilon} \tag{III.32}$$

Define the belief vector and information set as follows.

$$\psi = [\psi_0 \quad \psi_1 \quad \psi_2 \quad \psi_3]' \tag{III.33}$$

$$x_t = [1 \;\; i_{t-1} \;\; \log(\theta_t) \;\; \varepsilon_t]' \tag{III.34}$$

The (linearized) rational expectation for household's shadow price λ_{t+1} at time t is.

$$E_t(\lambda_{t+1}) = \bar{\lambda} \exp(\bar{\psi}' \cdot x_t) \tag{III.35}$$

Similarly, the (linearized) rational expectations for intermediate firms' shadow price Λ_{t+1} is.

$$E_t(\Lambda_{t+1}) = \bar{\Lambda} + \bar{\Psi}' \cdot x_t \tag{III.36}$$

where $\bar{\Psi}'$ is the coefficients of the forecasting model for the intermediate-good producers. Now, I define the locally rational equilibrium with the forecasting rules from the linearized rational expectations equilibrium.

<u>Definition 6</u>

Given an initial condition, a constant sequence of $\{B_t\}$ satisfying $B_t = \bar{B}$, and sequences of aggregate shocks, a *locally rational expectations equilibrium* is a stochastic sequence of prices and inflation $\{i_t, w_t, \Pi_t\}$, transfers $\{d_t, T_t\}$, household allocations $\{b_t, n_t, c_t, \lambda_t\}$, aggregates $\{Y_t, L_t, C_t, \Lambda_t, D_t, T_t\}$, and agent beliefs $\{\psi_t, \Psi_t\}$ such that

1. Given prices and inflation $\{i_{t-1}, i_t, w_t, \Pi_t\}$, household belief $\{\psi_t\}$, and household allocations $\{b_t, n_t, c_t, \lambda_t\}$ solve the household's problem.

$$\begin{aligned}
&c_t + b_t = n_t w_t + \left(\frac{1 + i_{t-1}}{1 + \Pi_t}\right) b_{t-1} + D_t - T_t \\
&c_t^{-\sigma} = \beta(1 + i_t) E_t^{\psi} \lambda_{t+1} \\
&\lambda_t = \frac{c_t^{-\sigma}}{1 + \Pi_t} \\
&n_t^{\chi} = c_t^{-\sigma} w_t \\
&E_t^{\psi} \lambda_{t+1} = \bar{\lambda} \exp(\psi_{t-1}' x_t)
\end{aligned}$$

2. Given prices $\{w_t\}$ and the consumption $\{c_t\}$, the intermediate firms choose $\{\Pi_t, Y_t, L_t, D_t\}$ maximize their profit.

$$c_t^{-\sigma}\left((\nu-1)Y_t + \left(\frac{1-\alpha-\alpha\nu}{\alpha^2}\right)\left(\frac{Y_t}{\theta_t}\right)^{\frac{1}{\alpha}} w_t + \phi\Pi_t(1+\Pi_t)\right) = \phi\beta E_t^{\Psi}\Lambda_{t+1}$$
$$\Lambda_t = c_t^{-\sigma}\Pi_t(1+\Pi_t)$$
$$D_t = Y_t - w_t L_t - \frac{\phi}{2}\Pi_t^2$$
$$Y_t = \theta_t L_t^{\alpha}$$
$$E_t^{\Psi}\Lambda_{t+1} = \bar{\Lambda} + \Psi_{t-1}' x_t$$

3. Given prices and inflation $\{i_{t-1}, \Pi_t\}$, government chooses $\{i_t, T_t\}$ such that.

$$\frac{1+i_{t-1}}{1+\Pi_t}B_{t-1} = B_t + T_t$$
$$i_t = \bar{i} + \phi_{\Pi}\Pi_t + \varepsilon_t$$

4. All markets clear as follows.

$$L_t = n_t$$
$$c_t = Y_t - \frac{\phi}{2}\Pi_t^2$$
$$b_t = \bar{B}$$

5. Agents belief update through the recursive least square algorithm as follows.

$$R_t = R_{t-1} + \gamma_t(x_t x_t' - R_{t-1})$$
$$\psi_t = \psi_{t-1} + \gamma_t R_t^{-1} x_t(\lambda_t - \psi_{t-1}' x_t)$$
$$\Psi_t = \Psi_{t-1} + \gamma_t R_t^{-1} x_t(\Lambda_t - \Psi_{t-1}' x_t)$$

Note that the household beliefs and the intermediate firms' beliefs enter the state space, and thus the state variables for this dynamic now are i_{t-1}, θ_t, ε_t, ψ_{t-1}, Ψ_{t-1} and the temporary equilibrium is defined as.

$$\mathcal{TE}(i_{t-1}, \theta_t, \varepsilon_t, \psi_{t-1}, \Psi_{t-1}) \to (i_t, w_t, \Pi_t, b_t, n_t, c_t, \lambda_t, Y_t, L_t, C_t, \Lambda_t, D_t, T_t, \psi_t, \Psi_t) \tag{III.37}$$

III.4.2 Heterogeneous Agent

This subsection introduces three different definitions of equilibria with heterogeneous agents: stationary recursive equilibrium, stochastic recursive equilibrium, and locally rational stochastic recursive equilibrium. Suppressing all of the aggregate shocks in the model by setting $\theta_t = 1$ and $\varepsilon_t = 0$ for all t, I define a stationary recursive equilibrium.

<u>Definition 7</u>

Given an initial condition, a constant sequence of $\{B_t\}$ satisfying $B_t = \bar{B}$, and suppressing aggregate productivity and monetary shocks, a *stationary recursive equilibrium* consists of a measure $\bar{\mu}$ over the state $s = (b, z)$; policy rules for the households $\bar{c}(b, z)$ where b is the last period's bond holding, $\bar{n}(b,z)$, $\bar{b}(b,z)$, and $\bar{\lambda}(b,z)$; prices $\bar{w}$ and $\bar{i}$; aggregate production, consumption, labor demand, and inflation: $\bar{Y}$, $\bar{C}$, $\bar{L}$, and $\bar{\Pi}$; intermediate firms shadow price and dividend $\bar{\Lambda}$ and $\bar{D}$ such that.

1. The household policy rules $\bar{c}(b,z)$, $\bar{n}(b,z)$, and $\bar{b}(b,z)$ solve recursive versions of Eq.(III.2)-(III.6) for all (b,z)

$$
\begin{aligned}
&\bar{c}(b,z) + \bar{b}(b,z) = z\bar{n}(b,z)\bar{w} + \left(\frac{1+\bar{i}}{1+\bar{\Pi}}\right) b + \bar{D} \\
&\bar{c}(b,z)^{-\sigma} \geq \beta(1+\bar{i})E\left(\bar{\lambda}(\bar{b}(b,z),z')\right) \\
&\bar{\lambda}(b,z) = \frac{\bar{c}(b,z)^{-\sigma}}{1+\bar{\Pi}} \\
&\bar{n}(b,z)^{\chi} = \bar{c}(b,z)^{-\sigma} z\bar{w}
\end{aligned}
$$

   where E is taken over z' against the stochastic process for $\{z_t\}$.

2. The intermediate-good producer maximizes their profit function and generate dividend.

$$
\begin{aligned}
&\bar{C}^{-\sigma}\left((\nu-1)\bar{Y} + \left(\frac{1-\alpha-\alpha\nu}{\alpha^2}\right)\bar{Y}^{\frac{1}{\alpha}}\bar{w} + \phi\bar{\Pi}(1+\bar{\Pi})\right) = \phi\beta\bar{\Lambda} \\
&\bar{\Lambda} = \bar{C}^{-\sigma}\bar{\Pi}(1+\bar{\Pi}) \\
&\bar{D} = \bar{Y} - \bar{w}\bar{L} - \frac{\phi}{2}\bar{\Pi}^2 \\
&\bar{Y} = \bar{L}^{\alpha}
\end{aligned}
$$

3. All markets clear

$$\int z\bar{n}(b,z)d\bar{\mu}(b,z) = \bar{L}$$
$$\int \bar{c}(b,z)d\bar{\mu}(b,z) = \bar{Y} - \frac{\phi}{2}\bar{\Pi}^2$$
$$\int \bar{b}(b,z)d\bar{\mu}(b,z) = \bar{B}$$

4. $\bar{\mu}$ is stationary under the households policy rules and the transition matrix for z: for any Borel set $\mathcal{B}$

$$\bar{\mu}(\mathcal{B}, z') = E(\bar{\mu}(\{b : \bar{b}(b,z) \in \mathcal{B}\}, z))$$

The equilibrium with aggregate shocks then extends in the standard manner by allowing policy rules, prices, and aggregates to additionally depend on the current distribution of agents μ and aggregate shocks θ and ε. The definition is given as follows.

<u>Definition 8</u>

Let household ω's individual states be (b_{-}, z). Given an initial condition, a constant sequence of $\{B_t\}$ satisfying $B_t = \bar{B}$, a *stochastic recursive equilibrium* consists of policy rules for the households $c(\omega;\mu,\theta,\varepsilon)$, $n(\omega;\mu,\theta,\varepsilon)$, $b(\omega;\mu,\theta,\varepsilon)$, and $\lambda(\omega;\mu,\theta,\varepsilon)$; prices $w(\mu,\theta,\varepsilon)$ and $i(\mu,\theta,\varepsilon)$; aggregate production, consumption, labor demand, and inflation: $Y(\mu,\theta,\varepsilon)$, $C(\mu,\theta,\varepsilon)$, $L(\mu,\theta,\varepsilon)$, and $\Pi(\mu,\theta,\varepsilon)$; intermediate firms shadow price and dividend $\Lambda(\mu,\theta,\varepsilon)$ and $D(\mu,\theta,\varepsilon)$, and a law of motion $H(\mu,\theta,\varepsilon)$ for μ such that

1. The policy rules $c(\omega;\mu,\theta,\varepsilon)$, $n(\omega;\mu,\theta,\varepsilon)$, $b(\omega;\mu,\theta,\varepsilon)$, and $\lambda(\omega;\mu,\theta,\varepsilon)$ solve recursive version of Eq.(III.2)-(III.6) for all ω, μ, and θ taking pricing functions $r(\mu,\theta,\varepsilon)$, $w(\mu,\theta,\varepsilon)$ and the law of motion $H(\mu,\theta,\varepsilon)$ as given

$$
\begin{aligned}
&c(\omega;\mu,\theta,\varepsilon)+b(\omega;\mu,\theta,\varepsilon)= \\
&z(\omega)n(\omega;\mu,\theta,\varepsilon)w(\mu,\theta,\varepsilon)+\left(\frac{1+i(\mu,\theta,\varepsilon)}{1+\Pi(\mu,\theta,\varepsilon)}\right)b_{-}(\omega)+D(\mu,\theta,\varepsilon) \\
&c(\omega;\mu,\theta,\varepsilon)^{-\sigma}\geq\beta(1+i(\mu,\theta,\varepsilon))E\left(\lambda(\omega';\mu',\theta',\varepsilon')|\omega;\mu,\theta,\varepsilon\right) \\
&\lambda(\omega;\mu,\theta,\varepsilon)=\frac{c(\omega;\mu,\theta,\varepsilon)^{-\sigma}}{1+\Pi(\mu,\theta,\varepsilon)} \\
&n^{\chi}(\omega;\mu,\theta,\varepsilon)=c(\omega;\mu,\theta,\varepsilon)^{-\sigma}z(\omega)w(\mu,\theta,\varepsilon)
\end{aligned}
$$

2. The intermediate-good producer maximizes their profit function and generates dividends.

$$
\begin{aligned}
&C(\mu,\theta,\varepsilon)^{-\sigma}\left((\nu-1)Y(\mu,\theta,\varepsilon)+\left(\frac{1-\alpha-\alpha\nu}{\alpha^2}\right)Y(\mu,\theta,\varepsilon)^{\frac{1}{\alpha}}w(\mu,\theta,\varepsilon)+\phi\Pi(\mu,\theta,\varepsilon)(1+\Pi(\mu,\theta,\varepsilon))\right) \\
&=\phi\beta E\Lambda(\mu',\theta',\varepsilon'|\mu,\theta,\varepsilon) \\
&\Lambda(\mu,\theta,\varepsilon)=C(\mu,\theta,\varepsilon)^{-\sigma}\Pi(\mu,\theta,\varepsilon)(1+\Pi(\mu,\theta,\varepsilon)) \\
&D(\mu,\theta,\varepsilon)=Y(\mu,\theta,\varepsilon)-\frac{\phi}{2}\Pi^2(\mu,\theta,\varepsilon) \\
&Y(\mu,\theta,\varepsilon)=\theta L(\mu,\theta,\varepsilon)^{\alpha}
\end{aligned}
$$

3. Given prices and inflation $\{i_{-},\Pi(\mu,\theta,\varepsilon)\}$, government chooses $\{i(\mu,\theta,\varepsilon),T(\mu,\theta,\varepsilon)\}$

$$
\begin{aligned}
T(\mu,\theta,\varepsilon)&=\frac{i_{-}-\Pi(\mu,\theta,\varepsilon)}{1+\Pi(\mu,\theta,\varepsilon)}\bar{B} \\
i(\mu,\theta,\varepsilon)&=\bar{i}+\phi_{\Pi}\Pi(\mu,\theta,\varepsilon)+\varepsilon
\end{aligned}
$$

4. All markets clear, and aggregation works as follows.

$$\int z(\omega)n(\omega;\mu,\theta,\varepsilon)d\omega = L(\mu,\theta,\varepsilon)$$
$$\int c(\omega;\mu,\theta,\varepsilon)d\omega = Y(\mu,\theta,\varepsilon) - \frac{\phi}{2}\Pi^2(\mu,\theta,\varepsilon)$$
$$\int b(\omega;\mu,\theta,\varepsilon)d\omega = \bar{B}$$
$$\int c(\omega;\mu,\theta,\varepsilon)d\omega = C(\mu,\theta,\varepsilon)$$

5. The law of motion for measure μ matches the properties of the stochastic processes and the profile of individual household policy rules.

The difficulty faced in solving a stochastic recursive equilibrium lies in the fact that policy rules and the law of motion depend on μ, a high dimensional object. The solution also imposes strong assumptions on households' knowledge about the economy's structure and how it evolves. As a comparison for the locally rational solution, which is defined as follows, I use the approach introduced by Reiter (2009) to approximate the rational expectations solution for the heterogeneous agents model.

Definition 9

Let household ω's individual states be (b_{-}, z, ψ) where ψ is the estimates of the coefficients in the forecasting rules used by agents. Given an initial condition, a constant sequence of $\{B_t\}$ satisfying $B_t = \bar{B}$, and sequences of aggregate shocks, a *locally rational stochastic recursive dynamics* consists of policy rules for the households $c(\omega;\theta,\varepsilon)$, $n(\omega;\theta,\varepsilon)$, $b(\omega;\theta,\varepsilon)$, and $\lambda(\omega;\theta,\varepsilon)$; prices $w(\theta,\varepsilon)$ and $i(\theta,\varepsilon)$; aggregate production, consumption, labor demand, and inflation: $Y(\theta,\varepsilon)$, $C(\theta,\varepsilon)$, $L(\theta,\varepsilon)$, and

$\Pi(\theta,\varepsilon)$; intermediate firms shadow price and dividend $\Lambda(\theta,\varepsilon)$ and $D(\theta,\varepsilon)$ and beliefs $\Psi(\theta,\varepsilon)$ such that

1. The policy rules $c(\omega;\theta,\varepsilon)$, $n(\omega;\theta,\varepsilon)$, $b(\omega;\theta,\varepsilon)$, and $\lambda(\omega;\theta,\varepsilon)$ solve recursive version of Eq.(III.2)-(III.6) for all ω, θ, and ε taking pricing functions $r(\theta,\varepsilon)$, $w(\theta,\varepsilon)$ as given

$$c(\omega;\theta,\varepsilon)+b(\omega;\theta,\varepsilon)=z(\omega)n(\omega;\theta,\varepsilon)w(\theta,\varepsilon)+\left(\frac{1+i(\theta,\varepsilon)}{1+\Pi(\theta,\varepsilon)}\right)b_{-}(\omega)+D(\theta,\varepsilon)$$
$$c(\omega;\theta,\varepsilon)^{-\sigma}\geq\beta(1+i(\theta,\varepsilon))E\left(\omega';\theta',\varepsilon'|\omega,\theta,\varepsilon\right)$$
$$\lambda(\omega;\theta,\varepsilon)=\frac{c(\omega;\theta,\varepsilon)^{-\sigma}}{1+\Pi(\theta,\varepsilon)}$$
$$n^{\chi}(\omega;\theta,\varepsilon)=c(\omega;\theta,\varepsilon)^{-\sigma}z(\omega)w(\theta,\varepsilon)$$

2. The intermediate-good producer maximizes their profit function and generates dividends.

$$C(\theta,\varepsilon)^{-\sigma}\left((\nu-1)Y(\theta,\varepsilon)+\left(\frac{1-\alpha-\alpha\nu}{\alpha^2}\right)Y(\theta,\varepsilon)^{\frac{1}{\alpha}}w(\theta,\varepsilon)+\phi\Pi(\theta,\varepsilon)(1+\Pi(\theta,\varepsilon))\right)$$
$$=\phi\beta E\Lambda(\theta',\varepsilon'|\theta,\varepsilon)$$
$$\Lambda(\theta,\varepsilon)=C(\theta,\varepsilon)^{-\sigma}\Pi(\theta,\varepsilon)(1+\Pi(\theta,\varepsilon))$$
$$D(\theta,\varepsilon)=Y(\theta,\varepsilon)-\frac{\phi}{2}\Pi^2(\theta,\varepsilon)$$
$$Y(\theta,\varepsilon)=\theta L(\theta,\varepsilon)^{\alpha}$$

3. All markets clear

$$\int z(\omega)n(\omega;\theta,\varepsilon)d\omega = L(\theta,\varepsilon)$$
$$\int c(\omega;\theta,\varepsilon)d\omega = Y(\theta,\varepsilon) - \frac{\phi}{2}\Pi^2(\theta,\varepsilon)$$
$$\int b(\omega;\theta,\varepsilon)d\omega = \bar{B}$$
$$\int c(\omega,\theta,\varepsilon)d\omega = C(\theta,\varepsilon)$$

4. Given prices and inflation $\{i_-, \Pi(\theta,\varepsilon)\}$, government chooses $\{i(\theta,\varepsilon), T(\theta,\varepsilon)\}$

$$T(\theta,\varepsilon) = \frac{i_- - \Pi(\theta,\varepsilon)}{1+\Pi(\theta,\varepsilon)}\bar{B}$$
$$i(\theta,\varepsilon) = \bar{i} + \phi_\Pi \Pi(\theta,\varepsilon) + \varepsilon$$

5. All households and intermediate-good producers update their estimates of coefficients in their forecasting model according to a recursive least square learning rule with exogenous sequences of gains.

$$R = R_- + \gamma(xx' - R_-)$$
$$\psi(\omega) = \psi_-(\omega) + \gamma R^{-1}x(\lambda - x'\psi_-(\omega))$$
$$\Psi = \Psi_- + \gamma R^{-1}x(\Lambda - x'\Psi_-)$$

We are ready to define the locally rational expectations equilibrium as a restricted perceptions equilibrium based on the definition of locally rational recursive dynamics. The folk theorem of the learning literature states that the long-run beliefs will converge to a restricted perceptions equilibrium (RPE) if the equilibrium is E- stable. Intuitively, RPE is characterized by self-confirming beliefs. Hold beliefs fixed at $\psi_t = \bar{\psi}$ for all t and all agents and $\Psi_t = \bar{\Psi}$ for the representative firm. Feed these fixed beliefs ψ_t and Ψ_t into the locally rational recursive dynamics without the belief evolving part. Let $\Lambda_h(\bar{\psi})$ and $\Lambda_f(\bar{\Psi})$ be the induced ergodic distribution of beliefs for households and the intermediate firm,. Let the shadow prices be λ_{t+1} and Λ_{t+1} for the households and the intermediate firm. The information set is X_t from the dynamics. We can construct the linear projection of $log(\lambda/\bar{\lambda})$ on X under measure $\Lambda(\bar{\psi})$ and . Effectively, the locally rational recursive dynamics defines a T-map for the agents beliefs with the form $\bar{\psi}' = T_h(\bar{\psi})$ and $\bar{\Psi}' = T_f(\bar{\Psi})$. Here

$$T_h(\bar{\psi}) = \mathbb{E}_{\Lambda_h(\bar{\psi})}[XX']^{-1}\mathbb{E}_{\Lambda_h(\bar{\psi})}[X \log(\lambda/\bar{\lambda})]$$
$$T_f(\bar{\Psi}) = \mathbb{E}_{\Lambda_f(\bar{\Psi})}[XX']^{-1}\mathbb{E}_{\Lambda_f(\bar{\Psi})}[X \log(\Lambda/\bar{\Lambda})]$$

Definition 10

A locally rational expectations equilibrium is a locally rational recursive dynamics with the ergodic distributions of the beliefs ψ^* such that

$$\psi^* = \mathbb{E}_{\Lambda_h(\psi^*)}[XX']^{-1}\mathbb{E}_{\Lambda_h(\psi^*)}[X \log(\lambda/\bar{\lambda})]$$
$$\Psi^* = \mathbb{E}_{\Lambda_a(\Psi^*)}[XX']^{-1}\mathbb{E}_{\Lambda_a(\Psi^*)}[X \log(\Lambda/\bar{\Lambda})]$$

This framework approximately nests the rational expectations equilibrium, and the LREE is a type of restricted perception equilibrium because the agents use a simplified learning rule, and the coefficients are self-referential from the agents' perspective.

III.5 Calibration and Computation

I choose three sets of parameters: (i) parameters related to household preferences, productions, and aggregate shocks; (ii) initial conditions; and (iii) stochastic processes for idiosyncratic shocks. The computation method follows the algorithm used in Evans, Li, and McGough (2019). I consider small supports for the aggregate shocks for the baseline model so that the nominal interest rate does not or rarely go below zero in simulations. The implication of introducing a zero lower bound will be explored in the extension of the paper in the future. For simplicity, I set the inflation trend to be zero. Note that a non-zero inflation trend can have significant implications for the model behavior.

III.5.1 Preferences, Productions and Aggregate Shocks

The parameters match standard representative agent calibrations such as Schmitt-Grohe and Uribe (2004) and Siu (2004). I set the CES parameters for household on consumption and labor at $\sigma = 2$, $\chi = 2$, and the discount factor β is set so that the targeted nominal interest rate is at $\bar{i} = 0.02$. The firms operate a decreasing return to scale technology so that the labor income accounts for 65% of the total output. To a first-order approximation, the steady-state markups of the intermediate firms are set to be 20%, which implies the elasticity of substitution among the intermediate goods is at $\nu = 6$. The parameter for the adjustment cost is set at $\phi = 20$ to match the slope of the Philips curve, as estimated by Sbordone (2002). The bond supply from the government is set so that the ratio of national debt to GDP is 75%. Targeted inflation is 0%. The inflation feedback parameter is set to be $\phi_\Pi = 2.0$, which means the central bank aggressively targets the inflation rate. For example, Coibion (2012) shows that the non-zero inflation trend subject to zero-bound interest rates has major implications on optimal monetary policy design. Ascari and

Sbordone (2014) found that a new Keynesian model with a non-zero inflation trend is associated with a less stable and more volatile economy and tends to destabilize inflation expectations. For illustration that learning can introduce movements to the aggregate variable when combined with agent-heterogeneity, we use a simple non-zero inflation trend for the calibration.

III.5.2 Initial Conditions and Learning

For the representative agent model, the initial conditions for each variable are at the steady-state level. The initial conditions are drawn from the stationary distribution of the stationary recursive equilibrium for the heterogeneous agent model. For the bounded rational computations, the initial beliefs are set to be $(0, -1, -1, -1)$ for both ψ_0 and Ψ_0. The exogenous gain process is set to be constant at level $\gamma_t = 0.01$.

III.5.3 Stochastic Processes

I calibrate the income process following Krueger, who estimated a process for disposable earnings after taxes and transfers. They estimated an annual persistence of innovations to be $\rho_\theta = 0.815$ with a standard deviation of $\sigma_\theta = 0.014$. I assume the idiosyncratic log productivity process is the sum of an AR(1) and i.i.d. component. The monetary policy shocks persistence is that $\rho_\varepsilon = 0.0625$, a set value associated with a moderately persistent monetary shock, and the standard deviation for the shock is $\sigma_\varepsilon = 0.0025$, which corresponds to 25 basis points.The details of calibrations are summarized in Table (1). Figure (1) shows the wealth distribution generated by the calibrated model. It shows that the stationary distribution for the bond holding across the agents is very skewed, and

Variable	Moment	RA	HA
σ	CES parameter	2.0000	2.0000
χ	Frisch elasticity	2.0000	2.0000
β	$\bar{i} = 0.02$	**0.9800**	**0.9320**
$\underline{b}$	Borrowing constraint	0.0000	0.0000
ν	20% markup	6.0000	6.0000
α	66% output $\rightarrow$ labor	0.7470	0.7470
ϕ	PC (Sbordone 2002)	20.000	20.000
ϕ_{Π}	Taylor feedback	2.0000	2.0000
$\bar{B}$	B2Y ratio = 75%	**0.6900**	**0.8190**
ρ_{θ}	Krueger et al. (2009)	0.8150	0.8150
σ_{θ}	Krueger et al. (2009)	0.0140	0.0140
ρ_{ε}	Gali (2015)	0.0625	0.0625
σ_{ε}	Gali (2015)	0.0025	0.0025
ρ_{π}	Idio.	-	0.9700
σ_{z^p}	Permanent idio.	-	0.1960
σ_{z^t}	I.I.D idio.	-	0.2300

TABLE 1 Calibrations

III.6 Simulation Results

This section presents the results from the multiple simulations I have implemented on the model with various settings. In general, there are four sets of results coming from representative-agent rational expectations equilibrium (RA-REE), representative-agent locally rational expectations (RA-LREE), heterogeneous-agent rational expectations equilibrium (HA-REE), and heterogeneous-agent locally rational expectations equilibrium (HA-LREE).

III.6.1 Representative Agent

Figure (2) is simulated with a representative agent in a locally rational expectations equilibrium and shows the convergence of beliefs of both households

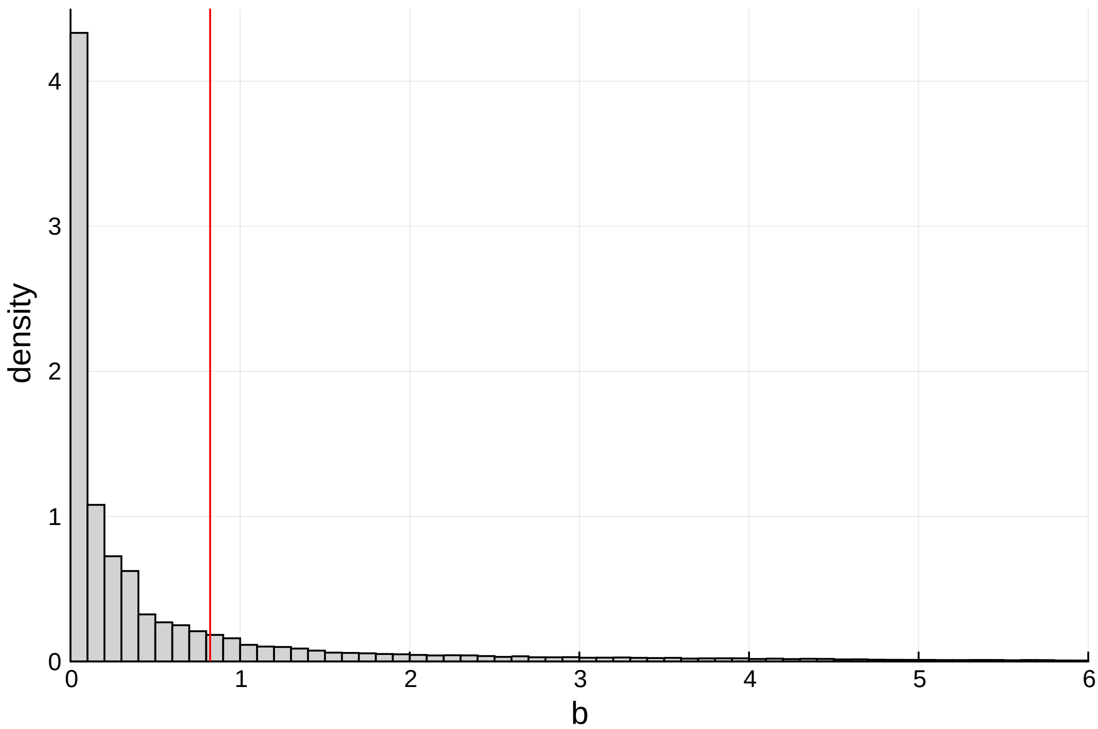
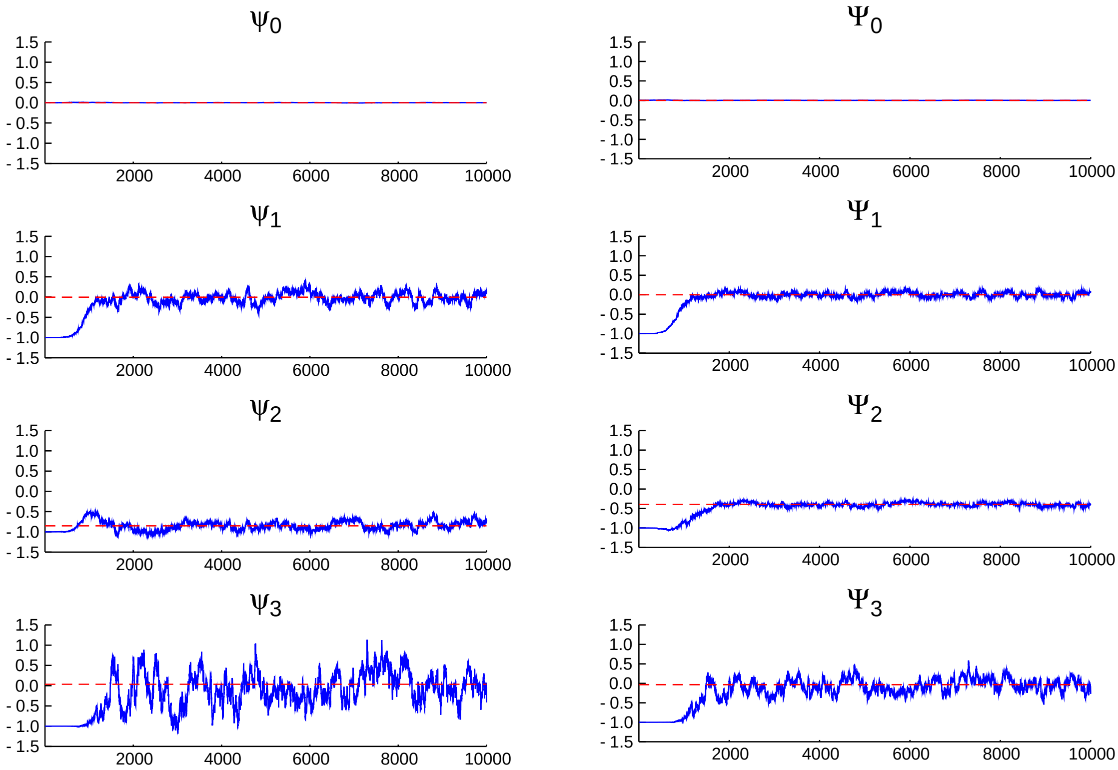


FIGURE 1. *Stationary Distribution for Bonds Holdings*
Note: This histogram contains the stationary distribution for bonds holdings across the households. The red vertical line represents the average bond holding. This histogram shows that the distribution is right-skewered with a mass point on the borrowing constraint.

and intermediate firms. The red dashed lines represent the coefficients obtained from running a regression of shadow prices on the information set using the simulated linearized rational expectations equilibrium. It is worth noting that both households and intermediate firms learn that the interest rate has an ambiguous coefficient in the learning rule for the future shadow prices. For a representative-agent environment, the learning rule that includes only the aggregate variables are correctly specified, and hence the coefficient estimates converge to the rational expectations equilibrium level.



(a) *Households* **(b)** *Intermediate Firms*

FIGURE 2. *Beliefs for Households and Intermediate Firms*

Note: This figure contains the belief evolution of a representative household. The red dashed line is the belief counterpart in a linearized rational expectations equilibrium. The constant learning gain is 0.01.

Consider the impulse responses to a positive innovation in aggregate productivity of ξ_θ in a representative-agent economy. Black lines in Figure (3) show how the economy responds to a 1.4 percentage under rational expectations equilibrium. The positive productivity shock boosts wage, labor, consumption, output and decreases nominal interest rate, real interest rate, and inflation. The size of the response has been converted to the percentage deviation from the steady-state. The interest rates and inflation rate are two exceptions. The unit measure for the two interest rates and inflation is one percentage point. Lump-sum tax transfer responds with an initial increase that is followed by a decrease. The initial increase in tax transfer is due to the decline in inflation when the shock arrives. As a result, the outstanding debt puts more pressure on the government as the interest payment has a higher real value—the follow-up decrease in tax results from a drop in the real interest rate.

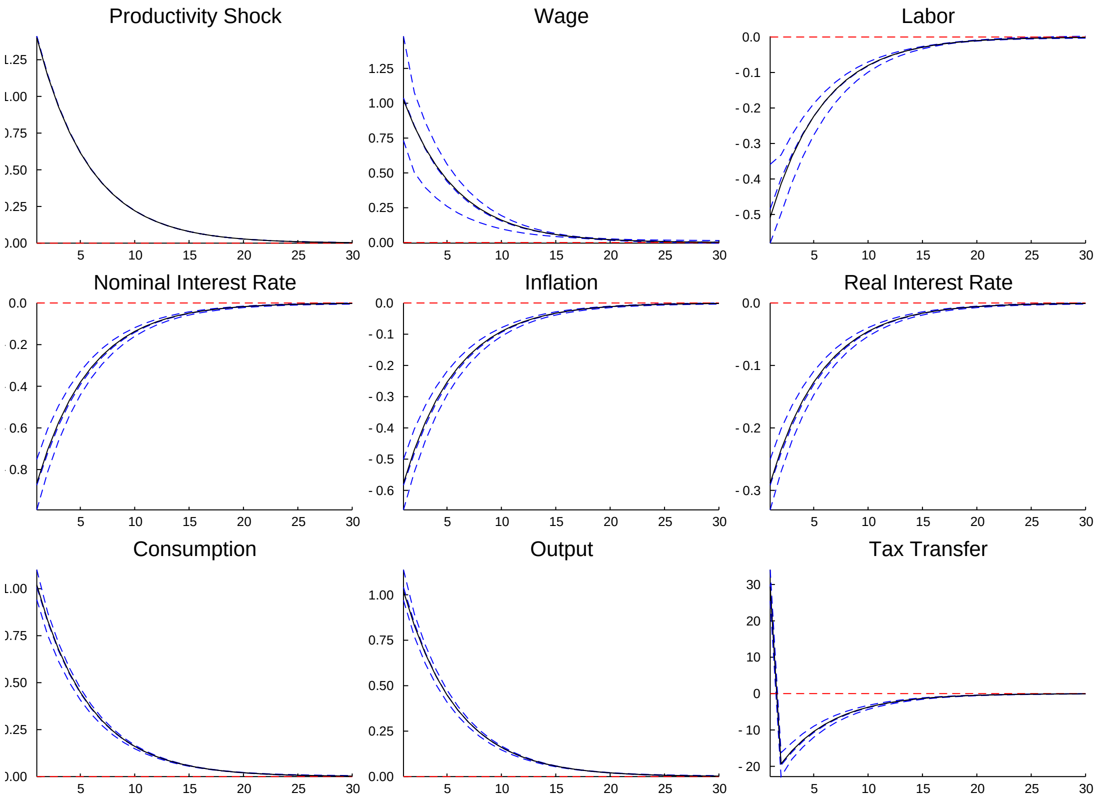


FIGURE 3. *IRFs - Representative Agent LREE (Productivity Shock)*
Note: This figure contains the impulse response functions from a positive productivity shock in a locally rational expectations equilibrium with a representative agent. The blue dashed lines are the IRF's from LREE with a 95% interval. The black lines are the IRF's from REE. Note that the median of IRF's from LREE aligns almost exactly with the IRF's from REE. The constant learning gain is 0.01.

I also consider the impulse responses to a positive innovation in the monetary policy rule ξ_ε in a representative-agent economy. Black lines in Figure (4) show how the economy responds to a 0.25 percentage-point shock under rational expectations equilibrium. The positive monetary shock decreases wage, labor, consumption, output, and inflation and increases nominal interest rates and real interest rates. Note that the persistence for monetary shocks is low, and thus the effects of innovation in monetary shock disappear after 5 periods (years). For all variables other than tax transfer, most of the response happens in the same period when the shock arrives, and the second period's impact is damped down very quickly. For tax transfer, the shock has a significant impact in both the first and second periods, but the reasons are different. The government takes a higher tax in the first period because of deflation caused by positive monetary shock. In the second period, the tax increases further due to higher real interest in the previous period.

Now I compare the impulse response functions simulated from LREE to the ones from REE with the representative agent. To simulate a set of IRF's with LREE, I draw the state variables from the ergodic distribution after the estimates for the coefficients of the forecasting rules are settled. I repeat 500 times and plot the responses at 0.25 percentile, median, and 99.75 percentile, represented by the blue dashed lines. As shown in Figure (3) and Figure (4), the IRF's simulated from LREE align with the IRF's simulated from REE. Farhi and Werning (2017) show that only when the model includes both agent heterogeneity with incomplete markets and bounded rationality is a departure from the fully rational benchmark model. Each of these two frictions, in isolation, would not affect the dynamics of the model. My result from this stage serves as a confirmation of their finding. It also shows that my definition of temporary equilibrium definition is correctly specified. Figure (2) is simulated with $10,000$ agents in a locally rational expectations equilibrium and shows the convergence of beliefs of both households and intermediate firms. This result matches the result from the previous chapter. The intuition here is that the agents are learning with the correctly specified forecasting rule, and hence the beliefs converge to the actual rational expectations equilibrium beliefs over time. I will show later that when heterogeneity is introduced in the model, the LREE and REE behave differently with a learning rule that only includes the aggregate shocks under a moderate constant learning gain. In a representative-agent model, agents' asset holding doesn't deviate too far from the steady-state level, and hence the forecasting rule doesn't need to include the individual variable.

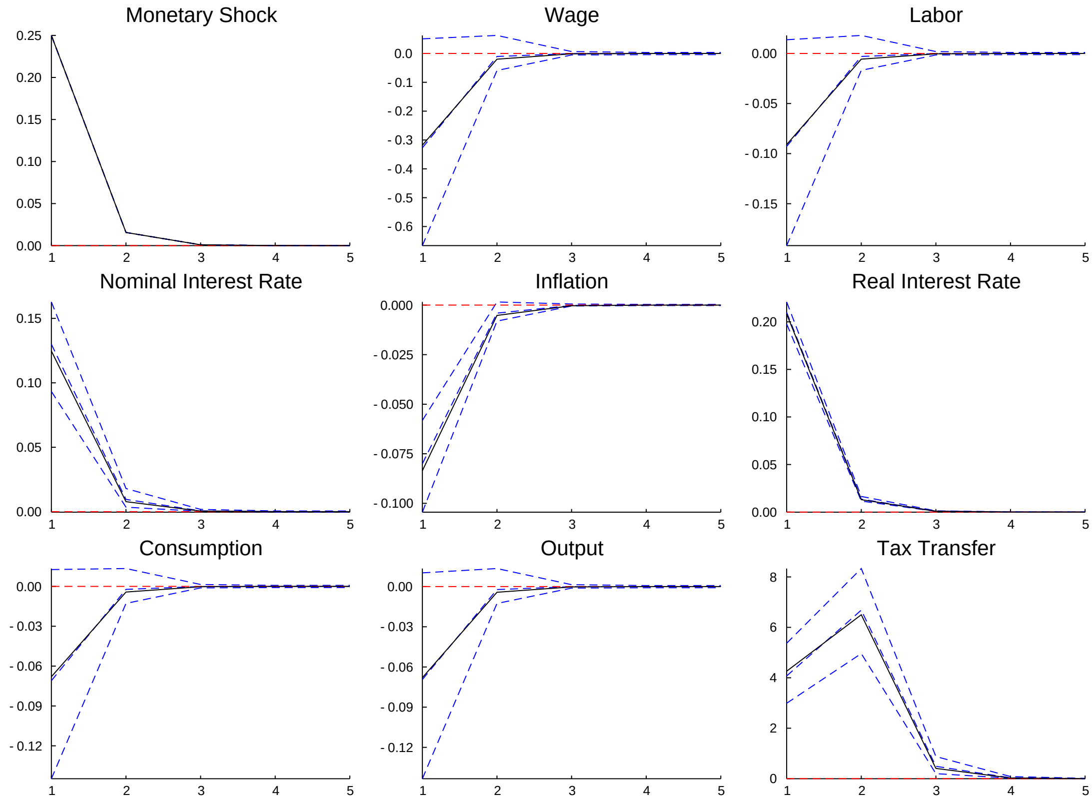


FIGURE 4. *IRFs - Representative Agent LREE (Monetary Shock)*
Note: This figure contains the impulse response functions from a positive monetary shock in a locally rational expectations equilibrium with a representative agent. The blue dashed lines are the IRF's from LREE with a 95% interval. The black lines are the IRF's from REE. The constant learning gain is 0.01.

III.6.2 Heterogeneous Agent

Now I simulate the impulse response functions from LREE with heterogeneous agents. The method used for simulation is based on the previous chapter. I draw from the ergonomic distribution of the state variables, including the profile of bonds holding, the profile of beliefs, lagged interest rate, and the variance-covariance matrix for the learning rule. I repeat 500 times and plot the responses at 0.25 percentile, median, and 99.75 percentile, represented by the three blue dashed lines, respectively. Note that the constant learning gain is 0.01, which is moderate. As a result, the rich agents use long historical data from the periods when they were poor and hence behave more like a poor-type agent in the rational expectations counterpart.

Figure (5) shows how endogenous variables respond to a one-standard deviation of negative productivity shock. It indicates that HA-REE predicts a stronger response of nominal interest rate, inflation, and real interest rate to a one-standard-deviation increase in productivity shock. HA-REE also predicts a weaker response of wage and hours compared to the HA-LREE. Figure (6) also shows that the responses predicted by HA-REE are similar to those predicted by HA-LREE with a one-standard-deviation of monetary shock. These results further confirm the conclusions from the last chapter. In a heterogeneous-agents setting, an adaptive learning rule that leaves out the individual variables induces deviations in impulse response functions from the REE.

III.7 Distributional Effects

I include the impulse response functions for the Gini index as a measurement for the second-order moment of the wealth distribution. The Gini index is a single number used for measuring the degree of inequality in a distribution. A higher concentration

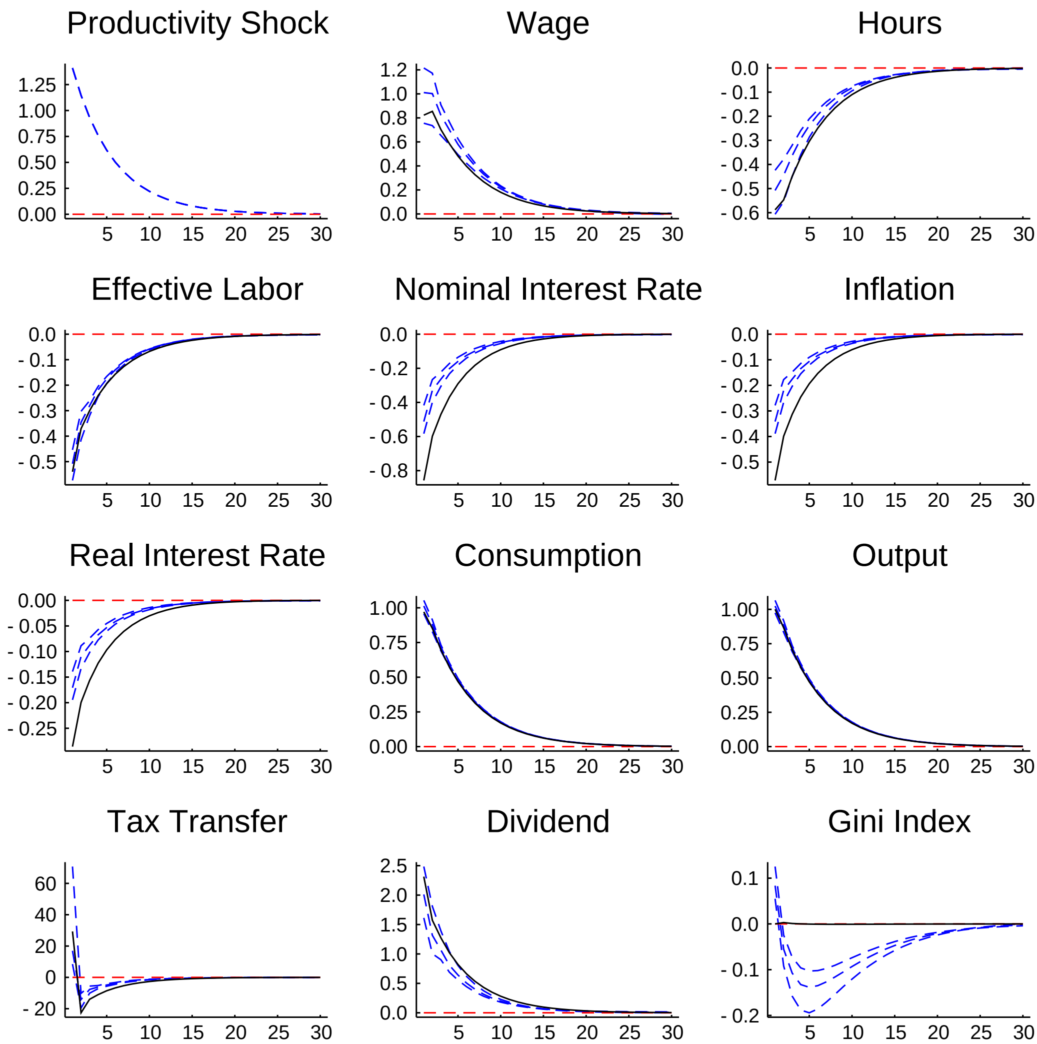


FIGURE 5. *IRFs - Heterogeneous Agent LREE (Productivity Shock)*
Note: This figure contains the impulse response functions from a positive productivity shock in a locally rational expectations equilibrium with a large number of (100, 000) heterogeneous agents. The blue dashed lines are the IRF's from LREE with a 95% interval. The constant learning gain is 0.01.

in wealth would translate to a higher Gini index. Although the magnitude of the Gini index does not match the US income data,[1] the directions in which the wealth

[1] A more detailed heterogeneous agent model can be used to match the wealth distribution better.

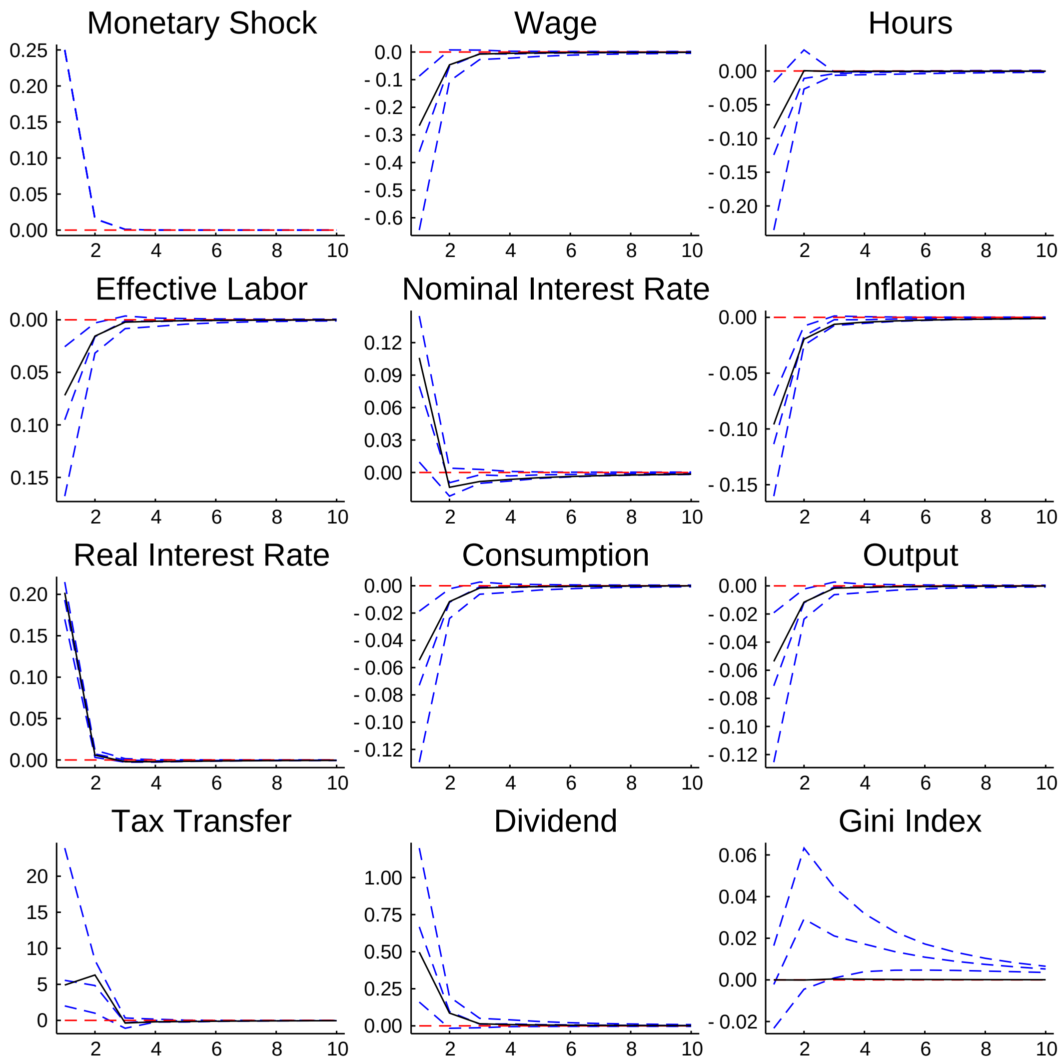


FIGURE 6. *IRFs - Heterogeneous Agent LREE (Monetary Shock)*
Note: This figure contains the impulse response functions from a positive monetary shock in a locally rational expectations equilibrium with a large number of heterogeneous agents. The blue dashed lines are the IRF's from LREE with a 95% interval. The constant learning gain is 0.01.

inequality goes are still indicative. When the HANK economy receives an unexpected one-standard-deviation positive productivity shock, the Gini index goes up when the shock arrives. An immediate decrease follows this initial increase in the first period, and the index goes back to the stationary distribution as the magnitude of the productivity shock damps down.

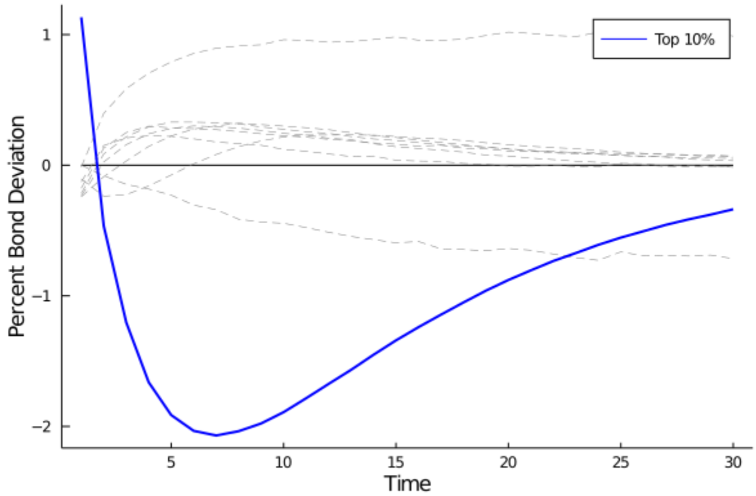
The wealthy agents mostly rely on the real interests from the bonds they hold from the last period, and the agents with less bond holding rely primarily on labor income. The initial increase in "inequality" arises from the unexpected deflation caused by productivity shock. Although the poor agents receive a higher labor income in the first period, the effect on inequality is compensated by the rich agents receiving a higher real interest caused by a lower inflation rate. The following-up decrease in the Gini index results from a higher wage rate and a lower real interest rate. I implement a similar experiment with a 25-basis-point positive monetary shock. In this case, the immediate response of the Gini index is ambiguous. After the first period, there is an increase in the Gini index. This initial inertia in the Gini index occurs because there is a decrease in the inflation rate and a wage increase. The following increase is due to both a higher real interest rate and a lower wage rate. I make two comments. First, the Gini index's movement indicates that inequality is exacerbated during a recession, which happens when the economy receives a negative productivity shock or a positive monetary shock. Second, the monetary shock has a long-lasting effect on wealth inequality, documented by Coibian (2017) et al. This paper provides the first theoretical result that matches what has been observed in the data.

To understand what is driving the result behind the movements in the Gini index, we need to dive into how different groups of agents from different asset brackets react to aggregate shocks. I separate the agents from the ergodic distribution from the LREE dynamics into ten different groups based on their asset holding. The movement in the bond holding is derived precisely the same way from the ergodic distribution as in the previous impulse response functions. The blue line represents the impulse responses of the wealthiest ten percent agents, whereas the other nine dashed lines represent the bottom 90% agents. Comparing the shapes of the blue lines to the bottom right panel of Figure (5) and Figure (6), one can see that most of the movement in the Gini coefficient can be explained by the wealthiest group of agents in the economy. This dynamic driven by the very upper end of the wealth distribution matches the empirical data. For example, the Congressional Budget Office (2011) found that the top wealthy households played a considerable role in income inequality dynamics since 1980. Specifically, the rapid growth of average market wealth for the top 1 percent of the population contributed to the increase in household income dispersion between 1979 and 2007. From the previous chapter, we can conclude that the behavior deviation for the rich agents in this learning environment comes from the low constant learning gain with a misspecified learning rule that does not include individual variables. These top rich agents behave more like poor-type agents, and their behaviors drive the movement in the wealth distribution.

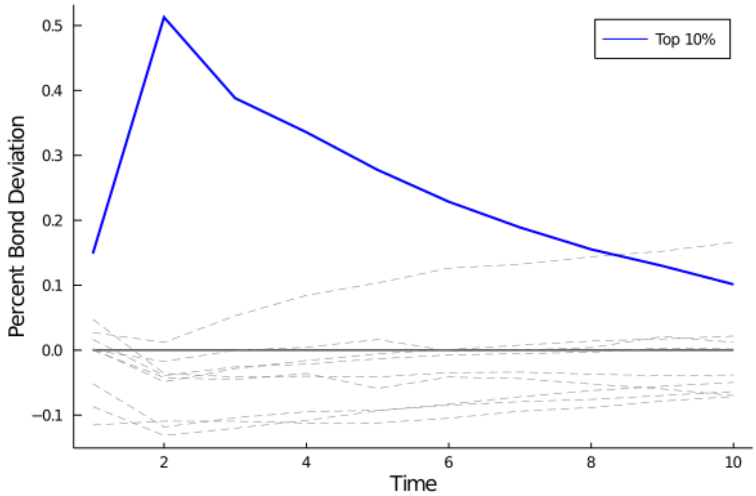
III.8 Conclusion

This chapter extends the local rationality concept from the previous chapter to a heterogeneous-agent New Keynesian model. The HANK-type models impose strong assumptions on what the agents understand about the model under a rational expectations equilibrium. I show that the rational expectations equilibrium can be learned through adaptive learning by locally rational agents.

The simulation results in this more complex environment are comparable to the real business cycle model. I primarily focus on the learning rule that only includes the aggregate variables and see how the economy behaves compared to the rational expectations. After a series of experiments, I show that the impulse responses are different in the locally rational expectations equilibrium compared to the rational expectations when the model presents heterogeneity. Specifically, I investigate how the wealth inequality responds to aggregate shocks and find that the HANK model



(a) *Impulse Response Functions - Productivity Shock*



(b) *Impulse Response Functions - Monetary Shock*

FIGURE 7. *IRFs from Top-Ten Percent Richest Agent*

Note: This top panel shows the impulse response function for the bond holding from the top-ten percent richest agent in the economy in response to a one-standard-deviation negative productivity shock. The bottom panel shows the impulse response function for the bond holding from the top-ten percent richest agent in the economy in response to a one-standard-deviation monetary shock. The dashed lines in both panels represent the other nine groups' impulse responses.

can only produce movement in the inequality when the agents are adaptive learners for their future shadow price. I further show that the distributional effects are driven mainly by the top-tier wealthy agents in the locally rational model. These agents behave more like poor-type agents because they include a long history of personal data, including the periods when they held less bond. As a result, these agents are not responding to the aggregate shocks the way they are supposed to with rational expectations equilibrium.

The theoretical findings match the empirical evidence that monetary shocks can have distributional effects on income and wealth. The HANK model in rational expectations equilibrium doesn't produce the same kind of movement in the distributions. In contrast, local rationality adds an extra layer of friction to the model and can have the distributional movement that matches data. Specifically, Coibian et al. (2007) find that a contractionary monetary distribution can make the distribution more unequal. This chapter supports the empirical evidence from a theoretical perspective based on adaptive learning. The future goal of this research agenda can study the implication of local rationality in a more realistic new Keynesian model that includes a non-zero inflation trend. The adaptive learning environment provides a natural benchmark to show that whether the model converges to rational expectations.

CHAPTER IV
E-STABLE SUNSPOT EQUILIBRIA WITH HETEROGENEOUS AGENTS

IV.1 Introduction

This paper investigates the implication of introducing multiple finite-state Markov extrinsic sunspot processes in a general univariate forward-looking model. In this model, each agent only observes a subset of the sunspots. There are adaptively stable Markov stationary sunspot equilibria (SSEs) near an indeterminate steady-state for both the linear and nonlinear cases. In the linear case, each sunspot process is associated with a knife-edged serial correlation condition, known as the resonant frequency condition. In the nonlinear case, the serial-correlation condition associated with each sunspot process is no longer knife-edged. In both cases, each serial correlation condition depends on the proportion of agents who observe that sunspot. As long as one sunspot process satisfies its serial correlation condition, adaptively stable SSEs exist. I illustrate the results using a standard version of the Samuelson overlapping generations model of money where agent-level beliefs are treated carefully. One of the criticisms for sunspot equilibrium literature is that no real-world sunspot has been identified.

In macroeconomic models of dynamic economies with forward-looking agents, steady states can be indeterminate. Indeterminacy occurs when an infinite number of rational expectations equilibria (REE) are associated with such a steady state. The concept of indeterminacy is closely related to the idea of a sunspot equilibrium. The idea is that, in the presence of indeterminacy, a rational expectations equilibrium path can exhibit fluctuations that depend on external shocks called sunspots. The sunspots are extrinsic in the sense that they are not based on model fundamentals. This dependency is self-fulfilling and arises when agents condition their expectations on sunspots, and the sunspots influence the economy only through agents' expectations. In a proper sunspot equilibrium, the allocation of resources depends on sunspots in a non-trivial manner. Fluctuations are speculative and driven solely by expectations. Sunspots in these solutions often follow a stationary stochastic process, and these self-fulfilling rational expectations solutions are commonly called stationary sunspot equilibria (SSE.) Contrary to the conventional wisdom that only intrinsic uncertainty should influence economic activity, the sunspots model explains volatility without going beyond the rational expectations framework. Several authors first documented the existence of rational expectations solutions driven by extrinsic stochastic processes. Shell (1977) provided the first sunspots model in an overlapping-generations exchange economy with fiat money. Azariadis (1981) was the first published paper to show that sunspots may be responsible for business cycles. Cass and Shell (1983) explored the conditions under which sunspots solutions arise and offered welfare analysis using an overlapping-generations economy. Azariadis and Guesnerie (1984) establish that the existence of two-period cycles is a sufficient condition for the existence of a two-state stationary sunspot equilibrium. Their conclusions were obtained in a particular class of overlapping generations economies. Guesnerie (1986) provided sufficient conditions for the existence of sunspot equilibria near deterministic cycles in a broader class of models with multiple commodities.

All of the early existence results for sunspot equilibria were initially obtained in simple stylized models, and the conclusions were not generalizable. The first generic result that provides criterion on indeterminacy was provided by Blanchard and Kahn (1980). They present a practical technique for determining whether a multivariate linear model has a unique equilibrium. The method is based on matrix eigenvalue decomposition and compares the number of explosive eigenvalues in the coefficient matrix to the number of variables that are not predetermined. An explosive eigenvalue is associated with a restriction since the expectation of future non-predetermined variables does not grow to infinity. All the restrictions collectively describe the law of motion for the non-predetermined variables in a way that is aligned with the rational expectations hypothesis. Depending on the number of restrictions implied by explosive eigenvalues and the number of non-predetermined variables, there might be no equilibrium, one unique equilibrium, or multiple equilibria. The method of Blanchard and Kahn is demonstrated to determine the existence and uniqueness of REE solutions. Still, the same technique can also be applied to establish the existence of sunspot equilibria in a linear model. Sunspot equilibria can be constructed in an easily analyzed vector autoregressive form, and the support of the sunspots can be either continuous or discrete. Sims (2000) uses generalized Schur decomposition to improve Blanchard and Kahn's technique to accommodate a broader collection of models researchers might encounter in practice. Woodford (1986) extends the results of Blanchard and Kahn to a general nonlinear model. He applies an implicit function theorem through a local analysis and shows that a nonlinear model's local equilibrium uniqueness is implied by uniqueness in the linearized model.

The existence of SSE alone does not justify its importance. A natural question to ask is whether agents will learn to believe in sunspots in the first place. Separate literature on equilibrium selection tries to answer this question. Woodford (1990) shows that, under some plausible assumptions, agents that follow adaptive learning rules may learn to coordinate their expectations and actions on sunspots. The stability result is obtained in a stylized model based on global analysis using the structure of the invariant set under learning and the index number theorem of Poincare-Hopf. However, the approach of Woodford cannot be used to locate stable sunspots. Evans and Honkapohja provide stability conditions for SSEs in several papers. Evans and Honkapohja's method works in a generic model class and can provide information about the stable sunspots' location. In particular, Evans and Honkapohja (1994) show that, in a general class of nonlinear models, E-stability gives the necessary and sufficient conditions for the local stability of finite-state SSEs near a deterministic cycle under adaptive learning. The stability of SSEs in a small neighborhood of cyclic equilibria is implied by the stability of the equilibria associated with the deterministic cycles. The proof uses the property that the determinate of a matrix is a continuous function in its eigenvalues. Evans and Honkapohja (2003a) consider a forward-looking linear model and provide conditions under which SSEs near a deterministic steady state are stable under learning. They also show that agents' representation in the learning process matters for the stability results, and the autoregressive solutions are never stable under learning. A resonant frequency condition must be satisfied for the SSE solution to be stable.

The stability results of SSEs near a deterministic steady state in a linear model extend to the nonlinear version of the model. In a companion paper of Evans and Honkapohja (2003b), they show that stability results carry over to the corresponding nonlinear model in a steady-state neighborhood. The proof relies on a local bifurcation, which arises when the differential equation governing the stability has one eigenvalue set to be zero. Evans and McGough (2005a) find that an SSE may be stable if the associated sunspot process's serial correlation exhibits the resonant frequency. In a separate paper, Evans and McGough (2011) also show that finite-state Markov sunspots' stability implies all sunspots are stable under learning with common factor representations. Evans and McGough (2018) study the existence and stability of near-rational sunspot equilibria (NRSE) in forward-looking nonlinear models where agents use the optimal linear forecasting model among similarly specified linear models sunspot process has continuous support.

They provide generic existence results for continuous-support sunspot equilibria in nonlinear models, and the solutions are constructive with simple recursive forms.

Sunspot equilibria remained a purely theoretical topic until several authors explored the possibility of fitting sunspot-driven business cycles into applied dynamic stochastic general equilibrium models. Benhabib and Farmer (1994) provide a simple condition for indeterminacy in a one-sector growth model. The condition requires increasing returns to scale for aggregate technology should be large enough to imply the aggregate labor demand curve is upward-sloping and steeper than the labor supply curve. Caballero and Lyons (1994) and Baxter and King (1991) estimate externalities to be large and in the plausible range of indeterminacy. Farmer and Guo (1994) develop a calibrated, nonconvex real business cycle model that well-matched the data. The model uses only sunspot processes as an exogenous stochastic driver to explain business cycle co-movements. The model matches the data better than the standard real business cycle (RBC) models with fundamental shocks.

Farmer and Guo demonstrate that the models with indeterminate equilibria can explain the macroeconomic data at business cycle frequencies that traditional RBC models cannot. Gali and Jordi (1994) developed an alternative way of introducing nonconvexity into RBC models. Instead of relying on the presence of large increasing returns, their model incorporates monopolistic competition and endogenous markups to allow for sunspot equilibria and sunspot-driven fluctuations. However, more estimates obtained later in the literature called into question these results by showing that the early estimates of externalities were overstated. See Basu and Fernald (1994) and Norrbin (1993) for the new estimates. Following that, researchers established different nonconvexities to generate indeterminacy with empirically plausible calibrations. For example, Benhabib and Farmer (1996) introduce mild increasing returns to scale by building sector-specific externalities into a two-sector model. Specifically, their model includes externalities in both the consumption goods sector and the investment goods sector. Their model does not need large external effects that give rise to an upward-sloping labor demand curve. Their two-sector model allows for indeterminacy within the regions of reasonable estimates at the industry level. However, several empirical researchers found that the returns to scale are roughly constant by refining the earlier findings of Hall (1990) on disaggregated US data. In response to this new finding, Benhabib, Meng, and Nishmura (2000) develop an RBC model with multiple sectors that generate indeterminacy without increasing returns-to-scale. Some authors also explored the possibility of introducing nonconvexity by assuming alternative utility functions. For example, Bennett and Farmer (2000) show that a one-sector growth model with preferences that are non-separable in consumption and leisure allows for indeterminate equilibria when demand and supply curves have the standard slopes. Hintermaier (2002) proves in a general setup that utility functions compatible with indeterminacy are not concave if the elasticity of scale is lower than the inverse of the labor share in production.

A separate but related literature has risen to investigate whether any of the indeterminate equilibria in these nonconvex RBC models are stable under learning. Evans and Honkapohja (2001) find that the sunspot equilibria studied by Farmer and Guo (1994) were not stable. Evans and McGough (2005b) study the sunspot solutions' stability properties under both the general form representation and the common factor representation in two alternative information assumptions. They find that there are large parameter regions in which sunspot solutions are stable for the reduced form. However, when the reduced form parameters are restricted to match the calibrated structural models, stable sunspot equilibria only exist for a tiny part of the standard indeterminacy region. The stability result is also subject to the timing assumption. Evans and McGough call this observation the stability puzzle in nonconvex economies. Evans and McGough (2005b) obtained the stability and instability regions numerically. Duffy and Xiao (2006) consider a host of sunspot-driven RBC type models and provide analytic conditions for sunspot equilibria to be stable under learning. They prove analytically that structural model parameter restrictions imply instability of the indeterminate solutions. McGough, Meng, and Xue (2013) study a one-sector RBC model with externalities. They find that the Benhabib-Farmer condition that the labor-demand curve is upward-sloping and steeper than the labor-supply curve is necessary for joint indeterminacy and E-stability.

In addition to RBC-type models, the idea of indeterminacy and sunspot-driven fluctuations also applies to other dynamic stochastic general equilibrium models. For example, extensive literature has arisen to warn of sunspot equilibria's consequences from a poorly designed monetary policy. The prospect of the agents coordinating on some external shocks causes inefficient fluctuations. Monetary policies should ensure sunspot-driven volatility does not arise. Many authors found that indeterminacy occurs if the monetary authority follows an interest rate rule that does not respond aggressively to inflation changes. King (2000) provides a detailed description of micro-founded New IS-LM models that incorporate expectation terms in both IS and Phillips Curve. The paper suggests the feedback parameter on inflation be large in the interest rule, which is in line with Taylor's Rule. Clarida, Gali, and Certler (2000) estimate the monetary policy rules before and after Paul Volcker was appointed chairman of the Board of Federal Reserve System. They find that the interest rule was accommodative in the pre-Volcker year and aggressive in the post-Volcker year. This paper applies the pre-Volcker rule to a calibrated New Keynesian (NK) model and finds the accommodative rule leaves open the possibility of sunspot-driven fluctuations. They argue that substantial volatility in inflation and output observed in the late sixties and seventies can be partially explained by the self-fulfilling changes in expectations. They also show that the NK model calibrated according to the post-Volcker rule is instead determined, matching the smaller variance of inflation and output observed in the eighties. Lubik and Schorfheide (2004) point out that determinacy is a property that cannot be established using single-equation methods. Instead, they estimate a fully specified rational expectations model using a Bayesian approach. They specify a prior probability distribution over parameters with equal weights on determinate and indeterminate regions. Using US data on the output gap, the interest rate, and the inflation rate, Lubik and Schorfheide compute these regions' posterior odds ratios. Their findings strongly confirm that the pre-Volcker rule was destabilizing.

The indeterminacy in these monetary models raises whether sunspot equilibria in the New Keynesian models are stable under learning. Honkapohja and Mitra (2004) were the first to consider a purely forward-looking AS equation with various interest rules, including those dependent on current, lagged, and expected inflation and output gap. They find that sunspot-driven equilibria they consider may be stable only if the interest rate rule depends on expected inflation and expected output gap. However, their initial conclusion only included the general form representation, which is a linear function of lagged endogenous variables and a sunspot variable taking the form of a martingale difference sequence. Evans and Honkapohja (2003a) find that previously-thought unstable sunspot equilibria can also be stable when represented as the common factor representation and argue that stability analysis must incorporate both general form and common factor representations. Another related literature concerns if agents can coordinate with different monetary policy designs, so the steady states are determinate. Bullard and Mitra (2002) study the stability property of a broad class of variants of the Taylor interest rate rule and find learnability of a unique rational expectations equilibrium is not guaranteed. They argue monetary policy should take into account the learnability constraints. Evans and Honkapohja (2003a) analyze learnability in a similar model and consider different ways of implementing optimal monetary policy under discretion.

So far, all of the existence and stability results derive within a representative agent framework. No heterogeneity exists across the agents - every agent shares the same information sets and beliefs and makes the same decisions. There are advantages to working with a representative agent. It is easy to work with one decision-maker instead of simultaneously analyzing many different decisions. However, rational expectations equilibria, including sunspot equilibria, can be mostly thought of as an outcome of a coordination game participated by a large number of rational agents. In an REE, each agent's decision is optimal, given what other agents' decisions are. In models of indeterminacy and sunspot equilibria, using a representative agent imposes two implicit assumptions on the economy: 1. Every agent is open to the idea that the sunspot variable may matter for the economy's outcomes. In the specification, every agent uses the same learning rule that depends on the sunspot variable. 2. There is only one sunspot process that all agents observe, and agents coordinate their actions on this one sunspot variable. In practice, there are two ways to deviate from the representative-agent framework. 3. Only a proportion of the agents believe in the sunspot, and the rest do not believe the economy fluctuates according to the sunspot. There are multiple sunspot processes and agents "disagree" on which of the sunspots matters. 4. Intuitively, either deviation would make a sunspot equilibrium less likely to exist, or if it does exist, more difficult to be stable. Departure from the representative-agent framework can serve as a "robust test" for sunspot equilibria' existence and stability. This paper proves that adaptively stable sunspot-driven equilibria can still exist under either deviation. I provide the necessary and sufficient conditions for sunspot equilibria' existence and stability near an indeterminate steady state in a general univariate forward-looking model. I show that the results obtained in Evans and Honkapohja (2003b) extend naturally to models with heterogeneous beliefs. I introduce multiple extrinsic finite-state Markov sunspot processes in a general univariate model. Expectational heterogeneity rises naturally, as each agent either does not observe any sunspots or observes only one of the sunspots. I prove the existence of restricted perception stationary sunspot equilibria (SSE) near an indeterminate steady state. Several insights are provided regarding the stability result. In a linear model, each sunspot process is associated with a knife-edged restriction on its serial correlation. In a nonlinear model, the condition is no longer knife-edged. The existence of E-stable SSE only requires one of the potentially many sunspot processes to satisfy its restriction. Suppose a smaller proportion of agents observe a sunspot process. In that case, it needs more substantial negative feedback from expectations at the steady-state to meet the RFC associated with that sunspot process. A standard version of the Samuelson overlapping generations model of money is used to illustrate the results. This paper also contributes to recent studies on the topic of bounded rationality with heterogeneous agents. Branch and Evans (2005) introduce intrinsic heterogeneity in expectation formation. In their model, agents choose from a list of misspecified econometric models. Honkapohja and Mitra (2006) show how different forms of heterogeneity in structure, forecasting models, and adaptive learning rules affect the conditions for convergence of adaptive learning towards REE.

This chapter shows that E-stable sunspot equilibria do not have to depend on only one sunspot. Agents can observe multiple stochastic signals, and the model can still present self-fulfilling fluctuations. This finding from the heterogeneous-agent environment helps induce the existence and stability of sunspot equilibria. However, the analytical result also shows that the region for E-stable sunspot equilibria to exist shrinks when we introduce more sunspots in the model. Generally, the model needs to have stronger negative feedback at the steady state when there are more random processes used by the agents as sunspots. In this sense, the finding of the chapter impedes the existence of stable sunspot equilibria. The dynamic selection result of the paper shows that agents, in general, would prefer the learning rule that includes sunspot variables because this type of learning rule nest the steady state learning rule. At worst, the learning rule with sunspots can do as well as the steady-state learning rule in terms of forecasting accuracy. This chapter balances all aspects and concludes that the heterogeneous-agent environment would help induce sunspot equilibria.

IV.2 Model

Consider the univariate nonstochastic model with a unit continuum of agents:

$$y_t = \int_{\Omega} H(E_t^{\omega}[G(y_{t+1})|I_t^{\omega}])d\omega. \tag{IV.1}$$

Here y is a univariate endogenous variable, and its law of motion is defined by the difference equation that involves a continuous of expectational terms index by $\omega \in \Omega$, where Ω is the set of all agents. I_t^{ω} is the information set that is available to agent ω at time t. $E_t^{\omega}G(y_{t+1})$ denotes the conditional expectation of $G(y_{t+1})$ held by agent ω at time t given I_t^{ω}. Note that $E_t^{\omega}G(y_{t+1})$ is the true mathematical conditional expectation of $G(y_{t+1})$. Functions $H : \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$ are both of differentiability class $\mathcal{C}^3$. Define function $F = H \circ G : \mathbb{R} \to \mathbb{R}$. Assume throughout that there exist a locally unique steady state $\hat{y}$ such that $\hat{y} = F(\hat{y})$. Consider sunspot processes with two states indexed by 1 and 2. Let there be N independent random variable $\{s^{t,k}\}_{k=1}^{N}$, each associated with an exogenous two-state Markov processes with transition probability matrices $\{\Pi^k\}_{k=1}^{N}$, and $s^{t,k} = \{s_{\tau}^k\}_{\tau=0}^{\tau=t}$ is the kth sunspot state up to time t. Denote $\Pi^k = (\pi_{i,j}^k)$ for all k. Every agent observes either one of processes or none of them. The individual information set I_t^{ω} depends on the agent's observability of the Markov processes. That is to say, either $I_t^{\omega} = s^{t,k}$

or $I_t^\omega = \varnothing$. Consider a stochastic process for y_t that depends on the N exogenous two-state Markov processes $\{s_t^k\}_{k=1}^N$. Denote $s_t = (s_t^1, s_t^2, \ldots, s_t^N)$ as the profile of sunspots at time t, and $\mathcal{S} = \{1, 2\}^N$ as the set of all possible values the profile vector can take, i.e. $s_t \in \mathcal{S}$. A rational expectations equilibrium is defined as follows.

Definition 11

$\{y_t\}$ is an REE if there exists a set $\{\bar{y}_s\}_{s\in\mathcal{S}} \in \mathbb{R}^{2^N}$ such that $y_t = \bar{y}_s$ if $s_t = s$ and that $\{y_t\}$ satisfies Eq.(IV.1) with E_t^ω is the mathematical expectation conditional on information set I_t^ω.

An immediate REE that follows the definition is $y_t = \hat{y}$ where $\hat{y}$ is the model's locally unique steady state. This solution is referred to as the fundamental solution. If $\bar{y}_{s_1} \neq \bar{y}_{s_2}$ for some $s_1, s_2 \in \mathcal{S}$, the REE is a stationary sunspot equilibrium (SSE). An interesting observation is that the cyclic order of an SSE can potentially be as large as 2^N, a number that grows exponentially in N.

IV.2.1 Adaptive Learning

To analyze the stability under adaptive learning, I replace the true mathematical expectation term E_t^ω in Eq.(IV.1) with $\hat{E}_t^\omega$, which is the subjective expectation held by agent ω at time t conditional on information I_t^ω. I categorize agents into two general types. One type believes that they are always in a steady state, and I call these agents the steady-state (SS) believers. The other type only observes one sunspot process and believes that they are in a two-state SSE with y_t taking values according to the observed sunspot. I call these agents the SSE-k believers where k indicates the sunspot process index they observe. SS believers use the average past value taken by y_t as the estimate for the steady-state. SSE-k believers use the average past value taken by y_t in each state of sunspot $s^{t,k}$ as the estimates for the values associated with each state. Formally, let ϕ_t^0 be the estimate of an SS believer and $\phi_t^k = (\phi_{1t}^k, \phi_{2t}^k)$

be the estimates of an SSE-k believer. SS believers and SSE-k believers use learning rules based on the following recursive equations:

$$\phi_t^0 = \phi_{t-1}^0 + t^{-1}(y_{t-1} - \phi_{t-1}^0),$$
$$\phi_{jt}^k = \begin{cases} \phi_{j,t-1}^k + (\# T_{j,t}^k)^{-1}(y_{t-1} - \phi_{j,t-1}^k) \text{ if } s_t^k = j \\ \phi_{j,t-1}^k \text{ if } s_t^k \neq j \end{cases}$$

for $j = 1, 2$. Here $T_{j,t}^k = \{\tau \in \{0, 1, \cdots, t-1\} | s_\tau^k = j\}$, and the operator $\#$ counts the number of elements in a set. At time t, an SS believer forms her expectation:

$$\hat{E}_t^\omega G(y_{t+1}) = G(\phi_t^0).$$

An SSE-k believer form her expectations:

$$\hat{E}_t^\omega G(y_{t+1}) = \begin{cases} \pi_{11}^k G(\phi_{1t}^k) + (1 - \pi_{11}^k) G(\phi_{2t}^k) \text{ if } s_t^k = 1 \\ (1 - \pi_{22}^k) F(\phi_{1t}^k) + \pi_{22}^k G(\phi_{2t}^k) \text{ if } s_t^k = 2 \end{cases}$$

Note these expectations are point expectations, and this works because the model is non-stochastic. The learning algorithm is closed by specifying that y_t is updated through the temporary equilibrium implied by Eq.(IV.1). The rest of the paper analyzes the existence of SSE solutions and their local stability under adaptive learning by deriving E-stability results. It has been established that E-stability governs stability under adaptive learning. See Evans and Honkapohja (2001), Chapter 12.

IV.3 General Existence and E-stability of SSE

This section presents the existence and stability results for the general model Eq.(1), featuring heterogeneous beliefs. I first present the existence and stability results for a general case with N sunspots and then give the simpler cases. I will comment where my results extend to the more general case with a mix of more sunspot observers. To set up the general results, I assume $N+1$ groups of agents denoted from Ω_0 to Ω_N that partition the total population Ω. Denote $\gamma_0, \gamma_1 \cdots \gamma_N$ as the proportions for each group and $\sum \gamma_i = 1$. The information set for each group is as follows:

$$I_t^{\omega} = \begin{cases} \varnothing \text{ if } \omega \in \Omega_0 \\ s^{t,k} \text{ if } \omega \in \Omega_k \end{cases}$$

Without loss of generality, I assume $\gamma_1 < \gamma_2 < \cdots < \gamma_N$. Agents in Ω_0 estimate the steady state to be α_0 where agents in Ω_k for $k > 0$ estimate $y_t = \alpha_{ki}$ when the kth sunspot is at state i. Let $w_{ki} = \alpha_{ki} - \hat{y}$ where $\hat{y}$ is the unique steady state for the model. Also let $\beta = F'(\hat{y})$ where $F = H \circ G$. Define two index sets: $\mathcal{I}_1 = \{i \in \{1, \cdots, N\} | |\beta| > \gamma_i^{-1}\}$, and $\mathcal{I}_2 = \{i \in \{1, \cdots, N\} | \beta < -\gamma_i^{-1}\}$.

Theorem 1

Stochastic sunspot equilibrium (SSE) exists if and only if $\mathcal{I}_1 \neq \varnothing$; E-stable SSE exists if and only if $\mathcal{I}_2 \neq \varnothing$.

I make a few comments about the existence and stability results. First, stable sunspots can exist when agents coordinate on different sunspots. Second, the existence region decreases when the largest population of sunspot observers reduces. Third, in the limit when there is only one type of SSE believers, i.e., $\gamma_N = 1$, the results match the previous literature[1]. The location SSE solutions depend on whether the model's linearity and the sunspot processes themselves. To make the results more concise, I define two mappings, $\mathcal{R}_l$ and $\mathcal{R}_n$, from a population index to a restriction that partially locates the sunspot solution.

$$\mathcal{R}_l(k) \to \begin{cases} \pi_{11}^k w_{k1} + \pi_{22}^k w_{k2} = 0 \text{ if } \pi_{11}^k + \pi_{22}^k - 1 = (\gamma_k \beta)^{-1} \\ w_{k1} = 0, w_{k2} = 0 \text{ if } \pi_{11}^k + \pi_{22}^k - 1 \neq (\gamma_k \beta)^{-1} \end{cases}$$

$$\mathcal{R}_n(k) \to \begin{cases} w_{k1} = \bar{w}_{k1}, w_{k2} = \bar{w}_{k2} \text{ if } \pi_{11}^k + \pi_{22}^k - 1 < (\gamma_k \beta)^{-1} \\ w_{k1} = 0, w_{k2} = 0 \text{ if } \pi_{11}^k + \pi_{22}^k - 1 \geq (\gamma_k \beta)^{-1} \end{cases}$$

where $\bar{w}_{k1}$ and $\bar{w}_{k2}$ are two non-zero real numbers.

IV.3.1 Linear Model

If both $H(y)$ and $G(y)$ are linear functions in y, then we call it a linear model. Note that $F(y) = H(G(y))$ is also a linear function. I present the following theorem for the location of the E-stable SSE.

Theorem 2

Given $\mathcal{I}_2 \neq \varnothing$, there exist a continuum of E-stable SSE solutions

$$\{(w_0, w_{11}, w_{12}, \cdots, w_{N1}, w_{N2}) \in \mathbb{R}^{2N+1} | w_0 = 0, \mathcal{R}_l(1), \cdots, \mathcal{R}_l(N)\}$$

at least for some $k \in \{1, \cdots, N\}$, the transition probability of the sunspot process satisfies the resonant frequency condition $\pi_{k1} + \pi_{k2} - 1 = (\gamma_k \beta)^{-1}$.

[1]Literature has found that in a similar model with only one type of sunspot observers, SSE exists if and only if $|\beta| < 1$, and E-stable SSE exists if and only if $\beta < -1$.

IV.3.2 Nonlinear Model

If either $H(y)$ or $G(y)$ are nonlinear functions in y, then we call the model is a nonlinear model. Recall that we assume $F = H \circ G : \mathbb{R} \to \mathbb{R}$ is three times continuously differentiable in a neighborhood of $\hat{y}$. Further $F'(\hat{y}) \neq 0$ and $F''(\hat{y}) \neq 0$. I present the following theorem.

Theorem 3

Given $\mathcal{I}_2 \neq \varnothing$, there exist a single= E-stable SSE solution

$$\{(w_0, w_{11}, w_{12}, \cdots, w_{N1}, w_{N2}) \in \mathbb{R}^{2N+1} | w_0 = 0, \mathcal{R}_n(1), \cdots, \mathcal{R}_l(N)\}$$

at least for some $k \in \{1, \cdots, N\}$, the transition probability of the sunspot process satisfies the serial correlation condition $\pi_{k1} + \pi_{k2} - 1 < (\gamma_k \beta)^{-1}$.

IV.4 Proof of the Existence and E-stability Results

The proofs of the theorems in the previous sections are illustrated in two simple cases of the model. The results are presented in two simple cases of the model. There is only one sunspot process in the first case, and each agent is either an SS believer or an SSE believer. There are two sunspot processes in the second case, and each agent is either an SSE-1 believer or an SSE-2 believer. I comment where the steps in the proof extend to the general case.

IV.4.1 SSE Believers v.s. SS Believers

Consider the case where there is only one sunspot process $\{s_t\}$ with transition matrix (π_{ij}). There is a mix of SS believers and SSE believers. The temporary equilibrium depends on "the weighted average belief". Eq.(IV.1) becomes

$$y_t = \sum_{i=0}^{i=1} \gamma_i H(E_t^i[G(y_{t+1})|I_t^i]), \tag{IV.2}$$

where γ_0 is the proportion of SS believers and γ_1 is the proportion of SSE believers. $\gamma_0 + \gamma_1 = 1$. Assume that SSE believers have perceived law of motion (PLM) $y_t = \alpha_{1i}$ where $i = s_t$, and that *SS believers* have restricted PLM $y_t = \alpha_0$. The mapping from the set of PLMs to the projected actual law of motion (ALM) is given by the following equation system. See Appendix I.3. for the derivation the projected ALM. Recall $F(\cdot) = H(G(\cdot))$.

$$T\begin{pmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_0 \end{pmatrix} = \begin{pmatrix} \gamma_1 H(\pi_{11}G(\alpha_{11}) + \pi_{12}G(\alpha_{12})) + \gamma_0 F(\alpha_0) \\ \gamma_1 H(\pi_{21}G(\alpha_{11}) + \pi_{22}G(\alpha_{12})) + \gamma_0 F(\alpha_0) \\ \gamma_1(\bar{p}_1 H(\pi_{11}G(\alpha_{11}) + \pi_{12}G(\alpha_{12})) + \bar{p}_2 H(\pi_{21}G(\alpha_{11}) + \pi_{22}G(\alpha_{12})) + \gamma_0 F(\alpha_0) \end{pmatrix}$$

where $\bar{p}_1 = \pi_{21}/(\pi_{21} + \pi_{12})$ and $\bar{p}_2 = \pi_{12}/(\pi_{21} + \pi_{12})$, and $(\bar{p}_1,\ \bar{p}_2)$ is the unique stationary distribution of the sunspot process s_t for state 1 and state 2. Let $\alpha = (\alpha_{11}, \alpha_{12}, \alpha_0)'$. The differential equation defining E-stability is $\frac{d\alpha}{d\tau} = T(\alpha) - \alpha$. For SSEs near a steady state, some useful results are implied by analysis of the linearization of the differential equation at the steady state.. Appendix I.4 shows that the linearized system at the steady state can be written as $\dot{w} = Aw + \Psi$, where

$$\dot{w} = \begin{pmatrix} \dot{w}_{11} \\ \dot{w}_{12} \\ \dot{w}_0 \end{pmatrix},\ w = \begin{pmatrix} w_{11} \\ w_{12} \\ w_0 \end{pmatrix},\ \Psi = \begin{pmatrix} \Psi_{11}(w_{11}, w_{12}, w_0) \\ \Psi_{12}(w_{11}, w_{12}, w_0) \\ \Psi_0(w_{11}, w_{12}, w_0) \end{pmatrix}$$

and the coefficient matrix of the linear part is $A = \beta\tilde{\Pi} - I$ with

$$\tilde{\Pi} = \begin{pmatrix} \gamma_1\pi_{11} & \gamma_1\pi_{12} & \gamma_0 \\ \gamma_1\pi_{21} & \gamma_1\pi_{22} & \gamma_0 \\ \gamma_1\bar{p}_1 & \gamma_1\bar{p}_2 & \gamma_0 \end{pmatrix} \tag{IV.3}$$

Here $\dot{w}_i = dw_i/d\tau$, and $\Psi_i(w_{11}, w_{12}, w_0)$ denote the nonlinear parts. Note that $\tilde{\Pi}$ in fact is a transition probability matrix. The eigenvalues of $\tilde{\Pi}$ are 0 , 1, and $\gamma_1(\pi_{11} + \pi_{22} - 1)$. Thus, the eigenvalues of the linear map A are $\lambda_1 = -1$, $\lambda_2 = \beta - 1$, $\lambda_3 = \gamma_1\beta(\pi_{11} + \pi_{22} - 1) - 1$.

Linear case

If the model is linear, $\Psi_i(w_{11}, w_{12}, w_0)$ contains only zeros. We only need to focus on the matrix A. If $|\beta| > (\gamma_1)^{-1}$, then $(\gamma_1\beta)^{-1} \in (-1, 1)$, then there exist (π_{ij}) such that $\pi_{11} + \pi_{22} = (\gamma_1\beta)^{-1} + 1 \in (0, 2)$ which implies $\lambda_3 = 0$, and A is not full rank. The model has a SSE solution only if matrix A is not full rank, and there exist a continuum of SSE solutions which are characterized by $A\bar{w} = 0$. Moreover, if $\beta < -(\gamma_1)^{-1}$ then both $\lambda_1 = 0$ and $\lambda_2 = \beta - 1$ are negative. If the Markov sunspot process satisfies restriction $\pi_{11} + \pi_{22} - 1 = (\gamma_1\beta)^{-1}$, then $\lambda_3 = \gamma_1\beta(\pi_{11} + \pi_{22} - 1) - 1 = 0$. There exist a continuum of SSE solutions which are characterized by $A\bar{w} = 0$, which implies

$$(\gamma_1\beta\pi_{11} - 1)\bar{w}_{11} + \gamma_1\beta(1 - \pi_{11})\bar{w}_{12} + \gamma_0\beta\bar{w}_2 = 0$$
$$\gamma_1\beta(1 - \pi_{22})\bar{w}_{11} + (\gamma_1\beta\pi_{22} - 1)\bar{w}_{12} + \gamma_0\beta\bar{w}_2 = 0$$

These two equations can simply to $(1 - \pi_{22})\bar{w}_{11} + (1 - \pi_{11})\bar{w}_{12} = 0$ and $\bar{w}_2 = 0$, which locates the continuum which represents the SSE solutions are E-stable under learning. Hence the special case of Theorem.(1) where $\gamma_2 = \cdots = \gamma_N = 0$ holds true for the linear case.

I make a few comments about the existence result for the linear model. First, note the condition for existence can also be written as $|\gamma_1\beta| > 1$, and recall the existence condition in the standard representative-agent model is $|\beta| > 1$. The proportion parameter γ_1 directly modifies the slope of the linear model in a multiplicative way. Intuitively, a γ_1 proportion of the agent population forms expectations of the future states, and their expectations affect today with a magnitude order of β. These SSE believers generate feedback at level $\gamma_1\beta$. Second, the resonant frequency condition is modified by the proportion parameter γ_1 compared to its counterpart in a standard representative-agent model. Third, the continuum set does not depend on γ_1, the proportion of agents who believe in sunspots. In the SSEs, the restrictions on w_{11} and w_{12} are identical to the restriction found in Evans and Honkapohja (2003a), and the only difference is that the steady-state believers think y_t is always at the steady-state of the model $\hat{y}$. Finally, the existence region shrinks as γ_1 becomes smaller, which aligns with the intuition that if fewer agents coordinate on the sunspot, SSEs are less likely to exist. The stability result extends naturally from Evans and Honkapohja (2003a). In the limit when there are only SSE believers, i.e., $\gamma_1 = 1$, theorem 2 matches the stability results found in previous literature. Note that if there is a mix of SSE believers and SS believers, the slope of the linear function β has to be more negative than the counterpart with representative agents for there to exist E-stable SSEs. A substantial negative slope β is required for the model to have stable SSE if the proportion of SSE believers, γ_1, is small.

Nonlinear case

The analysis for the nonlinear case is more complicated than the linear model. The proof relies on a local bifurcation analysis of the differential equation. The bifurcation arises when the linear part of the system has a zero eigenvalue, i.e. $\lambda_3 = 0$ or $\pi_{11} + \pi_{22} - 1 = (\gamma_1\beta)^{-1}$. Note that I am able to set eigenvalue λ_3 to be zero with the condition $\beta < -(\gamma_1)^{-1}$. Appendix I.5 proves that if $\beta < -(\gamma_1)^{-1}$, E-stable SSEs exist near the steady state. Treating π_{22} as a fixed number, I vary π_{11} to achieve bifurcation. Let $\bar{\pi}_{11} = 1 + (\gamma\beta)^{-1} - \pi_{22}$ and $v = \pi_{11} - \bar{\pi}_{11}$. The bifurcation occurs at $v = 0$. It follows that E-stable SSEs exist for $v < 0$. Note that $v < 0$ implies $\lambda_3 > 0$. If $v > 0$, the SSEs are not E-stable, and learning instead converges to the fundamental solution. Note that the sharp-edged resonant frequency condition is no longer needed. Specifically, if the transition matrix (π_{ij}) is such that $\pi_{11} + \pi_{22} - 1 < (\gamma_1\beta)^{-1}$ there exist an SSE near the steady-state. In particular, this result emphasizes the artificial nature of resonance frequency.

IV.4.2 SSE-1 Believers v.s. SSE-2 Believers

In this section, I consider the special case of the model where $\gamma_0 = 0$ and $N = 2$. I comment when appropriate how the stability result extends to the more general case where there are more processes and SSE believers. Two types of agents are called SSE-1 believers and SSE-2 believers. The temporary equilibrium depends on "the weighted average beliefs" of the two types of agents and writes as follows:

$$y_t = \sum_{i=1}^{2} \gamma_i H(\hat{E}_t^i G(y_{t+1})), \tag{IV.4}$$

The following equation system gives the mapping from the set of PLMs to the projected actual law of motion (ALM). Appendix I.6 derives the following T-map from the PLMs to the projected ALM

$$T\begin{pmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{21} \\ \alpha_{22} \end{pmatrix} = \begin{pmatrix} \gamma_1 H(\pi_{11}^1 G(\alpha_{11}) + \pi_{12}^1 G(\alpha_{12})) + \bar{p}_1^2 \gamma_2 H(\pi_{11}^2 G(\alpha_{21}) + \pi_{12}^2 G(\alpha_{22})) + \bar{p}_2^2 \gamma_2 H(\pi_{21}^2 G(\alpha_{21}) + \pi_{22}^2 G(\alpha_{22})) \\ \gamma_1 H(\pi_{21}^1 G(\alpha_{11}) + \pi_{22}^1 G(\alpha_{12})) + \bar{p}_1^2 \gamma_2 H(\pi_{11}^2 G(\alpha_{21}) + \pi_{12}^2 G(\alpha_{22})) + \bar{p}_2^2 \gamma_2 H(\pi_{21}^2 G(\alpha_{21}) + \pi_{22}^2 G(\alpha_{22})) \\ \bar{p}_1^1 \gamma_1 H(\pi_{11}^1 G(\alpha_{11}) + \pi_{12}^1 G(\alpha_{12})) + \bar{p}_2^1 \gamma_1 H(\pi_{21}^1 G(\alpha_{11}) + \pi_{22}^1 G(\alpha_{12})) + \gamma_2 H(\pi_{11}^2 G(\alpha_{21}) + \pi_{12}^2 G(\alpha_{22})) \\ \bar{p}_1^1 \gamma_1 H(\pi_{11}^1 G(\alpha_{11}) + \pi_{12}^1 G(\alpha_{12})) + \bar{p}_2^1 \gamma_1 H(\pi_{21}^1 G(\alpha_{11}) + \pi_{22}^1 G(\alpha_{12})) + \gamma_2 H(\pi_{21}^2 G(\alpha_{21}) + \pi_{22}^2 G(\alpha_{22})) \end{pmatrix}$$

Here $\bar{p}_1^j = \pi_{21}^j/(\pi_{21}^j + \pi_{12}^j)$ and $\bar{p}_2^j = \pi_{12}^j/(\pi_{21}^j + \pi_{12}^j)$, and $(\bar{p}_1^j,\ \bar{p}_2^j)$ are the stationary distributions of sunspot processes s_t^j for state 1 and state 2. Let $\alpha =$

$(\alpha_{11}, \alpha_{12}, \alpha_{12}, \alpha_{22})'$. The differential equation defining E-stability is $\frac{d\alpha}{d\tau} = T(\alpha) - \alpha$. The model can be transformed to deviation from steady state form with $w_i = \alpha_i - \hat{y}$ for $i \in \{11, 12, 21, 22\}$. Appendix I.7 shows that the linearized system at the steady state can be written as the deviation form $\dot{w} = Aw + \Psi$, where

$$\dot{w} = \begin{pmatrix} \dot{w}_{11} \\ \dot{w}_{12} \\ \dot{w}_{21} \\ \dot{w}_{22} \end{pmatrix}, \; w = \begin{pmatrix} w_{11} \\ w_{12} \\ w_{21} \\ w_{22} \end{pmatrix}, \; \Psi = \begin{pmatrix} \Psi_{11}(w_{11}, w_{12}, w_{21}, w_{22}) \\ \Psi_{12}(w_{11}, w_{12}, w_{21}, w_{22}) \\ \Psi_{21}(w_{11}, w_{12}, w_{21}, w_{22}) \\ \Psi_{22}(w_{11}, w_{12}, w_{21}, w_{22}) \end{pmatrix}$$

and the coefficient matrix of the linear part is $A = \beta\tilde{\Pi} - I$ with

$$\tilde{\Pi} = \begin{pmatrix} \gamma_1\pi^1_{11} & \gamma_1\pi^1_{12} & \gamma_2\bar{p}^2_1 & \gamma_2\bar{p}^2_2 \\ \gamma_1\pi^1_{21} & \gamma_1\pi^1_{22} & \gamma_2\bar{p}^2_1 & \gamma_2\bar{p}^2_2 \\ \gamma_1\bar{p}^1_1 & \gamma_1\bar{p}^1_2 & \gamma_2\pi^2_{11} & \gamma_2\pi^2_{12} \\ \gamma_1\bar{p}^1_1 & \gamma_1\bar{p}^1_2 & \gamma_2\pi^2_{21} & \gamma_2\pi^2_{22} \end{pmatrix} \tag{IV.5}$$

Here $\dot{w}_i = dw_i/d\tau$, and $\Psi_i(w_{11}, w_{12}, w_{21}, w_{22})$ denote the nonlinear parts. Note that $\tilde{\Pi}$ in fact qualifies as a transition probability matrix, and I will explore the implication of this observation in details later. The eigenvalues of $\tilde{\Pi}$ are 0 , 1, $\gamma_1(\pi^1_{11} + \pi^1_{22} - 1)$ and $\gamma_2(\pi^2_{11} + \pi^2_{22} - 1)$. Thus, the eigenvalues of the linear map A are $\lambda_1 = \gamma_1\beta(\pi^1_{11} + \pi^1_{22} - 1) - 1$, $\lambda_2 = \gamma_2\beta(\pi^2_{11} + \pi^2_{22} - 1) - 1$, $\lambda_3 = -1$, and $\lambda_4 = \beta - 1$.

Linear Case

When the model is linear, then $\forall i \in \mathcal{I}_1$, we have $(\gamma_i\beta)^{-1}$. There exist Π^i such that $\pi^i_{11} + \pi^i_{22} = (\gamma_i\beta)^{-1} + 1 \in (0, 2)$ which implies $\lambda_i = 0$ and A is not full rank. The model has a SSE solution only if matrix A is not full rank, and there exist a continuum of SSE solutions which are characterized by $A\bar{w} = 0$. Moreover, if $\beta < -(\gamma_1)^{-1}$ then both $\lambda_3 = 0$ and $\lambda_4 = \beta - 1$ are negative. If the Markov sunspot process satisfies restriction $\pi_{11}^i + \pi_{22}^i - 1 = (\gamma_i \beta)^{-1}$, then $\lambda_i = \gamma_i \beta(\pi_{11}^i + \pi_{22}^i - 1) - 1 = 0$. There exist a continuum of SSE solutions which are characterized by $A\bar{w} = 0$, which implies $(1 - \pi_{22}^i)\bar{w}_{11} + (1 - \pi_{11}^i)\bar{w}_{12} = 0$ for $i \in \mathcal{I}_2$ and $\bar{w}_j = 0$ for $j \notin \mathcal{I}_2$

Nonlinear Case

For the nonlinear case, Appendix I.9 shows some extra technical difficulties in proving the theorem. Specifically, one needs to set both eigenvalues λ_3 and λ_4 to be zero simultaneously for the bifurcation to arise. The proof of stability leads to a system of two differential equations interdependent in order 3 or higher in the center manifold. I prove that the system's stability result only depends on the linear and quadratic parts of the function, and thus we can analyze the two differential equations separately. The proof for the special case where $\gamma_0 = 0$ and $\gamma_i = 0$ $\forall i > 0$. Note that the coefficient matrix of the differential equations associated to E-stability would have N eigenvalues $\gamma_i \beta(\pi_{11}^2 + \pi_{22}^2 - 1) - 1$ where $i \in \{1, 2, \cdots, N\}$. It follows that there would be N serial correlation conditions, and if one of the conditions is satisfied, there exists a continuum of SSE solutions. The resonant frequency condition associated with a specific sunspot process depends on the proportion of the agents who observe that sunspot variable. For the nonlinear model's proof of stability, the bifurcation would happen at N points instead of 2 points. The proof of stability would also lead to a system of N differential equations that are still interdependent in order 3 or higher in the center manifold.

IV.5 An Example: OLG Model

This section provides a standard version of the Samuelson overlapping generations (OLG) model of money that treats heterogeneous beliefs with care. The purpose of this section is to demonstrate the general existence and E-stability results with simulations in a micro-founded model. Assume there is a continuum of agents who live two periods of time. Each agent supplies labor in their first period of life and consumes in the second period. The only asset in the economy is money, and the money aggregate supply $M = 1$, which is fixed over time. The utility maximization problem of agent ω is

$$\max_{n_t(\omega)} \ \mathbb{E}_{\omega,t} U(c_{t+1}(\omega)) - V(n_t(\omega))$$
$$\text{subject to}$$
$$p_{t+1}c_{t+1}(\omega) = p_t Q_t(\omega)$$
$$Q_t(\omega) = n_t(\omega)$$

Each agent is small, and the market is competitive, and thus each agent takes the price p_t as given. The expectation operator is sub-scripted with ω which means that the expectation is held specifically by agent ω. Assume functional forms $V(n) = \frac{n^{1+\varepsilon}}{1+\varepsilon}$ and $U(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$. Define the following a new variable $y_t = p_t^{-\frac{\varepsilon+1}{\varepsilon+\sigma}}$. The temporary equilibrium (TE) can be computed as follows. Appendix I.10 shows the derivation of the TE. $y_t = \int_\omega H(E_t^\omega G(y_{t+1}))d\omega$ where $H(y) = y^{\frac{1}{\varepsilon+\sigma}}$ and $G(y) = y^{\frac{(\varepsilon+\sigma)(1-\sigma)}{\varepsilon+1}}$. Define the compounded function $F(y) = H(G(y)) = y^\beta$, where $\beta = (1-\sigma)/(\varepsilon+1)$. Note that now the model is in the same form as Eq.(IV.1). I also consider the linearized version of the model, where the linearization happens at the steady state $\hat{y} = 1$. The linearized versions of functions H and G and the implied compounded function are written as $\tilde{H}(y) = 1+\frac{1}{\varepsilon+\sigma}(y-1)$, $\tilde{G}(y) = 1+\frac{(\varepsilon+\sigma)(1-\sigma)}{\varepsilon+1}(y-1)$, and $\tilde{F}(y) = 1+\beta(y-1)$.

IV.5.1 Simulations

I present five sets of configurations of the OLG model summarized in table (1). The first four simulations confirm the analytic results found in this paper for both linear and nonlinear cases. The last set of simulations shows a general example of 9 sunspot processes with SS believers and 9 groups of SSE believers. All of the simulations are implemented with a small constant gain of 0.05 instead of the decreasing gain of $\frac{1}{t}$ as stated in the model.

	Configuration		
Simulation	N	γ_0	Linearity
I	1	> 0	Linear
II	1	> 0	Nonlinear
III	2	$= 0$	Linear
IV	2	$= 0$	Nonlinear
V	9	> 0	Nonlinear

TABLE 1 Five Sets of Simulations

Simulation I

This set of simulations are shown in Figure (1). I make the following parameter choices for the first set of simulations. $\varepsilon = 1$ and $\sigma = 11$ so that $\beta = (1-\sigma)/(\varepsilon+1) = -5$. Also $\gamma_1 = 0.25$. The transition probability matrix for the sunspot process is set with the following values $\pi_{11} = 0.15 + v$ and $\pi_{22} = 0.05$. I consider three different values for v. $v^- = -0.1$, $v^0 = 0$, and $v^+ = 0.1$. The values for ν are set so that when $\nu = \nu^0$, the knife-edged resonant frequency condition $\gamma_1\beta(\pi_{11}+\pi_{22}-1)-1 = 0$ is satisfied. Initial values for belief coefficients $\alpha = (\alpha_{11}, \alpha_{12}, \alpha_0)$ are set to be (1.001, 0.999, 1.001) which is different from but in the neighborhood of the steady-state solution (1.0, 1.0, 1.0). In the left columns, the blue lines represent the evolution of SSE believers estimates α_{11}, α_{12}, and the red line represents the evolution of SS believers estimate α_0. The first row shows the case where $v < 0$, and there is an explosive root, which means there are no E-stable solutions. The second row corresponds to the case where the knife-edged resonant frequency condition is met, and the system converges to a point in the continuum specified by $(1-\pi_{22})(\alpha_{11}-1) = -(1-\pi_{11})(\alpha_{12}-1)$ and $\alpha_0 = 1$. The third

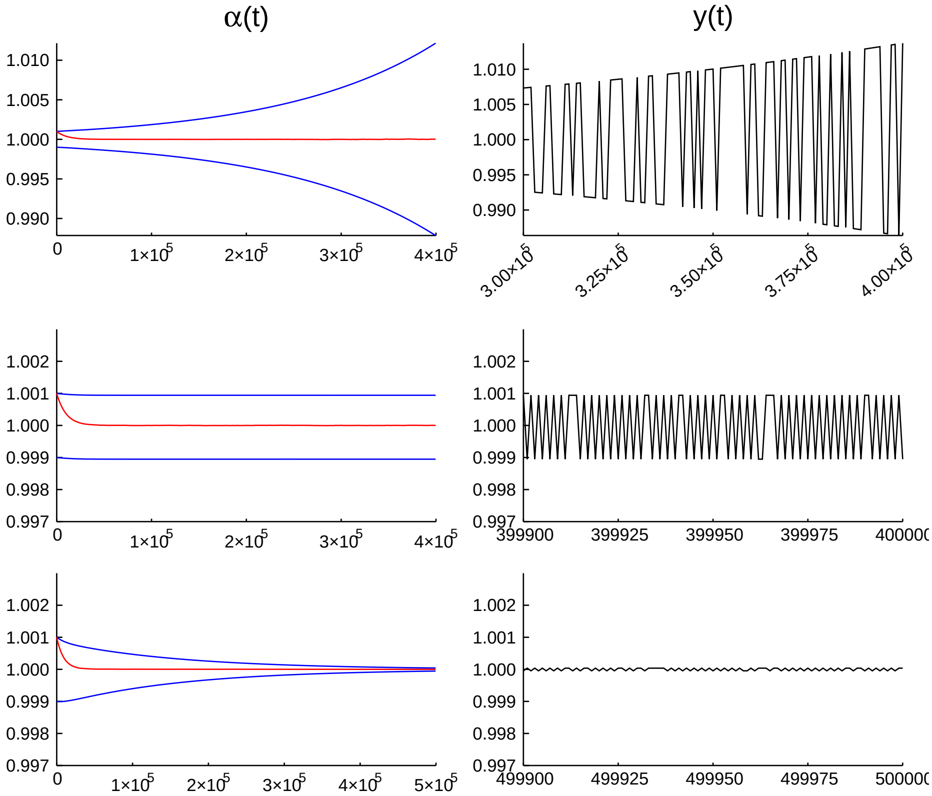


FIGURE 1. *Linear Model with SS Believers and SSE Believers*
Note: The first column shows the evolution for the belief components $\alpha = (\alpha_{11}, \alpha_{12}, \alpha_0)$, which are located from top to bottom in each graph on the left hand side. Three rows of simulations correspond to cases where v is v^-, v^0, and v^+ respectively.

row shows the case where $v > 0$, and the only E-stable solution is the fundamental solution $(1, 1, 1)$. The simulation matches the prediction by the existence and E-stability results.

Simulation II

This set of simulations from the nonlinear model is shown in Figure (2). Parameter choices are the same as simulation I. Two rows correspond to the case ν^-, and ν^+, respectively. With ν^-, there exists an E-stable SSE, which matches the simulation in the first row. Note that when the model is nonlinear, the restriction on the sunspot process's correlation is no longer knife-edged. With ν^+, the only E-stable solution is steady state, shown in the second row. Initial values for belief coefficients $\alpha = (\alpha_{11},\ \alpha_{12},\ \alpha_0)$ are set to be (1.01, 0.99, 1.01) which is different from but in the neighborhood of the steady state. In the left columns, the blue lines represent the evolution of α_{11}, α_{12}, and the red line represents the evolution of α_0.

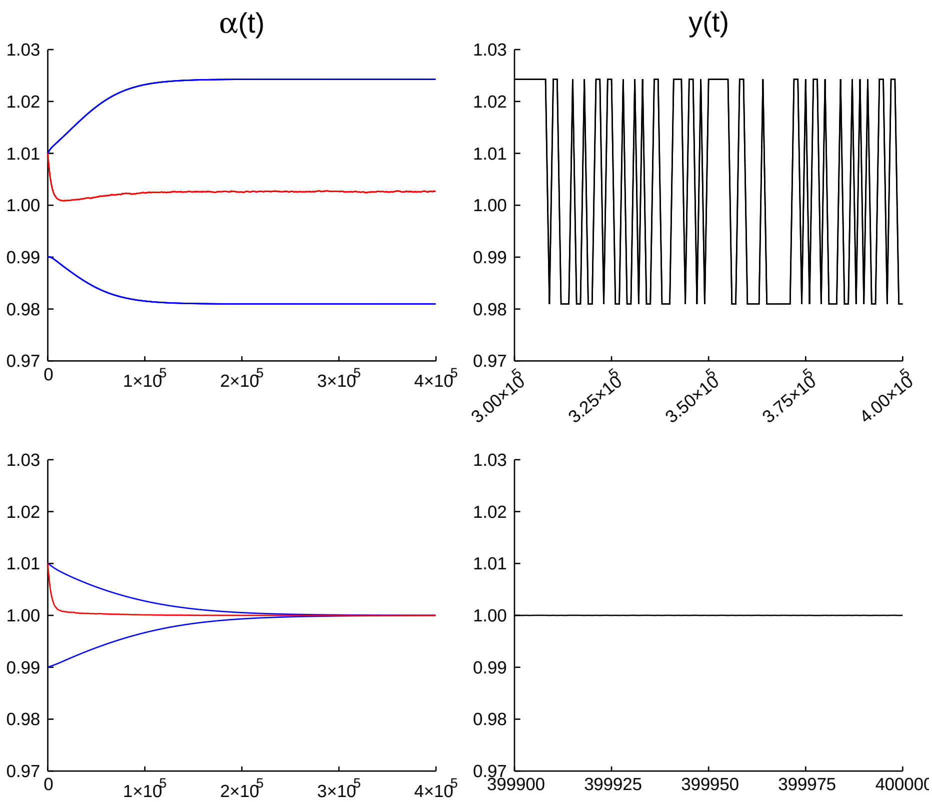


FIGURE 2. *Nonlinear Model with SS believers and SSE believers*
Note: The first column shows the evolution for the belief components α_{11}, α_{12}, α_0, which are from top to bottom in each graph on the left-hand side. Two rows of simulations correspond to cases where v is v^-, and v^+ respectively.

Simulation III

This set of simulations are shown in Figure (3) with two types of SSE believers: SSE-1 and SSE-2. The model is linearized around the steady state. I make the following parameter choices: $\varepsilon = 1$ and $\sigma = 21$ so that $\beta = -10$. Also the proportion of SSE-1 and SSE-2 agents are $\gamma_1 = 0.2$, $\gamma_2 = 0.8$. The transition probability matrix for the first sunspot process is $\pi_{11}^1 = 0.25 + v_1$, $\pi_{22}^1 = 0.25$ and for the other sunspot process is $\pi_{11}^2 = 0.5 + v_2$ and $\pi_{22}^2 = 0.375$. Consider $v_j^+ = 0.1$ and $v_j^0 = 0$ for $j = 1, 2$. Consider four combinations $(v_1^0,\, v_2^0)$, $(v_1^0,\, v_2^+)$, $(v_1^+,\, v_2^0)$ and $(v_1^+\, v_2^+)$, which are shown in Figure (3) from top to bottom. Initial values for belief coefficients $\alpha = (\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22})$ are set to be $(1.005, 0.995, 1.01, 0.99)$. The blue lines and red lines represent the belief components of SSE-1 and SSE-2 believers respectively. I ignore the cases that involve $v_j < 0$ because in these cases the dynamic is explosive according to E-stability principle, a result that has been found in the previous literature. The stable SSE are shown in the first three rows where at least one of ν's is zero. When $v_1 > 0$ and $v_2 > 0$, then the only E-stable solution is the fundamental solution.

Simulation IV

This set of simulations from the nonlinear model are shown in Figure (4). Parameter choices are the same as simulation III. Consider four cases: (v_1^-, v_2^-), (v_1^-, v_2^+), (v_1^+, v_2^-) and $(v_1^+\, v_2^+)$, which are shown in Figure (4) from top to bottom. Note that when v_1 and v_2 are both negative, there exists an E-stable SSE of order 4. When only one of the $v's$ is negative, and the other one is positive, there exists an E-stable of SSE of order 2. When neither $v's$ are negative, the only E-stable solution is the fundamental solution.

Simulation V

This simulation is shown in Figure (5) from a general case with SS believers and 9 groups of SSE believers in a nonlinear model. I make the following parameter choices: $\varepsilon = 1$ and $\sigma = 41$ so that $\beta = -20$. Each group of agents (including

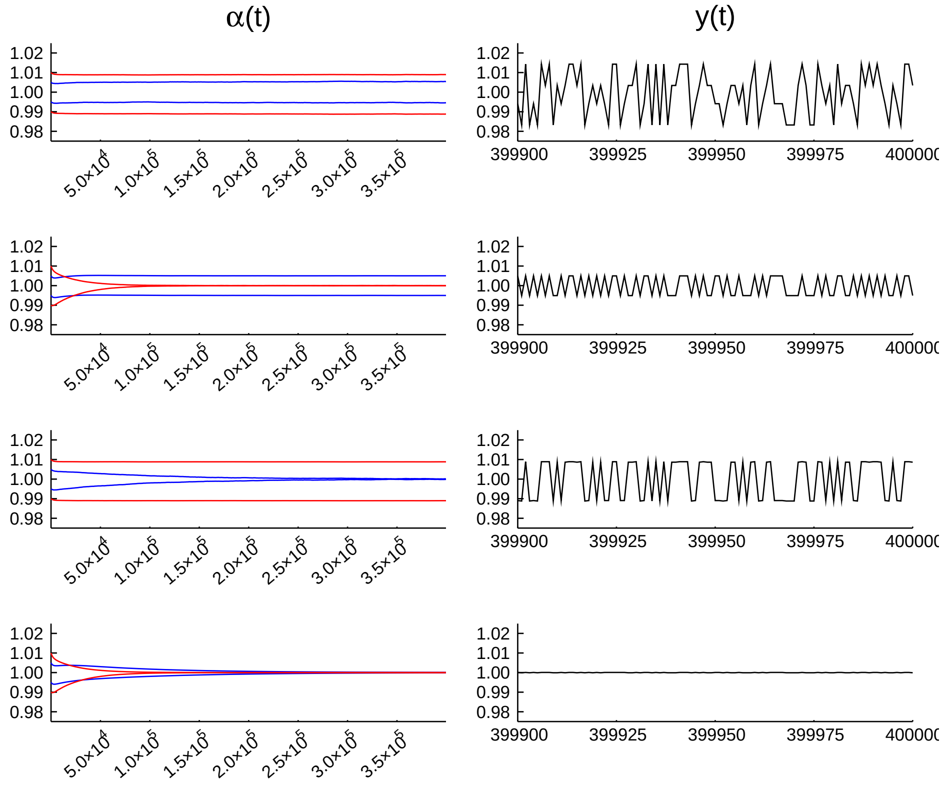


FIGURE 3. *Linear Model with SSE1 and SSE2 Believers*

the group of SS believers) accounts for 10% of the population, i.e. $\gamma_j = 0.1$ for all $j \in \{0, 1, \cdots, 10\}$. The probability transition matrix for the sunspot processes Π^j are set as $\pi^j_{11} = 0.225 + 0.025 * j + \nu_j$ and $\pi^j_{22} = 0.525 - 0.025 * j$. Note that the general theorems predict that SSE-j agents learn that the economy evolve according to the observed sunspot if $\nu_j < 0$, and learn that the economy is at the steady state if $\nu_j > 0$. To confirm this prediction, I divide the 9 groups of SSE believers into two categories. In the simulation, the first five groups of SSE believers have $v_j < -0.1$ (blue lines), and the last four groups of SSE believers have $v_j = 0.1$ (red lines). The black line in the middle represents the SS believer in the middle. I use a small constant gain equal to 0.00015 for the learning dynamic.

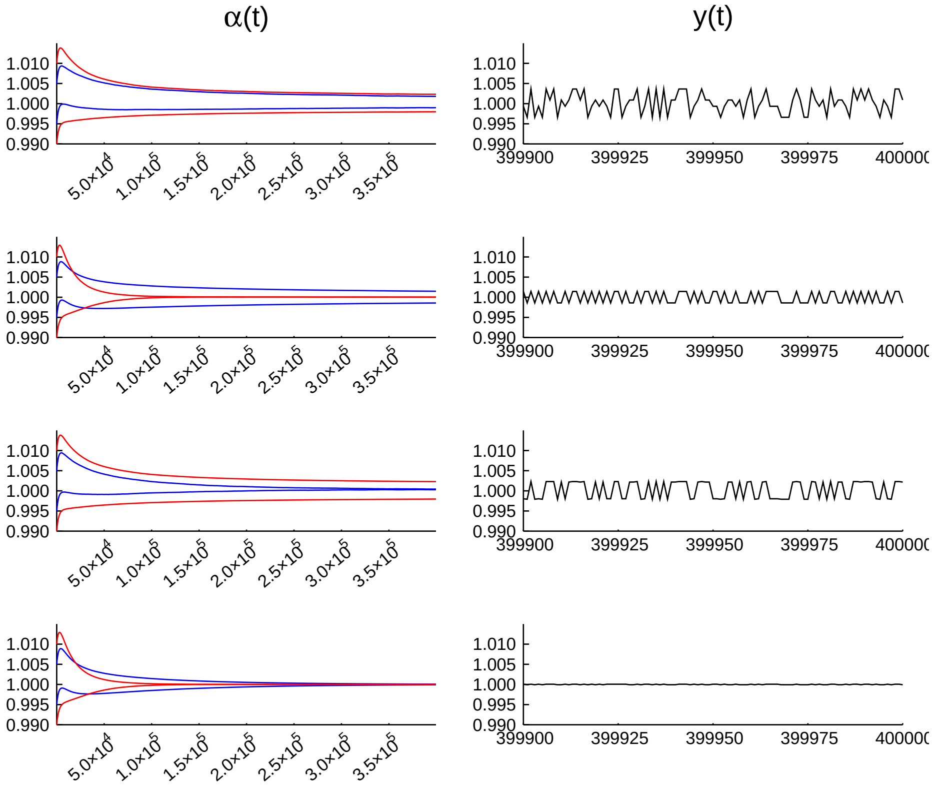


FIGURE 4. *Nonlinear Model with SSE1 and SSE2 Believers*

Note: Four rows correspond to four combinations $(v_1^0,\ v_2^0)$, $(v_1^0,\ v_2^+)$, $(v_1^+,\ v_2^0)$ and $(v_1^+\ v_2^+)$ respectively. The left column shows the evolution of beliefs $\alpha = (\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22})$ which are shown from top to bottom in each graph on the left hand side.

IV.6 Selection Dynamics

This section explores whether the agents favor the sunspot equilibrium by introducing model selection dynamics based on Branch and Evans (2006.) In the previous environment, agents are divided into different groups based on the forecast models they use. I also have exogenously set the proportion of each group. In a more realistic setting, agents may choose between a list of models and base their selection on their relative forecast performances. The advantage of adding the selection mechanism

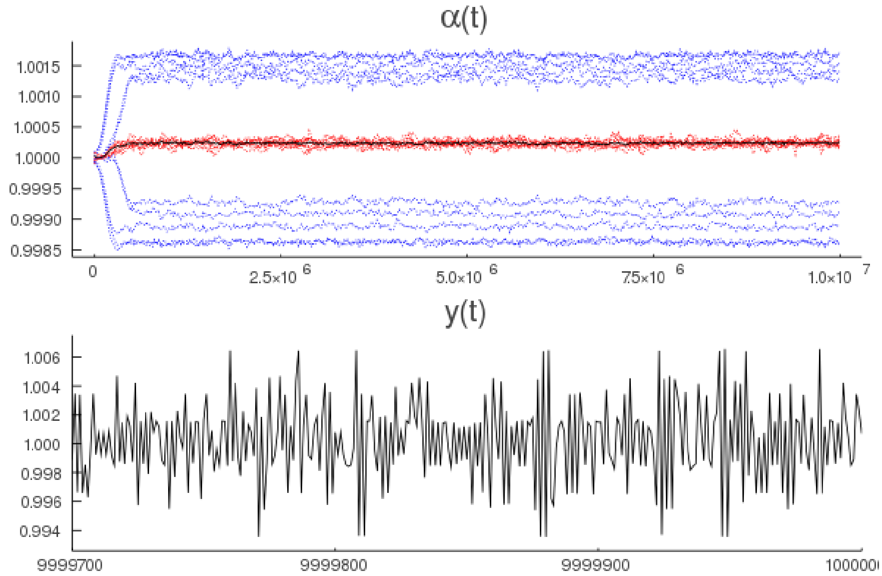


FIGURE 5. *Nonlinear Model with SS Believers and* 9 *Types of SSE Believers*
Note: The top graph shows the evolution of the agents' beliefs. The blue lines correspond to the groups of SSE believers with $v_j < 0$, and the orange lines correspond to the groups of SSE believers with $v_j > 0$

is that the model can now endogenously sort agents into different groups. In the model with SSE believers and SE believers, this experiment investigates whether agents will choose the SSE rule, SE rule, or both. This is the special case where $N = 1$. The selection mechanism pitch the sunspot equilibrium believers against the steady-state believers to compete based on forecasting accuracy.

The selection dynamics are added to the same standard version of the Samuelson overlapping generations (OLG) model of money that treats heterogeneous. Agents form real-time estimates formed via recursive least squares (RLS) and choose the forecasting rule based on unconditional mean squared errors for variable y. There is dual learning as agents recursively update their forecasting model parameters and evolve their predictor choice according to a dynamic predictor selection mechanism. Predictor proportions are updated according to the discrete choice probabilities. The fitness based on mean square errors of the two predictors $j \in \{0, 1\}$ are estimated by

$$\hat{\Phi}_{j,t} = \hat{\Phi}_{j,t-1} + \delta_t(-(y_t - \hat{E}_{j,t-1}y_t)^2 - \hat{\Phi}_{j,t-1})$$

where $0 < \delta_t < 1$. The mean squared errors map into predictor proportions according to the law of motion

$$\gamma_{j,t} = \frac{\exp\left[\xi\hat{\Phi}_{j,t}\right]}{\sum_{k=0}^{N}\exp\left[\xi\hat{\Phi}_{k,t}\right]}$$

The intensity parameter ξ governs how sensitive the agents are to the relative sizes of the accuracy measures. If $\xi = 0$, then $\gamma_{j,t} = \frac{1}{2}$ which means the proportion for each group is the same. If $\xi \to \infty$, then $\gamma_{j,t}$ is 1 or 0 depends on whether $\Phi_{j,t}$ is the largest or not.

I am interested in whether $\gamma_{j,t}$ converges to some level. Figure (6) illustrates numerically that sunspot equilibrium can be stable when the predictor proportion is determined endogenously under real-time learning. I make the following parameter choices for the first set of simulations. $\varepsilon = 1$ and $\sigma = 11$ so that $\beta = (1-\sigma)/(\varepsilon+1) = -5$. The transition probability matrix for the sunspot process is set with the following values $\pi_{11} = 0.15$ and $\pi_{22} = 0.05$. I initialize the proportion of SS believers $\gamma_{0,0} = 0$. The initial values for $(\alpha_{11}, \alpha_{12}, \alpha_2) = (1.2, 0.8, 1)$. This setup means that not only all agents start with using the steady-state forecasting rule, but also they start with the correct estimate. The proportion parameter is held constant until the 1000th period to gather a history of data so that accuracy measures can be computed for each forecasting model.

Figure (6) shows the simulation with a large intensity parameter $\xi = 400$. Also $\delta = 0.001$. The simulation converges to a state where all agents use the forecasting rule based on the sunspot, and the sunspot forecasting rule is consistently better than the steady-state rule. The intuition for this is that the threshold for the sunspot equilibrium to be stable is $\tilde{\gamma}_1 = 0.25$. When the proportion starts to change at 1000, $\gamma_{1,1000}$ jumped from 0 to around 0.4, which surpasses the threshold $\tilde{\gamma}_1$ for the sunspot equilibrium to be stable. A large proportion of agents start using the SSE rule at time 1000 because when all agents are forced to use the SS forecasting rule, both of the estimates for α_{11} and α_{12} will stay at 1.

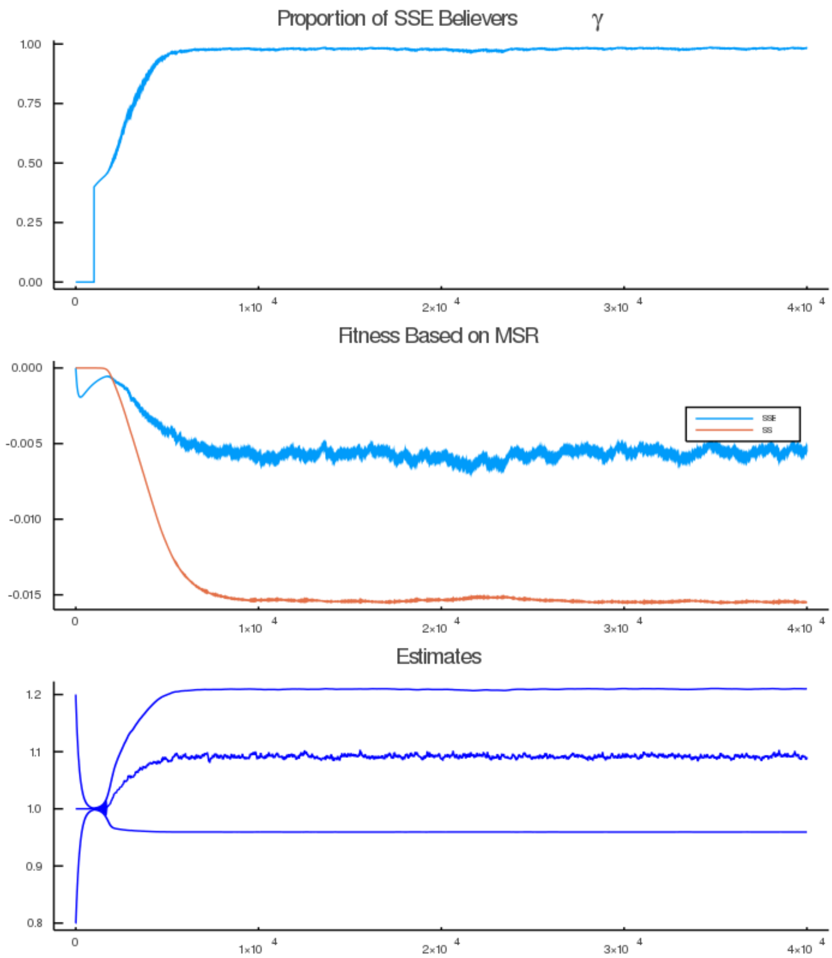


FIGURE 6. *Selection Dynamic I*

A key observation here is that the SSE forecast model nests the SS model. The intuition is also straightforward - the nesting model can perform as well as the nested model even when the economy is around the steady state. With a finite intensity parameter, half of the agents will eventually use the SSE model. As long as 1/2 passes the threshold implied by the model calibration, the SSE model will start to become a stable equilibrium under learning. Eventually, all agents will use the SSE model. Recall the calibration used in the simulation implies the threshold is 0.25.

What happens if the calibration is changed so that the threshold is larger than 0.5? I run the following simulation to demonstrate this scenario. The simulation in Figure (7) is based on a similar calibration except that now I change σ from 11 to $\frac{13}{3}$. Now the new proportion threshold $\tilde{\gamma}_1$ for the SSE to be stable is 0.75. Since the threshold for the SSE solution to be stable is higher now, the SSE believers eventually learn the steady state $\alpha_{11} = \alpha_{12} = 0$.

IV.7 Conclusion

Self-fulling outcomes of pessimism or optimism have significant macroeconomic implications. The studies of sunspot equilibria try to formalize this important phenomenon. One criticism of SSE literature is that self-fulling solutions are a highly coordinated outcome that is unlikely to rise in the real world. The literature on the existence and stability of sunspot equilibria has always assumed a representative agent. It fails to provide a robustness check that sunspot equilibria can exist when agents coordinate on different extrinsic information sets. This paper presents a direct response to this criticism. I show that the economy can evolve according to different sunspots, even if only a proportion of the agents participate in the coordination. They do not need to all coordinate on the same sunspot process. Besides, I provide the necessary and sufficient conditions under which SSE exists and is stable under

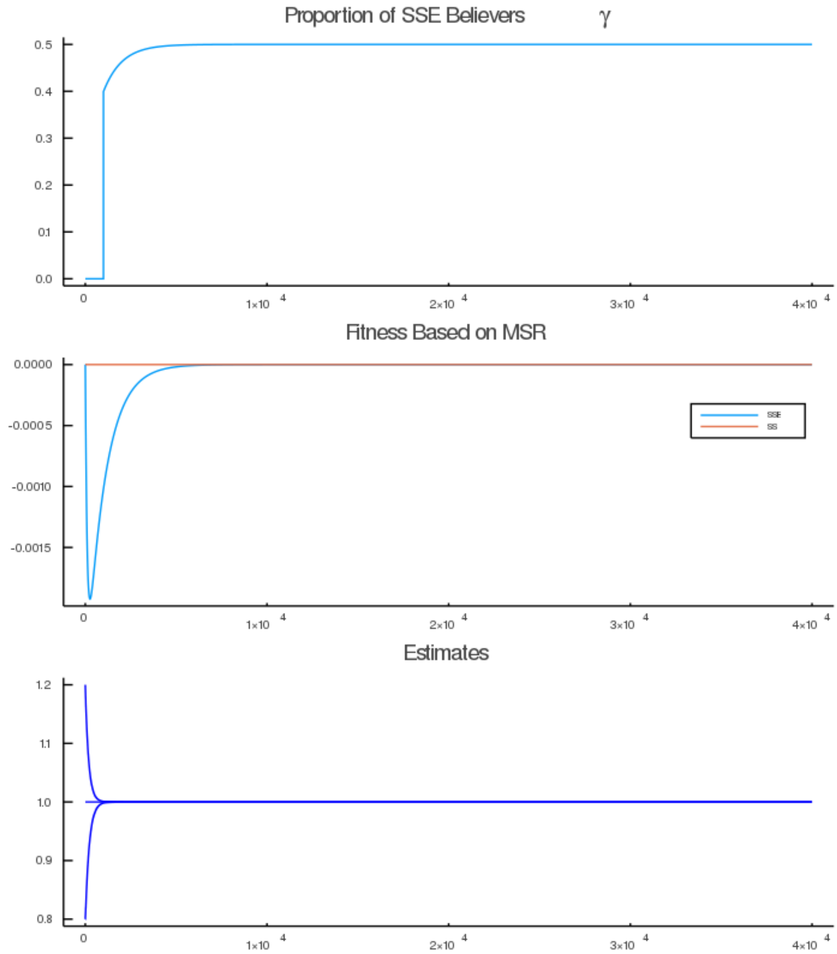


FIGURE 7. *Selection Dynamic II*

learning. This paper also confirms that agents choose the SSE rule over the SS rule with a suitable initial setup with model selection dynamics.

On the other hand, I find that the parameter space for SSE to exist is smaller in a model with belief heterogeneity than a representative-agent model. The stability region for these SSEs also shrinks accordingly. These findings have important implications. For the RBC-type models studied by Farmer and Guo (1994) that explain business cycle co-movements with SSE, this paper suggests that the existence region could be smaller than what has been found. The stability could be harder to come by. An open question is whether SSEs would still arise in calibrated non-convex RBC models such as the two-sector model in Benhabib and Farmer (1996) and the model with non-separable utilities in Bennett and Farmer (1999). For the New Keynesian literature that warns of the dangers of sunspot equilibria from a poorly-designed monetary policy, this paper suggests that the "safe region" where SSEs do not arise can be larger than what has been previously thought. The literature so far suggests the interest rule be such that the model has a slope of more than -1, so SSEs are not stable under learning. This paper finds that the threshold might be much lower than -1 if we are willing to assume that some agents do not observe any sunspots in the first place or that agents observe different sunspots.

This paper is the first to study indeterminacy when agents are heterogeneous in beliefs. The recent development in macroeconomics modeling has witnessed a shift from a representative agent framework to one that carefully treats agent-level heterogeneity, especially income/wealth heterogeneity in models featuring incomplete markets. See Mckay et al. (2016), Kaplan et al. (2018), and Bhandari et al. (2019.) One recommendation for future research is to study indeterminacy under interactions between income heterogeneity and belief heterogeneity.

APPENDIX

I.1 Derivation of the FOC of a Household

The inter-temporal condition for the profit maximization problem specified in Eq.(III.17) can be written as

$$\frac{Y_t}{P_t}\left(\frac{p_t(\omega)}{P_t}\right)^{-\nu}\left[(1-\nu)+\nu\mathcal{M}_t(p_t(\omega))\frac{P_t}{p_t(\omega)}-P_t\mathcal{M}_t'(p_t(\omega))\right]+\mathcal{C}_1(p_t(\omega),p_{t-1}(\omega))=\beta E_t\left[\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}}\mathcal{C}_2(p_{t+1}(\omega),p_t(\omega))\right] \tag{I.1}$$

Note that

$$\mathcal{M}_t(p_t(\omega))=\frac{w_t}{\alpha\theta_t}\left(\frac{p_t(\omega)Y_t}{\theta_t P_t}\right)^{\frac{1-\alpha}{\alpha}}$$
$$\mathcal{M}_t'(p_t(\omega))=\frac{\alpha-1}{\alpha^2}\frac{P_t w_t}{p_t^2(\omega)Y_t}\left(\frac{p_t(\omega)Y_t}{P_t\theta_t}\right)^{\frac{1}{\alpha}}$$
$$\mathcal{C}_1(p_t(\omega),p_{t-1}(\omega))=-\phi\frac{p_t(\omega)-p_{t-1}(\omega)}{p_{t-1}^2(\omega)}$$
$$\mathcal{C}_2(p_{t+1}(\omega),p_t(\omega))=\phi p_{t+1}(\omega)p_t(\omega)\frac{p_{t+1}(\omega)-p_t(\omega)}{p_t^3(\omega)}$$

Substituting these equations into Eq.(I.1) and assuming the equilibrium is symmetric

$$p_t(\omega)=P_t$$

for all t and ω, one obtains the optimality condition for the intermediate firm as follows

$$C_t^{-\sigma}\left((\nu-1)Y_t+\left(\frac{1-\alpha-\alpha\nu}{\alpha^2}\right)\left(\frac{Y_t}{\theta_t}\right)^{\frac{1}{\alpha}}w_t\right)+\phi\Lambda_t=\phi\beta E_t\Lambda_{t+1}$$

where Λ_t is the shadow price for the intermediate-good producers

$$\Lambda_t = C_t^{-\sigma}\Pi_t(1+\Pi_t)$$

I.2 Derivation of the FOC of an Intermediate Firm

The inter-temporal condition for the profit maximization problem specified in Eq.(III.17) can be written as

$$\frac{Y_t}{P_t}\left(\frac{p_t(\omega)}{P_t}\right)^{-\nu}\left[(1-\nu)+\nu\mathcal{M}_t(p_t(\omega))\frac{P_t}{p_t(\omega)}-P_t\mathcal{M}_t'(p_t(\omega))\right]+\mathcal{C}_1(p_t(\omega),p_{t-1}(\omega))=\beta E_t\left[\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}}\mathcal{C}_2(p_{t+1}(\omega),p_t(\omega))\right]$$

Note that

$$\mathcal{M}_t(p_t(\omega)) = \frac{w_t}{\alpha\theta_t}\left(\frac{p_t(\omega)Y_t}{\theta_t P_t}\right)^{\frac{1-\alpha}{\alpha}}$$

$$\mathcal{M}_t'(p_t(\omega)) = \frac{\alpha-1}{\alpha^2}\frac{P_t w_t}{p_t^2(\omega)Y_t}\left(\frac{p_t(\omega)Y_t}{P_t\theta_t}\right)^{\frac{1}{\alpha}}$$

$$\mathcal{C}_1(p_t(\omega),p_{t-1}(\omega)) = -\phi\frac{p_t(\omega)-p_{t-1}(\omega)}{p_{t-1}^2(\omega)}$$

$$\mathcal{C}_2(p_{t+1}(\omega),p_t(\omega)) = \phi p_{t+1}(\omega)p_t(\omega)\frac{p_{t+1}(\omega)-p_t(\omega)}{p_t^3(\omega)}$$

Substituting these equations into Eq.(I.1) and assuming the equilibrium is symmetric

$$p_t(\omega) = P_t$$

for all t and ω, one obtains the optimality condition for the intermediate firm as follows

$$C_t^{-\sigma}\left((\nu-1)Y_t+\left(\frac{1-\alpha-\alpha\nu}{\alpha^2}\right)\left(\frac{Y_t}{\theta_t}\right)^{\frac{1}{\alpha}}w_t\right)+\phi\Lambda_t = \phi\beta E_t\Lambda_{t+1}$$

where Λ_t is the shadow price for the intermediate-good producers

$\Lambda_t = C_t^{-\sigma}\Pi_t(1+\Pi_t)$

I.3 SS v.s. SSE: Derivation of the Projected ALM

SS believers and SSE believers form expectations based on their PLMs as follows.

$$\hat{E}_t^1 G(y_{t+1}) = \begin{cases} \pi_{11}G(\alpha_{11}) + \pi_{12}G(\alpha_{12}) \text{ if } s_t = 1 \\ \pi_{21}G(\alpha_{11}) + \pi_{22}G(\alpha_{12}) \text{ if } s_t = 2 \end{cases}$$
$$\hat{E}_t^2 G(y_{t+1}) = G(\alpha_0)$$

Combining the expectations and Eq.(1), one obtains the ALM. Recall $F(\cdot) = H(G(\cdot))$

$$y_t = \begin{cases} \gamma_1 H(\pi_{11}G(\alpha_{11}) + \pi_{12}G(\alpha_{12})) + \gamma_0 F(\alpha_0) \text{ if } s_t = 1 \\ \gamma_1 H(\pi_{21}G(\alpha_{11}) + \pi_{22}G(\alpha_{12})) + \gamma_0 F(\alpha_0) \text{ if } s_t = 2 \end{cases}$$

The SS believers do not observe the sunspot process and therefore regard the deviation from the steady state as white noise. Solving $\Pi\bar{p} = \bar{p}$. for $\bar{p} = (\bar{p}_1, \bar{p}_2)'$, the stationary distribution of the sunspot process, one obtains $\bar{p}_1 = \pi_{21}/(\pi_{21} + \pi_{12})$ and $\bar{p}_2 = \pi_{12}/(\pi_{21} + \pi_{12})$ From the SS believers' perspective, they see y_t evolves around the steady state as follows.

$$\begin{aligned} T_2(\alpha_0) =& \bar{p}_1(\gamma_1 H(\pi_{11}G(\alpha_{11}) + \pi_{12}G(\alpha_{12})) + \gamma_0 F(\alpha_0)) + \\ & \bar{p}_2(\gamma_1 H(\pi_{21}G(\alpha_{11}) + \pi_{22}G(\alpha_{12})) + \gamma_0 F(\alpha_0)) \end{aligned}$$

For the SSE believers, the ALM matches their PLM, and therefore no projection is needed. One derives the projected ALM as

$$T\begin{pmatrix}\alpha_{11}\\ \alpha_{12}\\ \alpha_0\end{pmatrix}=\begin{pmatrix}\gamma_1 H(\pi_{11}G(\alpha_{11})+\pi_{12}G(\alpha_{12}))+\gamma_0 F(\alpha_0)\\ \gamma_1 H(\pi_{21}G(\alpha_{11})+\pi_{22}G(\alpha_{12}))+\gamma_0 F(\alpha_0)\\ \gamma_1(\bar{p}_1 H(\pi_{11}G(\alpha_{11})+\pi_{12}G(\alpha_{12}))+\bar{p}_2 H(\pi_{21}G(\alpha_{11})+\pi_{22}G(\alpha_{12}))+\gamma_0 F(\alpha_0)\end{pmatrix}$$

I.4 SS v.s. SSE: Linearization of Differential Equation at the Steady State

Let $w_i=\alpha_i-\hat{y}$ and $\dot{w}_i=dw_i/d\tau$, and $\Psi_i(w_{11},w_{12},w_0)$ are the nonlinear part. Note $F'(\hat{y})=H'(G(\hat{y}))G'(\hat{y})$.

$$\begin{pmatrix}\dot{w}_{11}\\ \dot{w}_{12}\\ \dot{w}_0\end{pmatrix}=\begin{pmatrix}\gamma_1\pi_{11}F'(\hat{y})w_{11}+\gamma_1\pi_{12}F'(\hat{y})w_{12}+\gamma_0F'(\hat{y})w_0-w_{11}\\ \gamma_1\pi_{21}F'(\hat{y})w_{11}+\gamma_1\pi_{22}F'(\hat{y})w_{12}+\gamma_0F'(\hat{y})w_0-w_{12}\\ \gamma_1\bar{p}_1F'(\hat{y})w_{11}+\gamma_1\bar{p}_2F'(\hat{y})w_{12}+\gamma_0F'(\hat{y})w_0-w_0\end{pmatrix}+\begin{pmatrix}\Psi_{11}(w_{11},w_{12},w_0)\\ \Psi_{12}(w_{11},w_{12},w_0)\\ \Psi_2(w_{11},w_{12},w_0)\end{pmatrix}$$

The linear part can be written as

$$\begin{pmatrix}\gamma_1\pi_{11}F'(\hat{y})w_{11}+\gamma_1\pi_{12}F'(\hat{y})w_{12}+\gamma_0F'(\hat{y})w_0-w_{11}\\ \gamma_1\pi_{21}F'(\hat{y})w_{11}+\gamma_1\pi_{22}F'(\hat{y})w_{12}+\gamma_0F'(\hat{y})w_0-w_{12}\\ \gamma_1\bar{p}_1F'(\hat{y})w_{11}+\gamma_1\bar{p}_2F'(\hat{y})w_{12}+\gamma_0F'(\hat{y})w_0-w_0\end{pmatrix}=$$
$$\left(F'(\hat{y})\begin{pmatrix}\gamma_1\pi_{11}&\gamma_1\pi_{12}&\gamma_0\\ \gamma_1\pi_{21}&\gamma_1\pi_{22}&\gamma_0\\ \gamma_1\bar{p}_1&\gamma_1\bar{p}_2&\gamma_0\end{pmatrix}-I\right)\begin{pmatrix}w_{11}\\ w_{12}\\ w_0\end{pmatrix}$$

The linearized system at the steady state takes the following form

$$\begin{pmatrix} \dot{w}_{11} \\ \dot{w}_{12} \\ \dot{w}_0 \end{pmatrix} = (F'(\hat{y})\tilde{\Pi} - I) \begin{pmatrix} w_{11} \\ w_{12} \\ w_0 \end{pmatrix} + \begin{pmatrix} \Psi_{11}(w_{11}, w_{12}, w_0) \\ \Psi_{12}(w_{11}, w_{12}, w_0) \\ \Psi_2(w_{11}, w_{12}, w_0) \end{pmatrix}$$

where

$$\tilde{\Pi} = \begin{pmatrix} \gamma_1\pi_{11} & \gamma_1\pi_{12} & \gamma_0 \\ \gamma_1\pi_{21} & \gamma_1\pi_{22} & \gamma_0 \\ \gamma_1\bar{p}_1 & \gamma_1\bar{p}_2 & \gamma_0 \end{pmatrix}$$

I.5 Proof for SS v.s. SSE in Nonlinear Model

Define the following variables for bifurcation.

$$\begin{aligned} \bar{\pi}_{11} &= 1 + (\gamma_1\beta)^{-1} - \pi_{22} \\ v &= \pi_{11} - \bar{\pi}_{11} \end{aligned}$$

The dynamic system is now written as

$$\begin{aligned} \dot{w}_{11} &= \gamma_1(\bar{\pi}_{11} + v)F(\hat{y} + w_{11}) + \gamma_1(1 - \bar{\pi}_{11} - v)F(\hat{y} + w_{12}) + \gamma_0 F(\hat{y} + w_0) - w_{11} - \hat{y} \\ \dot{w}_{12} &= \gamma_1(1 - \pi_{22})F(\hat{y} + w_{11}) + \gamma_1\pi_{22}F(\hat{y} + w_{12}) + \gamma_0 F(\hat{y} + w_0) - w_{12} - \hat{y} \\ \dot{w}_0 &= \frac{\gamma_1(1 - \pi_{22})}{1 - (\gamma_1\beta)^{-1} + v}F(\hat{y} + w_{11}) + \frac{\gamma_1(1 - \bar{\pi}_{11} - v)}{1 - (\gamma_1\beta)^{-1} + v}F(\hat{y} + w_{12}) + \gamma_0 F(\hat{y} + w_0) - w_0 - \hat{y} \end{aligned}$$

At $w_{11} = w_{22} = w_0 = v = 0$, the coefficient matrix of the linear part is $A = \beta\tilde{\Pi} - I$, whose eigenvalues are -1, $F' - 1$ and 0 where $F' = \beta$. The diagonalization of A is given by

$$A = Q \begin{pmatrix} -1 & 0 & 0 \\ 0 & F' - 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q^{-1}, \text{ where } Q = \begin{pmatrix} b & 1 & a \\ b & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

with $a = \frac{1-\pi_{22}(\gamma_1 F')}{(1-\pi_{22})(\gamma_1 F')}$, and $b = 1 - \gamma_1^{-1}$. Let q^{ij} denote the elements of Q^{-1}. Define new variables x_i as follows

$$\begin{pmatrix} x_{11} \\ x_{12} \\ x_0 \end{pmatrix} = Q^{-1} \begin{pmatrix} w_{11} \\ w_{12} \\ w_0 \end{pmatrix}$$

This transformation implies $w_{11} = bx_{11} + x_{12} + ax_0$, $w_{12} = bx_{11} + x_{12} + x_0$, and $w_0 = x_{11} + x_{12}$. One obtains $\dot{x}_i = G_i(x_{11}, x_{12}, x_0,\ v)$ for $i \in \{11,\ 12,\ 2\}$ where

$$\begin{aligned}
&G_i(x_{11}, x_{12}, x_0, v) \\
=&q^{i1}\left[\gamma_1(\bar{\pi}_{11} + v)F(\hat{y} + w_{11}) + \gamma_1(1 - \bar{\pi}_{11} - v)F(\hat{y} + w_{12}) + \gamma_0 F(\hat{y} + w_0) - w_{11} - \hat{y}\right] \\
+&q^{i2}\left[\gamma_1(1 - \pi_{22})F(\hat{y} + w_{11}) + \gamma_1 \pi_{22} F(\hat{y} + w_{12}) + \gamma_0 F(\hat{y} + w_0) - w_{12} - \hat{y}\right] \\
+&q^{i3}\left[\frac{\gamma_1(1 - \pi_{22})}{1 - (\gamma_1\beta)^{-1} + v}F(\hat{y} + w_{11}) + \frac{\gamma_1(1 - \bar{\pi}_{11} - v)}{1 - (\gamma_1\beta)^{-1} + v}F(\hat{y} + w_{12}) + \gamma_0 F(\hat{y} + w_0) - w_0 - \hat{y}\right]
\end{aligned}$$

Apply the transformation for w_i, we have

$$\begin{aligned}
&G_i(x_{11}, x_{12}, x_0, v)\\
=&q^{i1}[\gamma_1(\bar{\pi}_{11}+v)F(\hat{y}+bx_{11}+x_{12}+ax_0)+\gamma_1(1-\bar{\pi}_{11}-v)F(\hat{y}+bx_{11}+x_{12}+x_0)+\\
&\quad\gamma_0 F(\hat{y}+x_{11}+x_{12})-(bx_{11}+x_{12}+ax_0)-\hat{y}]\\
+&q^{i2}[\gamma_1(1-\pi_{22})F(\hat{y}+bx_{11}+x_{12}+ax_0)+\gamma_1\pi_{22}F(\hat{y}+bx_{11}+x_{12}+x_0)+\\
&\quad\gamma_0 F(\hat{y}+x_{11}+x_{12})-(bx_{11}+x_{12}+x_0)-\hat{y}]\\
+&q^{i3}\left[\frac{\gamma_1(1-\pi_{22})}{1-(\gamma_1\beta)^{-1}+v}F(\hat{y}+bx_{11}+x_{12}+ax_0)+\frac{\gamma_1(1-\bar{\pi}_{11}-v)}{1-(\gamma_1\beta)^{-1}+v}F(\hat{y}+bx_{11}+x_{12}+x_0)+\right.\\
&\quad\gamma_0 F(\hat{y}+x_{11}+x_{12})-(x_{11}+x_{12})-\hat{y}]
\end{aligned}$$

Augmenting this system with $\dot{v}=0$ leads to a four-dimensional system for which the equations for $\dot{x}_0$ and $\dot{v}$ have zero linear parts and the equation for $\dot{x}_{11}$ and $\dot{x}_{12}$ have linear parts $-x_{11}$ and $(F'-1)x_{12}$ which are obviously stable. We now use the center manifold theory. In particular, the system has an invariant center manifold which can be represented by a three times continuously differentiable function $x_{11}=h_{11}(x_2,v)$ and $x_{12}=h_{12}(x_2,v)$ with $h_i(0,0)=0$ and $Dh_i(0,0)=0$ for $i\in\{11,12\}$. Local stability of the system is governed by local stability of the "projected system",

$$\begin{aligned}
\dot{x}_0 &= G_2(h_{11}(x_2,v), h_{12}(x_2,v), x_0, v)\\
\dot{v} &= 0
\end{aligned}$$

The second-order expansions are

$$
\begin{aligned}
F(\hat{y}+bx_{11}+x_{12}+ax_0) &\doteq F(\hat{y})+F'bx_{11}+F'x_{12}+F'ax_0+\\
&\frac{1}{2}F''(b^2x_{11}^2+x_{12}^2+a^2x_0^2+2bx_{11}x_{12}+2abx_{11}x_0+2ax_{12}x_0)\\
F(\hat{y}+bx_{11}+x_{12}+x_0) &\doteq F(\hat{y})+F'bx_{11}+F'x_{12}+F'x_0+\\
&\frac{1}{2}F''(b^2x_{11}^2+x_{12}^2+x_0^2+2bx_{11}x_{12}+2bx_{11}x_0+2x_{12}x_0)\\
F(\hat{y}+x_{11}+x_{12}) &\doteq F(\hat{y})+F'x_{11}+F'x_{12}+\frac{1}{2}F''(x_{11}^2+x_{12}^2+2x_{11}x_{12})\\
h_{11}(x_2,v) &= c_{11}x_2^2+d_{11}x_2v+f_{11}v^2+\mathcal{O}_{11}(||(x_0,v)||^3)\\
h_{12}(x_2,v) &= c_{12}x_2^2+d_{12}x_2v+f_{12}v^2+\mathcal{O}_{12}(||(x_0,v)||^3)
\end{aligned}
$$

where $F''=F''(\hat{y})$ and $\doteq$ denotes equality up to $\mathcal{O}(||(x_{11},x_{12},x_0)||^3)$. Also, note that $q^{31}=(a-1)^{-1}$, $q^{32}=(1-a)^{-1}$, and $q^{33}=0$. It follows that on the center manifold the differential equation for x_2 can be written as

$$
\dot{x}_2=\gamma_1F'vx_2+\frac{1}{2}\frac{F''}{F'}(1+a)x_2^2+\mathcal{O}(||(x_2,v)||^3) \tag{I.1}
$$

For the purpose of the theorem we are at liberty to choose π_{22} so that $a\neq-1$ which we now assume. The bifurcation occurs at $v=0$. It follows that E-stable SSEs exist for $v<0$. If $v>0$ the SSEs are not E-stable and learning instead converges to the fundamental solution. It is evident that the system exhibits a transcritical bifurcation at $v=0$. The 2SSEs are defined by the equations $\bar{x}_{11}=0$, $\bar{x}_{12}=0$, and $\bar{x}_0=-\frac{2\gamma_1(F')^2}{F''(1+a)}v$. In terms of the original variables, $\bar{w}_{11}=a\bar{x}_0$, $\bar{w}_{12}=\bar{x}_0$, and $\bar{w}_0=0$.

I.6 SSE1 v.s. SSE2: derivation of the T-map

SSE^1 believers and SSE^2 believers form expectations based on their PLMs as follows.

$$\hat{E}_t^j G(y_{t+1}) = \begin{cases} \pi_{11}^j G(\alpha_{j1}) + \pi_{12}^j G(\alpha_{j2}) \text{ if } s_t^j = 1 \\ \pi_{21}^j G(\alpha_{j1}) + \pi_{22}^j G(\alpha_{j2}) \text{ if } s_t^j = 2 \end{cases}$$

for $j \in \{1, 2\}$. Combining the expectations and Eq.(1), one obtains the ALM. Recall $F(\cdot) = H(G(\cdot))$

$$y_t = \begin{cases} \gamma_1 H(\pi_{11}^1 G(\alpha_{11}) + \pi_{12}^1 G(\alpha_{12})) + \gamma_2 H(\pi_{11}^2 G(\alpha_{21}) + \pi_{12}^2 G(\alpha_{22})) \text{ if } (s_t^1, s_t^2) = (1,1) \\ \gamma_1 H(\pi_{11}^1 G(\alpha_{11}) + \pi_{12}^1 G(\alpha_{12})) + \gamma_2 H(\pi_{21}^2 G(\alpha_{21}) + \pi_{22}^2 G(\alpha_{22})) \text{ if } (s_t^1, s_t^2) = (1,2) \\ \gamma_1 H(\pi_{21}^1 G(\alpha_{11}) + \pi_{22}^1 G(\alpha_{12})) + \gamma_2 H(\pi_{11}^2 G(\alpha_{21}) + \pi_{12}^2 G(\alpha_{22})) \text{ if } (s_t^1, s_t^2) = (2,1) \\ \gamma_1 H(\pi_{21}^1 G(\alpha_{11}) + \pi_{22}^1 G(\alpha_{12})) + \gamma_2 H(\pi_{21}^2 G(\alpha_{21}) + \pi_{22}^2 G(\alpha_{22})) \text{ if } (s_t^1, s_t^2) = (2,2) \end{cases}$$

SSE^1 believers and SSE^2 believers do not observe each other's sunspot process. The stationary distributions of sunspot processes Π^j are $(\bar{p}_1^j, \bar{p}_2^j)$ where $\bar{p}_1^j = \pi_{21}^j/(\pi_{21}^j + \pi_{12}^j)$ and $\bar{p}_2^j = \pi_{12}^j/(\pi_{21}^j + \pi_{12}^j)$. From the SSE^1 believers' perspective, they see y_t evolves according to s_t^1 as follows.

$$y_t = \begin{cases} \gamma_1 H(\pi_{11}^1 G(\alpha_{11}) + \pi_{12}^1 G(\alpha_{12})) + \bar{p}_1^2 \gamma_2 H(\pi_{11}^2 G(\alpha_{21}) + \pi_{12}^2 G(\alpha_{22})) + \bar{p}_2^2 \gamma_2 H(\pi_{21}^2 G(\alpha_{21}) + \pi_{22}^2 G(\alpha_{22})) \text{ if } s_t^1 = 1 \\ \gamma_1 H(\pi_{21}^1 G(\alpha_{11}) + \pi_{22}^1 G(\alpha_{12})) + \bar{p}_1^2 \gamma_2 H(\pi_{11}^2 G(\alpha_{21}) + \pi_{12}^2 G(\alpha_{22})) + \bar{p}_2^2 \gamma_2 H(\pi_{21}^2 G(\alpha_{21}) + \pi_{22}^2 G(\alpha_{22})) \text{ if } s_t^1 = 1 \end{cases}$$

From the SSE^2 believers' perspective, they see y_t evolves according to s_t^2 as follows.

$$y_t = \begin{cases} \bar{p}_1^1 \gamma_1 H(\pi_{11}^1 G(\alpha_{11}) + \pi_{12}^1 G(\alpha_{12})) + \bar{p}_2^1 \gamma_1 H(\pi_{21}^1 G(\alpha_{11}) + \pi_{22}^1 G(\alpha_{12})) + \gamma_2 H(\pi_{11}^2 G(\alpha_{21}) + \pi_{12}^2 G(\alpha_{22})) \text{ if } s_t^2 = 1 \\ \bar{p}_1^1 \gamma_1 H(\pi_{11}^1 G(\alpha_{11}) + \pi_{12}^1 G(\alpha_{12})) + \bar{p}_2^1 \gamma_1 H(\pi_{21}^1 G(\alpha_{11}) + \pi_{22}^1 G(\alpha_{12})) + \gamma_2 H(\pi_{21}^2 G(\alpha_{21}) + \pi_{22}^2 G(\alpha_{22})) \text{ if } s_t^2 = 2 \end{cases}$$

The T-map from the PLMs to the projected ALM is written as

$$
T \begin{pmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{21} \\ \alpha_{22} \end{pmatrix} =
$$

$$
\begin{pmatrix}
\gamma_1 H(\pi^1_{11} G(\alpha_{11}) + \pi^1_{12} G(\alpha_{12})) + \bar{p}^2_1 \gamma_2 H(\pi^2_{11} G(\alpha_{21}) + \pi^2_{12} G(\alpha_{22})) + \bar{p}^2_2 \gamma_2 H(\pi^2_{21} G(\alpha_{21}) + \pi^2_{22} G(\alpha_{22})) \\
\gamma_1 H(\pi^1_{21} G(\alpha_{11}) + \pi^1_{22} G(\alpha_{12})) + \bar{p}^2_1 \gamma_2 H(\pi^2_{11} G(\alpha_{21}) + \pi^2_{12} G(\alpha_{22})) + \bar{p}^2_2 \gamma_2 H(\pi^2_{21} G(\alpha_{21}) + \pi^2_{22} G(\alpha_{22})) \\
\bar{p}^1_1 \gamma_1 H(\pi^1_{11} G(\alpha_{11}) + \pi^1_{12} G(\alpha_{12})) + \bar{p}^1_2 \gamma_1 H(\pi^1_{21} G(\alpha_{11}) + \pi^1_{22} G(\alpha_{12})) + \gamma_2 H(\pi^2_{11} G(\alpha_{21}) + \pi^2_{12} G(\alpha_{22})) \\
\bar{p}^1_1 \gamma_1 H(\pi^1_{11} G(\alpha_{11}) + \pi^1_{12} G(\alpha_{12})) + \bar{p}^1_2 \gamma_1 H(\pi^1_{21} G(\alpha_{11}) + \pi^1_{22} G(\alpha_{12})) + \gamma_2 H(\pi^2_{21} G(\alpha_{21}) + \pi^2_{22} G(\alpha_{22}))
\end{pmatrix}
$$

I.7 SSE1 v.s. SSE2: linearization at the steady state

Let $w_i = \alpha_i - \hat{y}$ and $\dot{w}_i = dw_i/d\tau$, and $\Psi_i(w_{11}, w_{12}, w_{21}, w_{22})$ are the nonlinear part. Note $F'(\hat{y}) = H'(G(\hat{y}))G'(\hat{y})$.

$$
\begin{pmatrix} \dot{w}_{11} \\ \dot{w}_{12} \\ \dot{w}_{21} \\ \dot{w}_{22} \end{pmatrix} =
\begin{pmatrix}
\gamma_1 \pi^1_{11} F'(\hat{y}) w_{11} + \gamma_1 \pi^1_{12} F'(\hat{y}) w_{12} + \gamma_2 (\bar{p}^2_1 \pi^2_{11} + \bar{p}^2_2 \pi^2_{21}) F'(\hat{y}) w_{21} + \gamma_2 (\bar{p}^2_1 \pi^2_{12} + \bar{p}^2_2 \pi^2_{22}) F'(\hat{y}) w_{22} - w_{11} \\
\gamma_1 \pi^1_{21} F'(\hat{y}) w_{11} + \gamma_1 \pi^1_{22} F'(\hat{y}) w_{12} + \gamma_2 (\bar{p}^2_1 \pi^2_{11} + \bar{p}^2_2 \pi^2_{21}) F'(\hat{y}) w_{21} + \gamma_2 (\bar{p}^2_1 \pi^2_{12} + \bar{p}^2_2 \pi^2_{22}) F'(\hat{y}) w_{22} - w_{12} \\
\gamma_1 (\bar{p}^1_1 \pi^1_{11} + \bar{p}^1_2 \pi^1_{21}) F'(\hat{y}) w_{11} + \gamma_1 (\bar{p}^1_1 \pi^1_{12} + \bar{p}^1_2 \pi^1_{22}) F'(\hat{y}) w_{12} + \gamma_2 \pi^2_{11} F'(\hat{y}) w_{21} + \gamma_2 \pi^2_{12} F'(\hat{y}) w_{22} - w_{21} \\
\gamma_1 (\bar{p}^1_1 \pi^1_{11} + \bar{p}^1_2 \pi^1_{21}) F'(\hat{y}) w_{11} + \gamma_1 (\bar{p}^1_1 \pi^1_{12} + \bar{p}^1_2 \pi^1_{22}) F'(\hat{y}) w_{12} + \gamma_2 \pi^2_{21} F'(\hat{y}) w_{21} + \gamma_2 \pi^2_{22} F'(\hat{y}) w_{22} - w_{22}
\end{pmatrix} + \Psi
$$

where the nonlinear part is

$$
\Psi = \begin{pmatrix} \Psi_{11}(w_{11}, w_{12}, w_{21}, w_{22}) \\ \Psi_{12}(w_{11}, w_{12}, w_{21}, w_{22}) \\ \Psi_{21}(w_{11}, w_{12}, w_{21}, w_{22}) \\ \Psi_{22}(w_{11}, w_{12}, w_{21}, w_{22}) \end{pmatrix}
$$

and the linear part can be written as

$$\left(F'(\hat{y})\begin{pmatrix} \gamma_1\pi^1_{11} & \gamma_1\pi^1_{12} & \gamma_2\bar{p}^2_1 & \gamma_2\bar{p}^2_2 \\ \gamma_1\pi^1_{21} & \gamma_1\pi^1_{22} & \gamma_2\bar{p}^2_1 & \gamma_2\bar{p}^2_2 \\ \gamma_1\bar{p}^1_1 & \gamma_1\bar{p}^1_2 & \gamma_2\pi^2_{11} & \gamma_2\pi^2_{12} \\ \gamma_1\bar{p}^1_1 & \gamma_1\bar{p}^1_2 & \gamma_2\pi^2_{21} & \gamma_2\pi^2_{22} \end{pmatrix} - I\right)\begin{pmatrix} w_{11} \\ w_{12} \\ w_{21} \\ w_{22} \end{pmatrix}$$

The linearized system at the steady state takes the following form

$$\begin{pmatrix} \dot{w}_{11} \\ \dot{w}_{12} \\ \dot{w}_{21} \\ \dot{w}_{22} \end{pmatrix} = (F'(\hat{y})\tilde{\Pi} - I)\begin{pmatrix} w_{11} \\ w_{12} \\ w_{21} \\ w_{22} \end{pmatrix} + \Psi$$

where

$$\tilde{\Pi} = \begin{pmatrix} \gamma_1\pi^1_{11} & \gamma_1\pi^1_{12} & \gamma_2\bar{p}^2_1 & \gamma_2\bar{p}^2_2 \\ \gamma_1\pi^1_{21} & \gamma_1\pi^1_{22} & \gamma_2\bar{p}^2_1 & \gamma_2\bar{p}^2_2 \\ \gamma_1\bar{p}^1_1 & \gamma_1\bar{p}^1_2 & \gamma_2\pi^2_{11} & \gamma_2\pi^2_{12} \\ \gamma_1\bar{p}^1_1 & \gamma_1\bar{p}^1_2 & \gamma_2\pi^2_{21} & \gamma_2\pi^2_{22} \end{pmatrix}$$

I.8 Proof of Linear Model.

Throughout this proof, I assume $0 < \gamma_1 < \gamma_2 < 1$ without losing generality, and therefore $-\gamma_1^{-1} < -\gamma_2^{-1} < -1$. If $|\beta| > (\max(\gamma_1, \gamma_2))^{-1} = \gamma_2^{-1}$, then $(\gamma_2\beta)^{-1} \in (-1, 1)$. There exist (π^2_{ij}) such that $\pi^2_{11} + \pi^2_{22} = (\gamma_2\beta)^{-1} + 1 \in (0, 2)$ which implies eigenvalue $\lambda_4 = 0$. Note that A is not a full-rank matrix, and therefore the model has a SSE solution, and there exist a continuum of SSE solutions which are characterized by $A\bar{w} = 0$. If $\beta < -(\max(\gamma_1, \gamma_2))^{-1} = -\gamma_2^{-1}$ then both $\lambda_1 = 0$ and $\lambda_2 = \beta - 1$ are negative. Consider the following two cases

1. $-\gamma_1^{-1} < \beta < -\gamma_2^{-1}$. Note that $\lambda_3 = \gamma_1\beta(\pi_{11}^1 + \pi_{22}^1 - 1) - 1 < 0$. If the Markov sunspot process satisfies restriction $\pi_{11}^2 + \pi_{22}^2 - 1 = (\gamma_2\beta)^{-1}$, then $\lambda_4 = \gamma\beta(\pi_{11} + \pi_{22} - 1) - 1 = 0$. There exist a continuum of E-stable SSE solutions which are characterized by $A\bar{w} = 0$ which locate the continuum with restrictions $\bar{w}_{11} = \bar{w}_{12} = 0$ and $(1 - \pi_{22}^2)\bar{w}_{21} + (1 - \pi_{21}^2)\bar{w}_{12} = 0$.

2. $\beta < -\gamma_1^{-1} < -\gamma_2^{-1}$. If only one of the resonant frequency conditions $\pi_{11}^j + \pi_{22}^j - 1 = (\gamma_j\beta)^{-1}$ and the other sunspot process has serial correlation $\pi_{11}^{-j} + \pi_{22}^{-j} - 1 > (\gamma_{-j}\beta)^{-1}$, then $lambda_{j+2} = 0$ and $\lambda_{-j+2} < 0$. There exist a continuum of E-stable SSE solutions which are characterized by $A\bar{w} = 0$ which locate the continuum with restrictions $(1 - \pi_{22}^j)\bar{w}_{j1} + (1 - \pi_{j1}^j)\bar{w}_{j2} = 0$ and $\bar{w}_{-j1} = \bar{w}_{-j2} = 0$.If both resonant frequency conditions are satisfied then $\lambda_3 = \lambda_4 = 0$. There exist a continuum of E-stable SSE solutions which are characterized by $A\bar{w} = 0$ which locate the continuum with restrictions $(1 - \pi_{22}^1)\bar{w}_{11} + (1 - \pi_{21}^1)\bar{w}_{12} = 0$ and $(1 - \pi_{22}^2)\bar{w}_{21} + (1 - \pi_{21}^2)\bar{w}_{12} = 0$.

I.9 Proof of Nonlinear Model.

Throughout this proof, I assume $0 < \gamma_1 < \gamma_2 < 1$ without losing generality, and therefore $-\gamma_1^{-1} < -\gamma_2^{-1} < -1$. If $F' < -(\max(\gamma_1, \gamma_2))^{-1} = -\gamma_2^{-1}$ then both $\lambda_1 = 0$ and $\lambda_2 = F' - 1$ are negative. Let $F' = F'(\hat{y})$. First, consider the case where $-\gamma_1^{-1} < F' < -\gamma_2^{-1}$. Note that $\lambda_3 = \gamma_1 F'(\pi_{11}^1 + \pi_{22}^1 - 1) - 1 < 0$. Define the following variables for bifurcation.

$$\bar{\pi}_{11}^2 = 1 + (\gamma_2 F'(\hat{y}))^{-1} - \pi_{22}^2$$
$$v_2 = \pi_{11}^2 - \bar{\pi}_{11}^2$$

The following analysis is similar to the proof shown in Appendix **??**. Note that the differential equations for the transformed variables x_{11}, x_{12}, and x_{21} would have stable linear parts $-x_{11}$, $(F'-1)x_{12}$ and $(\gamma_1 F'(\pi_{11}^1+\bar{\pi}_{22}^1-1)-1)x_{21}$. The bifurcation occurs at $v_2 = 0$. It follows that E-stable SSEs exist for $v_2 < 0$. If $v_2 > 0$ the SSEs are not E-stable and learning instead converges to the fundamental solution. $v_2 < 0$ corresponds to $\pi_{11}^2 + \pi_{11}^2 - 1 < (\gamma_2 F'(\hat{y}))^{-1}$.
Now assume $F' < -\gamma_1^{-1} < -\gamma_2^{-1}$. If only one of the sunspot processes satisfies $\pi_{11}^j + \pi_{22}^j - 1 < (\gamma_j\beta)^{-1}$ and the other sunspot process has serial correlation $\pi_{11}^{-j} + \pi_{22}^{-j} - 1 > (\gamma_{-j}\beta)^{-1}$, one can define the following variables for bifurcation.

$$\begin{aligned}\bar{\pi}_{11}^j &= 1 + (\gamma_j F'(\hat{y}))^{-1} - \pi_{22}^j \\ v_j &= \pi_{11}^j - \bar{\pi}_{11}^j\end{aligned}$$

This case is also similar to the analysis shown in Appendix A.4. The bifurcation occurs at $v_j = 0$. It follows that E-stable SSEs exist for $v_j < 0$. If both sunspot processes satisfies $\pi_{11}^j + \pi_{22}^j - 1 < (\gamma_j\beta)^{-1}$, one can define the following variables for bifurcation.

$$\begin{aligned}\bar{\pi}_{11}^1 &= 1 + (\gamma_1 F'(\hat{y}))^{-1} - \pi_{22}^1 \quad \text{and} \quad v_1 = \pi_{11}^1 - \bar{\pi}_{11}^1 \\ \bar{\pi}_{11}^2 &= 1 + (\gamma_2 F'(\hat{y}))^{-1} - \pi_{22}^2 \quad \text{and} \quad v_2 = \pi_{11}^2 - \bar{\pi}_{11}^2\end{aligned}$$

The differential equation system is now written as

$$
\begin{aligned}
\dot{w}_{11} = {} & \gamma_1(\bar{\pi}^1_{11} + v_1)F(\hat{y} + w_{11}) + \gamma_1(1 - \bar{\pi}^1_{11} - v_1)F(\hat{y} + w_{12}) + \\
& \frac{\gamma_2(1 - \pi^2_{22})}{1 - (\gamma_2 F'(\hat{y}))^{-1} + v_2} F(\hat{y} + w_{21}) + \frac{\gamma_2(1 - \bar{\pi}^2_{11} - v_2)}{1 - (\gamma_2 F'(\hat{y}))^{-1} + v_2} F(\hat{y} + w_{22}) - w_{11} - \hat{y} \\
\dot{w}_{12} = {} & \gamma_1(1 - \pi^1_{22})F(\hat{y} + w_{11}) + \gamma_1 \pi^1_{22} F(\hat{y} + w_{12}) + \\
& \frac{\gamma_2(1 - \pi^2_{22})}{1 - (\gamma_2 F'(\hat{y}))^{-1} + v_2} F(\hat{y} + w_{21}) + \frac{\gamma_2(1 - \bar{\pi}^2_{11} - v_2)}{1 - (\gamma_2 F'(\hat{y}))^{-1} + v_2} F(\hat{y} + w_{22}) - w_{12} - \hat{y}
\end{aligned}
$$

$$
\begin{aligned}
\dot{w}_{21} = {} & \frac{\gamma_1(1 - \pi^1_{22})}{1 - (\gamma_1 F'(\hat{y}))^{-1} + v_1} F(\hat{y} + w_{11}) + \frac{\gamma_1(1 - \bar{\pi}^1_{11} - v_1)}{1 - (\gamma_1 F'(\hat{y}))^{-1} + v_1} F(\hat{y} + w_{12}) + \\
& \gamma_2(\bar{\pi}^2_{11} + v_2)F(\hat{y} + w_{21}) + \gamma_2(1 - \bar{\pi}^2_{11} - v_2)F(\hat{y} + w_{22}) - w_{22} - \hat{y} \\
\dot{w}_{22} = {} & \frac{\gamma_1(1 - \pi^1_{22})}{1 - (\gamma_1 F'(\hat{y}))^{-1} + v_1} F(\hat{y} + w_{11}) + \frac{\gamma_1(1 - \bar{\pi}^1_{11} - v_1)}{1 - (\gamma_1 F'(\hat{y}))^{-1} + v_1} F(\hat{y} + w_{12}) + \\
& \gamma_2(1 - \pi^2_{22})F(\hat{y} + w_{21}) + \gamma_2 \pi^2_{22} F(\hat{y} + w_{22}) - w_{22} - \hat{y}
\end{aligned}
$$

At $w_{11} = w_{12} = w_{21} = w_{22} = v_1 = v_2 = 0$, the coefficient matrix of the linear part is $A = F'(\hat{y})\tilde{\Pi} - I$, whose eigenvalues are -1, $F' - 1$, 0, and 0 where $F' = F'(\hat{y})$. The diagonalization of A is given by

$$
A = Q \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & F' - 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} Q^{-1}, \text{ where } Q = \begin{pmatrix} b & 1 & a_1 & 0 \\ b & 1 & 1 & 0 \\ 1 & 1 & 0 & a_2 \\ 1 & 1 & 0 & 1 \end{pmatrix}
$$

with $a_j = \frac{1 - \pi^j_{22}(\gamma_j F')}{(1 - \pi^j_{22})(\gamma_j F')}$ and $b = -\gamma_2/\gamma_1$. Let q^{ij} denote the elements of Q^{-1}. Define new variables x_i as $(x_{11}, x_{12}, x_{21}, x_{22})' = Q^{-1}(w_{11}, w_{12}, w_{21}, w_{22})'$. This transformation implies

$$
\begin{aligned}
w_{11} &= bx_{11} + x_{12} + a_1 x_{21} \\
w_{12} &= bx_{11} + x_{12} + x_{21} \\
w_{21} &= x_{11} + x_{12} + a_2 x_{22} \\
w_{22} &= x_{11} + x_{12} + x_{22}
\end{aligned}
$$

One obtains $\dot{x}_i = G_i(x_{11}, x_{12}, x_{21}, x_{22},\ v_1,\ v_2)$ as follows for $i \in \{11,\ 12,\ 21,\ 22\}$

$$G_i(x_{11}, x_{12}, x_{21}, x_{22}, v_1, v_2) = q^{i1}\dot{w}_{11} + q^{i2}\dot{w}_{12} + q^{i3}\dot{w}_{21} + q^{i4}\dot{w}_{22}$$

Apply the transformation for w_i, we have

$$
\begin{aligned}
&G_i(x_{11}, x_{12}, x_{21}, x_{22}, v_1, v_2) = \\
&q^{i1}[\gamma_1(\bar{\pi}^1_{11} + v_1)F(\hat{y} + bx_{11} + x_{12} + a_1x_{21}) + \gamma_1(1 - \bar{\pi}_{11} - v_1)F(\hat{y} + bx_{11} + x_{12} + x_{21}) + \\
&\qquad \frac{\gamma_2(1 - \pi^2_{22})}{1 - (\gamma_2 F'(\hat{y}))^{-1} + v_2} F(\hat{y} + x_{11} + x_{12} + a_2x_{22}) + \\
&\quad \frac{\gamma_2(1 - \bar{\pi}^2_{11} - v_2)}{1 - (\gamma_2 F'(\hat{y}))^{-1} + v_2} F(\hat{y} + x_{11} + x_{12} + a_2x_{22}) \\
&\qquad - (bx_{11} + x_{12} + a_1x_{21}) - \hat{y}] \\
&+q^{i2}[\gamma_1(1 - \pi^1_{22})F(\hat{y} + bx_{11} + x_{12} + a_1x_{21}) + \gamma_1\pi^1_{22}F(\hat{y} + bx_{11} + x_{12} + x_{21}) + \\
&\qquad \frac{\gamma_2(1 - \pi^2_{22})}{1 - (\gamma_2 F'(\hat{y}))^{-1} + v_2} F(\hat{y} + x_{11} + x_{12} + a_2x_{22}) + \\
&\quad \frac{\gamma_2(1 - \bar{\pi}^2_{11} - v_2)}{1 - (\gamma_2 F'(\hat{y}))^{-1} + v_2} F(\hat{y} + x_{11} + x_{12} + a_2x_{22}) \\
&\qquad - (bx_{11} + x_{12} + x_{21}) - \hat{y}] \\
&+q^{i3}[\frac{\gamma_1(1 - \pi^1_{22})}{1 - (\gamma_1 F'(\hat{y}))^{-1} + v_1} F(\hat{y} + bx_{11} + x_{12} + a_1x_{21}) + \\
&\quad \frac{\gamma_1(1 - \bar{\pi}^1_{11} - v_1)}{1 - (\gamma_1 F'(\hat{y}))^{-1} + v_1} F(\hat{y} + bx_{11} + x_{12} + x_{21}) + \\
&\qquad \gamma_2(\bar{\pi}^2_{11} + v_2)F(\hat{y} + x_{11} + x_{12} + a_2x_{22}) + \gamma_2(1 - \bar{\pi}^2_{11} - v_2)F(\hat{y} + x_{11} + x_{12} + x_{22}) \\
&\qquad - (x_{11} + x_{12} + a_2x_{22}) - \hat{y}] \\
&+q^{i4}[\frac{\gamma_1(1 - \pi^1_{22})}{1 - (\gamma_1 F'(\hat{y}))^{-1} + v_1} F(\hat{y} + bx_{11} + x_{12} + a_1x_{21}) + \\
&\quad \frac{\gamma_1(1 - \bar{\pi}^1_{11} - v_1)}{1 - (\gamma_1 F'(\hat{y}))^{-1} + v_1} F(\hat{y} + bx_{11} + x_{12} + x_{21}) + \\
&\qquad \gamma_2(1 - \pi^2_{22})F(\hat{y} + x_{11} + x_{12} + a_2x_{22}) + \gamma_2\pi^2_{22}F(\hat{y} + x_{11} + x_{12} + x_{22}) \\
&\qquad - (x_{11} + x_{12} + x_{22}) - \hat{y}]
\end{aligned}
$$

Augmenting this system with $\dot{v}_1 = 0$ and $\dot{v}_2 = 0$ leads to a six-dimensional system for which the equations for $\dot{x}_{21}$, $\dot{x}_{22}$, $\dot{v}_1$, $\dot{v}_2$ have zero linear parts and the equation for $\dot{x}_{11}$ and $\dot{x}_{12}$ have linear parts $-x_{11}$ and $(F' - 1)x_{12}$ which are obviously stable. We now use the center manifold theory. In particular, the system has an invariant center manifold which can be represented by a three times continuously differentiable function $x_{11} = h_{11}(x_{21}, x_{22}, v_1, v_2)$ and $x_{12} = h_{12}(x_{21}, x_{22}, v_1, v_2)$ with $h_i(0,0,0,0) = 0$ and $Dh_i(0,0,0,0) = 0$ for $i \in \{11, 12\}$. Local stability of the system is governed by local stability of the "projected system",

$$\begin{aligned}
\dot{x}_{21} &= G_{21}(h_{11}(x_{21}, x_{22}, v_1, v_2), h_{12}(x_{21}, x_{22}, v_1, v_2), x_{21}, x_{22}, v) \\
\dot{x}_{22} &= G_{22}(h_{11}(x_{21}, x_{22}, v_1, v_2), h_{12}(x_{21}, x_{22}, v_1, v_2), x_{21}, x_{22}, v) \\
\dot{v}_1 &= 0 \text{ and } \dot{v}_2 = 0
\end{aligned}$$

The second-order expansions are

$$\begin{aligned}
F(\hat{y} + bx_{11} + x_{12} + a_1 x_{21}) &\doteq F(\hat{y}) + F' b x_{11} + F' x_{12} + F' a_1 x_{21} + \\
&\quad \frac{1}{2} F''(b^2 x_{11}^2 + x_{12}^2 + a_1^2 x_{21}^2 + 2b x_{11} x_{12} + 2a_1 b x_{11} x_{21} + 2a_1 x_{12} x_{21}) \\
F(\hat{y} + bx_{11} + x_{12} + x_{21}) &\doteq F(\hat{y}) + F' b x_{11} + F' x_{12} + F' x_{21} + \\
&\quad \frac{1}{2} F''(b^2 x_{11}^2 + x_{12}^2 + x_{21}^2 + 2b x_{11} x_{12} + 2b x_{11} x_{21} + 2x_{12} x_{21}) \\
F(\hat{y} + x_{11} + x_{12} + a_2 x_{22}) &\doteq F(\hat{y}) + F' x_{11} + F' x_{12} + F' a_2 x_{22} + \\
&\quad \frac{1}{2} F''(x_{11}^2 + x_{12}^2 + a_2^2 x_{22}^2 + 2x_{11} x_{12} + 2a_2 b x_{11} x_{22} + 2a_2 x_{12} x_{22}) \\
F(\hat{y} + x_{11} + x_{12} + x_{22}) &\doteq F(\hat{y}) + F' x_{11} + F' x_{12} + F' x_{22} + \\
&\quad \frac{1}{2} F''(x_{11}^2 + x_{12}^2 + x_{22}^2 + 2x_{11} x_{12} + 2x_{11} x_{22} + 2x_{12} x_{22}) \\
h_{11}(x_{21}, x_{22}, v_1, v_2) &= c_{11} x_{21}^2 + d_{11} x_{22}^2 + f_{11}^1 v_1^2 + f_{11}^2 v_2^2 + \\
&\quad g_{11}^1 x_{21} v_1 + m_{11}^1 x_{22} v_1 + g_{11}^2 x_{21} v_2 + m_{11}^2 x_{22} v_2 + r_{11} x_{21} x_{22} + s_{11} v_1 v_2 + \\
&\quad \mathcal{O}_{11}(||(x_{21}, x_{22}, v_1, v_2)||^3) \\
h_{12}(x_{21}, x_{22}, v_1, v_2) &= c_{12} x_{21}^2 + d_{12} x_{22}^2 + f_{12}^1 v_1^2 + f_{12}^2 v_2^2 + \\
&\quad g_{12}^1 x_{21} v_1 + m_{12}^1 x_{22} v_1 + g_{12}^2 x_{21} v_2 + m_{12}^2 x_{22} v_2 + r_{12} x_{21} x_{22} + s_{12} v_1 v_2 + \\
&\quad \mathcal{O}_{12}(||(x_{21}, x_{22}, v_1, v_2)||^3)
\end{aligned}$$

where $F'' = F''(\hat{y})$ and $\doteq$ denotes equality up to $\mathcal{O}(||(x_{11}, x_{12}, x_{21}, x_{22})||^3)$. Also, note that $q^{31} = (a_1 - 1)^{-1}$, $q^{32} = (1 - a_1)^{-1}$, $q^{33} = 0$, $q^{34} = 0$, and $q^{41} = 0$, $q^{42} = 0$,

$q^{43} = (a_2-1)^{-1}$, $q^{44} = (1-a_2)^{-1}$. It follows that on the center manifold the differential equation for x_{21} and x_{22} are as follows

$$\dot{x}_{21} = \gamma_1 F' v_1 x_{21} + \frac{1}{2}\frac{F''}{F'}(1+a_1)x_{21}^2 + \mathcal{O}(||(x_{21}, x_{22}, v_1, v_2)||^3)$$
$$\dot{x}_{22} = \gamma_2 F' v_2 x_{22} + \frac{1}{2}\frac{F''}{F'}(1+a_2)x_{22}^2 + \mathcal{O}(||(x_{21}, x_{22}, v_1, v_2)||^3)$$

For the purpose of the theorem we are at liberty to choose π_{22}^1 and π_{22}^2 so that $a_1 \neq -1$ and $a_2 \neq -1$ which we now assume. Notice that the differential equations for $\dot{x}_{21}$ and $\dot{x}_{22}$ are in fact decoupled in the linear and quadratic part and only higher-order parts are coupled. It is evident that the two systems exhibit a transcritical bifurcation at $v_1 = 0$ and $v_2 = 0$ respectively. It follows that E-stable SSEs exist for $v_1 < 0$ and $v_2 < 0$. The 2SSEs are defined by the equations $\bar{x}_{11} = 0$, $\bar{x}_{12} = 0$, $\bar{x}_{21} = -\frac{2\gamma_1 (F')^2}{F''(1+a_1)} v_1$, and $\bar{x}_{22} = -\frac{2\gamma_2 (F')^2}{F''(1+a_2)} v_2$. In terms of the original variables we have $\bar{w}_{11} = a_1 \bar{x}_{21}$, $\bar{w}_{12} = \bar{x}_{21}$, $\bar{w}_{21} = a_2 \bar{x}_{22}$, and $\bar{w}_{22} = \bar{x}_{22}$.

I.10 Derivation of the Temporary Equilibrium

The temporary equilibrium is pinned down by the following three equations

$$(n_t(\omega))^{-\sigma} p_t^{1-\sigma} E_t^{\omega}(p_{t+1}^{\sigma-1}) = (n_t(\omega))^{\varepsilon} \tag{I.2}$$
$$p_t \int_{\omega} n_t(\omega) d\omega = 1 \tag{I.3}$$

where Eq.(I.2) is the first order condition for the agent of index ω, and Eq.(I.3) is the money market clearing condition. Solving Eq.(I.2) for $n_t(\omega)$, one obtains

$$n_t(\omega) = p_t^{\frac{1-\sigma}{\varepsilon+\sigma}} (E_t^{\omega}(p_{t+1}^{\sigma-1}))^{\frac{1}{\varepsilon+\sigma}} \tag{I.4}$$

Substitute Eq.(I.4) into Eq.(I.3), one obtains

$$(p_t)^{-\frac{\varepsilon+1}{\varepsilon+\sigma}} = \int_\omega (E_{\omega,t}(p_{t+1}^{\sigma-1}))^{\frac{1}{\varepsilon+\sigma}} d\omega \tag{I.5}$$

Define a new variable $y_t = p_t^{-\frac{\varepsilon+1}{\varepsilon+\sigma}}$. Eq.(I.5) can be written in terms of y_t as follows

$$\begin{aligned} y_t &= \int_\omega H(E_t^\omega G(y_{t+1}))d\omega \\ n_t(\omega) &= y_t^{\frac{\sigma-1}{\varepsilon+1}} H(E_t^\omega G(y_{t+1})) \end{aligned}$$

where $H(y) = y^{\frac{1}{\varepsilon+\sigma}}$ and $G(y) = y^{\frac{(\varepsilon+\sigma)(1-\sigma)}{\varepsilon+1}}$.

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