MODULES WITH GOOD FILTRATIONS OVER GENERALIZED SCHUR ALGEBRAS

by

ILAN WEINSCHELBAUM

A DISSERTATION

Presented to the Department of Mathematics
and the Division of Graduate Studies of the University of Oregon
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

June 2022

DISSERTATION APPROVAL PAGE

Student: Ilan Weinschelbaum

Title: Modules with Good Filtrations over Generalized Schur Algebras

This dissertation has been accepted and approved in partial fulfillment of the requirements for the Doctor of Philosophy degree in the Department of Mathematics by:

Alexander Kleshchev	Chair
Ben Elias	Core Member
Jon Brundan	Core Member
Victor Ostrik	Core Member
Ilya Bindeman	Institutional Representative

and

Krista Chronister	Vice Provost for Graduate Studies

Original approval signatures are on file with the University of Oregon Division of Graduate Studies.

Degree awarded June 2022

DISSERTATION ABSTRACT

Ilan Weinschelbaum

Doctor of Philosophy

Department of Mathematics

June 2022

Title: Modules with Good Filtrations over Generalized Schur Algebras

In this dissertation we examine generalized Schur algebras, as defined by Kleshchev and Muth. Given a quasi-hereditary superalgebra A, Kleshchev and Muth proved that for $n \geq d$, the generalized Schur algebra $T^A(n,d)$ is again quasi-hereditary. They described the bisuperalgebra struture on $T^A(n) := \bigoplus_d T^A(n,d)$. In particular, there is a coproduct which gives us a way to take a $T^A(n,d)$-module V and $T^A(n,r)$-module W and produce a $T^A(n,d+r)$-module $V \otimes W$. We will prove that if V and W each have standard (resp. costandard) filtrations, then so does $V \otimes W$. In the last chapter we will use this result to prove that in the case that A is the extended zigzag algebra Z, the extended zigzag Schur algebra $T^{\mathsf{Z}}(n,d)$ is Ringel self-dual for all $n \geq d$.

This dissertation contains previously unpublished co-authored material.

CURRICULUM VITAE

NAME OF AUTHOR: Ilan Weinschelbaum

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:
University of Oregon, Eugene, OR
Wesleyan University, Middletown, CT

DEGREES AWARDED:
Doctor of Philosophy, Mathematics, 2022, University of Oregon
Bachelor of Arts, Mathematics and Economics, 2015, Wesleyan University

AREAS OF SPECIAL INTEREST:
Representation Theory

PROFESSIONAL EXPERIENCE:

Graduate Employee, University of Oregon, 2015-2022

PUBLICATIONS:

A. Kleshchev and I. Weinschelbaum. Good filtrations for generalized Schur Algebras. *Arxiv preprint. arxiv:2202.08866*, 2022.

A. Kleshchev and I. Weinschelbaum. Ringel duality for extended zigzag Schur algebra. *in progress.*

ACKNOWLEDGEMENTS

First I would like to thank my advisor, Alexander Kleshchev. Your advice, both mathematical and non-mathematical, was crucial to my development in graduate school.

To my parents, my brother, my niece, my grandmother, and the rest of my family: thank you for everything. Your unflinching support, your boundless love, and your overwhelming confidence in me have been vital in getting to this place.

To my colleagues and friends from my time at the University of Oregon: thank you for traversing the muddy academic waters with me. Our time spent avoiding mathematics was just as helpful as our time spent discussing it. I would especially like to thank Daniel Hothem and Elijah Bodish for our fascinating algebraic conversations. To Fill Staley and Matt Lukac: I will forever remember our heated games of Catan as some of the sweetest memories of my time in graduate school.

To my cat, Brisket: you are a fierce troublemaker and a proud advocate for yourself. Thank you for reminding me that sometimes in life you must take risks.

Lastly, I thank my fierce, wonderful partner, Ellen Kress. Your extraordinary generosity, your indomitable spirit, and your laser-sharp focus are an inspiration to me. Your love and support have been a rejuvenating force when the academic struggles left me exhausted.

Para mi tío Gaby. Te extraño.

TABLE OF CONTENTS

LIST OF FIGURES

Figure | Page

CHAPTER I

INTRODUCTION AND PRELIMINARIES

1.1. Introduction

This dissertation contains previously unpublished co-authored material. The main results of this dissertation are the main results of [1, 2]. Those papers are joint work with Alexander Kleshchev, who served as the author's advisor. The material from these papers appears in chapters III, IV, and V of this dissertation. Chapter III contains material from both [1] and [2]. Chapter IV consists of material from [1]. Lastly, chapter V consists of material from [2].

Let $n \in \mathbb{Z}_{>0}$ and $d \in \mathbb{Z}_{\geq 0}$. Let $S(n,d)$ be the classical Schur Algebra as in [3]. It is well-known (see [4]) that $S(n,d)$ is quasi-hereditary. In fact, it is based quasi-hereditary in the sense of [5]. So, $S(n,d)$-mod is a Highest Weight Category in the sense of [6]. Furthermore, there is a coproduct on $S(n) := \bigoplus_{d\geq 0} S(n,d)$, which restricts to a map

$$S(n,r) \to \bigoplus_{r_1+r_2=r} S(n,r_1) \otimes S(n,r_2)$$

and thus if $V \in S(n,d)$-mod and $W \in S(n,r)$-mod then $V \otimes W$ is an $S(n,d+r)$-module via the coproduct.

It follows from the work of Donkin [7], Mathieu [8], and Wang [9] that if V and W both have standard filtrations (in the sense of [6]), then $V \otimes W$ does as well.

Let R be a principal ideal domain of characteristic 0 and let $\mathbb{F}$ be a field which is an R-module. In [10] and [11], Kleschev and Muth generalized a construction of Turner [12, 13, 14], in which they take a based quasi-hereditary $\mathbb{F}$-superalgebra A with an R-form A_R and conforming heredity data I, X, Y (see § 3.1.1 for details), and define the generalized Schur Algerbas

$$T^A(n)_{\mathbb{F}} := \bigoplus_{d \geq 0} T^A(n,d)_R \otimes_R \mathbb{F}.$$

Crucially, we must first define these algebras over R, and then extend scalars to $\mathbb{F}$ - so it is vital that A has the R-form A_R. In chapter 4, we generalize the aforementioned standard filtration result to these algebras.

Another classic result of Donkin (see [15]) is that the classical Schur algebra is a Ringel dual for itself (see § 2.3.3). This leads us to conjecture:

Conjecture 1.1. Let $n \in \mathbb{Z}_{>0}$ and $d \in N$ with $n \geq d$. Let A be a based quasi-hereditary algebra and $d \leq n$. If A' is a Ringel dual of A, then a Ringel dual of $T^A_{\mathfrak{a}}(n,d)$ is of the form $T^{A'}_{\mathfrak{a}'}(n,d)$ for some canonical choice of $\mathfrak{a}'$.

We check this conjecture in the case $A = \mathsf{Z}$, the extended zigzag algebra. We do so by first proving that Z is Ringel self-dual, and then proving that $T^{\mathsf{Z}}(n,d)_{\mathbb{F}}$ is Ringel self-dual.

The dissertation is organized as follows. The rest of chapter I is dedicated to preliminaries. In chapter II, we recap the definitions and results of Kleshchev and Muth about based quasi-hereditary algebras. In chapter III, we define the generalized Schur algebras, and describe several important results. In chapter IV, we prove that the tensor product of modules with standard filtrations again has a standard filtration (and analogously for modules with costandard filtrations). Lastly, in chapter V, we prove that $T^{\mathsf{Z}}(n,d)_{\mathbb{F}}$ is Ringel self-dual.

1.2. General Notation

For $m, n \in \mathbb{Z}$, we denote $[m, n] := \{k \in \mathbb{Z} \mid m \le k \le n\}$. If $n \in \mathbb{Z}_{>0}$, we also denote $[n] := \{1, 2, \dots, n\}$.

Throughout the paper, I denotes a finite partially ordered set. We often identify I with the set $\{0, 1, \dots, \ell\}$ for $\ell = |I| - 1$, so that the standard total order on integers refines the partial order on I.

For a set S, we often write elements of S^d as words $\boldsymbol{s} = s_1 \cdots s_d$ with $s_1, \dots, s_d \in S$. The symmetric group $\mathfrak{S}_d$ acts on the right on S^d by place permutations:

$$(s_1 \cdots s_d)\sigma = s_{\sigma 1} \cdots s_{\sigma d}.$$

An (arbitrary) ground field is denoted by $\mathbb{F}$. Often we will also need to work over a characteristic 0 principal ideal domain R such that $\mathbb{F}$ is a R-module, so that we can change scalars from R to $\mathbb{F}$ (in all examples of interest to us, one can use $R = \mathbb{Z}$). When the nature of the ground ring is not important, we will use $\Bbbk$ to denote either R or $\mathbb{F}$. On the other hand, when it is important to emphasize whether we are working over R or $\mathbb{F}$, we will use lower indices; for example for an R-algebra A_R and an A_R-module V_R, after extending scalars we have that $V_\mathbb{F} := \mathbb{F} \otimes_R V_R$ is a module over $A_\mathbb{F} := \mathbb{F} \otimes_R A_R$. We always assume R is purely even as an R-supermodule.

1.3. Superlinear Algebra

Let $V = \bigoplus_{\varepsilon \in \mathbb{Z}/2} V_\varepsilon$ be a $\Bbbk$-supermodule. If $v \in V_\varepsilon \setminus \{0\}$ for $\varepsilon \in \mathbb{Z}/2$, we say v is *homogeneous*, write $|v| = \varepsilon$, and refer to ε as the *parity* of v. In particular, when $|v| = \bar{0}$ we say v is *even* and when $|v| = \bar{1}$ we say that v is *odd*. If $S \subseteq$

V, we denote $S_{\bar{0}} := S \cap V_{\bar{0}}$ and $S_{\bar{1}} := S \cap V_{\bar{1}}$. If S consists of homogeneous elements then $S = S_{\bar{0}} \sqcup S_{\bar{1}}$. Let V and W be superspaces. For $\delta \in \mathbb{Z}/2$, a parity δ (homogeneous) linear map $f : V \to W$ is a linear map satisfying $f(V_\varepsilon) \subseteq W_{\varepsilon+\delta}$ for all ε. Superlinear maps follow the same even/odd conventions as vectors: if $|f| = \bar{0}$ we say f is *even* and if $|f| = \bar{1}$ we say f is *odd*.

Let $d \in \mathbb{Z}_{>0}$. The group $\mathfrak{S}_d$ acts on $V^{\otimes d}$ on the right with automorphisms, such that for all homogeneous $v_1, \dots, v_d \in V$ and $\sigma \in \mathfrak{S}_d$, we have

$$(v_1 \otimes \cdots \otimes v_d)^\sigma = (-1)^{\langle \sigma; \boldsymbol{v} \rangle} v_{\sigma 1} \otimes \cdots \otimes v_{\sigma d}, \tag{1.2}$$

where, setting $\boldsymbol{v} := v_1 \cdots v_d \in V^d$, we have put:

$$\langle \sigma; \boldsymbol{v} \rangle := \sharp\{(k,l) \in [d]^2 \mid k < l,\ \sigma^{-1}k > \sigma^{-1}l, \text{ and } v_k, v_l \in V_{\bar{1}}\}. \tag{1.3}$$

For $0 \leq c \leq d$, let ${}^{(c,d-c)}\mathscr{D}$ be the set of shortest coset representatives for $(\mathfrak{S}_c \times \mathfrak{S}_{d-c})\backslash\mathfrak{S}_d$. Given $w_1 \in V^{\otimes c}$ and $w_2 \in V^{\otimes(d-c)}$, we define the *star product*

$$w_1 * w_2 := \sum_{\sigma \in {}^{(c,d-c)}\mathscr{D}} (w_1 \otimes w_2)^\sigma \in V^{\otimes d}. \tag{1.4}$$

Let V and W be superspaces, $d \in \mathbb{Z}_{\geq 0}$, and $\boldsymbol{v} = v_1 \cdots v_d \in V^d$ and $\boldsymbol{w} = w_1 \cdots w_d \in W^d$ be d-tuples of homogeneous elements. We denote

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle := \sharp\{(k,l) \in [d]^2 \mid k > l,\ v_k \in V_{\bar{1}}, w_l \in W_{\bar{1}}\}. \tag{1.5}$$

Let now A be a unital $\Bbbk$-superalgebra. As usual, the tensor product $A^{\otimes d}$ is a superalgebra with respect to the following product: for homogeneous

$a_1, \dots, a_d, b_1, \dots, b_d \in A$, we set $\boldsymbol{a} := a_1 \cdots a_d$, $\boldsymbol{b} := b_1 \cdots b_d$ and define the product of (homogeneous) pure tensors via:

$$(a_1 \otimes \cdots \otimes a_d)(b_1 \otimes \dots b_d) = (-1)^{\langle \boldsymbol{a}, \boldsymbol{b} \rangle} a_1 b_1 \otimes \cdots \otimes a_d b_d,$$

and extend linearly.

For any superspace V, we consider the subspace of invariants

$$\Gamma^d V := \left(V^{\otimes d}\right)^{\mathfrak{S}_d} = \{w \in V^{\otimes d} \mid w^\sigma = w \text{ for all } \sigma \in \mathfrak{S}_d\}. \tag{1.6}$$

If A is a superalgebra, then $\Gamma^d A$ inherits the structure of a superalgebra from the tensor product $A^{\otimes d}$ of superalgebras. If V is an A-supermodule then $V^{\otimes d}$ is a supermodule over $A^{\otimes d}$ with the following action: for homogeneous $a_1, \dots, a_d \in A$ and $v_1, \dots, v_d \in V$, we set $\boldsymbol{a} := a_1 \cdots a_d$, $\boldsymbol{v} := v_1 \cdots v_d$ and define the action of (homogeneous) pure tensors via:

$$(a_1 \otimes \cdots \otimes a_d)(v_1 \otimes \dots v_d) = (-1)^{\langle \boldsymbol{a}, \boldsymbol{v} \rangle} a_1 v_1 \otimes \cdots \otimes a_d v_d,$$

Furthermore, observe that for all $\sigma \in \mathfrak{S}_d$, we have that

$$(\boldsymbol{a}\boldsymbol{v})^\sigma = \boldsymbol{a}^\sigma \boldsymbol{v}^\sigma \quad \text{and} \quad \big((a_1 \otimes \cdots \otimes a_d)(v_1 \otimes \dots v_d)\big)^\sigma = (a_1 \otimes \cdots \otimes a_d)^\sigma (v_1 \otimes \dots v_d)^\sigma.$$

So $\Gamma^d V$ is a supermodule over $\Gamma^d A$.

Let A be a unital $\Bbbk$-superalgebra and V, W be A-supermodules. A homogeneous A-*supermodule homomorphism* $f : V \to W$ is a homogeneous linear map $f : V \to W$ satisfying $f(av) = (-1)^{|f||a|} a f(v)$ for all (homogeneous) a, v. Let $\mathrm{Hom}_A(V, W)_{\bar{0}}$ be the $\Bbbk$-module of *even* A-supermodule homomorphisms $V \to W$, and let $\mathrm{Hom}_A(V,W)_{\bar{1}}$ be the $\Bbbk$-module of *odd* A-supermodule homomorphisms from V to W. Then the $\Bbbk$-supermodule of all A-supermodule homomorphisms $V \to W$ has the following superstructure:

$$\mathrm{Hom}_A(V,W) = \mathrm{Hom}_A(V,W)_{\bar{0}} \oplus \mathrm{Hom}_A(V,W)_{\bar{1}}.$$

We denote by A-mod the category of all finitely generated (left) A-supermodules and all A-supermodule homomorphisms. We denote by '$\cong$' an isomorphism in this category and by '$\simeq$' an *even isomorphism* in this category.

We have the parity change functor Π on A-mod: for $V \in A$-mod we have $\Pi V \in A$-mod with $(\Pi V)_\varepsilon = V_{\varepsilon+\bar{1}}$ for all $\varepsilon \in \mathbb{Z}/2$ and the new action $a \cdot v = (-1)^{|a|}av$ for $a \in A, v \in V$. We have $V \cong \Pi V$ via the (odd) identity map.

For any A-supermodule V, and simple A-supermodule L, we denote by $[V : L]$ the multiplicity of L or ΠL as a composition factor of V, without distinguishing between the two. For example $[L \oplus \Pi L : L] = 2$. Note that if $A = A_{\bar{0}}$ is a purely even superalgebra, it is just an algebra in the traditional sense - so this notation will also be used in the non-super setting.

All subspaces, ideals, submodules, etc. are assumed to be homogeneous. For example, given homogeneous elements $v_1, \dots, v_k$ of an A-supermodule V, we have the A-submodule $A\langle v_1, \dots, v_k\rangle \subseteq V$ generated by $v_1, \dots, v_k$.

1.4. Combinatorics

In this section we establish several combinatorial definitions and facts that we will use later in the text. See especially § 3.3.2 and § 5.3.

Partitions and compositions

We denote by Λ_+ the set of all partitions. For $\lambda \in \Lambda_+$, we have the conjugate partition λ', see [16, p.2]. The Young diagram of λ is

$$[\lambda] := \{(r, s) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \mid s \leq \lambda_r\}. \tag{1.7}$$

We refer to $(r, s) \in [\lambda]$ as the *nodes* of λ.

For $\lambda, \mu, \nu \in \Lambda_+$, we denote by $c^{\lambda}_{\mu,\nu}$ the corresponding Littlewood-Richardson coefficient, see [16, § I.9].

Let $n \in \mathbb{Z}_{>0}$. We denote $\Lambda(n) = \mathbb{Z}^n_{\geq 0}$ and interpret it as the set of *compositions* $\lambda = (\lambda_1, \dots, \lambda_n)$ with n non-negative parts. For $\lambda, \mu \in \Lambda(n)$, we define

$$\lambda + \mu := (\lambda_1 + \mu_1, \dots, \lambda_n + \mu_n).$$

For $1 \leq r \leq n$, we denote

$$\varepsilon_r := (0, \dots, 0, 1, 0, \dots, 0) \in \Lambda(n) \tag{1.8}$$

with 1 in position r. For $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda(n)$, set $|\lambda| := \lambda_1 + \cdots + \lambda_n$.

Denote

$$\Lambda_+(n) := \{\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda(n) \mid \lambda_1 \geq \cdots \geq \lambda_n\}.$$

Sometimes we collect equal parts of $\lambda \in \Lambda_+(n)$ to write it as

$$\lambda = \langle l_1^{a_1}, \dots, l_k^{a_k} \rangle \tag{1.9}$$

for $l_1 > \cdots > l_k \geq 0$ and $a_1, \ldots, a_k > 0$ with $a_1 + \cdots + a_k = n$. For example, if $\lambda = (3,3,2,2,1,0,0,0)$ in the traditional notation, we would write $\lambda = \langle 3^2, 2^2, 1^1.0^3 \rangle$ when equal parts are collected. We interpret $\Lambda_+(n)$ as a subset of Λ_+ in the obvious way. For $d \in \mathbb{Z}_{\geq 0}$, let

$$\Lambda(n,d) = \{\lambda \in \Lambda(n) \mid |\lambda| = d\} \quad \text{and} \quad \Lambda_+(n,d) = \{\lambda \in \Lambda_+(n) \mid |\lambda| = d\}.$$

Let S be a finite set. We will consider the set of *S-multicompositions* and *S-multipartitions*

$$\begin{aligned} \Lambda^S(n) &:= \Lambda(n)^S = \{\boldsymbol{\lambda} = (\lambda^{(s)})_{s\in S} \mid \lambda^{(s)} \in \Lambda(n) \text{ for all } s \in S\}, \\ \Lambda^S_+(n) &:= \Lambda_+(n)^S = \{\boldsymbol{\lambda} = (\lambda^{(s)})_{s\in S} \mid \lambda^{(s)} \in \Lambda_+(n) \text{ for all } s \in S\}. \end{aligned}$$

For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda^S(n)$ we define $\boldsymbol{\lambda} + \boldsymbol{\mu}$ to be $\boldsymbol{\nu} \in \Lambda^S(n)$ with $\nu^{(s)} = \lambda^{(s)} + \mu^{(s)}$ for all $s \in S$. For $\boldsymbol{\lambda} \in \Lambda^S(n)$, we define its Young diagram to be $[\boldsymbol{\lambda}] := \bigsqcup_{s\in S}[\lambda^{(s)}]$. For each $s \in S$, we refer to $\lambda^{(s)}$ as the (s)-component of $\boldsymbol{\lambda}$, and refer to $[\lambda^{(s)}]$ as the (s)-component of $[\boldsymbol{\lambda}]$. We also set

$$\|\boldsymbol{\lambda}\| := (|\lambda^{(s)}|)_{s\in S} \in \mathbb{Z}^S_{\geq 0}.$$

For $d \in \mathbb{Z}_{\geq 0}$, we set

$$\begin{aligned} \Lambda^S(n,d) &:= \{\boldsymbol{\lambda} \in \Lambda^S(n) \mid \textstyle\sum_{s\in S}|\lambda^{(s)}| = d\}, \\ \Lambda^S_+(n,d) &:= \{\boldsymbol{\lambda} \in \Lambda^S_+(n) \mid \textstyle\sum_{s\in S}|\lambda^{(s)}| = d\}. \end{aligned}$$

In the special case $S = I = \{0, \dots 0, \ell\}$, we also write $\boldsymbol{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell)})$ instead of $\boldsymbol{\lambda} = (\lambda^{(i)})_{i \in I} \in \Lambda^I(n)$. For $i \in I$, and $\lambda \in \Lambda(n, d)$, define

$$\boldsymbol{\iota}_i(\lambda) := (0, \dots, 0, \lambda, 0, \dots, 0) \in \Lambda^I(n, d), \tag{1.10}$$

with λ in the ith position.

Let $\leq$ be a partial order on S. We have a partial order $\trianglelefteq_S$ on the set $\mathbb{Z}_{\geq 0}^S$ with $(a_s)_{s \in S} \trianglelefteq_S (b_s)_{s \in S}$ if and only if $\sum_{t \geq s} a_t \leq \sum_{t \geq s} b_t$ for all $s \in S$. Let $\trianglelefteq$ be the usual *dominance partial order* on $\Lambda(n, d)$, i.e. $\lambda \trianglelefteq \mu$ if and only if $\sum_{r=1}^s \lambda_r \leq \sum_{r=1}^s \mu_r$ for all $s = 1, \dots, n$. We have a partial order $\leq_S$ on $\Lambda^S(n, d)$ defined as follows: $\boldsymbol{\lambda} \leq_S \boldsymbol{\mu}$ if and only if either $\|\boldsymbol{\lambda}\| \lhd_S \|\boldsymbol{\mu}\|$, or $\|\boldsymbol{\lambda}\| = \|\boldsymbol{\mu}\|$ and $\lambda^{(s)} \trianglelefteq \mu^{(s)}$ for all $s \in S$.

Symmetric functions

Let Sym be the ring of symmetric functions over $\mathbb{Z}$ in infinitely many variables $z_1, z_2, \dots$, and let $\{s_\lambda \in \mathrm{Sym} \mid \lambda \in \Lambda_+\}$ be the basis of Schur functions, see [16].

In [16], Macdonald proves that Sym is a Hopf algebra with coproduct

$$\Delta : \mathrm{Sym} \to \mathrm{Sym} \otimes \mathrm{Sym}, \;\; s_\lambda \mapsto \sum_{\mu, \nu \in \Lambda_+} c_{\mu,\nu}^\lambda \, s_\mu \otimes s_\nu,$$

where $c_{\mu,\nu}^\lambda$ are the Littlewood-Richardson coefficients, see [16, §I.5].

For $n \in \mathbb{Z}_{>0}$, let $\mathrm{Sym}(n) = \mathbb{Z}[z_1, \dots, z_n]^{\mathfrak{S}_n}$ be the ring of symmetric polynomials in $z_1, \dots, z_n$. There is a canonical homomorphism $\rho_n : \mathrm{Sym} \to \mathrm{Sym}(n)$ which sets $z_r = 0$ for all $r > n$, see [16, p.18]. For $\lambda \in \Lambda_+(n)$, let $s_\lambda(z_1, \dots, z_n) := \rho_n(s_\lambda) \in \mathrm{Sym}(n)$.

For a finite set S, we introduce S-fold tensor products $\mathrm{Sym}^S := \mathrm{Sym}^{\otimes S}$ and $\mathrm{Sym}^S(n) := \mathrm{Sym}(n)^{\otimes S}$. We have the canonical homomorphism

$$\rho_n^S = \rho_n^{\otimes S} : \mathrm{Sym}^S \to \mathrm{Sym}^S(n). \tag{1.11}$$

$\mathrm{Sym}(n)$, and therefore $\mathrm{Sym}^S(n)$ are $\mathbb{Z}_{\geq 0}$-graded by the degree of the polynomials. For $d \in \mathbb{Z}_{\geq 0}$, we denote by $\mathrm{Sym}^S(n,d)$ the degree d component of $\mathrm{Sym}^S(n)$.

Given $\boldsymbol{\nu} = (\nu^{(s)})_{s \in S} \in \Lambda_+^S$, we have an element

$$s_{\boldsymbol{\nu}} := \bigotimes_{t \in S} s_{\nu^{(t)}} \in \mathrm{Sym}^S.$$

If $\boldsymbol{\nu} \in \Lambda_+^S(n)$, we set

$$s_{\boldsymbol{\nu}}(z_1, \ldots, z_n) := \rho_n^S(s_{\boldsymbol{\nu}}) \in \mathrm{Sym}^S(n).$$

If $m = |S|$, iterating the coproduct (and using coassociativity and cocommutativity) we get the algebra homomorphism

$$\Delta^{m-1} : \mathrm{Sym} \to \mathrm{Sym}^S, \tag{1.12}$$

with Δ^0 interpreted as the identity map, see again [16] for details. For $\lambda \in \Lambda_+$ and $\boldsymbol{\nu} \in (\Lambda_+)^S$, we define the iterated Littlewood-Richardson coefficient $c_{\boldsymbol{\nu}}^{\lambda}$ from

$$\Delta^{m-1}(s_\lambda) = \sum_{\boldsymbol{\nu} \in \Lambda_+^S} c_{\boldsymbol{\nu}}^{\lambda} s_{\boldsymbol{\nu}}. \tag{1.13}$$

CHAPTER II

QUASIHEREDITARY ALGEBRAS

In [6], Cline, Parshall and Scott defined the notions of Quasihereditary Algebras and Highest Weight Categories. These ideas are related by a simple idea: if A is a quasi-hereditary algebra, then A-mod is a Highest Weight Category. In this chapter, we present a (brief) treatment of Highest Weight Categories, closely mirroring the presentation given by Donkin in the appendix of [17]. We will vary the ground ring in this chapter. We mostly care about the context when $\mathbb{k} = \mathbb{F}$, a field. But to properly construct the Generalized Schur Algebras in the next chapter, we will need to work with $\mathbb{k} = R$, a commutative principal ideal domain of characteristic 0. So both will be used at different times in this section.

2.1. Highest Weight Categories

The main reference in this section is [17]. We present results in slightly more generality, but the proof is essentially the same as in [17] in every case. Let A be a $\mathbb{k}$-(super)algebra which is free of finite rank as a $\mathbb{k}$-module. Let I be a finite, partially ordered set such that $\{L(i) \mid i \in I\}$ is a complete set of irreducible A-modules (up to isomorphism). We assume that for any simple A-module L, $\mathrm{End}_A(L) = \mathbb{k}$. Observe that if $\mathbb{k} = \mathbb{F}$, these assumptions are all satisfied if A is a finite-dimensional $\mathbb{k}$-algebra. For a free $\mathbb{k}$-module V, we will freely use the notation $\dim V$ to refer to either the rank of V (if $\mathbb{k} = R$), or the dimension of V (if $\mathbb{k} = \mathbb{F}$) - the distinction will not come up in this section. We note that we will use the traditional form of dimension, and not a superized version - i.e. $\mathbb{k}$ and $\Pi\mathbb{k}$ are both 1-dimensional, as opposed to (1,0)- or (0,1)-dimensional. Denote by $\leq$ the partial order on I. For each $i \in I$ let $P(i)$ be fixed minimal projective cover of $L(i)$, and $J(i)$ be a fixed minimal injective envelope of $L(i)$.

Definitions and Classical Results

For each $\Omega \subseteq I$, and each A-supermodule V, there is a unique maximal submodule, $U \subseteq V$ such that if $[U : L(j)] > 0$, then $j \in \Omega$. Denote this submodule by $O_\Omega(V)$. Similarly, there is a unique minimal submodule $U \subseteq V$ such that: if $[V/U : L(j)] > 0$, then $j \in \Omega$. Denote this submodule by $O^\Omega(V)$. Following the proof of [17, A1], we see that that O_Ω, O^Ω define functors from A-mod to the category of $\Bbbk$-supermodules, with O_Ω being left-exact, and O^Ω being right-exact.

For each $i \in I$, let $\Omega(i) = \{j \in I \mid j < i\}$. Fix $i \in I$. Let $M(i)$ be the unique maximal submodule of $P(i)$, and let $K(i) := O^{\Omega(i)}(M(i))$. We define the ***i**th Standard Module* by

$$\Delta(i) := P(i)/K(i) \tag{2.1}$$

Explicitly, $\Delta(i)$ is the largest quotient of $P(i)$ such that all of its composition factors are of the form $L(j)$ or $\Pi L(j)$ for $j \leq i$ and $[\Delta(i) : L(i)] = 1$.

The Costandard modules are defined similarly, with the roles of submodules and quotients swapped. More specifically, we define the ***i**th Costandard Module*, $\nabla(i)$, via the formula

$$\nabla(i)/L(i) = O_{\Omega(i)}(J(i)/L(i)) \tag{2.2}$$

and using the Correspondence Theorem. Explicitly, $\nabla(i)$ is the largest submodule of $J(i)$ such that all of its composition factors are of the form $L(j)$ or $\Pi L(j)$ for $j \leq i$ and $[\nabla(i) : L(i)] = 1$.

Even without further assumptions, the modules $\{\Delta(i)\}$ and $\{\nabla(i)\}$ have several amenable properties. The following are all proved in [17, Appendix A1].

Proposition 2.3. (i) *For all* $i \in I$, $\mathrm{End}_A(\Delta(i)) = \mathrm{End}_A(\nabla(i)) = \Bbbk$.

(ii) *For* $i, j \in I$, $\dim \mathrm{Hom}_A(\Delta(i), \nabla(j)) = \delta_{ij}$.

(iii) *The Grothendieck group of* A-mod *has* $\mathbb{Z}$*-bases:*

$$\{[L(i)] \mid i \in I\}, \quad \{[\Delta(i)] \mid i \in I\}, \quad \textit{and} \quad \{[\nabla(i)] \mid i \in I\}$$

Where we observe that in the Grothendieck group of A-mod, we equate $[L(i)]$ and $[\Pi L(i)]$ for all i, as $L(i)$ and $\Pi L(i)$ are isomorphic in A-mod (via an odd isomorphism).

Furthermore, by Proposition 2.3, for each A-supermodule V and each $i \in I$, we can define integers $(V : \Delta(i))$ and $(V : \nabla(i))$ from the equations

$$[V] = \sum_{i \in I} (V : \Delta(i))[\Delta(i)] \quad \text{and} \quad [V] = \sum_{i \in I} (V : \nabla(i))[\nabla(i)].$$

Let $V \in A$-mod. A *standard filtration* of V is an A-supermodule filtration $0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_l = V$ such that for every $r = 1, \ldots, l$, we have $W_r/W_{r-1} \cong \Delta(i_r)$ for some $i_r \in I$. We refer to $\Delta(i_1), \ldots, \Delta(i_l)$ as the factors of the filtration, and to $\Delta(i_1)$ (resp. $\Delta(i_l)$) as the bottom (resp. top) factor.

Let $V \in A$-mod. A *costandard filtration* of V is an A-supermodule filtration $0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_l = V$ such that for every $r = 1, \ldots, l$, we have $W_r/W_{r-1} \cong \nabla(i_r)$ for some $i_r \in I$. We again refer to $\nabla(i_1, \ldots, \nabla(i_l)$ as factors of the filtration, with $\nabla(i_1)$ (resp. $\nabla(i_l)$) called the bottom (resp. top) factor.

Definition 2.4. [17, Definition A2.1] We say that A-mod is a *Highest Weight Category* (with respect to the ordering $\leq$) if the following holds for all $i \in I$:

(i) $J(i)/\nabla(i)$ has a costandard filtration;

(ii) If $(J(i)/\nabla(i) : \nabla(j)) \neq 0$ for any $j \in I$, then $i < j$.

Later on, in § 2.2, we will explain why $\Delta(i)$ and $\nabla(i)$ are necessarily free $\Bbbk$-modules when A is based quasi-hereditary. In particular, modules with standard or costandard filtrations will also be free as $\Bbbk$-modules, so we may discuss their rank/dimension. We can now record the following important result.

Proposition 2.5. [17, Proposition A2.2] *Let $V, W \in A$-mod and $i, j \in J$.*

(i) *If V has a standard filtration and W has a costandard filtration, then*

$$\dim \operatorname{Ext}^j_A(V, W) = \begin{cases} \sum_{i \in I} (V : \Delta(i))(W : \nabla(i)) & \text{if } j = 0 \\ 0 & \text{otherwise} \end{cases}$$

(ii) *V has a standard (resp. costandard) filtration if and only if $\operatorname{Ext}^1_A(V, \nabla(i)) = 0$ (resp. $\operatorname{Ext}^1(\Delta(i), V) = 0$) for all $i \in I$.*

(iii) $(P(i) : \Delta(j)) = [\nabla(j), L(i)]$ *and* $(J(i) : \nabla(j)) = [\Delta(j), L(i)]$.

Observe that if V has a standard filtration and W has a costandard filtration, then for each $i \in I$, $(V : \Delta(i))$, and $(W : \nabla(i))$ do not depend on the choice of filtration, by Proposition 2.3. In fact, by Propositon 2.5(i), we have

$$(V : \Delta(i)) = \dim \operatorname{Hom}_A(V, \nabla(i)), \quad \text{and} \quad (W : \nabla(i)) = \dim \operatorname{Hom}_A(\Delta(i), W). \quad (2.6)$$

Tilting Modules and Ringel Duality

Again let A be a quasi-hereditary algebra over $\Bbbk$.

Definition 2.7. Let $T \in A$-mod. We say T is *tilting* if T has both a standard and a costandard filtration.

As proven in [17, $A4$], there is a very pleasing classification for tilting modules in A-mod.

Proposition 2.8. [17, Theorem A4.2] *For each $i \in I$, there is a unique (up to isomorphism) indecomposable tilting module* $\mathrm{T}(i)$*, such that* $[\mathrm{T}(i) : L(i)] = 1$ *and* $[\mathrm{T}(i) : L(j)] > 0$ *implies* $j \leq i$*. We call* $\mathrm{T}(i)$ *the* partial tilting module *of highest weight* i*. Furthermore, every tilting module is a direct sum of partial tilting modules.*

We remind the reader that we allow odd isomorphisms in A-mod. So, we have that $\mathrm{T}(i) \cong \Pi\,\mathrm{T}(i)$, but $\mathrm{T}(i) \not\simeq \Pi\,\mathrm{T}(i)$.

Definition 2.9. Let T be a tilting module such that every $\mathrm{T}(i)$ appears at least once as a summand. Then we say that T is a *full tilting module.* Furthermore, we call $A' := \mathrm{End}_A(\mathrm{T})$ a *Ringel Dual* of A.

It is important to note that A' is *not* uniquely defined up to isomorphism. However, it is well known that A' is defined up to (super) Morita equivalence.

As an easy example: if $A = \Bbbk = \mathbb{F}$, then we may take any $\Bbbk^{\oplus n}$ as a full tilting module for any n, each of which results in a different endomorphism algebra, and thus a different Ringel dual A'.

2.2. Based Quasihereditary Algebras

In this section, $\Bbbk$ may be $\mathbb{F}$ or R, and both versions will be used in the text. The main reference here in this section is [5]. Let A be a $\Bbbk$-superalgebra.

Heredity Data

Definition 2.10. [5, Definition 2.4] Let I be a finite partially ordered set (with order $\leq$) and let $X = \bigsqcup_{i \in I} X(i)$ and $Y = \bigsqcup_{i \in I} Y(i)$ be finite sets of homogeneous elements of A with distinguished elements $e_i \in X(i) \cap Y(i)$ for each $i \in I$. For each $i \in I$, we set $A^{>i} := \mathrm{span}\{xy \mid j > i,\, x \in X(j),\, y \in Y(j)\}$. We say that I, X, Y is *heredity data* if the following axioms hold:

(a) $B := \{xy \mid i \in I,\, x \in X(i),\, y \in Y(i)\}$ is a basis of A;

(b) For all $i \in I$, $x \in X(i)$, $y \in Y(i)$ and $a \in A$, we have

$$ax \equiv \sum_{x' \in X(i)} l^x_{x'}(a)x' \pmod{A^{>i}} \text{ and } ya \equiv \sum_{y' \in Y(i)} r^y_{y'}(a)y' \pmod{A^{>i}}$$

for some $l^x_{x'}(a), r^y_{y'}(a) \in \Bbbk$;

(c) For all $i, j \in I$ and $x \in X(i)$, $y \in Y(i)$ we have

$$xe_i = x,\ e_ix = \delta_{x,e_i}x,\ e_iy = y,\ ye_i = \delta_{y,e_i}y$$
$$e_jx = x \text{ or } 0,\ ye_j = y \text{ or } 0.$$

If A is endowed with heredity data I, X, Y, we call A *based quasi-hereditary*, and refer to B as a *heredity basis* of A. Furthermore, it follows by (c) that $e_i^2 = e_i$

for all $i \in I$. So we refer to the idempotents e_i as the *standard idempotents* of the heredity data.

From now on, A is a based quasi-hereditary superalgebra with heredity data I, X, Y. Set

$$B_{\mathfrak{a}} := \{xy \mid i \in I, x \in X(i)_{\bar{0}}, y \in Y(i)_{\bar{0}}\}, \quad B_{\mathfrak{c}} := \{xy \mid i \in I, x \in X(i)_{\bar{1}}, y \in Y(i)_{\bar{1}}\}.$$

Then,

$$B_{\bar{0}} = B_{\mathfrak{a}} \sqcup B_{\mathfrak{c}}, \tag{2.11}$$

and we may also write $B = B_{\bar{0}} \sqcup B_{\bar{1}}$.

The heredity data I, X, Y of A is called *conforming* if $B_{\mathfrak{a}}$ spans a unital subalgebra of A.

Lemma 2.12. [5, Lemmas 2.7, 2.8] *Let $i, j \in I$ and $x \in X(i)$, $y \in Y(i)$.*

(i) $e_i e_j = \delta_{i,j} e_i$

(ii) *If $j \not\leq i$, then $e_j x = y e_j = 0$.*

Corollary 2.13. *We have $X \cap Y = \{e_i \mid i \in I\}$.*

Proof. Let $z \in X \cap Y$. As $z \in X$ we have $z \in X(i)$ so $ze_i = z$ for some $i \in I$. As $z \in Y$, we have $z \in Y(j)$ so $e_j z = z$ for some $j \in I$. By Lemma 2.12(ii), $j = i$, and the result follows from Definition 2.10(c). □

It is shown in [5] that if $\Bbbk$ is noetherian, and the algebra A is known to be finitely generated and projective as a $\Bbbk$-module, then the existence of heredity data for A implies that A is quasi-hereditary in the sense of [18]. Observe that if $\Bbbk = \mathbb{F}$, then it is automatically noetherian, and A is automatically projective as an $\mathbb{F}$-supermodule. So in this case, the existence of heredity data implies that A is finitely generated (in fact, finite-dimensional) as an $\mathbb{F}$-supermodule. Thus, if $\Bbbk = \mathbb{F}$, then the existence of heredity data for A implies that A is quasi-hereditary in the sense of [6].

We can then take this further. Combining with [17, Proposition A3.7(ii)], we have

Proposition 2.14. *Let $\Bbbk = \mathbb{F}$, and A be a finite dimensional $\mathbb{F}$-algebra. If A has heredity data (I, X, Y), with I partially ordered by $\leq$, then A-mod is a highest weight category with respect to the ordering $\leq$.*

<u>Standard and Costandard Modules</u>

Definition 2.15. Let $0 \neq V \in A$-mod and $i \in I$. We call V a *highest weight module (of weight i)* if there exists a homogeneous $v \in V$ such that e_iV is spanned by v, $Av = V$, and $j > i$ implies $e_jV = 0$. In this case we refer to v as a *highest weight vector* of V.

Lemma 2.16. *Let $i \in I$, $0 \neq V \in A$-mod and $v \in V$ be a homogeneous vector. Suppose that $e_iv = v$, $Av = V$, and $yv = 0$ for all $y \in Y \setminus \{e_i\}$. Then V is a highest weight module of weight i.*

Proof. Since A is based quasi-hereditary, it follows from the assumption $yv = 0$ for all $y \in Y \setminus \{e_i\}$ that V is spanned by $\{xv \mid x \in X(i)\}$. For any $j \in I$, if $e_jV \neq 0$ then there is some $x \in X(i)$ such that $e_jx \neq 0$. By Definition 2.10(c), this implies that $e_jx = x$. But then Lemma 2.12(ii) implies that $j \leq i$. So, in particular, if $j > i$, then $e_jV = 0$, completing the proof. □

Fix $i \in I$. Observe that $A^{>i}$ is the ideal in A generated by $\{e_j \mid j > i\}$ and denote $\tilde{A} := A/A^{>i}$, $\tilde{a} := a + A^{>i} \in \tilde{A}$ for $a \in A$. By inflation, $\tilde{A}$-supermodules will be automatically considered as A-supermodules. In particular, the *standard module*

$$\Delta(i) := \tilde{A}\tilde{e}_i$$

is considered as an A-module. The modules $\{\Delta(i) \mid i \in I\}$ are the (left) standard modules of A-mod in the sense of § 2.1.1. So we may use these to speak of *standard filtrations* for A-supermodules, as in § 2.1.1 again.

We have that $\Delta(i)$ is a free $\Bbbk$-module with basis $\{v_x := \tilde{x} \mid x \in X(i)\}$ and the action $av_x = \sum_{x' \in X(i)} l^x_{x'}(a)v_{x'}$, cf. [5, §2.3]. Denoting

$$v_i := v_{e_i} \in \Delta(i),$$

we have $e_i v_i = v_i$, and $e_j \Delta(i) \neq 0$ implies $j \leq i$ thanks to Lemma 2.12. Moreover, for all for all $x \in X(i)$ we have $xv_i = v_x$ and $e_i v_x = \delta_{x,e_i} v_x$. In particular, $\Delta(i)$ is a highest weight module of weight i in the sense of Definition 2.15 (with even highest weight vector). If $V \in A$-mod is isomorphic to $\Delta(i)$, then using the fact that $e_i V$ is free of rank 1 as a $\Bbbk$-module, it is easy to see that either $V \simeq \Delta(i)$ or $V \simeq \Pi\Delta(i)$.

We also have the right standard A-module

$$\Delta^{\mathrm{op}}(i) := \tilde{e}_i \tilde{A},$$

and by symmetry every result we have about $\Delta(i)$ has its right analogue for $\Delta^{\mathrm{op}}(i)$, for example $\Delta^{\mathrm{op}}(i)$ is a free $\Bbbk$-module with basis $\{w_y := \tilde{y} \mid y \in Y(i)\}$.

These are the modules we use to define *right standard filtrations* similarly to the standard filtrations defined above.

Suppose now until the end of the subsection that $\Bbbk = \mathbb{F}$. Then each $L(i) := \operatorname{head}\Delta(i)$ is irreducible, and $\{L(i) \mid i \in I\}$ is a complete set of non-isomorphic irreducible A-supermodules. We also have that $L^{\mathrm{op}}(i) := \operatorname{head}\Delta^{\mathrm{op}}(i)$ is an irreducible right module, and $\{L^{\mathrm{op}}(i) \mid i \in I\}$ is a complete set of non-isomorphic irreducible right A-supermodules. These align with the modules we expect from § 2.1.1

By [5, Lemma 3.3], A is quasi-hereditary in the sense of Cline, Parshall and Scott, and A-mod is a highest weight category with standard modules $\{\Delta(i) \mid i \in I\}$, see [6, Theorem 3.6]. In particular, the projective cover $P(i)$ of $L(i)$ has a standard filtration with the top factor $\Delta(i)$ and all other factors of the form $\Delta(j)$ or $\Pi\Delta(j)$ for $j > i$. Moreover, $\Delta(i)$ is the largest quotient of $P(i)$ such that $[\Delta(i) : L(i)] = 1$ and $[\Delta(i) : L(j)] \neq 0$ implies $j \leq i$.

Proposition 2.17. (Universality of standard modules) *Let $\Bbbk = \mathbb{F}$, $i \in I$, and V be a highest weight module of weight i with highest weight vector v. Then there is an homogeneous surjection $\Delta(i) \twoheadrightarrow V$ of parity $|v|$; in particular $e_jV \neq 0$ implies $j \leq i$.*

Proof. Let e_iV be spanned by $v \in V$. There is a homogeneous surjective A-supermodule homomorphism $\varphi : Ae_i \twoheadrightarrow V,\ ae_i \mapsto av$ of parity $|\varphi| = |v|$. As $e_iL(i)$ is 1-dimensional and $e_iL(j) \neq 0$ implies $i \leq j$, we have that $Ae_i = P(i) \oplus P$, where P is a direct sum of supermodules isomorphic to $P(j)$ with $j > i$.

Note that $\operatorname{Hom}_A(\Delta(j), V) = 0$ for any $j > i$, so $\operatorname{Hom}_A(P(j), V) = 0$ for all $j > i$, and we deduce $\operatorname{Hom}_A(P, V) = 0$. Since each $P(j)$ has a standard filtration with factors isomorphic to $\Delta(r)$ for $r > j > i$, the map φ factors through $P(i)$ to give a surjection $P(i) \twoheadrightarrow V$. Moreover, $P(i)$ has a standard filtration with top factor $\Delta(i)$ and other factors isomorphic to $\Delta(j)$ with $j > i$, so the map further factors through the surjection $\Delta(i) \twoheadrightarrow V$. $\square$

The following is a useful criterion for V to have a standard filtration.

Corollary 2.18. *Let $\Bbbk = \mathbb{F}$, $V \in A$-mod, $v_1 \dots, v_t \in V$ be homogeneous elements, and set $V_s := A\langle v_1, \dots, v_s\rangle$ for $s = 1, \dots, t$. Suppose that the following conditions hold:*

1. $V_t = V$;
2. *for each $s = 1, \dots, t$ there exists $i_s \in I$ such that $e_{i_s} v_s - v_s \in V_{s-1}$ and $y v_s \in V_{s-1}$ for all $y \in Y \setminus \{e_{i_s}\}$;*
3. $\dim V = \sum_{s=1}^{t} \dim \Delta(i_s)$.

Then $V_s/V_{s-1} \simeq \Pi^{|v_s|}\Delta(i_s)$ for all $s = 1, \dots, t$. In particular, V has a standard filtration.

Proof. Observe that assumption (2) implies that each V_s/V_{s-1} is a highest weight module of weight i_s. Then, by Lemma 2.16 and Proposition 2.17, each V_s/V_{s-1} is a quotient of $\Pi^{|v_s|}\Delta(i_s)$. The result now follows by dimensions. $\square$

The highest weight category A-mod comes with costandard modules $\{\nabla(i) \mid i \in I\}$. Let $J(i)$ be the injective hull of $L(i)$ in A-mod for $i \in I$. As explained in § 2.1.1, one can define $\nabla(i)$ as the largest submodule of $J(i)$ such that $[\nabla(i) : L(i)] = 1$ and $[\nabla(i) : L(j)] > 0$ implies $j \leq i$ (c.f. [17, Appendix A1]). Using the heredity data of A, we may construct the module $\nabla(i)$, using the right standard modules above and dualizing.

Indeed: given a right A-supermodule V, there is a (left) A-supermodule structure on V^* with $af(v) = (-1)^{|a||f|+|a||v|}f(va)$ for $a \in A, f \in V^*, v \in V$. For example, note that $L^{\mathrm{op}}(i)^*$ is irreducible, $e_i L^{\mathrm{op}}(i)^* \neq 0$, and $e_j L^{\mathrm{op}}(i)^* \neq 0$ implies $j \leq i$; therefore $L^{\mathrm{op}}(i)^* \simeq L(i)$. Denoting by $P^{\mathrm{op}}(i)$ the projective cover of $L^{\mathrm{op}}(i)$, we deduce that $P^{\mathrm{op}}(i)^* \simeq J(i)$. This in turn implies easily:

$$\nabla(i) \simeq \Delta^{\mathrm{op}}(i)^*. \tag{2.19}$$

2.3. Example: The Classical Schur Algebra

One of the most important examples of a quasi-hereditary algebra is the (classical) Schur Algebra. The main references for this are [3] and [4].

<u>Definitions</u>

Let $n, d \in \mathbb{Z}_{\geq 0}$ with $n > 0$. Then we may consider the algebra $M_n(\Bbbk)$ as a (purely even) superalgebra over $\Bbbk$. Recalling the action of the symmetric group (1.2), and the space of invariants (1.6), the classical Schur Algebra is defined as

$$S(n, d) := \Gamma^d M_n(\Bbbk)$$

This algebra was first introduced by Issai Schur in his thesis [19]. Our approach is inspired by the results of [4], but will mimic the approach in Chapter 3, originally laid out by Kleshchev and Muth in [10],

For $r, s \in [n]$, we let $\xi_{r,s}$ be the matrix unit with 1 in the (r, s) position and 0's elsewhere. Recall (1.2) and the action of $\mathfrak{S}_d$ on $[n]^d$ from § 1.2 For $\boldsymbol{r} = r_1, \dots, r_d$, $\boldsymbol{s} = s_1, \dots, s_d \in [n]^d$, let $\mathfrak{S}_{\boldsymbol{r},\boldsymbol{s}}$ be the stabilizer of $(\boldsymbol{r}, \boldsymbol{s})$ for the diagonal action of $\mathfrak{S}_d$ on $[n]^d \times [n]^d$. Let ${}_{\boldsymbol{r},\boldsymbol{s}}\mathscr{D}$ be the set of shortest coset representatives for $\mathfrak{S}_{\boldsymbol{r},\boldsymbol{s}} \backslash \mathfrak{S}_d$. We define the element

$$\xi_{\boldsymbol{r},\boldsymbol{s}} = \sum_{\sigma \in \mathscr{D}} (\xi_{r_1,s_1} \otimes \xi_{r_2,s_2} \otimes \cdots \otimes \xi_{r_d,s_d})^{\sigma}$$

Fix a set, Z, of orbit representatives for the action of $\mathfrak{S}_d$ on $[n]^d \times [n]^d$]. It is a result of Schur (see [3]) that $\{\xi_{\boldsymbol{r},\boldsymbol{s}} \mid (\boldsymbol{r}, \boldsymbol{s}) \in Z\}$ is a basis for $S(n,d)$. However, this is not a heredity basis in the sense of Definition 2.10.

Heredity Data

Recall (1.4). For each composition $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(n,d)$ let $\boldsymbol{l}^{\lambda} = (1, \ldots, 1, 2, \ldots, 2, \ldots) \in [n]^d$ where there are λ_1 1's, λ_2 2's, etc. We have the *weight idempotent* ξ_λ defined by

$$\xi_\lambda := \xi_{\boldsymbol{l}^\lambda, \boldsymbol{l}^\lambda} = \xi_{1,1}^{\otimes \lambda_1} * \xi_{2,2}^{\otimes \lambda_2} * \cdots * \xi_{n,n}^{\otimes \lambda_n} \in S(n,d)$$

It is shown in [3] that the irreducible $S(n,d)$ modules are indexed by $\Lambda_+(n,d)$ - the set of partitions of d with at most n parts. So, certainly $\Lambda_+(n,d)$ will be the partially ordered set in our heredity data for $S(n,d)$, with partial order given by the dominance order $\trianglelefteq$. As expected, the weight idempotents $\{\xi_\lambda \mid \lambda \in \Lambda_+(n,d)\}$ will be the standard idempotents of the heredity data.

The heredity basis will be the basis of bideterminants defined by Green in [4]. For $\lambda \in \Lambda_+(n,d)$ recall the Young Diagram $[\lambda]$ from (1.7). A λ-*tableaux* is a function $T : [\lambda] \to [n]$. For a λ-tableau T, for each $i \in [n]$, define $[T : i] =$

$\#\{(r,s) \in [\lambda] \mid T(r,s) = i\}$. The *content* of T is the composition

$$\alpha(T) := \big([T:1], [T:2], \ldots, [T:n]\big) \in \Lambda(n) \tag{2.20}$$

A λ-tableau T is called *standard* if the following holds for all (r,s), $(r',s') \in [\lambda]$:

(a) if $s = s'$ and $r < r'$ then $T(r,s) \leq T(r',s')$;

(b) if $r = r'$ and s, s' then $T(r,s) < T(r',s')$.

We fix an arbitrary bijection $f : [\lambda] \xrightarrow{\sim} [d]$. For each $\boldsymbol{r} \in [n]^d$ we can interpret $\boldsymbol{r}$ as a function $[d] \to [n]$ by $i \mapsto r_i$ for $i \in [d]$. Then for each $\boldsymbol{r} \in [n]^d$ we define $T(\boldsymbol{r}) := f \circ \boldsymbol{r}$ (where $\boldsymbol{r}$ is considered as a function $[d] \to [n]$ as above). For $\lambda \in \Lambda_+(n,d)$ we say that $\boldsymbol{r} \in [n]^d$ is λ-*standard* if $T(\boldsymbol{r})$ is standard.

Define the set

$$\mathscr{B} := \{(\lambda, \boldsymbol{r}, \boldsymbol{s}) \mid \lambda \in \Lambda_+(n,d), \text{ and } \boldsymbol{r}, \boldsymbol{s} \text{ are both } \lambda\text{-standard}\}$$

And for $\lambda \in \Lambda_+(n,d)$ and $\boldsymbol{r}, \boldsymbol{s} \in [n]^d$, define the *bideterminant* $Y^{\lambda}_{\boldsymbol{r},\boldsymbol{s}}$ by

$$Y^{\lambda}_{\boldsymbol{r},\boldsymbol{s}} = \xi_{\boldsymbol{r},\boldsymbol{l}^{\lambda}}, \xi_{\boldsymbol{l}^{\lambda},\boldsymbol{s}}.$$

We have the following theorem of Green:

Theorem 2.21. [4, (16)] *the set* $\{Y^{\lambda}_{\boldsymbol{r},\boldsymbol{s}} \mid (\lambda, \boldsymbol{r}, \boldsymbol{s}) \in \mathscr{B}\}$ *is a basis for* $S(n,d)$.

For $\lambda \in \Lambda_+(n,d)$, let $X(\lambda) = \{\xi_{\boldsymbol{r},\boldsymbol{l}^{\lambda}} \mid \boldsymbol{r} \text{ is } \lambda\text{-standard}\}$ and $Y(\lambda) = \{\xi_{\boldsymbol{l}^{\lambda},\boldsymbol{r}} \mid \boldsymbol{r} \text{ is } \lambda\text{-standard}\}$. Furthermore, set $X = \sqcup_{\lambda \in \Lambda_+(n,d)} X(\lambda)$ and $Y = \sqcup_{\lambda \in \Lambda_+(n,d)} Y(\lambda)$. Then it can be checked directly that $(\Lambda_+(n,d), X, Y)$ is heredity data for $S(n,d)$. In fact, checking this is a special case of [11, Theorem 6.6] where $A = \Bbbk$.

The standard and costandard modules for $S(n,d)$ are described extensively in both [3] and [4]. We omit this here, as we will discuss the standards and costandard modules for Generalized Schur Algebras at length in chapter 3; with the classical Schur algebra being a special case.

Motivating Results

Fix $n \in \mathbb{Z}_{\geq 0}$ and define $S(n) := \bigoplus_{d\geq 0} S(n,d)$. There is a coproduct on $S(n)$ which it inherits from $\bigoplus_{d\geq 0} M_n(\Bbbk)^{\otimes d}$. Namely, on $M_n(\Bbbk)^{\otimes d}$ this coproduct is defined on pure tensors by

$$M_n(\Bbbk)^{\otimes d} \to \bigoplus_{c=0}^{d} M_n(\Bbbk)^{\otimes c} \otimes M_n(\Bbbk) M_n(\Bbbk)^{\otimes (d-c)}$$

$$\xi_1 \otimes \xi_d \mapsto \sum_{0\leq c\leq d} (\xi_1 \otimes \cdots \otimes \xi_c) \otimes (\xi_{c+1} \otimes \cdots \otimes \xi_d)$$

We will discuss this coproduct more in Chapter III, generalizing it to the Generalized Schur Algebras defined therein. In that chapter, we will discuss the coproduct in significantly more detail. Most importantly, if we take an $S(n,d)$-module V and an $S(n,r)$-module W, we may consider $V \otimes W$ as an $S(n,d+r)$-module via the coproduct described above.

In [9], Wang proves:

Theorem 2.22. *If V is an $S(n,d)$-module with a standard (resp. costandard) filtration, and W is an $S(n,r)$-module with a standard (resp. costandard) filtration, then $V \otimes W$ has a standard (resp. costandard) filtration.*

The main theorem of chapter IV is a generalization of Wang's result, relating to Generalized Schur Algebras.

In chapter V, we will take inspiration from Donkin. Namely, in [15] proves

Theorem 2.23. *There is a tilting module,* T *for* $S(n,d)$ *such that* $\mathrm{End}(T) \cong S(n,d)$*.*

This result inspires our main conjecture of chapter V. Namely, as stated in the introduction, we conjecture:

Conjecture 2.24. Let $n \in \mathbb{Z}_{>0}$ and $d \in N$ with $n \geq d$. Let A be a based quasi-hereditary algebra and $d \leq n$. If A' is a Ringel dual of A, then a Ringel dual of $T^A_{\mathfrak{a}}(n,d)$ is of the form $T^{A'}_{\mathfrak{a}'}(n,d)$ for some canonical choice of $\mathfrak{a}'$.

We will prove this conjecture in the case $A = \mathsf{Z}$, the extended zigzag algebra.

CHAPTER III

GENERALIZED SCHUR ALGEBRAS

In this chapter, we define the generalized Schur algebras, and describe some of their properties. In the last section, we re-examine the construction from a more linear algebraic perspective, defining new tools that will be used in Chapter V. This chapter contains previously unpublished co-authored material that appears in [1, 2].

3.1. Definition and Properties

Throughout the section, we fix $n \in \mathbb{Z}_{>0}$. We also fix a based quasi-hereditary superalgebra A_R over R with conforming heredity data I, X, Y.

Definition

Let S be a set and $d \in \mathbb{Z}_{\geq 0}$. Recall that the symmetric group $\mathfrak{S}_d$ acts on S^d by place permutations. For $\boldsymbol{s}, \boldsymbol{t} \in S^d$, we write $\boldsymbol{s} \sim \boldsymbol{t}$ if $\boldsymbol{s}\sigma = \boldsymbol{t}$ for some $\sigma \in \mathfrak{S}_d$. If $S_1, \dots, S_m$ are sets, then $\mathfrak{S}_d$ acts on $S_1^d \times \cdots \times S_m^d$ diagonally. For $(\boldsymbol{s}_1, \dots, \boldsymbol{s}_m), (\boldsymbol{t}_1, \dots, \boldsymbol{t}_m) \in S_1^d \times \cdots \times S_m^d$, we write $(\boldsymbol{s}_1, \dots, \boldsymbol{s}_m) \sim (\boldsymbol{t}_1, \dots, \boldsymbol{t}_m)$ if $(\boldsymbol{s}_1, \dots, \boldsymbol{s}_m)\sigma = (\boldsymbol{t}_1, \dots, \boldsymbol{t}_m)$ for some $\sigma \in \mathfrak{S}_d$. If $U \subseteq S_1^d \times \cdots \times S_m^d$ is a $\mathfrak{S}_d$-invariant subset, we denote by $U/\mathfrak{S}_d$ a complete set of the $\mathfrak{S}_d$-orbit representatives in U and we identify $U/\mathfrak{S}_d$ with the set of all $\mathfrak{S}_d$-orbits on U.

Let $H = H_{\bar{0}} \sqcup H_{\bar{1}}$ be a set of non-zero homogeneous elements of A_R. Define $\mathrm{Tri}^H(n, d)$ to be the set of all triples

$$(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{s}) = (a_1 \cdots a_d,\ r_1 \cdots r_d,\ s_1 \cdots s_d) \in H^d \times [n]^d \times [n]^d$$

such that for all $1 \leq k \neq l \leq d$ we have $(a_k, r_k, s_k) = (a_l, r_l, s_l)$ only if $a_k \in H_{\bar{0}}$. Then $\mathrm{Tri}^H(n,d) \subseteq H^d \times [n]^d \times [n]^d$ is a $\mathfrak{S}_d$-invariant subset, so we can choose a set $\mathrm{Tri}^H(n,d)/\mathfrak{S}_d$ of $\mathfrak{S}_d$-orbit representatives and identify it with the set of all $\mathfrak{S}_d$-orbits on $\mathrm{Tri}^H(n,d)$ as in the previous paragraph.

Sometimes we use a preferred choice of representatives for $\mathrm{Tri}^H(n,d)/\mathfrak{S}_d$ defined as follows. Fix a total order $<$ on $H \times [n] \times [n]$. We have a lexicographic order on $\mathrm{Tri}^H(n,d)$: $(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{s}) < (\boldsymbol{a}', \boldsymbol{r}', \boldsymbol{s}')$ if and only if there exists $l \in [d]$ such that $(a_k, r_k, s_k) = (a'_k, r'_k, s'_k)$ for all $k < l$ and $(a_l, r_l, s_l) < (a'_l, r'_l, s'_l)$. Denote

$$\mathrm{Tri}_0^H(n,d) = \{(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{s}) \in \mathrm{Tri}^H(n,d) \mid (\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{s}) \leq (\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{s})\sigma \text{ for all } \sigma \in \mathfrak{S}_d\}. \quad (3.1)$$

For $(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{s}) \in \mathrm{Tri}^H(n,d)$ and $\sigma \in \mathfrak{S}_d$, we define

$$\langle \boldsymbol{a}, \boldsymbol{r}, \boldsymbol{s} \rangle := \sharp\{(k,l) \in [d]^2 \mid k < l,\ a_k, a_l \in H_{\bar{1}},\ (a_k, r_k, s_k) > (a_l, r_l, s_l)\}. \quad (3.2)$$

Specializing to $H = B$, let $(\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s}) \in \mathrm{Tri}^B(n,d)$. For $b \in B$ and $r, s \in [n]$, we denote

$$[\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s} : b, r, s] := \sharp\{k \in [d] \mid (b_k, r_k, s_k) = (b, r, s)\},$$

and, recalling (2.11), we set

$$[\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s}]^!_{\mathfrak{c}} := \prod_{b \in B_{\mathfrak{c}},\, r,s \in [n]} [\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s} : b, r, s]!. \quad (3.3)$$

Let $M_n(A_R) = M_{n,0}(A_R)$ be the superalgebra of $n \times n$ matrices with entries in A_R. For $a \in A_R$, we denote by $\xi^a_{r,s} \in M_n(A_R)$ the matrix with a in the position (r,s) and zeros elsewhere. By definition, for all homogeneous $a \in A$, $|\xi^a_{r,s}| = |a|$.

For each $d \in \mathbb{Z}_{\geq 0}$ we have a superalgebra structure on $M_n(A_R)^{\otimes d}$, and thus on $\bigoplus_{d \geq 0} M_n(A_R)^{\otimes d}$ as well.

It is proben in [20, §4.1], that $\bigoplus_{d \geq 0} M_n(A_R)^{\otimes d}$ is a bisuperalgebra with the coproduct ∇ defined by

$$\begin{aligned} \nabla : \ M_n(A_R)^{\otimes d} &\to \bigoplus_{c=0}^{d} M_n(A_R)^{\otimes c} \otimes M_n(A_R)^{\otimes (d-c)} \\ \xi_1 \otimes \cdots \otimes \xi_d &\mapsto \sum_{c=0}^{d} (\xi_1 \otimes \cdots \otimes \xi_c) \otimes (\xi_{c+1} \otimes \cdots \otimes \xi_d) \end{aligned}$$

and product obtained from the tensor product of matrix superalgebras. Moreover, recalling (1.4), $\bigoplus_{d \geq 0} M_n(A_R)^{\otimes d}$ is also a bisuperalgebra with respect to ∇ and $*$, see [20, Lemma 3.12].

According to (1.2) $\mathfrak{S}_d$ acts on $M_n(A_R)^{\otimes d}$ with superalgebra automorphisms, and using the notation (1.6), we have the subsuperalgebra of invariants $\Gamma^d M_n(A_R) \subseteq M_n(A_R)^{\otimes d}$. For $(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{s}) \in \mathrm{Tri}^H(n, d)$, we have elements

$$\xi^{\boldsymbol{a}}_{\boldsymbol{r},\boldsymbol{s}} := \sum_{(\boldsymbol{c},\boldsymbol{t},\boldsymbol{u}) \sim (\boldsymbol{a},\boldsymbol{r},\boldsymbol{s})} (-1)^{\langle \boldsymbol{a},\boldsymbol{r},\boldsymbol{s}\rangle + \langle \boldsymbol{c},\boldsymbol{t},\boldsymbol{u}\rangle} \xi^{c_1}_{t_1,u_1} \otimes \cdots \otimes \xi^{c_d}_{t_d,u_d} \in \Gamma^d M_n(A_R).$$

We have the following R-basis of $\Gamma^d M_n(A_R)$:

$$\{\xi^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} \mid (\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s}) \in \mathrm{Tri}^B(n, d)/\mathfrak{S}_d\}. \tag{3.4}$$

For $(\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s}) \in \mathrm{Tri}^B(n, d)$, we also set

$$\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} := [\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s}]^!_{\mathfrak{c}}\, \xi^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}},$$

and

$$T(n,d)_R = T^A(n,d)_R := \operatorname{span}_R \left\{ \eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} \mid (\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}) \in \operatorname{Tri}^B(n,d) \right\} \subseteq \Gamma^d M_n(A_R).$$

Let

$$T(n)_R := \bigoplus_{d \geq 0} T(n,d)_R.$$

By [10, Proposition 3.12, Lemma 3.10], $T(n,d)_R$ is a unital R-subsuperalgebra of $M_n(A_R)^{\otimes d}$ with R-basis

$$\left\{ \eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} \mid (\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}) \in \operatorname{Tri}^B(n,d)/\mathfrak{S}_d \right\}. \tag{3.5}$$

Moreover, by in [10], Kleshchev and Muth prove that there are two different bisuperalgebra structures on $T(n)_R$. We have the following lemmas.

Lemma 3.6. [10, Corollary 3.24] *$T(n)_R$ is a sub-bisuperalgebra of $\bigoplus_{d\geq 0} M_n(A_R)^{\otimes d}$ with respect to ∇ and the usual product.*

Lemma 3.7. [10, Corollary 4.4] *$T(n)_R$ is a sub-bisuperalgebra of $\bigoplus_{d\geq 0} M_n(A_R)^{\otimes d}$ with respect to the coproduct ∇ and the product $*$*

Extending scalars from R to $\mathbb{F}$, we now define the $\mathbb{F}$-superalgebra

$$T(n,d)_{\mathbb{F}} = T^A(n,d)_{\mathbb{F}} := \mathbb{F} \otimes_R T(n,d)_R.$$

We denote $1_{\mathbb{F}} \otimes \eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} \in T(n,d)_{\mathbb{F}}$ again by $\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}$, the map $\mathrm{id}_{\mathbb{F}} \otimes \nabla$ again by ∇, etc. In fact, when working over the field, we will often drop the index and write simply

$$T(n,d) := T(n,d)_{\mathbb{F}}. \tag{3.8}$$

If W_1 is a $T(n,d_1)$-supermodule and W_2 is a $T(n,d_2)$-supermodule, we consider $W_1 \otimes W_2$ as a $T(n,d_1+d_2)$-supermodule via the coproduct ∇.

Properties of Product and Coproduct

In this section we work over R. Define the structure constants $\kappa^b_{a,c} \in R$ of A_R from $ac = \sum_{b\in B} \kappa^b_{a,c} b$ for $a, c \in A_R$. More generally, for $\boldsymbol{b} = (b_1, \dots, b_d) \in B^d$ and $\boldsymbol{a} = (a_1, \dots, a_d)$, $\boldsymbol{c} = (c_1, \dots, c_d) \in A_R^d$, we define

$$\kappa^{\boldsymbol{b}}_{\boldsymbol{a},\boldsymbol{c}} := \kappa^{b_1}_{a_1,c_1} \cdots \kappa^{b_d}_{a_d,c_d} \in R.$$

Recall the notation (3.2), (1.5). The following generalization of [3, (2.3b)] follows from [20, (6.14)], cf. [10, Proposition 3.6].

Proposition 3.9. *Let* $(\boldsymbol{a},\boldsymbol{p},\boldsymbol{q})$, $(\boldsymbol{c},\boldsymbol{u},\boldsymbol{v}) \in \mathrm{Tri}^B(n,d)$. *Then in* $T(n,d)_R$ *we have*

$$\eta^{\boldsymbol{a}}_{\boldsymbol{p},\boldsymbol{q}}\, \eta^{\boldsymbol{c}}_{\boldsymbol{u},\boldsymbol{v}} = \sum_{[\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]\in \mathrm{Tri}^B(n,d)/\mathfrak{S}_d} g^{\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}}_{\boldsymbol{a},\boldsymbol{p},\boldsymbol{q};\boldsymbol{c},\boldsymbol{u},\boldsymbol{v}}\, \eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}$$

where

$$g^{\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}}_{\boldsymbol{a},\boldsymbol{p},\boldsymbol{q};\boldsymbol{c},\boldsymbol{u},\boldsymbol{v}} = \frac{[\boldsymbol{a},\boldsymbol{p},\boldsymbol{q}]^!_{\mathfrak{c}} \cdot [\boldsymbol{c},\boldsymbol{u},\boldsymbol{v}]^!_{\mathfrak{c}}}{[\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]^!_{\mathfrak{c}}} \sum_{\boldsymbol{a}',\boldsymbol{c}',\boldsymbol{t}} (-1)^{\langle \boldsymbol{a},\boldsymbol{p},\boldsymbol{q}\rangle + \langle \boldsymbol{c},\boldsymbol{u},\boldsymbol{v}\rangle + \langle \boldsymbol{a}',\boldsymbol{r},\boldsymbol{t}\rangle + \langle \boldsymbol{c}',\boldsymbol{t},\boldsymbol{s}\rangle + \langle \boldsymbol{a}',\boldsymbol{c}'\rangle}\, \kappa^{\boldsymbol{b}}_{\boldsymbol{a}',\boldsymbol{c}'},$$

the sum being over all $\boldsymbol{a}',\boldsymbol{c}' \in B^d$ *and* $\boldsymbol{t} \in [n]$ *such that* $(\boldsymbol{a}',\boldsymbol{r},\boldsymbol{t}) \sim (\boldsymbol{a},\boldsymbol{p},\boldsymbol{q})$ *and* $(\boldsymbol{c}',\boldsymbol{t},\boldsymbol{s}) \sim (\boldsymbol{c},\boldsymbol{u},\boldsymbol{v})$.

Lemma 3.10. [10, Lemma 4.6] *Let* $q \in \mathbb{Z}_{>0}$, $d_1, \dots, d_q \in \mathbb{Z}_{\geq 0}$ *with* $d_1 + \cdots + d_q = d$, *and for* $m = 1, \dots, q$, *we have* $(\boldsymbol{b}^m, \boldsymbol{r}^m, \boldsymbol{s}^m) \in \mathrm{Tri}^B(n, d_m)$ *with* $\boldsymbol{b}^m = b^m_1 \cdots b^m_{d_m}$, $\boldsymbol{r}^m = r^m_1 \cdots r^m_{d_m}$, $\boldsymbol{s}^m = s^m_1 \cdots s^m_{d_m}$. *If* $(b^m_t, r^m_t, s^m_t) \neq (b^l_u, r^l_u, s^l_u)$ *for all*

$1 \le m \ne l \le q$, $1 \le t \le d_m$ *and* $1 \le u \le d_l$, *then*

$$\eta^{\boldsymbol{b}^1\cdots\boldsymbol{b}^q}_{\boldsymbol{r}^1\cdots\boldsymbol{r}^q,\boldsymbol{s}^1\cdots\boldsymbol{s}^q} = \eta^{\boldsymbol{b}^1}_{\boldsymbol{r}^1,\boldsymbol{s}^1} * \cdots * \eta^{\boldsymbol{b}^q}_{\boldsymbol{r}^q,\boldsymbol{s}^q}.$$

To describe ∇ on basis elements, let $\mathcal{T} = (\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s}) \in \mathrm{Tri}^B_0(n, d)$. We write $\eta_{\mathcal{T}} := \eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}$ and $\mathcal{T}\sigma := (\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s})\sigma$ for $\sigma \in \mathfrak{S}_d$. We have that the stabilizer $\mathfrak{S}_{\mathcal{T}} := \{\sigma \in \mathfrak{S}_d \mid \mathcal{T}\sigma = \mathcal{T}\}$ is a standard parabolic subgroup. Let ${}^{\mathcal{T}}\mathscr{D}$ be the set of the shortest coset representatives in $\mathfrak{S}_{\mathcal{T}}\backslash\mathfrak{S}_d$. We also set

$$[\mathcal{T}]^!_{\mathfrak{c}} := [\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s}]^!_{\mathfrak{c}}. \tag{3.11}$$

If $d = d_1 + d_2$, $\mathcal{T}^1 = (\boldsymbol{b}^1, \boldsymbol{r}^1, \boldsymbol{s}^1) \in \mathrm{Tri}^B(n, d_1)$ and $\mathcal{T}^2 = (\boldsymbol{b}^2, \boldsymbol{r}^2, \boldsymbol{s}^2) \in \mathrm{Tri}^B(n, d_2)$, we denote $\mathcal{T}^1\mathcal{T}^2 := (\boldsymbol{b}^1\boldsymbol{b}^2, \boldsymbol{r}^1\boldsymbol{r}^2, \boldsymbol{s}^1\boldsymbol{s}^2) \in B^d \times [n]^d \times [n]^d$. Recall the notation (3.1). For $\mathcal{T} \in \mathrm{Tri}^B_0(n, d)$ define

$$\mathrm{Spl}(\mathcal{T}) := \bigsqcup_{0 \le c \le d} \big\{(\mathcal{T}^1, \mathcal{T}^2) \in \mathrm{Tri}^B_0(n, c) \times \mathrm{Tri}^B_0(n, d-c) \mid \mathcal{T}^1\mathcal{T}^2 \sim \mathcal{T}\big\}.$$

For $(\mathcal{T}^1, \mathcal{T}^2) \in \mathrm{Spl}(\mathcal{T})$, let $\sigma^{\mathcal{T}}_{\mathcal{T}^1,\mathcal{T}^2}$ be the unique element of ${}^{\mathcal{T}}\mathscr{D}$ such that $\mathcal{T}\sigma^{\mathcal{T}}_{\mathcal{T}^1,\mathcal{T}^2} = \mathcal{T}^1\mathcal{T}^2$. Recalling the notation (1.3), we have:

Lemma 3.12. [10, Corollary 3.24] *If* $\mathcal{T} = (\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s}) \in \mathrm{Tri}^B_0(n, d)$ *then*

$$\nabla(\eta_{\mathcal{T}}) = \sum_{(\mathcal{T}^1,\mathcal{T}^2)\in\mathrm{Spl}(\mathcal{T})} (-1)^{\langle \sigma^{\mathcal{T}}_{\mathcal{T}^1,\mathcal{T}^2};\boldsymbol{b}\rangle} \frac{[\mathcal{T}]^!_{\mathfrak{c}}}{[\mathcal{T}^1]^!_{\mathfrak{c}}[\mathcal{T}^2]^!_{\mathfrak{c}}} \eta_{\mathcal{T}^1} \otimes \eta_{\mathcal{T}^2}.$$

3.2. Special Elements and Characters

In this section, we define the formal characters for $T^A(n,d)$-modules, and describe several important elements we use to study them. To do this, we first describe the combinatorics of colored tableaux.

Throughout the section, let A be a $\Bbbk$-superalgebra with (not necessarily conforming) heredity data I, X, Y as in § 2.2.

Tableaux

We introduce *colored alphabets*

$$\mathscr{A}_X := [n] \times X \quad \text{and} \quad \mathscr{A}_{X(i)} := [n] \times X(i).$$

so that $\mathscr{A}_X = \bigsqcup_{i \in I} \mathscr{A}_{X(i)}$. An element $(l, x) \in \mathscr{A}_X$ is often written as l^x. If $L = l^x \in \mathscr{A}_X$, we denote $\mathrm{color}(L) := x$. For all $i \in I$, we fix arbitrary total orders '$<$' on the sets $\mathscr{A}_{X(i)}$ such that for $r, s \in [n]$, if $r < s$ (in the standard order on $[n]$) then $r^x < s^x$ for all $x \in X(i)$. All definitions of this subsection which involve X have obvious analogues for Y, for example, we have the colored alphabets $\mathscr{A}_Y$ and $\mathscr{A}_{Y(i)}$.

Let $\boldsymbol{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell)}) \in \Lambda^I(n,d)$. Fix $i \in I$. Recall the Young diagram of a partition from (1.7). A *standard $X(i)$-colored $\lambda^{(i)}$-tableau* is a function $T : [\lambda^{(i)}] \to \mathscr{A}_{X(i)}$ such that the following two conditions are satisfied for any pair of nodes $(r,s), (r',s') \in [\lambda^{(i)}]$:

(R) If $r = r'$ and $s < s'$, then $T(r,s) \le T(r',s')$, and the equality is allowed only if $\mathrm{color}(T(r,s)) \in X(i)_{\bar{0}}$.

(C) If $s = s'$ and $r < r'$, then $T(r,s) \le T(r',s')$, and equality is allowed only if $\mathrm{color}(T(r,s)) \in X(i)_{\bar{1}}$.

We denote by $\mathrm{Std}^{X(i)}(\lambda^{(i)})$ the set of all standard $X(i)$-colored $\lambda^{(i)}$-tableaux. Recalling the idempotents $e_i \in X(i) \cap Y(i)$, the *initial* $\lambda^{(i)}$*-tableau* $T^{\lambda^{(i)}}$ is given by

$$T^{\lambda^{(i)}} : [\lambda^{(i)}] \to \mathscr{A}_{X(i)},\ (r,s) \mapsto r^{e_i}.$$

Note that $T^{\lambda^{(i)}}$ is in both $\mathrm{Std}^{X(i)}(\lambda^{(i)})$ and $\mathrm{Std}^{Y(i)}(\lambda^{(i)})$.

Let $T \in \mathrm{Std}^{X(i)}(\lambda^{(i)})$. Denote $d_i := |\lambda^{(i)}|$. Reading the entries of T along the rows from left to right starting from the first row, we get a sequence $l_1^{x_1} \cdots l_{d_i}^{x_{d_i}} \in \mathscr{A}_{X(i)}^{d_i}$. (Fixing this reading is analogous to choosing a fixed bijection $[d] \to [\lambda]$ as we did for the classical Schur algebra in section § 2.3.) We denote $\boldsymbol{l}^T := l_1 \cdots l_{d_i}$ and $\boldsymbol{x}^T := x_1 \cdots x_{d_i}$.

For a function $\boldsymbol{T} : [\boldsymbol{\lambda}] \to \mathscr{A}_X$ and $i \in I$, we set $T^{(i)} := \boldsymbol{T}|_{[\lambda^{(i)}]}$ to be the restriction of $\boldsymbol{T}$ to $[\lambda^{(i)}]$. We write $\boldsymbol{T} = (T^{(0)}, \ldots, T^{(\ell)})$, keeping in mind that the restrictions $T^{(i)}$ determine $\boldsymbol{T}$ uniquely. A *standard* X*-colored* $\boldsymbol{\lambda}$*-tableau* is a function $\boldsymbol{T} : [\boldsymbol{\lambda}] \to \mathscr{A}_X$ such that $T^{(i)} \in \mathrm{Std}^{X(i)}(\lambda^{(i)})$ for all $i \in I$. We denote by $\mathrm{Std}^X(\boldsymbol{\lambda})$ the set of all standard X-colored $\boldsymbol{\lambda}$-tableaux. For example, we have the *initial* $\boldsymbol{\lambda}$*-tableau*

$$\boldsymbol{T}^{\boldsymbol{\lambda}} = (T^{\lambda^{(0)}}, \ldots, T^{\lambda^{(\ell)}}) \in \mathrm{Std}^X(\boldsymbol{\lambda}) \cap \mathrm{Std}^Y(\boldsymbol{\lambda}).$$

For $\boldsymbol{T} \in \mathrm{Std}^X(\boldsymbol{\lambda})$, we denote

$$\boldsymbol{l}^{\boldsymbol{T}} := \boldsymbol{l}^{T^{(0)}} \cdots \boldsymbol{l}^{T^{(\ell)}} \in [n]^d, \quad \boldsymbol{x}^{\boldsymbol{T}} := \boldsymbol{x}^{T^{(0)}} \cdots \boldsymbol{x}^{T^{(\ell)}} \in X^d, \quad \text{and} \quad \boldsymbol{l}^{\boldsymbol{\lambda}} := \boldsymbol{l}^{\boldsymbol{T}^{\boldsymbol{\lambda}}}.$$

The sequence $\boldsymbol{y}^{\boldsymbol{T}}$ for $\boldsymbol{T} \in \mathrm{Std}^Y(\boldsymbol{\lambda})$ is defined similarly to $\boldsymbol{x}^{\boldsymbol{T}}$.

Let $\boldsymbol{\lambda} \in \Lambda^I_+(n,d)$ and $\boldsymbol{T} \in \mathrm{Std}^X(\boldsymbol{\lambda})$, with $\boldsymbol{l}^{\boldsymbol{T}} = l_1 \cdots l_d$ and $\boldsymbol{x}^{\boldsymbol{T}} = x_1 \cdots x_d$. Suppose that there exist $i_1, \dots, i_d \in I$ such that $e_{i_1} x_1 = x_1, \dots, e_{i_d} x_d = x_d$. Recalling (1.8) and (1.10), we define the *left weight* of $\boldsymbol{T}$ to be

$$\boldsymbol{\alpha}(\boldsymbol{T}) := \sum_{c=1}^{d} \boldsymbol{\iota}_{i_c}(\varepsilon_{l_c}) \in \Lambda^I(n,d).$$

cf. (2.20). For $\boldsymbol{\mu} \in \Lambda^I(n,d)$, we denote

$$\mathrm{Std}^X(\boldsymbol{\lambda}, \boldsymbol{\mu}) := \{\boldsymbol{T} \in \mathrm{Std}^X(\boldsymbol{\lambda}) \mid \boldsymbol{\alpha}(\boldsymbol{T}) = \boldsymbol{\mu}\}. \tag{3.13}$$

Idempotents and Characters

Let $\lambda \in \Lambda(n,d)$. Set $\boldsymbol{l}^\lambda := 1^{\lambda_1} \cdots n^{\lambda_n}$. For an idempotent $e \in A$ we have an idempotent $\eta^e_\lambda := \eta^{e^d}_{\boldsymbol{l}^\lambda, \boldsymbol{l}^\lambda} \in T(n,d)$. Let $e_0, \dots, e_\ell \in A$ be the standard idempotents. For each $\boldsymbol{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell)}) \in \Lambda^I(n,d)$, we have the idempotent

$$\eta_{\boldsymbol{\lambda}} := \eta^{e_0}_{\lambda^{(0)}} * \cdots * \eta^{e_\ell}_{\lambda^{(\ell)}} \in T^A(n,d).$$

Where the fact that $\eta_{\boldsymbol{\lambda}} \in T^A(n,d)$ follows because each standard idempotent $e_i \in B_{\mathfrak{a}}$. The idempotents $\eta_{\boldsymbol{\lambda}}$ are obviously orthogonal.

Definition 3.14. For any $T^A(n,d)$-supermodule V, $\boldsymbol{\mu} \in \Lambda^I_+(n,d)$ and $0 \neq v \in V$, we say that v is a *weight vector of weight* $\boldsymbol{\mu}$ if $v \in \eta_{\boldsymbol{\mu}} V$.

For $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda(n)$, define the monomial $z^\lambda := z_1^{\lambda_1} \cdots z_n^{\lambda_n} \in \mathbb{Z}[z_1, \dots, z_n]$. For $\boldsymbol{\lambda} \in \Lambda^I(n)$, we now set

$$z^{\boldsymbol{\lambda}} := z^{\lambda^{(0)}} \otimes z^{\lambda^{(1)}} \otimes \cdots \otimes z^{\lambda^{(\ell)}} \in \mathbb{Z}[z_1, \dots, z_n]^{\otimes I}.$$

Following [10, §5A], see especially [10, Lemma 5.9], for a $T(n,d)$-module V, we define its *formal character*

$$\operatorname{ch} V := \sum_{\boldsymbol{\mu}\in\Lambda^I(n,d)} (\dim \eta_{\boldsymbol{\mu}} V) z^{\boldsymbol{\mu}} \in \operatorname{Sym}^I(n,d).$$

If $\sum_{i\in I} e_i = 1_A$, then $1_{T^A(n,d)} = \sum_{\boldsymbol{\lambda}\in\Lambda^I(n,d)} \eta_{\boldsymbol{\lambda}}$, but we do not need to assume this. So in general we might have $\sum_{\boldsymbol{\mu}\in\Lambda^I(n,d)} \eta_{\boldsymbol{\mu}} V \subsetneq V$. It is worth noting that later on it will be shown that each simple supermodule has nonzero character. So, it is impossible for this character to 'miss' a simple factor. This character also has the property we value most, namely:

Lemma 3.15. *[10, Lemma 5.10] If* $W_1 \in T(n,d_1)$-mod *and* $W_2 \in T(n,d_2)$-mod*, then* $\operatorname{ch}(W_1 \otimes W_2) = \operatorname{ch}(W_1)\operatorname{ch}(W_2)$.

<u>Other Important Elements</u>

The group $\mathfrak{S}_n$ acts on $\Lambda(n)$ on the left via

$$\sigma\lambda := (\lambda_{\sigma^{-1}1}, \dots, \lambda_{\sigma^{-1}n}). \tag{3.16}$$

The group $\mathfrak{S}_n^I := \prod_{i\in I}\mathfrak{S}_n$ acts on $\Lambda^I(n)$ via $\boldsymbol{\sigma}\boldsymbol{\lambda} := (\sigma^{(0)}\lambda^{(0)}, \dots, \sigma^{(\ell)}\lambda^{(\ell)})$, for $\boldsymbol{\sigma} = (\sigma^{(0)}, \dots, \sigma^{(\ell)}) \in \mathfrak{S}_n^I$ and $\boldsymbol{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell)}) \in \Lambda^I(n)$. For $a \in A$ and $\sigma \in \mathfrak{S}_n$, let $\xi_\sigma^a := \sum_{r=1}^n \xi_{\sigma(r),r}^a \in M_n(A)$. For $\boldsymbol{\sigma} = (\sigma^{(0)}, \dots, \sigma^{(\ell)}) \in \mathfrak{S}_n^I$, we set

$$\xi_d(\boldsymbol{\sigma}) := \sum_{d_0+\cdots+d_\ell=d} (\xi_{\sigma^{(0)}}^{e_0})^{\otimes d_0} * \cdots * (\xi_{\sigma^{(\ell)}}^{e_\ell})^{\otimes d_\ell} \in T^A(n,d). \tag{3.17}$$

where the fact that each $\xi_d(\boldsymbol{\sigma}) \in T^A(n,d)$ again follows because each standard idempotent $e_i \in B_{\mathfrak{a}}$.

These elements will play a role analogous to the Weyl group in classical type-A Lie theory, cf. [21]. In particular, they have some amenable properties. Namely, it allows us to move between weights as we expect. More precisely, we have:

Lemma 3.18. *[10, Lemmas 5.6, 5.7] For all* $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathfrak{S}_n^I$ *and* $\boldsymbol{\lambda} \in \Lambda^I(n,d)$*, we have* $\xi_d(\boldsymbol{\sigma})\xi_d(\boldsymbol{\tau}) = \xi_d(\boldsymbol{\sigma\tau})$ *and* $\xi_d(\boldsymbol{\sigma})\eta_{\boldsymbol{\lambda}}\xi_d(\boldsymbol{\sigma}^{-1}) = \eta_{\boldsymbol{\sigma\lambda}}$.

An immediate result of this Lemma is the following.

Corollary 3.19. *For* $\boldsymbol{\sigma} \in \mathfrak{S}_n^I$, $\boldsymbol{\lambda} \in \Lambda^I(n,d)$ *and* $V \in T(n,d)$-mod*, we have* $\xi_d(\boldsymbol{\sigma})\,\eta_{\boldsymbol{\lambda}} V = \eta_{\boldsymbol{\sigma\lambda}} V$.

Lemma 3.20. *For* $\boldsymbol{\sigma} \in \mathfrak{S}_n^I$*, we have* $\nabla(\xi_d(\boldsymbol{\sigma})) = \sum_{c=0}^{d} \xi_c(\boldsymbol{\sigma}) \otimes \xi_{d-c}(\boldsymbol{\sigma})$.

Proof. By definition,

$$\nabla((\xi_{\sigma^{(i)}}^{e_i})^{\otimes d_i}) = \sum_{c_i=0}^{d_i} \big((\xi_{\sigma^{(i)}}^{e_i})^{\otimes c_i}\big) \otimes \big((\xi_{\sigma^{(i)}}^{e_i})^{\otimes d_i - c_i}\big).$$

By Lemma 3.7, we have

$$\nabla(\xi_d(\boldsymbol{\sigma})) = \sum_{d_0+\cdots+d_\ell=d} \nabla\big((\xi_{\sigma^{(0)}}^{e_0})^{\otimes d_0}\big) * \cdots * \nabla\big((\xi_{\sigma^{(\ell)}}^{e_\ell})^{\otimes d_\ell}\big),$$

and the result follows. ∎

We end the subsection collecting some results that are useful for the case $n < d$. In that case, we choose a large N and truncate using the following idempotents. For $N \geq n$, set $E_n^N := \sum_{r=1}^{n} \xi_{r,r}^1 \in M_N(A)$ and consider the idempotent

$$\eta_n^N(d) := (E_n^N)^{\otimes d} \in T^A(N,d). \tag{3.21}$$

We get the following lemmas, which will be used to examine the $n < d$ case in § 4.3.

Lemma 3.22. [10, Lemma 5.15] *Let $N \geq n$. Then we have a unital superalgebra isomorphism*

$$T(n,d) \xrightarrow{\sim} \eta_n^N(d)T(N,d)\eta_n^N(d), \;\; \eta_{\boldsymbol{r},\boldsymbol{s}}^{\boldsymbol{b}} \mapsto \eta_{\boldsymbol{r},\boldsymbol{s}}^{\boldsymbol{b}}$$

Lemma 3.23. [10, Proposition 5.19] *Let $d_1, d_2 \in \mathbb{Z}_{\geq 0}$ with $d_1+d_2 = d$, $n \leq N$, $V_1 \in T(N,d_1)$-mod and $V_2 \in T(N,d_2)$-mod. Then there is a functorial isomorphism of $T(n,d)$-modules $\eta_n^N(d)(V_1 \otimes V_2) \simeq (\eta_n^N(d_1)V_1) \otimes (\eta_n^N(d_2)V_2)$.*

3.3. Quasi-hereditary structure on $T(n,d)$

Throughout the section, let A be a based quasi-hereditary superalgebra with conforming heredity data I, X, Y. Throughout this section, we assume that $d \leq n$. Then, by [11, Theorem 6.6], $T(n,d) = T^A(n,d)$ is a based quasi-hereditary algebra.

Heredity Data and Standard Modules

We now describe the heredity data $\Lambda_+^I(n,d), \mathcal{X}(n,d), \mathcal{Y}(n,d)$ for $T(n,d)$ following [11, §6]. We have already defined the partially ordered set $\Lambda_+^I(n,d)$ of I-multipartitions with partial order $\leq_I$, see § 1.4.1. For $\boldsymbol{\lambda} \in \Lambda_+^I(n,d)$ the corresponding sets $\mathcal{X}(\boldsymbol{\lambda}) = \{\mathcal{X}_{\boldsymbol{S}} \mid \boldsymbol{S} \in \mathrm{Std}^X(\boldsymbol{\lambda})\}$ and $\mathcal{Y}(\boldsymbol{\lambda}) = \{\mathcal{Y}_{\boldsymbol{T}} \mid \boldsymbol{T} \in \mathrm{Std}^Y(\boldsymbol{\lambda})\}$ are labeled by the standard X-colored and Y-colored $\boldsymbol{\lambda}$-tableaux, respectively. Recalling the notation $\boldsymbol{x}^{\boldsymbol{S}}$, $\boldsymbol{l}^{\boldsymbol{S}}$, etc. from § 3.2.1, for $\boldsymbol{S} \in \mathrm{Std}^X(\boldsymbol{\lambda})$ and $\boldsymbol{T} \in \mathrm{Std}^Y(\boldsymbol{\lambda})$, we define the elements $\mathcal{X}_{\boldsymbol{S}}$ and $\mathcal{Y}_{\boldsymbol{T}}$ as follows:

$$\mathcal{X}_{\boldsymbol{S}} := \eta_{\boldsymbol{l}^{\boldsymbol{S}},\boldsymbol{l}^{\boldsymbol{\lambda}}}^{\boldsymbol{x}^{\boldsymbol{S}}}, \quad \text{and} \quad \mathcal{Y}_{\boldsymbol{T}} := \eta_{\boldsymbol{l}^{\boldsymbol{\lambda}},\boldsymbol{l}^{\boldsymbol{T}}}^{\boldsymbol{y}^{\boldsymbol{T}}}.$$

For any $\boldsymbol{\lambda} \in \Lambda^I_+(n,d)$, we have $\mathcal{X}_{\boldsymbol{T}^{\boldsymbol{\lambda}}} = \mathcal{Y}_{\boldsymbol{T}^{\boldsymbol{\lambda}}} = \eta_{\boldsymbol{\lambda}}$, so $\mathcal{X}(\boldsymbol{\lambda}) \cap \mathcal{Y}(\boldsymbol{\lambda}) = \{\eta_{\boldsymbol{\lambda}}\}$, and $\{\eta_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \Lambda^I_+(n,d)\}$ are the standard idempotents of the heredity data.

Let $\boldsymbol{\lambda} \in \Lambda^I_+(n,d)$. Following § 2.2, the standard module $\Delta(\boldsymbol{\lambda})$ has basis

$$\{v_{\boldsymbol{T}} := \mathcal{X}_{\boldsymbol{T}} v_{\boldsymbol{\lambda}} \mid \boldsymbol{T} \in \mathrm{Std}^X(\boldsymbol{\lambda})\}, \tag{3.24}$$

where $v_{\boldsymbol{\lambda}}$ is the (unique up to scalar) vector of weight $\boldsymbol{\lambda}$ in $\Delta(\boldsymbol{\lambda})$. Moreover, if $\boldsymbol{T} \in \mathrm{Std}^X(\boldsymbol{\lambda}, \boldsymbol{\mu})$ for some $\boldsymbol{\mu} \in \Lambda^I(n,d)$, see (3.13), then

$$v_{\boldsymbol{T}} \in \eta_{\boldsymbol{\mu}} \Delta(\boldsymbol{\lambda}) \tag{3.25}$$

i.e. $v_{\boldsymbol{T}}$ is a weight vector of weight $\boldsymbol{\mu}$.

Corollary 3.19 immediately implies:

Lemma 3.26. *For $\boldsymbol{\sigma} \in \mathfrak{S}^I_n$ and $\boldsymbol{\lambda} \in \Lambda^I_+(n,d)$ such that $\boldsymbol{\sigma}\boldsymbol{\lambda} = \boldsymbol{\lambda}$, we have $\xi_d(\boldsymbol{\sigma}) v_{\boldsymbol{\lambda}} = \pm v_{\boldsymbol{\lambda}}$.*

If follows from [11, Theorem 6.6] that the formal character of the standard module $\Delta(\boldsymbol{\lambda})$ is of the form

$$\mathrm{ch}\,\Delta(\boldsymbol{\lambda}) = z^{\boldsymbol{\lambda}} + \sum_{\boldsymbol{\mu} <_I \boldsymbol{\lambda}} c_{\boldsymbol{\mu}} z^{\boldsymbol{\mu}}. \tag{3.27}$$

This implies:

Lemma 3.28. *The formal characters $\{\mathrm{ch}\,\Delta(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \Lambda^I_+(n,d)\}$ are linearly independent. In particular:*

(i) *if* $V \in T(n,d)$-mod *has a standard filtration and* $\operatorname{ch} V = \sum_{\boldsymbol{\lambda}\in\Lambda^I(n,d)} m_{\boldsymbol{\lambda}} \operatorname{ch}\Delta(\boldsymbol{\lambda})$ *then every* $\Delta(\boldsymbol{\lambda})$ *appears as a subquotient of the filtration exactly* $m_{\boldsymbol{\lambda}}$ *times.*

(ii) *if* $\boldsymbol{\lambda} \in \Lambda^I_+(n,d)$, $\boldsymbol{\mu} \in \Lambda^I_+(n,c)$ *and* $\Delta(\boldsymbol{\lambda})\otimes\Delta(\boldsymbol{\mu})$ *has a standard filtration, then* $\Delta(\boldsymbol{\lambda}+\boldsymbol{\mu})$ *appears in this filtration once and all other subquotients* $\Delta(\boldsymbol{\nu})$ *of the filtration satisfy* $\boldsymbol{\nu} <_I \boldsymbol{\lambda}+\boldsymbol{\nu}$.

One other easy consequence of this is the aforementioned result that no simple module has zero character. Indeed, for $\boldsymbol{\lambda} \in \Lambda^I_+(n,d)$, $L(\boldsymbol{\lambda})$ is the head of $\Delta(\boldsymbol{\lambda})$, and thus has exactly one (up to scalar) weight vector of weight $\boldsymbol{\lambda}$, which must appear in the character of $L(\boldsymbol{\lambda})$.

Another, less obvious consequence is the following useful criterion.

Lemma 3.29. *Let* $\boldsymbol{\lambda} \in \Lambda^I_+(n,d)$, $\boldsymbol{r},\boldsymbol{s} \in [n]^d$ *and* $y_1,\dots,y_d \in Y$ *with at least one* $y_r \notin X$. *Suppose that* $v \in \eta_{\boldsymbol{\nu}}\Delta(\boldsymbol{\lambda})$ *for some* $\boldsymbol{\nu} \in \Lambda^I(n,d)$ *with* $\|\boldsymbol{\nu}\| = \|\boldsymbol{\lambda}\|$. *Then* $\eta^{y_1\cdots y_d}_{\boldsymbol{r},\boldsymbol{s}} v = 0$.

Proof. Suppose $\eta^{y_1\cdots y_d}_{\boldsymbol{r},\boldsymbol{s}} v \neq 0$. Then $\eta^{y_1\cdots y_d}_{\boldsymbol{r},\boldsymbol{s}}\eta_{\boldsymbol{\nu}} = \eta^{y_1\cdots y_d}_{\boldsymbol{r},\boldsymbol{s}}$. So there exist $i_1,\dots,i_d \in I$ such that $y_1 e_{i_1} = y_1, \dots, y_d e_{i_d} = y_d$ and for all $i \in I$ we have $\sharp\{k \mid i_k = i\} = |\nu^{(i)}| = |\lambda^{(i)}|$. On the other hand, there exist $j_1,\dots,j_d$ such that $e_{j_1}y_1 = y_1, \dots, e_{j_d}y_d = y_d$. By Lemma 2.12, $j_1 \geq i_1,\dots,j_d \geq i_d$, and by the assumption that at least one $y_r \notin X$, we have that at least one $j_r > i_r$. So $\eta^{y_1\cdots y_d}_{\boldsymbol{r},\boldsymbol{s}} v \in \eta_{\boldsymbol{\mu}}\Delta(\lambda)$ for $\boldsymbol{\mu}$ satisfying $\|\boldsymbol{\mu}\| \rhd_I \|\boldsymbol{\lambda}\|$, hence $\boldsymbol{\mu} >_I \boldsymbol{\lambda}$, which contradicts (3.27). □

<u>Character Formula</u>

Throughout the subsection we continue to assume that $d \leq n$. Recall(1.10). We will rely heavily on the following fundamental result of [11].

Lemma 3.30. [11, Theorem 6.17(i)] *Let* $\boldsymbol{\lambda} = (\lambda^{(i)})_{i\in I} \in \Lambda_+^I(n,d)$. *Then*

$$\Delta(\boldsymbol{\lambda}) \simeq \bigotimes_{i\in I} \Delta(\boldsymbol{\iota}_i(\lambda^{(i)})).$$

For each $i, j \in I$, set

$${}_jX(i) := \{x \in X(i) \mid e_j x = x\}.$$

Note that ${}_jX(i) \neq \varnothing$ only if $j \leq i$. For $\boldsymbol{\nu} = (\boldsymbol{\nu}^{(x)})_{x\in X(i)} \in \Lambda_+^{X(i)}$ and $j \in I$, we define

$${}_j\boldsymbol{\nu} = (\boldsymbol{\nu}^{(x)})_{x\in {}_jX(i)} \in \Lambda_+^{{}_jX(i)}.$$

Fix $i \in I$ until the end of the subsection. We define an algebra homomorphism

$$\chi : \mathrm{Sym}^{X(i)} \to \mathrm{Sym}^I, \ \bigotimes_{x\in X(i)} f_x \mapsto \bigotimes_{j\in I} \prod_{x\in {}_jX(i)} f_x,$$

cf. [11, (7.41)]. By the Littlewood-Richardson rule, for $\boldsymbol{\nu} \in \Lambda_+^{X(i)}$, we have

$$\chi\left(\boldsymbol{s}_{\boldsymbol{\nu}}\right) = \sum_{\boldsymbol{\gamma}\in\Lambda_+^I} \prod_{j\in I} c_{{}_j\boldsymbol{\nu}}^{\gamma^{(j)}} \boldsymbol{s}_{\boldsymbol{\gamma}}. \tag{3.31}$$

For a multipartition $\boldsymbol{\nu} \in \Lambda_+^{X(i)}(n,d)$, we define its *superconjugate* multipartition

$$\boldsymbol{\nu}^{\mathsf{con}} := (\tilde{\nu}^{(x)})_{x\in X(i)},$$

where $\tilde{\nu}^{(x)} := \nu^{(x)}$ if x is even and $\tilde{\nu}^{(x)}$ is the conjugate partition $(\nu^{(x)})'$ if x is odd. Using [16, (2.7),(3.8)], we have the algebra homomorphism

$$\mathsf{con} : \mathrm{Sym}^{X(i)} \to \mathrm{Sym}^{X(i)}, s_{\boldsymbol{\nu}} \mapsto s_{\boldsymbol{\nu}^{\mathsf{con}}}.$$

Let $t := |X(i)|$. By choosing a total order on $X(i)$ we will identify $\Lambda_+^{X(i)}$ with Λ_+^t, $\mathrm{Sym}^{X(i)}$ with $\mathrm{Sym}^{\otimes t}$, etc. In particular, we have a well-defined map $\mathsf{con} : \mathrm{Sym}^{\otimes t} \to \mathrm{Sym}^{\otimes t}$. Recalling (1.11) and (1.12), we now have:

Theorem 3.32. *Let $d \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{>0}$, with $d \leq n$, and $\lambda \in \Lambda_+(n,d) \subseteq \Lambda_+$ and $i \in I$. Then*

$$\mathrm{ch}\,\Delta(\boldsymbol{\iota}_i(\lambda)) = \rho_n^I \circ \chi \circ \mathsf{con} \circ \Delta^{t-1}(s_\lambda).$$

Proof. By [11, Proposition 7.45], we have

$$\begin{aligned}
\mathrm{ch}\,\Delta(\boldsymbol{\iota}_i(\lambda)) &= \sum_{\boldsymbol{\gamma} \in \Lambda_+^I(n)} \sum_{\boldsymbol{\nu} \in \Lambda_+(n)^t} c^{\lambda}_{\boldsymbol{\nu}^{\mathsf{con}}} \left(\prod_{j \in I} c^{\gamma^{(j)}}_{{}_j\boldsymbol{\nu}} \right) s_{\boldsymbol{\gamma}}(z_1, \dots, z_n) \\
&= \rho_n^I \left(\sum_{\boldsymbol{\gamma} \in \Lambda_+^I} \sum_{\boldsymbol{\nu} \in \Lambda_+^t} c^{\lambda}_{\boldsymbol{\nu}^{\mathsf{con}}} \left(\prod_{j \in I} c^{\gamma^{(j)}}_{{}_j\boldsymbol{\nu}} \right) s_{\boldsymbol{\gamma}} \right) \\
&= \rho_n^I \circ \chi \left(\sum_{\boldsymbol{\nu} \in \Lambda_+^t} c^{\lambda}_{\boldsymbol{\nu}^{\mathsf{con}}} s_{\boldsymbol{\nu}} \right) \\
&= \rho_n^I \circ \chi \circ \mathsf{con} \left(\sum_{\boldsymbol{\nu} \in \Lambda_+^t} c^{\lambda}_{\boldsymbol{\nu}} s_{\boldsymbol{\nu}} \right) \\
&= \rho_n^I \circ \chi \circ \mathsf{con} \circ \Delta^{t-1}(s_\lambda),
\end{aligned}$$

where we have used (3.31) for the third equality and (1.13) for the last equality.

□

Theorem 3.33. *Let $\lambda \in \Lambda_+(n,d)$, $\mu \in \Lambda_+(n,e)$ and $i \in I$. If $d+e \leq n$ then*

$$\operatorname{ch}\big(\Delta(\boldsymbol{\iota}_i(\lambda)) \otimes \Delta(\boldsymbol{\iota}_i(\mu))\big) = \sum_{\nu \in \Lambda_+(n,d+e)} c^{\nu}_{\lambda,\mu} \operatorname{ch}\Delta(\boldsymbol{\iota}_i(\nu)).$$

Proof. By Lemma 3.15, Theorem 3.32 and the Littlewood-Richardson rule, we have the following:

$$\begin{aligned}\operatorname{ch}\big(\Delta(\boldsymbol{\iota}_i(\lambda)) \otimes \Delta(\boldsymbol{\iota}_i(\mu))\big) &= \big(\operatorname{ch}\Delta(\boldsymbol{\iota}_i(\lambda))\big)\big(\operatorname{ch}\Delta(\boldsymbol{\iota}_i(\mu))\big) \\ &= \big(\rho_n^I \circ \chi \circ \mathsf{con} \circ \mathsf{\Delta}^{t-1}(s_\lambda)\big)\,\big(\rho_n^I \circ \chi \circ \mathsf{con} \circ \mathsf{\Delta}^{t-1}(s_\mu)\big) \\ &= \rho_n^I \circ \chi \circ \mathsf{con} \circ \mathsf{\Delta}^{t-1}(s_\lambda s_\mu) \\ &= \rho_n^I \circ \chi \circ \mathsf{con} \circ \mathsf{\Delta}^{t-1}\left(\sum_{\nu \in \Lambda_+(n,d+e)} c^{\nu}_{\lambda,\mu} s_\nu\right) \\ &= \sum_{\nu \in \Lambda_+(n,d+e)} c^{\nu}_{\lambda,\mu} \operatorname{ch}\Delta(\boldsymbol{\iota}_i(\nu)),\end{aligned}$$

as required. □

For $\boldsymbol{\lambda} = (\lambda^{(j)})_{j\in I} \in \Lambda^I_+(n,d)$, $\boldsymbol{\mu} = (\mu^{(j)})_{j\in I} \in \Lambda^I_+(n,e)$ and $\boldsymbol{\nu} = (\nu^{(j)})_{j\in I} \in \Lambda^I_+(n,d+e)$ we define

$$c^{\boldsymbol{\nu}}_{\boldsymbol{\lambda},\boldsymbol{\mu}} := \prod_{j\in I} c^{\nu^{(j)}}_{\lambda^{(j)},\mu^{(j)}}. \tag{3.34}$$

Corollary 3.35. *Let $\boldsymbol{\lambda} \in \Lambda^I_+(n,d)$ and $\boldsymbol{\mu} \in \Lambda^I_+(n,e)$. If $d+e \leq n$ then*

$$\operatorname{ch}(\Delta(\boldsymbol{\lambda}) \otimes \Delta(\boldsymbol{\mu})) = \sum_{\boldsymbol{\nu} \in \Lambda^I_+(n,d+e)} c^{\boldsymbol{\nu}}_{\boldsymbol{\lambda},\boldsymbol{\mu}} \operatorname{ch}\Delta(\boldsymbol{\nu}).$$

Proof. This follows from Theorem 3.33 and Lemma 3.30. □

3.4. Modified Divided Powers

We can also view the construction of the algebra $T^A(n,d)$ as an application of another construction - namely the Modified Divided Power. In this section we will develop this construction, explore some of its amenable properties, and ultimately recover $T^A(n,d)_R$ by applying the construction in a special situation. In the last chapter, we will revisit these tools to construct modules for $T^{\mathsf{Z}}(n,d) := T^{\mathsf{Z}}(n,d)_R \otimes_R \mathbb{F}$ where Z is the Extended Zig-Zag algebra, see § 5.1.

Throughout this section, $d \in \mathbb{Z}_{\geq 0}$ is fixed.

Modified Divided Power $\tilde{\boldsymbol{\Gamma}}^{\boldsymbol{d}}\boldsymbol{V}$

Definition 3.36. A *calibrated* $\Bbbk$-supermodule is a free $\Bbbk$-supermodule V of finite rank, with a fixed supermodule decomposition $V_{\bar{0}} = V_{\mathfrak{a}} \oplus V_{\mathfrak{c}}$ such that $V_{\mathfrak{a}}$, $V_{\mathfrak{c}}$ are also free $\Bbbk$-supermodules.

Except for the final subsection, in this section we will exclusively work with the case $\Bbbk = R$, as we need characteristic 0 to construct several things in this subsection.

Let $V = V_{\mathfrak{a}} \oplus V_{\mathfrak{c}} \oplus V_{\bar{1}}$ be a calibrated R-supermodule. Choose bases $B^V_{\mathfrak{a}}$ for $V_{\mathfrak{a}}$, $B^V_{\mathfrak{c}}$ for $V_{\mathfrak{c}}$, and $B^V_{\bar{1}}$ for $V_{\bar{1}}$, so that $B^V_{\bar{0}} := B^V_{\mathfrak{a}} \sqcup B^V_{\mathfrak{c}}$ is a basis of $V_{\bar{0}}$ and $B^V = B^V_{\mathfrak{a}} \sqcup B^V_{\mathfrak{c}} \sqcup B^V_{\bar{1}}$ is a basis of V.

For $\boldsymbol{b} \in (B^V)^d$, we define

$$\langle \boldsymbol{b} \rangle := \sharp\{(k,l) \in [d]^2 \mid k < l,\ b_k, b_l \in B^V_{\bar{1}},\ b_k > b_l\}.$$

Setting

$$[\boldsymbol{b} : b] := \sharp\{k \in [d] \mid b_k = b\} \tag{3.37}$$

for all $b \in B^V$, we let $[\boldsymbol{b}]^!_{\mathfrak{c}} := \prod_{b \in B^V_{\mathfrak{c}}} [\boldsymbol{b} : b]!$, cf. (3.3).

Define $\mathrm{Seq}(B^V, d)$ to be the set of all d-tuples $\boldsymbol{b} = b_1 \cdots b_d \in (B^V)^d$ such that $b_k = b_l$ for some $1 \leq k \neq l \leq d$ only if $b_k \in B^V_{\bar{0}}$. Then $\mathrm{Seq}(B^V, d) \subseteq (B^V)^d$ is a $\mathfrak{S}_d$-invariant subset, so we can choose a corresponding set $\mathrm{Seq}(B^V, d)/\mathfrak{S}_d$ of $\mathfrak{S}_d$-orbit representatives and identify it with the set of all $\mathfrak{S}_d$-orbits on $\mathrm{Seq}(B^V, d)$, cf. § 3.1.1. Fix a total order $<$ on B^V.

Recall the invariant space $\Gamma^d V$ from § 1.3. For $\boldsymbol{b} = b_1 \cdots b_d \in \mathrm{Seq}(B^V, d)$, we have elements

$$x_{\boldsymbol{b}} := \sum_{\boldsymbol{b}' = b'_1 \cdots b'_d \sim \boldsymbol{b}} (-1)^{\langle \boldsymbol{b} \rangle + \langle \boldsymbol{b}' \rangle} b'_1 \otimes \cdots \otimes b'_d \in \Gamma^d V \qquad \text{and} \qquad y_{\boldsymbol{b}} := [\boldsymbol{b}]^!_{\mathfrak{c}} x_{\boldsymbol{b}} \in \Gamma^d V.$$

We define the *modified divided power* $\tilde{\Gamma}^d V$ by

$$\tilde{\Gamma}^d V := \mathrm{span}_R \{ y_{\boldsymbol{b}} \mid \boldsymbol{b} \in \mathrm{Seq}(B^V, d) \} \subseteq \Gamma^d V.$$

Note that $\{x_{\boldsymbol{b}} \mid \boldsymbol{b} \in \mathrm{Seq}(B^V, d)/\mathfrak{S}_d\}$ is a basis of $\Gamma^d V$ and $\{y_{\boldsymbol{b}} \mid \boldsymbol{b} \in \mathrm{Seq}(B^V, d)/\mathfrak{S}_d\}$ is a basis of $\tilde{\Gamma}^d V$. We point out that in general $\tilde{\Gamma}^d V$ depends on $V_{\mathfrak{a}}$. An argument as in [10, Proposition 4.11] shows that it does not depend on $V_{\mathfrak{c}}$, but we are not going to need this fact.

Remark 3.38. In general, if we start with a calibrated $\mathbb{F}$-supermodule, $V_{\mathbb{F}}$, simply applying the same construction over $\mathbb{F}$ is not always possible. If $0 < \mathrm{char}\, \mathbb{F} < d$, then several of the definitions don't make sense. So, to avoid these issues we define $\tilde{\Gamma}^d V_{\mathbb{F}} := \tilde{\Gamma}^d V_R \otimes_R \mathbb{F}$, where V_R is a calibrated R-supermodule such that $V_{\mathbb{F},\mathfrak{a}} = V_{R,\mathfrak{a}} \otimes_R \mathbb{F}$, $V_{\mathbb{F},\mathfrak{c}} = V_{R,\mathfrak{c}} \otimes_R \mathbb{F}$, and $V_{\mathbb{F},\bar{1}} = V_{R,\bar{1}} \otimes_R \mathbb{F}$.

If V and W are calibrated R-supermodules, then we make $V \oplus W$ into a calibrated supermodule by taking $(V \oplus W)_{\mathfrak{a}} := V_{\mathfrak{a}} \oplus W_{\mathfrak{a}}$ and $(V \oplus W)_{\mathfrak{c}} := V_{\mathfrak{c}} \oplus W_{\mathfrak{c}}$. One immediate result of this construction is the following.

Lemma 3.39. *For calibrated R-supermodules V, W, and $d \in \mathbb{Z}_{\geq 0}$, we have an isomorphism of R-supermodules*

$$\bigoplus_{d_1+d_2=d} (\tilde{\Gamma}^{d_1} V) \otimes (\tilde{\Gamma}^{d_2} W) \xrightarrow{\sim} \tilde{\Gamma}^d (V \oplus W), \ y \otimes y' \mapsto y * y'.$$

Proof. Observe that $B_{\mathfrak{a}}^{V \oplus W} = B_{\mathfrak{a}}^{V} \sqcup B_{\mathfrak{a}}^{W}$, and similarly for $B_{\mathfrak{c}}^{V \oplus W}$ and $B_{\bar{1}}^{V \oplus W}$. It follows that $y_{\boldsymbol{b}^1} * y_{\boldsymbol{b}^2} = y_{\boldsymbol{b}^1 \boldsymbol{b}^2}$ for all $\boldsymbol{b}^1 \in \mathrm{Seq}(B^V, d_1)$, $\boldsymbol{b}^2 \in \mathrm{Seq}(B^W, d_2)$. Comparing bases then gives the result. □

Bilinear Form on $\tilde{\Gamma}^d V$

We continue with the assumptions of the previous subsection. In particular, V is a calibrated R-supermodule. Suppose in addition that V has an (R-valued) even non-degenerate bilinear form $(\cdot, \cdot)$ and a dual basis $B^{V,*} = \{b^* \mid b \in B^V\}$ for V such that $(b, c^*) = \delta_{b,c}$ for all $b, c \in B^V$, such that the following conditions hold:

- $B^{V,*} = B_{\mathfrak{a}}^{V,*} \sqcup B_{\mathfrak{c}}^{V,*} \sqcup B_{\bar{1}}^{V,*}$ where $B_{\mathfrak{a}}^{V,*}$ is a basis for $V_{\mathfrak{a}}$, $B_{\mathfrak{c}}^{V,*}$ is a basis for $V_{\mathfrak{c}}$, and $B_{\bar{1}}^{V,*}$ is a basis for $V_{\bar{1}}$;
- $b \in B_{\mathfrak{a}}^{V}$ if and only if $b^* \in B_{\mathfrak{c}}^{V,*}$;
- $b \in B_{\mathfrak{c}}^{V}$ if and only if $b^* \in B_{\mathfrak{a}}^{V,*}$;
- $b \in B_{\bar{1}}^{V}$ if and only if $b^* \in B_{\bar{1}}^{V,*}$.

Observe that this implies that $(\cdot, \cdot)$ restricts to a perfect pairing on $V_{\mathfrak{a}} \times V_{\mathfrak{c}}$, on $V_{\mathfrak{c}} \times V_{\mathfrak{a}}$, and on $V_{\bar{1}} \times V_{\bar{1}}$.

The form extends to the form $(\cdot,\cdot)_{\otimes}$ on $V^{\otimes d}$:

$$(v_1 \otimes \cdots \otimes v_d, w_1 \otimes \cdots \otimes w_d)_{\otimes} = (-1)^{\langle \boldsymbol{v},\boldsymbol{w}\rangle}(v_1, w_1)\cdots(v_d, w_d). \tag{3.40}$$

for all $\boldsymbol{v}, \boldsymbol{w} \in V^d$. Note that for any $\sigma \in \mathfrak{S}_d$, we have

$$((v_1 \otimes \cdots \otimes v_d)^{\sigma}, (w_1 \otimes \cdots \otimes w_d)^{\sigma})_{\otimes} = (v_1 \otimes \cdots \otimes v_d, w_1 \otimes \cdots \otimes w_d)_{\otimes}. \tag{3.41}$$

Moreover, for $\boldsymbol{b} \in (B^V)^d$ and $\boldsymbol{c} \in (B^V)^d$, we have

$$(b_1 \otimes \cdots \otimes b_d, c_1^* \otimes \cdots \otimes c_d^*)_{\otimes} = (-1)^{\langle \boldsymbol{b},\boldsymbol{c}\rangle}\delta_{\boldsymbol{b},\boldsymbol{c}}. \tag{3.42}$$

Lemma 3.43. *Let $\boldsymbol{b}, \boldsymbol{c} = c_1 \cdots c_d \in \mathrm{Seq}(B^V, d)$. Then $\boldsymbol{c}^* := c_1^* \cdots c_d^* \in \mathrm{Seq}(B^{V,*}, d)$, and $(y_{\boldsymbol{b}}, y_{\boldsymbol{c}^*})_{\otimes} = \pm d!\,\delta_{\boldsymbol{b}\sim\boldsymbol{c}}$.*

Proof. By (3.42), $(y_{\boldsymbol{b}}, y_{\boldsymbol{c}^*})_{\otimes} \neq 0$ only if $\boldsymbol{b} \sim \boldsymbol{c}$. So we may assume that $\boldsymbol{c} = \boldsymbol{b}$ and that the stabilizer $\mathfrak{S}_{\boldsymbol{b}}$ is a standard parabolic subgroup. As no odd element repeats in $\boldsymbol{b}$, we have

$$\begin{aligned}|\mathfrak{S}_{\boldsymbol{b}}| &= \Big(\prod_{b \in B^V_{\mathfrak{a}}} [\boldsymbol{b} : b]!\Big)\Big(\prod_{b \in B^V_{\mathfrak{c}}} [\boldsymbol{b} : b]!\Big) = \Big(\prod_{b \in B^V_{\mathfrak{a}}} [\boldsymbol{c}^* : b^*]!\Big)\Big(\prod_{b \in B^V_{\mathfrak{c}}} [\boldsymbol{b} : b]!\Big) \\ &= \Big(\prod_{c \in B^{V,*}_{\mathfrak{c}}} [\boldsymbol{c}^* : c]!\Big)\Big(\prod_{b \in B^V_{\mathfrak{c}}} [\boldsymbol{b} : b]!\Big).\end{aligned}$$

So, using (3.41) and (3.42), we have that $(y_{\boldsymbol{b}}, y_{\boldsymbol{c}^*})_\otimes$ equals

$$
\begin{aligned}
&\Big(\big(\prod_{b\in B^V_{\mathfrak{c}}}[\boldsymbol{b}:b]!\big)x_{\boldsymbol{b}}, \big(\prod_{c\in B^{V,*}_{\mathfrak{c}}}[\boldsymbol{c}^*:c]!\big)x_{\boldsymbol{c}^*}\Big)_\otimes \\
&=\Big(\big(\prod_{b\in B^V_{\mathfrak{c}}}[\boldsymbol{b}:b]!\big)\sum_{\sigma\in\mathfrak{S}_d/\mathfrak{S}_{\boldsymbol{b}}}(b_1\otimes\cdots\otimes b_d)^\sigma, \big(\prod_{c\in B^{V,*}_{\mathfrak{c}}}[\boldsymbol{c}^*:c]!\big)\sum_{\sigma\in\mathfrak{S}_d/\mathfrak{S}_{\boldsymbol{b}}}(c_1^*\otimes\cdots\otimes c_d^*)^\sigma\Big)_\otimes \\
&=\big(\prod_{b\in B^V_{\mathfrak{c}}}[\boldsymbol{b}:b]!\big)\big(\prod_{c\in B^{V,*}_{\mathfrak{c}}}[\boldsymbol{c}^*:c]!\big)[\mathfrak{S}_d:\mathfrak{S}_{\boldsymbol{b}}](b_1\otimes\cdots\otimes b_d, c_1^*\otimes\cdots\otimes c_d^*)_\otimes \\
&=\pm\, d!
\end{aligned}
$$

which completes the proof. □

In view of the lemma, we have $(z,w)_\otimes$ is divisible by $d!$ for all $z,w\in\tilde{\Gamma}^dV$. So we can define a new form on $\tilde{\Gamma}^dV$ by setting

$$(z,w)_\sim := \frac{1}{d!}(z,w)_\otimes \tag{3.44}$$

for all $v,w\in\tilde{\Gamma}^dV$. The following is now clear from the lemma:

Proposition 3.45. *The bilinear form $(\cdot,\cdot)_\sim$ on $\tilde{\Gamma}^dV$ is even and non-degenerate. Moreover, it is supersymmetric (resp. skew-supersymmetric) if $(\cdot,\cdot)$ is so.*

$\tilde{\boldsymbol{\Gamma}}^{\boldsymbol{d}}\boldsymbol{V}$ as a module over $\tilde{\boldsymbol{\Gamma}}^{\boldsymbol{d}}\boldsymbol{A}$

Let $A = A_R$ be a based quasi-hereditary algebra over R with conforming heredity data I, X, Y. In particular we have a unital subalgebra $\mathfrak{a}\subseteq A_{\bar{0}}$ and heredity basis $B = B_{\mathfrak{a}}\sqcup B_{\mathfrak{c}}\sqcup B_{\bar{1}}$. Observe that $A=\mathfrak{a}\oplus\mathfrak{c}\oplus A_{\bar{1}}$ is a calibrated R-supermodule, so we may speak of $\mathrm{Seq}(B,d)$, etc. as in the previous subsections.

Using this data, it is easy to see that $\tilde{\Gamma}^d A = T^A(1, d_R)$, as in § 3.1.1. In particular, $\tilde{\Gamma}^d A$ is an R-superalgebra.

Let $V = V_{\mathfrak{a}} \oplus V_{\mathfrak{c}} \oplus V_{\bar{1}}$ be a calibrated R-supermodule as in the previous subsection, and assume in addition that V is an A-supermodule. Then $\Gamma^d V$ is naturally a $\Gamma^d A$-supermodule, see § 1.3. So upon restriction to the subalgebra $\tilde{\Gamma}^d A \subseteq \Gamma^d A$, we view $\Gamma^d V$ as a $\tilde{\Gamma}^d A$-supermodule. In this subsection we show that under a natural additional assumption, $\tilde{\Gamma}^d V \subseteq \Gamma^d V$ is a $\tilde{\Gamma}^d A$-subsupermodule.

For $a \in A$ and $b, c \in B^V$, we define the structure constants $\kappa^b_{a,c}$ from $ac = \sum_{b \in B^V} \kappa^b_{a,c} b$, cf § 3.1.2. For $\boldsymbol{a} = a_1 \cdots a_d \in \mathrm{Seq}(B, d)$ and $\boldsymbol{b} = b_1 \cdots b_d, \boldsymbol{c} = c_1 \cdots c_d \in \mathrm{Seq}(B^V, d)$, we also set $\kappa^{\boldsymbol{b}}_{\boldsymbol{a},\boldsymbol{c}} := \kappa^{b_1}_{a_1,c_1} \cdots \kappa^{b_d}_{a_d,c_d}$. We want to describe the structure constants $f^{\boldsymbol{b}}_{\boldsymbol{a},\boldsymbol{c}}$ defined from

$$\xi^{\boldsymbol{a}} x_{\boldsymbol{c}} = \sum_{\boldsymbol{b} \in \mathrm{Seq}(B^V, d)/\mathfrak{S}_d} f^{\boldsymbol{b}}_{\boldsymbol{a},\boldsymbol{c}} x_{\boldsymbol{b}}.$$

Recall the stabilizer $\mathfrak{S}_{\boldsymbol{s}}$ from § 1.2. The following lemma is an analogue of [10, Corollary 3.7], and its proof is essentially the same as the $n = 1$ case of that Corollary:

Lemma 3.46. *Let $\boldsymbol{a} \in \mathrm{Seq}(B, d)$ and $\boldsymbol{b}, \boldsymbol{c} \in \mathrm{Seq}(B^V, d)$. Let X be the set of all pairs $(\boldsymbol{a}', \boldsymbol{c}') \in \mathrm{Seq}(B, d) \times \mathrm{Seq}(B^V, d)$ such that $\boldsymbol{a}' \sim \boldsymbol{a}$, $\boldsymbol{c}' \sim \boldsymbol{c}$ and $|a'_k| + |c'_k| = |b_k|$ for all $k = 1, \ldots, d$. We fix a set $X/\mathfrak{S}_{\boldsymbol{b}}$ of orbit representatives for the diagonal action of $\mathfrak{S}_{\boldsymbol{b}}$ on X. Then*

$$f^{\boldsymbol{b}}_{\boldsymbol{a},\boldsymbol{c}} = \sum_{(\boldsymbol{a}', \boldsymbol{c}') \in X/\mathfrak{S}_{\boldsymbol{b}}} (-1)^{\langle \boldsymbol{a} \rangle + \langle \boldsymbol{a}' \rangle + \langle \boldsymbol{c} \rangle + \langle \boldsymbol{c}' \rangle + \langle \boldsymbol{a}', \boldsymbol{c}' \rangle} [\mathfrak{S}_{\boldsymbol{b}} : (\mathfrak{S}_{\boldsymbol{b}} \cap \mathfrak{S}_{\boldsymbol{a}'} \cap \mathfrak{S}_{\boldsymbol{c}'})] \kappa^{\boldsymbol{b}}_{\boldsymbol{a}', \boldsymbol{c}'}$$

Proof. Clearly, we have

$$f^{\boldsymbol{b}}_{\boldsymbol{a},\boldsymbol{c}} = \sum (-1)^{\langle \boldsymbol{a}\rangle + \langle \boldsymbol{a}'\rangle + \langle \boldsymbol{c}\rangle + \langle \boldsymbol{c}'\rangle + \langle \boldsymbol{a}',\boldsymbol{c}'\rangle} \kappa^{\boldsymbol{b}}_{\boldsymbol{a}',\boldsymbol{c}'},$$

the sum being over all $(\boldsymbol{a}', \boldsymbol{c}') \in \mathrm{Seq}(B, d) \times \mathrm{Seq}(B^V, d)$ such that $\boldsymbol{a}' \sim \boldsymbol{a}$ and $\boldsymbol{c}' \sim \boldsymbol{c}$, cf. [20, (3.14)]. It remains to note that $\kappa^{\boldsymbol{b}}_{\boldsymbol{a}',\boldsymbol{c}'} = 0$ unless $(\boldsymbol{a}', \boldsymbol{c}') \in X$, and for $(\boldsymbol{a}', \boldsymbol{c}'), (\boldsymbol{a}'', \boldsymbol{c}'') \in X$ in the same $\mathfrak{S}_{\boldsymbol{b}}$-orbit the corresponding summands are equal to each other, cf. the proof of [10, Corollary 3.7]. □

We now introduce our reasonable assumptions on the calibrated R-supermodules.

Definition 3.47. Suppose A is an R-superalgebra with an even subalgebra $\mathfrak{a} \subseteq A_{\bar{0}}$. Let V be a calibrated R-supermodule that is also an A-supermodule. If $\mathfrak{a}V_{\mathfrak{a}} \subseteq V_{\mathfrak{a}}$ then we say that V is a (left) $(A, \mathfrak{a})$-*calibrated supermodule.*

There is of course a right-sided analogue to this definition, and all of the left-sided results of this section have right-sided analogues whose proofs are the same. Once the left-sided results are proven, we will freely use the right-sided results as well.

Lemma 3.48. *If V is an $(A, \mathfrak{a})$-calibrated supermodule, then $\tilde{\Gamma}^d V \subseteq \Gamma^d V$ is a $\tilde{\Gamma}^d A$-submodule.*

Proof. We briefly work over the field of quotients of R, as we do not *a priori* know that we can divide these coefficients. For each $\boldsymbol{a} \in \mathrm{Seq}(B, d)$ and $\boldsymbol{c} \in \mathrm{Seq}(B^V, d)$, we have:

$$\eta^{\boldsymbol{a}} y_{\boldsymbol{c}} = ([\boldsymbol{a}]^!_{\mathfrak{c}} \xi^{\boldsymbol{a}})([\boldsymbol{c}]^!_{\mathfrak{c}} x_{\boldsymbol{c}}) = \sum_{\boldsymbol{b} \in \mathrm{Seq}(B^V,d)/\mathfrak{S}_d} [\boldsymbol{a}]^!_{\mathfrak{c}} [\boldsymbol{c}]^!_{\mathfrak{c}} f^{\boldsymbol{b}}_{\boldsymbol{a},\boldsymbol{c}} x_{\boldsymbol{b}} = \sum_{\boldsymbol{b} \in \mathrm{Seq}(B^V,d)/\mathfrak{S}_d} \frac{[\boldsymbol{a}]^!_{\mathfrak{c}} [\boldsymbol{c}]^!_{\mathfrak{c}} f^{\boldsymbol{b}}_{\boldsymbol{a},\boldsymbol{c}}}{[\boldsymbol{b}]^!_{\mathfrak{c}}} y_{\boldsymbol{b}}.$$

So in view of Lemma 3.46, it suffices to prove that for fixed $\boldsymbol{a} \in \mathrm{Seq}(B, d)$, $\boldsymbol{b}, \boldsymbol{c} \in \mathrm{Seq}(B^V, d)$ and $(\boldsymbol{a}', \boldsymbol{c}') \in X$ satisfying $\kappa^{\boldsymbol{b}}_{\boldsymbol{a}', \boldsymbol{c}'} \neq 0$, the integer

$$M^{\boldsymbol{b}}_{\boldsymbol{a},\boldsymbol{c}} := [\boldsymbol{a}]^!_{\mathfrak{c}}[\boldsymbol{c}]^!_{\mathfrak{c}}[\mathfrak{S}_{\boldsymbol{b}} : (\mathfrak{S}_{\boldsymbol{b}} \cap \mathfrak{S}_{\boldsymbol{a}'} \cap \mathfrak{S}_{\boldsymbol{c}'})]$$

is divisible by $[\boldsymbol{b}]^!_{\mathfrak{c}}$, where X consists of all pairs $(\boldsymbol{a}', \boldsymbol{c}') \in \mathrm{Seq}(B, d) \times \mathrm{Seq}(B^V, d)$ such that $\boldsymbol{a}' \sim \boldsymbol{a}$, $\boldsymbol{c}' \sim \boldsymbol{c}$, and $|a'_k| + |c'_k| = |b_k|$ for all $k \in [d]$.

For $a \in B$ and $b, c \in B^V$, let

$$m^b_{a,c} = \#\{k \in [1, d] \mid a'_k = a, c'_k = c, b_k = b\}.$$

Then, recalling the notation (3.37), we have

$$|\mathfrak{S}_{\boldsymbol{b}} \cap \mathfrak{S}_{\boldsymbol{a}'} \cap \mathfrak{S}_{\boldsymbol{c}'}| = \prod_{a\in B, b,c\in B^V} m^b_{a,c}!,$$

$$[\boldsymbol{a} : a] = [\boldsymbol{a}' : a] = \sum_{b,c\in B^V} m^b_{a,c}; \; [\boldsymbol{c} : c] = [\boldsymbol{c}' : c] = \sum_{a\in B, b\in B^V} m^b_{a,c}; \; [\boldsymbol{b} : b] = \sum_{a\in B, c\in B^V} m^b_{a,c}.$$

In particular, for all $b, c \in B^V$ and $a \in B$, we have integers

$$z_b := \frac{[\boldsymbol{b} : b]!}{\prod_{a\in B, c\in B^V} m^b_{a,c}!}, \; Z_c := \frac{[\boldsymbol{c} : c]!}{\prod_{a\in B_{\mathfrak{a}}, b\in B^V_{\mathfrak{c}}} m^b_{a,c}!}, \; Z_a := \frac{[\boldsymbol{a} : a]!}{\prod_{b\in B^V_{\mathfrak{c}}, c\in B^V_{\bar{0}}} m^b_{a,c}!}.$$

Denoting $C = \prod_{b\in B^V_{\mathfrak{a}} \sqcup B^V_{\bar{1}}} z_b$, we have

$$[\mathfrak{S}_{\boldsymbol{b}} : \mathfrak{S}_{\boldsymbol{b}} \cap \mathfrak{S}'_{\boldsymbol{a}} \cap \mathfrak{S}'_{\boldsymbol{c}}] = \frac{\prod_{b\in B^V} [\boldsymbol{b} : b]!}{\prod_{a\in B, b,c\in B^V} m^b_{a,c}!} = \prod_{b\in B^V} z_b = C \prod_{b\in B^V_{\mathfrak{c}}} z_b.$$

Let $b \in B^V_{\mathfrak{c}}$. If $a \in B_{\bar{1}}$ or $c \in B^V_{\bar{1}}$, then $m^b_{a,c} \leq 1$ because there are no repeated odd elements in tuples in $\mathrm{Seq}(B, d)$ or $\mathrm{Seq}(B^V, d)$. Also observe that if

$a \in B_{\mathfrak{a}}$ and $c \in B_{\mathfrak{a}}^V$, then $ac \in V_{\mathfrak{a}}$ by assumption, so, since $b \in B_{\mathfrak{c}}^V$, we have $\kappa_{a,c}^b = 0$, hence $m_{a,c}^b = 0$. So

$$z_b = \frac{[\boldsymbol{b} : b]!}{\left(\prod_{a\in B_{\mathfrak{c}}, c\in B_{\bar{0}}^V} m_{a,c}^b!\right)\left(\prod_{a\in B_{\mathfrak{a}}, c\in B_{\mathfrak{c}}^V} m_{a,c}^b!\right)}.$$

Thus we have

$$\begin{aligned}
M_{\boldsymbol{a},\boldsymbol{c}}^{\boldsymbol{b}} &= \left(\prod_{a\in B_{\mathfrak{c}}} [\boldsymbol{a} : a]!\right)\left(\prod_{c\in B_{\mathfrak{c}}^V} [\boldsymbol{c} : c]!\right) \cdot C \prod_{b\in B_{\mathfrak{c}}^V} \frac{[\boldsymbol{b} : b]!}{\left(\prod_{a\in B_{\mathfrak{c}}, c\in B_{\bar{0}}^V} m_{a,c}^b!\right)\left(\prod_{a\in B_{\mathfrak{a}}, c\in B_{\mathfrak{c}}^V} m_{a,c}^b!\right)} \\
&= C\left(\prod_{a\in B_{\mathfrak{c}}} \frac{[\boldsymbol{a} : a]!}{\prod_{b\in B_{\mathfrak{c}}^V, c\in B_{\bar{0}}^V} m_{a,c}^b!}\right)\left(\prod_{c\in B_{\mathfrak{c}}^V} \frac{[\boldsymbol{c} : c]!}{\prod_{a\in B_{\mathfrak{a}}, b\in B_{\mathfrak{c}}^V} m_{a,c}^b!}\right)\left(\prod_{b\in B_{\mathfrak{c}}} [\boldsymbol{b} : b]!\right) \\
&= C\left(\prod_{a\in B_{\mathfrak{c}}} Z_a\right)\left(\prod_{c\in B_{\mathfrak{c}}^V} Z_c\right)[\boldsymbol{b}]_{\mathfrak{c}}^!,
\end{aligned}$$

which completes the proof. □

Let $e \in \mathfrak{a}$ be an idempotent such that $be = b$ or $be = 0$ for all $b \in B$. (In this case, as in [10, §5], we say that B is *right e-admissible*.) Let $Be = \{b \in B \mid be = b\}$, $B_{\mathfrak{a}}e := \{b \in B_{\mathfrak{a}} \mid be = b\}$ and $B_{\mathfrak{c}}e := \{b \in B_{\mathfrak{c}} \mid be = b\}$. We have an idempotent $\eta^{e^d} = e^{\otimes d} \in \tilde{\Gamma}^d A$. In the special case where $V = Ae$, we always take $V_{\mathfrak{a}} := \mathfrak{a}e$ with basis $B_{\mathfrak{a}}e$ and $V_{\mathfrak{c}} := \mathfrak{c}e$ with basis $B_{\mathfrak{c}}$. In this case we can describe the $\tilde{\Gamma}^d A$-module $\tilde{\Gamma}^d V$ explicitly as follows:

Lemma 3.49. *Let $e \in \mathfrak{a}$ be an idempotent such that the basis B is right e-admissible. Then $\tilde{\Gamma}^d(Ae) \cong (\tilde{\Gamma}^d A)\eta^{e^d}$.*

Proof. Note that, since B is right e-admissible, $\tilde{\Gamma}^d(Ae)$ has basis $\{y_{\boldsymbol{b}} \mid \boldsymbol{b} \in \mathrm{Seq}(Be, d)/\mathfrak{S}_d\}$ and $(\tilde{\Gamma}^d A)\eta^{e^d}$ has basis $\{\eta^{\boldsymbol{b}} \mid \boldsymbol{b} \in \mathrm{Seq}(Be, d)/\mathfrak{S}_d\}$. There is a

$\tilde{\Gamma}^d A$-module map $\varphi : (\tilde{\Gamma}^d A)\eta^{e^d} \to \tilde{\Gamma}^d(Ae)$ with $\varphi(\eta^{e^d}) = y_{e^d}$. It remains to notice that $\varphi(\eta^{\boldsymbol{b}}) = y_{\boldsymbol{b}}$ for all $\boldsymbol{b} \in \mathrm{Seq}(Be, d)/\mathfrak{S}_d$. □

It follows from the discussion at the start of this subsection that $\oplus_{d\geq 0}\tilde{\Gamma}^d A = \oplus_{d\geq 0} T^A(1,d)_R$. In fact, for $(A,\mathfrak{a})$-calibrated supermodules V and W, $(\tilde{\Gamma}^{d_1}V) \otimes (\tilde{\Gamma}^{d_2}W)$ is a $\tilde{\Gamma}^{d_1+d_2}A$-supermodule, see § 3.1.2. Recalling the isomorphism from Lemma 3.39, we now obtain:

Lemma 3.50. *If V, W are $(A,\mathfrak{a})$-calibrated, then we have an isomorphism of $\tilde{\Gamma}^d A$-modules*

$$\bigoplus_{d_1+d_2=d} (\tilde{\Gamma}^{d_1}V) \otimes (\tilde{\Gamma}^{d_2}W) \xrightarrow{\sim} \tilde{\Gamma}^d(V \oplus W),\ y \otimes y' \mapsto y * y'.$$

Proof. The fact this is a homomorphism of $\tilde{\Gamma}^d A_R$ modules follows from Lemma 3.7. Bijectivity follows from Lemma 3.39. □

Suppose there is an even anti-involution $\tau : A \to A$, where we adopt the convention $\tau(ab) = \tau(b)\tau(a)$ (no sign). Then τ is an isomorphism $A \to A^{\mathrm{op}}$, where A^{op} is defined via $a * b := ba$ (no sign again). We make the additional assumption that $\tau(\mathfrak{a}) = \mathfrak{a}$, in which case $\tau^{\otimes d}$ restricts to an anti-involution τ_d on $\tilde{\Gamma}^d A$, see [10, (4.12)].

Given $W \in \tilde{\Gamma}^d A$-mod, its τ_d-dual W^{τ_d} is defined as W^* with the action $(xf)(w) = f(\tau_d(x)w)$ for all $f \in W^*, w \in W, x \in \tilde{\Gamma}^d A$. Note that $W \simeq W^{\tau_d}$ if and only if there is a non-degenerate τ_d-contravariant form $(\cdot,\cdot)$ on W, where τ_d-contravariance follows the convention $(xv, w) = (-1)^{|x||v|}(v, \tau_d(x)w)$ for all $x \in \tilde{\Gamma}^d A$ and $v, w \in W$.

Let V be an $(A,\mathfrak{a})$-calibrated supermodule. Suppose that V has an even, non-degenerate, τ-contravariant form $(\cdot,\cdot)$ For homogeneous $\boldsymbol{a} = a_1 \cdots a_d \in A^d$ and $\boldsymbol{v} = v_1 \cdots v_d \in V^d$, we set $\boldsymbol{a} \cdot \boldsymbol{v} := (a_1v_1)\cdots(a_dv_d) \in V^d$. Then $(-1)^{\langle \boldsymbol{a}\cdot\boldsymbol{v}, \boldsymbol{w}\rangle} =$

$(-1)^{\langle \boldsymbol{a},\boldsymbol{w}\rangle+\langle \boldsymbol{v},\boldsymbol{w}\rangle}$. Using this, it is easy to establish that $(\cdot,\cdot)_\otimes$ is a $\tau^{\otimes d}$-contravariant form on $V^{\otimes d}$, cf. (3.40). Recalling Proposition 3.45, we deduce:

Lemma 3.51. *Let $(\cdot,\cdot)$ be an even non-degenerate τ-contravariant bilinear form on the A-module V as above, which upon restriction yields perfect pairings on $V_{\mathfrak{a}} \times V_{\mathfrak{c}}$ and $V_{\mathfrak{c}} \times V_{\mathfrak{a}}$. Then $(\cdot,\cdot)_\sim$ is a non-degenerate τ_d-contravariant form on the $\tilde{\Gamma}^d A$-module $\tilde{\Gamma}^d V$.*

From $\boldsymbol{A}$ to $\boldsymbol{M_n(A)}$

Throughout the subsection $n \in \mathbb{Z}_{>0}$ is fixed.

In this subsection we will need to be careful with the ground ring and write R or $\mathbb{F}$ in the indices for algebras and modules when necessary. So let again A_R be a based quasi-hereditary algebra over R with conforming heredity data I, X, Y, in particular we have a subalgebra $\mathfrak{a}_R \subseteq A_{R,\bar{0}}$ and heredity basis $B = B_{\mathfrak{a}} \sqcup B_{\mathfrak{c}} \sqcup B_{\bar{1}}$. Taking the R-supermodule decomposition $M_n(A_R) = M_n(\mathfrak{a}_R) \oplus M_n(\mathfrak{c}_R) \oplus M_n(A_{R,\bar{1}})$, we see that this is a calibrated R-supermodule, and we recover $T^A(n,d)_R$ as $\tilde{\Gamma}^d M_n(A_R)$.

Let V_R be an $(A_R, \mathfrak{a}_R)$-calibrated supermodule with R-basis $B^V = B^V_{\mathfrak{a}} \sqcup B^V_{\mathfrak{c}} \sqcup B^V_{\bar{1}}$, as in Lemma 3.48. The superspace of column vectors $\mathrm{Col}_n(V_R) = V_R^{\oplus n}$ is a left supermodule over $M_n(A_R)$ in a natural way. If V_R is a right A_R-module, we will also consider the right $M_n(A_R)$-module $\mathrm{Row}_n(V_R) = V_R^{\oplus n}$ of row vectors. We will always take $\mathrm{Col}_n(V_R)_{\mathfrak{a}} := \mathrm{Col}_n(V_{R,\mathfrak{a}})$ and $\mathrm{Col}_n(V_R)_{\mathfrak{c}} := \mathrm{Col}_n(V_{R,\mathfrak{c}})$ and analogously for $\mathrm{Row}_n(V_R)$, making $\mathrm{Col}_n(V_R)$ a $(M_n(A_R), M_n(\mathfrak{a}_R))$-calibrated supermodule, and $\mathrm{Row}_n(V_r)$ a right $(M_n(A_R), M_n(\mathfrak{a}_R))$-calibrated supermodule. Then by Lemma 3.48, we have $\tilde{\Gamma}^d \,\mathrm{Col}_n(V_R)$ is a left $T^A(n,d)_R$-module. Extending scalars to $\mathbb{F}$ we get the left module $\tilde{\Gamma}^d \,\mathrm{Col}_n(V_R)_{\mathbb{F}}$ over $T^A(n,d) = \mathbb{F} \otimes_R T^A(n,d)_R$. Note that in general $\tilde{\Gamma}^d \operatorname{Col}_n(V_R)_{\mathbb{F}}$ is not the same as $\tilde{\Gamma}^d \operatorname{Col}_n(V_{\mathbb{F}})$, where $V_{\mathbb{F}} = \mathbb{F} \otimes_R V_R$, cf. Remark 3.38. Similarly, we have a right $T^A(n,d)_R$-module $\tilde{\Gamma}^d \operatorname{Row}_n(V_R)$ and a right $T^A(n,d)$-module $\tilde{\Gamma}^d \operatorname{Row}_n(V_R)_{\mathbb{F}}$.

Lemma 3.52. *Let $e \in \mathfrak{a}_R$ be an idempotent such that the basis B is right e-admissible. Then $\tilde{\Gamma}^d(\operatorname{Col}_n(A_R e)) \simeq T(n,d)_R \eta^{e^d}_{1^d,1^d}$.*

Proof. First notice that $\operatorname{Col}_n(A_R e) \simeq M_n(A_R)\xi^e_{1,1}$ as $M_n(A_R)$-supermodules. The result now follows by applying Lemma 3.49. $\square$

CHAPTER IV

TENSOR PRODUCTS OF STANDARD MODULES

This chapter contains previously unpublished co-authored material, which appears in [1].

We again fix a based quasi-hereditary superalgebra A_R over R with conforming heredity data I, X, Y. Recalling the convention (3.8), we have the $\mathbb{F}$-superalgebra $T(n,d) := T^A(n,d)_R \otimes_R \mathbb{F}$. Under the assumption $d \leq n$, this superalgebra is based quasi-hereditary with heredity data $\Lambda^I_+(n,d), \mathcal{X}(n,d), \mathcal{Y}(n,d)$, see § 3.3. We present all results of this section in terms of left modules. However the analogous results for right modules (and bimodules) are also true, and are proven in a nearly identical manner.

The main goal of this section is to prove the following.

Theorem 4.1. *Let $n \in \mathbb{Z}_{>0}$ and $c, d \in \mathbb{Z}_{\geq 0}$ such that $d + c \leq n$. Let $\boldsymbol{\lambda} \in \Lambda^I_+(n,c)$ and $\boldsymbol{\mu} \in \Lambda^I_+(n,d)$. The $T(n,d+c)$-module $\Delta(\boldsymbol{\lambda}) \otimes \Delta(\boldsymbol{\mu})$ has a standard filtration, and the $T(n,d+c)$-module $\nabla(\boldsymbol{\lambda}) \otimes \nabla(\boldsymbol{\mu})$ has a costandard filtration.*

In fact, we will be able to handle the case of small n as well, but we cannot guarantee that $T^A(n,d)$ is quasi-hereditary in this case, see § 4.3.

4.1. Reduction

We begin to prove Theorem 4.1 by reducing to the case of 'one color' and 'fundamental dominant weights', cf. [9, (3.5)], [7, Proposition 3.5.4(i)], [21]. For integer $0 \leq c \leq n$, recalling (1.8), we define

$$\omega_c := \varepsilon_1 + \cdots + \varepsilon_c \in \Lambda_+(n,c).$$

Proposition 4.2. *Suppose that for all $n \in \mathbb{Z}_{>0}$, $d, c \in \mathbb{Z}_{\geq 0}$ with $d + c \leq n$, $\lambda \in \Lambda_+(n, d)$, and $i \in I$, the tensor product $\Delta(\boldsymbol{\iota}_i(\lambda)) \otimes \Delta(\boldsymbol{\iota}_i(\omega_c))$ has a standard filtration. Then for all $n \in \mathbb{Z}_{>0}$, $d, c \in \mathbb{Z}_{\geq 0}$ with $d + c \leq n$, $\boldsymbol{\lambda} \in \Lambda^I_+(n, d)$, $\boldsymbol{\mu} \in \Lambda^I_+(n, c)$, the tensor product $\Delta(\boldsymbol{\lambda}) \otimes \Delta(\boldsymbol{\mu})$ has a standard filtration.*

Proof. We apply induction on the total degree $d + c$, the base case $d + c = 0$ being trivial, since $T(n, 0) \cong \mathbb{F}$. Let $d + c > 0$. Take $\boldsymbol{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell)}) \in \Lambda^I_+(n, d)$ and $\boldsymbol{\mu} = (\mu^{(0)}, \dots, \mu^{(\ell)}) \in \Lambda^I_+(n, c)$. For all $i \in I$, set $d_i = |\lambda^{(i)}|$ and $c_i := |\mu^{(i)}|$. By Theorem 3.30, we have

$$\Delta(\boldsymbol{\lambda}) \otimes \Delta(\boldsymbol{\mu}) \simeq \bigotimes_{i \in I} \Big(\Delta(\boldsymbol{\iota}_i(\lambda^{(i)})) \otimes \Delta(\boldsymbol{\iota}_i(\mu^{(i)})) \Big). \tag{4.3}$$

Suppose there exist distinct $j, k \in I$ with $d_j, d_k > 0$. Then $d_i < d$ for all $i \in I$. By the inductive assumption, for all $i \in I$, we then have that $\Delta(\boldsymbol{\iota}_i(\lambda^{(i)})) \otimes \Delta(\boldsymbol{\iota}_i(\mu^{(i)}))$ has a standard filtration. It follows from Lemma 3.28 and Theorem 3.33 that in this filtration only subquotients of the form $\Delta(\boldsymbol{\iota}_i(\nu^{(i)}))$ with $\nu^{(i)} \in \Lambda_+(n, d_i + c_i)$ appear. Hence the right hand side of (4.3) has a filtration with subquotients of the form $\bigotimes_{i \in I} \Delta(\boldsymbol{\iota}_i(\nu^{(i)})) \simeq \Delta(\boldsymbol{\nu})$, where the isomorphism is given by Lemma 3.30.

Thus we may assume that there exists a unique i with $d_i = d$ and $d_k = 0$ for all $k \neq i$, i.e. $\boldsymbol{\lambda} = \boldsymbol{\iota}_i(\lambda)$ for some $i \in I$ and $\lambda \in \Lambda_+(n, d)$. Similarly we may assume that $\boldsymbol{\mu} = \boldsymbol{\iota}_j(\mu)$ for some $j \in I$ and $\mu \in \Lambda_+(n, c)$. Moreover, we may assume that $j = i$ since otherwise $\Delta(\boldsymbol{\lambda}) \otimes \Delta(\boldsymbol{\mu}) = \Delta(\boldsymbol{\iota}_i(\lambda)) \otimes \Delta(\boldsymbol{\iota}_j(\mu))$ is a standard module, again by Lemma 3.30.

We now also apply induction on the dominance order on μ. If μ is minimal in the dominance order, then $\mu = \omega_c$ and we are done by assumption. Otherwise, we can write $\mu = \gamma + \omega_r$ for $\gamma \in \Lambda_+(n, s)$ with $0 < s, r < c$. By the inductive assumption on the degree, we have that $\Delta(\boldsymbol{\iota}_i(\gamma)) \otimes \Delta(\boldsymbol{\iota}_i(\omega_r))$ has a standard filtration. By Lemma 3.28 and Theorem 3.33, in this filtration $\Delta(\boldsymbol{\iota}_i(\mu))$ appears once and other standard subquotients are of the form $\Delta(\boldsymbol{\iota}_i(\nu))$ with $\nu \lhd \mu$. By [17, Proposition A2.2(i)], there is a short exact sequence

$$0 \to \Delta(\boldsymbol{\iota}_i(\mu)) \to \Delta(\boldsymbol{\iota}_i(\gamma)) \otimes \Delta(\boldsymbol{\iota}_i(\omega_r)) \to Q \to 0,$$

where Q has a standard filtration with subquotients of the form $\Delta(\boldsymbol{\iota}_i(\nu))$ with $\nu \lhd \mu$. Tensoring with $\Delta(\boldsymbol{\iota}_i(\lambda))$ we get a short exact sequence

$$0 \to \Delta(\boldsymbol{\iota}_i(\lambda)) \otimes \Delta(\boldsymbol{\iota}_i(\mu)) \to \Delta(\boldsymbol{\iota}_i(\lambda)) \otimes \Delta(\boldsymbol{\iota}_i(\gamma)) \otimes \Delta(\boldsymbol{\iota}_i(\omega_r)) \to \Delta(\boldsymbol{\iota}_i(\lambda)) \otimes Q \to 0.$$

By induction on the dominance order, $\Delta(\boldsymbol{\iota}_i(\lambda)) \otimes Q$ has a standard filtration. By induction on the degree, using Lemma 3.28 and Theorem 3.33, we have that $\Delta(\boldsymbol{\iota}_i(\lambda)) \otimes \Delta(\boldsymbol{\iota}_i(\gamma))$ has a standard filtration with subquotients of the form $\Delta(\boldsymbol{\iota}_i(\kappa))$ with $\kappa \unlhd \lambda + \gamma$. Hence by inductive assumption, the middle term has a standard filtration So by [17, Proposition A2.2(v)], $\Delta(\boldsymbol{\iota}_i(\lambda)) \otimes \Delta(\boldsymbol{\iota}_i(\mu))$ has a standard filtration. □

4.2. The Filtration

In view of Proposition 4.2, we now fix $i \in I$, $\lambda \in \Lambda_+(n, d)$, $c \in \mathbb{Z}_{>0}$ such that $d + c \leq n$, and set

$$\boldsymbol{\lambda} := \boldsymbol{\iota}_i(\lambda), \quad \boldsymbol{\mu} := \boldsymbol{\iota}_i(\omega_c).$$

We have highest weight vectors $v_{\boldsymbol{\lambda}} \in \Delta(\boldsymbol{\lambda})$ and $v_{\boldsymbol{\mu}} \in \Delta(\boldsymbol{\mu})$.

Recalling the action of $\mathfrak{S}_n$ on $\Lambda(n)$ from (3.16), we denote

$$\mathfrak{S}_\lambda := \{\sigma \in \mathfrak{S}_n \mid \sigma\lambda = \lambda\}$$

Recall the notation 1.9. If $\lambda = \langle l_1^{a_1}, \dots, l_k^{a_k}\rangle$ for $l_1 > \dots > l_k \geq 0$ and $a_1, \dots, a_k > 0$ with $a_1 + \dots + a_k = n$ then $\mathfrak{S}_\lambda = \mathfrak{S}_{a_1} \times \dots \times \mathfrak{S}_{a_k}$.

Let $\Omega := \{P \subseteq [n] \mid |P| = c\}$. The group $\mathfrak{S}_n$ acts on Ω via $\sigma P = \{\sigma p_1, \dots, \sigma p_c\}$ for $P = \{p_1, \dots, p_c\} \in \Omega$ and $\sigma \in \mathfrak{S}_n$. Denote

$$\varepsilon_P := \varepsilon_{p_1} + \dots + \varepsilon_{p_c} \in \Lambda(n, c).$$

Note that $\sigma(\varepsilon_P) = \varepsilon_{\sigma P}$ for all $\sigma \in \mathfrak{S}_n$ and $P \in \Omega$. We denote

$$\Omega_\lambda := \{P \in \Omega \mid \lambda + \varepsilon_P \in \Lambda_+(n, d + c)\}.$$

Given $P = \{p_1, \dots, p_c\}$ and $Q = \{q_1, \dots, q_c\}$ in Ω, with $1 \leq p_1 < \dots < p_c \leq n$ and $1 \leq q_1 < \dots < q_c \leq n$, we write $P < Q$ if and only if $(p_1, \dots, p_c) < (q_1, \dots, q_c)$ lexicographically. This yields the total order on Ω. Let $\Omega_\lambda = \{P_1, P_2, \dots, P_t\}$ with

$$P_1 = \{1, 2, \dots, c\} < P_2 < \dots < P_t.$$

The following is easy to see:

Lemma 4.4. *Let* $1 \leq r \leq t$. *Then*

(i) P_r *is the minimal element of the orbit* $\mathfrak{S}_\lambda \cdot P_r$*;*

(ii) $\Omega = \bigsqcup_{r=1}^t \mathfrak{S}_\lambda \cdot P_r$.

Proof. Collecting equal parts, write $\lambda = \langle l_1^{a_1}, \dots, l_k^{a_k} \rangle$ for $l_1 > \cdots > l_k \geq 0$ and $a_1, \dots, a_k > 0$ with $a_1 + \cdots + a_k = n$, so that $\mathfrak{S}_\lambda = \mathfrak{S}_{a_1} \times \cdots \times \mathfrak{S}_{a_k}$ and $A_1 := [1, a_1], A_2 := [a_1 + 1, a_1 + a_2], \dots, A_k := [n - a_k + 1, n]$ are the orbits of $\mathfrak{S}_\lambda$ on $[n]$. Now $P, Q \in \Omega$ are in the same $\mathfrak{S}_\lambda$-orbit if and only if $|P \cap A_s| = |Q \cap A_s|$ for all $s = 1, \dots, k$, and it is clear that each orbit has a unique element from Ω_λ which is the lexicographically minimal element of the orbit. □

Let $P = \{p_1, \dots, p_c\} \in \Omega$ with $p_1 < \cdots < p_c$. There is a unique tableau $\boldsymbol{T}^P \in \mathrm{Std}^X(\boldsymbol{\mu})$ with $\boldsymbol{l}^{\boldsymbol{T}^P} = p_1 \cdots p_c$ and $\boldsymbol{x}^{\boldsymbol{T}^P} = e_i^c$. We denote the corresponding standard basis vector

$$w_P := v_{\boldsymbol{T}^P} = \eta^{e_i^c}_{p_1 \cdots p_c, 12 \cdots c} v_{\boldsymbol{\mu}} \in \Delta(\boldsymbol{\mu}),$$

see (3.24). Note that the vectors w_P *do not* exhaust the standard basis of $\Delta(\boldsymbol{\mu})$.

Lemma 4.5. *Let $\nu \in \Lambda(n, c)$. If $\eta_{\boldsymbol{\iota}_i(\nu)} \Delta(\boldsymbol{\mu}) \neq 0$, then ν is of the form ε_P and w_P spans $\eta_{\boldsymbol{\iota}_i(\nu)} \Delta(\boldsymbol{\mu})$.*

Proof. By (3.24), (3.25), the weight space $\eta_{\boldsymbol{\iota}_i(\nu)} \Delta(\boldsymbol{\mu}) \neq 0$ is spanned by the basis elements $v_{\boldsymbol{T}}$ such that $\boldsymbol{T} \in \mathrm{Std}^X(\boldsymbol{\mu}, \boldsymbol{\iota}_i(\nu))$. As $\boldsymbol{\mu} = \boldsymbol{\iota}_i(\omega_c)$, we deduce, using the property (c) of Definition 2.10, that $\boldsymbol{T} = \boldsymbol{T}_P$ for some $P \in \Omega$, i.e. $v_{\boldsymbol{T}} = w_P$. □

For $\sigma \in \mathfrak{S}_n$ denote

$$\boldsymbol{\iota}_i(\sigma) := (1, \dots, 1, \sigma, 1, \dots, 1) \in \mathfrak{S}_n^I,$$

with σ in the ith position. Recalling (3.17), we have an element

$$\xi_c(\boldsymbol{\iota}_i(\sigma)) = (\xi_\sigma^{e_i})^{\otimes c} \in T^A(n, c).$$

Lemma 4.6. *Let $P = \{p_1, \dots, p_c\} \in \Omega$ with $p_1 < \dots < p_c$, and $\sigma \in \mathfrak{S}_n$ such that $\sigma p_1 < \dots < \sigma p_c$. Then $\xi_c(\boldsymbol{\iota}_i(\sigma))w_P = w_{\sigma P}$.*

Proof. By definition of $\xi_\sigma^{e_i}$, we have in $T(n,c)$:

$$(\xi_\sigma^{e_i})^{\otimes c}\, \eta^{e_i^c}_{p_1\cdots p_c, 12\cdots c} = \eta^{e_i^c}_{\sigma p_1 \cdots \sigma p_c, 12\cdots c}.$$

So

$$\xi_c(\boldsymbol{\iota}_i(\sigma))w_P = (\xi_\sigma^{e_i})^{\otimes c}\, \eta^{e_i^c}_{p_1\cdots p_c, 12\cdots c}\, v_{\boldsymbol{\mu}} = \eta^{e_i^c}_{\sigma p_1 \cdots \sigma p_c, 12\cdots c}\, v_{\boldsymbol{\mu}} = w_{\sigma P},$$

as required. □

Corollary 4.7. *Let $P \in \Omega$. Then $T(n,c)w_P = \Delta(\boldsymbol{\mu})$.*

Proof. Write $P = \{p_1, \dots, p_c\}$ with $p_1 < \dots p_c$. Take $\sigma \in \mathfrak{S}_n$ with $\sigma(p_a) = a$ for $a = 1, \dots, c$. By Lemma 4.6, we have $\xi_c(\boldsymbol{\iota}_i(\sigma))w_P = w_{\{1,\dots,c\}} = v_{\boldsymbol{\mu}}$, and the result follows since $T(n,c)v_{\boldsymbol{\mu}} = \Delta(\boldsymbol{\mu})$. □

Lemma 4.8. *We have $T(n, d+c)(v_{\boldsymbol{\lambda}} \otimes w_{P_t}) = \Delta(\boldsymbol{\lambda}) \otimes \Delta(\boldsymbol{\mu})$.*

Proof. Returning to our normal notation, write $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \geq \dots \geq \lambda_n \geq 0$. Let h be maximal with $\lambda_h > 0$, so $\boldsymbol{l}^{\boldsymbol{\lambda}} = 1^{\lambda_1} \cdots h^{\lambda_h}$. Since $n \geq d + c$, we have $P_t = \{h+1, \dots, h+c\}$.

By (3.24), $\Delta(\boldsymbol{\lambda})$ is spanned by elements of the form $\eta^{\boldsymbol{x}}_{\boldsymbol{r},\boldsymbol{l}^{\boldsymbol{\lambda}}} v_{\boldsymbol{\lambda}}$ for $(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{l}^{\boldsymbol{\lambda}}) \in \mathrm{Tri}^X(n,d)$. Let $\mathcal{T}' \in \mathrm{Tri}^X_0(n,d)$ with $\mathcal{T}' \sim (\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{l}^{\boldsymbol{\lambda}})$, see (3.1). Then

$$[\mathcal{T}']^!_{\mathfrak{c}} = 1, \tag{4.9}$$

since $x = xe_i \in B_{\mathfrak{a}}$ for all $x \in X(i)$, see (2.11) and (3.11).

On the other hand, for $(\boldsymbol{b},\boldsymbol{t},\boldsymbol{u}) \in \mathrm{Tri}^B(n,c)$, we have that $\eta^{\boldsymbol{b}}_{\boldsymbol{t},\boldsymbol{u}} w_{P_t} = 0$ unless $\eta^{\boldsymbol{b}}_{\boldsymbol{t},\boldsymbol{u}} \eta_{\boldsymbol{\iota}_i(\varepsilon_{P_t})} = \eta^{\boldsymbol{b}}_{\boldsymbol{t},\boldsymbol{u}}$, and, in view of Corollary 4.7, $\Delta(\boldsymbol{\mu})$ is spanned by all $\eta^{\boldsymbol{b}}_{\boldsymbol{t},\boldsymbol{u}} w_{P_t}$ with $\boldsymbol{u} \sim (h+1)\cdots(h+c)$.

Let $(\boldsymbol{x},\boldsymbol{r},\boldsymbol{l}^{\boldsymbol{\lambda}}) \in \mathrm{Tri}^X(n,d)$ and $(\boldsymbol{b},\boldsymbol{t},\boldsymbol{u}) \in \mathrm{Tri}^B(n,c)$ satisfy $\boldsymbol{u} \sim (h+1)\cdots(h+c)$, and $\mathcal{T} \in \mathrm{Tri}^B_0(n,d+c)$ be the initial triple with $\mathcal{T} \sim (\boldsymbol{xb},\boldsymbol{rt},\boldsymbol{l}^{\boldsymbol{\lambda}}\boldsymbol{u})$. Let $(\mathcal{T}^1,\mathcal{T}^2) \in \mathrm{Spl}(\mathcal{T})$ with $\mathcal{T}^1 \in \mathrm{Tri}^B_0(n,d)$ and $\mathcal{T}^2 \in \mathrm{Tri}^B_0(n,c)$. Suppose $\mathcal{T}^1 = (\boldsymbol{a},\boldsymbol{v},\boldsymbol{s}) \not\sim (\boldsymbol{x},\boldsymbol{r},\boldsymbol{l}^{\boldsymbol{\lambda}})$. Since $\boldsymbol{l}^{\boldsymbol{\lambda}} = 1^{\lambda_1}\cdots h^{\lambda_h}$ and $\boldsymbol{u} \sim (h+1)\cdots(h+c)$, we necessarily have that $s_k \in \{h+1,\dots,h+c\}$ for some $1 \le k \le d$. Hence $\eta_{\mathcal{T}^1} v_{\boldsymbol{\lambda}} = 0$. Now, by Lemma 3.12 and (4.9),

$$\eta^{\boldsymbol{xb}}_{\boldsymbol{rt},\boldsymbol{l}^{\boldsymbol{\lambda}}\boldsymbol{u}}(v_{\boldsymbol{\lambda}} \otimes w_{P_t}) = (\eta^{\boldsymbol{x}}_{\boldsymbol{r},\boldsymbol{l}^{\boldsymbol{\lambda}}} v_{\boldsymbol{\lambda}}) \otimes (\eta^{\boldsymbol{b}}_{\boldsymbol{t},\boldsymbol{u}} w_{P_t}),$$

which implies the lemma. □

For $r = 0,1,\dots,t$, we denote

$$M_r := T(n,d)\langle v_{\boldsymbol{\lambda}} \otimes w_{P_s} \mid 1 \le s \le r\rangle \subseteq \Delta(\boldsymbol{\lambda}) \otimes \Delta(\boldsymbol{\mu}).$$

In view of Lemma 4.8, we have a filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t = \Delta(\boldsymbol{\lambda}) \otimes \Delta(\boldsymbol{\mu}). \tag{4.10}$$

Our goal is to show that $M_r/M_{r-1} \simeq \Delta(\boldsymbol{\iota}_i(\lambda + \varepsilon_{P_r}))$ for all $r = 1,\dots,t$, to get the required standard filtration of $\Delta(\boldsymbol{\lambda}) \otimes \Delta(\boldsymbol{\mu})$.

Lemma 4.11. *If $1 \le r \le t$ and $P \in \mathfrak{S}_\lambda \cdot P_r$, then $v_{\boldsymbol{\lambda}} \otimes w_P \in M_r$.*

Proof. Write $P = \{p_1, \dots, p_c\}$ with $p_1 < \cdots < p_c$. Let $\sigma \in \mathfrak{S}_\lambda$ be such that $\sigma P_r = P$ and $\sigma p_1 < \cdots < \sigma p_c$. Note using Lemmas 3.26, 4.6 and 3.20 that

$$v_{\boldsymbol{\lambda}} \otimes w_P = \pm(\xi_d(\boldsymbol{\iota}_i(\sigma))v_{\boldsymbol{\lambda}}) \otimes (\xi_c(\boldsymbol{\iota}_i(\sigma))w_{P_r}) = \pm\xi_{d+c}(\boldsymbol{\iota}_i(\sigma))(v_{\boldsymbol{\lambda}} \otimes w_{P_r}) \in M_r,$$

as required. □

Lemma 4.12. *Let $1 \le r \le t$ and $E \in \mathcal{Y}(n, d+c)$. If $E(v_{\boldsymbol{\lambda}} \otimes w_{P_r}) \notin M_{r-1}$ then $E = \eta_{\boldsymbol{\iota}_i(\lambda+\varepsilon_{P_r})}$. In particular, M_r/M_{r-1} is a highest weight module of weight $\boldsymbol{\iota}_i(\lambda + \varepsilon_{P_r})$.*

Proof. Suppose $E(v_{\boldsymbol{\lambda}} \otimes w_{P_r}) \notin M_{r-1}$. Write $E = \mathcal{Y}_{\boldsymbol{T}} := \eta^{\boldsymbol{y^T}}_{\boldsymbol{l^\nu},\boldsymbol{l^T}}$, with $\boldsymbol{T} \in \mathrm{Std}^Y(\boldsymbol{\nu})$ for some $\boldsymbol{\nu} \in \Lambda^I_+(n, d+c)$. By Lemma 3.12, $\nabla(\eta^{\boldsymbol{y^T}}_{\boldsymbol{l^\nu},\boldsymbol{l^T}})$ is a linear combination of elements of the form $\eta^{\boldsymbol{y}}_{\boldsymbol{r},\boldsymbol{s}} \otimes \eta^{\boldsymbol{y'}}_{\boldsymbol{r'},\boldsymbol{s'}}$ such that $\boldsymbol{yy'} \sim \boldsymbol{y}^T$. By Lemma 3.29, $\eta^{\boldsymbol{y}}_{\boldsymbol{r},\boldsymbol{s}} v_{\boldsymbol{\lambda}} \neq 0$ only if $\boldsymbol{y} = e_i^d$, and $\eta^{\boldsymbol{y'}}_{\boldsymbol{r'},\boldsymbol{s'}} w_{P_r} \neq 0$ only if $\boldsymbol{y'} = e_i^c$. We conclude that $\boldsymbol{y^T} = e_i^{d+c}$, and so E can be written in the form $E = \eta^{e_i^{d+c}}_{\boldsymbol{r},\boldsymbol{s}}$ with $r_k \le s_k$ for all k.

If $r_k < s_k$ for some k, then in view of Lemma 3.12 and Lemma 4.5, $E(v_{\boldsymbol{\lambda}} \otimes w_{P_r})$ is a multiple of $v_{\boldsymbol{\lambda}} \otimes w_P$ for some $P < P_r$. By Lemma 4.4, $P \in \mathfrak{S}_\lambda \cdot P_s$ for some $s < r$, hence $v_{\boldsymbol{\lambda}} \otimes w_P \in M_s \subseteq M_{r-1}$ by Lemma 4.11, giving a contradiction. So $r_k = s_k$ for all k. Then E is of the form $\eta_{\boldsymbol{\nu}}$, and $E(v_{\boldsymbol{\lambda}} \otimes w_{P_r}) \neq 0$ implies $\boldsymbol{\nu} = \boldsymbol{\iota}_i(\lambda + \varepsilon_{P_r})$.

The second statement now follows from Lemma 2.16. □

Theorem 4.13. *We have a filtration of $T(n, d+c)$-modules*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t = \Delta(\boldsymbol{\lambda}) \otimes \Delta(\boldsymbol{\mu})$$

such that $M_r/M_{r-1} \simeq \Delta(\boldsymbol{\iota}_i(\lambda + \varepsilon_{P_r}))$ for all $r = 1, \dots, t$.

Proof. We consider the filtration (4.10). By Lemma 4.12, each M_r/M_{r-1} is a highest weight module of weight $\boldsymbol{\iota}_i(\lambda+\varepsilon_{P_r})$. Moreover, recalling that $\boldsymbol{\lambda} = \boldsymbol{\iota}_i(\lambda)$ and $\boldsymbol{\mu} = \boldsymbol{\iota}_i(\omega_c)$, by Theorem 3.33, we have

$$\operatorname{ch}\big(\Delta(\boldsymbol{\lambda})\otimes\Delta(\boldsymbol{\mu})\big) = \sum_{\nu\in\Lambda_+(n,d+e)} c^{\nu}_{\lambda,\omega_c}\operatorname{ch}\Delta(\boldsymbol{\iota}_i(\nu)) = \sum_{r=1}^{t}\operatorname{ch}\Delta(\boldsymbol{\iota}_i(\lambda+\varepsilon_{P_r})),$$

where we have used Pieri's rule for the last equality. Therefore, using linear independence of characters, we get

$$\dim\big(\Delta(\boldsymbol{\lambda})\otimes\Delta(\boldsymbol{\mu})\big) = \sum_{r=1}^{t}\dim\Delta(\boldsymbol{\iota}_i(\lambda+\varepsilon_{P_r})).$$

An application of Corollary 2.18 yields that each M_r/M_{r-1} must be isomorphic to $\Delta(\boldsymbol{\iota}_i(\lambda+\varepsilon_{P_r}))$. □

Recall (2.6) and (3.34).

Corollary 4.14. *Let $n\in\mathbb{Z}_{>0}$, $d,c\in\mathbb{Z}_{\geq 0}$ with $d+c\leq n$, $\boldsymbol{\lambda}\in\Lambda^I_+(n,d)$, $\boldsymbol{\mu}\in\Lambda^I_+(n,c)$ and $\boldsymbol{\nu}\in\Lambda^I_+(n,d+c)$. Then the tensor product $\Delta(\boldsymbol{\lambda})\otimes\Delta(\boldsymbol{\mu})$ has a standard filtration, and*

$$(\Delta(\boldsymbol{\lambda})\otimes\Delta(\boldsymbol{\mu}):\Delta(\boldsymbol{\nu})) = c^{\boldsymbol{\nu}}_{\boldsymbol{\lambda},\boldsymbol{\mu}}.$$

Proof. The first statement follows from Proposition 4.2 and Theorem 4.13. The second statement now follows from Corollary 3.35 using linear independence of formal characters. □

By a symmetric argument (switching the roles of $\mathcal{X}$ and $\mathcal{Y}$ everywhere), we also have the right module version of Corollary 4.14, which claims that the right

$T(n, d+c)$-module $\Delta^{\mathrm{op}}(\boldsymbol{\lambda}) \otimes \Delta^{\mathrm{op}}(\boldsymbol{\mu})$ has a Δ^{op}-filtration. In view of (2.19), by dualizing, we now get:

Corollary 4.15. *Let $n \in \mathbb{Z}_{>0}$, $d, c \in \mathbb{Z}_{\geq 0}$ with $d + c \leq n$, $\boldsymbol{\lambda} \in \Lambda^I_+(n, d)$, $\boldsymbol{\mu} \in \Lambda^I_+(n, c)$ and $\boldsymbol{\nu} \in \Lambda^I_+(n, d+c)$. Then the tensor product $\nabla(\boldsymbol{\lambda}) \otimes \nabla(\boldsymbol{\mu})$ has a costandard filtration.*

Remark 4.16. Note that in Theorem 4.13, the factors of the standard filtration are isomorphic to standard modules via *even* isomorphisms. Using this fact and (an appropriate strengthening of) Proposition 4.2, one can similarly strengthen Corollaries 4.14 and 4.15.

4.3. The Case of Small n

Let $d \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{>0}$. If $n < d$, the algebra $T(n, d)$ does not have to be quasi-hereditary, but it still has a natural family of 'standard' and 'costandard' modules which play an important role. For example, if A has a standard anti-involution then $T(n, d)$ is cellular with 'standard' modules being the cell modules, see [11, Lemma 6.25]. These 'standard' (resp. 'costandard') modules are obtained by an idempotent truncation from the standard modules $\Delta(\boldsymbol{\lambda})$ (resp. costandard modules $\nabla(\boldsymbol{\lambda})$) over $T(N, d)$ for any $N \geq d$. This section explores this case.

Throughout the subsection we assume that $N \geq n$. In view of Lemma 3.22, we now always identify the algebras $T(n, d)$ and $\eta^N_n(d) T(N, d) \eta^N_n(d)$. For any $T(N, d)$-module V, we consider $\eta^N_n(d)V$ as a module over $T(n, d) = \eta^N_n(d) T(N, d) \eta^N_n(d)$.

We always consider $\Lambda^I_+(n, d)$ as a subset of $\Lambda^I_+(N, d)$ by adding $N - n$ zeroes to every component $\lambda^{(i)}$ of $\boldsymbol{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell)}) \in \Lambda^I_+(n, d)$. Note that this embedding is a bijection if $n \geq d$. However, when we consider $\boldsymbol{\lambda} \in \Lambda^I_+(n, d)$ as an element of $\Lambda^I_+(N,d)$ the set $\mathrm{Std}^X(\boldsymbol{\lambda})$ of standard X-colored $\boldsymbol{\lambda}$-tableaux changes, so in this subsection we will use the more detailed notation $\mathrm{Std}^X_n(\boldsymbol{\lambda})$ to indicate that the entries of the tableaux are of the form r^x with $r \in [n]$. We will also use the more detailed notation $\Delta_n(\boldsymbol{\lambda})$ for the standard $T(n,d)$-module $\Delta(\boldsymbol{\lambda})$ which so far has only been defined for all $\boldsymbol{\lambda} \in \Lambda^I_+(n,d)$ when $n \geq d$. Recall from (3.24) that for $n \geq d$ we have that $\Delta_n(\boldsymbol{\lambda})$ has basis $\{v_{\boldsymbol{T}} := \mathcal{X}_{\boldsymbol{T}} v_{\boldsymbol{\lambda}} \mid \boldsymbol{T} \in \mathrm{Std}^X_n(\boldsymbol{\lambda})\}$.

Let $N \geq d$. Fix $\boldsymbol{\lambda} \in \Lambda^I_+(N,d)$. Recall the idempotent $\eta^N_n(d)$ of (3.21). It is easy to see that for $\boldsymbol{T} \in \mathrm{Std}^X_N(\boldsymbol{\lambda})$, we have

$$\eta^N_n(d) v_{\boldsymbol{T}} = \begin{cases} v_{\boldsymbol{T}} & \text{if } \boldsymbol{T} \in \mathrm{Std}^X_n(\boldsymbol{\lambda}), \\ 0 & \text{otherwise.} \end{cases} \tag{4.17}$$

If $n \geq d$ it follows from (4.17) that $\dim \Delta_n(\boldsymbol{\lambda}) = \dim \eta^N_n(d)\Delta_N(\boldsymbol{\lambda})$. Since the $T(n,d)$-module $\eta^N_n(d)\Delta_N(\boldsymbol{\lambda})$ is easily seen to be a highest weight module of weight $\boldsymbol{\lambda}$, Proposition 2.17 now yields an isomorphism of $T(n,d)$-modules

$$\Delta_n(\boldsymbol{\lambda}) \simeq \eta^N_n(d)\Delta_N(\boldsymbol{\lambda}). \tag{4.18}$$

Now for $n < d \leq N$ and $\boldsymbol{\lambda} \in \Lambda^I_+(N,d)$, we define the 'standard' module

$$\Delta_n(\boldsymbol{\lambda}) := \eta^N_n(d)\Delta_N(\boldsymbol{\lambda}).$$

By (4.18), this definition does not depend on the choice of $N \geq d$. However, note that some of the $\Delta_n(\boldsymbol{\lambda})$'s might be zero. Define

$$\mathscr{P}^X_+(n,d) := \{\boldsymbol{\lambda} \in \Lambda^I_+(N,d) \mid \mathrm{Std}^X_n(\boldsymbol{\lambda}) \neq \varnothing\}.$$

Note that $\mathscr{P}^X_+(n,d)$ does not depend on the choice of $N \geq d$. Moreover,

$$\Lambda^I_+(n,d) \subseteq \mathscr{P}^X_+(n,d) \subseteq \Lambda^I_+(N,d),$$

with containments being equalities when $n \geq d$. By (4.17), we have:

Lemma 4.19. *Let $N \geq d > n$ and $\boldsymbol{\lambda} \in \Lambda^I_+(N,d)$. Then $\Delta_n(\boldsymbol{\lambda}) \neq 0$ if and only if $\boldsymbol{\lambda} \in \mathscr{P}^X_+(n,d)$.*

The story for the costandard modules $\nabla_n(\boldsymbol{\lambda}) := \eta^N_n(d)\nabla_N(\boldsymbol{\lambda})$ is entirely similar, the non-zero ones being labeled by $\mathscr{P}^Y_+(n,d) := \{\boldsymbol{\lambda} \in \Lambda^I_+(N,d) \mid \mathrm{Std}^Y_n(\boldsymbol{\lambda}) \neq \varnothing\}$.

Theorem 4.20. *Let $\boldsymbol{\lambda} \in \mathscr{P}^X_+(n,d)$ and $\boldsymbol{\mu} \in \mathscr{P}^X_+(n,c)$. Then the $T(n,d+c)$-module $\Delta_n(\boldsymbol{\lambda}) \otimes \Delta_n(\boldsymbol{\mu})$ has a filtration with factors of the form $\Delta_n(\boldsymbol{\nu})$ with $\boldsymbol{\nu} \in \mathscr{P}^X_+(n,d+c)$. Similarly for $\boldsymbol{\lambda} \in \mathscr{P}^Y_+(n,d)$ and $\boldsymbol{\mu} \in \mathscr{P}^Y_+(n,c)$, the $T(n,d+c)$-module $\nabla_n(\boldsymbol{\lambda}) \otimes \nabla_n(\boldsymbol{\mu})$ has a filtration with factors of the form $\nabla_n(\boldsymbol{\nu})$ with $\boldsymbol{\nu} \in \mathscr{P}^Y_+(n,d+c)$.*

Proof. We prove the result for the Δ's, the proof for ∇'s being similar. Choose $N \geq d+c$. By Corollary 4.14, $\Delta_N(\boldsymbol{\lambda}) \otimes \Delta_N(\boldsymbol{\mu})$ has a filtration with factors of the form $\Delta_N(\boldsymbol{\nu})$ with $\boldsymbol{\nu} \in \Lambda^I_+(N,d+c)$. Applying the exact functor

$$T(N,d+c)\text{-mod} \to T(n,d+c)\text{-mod},\ V \mapsto \eta^N_n(d)V$$

to this filtration and using Lemma 3.23, we get the required result. ☐

CHAPTER V

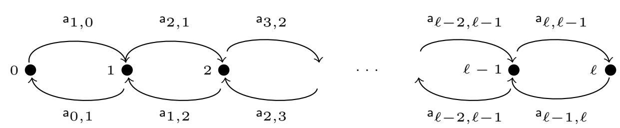
EXAMPLES OF RINGEL DUALITY

In this chapter, we will prove that the extended zigzag Schur algebra is Ringel self-dual. This chapter contains previously unpublished co-authored material that appears in [2].

5.1. The Extended Zigzag Algebra

In this subsection we work over $\Bbbk$. Fix $\ell \geq 1$ and set $I := \{0, 1, \ldots, \ell\}$, $J := I \setminus \{\ell\}$. Let Γ be the quiver with vertex set I and arrows $\{\mathsf{a}_{j,j+1}, \mathsf{a}_{j+1,j} \mid j \in J\}$ as in the figure below:

FIGURE 1. Extended Zigzag Quiver



The *extended zigzag algebra* Z is the path algebra $\Bbbk\Gamma$ modulo the following relations:

1. All paths of length three or greater are zero.

2. All paths of length two that are not cycles are zero.

3. All length-two cycles based at the same vertex are equivalent.

4. $\mathsf{a}_{\ell,\ell-1}\mathsf{a}_{\ell-1,\ell} = 0$.

Length zero paths yield the standard idempotents $\{\mathsf{e}_i \mid i \in I\}$ with $\mathsf{e}_i\mathsf{a}_{i,j}\mathsf{e}_j = \mathsf{a}_{i,j}$ for all admissible i, j. The algebra Z is graded by the path length:

$$\mathsf{Z} = \mathsf{Z}^0 \oplus \mathsf{Z}^1 \oplus \mathsf{Z}^2.$$

We consider Z as a superalgebra with

$$\mathsf{Z}_{\bar{0}} = \mathsf{Z}^0 \oplus \mathsf{Z}^2 \quad \text{and} \quad \mathsf{Z}_{\bar{1}} = \mathsf{Z}^1.$$

Define $\mathsf{c}_j := \mathsf{a}_{j,j+1}\mathsf{a}_{j+1,j}$ for all $j \in J$. The algebra Z has an anti-involution τ with $\tau(\mathsf{e}_i) = \mathsf{e}_i,\ \tau(\mathsf{a}_{ij}) = \mathsf{a}_{ji},\ \tau(\mathsf{c}_j) = \mathsf{c}_j$.

We consider the total order on I given by $0 < 1 < \cdots < \ell$. For $i \in I$, we set

$$\mathsf{X}(i) := \begin{cases} \{\mathsf{e}_i, \mathsf{a}_{i-1,i}\} & \text{if } i > 0, \\ \{\mathsf{e}_0\} & \text{if } i = 0, \end{cases} \qquad \mathsf{Y}(i) := \begin{cases} \{\mathsf{e}_i, \mathsf{a}_{i,i-1}\} & \text{if } i > 0, \\ \{\mathsf{e}_0\} & \text{if } i = 0. \end{cases} \tag{5.1}$$

With respect to this data we have:

Lemma 5.2. [5, Lemma 4.14] *The graded superalgebra Z is a basic based quasi-hereditary with conforming heredity data $I, \mathsf{X}, \mathsf{Y}$ and standard anti-involution τ. For the corresponding heredity basis B we have* $\mathsf{B}_{\bar{1}} = \{\mathsf{a}_{j,j+1}, \mathsf{a}_{j+1,j} \mid j \in J\},\ \mathsf{B}_{\mathfrak{a}} = \{\mathsf{e}_i \mid i \in I\},\ \mathsf{B}_{\mathfrak{c}} = \{\mathsf{c}_j \mid j \in J\}$.

For $i \in I$, let $\mathsf{L}(i) = \mathbb{k} \cdot \mathsf{v}_i$ with $|\mathsf{v}_i| = \bar{0}$ and the action $\mathsf{e}_i\mathsf{v}_i = \mathsf{v}_i$, $\mathsf{b}\mathsf{v}_i = 0$ for all $\mathsf{b} \in \mathsf{B} \setminus \{\mathsf{e}_i\}$. This makes $\mathsf{L}(i)$ a Z-supermodule, and, up to isomorphism, $\{\mathsf{L}(i) \mid i \in I\}$ is a complete set of irreducible Z-supermodules (recall that we allow for odd isomorphisms). The standard modules $\Delta(i)$ have similarly explicit description: $\Delta(0) = \mathsf{L}(0)$, and for $i > 0$, $\Delta(i)$ has basis $\{\mathsf{v}_i, \mathsf{w}_{i-1}\}$ with $|\mathsf{v}_i| = \bar{0}$, $|\mathsf{w}_{i-1}| = \bar{1}$

and the only non-zero actions of elements of B are: $\mathsf{e}_i\mathsf{v}_i = \mathsf{v}_i$, $\mathsf{e}_{i-1}\mathsf{w}_{i-1} = \mathsf{w}_{i-1}$, $\mathsf{a}_{i-1,i}\mathsf{v}_i = \mathsf{w}_{i-1}$. Dually, $\nabla(0) = \mathsf{L}(0)$, and for $i > 0$, $\nabla(i)$ has basis $\{\mathsf{v}_i^*, \mathsf{w}_{i-1}^*\}$ with $|\mathsf{v}_i^*| = \bar{0}$, $|\mathsf{w}_{i-1}^*| = \bar{1}$ and the only non-zero actions of elements of B are: $\mathsf{e}_i\mathsf{v}_i^* = \mathsf{v}_i^*$, $\mathsf{e}_{i-1}\mathsf{w}_{i-1}^* = \mathsf{w}_{i-1}^*$, $\mathsf{a}_{i,i-1}\mathsf{w}_{i-1}^* = \mathsf{v}_i^*$.

Moreover recalling Proposition 2.8, we may describe the partial tilting modules as well. We have $\mathsf{T}(0) = \mathsf{L}(0) = \Delta(0) = \nabla(0)$ and we claim that $\mathsf{T}(i) = \Pi\mathsf{Ze}_{i-1}$ for $i > 0$. We will prove the following

Lemma 5.3. *For $i > 0$, there is an even isomorphism of Z supermodules $\mathsf{Ze}_{i-1} \simeq \Pi\mathsf{T}(i)$.*

Proof. Observe that Ze_0 has basis $\{\mathsf{e}_0, \mathsf{a}_{1,0}, \mathsf{c}_0\}$; and, for $i > 1$, Ze_{i-1} has basis $\{\mathsf{e}_{i-1}, \mathsf{a}_{i-2,i-1}, \mathsf{a}_{i,i-1}\mathsf{c}_{i-1}\}$. With these bases it is easy to see that for all $i \geq 1$, Ze_{i-1} has a standard filtration with $\Pi\Delta(i) \subseteq \Pi\mathsf{T}(i)$ and $\Pi\mathsf{T}(i)/\Pi\Delta(i) \simeq \Delta(i-1)$; and a costandard filtration with $\nabla(i-1) \subseteq \Pi\mathsf{T}(i)$ and $\Pi\mathsf{T}(i)/\nabla(i-1) \simeq \Pi\nabla(i)$. For example, $\Pi\Delta(i) = \text{span}_{\mathbb{k}}\{a_{i,i-1}, c_{i-1}\}$ for $i > 1$, and this easily checked to be a submodule. The other subquotients are all found similarly.

So, Ze_{i-1} is a tilting module, and since it is indecomposable, it is a partial tilting module. Since the largest j such that $(\mathsf{Ze}_{i-1} : \Delta(j)) > 0$ is $j = i$, it follows that Ze_{i-1} is isomorphic to $\mathsf{T}(i)$. However, $\mathsf{T}(i)$ has $\Delta(i)$ as a submodule, not $\Pi\Delta(i)$. Hence $\mathsf{Ze}_{i-1} \simeq \Pi\mathsf{T}(i)$. □

It will actually be more convenient for us to work with the tilting modules $\Pi\mathsf{T}(i)$. Thus $\Pi\mathsf{T}(0) = \Pi\mathsf{L}(0)$ is 1-dimensional with basis $\{\mathsf{v}_0\}$ where $|\mathsf{v}_0| = \bar{1}$, and for $i > 0$, we fix the basis for $\Pi\mathsf{T}(i)$ from the proof of Lemma 5.3.

We have a full tilting module

$$\mathsf{T} := \bigoplus_{i \in I} \Pi \mathsf{T}(i) \tag{5.4}$$

and the Ringel dual algebra $\mathsf{Z}' := \mathrm{End}_{\mathsf{Z}}(\mathsf{T})^{\mathrm{op}}$. For any $i \in I$, let $\iota_i : \Pi\mathsf{T}(i) \to \mathsf{T}$ and $\pi_i : \mathsf{T} \to \Pi\mathsf{T}(i)$ be the natural embedding and projection. Observe that for all i, ι_i and π_i are even homomorphisms. We have the right multiplication maps $\rho_{\mathsf{c}_i} : \mathsf{Ze}_i \to \mathsf{Ze}_i$, $\rho_{\mathsf{a}_{ij}} : \mathsf{Ze}_i \to \mathsf{Ze}_j$, which are odd homomorphisms. Let $f : \Pi\mathsf{T}(0) = \Pi\mathsf{L}(0) \hookrightarrow \Pi\mathsf{T}(1) = \mathsf{Ze}_0$ be the embedding given by $\mathsf{v}_0 \mapsto \mathsf{c}_0$, and let $g : \Pi\mathsf{T}(1) = \mathsf{Ze}_0 \twoheadrightarrow \Pi\mathsf{T}(0) = \Pi\mathsf{L}(0)$ be the surjection such that $\mathsf{e}_0 \mapsto \mathsf{v}_0$. Observe that both f and g are odd homomorphisms. Define the following elements of Z':

- $\mathsf{e}'_i := \pi_{\ell-i}$ for all $i \in I$;
- $\mathsf{c}'_i := \iota_{\ell-i} \circ \rho_{\mathsf{c}_{\ell-i-1}} \circ \pi_{\ell-i}$ for all $i \in J$;
- $\mathsf{a}'_{i+1,i} := \iota_{\ell-i} \circ \rho_{\mathsf{a}_{\ell-i-2,\ell-i-1}} \circ \pi_{\ell-i-1}$ and $\mathsf{a}'_{i,i+1} := \iota_{\ell-i-1} \circ \rho_{\mathsf{a}_{\ell-i-1,\ell-i-2}} \circ \pi_{\ell-i}$, for all $i = 0, \dots, \ell-2$;
- $\mathsf{a}'_{\ell,\ell-1} := \iota_1 \circ f \circ \pi_0$ and $\mathsf{a}'_{\ell-1,\ell} := \iota_0 \circ g \circ \pi_1$.

Notice that for all admissible i, the e'_i and c'_i are even homomorphisms, and the $\mathsf{a}_{i\pm1,i}$ are odd.

Observe that Z' has basis given by

$$\{\mathsf{e}'_i \mid i \in I\} \sqcup \{\mathsf{a}'_{i,i+1}, \mathsf{a}'_{i+1,i}, \mathsf{c}'_i \mid i \in J\}$$

The following is now easy to check:

Lemma 5.5. *Mapping* $\mathsf{e}_i \mapsto \mathsf{e}'_i$, $\mathsf{a}_{i,j} \mapsto \mathsf{a}'_{i,j}$, $\mathsf{c}_i \mapsto \mathsf{c}'_i$ *is an isomorphism of superalgebras* $\mathsf{Z} \xrightarrow{\sim} \mathsf{Z}'$. *In other words,* Z *is Ringel self-dual.*

We now aim to describe the structure of T as a right Z'-module, which will be fundamental in proving the Ringel self-duality of the extended zigzag Schur algebra, see § 5.3.

We use the isomorphism $\mathsf{Z} \xrightarrow{\sim} \mathsf{Z}'$ of Lemma 5.5 to transport the heredity data $I, \mathsf{X}, \mathsf{Y}$ from Z onto a heredity data $I', \mathsf{X}', \mathsf{Y}'$ for Z' so that $I' = I$ *with the same order*, and

$$\mathsf{X}'(i) := \begin{cases} \{\mathsf{e}'_i, \mathsf{a}'_{i-1,i}\} & \text{if } i > 0, \\ \{\mathsf{e}'_0\} & \text{if } i = 0, \end{cases} \qquad \mathsf{Y}'(i) := \begin{cases} \{\mathsf{e}'_i, \mathsf{a}'_{i,i-1}\} & \text{if } i > 0, \\ \{\mathsf{e}'_0\} & \text{if } i = 0. \end{cases}$$

(We point out that this heredity structure on the Ringel dual is different from the one coming from [17, A4] where the partial order on I is opposite to the original one.) With this hereditary data, we have the right modules $\mathsf{L}'(i), \Delta'(i), \nabla'(i)$ and $\mathsf{T}'(i)$. For example, $\mathsf{T}'(0) \simeq \mathsf{L}'(0)$ (with e'_j acting as $\delta_{ij}\,\mathrm{id}$) and $\mathsf{T}'(i) \simeq \Pi \mathsf{e}'_{i-1}\mathsf{Z}'$ for $i > 0$. Then it is easy to check that, as a right Z'-module, T decomposes as follows:

$$\mathsf{T}_{\mathsf{Z}'} = \bigoplus_{i \in I} \Pi \mathsf{T}'(i),$$

where the summands are defined explicitly as follows:

- $\Pi \mathsf{T}'(0) = \Bbbk \cdot \mathsf{a}_{\ell,\ell-1} \subseteq \mathsf{Z}\mathsf{e}_{\ell-1} = \Pi \mathsf{T}(\ell) \subseteq \mathsf{T}$;
- $\Pi \mathsf{T}'(\ell) = \mathrm{span}_{\Bbbk}(\mathsf{v}_0, \mathsf{e}_0, \mathsf{c}_0, \mathsf{a}_{0,1}) \subseteq \Pi \mathsf{T}(0) \oplus \Pi \mathsf{T}(1) \oplus \Pi \mathsf{T}(2) \subseteq \mathsf{T}$ (dropping $\mathsf{a}_{0,1}$ if $\ell = 1$);

- for for $i \notin 0, \ell$, we set $\Pi\mathsf{T}'(i) = \mathrm{span}_{\Bbbk}(\mathsf{e}_{\ell-i}, \mathsf{a}_{\ell-i,\ell-i-1}, \mathsf{a}_{\ell-i,\ell-i+1}, \mathsf{c}_{\ell-i}) \subseteq \Pi\mathsf{T}(\ell-i+1) \oplus \Pi\mathsf{T}(\ell-i) \oplus \Pi\mathsf{T}(\ell-i+2) \subseteq \mathsf{T}$ (dropping $\mathsf{a}_{\ell-i,\ell-i+1}$ if $i = 1$).

Obviously T is an $(\mathsf{Z}, \mathsf{Z}')$ bimodule. Recall the subalgebra $\mathfrak{a} = \mathrm{span}(B_{\mathfrak{a}}) = \mathrm{span}_{\Bbbk}(\mathsf{e}_i \mid i \in I) \subseteq \mathsf{Z}_{\bar{0}}$ and the analogous subalgebra $\mathfrak{a}' = \mathrm{span}_{\Bbbk}(\mathsf{e}'_i \mid i \in I) \subseteq \mathsf{Z}'_{\bar{0}}$. We have $\Bbbk$-module decomposition $\mathsf{T}_{\bar{0}} = \mathsf{T}_{\mathfrak{a}} \oplus \mathsf{T}_{\mathfrak{c}}$ where $\mathsf{T}_{\mathfrak{a}} = \mathrm{span}_{\Bbbk}(\mathsf{e}_j \mid j \in J)$ and $\mathsf{T}_{\mathfrak{c}} = \mathrm{span}_{\Bbbk}(\mathsf{c}_j \mid j \in J)$, making T into a calibrated $\Bbbk$-supermodule. Then it is clear from the explicit construction above that

$$\mathfrak{a} \cdot (\mathsf{T}_{\mathfrak{a}}) \cdot \mathfrak{a}' \subseteq \mathsf{T}_{\mathfrak{a}}. \tag{5.6}$$

And so, $\mathsf{T}_{\mathfrak{a}}$ is both a left $(\mathsf{Z}, \mathfrak{a})$-calibrated supermodule and a right $(\mathsf{Z}', \mathfrak{a}')$-calibrated supermodule. For any $i \in I$, we make $\Pi\mathsf{T}(i)$ into an $(\mathsf{Z}, \mathfrak{a})$-calibrated supermodule by setting

$$\Pi\mathsf{T}(i)_{\mathfrak{a}} := \Pi\mathsf{T}(i) \cap \mathsf{T}_{\mathfrak{a}} \quad \text{and} \quad \Pi\mathsf{T}(i)_{\mathfrak{c}} := \Pi\mathsf{T}(i) \cap \mathsf{T}_{\mathfrak{c}}.$$

Similarly, we make $\Pi\mathsf{T}'(i)$ into an $(\mathsf{Z}', \mathfrak{a}')$-calibrated supermodule by setting

$$\Pi\mathsf{T}'(i)_{\mathfrak{a}} := \Pi\mathsf{T}'(i) \cap \mathsf{T}_{\mathfrak{a}} \quad \text{and} \quad \Pi\mathsf{T}'(i)_{\mathfrak{c}} := \Pi\mathsf{T}'(i) \cap \mathsf{T}_{\mathfrak{c}}.$$

5.2. The Extended Zigzag Schur Algebra

Throughout the rest of the chapter, fix $d \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{>0}$ with $d \leq n$. In this section we will need to work over R and $\mathbb{F}$ at different times, so we will be specific about the base ring, using subscripts. First we work integrally, so we begin with the algebra Z_R over R and its tilting module T_R constructed in § 5.1.

Applying the construction of § 3.1.1 to the algebra Z_R (and extending scalars to $\mathbb{F}$) we get the *extended zigzag Schur Algebra* $T^{\mathsf{Z}}(n,d) := \mathbb{F} \otimes_R T^{\mathsf{Z}}(n,d)_R$.

As per § 3.3, $T^{\mathsf{Z}}(n,d)$ is based quasi-hereditary. We will now construct a full tilting module for $T^{\mathsf{Z}}(n,d)$ using the tools of § 3.4.

Recall that T_R is also a right module over the Ringel dual Z'_R, and recall the constructions of § 3.4.4. In particular, we have the left $M_n(\mathsf{Z}_R)$-module structure on $\mathrm{Col}_n(\mathsf{T}_R)$ and a right $M_n(\mathsf{Z}'_R)$-module structure on $\mathrm{Row}_n(\mathsf{T}_R)$. As in § 3.4.4, we make these into calibrated R-supermodules by setting. $\mathrm{Col}_n(\mathsf{T}_R)_{\mathfrak{a}} := \mathrm{Col}_n(\mathsf{T}_{R,\mathfrak{a}})$ and $\mathrm{Row}_n(\mathsf{T}_R)_{\mathfrak{a}} := \mathrm{Row}_n(\mathsf{T}_{R,\mathfrak{a}})$. In fact, by (5.6), we have that $M_n(\mathfrak{a})\,\mathrm{Col}_n(\mathsf{T}_R)_{\mathfrak{a}} \subseteq \mathrm{Col}_n(\mathsf{T}_R)_{\mathfrak{a}}$ and $\mathrm{Row}_n(\mathsf{T}_R)_{\mathfrak{a}} M_n(\mathfrak{a}) \subseteq \mathrm{Row}_n(\mathsf{T}_R)_{\mathfrak{a}}$. Thus $\mathrm{Col}_n(\mathsf{T}_R)$ is a left $(M_n(\mathsf{Z}_R), M_n(\mathfrak{a}_R))$-calibrated supermodule, and $\mathrm{Row}_n(\mathsf{T}_R)$ is a right $(M_n(\mathsf{Z}'_R), M_n(\mathfrak{a}'_R))$-calibrated supermodule.

So by Lemma 3.48 (and its right module analogue), the modified divided power $\tilde{\Gamma}^d\,\mathrm{Col}_n(\mathsf{T}_R)$ is a left module over $T^{\mathsf{Z}}(n,d)_R$ and the modified divided power $\tilde{\Gamma}^d\,\mathrm{Row}_n(\mathsf{T}_R)$ is a right module over $T^{\mathsf{Z}'}(n,d)_R$. Similarly, for every $i \in I$, we have left $T^{\mathsf{Z}}(n,d)_R$-modules $\tilde{\Gamma}^d\,\mathrm{Col}_n(\Pi\mathsf{T}(i)_R)$ and right $T^{\mathsf{Z}'}(n,d)_R$-modules $\mathrm{Row}_n(\Pi\mathsf{T}'(i)_R)$. Extending scalars, we have a left module

$$\mathscr{T}_i^d := \mathbb{F} \otimes_R \tilde{\Gamma}^d\,\mathrm{Col}_n(\Pi\mathsf{T}(i)_R)$$

over $T^{\mathsf{Z}}(n,d)$. These modules will be vital in constructing a *left* full tilting module for $T^{\mathsf{Z}}(n,d)$. However, all subsequent left-sided results about column modules have right-sided analogues for row modules, which are proven in an identical way. We will not prove these results, but we will mention them when they are needed.

Recall that for $d \leq n$, the algebra $T^{\mathsf{Z}}(n,d)$ is quasi-hereditary with respect to the poset $\Lambda_+^I(n,d)$ with partial order $\leq_I$, so it has its own standard modules

$\{\Delta(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \Lambda^I_+(n,d)\}$, costandard modules $\{\nabla(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \Lambda^I_+(n,d)\}$ and indecomposable tilting modules $\{\mathrm{T}(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \Lambda^I_+(n,d)\}$. Moreover, by [11, Proposition 6.20], the anti-involution τ on Z extends to the anti-involution $\tau_{n,d} : T^{\mathsf{Z}}(n,d) \to T^{\mathsf{Z}}(n,d),\ \eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} \mapsto \eta^{\tau(\boldsymbol{b})}_{\boldsymbol{s},\boldsymbol{r}}$ where for $\boldsymbol{b} = \mathsf{b}_1 \cdots \mathsf{b}_d \in \mathsf{B}^d$ we denote $\tau(\boldsymbol{b}) := \tau(\mathsf{b}_1)\cdots\tau(\mathsf{b}_d)$. We then have for all $\boldsymbol{\lambda}$:

$$\Delta(\boldsymbol{\lambda})^{\tau_{n,d}} \simeq \nabla(\boldsymbol{\lambda}) \tag{5.7}$$

Since $\tau_{n,d}(\eta_{\boldsymbol{\mu}}) = \eta_{\boldsymbol{\mu}}$ for all $\boldsymbol{\mu} \in \Lambda^I(n,d)$, we deduce:

Lemma 5.8. *For all $\boldsymbol{\lambda} \in \Lambda^I_+(n,d)$, we have* $\mathrm{ch}\,\Delta(\boldsymbol{\lambda}) = \mathrm{ch}\,\nabla(\boldsymbol{\lambda})$.

Proposition 5.9. *The left $T^{\mathsf{Z}}(n,d)$-module $\mathscr{T}_i^d$ is tilting and has highest weight $\boldsymbol{\iota}_i(1^d)$ (with respect to $\leq_I$).*

Proof. Suppose first that $i = 0$. Since $\boldsymbol{\iota}_0(1^d)$ is minimal in $\Lambda^+_I(n,d)$ it follows that $\mathrm{T}(\boldsymbol{\iota}_0(1^d)) \simeq \Delta(\boldsymbol{\iota}_0(1^d)) \simeq L(\boldsymbol{\iota}_0(1^d))$. Using the assumption $d \leq n$, we note that $\mathscr{T}_0^d$ is a highest weight module of weight $\boldsymbol{\iota}_0(1^d)$ (see Definition 2.15), and thus using Proposition 2.17 we deduce that

$$\mathscr{T}_0^d \cong \mathrm{T}(\boldsymbol{\iota}_0((1^d))) \simeq \Delta(\boldsymbol{\iota}_0((1^d))). \tag{5.10}$$

We point out the parity of isomorphism $\mathscr{T}_0^d \xrightarrow{\sim} \mathrm{T}(\boldsymbol{\iota}_0((1^d)))$ depends on the parity of d. Indeed, the highest weight vector in $\mathscr{T}_0^d$ is $\mathsf{v}_0^{\otimes d}$, whose parity is the same as the parity of d.

Let now $i > 0$. Since $\Pi\mathsf{T}(i)_R = \mathsf{Z}_R\mathsf{e}_{i-1}$, we have $\mathrm{Col}_n(\Pi\mathsf{T}(i)_R) \simeq M_n(\mathsf{Z}_R)\xi^{\mathsf{e}_{i-1}}_{1,1}$, and so by Lemma 3.52, we have

$$\mathscr{T}_i^d \simeq T^{\mathsf{Z}}(n,d)\eta^{\mathsf{e}^d_{i-1}}_{1^d,1^d} = T^{\mathsf{Z}}(n,d)\eta_{\boldsymbol{\iota}_{i-1}(d)}. \tag{5.11}$$

In particular, $\mathscr{T}_i^d$ is projective, and thus has a standard filtration. To prove that $\mathscr{T}_i^d$ also has a costandard filtration, it suffices to show that it is $\tau_{n,d}$-self-dual, or equivalently possesses a non-degenerate $\tau_{n,d}$-contravariant bilinear form.

To construct this form we work over R, and apply the tools of § 3.4.2. Recall that for $i > 1$, we have $\Pi\mathsf{T}(i)_R = \mathsf{Z}_R\mathsf{e}_{i-1}$ has basis $B^{\Pi\mathsf{T}(i)} := \{\mathsf{e}_{i-1}, \mathsf{c}_{i-1}, \mathsf{a}_{i-2,i-1}, \mathsf{a}_{i,i-1}\}$. Consider the bilinear form $(\cdot,\cdot)$ on $\Pi\mathsf{T}(i)_R$ such that

$$(\mathsf{e}_{i-1}, \mathsf{c}_{i-1}) = -(\mathsf{c}_{i-1}, \mathsf{e}_{i-1}) = (\mathsf{a}_{i-2,i-1}, \mathsf{a}_{i-2,i-1}) = (\mathsf{a}_{i,i-1}, \mathsf{a}_{i,i-1}) = 1$$

and all the other pairings of basis elements are 0.

Note that that this form is non-degenerate and τ-contravariant (and superskewsymmetric). Extending this form in the obvious way to $\mathrm{Col}_n(\Pi\mathsf{T}(i)_R)$ results in a non-degenerate $\tau_{n,1}$-contravariant form again denoted $(\cdot,\cdot)$.

Notice that setting $B_{\mathfrak{a}}^{\Pi\mathsf{T}(i)} = \{\mathsf{e}_{i-1}\}$, $B_{\mathfrak{c}}^{\Pi\mathsf{T}(i)} = \{\mathsf{c}_{i-1}\}$, and $B_{\bar{1}}^{\Pi\mathsf{T}(i)} = \{\mathsf{a}_{i-2,i-1}, \mathsf{a}_{i,i-1}\}$, the basis $B^{\Pi\mathsf{T}(i)} = B_{\mathfrak{a}}^{\Pi\mathsf{T}(i)} \sqcup B_{\mathfrak{c}}^{\Pi\mathsf{T}(i)} \sqcup B_{\bar{1}}^{\Pi\mathsf{T}(i)}$ satisfies the hypotheses of Lemma 3.51 with respect to the form $(\cdot,\cdot)$.

Denote by $v_r^{\mathsf{b}} \in \mathrm{Col}_n(\Pi\mathsf{T}(i)_R)$ the column vector with $\mathsf{b} \in \Pi\mathsf{T}(i)_R$ in the rth position and 0s elsewhere. Then the set $\{v_r^{\mathsf{b}} \mid r \in [n], \mathsf{b} \in B^{\Pi\mathsf{T}(i)}\}$ is a basis for $\mathrm{Col}_n(\Pi\mathsf{T}(i)_R)$. It is clear that this also satisfies the hypotheses of Lemma 3.51 with respect to the extended form $(\cdot,\cdot)$.

So applying Lemma 3.51, we see that $(\cdot,\cdot)_\sim$ is a superskewsymmetric, non-degenerate, $\tau_{n,d}$-contravariant form on $\tilde{\Gamma}^d \mathrm{Col}_n(\Pi\mathsf{T}(i)_R)$. Extending the scalars to $\mathbb{F}$, we deduce that, $\mathscr{T}_i^d$ is $\tau_{n,d}$-self-dual. In particular, this proves that $\mathscr{T}_i^d$ is a tilting module.

To see that the highest weight of $\mathscr{T}_i^d$ is as claimed, observe that the highest weight vector $\mathsf{a}_{i,i-1} \in \Pi\mathsf{T}(i)_R$ is odd, and so the vector $v_1^{\mathsf{a}_{i,i-1}} * \cdots * v_d^{\mathsf{a}_{i,i-1}} \in$

$\tilde{\Gamma}^d \operatorname{Col}_n(\Pi\mathsf{T}(i)_R)$ of weight $\boldsymbol{\iota}_i(1^d)$ has the highest left weight possible among the weight vectors of $\tilde{\Gamma}^d \operatorname{Col}_n(\Pi\mathsf{T}(i)_R)$. This completes the $i > 1$ case.

The case $i = 1$ is similar to the case $i > 1$ but $\Pi\mathsf{T}(1)_R = \mathsf{Z}_R\mathsf{e}_0$ has basis $B^{\Pi\mathsf{T}(1)} := \{\mathsf{e}_0, \mathsf{c}_0, \mathsf{a}_{1,0}\}$, and we use the form such that

$$(\mathsf{e}_0, \mathsf{c}_0) = -(\mathsf{c}_0, \mathsf{e}_0) = (\mathsf{a}_{1,0}, \mathsf{a}_{1,0}) = 1$$

are the only non-trivial pairings of basis elements. From here, the proof is identical to the above case. □

Now consider $M_n(\mathsf{T}_R)$ as an $(M_n(\mathsf{Z}_R), M_n(\mathsf{Z}'_R))$-bimodule in the obvious way. Taking $M_n(\mathsf{T}_R)_{\mathfrak{a}} := M_n(\mathsf{T}_{R,\mathfrak{a}})$ and $M_n(\mathsf{T}_R)_{\mathfrak{c}} := M_n(\mathsf{T}_{R,\mathfrak{c}})$ we make $M_n(\mathsf{T}_R)$ into both a left $(M_n\mathsf{Z}_R, M_n(\mathfrak{a}_R))$-calibrated supermodule and a right $(M_n\mathsf{Z}'_R, M_n(\mathfrak{a}'_R))$-calibrated supermodule. Since the left action of $M_n(\mathsf{Z}_R)$ and the right action of $M_n(\mathsf{Z}'_R)$ commute, in view of Lemma 3.48 (and its right module analogue), the modified divided power $\tilde{\Gamma}^d M_n(\mathsf{T}_R)$ is a $(T^{\mathsf{Z}}(n,d)_R, T^{\mathsf{Z}'}(n,d)_R)$-bimodule. We now extend the scalars from R to $\mathbb{F}$ to get the $(T^{\mathsf{Z}}(n,d), T^{\mathsf{Z}'}(n,d))$-bimodule

$$\mathscr{T} := \mathbb{F} \otimes_R \tilde{\Gamma}^d M_n(\mathsf{T}_R).$$

The rest of this section is dedicated to proving that $\mathscr{T}$ is a left full tilting module for $T^{\mathsf{Z}}(n,d)$. However, we recall that all of the left-sided results have right-sided analogues, so our proofs will also serve to show that $\mathscr{T}$ is a right full tilting module for $T^{\mathsf{Z}'}(n,d)$.

For each composition $\mu \in \Lambda(n,d)$ define $\mathscr{T}_i^{\mu} := \mathscr{T}_i^{\mu_1} \otimes \cdots \otimes \mathscr{T}_i^{\mu_n}$. Furthermore, for each multicomposition $\boldsymbol{\mu} = (\mu^{(0)}, \dots, \mu^{(\ell)}) \in \Lambda^I(n,d)$ define $\mathscr{T}^{\boldsymbol{\mu}} := \mathscr{T}_0^{\mu^{(0)}} \otimes \cdots \otimes \mathscr{T}_\ell^{\mu^{(\ell)}}$.

Recall that $\mathsf{T}_R \simeq \bigoplus_{i\in I} \Pi\mathsf{T}(i)_R$. So, as left modules over $T^{\mathsf{Z}}(n,1)_R = M_n(\mathsf{Z}_R)$, we have

$$M_n(\mathsf{T}_R) \simeq \mathrm{Col}_n(\mathsf{T}_R)^{\oplus n} \simeq \bigoplus_{i\in I} \mathrm{Col}_n(\Pi\mathsf{T}(i)_R)^{\oplus n}.$$

Now, using Lemma 3.50 and the decomposition 5.4, and extending scalars, we have as left $T^{\mathsf{Z}}(n,d)$-modules:

$$\mathscr{T} \simeq \bigoplus_{\boldsymbol{\mu}\in\Lambda^I(n,d)} \mathscr{T}^{\boldsymbol{\mu}}. \tag{5.12}$$

For $\boldsymbol{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell)}) \in \Lambda^I_+(n,d)$, we define the *conjugate multipartition*

$$\boldsymbol{\lambda}' := ((\lambda^{(0)})', \dots, (\lambda^{(\ell)})') \in \Lambda^I_+(n,d). \tag{5.13}$$

Proposition 5.14. *As a left $T^{\mathsf{Z}}(n,d)$-module, $\mathscr{T}$ is a full tilting module.*

Proof. Note that each $\mathscr{T}^{\boldsymbol{\mu}}$ is tilting by Proposition 5.9 and Theorem 4.1. So $\mathscr{T}$ is tilting by (5.12). To show that $\mathscr{T}$ is full tilting, it suffices for each $\boldsymbol{\lambda} \in \Lambda^+_I(n,d)$ to find a summand $\mathscr{T}^{\boldsymbol{\mu}}$ in (5.12) which has highest weight $\boldsymbol{\lambda}$. Fix $\boldsymbol{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell)}) \in \Lambda^+_I(n,d)$ and take $\boldsymbol{\mu} = \boldsymbol{\lambda}'$. By Proposition 5.9 again, the highest weight of $\mathscr{T}_i^s$ is $\boldsymbol{\iota}_i(1^s)$ for each $s \in \mathbb{Z}_{>0}$. So

$$\sum_{i=0}^{\ell}\sum_{r=1}^{n} \boldsymbol{\iota}_i(1^{\mu_r^{(i)}}) = \sum_{i=0}^{\ell} \boldsymbol{\iota}_i(\lambda^{(i)}) = \boldsymbol{\lambda}$$

is the highest weight of $\mathscr{T}^{\boldsymbol{\mu}}$. □

Corollary 5.15. *As a left $T^{\mathsf{Z}}(n,d)$-module and as a right $T^{\mathsf{Z}'}(n,d)$-module, $\mathscr{T}$ is faithful.*

Proof. As a left $T^{\mathsf{Z}}(n,d)$-module, $\mathscr{T}$ is faithful since it is full tilting by Proposition 5.14 (a full tilting module is faithful for example by [22, Lemma 6]).

The second statement follows similarly from the right module analogue of that proposition. $\square$

We point out the right-sided formulation in Corollary 5.15 because we will need it in the next section.

5.3. Extended Zigzag Schur Algebra is Ringel Self-Dual

In view of Corollary 5.15, we have an embedding of $T^{\mathsf{Z}'}(n,d)$ into $\mathrm{End}_{T^{\mathsf{Z}}(n,d)}(\mathscr{T})^{\mathrm{op}}$. To prove that this embedding is an isomorphism, we now count the dimension of $\mathrm{End}_{T^{\mathsf{Z}}(n,d)}(\mathscr{T})$.

Recalling (3.13), for $\boldsymbol{\lambda} \in \Lambda_+^I(n,d)$ and $\boldsymbol{\mu} \in \Lambda^I(n,d)$, let

$$k_{\boldsymbol{\lambda},\boldsymbol{\mu}} := |\,\mathrm{Std}^{\mathsf{X}}(\boldsymbol{\lambda},\boldsymbol{\mu})|.$$

By (3.24), (3.25) and Lemma 5.8, we have

$$k_{\boldsymbol{\lambda},\boldsymbol{\mu}} = \dim \eta_{\boldsymbol{\mu}}\Delta(\boldsymbol{\lambda}) = \dim \eta_{\boldsymbol{\mu}}\nabla(\boldsymbol{\lambda}). \tag{5.16}$$

Let $i \in I$. If $i \neq 0$, we define

$$\boldsymbol{\beta}_i(d,s) := \boldsymbol{\iota}_{i-1}((s)) + \boldsymbol{\iota}_i((1^{d-s})) \in \Lambda_+^I(n,d)$$

for all $0 \leq s \leq d$ We also define

$$\boldsymbol{\beta}_0(d,0) := \boldsymbol{\iota}_0((1^d)).$$

We define by $\Xi_{d,i}$ to be the set of all $\boldsymbol{\beta}_i(d,s)$'s, i.e.

$$\Xi_{d,i} := \begin{cases} \{\boldsymbol{\beta}_i(d,s) \mid 0 \le s \le d\} & \text{if } i \ne 0, \\ \{\boldsymbol{\beta}_0(d,0)\} & \text{if } i = 0. \end{cases}$$

Lemma 5.17. *Let $\boldsymbol{\beta} \in \Lambda_+^I(n,d)$ and $i \in I$. Then*

$$(\mathscr{T}_i^d : \Delta(\boldsymbol{\beta})) = \begin{cases} 1 & \textit{if } \boldsymbol{\beta} \in \Xi_{d,i}, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. By (5.10), we have $\mathscr{T}_0^d \cong \Delta(\boldsymbol{\iota}_0((1^d)))$, so we may assume that $i \ne 0$. Then by (5.11), we have $\mathscr{T}_i^d \simeq T(n,d)\eta_{\boldsymbol{\iota}_{i-1}((d))}$. Now, using (2.6) and (5.16), we get

$$\begin{aligned} (\mathscr{T}_i^d : \Delta(\boldsymbol{\beta})) &= \dim \operatorname{Hom}_{T(n,d)}(\mathscr{T}_i^d, \nabla(\boldsymbol{\beta})) \\ &= \dim \operatorname{Hom}_{T(n,d)}(T(n,d)\eta_{\boldsymbol{\iota}_{i-1}((d))}, \nabla(\boldsymbol{\beta})) \\ &= \dim \eta_{\boldsymbol{\iota}_{i-1}((d))}\nabla(\boldsymbol{\beta}) \\ &= k_{\boldsymbol{\beta},\boldsymbol{\iota}_{i-1}((d))}. \end{aligned}$$

It remains to observe that $k_{\boldsymbol{\beta},\boldsymbol{\iota}_{i-1}((d))} = 1$ if $\boldsymbol{\beta} = \boldsymbol{\beta}_i(d,s)$ for some $0 \le s \le d$ and $k_{\boldsymbol{\beta},\boldsymbol{\iota}_{i-1}((d))} = 0$ otherwise. □

Let $0 \le r \le d$. Recalling (3.34), our next goal is to compute the Littlewood-Richardson coefficient $c^{\boldsymbol{\lambda}}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ for all $\boldsymbol{\lambda} \in \Lambda_+^I(n,d)$, $\boldsymbol{\alpha} \in \Lambda_+^I(n,d-r)$ and $\boldsymbol{\beta} \in \Xi_{r,i}$. Let $i \in I$ and $\boldsymbol{\beta} = \boldsymbol{\beta}_i(r,s) \in \Xi_{r,i}$, in particular, $0 \le s \le r$, and $s = 0$ if $i = 0$. We define $\Omega^{\boldsymbol{\lambda}}_{\boldsymbol{\beta}}$ to be the set of all $\boldsymbol{\alpha} = (\alpha^{(0)}, \dots, \alpha^{(\ell)}) \in \Lambda_+^I(n,d-r)$ such that $\alpha^{(j)} = \lambda^{(j)}$ for all $j \notin \{i-1, i\}$, $[\alpha^{(i-1)}]$ is obtained from $[\lambda^{(i-1)}]$ by removing s nodes from distinct columns, and $[\alpha^{(i)}]$ is obtained from $[\lambda^{(i)}]$ by removing $r-s$ nodes from distinct rows (if $i=0$, then the condition on $[\alpha^{(i-1)}]$ should be dropped).

Lemma 5.18. *Let* $0 \le r \le d$, $i \in I$, $\boldsymbol{\lambda} \in \Lambda_+^I(n,d)$, $\boldsymbol{\alpha} \in \Lambda_+^I(n,d-r)$ *and* $\boldsymbol{\beta} = \Xi_{r,i}$. *Then*

$$c^{\boldsymbol{\lambda}}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \begin{cases} 1 & \text{if } \boldsymbol{\alpha} \in \Omega^{\boldsymbol{\lambda}}_{\boldsymbol{\beta}} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Recall these Littlewood-Richardsoon coefficients from (3.34). The result is an immediate consequence of the Littlewood-Richardson rule. □

For each $\boldsymbol{\mu} \in \Lambda^I(n,d)$, define $\overleftarrow{\boldsymbol{\mu}} = (\overleftarrow{\mu}^{(0)}, \dots, \overleftarrow{\mu}^{(\ell)}) \in \Lambda^I(n,d)$ by setting $\overleftarrow{\mu}^{(i)} := \mu^{(\ell-i)}$ for all $i \in I$. Recall the conjugate multipartition from (5.13).

Proposition 5.19. *Let* $\boldsymbol{\lambda} \in \Lambda_+^I(n,d)$ *and* $\boldsymbol{\mu} \in \Lambda^I(n,d)$. *Then* $(\mathscr{T}^{\boldsymbol{\mu}} : \Delta(\boldsymbol{\lambda})) = k_{\overleftarrow{\boldsymbol{\lambda}}',\overleftarrow{\boldsymbol{\mu}}}$.

Proof. We proceed by induction on the number of non-zero parts of $\boldsymbol{\mu}$. To start the induction, we suppose that $\boldsymbol{\mu}$ has only one row, in which case $\mathscr{T}^{\boldsymbol{\mu}} \cong \mathscr{T}_i^d$ for some i, and the result follows from Lemma 5.17. So we may assume that $\boldsymbol{\mu}$ has at least two rows (possibly in different colors).

Now let i be maximal such that $\mu^{(i)} \neq \varnothing$ and pick the largest t such that $\mu_t^{(i)} \neq 0$. Denote $r := \mu_t^{(i)}$. Let $\nu^{(i)}$ be $\mu^{(i)}$ with last non-zero row removed, i.e.:

$$\nu^{(i)} := (\mu_1^{(i)}, \dots, \mu_{t-1}^{(i)}, 0, \dots, 0) \in \Lambda(n, |\mu^{(i)}| - r),$$

and $\nu^{(j)} := \mu^{(j)}$ for all $j \neq i$. Set $\boldsymbol{\nu} := (\nu^{(0)}, \dots, \nu^{(\ell)}) \in \Lambda^I(n, d-r)$. Then by definition, $\mathscr{T}^{\boldsymbol{\mu}} \cong \mathscr{T}^{\boldsymbol{\nu}} \otimes \mathscr{T}_i^r$.

By (5.1), we have $\mathsf{X}(0) = \{\mathsf{e}_0\}$ and $\mathsf{X}(i) = \{\mathsf{e}_i, \mathsf{a}_{i-1,i}\}$ for $i \neq 0$. Recalling § 3.2.1, for $i \neq 0$, we put the total order on $\mathscr{A}_{\mathsf{X}(i)}$ given by: $1^{\mathsf{e}_i} < \cdots < n^{\mathsf{e}_i} < 1^{\mathsf{a}_{i-1,i}} < \cdots < n^{\mathsf{a}_{i-1,i}}$. And we endow $\mathscr{A}_{\mathsf{X}(0)}$ with the order $1^{\mathsf{e}_0} < \cdots < n^{\mathsf{e}_0}$.

By Corollary 4.14, the inductive hypothesis, Lemma 5.17, and Lemma 5.18 we have

$$\begin{aligned}
(\mathscr{T}^{\boldsymbol{\mu}} : \Delta(\boldsymbol{\lambda})) &= (\mathscr{T}^{\boldsymbol{\nu}} \otimes \mathscr{T}_i^r : \Delta(\boldsymbol{\lambda})) \\
&= \sum_{\boldsymbol{\alpha} \in \Lambda_+^I(n,d-r)} \sum_{\boldsymbol{\beta} \in \Lambda_+^I(n,r)} c^{\boldsymbol{\lambda}}_{\boldsymbol{\alpha},\boldsymbol{\beta}} (\mathscr{T}^{\boldsymbol{\nu}} : \Delta(\boldsymbol{\alpha}))(\mathscr{T}_i^r : \Delta(\boldsymbol{\beta})) \\
&= \sum_{\boldsymbol{\alpha} \in \Lambda_+^I(n,d-r)} \sum_{\boldsymbol{\beta} \in \Xi_{r,i}} c^{\boldsymbol{\lambda}}_{\boldsymbol{\alpha},\boldsymbol{\beta}} k_{\overleftarrow{\boldsymbol{\alpha}}', \overleftarrow{\boldsymbol{\nu}}} \\
&= \sum_{\boldsymbol{\beta} \in \Xi_{r,i}} \sum_{\boldsymbol{\alpha} \in \Omega^{\boldsymbol{\lambda}}_{\boldsymbol{\beta}}} k_{\overleftarrow{\boldsymbol{\alpha}}', \overleftarrow{\boldsymbol{\nu}}} \\
&= \sum_{\boldsymbol{\beta} \in \Xi_{r,i}} \sum_{\boldsymbol{\alpha} \in \Omega^{\boldsymbol{\lambda}}_{\boldsymbol{\beta}}} |\operatorname{Std}^{\mathsf{X}}(\overleftarrow{\boldsymbol{\alpha}}', \overleftarrow{\boldsymbol{\nu}})|.
\end{aligned}$$

Since $k_{\overleftarrow{\boldsymbol{\lambda}}', \overleftarrow{\boldsymbol{\mu}}} = |\operatorname{Std}^{\mathsf{X}}(\overleftarrow{\boldsymbol{\lambda}}', \overleftarrow{\boldsymbol{\mu}})|$ it remains to prove that there is a bijection

$$\bigsqcup_{\boldsymbol{\beta} \in \Xi_{r,i}} \bigsqcup_{\boldsymbol{\alpha} \in \Omega^{\boldsymbol{\lambda}}_{\boldsymbol{\beta}}} \operatorname{Std}^{\mathsf{X}}(\overleftarrow{\boldsymbol{\alpha}}', \overleftarrow{\boldsymbol{\nu}}) \xrightarrow{\sim} \operatorname{Std}^{\mathsf{X}}(\overleftarrow{\boldsymbol{\lambda}}', \overleftarrow{\boldsymbol{\mu}}).$$

Suppose $i = 0$. Recall that $\Xi_{r,0} = \{\boldsymbol{\iota}_0(1^r)\}$. Observe that $\Omega^{\boldsymbol{\lambda}}_{\boldsymbol{\iota}_0(1^r)}$ is the set of all $\boldsymbol{\alpha} \in \Lambda_+^I(n,d)$ such that $\boldsymbol{\alpha} = \boldsymbol{\iota}_0(\alpha)$, for $\alpha \in \Lambda_+(n,d)$ with $[\alpha]$ obtained from $[\lambda^{(0)}]$ by removing r nodes from distinct rows. This condition is met if and only if the (ℓ)-component of $[\overleftarrow{\boldsymbol{\alpha}}']$ is obtained from the (ℓ)-component of $[\overleftarrow{\boldsymbol{\lambda}}']$ by removing r nodes from distinct columns. So the bijection

$$\bigsqcup_{\boldsymbol{\beta} \in \Xi_{r,i}} \bigsqcup_{\boldsymbol{\alpha} \in \Omega^{\boldsymbol{\lambda}}_{\boldsymbol{\beta}}} \operatorname{Std}^{\mathsf{X}}(\overleftarrow{\boldsymbol{\alpha}}', \overleftarrow{\boldsymbol{\nu}}) = \bigsqcup_{\boldsymbol{\alpha} \in \Omega^{\boldsymbol{\lambda}}_{\boldsymbol{\iota}_0(1^r)}} \xrightarrow{\sim} \operatorname{Std}^{\mathsf{X}}(\overleftarrow{\boldsymbol{\lambda}}', \overleftarrow{\boldsymbol{\mu}}).$$

follows from the classical argument. We don't repeat it here, since it is an easier version of the argument that follows for the $i > 0$ case. So we move on to the next case.

Now suppose $i > 0$. Let $\boldsymbol{\beta} \in \Xi_{r,i}$, $\boldsymbol{\alpha} \in \Omega^{\boldsymbol{\lambda}}_{\boldsymbol{\beta}}$ and $\boldsymbol{T} \in \mathrm{Std}^{\mathsf{X}}(\overleftarrow{\boldsymbol{\alpha}}', \overleftarrow{\boldsymbol{\nu}})$. By definition, $\boldsymbol{\beta}$ is of the form $\boldsymbol{\beta}_i(r,s)$. Moreover, the Young diagram $[\boldsymbol{\alpha}]$ is obtained by removing s nodes from distinct columns of the $(i-1)$-component $[\boldsymbol{\lambda}]$, and removing $r-s$ nodes from distinct rows of the (i)-component $[\lambda^{(i)}]$ of $[\boldsymbol{\lambda}]$. Therefore $[\overleftarrow{\boldsymbol{\alpha}}']$ is obtained by removing s nodes $N_1, \ldots, N_s$ from distinct rows of the $(\ell - i + 1)$-component of $[\overleftarrow{\boldsymbol{\lambda}}']$ and $r-s$ nodes $M_1, \ldots, M_{r-s}$ from distinct columns of the $(\ell - i)$-component of $[\overleftarrow{\boldsymbol{\lambda}}']$. Now extend $\boldsymbol{T}$ to the tableau $\hat{\boldsymbol{T}} \in \mathrm{Std}^{\mathsf{X}}(\overleftarrow{\boldsymbol{\lambda}}', \overleftarrow{\boldsymbol{\mu}})$ by setting

$$\hat{\boldsymbol{T}}(N_1) = \cdots = \hat{\boldsymbol{T}}(N_s) = t^{\mathsf{a}_{\ell-i,\ell-i+1}} \qquad \text{and} \qquad \hat{\boldsymbol{T}}(M_1) = \cdots = \hat{\boldsymbol{T}}(M_{r-s}) = t^{\mathsf{e}_{\ell-i}}.$$

The tableaux $\hat{\boldsymbol{T}}$ is indeed standard since, by maximality of i and t, we have $\boldsymbol{T}(N) < t^{\mathsf{a}_{\ell-i,\ell-i+1}}$ for all N in the $(\ell - i + 1)$-component of $[\overleftarrow{\boldsymbol{\alpha}}']$ and $\boldsymbol{T}(N) < t^{\mathsf{e}_{\ell-i}}$ for all N in the $(\ell - i)$-component of $[\overleftarrow{\boldsymbol{\alpha}}']$. The map $\boldsymbol{T} \mapsto \hat{\boldsymbol{T}}$ is clearly injective. To see that it is surjective, it suffices to show that for any $\boldsymbol{S} \in \mathrm{Std}^{\mathsf{X}}(\overleftarrow{\boldsymbol{\lambda}}', \overleftarrow{\boldsymbol{\mu}})$ there exists $\boldsymbol{\beta} \in \Xi_{r,i}$ and $\boldsymbol{\alpha} \in \Omega^{\boldsymbol{\lambda}}_{\boldsymbol{\beta}}$ with

$$[\overleftarrow{\boldsymbol{\lambda}}'] \setminus \{N \mid \boldsymbol{S}(N) \in \{t^{\mathsf{e}_{\ell-i}}, t^{\mathsf{a}_{\ell-i,\ell-i+1}}\}\} = [\overleftarrow{\boldsymbol{\alpha}}'].$$

Indeed, there are exactly $\overleftarrow{\mu}_t^{(\ell-i)} = \mu_t^{(i)} = r$ nodes N in the Young diagram $[\overleftarrow{\boldsymbol{\lambda}}']$ such that $\boldsymbol{S}(N) \in \{t^{\mathsf{e}_{\ell-i}}, t^{\mathsf{a}_{\ell-i,\ell-i+1}}\}$. So for some $0 \leq s \leq r$, we can write

$$\{N \in [\overleftarrow{\boldsymbol{\lambda}}'] \mid \boldsymbol{S}(N) = t^{\mathsf{a}_{\ell-i,\ell-i+1}}\} = \{N_1, \dots, N_s\},$$
$$\{N \in [\overleftarrow{\boldsymbol{\lambda}}'] \mid \boldsymbol{S}(N) = t^{\mathsf{e}_{\ell-i}}\} = \{M_1, \dots, M_{r-s}\}.$$

By maximality of i and t, we have that the nodes $N_1, \dots, N_s$ are at the ends of distinct rows of the $(\ell - i + 1)$-component of $[\overleftarrow{\boldsymbol{\lambda}}']$ and the nodes $M_1, \dots, M_{r-s}$ are at the ends of distinct columns of the $(\ell - i)$-component of $[\overleftarrow{\boldsymbol{\lambda}}']$. It remains to note that removing these nodes produces a shape $[\overleftarrow{\boldsymbol{\alpha}}']$ with $\boldsymbol{\alpha} \in \Omega^{\boldsymbol{\lambda}}_{\boldsymbol{\beta}_i(r,s)}$.

□

Theorem 5.20. *Let $d \leq n$. We have* $\mathrm{End}_{T^{\mathsf{Z}}(n,d)}(\mathscr{T})^{op} \cong T^{\mathsf{Z}}(n,d)$*. In particular, $T^{\mathsf{Z}}(n,d)$ is Ringel self-dual.*

Proof. By Corollary 5.15, $T^{\mathsf{Z}}(n,d)$ embeds into $\mathrm{End}_{T^{\mathsf{Z}}(n,d)}(\mathscr{T})^{\mathrm{op}}$. So it suffices to show that $\dim T^{\mathsf{Z}}(n,d) = \dim \mathrm{End}_{T^{\mathsf{Z}}(n,d)}(\mathscr{T})$.

In view of (5.7), we have that each $\mathscr{T}^{\boldsymbol{\mu}}$ is $\tau_{n,d}$-self-dual and $(\mathscr{T}^{\boldsymbol{\mu}} : \Delta(\boldsymbol{\lambda})) = (\mathscr{T}^{\boldsymbol{\mu}} : \nabla(\boldsymbol{\lambda}))$ for all $\boldsymbol{\lambda} \in \Lambda^I_+(n,d)$. We now have:

$$
\begin{aligned}
\dim \mathrm{End}_{T^{\mathsf{z}}(n,d)}(\mathscr{T}) &= \sum_{\boldsymbol{\mu},\boldsymbol{\nu} \in \Lambda^I(n,d)} \dim \mathrm{Hom}_{T(n,d)}(\mathscr{T}^{\boldsymbol{\mu}}, \mathscr{T}^{\boldsymbol{\nu}}) \\
&= \sum_{\boldsymbol{\lambda} \in \Lambda^I_+(n,d)} \sum_{\boldsymbol{\mu},\boldsymbol{\nu} \in \Lambda^I(n,d)} (\mathscr{T}^{\boldsymbol{\mu}} : \Delta(\boldsymbol{\lambda}))(\mathscr{T}^{\boldsymbol{\nu}} : \nabla(\boldsymbol{\lambda})) \\
&= \sum_{\boldsymbol{\lambda} \in \Lambda^I_+(n,d)} \sum_{\boldsymbol{\mu},\boldsymbol{\nu} \in \Lambda^I(n,d)} (\mathscr{T}^{\boldsymbol{\mu}} : \Delta(\boldsymbol{\lambda}))(\mathscr{T}^{\boldsymbol{\nu}} : \Delta(\boldsymbol{\lambda})) \\
&= \sum_{\boldsymbol{\lambda} \in \Lambda^I_+(n,d)} \sum_{\boldsymbol{\mu},\boldsymbol{\nu} \in \Lambda^I(n,d)} k_{\overleftarrow{\boldsymbol{\lambda}'},\overleftarrow{\boldsymbol{\mu}}} k_{\overleftarrow{\boldsymbol{\lambda}'},\overleftarrow{\boldsymbol{\nu}}} \\
&= \sum_{\boldsymbol{\lambda} \in \Lambda^I_+(n,d)} \sum_{\boldsymbol{\mu},\boldsymbol{\nu} \in \Lambda^I(n,d)} k_{\boldsymbol{\lambda},\boldsymbol{\mu}} k_{\boldsymbol{\lambda},\boldsymbol{\nu}} \\
&= \dim T^{\mathsf{Z}}(n,d),
\end{aligned}
$$

where we have used (5.12) for the first equality, (2.6) for the second equality, Proposition 5.19 for the fourth equality and [11, Theorem 5.17] for the last equality. $\square$

REFERENCES CITED

[1] A. Kleshchev and I. Weinschelbaum. Good filtrations for generalized schur algebras. *arXiv: Representation Theory*, 2022. arXiv:2202.08866.

[2] A. Kleshchev and I. Weinschelbaum. Ringel duality for extended zigzag schur algebra. in progress.

[3] J. A. Green. *Polynomial representations of GL_n*, volume 830 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2 edition, 2007. ISBN 978-3-540-83176-1.

[4] J. A. Green. Combinatorics and the schur algebra. *Journal of Pure and Applied Algebra*, 88(1-3):89–106, 1993.

[5] A. Kleshchev and R. Muth. Based quasi-hereditary algebras. *Journal of Algebra*, 558:504–522, 2020.

[6] B. Parshall E. Cline and L. Scott. Finite-dimensional algebras and highest weight categories. *Journal für die reine und angewandte Mathematik*, 1988 (391):85–89, 2009.

[7] S. Donkin. *Rational Representations of Algebraic Groups*, volume 1140 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1985. ISBN 978-3-540-15668-0.

[8] O. Mathieu. Filtrations of g-modules. *Annales Scientifiques de l'École Normale Supérieure*, 23(4):625–644, 1990.

[9] Wang Jian-Pan. Sheaf cohomology on gb and tensor products of weyl modulus. *Journal of Algebra*, 77(1):162–185, 1982.

[10] A. Kleshchev and R. Muth. Generalized schur algebras. *Algebra & Number Theory*, 14:501–544, 2020.

[11] A. Kleshchev and R. Muth. Schurifying quasi-hereditary algebras. *arXiv: Representation Theory*, 2018. arXiv:1810.02849.

[12] W. Turner. Rock blocks. *Memoirs of the American Mathematical Society*, 202 (947), 2009.

[13] W. Turner. Tilting equivalences: from hereditary algebras to symmetric groups. *Journal of Algebra*, 319(10):3975–4007, 2008.

[14] W. Turner. Bialgebras and caterpillars. *Quarterly Journal of Mathemtics*, 59 (3):379–388, 2008.

[15] S. Donkin. On schur algebras and related algebras. iv. the blocks of the schur algebras. *Journal of Algebra*, 168:400–429, 1994.

[16] I. G. Macdonald. *Symmetric Functions and Hall Polynomials*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2 edition, 1995. ISBN 0-19-853489-2.

[17] S. Donkin. *The q-Schur Algebra*. London Mathematical Society Lecture Note Series). Cambridge University Press, Cambridge, 1998. ISBN 9780511600708.

[18] R. Rouquier. q-schur algebras and complex reflection groups. *Moscow Mathematical Journal*, 8(1):119–158, 2008.

[19] I. Schur. *Über eine Klasse von Matrizen, die sich einer gegebenen Matrix zuordinen lassen*. PhD thesis, Berlin, Berlin, 1901.

[20] A. Evseev and A. Kleshchev. Turner doubles and generalized schur algebras. *Advances in Mathematics*, 317:665–717, 2017.

[21] J. Humphreys. *Introduction to Lie Algebras and Representation Theory*, volume 9 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, 1972. ISBN 0-387-90053-5.

[22] C. M. Ringel. The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences. *Mathematische Zeitschrift*, 1991(208):209–223, 1991.