THE COMBINATORICS OF LOG-COULOMB GASES IN P-FIELDS

by

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DISSERTATION ABSTRACT

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This thesis is based on the article [16], which studies the integral

$$\int_{K^N} \rho(x_1, \ldots, x_N) \big(\max_{i<j} |x_i - x_j| \big)^a \big(\min_{i<j} |x_i - x_j| \big)^b \prod_{i<j} |x_i - x_j|^{s_{ij}} \, dx_1 \ldots dx_N$$

where K is an arbitrary p-field, ρ is a well-behaved function that depends only on the norm of $(x_1, \ldots, x_N) \in K^N$, and a, b, s_{ij} are certain complex numbers. A mixture of analysis and combinatorics is used to find two explicit formulas for the integral (one for $b \neq 0$ and one for $b = 0$) and an explicit description of all $s_{ij} \in \mathbb{C}$ for which it converges absolutely (for fixed ρ, a, and b). The integral's role as the canonical partition function for a log-Coulomb gas (in K) is highlighted throughout, leading to a p-field analogue of Mehta's Integral Formula and formulas for the joint moments of the gas' diameter and minimum particle spacing. The notion of log-Coulomb gas in $\mathbb{P}^1(K)$ is also addressed and related to that in K in a concrete way: The grand canonical partition function for a log-Coulomb gas in $\mathbb{P}^1(K)$ is the $(q+1)$th power of the grand canonical partition function for a log-Coulomb gas in the open unit ball of K.

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CHAPTER I

INTRODUCTION, BACKGROUND, AND RESULTS

1.1. Introduction

This thesis is the result of an investigation of the following problem: *Find an explicit formula for the (Haar) integral*

$$Z_N(\beta) = \int_{\mathbb{Z}_p^N} \prod_{i<j} |x_i - x_j|_p^{\beta} \, dx_1 \dots dx_N$$

where p is a prime number, N is a positive integer, and β is a complex number. This integral is relevant in two seemingly disjoint areas of mathematics. On one hand, it is the direct p-adic analogue of the classical *Mehta Integral*, which plays an important role in random matrix theory and defines the canonical partition function for the statistical mechanical model known as *log-Coulomb gas*. On the other hand, Z_N is the local zeta function attached to the Vandermonde polynomial $V(x_1, \dots, x_N) = \prod_{i<j}(x_i - x_j)$ and it encodes the sequence $(N_m(V))_{m=0}^{\infty}$ defined by

$$N_m(V) := \#\{(x_1, \dots, x_N) \in (\mathbb{Z}/p^m\mathbb{Z})^N : V(x_1, \dots, x_N) \equiv 0 \mod p^m\}.$$

More precisely, the generating function $P_V(t) = \sum_{m=0}^{\infty}(N_m(V)/p^{mN})t^m$ is analytic for $|t| < 1$ and satisfies $P_V(p^{-\beta}) = \frac{1-p^{-\beta}Z_N(\beta)}{1-p^{-\beta}}$ whenever $\mathrm{Re}(\beta) > 0$ [3], so finding a formula for $Z_N(\beta)$ is essentially equivalent to finding one for all $N_m(V)$.

Despite their apparent differences, the statistical-physical and arithmetical interpretations of Z_N share a common combinatorial theme. This stems from the fact that all p-fields (such as $\mathbb{Q}_p$) have finite residue fields and canonical absolute values (such as $|\cdot|_p$) with countable images. In fact, these two properties are responsible for the common idea behind almost all of the results in this thesis:

Main Idea: *The integration domain $\mathbb{Z}_p^N$ can be broken into finitely many subsets that are indexed and explicitly described by chains in the partition lattice for the set $[N] = \{1, \ldots, N\}$. The integral over each of these subsets can be computed explicitly using a mixture of counting and geometric series summation.*

By making this idea precise, we will find explicit formulas for $Z_N(\beta)$, its projective analogue, and more general p-field integrals of the form

$$\int_{K^N} \rho(x_1, \ldots, x_N) \big(\max_{i<j} |x_i - x_j| \big)^a \big(\min_{i<j} |x_i - x_j| \big)^b \prod_{i<j} |x_i - x_j|^{s_{ij}} \, dx_1 \ldots dx_N.$$

All of them turn out to be finite sums over the same set of chains of partitions, and they turn out to be valid for all p-fields K simultaneously. We will spend the next few sections of this chapter developing relevant background on log-Coulomb gases, local fields, local zeta functions, and the metric and measure structures on the projective lines of p-fields. In the last section of the chapter we will define the chains of partitions mentioned above and conclude with precise statements and some consequences of our main results.

1.2. log-Coulomb gases and canonical partition functions

Let X be a topological space with a metric d and a finite positive Borel measure λ such that $\lambda^N(\{(x_1, \ldots, x_N) \in X^N : x_i = x_j \text{ for some } i \neq j\}) = 0$ for every $N \geq 1$. A *log-Coulomb gas* with N particles in X is a statistical model described as follows: Consider N particles with fixed charge values $\mathfrak{q}_1, \ldots, \mathfrak{q}_N \in \mathbb{R}$ and corresponding variable locations $x_1, \ldots, x_N \in X$. Whether or not the charge values are distinct, we assume the particles are distinguished by the labels $1, \ldots, N$, so that unique configurations of the system correspond to unique tuples $(x_1, \ldots, x_N) \in X^N$. We call each tuple a *microstate* of the system, and each microstate has an *energy* defined by

$$E(x_1, \ldots, x_N) := \begin{cases} -\sum_{1 \le i < j \le N} \mathfrak{q}_i \mathfrak{q}_j \log d(x_i, x_j) & \text{if } x_i \neq x_j \text{ for all } i < j, \\ \infty & \text{otherwise.} \end{cases} \tag{1.2.1}$$

Note that $E^{-1}(\infty)$ has measure zero in X^N by our choice of λ, and that E is identically zero if $N = 1$. We assume the system is in thermal equilibrium with a heat reservoir at *inverse temperature* $\beta > 0$, so that the microstates are distributed according to the density $e^{-\beta E(x_1, \ldots, x_N)} = \prod_{i<j} d(x_i, x_j)^{\mathfrak{q}_i \mathfrak{q}_j \beta}$. The *canonical partition function* $\beta \mapsto Z_N(X, \beta)$ is defined as the total mass of this density, namely

$$Z_N(X, \beta) := \int_{X^N} \prod_{1 \le i < j \le N} d(x_i, x_j)^{\mathfrak{q}_i \mathfrak{q}_j \beta} \, d\lambda(x_1) \ldots d\lambda(x_N), \tag{1.2.2}$$

and it readily describes explicit relationships between the system's temperature and observable parameters. For instance, the system's dimensionless free energy, mean energy, and energy fluctuation (variance) are respectively given by $-\log Z_N(X, \beta)$, $-\partial/\partial\beta \log Z_N(X, \beta)$, and $\partial^2/\partial\beta^2 \log Z_N(X, \beta)$, all of which are functions of β (and hence of temperature). In general, a closed formula for the integral $Z_N(X, \beta)$ and a description of its "analytic domain" (i.e., the largest open set of complex β for which the integral converges absolutely) are useful for precisely understanding the system's macroscopic behavior as a function of its temperature. We will now discuss three examples in which the desired closed formulas and explicit analytic domains for $Z_N(X, \beta)$ can be found.

Example 1.2.1. Let $X = \mathbb{R}$ with the standard metric d, the standard Gaussian measure λ (i.e., $d\lambda(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$), and $N \geq 2$ charges with $\mathfrak{q}_1 = \cdots = \mathfrak{q}_N = 1$. In this case $Z_N(\mathbb{R}, \beta)$ is known as *Mehta's integral* [7]. It converges absolutely if and only if $\mathrm{Re}(\beta) > -2/N$, and in this case it converges to

$$\prod_{j=1}^{N} \frac{\Gamma(1 + j\beta/2)}{\Gamma(1 + \beta/2)}. \tag{1.2.3}$$

Before moving to the next examples, it is important to note that Mehta's integral $Z_N(\mathbb{R}, \beta)$ is traditionally written in the form

$$\frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{1}{2}(x_1^2 + \cdots + x_N^2)} \prod_{i<j} |x_i - x_j|^{\beta}\, dx_1 \ldots dx_N.$$

We will prefer to treat the Gaussian factor $e^{-\frac{1}{2}(x_1^2+\cdots+x_N^2)}$ as part of the measure in this thesis, though it is traditionally incorporated into (1.2.2) by adding a harmonic potential term $\frac{1}{2\beta}(x_1^2 + \cdots + x_N^2)$ to the microstate energy $E(x_1, \ldots, x_N)$ in (1.2.1) and using the scaled Lebesgue measure $d\lambda(x) = \frac{1}{\sqrt{2\pi}}\, dx$ instead of the Gaussian measure. In the early 1960's, Mehta and Dyson showed that the integrand describes the distribution of eigenvalues $x_1, \ldots, x_N$ (with multiplicity) for the $N \times N$ Gaussian orthogonal, unitary, and symplectic random matrix ensembles at the respective special values $\beta = 1$, $\beta = 2$, and $\beta = 4$. Bombieri extended the formula (1.2.3) to $\mathrm{Re}(\beta) > -2/N$ roughly a decade later [7].

As far as Fourier analysis is concerned, the $\mathbb{Q}_p^N$-analogue of the normalized Gaussian $\frac{1}{(\sqrt{2\pi})^N} e^{-\frac{1}{2}(x_1^2+\cdots+x_N^2)}$ is the indicator function $\mathbf{1}_{\mathbb{Z}_p^N}(x_1, \ldots, x_N)$. More precisely, both functions are centrally symmetric probability densities and are equal to their respective real and p-adic Fourier transforms [12]. If we now let dx stand for the standard Haar measure on $\mathbb{Q}_p$ (i.e., the one that gives $\mathbb{Z}_p$ measure 1), then we may understand the measure λ satisfying $d\lambda = \mathbf{1}_{\mathbb{Z}_p}(x)\,dx$ as the p-adic analogue of the standard Gaussian measure on $\mathbb{R}$, and henceforth the integral $Z_N(\beta)$ from the beginning of this chapter can be understood as the *p-adic Mehta integral*:

$$Z_N(\mathbb{Q}_p, \beta) = \int_{\mathbb{Q}_p^N} \mathbf{1}_{\mathbb{Z}_p^N}(x_1, \ldots, x_N) \prod_{i<j} |x_i - x_j|_p^\beta \, dx_1 \ldots dx_N.$$

That is, $Z_N(\beta) = Z_N(X, \beta)$ is the canonical partition function for a log-Coulomb gas in $X = \mathbb{Q}_p$ with the standard p-adic metric d, the p-adic "Gaussian" measure λ, and the charge values $\mathfrak{q}_1 = \cdots = \mathfrak{q}_N = 1$. Like the harmonic potential in the classical setting, the indicator $\mathbf{1}_{\mathbb{Z}_p^N}$ can be attributed to "adding an ∞ term" to the microstate energy $E(x_1, \ldots, x_N)$ whenever $(x_1, \ldots, x_N) \in \mathbb{Q}_p^N \setminus \mathbb{Z}_p^N$, and this amounts to saying that the gas is confined to an infinite potential well in $\mathbb{Z}_p$. We will see later that the indicator function may be replaced by more general functions $\rho : \mathbb{Q}_p^N \to \mathbb{R}$ which correspond to other kinds of potentials. Unlike $Z_N(\mathbb{R}, \beta)$, we will find that the formula for $Z_N(\mathbb{Q}_p, \beta)$ generalizes easily to gases with multiple components (meaning $\mathfrak{q}_1, \ldots, \mathfrak{q}_N$ may be distinct). However, even in the one component case $\mathfrak{q}_1 = \cdots = \mathfrak{q}_N = 1$, the explicit formulas for $Z_N(\mathbb{Q}_p, \beta)$ and its relatives become complicated very rapidly as N increases. Thus we will only consider $N = 3$ in the next two examples:

Example 1.2.2. Let $X = \mathbb{Z}_p$ with the standard p-adic metric, standard Haar measure, and $N = 3$ charges with values $\mathfrak{q}_1 = 1$, $\mathfrak{q}_2 = 2$, and $\mathfrak{q}_3 = 3$. Then one of our main results implies that $Z_3(\mathbb{Z}_p, \beta)$ converges absolutely if and only if $\mathrm{Re}(\beta) > -1/6$, and in this case it converges to

$$\frac{(p-1)(p-2)p^{11\beta}}{p^{2+11\beta} - 1} + \frac{(p-1)^2 p^{11\beta}}{p^{2+11\beta} - 1} \left[\frac{1}{p^{1+2\beta} - 1} + \frac{1}{p^{1+3\beta} - 1} + \frac{1}{p^{1+6\beta} - 1} \right].$$

In fact, we will show that multi-component canonical partition functions can be defined and computed in the p-adic projective setting as well:

Example 1.2.3. Let $X = \mathbb{P}^1(\mathbb{Q}_p)$ with the spherical metric δ, the unique $PGL_2(\mathbb{Z}_p)$-invariant Borel probability measure λ, and $N = 3$ charges with values $\mathfrak{q}_1 = 1$, $\mathfrak{q}_2 = 2$, and $\mathfrak{q}_3 = 3$ as above. Another of our main results implies that $Z_3(\mathbb{P}(\mathbb{Q}_p), \beta)$ also converges absolutely if and only if $\mathrm{Re}(\beta) > -1/6$, and in this case it converges to

$$\frac{(p-1)(p^{3+11\beta}-2)}{(p+1)^2(p^{2+11\beta}-1)} + \frac{(p-1)(p^{3+11\beta}-1)}{(p+1)^2(p^{2+11\beta}-1)}\left[\frac{1}{p^{1+2\beta}-1} + \frac{1}{p^{1+3\beta}-1} + \frac{1}{p^{1+6\beta}-1}\right].$$

The evident similarities between Examples 1.2.2 and 1.2.3 hint at an interesting relationship between log-Coulomb gases in $\mathbb{Q}_p$ and those in the projective line $\mathbb{P}^1(\mathbb{Q}_p)$. This relationship will be made explicit at the end of this chapter and proved in Chapter 5. Moreover, the large expression in Example 1.2.3 is invariant under the involution $p \mapsto p^{-1}$, and the same becomes true for the large expression in Example 1.2.2 after it is scaled by $p^{-\frac{11}{2}\beta}$. This type of symmetry is a familiar—though not yet fully understood—phenomenon in the theory of local zeta functions [5], and it will make another brief appearance in the Appendix. We will now review the necessary background on local fields, projective lines, and local zeta functions (which include $Z_N(\mathbb{R}, \beta)$ and $Z_N(\mathbb{Q}_p, \beta)$), then conclude this chapter with the statement of our main theorem before returning to the subject of log-Coulomb gases in Chapter 2.

1.3. Local fields and their projective lines

A topological field K is called a *local field* if it is Hausdorff, non-discrete, and locally compact. Among the best known examples are $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Q}_p$, but we allow K to remain arbitrary for the moment, and recall that an isomorphism of local fields $K \cong K'$ is both an algebraic isomorphism and a homeomorphism. Following [17], recall that every local field K admits an additive Haar measure μ which is unique up to normalization. Given a measurable set $M \subset K$ with $0 < \mu(M) < \infty$, it can be shown that the function $|\cdot| : K \to \mathbb{R}_{\geq 0}$ defined by

$$|x| := \begin{cases} \sqrt{\mu(xM)/\mu(M)} & \text{if } K \cong \mathbb{C}, \\ \mu(xM)/\mu(M) & \text{otherwise,} \end{cases}$$

satisfies the axioms of an absolute value on K. In fact, $|\cdot|$ is independent of M and the normalization of μ, the metric topology generated by $|\cdot|$ coincides with the intrinsic topology on K, and K is complete with respect to $|\cdot|$. Thus, $|\cdot|$ is aptly called the *canonical absolute value* on K, and we will fix a normalization of μ once and for all by specifying the measure of the closed unit ball:

$$\mu(\{x \in K : |x| \leq 1\}) := \begin{cases} \pi & \text{if } K \cong \mathbb{C}, \\ 2 & \text{if } K \cong \mathbb{R}, \\ 1 & \text{otherwise.} \end{cases}$$

At first glance, the condition "K is a local field not isomorphic to $\mathbb{R}$ or $\mathbb{C}$" seems rather vague, but the following summary shows that it is quite specific.

Theorem 1.3.1 (The main dichotomy and properties of local fields [17]). *Suppose K is a local field with canonical absolute value $|\cdot|$. There are two main possibilities:*

1. *K is archimedean, meaning the image of the canonical ring homomorphism $\mathbb{Z} \to K$ is unbounded with respect to $|\cdot|$. In this case $K \cong \mathbb{R}$ or $K \cong \mathbb{C}$, $|\cdot|$ is respectively identified with the usual absolute value on $\mathbb{R}$ or $\mathbb{C}$, and μ is respectively identified with the standard Lebesgue measure on $\mathbb{R}$ or $\mathbb{C}$.*

2. *K is nonarchimedean, meaning the image of $\mathbb{Z} \to K$ is bounded. In this case $|\cdot|$ satisfies the strong triangle inequality: $|x+y| \leq \max\{|x|,|y|\}$ for all $x, y \in K$. Consequently, K is totally disconnected and the unit balls*

$$R := \{x \in K : |x| \leq 1\} \qquad \textit{and} \qquad P := \{x \in K : |x| < 1\}$$

are both open, compact, and closed under addition and multiplication. In fact, R is a local PID, its unique maximal ideal is P, its group of units is

$$R^\times = R \setminus P = \{x \in K : |x| = 1\},$$

and its residue field *R/P is isomorphic to $\mathbb{F}_q$ for some prime power q. There is a canonical isomorphism of $(R/P)^\times$ onto the group of $(q-1)$th roots of unity $U_{q-1} \subset K^\times$. It extends to a bijection $R/P \to \{0\} \sqcup U_{q-1}$ with inverse $x \mapsto x + P$, so $\{0\} \sqcup U_{q-1}$ is a canonical set of representatives for the cosets of $P \subset R$. On the other hand, there is a* canonical valuation *$v : K \to \mathbb{Z} \cup \{\infty\}$ given by*

$$v(x) := \begin{cases} -\log_q |x| & \textit{if } x \neq 0, \\ \infty & \textit{if } x = 0, \end{cases}$$

which satisfies $v(x+y) \geq \min\{v(x), v(y)\}$ *for all* $x, y \in K$ *and restricts to a surjective homomorphism* $K^\times \to \mathbb{Z}$*. Then* $|x| = q^{-v(x)}$ *is an integer power of* q *for each* $x \in K^\times$*, and the surjectivity of* v *implies that* K *has* uniformizers*, i.e., elements* $\pi \in K$ *satisfying* $v(\pi) = 1$*. Fixing a uniformizer* $\pi \in K$ *provides a concrete approach to the elements, balls, and Haar measure* μ*:*

(a) *If* $x \in K^\times$ *and* $m = v(x)$*, there is a unique* $u \in R^\times$ *such that* $x = u\pi^m$ *and a unique sequence* $(d(n))_{n=m}^{\infty}$ *such that* $d(m) \neq 0$ *and*

$$x = \sum_{n=m}^{\infty} d(n)\pi^n.$$

(b) *The open balls in* K *are precisely the sets of the form* $y + \pi^m R$ *with* $y \in K$ *and* $m \in \mathbb{Z}$*, and every such ball is compact with measure equal to its radius, i.e.,* $\mu(y + \pi^m R) = |\pi^m| = q^{-m}$*. In particular, if* $m_1, m_2 \in \mathbb{Z}$ *and* $m_1 \leq m_2$*, then* $\pi^{m_2}R$ *is a subgroup of* $\pi^{m_1}R$ *with index* $q^{m_2 - m_1}$*.*

Indeed, a local field K that is not isomorphic to $\mathbb{R}$ or $\mathbb{C}$ has a surprisingly rich structure. The properties above are actually strong enough to classify all such K:

Corollary 1.3.2 (The classification of nonarchimedean local fields [17])**.** *Suppose* K *is a nonarchimedean local field with canonical absolute value* $|\cdot|$ *and* R*,* P*, and* q *as above. Since* q *is a prime power, there is a unique prime* p *such that* $q = p^f$ *for some integer* $f \geq 1$*, and there are only two possibilities:*

1. $\underline{\mathrm{char}(K) = 0}$*, in which case* K *is isomorphic to a finite extension of* $\mathbb{Q}_p$ *and hence called a* p-adic field*. In particular, if* $K \cong \mathbb{Q}_p$*, then* $|\cdot|$ *is identified with* $|\cdot|_p$*,* $R \cong \mathbb{Z}_p$*,* $P \cong p\mathbb{Z}_p$*,* $R/P \cong \mathbb{F}_p$*, and hence* $q = p$*.*

2. <u>$\mathrm{char}(K) = p$</u>, *in which case* $K \cong \mathbb{F}_q((t))$, $R \cong \mathbb{F}_q[[t]]$, $P \cong t\mathbb{F}_q[[t]]$, *and* K *is called a* function field.

In light of the role played by p in the classification, we will follow Weil and use the term "*p-field*" as a shorthand for "nonarchimedean local field" from now on. It should also be emphasized that for any p-field K, the absolute value $|\cdot|$, the valuation v, the subsets R, P and $R^\times$, the group isomorphism $R/P \cong U_{q-1}$, and hence the prime power q are all canonical (i.e., "built in") features of K. There is no canonical uniformizer in K and and no canonical choice of Haar measure on K, but the *set* of uniformizers and the *set* of Haar measures are both canonical: If $\pi \in K$ is a fixed uniformizer and μ is the aforementioned Haar measure (the unique one satisfying $\mu(R) = 1$), then the uniformizers in K are precisely the elements of the form $u\pi$ with $u \in R^\times$ and the additive Haar measures on K are precisely the measures of the form $c\mu$ with $c \in \mathbb{R}_{>0}$. With these facts in mind, it will be convenient to settle on notation that will be used frequently from here on out:

Notation 1.3.3. Whenever K is declared to be a p-field, the symbols $|\cdot|$, v, R, P, q, and U_{q-1} will be understood to be the items defined above, and we will assume π stands for a fixed uniformizer of K (in particular, $\pi = t$ and $v = \mathrm{ord}_t$ if $K = \mathbb{F}_p((t))$, or $\pi = p$ and $v = \mathrm{ord}_p$ if $K = \mathbb{Q}_p$). For any local field K, we reserve the symbol dx for integration against the Haar measure μ, and for each positive integer N we define the standard norm $\|\cdot\|$ on the N-fold product K^N via

$$\|(x_1, \ldots, x_N)\| := \begin{cases} \sqrt{\sum_{i=1}^N |x_i|^2} & \text{if } K \text{ is archimedean,} \\ \max\limits_{1 \le i \le N} |x_i| & \text{if } K \text{ is nonarchimedean.} \end{cases}$$

Note that $\|\cdot\|$ has the same image as $|\cdot|$ in either case.

The following lemma is a straightforward consequence of the definition of $\|\cdot\|$ and the strong triangle inequality for p-fields:

Lemma 1.3.4. *If K is a p-field and N is any positive integer, then the inequality*

$$\|(x_1,\ldots,x_N)+(y_1,\ldots,y_N)\| \leq \max\{\|(x_1,\ldots,x_N)\|,\|(y_1,\ldots,y_N)\|\}$$

holds for all $(x_1,\ldots,x_N),(y_1,\ldots,y_N) \in K^N$, and it becomes equality whenever $\|(x_1,\ldots,x_N)\| \neq \|(y_1,\ldots,y_N)\|$. Moreover, $\|\cdot\|$ decomposes $K^N \setminus \{(0,\ldots,0)\}$ into countably many fibers of the form

$$\{(x_1,\ldots,x_N) \in K^N : \|(x_1,\ldots,x_N)\| = q^{-m}\} = \pi^m R^N \setminus \pi^{m+1} R^N = \pi^m (R^N \setminus \pi R^N).$$

We now recall some useful facts about projective lines from [6] in our present notation. If K is a local field, recall that its *projective line* is the quotient space $\mathbb{P}^1(K) := (K^2 \setminus \{(0,0)\})/\sim$, where $(x_0,x_1) \sim (y_0,y_1)$ if and only if $y_0 = \lambda x_0$ and $y_1 = \lambda x_1$ for some $\lambda \in K^\times$. Thus we regard $\mathbb{P}^1(K)$ concretely as the set of symbols $[x_0 : x_1]$ such that $(x_0,x_1) \in K^2 \setminus \{(0,0)\}$, subject to $[\lambda x_0 : \lambda x_1] = [x_0 : x_1]$ for all $\lambda \in K^\times$ and endowed with the topology induced by the quotient map $(x_0,x_1) \mapsto [x_0 : x_1]$. Note that $[x_0 : x_1] \neq [1 : 0]$ if and only if $x_1 \neq 0$, so $x = x_0/x_1$ is the unique element of K satisfying $[x : 1] = [x_0 : x_1]$, and the rule $\iota(x) := [x : 1]$ defines a homeomorphism $\iota : K \to \mathbb{P}^1(K) \setminus \{[1 : 0]\}$. The projective line is compact and metrizable by the *spherical metric* $\delta : \mathbb{P}^1(K) \times \mathbb{P}^1(K) \to [0,1]$, which is defined via

$$\delta([x_0 : x_1],[y_0 : y_1]) := \frac{|x_0 y_1 - x_1 y_0|}{\|(x_0,x_1)\| \cdot \|(y_0,y_1)\|}. \tag{1.3.4}$$

Suppose K is a p-field and leave it fixed for the rest of this section. The image of δ is clearly $\{0\} \cup \{q^{-m} : m \in \mathbb{Z}_{\geq 0}\}$, every open set in $\mathbb{P}^1(K)$ is a union of balls of the form

$$B_m[x_0 : x_1] := \{[y_0 : y_1] \in \mathbb{P}^1(K) : \delta([x_0 : x_1], [y_0 : y_1]) \leq q^{-m}\} \tag{1.3.5}$$

with $[x_0 : x_1] \in \mathbb{P}^1(K)$ and $m \in \mathbb{Z}_{\geq 0}$, and every such ball is open and compact. The homeomorphism $\iota : K \to \mathbb{P}^1(K) \setminus \{[1 : 0]\}$ also relates the metric structures of K and $\mathbb{P}^1(K)$ in an explicit and convenient way: For any $x, y \in K$, (1.3.4) implies

$$\delta(\iota(x), \iota(y)) = \begin{cases} |x - y| & \text{if } x, y \in R, \\ 1 & \text{if } x \in R \text{ and } y \notin R, \\ |x^{-1} - y^{-1}| & \text{if } x, y \notin R, \end{cases} \tag{1.3.6}$$

and $\delta(\iota(x), [1 : 0]) = (\max\{1, |x|\})^{-1}$ for all $x \in K$. By the definition in (1.3.5), the rule (1.3.6), and the strong triangle "equality" for $|\cdot|$ (i.e., Lemma 1.3.4 for $N = 1$), one easily verifies that

$$\iota(y + \pi^m R) = \begin{cases} B_v(\iota(y)) & \text{if } y \in R, \\ B_{m-2v(y)}(\iota(y)) & \text{if } y \notin R, \end{cases} \tag{1.3.7}$$

whenever $y \in K$ and $m \in \mathbb{Z}_{>0}$. That is, ι sends the open ball of radius $r \in (0, 1)$ centered at $y \in K$ onto the open ball of radius $r/\max\{1, |y|^2\}$ centered at $\iota(y) \in \mathbb{P}^1(K) \setminus \{[1 : 0]\}$, so $\iota : K \to \mathbb{P}^1(K) \setminus \{[1 : 0]\}$ restricts to an isometry on R and a contraction on $K \setminus R$.

Though $\mathbb{P}^1(K)$ lacks a natural Haar measure (since it is not a group), it is a homogeneous space for the *projective linear group* $PGL_2(R)$, which is defined to be the quotient of $GL_2(R) = \{A \in M_2(R) : \det(A) \in R^\times\}$ by its center $Z = \{\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right) : \lambda \in R^\times\} \cong R^\times$. Indeed, it is straightforward to check that the rule $\phi[x_0 : x_1] := [ax_0 + bx_1 : cx_0 + dx_1]$, where $\phi \in PGL_2(R)$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(R)$ is any representative of ϕ, gives a well-defined transitive action of $PGL_2(R)$ on $\mathbb{P}^1(K)$. This action is compatible with δ and endows $\mathbb{P}^1(K)$ with a nice measure:

Lemma 1.3.5 ($PGL_2(R)$-invariance [6])**.** *The spherical metric satisfies*

$$\delta(\phi[x_0 : x_1], \phi[y_0 : y_1]) = \delta([x_0 : x_1], [y_0 : y_1])$$

for all $\phi \in PGL_2(R)$ and all $[x_0 : x_1], [y_0 : y_1] \in \mathbb{P}^1(K)$. There is also a unique Borel probability measure ν on $\mathbb{P}^1(K)$ satisfying $\nu(\phi(M)) = \nu(M)$ for all $\phi \in PGL_2(R)$ and all Borel subsets $M \subset \mathbb{P}^1(K)$. In particular, for each $m \in \mathbb{Z}_{\geq 0}$ the relation $\phi(B_m[x_0 : x_1]) = B_m(\phi[x_0 : x_1])$ defines a transitive $PGL_2(R)$ action on the set of balls of radius q^{-m}, and thus the measure of a ball $B_m[x_0 : x_1] \subset \mathbb{P}^1(K)$ depends only on m.

The map ι also relates the measures on K and $\mathbb{P}^1(K)$ in a simple way: Given $m > 0$ and a complete set of representatives $y_1, \ldots, y_{q^m} \in R$ for the cosets of $\pi^m R \subset R$, applying (1.3.7) to the partition $R = (y_1 + \pi^m R) \sqcup \cdots \sqcup (y_{q^m} + \pi^m R)$ yields

$$\iota(R) = B_m[y_1 : 1] \sqcup \cdots \sqcup B_m[y_{q^m} : 1].$$

Therefore $PGL_2(R)$-invariance of ν implies that the measure of $\iota(R)$ is q^m times the measure of $B_m[0:1] = \iota(\pi^m R)$. On the other hand,

$$\iota(K \setminus R) = \iota(\{x : |x| \geq q\}) = \{\iota(x) : \delta(\iota(x), [1:0]) \leq q^{-1}\} = B_1[1:0] \setminus \{[1:0]\}$$

implies $\mathbb{P}^1(K) = \iota(R) \sqcup \iota(K \setminus R) \sqcup \{[1:0]\} = \iota(R) \sqcup B_1[1:0]$, which has measure 1. But $\iota(R)$ has q times the measure of $B_1[1:0]$, so the measure of $\iota(R)$ must be $q/(q+1)$ and hence every ball $B_v[x_0 : x_1] \subset \mathbb{P}^1(K)$ with $m > 0$ has measure $q^{-m} \cdot q/(q+1)$. Combining this with (1.3.7), we conclude that the measure ν on $\mathbb{P}^1(K) \setminus \{[1:0]\}$ pulls back along ι to an explicit measure $\nu \circ \iota$ on K, i.e.,

$$\nu(\iota(M)) = \frac{q}{q+1} \int_M \left(\max\{1, |x|^2\}\right)^{-1} dx \tag{1.3.8}$$

for all Borel subsets $M \subset K$.

Remark 1.3.6. Recall that $\{0\} \sqcup U_{q-1}$ is a full set of representatives for the cosets of $P \subset R$. Thus if we fix a primitive root $\xi \in U_{q-1}$, we may write $\{0\} \cup U_{q-1} = \{0, 1, \xi, \ldots, \xi^{q-2}\}$ and get an explicit partition of R into q cosets of P:

$$R = P \sqcup \underbrace{(1+P) \sqcup (\xi + P) \sqcup \cdots \sqcup (\xi^{q-2} + P)}_{R^\times}. \tag{1.3.9}$$

Note that two elements $x, y \in R$ satisfy $|x - y| = 1$ if and only if x and y belong to different cosets, and each coset is a ball with measure and radius q^{-1}. Applying ι to (1.3.9) and using the rule (1.3.7) allows $\mathbb{P}^1(K) = \iota(R) \sqcup B_1[1:0]$ to be refined into an analogous partition with $q+1$ parts:

$$\mathbb{P}^1(K) = B_1[0:1] \sqcup \underbrace{B_1[1:1] \sqcup B_1[\xi:1] \sqcup \cdots \sqcup B_1[\xi^{q-2}:1]}_{\iota(R^\times)} \sqcup B_1[1:0]. \qquad (1.3.10)$$

Indeed, two elements $[x_0 : x_1], [y_0 : y_1] \in \mathbb{P}^1(K)$ satisfy $\delta([y_0 : y_1], [y_0 : y_1]) = 1$ if and only if $[x_0 : x_1]$ and $[y_0 : y_1]$ belong to different parts, and each part is a ball with measure $1/(q+1)$ and radius q^{-1}. Moreover, ι sends $R^\times$ onto the "equator" $\iota(R^\times)$, i.e., the set of points in $\mathbb{P}^1(K)$ with δ-distance 1 from both the "south pole" $[0 : 1]$ and the "north pole" $[1 : 0]$. The "reflection" $\phi \in PGL_2(R)$ represented by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in GL_2(R)$ fixes $B_1[1 : 1]$, interchanges $B_1[0 : 1]$ and $B_1[1 : 0]$, and interchanges $B_1[\xi^k : 1]$ and $B_1[\xi^{q-1-k} : 1]$ for $0 < k < q-1$.

1.4. Local zeta functions

We have already seen three examples of local zeta functions: The classical Mehta Integral $Z_N(\mathbb{R}, \beta)$ in Example 1.2.1, its p-adic analogue $Z_N(\mathbb{Q}_p, \beta)$, and its multi-component variant in Example 1.2.2. In particular, we saw that the last of these converges absolutely to a rational expression of p and $p^{-\beta}$ whenever β has sufficiently large real part. The celebrated *Igusa's Theorem* shows that this notion of rationality holds for a very wide class of local zeta functions, and it will be apparent in our main results. Though our methods will be independent of Igusa's Theorem, it is worthwhile to recall what local zeta functions are, what the theorem states and implies, and how it generalizes to a "multivariate" version.

Definition 1.4.1. Suppose K is a local field and N is a positive integer.

(a) If $K \cong \mathbb{R}$ or $K \cong \mathbb{C}$, one defines smooth functions Φ on K^N as usual. For such K, a *Schwartz-Bruhat* function is a smooth function $\Phi : K^N \to \mathbb{C}$ that satisfies a "rapid decay" condition, namely

$$\sup_{(x_1,\ldots,x_N)\in K^N} |(\boldsymbol{\partial}\Phi)(x_1,\ldots,x_N)| < \infty$$

for all operators $\boldsymbol{\partial} = x_1^{m_1}(\partial/\partial x_1)^{n_1}\cdots x_N^{m_N}(\partial/\partial x_N)^{n_N}$ with $m_i, n_i \in \mathbb{Z}_{\geq 0}$.

(b) If K is a p-field, a function $\Phi : K^N \to \mathbb{C}$ is *Schwartz-Bruhat* if it is locally constant (an analogue of "smooth") and with compact support (an analogue of "rapid decay").

The $\mathbb{C}$-vector space of Schwartz-Bruhat functions plays a fundamental role in Fourier analysis and the theory of distributions on K^N (for any local field K). Basic examples of Schwartz-Bruhat functions include the familiar Gaussian on $\mathbb{R}^N$ and the indicator $\mathbf{1}_{\mathbb{Z}_p^N}$ on $\mathbb{Q}_p^N$, and both will continue to play a role in this section.

Definition 1.4.2. Fix a local field K and finitely many polynomials $f_1,\ldots,f_k \in K[x_1,\ldots,x_N]$ in $N \geq 1$ variables, write $\boldsymbol{s} = (s_1,\ldots,s_k)$ for a generic element of $\mathbb{C}^k$, and suppose $\Phi : K^N \to \mathbb{C}$ is a Schwartz-Bruhat function. The associated *multivariate local zeta function* is the holomorphic function defined on the open region $\mathcal{H}^k = \{\boldsymbol{s} \in \mathbb{C}^k : \mathrm{Re}(s_j) > 0 \text{ for all } j\}$ by

$$Z_\Phi(\boldsymbol{s},\boldsymbol{f}) := \int_{K^N} \Phi(x_1,\ldots,x_N)\prod_{j=1}^{k}|f_j(x_1,\ldots,x_N)|^{s_j}\,dx_1\ldots dx_N.$$

It is easy to verify that $Z_\Phi(\cdot, \boldsymbol{f})$ is holomorphic at every $\boldsymbol{s} \in \mathcal{H}^k$, though it is generally difficult to find its closed form and describe its meromorphic continuation. *Igusa's Theorem* partially solves this problem in the univariate case (i.e., $k = 1$) when K is a p-adic field (a p-field with $\mathrm{char}(K) = 0$):

Proposition 1.4.3 ([10]). *Let K be a p-adic field, suppose $\Phi : K^N \to \mathbb{C}$ is a Schwartz-Bruhat function, and suppose $f \in K[x_1, \ldots, x_N]$. Then there is a rational function $r \in \mathbb{C}(t)$ such that the local zeta function defined by*

$$Z_\Phi(s, f) = \int_{K^N} \Phi(x_1, \ldots, x_N)|f(x_1, \ldots, x_N)|^s \, dx_1 \ldots dx_N$$

satisfies $Z_\Phi(s, f) = r(q^{-s})$ for $\mathrm{Re}(s) > 0$. In particular, the meromorphic continuation of $Z_\Phi(s, f)$ is given by $r(q^{-s})$.

The general theorem is established in [9] and [10] and gives a similar result when $|\cdot|^s$ is replaced by any continuous homomorphism $K^\times \to \mathbb{C}^\times$ (we need not deal with these here), but Igusa's proof relies on the existence of a certain type of resolution of singularities for f. Existence of such a resolution is guaranteed by [8] if $\mathrm{char}(K) = 0$, but otherwise depends more subtly on K and f. Though Igusa's Theorem does not address the $\mathrm{char}(K) > 0$ case, it is quite powerful. For instance, the generating function $P_f(t) = \sum_{m=0}^{\infty} (N_m(f)/p^{mN})t^m$ for the sequence defined by

$$N_m(f) := \#\{(x_1, \ldots, x_N) \in (\mathbb{Z}/p^m\mathbb{Z})^N : f(x_1, \ldots, x_N) \equiv 0 \mod p^m\}$$

is analytic for $|t| < 1$, and if $\Phi = \mathbf{1}_{\mathbb{Z}_p^N}$ it satisfies $P_f(p^{-s}) = \frac{1-p^{-s}Z_\Phi(s,f)}{1-p^{-s}}$ whenever $\mathrm{Re}(s) > 0$ [3]. Now Proposition 1.4.3 implies $P_f(t)$ is rational in t, and hence $P_f(t)$ is the sum of a polynomial and finitely many geometric series in powers of t.

Using similar resolution techniques, Loeser generalized Igusa's Theorem to $k > 1$ in [11], which implies the following analogue of Proposition 1.4.3:

Proposition 1.4.4 ([11]). *Let K be a p-adic field. If $\Phi : K^N \to \mathbb{C}$ is a Schwartz-Bruhat function and $\boldsymbol{f} = (f_1, \ldots, f_k)$ with $f_j \in K[x_1, \ldots, x_N]$, then there is a rational function $r \in \mathbb{C}(t_1, \ldots, t_k)$ such that the local zeta function defined by*

$$Z_\Phi(\boldsymbol{s}, \boldsymbol{f}) := \int_{K^N} \Phi(x_1, \ldots, x_N) \prod_{j=1}^{k} |f_j(x_1, \ldots, x_N)|^{s_j} \, dx_1 \ldots dx_N$$

satisfies $Z_\Phi(\boldsymbol{s}, \boldsymbol{f}) = r(q^{-s_1}, \ldots, q^{-s_k})$ for all $\boldsymbol{s} \in \mathcal{H}^k$.

If $\operatorname{supp}(\Phi)$ is no longer assumed to be compact, then $Z_\Phi(\cdot, \boldsymbol{f})$ is no longer a proper local zeta function in the sense Definition 1.4.2, but it may still admit a meromorphic continuation of a similar rational form. Such an example was recently investigated in [1] with applications to p-adic string theory. Therein it is shown that for $N \geq 4$ the *p-adic open string N-point zeta function*, defined by

$$Z^{(N)}(\boldsymbol{s}) := \int_{\mathbb{Q}_p^{N-3}} \prod_{i=2}^{N-2} |x_i|^{s_{1i}} |1 - x_i|^{s_{i(N-1)}} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|^{s_{ij}} \, dx_1 \ldots dx_N,$$

coincides with a rational function in $p^{-s_{ij}}$ for all $1 \leq i < j \leq N-1$ on a nonempty open domain of tuples $\boldsymbol{s} = (s_{ij})_{1 \leq i < j \leq N-1} \in \mathbb{C}^{\binom{N-1}{2}}$, despite the unbounded support of the integrand. Unlike Igusa's original method, a formula for $Z^{(N)}(\boldsymbol{s})$ was found by decomposing $\mathbb{Q}_p^{N-3}$ into finitely many sets, integrating over each one, and summing the results. This method does not require $\operatorname{char}(K) = 0$ and generalizes to all p-fields, while also providing a description of the domain and poles of $Z^{(N)}$ in terms of the decomposition of $\mathbb{Q}_p^{N-3}$. Without placing any restrictions on $\operatorname{char}(K)$ or q, we will use a similar method to prove our main results.

1.5. Norm Densities and log-Coulomb gases in K and $\mathbb{P}^1(K)$

Recall that both of the integrals $Z_N(\mathbb{R},\beta)$ and $Z_N(\mathbb{Q}_p,\beta)$ from Section 1.2 are examples of the *generalized Mehta integral*

$$\int_{K^N} \rho(\|x\|) \prod_{i<j} |x_i - x_j|^\beta \, dx_1 \ldots dx_N, \tag{1.5.11}$$

where $\|x\|$ is shorthand for $\|(x_1,\ldots,x_N)\|$. Indeed, if we let $K = \mathbb{R}$ (with the appropriate dx, $|\cdot|$, and $\|\cdot\|$) and let $\rho(t) = \frac{1}{(2\pi)^{N/2}} e^{-t^2/2}$, then the integral becomes $Z_N(\mathbb{R},\beta)$. Similarly, if we let $K = \mathbb{Q}_p$ (with the appropriate dx, $|\cdot|$, and $\|\cdot\|$) and let $\rho = \mathbf{1}_{[0,1]}$, then the integral becomes $Z_N(\mathbb{Q}_p,\beta)$.

We will work exclusively with p-fields K from now on, but will further generalize (1.5.11) in several ways. To help our arguments and results work for all p-fields K, we note that the set

$$\mathcal{N} := \{0,1,2,3,\ldots\} \cup \{1/2, 1/3, 1/4, \ldots\}$$

always contains the image $\|K^N\|$ (no matter which p-field K is) and make the following definition:

Definition 1.5.1. A *norm-density* is a function $\rho : \mathcal{N} \to \mathbb{C}$ satisfying the mild growth conditions

$$\limsup_{n\to\infty} \frac{\log|\rho(\frac{1}{n})|_{\mathbb{C}}}{\log(n)} \leq 1 \qquad \text{and} \qquad \limsup_{n\to\infty} \frac{\log|\rho(n)|_{\mathbb{C}}}{\log(n)} = -\infty, \tag{1.5.12}$$

where $|\cdot|_{\mathbb{C}}$ denotes the canonical absolute value on $\mathbb{C}$ and $\log : [0,\infty] \to [-\infty,\infty]$ is the extended natural logarithm (i.e., $\log(0) := -\infty$ and $\log(\infty) := \infty$).

Note that the function $x \mapsto \rho(\|x\|)$ has modest growth as $\|x\| \to 0$ and fast decay as $\|x\| \to \infty$, regardless of our choice of K^N. Examples of norm-densities include $\rho(t) = e^{-t}$, $\rho(t) = e^{-t^2/2}$, $\rho(t) = -\log(t)\mathbf{1}_{[0,1]}(t)$, and $\rho(t) = \mathbf{1}_{[0,1]}(t)$.

Definition 1.5.2. Suppose K is a p-field, suppose ρ is a norm-density, let a and b be complex numbers, and let N be a positive integer. For all suitable $\boldsymbol{s} = (s_{ij})_{1 \leq i < j \leq N} \in \mathbb{C}^{\binom{N}{2}}$, define

$$Z_N^\rho(K, a, b, \boldsymbol{s}) := \int_{K^N} \rho(\|x\|) \big(\max_{i<j} |x_i - x_j| \big)^a \big(\min_{i<j} |x_i - x_j| \big)^b \prod_{i<j} |x_i - x_j|^{s_{ij}} \, dx_1 \dots dx_N.$$

Our first main theorem establishes an explicit formula for $Z_N^\rho(K, a, b, \boldsymbol{s})$ and an explicit description of its domain ($\boldsymbol{s}$ values for which it converges absolutely), both in terms of combinatorial objects that will be defined in the next section. It is not hard to show that the integral

$$Z_N^\rho(K, 0, 0, \boldsymbol{s}) = \int_{K^N} \rho(\|x\|) \prod_{i<j} |x_i - x_j|^{s_{ij}} \, dx_1 \dots dx_N$$

converges absolutely when all $s_{ij} = 0$, so if the norm-density ρ is positive and not identically zero, then $\rho(\|x\|)\, dx_1 \dots dx_N$ is a finite positive Borel measure on K^N. In this case $Z_N^\rho(K, 0, 0, \boldsymbol{s})$ becomes the canonical partition function for a log-Coulomb gas in K when it is evaluated at $s_{ij} = \mathfrak{q}_i \mathfrak{q}_j \beta$ for all $i < j$ (for some choice of charge values $\mathfrak{q}_1, \dots, \mathfrak{q}_N \in \mathbb{R}$). For suitable $\beta \in \mathbb{R}$, the expectation of the random variable $\big(\max_{i<j} |x_i - x_j|\big)^a \big(\min_{i<j} |x_i - x_j|\big)^b$ against the probability density $\frac{1}{Z_N^\rho(K,0,0,\boldsymbol{s})} \rho(\|x\|) \prod_{i<j} |x_i - x_j|^{\mathfrak{q}_i \mathfrak{q}_j \beta}$ is given by

$$\mathbb{E}\left[\big(\max_{i<j} |x_i - x_j| \big)^a \big(\min_{i<j} |x_i - x_j| \big)^b \right] = \frac{Z_N^\rho(K, a, b, \boldsymbol{s})}{Z_N^\rho(K, 0, 0, \boldsymbol{s})}. \tag{1.5.13}$$

In particular, taking $a, b \in \mathbb{Z}_{\geq 0}$ in (1.5.13) gives the joint moments of the gas' diameter $\max_{i<j} |x_i - x_j|$ and minimum particle spacing $\min_{i<j} |x_i - x_j|$. Though these canonical partition functions and joint moments have not been computed before, the study of Coulomb gases (which include log-Coulomb gases) in such p-fields K, or more generally K^d with $d \geq 1$, has become increasingly active in the last decade. Following the classical $\mathbb{R}^d$ analogue in [13], the article [18] gives Coulomb gases in K^d a natural motivation by realizing their potentials as fundamental solutions to the pseudodifferential analogue of Poisson's electrostatic equation. The same article also realizes the Haar measure μ^d (restricted to R^d) as the equilibrium measure for the confining potential $-\log \mathbf{1}_{[0,1]}(\|x\|)$ in the non-log-Coulomb case. In the log-Coulomb case, the more recent article [19] uses graph-theoretic machinery to establish and analyze formulas for $Z_N^{\rho}(K, 0, 0, \boldsymbol{s})$ with $\rho = \mathbf{1}_{[0,1]}$ and $s_{ij} = \mathfrak{q}_i \mathfrak{q}_j \beta$, with arbitrary $\mathfrak{q}_1, \ldots, \mathfrak{q}_N \in \mathbb{R}$. Our results are closely related to [19] but were discovered by "unwinding" a recurrence for $Z_N(\mathbb{Q}_p, \beta)$ that was established in [14]. A related recurrence can be found in the Appendix.

In Section 1.2 we recognized the p-adic Mehta Integral $Z_N(\mathbb{Q}_p, \beta)$ as an example of $Z_N(X, \beta)$ with $X = \mathbb{Q}_p$ and $d\lambda(x) = \mathbf{1}_{\mathbb{Z}_p^N}(x)\, dx$. However, we could as well have chosen $X = \mathbb{Z}_p$ and $d\lambda(x) = dx$ and imagined the gas to be inherently in $\mathbb{Z}_p$ (rather than probabilistically confined to $\mathbb{Z}_p$ by an infinite potential well), which is a reasonable argument for writing the Mehta integral as $Z_N(\mathbb{Z}_p, \beta)$ instead. This interpretation will be useful for highlighting relationships between the canonical partition functions for log-Coulomb gases in R, P, and $\mathbb{P}^1(K)$, which motivates the following definition:

Definition 1.5.3. If $N \geq 0$, $m \in \mathbb{Z}$, and $\boldsymbol{s} = (s_{ij})_{1 \leq i < j \leq N} \in \mathbb{C}^{\binom{N}{2}}$ (the empty tuple if $N = 0$ or $N = 1$), define $\mathcal{Z}_0(\pi^m R, \boldsymbol{s}) = \mathcal{Z}_0(\mathbb{P}^1(K), \boldsymbol{s}) = 1$,

$$\mathcal{Z}_N(\pi^m R, \boldsymbol{s}) = \int_{\pi^m R^N} \prod_{i<j} |x_i - x_j|^\beta \, dx_1 \dots dx_N, \quad \text{and}$$
$$\mathcal{Z}_N(\mathbb{P}^1(K), \boldsymbol{s}) = \int_{(\mathbb{P}^1(K))^N} \prod_{i<j} \delta([x_{i,0} : x_{i,1}], [x_{j,0} : x_{j,1}])^{s_{ij}} \, d[x_{1,0} : x_{1,1}] \dots d[x_{N,0} : x_{N,1}]$$

for $N \geq 1$, where $d[x_0 : x_1]$ stands for the unique $PGL_2(R)$-invariant Borel probability measure ν on $\mathbb{P}^1(K)$. Note that the first integral is equal to $\mathcal{Z}_N(R, \boldsymbol{s})$ if $m = 0$ and equal to $\mathcal{Z}_N(P, \boldsymbol{s})$ if $m = 1$. We will reserve the notation $Z_N(R, \beta)$, $Z_N(P, \beta)$ and $Z_N(\mathbb{P}^1(K), \beta)$ for the one-component canonical partition functions (where $\mathfrak{q}_1 = \cdots = \mathfrak{q}_N = 1$) obtained by respectively evaluating $\mathcal{Z}_N(R, \boldsymbol{s})$, $\mathcal{Z}_N(P, \boldsymbol{s})$ and $\mathcal{Z}_N(\mathbb{P}^1(K), \boldsymbol{s})$ at $s_{ij} = \beta$ for all $i < j$.

1.6. Splitting chains and the Main Theorem

There are two main factors comprising $Z_N^\rho(K, a, b, \boldsymbol{s})$, and they can be defined in their own right. Thus, until the statement of the main theorem, we will allow N and q to be arbitrary integers satisfying $N \geq 2$ and $q \geq 2$. The first of the two main factors is the *root function*, defined on a convex domain called the *root polytope* as follows:

Definition 1.6.1. For $N \geq 2$ and $a, b \in \mathbb{C}$ we define the *root polytope* $\mathcal{RP}_N(a, b)$ by

$$\mathcal{RP}_N(a, b) := \left\{ \boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}} : \operatorname{Re}\left(N - 1 + a + b + \sum_{i<j} s_{ij} \right) > 0 \right\}.$$

For such N, a, b, an integer $q \geq 2$, and a norm-density ρ, the associated *root function* $\mathcal{RP}_N(a,b) \to \mathbb{C}$ is defined by

$$\boldsymbol{s} \mapsto H_q^\rho\left(N + a + b + \sum_{i<j} s_{ij}\right) \qquad \text{where} \qquad H_q^\rho(z) := \frac{1 - q^{-z}}{1 - q^{-(z-1)}} \cdot \sum_{m \in \mathbb{Z}} \rho(q^m) q^{mz}.$$

The second factor is more complicated and requires some combinatorial language. Recall that a *partition* of the set $[N] := \{1, 2, \dots, N\}$ is a set $\pitchfork$ of nonempty pairwise disjoint subsets $\lambda \subset [N]$ satisfying $\bigcup_{\lambda \in \pitchfork} \lambda = [N]$. If $\pitchfork_1$ and $\pitchfork_2$ are partitions of $[N]$, we write $\pitchfork_2 \leq \pitchfork_1$ and call $\pitchfork_2$ a *refinement* of $\pitchfork_1$ if each part $\lambda_2 \in \pitchfork_2$ is contained in some part $\lambda_1 \in \pitchfork_1$. We write $\pitchfork_2 < \pitchfork_1$ and call $\pitchfork_2$ a *proper refinement* of $\pitchfork_1$ if both $\pitchfork_2 \leq \pitchfork_1$ and $\pitchfork_2 \neq \pitchfork_1$. The relation $\leq$ makes the collection of all partitions of $[N]$ into a partially ordered lattice with height N, unique maximal element $\overline{\pitchfork} := \{[N]\}$, and unique minimal element $\underline{\pitchfork} := \{\{1\}, \{2\}, \dots, \{N\}\}$. The *rank* of a partition $\pitchfork$ of $[N]$ is the integer

$$\operatorname{rank}(\pitchfork) := N - \#\pitchfork = \sum_{\lambda \in \pitchfork} (\#\lambda - 1).$$

Definition 1.6.2. Suppose $N \geq 2$. As needed, empty sums are defined to be 0.

(a) For each nonempty subset $\lambda \subset [N]$, define the *part exponent* $e_\lambda : \mathbb{C}^{\binom{N}{2}} \to \mathbb{C}$ by

$$e_\lambda(\boldsymbol{s}) := \sum_{\substack{i<j \\ i,j \in \lambda}} \left(s_{ij} + \frac{2}{\#\lambda}\right) = (\#\lambda - 1) + \sum_{\substack{i<j \\ i,j \in \lambda}} s_{ij}.$$

(b) For each partition $\pitchfork$ of $[N]$, define the *partition exponent* $E_\pitchfork : \mathbb{C}^{\binom{N}{2}} \to \mathbb{C}$ by

$$E_\pitchfork(\boldsymbol{s}) := \sum_{\lambda \in \pitchfork} e_\lambda(\boldsymbol{s}) = \operatorname{rank}(\pitchfork) + \sum_{\lambda \in \pitchfork} \sum_{\substack{i<j \\ i,j \in \lambda}} s_{ij}.$$

Definition 1.6.3 (Splitting chains)**.** A finite tuple $\boldsymbol{\pitchfork} = (\pitchfork_0, \pitchfork_1, \ldots, \pitchfork_L)$ of partitions of $[N]$ satisfying

$$\overline{\pitchfork} = \pitchfork_0 > \pitchfork_1 > \pitchfork_2 > \cdots > \pitchfork_L = \underline{\pitchfork}$$

shall be called a *splitting chain* of order N. We write $\mathcal{S}_N$ for the set of all splitting chains of order N, and we attach the following terminology and notation to each $\boldsymbol{\pitchfork} \in \mathcal{S}_N$ with $N \geq 2$:

(a) The positive integer $L(\boldsymbol{\pitchfork}) := L$ is the *length* of $\boldsymbol{\pitchfork}$ and the partitions $\pitchfork_0, \pitchfork_1, \ldots, \pitchfork_{L(\boldsymbol{\pitchfork})-1}$ are the *levels* of $\boldsymbol{\pitchfork}$. Call each non-singleton part $\lambda \in \pitchfork_0 \cup \pitchfork_1 \cup \cdots \cup \pitchfork_{L(\boldsymbol{\pitchfork})-1}$ a *branch* of $\boldsymbol{\pitchfork}$ and write $\mathcal{B}(\boldsymbol{\pitchfork})$ for the set of all branches of $\boldsymbol{\pitchfork}$, i.e.,

$$\mathcal{B}(\boldsymbol{\pitchfork}) := (\pitchfork_0 \cup \pitchfork_1 \cup \cdots \cup \pitchfork_{L(\boldsymbol{\pitchfork})-1}) \setminus \underline{\pitchfork}.$$

(b) Since $\boldsymbol{\pitchfork}$ must terminate at $\pitchfork_{L(\boldsymbol{\pitchfork})} = \underline{\pitchfork}$, each branch appears in a final level $\pitchfork_\ell$ before it refines into two or more parts in $\pitchfork_{\ell+1}$. Thus for each $\lambda \in \mathcal{B}(\boldsymbol{\pitchfork})$ we define the *depth* $\ell_{\boldsymbol{\pitchfork}}(\lambda) \in \{0, 1, \ldots, L(\boldsymbol{\pitchfork}) - 1\}$ and *degree* $\deg_{\boldsymbol{\pitchfork}}(\lambda) \in \{2, 3, \ldots, N\}$ respectively by

$$\ell_{\boldsymbol{\pitchfork}}(\lambda) := \max\{\ell : \lambda \in \pitchfork_\ell\} \qquad \text{and} \qquad \deg_{\boldsymbol{\pitchfork}}(\lambda) := \#\{\lambda' \in \pitchfork_{\ell_{\boldsymbol{\pitchfork}}(\lambda)+1} : \lambda' \subset \lambda\}.$$

(c) Using the falling factorial notation $(z)_n = z \cdot (z-1) \cdot (z-2) \cdot \ldots \cdot (z-n+1)$ for integers $n \geq 1$, we define the *multiplicity polynomial* $M_{\boldsymbol{\pitchfork}}(t) \in \mathbb{Z}[t]$ by

$$M_{\boldsymbol{\pitchfork}}(t) := \prod_{\lambda \in \mathcal{B}(\boldsymbol{\pitchfork})} (t-1)_{\deg_{\boldsymbol{\pitchfork}}(\lambda)-1}.$$

It is a key observation that $\mathcal{S}_N$ is finite for each $N \geq 2$. This is easily seen from the definition, since every $\pitchfork \in \mathcal{S}_N$ must satisfy $0 < L(\pitchfork) < N$ and there are at most finitely many $\pitchfork \in \mathcal{S}_N$ of a given length. One should also note that the multiplicity polynomial for a splitting chain $\pitchfork \in \mathcal{S}_N$ factors as

$$M_{\pitchfork}(t) = \prod_{d=1}^{N-1} (t-d)^{p_{\pitchfork}(d)} \qquad \text{where} \qquad p_{\pitchfork}(d) = \#\{\lambda \in \mathcal{B}(\pitchfork) : \deg_{\pitchfork}(\lambda) > d\},$$

because the falling factorial $(t-1)_{\deg_{\pitchfork}(\lambda)-1}$ contributes a factor of $(t-d)$ if and only if $\deg_{\pitchfork}(\lambda) > d$, and there are precisely $p_{\pitchfork}(d)$ such falling factorials in the definition of $M_{\pitchfork}(t)$. Thus, given an integer $q \geq 2$, we have $M_{\pitchfork}(q) > 0$ if $\deg_{\pitchfork}(\lambda) \leq q$ for all $\lambda \in \mathcal{B}(\pitchfork)$, and $M_{\pitchfork}(q) = 0$ otherwise. Multiplicity polynomials and the exponents in Definition 1.6.2 together form the *branch/level polytopes* and *branch/level functions* defined below. As we shall soon see, the sum of level functions over all $\pitchfork \in \mathcal{S}_N$ is the second main factor in our formula for $Z_N^{\rho}(K, a, b, \boldsymbol{s})$.

Definition 1.6.4. Suppose $N \geq 2$ and $q \geq 2$ are integers and suppose $\pitchfork \in \mathcal{S}_N$. As needed, products and intersections over empty index sets are respectively defined to be 1 and $\mathbb{C}^{\binom{N}{2}}$.

(a) The *branch polytope* $\mathcal{BP}_{\pitchfork}$ and *branch function* $I_{\pitchfork,q} : \mathcal{BP}_{\pitchfork} \to \mathbb{C}$ are respectively defined by

$$\mathcal{BP}_{\pitchfork} := \bigcap_{\lambda \in \mathcal{B}(\pitchfork) \setminus \overline{\pitchfork}} \left\{ \boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}} : \mathrm{Re}(e_\lambda(\boldsymbol{s})) > 0 \right\} \qquad \text{and}$$

$$I_{\pitchfork,q}(\boldsymbol{s}) := \frac{M_{\pitchfork}(q)}{q^{N-1}} \cdot \prod_{\lambda \in \mathcal{B}(\pitchfork) \setminus \overline{\pitchfork}} \frac{1}{q^{e_\lambda(\boldsymbol{s})} - 1}.$$

(b) Given $b \in \mathbb{C}$, the *level polytope* $\mathcal{LP}_{⫛}(b)$ and *level function* $J_{⫛,q}(b,\cdot) : \mathcal{LP}_{⫛}(b) \to \mathbb{C}$ are respectively defined by

$$\mathcal{LP}_{⫛}(b) := \bigcap_{\ell=1}^{L(⫛)-1} \left\{ \boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}} : \operatorname{Re}(b + E_{⫛_\ell}(\boldsymbol{s})) > 0 \right\} \qquad \text{and}$$

$$J_{⫛,q}(b,\boldsymbol{s}) := \frac{M_{⫛}(q)}{q^{N-1}} \cdot \prod_{\ell=1}^{L(⫛)-1} \frac{1}{q^{b+E_{⫛_\ell}(\boldsymbol{s})} - 1}.$$

Note that $I_{⫛,q}(\boldsymbol{s})$ and $J_{⫛,q}(b,\boldsymbol{s})$ are rational functions in the variables q, q^{-b}, and $q^{-s_{ij}}$ for all $i < j$, with $\mathbb{Q}$ coefficients determined by $⫛$ alone. Given $⫛$ and an integer $q \geq 2$, there are two possibilities:

(i) If $\deg_{⫛}(\lambda) \leq q$ for all $\lambda \in \mathcal{B}(⫛)$, then $M_{⫛}(q) > 0$, and hence $I_{⫛,q}(\boldsymbol{s})$ and $J_{⫛,q}(b,\boldsymbol{s})$ are never zero.

(ii) If $\deg_{⫛}(\lambda) > q$ for some $\lambda \in \mathcal{B}(⫛)$, then $M_{⫛}(q) = 0$, and hence $I_{⫛,q}$ and $J_{⫛,q}(b,\cdot)$ are identically zero on $\mathcal{BP}_{⫛}$ and $\mathcal{LP}_{⫛}(b)$ respectively.

In any case, $I_{⫛,q}$ and $J_{⫛,q}(b,\cdot)$ are holomorphic on their respective polytopes $\mathcal{BP}_{⫛}$ and $\mathcal{LP}_{⫛}(b)$, which are both open and convex. These polytopes are related by the following lemma, which is the last ingredient we need to state the main theorem.

Lemma 1.6.5. *We say that a splitting chain $⫛$ is* reduced *if for each $\lambda \in \mathcal{B}(⫛)$ there is a unique level $⫛_\ell$ containing λ (namely, the level $⫛_{\ell_{⫛}(\lambda)}$). We write*

$$\mathcal{R}_N := \{⫛ \in \mathcal{S}_N : ⫛ \text{ is reduced}\}$$

and define an equivalence relation $\simeq$ on $\mathcal{S}_N$ by writing $⫛ \simeq ⫛'$ if and only if $\mathcal{B}(⫛) = \mathcal{B}(⫛')$.

(a) *If* $\pitchfork \simeq \pitchfork'$*, then the branch degrees, part exponents, multiplicity polynomials, and branch polytopes for* $\pitchfork$ *and* $\pitchfork'$ *respectively coincide.*

(b) *For each* $\pitchfork \in \mathcal{S}_N$ *there is a unique* $\pitchfork^* \in \mathcal{R}_N$ *such that* $\pitchfork \simeq \pitchfork^*$*. We call this* $\pitchfork^*$ *the* reduction *of* $\pitchfork$ *and regard* $\mathcal{R}_N$ *as a complete set of representatives for* $\mathcal{S}_N$ *modulo* $\simeq$*.*

(c) *For each* $\pitchfork^* \in \mathcal{R}_N$ *we have*

$$\bigcap_{\substack{\pitchfork \in \mathcal{S}_N \\ \pitchfork \simeq \pitchfork^*}} \mathcal{LP}_{\pitchfork}(0) = \mathcal{BP}_{\pitchfork^*},$$

and therefore

$$\bigcap_{\pitchfork \in \mathcal{S}_N} \mathcal{LP}_{\pitchfork}(0) = \bigcap_{\pitchfork^* \in \mathcal{R}_N} \mathcal{BP}_{\pitchfork^*}.$$

Our main theorem shows that $Z_N^{\rho}(K, a, b, \boldsymbol{s})$ and $\mathcal{Z}_N(\mathbb{P}^1(K), \boldsymbol{s})$ can be expressed neatly in terms of root, level, and branch functions, and that their domains of absolute convergence are simply intersections of root, level, and branch polytopes:

Theorem 1.6.6 (Main Theorem)**.** *Fix* $N \geq 2$ *and* $a, b \in \mathbb{C}$ *and define the convex open polytope*

$$\Omega_N(a, b) := \mathcal{RP}_N(a, b) \cap \bigcap_{\pitchfork \in \mathcal{S}_N} \mathcal{LP}_{\pitchfork}(b).$$

(a) *If* K *is a* p*-field and* ρ *is a norm-density that is not identically zero, then the integral* $Z_N^{\rho}(K, a, b, \boldsymbol{s})$ *converges absolutely for all* $\boldsymbol{s} \in \Omega_N(a, b)$*, and* $\Omega_N(a, b)$ *is the largest open subset of* $\mathbb{C}^{\binom{N}{2}}$ *with this property.*

(b) *If* K *and* ρ *are as above, then on each compact subset of* $\Omega_N(a,b)$ *the integral is given by the uniformly convergent sum*

$$Z_N^{\rho}(K,a,b,\boldsymbol{s}) = H_q^{\rho}\left(N+a+b+\sum_{i<j} s_{ij}\right)\cdot \sum_{\pitchfork\in\mathcal{S}_N} J_{\pitchfork,q}(b,\boldsymbol{s}).$$

(c) *Given* $b=0$, K *and* ρ *as above, and* $\pitchfork^* \in \mathcal{R}_N$, *we have*

$$\sum_{\substack{\pitchfork\in\mathcal{S}_N\\ \pitchfork\simeq\pitchfork^*}} J_{\pitchfork,q}(0,\boldsymbol{s}) = I_{\pitchfork^*,q}(\boldsymbol{s}) \quad \textit{for all } \boldsymbol{s}\in\mathcal{BP}_{\pitchfork^*}.$$

Thus on each compact subset of $\Omega_N(a,0)$ *the integral is given by the uniformly convergent sum*

$$Z_N^{\rho}(K,a,0,\boldsymbol{s}) = H_q^{\rho}\left(N+a+\sum_{i<j} s_{ij}\right)\cdot \sum_{\pitchfork^*\in\mathcal{R}_N} I_{\pitchfork^*,q}(\boldsymbol{s}).$$

(d) *Given* K *as above, the integral* $\mathcal{Z}_N(\mathbb{P}^1(K),\boldsymbol{s})$ *converges absolutely if and only if* $\boldsymbol{s}\in\Omega_N(0,0)$, *and for such* $\boldsymbol{s}$ *it is given by the finite sum*

$$\mathcal{Z}_N(\mathbb{P}^1(K),\boldsymbol{s}) = \frac{(q/(q+1))^{N-1}}{q^{e_{[N]}(\boldsymbol{s})}-1}\cdot \sum_{\pitchfork^*\in\mathcal{R}_N} \frac{q^{e_{[N]}(\boldsymbol{s})+1}+1-\deg_{\pitchfork^*}([N])}{q+1-\deg_{\pitchfork^*}([N])}\cdot I_{\pitchfork^*,q}(\boldsymbol{s}).$$

The denominator $q+1-\deg_{\pitchfork^*}([N])$ *in the summand for* $\pitchfork^*\in\mathcal{R}_N$ *is also a factor of* $M_{\pitchfork^*}(q)$ *(and hence can be cancelled out of* $I_{\pitchfork^*,q}(\boldsymbol{s})$*), so the apparent singularity at* $q=\deg_{\pitchfork^*}([N])-1$ *is removable.*

There are several features of Theorem 1.6.6 that are worth emphasizing here. Note that part (a) is independent of K and ρ, and that the rest of the theorem depends on K only via q. That is, the region of absolute convergence

$\Omega_N(a,b)$ is always the same, no matter what the p-field K is, and the formulas for $Z_N^\rho(K,a,b,\boldsymbol{s})$ and $\mathcal{Z}_N(\mathbb{P}^1(K),\boldsymbol{s})$ are "uniform in K" in the sense that there are no extra cases or constraints for q. Moreover, the dependence of $Z_N^\rho(K,a,b,\boldsymbol{s})$ on ρ and a is carried entirely by the root functions appearing in parts (b) and (c). Setting $a = b = 0$, we see that $Z_N^\rho(K,0,0,\boldsymbol{s})$ and $\mathcal{Z}_N(\mathbb{P}^1(K),\boldsymbol{s})$ have the same region of absolute convergence (namely, $\Omega_N(0,0)$, unless ρ is identically zero), and both are given by similar sums over $\mathcal{R}_N$. The term in $Z_N^\rho(K,0,0,\boldsymbol{s})$ corresponding to $\pitchfork^* \in \mathcal{R}_N$ can be rearranged into the explicit form

$$
\begin{aligned}
H_q^\rho\left(N+\sum_{i<j} s_{ij}\right) I_{\pitchfork^*,q}(\boldsymbol{s}) &= \frac{1-q^{-(N+\sum_{i<j} s_{ij})}}{1-q^{-(N-1+\sum_{i<j} s_{ij})}} \cdot \sum_{m\in\mathbb{Z}} \rho(q^m) q^{m(N+\sum_{i<j} s_{ij})} \\
&\qquad \cdot \frac{M_{\pitchfork^*}(q)}{q^{N-1}} \cdot \prod_{\lambda\in\mathcal{B}(\pitchfork^*)\setminus\overline{\pitchfork}} \frac{1}{q^{e_\lambda(\boldsymbol{s})}-1} \\
&= \frac{q^{\sum_{i<j} s_{ij}} - q^{-N}}{q^{\sum_{i<j} s_{ij}} - q^{-(N-1)}} \cdot \sum_{m\in\mathbb{Z}} \rho(q^m) q^{m(N+\sum_{i<j} s_{ij})} \\
&\qquad \cdot \frac{(q-1)_{\deg_{\pitchfork^*}([N])-1}}{q^{N-1}} \cdot \prod_{\lambda\in\mathcal{B}(\pitchfork^*)\setminus\overline{\pitchfork}} \frac{(q-1)_{\deg_{\pitchfork^*}(\lambda)-1}}{q^{e_\lambda(\boldsymbol{s})}-1} \\
&= (q^{\sum_{i<j} s_{ij}} - q^{-N}) \cdot \sum_{m\in\mathbb{Z}} \rho(q^m) q^{m(N+\sum_{i<j} s_{ij})} \cdot \prod_{\lambda\in\mathcal{B}(\pitchfork^*)} \frac{(q-1)_{\deg_{\pitchfork^*}(\lambda)-1}}{q^{e_\lambda(\boldsymbol{s})}-1}
\end{aligned}
$$

and similarly, the term in $\mathcal{Z}_N(\mathbb{P}^1(K),\boldsymbol{s})$ corresponding to the same $\pitchfork^* \in \mathcal{R}_N$ can be written explicitly as

$$
\begin{aligned}
&\frac{(q/(q+1))^{N-1}}{q^{e_{[N]}(\boldsymbol{s})}-1} \cdot \frac{q^{e_{[N]}(\boldsymbol{s})+1} + 1 - \deg_{\pitchfork^*}([N])}{q+1-\deg_{\pitchfork^*}([N])} \cdot I_{\pitchfork^*,q}(\boldsymbol{s}) \\
&\qquad = \frac{1}{(q+1)^{N-1}} \cdot \frac{q^{e_{[N]}(\boldsymbol{s})+1} + 1 - \deg_{\pitchfork^*}([N])}{q+1-\deg_{\pitchfork^*}([N])} \cdot \prod_{\lambda\in\mathcal{B}(\pitchfork^*)} \frac{(q-1)_{\deg_{\pitchfork^*}(\lambda)-1}}{q^{e_\lambda(\boldsymbol{s})}-1}.
\end{aligned}
$$

Note that both of these terms share a common factor of $\prod_{\lambda\in\mathcal{B}(\pitchfork^*)} \frac{(q-1)_{\deg_{\pitchfork^*}(\lambda)-1}}{q^{e_\lambda(\boldsymbol{s})}-1}$, which generalizes the similarity between Examples 1.2.2 and 1.2.3. In addition to being independent of K and ρ, the next proposition implies that $\Omega_N(a,b)$ is independent of splitting chains altogether:

Proposition 1.6.7. *If $N \geq 2$ and $b \in \mathbb{C}$, the intersection of level polytopes $\bigcap_{\pitchfork\in\mathcal{S}_N} \mathcal{LP}_{\pitchfork}(b)$ is equal to the following intersection over all partitions $\pitchfork$ of $[N]$ satisfying $\underline{\pitchfork} < \pitchfork < \overline{\pitchfork}$:*

$$\bigcap_{\pitchfork\in\mathcal{S}_N} \mathcal{LP}_{\pitchfork}(b) = \bigcap_{\underline{\pitchfork}<\pitchfork<\overline{\pitchfork}} \left\{ \boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}} : \mathrm{Re}(b + E_{\pitchfork}(\boldsymbol{s})) > 0 \right\}.$$

Similarly, the intersection of branch polytopes $\bigcap_{\pitchfork^*\in\mathcal{R}_N} \mathcal{BP}_{\pitchfork^*}$ is equal to an intersection over all proper subsets $\lambda \subsetneq [N]$ of size $\#\lambda > 1$:

$$\bigcap_{\pitchfork^*\in\mathcal{R}_N} \mathcal{BP}_{\pitchfork^*} = \bigcap_{\substack{\lambda\subsetneq[N]\\ \#\lambda>1}} \left\{ \boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}} : \mathrm{Re}(e_\lambda(\boldsymbol{s})) > 0 \right\}.$$

Therefore

$$\Omega_N(0,b) = \bigcap_{\underline{\pitchfork}<\pitchfork\leq\overline{\pitchfork}} \left\{ \boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}} : \mathrm{Re}(b + E_{\pitchfork}(\boldsymbol{s})) > 0 \right\} \qquad \textit{and}$$
$$\Omega_N(0,0) = \bigcap_{\substack{\lambda\subset[N]\\ \#\lambda>1}} \left\{ \boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}} : \mathrm{Re}(e_\lambda(\boldsymbol{s})) > 0 \right\}.$$

The last claim in this proposition follows immediately from its first two claims and the definition of $\Omega_N(a,b)$ in Theorem 1.6.6. The proofs of the first two claims will occur inside the proofs of the main theorem in Chapters 3 and 4, and we will indicate where they happen.

1.7. Simple examples and a note about poles

Of the "parameters" ρ, K, a, b, and N, the last has the most complicated role by far. Thus we work only with the $N = 2$ and $N = 3$ examples for now. In order to streamline notation, we will write each partition $\pitchfork = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ as a string of parts, i.e., $\pitchfork = \lambda_1\lambda_2\ldots\lambda_k$.

Example 1.7.1. Fix $a, b \in \mathbb{C}$ and a norm-density ρ.

- If $N = 2$, then $\binom{N}{2} = 1$, so each $\boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}}$ is simply a number $s \in \mathbb{C}$. The only splitting chain in $\mathcal{S}_2$ is $\boldsymbol{\pitchfork} = (\{1,2\}, \{1\}\{2\})$, and by Definition 1.6.4 it has

$$\mathcal{LP}_{\boldsymbol{\pitchfork}}(b) = \mathbb{C} \qquad \text{and} \qquad J_{\boldsymbol{\pitchfork},q}(b, \boldsymbol{s}) = I_{\boldsymbol{\pitchfork},q}(\boldsymbol{s}) = \frac{q-1}{q} \qquad \text{for all } q \geq 2.$$

  Thus for all p-fields K with residue cardinality q and all s in the region

$$\Omega_2(a,b) = \mathcal{RP}_2(a,b) \cap \mathbb{C} = \mathcal{RP}_2(a,b) = \{s \in \mathbb{C} : \mathrm{Re}(1 + a + b + s) > 0\},$$

  the integral $Z_2^\rho(K, a, b, \boldsymbol{s})$ converges absolutely to the value

$$Z_2^\rho(K, a, b, \boldsymbol{s}) = \frac{q-1}{q} \cdot \frac{1 - q^{-(2+a+b+s)}}{1 - q^{-(1+a+b+s)}} \cdot \sum_{m \in \mathbb{Z}} \rho(q^m) q^{m(2+a+b+s)}.$$

- If $N = 3$, then we have $\boldsymbol{s} = (s_{12}, s_{13}, s_{23}) \in \mathbb{C}^3$ with root polytope

$$\mathcal{RP}_3(a,b) = \{\boldsymbol{s} \in \mathbb{C}^3 : \mathrm{Re}(2 + a + b + s_{12} + s_{13} + s_{23}) > 0\},$$

  and Definition 1.6.4 provides the following table for all four splitting chains in $\mathcal{S}_3$:

$\pitchfork \in \mathcal{S}_3$	$J_{\pitchfork,q}(b,\boldsymbol{s})$	$\mathcal{LP}_{\pitchfork}(b)$
$\pitchfork_0 = \{1,2,3\}$ $\pitchfork_1 = \{1\}\{2\}\{3\}$	$\dfrac{(q-1)(q-2)}{q^2}$	$\mathbb{C}^3$
$\pitchfork_0 = \{1,2,3\}$ $\pitchfork_1 = \{1,2\}\{3\}$ $\pitchfork_2 = \{1\}\{2\}\{3\}$	$\dfrac{(q-1)^2}{q^2} \cdot \dfrac{1}{q^{1+b+s_{12}}-1}$	$\{\boldsymbol{s} \in \mathbb{C}^3 : \mathrm{Re}(1+b+s_{12}) > 0\}$
$\pitchfork_0 = \{1,2,3\}$ $\pitchfork_1 = \{1,3\}\{2\}$ $\pitchfork_2 = \{1\}\{2\}\{3\}$	$\dfrac{(q-1)^2}{q^2} \cdot \dfrac{1}{q^{1+b+s_{13}}-1}$	$\{\boldsymbol{s} \in \mathbb{C}^3 : \mathrm{Re}(1+b+s_{13}) > 0\}$
$\pitchfork_0 = \{1,2,3\}$ $\pitchfork_1 = \{1\}\{2,3\}$ $\pitchfork_2 = \{1\}\{2\}\{3\}$	$\dfrac{(q-1)^2}{q^2} \cdot \dfrac{1}{q^{1+b+s_{23}}-1}$	$\{\boldsymbol{s} \in \mathbb{C}^3 : \mathrm{Re}(1+b+s_{23}) > 0\}$

Thus, for all p-fields K with residue cardinality q and for all $\boldsymbol{s}$ in the region

$$\begin{aligned}\Omega_3(a,b) = \{\boldsymbol{s} \in \mathbb{C}^3 : \mathrm{Re}(2+a+b+s_{12}+s_{13}+s_{23}) > 0\} \\ \cap \bigcap_{1\le i<j\le 3} \{\boldsymbol{s} \in \mathbb{C}^3 : \mathrm{Re}(1+b+s_{ij}) > 0\},\end{aligned}$$

the integral $Z_3^\rho(K,a,b,\boldsymbol{s})$ converges absolutely to the value

$$Z_3^\rho(K,a,b,\boldsymbol{s}) = \frac{1-q^{-(3+a+b+s_{12}+s_{13}+s_{23})}}{1-q^{-(2+a+b+s_{12}+s_{13}+s_{23})}} \cdot \sum_{m\in\mathbb{Z}} \rho(q^m) q^{m(3+a+b+s_{12}+s_{13}+s_{23})}$$
$$\cdot \left(\frac{(q-1)(q-2)}{q^2} + \frac{(q-1)^2}{q^2} \left[\frac{1}{q^{1+b+s_{12}}-1} + \frac{1}{q^{1+b+s_{13}}-1} + \frac{1}{q^{1+b+s_{23}}-1} \right] \right).$$

Remark 1.7.2. Note that every splitting chain of order 2 or 3 is reduced and $J_{\boldsymbol{\pitchfork}}(0,\boldsymbol{s}) = I_{\boldsymbol{\pitchfork}}(\boldsymbol{s})$ for all $\boldsymbol{\pitchfork} \in \mathcal{S}_2$ and all $\boldsymbol{\pitchfork} \in \mathcal{S}_3$. Therefore part (c) of Theorem 1.6.6 is redundant when $N = 2$ or $N = 3$, for in these cases it coincides with part (b) applied to $b = 0$. If $N \geq 4$ we have $\mathcal{R}_N \subsetneq \mathcal{S}_N$, because there is at least one non-reduced splitting chain $\boldsymbol{\pitchfork} = (\pitchfork_0, \pitchfork_1, \pitchfork_2, \pitchfork_3) \in \mathcal{S}_N$ such as the one given by

$$\begin{aligned}
\pitchfork_0 &= \{1,2,3,4,\ldots,N\},\\
\pitchfork_1 &= \{1,2\}\{3,4,\ldots,N\},\\
\pitchfork_2 &= \{1,2\}\{3\}\{4\}\ldots\{N\},\\
\pitchfork_3 &= \{1\}\{2\}\{3\}\{4\}\ldots\{N\}.
\end{aligned}$$

Finding closed forms for the cardinalities of $\mathcal{S}_N$ and $\mathcal{R}_N$ for general N is nontrivial, but they can be bounded below as follows. Given $\boldsymbol{\pitchfork} \in \mathcal{R}_N$ and $i \in [N]$, we may construct a particular $\boldsymbol{\pitchfork}' \in \mathcal{R}_{N+1}$: For each $\ell \in \{0,1,2,\ldots,L(\boldsymbol{\pitchfork})\}$, let $\pitchfork'_\ell$ be the partition of $[N+1]$ obtained from $\pitchfork_\ell$ by replacing the unique part $\lambda \in \pitchfork_\ell$ containing i by the larger part $\lambda \cup \{N+1\}$. If we then set $\pitchfork'_{L(\boldsymbol{\pitchfork})+1} := \underline{\pitchfork}$, it is easily verified that $\boldsymbol{\pitchfork}' = (\pitchfork'_0, \pitchfork'_1, \ldots, \pitchfork'_{L(\boldsymbol{\pitchfork})+1})$ is a reduced splitting chain of order $N+1$. Thus $(\boldsymbol{\pitchfork}, i) \mapsto \boldsymbol{\pitchfork}'$ defines a function $\mathcal{R}_N \times [N] \to \mathcal{R}_{N+1}$, which is injective because it has a left inverse: The integer i can be recovered from $\boldsymbol{\pitchfork}'$ because it is the only element of $[N]$ satisfying $\{i, N+1\} \in \boldsymbol{\pitchfork}'_{L(\boldsymbol{\pitchfork})}$, and then $\boldsymbol{\pitchfork}$ can be recovered from $\boldsymbol{\pitchfork}'$ by simply removing $\pitchfork_{L(\boldsymbol{\pitchfork})+1}$ and all copies of $N+1$ from $\boldsymbol{\pitchfork}'$. Thus we have $\#\mathcal{R}_N \cdot N \leq \#\mathcal{R}_{N+1}$ for all $N \geq 2$, and we saw before that $\#\mathcal{R}_2 = \#\mathcal{S}_2 = 1$, $\#\mathcal{R}_3 = \#\mathcal{S}_3 = 4$, and $\mathcal{R}_N \subsetneq \mathcal{S}_N$ for all $N \geq 4$. Induction on N then gives the following bounds:

$$(N-1)! \leq \#\mathcal{R}_N \leq \#\mathcal{S}_N \qquad \text{for all } N \geq 2.$$

The left inequality is strict for $N \geq 3$ and both are strict for $N \geq 4$.

The preceding remark implies that the branch function sum $\sum_{\boldsymbol{\pitchfork}^* \in \mathcal{R}_N} I_{\boldsymbol{\pitchfork}^*;q}(\boldsymbol{s})$ has strictly fewer terms than the level function sum $\sum_{\boldsymbol{\pitchfork} \in \mathcal{S}_N} J_{\boldsymbol{\pitchfork},q}(b, \boldsymbol{s})$ when $N \geq 4$, and hence part (c) of Theorem 1.6.6 becomes a simplification of part (b) applied to $b = 0$. Though $N = 4$ is the least N for which this simplification is noticeable, the sums of branch functions and level functions respectively have $\#\mathcal{R}_4 = 26$ terms and $\#\mathcal{S}_4 = 32$ terms in this case, so the rather large computation of $Z_4^{\rho}(K, a, b, \boldsymbol{s})$ will be postponed until the Appendix. For now we consider only three elements from $\mathcal{S}_4$ to discuss how part (c) of Theorem 1.6.6 simplifies the $b = 0$ case of part (b).

Example 1.7.3. Consider the three splitting chains $\boldsymbol{\pitchfork}^*, \boldsymbol{\pitchfork}', \boldsymbol{\pitchfork}'' \in \mathcal{S}_4$ defined by

$$\begin{aligned} \pitchfork_0^* &= \{1,2,3,4\}, \\ \pitchfork_1^* &= \{1,2\}\{3,4\}, \\ \pitchfork_2^* &= \{1\}\{2\}\{3\}\{4\}, \end{aligned} \qquad \begin{aligned} \pitchfork_0' &= \{1,2,3,4\}, \\ \pitchfork_1' &= \{1,2\}\{3,4\}, \\ \pitchfork_2' &= \{1,2\}\{3\}\{4\}, \\ \pitchfork_3' &= \{1\}\{2\}\{3\}\{4\}, \end{aligned} \qquad \text{and} \qquad \begin{aligned} \pitchfork_0'' &= \{1,2,3,4\}, \\ \pitchfork_1'' &= \{1,2\}\{3,4\}, \\ \pitchfork_2'' &= \{1\}\{2\}\{3,4\}, \\ \pitchfork_3'' &= \{1\}\{2\}\{3\}\{4\}. \end{aligned}$$

Recalling Lemma 1.6.5, $\pitchfork^*$ is the common reduction of all three, and it is easily verified that no other $\pitchfork \in \mathcal{S}_4 \setminus \{\pitchfork^*, \pitchfork', \pitchfork''\}$ satisfies $\pitchfork \simeq \pitchfork^*$. By Definition 1.6.4, the splitting chains $\pitchfork^*$, $\pitchfork'$, and $\pitchfork''$ contribute the following level functions to the sum $\sum_{\pitchfork \in \mathcal{S}_4} J_{\pitchfork,q}(b, \boldsymbol{s})$ in part (b) of Theorem 1.6.6:

$$\begin{aligned}
J_{\pitchfork^*,q}(b, \boldsymbol{s}) &= \frac{(q-1)^3}{q^3} \cdot \frac{1}{q^{2+b+s_{12}+s_{34}} - 1}, \\
J_{\pitchfork',q}(b, \boldsymbol{s}) &= \frac{(q-1)^3}{q^3} \cdot \frac{1}{q^{2+b+s_{12}+s_{34}} - 1} \cdot \frac{1}{q^{1+b+s_{12}} - 1}, \\
J_{\pitchfork'',q}(b, \boldsymbol{s}) &= \frac{(q-1)^3}{q^3} \cdot \frac{1}{q^{2+b+s_{12}+s_{34}} - 1} \cdot \frac{1}{q^{1+b+s_{34}} - 1}.
\end{aligned}$$

In particular, their total contribution to the sum can be written as

$$\sum_{\substack{\pitchfork \in \mathcal{S}_4 \\ \pitchfork \simeq \pitchfork^*}} J_{\pitchfork,q}(b, \boldsymbol{s}) = \frac{(q-1)^3}{q^3} \cdot \frac{1}{q^{1+b+s_{12}} - 1} \cdot \frac{1}{q^{1+b+s_{34}} - 1} \cdot \frac{q^{2+2b+s_{12}+s_{34}} - 1}{q^{2+b+s_{12}+s_{34}} - 1}. \quad (1.7.14)$$

Equation (1.7.14) hints at an interesting analytic feature of the parameter b. Indeed, if $q \geq 2$ and $b \in \mathbb{C}$ are fixed, then each of the summands $J_{\pitchfork^*,q}(b, \boldsymbol{s})$, $J_{\pitchfork',q}(b, \boldsymbol{s})$, and $J_{\pitchfork'',q}(b, \boldsymbol{s})$ is meromorphic in $\boldsymbol{s} = (s_{12}, s_{13}, s_{14}, s_{23}, s_{24}, s_{34}) \in \mathbb{C}^6$, and each of their sets of poles contains the infinite set

$$\begin{aligned}
C(b) = \Bigg\{ \boldsymbol{s} \in \mathbb{C}^6 : 2 + b + s_{12} + s_{34} &\in \frac{2\pi i \mathbb{Z}}{\log(q)}, \\
1 + b + s_{12} &\notin \frac{2\pi i \mathbb{Z}}{\log(q)}, \\
1 + b + s_{34} &\notin \frac{2\pi i}{\log(q)} \Bigg\}.
\end{aligned}$$

If b is not an integer multiple of $2\pi i / \log(q)$, the poles for the sum in (1.7.14) also include $C(b)$. However, if b *is* an integer multiple of $2\pi i / \log(q)$, then

$(q^{2+2b+s_{12}+s_{34}}-1)/(q^{2+b+s_{12}+s_{34}}-1) = 1$ and *none* of the $\boldsymbol{s} \in C(b)$ are poles for the sum in (1.7.14). In particular, $C(0)$ is a common set of poles for all of the level functions $J_{\pitchfork^*,q}(0,\boldsymbol{s})$, $J_{\pitchfork',q}(0,\boldsymbol{s})$, and $J_{\pitchfork'',q}(0,\boldsymbol{s})$, but all such poles "cancel" when the level functions are added together:

$$\sum_{\substack{\pitchfork \in \mathcal{S}_4 \\ \pitchfork \simeq \pitchfork^*}} J_{\pitchfork,q}(0,\boldsymbol{s}) = \frac{(q-1)^3}{q^3} \cdot \frac{1}{q^{1+s_{12}}-1} \cdot \frac{1}{q^{1+s_{34}}-1} = I_{\pitchfork^*,q}(\boldsymbol{s}).$$

Thus, by collapsing the sum $\sum_{\pitchfork \in \mathcal{S}_4} J_{\pitchfork,q}(0,\boldsymbol{s})$ from part (b) of Theorem 1.6.6 into its "reduced" form $\sum_{\pitchfork \in \mathcal{R}_4} I_{\pitchfork,q}(\boldsymbol{s})$, part (c) shows that many level function poles "cancel" in the $b=0$ case.

Remark 1.7.4. For simple choices of ρ, the root function sums to a closed form. In this case Theorem 1.6.6 provides meromorphic continuations of both $\boldsymbol{s} \mapsto Z_N^\rho(K,a,b,\boldsymbol{s})$ and $\boldsymbol{s} \mapsto Z_N^\rho(K,a,0,\boldsymbol{s})$, and their candidate poles may be easily described. For example, suppose $\rho = \mathbf{1}_{[0,1]}$. It is easily verified from Definition 1.6.1 and Theorem 1.6.6 that $Z_N^\rho(K,a,b,\boldsymbol{s})$ coincides with the sum

$$\frac{q^{a+b+\sum_{i<j} s_{ij}}}{q^{N-1+a+b+\sum_{i<j} s_{ij}}-1} \cdot \sum_{\pitchfork \in \mathcal{S}_N} M_\pitchfork(q) \cdot \prod_{\ell=1}^{L(\pitchfork)-1} \frac{1}{q^{b+E_{\pitchfork_\ell}(\boldsymbol{s})}-1} \tag{1.7.15}$$

on the convex open region $\Omega_N(a,b)$. Since each summand is meromorphic in $\mathbb{C}^{\binom{N}{2}}$ with set of poles

$$\mathscr{L}_{\pitchfork,q} := \left\{ \boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}} : N-1+a+b+\sum_{i<j} s_{ij} \in \frac{2\pi i \mathbb{Z}}{\log(q)} \right\}$$
$$\cup \bigcup_{\ell=1}^{L(\pitchfork)-1} \left\{ \boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}} : b + E_{\pitchfork_\ell}(\boldsymbol{s}) \in \frac{2\pi i \mathbb{Z}}{\log(q)} \right\},$$

then (1.7.15) defines the meromorphic continuation of $Z_N^\rho(K, a, b, \boldsymbol{s})$ to $\mathbb{C}^{\binom{N}{2}}$, and its poles are contained in the union $\bigcup_{\pitchfork \in \mathcal{S}_N} \mathscr{L}_{\pitchfork,q}$. Similarly, Definition 1.6.1 and part (c) of Theorem 1.6.6 show that $Z_N^\rho(K, a, 0, \boldsymbol{s})$ coincides with the sum

$$\frac{q^{a+\sum_{i<j} s_{ij}}}{q^{N-1+a+\sum_{i<j} s_{ij}} - 1} \cdot \sum_{\pitchfork^* \in \mathcal{R}_N} M_{\pitchfork^*}(q) \cdot \prod_{\lambda \in \mathcal{B}(\pitchfork^*) \setminus \overline{\pitchfork}} \frac{1}{q^{e_\lambda(\boldsymbol{s})} - 1} \tag{1.7.16}$$

on a convex open region, and each summand is meromorphic in $\mathbb{C}^{\binom{N}{2}}$ with set of poles

$$\mathscr{B}_{\pitchfork^*,q} := \left\{ \boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}} : N - 1 + a + \sum_{i<j} s_{ij} \in \frac{2\pi i \mathbb{Z}}{\log(q)} \right\} \cup \bigcup_{\lambda \in \mathcal{B}(\pitchfork^*) \setminus \overline{\pitchfork}} \left\{ \boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}} : e_\lambda(\boldsymbol{s}) \in \frac{2\pi i \mathbb{Z}}{\log(q)} \right\}.$$

Therefore (1.7.16) defines the meromorphic continuation of $Z_N^\rho(K, a, 0, \boldsymbol{s})$ to $\mathbb{C}^{\binom{N}{2}}$, and its poles are contained in the union $\bigcup_{\pitchfork^* \in \mathcal{R}_N} \mathscr{B}_{\pitchfork^*,q}$. Though the poles of each summand in (1.7.15) and (1.7.16) are easily described, we saw in Example 1.7.3 that pole cancellation is possible when summands are brought together. As is true for general local zeta functions, determining precisely which of the poles in the candidate sets $\bigcup_{\pitchfork \in \mathcal{S}_N} \mathscr{L}_{\pitchfork,q}$ and $\bigcup_{\pitchfork^* \in \mathcal{R}_N} \mathscr{B}_{\pitchfork^*,q}$ cancel is a highly nontrivial task.

1.8. Outline of the remaining chapters

In Chapter 2 we will focus on the specialization of Theorem 1.6.6 to the log-Coulomb gas setting (i.e., where $s_{ij} = \mathfrak{q}_i \mathfrak{q}_j \beta$ for all $i < j$). After discussing the moments of the gas' diameter and minimum particle spacing, we will conclude with a section on grand canonical partition functions and the so-called *qth and*

$(q+1)$*th Power Laws*. The qth Power Law (for log-Coulomb gas in R) was proved recently by Sinclair [14], and the $(q+1)$th Power Law (for log-Coulomb gas in $\mathbb{P}^1(K)$) may be considered another of our main results. However, it requires a significantly shorter proof of the Main Theorem (Theorem 1.6.6), which is distributed throughout Chapters 3-5 as follows: Chapter 3 uses the levels of splitting chains prove parts (a) and (b) in the $\rho = \mathbf{1}_{[0,1]}$ case and establishes the first claim of Proposition 1.6.7 in the process. Chapter 4 uses the branches of splitting chains to prove Lemma 1.6.5 and part (c) and establishes the second claim in Proposition 1.6.7. It then concludes the proof of parts (a) and (b) for general ρ. Chapter 5 establishes a decomposition of $(\mathbb{P}^1(K))^N$ that leads to a proof of part (d) and the $(q+1)$th Power Law. Finally, the Appendix contains the full explicit computation of $Z_4^\rho(K, a, b, \boldsymbol{s})$, followed by a quadratic recurrence (in N) that allows for efficient computation of the canonical partition functions $Z_N(R, \beta)$ and $Z_N(\mathbb{P}^1(K), \beta)$.

CHAPTER II

CONSEQUENCES FOR LOG-COULOMB GAS IN K

2.1. The $s_{ij} = \mathfrak{q}_i\mathfrak{q}_j\beta$ specialization and one-component symmetries

Formulas for the multi-component p-field analogue of Mehta's Integral and the expected value in (1.5.13) are easily obtained by evaluating the formulas in Theorem 1.6.6 at special values of $\boldsymbol{s}$. With this in mind, we define several new items that are closely related to those in Definitions 1.6.1 and 1.6.4.

Definition 2.1.1. Suppose $a, b \in \mathbb{C}$ and $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_N > 0$ where $N \geq 2$, and let $\boldsymbol{c} := (\mathfrak{q}_i\mathfrak{q}_j)_{i<j}$.

(a) Define the *root abscissa* $\mathcal{RP}^{\mathbf{c}}_N(a,b)$ by

$$\mathcal{RP}^{\mathbf{c}}_N(a,b) := -\frac{N-1+\operatorname{Re}(a+b)}{\sum_{i<j}\mathfrak{q}_i\mathfrak{q}_j}.$$

(b) For each $\pitchfork \in \mathcal{S}_N$, define the *branch abscissa* $\mathcal{BP}^{\mathbf{c}}_{\pitchfork}$ by

$$\mathcal{BP}^{\mathbf{c}}_{\pitchfork} := -\inf_{\lambda\in\mathcal{B}(\pitchfork)\setminus\overline{\pitchfork}}\left\{\frac{\#\lambda-1}{\varepsilon_\lambda(\boldsymbol{c})}\right\} \qquad \text{where} \qquad \varepsilon_\lambda(\boldsymbol{c}) := \sum_{\substack{i<j\\ i,j\in\lambda}}\mathfrak{q}_i\mathfrak{q}_j.$$

(c) For each $\pitchfork \in \mathcal{S}_N$, define the *level abscissa* $\mathcal{LP}^{\mathbf{c}}_{\pitchfork}$ by

$$\mathcal{LP}^{\mathbf{c}}_{\pitchfork}(b) := -\inf_{1\leq \ell\leq L(\pitchfork)-1}\left\{\frac{\operatorname{rank}(\pitchfork_\ell)+\operatorname{Re}(b)}{\mathcal{E}_{\pitchfork_\ell}(\boldsymbol{c})}\right\} \quad \text{where} \quad \mathcal{E}_{\pitchfork_\ell}(\boldsymbol{c}) := \sum_{\lambda\in\pitchfork_\ell}\varepsilon_\lambda(\boldsymbol{c}).$$

If $\beta \in \mathbb{C}$ and $\boldsymbol{c}$ is defined as above, then Definitions 1.6.1, 1.6.3, 1.6.4, and 2.1.1 together imply

$$\begin{aligned} \beta\boldsymbol{c} \in \mathcal{RP}_N(a,b) \qquad &\Longleftrightarrow \qquad \mathrm{Re}(\beta) > \mathcal{RP}_N^{\mathbf{c}}(a,b), \\ \beta\boldsymbol{c} \in \mathcal{BP}_{\mathfrak{m}} \qquad &\Longleftrightarrow \qquad \mathrm{Re}(\beta) > \mathcal{BP}_{\mathfrak{m}}^{\mathbf{c}}, \\ \beta\boldsymbol{c} \in \mathcal{LP}_{\mathfrak{m}}(b) \qquad &\Longleftrightarrow \qquad \mathrm{Re}(\beta) > \mathcal{LP}_{\mathfrak{m}}^{\mathbf{c}}(b), \end{aligned}$$

and hence the convergence criteria for $\boldsymbol{s}$ in Theorem 1.6.6 become criteria for β when $\boldsymbol{s} = \beta\boldsymbol{c}$. The following corollary comes straight from this observation and Theorem 1.6.6:

Corollary 2.1.2. *Fix $N \geq 2$, $a, b \in \mathbb{C}$, a nonzero norm-density ρ, and $\boldsymbol{c} = (\mathfrak{q}_i\mathfrak{q}_j)_{i<j}$ where $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_N > 0$.*

(a) If K is any p-field, the integral $Z_N^\rho(K, a, b, \beta\boldsymbol{c})$ converges absolutely to

$$H_q^\rho\left(N + a + b + \sum_{i<j} \mathfrak{q}_i\mathfrak{q}_j\beta\right) \cdot \sum_{\mathfrak{m}\in\mathcal{S}_N} J_{\mathfrak{m},q}(b, \beta\boldsymbol{c})$$

if and only if

$$\mathrm{Re}(\beta) > \sup\left\{\mathcal{RP}_N^{\mathbf{c}}(a,b),\ \sup_{\mathfrak{m}\in\mathcal{S}_N} \mathcal{LP}_{\mathfrak{m}}^{\mathbf{c}}(b)\right\}.$$

(b) For the same K, if $b = 0$, the integral $Z_N^\rho(K, a, 0, \beta\boldsymbol{c})$ converges absolutely to

$$H_q^\rho\left(N + a + \sum_{i<j} \mathfrak{q}_i\mathfrak{q}_j\beta\right) \cdot \sum_{\mathfrak{m}^*\in\mathcal{R}_N} I_{\mathfrak{m}^*,q}(\beta\boldsymbol{c})$$

if and only if

$$\mathrm{Re}(\beta) > \sup\left\{\mathcal{RP}_N^{\mathbf{c}}(a,0),\ \sup_{\mathfrak{m}^*\in\mathcal{R}_N} \mathcal{BP}_{\mathfrak{m}^*}^{\mathbf{c}}\right\}.$$

Before concluding this section with formulas for the analogue of Mehta's integral and the expectation in (1.5.13), we remark on the one-component case, namely $\mathfrak{q}_1 = \mathfrak{q}_2 = \cdots = \mathfrak{q}_N = 1$. In this case $\boldsymbol{c} = \mathbf{1}$ is simply the $\binom{N}{2}$-tuple of 1's, and for each $\boldsymbol{\pitchfork} \in \mathcal{S}_N$ it is easily verified that

$$e_\lambda(\beta\mathbf{1}) = \#\lambda - 1 + \varepsilon_\lambda(\mathbf{1})\beta = \binom{\#\lambda}{2}\left(\beta + \frac{2}{\#\lambda}\right)$$

for all $\lambda \in \mathcal{B}(\boldsymbol{\pitchfork})$ and

$$E_{\pitchfork_\ell}(\beta\mathbf{1}) = \sum_{\lambda\in\pitchfork_\ell} e_\lambda(\beta\mathbf{1}) = \sum_{\lambda\in\pitchfork_\ell} \binom{\#\lambda}{2}\left(\beta + \frac{2}{\#\lambda}\right)$$

for all $\ell \in \{0, 1, \ldots, L(\boldsymbol{\pitchfork}) - 1\}$. The exponents above have no dependence on the particular labels $1, 2, \ldots, N$, so we shall take a moment to discuss a relationship between $\mathcal{S}_N$ and the symmetric group action on the label set $\{1, 2, \ldots, N\}$.

Definition 2.1.3. Denote the symmetric group on $[N] = \{1, 2, \ldots, N\}$ by $\mathrm{Sym}([N])$. Given $\sigma \in \mathrm{Sym}([N])$ and a nonempty subset $\lambda = \{i_1, i_2, \ldots, i_k\} \subset [N]$, we write $\sigma(\lambda) := \{\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_k)\}$, for a partition $\pitchfork = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of $[N]$ we write $\sigma(\pitchfork) := \{\sigma(\lambda_1), \sigma(\lambda_2), \ldots, \sigma(\lambda_n)\}$, and finally, for each $\boldsymbol{\pitchfork} = (\pitchfork_0, \pitchfork_1, \ldots, \pitchfork_{L(\boldsymbol{\pitchfork})}) \in \mathcal{S}_N$ we write $\sigma(\boldsymbol{\pitchfork}) := (\sigma(\pitchfork_0), \sigma(\pitchfork_1), \ldots, \sigma(\pitchfork_{L(\boldsymbol{\pitchfork})}))$.

If $\mathrm{Aut}(\mathcal{S}_N)$ denotes the group of bijections $\mathcal{S}_N \to \mathcal{S}_N$, the homomorphism $\mathrm{Sym}([N]) \to \mathrm{Aut}(\mathcal{S}_N)$ given by $\sigma \mapsto (\boldsymbol{\pitchfork} \mapsto \sigma(\boldsymbol{\pitchfork}))$ is an action of $\mathrm{Sym}([N])$ on $\mathcal{S}_N$. The following properties of this action are clear from Definitions 1.6.3 and 1.6.4: If $\boldsymbol{\pitchfork} \in \mathcal{S}_N$ and $\sigma \in \mathrm{Sym}([N])$, then

– $L(\sigma(\boldsymbol{\pitchfork})) = L(\boldsymbol{\pitchfork})$, and $\sigma(\boldsymbol{\pitchfork}) = \boldsymbol{\pitchfork}$ if and only if $\sigma(\pitchfork_\ell) = \pitchfork_\ell$ for all $\ell \in \{0, 1, \ldots, L(\boldsymbol{\pitchfork})\}$,

– $\sigma(\lambda) \in \mathcal{B}(\sigma(\pitchfork))$ if and only if $\lambda \in \mathcal{B}(\pitchfork)$,

– for each $\lambda \in \mathcal{B}(\pitchfork)$ we have $\#\sigma(\lambda) = \#\lambda$, $\ell_{\sigma(\pitchfork)}(\sigma(\lambda)) = \ell_{\pitchfork}(\lambda)$, and $\deg_{\sigma(\pitchfork)}(\sigma(\lambda)) = \deg_{\pitchfork}(\lambda)$, so

– $M_{\sigma(\pitchfork)}(t) = M_{\pitchfork}(t)$, $e_{\sigma(\lambda)}(\beta\mathbf{1}) = e_{\lambda}(\beta\mathbf{1})$ for all $\lambda \in \mathcal{B}(\pitchfork)$, and hence $E_{\sigma(\pitchfork_\ell)}(\beta\mathbf{1}) = E_{\pitchfork_\ell}(\beta\mathbf{1})$ for all $\ell \in \{0, 1, \ldots, L(\pitchfork) - 1\}$.

In particular, the $\mathrm{Sym}([N])$ action on $\mathcal{S}_N$ restricts to one on $\mathcal{R}_N$.

Definition 2.1.4. For each splitting chain $\pitchfork \in \mathcal{S}_N$, define the *orbit* by $\mathrm{Orb}(\pitchfork) := \{\sigma(\pitchfork) : \sigma \in \mathrm{Sym}([N])\}$, the *stabilizer* by $\mathrm{Stab}(\pitchfork) := \{\sigma \in \mathrm{Sym}([N]) : \sigma(\pitchfork) = \pitchfork\}$, and the *weight* by $W(\pitchfork) := \#\,\mathrm{Orb}(\pitchfork) = \frac{N!}{\#\,\mathrm{Stab}(\pitchfork)}$.

Definitions 1.6.4 and 2.1.4 and the properties of the action immediately imply the following:

Lemma 2.1.5. *Suppose $q \geq 2$, $b \in \mathbb{C}$, and $\pitchfork \in \mathcal{S}_N$.*

(a) For each β in the domain of $\beta \mapsto I_{\pitchfork,q}(\beta\mathbf{1})$ we have

$$\sum_{\pitchfork' \in \mathrm{Orb}(\pitchfork)} I_{\pitchfork',q}(\beta\mathbf{1}) = W(\pitchfork) I_{\pitchfork,q}(\beta\mathbf{1}) = \frac{W(\pitchfork)(q-1)_{\deg_{\pitchfork}([N])-1}}{q^{N-1}} \cdot \prod_{\lambda \in \mathcal{B}(\pitchfork) \setminus \overline{\pitchfork}} \frac{(q-1)_{\deg_{\pitchfork}(\lambda)-1}}{q^{\binom{\#\lambda}{2}\left(\beta + \frac{2}{\#\lambda}\right)} - 1}.$$

(b) For each β in the domain of $\beta \mapsto J_{\pitchfork,q}(b, \beta\mathbf{1})$ we have

$$\sum_{\pitchfork' \in \mathrm{Orb}(\pitchfork)} J_{\pitchfork',q}(b, \beta\mathbf{1}) = W(\pitchfork) J_{\pitchfork,q}(b, \beta\mathbf{1}) = \frac{W(\pitchfork) M_{\pitchfork}(q)}{q^{N-1}} \cdot \prod_{\ell=1}^{L(\pitchfork)-1} \frac{1}{q^{b + \sum_{\lambda \in \pitchfork_\ell} \binom{\#\lambda}{2}\left(\beta + \frac{2}{\#\lambda}\right)} - 1}.$$

If $\mathcal{C}_N \subset \mathcal{S}_N$ is a complete set of orbit representatives for the action of $\mathrm{Sym}([N])$ on $\mathcal{S}_N$, then $\mathcal{C}_N \cap \mathcal{R}_N$ is a complete set of orbit representatives for the restricted action on $\mathcal{R}_N$. Then by part (a) of Lemma 2.1.5, the sum over $\pitchfork \in \mathcal{S}_N$ appearing in the main formula for $Z_N^\rho(K, a, b, \beta\mathbf{1})$ can be grouped into a weighted sum over $\mathcal{C}_N$. Similarly, part (b) of Lemma 2.1.5 implies that the sum over $\pitchfork^* \in \mathcal{R}_N$ in the formula for $Z_N^\rho(K, a, 0, \beta\mathbf{1})$ can be grouped into a weighted sum over $\mathcal{C}_N \cap \mathcal{R}_N$. From the viewpoint of log-Coulomb gas, the appearance of these weighted sums has an intuitive explanation: The condition $\mathfrak{q}_1 = \mathfrak{q}_2 = \cdots = \mathfrak{q}_N = 1$ makes the particles of the gas identical and imposes symmetries on the set of microstates $x \in K^N$. Each $\pitchfork \in \mathcal{C}_N$ represents a distinct symmetry class of microstates, the factor $\frac{W(\pitchfork)M_{\pitchfork}(q)}{q^{N-1}}$ can be regarded as its weight, and the two products of rational functions of $q^{-\beta}$ appearing in Lemma 2.1.5 are its respective contributions to the functions $\beta \mapsto Z_N^\rho(K, a, 0, \beta\mathbf{1})$ and $\beta \mapsto Z_N^\rho(K, a, b, \beta\mathbf{1})$. In particular, each symmetry class contributes a weighted term to the canonical partition function $\beta \mapsto Z_N^\rho(K, 0, 0, \beta\mathbf{1})$. It is also worth noting that the condition on $\mathrm{Re}(\beta)$ in part (b) of Corollary 2.1.2 simplifies further when $a = b = 0$ and $\boldsymbol{c} = \mathbf{1}$. Indeed, for general $\boldsymbol{c} = (\mathfrak{q}_i\mathfrak{q}_j)_{i<j}$ we have

$$\sup\left\{\mathcal{RP}_N^{\mathbf{c}}(0,0),\ \sup_{\pitchfork^* \in \mathcal{R}_N} \mathcal{BP}_{\pitchfork^*}^{\mathbf{c}}\right\} = -\inf_{\pitchfork^* \in \mathcal{R}_N}\left\{\inf_{\lambda \in \mathcal{B}(\pitchfork^*)}\left\{\frac{\#\lambda - 1}{\sum_{\substack{i<j \\ i,j \in \lambda}} \mathfrak{q}_i\mathfrak{q}_j}\right\}\right\}, \tag{2.1.1}$$

so if $\pitchfork^* \in \mathcal{R}_N$ and $\boldsymbol{c} = \mathbf{1}$ we have

$$\frac{\#\lambda - 1}{\sum_{\substack{i<j \\ i,j \in \lambda}} \mathfrak{q}_i\mathfrak{q}_j} = \frac{\#\lambda - 1}{\binom{\#\lambda}{2}} = \frac{2}{\#\lambda} \qquad \text{for all } \lambda \in \mathcal{B}(\pitchfork^*).$$

Thus if $\boldsymbol{c} = \mathbf{1}$, the inner infima in (2.1.1) are all $\frac{2}{N}$, so the quantity in (2.1.1) is simply $-\frac{2}{N}$. This fact and Lemma 2.1.5 yield the p-field analogue of Mehta's integral formula:

Theorem 2.1.6 (Mehta's integral formula for p-fields)**.** *Suppose K is a p-field, suppose ρ is a nonzero norm-density, let $\boldsymbol{c} = (\mathfrak{q}_i\mathfrak{q}_j)_{i<j}$ where $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_N > 0$, and consider the generalized Mehta Integral:*

$$Z_N^\rho(K, 0, 0, \beta\boldsymbol{c}) = \int_{K^N} \rho(\|x\|) \prod_{i<j} |x_i - x_j|^{\mathfrak{q}_i\mathfrak{q}_j\beta}\, dx_1 \ldots dx_N$$

(a) The integral converges absolutely if and only if

$$\mathrm{Re}(\beta) > -\inf_{\mathfrak{m}^* \in \mathcal{R}_N} \left\{ \inf_{\lambda \in \mathcal{B}(\mathfrak{m}^*)} \left\{ \frac{\#\lambda - 1}{\sum_{\substack{i<j \\ i,j \in \lambda}} \mathfrak{q}_i\mathfrak{q}_j} \right\} \right\},$$

and in this case it converges to

$$(q^{\sum_{i<j} \mathfrak{q}_i\mathfrak{q}_j\beta} - q^{-N}) \cdot \sum_{m \in \mathbb{Z}} \rho(q^m) q^{m(N + \sum_{i<j} \mathfrak{q}_i\mathfrak{q}_j\beta)} \cdot \sum_{\mathfrak{m}^* \in \mathcal{R}_N} \prod_{\lambda \in \mathcal{B}(\mathfrak{m}^*)} \frac{(q-1)_{\deg_{\mathfrak{m}^*}(\lambda)-1}}{q^{e_\lambda(\beta\boldsymbol{c})} - 1}.$$

(b) In particular, if $\mathfrak{q}_1 = \cdots = \mathfrak{q}_N = 1$, the integral converges absolutely if and only if $\mathrm{Re}(\beta) > -\frac{2}{N}$. In this case it converges to

$$(q^{\binom{N}{2}\beta} - q^{-N}) \cdot \sum_{m \in \mathbb{Z}} \rho(q^m) q^{m(N + \binom{N}{2}\beta)} \cdot \sum_{\mathfrak{m}^* \in \mathcal{C}_N \cap \mathcal{R}_N} W(\mathfrak{m}^*) \prod_{\lambda \in \mathcal{B}(\mathfrak{m}^*)} \frac{(q-1)_{\deg_{\mathfrak{m}^*}(\lambda)-1}}{q^{\binom{\#\lambda}{2}\left(\beta + \frac{2}{\#\lambda}\right)} - 1}$$

where $\mathcal{C}_N \subset \mathcal{S}_N$ is a set of orbit representatives the $\mathrm{Sym}([N])$ *action on $\mathcal{S}_N$.*

If the norm-density ρ above is not identically zero and nonnegative, then $Z_N^\rho(K, 0, 0, \beta\boldsymbol{c}) \in (0, \infty)$ for all $\beta > 0$ and $\frac{1}{Z_N^\rho(K,0,0,\beta\boldsymbol{c})} \rho(\|x\|) \prod_{i<j} |x_i - x_j|^{\mathfrak{q}_i\mathfrak{q}_j\beta}$ is a

well-defined probability density on the microstates $x \in K^N$. Moreover, none of the abscissae in Definition 2.1.1 are positive if both $\mathrm{Re}(b) \geq -1$ and $\mathrm{Re}(a+b) \geq 1-N$, in which case the conditions on $\mathrm{Re}(\beta)$ in Corollary 2.1.2 are met by all $\beta > 0$. This observation and (1.5.13) imply the following corollary:

Corollary 2.1.7. *Suppose K is a p-field, suppose ρ is a norm-density, and let $\boldsymbol{c} = (\mathfrak{q}_i\mathfrak{q}_j)_{i<j}$ where $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_N > 0$.*

(a) If $\mathrm{Re}(b) \geq -1$ and $\mathrm{Re}(a+b) \geq 1-N$, then for any inverse temperature $\beta > 0$ we have

$$\mathbb{E}\left[\left(\max_{i<j}|x_i - x_j|\right)^a\left(\min_{i<j}|x_i - x_j|\right)^b\right]$$
$$= \frac{H_q^\rho\left(N + a + b + \sum_{i<j}\mathfrak{q}_i\mathfrak{q}_j\beta\right)\cdot\sum_{\mathfrak{m}^*\in\mathcal{S}_N} J_{\mathfrak{m},q}(b, \beta\boldsymbol{c})}{H_q^\rho\left(N + \sum_{i<j}\mathfrak{q}_i\mathfrak{q}_j\beta\right)\cdot\sum_{\mathfrak{m}^*\in\mathcal{S}_N} J_{\mathfrak{m},q}(0, \beta\boldsymbol{c})}$$
$$= \frac{H_q^\rho\left(N + a + b + \sum_{i<j}\mathfrak{q}_i\mathfrak{q}_j\beta\right)\cdot\sum_{\mathfrak{m}^*\in\mathcal{S}_N} J_{\mathfrak{m},q}(b, \beta\boldsymbol{c})}{H_q^\rho\left(N + \sum_{i<j}\mathfrak{q}_i\mathfrak{q}_j\beta\right)\cdot\sum_{\mathfrak{m}^*\in\mathcal{R}_N} I_{\mathfrak{m}^*,q}(\beta\boldsymbol{c})}.$$

(b) In particular, if $b = 0$ and $\mathrm{Re}(a) \geq 1-N$, then for any inverse temperature $\beta > 0$ we have

$$\mathbb{E}\left[\left(\max_{i<j}|x_i - x_j|\right)^a\right] = \frac{H_q^\rho\left(N + a + \sum_{i<j}\mathfrak{q}_i\mathfrak{q}_j\beta\right)}{H_q^\rho\left(N + \sum_{i<j}\mathfrak{q}_i\mathfrak{q}_j\beta\right)}.$$

As we mentioned in Section 1.5, applying part (a) of Corollary 2.1.7 to $a, b \in \mathbb{Z}_{\geq 0}$ gives the joint moments of the random variables $\max_{i<j}|x_i - x_j|$ and $\min_{i<j}|x_i - x_j|$. In particular, the average value in part (b) of Corollary 2.1.7

can be computed without the use of branch or level functions, and thus admits a simple closed form for suitably chosen ρ. The next example demonstrates this and addresses the low-temperature limit (i.e., $\beta \to \infty$) in the $b = 0$ case.

Example 2.1.8. Recall that $\|K^N \setminus \{0\}\| = q^{\mathbb{Z}}$ and let ρ be the norm-density defined by $\rho(t) = \mathbf{1}_{[0,q^M]}(t)$ where $M \in \mathbb{Z}$. Since $\rho(\|x\|) = 1$ if and only if all x_i are in the ball $\pi^{-M}R = \{y \in K : |y| \leq q^M\}$ and otherwise $\rho(\|x\|) = 0$, ρ guarantees that the charges are almost surely confined to this ball, and by Definition 1.6.1 we have

$$H_q^{\rho}(z) = \frac{1 - q^{-z}}{1 - q^{-(z-1)}} \cdot \sum_{m=-M}^{\infty} (q^{-z})^m = \frac{q^{Mz}}{1 - q^{-(z-1)}} \qquad \text{for } \operatorname{Re}(z) > 1.$$

Then for $\operatorname{Re}(a) \geq 1 - N$ part (b) of Corollary 2.1.7 gives the explicit formula

$$\mathbb{E}\left[\left(\max_{i<j} |x_i - x_j|\right)^a\right] = \frac{\left(\frac{q^{M(N+a+\sum_{i<j} \mathfrak{q}_i\mathfrak{q}_j\beta)}}{1-q^{-(N-1+a+\sum_{i<j} \mathfrak{q}_i\mathfrak{q}_j\beta)}}\right)}{\left(\frac{q^{M(N+\sum_{i<j} \mathfrak{q}_i\mathfrak{q}_j\beta)}}{1-q^{-(N-1+\sum_{i<j} \mathfrak{q}_i\mathfrak{q}_j\beta)}}\right)} = q^{Ma} \cdot \frac{q^{N-1+\sum_{i<j} \mathfrak{q}_i\mathfrak{q}_j\beta} - 1}{q^{N-1+\sum_{i<j} \mathfrak{q}_i\mathfrak{q}_j\beta} - q^{-a}},$$

from which the following asymptotic estimate is clear:

$$\mathbb{E}\left[\left(\max_{i<j} |x_i - x_j|\right)^a\right] \sim q^{Ma} \qquad \text{as } N \to \infty \text{ or } \beta \to \infty.$$

(By taking $N \to \infty$, we are assuming here that a charge $\mathfrak{q}_i > 0$ has been specified for every $i \in \mathbb{N}$.) Since $\max_{i<j} |x_i - x_j| \leq q^M$ almost surely, this estimate implies that a gas comprised of many particles and/or held at a low temperature has a relatively high probability of attaining microstates $x \in K^N$ with $\max_{i<j} |x_i - x_j| = q^M$. Roughly speaking, this says the gas is very likely to spread out as widely as possible if it is cold and/or if it has many particles.

Remark 2.1.9. The previous example hints at a more general feature of low-temperature limits: Suppose ρ is a compactly supported nonzero norm-density. There is a greatest $M \in \mathbb{Z}$ for which $\rho(q^M) \neq 0$, so given $\delta > 1$ the scaled sum $\frac{H_q^\rho(z)}{q^{Mz}} = \frac{1-q^{-z}}{1-q^{-(z-1)}} \cdot \sum_{m=-M}^{\infty} \rho(q^{-m}) q^{-(m+M)z}$ converges uniformly for $\mathrm{Re}(z) \geq \delta$ by (1.5.12). Therefore we may take $z \to \infty$ term-by-term to obtain $\lim_{z\to\infty} \frac{H_q^\rho(z)}{q^{Mz}} = \rho(q^M)$, and the ratio of root functions in part (a) of Corollary 2.1.7 satisfies

$$\lim_{\beta\to\infty} \frac{H_q^\rho\big(N + a + b + \sum_{i<j} \mathfrak{q}_i\mathfrak{q}_j\beta\big)}{H_q^\rho\big(N + \sum_{i<j} \mathfrak{q}_i\mathfrak{q}_j\beta\big)} = \lim_{\beta\to\infty} \frac{q^{M(a+b)} \cdot \frac{H_q^\rho\left(N+a+b+\sum_{i<j} \mathfrak{q}_i\mathfrak{q}_j\beta\right)}{q^{M(N+a+b+\sum_{i<j} \mathfrak{q}_i\mathfrak{q}_j\beta)}}}{\frac{H_q^\rho\left(N+\sum_{i<j} \mathfrak{q}_i\mathfrak{q}_j\beta\right)}{q^{M(N+\sum_{i<j} \mathfrak{q}_i\mathfrak{q}_j\beta)}}} = q^{M(a+b)}. \tag{2.1.2}$$

The ratio of sums in part (a) of Corollary 2.1.7 also converges as $\beta \to \infty$. More precisely, for each $\pitchfork \in \mathcal{S}_N$, define

$$Q_{\pitchfork}(\boldsymbol{c}) := \sum_{\ell=1}^{L(\pitchfork)-1} \mathcal{E}_{\pitchfork_\ell}(\boldsymbol{c}) = \sum_{\ell=1}^{L(\pitchfork)-1} \sum_{\lambda\in\pitchfork_\ell} \sum_{\substack{i,j\in\lambda \\ i<j}} \mathfrak{q}_i\mathfrak{q}_j.$$

For any $q \geq 2$, the set $\mathcal{S}_{N,q} = \{\pitchfork \in \mathcal{S}_N : M_{\pitchfork}(q) > 0\}$ contains the splitting chain

$$\pitchfork = ([N],\ [N-1]\{N\},\ [N-2]\{N-1\}\{N\},\ \ldots, \{1\}\{2\}\ldots\{N\}),$$

so $\mathcal{S}_{N,q} \neq \varnothing$ and hence a non-negative minimum $Q_{N,q}^{\min}(\boldsymbol{c}) := \min\{Q_{\pitchfork}(\boldsymbol{c}) : \pitchfork \in \mathcal{S}_{N,q}\}$ exists. Taking $\beta \to \infty$ gives

$$\begin{aligned} J_{\pitchfork,q}(b, \beta\boldsymbol{c}) &= \frac{M_{\pitchfork}(q)}{q^{N-1}} \cdot \prod_{\ell=1}^{L(\pitchfork)-1} \frac{1}{q^{b+E_{\pitchfork_\ell}(\beta\boldsymbol{c})} - 1} \\ &\sim \frac{M_{\pitchfork}(q)}{q^{N-1}} \cdot q^{-\sum_{\ell=1}^{L(\pitchfork)-1}(b+E_{\pitchfork_\ell}(\beta\boldsymbol{c}))} = \frac{M_{\pitchfork}(q)}{q^{N-1+\sum_{\ell=1}^{L(\pitchfork)-1}(b+\mathrm{rank}(\pitchfork_\ell))}} \cdot q^{-\beta Q_{\pitchfork}(\boldsymbol{c})}, \end{aligned}$$

and therefore

$$\sum_{\pitchfork\in\mathcal{S}_N} J_{\pitchfork,q}(b,\beta\boldsymbol{c}) \sim \sum_{\substack{\pitchfork\in\mathcal{S}_{N,q}\\ Q_\pitchfork(\boldsymbol{c})=Q_N^{\min}(\boldsymbol{c})}} \frac{M_\pitchfork(q)}{q^{N-1+\sum_{\ell=1}^{L(\pitchfork)-1}(b+\operatorname{rank}(\pitchfork_\ell))}} \cdot q^{-\beta Q_{N,q}^{\min}(\boldsymbol{c})}.$$

The factors $q^{-(N-1)}$ and $q^{-\beta Q_{N,q}^{\min}(\boldsymbol{c})}$ are independent of b and common to all terms in the right-hand sum, so we may abbreviate the above summation by $\sum'$ and obtain

$$\lim_{\beta\to\infty} \frac{\sum_{\pitchfork\in\mathcal{S}_N} J_{\pitchfork,q}(b,\beta\boldsymbol{c})}{\sum_{\pitchfork\in\mathcal{S}_N} J_{\pitchfork,q}(0,\beta\boldsymbol{c})} = \frac{\sum' M_\pitchfork(q) q^{-\sum_{\ell=1}^{L(\pitchfork)-1}(\operatorname{rank}(\pitchfork_\ell)+b)}}{\sum' M_\pitchfork(q) q^{-\sum_{\ell=1}^{L(\pitchfork)-1}\operatorname{rank}(\pitchfork_\ell)}}. \tag{2.1.3}$$

Combining this with part (a) of Corollary 2.1.7 and (2.1.2) gives the low-temperature limit of any joint moment:

$$\lim_{\beta\to\infty} \mathbb{E}\left[\left(\max_{i<j}|x_i-x_j|\right)^a \left(\min_{i<j}|x_i-x_j|\right)^b\right] = q^{M(a+b)} \cdot \frac{\sum' M_\pitchfork(q) q^{-\sum_{\ell=1}^{L(\pitchfork)-1}(\operatorname{rank}(\pitchfork_\ell)+b)}}{\sum' M_\pitchfork(q) q^{-\sum_{\ell=1}^{L(\pitchfork)-1}\operatorname{rank}(\pitchfork_\ell)}}. \tag{2.1.4}$$

Explicit computation of (2.1.4) is generally impractical as it depends on N, q, and $\boldsymbol{c}$ in very complicated ways. Still, it is interesting that the ratio of sums in (2.1.4) is a weighted average of the finite set of values

$$\{q^{-b(L(\pitchfork)-1)} : \pitchfork \in \mathcal{S}_N \text{ with } M_\pitchfork(q) > 0 \text{ and } Q_\pitchfork(\boldsymbol{c}) = Q_{N,q}^{\min}(\boldsymbol{c})\},$$

with each weight $M_\pitchfork(q) q^{-\sum_{\ell=1}^{L(\pitchfork)-1}\operatorname{rank}(\pitchfork_\ell)}$ independent of a, b, and ρ. Moreover, if $q \geq N$, then the splitting chain $\pitchfork = ([N],\ \{1\}\{2\}\ldots\{N\}) \in \mathcal{S}_N$ has $M_\pitchfork(q) = (q-1)_{N-1} > 0$ and $Q_\pitchfork(\boldsymbol{c}) = Q_{N,q}^{\min}(\boldsymbol{c}) = 0$, and in fact it is the only one satisfying $Q_\pitchfork(\boldsymbol{c}) = Q_{N,q}^{\min}(\boldsymbol{c})$. Therefore $\lim_{\beta\to\infty}\sum_{\pitchfork\in\mathcal{S}_N} J_{\pitchfork,q}(b,\beta\boldsymbol{c}) = (q-1)_{N-1} > 0$ whenever $q \geq N$ and we obtain a final corollary:

Corollary 2.1.10. *Suppose K is a p-field such that $q \geq N$ and suppose* $\mathrm{Re}(b) \geq -1$ *and* $\mathrm{Re}(a+b) \geq 1-N$*. Then if ρ is a compactly supported nonzero norm-density and M is the largest integer satisfying $\rho(q^M) \neq 0$, we have*

$$\lim_{\beta \to \infty} \mathbb{E}\left[\big(\max_{i<j} |x_i - x_j|\big)^a \big(\min_{i<j} |x_i - x_j|\big)^b\right] = q^{M(a+b)}.$$

2.2. Grand canonical partition functions and the Power Laws

So far, we have only considered log-Coulomb gases with N labeled (and hence distinguishable) particles. Our second main result concerns the situation in which all particles are identical with charge $\mathfrak{q}_i = 1$ for all i, in which case the microstates $(x_1, \ldots, x_N) \in X^N$ are "unlabeled" and hence unique only up to permutations of their entries. Since the energy $E(x_1, \ldots, x_N)$ and measure on X^N are invariant under such permutations, each unlabeled microstate makes the contribution $e^{-\beta E(x_1,\ldots,x_N)} dx_1 \ldots dx_N$ to the integral $Z_N(X, \beta)$ in (1.2.2) precisely $N!$ times. Therefore the canonical partition function for the unlabeled microstates is given by $Z_N(X,\beta)/N!$. We further assume that the system exchanges particles with the heat reservoir with chemical potential μ and define the *fugacity parameter* $f = e^{\mu\beta}$. In this situation the particle number $N \geq 0$ is treated as a random variable and the canonical partition function is replaced by the *grand canonical partition function*

$$Z(f, X, \beta) := \sum_{N=0}^{\infty} Z_N(X, \beta) \frac{f^N}{N!} \tag{2.2.5}$$

with the familiar convention $Z_0(X, \beta) = 1$. Many properties of the system can be deduced from the grand canonical partition function. For instance, if $\beta > 0$ is fixed and $Z_N(X, \beta)$ is sub-exponential in N, then $Z(f, X, \beta)$ is analytic in f and the expected number of particles in the system is given by $f \frac{\partial}{\partial f} \ln(Z(f, X, \beta))$. The canonical partition function for each $N \geq 0$ can also be recovered by evaluating the Nth derivative of $Z(f, X, \beta)$ with respect to f at $f = 0$.

We are interested in the examples $Z(f, R, \beta)$, $Z(f, P, \beta)$, and $Z(f, \mathbb{P}^1(K), \beta)$, which turn out to share several common properties and interesting relationships. By setting $s_{ij} = \beta$ in Definition 1.5.3, one sees that $|Z_N(R, \beta)|_{\mathbb{C}}$, $|Z_N(P, \beta)|_{\mathbb{C}}$, and $|Z_N(\mathbb{P}^1(K), \beta)|_{\mathbb{C}}$ are bounded above by 1 for all $N \geq 0$ and all $\beta > 0$, and hence $Z(f, R, \beta)$, $Z(f, P, \beta)$, and $Z(f, \mathbb{P}^1(K), \beta)$ are analytic in f when $\beta > 0$. Sinclair recently found an elegant relationship between the first two, which is closely related to the partition of R into cosets of P (as in (1.3.9)):

Proposition 2.2.1 (The qth Power Law [14]). *For $\beta > 0$ we have*

$$Z(f, R, \beta) = (Z(f, P, \beta))^q.$$

Roughly speaking, the qth Power Law states that a log-Coulomb gas in R exchanging energy and particles with a heat reservoir "factors" into q identical sub-gases (one in each coset of P) that exchange energy and particles with the reservoir. For $\beta > 0$, note that the series equation $Z(f, R, \beta) = (Z(f, P, \beta))^q$ is equivalent to the coefficient identities

$$\frac{Z_N(R, \beta)}{N!} = \sum_{\substack{N_0 + \cdots + N_{q-1} = N \\ N_0, \ldots, N_{q-1} \geq 0}} \prod_{k=0}^{q-1} \frac{Z_{N_k}(P, \beta)}{N_k!} \qquad \text{for all } \beta > 0 \text{ and } N \geq 0. \tag{2.2.6}$$

The $\beta = 1$ case of (2.2.6) is given in [2], in which the positive number $Z_N(R, 1)/N!$ is recognized as the probability that a random monic polynomial in $R[x]$ splits completely in R. The more general $\beta > 0$ case given in [14] makes explicit use of the partition of R into cosets of P (as in (1.3.9)). In Chapter 5 we will use the analogous partition of $\mathbb{P}^1(K)$ into $q+1$ balls (recall (1.3.10)) to show that

$$\frac{Z_N(\mathbb{P}^1(K),\beta)}{N!} = \sum_{\substack{N_0+\cdots+N_q=N \\ N_0,\ldots,N_q\geq 0}} \prod_{k=0}^{q} \left(\frac{q}{q+1}\right)^{N_k} \frac{Z_{N_k}(P,\beta)}{N_k!} \qquad \text{for all } \beta>0 \text{ and } N\geq 0, \tag{2.2.7}$$

which immediately implies our second main result:

Theorem 2.2.2 (The $(q+1)$th Power Law)**.** *For all $\beta > 0$ we have*

$$Z(f,\mathbb{P}^1(K),\beta) = (Z(\tfrac{qf}{q+1},P,\beta))^{q+1}.$$

Like the qth Power Law, the $(q+1)$th Power Law roughly states that a log-Coulomb gas in $\mathbb{P}^1(K)$ exchanging energy and particles with a heat reservoir "factors" into $q+1$ identical sub-gases in the balls $B_1[0:1]$, $B[1:1]$, $B[\xi:1]$, $\ldots$, $B[\xi^{q-2}:1]$, $B_1[1:0]$ (all of which are homeomorphic to P), with fugacity $\frac{qf}{q+1}$. As a final note, The qth Power Law also allows the $(q+1)$th Power Law to be written more crudely as

$$Z(f,\mathbb{P}^1(K),\beta) = Z(\tfrac{qf}{q+1},R,\beta)\cdot Z(\tfrac{qf}{q+1},P,\beta), \tag{2.2.8}$$

which is to say that the gas in $\mathbb{P}^1(K)$ "factors" into two sub-gases: one in $\iota(R)$ and one in $B[1:0]$ (which are respectively homeomorphic to R and P), both with fugacity $\frac{qf}{q+1}$.

CHAPTER III

SERIES REPRESENTATIONS, TREES, AND LEVEL PAIRS

From now on K will be a fixed p-field with μ, $|\cdot|$, $\|\cdot\|$, R, P, and π as defined in Section 1.3, and D will stand for any fixed set of representatives for the cosets of P in R (such as $D = \{0\} \cup U_{q-1}$). The results in this chapter will depend largely on the following proposition, which is a straightforward consequence of Theorem 1.3.1:

Proposition 3.0.1. *For each $x \in R$ there is a unique sequence $(d(0), d(1), d(2), \dots)$ in D such that*

$$x = \sum_{n=0}^{\infty} \pi^n d(n).$$

It converges absolutely with respect to $|\cdot|$ and satisfies $v(x) = \inf\{n : d(n) \neq 0\}$. If $(d'(0), d'(1), d'(2), \dots)$ is the corresponding sequence for another element $y \in R$, then we have $v(x-y) = \inf\{n : d(n) \neq d'(n)\}$, and the following are equivalent:

(i) $|x - y| \leq q^{-m}$,

(ii) $v(x - y) \geq m$,

(iii) $\inf\{n : d(n) \neq d'(n)\} \geq m$,

(iv) $x \equiv y \mod \pi^m$.

Moreover, for each m, the collection of partial sums $\{\sum_{n=0}^{m-1} \pi^n d(n) : d(n) \in D\}$ is a full set of representatives for the quotient $R/\pi^m R$.

In Section 3.1 we will use Proposition 3.0.1 to explain how elements of R^N may be visualized as trees, leading to a relationship with splitting chains in Section 3.2. This will allow us to express certain integrals in terms of level functions in Proposition 3.3.3 in Section 3.3. We conclude this chapter with the proof of Proposition 3.3.4, which implies parts (a) and (b) of Theorem 1.6.6 in the special case $\rho = \mathbf{1}_{[0,1]}$.

3.1. Series representations and trees

Fix an integer $N \geq 2$ and henceforth write x (and y, z, etc.) for tuples $(x_1, \dots, x_N) \in R^N$. With Proposition 3.0.1 in hand, our next task is to give a consistent method for visualizing and organizing the elements of $R^N \setminus V_0$, where $V_0 := \{x \in R^N : x_i = x_j \text{ for some } i < j\}$. Given $x = (x_1, \dots, x_N) \in R^N$, Proposition 3.0.1 provides a unique sequence $(d_i(0), d_i(1), d_i(2), \dots)$ in D satisfying $x_i = \sum_{n=0}^{\infty} \pi^n d_i(n)$ for each entry x_i. This gives a unique series representation for x, namely

$$x = \sum_{n=0}^{\infty} \pi^n d(n) \quad \text{ where } \quad d(n) = (d_1(n), d_2(n), \dots, d_N(n)) \in D^N,$$

and this series converges absolutely in R^N. Moreover, given $m \in \mathbb{N}$, the set of finite sums $\{\sum_{n=0}^{m-1} \pi^n d(n) : d(n) \in D^N\}$ is a complete set of representatives for the quotient $R^N/\pi^m R^N$, so we will abuse notation and write

$$R^N/\pi^m R^N = \left\{ \sum_{n=0}^{m-1} \pi^n d(n) : d(n) \in D^N \right\}.$$

Given $x = \sum_{n=0}^{\infty} \pi^n d(n) \in R^N$ and $m \in \mathbb{N}$, it is clear that the unique elements $y \in R^N/\pi^m R^N$ and $z \in \pi^m R^N$ satisfying $x = y + z$ are respectively $y = \sum_{n=0}^{m-1} \pi^n d(n)$ and $z = \sum_{n=m}^{\infty} \pi^n d(n)$. Our next definition makes use of this and the following observation: $x \in R^N \setminus V_0$ if and only if $x \in R^N$ and $\sup_{i<j} v(x_i - x_j) < \infty$.

Definition 3.1.1. We call an element $y \in R^N \setminus V_0$ a *tree* of length $m \in \mathbb{N}$ if $y \in R^N/\pi^m R^N$ and $\max_{i<j} v(y_i - y_j) + 1 = m$.

Given $x = \sum_{n=0}^{\infty} \pi^n d(n) \in R^N \setminus V_0$ with $m = \max_{i<j} v(x_i - x_j) + 1$, note that $y = \sum_{n=0}^{m-1} \pi^n d(n)$ is the unique partial sum of x satisfying Definition 3.1.1, so y will accordingly be called the *tree part* of x. The reason for the name "tree" is clarified by the next example, which will be revisited in later proofs.

Example 3.1.2. Suppose $N = 9$ and $K = \mathbb{Q}_5$ with uniformizer $\pi = 5$ and digit set $D = \{0, 1, 2, 3, 4\}$. The tree $y = \sum_{n=0}^{7} 5^n d(n)$ corresponding to the digit vectors $d(0), d(1), \ldots, d(7)$ in Figure 1 can be visualized as a rooted tree. The root represents the value 0, and the nodes traversed by the path from the root down to the leaf y_i represent the consecutive partial sums of $y_i = \sum_{n=0}^{7} 5^n d_i(n)$. It should be noted that for general trees $y \in R^N \setminus V_0$, the corresponding diagram need not have y_i in index order at the bottom. The tree in this example was only chosen this way only to make the diagram in Figure 1 easily discernible from the digits.

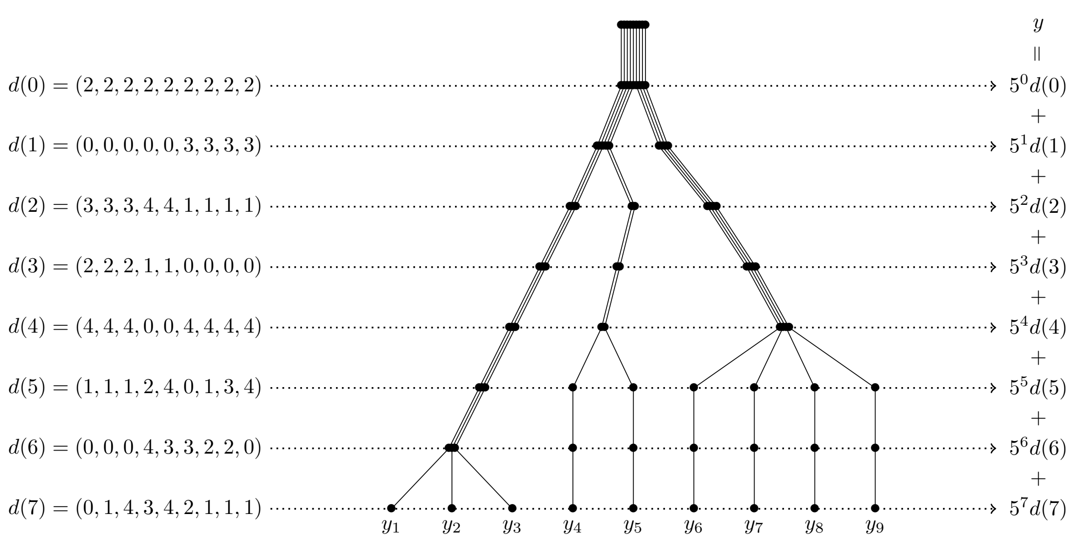

FIGURE 1. The diagram for a tree $y \in \mathbb{Z}_5^9$ of length 8.

3.2. Useful properties of level pairs

The connection between splitting chains and elements of $R^N \setminus V_0$ begins with the following definition:

Definition 3.2.1. If $\boldsymbol{\pitchfork} \in \mathcal{S}_N$ and $\boldsymbol{n} = (\eta_0, \eta_1, \ldots, \eta_{L(\boldsymbol{\pitchfork})-1}) \in \mathbb{N}^{L(\boldsymbol{\pitchfork})}$, we call the pair $(\boldsymbol{\pitchfork}, \boldsymbol{n})$ a *level pair*.

Given $x \in R^N \setminus V_0$, we may associate a unique level pair to x as follows. Its tree part y has some length m, so we have $m = \max_{i<j}\{v(y_i - y_j)\} + 1$ and hence there is a unique positive integer L and unique integers $m_0, m_1, \ldots, m_{L+1}$ satisfying $-1 =: m_0 < m_1 < \cdots < m_{L+1} := m_L + 1 = m$ and

$$\{v(y_i - y_j) : 1 \leq i < j \leq N\} = \{m_1, m_2, m_3, \ldots, m_L\}.$$

Then for each $\ell \in \{0, 1, 2, \ldots, L\}$ we define an equivalence relation $\sim_\ell$ on $[N]$ via

$$i \sim_\ell j \qquad \iff \qquad y_i \equiv y_j \mod \pi^{m_{\ell+1}}$$

and let $\pitchfork_\ell$ be the partition of $[N]$ comprised of $\sim_\ell$-equivalence classes. Since $\min_{i<j}\{v(y_i - y_j)\} = m_1$, Proposition 3.0.1 implies $y_i \equiv y_j \mod \pi^{m_1}$ for all $i < j$ and hence $\pitchfork_0 = \{[N]\} = \overline{\pitchfork}$. Similarly, $\max_{i<j}\{v(y_i - y_j)\} = m_L < m_{L+1}$ implies $y_i \not\equiv y_j \mod \pi^{m_{L+1}}$ for all $i < j$ and hence $\pitchfork_L = \{\{1\}, \{2\}, \ldots, \{N\}\} = \underline{\pitchfork}$. For each $\ell \in \{0, 1, \ldots, L-1\}$, note that every pair $i < j$ satisfying $i \sim_{\ell+1} j$ also satisfies $i \sim_\ell j$, and hence $\pitchfork_{\ell+1} \leq \pitchfork_\ell$. In particular, since $v(y_i - y_j) = m_{\ell+1}$ for at least one pair $i < j$, then this pair satisfies $i \sim_\ell j$ and $i \not\sim_{\ell+1} j$, so in fact we have $\pitchfork_{\ell+1} < \pitchfork_\ell$. Then $\overline{\pitchfork} = \pitchfork_0 > \pitchfork_1 > \pitchfork_2 > \cdots > \pitchfork_L = \underline{\pitchfork}$, meaning $\boldsymbol{\pitchfork} = (\pitchfork_0, \pitchfork_1, \pitchfork_2, \ldots, \pitchfork_L)$ is a splitting chain of order N and length $L(\boldsymbol{\pitchfork}) = L$.

Finally, define $\boldsymbol{n} = (\eta_0, \eta_1, \ldots, \eta_{L-1}) \in \mathbb{N}^L$ via $\eta_\ell := m_{\ell+1} - m_\ell$. Thus $(\pitchfork, \boldsymbol{n})$ is a level pair determined completely by x, so we call it the *level pair associated to* x.

For any $x \in \mathbb{Z}_5^9$ with tree part y as in Example 3.1.2, the level pair $(\pitchfork, \boldsymbol{n})$ associated to x can be seen in the tree diagram as in Figure 2 below. It is comprised of the splitting chain $\pitchfork = (\pitchfork_0, \pitchfork_1, \pitchfork_2, \pitchfork_3, \pitchfork_4) \in \mathcal{S}_9$ described at right and the tuple $\boldsymbol{n} = (2, 1, 3, 2)$ described at left. We have also included the (boxed) integers m_0, m_1, m_2, m_3, m_4 to make it clear that $m_0 = -1$ and $m_{\ell+1} = -1 + \eta_0 + \cdots + \eta_\ell$ for $0 \le \ell \le 3$:

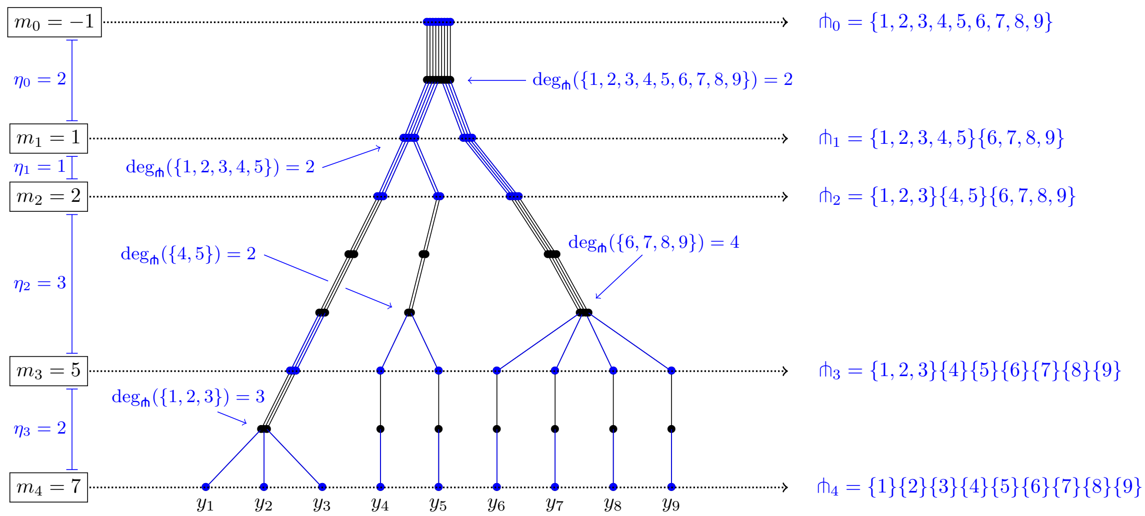

FIGURE 2. The level pair $(\pitchfork, \boldsymbol{n})$ associated to the tree in Example 3.1.2

The level pair associated to x should be regarded as a compact summary of key features of the diagram for the tree part of x. More precisely, for each $\ell \in \{0, 1, \ldots, L(\pitchfork)-1\}$ we have $y_i - y_j \in \pi^{m_{\ell+1}} R$ (where $m_{\ell+1} = -1+\eta_0+\eta_1+\cdots+\eta_\ell$) if and only if i and j are contained in the same $\lambda \in \pitchfork_\ell$. The proper refinement $\pitchfork_\ell > \pitchfork_{\ell+1}$ reflects the fact that at least one $\lambda \in \pitchfork_\ell$ breaks into $\deg_\pitchfork(\lambda) > 1$ parts in $\pitchfork_{\ell+1}$, because at least one pair $i, j \in \lambda$ satisfies $y_i \not\equiv y_j \mod \pi^{m_{\ell+1}+1}$, and hence the paths for y_i and y_j in the diagram split at level $m_{\ell+1}$ (see Figure 2). The integers

$m_1, m_2, \ldots, m_{L(\pitchfork)}$ mark the levels where these splittings happen, and the integers $\eta_0, \eta_1, \ldots, \eta_{L(\pitchfork)-1}$ appearing in the tuple $\boldsymbol{n}$ are the spacings between those m_ℓ.

Definition 3.2.2. For each level pair $(\pitchfork, \boldsymbol{n})$ define

$$\mathcal{T}(\pitchfork, \boldsymbol{n}) := \{x \in R^N \setminus V_0 : (\pitchfork, \boldsymbol{n}) \text{ is the level pair associated to } x\}.$$

There are three key properties of the sets $\mathcal{T}(\pitchfork, \boldsymbol{n})$ that will be used in our proof. The first is the following decomposition of R^N, which is immediate from Definition 3.2.2 because each $x \in R^N \setminus V_0$ has exactly one associated level pair $(\pitchfork, \boldsymbol{n})$:

$$R^N = V_0 \sqcup \bigsqcup_{\pitchfork \in \mathcal{S}_N} \bigsqcup_{\boldsymbol{n} \in \mathbb{N}^{L(\pitchfork)}} \mathcal{T}(\pitchfork, \boldsymbol{n}). \tag{3.2.1}$$

In particular, note that the union is countable because $\mathcal{S}_N$ is finite and $\mathbb{N}^{L(\pitchfork)}$ is countable for each $\pitchfork \in \mathcal{S}_N$, and note that some $\mathcal{T}(\pitchfork, \boldsymbol{n})$ may be empty. The second key property of $\mathcal{T}(\pitchfork, \boldsymbol{n})$ is the following lemma:

Lemma 3.2.3. *Each $\mathcal{T}(\pitchfork, \boldsymbol{n})$ is compact and open with measure*

$$\mu^N(\mathcal{T}(\pitchfork, \boldsymbol{n})) = M_{\pitchfork}(q) \cdot \prod_{\ell=0}^{L(\pitchfork)-1} q^{-\operatorname{rank}(\pitchfork_\ell)\eta_\ell}.$$

In particular, $\mathcal{T}(\pitchfork, \boldsymbol{n}) = \varnothing$ if and only if $M_{\pitchfork}(q) = 0$.

Proof. Fix a level pair $(\pitchfork, \boldsymbol{n})$. Using the tuple $\boldsymbol{n} = (\eta_0, \eta_1, \ldots, \eta_{L(\pitchfork)-1}) \in \mathbb{N}^{L(\pitchfork)}$, we define the familiar integers $m_0, m_1, \ldots, m_{L(\pitchfork)+1}$ by $m_0 := -1$,

$$m_{\ell'+1} := -1 + \sum_{\ell=0}^{\ell'} \eta_\ell \qquad \text{for } 0 \le \ell' \le L(\pitchfork) - 1,$$

and

$$m_{L(\pitchfork)+1} := m_{L(\pitchfork)} + 1 = \sum_{\ell=0}^{L(\pitchfork)-1} \eta_\ell,$$

and note that $\eta_\ell = m_{\ell+1} - m_\ell$ for all $\ell \in \{0, 1, \dots, L(\pitchfork) - 1\}$. By the discussion following Definition 3.2.1, note that $x \in \mathcal{T}(\pitchfork, \boldsymbol{n})$ if and only if $x \in y + \pi^{m_{L(\pitchfork)+1}} R^N$, where y is a tree with the following properties:

(i) y is a finite sum of the form $y = \sum_{n=0}^{m_{L(\pitchfork)}} \pi^n d(n)$,

(ii) $\{v(y_i - y_j) : 1 \leq i < j \leq N\} = \{m_1, m_2, \dots, m_{L(\pitchfork)}\}$, and

(iii) for $\lambda \in \pitchfork_\ell$, $i, j \in \lambda$ if and only if $y_i \equiv y_j \mod \pi^{m_{\ell+1}}$.

Since $y + \pi^{m_{L(\pitchfork)+1}} R^N$ is open and compact with measure

$$\mu^N(y + \pi^{m_{L(\pitchfork)+1}} R^N) = \mu^N(\pi^{m_{L(\pitchfork)+1}} R^N) = q^{-N m_{L(\pitchfork)+1}} = \prod_{\ell=0}^{L(\pitchfork)-1} q^{-N\eta_\ell},$$

it remains to find the number of trees y satisfying (i)-(iii) and multiply the measure above by this number. According to (i), every such y corresponds to a unique finite sequence of digit tuples $d(0), d(1), \dots, d(m_{L(\pitchfork)}) \in D^N$, so we will count all valid y by counting sequences. The terms in such a sequence may be chosen independently, so we will start by counting valid $d(n) \in D^N$ for each $n \in \{0, 1, \dots, m_{L(\pitchfork)}\}$ in two cases, maintaining conditions (i)-(iii) as we go:

(I) Suppose $m_\ell < n < m_{\ell+1}$ for some $\ell \in \{0, 1, \dots, L(\pitchfork) - 1\}$. For each $\lambda \in \pitchfork_\ell$ we must have $y_i \equiv y_j \mod \pi^{m_{\ell+1}}$ for all $i, j \in \lambda$. By Proposition 3.0.1, we must therefore choose $d(n) \in D^N$ in such a way that for every $\lambda \in \pitchfork_\ell$, we have $\inf\{n : d_i(n) \neq d_j(n)\} = v(y_i - y_j) \geq m_{\ell+1}$ for all $i, j \in \lambda$. As $n < m_{\ell+1}$, this means we must ensure $d_i(n) = d_j(n)$ for all $i, j \in \lambda$. Thus, for each $\lambda \in \pitchfork_\ell$

we must choose one value $d_\lambda \in D$ and set $d_i(n) = d_\lambda$ for all $i \in \lambda$. This must be done for $\#\pitchfork_\ell$ parts λ with $\#D = q$ choices per part, so we have $q^{\#\pitchfork_\ell}$ valid choices of $d(n)$.

(II) Suppose $n = m_{\ell+1}$ for some $\ell \in \{0, 1 \ldots, L(\pitchfork) - 1\}$. Recall that $\pitchfork_{\ell+1}$ is a proper refinement of $\pitchfork_\ell$, note that $\pitchfork_\ell$ decomposes into the two disjoint sets

$$\pitchfork_\ell' := \{\lambda \in \pitchfork_\ell : \lambda \in \pitchfork_{\ell+1}\} \qquad \text{and}$$
$$\pitchfork_\ell'' := \{\lambda \in \pitchfork_\ell : \lambda \text{ is a union of at least two } \lambda' \in \pitchfork_{\ell+1}\},$$

and note that the latter is actually $\pitchfork_\ell'' = \{\lambda \in \mathcal{B}(\pitchfork) : \ell_\pitchfork(\lambda) = \ell\}$ by part (b) of Definition 1.6.3. We use the decomposition $\pitchfork_\ell = \pitchfork_\ell' \sqcup \pitchfork_\ell''$ to break the problem of counting valid digit tuples $d(m_{\ell+1}) \in D^N$ into two corresponding subcases:

- If $\lambda \in \pitchfork_\ell'$, then $\lambda \in \pitchfork_{\ell+1}$, and this means any $i, j \in \lambda$ must satisfy $y_i \equiv y_j \mod \pi^{m_{\ell+2}}$. Recalling Proposition 3.0.1, this means we must have $\inf\{n : d_i(n) \neq d_j(n)\} = v(y_i - y_j) \geq m_{\ell+2}$, so we need only choose one value $d_\lambda \in D$ and set $d_i(m_{\ell+1}) = d_\lambda$ for all $i \in \lambda$ just as in (I). Thus for each $\lambda \in \pitchfork_\ell'$ we have $q = \#D$ valid ways to choose the partial digit tuple $(d_i(m_{\ell+1}))_{i\in\lambda}$.
- If $\lambda \in \pitchfork_\ell''$, then the number of parts $\lambda' \in \pitchfork_{\ell+1}$ comprising λ is precisely $\deg_\pitchfork(\lambda)$. Given one such $\lambda' \subset \lambda$, every pair $i, j \in \lambda'$ must satisfy $y_i \equiv y_j \mod \pi^{m_{\ell+2}}$, or equivalently $\inf\{n : d_i(n) \neq d_j(n)\} = v(y_i - y_j) \geq m_{\ell+2}$. Thus by Proposition 3.0.1 again, for every pair $i, j \in \lambda'$ we must have $d_i(m_{\ell+1}) = d_j(m_{\ell+1})$. On the other hand, if $\lambda', \lambda'' \in \pitchfork_{\ell+1}$ are distinct parts contained in λ and we have $i \in \lambda'$ and $j \in \lambda''$, then both $y_i \equiv y_j \mod \pi^{m_{\ell+1}}$ and $y_i \not\equiv y_j \mod \pi^{m_{\ell+2}}$ must be satisfied. By Proposition

3.0.1 and the necessary condition $v(y_i - y_j) \in \{m_1, m_2, \ldots, m_{L(\pitchfork)}\}$, we must ensure $\inf\{n : d_i(n) \neq d_j(n)\} = v(y_i - y_j) = m_{\ell+1}$ and hence $d_i(m_{\ell+1}) \neq d_j(m_{\ell+1})$. Therefore we must choose an ordered set of $\deg_{\pitchfork}(\lambda)$ distinct values $d_{\lambda'} \in D$ (one for each part $\lambda' \in \pitchfork_{\ell+1}$ contained in λ, and ordered because these λ' are distinct), and for each $\lambda' \subset \lambda$ we must set $d_i(m_{\ell+1}) = d_{\lambda'}$ for all $i \in \lambda'$. Thus, for each $\lambda \in \pitchfork_\ell''$ the number of valid ways to choose the partial digit tuple $(d_i(m_{\ell+1}))_{i\in\lambda}$ is the number of ways of choosing these $d_{\lambda'}$, namely

$$\binom{\#D}{\deg_{\pitchfork}(\lambda)} \cdot (\deg_{\pitchfork}(\lambda))! = (q)_{\deg_{\pitchfork}(\lambda)} = q \cdot (q-1)_{\deg_{\pitchfork}(\lambda)-1}.$$

The two subcases now combine to conclude case (II) as follows: The entries in the tuple $d(m_{\ell+1}) = (d_1(m_{\ell+1}), \ldots, d_N(m_{\ell+1}))$ are partitioned according to $\pitchfork_\ell = \pitchfork_\ell' \sqcup \pitchfork_\ell''$, so the number of valid such tuples is simply the product

$$\begin{aligned}\prod_{\lambda\in\pitchfork_\ell} \#\{\text{valid ways to choose } (d_i(m_{\ell+1}))_{i\in\lambda}\} &= \prod_{\lambda\in\pitchfork_\ell'} q \cdot \prod_{\lambda\in\pitchfork_\ell''} (q \cdot (q-1)_{\deg_{\pitchfork}(\lambda)-1}) \\ &= q^{\#\pitchfork_\ell'} \cdot q^{\#\pitchfork_\ell''} \cdot \prod_{\lambda\in\pitchfork_\ell''} (q-1)_{\deg_{\pitchfork}(\lambda)-1} \\ &= q^{\#\pitchfork_\ell} \cdot \prod_{\substack{\lambda\in\mathcal{B}(\pitchfork) \\ \ell_{\pitchfork}(\lambda)=\ell}} (q-1)_{\deg_{\pitchfork}(\lambda)-1}.\end{aligned}$$

Finally, we combine cases (I) and (II): For each $0 \leq \ell \leq L(\pitchfork) - 1$, case (I) provides $q^{\#\pitchfork_\ell(m_{\ell+1}-m_\ell-1)} = q^{\#\pitchfork_\ell(\eta_\ell-1)}$ valid choices for the partial list of tuples $d(m_\ell+1), d(m_\ell+2), \ldots, d(m_{\ell+1}-1)$, and the final product from case (II) is the number of valid ways to choose $d(m_{\ell+1})$ and hence extend the list to one of the form $d(m_\ell+1), d(m_\ell+2), \ldots, d(m_{\ell+1}-1), d(m_{\ell+1})$. Concatenating these lists for

$\ell \in \{0, 1, \ldots, L(\pitchfork) - 1\}$, we conclude that there are

$$\prod_{\ell=0}^{L(\pitchfork)-1} \left(q^{\#\pitchfork_\ell(\eta_\ell - 1)} \cdot q^{\#\pitchfork_\ell} \cdot \prod_{\substack{\lambda \in \mathcal{B}(\pitchfork) \\ \ell_\pitchfork(\lambda) = \ell}} (q-1)_{\deg_\pitchfork(\lambda) - 1} \right) = M_\pitchfork(q) \cdot \prod_{\ell=0}^{L(\pitchfork)-1} q^{\#\pitchfork_\ell \eta_\ell}$$

ways to choose a sequence of digit tuples $d(0), d(1), \ldots, d(m_{L(\pitchfork)})$ such that $y = \sum_{n=0}^{m_{L(\pitchfork)}} \pi^n d(n)$ satisfies (i)-(iii). Thus $\mathcal{T}(\pitchfork, \boldsymbol{n})$ is a disjoint union of $M_\pitchfork(q) \cdot \prod_{\ell=0}^{L(\pitchfork)-1} q^{\#\pitchfork_\ell \eta_\ell}$ sets of the form $y + \pi^{m_{L(\pitchfork)+1}} R^N$, so clearly $\mathcal{T}(\pitchfork, \boldsymbol{n}) = \varnothing$ if and only if $M_\pitchfork(q) = 0$, and $\mathcal{T}(\pitchfork, \boldsymbol{n})$ is open and compact with measure

$$\mu^N(\mathcal{T}(\pitchfork, \boldsymbol{n})) = M_\pitchfork(q) \cdot \prod_{\ell=0}^{L(\pitchfork)-1} q^{\#\pitchfork_\ell \eta_\ell} \cdot \prod_{\ell=0}^{L(\pitchfork)-1} q^{-N\eta_\ell} = M_\pitchfork(q) \cdot \prod_{\ell=0}^{L(\pitchfork)-1} q^{-\operatorname{rank}(\pitchfork_\ell)\eta_\ell}.$$

□

The final key property of the sets $\mathcal{T}(\pitchfork, \boldsymbol{n})$ is that all factors of the integrand in Definition 1.5.2 (except possibly ρ) are constant on each one. More precisely:

Lemma 3.2.4. *If $(\pitchfork, \boldsymbol{n})$ is a level pair, $a, b \in \mathbb{C}$, $\boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}}$, and $x \in \mathcal{T}(\pitchfork, \boldsymbol{n})$, then*

$$\begin{aligned} &\big(\max_{i<j} |x_i - x_j|\big)^a \big(\min_{i<j} |x_i - x_j|\big)^b \prod_{i<j} |x_i - x_j|^{s_{ij}} \\ &\qquad = q^{-(a+b+\sum_{i<j} s_{ij})(\eta_0 - 1)} \cdot \prod_{\ell=1}^{L(\pitchfork)-1} q^{-(b + E_{\pitchfork_\ell}(\boldsymbol{s}) - \operatorname{rank}(\pitchfork_\ell))\eta_\ell}. \end{aligned}$$

Proof. Just as in the proof of Lemma 3.2.3, we use the given tuple $\boldsymbol{n} = (\eta_0, \eta_1, \ldots, \eta_{L(\pitchfork)-1})$ to define integers $m_0, m_1, \ldots, m_{L(\pitchfork)+1}$ via $m_0 := -1$,

$$m_{\ell'+1} := -1 + \sum_{\ell=0}^{\ell'} \eta_\ell \qquad \text{for} \qquad \ell' \in \{0, 1, \ldots, L(\pitchfork) - 1\}$$

and $m_{L(\pitchfork)+1} := m_{L(\pitchfork)} + 1$, and have $\eta_\ell = m_{\ell+1} - m_\ell$ for all $\ell \in \{0, 1, \ldots, L(\pitchfork) - 1\}$. Now if y is the tree part of x, we have $m_{L(\pitchfork)} = \max_{i<j}\{v(y_i - y_j)\}$ and $x = y + z$ with $z \in \pi^{m_{L(\pitchfork)+1}} R^N$, so $\min_{i<j}\{v(z_i - z_j)\} > m_{L(\pitchfork)}$ and hence $v(y_i - y_j) = v(x_i - x_j)$ for all $i < j$ by the strong triangle equality. Therefore

$$\big(\max_{i<j} |x_i - x_j|\big)^a \big(\min_{i<j} |x_i - x_j|\big)^b \prod_{i<j} |x_i - x_j|^{s_{ij}}$$
$$= \big(\max_{i<j} |y_i - y_j|\big)^a \big(\min_{i<j} |y_i - y_j|\big)^b \prod_{i<j} |y_i - y_j|^{s_{ij}},$$

where

(i) y is a finite sum of the form $y = \sum_{n=0}^{m_{L(\pitchfork)}} \pi^n d(n)$,

(ii) $\{v(y_i - y_j) : 1 \le i < j \le N\} = \{m_1, m_2, \ldots, m_{L(\pitchfork)}\}$, and

(iii) for $\lambda \in \pitchfork_\ell$, $i, j \in \lambda$ if and only if $y_i \equiv y_j \mod \pi^{m_{\ell+1}}$

as in the proof of Lemma 3.2.3. Now

$$\left(\max_{i<j} |y_i - y_j|\right)^a = q^{-a\cdot\min_{i<j} v(y_i - y_j)} = q^{-am_1} = q^{-a(\eta_0 - 1)},$$
$$\left(\min_{i<j} |y_i - y_j|\right)^b = q^{-b\cdot\max_{i<j} v(y_i - y_j)} = q^{-bm_{L(\pitchfork)}} = q^{-b(\eta_0 - 1)} \cdot \prod_{\ell=1}^{L(\pitchfork)-1} q^{-b\eta_\ell},$$

and

$$\begin{aligned}
\sum_{i<j} s_{ij} v(y_i - y_j) &= \sum_{\ell=1}^{L(\pitchfork)} \sum_{\substack{i<j \\ v(y_i-y_j)=m_\ell}} s_{ij} m_\ell \\
&= \sum_{\ell=1}^{L(\pitchfork)} \sum_{\substack{i<j \\ v(y_i-y_j)=m_\ell}} s_{ij}(-1+\eta_0+\eta_1+\cdots+\eta_{\ell-1}) \\
&= \sum_{\substack{i<j \\ v(y_i-y_j)=m_1}} s_{ij}(-1+\eta_0) \\
&\quad + \sum_{\substack{i<j \\ v(y_i-y_j)=m_2}} s_{ij}(-1+\eta_0+\eta_1) \\
&\qquad \vdots \\
&\qquad + \sum_{\substack{i<j \\ v(y_i-y_j)=m_{L(\pitchfork)}}} s_{ij}(-1+\eta_0+\eta_1+\cdots+\eta_{L(\pitchfork)-1}),
\end{aligned}$$

so exchanging the order of summation in the above sum of sums gives

$$\sum_{i<j} s_{ij} v(y_i - y_j) = \left[\sum_{\substack{i<j \\ v(y_i-y_j)\geq m_1}} s_{ij}\right](\eta_0 - 1) + \sum_{\ell=1}^{L(\pitchfork)-1} \left[\sum_{\substack{i<j \\ v(y_i-y_j)\geq m_{\ell+1}}} s_{ij}\right]\eta_\ell.$$

Since $v(y_i - y_j) \geq m_1$ for all $i < j$, the first term in brackets is simply $\sum_{i<j} s_{ij}$. For the other terms in brackets, recall

$$\begin{aligned}
v(y_i - y_j) \geq m_{\ell+1} \qquad &\Longleftrightarrow \qquad y_i \equiv y_j \mod \pi^{m_{\ell+1}} \\
&\Longleftrightarrow \qquad i, j \in \lambda \text{ for some } \lambda \in \pitchfork_\ell
\end{aligned}$$

by Proposition 3.0.1 and property (iii) of y. Therefore

$$\sum_{\substack{i<j \\ v(y_i-y_j)\geq m_{\ell+1}}} s_{ij} = \sum_{\lambda\in\pitchfork_\ell}\sum_{\substack{i<j \\ i,j\in\lambda}} s_{ij} = E_{\pitchfork_\ell}(\boldsymbol{s}) - \operatorname{rank}(\pitchfork_\ell)$$

by part (c) of Definition 1.6.3, and hence

$$\sum_{i<j} s_{ij} v(y_i - y_j) = \left[\sum_{i<j} s_{ij}\right](\eta_0 - 1) + \sum_{\ell=1}^{L(\pitchfork)-1}\left[E_{\pitchfork_\ell}(\boldsymbol{s}) - \operatorname{rank}(\pitchfork_\ell)\right]\eta_\ell$$

implies

$$\prod_{i<j}|y_i - y_j|^{s_{ij}} = q^{-(\sum_{i<j} s_{ij})(\eta_0 - 1)} \cdot \prod_{\ell=1}^{L(\pitchfork)-1} q^{-(E_{\pitchfork_\ell}(\boldsymbol{s}) - \operatorname{rank}(\pitchfork_\ell))\eta_\ell}.$$

Combining this with the max and min factors gives the desired result:

$$\begin{aligned}
&\big(\max_{i<j}|x_i - x_j|\big)^a\big(\min_{i<j}|x_i - x_j|\big)^b\prod_{i<j}|x_i - x_j|^{s_{ij}} \\
&\qquad = \big(\max_{i<j}|y_i - y_j|\big)^a\big(\min_{i<j}|y_i - y_j|\big)^b\prod_{i<j}|y_i - y_j|^{s_{ij}} \\
&\qquad = q^{-(a+b+\sum_{i<j} s_{ij})(\eta_0 - 1)} \cdot \prod_{\ell=1}^{L(\pitchfork)-1} q^{-(b+E_{\pitchfork_\ell}(\boldsymbol{s}) - \operatorname{rank}(\pitchfork_\ell))\eta_\ell}.
\end{aligned}$$

□

3.3. Integration with level pairs

Though Lemmas 3.2.3 and 3.2.4 are useful on their own, their combination is especially important. Indeed, Lemma 3.2.3 provides an explicit formula for the measure of $\mathcal{T}(\pitchfork, \boldsymbol{n})$, on which the constant value taken by

$$x \mapsto \big(\max_{i<j} |x_i - x_j|\big)^a \big(\min_{i<j} |x_i - x_j|\big)^b \prod_{i<j} |x_i - x_j|^{s_{ij}}$$

is given in Lemma 3.2.4. Thus the integral of this function over a given set $\mathcal{T}(\pitchfork, \boldsymbol{n})$ is simply the product of the function value and the value of $\mu^N(\mathcal{T}(\pitchfork, \boldsymbol{n}))$:

Corollary 3.3.1. *If $a, b \in \mathbb{C}$, then for every $\boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}}$ we have*

$$\int_{\mathcal{T}(\pitchfork,\boldsymbol{n})} \big(\max_{i<j} |x_i - x_j|\big)^a \big(\min_{i<j} |x_i - x_j|\big)^b \prod_{i<j} |x_i - x_j|^{s_{ij}} \, dx_1 \dots dx_N$$
$$= q^{-(N-1+a+b+\sum_{i<j} s_{ij})(\eta_0 - 1)} \cdot \frac{M_{\pitchfork}(q)}{q^{N-1}} \cdot \prod_{\ell=1}^{L(\pitchfork)-1} q^{-(b+E_{\pitchfork_\ell}(\boldsymbol{s}))\eta_\ell}.$$

Note that this quantity is entire in each of the variables a, b, and s_{ij}, and all mixed partial derivatives in those variables commute with each other and the integral sign.

Remark 3.3.2. Note that Corollary 3.3.1 actually generalizes Lemma 3.2.3, as the latter can be recovered by setting $s_{ij} = a = b = 0$ in integral formula above. Moreover, the exponential factors in the formula are completely determined by the level pair $(\pitchfork, \boldsymbol{n})$, which encodes the common features of the tree diagrams for $x \in \mathcal{T}(\pitchfork, \boldsymbol{n})$ (recall 2). In particular, we may regard $\pitchfork_0 = \{[N]\}$ and η_0 as "root data" that determine the factor

$$q^{-(a+b+E_{\pitchfork_0}(\boldsymbol{s}))(\eta_0-1)} = q^{-(N-1+a+b+\sum_{i<j} s_{ij})(\eta_0-1)},$$

and note that

$$|q^{-(N-1+a+b+\sum_{i<j} s_{ij})}|_{\mathbb{C}} < 1 \qquad \Longleftrightarrow \qquad \boldsymbol{s} \in \mathcal{RP}_N(a,b). \tag{3.3.2}$$

This is precisely the reason we named $\mathcal{RP}_N(a,b)$ the "root polytope". Similarly, for each $\ell \in \{1,2,\ldots,L(\pitchfork)-1\}$, recall that $\pitchfork_\ell$ describes how the N paths representing $(x_1,x_2,\ldots,x_N) = x \in \mathcal{T}(\pitchfork,\boldsymbol{n})$ branch in a particular level in the tree diagram, and that η_ℓ measures the vertical distance between the tree diagram levels corresponding to $\pitchfork_\ell$ and $\pitchfork_{\ell+1}$. Thus we regard $\pitchfork_\ell$ and η_ℓ as the ℓth "level data", which determine the exponential factor $q^{-(b+E_{\pitchfork_\ell}(\boldsymbol{s}))\eta_\ell}$. Accordingly, we named $\mathcal{LP}_{\pitchfork}(b)$ the "level polytope" in Definition 1.6.4 because

$$|q^{-(b+E_{\pitchfork_\ell}(\boldsymbol{s}))}|_{\mathbb{C}} < 1 \quad \text{for all} \quad \ell \in \{1,2,\ldots,L(\pitchfork)-1\} \qquad \Longleftrightarrow \qquad \boldsymbol{s} \in \mathcal{LP}_{\pitchfork}(b). \tag{3.3.3}$$

In the following proposition, we will finally see how the exponential factors corresponding to the root and how level polytopes combine to form the root and level functions. It is the most important result in this chapter.

Proposition 3.3.3. *Suppose $a,b \in \mathbb{C}$ and define $R^N_{\pitchfork} := \bigsqcup_{\boldsymbol{n}\in\mathbb{N}^{L(\pitchfork)}} \mathcal{T}(\pitchfork,\boldsymbol{n})$ for each $\pitchfork \in \mathcal{S}_N$. If $M_{\pitchfork}(q) > 0$, then the integral*

$$\int_{R^N_{\pitchfork}} \big(\max_{i<j}|x_i-x_j|\big)^a \big(\min_{i<j}|x_i-x_j|\big)^b \prod_{i<j}|x_i-x_j|^{s_{ij}}\, dx_1\ldots dx_N$$

converges absolutely if and only if $\boldsymbol{s} \in \mathcal{RP}_N(a,b) \cap \mathcal{LP}_{\pitchfork}(b)$, and for such $\boldsymbol{s}$ it converges to

$$\frac{1}{1-q^{-(N-1+a+b+\sum_{i<j}s_{ij})}} \cdot J_{\pitchfork}(b,\boldsymbol{s}).$$

Otherwise $M_{\pitchfork}(q) = 0$, in which case $R^N_{\pitchfork} = \varnothing$ and the integral is simply zero.

Proof. The $M_{\pitchfork}(q) = 0$ case is immediate from Lemma 3.2.3, so suppose $M_{\pitchfork}(q) > 0$ and $\boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}}$. Then Corollary 3.3.1 and Fubini's Theorem for sums of nonnegative terms imply

$$\begin{aligned}
&\int_{R_{\pitchfork}^N} \left| \big(\max_{i<j} |x_i - x_j|\big)^a \big(\min_{i<j} |x_i - x_j|\big)^b \prod_{i<j} |x_i - x_j|^{s_{ij}} \right|_{\mathbb{C}} dx_1 \dots dx_N \\
&\quad = \sum_{\boldsymbol{n} \in \mathbb{N}^{L(\pitchfork)}} \int_{\mathcal{T}(\pitchfork, \boldsymbol{n})} \big(\max_{i<j} |x_i - x_j|\big)^{\mathrm{Re}(a)} \big(\min_{i<j} |x_i - x_j|\big)^{\mathrm{Re}(b)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot \prod_{i<j} |x_i - x_j|^{\mathrm{Re}(s_{ij})} \, dx_1 \dots dx_N \\
&\quad = \sum_{\boldsymbol{n} \in \mathbb{N}^{L(\pitchfork)}} q^{-\mathrm{Re}(N-1+a+b+\sum_{i<j} s_{ij})(\eta_0 - 1)} \cdot \frac{M_{\pitchfork}(q)}{q^{N-1}} \prod_{\ell=1}^{L(\pitchfork)-1} q^{-\mathrm{Re}(b+E_{\pitchfork_\ell}(\boldsymbol{s}))\eta_\ell} \\
&\quad = \sum_{\eta_0=1}^{\infty} |q^{-(N-1+a+b+\sum_{i<j} s_{ij})}|_{\mathbb{C}}^{(\eta_0-1)} \cdot \frac{M_{\pitchfork}(q)}{q^{N-1}} \cdot \prod_{\ell=1}^{L(\pitchfork)-1} \sum_{\eta_\ell=1}^{\infty} |q^{-(b+E_{\pitchfork_\ell}(\boldsymbol{s}))}|_{\mathbb{C}}^{\eta_\ell}.
\end{aligned}$$

Therefore the integral on the first line converges if and only if all of the geometric series in the product on the last line converge. But this is the case if and only if $\boldsymbol{s} \in \mathcal{RP}_N(a,b) \cap \mathcal{LP}_{\pitchfork}(b)$ by (3.3.2) and (3.3.3), so we have established the first claim. Moreover, if $\boldsymbol{s} \in \mathcal{RP}_N(a,b) \cap \mathcal{LP}_{\pitchfork}(b)$ then the function

$$x \mapsto \mathbf{1}_{R_{\pitchfork}^N}(x) \left| \big(\max_{i<j} |x_i - x_j|\big)^a \big(\min_{i<j} |x_i - x_j|\big)^b \prod_{i<j} |x_i - x_j|^{s_{ij}} \right|_{\mathbb{C}}$$

is in $L^1(K^N, \mu^N)$ and dominates every partial sum of the function

$$x \mapsto \sum_{\boldsymbol{n} \in \mathbb{N}^{L(\pitchfork)}} \mathbf{1}_{\mathcal{T}(\pitchfork, \boldsymbol{n})}(x) \big(\max_{i<j} |x_i - x_j|\big)^a \big(\min_{i<j} |x_i - x_j|\big)^b \prod_{i<j} |x_i - x_j|^{s_{ij}},$$

so the Dominated Convergence Theorem, Corollary 3.3.1, and Fubini's Theorem for absolutely convergent sums together imply

$$\begin{aligned}
&\int_{R^N_{\pitchfork}} \big(\max_{i<j}|x_i-x_j|\big)^a \big(\min_{i<j}|x_i-x_j|\big)^b \prod_{i<j}|x_i-x_j|^{s_{ij}}\,dx_1\ldots dx_N \\
&\quad = \sum_{\boldsymbol{n}\in\mathbb{N}^{L(\pitchfork)}} \int_{\mathcal{T}(\pitchfork,\boldsymbol{n})} \big(\max_{i<j}|x_i-x_j|\big)^a \big(\min_{i<j}|x_i-x_j|\big)^b \\
&\qquad\qquad \cdot \prod_{i<j}|x_i-x_j|^{s_{ij}}\,dx_1\ldots dx_N \\
&\quad = \sum_{\boldsymbol{n}\in\mathbb{N}^{L(\pitchfork)}} q^{-(N-1+a+b+\sum_{i<j}s_{ij})(\eta_0-1)} \cdot \frac{M_{\pitchfork}(q)}{q^{N-1}} \cdot \prod_{\ell=1}^{L(\pitchfork)-1} q^{-(b+E_{\pitchfork_\ell}(\boldsymbol{s}))\eta_\ell} \\
&\quad = \sum_{\eta_0=1}^{\infty} q^{-(N-1+a+b+\sum_{i<j}s_{ij})(\eta_0-1)} \cdot \frac{M_{\pitchfork}(q)}{q^{N-1}} \cdot \prod_{\ell=1}^{L(\pitchfork)-1} \sum_{\eta_\ell=1}^{\infty} q^{-(b+E_{\pitchfork_\ell}(\boldsymbol{s}))\eta_\ell} \\
&\quad = \frac{1}{1-q^{-(N-1+a+b+\sum_{i<j}s_{ij})}} \cdot J_{\pitchfork,q}(b,\boldsymbol{s}).
\end{aligned}$$

□

Proposition 3.3.3 is the key ingredient in the next proposition, which is the foundation of parts (a) and (b) of Theorem 1.6.6.

Proposition 3.3.4. *Suppose K is a p-field and suppose $a, b \in \mathbb{C}$. Then the integral*

$$\int_{R^N} \big(\max_{i<j}|x_i-x_j|\big)^a \big(\min_{i<j}|x_i-x_j|\big)^b \prod_{i<j}|x_i-x_j|^{s_{ij}}\,dx_1\ldots dx_N$$

converges absolutely if and only if $\boldsymbol{s}$ belongs to $\Omega_N(a,b)$, and for such $\boldsymbol{s}$ it converges to

$$\frac{1}{1-q^{-(N-1+a+b+\sum_{i<j}s_{ij})}} \cdot \sum_{\pitchfork\in\mathcal{S}_N} J_{\pitchfork,q}(b,\boldsymbol{s}).$$

Proof. First, note that the decomposition in (3.2.1) can be rewritten as

$$R^N = V_0 \sqcup \bigsqcup_{\pitchfork \in \mathcal{S}_N} R^N_{\pitchfork}, \tag{3.3.4}$$

and that for each integer $m \geq 1$ we have

$$V_0 = \bigcup_{1 \leq i < j \leq N} \{x \in R^N : x_i = x_j\} \subset \bigcup_{1 \leq i < j \leq N} \bigsqcup_{\substack{y \in R^N/\pi^m R^N \\ y_i = y_j}} (y + \pi^m R^N).$$

For each pair $\{i, j\}$ satisfying $1 \leq i < j \leq N$, we have

$$\#\{y \in R^N/\pi^m R^N : y_i = y_j\} = q^{(N-1)m} \qquad \text{and} \qquad \mu^N(y + \pi^m R^N) = q^{-Nm}$$

by Proposition 3.0.1 and Theorem 1.3.1. Thus V_0 is contained in a union of $\binom{N}{2}$ sets of μ^N-measure $q^{(N-1)m} \cdot q^{-Nm} = q^{-m}$, and since $m \geq 1$ can be arbitrarily large it follows that $\mu^N(V_0) = 0$. This fact and (3.3.4) together imply

$$\begin{aligned} &\int_{R^N} \bigl(\max_{i<j} |x_i - x_j|\bigr)^a \bigl(\min_{i<j} |x_i - x_j|\bigr)^b \prod_{i<j} |x_i - x_j|^{s_{ij}}\, dx_1 \dots dx_N \\ &\quad = \sum_{\pitchfork \in \mathcal{S}_N} \int_{R^N_{\pitchfork}} \bigl(\max_{i<j} |x_i - x_j|\bigr)^a \bigl(\min_{i<j} |x_i - x_j|\bigr)^b \prod_{i<j} |x_i - x_j|^{s_{ij}}\, dx_1 \dots dx_N. \end{aligned}$$

According to Proposition 3.3.3, the integral over $R^N_{\pitchfork}$ converges absolutely if and only if $M_{\pitchfork}(q) = 0$ (in which case $R^N_{\pitchfork} = \varnothing$) or $\boldsymbol{s} \in \mathcal{RP}_N(a, b) \cap \mathcal{LP}_{\pitchfork}(b)$. Therefore the integral over R^N converges if and only if $\boldsymbol{s}$ is in the polytope

$$\mathcal{RP}_N(a, b) \cap \bigcap_{\substack{\pitchfork \in \mathcal{S}_N \\ M_{\pitchfork}(q) > 0}} \mathcal{LP}_{\pitchfork}(b).$$

Recalling the definition of $\Omega_N(a,b)$ in part (a) of Theorem 1.6.6, it remains to show that the condition "$M_{\boldsymbol{\pitchfork}}(q) > 0$" in the intersection above is extraneous. If $N = 2$, the only splitting chain in $\mathcal{S}_2$ is $\boldsymbol{\pitchfork} = ([N],\ \{1\}\{2\})$, which has $M_{\boldsymbol{\pitchfork}}(q) = q - 1 > 0$ because $q \geq 2$. Thus if $N = 2$ the condition "$M_{\boldsymbol{\pitchfork}}(q) > 0$" is automatic and the proof is complete. Now suppose $N > 2$. It suffices to show that

$$\bigcap_{\substack{\boldsymbol{\pitchfork} \in \mathcal{S}_N \\ M_{\boldsymbol{\pitchfork}}(q) > 0}} \mathcal{LP}_{\boldsymbol{\pitchfork}}(b) \subset \bigcap_{\boldsymbol{\pitchfork} \in \mathcal{S}_N} \mathcal{LP}_{\boldsymbol{\pitchfork}}(b) \tag{3.3.5}$$

because the reverse containment is obvious. To this end, let $\pitchfork^\circ$ be an arbitrary partition of $[N]$ other than $\overline{\pitchfork} = \{[N]\}$ or $\underline{\pitchfork} = \{\{1\},\{2\},\ldots,\{N\}\}$. We will construct a splitting chain $\boldsymbol{\pitchfork}^\circ \in \mathcal{S}_N$ that has $\pitchfork^\circ$ as a level, satisfies $M_{\boldsymbol{\pitchfork}^\circ}(q) > 0$ for any $q \geq 2$, and has length $L(\boldsymbol{\pitchfork}^\circ) \geq 2$ as follows. Put $k = \#\pitchfork^\circ - 1$ and define $\pitchfork_k := \pitchfork^\circ$. Then $k \geq 1$ and we may write $\pitchfork_k = \{\lambda_1, \lambda_2, \ldots, \lambda_{k+1}\}$ where $\#\lambda_1 \geq 2$ and $\#\lambda_1 \geq \#\lambda_2 \geq \cdots \geq \#\lambda_{k+1}$. Now for each $\ell \in \{0,1,2,\ldots,k-1\}$, define

$$\pitchfork_\ell := \{\lambda_1, \lambda_2, \ldots, \lambda_\ell, (\lambda_{\ell+1} \cup \lambda_{\ell+2} \cup \cdots \cup \lambda_{k+1})\}$$

and note that $\overline{\pitchfork} = \pitchfork_0 > \pitchfork_1 > \cdots > \pitchfork_k$ where each refinement is given by splitting a single part into two parts. For $\ell \geq k+1$, recursively define $\pitchfork_\ell$ to be any refinement of $\pitchfork_{\ell-1}$ such that each non-singleton part $\lambda \in \pitchfork_{\ell-1}$ splits into $\lambda' = \lambda \setminus \{i\} \in \pitchfork_\ell$ and $\{i\} \in \pitchfork_\ell$ for some $i \in \lambda$. The largest part $\lambda_1 \in \pitchfork_k$ will fully refine into singletons after $\#\lambda_1 - 1$ steps in the recursion, by which time all other parts will have also refined into singletons. Therefore the recursion must stop at $\ell = k + \#\lambda_1 - 1$ with $\pitchfork_k > \pitchfork_{k+1} > \cdots > \pitchfork_{k+\#\lambda_1-1} = \underline{\pitchfork}$, where each refinement is given by refining non-singleton parts into exactly two parts. Thus we have constructed a splitting chain

$\boldsymbol{\pitchfork}^\circ = (\pitchfork_0, \pitchfork_1, \ldots, \pitchfork_{k+\#\lambda_1-1}) \in \mathcal{S}_N$ that has the given partition $\pitchfork^\circ$ as its kth level, has length $L(\boldsymbol{\pitchfork}^\circ) = k + \#\lambda_1 - 1 \geq k+1 = \#\pitchfork \geq 2$, and has $\deg_{\boldsymbol{\pitchfork}^\circ}(\lambda) = 2 \leq q$ for all branches $\lambda \in \mathcal{B}(\boldsymbol{\pitchfork}^\circ)$. The last property implies $M_{\boldsymbol{\pitchfork}^\circ}(q) > 0$, so we have

$$\bigcap_{\substack{\boldsymbol{\pitchfork} \in \mathcal{S}_N \\ M_{\boldsymbol{\pitchfork}}(q) > 0}} \mathcal{LP}_{\boldsymbol{\pitchfork}}(b) \subset \mathcal{LP}_{\boldsymbol{\pitchfork}^\circ}(b) = \bigcap_{\ell=1}^{L(\boldsymbol{\pitchfork}^\circ)-1} \left\{ \boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}} : \operatorname{Re}(b + E_{\pitchfork_\ell}(\boldsymbol{s})) > 0 \right\}$$

$$\subset \left\{ \boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}} : \operatorname{Re}(b + E_{\pitchfork^\circ}(\boldsymbol{s})) > 0 \right\}.$$

This argument works for every partition $\pitchfork^\circ$ with $\underline{\pitchfork} < \pitchfork^\circ < \overline{\pitchfork}$, so it follows that

$$\bigcap_{\substack{\boldsymbol{\pitchfork} \in \mathcal{S}_N \\ M_{\boldsymbol{\pitchfork}}(q) > 0}} \mathcal{LP}_{\boldsymbol{\pitchfork}}(b) \subset \bigcap_{\substack{\text{partitions } \pitchfork \\ \underline{\pitchfork} < \pitchfork < \overline{\pitchfork}}} \left\{ \boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}} : \operatorname{Re}(b + E_{\pitchfork}(\boldsymbol{s})) > 0 \right\}.$$

On the other hand, for every splitting chain $\boldsymbol{\pitchfork} \in \mathcal{S}_N$, each level $\pitchfork_\ell$ with $1 \leq \ell \leq L(\boldsymbol{\pitchfork}) - 1$ is a partition of $[N]$ satisfying $\underline{\pitchfork} < \pitchfork_\ell < \overline{\pitchfork}$, so

$$\bigcup_{\boldsymbol{\pitchfork} \in \mathcal{S}_N} \{\pitchfork_1, \pitchfork_2, \ldots, \pitchfork_{L(\boldsymbol{\pitchfork})-1}\} \subset \{\text{partitions } \pitchfork : \underline{\pitchfork} < \pitchfork < \overline{\pitchfork}\}.$$

This implies

$$\bigcap_{\substack{\boldsymbol{\pitchfork} \in \mathcal{S}_N \\ M_{\boldsymbol{\pitchfork}}(q) > 0}} \mathcal{LP}_{\boldsymbol{\pitchfork}}(b) \subset \bigcap_{\substack{\text{partitions } \pitchfork \\ \underline{\pitchfork} < \pitchfork < \overline{\pitchfork}}} \left\{ \boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}} : \operatorname{Re}(b + E_{\pitchfork}(\boldsymbol{s})) > 0 \right\}$$

$$\subset \bigcap_{\boldsymbol{\pitchfork} \in \mathcal{S}_N} \bigcap_{\ell=1}^{L(\boldsymbol{\pitchfork})-1} \left\{ \boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}} : \operatorname{Re}(b + E_{\pitchfork_\ell}(\boldsymbol{s})) > 0 \right\} = \bigcap_{\boldsymbol{\pitchfork} \in \mathcal{S}_N} \mathcal{LP}_{\boldsymbol{\pitchfork}}(b),$$

so (3.3.5) holds and the proof is complete. Note that we also just proved the first claim in Proposition 1.6.7. $\square$

CHAPTER IV

BRANCH PAIRS AND THE CONCLUSION FOR Z_N^{ρ}

The goals of this chapter are to prove Lemma 1.6.5, find a correspondence between level pairs and *branch pairs* (to be defined shortly), and use them to write some of the previous integrals in a simpler way. Defining and proving the correspondence is arguably the most technical part of this thesis and will take the majority of Section 4.1. In Section 4.2 we will use the correspondence to prove Proposition 4.2.4, which is a "branch-centric" analogue of Proposition 3.3.4. Finally, in Section 4.3 we will extend Propositions 3.3.4 and 4.2.4 to general norm-densities and conclude the proof of parts (a)-(c) of Theorem 1.6.6.

4.1. Reduced splitting chains, branch pairs, and a correspondence

Before defining branch pairs, we will restate and prove Lemma 1.6.5.

Lemma 4.1.1 (Lemma 1.6.5)**.** *We say that a splitting chain $\pitchfork$ is* reduced *if for each $\lambda \in \mathcal{B}(\pitchfork)$ there is a unique level $\pitchfork_\ell$ containing λ (namely, the level $\pitchfork_{\ell_{\pitchfork}(\lambda)}$). We write $\mathcal{R}_N := \{\pitchfork \in \mathcal{S}_N : \pitchfork \text{ is reduced}\}$ and define an equivalence relation $\simeq$ on $\mathcal{S}_N$ by writing $\pitchfork \simeq \pitchfork'$ if and only if $\mathcal{B}(\pitchfork) = \mathcal{B}(\pitchfork')$.*

(a) If $\pitchfork \simeq \pitchfork'$, then the branch degrees, part exponents, multiplicity polynomials, and branch polytopes for $\pitchfork$ and $\pitchfork'$ respectively coincide.

(b) For each $\pitchfork \in \mathcal{S}_N$ there is a unique $\pitchfork^* \in \mathcal{R}_N$ such that $\pitchfork \simeq \pitchfork^*$. We call this $\pitchfork^*$ the reduction *of $\pitchfork$ and regard $\mathcal{R}_N$ as a complete set of representatives for $\mathcal{S}_N$ modulo $\simeq$.*

(c) For each $\boldsymbol{\pitchfork}^* \in \mathcal{R}_N$ *we have*

$$\bigcap_{\substack{\boldsymbol{\pitchfork} \in \mathcal{S}_N \\ \boldsymbol{\pitchfork} \simeq \boldsymbol{\pitchfork}^*}} \mathcal{LP}_{\boldsymbol{\pitchfork}}(0) = \mathcal{BP}_{\boldsymbol{\pitchfork}^*},$$

and therefore

$$\bigcap_{\boldsymbol{\pitchfork} \in \mathcal{S}_N} \mathcal{LP}_{\boldsymbol{\pitchfork}}(0) = \bigcap_{\boldsymbol{\pitchfork}^* \in \mathcal{R}_N} \mathcal{BP}_{\boldsymbol{\pitchfork}^*}.$$

Proof.

(a) Suppose $\boldsymbol{\pitchfork}, \boldsymbol{\pitchfork}' \in \mathcal{S}_N$ and $\boldsymbol{\pitchfork} \simeq \boldsymbol{\pitchfork}'$. Then $\mathcal{B}(\boldsymbol{\pitchfork}) = \mathcal{B}(\boldsymbol{\pitchfork}')$ and our only task is to prove that $\deg_{\boldsymbol{\pitchfork}}(\lambda) = \deg_{\boldsymbol{\pitchfork}'}(\lambda)$ for all $\lambda \in \mathcal{B}(\boldsymbol{\pitchfork})$, for then the rest of (a) will follow immediately from part (c) of Definition 1.6.3 part (b) of Definition 1.6.4. To this end, suppose $\lambda \in \mathcal{B}(\boldsymbol{\pitchfork})$. Any branch $\lambda' \in \mathcal{B}(\boldsymbol{\pitchfork})$ contained in both $\pitchfork_{\ell_{\boldsymbol{\pitchfork}}(\lambda)+1}$ and λ must not appear in any of the levels $\pitchfork_0, \pitchfork_1, \ldots, \pitchfork_{\ell_{\boldsymbol{\pitchfork}}(\lambda)}$ because $\pitchfork_{\ell_{\boldsymbol{\pitchfork}}(\lambda)+1}$ properly refines all of them and by definition, $\ell_{\boldsymbol{\pitchfork}}(\lambda) = \max\{\ell \in \{0, 1, \ldots, L(\boldsymbol{\pitchfork}) - 1\} : \lambda \in \pitchfork_\ell\}$. Moreover, no branch $\lambda'' \subsetneq \lambda'$ can appear in $\pitchfork_{\ell_{\boldsymbol{\pitchfork}}(\lambda)+1}$ because $\lambda' \in \pitchfork_{\ell_{\boldsymbol{\pitchfork}}(\lambda)+1}$. Therefore $\{\lambda' \in \pitchfork_{\ell_{\boldsymbol{\pitchfork}}(\lambda)+1} : \lambda' \subset \lambda\}$ is comprised of precisely the largest branches in $\mathcal{B}(\boldsymbol{\pitchfork})$ that are properly contained in λ, along with any remaining singletons $\{i\} \subset \lambda$. Thus $\{\lambda' \in \pitchfork_{\ell_{\boldsymbol{\pitchfork}}(\lambda)+1} : \lambda' \subset \lambda\}$ is completely determined by $\mathcal{B}(\boldsymbol{\pitchfork})$ and λ. But $\mathcal{B}(\boldsymbol{\pitchfork}) = \mathcal{B}(\boldsymbol{\pitchfork}')$, so

$$\{\lambda' \in \pitchfork_{\ell_{\boldsymbol{\pitchfork}}(\lambda)+1} : \lambda' \subset \lambda\} = \{\lambda' \in \pitchfork'_{\ell_{\boldsymbol{\pitchfork}'}(\lambda)+1} : \lambda' \subset \lambda\}$$

and we conclude that $\deg_{\boldsymbol{\pitchfork}}(\lambda) = \deg_{\boldsymbol{\pitchfork}'}(\lambda)$.

(b) Suppose $\boldsymbol{\pitchfork} \in \mathcal{S}_N$ and note that $\mathcal{B}(\boldsymbol{\pitchfork})$ is partially ordered by $\subset$ with unique largest element $[N]$. We will construct an element $\boldsymbol{\pitchfork}^* \in \mathcal{R}_N$ satisfying $\mathcal{B}(\boldsymbol{\pitchfork}^*) = \mathcal{B}(\boldsymbol{\pitchfork})$. Begin by letting $\pitchfork_0^* := \{[N]\}$, and continue recursively for $\ell \geq 0$ as follows: Define a partition $\pitchfork_{\ell+1}^*$ of $[N]$ by taking the largest branches remaining in $\mathcal{B}(\boldsymbol{\pitchfork}^*) \setminus (\pitchfork_0^* \cup \pitchfork_1^* \cup \cdots \cup \pitchfork_\ell^*)$ and any leftover singletons in $[N]$. At the first $\ell \geq 0$ for which $\mathcal{B}(\boldsymbol{\pitchfork}) \setminus (\pitchfork_0^* \cup \pitchfork_1^* \cup \cdots \cup \pitchfork_\ell^*) = \varnothing$, end the recursion and let $L^* := \ell + 1$ and $\pitchfork_{L^*}^* := \underline{\pitchfork}$. Then by construction we have $\pitchfork_{\ell+1}^* < \pitchfork_\ell^*$ because each part of $\pitchfork_{\ell+1}^*$ is contained in a part of $\pitchfork_\ell^*$ and at least one part of $\pitchfork_{\ell+1}^*$ is properly contained in one of those in $\pitchfork_\ell^*$. Thus $\boldsymbol{\pitchfork}^* = (\pitchfork_0^*, \pitchfork_1^*, \ldots, \pitchfork_{L^*}^*)$ is a splitting chain of order N and length $L^* \leq L(\boldsymbol{\pitchfork})$ with $\mathcal{B}(\boldsymbol{\pitchfork}^*) = \left(\bigcup_{\ell=0}^{L^*-1} \pitchfork_\ell^*\right) \setminus \underline{\pitchfork} = \mathcal{B}(\boldsymbol{\pitchfork})$. Moreover, $\boldsymbol{\pitchfork}^*$ is reduced because each $\lambda \in \mathcal{B}(\boldsymbol{\pitchfork}^*)$ is contained in exactly one $\pitchfork_\ell^*$, and $\boldsymbol{\pitchfork}^*$ is unique because it was completely determined by $\mathcal{B}(\boldsymbol{\pitchfork})$.

(c) Suppose $\boldsymbol{\pitchfork}^* \in \mathcal{R}_N$. The first claim is obvious from Definition 1.6.4 if $\mathcal{B}(\boldsymbol{\pitchfork}^*) \setminus \overline{\pitchfork} = \varnothing$, so suppose otherwise and choose an arbitrary branch $\lambda^\circ \in \mathcal{B}(\boldsymbol{\pitchfork}^*) \setminus \overline{\pitchfork}$. We will construct a splitting chain $\boldsymbol{\pitchfork}^\circ \in \mathcal{S}_N$ such that $\boldsymbol{\pitchfork}^\circ \simeq \boldsymbol{\pitchfork}^*$ and such that $\boldsymbol{\pitchfork}^\circ$ has a level containing λ° and no other branches. The set $\mathcal{B}' := \{\lambda \in \mathcal{B}(\boldsymbol{\pitchfork}^*) : \lambda \not\subset \lambda^\circ\}$ is partially ordered by $\subset$ with unique largest element $[N]$, so we may apply the same algorithm in the proof of part (b) to obtain the unique reduced splitting chain $\boldsymbol{\pitchfork}' = (\pitchfork_0', \pitchfork_1', \ldots, \pitchfork_L')$ satisfying $\mathcal{B}(\boldsymbol{\pitchfork}') = \mathcal{B}'$. There is a smallest branch in $\mathcal{B}(\boldsymbol{\pitchfork}')$ that contains λ°, say λ', and there are no subsets of λ° in $\mathcal{B}(\boldsymbol{\pitchfork}')$. Thus if $\lambda^\circ = \{i_1, i_2, \ldots, i_n\}$, the singletons $\{i_1\}, \{i_2\}, \ldots, \{i_n\}$ must appear in $\pitchfork_\ell'$ for all $\ell > \ell_{\boldsymbol{\pitchfork}'}(\lambda')$. Now let $\pitchfork_0, \pitchfork_1, \ldots, \pitchfork_{L'}$ be the partitions satisfying $\pitchfork_\ell = \pitchfork_\ell'$ for $0 \leq \ell \leq \ell_{\boldsymbol{\pitchfork}'}(\lambda')$, and for $\ell > \ell_{\boldsymbol{\pitchfork}'}(\lambda')$ take $\pitchfork_\ell$ to be equal to $\pitchfork_\ell'$ but with $\{i_1\}\{i_2\} \ldots \{i_n\}$ replaced by

$\lambda^\circ = \{i_1, i_2, \ldots, i_n\}$. This yields partitions

$$\overline{\pitchfork} = \pitchfork_0 > \pitchfork_1 > \cdots > \pitchfork_{L'}$$

with $\mathcal{B}(\boldsymbol{\pitchfork}^*) \setminus (\pitchfork_0 \cup \pitchfork_1 \cup \cdots \cup \pitchfork_{L'}) = \{\lambda \in \mathcal{B}(\boldsymbol{\pitchfork}^*) : \lambda \subsetneq \lambda^\circ\}$ where λ° is the only non-singleton part in $\pitchfork_{L'}$. We continue recursively for $\ell \geq L'$, defining $\pitchfork_{\ell+1}$ to be the partition comprised of the largest branches remaining in $\mathcal{B}(\boldsymbol{\pitchfork}^*) \setminus (\pitchfork_0 \cup \pitchfork_1 \cup \cdots \cup \pitchfork_\ell)$ and any leftover singletons in $[N]$. We end the recursion at the first $\ell \geq L'$ such that $\mathcal{B}(\boldsymbol{\pitchfork}^*) \setminus (\pitchfork_0 \cup \pitchfork_1 \cup \cdots \cup \pitchfork_\ell) = \varnothing$ and set $L := \ell + 1$ and $\pitchfork_L := \underline{\pitchfork}$. The result is a splitting chain $\boldsymbol{\pitchfork}^\circ = (\pitchfork_0, \pitchfork_1, \ldots, \pitchfork_L)$ with $\mathcal{B}(\boldsymbol{\pitchfork}^\circ) = \mathcal{B}(\boldsymbol{\pitchfork}^*)$ (i.e., $\boldsymbol{\pitchfork}^\circ \simeq \boldsymbol{\pitchfork}^*$) and a level $\pitchfork_{L'}$ whose only non-singleton part is λ°, and hence $E_{\pitchfork_{L'}}(\boldsymbol{s}) = e_{\lambda^\circ}(\boldsymbol{s})$. Thus for $\lambda^\circ \in \mathcal{R}_N$ we have a splitting chain $\boldsymbol{\pitchfork}^\circ \simeq \boldsymbol{\pitchfork}^*$ satisfying

$$\begin{aligned}\mathcal{LP}_{\boldsymbol{\pitchfork}^\circ}(0) &= \bigcap_{\ell=1}^{L-1} \left\{\boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}} : \mathrm{Re}(E_{\pitchfork_\ell}(\boldsymbol{s})) > 0\right\} \\ &\subset \left\{\boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}} : \mathrm{Re}(E_{\pitchfork_{L'}}(\boldsymbol{s})) > 0\right\} = \left\{\boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}} : \mathrm{Re}(e_{\lambda^\circ}(\boldsymbol{s})) > 0\right\},\end{aligned}$$

and hence

$$\bigcap_{\substack{\boldsymbol{\pitchfork} \in \mathcal{S}_N \\ \boldsymbol{\pitchfork} \simeq \boldsymbol{\pitchfork}^*}} \mathcal{LP}_{\boldsymbol{\pitchfork}}(0) \subset \left\{\boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}} : \mathrm{Re}(e_{\lambda^\circ}(\boldsymbol{s})) > 0\right\}.$$

Since this argument works for any $\lambda^\circ \in \mathcal{B}(\boldsymbol{\pitchfork}^*) \setminus \overline{\pitchfork}$, it follows that

$$\bigcap_{\substack{\boldsymbol{\pitchfork} \in \mathcal{S}_N \\ \boldsymbol{\pitchfork} \simeq \boldsymbol{\pitchfork}^*}} \mathcal{LP}_{\boldsymbol{\pitchfork}}(0) \subset \bigcap_{\lambda \in \mathcal{B}(\boldsymbol{\pitchfork}) \setminus \overline{\pitchfork}} \left\{\boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}} : \mathrm{Re}(e_\lambda(\boldsymbol{s})) > 0\right\} = \mathcal{BP}_{\boldsymbol{\pitchfork}^*}.$$

To show the reverse containment, suppose $\boldsymbol{s} \in \mathcal{BP}_{\boldsymbol{\pitchfork}^*}$, so that $\mathrm{Re}(e_\lambda(\boldsymbol{s})) > 0$ for all $\lambda \in \mathcal{B}(\boldsymbol{\pitchfork}^*) \setminus \overline{\pitchfork}$. For any splitting chain $\boldsymbol{\pitchfork} \simeq \boldsymbol{\pitchfork}^*$ and any level $\pitchfork_\ell$ with $1 \leq \ell \leq L(\boldsymbol{\pitchfork}) - 1$, the level exponent $E_{\pitchfork_\ell}(\boldsymbol{s}) = \sum_{\lambda \in \pitchfork_\ell} e_\lambda(\boldsymbol{s})$ is a sum over at least one $\lambda \in \mathcal{B}(\boldsymbol{\pitchfork}) \cap \pitchfork_\ell \subset \mathcal{B}(\boldsymbol{\pitchfork}^*) \setminus \overline{\pitchfork}$ and hence $\mathrm{Re}(E_{\pitchfork_\ell}(\boldsymbol{s})) > 0$. It follows that $\boldsymbol{s} \in \mathcal{LP}_{\boldsymbol{\pitchfork}}(0)$ for all $\boldsymbol{\pitchfork} \simeq \boldsymbol{\pitchfork}^*$, and we conclude that

$$\bigcap_{\substack{\boldsymbol{\pitchfork} \in \mathcal{S}_N \\ \boldsymbol{\pitchfork} \simeq \boldsymbol{\pitchfork}^*}} \mathcal{LP}_{\boldsymbol{\pitchfork}}(0) = \mathcal{BP}_{\boldsymbol{\pitchfork}^*}.$$

Finally, since this holds for all $\boldsymbol{\pitchfork}^* \in \mathcal{R}_N$, part (b) implies $\bigcap_{\boldsymbol{\pitchfork} \in \mathcal{S}_N} \mathcal{LP}_{\boldsymbol{\pitchfork}}(0) = \bigcap_{\boldsymbol{\pitchfork}^* \in \mathcal{R}_N} \mathcal{BP}_{\boldsymbol{\pitchfork}}$.

$\square$

It is worth noting here that the recursive algorithm in the proof of part (b) of Lemma 1.6.5 can be used to find the reduction of any splitting chain. We now apply this algorithm to the splitting chain $\boldsymbol{\pitchfork} \in \mathcal{S}_9$ from Figure 2 in Section 3.2.

Example 4.1.2. Recall $\boldsymbol{\pitchfork} = (\pitchfork_0, \pitchfork_1, \pitchfork_2, \pitchfork_3, \pitchfork_4) \in \mathcal{S}_9$ from Figure 2, where

$$\begin{aligned}
\pitchfork_0 &= \{1,2,3,4,5,6,7,8,9\}, \\
\pitchfork_1 &= \{1,2,3,4,5\}\{6,7,8,9\}, \\
\pitchfork_2 &= \{1,2,3\}\{4,5\}\{6,7,8,9\}, \\
\pitchfork_3 &= \{1,2,3\}\{4\}\{5\}\{6\}\{7\}\{8\}\{9\}, \\
\pitchfork_4 &= \{1\}\{2\}\{3\}\{4\}\{5\}\{6\}\{7\}\{8\}\{9\}.
\end{aligned}$$

Before starting the algorithm, note that its branch set is

$$\mathcal{B}(\boldsymbol{\pitchfork}) = \big\{\{1,2,3,4,5,6,7,8,9\},\ \{1,2,3,4,5\},\ \{6,7,8,9\},\ \{1,2,3\},\ \{4,5\}\big\}.$$

We initialize the algorithm by letting $\pitchfork_0^* := \{1,2,3,4,5,6,7,8,9\}$, and the recursive part runs as follows:

- $\ell = 0:$ The maximal branches remaining in

$$\mathcal{B}(\boldsymbol{\pitchfork}) \setminus \pitchfork_0^* = \big\{\{1,2,3,4,5\},\ \{6,7,8,9\}, \{1,2,3\}, \{4,5\}\big\}$$

  (partially ordered via $\subset$) are the incomparable sets $\{1,2,3,4,5\}$ and $\{6,7,8,9\}$, so we define the partition

$$\pitchfork_1^* := \{1,2,3,4,5\}\{6,7,8,9\}.$$

- $\ell = 1:$ The maximal branches remaining in $\mathcal{B}(\boldsymbol{\pitchfork}) \setminus (\pitchfork_0^* \cup \pitchfork_1^*) = \{\{1,2,3\},\ \{4,5\}\}$ are the incomparable sets $\{1,2,3\}$ and $\{4,5\}$, so by including leftover singletons $\{i\} \subset [9]$ we define the partition

$$\pitchfork_2^* := \{1,2,3\}\{4,5\}\{6\}\{7\}\{8\}\{9\}.$$

- $\ell = 2:$ We now have $\mathcal{B}(\boldsymbol{\pitchfork}) \setminus (\pitchfork_0^* \cup \pitchfork_1^* \cup \pitchfork_2^*) = \varnothing$, so end the recursion.

Finally, since the recursion stopped at step $\ell = 2$, we set $L^* := \ell + 1 = 3$, define the last partition via

$$\pitchfork_3^* := \underline{\pitchfork} = \{1\}\{2\}\{3\}\{4\}\{5\}\{6\}\{7\}\{8\}\{9\},$$

and note that the algorithm is done. It is straightforward to verify that the resulting tuple $\boldsymbol{\pitchfork}^* := (\pitchfork_0^*, \pitchfork_1^*, \pitchfork_2^*, \pitchfork_3^*)$ is a reduced splitting chain of order 9 with $\boldsymbol{\pitchfork} \simeq \boldsymbol{\pitchfork}^*$ and $L(\boldsymbol{\pitchfork}^*) \leq L(\boldsymbol{\pitchfork})$.

We are now ready to define and discuss branch pairs:

Definition 4.1.3. If $\pitchfork^* \in \mathcal{R}_N$ and $\boldsymbol{k} = (k_\lambda)$ is a tuple of positive integers indexed by $\lambda \in \mathcal{B}(\pitchfork^*)$, we call $[\pitchfork^*, \boldsymbol{k}]$ a *branch pair.*

The following theorem establishes a useful and explicit correspondence between the set of all branch pairs $[\pitchfork^*, \boldsymbol{k}]$ with a particular $\pitchfork^* \in \mathcal{R}_N$ and the set of all level pairs $(\pitchfork, \boldsymbol{n})$ such that $\pitchfork$ has reduction $\pitchfork^*$:

Theorem 4.1.4. *Suppose $\pitchfork^* \in \mathcal{R}_N$. There is a bijection*

$$\left\{[\pitchfork^*, \boldsymbol{k}] : \boldsymbol{k} = (k_\lambda) \in \mathbb{N}^{\mathcal{B}(\pitchfork^*)}\right\} \longleftrightarrow \bigsqcup_{\substack{\pitchfork \in \mathcal{S}_N \\ \pitchfork \simeq \pitchfork^*}} \left\{(\pitchfork, \boldsymbol{n}) : \boldsymbol{n} = (\eta_0, \eta_1, \ldots, \eta_{L(\pitchfork)-1}) \in \mathbb{N}^{L(\pitchfork)}\right\}$$

such that if $[\pitchfork^*, \boldsymbol{k}]$ and $(\pitchfork, \boldsymbol{n})$ correspond, then we have $k_{[N]} = \eta_0$ and for each $\lambda \in \mathcal{B}(\pitchfork) \setminus \overline{\pitchfork}$ we have

$$k_\lambda = \sum_{\ell = \ell_{\pitchfork}(\lambda^*)+1}^{\ell_{\pitchfork}(\lambda)} \eta_\ell \tag{4.1.1}$$

where $\lambda^* \in \mathcal{B}(\pitchfork)$ is the smallest branch properly containing λ.

Proof. Fix $\pitchfork^* \in \mathcal{R}_N$ and let $\boldsymbol{k} = (k_\lambda)$ be an arbitrary tuple of positive integers indexed by $\lambda \in \mathcal{B}(\pitchfork^*)$. We associate a unique level pair to $[\pitchfork^*, \boldsymbol{k}]$ as follows. The set

$$\mathcal{M} := \left\{ -1 + \sum_{\substack{\lambda' \in \mathcal{B}(\pitchfork^*) \\ \lambda' \supset \lambda}} k_{\lambda'} : \lambda \in \mathcal{B}(\pitchfork^*) \right\}$$

is comprised of finitely many, say L, nonnegative integers. Put $m_0 := -1$ and let $\{m_1, m_2, \ldots, m_L\}$ be the enumeration of $\mathcal{M}$ satisfying $m_0 < m_1 < m_2 < \cdots < m_L$.

For each $\lambda \in \mathcal{B}(\boldsymbol{\pitchfork}^*)$ define

$$\ell_{[\boldsymbol{\pitchfork}^*,\boldsymbol{k}]}(\lambda) := \text{the unique } \ell \in \{0,1,\ldots,L-1\} \text{ such that } \sum_{\substack{\lambda' \in \mathcal{B}(\boldsymbol{\pitchfork}^*) \\ \lambda' \supset \lambda}} k_{\lambda'} = m_{\ell+1} + 1.$$

Then by the definition of $\mathcal{M} = \{m_1, m_2, \ldots, m_L\}$, for each $\ell \in \{0,1,\ldots,L-1\}$ there is at least one $\lambda \in \mathcal{B}(\boldsymbol{\pitchfork}^*)$ satisfying $\ell_{[\boldsymbol{\pitchfork}^*,\boldsymbol{k}]}(\lambda) = \ell$, and $\lambda = [N]$ is the unique branch satisfying $\ell_{[\boldsymbol{\pitchfork}^*,\boldsymbol{k}]}(\lambda) = 0$. Moreover, we have $\ell_{[\boldsymbol{\pitchfork}^*,\boldsymbol{k}]}(\lambda') < \ell_{[\boldsymbol{\pitchfork}^*,\boldsymbol{k}]}(\lambda)$ whenever $\lambda, \lambda' \in \mathcal{B}(\boldsymbol{\pitchfork}^*)$ satisfy $\lambda \subsetneq \lambda'$. We now construct L partitions $\pitchfork_0, \pitchfork_1, \ldots, \pitchfork_{L-1}$ of $[N]$ as follows. Let $\pitchfork_0 := \{[N]\}$, and for each $\ell \in \{1,\ldots,L-1\}$ let $\mathcal{B}_\ell(\boldsymbol{\pitchfork}^*)$ be the subset of $\mathcal{B}(\boldsymbol{\pitchfork}^*)$ defined by

$$\lambda \in \mathcal{B}_\ell(\boldsymbol{\pitchfork}^*) \quad \Longleftrightarrow \quad \begin{array}{l} \ell_{[\boldsymbol{\pitchfork}^*,\boldsymbol{k}]}(\lambda) \geq \ell \text{ and } \ell_{[\boldsymbol{\pitchfork}^*,\boldsymbol{k}]}(\lambda^*) < \ell, \text{ where } \lambda^* \text{ is the} \\ \text{smallest branch in } \mathcal{B}(\boldsymbol{\pitchfork}^*) \text{ satisfying } \lambda \subsetneq \lambda^*, \end{array}$$

let $\pitchfork_\ell$ be the partition of $[N]$ comprised of all $\lambda \in \mathcal{B}_\ell(\boldsymbol{\pitchfork}^*)$ and all $\{i\} \subset [N] \setminus \bigcup_{\lambda \in \mathcal{B}_\ell(\boldsymbol{\pitchfork}^*)} \lambda$, and finally let $\pitchfork_L := \underline{\pitchfork}$. Now if $\ell \in \{1,2,\ldots,L\}$ and $\lambda \in \pitchfork_\ell$, then either λ is a singleton or $\lambda \in \mathcal{B}_\ell(\boldsymbol{\pitchfork}^*)$. In the latter case we have $\ell_{[\boldsymbol{\pitchfork}^*,\boldsymbol{k}]}(\lambda^*) < \ell \leq \ell_{[\boldsymbol{\pitchfork}^*,\boldsymbol{k}]}(\lambda)$ where λ^* is the smallest branch in $\mathcal{B}(\boldsymbol{\pitchfork}^*)$ satisfying $\lambda \subsetneq \lambda^*$. If $\ell_{[\boldsymbol{\pitchfork}^*,\boldsymbol{k}]}(\lambda^*) = \ell - 1$, then $\lambda^* \in \pitchfork_{\ell-1}$. Otherwise $\ell_{[\boldsymbol{\pitchfork}^*,\boldsymbol{k}]}(\lambda^*) < \ell - 1$, in which case $\lambda \in \pitchfork_{\ell-1}$, so in any case each $\lambda \in \pitchfork_\ell$ is contained in some part of $\pitchfork_{\ell-1}$ and hence $\pitchfork_\ell \leq \pitchfork_{\ell-1}$. Moreover, there is at least one part $\lambda' \in \pitchfork_{\ell-1}$ with $\ell_{[\boldsymbol{\pitchfork}^*,\boldsymbol{k}]}(\lambda') = \ell - 1$, so $\lambda' \notin \mathcal{B}_\ell(\boldsymbol{\pitchfork}^*)$ implies $\lambda' \notin \pitchfork_\ell$ and hence $\pitchfork_\ell < \pitchfork_{\ell-1}$. Now $\boldsymbol{\pitchfork} := (\pitchfork_0, \pitchfork_1, \ldots, \pitchfork_L)$ is a tuple of partitions of $[N]$ satisfying $\pitchfork_0 > \pitchfork_1 > \cdots > \pitchfork_L = \underline{\pitchfork}$, so $\boldsymbol{\pitchfork}$ is a splitting chain of order N and length $L(\boldsymbol{\pitchfork}) = L$. It is clear from the construction of $\boldsymbol{\pitchfork}$ that $\mathcal{B}(\boldsymbol{\pitchfork}) = \bigcup_{\ell=0}^{L-1} \mathcal{B}_\ell(\boldsymbol{\pitchfork}^*) = \mathcal{B}(\boldsymbol{\pitchfork}^*)$, and that each branch $\lambda \in \mathcal{B}(\boldsymbol{\pitchfork}) = \mathcal{B}(\boldsymbol{\pitchfork}^*)$ has depth $\ell_{\boldsymbol{\pitchfork}}(\lambda) = \ell_{[\boldsymbol{\pitchfork}^*,\boldsymbol{k}]}(\lambda)$. Thus if we define $\boldsymbol{n} := (\eta_0, \eta_1, \ldots, \eta_{L-1}) \in \mathbb{N}^L$ by $\eta_\ell := m_{\ell+1} - m_\ell$, it follows that $(\boldsymbol{\pitchfork}, \boldsymbol{n})$ is a level pair such that $\pitchfork \simeq \pitchfork^*$ and every $\lambda \in \mathcal{B}(\pitchfork)$ satisfies

$$\sum_{\substack{\lambda' \in \mathcal{B}(\pitchfork) \\ \lambda' \supset \lambda}} k_{\lambda'} = m_{\ell_{[\pitchfork^*, \boldsymbol{k}]}(\lambda)+1} + 1 = \sum_{\ell=0}^{\ell_{[\pitchfork^*, \boldsymbol{k}]}(\lambda)} (m_{\ell+1} - m_\ell) = \sum_{\ell=0}^{\ell_{\pitchfork}(\lambda)} \eta_\ell.$$

Then $k_{[N]} = \eta_0$, and if $\lambda \in \mathcal{B}(\pitchfork) \setminus \overline{\pitchfork}$ and λ^* is the smallest branch in $\mathcal{B}(\pitchfork)$ properly containing λ we have

$$k_\lambda = \sum_{\substack{\lambda' \in \mathcal{B}(\pitchfork) \\ \lambda' \supset \lambda}} k_{\lambda'} - \sum_{\substack{\lambda' \in \mathcal{B}(\pitchfork) \\ \lambda' \supset \lambda^*}} k_{\lambda'} = \sum_{\ell=0}^{\ell_{\pitchfork}(\lambda)} \eta_\ell - \sum_{\ell=0}^{\ell_{\pitchfork}(\lambda^*)} \eta_\ell = \sum_{\ell=\ell_{\pitchfork}(\lambda^*)+1}^{\ell_{\pitchfork}(\lambda)} \eta_\ell.$$

Therefore by setting $F([\pitchfork^*, \boldsymbol{k}]) := (\pitchfork, \boldsymbol{n})$ we obtain a well-defined map

$$\begin{aligned} F : &\left\{ [\pitchfork^*, \boldsymbol{k}] : \boldsymbol{k} = (k_\lambda) \in \mathbb{N}^{\mathcal{B}(\pitchfork^*)} \right\} \\ &\qquad \longrightarrow \bigsqcup_{\substack{\pitchfork \in \mathcal{S}_N \\ \pitchfork \simeq \pitchfork'}} \left\{ (\pitchfork, \boldsymbol{n}) : \boldsymbol{n} = (\eta_0, \eta_1, \ldots, \eta_{L(\pitchfork)-1}) \in \mathbb{N}^{L(\pitchfork)} \right\} \end{aligned}$$

satisfying (4.1.1). We will now show that F is a bijection by constructing an inverse. Let $\pitchfork \in \mathcal{S}_N$ be any splitting chain with reduction $\pitchfork^*$, let $\boldsymbol{n} = (\eta_0, \eta_1, \ldots, \eta_{L(\pitchfork)-1})$ be an arbitrary tuple of $L(\pitchfork)$ positive integers, and define $G((\pitchfork, \boldsymbol{n})) := [\pitchfork^*, \boldsymbol{k}]$ by defining $k_\lambda \in \mathbb{N}$ for each $\lambda \in \mathcal{B}(\pitchfork^*) = \mathcal{B}(\pitchfork)$ via

$$k_\lambda := \begin{cases} \eta_0 & \text{if } \lambda = [N], \\ \displaystyle\sum_{\ell=\ell_{\pitchfork}(\lambda^*)+1}^{\ell_{\pitchfork}(\lambda)} \eta_\ell & \text{if } \lambda^* \in \mathcal{B}(\pitchfork) \text{ is the smallest branch properly containing } \lambda. \end{cases}$$

Therefore we have a well-defined map

$$G : \bigsqcup_{\substack{\pitchfork \in \mathcal{S}_N \\ \pitchfork \simeq \pitchfork^*}} \left\{(\pitchfork, \boldsymbol{n}) : \boldsymbol{n} = (\eta_0, \eta_1, \dots, \eta_{L(\pitchfork)-1}) \in \mathbb{N}^{L(\pitchfork)}\right\}$$
$$\longrightarrow \left\{[\pitchfork^*, \boldsymbol{k}] : \boldsymbol{k} = (k_\lambda) \in \mathbb{N}^{\mathcal{B}(\pitchfork^*)}\right\},$$

and it is immediate from (4.1.1) and the definition of G that $G \circ F([\pitchfork^*, \boldsymbol{k}]) = [\pitchfork^*, \boldsymbol{k}]$ for every $\boldsymbol{k} = (k_\lambda)$ indexed by $\lambda \in \mathcal{B}(\pitchfork^*)$. It remains to show that $F \circ G((\pitchfork, \boldsymbol{n})) = (\pitchfork, \boldsymbol{n})$ for all level pairs in

$$\bigsqcup_{\substack{\pitchfork \in \mathcal{S}_N \\ \pitchfork \simeq \pitchfork^*}} \left\{(\pitchfork, \boldsymbol{n}) : \boldsymbol{n} = (\eta_0, \eta_1, \dots, \eta_{L(\pitchfork)-1}) \in \mathbb{N}^{L(\pitchfork)}\right\}.$$

To this end, let $(\pitchfork', \boldsymbol{n}')$ be such a level pair and suppose $[\pitchfork^*, \boldsymbol{k}] = G((\pitchfork', \boldsymbol{n}'))$, so that

$$k_\lambda = \begin{cases} \eta'_0 & \text{if } \lambda = [N], \\ \displaystyle\sum_{\ell = \ell_{\pitchfork'}(\lambda^*)+1}^{\ell_{\pitchfork'}(\lambda)} \eta'_\ell & \text{if } \lambda^* \in \mathcal{B}(\pitchfork') \text{ is the smallest branch properly containing } \lambda, \end{cases} \tag{4.1.2}$$

for each $\lambda \in \mathcal{B}(\pitchfork')$. Now suppose $(\pitchfork, \boldsymbol{n}) = F([\pitchfork^*, \boldsymbol{k}])$ and recall the following details from our definition of F. The strictly increasing set of integers $\mathcal{M} = \{m_1, m_2, \dots, m_L\}$ is defined by

$$\mathcal{M} = \left\{ -1 + \sum_{\substack{\lambda' \in \mathcal{B}(\pitchfork^*) \\ \lambda' \supset \lambda}} k_{\lambda'} : \lambda \in \mathcal{B}(\pitchfork^*) \right\}$$

and satisfies $\eta_\ell = m_{\ell+1} - m_\ell$ for all $\ell \in \{0, 1, \ldots, L-1\}$, where $m_0 = -1$. Moreover, recall that $\boldsymbol{\pitchfork} = (\pitchfork_0, \pitchfork_1, \ldots, \pitchfork_L)$ is then completely determined using the integers defined for each $\lambda \in \mathcal{B}(\boldsymbol{\pitchfork}^*)$ by

$$\ell_{[\boldsymbol{\pitchfork}^*, \boldsymbol{k}]}(\lambda) = \text{the unique } \ell \in \{0, 1, \ldots, L-1\} \text{ such that } \sum_{\substack{\lambda' \in \mathcal{B}(\boldsymbol{\pitchfork}^*) \\ \lambda' \supset \lambda}} k_{\lambda'} = m_{\ell+1} + 1,$$

and we saw that $L(\boldsymbol{\pitchfork}) = L$, $\mathcal{B}(\boldsymbol{\pitchfork}) = \mathcal{B}(\boldsymbol{\pitchfork}^*)$, and $\ell_{\boldsymbol{\pitchfork}}(\lambda) = \ell_{[\boldsymbol{\pitchfork}^*, \boldsymbol{k}]}(\lambda)$ for all $\lambda \in \mathcal{B}(\boldsymbol{\pitchfork}) = \mathcal{B}(\boldsymbol{\pitchfork}^*)$. Now since $\mathcal{B}(\boldsymbol{\pitchfork}^*) = \mathcal{B}(\boldsymbol{\pitchfork}')$ and each integer k_λ with $\lambda \in \mathcal{B}(\boldsymbol{\pitchfork}')$ is given by (4.1.2), we have

$$\begin{aligned}
\{m_1, m_2, \ldots, m_L\} = \mathcal{M} &= \left\{ -1 + \sum_{\substack{\lambda' \in \mathcal{B}(\boldsymbol{\pitchfork}') \\ \lambda' \supset \lambda}} k_{\lambda'} : \lambda \in \mathcal{B}(\boldsymbol{\pitchfork}') \right\} \\
&= \left\{ -1 + \sum_{\ell=0}^{\ell_{\boldsymbol{\pitchfork}'}(\lambda)} \eta'_\ell : \lambda \in \mathcal{B}(\boldsymbol{\pitchfork}') \right\}.
\end{aligned}$$

In particular, for each $\lambda \in \mathcal{B}(\boldsymbol{\pitchfork}) = \mathcal{B}(\boldsymbol{\pitchfork}^*) = \mathcal{B}(\boldsymbol{\pitchfork}')$ we have

$$\sum_{\ell=0}^{\ell_{\boldsymbol{\pitchfork}}(\lambda)} \eta_\ell = m_{\ell_{\boldsymbol{\pitchfork}}(\lambda)+1} + 1 = \sum_{\substack{\lambda' \in \mathcal{B}(\boldsymbol{\pitchfork}^*) \\ \lambda' \supset \lambda}} k_{\lambda'} = \sum_{\substack{\lambda' \in \mathcal{B}(\boldsymbol{\pitchfork}') \\ \lambda' \supset \lambda}} k_{\lambda'} = \sum_{\ell=0}^{\ell_{\boldsymbol{\pitchfork}'}(\lambda)} \eta'_\ell. \tag{4.1.3}$$

Since $\boldsymbol{\pitchfork}'$ is a splitting chain, it must satisfy $\{[N]\} = \pitchfork'_0 > \pitchfork'_1 > \cdots > \pitchfork'_{L(\boldsymbol{\pitchfork}')} = \underline{\pitchfork}$, and hence for each level index $\ell' \in \{0, 1, 2, \ldots, L(\boldsymbol{\pitchfork}') - 1\}$ we may select a branch $\lambda^{(\ell')} \in \mathcal{B}(\boldsymbol{\pitchfork}') \cap \pitchfork'_{\ell'}$ satisfying $\ell_{\boldsymbol{\pitchfork}'}(\lambda^{(\ell')}) = \ell'$ and have

$$L(\boldsymbol{\pitchfork}') - 1 = \ell_{\boldsymbol{\pitchfork}'}(\lambda^{(L(\boldsymbol{\pitchfork}')-1)}) = \max\{\ell_{\boldsymbol{\pitchfork}'}(\lambda) : \lambda \in \mathcal{B}(\boldsymbol{\pitchfork}')\}.$$

Now since each η'_ℓ is positive, it follows that

$$\{m_1, m_2, \ldots, m_L\} = \left\{ -1 + \sum_{\ell=0}^{\ell_{\pitchfork'}(\lambda)} \eta'_\ell : \lambda \in \mathcal{B}(\pitchfork') \right\}$$
$$= \left\{ -1 + \sum_{\ell=0}^{\ell'} \eta'_\ell : \ell' \in \{0, 1, \ldots, L(\pitchfork') - 1\} \right\}.$$

But the values $m_1, m_2, \ldots, m_L$ strictly increase and the sums $-1 + \sum_{\ell=0}^{\ell'} \eta'_\ell$ also strictly increase with ℓ', so it must be the case that $L(\pitchfork') = L = L(\pitchfork)$ and moreover,

$$m_{\ell'+1} = -1 + \sum_{\ell=0}^{\ell'} \eta'_\ell \quad \text{for all } \ell' \in \{0, 1, \ldots, L(\pitchfork') - 1\}.$$

Thus $\eta'_0 = m_1 + 1 = \eta_0$, and for every $\ell' \in \{1, \ldots, L(\pitchfork) - 1\}$ we have

$$\eta_{\ell'} = m_{\ell'+1} - m_{\ell'} = \left(-1 + \sum_{\ell=0}^{\ell'} \eta'_\ell \right) - \left(-1 + \sum_{\ell=0}^{\ell'-1} \eta'_\ell \right) = \eta'_{\ell'},$$

so we conclude that $\boldsymbol{n} = \boldsymbol{n}'$. Now (4.1.3) and positivity of $\eta_\ell = \eta'_\ell$ imply $\ell_{\pitchfork'}(\lambda) = \ell_{\pitchfork}(\lambda) = \ell_{[\pitchfork^*, \boldsymbol{k}]}(\lambda)$ for all $\lambda \in \mathcal{B}(\pitchfork') = \mathcal{B}(\pitchfork^*) = \mathcal{B}(\pitchfork)$, so each partition $\pitchfork_\ell$ defined via the set $\mathcal{B}_\ell(\pitchfork^*)$ above is precisely $\pitchfork'_\ell$. Therefore $\pitchfork = \pitchfork'$, so

$$F \circ G((\pitchfork', \boldsymbol{n}')) = F([\pitchfork^*, \boldsymbol{k}]) = (\pitchfork, \boldsymbol{n}) = (\pitchfork', \boldsymbol{n}')$$

and we conclude that $G = F^{-1}$. $\square$

To make the correspondence more intuitive, we recall that the splitting pair $(\pitchfork, \boldsymbol{n})$ associated to the tree in Example 3.1.2 had $\boldsymbol{n} = (2, 1, 3, 2)$ in Figure 2. By Theorem 4.1.4, $(\pitchfork, \boldsymbol{n})$ corresponds to $[\pitchfork^*, \boldsymbol{k}]$ where $\pitchfork^*$ is the reduction computed in

Example 4.1.2 and $\boldsymbol{k}$ is displayed in the diagram below. Note that these $\boldsymbol{k}$ and $\boldsymbol{n}$ indeed satisfy (4.1.1).

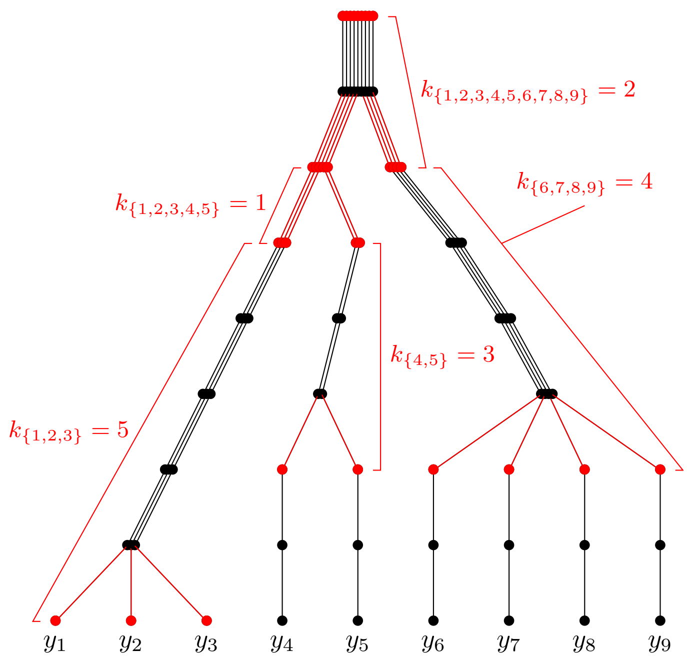

FIGURE 3. The branch pair $[\pitchfork^*, \boldsymbol{k}]$ associated to the tree in Example 3.1.2

4.2. Integration with branch pairs

With Lemma 1.6.5 and Theorem 4.1.4 in hand, we may now give a "branch-centric" reinterpretation of Proposition 3.3.1 in the $b = 0$ case.

Corollary 4.2.1. *If $a \in \mathbb{C}$, $[\pitchfork^*, \boldsymbol{k}]$ is a branch pair, and $(\pitchfork, \boldsymbol{n})$ is the level pair corresponding to $[\pitchfork^*, \boldsymbol{k}]$, then for every $\boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}}$ we have*

$$\int_{\mathcal{T}(\pitchfork,\boldsymbol{n})} \left(\max_{i<j} |x_i - x_j|\right)^a \prod_{i<j} |x_i - x_j|^{s_{ij}}\, dx_1 \dots dx_N$$
$$= q^{-(N-1+a+\sum_{i<j} s_{ij})(k_{[N]}-1)} \cdot \frac{M_{\pitchfork^*}(q)}{q^{N-1}} \cdot \prod_{\lambda \in \mathcal{B}(\pitchfork^*)\setminus \overline{\pitchfork}} q^{-e_\lambda(\boldsymbol{s}) k_\lambda}.$$

Proof. If $b = 0$, Proposition 3.3.1 gives

$$\int_{\mathcal{T}(\pitchfork, \boldsymbol{n})} \big(\max_{i<j} |x_i - x_j|\big)^a \prod_{i<j} |x_i - x_j|^{s_{ij}} \, dx_1 \dots dx_N$$
$$= q^{-(N-1+a+\sum_{i<j} s_{ij})(\eta_0 - 1)} \cdot \frac{M_{\pitchfork}(q)}{q^{N-1}} \cdot \prod_{\ell=1}^{L(\pitchfork)-1} q^{-E_{\pitchfork_\ell}(\boldsymbol{s})\eta_\ell}.$$

Since $\pitchfork \simeq \pitchfork^*$, part (a) of Lemma 1.6.5 implies $M_{\pitchfork^*}(q) = M_{\pitchfork}(q)$ and $\mathcal{B}(\pitchfork^*) = \mathcal{B}(\pitchfork)$. We also have $k_{[N]} = \eta_0$ by Theorem 4.1.4, so it suffices to show that

$$\sum_{\ell=1}^{L(\pitchfork)-1} E_{\pitchfork_\ell}(\boldsymbol{s})\eta_\ell = \sum_{\lambda \in \mathcal{B}(\pitchfork) \setminus \overline{\pitchfork}} e_\lambda(\boldsymbol{s}) k_\lambda. \tag{4.2.4}$$

To see why (4.2.4) is true, note that

$$E_{\pitchfork_\ell}(\boldsymbol{s}) = \sum_{\lambda \in \mathcal{B}(\pitchfork) \cap \pitchfork_\ell} e_\lambda(\boldsymbol{s}),$$

and for $\ell \in \{1, 2, \dots, L(\pitchfork) - 1\}$ we have $\lambda \in \mathcal{B}(\pitchfork) \cap \pitchfork_\ell$ if and only if $\ell_{\pitchfork}(\lambda^*) + 1 \leq \ell \leq \ell_{\pitchfork}(\lambda)$, where λ^* denotes the smallest branch in $\mathcal{B}(\pitchfork)$ properly containing λ. Therefore if $\lambda \in \mathcal{B}(\pitchfork) \setminus \overline{\pitchfork}$, then the branch exponent $e_\lambda(\boldsymbol{s})$ is a summand of $E_{\pitchfork_\ell}(\boldsymbol{s})$ if and only if $\ell_{\pitchfork}(\lambda^*) + 1 \leq \ell \leq \ell_{\pitchfork}(\lambda)$, so we have

$$\sum_{\ell=1}^{L(\pitchfork)-1} E_{\pitchfork_\ell}(\boldsymbol{s})\eta_\ell = \sum_{\lambda \in \mathcal{B}(\pitchfork) \setminus \overline{\pitchfork}} \left(\sum_{\ell=\ell_{\pitchfork}(\lambda^*)+1}^{\ell_{\pitchfork}(\lambda)} e_\lambda(\boldsymbol{s})\eta_\ell \right) = \sum_{\lambda \in \mathcal{B}(\pitchfork) \setminus \overline{\pitchfork}} e_\lambda(\boldsymbol{s}) \left(\sum_{\ell=\ell_{\pitchfork}(\lambda^*)+1}^{\ell_{\pitchfork}(\lambda)} \eta_\ell \right).$$

But $k_\lambda = \sum_{\ell=\ell_{\pitchfork}(\lambda^*)+1}^{\ell_{\pitchfork}(\lambda)} \eta_\ell$ by (4.1.1) in Theorem 4.1.4, so (4.2.4) is proved and the corollary follows.

□

The following remark should be understood of the "branch-centric" analogue of Remark 3.3.2.

Remark 4.2.2. Note that the integral formula in Corollary 4.2.1 provides yet another method for computing $\mu^N(\mathcal{T}(\pitchfork, \boldsymbol{n}))$, but now in terms of the branch pair $[\pitchfork^*, \boldsymbol{k}]$ corresponding to $(\pitchfork, \boldsymbol{n})$. Indeed, setting $s_{ij} = a = 0$ for all $i < j$ gives $e_\lambda(\boldsymbol{s}) = \#\lambda - 1$ by part (b) of Definition 1.6.4, and then the formula in Corollary 4.2.1 simplifies very nicely:

$$\mu^N(\mathcal{T}(\pitchfork, \boldsymbol{n})) = M_{\pitchfork^*}(q) \cdot \prod_{\lambda \in \mathcal{B}(\pitchfork^*)} q^{-(\#\lambda - 1)k_\lambda}. \tag{4.2.5}$$

The exponential factors in the formula in Corollary 4.2.1 are completely determined by the branch pair $[\pitchfork^*, \boldsymbol{k}]$ corresponding to the level pair $(\pitchfork, \boldsymbol{n})$. Since $k_{[N]} = \eta_0$ in this case, the leftmost factor $q^{-(N-1+a+\sum_{i<j} s_{ij})(k_{[N]}-1)}$ pertains to "root data" and the root polytope (just as in Remark 3.3.2), with $b = 0$. The "branch data" that determine the factor $q^{-e_\lambda(\boldsymbol{s})k_\lambda}$ is comprised of the branch $\lambda \in \mathcal{B}(\pitchfork^*) \setminus \overline{\pitchfork} = \mathcal{B}(\pitchfork) \setminus \overline{\pitchfork}$ and the integer k_λ, which have clear visual interpretations in the tree diagram for any $x \in \mathcal{T}(\pitchfork, \boldsymbol{n})$ (recall Figure 3). In analogy with (3.3.3) in Remark 3.3.2, we have

$$|q^{-e_\lambda(\boldsymbol{s})}|_{\mathbb{C}} < 1 \quad \text{for all} \quad \lambda \in \mathcal{B}(\pitchfork^*) \setminus \overline{\pitchfork} \qquad \Longleftrightarrow \qquad \boldsymbol{s} \in \mathcal{BP}_{\pitchfork^*}, \tag{4.2.6}$$

which is precisely why we call $\mathcal{BP}_{\pitchfork^*}$ the branch polytope.

We now give the "branch-centric" analogue of Proposition 3.3.3, which will have a similar proof and a similar purpose. Just as for level functions in Proposition 3.3.3, this is where branch functions enter the picture.

Proposition 4.2.3. *Suppose $\pitchfork^* \in \mathcal{R}_N$ and $a \in \mathbb{C}$. If $M_{\pitchfork^*}(q) > 0$, then for every $\pitchfork \simeq \pitchfork^*$ the integral*

$$\int_{R^N_{\pitchfork}} \big(\max_{i<j} |x_i - x_j|\big)^a \prod_{i<j} |x_i - x_j|^{s_{ij}}\, dx_1 \dots dx_N$$

converges absolutely for all $\boldsymbol{s} \in \mathcal{RP}_N(a,0) \cap \mathcal{BP}_{\pitchfork^*}$, and for such $\boldsymbol{s}$ we have

$$\sum_{\substack{\pitchfork \in \mathcal{S}_N \\ \pitchfork \simeq \pitchfork^*}} \int_{R^N_{\pitchfork}} \big(\max_{i<j} |x_i - x_j|\big)^a \prod_{i<j} |x_i - x_j|^{s_{ij}}\, dx_1 \dots dx_N = \frac{1}{1 - q^{-(N-1+a+\sum_{i<j} s_{ij})}} \cdot I_{\pitchfork^*;q}(\boldsymbol{s}).$$

Otherwise $M_{\pitchfork^*}(q) = 0$, in which case $R^N_{\pitchfork} = \varnothing$ for all $\pitchfork \simeq \pitchfork^*$ and all integrals above are zero.

Proof. The $M_{\pitchfork^*}(q) = 0$ case is immediate from (4.2.5) and the definition of $R^N_{\pitchfork}$, so suppose $M_{\pitchfork^*}(q) > 0$. The first claim follows from part (c) of Lemma 1.6.5 and Proposition 3.3.3. To prove the second claim, suppose $\boldsymbol{s} \in \mathcal{RP}_N(a,0) \cap \mathcal{BP}_{\pitchfork^*}$, note the function

$$x \mapsto \sum_{\substack{\pitchfork \in \mathcal{S}_N \\ \pitchfork \simeq \pitchfork^*}} \mathbf{1}_{R^N_{\pitchfork}}(x) \left| \big(\max_{i<j} |x_i - x_j|\big)^a \prod_{i<j} |x_i - x_j|^{s_{ij}} \right|_{\mathbb{C}}$$

is in $L^1(K^N, \mu^N)$ by Proposition 3.3.3, and that it dominates every partial sum of the function

$$x \mapsto \sum_{\substack{\pitchfork \in \mathcal{S}_N \\ \pitchfork \simeq \pitchfork^*}} \sum_{\boldsymbol{n} \in \mathbb{N}^{L(\pitchfork)}} \mathbf{1}_{\mathcal{T}(\pitchfork, \boldsymbol{n})}(x) \big(\max_{i<j} |x_i - x_j|\big)^a \prod_{i<j} |x_i - x_j|^{s_{ij}}.$$

Then the Dominated Convergence Theorem, Theorem 4.1.4, Corollary 4.2.1, Fubini's Theorem for absolutely convergent sums, (3.2.1), (3.3.2), and (4.2.6) imply

$$
\begin{aligned}
&\sum_{\substack{\pitchfork\in\mathcal{S}_N\\ \pitchfork\simeq\pitchfork^*}} \int_{R^N_{\pitchfork}} \big(\max_{i<j}|x_i-x_j|\big)^a \prod_{i<j}|x_i-x_j|^{s_{ij}}\,dx_1\ldots dx_N \\
&\quad= \sum_{\substack{\pitchfork\in\mathcal{S}_N\\ \pitchfork\simeq\pitchfork^*}} \sum_{\boldsymbol{n}\in\mathbb{N}^{L(\pitchfork)}} \int_{\mathcal{T}(\pitchfork,\boldsymbol{n})} \big(\max_{i<j}|x_i-x_j|\big)^a \prod_{i<j}|x_i-x_j|^{s_{ij}}\,dx_1\ldots dx_N \\
&\quad= \sum_{\boldsymbol{k}\in\mathbb{N}^{\mathcal{B}(\pitchfork^*)}} q^{-(N-1+a+\sum_{i<j}s_{ij})(k_{[N]}-1)}\cdot\frac{M_{\pitchfork^*}(q)}{q^{N-1}}\cdot\prod_{\lambda\in\mathcal{B}(\pitchfork^*)\setminus\overline{\pitchfork}} q^{-e_\lambda(\boldsymbol{s})k_\lambda} \\
&\quad= \sum_{k_{[N]}=1}^{\infty} q^{-(N-1+a+\sum_{i<j}s_{ij})(k_{[N]}-1)}\cdot\frac{M_{\pitchfork^*}(q)}{q^{N-1}}\cdot\prod_{\lambda\in\mathcal{B}(\pitchfork^*)\setminus\overline{\pitchfork}}\sum_{k_\lambda=1}^{\infty} q^{-e_\lambda(\boldsymbol{s})k_\lambda} \\
&\quad= \frac{1}{1-q^{-(N-1+a+\sum_{i<j}s_{ij})}}\cdot I_{\pitchfork^*,q}(\boldsymbol{s}).
\end{aligned}
$$

□

At this point, we can easily prove the first statement in part (c) of Theorem 1.6.6: Given $\pitchfork^*\in\mathcal{R}_N$ with $M_{\pitchfork^*}(q)>0$ and $a=b=0$, the two formulas in Propositions 3.3.3 and 4.2.3 imply

$$
\sum_{\substack{\pitchfork\in\mathcal{S}_N\\ \pitchfork\simeq\pitchfork^*}} J_{\pitchfork,q}(0,\boldsymbol{s}) = (1-q^{-(N-1+\sum_{i<j}s_{ij})})\cdot\sum_{\substack{\pitchfork\in\mathcal{S}_N\\ \pitchfork\simeq\pitchfork^*}}\int_{R^N_{\pitchfork}}\prod_{i<j}|x_i-x_j|^{s_{ij}}\,dx_1\ldots dx_N = I_{\pitchfork^*,q}(\boldsymbol{s})
$$

for all $\boldsymbol{s}\in\mathcal{RP}_N(0,0)\cap\mathcal{BP}_{\pitchfork^*}$. The leftmost and rightmost expressions above are both holomorphic in the open set $\mathcal{BP}_{\pitchfork^*}$, which is simply connected because it is convex. Therefore since the two expressions agree on $\mathcal{RP}_N(0,0)\cap\mathcal{BP}_{\pitchfork^*}$, they must in fact agree on all of $\mathcal{BP}_{\pitchfork^*}$. Otherwise $M_{\pitchfork^*}(q)=0$ and all three expressions above are identically zero on $\mathcal{BP}_{\pitchfork^*}$, so the first statement in part (c) of Theorem 1.6.6 is proved in all cases. Finally, we obtain the analogue of Proposition 3.3.4, which is immediate from Proposition 4.2.3 and part (c) of Lemma 1.6.5:

Corollary 4.2.4. *Suppose K is a p-field and suppose $a \in \mathbb{C}$. The integral*

$$\int_{R^N} \big(\max_{i<j} |x_i - x_j|\big)^a \prod_{i<j} |x_i - x_j|^{s_{ij}} \, dx_1 \dots dx_N$$

converges absolutely for all $\boldsymbol{s} \in \mathcal{RP}_N(a,0) \cap \bigcap_{\boldsymbol{\pitchfork}^* \in \mathcal{R}_N} \mathcal{BP}_{\boldsymbol{\pitchfork}^*} = \Omega_N(a,0)$, and for such $\boldsymbol{s}$ it converges to

$$\frac{1}{1 - q^{-(N-1+a+\sum_{i<j} s_{ij})}} \cdot \sum_{\boldsymbol{\pitchfork}^* \in \mathcal{R}_N} I_{\boldsymbol{\pitchfork}^*,q}(\boldsymbol{s}).$$

It is worth settling the second claim of Proposition 1.6.7 here before returning to this chapter's main proof. To this end, recall that

$$\bigcap_{\boldsymbol{\pitchfork}^* \in \mathcal{R}_N} \mathcal{BP}_{\boldsymbol{\pitchfork}^*} = \bigcap_{\boldsymbol{\pitchfork}^* \in \mathcal{R}_N} \bigcap_{\lambda \in \mathcal{B}(\boldsymbol{\pitchfork}^*) \setminus \overline{\pitchfork}} \left\{ \boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}} : \mathrm{Re}(e_\lambda(\boldsymbol{s})) > 0 \right\}$$

by part (a) of Definition 1.6.4. Given a non-singleton subset $\lambda \subsetneq [N]$, let $\boldsymbol{\pitchfork}^* = (\pitchfork_0^*, \pitchfork_1^*, \pitchfork_2^*)$ be the unique splitting chain such that the level $\pitchfork_1^*$ is comprised of λ and all of the singletons $\{i\} \in [N] \setminus \lambda$. Clearly $\boldsymbol{\pitchfork}^* \in \mathcal{R}_N$ and $\lambda \in \mathcal{B}(\boldsymbol{\pitchfork}^*) \setminus \overline{\pitchfork}$, so it follows that

$$\{\lambda \subsetneq [N] : \#\lambda > 1\} \subset \bigcup_{\boldsymbol{\pitchfork}^* \in \mathcal{R}_N} (\mathcal{B}(\boldsymbol{\pitchfork}^*) \setminus \overline{\pitchfork}).$$

The reverse containment is clear from Definition 1.6.3, so the union at right is simply $\{\lambda \subsetneq [N] : \#\lambda > 1\}$, meaning

$$\bigcap_{\boldsymbol{\pitchfork}^* \in \mathcal{R}_N} \mathcal{BP}_{\boldsymbol{\pitchfork}^*} = \bigcap_{\substack{\lambda \subsetneq [N] \\ \#\lambda > 1}} \left\{ \boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}} : \mathrm{Re}(e_\lambda(\boldsymbol{s})) > 0 \right\}$$

and hence Proposition 1.6.7 is proved.

4.3. The final step

We need one more lemma to finish the proofs of parts (a)-(c) of Theorem 1.6.6:

Lemma 4.3.1. *Suppose K is a p-field, suppose $a, b \in \mathbb{C}$, suppose ρ is a norm density, and define*

$$Z_N(K, a, b, \boldsymbol{s}) := \int_{R^N} \big(\max_{i<j} |x_i - x_j|\big)^a \big(\min_{i<j} |x_i - x_j|\big)^b \prod_{i<j} |x_i - x_j|^{s_{ij}} \, dx_1 \dots dx_N$$

for all $\boldsymbol{s} \in \Omega_N(a, b)$. Then for all such $\boldsymbol{s}$ we have

$$Z_N^\rho(K, a, b, \boldsymbol{s}) = \left(\sum_{m \in \mathbb{Z}} \frac{\rho(q^{-m})}{q^{m(N+a+b+\sum_{i<j} s_{ij})}} \right) \left(1 - \frac{1}{q^{N+a+b+\sum_{i<j} s_{ij}}} \right) Z_N(K, a, b, \boldsymbol{s}),$$

and the sum over $m \in \mathbb{Z}$ converges absolutely uniformly on each compact subset of $\Omega_N(a, b)$.

Proof. We first prove the following claim: For each $m \in \mathbb{Z}$ and every $\boldsymbol{s} \in \Omega_N(a, b)$ the integral

$$\int_{(\pi^m R)^N \setminus (\pi^{m+1} R)^N} \rho(\|x\|) \big(\max_{i<j} |x_i - x_j|\big)^a \big(\min_{i<j} |x_i - x_j|\big)^b \prod_{i<j} |x_i - x_j|^{s_{ij}} \, dx_1 \dots dx_N$$

converges absolutely to

$$\frac{\rho(q^{-m})}{q^{m(N+a+b+\sum_{i<j} s_{ij})}} \left(1 - \frac{1}{q^{N+a+b+\sum_{i<j} s_{ij}}} \right) Z_N(K, a, b, \boldsymbol{s}).$$

To see why this claim holds, note that $Z_N(K, a, b, \boldsymbol{s})$ is defined for all $\boldsymbol{s} \in \Omega_N(a, b)$ by Proposition 3.3.4. Then for any $m \in \mathbb{Z}$, the change of variables $R^N \to (\pi^m R)^N$

defined by $x \mapsto \pi^m y$ gives

$$\begin{aligned}
\int_{(\pi^m R)^N} &\big(\max_{i<j} |x_i - x_j|\big)^a \big(\min_{i<j} |x_i - x_j|\big)^b \prod_{i<j} |x_i - x_j|^{s_{ij}}\, dx_1 \ldots dx_N \\
&= \frac{1}{q^{mN}} \int_{R^N} \big(\max_{i<j} |\pi^m y_i - \pi^m y_j|\big)^a \big(\min_{i<j} |\pi^m y_i - \pi^m y_j|\big)^b \\
&\qquad \cdot \prod_{i<j} |\pi^m y_i - \pi^m y_j|^{s_{ij}}\, dy_1 \ldots dy_N \\
&= \frac{1}{q^{m(N+a+b+\sum_{i<j} s_{ij})}} \int_{R^N} \big(\max_{i<j} |y_i - y_j|\big)^a \big(\min_{i<j} |y_i - y_j|\big)^b \\
&\qquad \cdot \prod_{i<j} |y_i - y_j|^{s_{ij}}\, dy_1 \ldots dy_N \\
&= \frac{1}{q^{m(N+a+b+\sum_{i<j} s_{ij})}} \cdot Z_N(K, a, b, \boldsymbol{s})
\end{aligned}$$

for all $\boldsymbol{s} \in \Omega_N(a, b)$. But the norm $\|x\| = \max_{1 \le i \le N} |x_i|$ takes the constant value q^{-m} at every $x \in (\pi^m R)^N \setminus (\pi^{m+1} R)^N$, so for every $m \in \mathbb{Z}$ and every $\boldsymbol{s} \in \Omega_N(a, b)$ we have

$$\begin{aligned}
\int_{(\pi^m R)^N \setminus (\pi^{m+1} R)^N} &\rho(\|x\|)\big(\max_{i<j} |x_i - x_j|\big)^a \big(\min_{i<j} |x_i - x_j|\big)^b \prod_{i<j} |x_i - x_j|^{s_{ij}}\, dx_1 \ldots dx_N \\
&= \rho(q^{-m}) \Bigg(\frac{1}{q^{m(N+a+b+\sum_{i<j} s_{ij})}} \cdot Z_N(K, a, b, \boldsymbol{s}) \\
&\qquad - \frac{1}{q^{(m+1)(N+a+b+\sum_{i<j} s_{ij})}} \cdot Z_N(K, a, b, \boldsymbol{s}) \Bigg) \\
&= \frac{\rho(q^{-m})}{q^{m(N+a+b+\sum_{i<j} s_{ij})}} \left(1 - \frac{1}{q^{N+a+b+\sum_{i<j} s_{ij}}} \right) Z_N(K, a, b, \boldsymbol{s})
\end{aligned}$$

and the desired claim is proved. In particular, since $(\mathrm{Re}(s_{ij}))_{i<j} \in \Omega_N(\mathrm{Re}(a), \mathrm{Re}(b))$ whenever $\boldsymbol{s} \in \Omega_N(a, b)$, note that the claim also holds if $\rho(\cdot)$, a, b, and s_{ij} are

replaced by $|\rho(\cdot)|_{\mathbb{C}}$, $\mathrm{Re}(a)$, $\mathrm{Re}(b)$, and $\mathrm{Re}(s_{ij})$. Now for the main claim, note that

$$\begin{aligned}&\rho(\|x\|)\big(\max_{i<j}|x_i-x_j|\big)^a\big(\min_{i<j}|x_i-x_j|\big)^b\prod_{i<j}|x_i-x_j|^{s_{ij}}\\&=\sum_{m\in\mathbb{Z}}\rho(q^{-m})\big(\max_{i<j}|x_i-x_j|\big)^a\big(\min_{i<j}|x_i-x_j|\big)^b\prod_{i<j}|x_i-x_j|^{s_{ij}}\mathbf{1}_{(\pi^mR)^N\setminus(\pi^{m+1}R)^N}(x)\end{aligned}$$

for all $x\in K^N\setminus\{0\}$, and therein each partial sum is dominated by the function

$$\begin{aligned}x\mapsto&\left|\rho(\|x\|)\big(\max_{i<j}|x_i-x_j|\big)^a\big(\min_{i<j}|x_i-x_j|\big)^b\prod_{i<j}|x_i-x_j|^{s_{ij}}\right|_{\mathbb{C}}\\&=\sum_{m\in\mathbb{Z}}\Bigg(|\rho(q^{-m})|_{\mathbb{C}}\big(\max_{i<j}|x_i-x_j|\big)^{\mathrm{Re}(a)}\big(\min_{i<j}|x_i-x_j|\big)^{\mathrm{Re}(b)}\\&\qquad\cdot\prod_{i<j}|x_i-x_j|^{\mathrm{Re}(s_{ij})}\mathbf{1}_{(\pi^mR)^N\setminus(\pi^{m+1}R)^N}(x)\Bigg).\end{aligned}$$

Now Fubini's Theorem for sums of nonnegative terms and the claim we just proved give

$$\begin{aligned}&\int_{K^N}\left|\rho(\|x\|)\big(\max_{i<j}|x_i-x_j|\big)^a\big(\min_{i<j}|x_i-x_j|\big)^b\prod_{i<j}|x_i-x_j|^{s_{ij}}\right|_{\mathbb{C}}dx_1\dots dx_N\\&=\sum_{m\in\mathbb{Z}}\Bigg(\int_{(\pi^mR)^N\setminus(\pi^{m+1}R)^N}|\rho(q^{-m})|_{\mathbb{C}}\big(\max_{i<j}|x_i-x_j|\big)^{\mathrm{Re}(a)}\big(\min_{i<j}|x_i-x_j|\big)^{\mathrm{Re}(b)}\\&\qquad\cdot\prod_{i<j}|x_i-x_j|^{\mathrm{Re}(s_{ij})}\,dx_1\dots dx_N\Bigg)\\&=\sum_{m\in\mathbb{Z}}\Bigg(\frac{|\rho(q^{-m})|_{\mathbb{C}}}{q^{m(\mathrm{Re}(N+a+b+\sum_{i<j}s_{ij}))}}\left(1-\frac{1}{q^{\mathrm{Re}(N+a+b+\sum_{i<j}s_{ij})}}\right)\\&\qquad\cdot Z_N(K,\mathrm{Re}(a),\mathrm{Re}(b),(\mathrm{Re}(s_{ij}))_{i<j})\Bigg)\end{aligned}$$

for every $\boldsymbol{s}\in\Omega_N(a,b)$. Now suppose C is any compact subset of $\Omega_N(a,b)$. Since C is therefore a compact subset of the root polytope

$$\mathcal{RP}_N(a,b) = \{\boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}} : \mathrm{Re}(N - 1 + a + b + \sum_{i<j} s_{ij}) > 0\},$$

there exist real numbers σ_1 and σ_2 satisfying

$$\begin{aligned} \limsup_{n\to\infty} \frac{\log|\rho(\frac{1}{n})|_{\mathbb{C}}}{\log(n)} &\leq 1 \\ &< \sigma_1 \leq \mathrm{Re}\left(N + a + b + \sum_{i<j} s_{ij}\right) \leq \sigma_2 \\ &< \infty = -\limsup_{n\to\infty} \frac{\log|\rho(n)|_{\mathbb{C}}}{\log(n)} \end{aligned}$$

for all $\boldsymbol{s} \in C$. To show that the preceding sum over $m \in \mathbb{Z}$ converges uniformly on C, it suffices to verify the convergence of the two series

$$\sum_{m=0}^{\infty} \frac{|\rho(q^{-m})|_{\mathbb{C}}}{q^{m\sigma_1}} \qquad \text{and} \qquad \sum_{m=1}^{\infty} |\rho(q^m)|_{\mathbb{C}} q^{m\sigma_2}.$$

Indeed, if $\log : [0,\infty] \to [-\infty,\infty]$ is the extended logarithm we have

$$\begin{aligned} \log\left(\limsup_{m\to\infty} \sqrt[m]{\frac{|\rho(q^{-m})|_{\mathbb{C}}}{q^{m\sigma_1}}}\right) &= \log(q) \cdot \left(\limsup_{m\to\infty} \frac{\log|\rho(q^{-m})|_{\mathbb{C}}}{\log(q^m)} - \sigma_1\right) \\ &\leq \log(q) \cdot \left(\limsup_{n\to\infty} \frac{\log|\rho(\frac{1}{n})|_{\mathbb{C}}}{\log(n)} - \sigma_1\right) < 0 \end{aligned}$$

and

$$\begin{aligned} \log\left(\limsup_{m\to\infty} \sqrt[m]{|\rho(q^m)|_{\mathbb{C}} q^{m\sigma_2}}\right) &= \log(q) \cdot \left(\limsup_{m\to\infty} \frac{\log|\rho(q^m)|_{\mathbb{C}}}{\log(q^m)} + \sigma_2\right) \\ &\leq \log(q) \cdot \left(\limsup_{n\to\infty} \frac{\log|\rho(n)|_{\mathbb{C}}}{\log(n)} + \sigma_2\right) < 0, \end{aligned}$$

so the series both converge by the root test and we conclude that our series expansion for

$$\int_{K^N} \left| \rho(\|x\|)\big(\max_{i<j}|x_i - x_j|\big)^a \big(\min_{i<j}|x_i - x_j|\big)^b \prod_{i<j}|x_i - x_j|^{s_{ij}} \right|_{\mathbb{C}} dx_1 \ldots dx_N$$

converges uniformly on C. Thus by the dominated convergence theorem we have

$$\begin{aligned} &Z_N^\rho(K,a,b,\boldsymbol{s}) \\ &\quad= \int_{K^N} \rho(\|x\|)\big(\max_{i<j}|x_i - x_j|\big)^a \big(\min_{i<j}|x_i - x_j|\big)^b \prod_{i<j}|x_i - x_j|^{s_{ij}} dx_1 \ldots dx_N \\ &\quad= \left(\sum_{m\in\mathbb{Z}} \frac{\rho(q^{-m})}{q^{m(N+a+b+\sum_{i<j}s_{ij})}}\right)\left(1 - \frac{1}{q^{N+a+b+\sum_{i<j}s_{ij}}}\right) Z_N(K,a,b,\boldsymbol{s}), \end{aligned}$$

and we conclude that the sum over $m \in \mathbb{Z}$ converges absolutely uniformly on C.

$\square$

Finally, we combine Lemma 4.3.1 with Propositions 3.3.4 and 4.2.3 to finish the proof of Theorem 1.6.6:

Proof of Theorem 1.6.6.

(a) Since ρ is not identically zero, there exists $m \in \mathbb{Z}$ such that $\rho(q^{-m}) \neq 0$. Moreover, the quantity $1 - \frac{1}{q^{N+a+b+\sum_{i<j}s_{ij}}}$ attains nonzero values on every open subset $U \subset \mathbb{C}^{\binom{N}{2}}$, so the quantity

$$\frac{\rho(q^{-m})}{q^{m(N+a+b+\sum_{i<j}s_{ij})}}\left(1 - \frac{1}{q^{N+a+b+\sum_{i<j}s_{ij}}}\right) Z_N(K,a,b,\boldsymbol{s})$$

appearing in the proof above may converge absolutely at every point of an open set $U \subset \mathbb{C}^{\binom{N}{2}}$ only if the integral $Z_N(K,a,b,\boldsymbol{s})$ does. But Proposition

3.3.4 says that the integral defining $Z_N(K, a, b, \boldsymbol{s})$ converges absolutely if and only if $\boldsymbol{s} \in \Omega_N(a, b)$, and we know that the parenthetical sum over $m \in \mathbb{Z}$ in Lemma 4.3.1 converges absolutely uniformly on $\Omega_N(a, b)$. Thus $Z_N^\rho(K, a, b, \boldsymbol{s})$ converges absolutely for every $\boldsymbol{s} \in \Omega_N(a, b)$, and $\Omega_N(a, b)$ is the largest open set with this property.

(b) If C is a compact subset of $\Omega_N(a, b)$, then $Z_N(K, a, b, \boldsymbol{s})$ restricts to a continuous and hence bounded function on C, and note that the same is true for the function $\boldsymbol{s} \mapsto 1 - \frac{1}{q^{N+a+b+\sum_{i<j} s_{ij}}}$. We already showed that the parenthetical sum in Lemma 4.3.1 converges uniformly on C, so by Lemma 4.3.1, Proposition 3.3.4, and Definition 1.6.1 we have

$$\begin{aligned} Z_N^\rho(K, a, b, \boldsymbol{s}) &= \left(\sum_{m \in \mathbb{Z}} \rho(q^m) q^{m(N+a+b+\sum_{i<j} s_{ij})} \right) \left(1 - \frac{1}{q^{N+a+b+\sum_{i<j} s_{ij}}} \right) \\ &\quad \cdot \frac{1}{1 - q^{-(N-1+a+b+\sum_{i<j} s_{ij})}} \cdot \sum_{\pitchfork \in \mathcal{S}_N} J_{\pitchfork, q}(b, \boldsymbol{s}) \\ &= H_q^\rho \left(N + a + b + \sum_{i<j} s_{ij} \right) \cdot \sum_{\pitchfork \in \mathcal{S}_N} J_{\pitchfork, q}(b, \boldsymbol{s}), \end{aligned}$$

and the sum converges uniformly on C.

(c) We already proved the first claim relating level and branch functions immediately after the proof of Proposition 4.2.3. If C is a compact subset of $\mathcal{RP}_N(a, 0) \cap \bigcap_{\pitchfork^* \in \mathcal{R}_N} \mathcal{BP}_{\pitchfork^*}$, then $Z_N(K, a, 0, \boldsymbol{s})$ (i.e., the value of the integral from Proposition 4.2.4) restricts to a continuous and hence bounded function on C. But

$$\mathcal{RP}_N(a, 0) \cap \bigcap_{\pitchfork^* \in \mathcal{R}_N} \mathcal{BP}_{\pitchfork^*} = \Omega_N(a, 0),$$

so Lemma 4.3.1, Proposition 4.2.4, and Definition 1.6.1 similarly imply

$$\begin{aligned} Z_N^\rho(K, a, 0, \boldsymbol{s}) &= \left(\sum_{m\in\mathbb{Z}} \rho(q^m) q^{m(N+a+\sum_{i<j} s_{ij})}\right)\left(1 - \frac{1}{q^{N+a+\sum_{i<j} s_{ij}}}\right) \\ &\qquad \cdot \frac{1}{1-q^{-(N-1+a+\sum_{i<j} s_{ij})}} \cdot \sum_{\pitchfork^*\in\mathcal{R}_N} I_{\pitchfork^*,q}(\boldsymbol{s}) \\ &= H_q^\rho\left(N + a + \sum_{i<j} s_{ij}\right) \cdot \sum_{\pitchfork^*\in\mathcal{R}_N} I_{\pitchfork^*,q}(\boldsymbol{s}), \end{aligned}$$

and the sum converges uniformly on C.

$\square$

CHAPTER V

THE INTEGRAL OVER $(\mathbb{P}^1(K))^N$

In this chapter we will establish part (d) of Theorem 1.6.6 and Theorem 2.2.2 (the $(q+1)$*th Power Law*). The rough idea behind both is a decomposition of $(\mathbb{P}^1(K))^N$ into $(q+1)^N$ cells that are isometrically homeomorphic to P^N. We will setup prerequisite notation and results in Section 5.1, then use the decomposition to relate the integrals $\mathcal{Z}_N(\mathbb{P}^1(K), \boldsymbol{s})$ and $\mathcal{Z}_N(P, \boldsymbol{s})$ (recall Definition 1.5.3) in Section 5.2. With this relationship in hand, we will conclude the chapter with proofs of Theorem 1.6.6 and Theorem 2.2.2 in their own sections.

5.1. I-analogues of integrals and splitting chains

We begin with a p-field K and an integer $N \geq 2$ that shall remain fixed for the rest of this chapter, and recall that symbol $\boldsymbol{s}$ stands for a complex tuple $(s_{ij})_{1\leq i<j\leq N}$. To better organize the forthcoming arguments, we fix the following notation as well:

Notation 5.1.1. Let I be a subset of $[N] = \{1, \ldots, N\}$.

- For any set X we write X^I for the product $\prod_{i\in I} X = \{x_I = (x_i)_{i\in I} : x_i \in X\}$ and assume X^I has the product topology if X is a topological space.
- We write dx_I for the product Haar measure on K^I satisfying $\int_{R^I} dx_I = 1$, and we make this consistent for $I = \varnothing$ by giving the singleton space $K^\varnothing = R^\varnothing = \{0\}$ measure 1. We also write $d[x_0 : x_1]_I$ for the product measure on $(\mathbb{P}^1(K))^I$ (where $d[x_0 : x_1]$ is the measure from Lemma 1.3) with the same measure 1 convention when $I = \varnothing$.

– For a measurable subset $X \subset K$ we set $\mathcal{Z}_{\varnothing}(X, \boldsymbol{s}) := 1$ and

$$\mathcal{Z}_I(X, \boldsymbol{s}) := \int_{X^I} \prod_{\substack{i<j \\ i,j \in I}} |x_i - x_j|^{s_{ij}} \, dx_I \qquad \text{if} \quad I \neq \varnothing.$$

Note that $\mathcal{Z}_I(X, \boldsymbol{s})$ is constant with respect to those s_{ij} with i or j not in I, and it is equal to $\mathcal{Z}_N(X, \boldsymbol{s})$ if $I = [N]$.

– We write $(I_0, \ldots, I_q) \vdash [N]$ (recall $q = \#(R/P)$) for an *ordered* partition of $[N]$ into at most $q+1$ parts. That is, $(I_0, \ldots, I_q) \vdash [N]$ means $I_0, \ldots, I_q$ are $q+1$ disjoint ordered subsets of $[N]$ with union equal to $[N]$, where some I_k may be empty.

– We generalize the explicit form of $\Omega_N(0,0)$ in Proposition 1.6.7 via

$$\Omega_I := \bigcap_{\substack{\lambda \subset I \\ \#\lambda > 1}} \left\{ \boldsymbol{s} \in \mathbb{C}^{\binom{N}{2}} : \operatorname{Re}(e_\lambda(\boldsymbol{s})) > 0 \right\}.$$

We will also need I-analogues of splitting chains:

Definition 5.1.2. Suppose $I \subset [N]$. An *I-splitting chain* of length $L \geq 0$ is a tuple $\boldsymbol{\pitchfork} = (\pitchfork_0, \ldots, \pitchfork_L)$ of partitions of I satisfying

$$\{I\} = \pitchfork_0 > \pitchfork_1 > \pitchfork_2 > \cdots > \pitchfork_L = \{\{i\} : i \in I\}.$$

If $\#I \geq 2$, we define $\mathcal{B}(\boldsymbol{\pitchfork})$, $\ell_{\boldsymbol{\pitchfork}}(\lambda)$, and $\deg_{\boldsymbol{\pitchfork}}(\lambda) \in \{2, 3, \ldots, \#I\}$ just as in Definition 1.6.3. Otherwise $\mathcal{B}(\boldsymbol{\pitchfork})$ will be treated as the empty set and there is no need to define $\ell_{\boldsymbol{\pitchfork}}$ or $\deg_{\boldsymbol{\pitchfork}}$. An I-splitting chain $\boldsymbol{\pitchfork}$ is *reduced* if each $\lambda \in \mathcal{B}(\boldsymbol{\pitchfork})$ satisfies $\lambda \in \pitchfork_\ell \iff \ell = \ell_{\boldsymbol{\pitchfork}}(\lambda)$, and we write $\mathcal{R}_I$ for the set of reduced I-splitting chains.

Note that $\mathcal{R}_\varnothing = \varnothing$ because $I = \varnothing$ has no partitions, $\Omega_\varnothing = \mathbb{C}^{N(N-1)/2}$ because $\Omega_\varnothing$ is an intersection of subsets of $\mathbb{C}^{N(N-1)/2}$ over an empty index set, and $e_\varnothing(\boldsymbol{s}) = -1$ for a similar reason. For each singleton $\{i\}$, the set $\mathcal{R}_{\{i\}}$ is comprised of a single splitting chain of length zero, we have $\Omega_{\{i\}} = \mathbb{C}^{N(N-1)/2}$ for the same reason as the $I = \varnothing$ case, and similarly $e_{\{i\}}(\boldsymbol{s}) = 0$. At the other extreme, taking $I = [N]$ in Definition 5.1.2 recovers Definition 1.6.3 and $\Omega_I = \Omega_N(0,0)$.

Proposition 5.1.3. *For any $m \in \mathbb{Z}$ and any nonempty subset $I \subset [N]$, the integral $\mathcal{Z}_I(\pi^m R, \boldsymbol{s})$ converges absolutely if and only if $\boldsymbol{s} \in \Omega_I$, and in this case*

$$\mathcal{Z}_I(\pi^m R, \boldsymbol{s}) = \frac{1}{q^{(m-1)(e_I(\boldsymbol{s})+1)+\#I}} \sum_{\pitchfork \in \mathcal{R}_I} \prod_{\lambda \in \mathcal{B}(\pitchfork)} \frac{(q-1)_{\deg_\pitchfork(\lambda)-1}}{q^{e_\lambda(\boldsymbol{s})} - 1}.$$

Proof. First suppose I is a singleton, so that the product inside the integral $Z_I(\pi^m R, \boldsymbol{s})$ is empty and hence

$$Z_I(\pi^m R, \boldsymbol{s}) = \int_{(\pi^m R)^I} 1 \, dx_I = \int_{\pi^m R} dx = q^{-m}.$$

This integral is constant, and hence absolutely convergent, for all $\boldsymbol{s} \in \mathbb{C}^{N(N-1)/2} = \Omega_I$. On the other hand, $\mathcal{R}_I$ consists of a single I-splitting chain, namely the one-tuple $\pitchfork = (\{I\})$. Then $\mathcal{B}(\pitchfork) = \varnothing$ and $e_I(\boldsymbol{s}) = 0$ imply

$$\frac{1}{q^{(m-1)(e_I(\boldsymbol{s})+1)+\#I}} \sum_{\pitchfork \in \mathcal{R}_I} \prod_{\lambda \in \mathcal{B}(\pitchfork)} \frac{(q-1)_{\deg_\pitchfork(\lambda)-1}}{q^{e_\lambda(\boldsymbol{s})} - 1} = \frac{1}{q^{(m-1)\cdot 1+1}} \prod_{\lambda \in \varnothing} \frac{(q-1)_{\deg_\pitchfork(\lambda)-1}}{q^{e_\lambda(\boldsymbol{s})} - 1} = q^{-m}$$

as well, so the claim holds for any singleton subset $I \subset [N]$. Now suppose I is not a singleton. By relabeling I we may assume $I = [n]$ where $2 \le n \le N$, in which case

$\mathcal{Z}_I(\pi^m R, \boldsymbol{s})$ is equal to

$$Z_n^\rho(K, 0, 0, (s_{ij})_{1\le i<j\le n}) = \int_{K^n} \rho(\|x\|) \prod_{1\le i<j\le n} |x_i - x_j|^{s_{ij}}\, dx_1 \dots dx_n$$

where $\rho = \mathbf{1}_{[0,q^{-m}]}$. By part (a) of Theorem 1.6.6, this integral converges absolutely if and only if $\boldsymbol{s} \in \Omega_n(0,0) \times \mathbb{C}^{\binom{N}{2}-\binom{n}{2}} = \Omega_{[n]}$ (the factors of $\mathbb{C}$ stand for those s_{ij} in $\boldsymbol{s}$ with $\{i,j\} \subset [N]$ and $\{i,j\} \not\subset [n]$). Therefore every $\boldsymbol{s} \in \Omega_{[n]}$ satisfies $(s_{ij})_{1\le i<j\le n} \in \Omega_n(0,0)$, so Definition 1.6.1 and part (c) of Theorem 1.6.6 imply

$$\begin{aligned}
\mathcal{Z}_I(\pi^m R, \boldsymbol{s}) &= Z_n^\rho(K, 0, 0, (s_{ij})_{1\le i<j\le n}) \\
&= H_q^\rho\left(n + \sum_{i<j} s_{ij}\right) \cdot \sum_{\pitchfork \in \mathcal{R}_n} I_{\pitchfork,q}(\boldsymbol{s}) \\
&= \frac{q^{-m(n+\sum_{i<j} s_{ij})}}{1 - q^{-(n+\sum_{i<j} s_{ij}-1)}} \cdot \sum_{\pitchfork \in \mathcal{R}_n} \frac{(q-1)_{\deg_\pitchfork([n])-1}}{q^{n-1}} \prod_{\lambda \in \mathcal{B}(\pitchfork)\setminus\overline{\pitchfork}} \frac{(q-1)_{\deg_\pitchfork(\lambda)-1}}{q^{e_\lambda(\boldsymbol{s})} - 1} \\
&= \frac{q^{-m(e_{[n]}(\boldsymbol{s})+1)}}{1 - q^{-e_{[n]}(\boldsymbol{s})}} \cdot \sum_{\pitchfork \in \mathcal{R}_n} \frac{(q-1)_{\deg_\pitchfork([n])-1}}{q^{n-1}} \prod_{\lambda \in \mathcal{B}(\pitchfork)\setminus\overline{\pitchfork}} \frac{(q-1)_{\deg_\pitchfork(\lambda)-1}}{q^{e_\lambda(\boldsymbol{s})} - 1} \\
&= \frac{1}{q^{(m-1)(e_I(\boldsymbol{s})+1)+n}} \sum_{\pitchfork \in \mathcal{R}_n} \prod_{\lambda \in \mathcal{B}(\pitchfork)} \frac{(q-1)_{\deg_\pitchfork(\lambda)-1}}{q^{e_\lambda(\boldsymbol{s})} - 1}.
\end{aligned}$$

Since the claim holds for $I = [n]$, we conclude that it holds for any non-singleton subset $I \subset [N]$ and the proof is complete.

$\square$

Our proof of part (d) of Theorem 1.6.6 will be essentially a combination of the $m = 1$ case of Proposition 5.1.3 with the main result of the next section.

5.2. The decomposition of $\mathcal{Z}_N(\mathbb{P}^1(K), \boldsymbol{s})$

Our only task in this section is to prove the following theorem:

Theorem 5.2.1. *For each $N \geq 2$, the integral $\mathcal{Z}_N(\mathbb{P}^1(K), \boldsymbol{s})$ converges absolutely if and only if $\boldsymbol{s} \in \Omega_N$, and in this case*

$$\mathcal{Z}_N(\mathbb{P}^1(K), \boldsymbol{s}) = \left(\frac{q}{q+1}\right)^N \sum_{(I_0,\ldots,I_q)\vdash[N]} \prod_{k=0}^{q} \mathcal{Z}_{I_k}(P, \boldsymbol{s}).$$

Proof. The partition of $\mathbb{P}^1(K)$ in (1.3.10) can be rewritten in the form

$$\mathbb{P}^1(K) = \bigsqcup_{k=0}^{q} \phi_k(B_1[0:1]),$$

where $\phi_k \in PGL_2(R)$ is the element represented by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ if $k = 0$, $\left(\begin{smallmatrix} 1 & \xi^{k-1} \\ 0 & 1 \end{smallmatrix}\right)$ if $0 < k < q$, or $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ if $k = q$. This leads to a partition of the N-fold product,

$$(\mathbb{P}^1(K))^N = \bigsqcup_{(I_0,\ldots,I_q)\vdash[N]} C(I_0, \ldots, I_q),$$

where each part is a "cell" of the form

$$C(I_0, \ldots, I_q) := \Big\{([x_{1,0} : x_{1,1}], \ldots, [x_{N,0} : x_{N,1}]) \in (\mathbb{P}^1(K))^N : \\ [x_{i,0} : x_{i,1}] \in \phi_k(B_1[0:1]) \iff i \in I_k\Big\} = \prod_{k=0}^{q} (\phi_k(B_1[0:1]))^{I_k}.$$

Accordingly, the integral $\mathcal{Z}_N(\mathbb{P}^1(K), \boldsymbol{s})$ breaks into a sum of integrals of the form

$$\int_{C(I_0,\ldots,I_q)} \prod_{1\leq i<j\leq N} \delta([x_{i,0} : x_{i,1}], [x_{j,0} : x_{j,1}])^{s_{ij}} \, d[x_{1,0} : x_{1,1}] \ldots d[x_{N,0} : x_{N,1}], \quad (5.2.1)$$

summed over all $(I_0, \ldots, I_q) \vdash [N]$. Since each cell $C(I_0, \ldots, I_q)$ has positive measure, the integral $\mathcal{Z}_N(\mathbb{P}^1(K), \boldsymbol{s})$ converges absolutely if and only if the integral in (5.2.1) converges absolutely for every $(I_0, \ldots, I_q) \vdash [N]$. Recall that two points in $\mathbb{P}^1(K)$ are δ-distance 1 apart if and only if they are in different balls in the decomposition (1.3.10). Thus, by the definition of $C(I_0, \ldots, I_q)$, it follows that the entries of each tuple $([x_{1,0} : x_{1,1}], \ldots, [x_{N,0} : x_{N,1}]) \in C(I_0, \ldots, I_q)$ satisfy $\delta([x_{i,0} : x_{i,1}], [x_{j,0} : x_{j,1}])^{s_{ij}} = 1$ if and only if $i \in I_k$, $j \in I_{k'}$, and $k \neq k'$. Therefore the integrand in (5.2.1) factors as

$$\prod_{1 \leq i < j \leq N} \delta([x_{i,0} : x_{i,1}], [x_{j,0} : x_{j,1}])^{s_{ij}} = \prod_{k=0}^{q} \prod_{\substack{i<j \\ i,j \in I_k}} \delta([x_{i,0} : x_{i,1}], [x_{j,0} : x_{j,1}])^{s_{ij}},$$

and the measure on $C(I_0, \ldots, I_q)$ factors in a similar way, namely $\prod_{k=0}^{q} d[x_0 : x_1]_{I_k}$, where each factor has the form $d[x_0 : x_1]_{I_k} := \prod_{i \in I_k} d[x_{i,0} : x_{i,1}]$. Now Fubini's Theorem for positive functions and $PGL_2(R)$-invariance give

$$\begin{aligned}
&\int_{C(I_0,\ldots,I_q)} \left| \prod_{1 \leq i<j \leq N} \delta([x_{i,0} : x_{i,1}], [x_{j,0} : x_{j,1}])^{s_{ij}} \right|_{\mathbb{C}} d[x_{1,0} : x_{1,1}] \ldots d[x_{N,0} : x_{N,1}] \\
&\qquad = \prod_{k=0}^{q} \int_{(\phi_k(B_1[0:1]))^{I_k}} \left| \prod_{\substack{i<j \\ i,j \in I_k}} \delta([x_{i,0} : x_{i,1}], [x_{j,0} : x_{j,1}])^{s_{ij}} \right|_{\mathbb{C}} d[x_0 : x_1]_{I_k} \\
&\qquad = \prod_{k=0}^{q} \int_{(B_1[0:1])^{I_k}} \left| \prod_{\substack{i<j \\ i,j \in I_k}} \delta([x_{i,0} : x_{i,1}], [x_{j,0} : x_{j,1}])^{s_{ij}} \right|_{\mathbb{C}} d[x_0 : x_1]_{I_k},
\end{aligned}$$

so the integral in (5.2.1) converges absolutely if and only if all $q+1$ of the integrals

$$\int_{(B_1[0:1])^{I_k}} \prod_{\substack{i<j \\ i,j \in I_k}} \delta([x_{i,0} : x_{i,1}], [x_{j,0} : x_{j,1}])^{s_{ij}} \, d[x_0 : x_1]_{I_k} \tag{5.2.2}$$

converge absolutely. The change of variables $P^{I_k} \to (B_1[0 : 1])^{I_k}$ given by the isometry $\iota : P \to B_1[0 : 1]$ in each coordinate, along with (1.3.6), (1.3.7), and (1.3.8), allows the integral in (5.2.2) to be rewritten as $(\frac{q}{q+1})^{\#I_k} \mathcal{Z}_{I_k}(P, \boldsymbol{s})$. Thus, Proposition 5.1.3 implies that the integral in (5.2.2) converges absolutely if and only if $\boldsymbol{s} \in \Omega_{I_k}$. It follows that the integral over $C(I_0, \dots, I_q)$ in (5.2.1) converges absolutely if and only if $\boldsymbol{s} \in \Omega_{I_0} \cap \dots \cap \Omega_{I_q}$, and in this case Fubini's Theorem for absolutely integrable functions, $PGL_2(R)$-invariance, and the change of variables above allow it to be rewritten as

$$\int_{C(I_0,\dots,I_q)} \prod_{1 \le i < j \le N} \delta([x_{i,0} : x_{i,1}], [x_{j,0} : x_{j,1}])^{s_{ij}} \, d[x_{1,0} : x_{1,1}] \dots d[x_{N,0} : x_{N,1}]$$

$$= \prod_{k=0}^{q} \int_{(B_1[0:1])^{I_k}} \prod_{\substack{i<j \\ i,j \in I_k}} \delta([x_{i,0} : x_{i,1}], [x_{j,0} : x_{j,1}])^{s_{ij}} \, d[x_0 : x_1]_{I_k}$$

$$= \prod_{k=0}^{q} \left(\frac{q}{q+1} \right)^{\#I_k} \mathcal{Z}_{I_k}(P, \boldsymbol{s}) = \left(\frac{q}{q+1} \right)^{N} \prod_{k=0}^{q} \mathcal{Z}_{I_k}(P, \boldsymbol{s}).$$

Finally, since $\mathcal{Z}_N(\mathbb{P}^1(K), \boldsymbol{s})$ is the sum of these integrals over all $(I_0, \dots, I_q) \vdash [N]$, it converges absolutely if and only if

$$\boldsymbol{s} \in \bigcap_{(I_1,\dots,I_q) \vdash [N]} (\Omega_{I_1} \cap \dots \cap \Omega_{I_q}) = \bigcap_{\substack{I \subset [N] \\ \#I > 1}} \Omega_I.$$

The last equality of intersections holds because each subset $I \subset [N]$ with $\#I > 1$ appears as a part in at least one of the ordered partitions $(I_1, \dots, I_q) \vdash [N]$, and none of the parts with $\#I_k \le 1$ affect the intersection (because $\Omega_{I_k} = \mathbb{C}^{N(N-1)/2}$ for such I_k). The intersection of Ω_I over all $I \subset [N]$ with $\#I > 1$ is clearly equal to $\Omega_{[N]} = \Omega_N$ by Definition 5.1.2, so the proof is complete.

□

5.3. The last piece of the Main Theorem

Theorem 5.2.1 established that the integral $\mathcal{Z}_N(\mathbb{P}^1(K), \boldsymbol{s})$ converges absolutely if and only if $\boldsymbol{s} \in \Omega_N$, and for such $\boldsymbol{s}$ it gave

$$\mathcal{Z}_N(\mathbb{P}^1(K), \boldsymbol{s}) = \left(\frac{q}{q+1}\right)^N \sum_{(I_0,\ldots,I_q)\vdash[N]} \prod_{k=0}^{q} \mathcal{Z}_{I_k}(P, \boldsymbol{s}). \tag{5.3.3}$$

It remains to show that the righthand sum can be converted into the sum over $\pitchfork \in \mathcal{R}_N$ proposed in part (d) of Theorem 1.6.6.

Proof of part (d) of Theorem 1.6.6. We begin by breaking the terms of the sum in (5.3.3) into two main groups. The simpler group is indexed by those $(I_0, \ldots, I_q)$ with $I_j = [N]$ for some j and $I_k = \varnothing$ for all $k \neq j$, in which case $\mathcal{Z}_{I_j}(P, \boldsymbol{s}) = \mathcal{Z}_N(P, \boldsymbol{s})$ and $\mathcal{Z}_{I_k}(P, \boldsymbol{s}) = 1$ for all $k \neq j$. Therefore each of the group's $q+1$ terms (one for each $j \in \{0, \ldots, q\}$) contributes the quantity $\prod_{k=0}^{q} \mathcal{Z}_{I_k}(P, \boldsymbol{s}) = \mathcal{Z}_N(P, \boldsymbol{s})$ to the sum in (5.3.3) for a total contribution with value

$$(q+1)\mathcal{Z}_N(P, \boldsymbol{s}) = \frac{q+1}{q^N} \sum_{\pitchfork\in\mathcal{R}_N} \prod_{\lambda\in\mathcal{B}(\pitchfork)} \frac{(q-1)_{\deg_{\pitchfork}(\lambda)-1}}{q^{e_\lambda(\boldsymbol{s})} - 1} \tag{5.3.4}$$

by the $m = 1$ and $I = [N]$ case of Proposition 5.1.3. The other group of terms is indexed by the ordered partitions $(I_0, \ldots, I_q) \vdash [N]$ satisfying $I_0, \ldots, I_q \subsetneq [N]$. To deal with them carefully, we fix one such $(I_0, \ldots, I_q)$ for the moment, and note that the number d of nonempty parts I_k must be at least 2. Thus we have indices $k_1, \ldots, k_d \in \{0, \ldots, q\}$ with $I_{k_j} \neq \varnothing$, and for every $k \in \{0, \ldots, q\} \setminus \{k_1, \ldots, k_d\}$ we have $I_k = \varnothing$ and hence $\mathcal{Z}_{I_k}(P, \boldsymbol{s}) = 1$. For the nonempty sets I_{k_j}, Proposition 5.1.3 expands $\mathcal{Z}_{I_{k_j}}(P, \boldsymbol{s})$ as a sum over $\mathcal{R}_{I_{k_j}}$ (whose elements shall be denoted $\pitchfork_j$ instead of $\boldsymbol{\pitchfork}$) and hence

$$\begin{aligned}\prod_{k=0}^{q} \mathcal{Z}_{I_k}(P,\boldsymbol{s}) &= \prod_{j=1}^{d} \frac{1}{q^{\# I_{k_j}}} \sum_{\boldsymbol{\pitchfork}_j \in \mathcal{R}_{I_{k_j}}} \prod_{\lambda \in \mathcal{B}(\boldsymbol{\pitchfork}_j)} \frac{(q-1)_{\deg_{\boldsymbol{\pitchfork}_j}(\lambda)-1}}{q^{e_\lambda(\boldsymbol{s})}-1} \\ &= \frac{1}{q^N} \sum_{(\boldsymbol{\pitchfork}_1,\ldots,\boldsymbol{\pitchfork}_d) \in \mathcal{R}_{I_{k_1}} \times \cdots \times \mathcal{R}_{I_{k_d}}} \prod_{j=1}^{d} \prod_{\lambda \in \mathcal{B}(\boldsymbol{\pitchfork}_j)} \frac{(q-1)_{\deg_{\boldsymbol{\pitchfork}_j}(\lambda)-1}}{q^{e_\lambda(\boldsymbol{s})}-1} \\ &= \frac{1}{q^N} \sum_{(\boldsymbol{\pitchfork}_1,\ldots,\boldsymbol{\pitchfork}_d) \in \mathcal{R}_{I_{k_1}} \times \cdots \times \mathcal{R}_{I_{k_d}}} \prod_{\lambda \in \mathcal{B}(\boldsymbol{\pitchfork}_1) \sqcup \cdots \sqcup \mathcal{B}(\boldsymbol{\pitchfork}_d)} \frac{(q-1)_{\deg_{\boldsymbol{\pitchfork}_j}(\lambda)-1}}{q^{e_\lambda(\boldsymbol{s})}-1}.\end{aligned}$$

We now make use of a simple correspondence between the tuples $(\boldsymbol{\pitchfork}_1,\ldots,\boldsymbol{\pitchfork}_d) \in \mathcal{R}_{I_{k_1}} \times \cdots \times \mathcal{R}_{I_{k_d}}$ and the reduced splitting chains $\boldsymbol{\pitchfork} = (\pitchfork_0, \pitchfork_1, \ldots, \pitchfork_L) \in \mathcal{R}_N$ satisfying $\pitchfork_1 = \{I_{k_1},\ldots,I_{k_d}\}$. To establish it, note that each $\boldsymbol{\pitchfork} \in \mathcal{R}_N$ corresponds uniquely to its branch set $\mathcal{B}(\boldsymbol{\pitchfork})$ by part (b) of Lemma 1.6.5, which generalizes in an obvious way to reduced I-splitting chains (for any nonempty $I \subset [N]$). Now if $\boldsymbol{\pitchfork} = (\pitchfork_0, \pitchfork_1, \ldots, \pitchfork_L) \in \mathcal{R}_N$ satisfies $\pitchfork_1 = \{I_{k_1},\ldots,I_{k_d}\}$, the corresponding branch set $\mathcal{B}(\boldsymbol{\pitchfork})$ decomposes as

$$\mathcal{B}(\boldsymbol{\pitchfork}) = \{[N]\} \sqcup \bigsqcup_{j=1}^{d} \{\lambda \in \mathcal{B}(\boldsymbol{\pitchfork}) : \lambda \subset I_{k_j}\}.$$

Each of the sets $\{\lambda \in \mathcal{B}(\boldsymbol{\pitchfork}) : \lambda \subset I_{k_j}\}$ is the branch set $\mathcal{B}(\boldsymbol{\pitchfork}_j)$ for a unique $\boldsymbol{\pitchfork}_j \in \mathcal{R}_{I_{k_j}}$, so in this sense $\boldsymbol{\pitchfork}$ "breaks" into a unique tuple $(\boldsymbol{\pitchfork}_1,\ldots,\boldsymbol{\pitchfork}_d) \in \mathcal{R}_{I_{k_1}} \times \cdots \times \mathcal{R}_{I_{k_d}}$. On the other hand, any tuple $(\boldsymbol{\pitchfork}_1,\ldots,\boldsymbol{\pitchfork}_d) \in \mathcal{R}_{I_{k_1}} \times \cdots \times \mathcal{R}_{I_{k_d}}$ can be "assembled" as follows. Since $\{I_{k_1},\ldots,I_{k_d}\}$ is a partition of $[N]$, taking the union of the d branch sets $\mathcal{B}(\boldsymbol{\pitchfork}_1),\ldots,\mathcal{B}(\boldsymbol{\pitchfork}_d)$ and the singleton $\{[N]\}$ forms the branch set $\mathcal{B}(\boldsymbol{\pitchfork})$ for a unique $\boldsymbol{\pitchfork} \in \mathcal{R}_N$. It is clear that "breaking" and "assembling" are inverses, giving a correspondence $\mathcal{R}_N \longleftrightarrow \mathcal{R}_{I_{k_1}} \times \cdots \times \mathcal{R}_{I_{k_d}}$ under which each identification

$\pitchfork \longleftrightarrow (\pitchfork_1, \dots, \pitchfork_d)$ amounts to a branch set equation, i.e.,

$$\mathcal{B}(\pitchfork) \setminus \{[N]\} = \mathcal{B}(\pitchfork_1) \sqcup \cdots \sqcup \mathcal{B}(\pitchfork_d).$$

In particular, each $\lambda \in \mathcal{B}(\pitchfork) \setminus \{[N]\}$ is contained in exactly one $\mathcal{B}(\pitchfork_j)$, and $\deg_{\pitchfork}(\lambda) = \deg_{\pitchfork_j}(\lambda)$ by Definition 1.6.3 in this case. These facts allow the sum over $\mathcal{R}_{I_1} \times \cdots \times \mathcal{R}_{I_d}$ above to be rewritten as a sum over all $\pitchfork \in \mathcal{R}_N$ with $\pitchfork_1 = \{I_{k_1}, \dots, I_{k_q}\}$, and each product over $\lambda \in \mathcal{B}(\pitchfork_{k_1}) \sqcup \cdots \sqcup \mathcal{B}(\pitchfork_{k_d})$ inside it is simply a product over $\lambda \in \mathcal{B}(\pitchfork) \setminus \{[N]\}$. We conclude that an ordered partition $(I_0, \dots, I_q) \vdash [N]$ with $I_0, \dots, I_q \subsetneq [N]$ contributes the quantity

$$\prod_{k=0}^{q} \mathcal{Z}_{I_k}(P, \boldsymbol{s}) = \frac{1}{q^N} \sum_{\substack{\pitchfork \in \mathcal{R}_N \\ \pitchfork_1 = \{I_{k_1}, \dots, I_{k_d}\}}} \prod_{\lambda \in \mathcal{B}(\pitchfork) \setminus \{[N]\}} \frac{(q-1)_{\deg_{\pitchfork}(\lambda) - 1}}{q^{e_\lambda(\boldsymbol{s})} - 1} \tag{5.3.5}$$

to the sum in (5.3.3), where $\{I_{k_1}, \dots, I_{k_d}\}$ is the (unordered) subset of nonempty parts in that particular ordered partition. We must now total the contribution in (5.3.5) over all possible $(I_0, \dots, I_q) \vdash [N]$ with $I_0, \dots, I_q \subsetneq [N]$. Given a partition $\{\lambda_1, \dots, \lambda_d\} \vdash [N]$ with $d \geq 2$, note that there are precisely $(q+1)_d = (q+1) \cdot (q)_{d-1}$ ordered partitions $(I_0, \dots, I_q) \vdash [N]$ such that $\{I_{k_1}, \dots, I_{k_d}\} = \{\lambda_1, \dots, \lambda_d\}$. Therefore summing (5.3.5) over all $(I_0, \dots, I_q) \vdash [N]$ with $I_0, \dots, I_q \subsetneq [N]$ gives

$$\begin{aligned}
\sum_{\substack{(I_0, \dots, I_q) \vdash [N] \\ I_0, \dots, I_q \subsetneq [N]}} \prod_{k=0}^{q} \mathcal{Z}_{I_k}(P, \boldsymbol{s}) &= \frac{1}{q^N} \sum_{\substack{(I_0, \dots, I_q) \vdash [N] \\ I_0, \dots, I_q \subsetneq [N]}} \sum_{\substack{\pitchfork \in \mathcal{R}_N \\ \pitchfork_1 = \{I_{k_1}, \dots, I_{k_d}\}}} \prod_{\lambda \in \mathcal{B}(\pitchfork) \setminus \{[N]\}} \frac{(q-1)_{\deg_{\pitchfork}(\lambda) - 1}}{q^{e_\lambda(\boldsymbol{s})} - 1} \\
&= \frac{q+1}{q^N} \sum_{\substack{\{\lambda_1, \dots, \lambda_d\} \vdash [N] \\ d \geq 2}} (q)_{d-1} \sum_{\substack{\pitchfork \in \mathcal{R}_N \\ \pitchfork_1 = \{\lambda_1, \dots, \lambda_d\}}} \prod_{\lambda \in \mathcal{B}(\pitchfork) \setminus \{[N]\}} \frac{(q-1)_{\deg_{\pitchfork}(\lambda) - 1}}{q^{e_\lambda(\boldsymbol{s})} - 1},
\end{aligned}$$

Given a partition $\{\lambda_1,\ldots,\lambda_d\} \vdash [N]$, those splitting chains $\boldsymbol{\pitchfork} \in \mathcal{R}_N$ with $\pitchfork_1 = \{\lambda_1,\ldots,\lambda_d\}$ all have $\deg_{\boldsymbol{\pitchfork}}([N]) = \#\pitchfork_1 = d$ by Definition 1.6.3. Moreover, no $\boldsymbol{\pitchfork} \in \mathcal{R}_N$ is missed or repeated in the sum of sums above, so it can be rewritten as

$$\begin{aligned}\sum_{\substack{(I_0,\ldots,I_q)\vdash[N]\\ I_0,\ldots,I_q\subsetneq[N]}} \prod_{k=0}^{q} \mathcal{Z}_{I_k}(P,\boldsymbol{s}) &= \frac{q+1}{q^N}\sum_{\boldsymbol{\pitchfork}\in\mathcal{R}_N}(q)_{\deg_{\boldsymbol{\pitchfork}}([N])-1}\prod_{\lambda\in\mathcal{B}(\boldsymbol{\pitchfork})\setminus\{[N]\}}\frac{(q-1)_{\deg_{\boldsymbol{\pitchfork}}(\lambda)-1}}{q^{e_\lambda(\boldsymbol{s})}-1}\\
&= \frac{q+1}{q^N}\sum_{\boldsymbol{\pitchfork}\in\mathcal{R}_N}\frac{(q)_{\deg_{\boldsymbol{\pitchfork}}([N])-1}}{(q-1)_{\deg_{\boldsymbol{\pitchfork}}([N])-1}}\cdot(q^{e_{[N]}(\boldsymbol{s})}-1)\prod_{\lambda\in\mathcal{B}(\boldsymbol{\pitchfork})}\frac{(q-1)_{\deg_{\boldsymbol{\pitchfork}}(\lambda)-1}}{q^{e_\lambda(\boldsymbol{s})}-1}\\
&= \frac{q+1}{q^N}\sum_{\boldsymbol{\pitchfork}\in\mathcal{R}_N}\frac{q^{N+\sum_{i<j}s_{ij}}-q}{q+1-\deg_{\boldsymbol{\pitchfork}}([N])}\prod_{\lambda\in\mathcal{B}(\boldsymbol{\pitchfork})}\frac{(q-1)_{\deg_{\boldsymbol{\pitchfork}}(\lambda)-1}}{q^{e_\lambda(\boldsymbol{s})}-1}.\end{aligned}$$

Note that the summand for each $\boldsymbol{\pitchfork} \in \mathcal{R}_N$ is still defined for any prime power q since the denominators $(q-1)_{\deg_{\boldsymbol{\pitchfork}}([N])-1}$ and $q+1-\deg_{\boldsymbol{\pitchfork}}([N])$ (which vanish when $q = \deg_{\boldsymbol{\pitchfork}}([N])-1$) are cancelled by the numerator $(q-1)_{\deg_{\boldsymbol{\pitchfork}}([N])-1}$ appearing in the product over $\lambda \in \mathcal{B}(\boldsymbol{\pitchfork})$. Finally, we obtain the righthand side of (5.3.3) by combining the sum directly above with that in (5.3.4) and multiplying through by $(\frac{q}{q+1})^N$. This yields the desired formula for $\mathcal{Z}_N(\mathbb{P}^1(K),\boldsymbol{s})$:

$$\begin{aligned}\mathcal{Z}_N(\mathbb{P}^1(K),\boldsymbol{s}) &= \frac{1}{(q+1)^{N-1}}\sum_{\boldsymbol{\pitchfork}\in\mathcal{R}_N}\left(1+\frac{q^{N+\sum_{i<j}s_{ij}}-q}{q+1-\deg_{\boldsymbol{\pitchfork}}([N])}\right)\prod_{\lambda\in\mathcal{B}(\boldsymbol{\pitchfork})}\frac{(q-1)_{\deg_{\boldsymbol{\pitchfork}}(\lambda)-1}}{q^{e_\lambda(\boldsymbol{s})}-1}\\
&= \frac{1}{(q+1)^{N-1}}\sum_{\boldsymbol{\pitchfork}\in\mathcal{R}_N}\frac{q^{N+\sum_{i<j}s_{ij}}+1-\deg_{\boldsymbol{\pitchfork}}([N])}{q+1-\deg_{\boldsymbol{\pitchfork}}([N])}\prod_{\lambda\in\mathcal{B}(\boldsymbol{\pitchfork})}\frac{(q-1)_{\deg_{\boldsymbol{\pitchfork}}(\lambda)-1}}{q^{e_\lambda(\boldsymbol{s})}-1}.\end{aligned}$$

$\square$

5.4. The proof of the $(q+1)$th Power Law

Our final task is to prove the $(q+1)$th Power Law, which we noted in Section 2.2 is equivalent to the equations in (2.2.7). That is, it remains to prove

$$\frac{Z_N(\mathbb{P}^1(K),\beta)}{N!} = \sum_{\substack{N_0+\cdots+N_q=N \\ N_0,\ldots,N_q\geq 0}} \prod_{k=0}^{q} \left(\frac{q}{q+1}\right)^{N_k} \frac{Z_{N_k}(P,\beta)}{N_k!} \qquad \text{for all } \beta>0 \text{ and } N\geq 0.$$

Proof. Fix $N \geq 0$ and $\beta > 0$, and fix $\boldsymbol{s}$ via $s_{ij} = \beta$ for all $i < j$, so that $\mathcal{Z}_N(\mathbb{P}^1(K),\boldsymbol{s}) = Z_N(\mathbb{P}^1(K),\beta)$ and $\mathcal{Z}_I(P,\boldsymbol{s}) = Z_{\#I}(P,\beta)$ for any subset $I \subset [N]$. The formula in Theorem 5.2.1 relates these functions of β via

$$\begin{aligned} Z_N(\mathbb{P}^1(K),\beta) &= \mathcal{Z}_N(\mathbb{P}^1(K),\boldsymbol{s}) \\ &= \left(\frac{q}{q+1}\right)^{N} \sum_{(I_0,\ldots,I_q)\vdash[N]} \prod_{k=0}^{q} \mathcal{Z}_{\#I_k}(P,\boldsymbol{s}) \\ &= \sum_{(I_0,\ldots,I_q)\vdash[N]} \prod_{k=0}^{q} \left(\frac{q}{q+1}\right)^{\#I_k} Z_{\#I_k}(P,\beta). \end{aligned}$$

For each choice of $q+1$ ordered integers $N_0,\ldots,N_q\geq 0$ satisfying $N_0+\cdots+N_q = N$, there are precisely

$$\binom{N}{N_0,\ldots,N_q} = \frac{N!}{N_0!\cdots N_q!}$$

ordered partitions $(I_0,\ldots,I_q) \vdash [N]$ satisfying $\#I_0 = N_0,\ldots,\#I_q = N_q$. Finally, grouping ordered partitions according to all possible ordered integer choices establishes the desired equation:

$$\begin{aligned}\frac{Z_N(\mathbb{P}^1(K),\boldsymbol{s})}{N!} &= \frac{1}{N!}\cdot\sum_{(I_0,\ldots,I_q)\vdash[N]}\prod_{k=0}^{q}\left(\frac{q}{q+1}\right)^{\#I_k} Z_{\#I_k}(P,\beta)\\ &= \frac{1}{N!}\cdot\sum_{\substack{N_0+\cdots+N_q=N\\ N_0,\ldots,N_q\geq 0}}\binom{N}{N_0,\ldots,N_q}\prod_{k=0}^{q}\left(\frac{q}{q+1}\right)^{N_k} Z_{N_k}(P,\beta)\\ &= \sum_{\substack{N_0+\cdots+N_q=N\\ N_0,\ldots,N_q\geq 0}}\prod_{k=0}^{q}\left(\frac{q}{q+1}\right)^{N_k}\frac{Z_{N_k}(P,\beta)}{N_k!}.\end{aligned}$$

□

APPENDIX

A.1. Explicit computation of $Z_4^\rho(K, a, b, \boldsymbol{s})$

Let $N = 4$, fix $a, b \in \mathbb{C}$, let ρ be any norm-density. We will tabulate all splitting chains of order 4 and use Theorem 1.6.6 to compute the integral

$$\int_{K^4} \rho(\|x\|)\big(\max_{i<j} |x_i - x_j|\big)^a \big(\min_{i<j} |x_i - x_j|\big)^b \prod_{1\le i<j\le 4} |x_i - x_j|^{s_{ij}}\, dx_1 dx_2 dx_3 dx_4$$

explicitly. Writing $\boldsymbol{s}$ for the 6-tuple $\boldsymbol{s} = (s_{12}, s_{13}, s_{14}, s_{23}, s_{24}, s_{34}) \in \mathbb{C}^6$, we have a root polytope

$$\mathcal{RP}_4(a,b) = \{\boldsymbol{s} \in \mathbb{C}^6 : \mathrm{Re}(3 + a + b + \sum_{1\le i<j\le 4} s_{ij}) > 0\},$$

on which

$$H_q^\rho\Bigg(4+a+b+\sum_{1\le i<j\le 4} s_{ij}\Bigg) = \frac{1 - q^{-(4+a+b+\sum_{1\le i<j\le 4} s_{ij})}}{1 - q^{-(3+a+b+\sum_{1\le i<j\le 4} s_{ij})}} \cdot \sum_{m\in\mathbb{Z}} \rho(q^m) q^{m(4+a+b+\sum_{1\le i<j\le 4} s_{ij})}$$

is defined and holomorphic. There are 32 splitting chains $\boldsymbol{\pitchfork} \in \mathcal{S}_4$, so we will save table space below by suppressing the partition labels "$\pitchfork_\ell =$" and by only writing all of the polytope conditions "$\mathrm{Re}(b + E_{\pitchfork_\ell}(\boldsymbol{s})) > 0$" at the end (not along the way). Given $\boldsymbol{\pitchfork} \in \mathcal{S}_4$, the level $\pitchfork_1$ can either contain one part of size 3 (and a singleton), one part of size 2 (and two singletons), two parts of size 2, or four singletons. Thus it will be practical to sort $\boldsymbol{\pitchfork} \in \mathcal{S}_4$ according to the form of $\pitchfork_1$:

(1) There are four $\boldsymbol{\pitchfork} \in \mathcal{S}_4$ with $\pitchfork_1 = \{1,2,3\}\{4\}$. Unsurprisingly, they form a table very similar to the one for $\mathcal{S}_3$ in Example 1.7.1:

$\boldsymbol{\pitchfork}$	$J_{\boldsymbol{\pitchfork},q}(b,\boldsymbol{s})$
$\{1,2,3,4\}$ $\{1,2,3\}\{4\}$ $\{1\}\{2\}\{3\}\{4\}$	$\frac{(q-1)^2(q-2)}{q^3} \cdot \frac{1}{q^{2+b+s_{12}+s_{13}+s_{23}}-1}$
$\{1,2,3,4\}$ $\{1,2,3\}\{4\}$ $\{1,2\}\{3\}\{4\}$ $\{1\}\{2\}\{3\}\{4\}$	$\frac{(q-1)^3}{q^3} \cdot \frac{1}{q^{2+b+s_{12}+s_{13}+s_{23}}-1} \cdot \frac{1}{q^{1+b+s_{12}}-1}$
$\{1,2,3,4\}$ $\{1,2,3\}\{4\}$ $\{1,3\}\{2\}\{4\}$ $\{1\}\{2\}\{3\}\{4\}$	$\frac{(q-1)^3}{q^3} \cdot \frac{1}{q^{2+b+s_{12}+s_{13}+s_{23}}-1} \cdot \frac{1}{q^{1+b+s_{13}}-1}$
$\{1,2,3,4\}$ $\{1,2,3\}\{4\}$ $\{2,3\}\{1\}\{4\}$ $\{1\}\{2\}\{3\}\{4\}$	$\frac{(q-1)^3}{q^3} \cdot \frac{1}{q^{2+b+s_{12}+s_{13}+s_{23}}-1} \cdot \frac{1}{q^{1+b+s_{23}}-1}$

All four of the splitting chains in the table are reduced and each satisfies $J_{\boldsymbol{\pitchfork},q}(0,\boldsymbol{s}) = I_{\boldsymbol{\pitchfork},q}(\boldsymbol{s})$. There are also four $\boldsymbol{\pitchfork} \in \mathcal{S}_4$ satisfying $\pitchfork_1 = \{1,2,4\}\{3\}$, and their table is obtained by simply transposing the indices 3 and 4 in the table above. Similarly, there are another four $\boldsymbol{\pitchfork} \in \mathcal{S}_4$ satisfying $\pitchfork_1 = \{1,3,4\}\{2\}$ and another four satisfying $\pitchfork_1 = \{2,3,4\}\{1\}$. Thus there are 16 distinct $\boldsymbol{\pitchfork} \in \mathcal{S}_4$ such that $\pitchfork_1$ has a part of size 3, and all of them are reduced.

(2) There are six $\boldsymbol{\pitchfork} \in \mathcal{S}_4$ such that $\pitchfork_1$ contains a single part of size 2. All six are reduced and satisfy $J_{\boldsymbol{\pitchfork},q}(0,\boldsymbol{s}) = I_{\boldsymbol{\pitchfork},q}(\boldsymbol{s})$:

$\boldsymbol{\pitchfork}$	$J_{\boldsymbol{\pitchfork},q}(b,\boldsymbol{s})$
$\{1,2,3,4\}$ $\{1,2\}\{3\}\{4\}$ $\{1\}\{2\}\{3\}\{4\}$	$\frac{(q-1)^2(q-2)}{q^3} \cdot \frac{1}{q^{1+b+s_{12}}-1}$
$\{1,2,3,4\}$ $\{1,3\}\{2\}\{4\}$ $\{1\}\{2\}\{3\}\{4\}$	$\frac{(q-1)^2(q-2)}{q^3} \cdot \frac{1}{q^{1+b+s_{13}}-1}$
$\{1,2,3,4\}$ $\{1,4\}\{2\}\{3\}$ $\{1\}\{2\}\{3\}\{4\}$	$\frac{(q-1)^2(q-2)}{q^3} \cdot \frac{1}{q^{1+b+s_{14}}-1}$
$\{1,2,3,4\}$ $\{2,3\}\{1\}\{4\}$ $\{1\}\{2\}\{3\}\{4\}$	$\frac{(q-1)^2(q-2)}{q^3} \cdot \frac{1}{q^{1+b+s_{23}}-1}$
$\{1,2,3,4\}$ $\{2,4\}\{1\}\{3\}$ $\{1\}\{2\}\{3\}\{4\}$	$\frac{(q-1)^2(q-2)}{q^3} \cdot \frac{1}{q^{1+b+s_{24}}-1}$
$\{1,2,3,4\}$ $\{3,4\}\{1\}\{2\}$ $\{1\}\{2\}\{3\}\{4\}$	$\frac{(q-1)^2(q-2)}{q^3} \cdot \frac{1}{q^{1+b+s_{34}}-1}$

(3) There are three splitting chains $\boldsymbol{\pitchfork} \in \mathcal{S}_4$ with $\pitchfork_1 = \{1,2\}\{3,4\}$:

$\boldsymbol{\pitchfork}$	$J_{\boldsymbol{\pitchfork},q}(b,\boldsymbol{s})$
$\{1,2,3,4\}$ $\{1,2\}\{3,4\}$ $\{1\}\{2\}\{3\}\{4\}$	$\frac{(q-1)^3}{q^3} \cdot \frac{1}{q^{2+b+s_{12}+s_{34}}-1}$
$\{1,2,3,4\}$ $\{1,2\}\{3,4\}$ $\{1,2\}\{3\}\{4\}$ $\{1\}\{2\}\{3\}\{4\}$	$\frac{(q-1)^3}{q^3} \cdot \frac{1}{q^{2+b+s_{12}+s_{34}}-1} \cdot \frac{1}{q^{1+b+s_{12}}-1}$
$\{1,2,3,4\}$ $\{1,2\}\{3,4\}$ $\{1\}\{2\}\{3,4\}$ $\{1\}\{2\}\{3\}\{4\}$	$\frac{(q-1)^3}{q^3} \cdot \frac{1}{q^{2+b+s_{12}+s_{34}}-1} \cdot \frac{1}{q^{1+b+s_{34}}-1}$

There are also three $\boldsymbol{\pitchfork} \in \mathcal{S}_4$ satisfying $\pitchfork_1 = \{1,3\}\{2,4\}$, and their table is obtained by simply transposing the indices 2 and 3 in the table above. Similarly, there are another three $\boldsymbol{\pitchfork} \in \mathcal{S}_4$ satisfying $\pitchfork_1 = \{1,4\}\{2,3\}$. Thus there are nine distinct $\boldsymbol{\pitchfork} \in \mathcal{S}_4$ such that $\pitchfork_1$ has a pair of parts of size 2, but only three of them are reduced (the three that have length 2).

(4) Finally, we have only one $\pitchfork \in \mathcal{S}_4$ with $\pitchfork_1 = \{1\}\{2\}\{3\}\{4\}$. It is $\pitchfork = (\{1,2,3,4\},\ \{1\}\{2\}\{3\}\{4\})$, which is clearly reduced and has

$$J_{\pitchfork,q}(b,\boldsymbol{s}) = \frac{(q-1)(q-2)(q-3)}{q^3}.$$

Combining the level polytopes for all $\pitchfork \in \mathcal{S}_4$ with the root polytope condition $\mathrm{Re}(3+a+b+\sum_{1\le i<j\le 4}) > 0$, we conclude that $\boldsymbol{s} \in \Omega_4(a,b)$ if and only if

$$\begin{aligned}
\mathrm{Re}(1+b+s_{ij}) > 0 \qquad &\text{for} \qquad 1 \le i < j \le 4,\\
\mathrm{Re}(2+b+s_{12}+s_{34}) > 0,&\\
\mathrm{Re}(2+b+s_{13}+s_{24}) > 0,&\\
\mathrm{Re}(2+b+s_{14}+s_{23}) > 0,&\\
\mathrm{Re}(2+b+s_{ij}+s_{ik}+s_{jk}) > 0 \quad &\text{for} \quad 1 \le i < j < k \le 4, \quad \text{and}\\
\mathrm{Re}(3+a+b+s_{12}+s_{13}+s_{14}+s_{23}+s_{24}+s_{34}) > 0.&
\end{aligned}$$

On every compact subset of $C \subset \Omega_4(a,b)$, the integral

$$\int_{K^4} \rho(\|x\|)\big(\max_{i<j}|x_i-x_j|\big)^a\big(\min_{i<j}|x_i-x_j|\big)^b \prod_{1\le i<j\le 4}|x_i-x_j|^{s_{ij}}\,dx_1dx_2dx_3dx_4$$

converges absolutely to the product of the root function value

$$H_q^\rho\Bigg(4+a+b+\sum_{1\le i<j\le 4}s_{ij}\Bigg) = \frac{1-q^{-(4+a+b+\sum_{1\le i<j\le 4}s_{ij})}}{1-q^{-(3+a+b+\sum_{1\le i<j\le 4}s_{ij})}}\cdot\sum_{m\in\mathbb{Z}}\rho(q^m)q^{m(4+a+b+\sum_{1\le i<j\le 4}s_{ij})}$$

and the following sum of level function values:

$$
\begin{aligned}
&\frac{(q-1)(q-2)(q-3)}{q^3} + \frac{(q-1)^2(q-2)}{q^3}\Bigg[\frac{1}{q^{2+b+s_{12}+s_{13}+s_{23}}-1} + \frac{1}{q^{2+b+s_{12}+s_{14}+s_{24}}-1} \\
&\qquad + \frac{1}{q^{2+b+s_{13}+s_{14}+s_{34}}-1} + \frac{1}{q^{2+b+s_{23}+s_{24}+s_{34}}-1} \\
&\qquad + \frac{1}{q^{1+b+s_{12}}-1} + \frac{1}{q^{1+b+s_{13}}-1} + \frac{1}{q^{1+b+s_{14}}-1} \\
&\qquad + \frac{1}{q^{1+b+s_{12}}-1} + \frac{1}{q^{1+b+s_{13}}-1} + \frac{1}{q^{1+b+s_{14}}-1}\Bigg] \\
&+ \frac{(q-1)^3}{q^3}\Bigg[\frac{1}{q^{2+b+s_{12}+s_{34}}-1}\left(1 + \frac{1}{q^{1+b+s_{12}}-1} + \frac{1}{q^{1+b+s_{34}}-1}\right) \\
&\qquad + \frac{1}{q^{2+b+s_{13}+s_{24}}-1}\left(1 + \frac{1}{q^{1+b+s_{13}}-1} + \frac{1}{q^{1+b+s_{24}}-1}\right) \\
&\qquad + \frac{1}{q^{2+b+s_{14}+s_{23}}-1}\left(1 + \frac{1}{q^{1+b+s_{14}}-1} + \frac{1}{q^{1+b+s_{23}}-1}\right)\Bigg] \\
&+ \frac{(q-1)^3}{q^3}\Bigg[\frac{1}{q^{2+b+s_{12}+s_{13}+s_{23}}-1}\left(\frac{1}{q^{1+b+s_{12}}-1} + \frac{1}{q^{1+b+s_{13}}-1} + \frac{1}{q^{1+b+s_{23}}-1}\right) \\
&\qquad + \frac{1}{q^{2+b+s_{12}+s_{14}+s_{24}}-1}\left(\frac{1}{q^{1+b+s_{12}}-1} + \frac{1}{q^{1+b+s_{14}}-1} + \frac{1}{q^{1+b+s_{24}}-1}\right) \\
&\qquad + \frac{1}{q^{2+b+s_{13}+s_{14}+s_{34}}-1}\left(\frac{1}{q^{1+b+s_{13}}-1} + \frac{1}{q^{1+b+s_{14}}-1} + \frac{1}{q^{1+b+s_{34}}-1}\right) \\
&\qquad + \frac{1}{q^{2+b+s_{23}+s_{24}+s_{34}}-1}\left(\frac{1}{q^{1+b+s_{23}}-1} + \frac{1}{q^{1+b+s_{24}}-1} + \frac{1}{q^{1+b+s_{34}}-1}\right)\Bigg].
\end{aligned}
$$

The terms inside $\{\dots\}$ are grouped to emphasize several facts: The first group $\frac{(q-1)(q-2)(q-3)}{q^3}$ vanishes unless $q \geq 4$. The second group $\frac{(q-1)^2(q-2)}{q^3}\big[\dots\big]$ vanishes unless $q \geq 3$. The third group $\frac{(q-1)^3}{q^3}\big[\dots\big]$ is nonzero for all $q \geq 2$ (and hence all K), but collapses from nine terms down to three by part (c) of Theorem 1.6.6 when $b = 0$ (recall Example 1.7.3). The last group $\frac{(q-1)^3}{q^3}\big[\dots\big]$ is also nonzero for all $q \geq 2$ and corresponds to the 12 splitting chains of length 3 from case (1).

A.2. Functional equations and a quadratic recurrence

Although Theorem 1.6.6 provides explicit formulas for the canonical partition functions $Z_N(R,\beta)$ and $Z_N(\mathbb{P}^1(K),\beta)$, they are generally not efficient for computation as they require a tabulation of reduced splitting chains of order N. For a practical alternative, we take advantage of both Power Laws and the following ideas from [2] and [14]: Apply $Z(f,P,\beta)\cdot\frac{\partial}{\partial f}$ to the qth Power Law $Z(f,R,\beta)=(Z(f,P,\beta))^q$ to get

$$Z(f,P,\beta)\cdot\frac{\partial}{\partial f}Z(f,R,\beta)=q\cdot Z(f,R,\beta)\cdot\frac{\partial}{\partial f}Z(f,P,\beta),$$

then expand both sides as power series in f to obtain the coefficient equations

$$\sum_{k=1}^{N}\frac{Z_{N-k}(P,\beta)}{(N-k)!}\frac{Z_k(R,\beta)}{(k-1)!}=q\cdot\sum_{k=1}^{N}\frac{Z_{N-k}(R,\beta)}{(N-k)!}\frac{Z_k(P,\beta)}{(k-1)!}\qquad\text{for all } N\geq 1.\quad\text{(A.2.1)}$$

The identities $Z_j(P,\beta)=q^{-j-\binom{j}{2}\beta}Z_j(R,\beta)$ follow easily from Definition 1.5.3 and eliminate all instances of $Z_j(P,\beta)$ in (A.2.1) while introducing powers of the form $q^{-j-\binom{j}{2}\beta}$. For $N\geq 2$, a careful rearrangement of these powers, the factorials, and the terms in (A.2.1) yields the explicit recurrence

$$\frac{Z_N(R,\beta)}{N!q^{\frac{1}{2}\binom{N}{2}\beta}}=\sum_{k=1}^{N-1}\frac{k}{N}\cdot\frac{\sinh\left(\frac{\log(q)}{2}\left[\left(N+\binom{N}{2}\beta\right)\left(1-\frac{2k}{N}\right)+1\right]\right)}{\sinh\left(\frac{\log(q)}{2}\left[\left(N+\binom{N}{2}\beta\right)-1\right]\right)}\cdot\frac{Z_{N-k}(R,\beta)}{(N-k)!q^{\frac{1}{2}\binom{N-k}{2}\beta}}\cdot\frac{Z_k(R,\beta)}{k!q^{\frac{1}{2}\binom{k}{2}\beta}}.$$

The expression at left is identically 1 if $N=0$ or $N=1$, so induction confirms that it is polynomial in ratios of hyperbolic sines for all $N\geq 0$. In particular, its dependence on q is carried only by the factor $\log(q)$ appearing inside the hyperbolic sines, which motivates the following lemma:

Lemma A.2.1 (The Quadratic Recurrence). *Set* $F_0(t,\beta) = F_1(t,\beta) = 1$ *for all* $\beta \in \mathbb{C}$ *and all* $t \in \mathbb{R}$. *For* $N \geq 2$, $\mathrm{Re}(\beta) > -2/N$, *and* $t \in \mathbb{R}$, *define* $F_N(t,\beta)$ *by the recurrence*

$$F_N(t,\beta) := \begin{cases} \displaystyle\sum_{k=1}^{N-1} \frac{k}{N} \cdot \frac{\sinh\left(\frac{t}{2}\left[\left(N+\binom{N}{2}\beta\right)\left(1-\frac{2k}{N}\right)+1\right]\right)}{\sinh\left(\frac{t}{2}\left[\left(N+\binom{N}{2}\beta\right)-1\right]\right)} \cdot F_{N-k}(t,\beta) \cdot F_k(t,\beta) & \text{if } t \neq 0, \\ \\ \displaystyle\sum_{k=1}^{N-1} \frac{k}{N} \cdot \frac{\left(N+\binom{N}{2}\beta\right)\left(1-\frac{2k}{N}\right)+1}{\left(N+\binom{N}{2}\beta\right)-1} \cdot F_{N-k}(0,\beta) \cdot F_k(0,\beta) & \text{if } t = 0. \end{cases}$$

(a) For fixed $N \geq 2$ *and fixed* t*, the function* $\beta \mapsto F_N(t,\beta)$ *is holomorphic for* $\mathrm{Re}(\beta) > -2/N$.

(b) For fixed $N \geq 2$ *and fixed* β*, the function* $t \mapsto F_N(t,\beta)$ *is defined, even, and smooth on* $\mathbb{R}$.

Both parts of the Quadratic Recurrence are straightforward to verify by induction. An interesting and immediate consequence of The Quadratic Recurrence and the preceding discussion is the formula

$$Z_N(R,\beta) = N!q^{\frac{1}{2}\binom{N}{2}\beta} F_N(\log(q),\beta),$$

which offers a computationally efficient alternative to part (b) of Theorem 2.1.6 and extends $Z_N(R,\beta)$ to a smooth function of $q \in (0,\infty)$ in an obvious way. Moreover, the extended function transforms nicely under the involution $q \mapsto q^{-1}$:

$$\begin{aligned} Z_N(R,\beta)\big|_{q\mapsto q^{-1}} &= N!q^{-\frac{1}{2}\binom{N}{2}\beta} F_N\left(\log(q^{-1}),\beta\right) \\ &= N!q^{-\frac{1}{2}\binom{N}{2}\beta} F_N\left(\log(q),\beta\right) = q^{-\binom{N}{2}\beta} Z_N(R,\beta). \end{aligned}$$

The Quadratic Recurrence serves the projective analogue as well. Expanding (2.2.8) into powers of f yields the coefficient equations

$$\frac{Z_N(\mathbb{P}^1(K),\beta)}{N!} = \sum_{k=0}^{N} \left(\frac{q}{q+1}\right)^N \frac{Z_{N-k}(R,\beta)}{(N-k)!}\frac{Z_k(P,\beta)}{k!} \qquad \text{for all } N \geq 0, \qquad (\text{A.2.2})$$

and the identities $Z_j(P,\beta) = q^{-j-\binom{j}{2}\beta} Z_j(R,\beta)$ and $Z_j(R,\beta) = j!q^{\frac{1}{2}\binom{j}{2}\beta} F_j(\log(q),\beta)$ allow the kth summand to be rewritten as

$$\left(\frac{q}{q+1}\right)^N \frac{Z_{N-k}(R,\beta)}{(N-k)!}\frac{Z_k(P,\beta)}{k!} = \frac{q^{\frac{1}{2}\left(N+\binom{N}{2}\beta\right)\left(1-\frac{2k}{N}\right)}}{\left(2\cosh\left(\frac{\log(q)}{2}\right)\right)^N} \cdot F_{N-k}(\log(q),\beta)\cdot F_k(\log(q),\beta)$$

for all $N \geq 1$. Thus, adding two copies of the sum in (A.2.2) together, pairing the kth term of the first copy with the $(N-k)$th term of the second copy, and dividing by 2 gives

$$\frac{Z_N(\mathbb{P}^1(K),\beta)}{N!} = \sum_{k=0}^{N} \frac{\cosh\left(\frac{\log(q)}{2}\left(N+\binom{N}{2}\beta\right)\left(1-\frac{2k}{N}\right)\right)}{\left(2\cosh\left(\frac{\log(q)}{2}\right)\right)^N} \cdot F_{N-k}(\log(q),\beta)\cdot F_k(\log(q),\beta)$$

which is also valid for all $N \geq 1$ and $\mathrm{Re}(\beta) > -2/N$. Through this formula, $Z_N(\mathbb{P}^1(K),\beta)$ clearly extends to a smooth function of $q \in (0,\infty)$ and is invariant under the involution $q \mapsto q^{-1}$. We conclude this section with the following summary:

Theorem A.2.2 (Efficient Formulas and Functional Equations)**.** *Suppose $N \geq 2$ and $\mathrm{Re}(\beta) > -2/N$, and define $(F_k(t,\beta))_{k=0}^N$ as in A.2.1. The Nth canonical partition functions are given by the formulas*

$$Z_N(R,\beta) = N! q^{\frac{1}{2}\binom{N}{2}\beta} F_N(\log(q),\beta) \qquad \textit{and}$$

$$Z_N(\mathbb{P}(K),\beta) = N! \sum_{k=0}^{N} \frac{\cosh\left(\frac{\log(q)}{2}\left(N+\binom{N}{2}\beta\right)\left(1-\frac{2k}{N}\right)\right)}{\left(2\cosh\left(\frac{\log(q)}{2}\right)\right)^N} \cdot F_{N-k}(\log(q),\beta) \cdot F_k(\log(q),\beta),$$

which extend $Z_N(R,\beta)$ and $Z_N(\mathbb{P}^1(K),\beta)$ to smooth functions of $q \in (0,\infty)$ satisfying

$$Z_N(R,\beta)\big|_{q\mapsto q^{-1}} = q^{-\binom{N}{2}\beta} Z_N(R,\beta) \qquad \textit{and} \qquad Z_N(\mathbb{P}^1(K),\beta)\big|_{q\mapsto q^{-1}} = Z_N(\mathbb{P}^1(K),\beta).$$

It should be noted here that the first $q \mapsto q^{-1}$ functional equation is a special case of the one proved in [4], and that both functional equations closely resemble the ones in [15].

REFERENCES CITED

[1] M. Bocardo-Gaspar, H. García-Compeán, and W. A. Zúñiga Galindo. Regularization of p-adic string amplitudes, and multivariate local zeta functions. *Lett. Math. Phys.*, 109(5):1167–1204, 2019.

[2] J. Buhler, D. Goldstein, D. Moews, and J. Rosenberg. The probability that a random monic p-adic polynomial splits. *Experiment. Math.*, 15(1):21–32, 2006.

[3] J. Denef. Report on Igusa's local zeta function. *Astérisque*, (201-203):Exp. No. 741, 359–386 (1992), 1991. Séminaire Bourbaki, Vol. 1990/91.

[4] J. Denef and D. Meuser. A functional equation of Igusa's local zeta function. *Amer. J. Math.*, 113(6):1135–1152, 1991.

[5] M. du Sautoy and F. Grunewald. Zeta functions of groups and rings. In *International Congress of Mathematicians. Vol. II*, pages 131–149. Eur. Math. Soc., Zürich, 2006.

[6] P. Fili and C. Petsche. Energy integrals over local fields and global height bounds. *Int. Math. Res. Not. IMRN*, (5):1278–1294, 2015.

[7] P. J. Forrester and S. O. Warnaar. The importance of the Selberg integral. *Bull. Amer. Math. Soc. (N.S.)*, 45(4):489–534, 2008.

[8] H. Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II. *Ann. of Math. (2)* **79** *(1964), 109–203; ibid. (2)*, 79:205–326, 1964.

[9] J.-i. Igusa. Complex powers and asymptotic expansions. I. Functions of certain types. *J. Reine Angew. Math.*, 268/269:110–130, 1974. Collection of articles dedicated to Helmut Hasse on his seventy-fifth birthday, II.

[10] J.-i. Igusa. Complex powers and asymptotic expansions. II. Asymptotic expansions. *J. Reine Angew. Math.*, 278/279:307–321, 1975.

[11] F. Loeser. Fonctions zêta locales d'igusa à plusieurs variables, intégration dans les fibres, et discriminants. *Annales scientifiques de l'École Normale Supérieure*, 4e série, 22(3):435–471, 1989.

[12] D. Ramakrishnan and R. J. Valenza. *Fourier analysis on number fields*, volume 186 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999.

[13] S. Serfaty. *Coulomb gases and Ginzburg-Landau vortices.* Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2015.

[14] C. D. Sinclair. Non-Archimedean Electrostatics. *arXiv e-prints*, page arXiv:2002.07121, Feb. 2020.

[15] C. Voll. Functional equations for zeta functions of groups and rings. *Ann. of Math. (2)*, 172(2):1181–1218, 2010.

[16] J. Webster. log-coulomb gas with norm-density in p-fields. *P-Adic Num Ultrametr Anal Appl*, 13(1):1–43, 2021.

[17] A. Weil. *Basic number theory.* Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the second (1973) edition.

[18] W. A. Zúñiga Galindo and S. M. Torba. Non-Archimedean Coulomb gases. *J. Math. Phys.*, 61(1):013504, 16, 2020.

[19] W. A. Zúñiga-Galindo, B. A. Zambrano-Luna, and E. León-Cardenal. Graphs, local zeta functions, Log-Coulomb Gases, and phase transitions at finite temperature. *arXiv e-prints*, page arXiv:2003.08532, Mar. 2020.