ENSEMBLE AVERAGES OF ASSORTED LOG-GAS MODELS

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DISSERTATION ABSTRACT

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We use techniques in the shuffle and exterior algebras to present the partition functions for several log-gas models in terms of either the Hyperpfaffian or the Berezin integral of an appropriate alternating tensor. Our methods generalize the de Bruijn integral identities from classical β-ensembles ($\beta = 1, 2, 4$) to iterated integrals of more general determinantal integrands, such as those arising from multicomponent and constellation ensembles. In the latter case, adjusting the distances between parallel lines or concentric circles also gives an interpolation between the limiting ensembles, such as one-dimensional β-ensembles with $\beta = K$ and $\beta = K^2$.

This dissertation includes unpublished coauthored material.

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CHAPTER I

INTRODUCTION

This dissertation includes unpublished coauthored material in chapters I, II, III, and V. In this chapter, sections 1.1 and 1.2 appear nearly as is in [35], a work coauthored with Jonathan M. Wells.

Within the intersection of probability theory and mathematical physics, random matrix theory is the study of the eigenvalue statistics obtained from different classes of random matrices. One particularly straightforward (and therefore well-studied) way of producing a random matrix is to specify a probability distribution for each of the matrix's entries. For example, classical random matrix theory has investigated random Hermitian matrices whose entries are independent (up to symmetry), identically distributed real, complex, or quaternionic Gaussian random variables. Alternatively, one could specify a probability distribution on an existing collection (typically group) of matrices. In either case, we obtain an induced distribution for the eigenvalues of these random matrices. While the eigenvalues of a single matrix tell us about *that* matrix, the eigenvalue statistics of random matrices tell us about the methods used to generate those matrices.

Since the beginning of the 20^{th} century, random matrix theory has primarily been applied in one of two ways. First, create a matrix which is meaningful to your data, such as a correlation matrix, a sample covariance matrix, or the transition matrix for a Markov chain. Next, compute its eigenvalues and compare the distribution to known examples in random matrix theory. If your sample eigenvalues are similar enough to the known examples, you might conclude your data is "all noise" as the matrix elements do not seem to be significantly correlated.

In contrast, if there are eigenvalues larger than those predicted by the known examples, this is suggestive of some "signal" in the form of correlations in the data. This first type of analysis has been used to study forest dynamics, wireless communications, machine learning, neural networks, and mutations of infectious diseases.

Suppose instead your data behaves like the eigenvalues themselves. Notably, the eigenvalues of random matrices tend to behave like charged particles, repelling each other and avoiding clustering. Similar behavior has been observed from perched birds, parked cars, and even the zeroes of the Riemann zeta function (and other L-functions). In the context of nuclear physics, random Hermitian matrices approximate the Hamiltonians of nuclear systems, while the eigenvalues predict the average behavior of the energy levels. Moreover, the exact same eigenvalue densities which arise in random matrix theory (see section 1.1) can also be seen in the studies of random tilings and roots of random polynomials.

The broad applicability of random matrix theory is owed to a collection of universal results akin to the classical Central Limit Theorem. However, the practicality of these results comes from being able to express the densities of eigenvalues in terms of "known" functions whose asymptotics are well-understood. The ongoing expansion of random matrix theory is aimed at discovering and "solving" new models in distinct universality classes. The realm of log-gases (see section 1.3) certainly promises a wide variety of interesting models wherein charged particles mimic and generalize the behavior of eigenvalues. The main results of this volume represent the first (largely algebraic) steps toward solvability for more models of this kind.

1.1. Classical Methods

The β-ensembles are a well-studied collection of random matrices whose eigenvalue densities take a common form, indexed by a non-negative, real parameter β. Suppose μ is a continuous probability measure on $\mathbb{R}$ with Radon-Nikodym derivative $\frac{d\mu}{dx} = w(x)$. For each $\beta \in \mathbb{R}_{>0}$, consider the N-point process specified by the joint probability density

$$\rho_N(x_1, \ldots, x_N) = \frac{1}{Z_N(\beta)N!} \prod_{j<k} |x_k - x_j|^\beta \prod_j w(x_j)$$

where $Z_N(\beta)$ denotes the *partition function* of β, and $Z_N(\beta)N!$ is the normalizing constant required for ρ_N to be a probability density function. Explicitly,

$$Z_N(\beta) = \frac{1}{N!} \int_{\mathbb{R}^N} \prod_{j<k} |x_k - x_j|^\beta \prod_j w(x_j)\, dx_1 \ldots, dx_N.$$

The integral that appears above is closely related to the Mehta integral [12], the Selberg integral [28] and its generalization, the Aomoto integral [4]. In [21] and [22], Luque and Thibon presented an evaluation of these integrals in terms of hyperdeterminants, which were first introduced by Cayley in [7]. Moreover, the eigenvalue density function ρ_N above can be identified with the Boltzmann factor of an electrostatic system of log-gas particles (see section 1.3), as first observed by Dyson [11], and further developed by Forrester in [15].

The *classical* β-ensembles are those with $\beta = 1, 2, 4$ and $w(x) = e^{-x^2/2}$, corresponding to Hermitian matrices with real, complex, or quaternionic Gaussian entries (respectively). The $\beta = 1$ case was first investigated in the 1950s by Wigner in the context of nuclear physics [36] following Wigner's discovery of a

similar ensemble of real-valued matrices used by Wishart in the 1920s in the field of multivariate statistics [37]. In the subsequent decade, Dyson and Mehta [12] unified a previously disparate collection of random matrix models under the umbrella of random Hermitian matrices (with $\beta = 1, 2, 4$ corresponding to the dimensions of three associative division algebras over $\mathbb{R}$). In [10], Dumitriu and Edelman provide tridiagonal matrix models for β-ensembles of arbitrary positive β, which are then used by Ramírez, Rider, and Virág in [26] to obtain the asymptotic distribution of the largest eigenvalue.

For each $1 \leq n \leq N$, define the n^{th} correlation function by

$$R_n(x_1, \ldots, x_n) = \frac{N!}{(N-n)!} \int_{\mathbb{R}^{N-n}} \rho_N(x_1, \ldots, x_n, y_1, \ldots, y_{N-n})\, dy_1 \cdots dy_{N-n}.$$

It turns out that the correlation functions for the classic β-ensembles take a particularly nice algebraic form. For example, when $\beta = 2$, it can be shown using only elementary matrix operations and Fubini's Theorem that

$$R_n(x_1, \ldots, x_n) = \det(K(x_j, x_k)_{1 \leq j,k \leq n}),$$

where the *kernel* $K(x, y)$ is a certain square integrable function $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ that can most easily be expressed in terms of a family of polynomials which are orthogonal with respect to the measure μ. For this reason, we say the classical $\beta = 2$ ensemble is an example of a *determinantal* point process. The details of this derivation are given in [24]. Similarly, when $\beta = 1$ or 4,

$$R_n(x_1, \ldots, x_n) = \text{Pf}(K_\beta(x_j, x_k)_{1 \leq j,k \leq n}),$$

where $\mathrm{Pf}(A) = \sqrt{\det(A)}$ denotes the Pfaffian (see section 2.3) of an antisymmetric matrix A, and where $K(x, y)$ is a certain 2×2 matrix-valued function whose entries are square-integrable, and which satisfies $K(x, y)^T = -K(y, x)$. We then say the classical $\beta = 1$ and $\beta = 4$ ensembles are examples of *Pfaffian* point processes. This result was first shown for circular ensembles by Dyson in [11], then for Gaussian ensembles by Mehta in [24] and then for general weights (μ) by Mehta and Mahoux in [23], except for the case $\beta = 1$ and N odd. Finally, the last remaining case was given by Adler, Forrester, and Nagao in [1]. An investigation of *hyperdeterminantal* point processes, another generalization of the determinantal point process, tracing its roots to Cayley's hyperdeterminants, can be found in [14].

The immediate advantage of these determinantal and Pfaffian expressions for the correlation functions is that these matrix kernels do not essentially increase in complexity as N grows large, since the dimensions of the matrix kernel are stable, and the entries are expressed as a sum whose asymptotics are well-understood.

1.2. Hyperpfaffian Partition Functions

Derivations of the determinantal and Pfaffian expressions of the correlation functions have been presented in numerous ways over the past several decades. Of particular note is the method of Tracy and Widom [33], who first show that the partition function is determinantal or Pfaffian, and then use matrix identities and generating functions to obtain a corresponding form for the correlation functions.

But recognizing the partition function $Z_N(\beta)$ as the determinant or Pfaffian of a matrix of integrals of appropriately chosen orthogonal polynomials is essential and nontrivial. One way to do this is to apply the Andreief determinant identity [3] to the iterated integral which defines $Z_N(\beta)$. This is immediate when $\beta = 2$, and

viewing the Pfaffian as the square root of a determinant, this identity can also be applied (with some additional finesse) when $\beta = 1$ or 4. However, viewing the Pfaffian in the context of the exterior algebra allows us to extend the Andreief determinant identity to analogous Pfaffian identities, referred to as the de Bruijn integral identities [9].

In 2002, Luque and Thibon [20] used techniques in the shuffle algebra to show that when $\beta = L^2$ is an even square integer, the partition function $Z_N(\beta)$ can be written as a Hyperpfaffian of an L-form (see section 2.3) whose coefficients are integrals of Wronskians (see section 2.6) of suitable polynomials. Then in 2011, Sinclair [31] used other combinatorial methods to show that the result also holds when $\beta = L^2$ is an odd square integer.

In his 2013 dissertation, Shum [29] considered 2-fold *constellation* ensembles (both linear and circular) in which a $\beta = 1$ ensemble is copied onto a parallel line (or concentric circle) in the complex plane. He demonstrated these ensembles to be completely solvable Pfaffian point processes and then showed how these ensembles give an interpolation between the classical $\beta = 2$ and $\beta = 4$ ensembles (by adjusting the distance between parallel lines or concentric circles). In this volume, the many new variations on the constellation setup (see chapter IV) allow for many more interpolations, including but not limited to an interpolation between $\beta = L$ and $\beta = L^2$ ensembles. Thus, the partition functions of integer β-ensembles can all be written as a limit of Hyperpfaffians, even when β is a square-free integer.

Furthermore, we consider *multicomponent* ensembles (see chapter V) in which the joint probability density functions generally have the form

$$\rho_N(x_1, \ldots, x_N) = \frac{1}{Z_f(\beta)} \prod_{j<k} |x_k - x_j|^{\beta f(j,k)} \prod_j w_j(x_j),$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ specifies (possibly) different exponents for each factor in the product. This setup was first investigated for particles of charge 1 and 2 on the complex unit circle by Forrester in [17] and [16], and then considered on the real line by Rider, Sinclair and Xu in [27]. This model is closely associated to the eigenvalue densities for the real Ginibre ensembles as discussed by Forrester and Nagao in [19], by Akemann and Kanzieper in [2], and then by Borodin and Sinclair in [6]. The limiting behavior for the two species model in the circular case was later studied by Shum and Sinclair in [30]. A recent paper by Forrester and Li [18] extends these results further to express the skew orthogonal polynomials for classical weight functions in terms of hypergeometric polynomials.

In a fully general log-gas, each particle is allowed a possibly distinct charge L_j (so that $f(j,k) = L_j L_k$). In [32], Sinclair showed the partition function $Z_f(\beta)$ has a Berezin integral (see section 2.4) expression provided each $\sqrt{\beta}L_j$ is an even integer. In section 5.2, we extend to arbitrary positive integers $\sqrt{\beta}L_j \in \mathbb{Z}_{>0}$ using shuffle algebra techniques, analogous to the methods of Thibon and Luque. Note, single-component β-ensembles are a subset of multicomponent ensembles in which $f(i,j) = 1$, and the Berezin integral is a generalization of the Hyperpfaffian. Thus, one consequence of this work is a new, all-encompassing derivation of the Hyperpfaffian partition functions for single-component β-ensembles in which $\beta = L^2$ is any square integer, even or odd.

As previously mentioned, our methods are predicated on being able to write the joint density ρ_N as a determinant (see section 2.7) without absolute value (see section 5.3). More generally, we can replace the partition function $Z_N(\beta)$ (or $Z_f(\beta)$) with an iterated integral of any determinant fitting relatively few criteria. Theorem 3.1, given at the beginning of chapter III, is the most broad generalization

of the de Bruijn integral identities to date, promising wider application even outside the realm of random matrix theory (and log-gas models).

1.3. The Elementary Log-Gas Setup

Suppose a finite number of charged particles are placed on an infinite wire represented by the real line. The charges of the particles are assumed to be the same positive integer L, and the particles are assumed to repel each other with logarithmic interactions. Additionally, we assume any two particles of the same charge are indistinguishable. The wire is imbued with a potential which discourages the particles from escaping to infinity in either direction, and heat is applied to the system according to the inverse temperature β parameter. In fact, if T is the temperature of the system, then $\beta = 1/k_{\mathrm{B}}T$, where k_{B} is the Boltzmann constant.

Let $\vec{x} \in \mathbb{R}^N$ be the *location vector* of the system, with each x_j giving the location of a particle of charge L. Under the assumption of logarithmic interactions, the contribution of potential energy to the system by two particles at locations x_j and x_k is given by $-\beta L^2 \log|x_k - x_j|$. If U is the potential on the system, then at inverse temperature β, the total potential energy of the system is given by

$$E_N(\vec{x}) = \beta L \sum_{j=1}^{N} U(x_j) - \beta L^2 \sum_{j<k} \log|x_k - x_j|.$$

The first iterated sum accounts for the influence of the potential on each particle, while the second sum accounts for the interactions between particles.

With this setup, the relative density of states (corresponding to varying location vectors $\vec{x}$) is given by the Boltzmann factor

$$\begin{aligned}\Omega_N(\vec{x}) &= \exp\left(-E_N(\vec{x})\right) \\ &= \prod_{j=1}^{N} \exp\left(-\beta L U(x_j)\right) \times \prod_{j<k} |x_k - x_j|^{\beta L^2}.\end{aligned}$$

Thus, the probability of finding the system in a state corresponding to a location vector $\vec{x}$ is given by the joint probability density function

$$\rho_N(\vec{x}) = \frac{\Omega_N(\vec{x})}{Z_N(\beta) N!},$$

where the partition function (of the single-component log-gas) $Z_N(\beta)$ is given by

$$\begin{aligned}Z_N(\beta) &= \frac{1}{N!} \int_{\mathbb{R}^N} \Omega_N(\vec{x})\, dx_1 \cdots dx_N \\ &= \int_{-\infty < x_1 < \cdots < x_N < \infty} \Delta(\vec{x})^{\beta L^2}\, d\mu(x_1) \cdots d\mu(x_N),\end{aligned}$$

in which $d\mu(x) = w(x)\, dx = e^{-\beta L U(x)}\, dx$ and $\Delta(\vec{x})$ denotes the Vandermonde determinant (see section 2.7) evaluated at the variables $x_1, \ldots, x_N$. The $N!$ is there because the N particles are indistinguishable. In the second line, we drop the $N!$ by changing the domain of integration to one in which the variables are totally ordered. Because of this total ordering, each $x_k - x_j > 0$ so that we are able to drop the absolute value as well. At this point, it is necessary to assume the potential U is one for which $Z_N(\beta)$ is finite.

Note, unit charges (meaning $L = 1$) at inverse temperature $\beta = b^2$ have the same Boltzmann factor (and resulting density function) as charge $L = b$

particles at inverse temperature $\beta = 1$ (subject to different but related potentials $U(x)$). In general, replacing β with $\beta' = \beta/b^2$ and replacing L with $L' = bL$ leaves $\beta L^2 = \beta' L'^2$ unchanged. Then replacing U with new potential bU returns the previous $\Omega_N(\vec{x})$. Thus, for computational purposes, we can change to $\beta = 1$ (provided $\sqrt{\beta}L \in \mathbb{Z}$ for the original β) and only allow the charges of the particles to vary.

Moreover, the density ρ_N and partition function Z_N are of the same form as those seen for the β-ensembles in section 1.1. The repulsion between eigenvalues (or charged particles) comes from the fact ρ_N is small whenever the pairwise distances $|x_k - x_j|$ are small, and the exponents, either β or L^2 determine how strong the repulsion is. Finally, the weight function w roughly tells us how individual eigenvalues (or charged particles) would be distributed if not subjected to the influence of the others.

Note, however, the weight functions w which appear for the β-ensembles are induced from the methods by which we construct our random matrices (such as the distributions on individual matrix entries). Not all generic weight functions are obtainable from known matrix models. In contrast, the log-gas perspective permits a wider range of weight functions (obtained directly from a choice of potential U).

The partition function Z_N and its analogues are the central objects of interest to this volume. Though we assume $\beta = 1$ for computational purposes, Z_N is inherently a function of β, among other parameters. The potential U dictates the external forces experienced by each particle individually, affecting the measures μ against which we are integrating. The charge L and the inverse temperature β together influence the strength of the interactions between the particles, affecting the exponents on the interaction terms in the Boltzmann factor.

Recall, this Z_N is an iterated integral in N many variables. As in the works of Sinclair, our goal here is not to compute these integrals for any particular choice of several parameters. Instead, we demonstrate, in general, how to write Z_N and its analogues as a Hyperpfaffian, or Berezin integral in the multicomponent case, of an alternating tensor whose coefficients are only single or double integrals of Wronskians whose entries are (potentially orthogonal, skew orthogonal, or biorthogonal) polynomials.

This model of identically charged log-gas models could be called a single-component linear (or one-dimensional) ensemble. In contrast, the *multicomponent* ensembles of chapter V are obtained by allowing the particles to have possibly distinct charges $L_1, \ldots, L_N$. In chapter VI, *circular* ensembles are obtained by placing the particles on the unit circle rather than the real line. Finally, the *constellation* ensembles of chapter IV are obtained by copying a one-dimensional arrangement of particles onto parallel lines (or concentric circles) in the complex plane.

1.4. Example Formulae

Let $\vec{p} = \{p_j\}_{j=1}^{2N}$ be any family of polynomials such that each p_j is a monic polynomial of degree $j-1$. Let A be the matrix whose entries are defined by

$$\begin{aligned} A_{j,k} &= \int_{-\infty}^{\infty} \det \begin{bmatrix} p_j(x) & p_j'(x) \\ p_k(x) & p_k'(x) \end{bmatrix} w(x)\,dx \\ &= \int_{-\infty}^{\infty} \big(p_j(x)p_k'(x) - p_j'(x)p_k(x)\big)\, w(x)\,dx. \end{aligned}$$

When $\beta = 4$ (such as in the case of charge $L = 2$ particles on the real line), the partition function $Z_N(4)$ is given by

$$Z_N(4) = \mathrm{Pf}A,$$

the Pfaffian of the skew symmetric matrix A. At higher (square) integer values of β, the Pfaffian is replaced with a Hyperpfaffian.

At face value, this says we can compute the iterated integral which defines $Z_N(\beta)$ by instead computing integrals of univariate functions. Therein, we have the freedom to choose polynomials $\vec{p}$ which are "nice" (see section 2.7 for why) such as skew orthogonal polynomials for which $A_{j,k}$ is often 0. Finally, the Pfaffian, like a determinant, has its own structure and accompanying identities which can be exploited further as in the methods of Tracy and Widom.

Next, Let L be a positive even integer. Let V be an $NL \times NL$ matrix whose entries in the first L columns are real-valued functions of the variable x_1. Further suppose the entries in the next L columns are the same functions evaluated at the variable x_2, and so on up through x_N in the last L columns. Explicitly, the entries are

$$V_{j,(n-1)L+k} = f_{j,k}(x_n)$$

for some family of real-valued functions $\{f_{j,k}\}_{j,k=1}^{NL,L}$. Let A be the L-dimensional array whose $(n_1, \ldots, n_L)$-entry $A_{n_1,\ldots,n_L}$ is given by

$$A_{n_1,\ldots,n_L} = \int_a^b \det\left[f_{n_j,k}(x)\right]_{j,k=1}^{L} dx.$$

Note, these integrands are simply determinants of the $L \times L$ univariate submatrices of V, obtained by taking L-many rows from the L-many columns which share a variable. The following theorem is a special case of Theorem 3.1, the main integral identity given at the beginning of chapter III:

Theorem 1.1. *Let V and A be defined as above. Then,*

$$\int_{a<x_1<\cdots<x_N<b} \det V \, dx_1 \cdots dx_N = \mathrm{PF}A,$$

where PFA *denotes the Hyperpfaffian of A.*

Now divorced from the context of computing partition functions, this is a general statement about integrating multivariate determinants. Recall, in this setup, we assumed the same functions for each of the N many variables (as in an alternant matrix, for example), which occurs with the β-ensembles and equivalent log-gas model of section 1.3. A more general version of this theorem holds even when there is no resemblance between any of the columns of V. Furthermore, we can drop the requirement that the entries of V be univariate functions. The main results of chapters IV, V, and VI are analogous formulae with the particulars of different ensembles (constellation, multicomponent, and circular ensembles, respectively) substituted in.

CHAPTER II

PRELIMINARY DEFINITIONS

This chapter contains unpublished coauthored material. In particular, sections 2.4-2.7 appear nearly as is in [35].

In this chapter, we introduce a mix of conventions and definitions which simplify the statement of our main results. First, for any positive integer N, let $\underline{N}$ denote the set $\{1, \ldots, N\}$. Assuming positive integers $K \leq N$, let $\mathfrak{t} : \underline{K} \nearrow \underline{N}$ denote a strictly increasing function from $\underline{K}$ to $\underline{N}$, meaning

$$1 \leq \mathfrak{t}(1) < \mathfrak{t}(2) < \cdots < \mathfrak{t}(K) \leq N.$$

It will be convenient to use these increasing functions to track indices used in denoting submatrices and elements of tensor and exterior algebras, among other things (often in place of, but sometimes in conjunction with, permutations). For example, given an $N \times N$ matrix V, $V_{\mathfrak{t}}$ might denote the $K \times K$ submatrix composed of the rows $\mathfrak{t}(1), \ldots, \mathfrak{t}(K)$, taken from the first K columns of V. More conventions related to indexing and permutations (which are relevant to the proofs) are introduced in section 3.1 but are not necessary for the statements of our main results.

2.1. The Tensor Algebra

Let R be a commutative ring with unity, and let V be an R-module. The *tensor product* $T^2(V) = V \otimes V$ is formed by taking the quotient of the free abelian

group on $V \times V$ by the ideal $\mathcal{I}$ generated by elements of the form

$$(v_1 + v_2, w_1) - (v_1, w_1) - (v_2, w_1), \quad (v_1, w_1 + w_2) - (v_1, w_1) - (v_1, w_2),$$

$$(rv_1, w_1) - (v_1, rw_1)$$

for $r \in R$, $v_j, w_j \in V$. Then for each integer $k \geq 1$, define the k^{th} *tensor power* of V by

$$T^k(V) = V \otimes V \otimes \cdots \otimes V \quad (k \text{ factors})$$

with $T^0(V) = R$. We call elements of $T^k(V)$ k-tensors. Define the *tensor algebra* $T(V)$ by

$$T(V) = \bigoplus_{k=0}^{\infty} T^k(V),$$

and observe that $T(V)$ is indeed an R-algebra with multiplication

$$(v_1 \otimes \cdots \otimes v_k) \otimes (w_1 \otimes \cdots \otimes w_l) = v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_l.$$

The following theorem is well-known:

Theorem 2.1. *If V is a rank d free R-module with basis $X = \{\varepsilon_1, \ldots, \varepsilon_d\}$, then $T^k(V)$ has a basis*

$$\{\varepsilon_{j_1} \otimes \cdots \otimes \varepsilon_{j_k} \,|\, 1 \leq i_1, \ldots, i_k \leq d\}.$$

For a set X and a ring R, let $R\langle X \rangle$ denote the free unital algebra on X over R with multiplicative unit e. We may identify the tensor algebra $T(V)$ of a free R-module V with the free R-algebra $R\langle X \rangle$, where X is an R-basis for V. The identification is given by $v \otimes w = vw$ for $v, w \in X$.

2.2. The Exterior Algebra

Suppose V is a real vector space of dimension d with basis $\varepsilon_1, \ldots, \varepsilon_d$, and let $T(V)$ be the tensor algebra of V over $\mathbb{R}$. The *exterior algebra* of V is obtained by taking the quotient of $T(V)$ by the ideal $\mathcal{I}$ generated by elements of the form $v \otimes v$ for $v \in V$. The exterior algebra $T(V)/\mathcal{I}$ is denoted by $\bigwedge(V)$, and the image of $v_1 \otimes \cdots \otimes v_k$ in $\bigwedge(V)$ is denoted by $v_1 \wedge \cdots \wedge v_k$.

Note, $\mathcal{I}$ is generated by homogeneous elements and is thus a graded ideal. Hence, $\bigwedge(V)$ is a graded algebra, and the k^{th} homogeneous component $\bigwedge^k(V) = T^k(V)/\mathcal{I}$ is called the k^{th} *exterior power* of V. Elements of $\bigwedge^k(V)$ are called antisymmetric (or alternating) k-tensors, or k-forms. The multiplication

$$(v_1 \wedge \cdots \wedge v_k) \wedge (w_1 \wedge \cdots \wedge w_l) = v_1 \wedge \cdots \wedge v_k \wedge w_1 \wedge \cdots \wedge w_l.$$

in the exterior algebra is called the *wedge product* (or exterior product). Multiplication is anticommutative in that $v \wedge w = -w \wedge v$ because

$$0 = (v + w) \wedge (v + w) = v \wedge v + w \wedge w + v \wedge w + w \wedge v = v \wedge w + w \wedge v$$

for all $v, w \in V$.

Let $X = \{\varepsilon_1, \ldots, \varepsilon_d\}$ be a basis for V. For any injection $\mathfrak{t} : \underline{k} \to \underline{d}$, let $\varepsilon_{\mathfrak{t}} \in \bigwedge^k(V)$ denote

$$\varepsilon_{\mathfrak{t}} = \varepsilon_{\mathfrak{t}(1)} \wedge \varepsilon_{\mathfrak{t}(2)} \wedge \cdots \wedge \varepsilon_{\mathfrak{t}(k)}.$$

As in Theorem 2.1, $\{\varepsilon_{\mathfrak{t}} \,|\, \mathfrak{t} : \underline{k} \nearrow \underline{d}\}$ is a basis for $\bigwedge^k(V)$. In particular, $\bigwedge^k(V)$ has dimension $\binom{d}{k}$. Also, $\bigwedge^d(V)$ is a one-dimensional subspace we call the

determinantal line, spanned by

$$\varepsilon_{\text{vol}} = \varepsilon_{\text{id}} = \varepsilon_1 \wedge \varepsilon_2 \wedge \cdots \wedge \varepsilon_d,$$

which we call the *volume form* in $T(V)$.

More generally, for any commutative ring with unity R and finite-rank free R-module V, we may define the exterior algebra $\bigwedge_R(V)$ just as above by taking a suitable quotient of the tensor algebra $T(V)$. All of the aforementioned properties of the exterior algebra still hold, where subspace and dimension are replaced with submodule and rank, as appropriate.

2.3. Pfaffians and Hyperpfaffians

Let A be a $2N \times 2N$ antisymmetric matrix. Define the *Pfaffian* of A, $\text{Pf}(A)$, by

$$\text{Pf}(A) = \frac{1}{2^N N!} \sum_{\sigma \in S_{2N}} \text{sgn}(\sigma) \prod_{j=1}^{N} A_{\sigma(2j-1),\sigma(2j)}.$$

To each antisymmetric matrix A, associate a 2-form $\omega_A \in \bigwedge^2(V)$ given by

$$\omega_A = \sum_{j<k} A_{j,k} \varepsilon_j \wedge \varepsilon_k.$$

Similarly, to each 2-form $\omega \in \bigwedge^2(V)$ with $\omega = \sum_{j<k} a_{j,k} \varepsilon_j \wedge \varepsilon_k$, associate the antisymmetric matrix $A(\omega)$ given by

$$A(\omega)_{j,k} = \begin{cases} a_{j,k} & \text{if } j < k \\ -a_{j,k} & \text{if } j > k \\ 0 & \text{if } j = k \end{cases}.$$

Together, these give a bijection between 2-forms and antisymmetric matrices. We define the Pfaffian $\mathrm{Pf}(\omega)$ of a 2-form ω to be the Pfaffian of the associated antisymmetric matrix.

Let $A = \{A_{\mathfrak{t}} \,|\, \mathfrak{t} : \underline{L} \to \underline{NL}\}$ be an L-dimensional array of values in R with the property that

$$A_{\mathfrak{t}\circ\sigma} = \mathrm{sgn}(\sigma) A_{\mathfrak{t}}$$

for each $\sigma \in S_L$. By way of analogy with antisymmetric matrices, we will call A an *antisymmetric* L-dimensional array. Define the *Hyperpfaffian* of A, $\mathrm{PF}(A)$, by

$$\mathrm{PF}(A) = \frac{1}{(L!)^N N!} \sum_{\sigma \in S_{NL}} \mathrm{sgn}(\sigma) \prod_{j=1}^{N} A_{\sigma((j-1)L+1),\ldots,\sigma(jL)}.$$

As before, to each antisymmetric L-dimensional array A, associate an L-form $\omega_A \in \bigwedge^L(V)$ given by

$$\omega_A = \sum_{\mathfrak{t}:\underline{L}\nearrow\underline{NL}} A_{\mathfrak{t}} \varepsilon_{\mathfrak{t}}.$$

Similarly, to each L-form $\omega \in \bigwedge^L(V)$ with $\omega = \sum_{\mathfrak{t}} a_{\mathfrak{t}} \varepsilon_{\mathfrak{t}}$, associate the antisymmetric L-dimensional array $A(\omega)$ given by

$$A(\omega)_{\mathfrak{t}\circ\sigma} = \mathrm{sgn}(\sigma) a_{\mathfrak{t}} \quad \text{for } \mathfrak{t} : \underline{L} \nearrow \underline{NL} \text{ and } \sigma \in S_L.$$

Again, this gives a bijection between L-forms and antisymmetric L-dimensional arrays. We define the Hyperpfaffian $\mathrm{PF}(\omega)$ of an L-form ω to be the Hyperpfaffian of the associated array.

2.4. The Berezin Integral

For each $0 < n \leq N$, define $\frac{\partial}{\partial \varepsilon_n} : \bigwedge^K(\mathbb{R}^N) \to \bigwedge^{K-1}(\mathbb{R}^N)$ on basis elements by

$$\frac{\partial}{\partial \varepsilon_n}\varepsilon_{\mathfrak{t}} = \begin{cases} (-1)^k \varepsilon_{\mathfrak{t}(1)} \wedge \cdots \wedge \varepsilon_{\mathfrak{t}(k-1)} \wedge \varepsilon_{\mathfrak{t}(k+1)} \wedge \cdots \wedge \varepsilon_{\mathfrak{t}(K)} & \text{if } k = \mathfrak{t}^{-1}(n) \\ 0 & \text{otherwise} \end{cases},$$

and then extend linearly. If $n \in \mathfrak{t}(\underline{K})$, meaning ε_n appears as a factor in $\varepsilon_{\mathfrak{t}}$, then $\frac{\partial \varepsilon_{\mathfrak{t}}}{\partial \varepsilon_n}$ is the result of permuting ε_n to the front and then removing it, picking up a sign associated with changing the order in which the basis elements occur. If $\varepsilon_{\mathfrak{t}}$ does not have ε_n as a factor, then $\frac{\partial \varepsilon_{\mathfrak{t}}}{\partial \varepsilon_n} = 0$. Given an injection $\mathfrak{s} : \underline{L} \to \underline{N}$, we define the Berezin integral [5] (with respect to $\varepsilon_{\mathfrak{s}}$) as a linear operator $\bigwedge(\mathbb{R}^N) \to \bigwedge(\mathbb{R}^N)$ given by

$$\int \varepsilon_{\mathfrak{t}} \, d\varepsilon_{\mathfrak{s}} = \int \varepsilon_{\mathfrak{t}} \, d\varepsilon_{\mathfrak{s}(1)} \, d\varepsilon_{\mathfrak{s}(2)} \cdots d\varepsilon_{\mathfrak{s}(L)} = \frac{\partial}{\partial \varepsilon_{\mathfrak{s}(L)}} \cdots \frac{\partial}{\partial \varepsilon_{\mathfrak{s}(2)}} \frac{\partial}{\partial \varepsilon_{\mathfrak{s}(1)}} \varepsilon_{\mathfrak{t}}.$$

Our main results are stated in terms of Berezin integrals with respect to the volume form $\varepsilon_{\text{vol}} \in \bigwedge^N(\mathbb{R}^N)$. Note, if $\varepsilon_{\mathfrak{t}} \in \bigwedge^K(\mathbb{R}^N)$ for any $K < N$, then

$$\int \varepsilon_{\mathfrak{t}} \, d\varepsilon_{\text{vol}} = 0$$

because $\varepsilon_{\mathfrak{t}}$ is missing some ε_k as a factor. Thus, the Berezin integral with respect to ε_{vol} is a projection operator $\bigwedge(\mathbb{R}^N) \to \bigwedge^N(\mathbb{R}^N) \cong \mathbb{R}$. In particular, if $\sigma \in S_N$, then

$$\int \varepsilon_{\sigma} \, d\varepsilon_{\text{vol}} = \text{sgn}(\sigma).$$

2.5. Exponentials of Forms

For $\omega \in \bigwedge(\mathbb{R}^N)$ and positive integer m, we write

$$\omega^{\wedge m} = \omega \wedge \cdots \wedge \omega,$$

with ω appearing as a factor m times. By convention, $\omega^{\wedge 0} = 1$. We then define the exponential

$$\exp(\omega) = \sum_{m=0}^{\infty} \frac{\omega^{\wedge m}}{m!}.$$

Moreover, suppose $\omega = \omega_1 + \omega_2 + \cdots + \omega_J$ where each $\omega_j \in \bigwedge^{L_j}(\mathbb{R}^N)$ and each L_j even, then (we say each ω_j is a homogeneous even form of length L_j and) it is easily verified

$$\exp(\omega) = \exp(\omega_1 + \cdots + \omega_J) = \exp(\omega_1) \wedge \cdots \wedge \exp(\omega_J).$$

In chapter V, section 6.4, and section 6.7, the forms we are exponentiating are non-homogeneous. However, we get a homogeneous form in the case when we only have one species of particle, such as in chapter IV. In that case, exactly one summand in the exponential will live at the determinantal line. Assuming $\omega \in \bigwedge^{L}(\mathbb{R}^N)$ with $LM = N$, we get

$$\int \exp(\omega)\, d\varepsilon_{\text{vol}} = \int \sum_{m=0}^{\infty} \frac{\omega^{\wedge m}}{m!}\, d\varepsilon_{\text{vol}} = \int \frac{\omega^{\wedge M}}{M!}\, d\varepsilon_{\text{vol}} = \text{PF}(\omega),$$

where $\text{PF}(\omega)$ is the Hyperpfaffian of ω, the real number coefficient on ε_{vol} in $\frac{\omega^{\wedge M}}{M!}$. Thus, this Berezin integral is the appropriate generalization of the Hyperpfaffian. To avoid confusing this Berezin integral with other integrals which appear in our

computations, we will write

$$\mathrm{BE}_{\mathrm{vol}}(\omega) = \int \exp(\omega)\, d\varepsilon_{\mathrm{vol}},$$

where the subscript on the left hand side indicates which form we are integrating with respect to.

The partition function of a one-dimensional (non-constellation) ensemble with a single species (non-multicomponent) has been shown to have a Hyperpfaffian expression (for certain β) [31]. In chapter IV, although the constellation setup takes our particles into the complex plane, we are able to maintain homogeneous forms for which the Hyperpfaffian is defined. In chapter V, as we generalize to (multicomponent) ensembles with multiple species (and therefore non-homogeneous forms), we replace the Hyperpfaffian with the more general Berezin integral (of an exponential).

2.6. Wronskians

For any non-negative integer l, define the l^{th} modified differential operator D^l by

$$D^l f(x) = \frac{1}{l!}\frac{d^l f}{dx^l},$$

with $D^0 f(x) = f(x)$. Define the modified Wronskian, $\mathrm{Wr}(\vec{f}, x)$, of a family, $\vec{f} = \{f_n\}_{n=1}^{L}$, of L many sufficiently differentiable functions by

$$\mathrm{Wr}(\vec{f}, x) = \det\left[D^{l-1} f_n(x)\right]_{n,l=1}^{L}.$$

We call this the *modified* Wronskian because it differs from the typical Wronskian (used in the study of elementary differential equations to test for linear dependence of solutions) by a combinatorial factor of $\prod_{l=1}^{L} l!$.

A *complete* N-family of monic polynomials is a collection $\vec{p} = \{p_n\}_{n=1}^{N}$ such that each p_n is monic of degree $n-1$. Given $\mathfrak{t} : \underline{L} \nearrow \underline{N}$, denote $\vec{p}_{\mathfrak{t}} = \{p_{\mathfrak{t}(k)}\}_{k=1}^{L}$. Then the (modified) Wronskian of $\vec{p}_{\mathfrak{t}}$ is given by

$$\mathrm{Wr}(\vec{p}_{\mathfrak{t}}, x) = \det\left[D^{l-1} p_{\mathfrak{t}(k)}(x)\right]_{k,l=1}^{L}.$$

Similarly, define the proto-Wronskian, $\mathrm{Pr}_{\vec{y}}(\vec{f}, x)$, (with respect to translation vector $\vec{y}$) by

$$\mathrm{Pr}_{\vec{y}}(\vec{f}, x) = \det\left[f_n(x + iy_k)\right]_{n,k=1}^{K}.$$

We call this the *proto*-Wronskian because

$$\lim_{\vec{y}\to 0} \frac{\mathrm{Pr}_{\vec{y}}(\vec{f}, x)}{\Delta(i\vec{y})} = \mathrm{Wr}(\vec{f}, x),$$

where $\Delta(i\vec{y})$ denotes the Vandermonde determinant evaluated at the variables $iy_1, \ldots, iy_K$. A proof of this is given in section 2.8.

The Wronskian, which appears when studying one-dimensional (possibly multicomponent or circular) ensembles, has columns generated by taking higher derivatives of each f_n. The number of columns is equal to the charge of the particles under consideration. The proto-Wronskian, which appears when studying constellation ensembles, has columns generated by instead evaluating each f_n at different translations $x + iy_k$. The number of columns K is equal to the number of parallel lines under consideration (see chapter IV).

When the charge of each particle is $L \neq 1$, it is necessary to conflate these two structures. To that end, for $\vec{f} = \{f_m\}_{m=1}^{LK}$, define

$$\mathrm{Wr} \otimes \mathrm{Pr}_{\vec{y}}(\vec{f}, x) = \det \left[\left[D^{l-1} f_{(n-1)L+j}(x + iy_k) \right]_{j,l=1}^{L} \right]_{n,k=1}^{K}.$$

The first column of the associated matrix is LK many functions evaluated at $x + iy_1$. The second column is the first derivatives of those functions evaluated at the same $x + iy_1$, and so on until the first L many columns have been exhausted. The next L many columns are the same functions and derivatives evaluated at $x + iy_2$, and so on until all y_k have been exhausted. The resulting $LK \times LK$ matrix will have $L \times L$ Wronskian blocks evaluated at one of the K many $x+iy_k$. In section 2.8, we will show

$$\lim_{\vec{y} \to 0} \frac{\mathrm{Wr} \otimes \mathrm{Pr}_{\vec{y}}(\vec{f}, x)}{\Delta(i\vec{y})^{L^2}} = \mathrm{Wr}(\vec{f}, x).$$

Suppose, for example, $L = 3$, $K = 2$, and $\vec{f} = \{x^{n-1}\}_{n=1}^{6}$ (which happens when there are 2 parallel lines of charge 3 particles). Then

$\mathrm{Wr} \otimes \mathrm{Pr}_{\vec{y}}(\vec{f}, x) =$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ x+iy_1 & 1 & 0 & x+iy_2 & 1 & 0 \\ (x+iy_1)^2 & 2(x+iy_1) & 1 & (x+iy_2)^2 & 2(x+iy_2) & 1 \\ (x+iy_1)^3 & 3(x+iy_1)^2 & 3(x+iy_1) & (x+iy_2)^3 & 3(x+iy_2)^2 & 3(x+iy_2) \\ (x+iy_1)^4 & 4(x+iy_1)^3 & 6(x+iy_1)^2 & (x+iy_2)^4 & 4(x+iy_2)^3 & 6(x+iy_2)^2 \\ (x+iy_1)^5 & 5(x+iy_1)^4 & 10(x+iy_1)^3 & (x+iy_2)^5 & 5(x+iy_2)^4 & 10(x+iy_2)^3 \end{bmatrix}.$$

The first three columns correspond to charge 3 particles on the line $\mathbb{R} + iy_1$, and the last three columns correspond to charge 3 particles on the line $\mathbb{R} + iy_2$.

2.7. Confluent Determinants

Fix $\vec{L} = (L_1, \dots L_M) \in (Z_{>0})^M$, and let $N = \sum_{m=1}^{M} L_m$. Let $\vec{f} = \{f_n\}_{n=1}^{N}$ be a family (not necessarily complete) of $\max(L_1, \dots, L_M) - 1$ times differentiable functions. Define the confluent alternant (with respect to shape $\vec{L}$) to be the $N \times N$ matrix

$$V_{\vec{f}}^{\vec{L}}(\vec{x}) = \left[V_{\vec{f}}^{L_1}(x_1) \quad V_{\vec{f}}^{L_2}(x_2) \quad \cdots \quad V_{\vec{f}}^{L_M}(x_M) \right],$$

where each $V_{\vec{f}}^{L_m}(x_m)$ is an $N \times L_m$ matrix defined by

$$V_{\vec{f}}^{L_m}(x_m) = \left[D^{l-1} f_n(x_m) \right]_{n,l=1}^{N,L_m}.$$

Then each variable x_m appears in L_m many consecutive columns, generated from $\vec{f}$ by taking derivatives. Note, any increasing function $\mathfrak{t} : \underline{L_m} \nearrow \underline{N}$ defines an $L_m \times L_m$ submatrix with Wronskian determinant corresponding to the polynomials $\vec{f_{\mathfrak{t}}} = \{f_{\mathfrak{t}(l)}\}_{l=1}^{L_m}$. Explicitly,

$$\det V_{\vec{f},\mathfrak{t}}^{L_m}(x_m) = \mathrm{Wr}(\vec{f_{\mathfrak{t}}}, x_m).$$

Let $\vec{g} = \{x^{n-1}\}_{n=1}^{N}$. If $\vec{p}$ is any complete N-family of monic polynomials, then

$$\det V_{\vec{g}}^{\vec{L}}(\vec{x}) = \det V_{\vec{p}}^{\vec{L}}(\vec{x})$$

because $V_{\vec{p}}^{\vec{L}}(\vec{x})$ can be obtained from $V_{\vec{g}}^{\vec{L}}(\vec{x})$ by performing elementary column operations. This is only because the p_j are assumed to be monic, and $\vec{p}$ is complete,

containing a p_j of each degree. We call $V_{\vec{g}}^{\vec{L}}(\vec{x})$ the confluent Vandermonde matrix (with respect to shape $\vec{L}$, in variables $\vec{x}$). We omit the $\vec{f}$ subscript when it is clear from context which family of functions is being used.

If all L_m are the same L, we write $V^L(\vec{x})$ for what we call the L^{th} confluent Vandermonde matrix (in variables $\vec{x}$). Observe, the 1^{st} confluent Vandermonde matrix is the ordinary Vandermonde matrix (in M many variables) whose determinant is

$$\Delta(\vec{x}) = \det V_{\vec{g}}^{1}(\vec{x}) = \prod_{1 \leq n < m \leq M} (x_m - x_n).$$

More generally, it is known [25]

$$\det V_{\vec{p}}^{\vec{L}}(\vec{x}) = \prod_{1 \leq n < m \leq M} (x_m - x_n)^{L_m L_n}$$

for any complete N-family of monic polynomials $\vec{p}$. In particular,

$$\det V_{\vec{p}}^{L}(\vec{x}) = \prod_{1 \leq n < m \leq M} (x_m - x_n)^{L^2} = \Delta(\vec{x})^{L^2}.$$

In the previous, more general case, we will write $\Delta^{\vec{L}}(\vec{x}) = \det V_{\vec{p}}^{\vec{L}}(\vec{x})$ to denote the confluent Vandermonde determinant with different exponents $L_m L_n$ on each difference $x_m - x_n$.

Recall from section 1.2, we desire to write the density functions of our ensembles as determinants. In the case of a single-component log-gas of charge L particles, we are able to write the density function as the determinant of a single $ML \times ML$ confluent Vandermonde matrix rather than the L^2 power of an $M \times M$ ordinary Vandermonde matrix. In the case of a multicomponent log-gas (see section 5.1), with charges $L_1, \ldots, L_M$, the different exponents $L_m L_n$ on each

difference $x_m - x_n$ mean the density function is no longer one power of an ordinary Vandermonde determinant. However, the density function can still be written as the determinant of a single confluent Vandermonde matrix (corresponding to shape $\vec{L}$).

As an example, consider $\vec{L} = (2, 3, 1)$ and $\vec{g} = \{x^{n-1}\}_{n=1}^{N}$. For simplicity, we will use the variables $\vec{x} = (a, b, c)$. Then the three columns corresponding to b are

$$V^3(b) = \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ b^2 & 2b & 1 \\ b^3 & 3b^2 & 3b \\ b^4 & 4b^3 & 6b^2 \\ b^5 & 5b^4 & 10b^3 \\ b^6 & 6b^5 & 15b^4 \end{bmatrix}.$$

In the third column, we have not just the second derivative but also a denominator of 2!. One consequence of these $l!$ denominators in D^{l-1} is that we get 1's on the top diagonal. Together, the full 6×6 confluent Vandermonde matrix is

$$V^{\vec{L}}(\vec{x}) = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ a & 1 & b & 1 & 0 & c \\ a^2 & 2a & b^2 & 2b & 1 & c^2 \\ a^3 & 3a^2 & b^3 & 3b^2 & 3b & c^3 \\ a^4 & 4a^3 & b^4 & 4b^3 & 6b^2 & c^4 \\ a^5 & 5a^4 & b^5 & 5b^4 & 10b^3 & c^5 \\ a^6 & 6a^5 & b^6 & 6b^5 & 15b^4 & c^6 \end{bmatrix}$$

whose determinant is

$$\det V^{\vec{L}}(\vec{x}) = (b-a)^6(c-a)^2(c-b)^3$$

2.8. Proto-Confluence

For completeness, we will give a proof of the confluent Vandermonde determinant identity. This proof uses the following lemma:

Lemma 2.1. *Suppose f is an n times differentiable function, and let $\nabla_h^n[f](x)$ be the n-step finite forward difference formula for f at x defined by*

$$\nabla_h^n[f](x) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}f(x+(n-k)h).$$

Then

$$\lim_{h\to 0}\frac{\nabla_h^n[f](x)}{h^n} = f^{(n)}(x).$$

To prove this, it is straightforward to show by induction on n,

$$\nabla_h^{n+1}[f](x) = \nabla_h^n[f](x+h) - \nabla_h^n[f](x),$$

and then show

$$\lim_{h\to 0}\frac{\nabla_h^n[f](x+h) - \nabla_h^n[f](x)}{h^{n+1}} = f^{(n+1)}(x).$$

Note, this also holds for f holomorphic with $x, h \in \mathbb{C}$.

Next, let $\vec{x} \in \mathbb{R}^M$, and define $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^M) \in \mathbb{R}^N$ by

$$\mathbf{x}^m = (x_m, x_m + h, x_m + 2h, \ldots, x_m + (L_m - 1)h) \in \mathbb{R}^{L_m}.$$

Define

$$B_{\vec{f}}^{\vec{L}}(h) = \left[B_{\vec{f}}^{L_1}(h) \quad B_{\vec{f}}^{L_2}(h) \quad \cdots \quad B_{\vec{f}}^{L_M}(h) \right],$$

where each $B_{\vec{f}}^{L_m}(h)$ is an $N \times L_m$ matrix defined by

$$B_{\vec{f}}^{L_m}(h) = \left[\frac{\nabla_h^l[f_n](x_m)}{h^{l-1}(l-1)!} \right]_{n,l=1}^{N,L_m}.$$

Note, $\nabla_h^l[f_n](x_m)$ is a linear combination of $f_n(x_m + (l-1)h)$ for $1 \leq l \leq L_m$. Thus, by taking linear combinations of columns,

$$\det V_{\vec{f}}^1(\mathbf{x}) = C_M^{\vec{L}}(h) \det B_{\vec{f}}^{\vec{L}}(h),$$

where

$$C_M^{\vec{L}}(h) = \prod_{m=1}^{M} \left[h^{\binom{L_m}{2}} \prod_{l=1}^{L_m} (l-1)! \right] = \prod_{m=1}^{M} \Delta(h\underline{L_m}).$$

By Lemma 2.1 (acting on each entry in $B_{\vec{f}}^{\vec{L}}(h)$), we have

$$\det V_{\vec{f}}^{\vec{L}}(\vec{x}) = \lim_{h \to 0} \det B_{\vec{f}}^{\vec{L}}(h) = \lim_{h \to 0} \frac{\det V_{\vec{f}}^1(\vec{\mathbf{x}})}{C_M^{\vec{L}}(h)}.$$

In particular, if $\vec{p}$ is a complete N-family of monic polynomials, then

$$\begin{aligned}
\det V_{\vec{p}}^{\vec{L}}(\vec{x}) &= \lim_{h\to 0} \frac{\det V_{\vec{p}}^{1}(\mathbf{x})}{C_M^{\vec{L}}(h)} \\
&= \lim_{h\to 0} \left[\prod_{1\le n<m\le M} \left(\prod_{l=1}^{L_m-1} \prod_{k=1}^{L_n-1} (x_m - x_n + (l-k)h) \right) \right] \\
&\qquad \times \left[\prod_{m=1}^{M} \left(\prod_{l=1}^{L_m-1} \prod_{k=1}^{L_m-1} (l-k)h \right) \right] \Bigg/ \prod_{k=1}^{M} \left[h^{\binom{L_m}{2}} \prod_{l=1}^{L_m} (l-1)! \right] \\
&= \prod_{1\le n<m\le M} (x_m - x_n)^{L_m L_n} \\
&= \Delta^{\vec{L}}(\vec{x}).
\end{aligned}$$

Because $V_{\vec{f}}^{1}(\mathbf{x})$, an ordinary alternant evaluated at the translated variables $\mathbf{x}$, gives the confluent alternant (with respect to shape $\vec{L}$) in the limit, we can call $V_{\vec{f}}^{1}(\mathbf{x})$ a *proto-confluent* alternant (with respect to a translation vector $\vec{y}$) in the variables $\vec{x}$.

As mentioned in section 2.6, the derivative columns of the confluent Vandermonde matrix (and its Wronskian minors) correspond to the charges of the particles in a log-gas. Similarly, the translated variables of the proto-confluent Vandermonde matrix (and its proto-Wronskian minors) correspond to the additional copies of each particle across parallel lines in a constellation ensemble (see section 4.1). Taking the limit as $h \to 0$, the parallel lines collapse onto each other so that the additional copies of each particle combine into particles of higher charge, corresponding to higher derivative columns (see section 4.5).

CHAPTER III

GENERALIZED DE BRUIJN IDENTITIES

This chapter contains unpublished coauthored material. In particular, sections 3.1-3.5 appear nearly as is in [35].

Let $N = L_1 + \cdots + L_J$. Define $K_j = \sum_{k=1}^{j} L_k$. Let $A(\vec{x})$ be an $N \times N$ matrix whose entries are single-variable integrable functions of variables $\vec{x} = (x_1, \ldots, x_J)$. Explicitly, the first L_1 many columns are functions of x_1, the second L_2 many columns are functions of x_2, and so on up through x_J. For $\mathfrak{t} : \underline{L_j} \nearrow \underline{N}$, let $A_{\mathfrak{t}}(x_j)$ denote the $L_j \times L_j$ submatrix of $A(\vec{x})$ given by

$$A_{\mathfrak{t}}(x_j) = \left[A(\vec{x})_{\mathfrak{t}(l),n+K_{j-1}}\right]_{l,n=1}^{L_j},$$

equivalently obtained from $A(\vec{x})$ by taking the rows $\mathfrak{t}(1), \ldots, \mathfrak{t}(L_j)$ from the L_j many columns in the same variable x_j. Define

$$\gamma_j^A = \sum_{\mathfrak{t}:\underline{L_j}\nearrow\underline{N}} \int_{\mathbb{R}} \det A_{\mathfrak{t}}(x_j)\, dx_j\, \varepsilon_{\mathfrak{t}},$$

and define

$$\eta_{j,k}^A = \sum_{\mathfrak{t}:\underline{L_j}\nearrow\underline{N}} \sum_{\mathfrak{s}:\underline{L_k}\nearrow\underline{N}} \int\int_{x_j<x_k} \det A_{\mathfrak{t}}(x_j) \det A_{\mathfrak{s}}(x_k)\, dx_j\, dx_k\, \varepsilon_{\mathfrak{t}} \wedge \varepsilon_{\mathfrak{s}}.$$

Then the general algebraic framework for the partition functions featured in this volume can be summarized in the following theorem:

Theorem 3.1. *Suppose the first r many L_j are even, then*

$$\int_{-\infty<x_1<\cdots<x_J<\infty} \det A(\vec{x})\, dx_1 \cdots dx_J = \int \omega\, d\varepsilon_{\mathrm{vol}},$$

where ω is defined as follows:

1. *If N is even, then*

$$\omega = \frac{1}{\left(r + \frac{J-r}{2}\right)!} \bigwedge_{j=1}^{r} \gamma_j^A \wedge \bigwedge_{m=1}^{(J-r)/2} \eta^A_{r+2m-1,r+2m}.$$

2. *If N is odd, then*

$$\omega = \frac{1}{\left(r+1+\frac{J-r-1}{2}\right)!} \bigwedge_{j=1}^{r} \gamma_j^A \wedge \bigwedge_{m=1}^{(J-r-1)/2} \eta^A_{r+2m-1,r+2m} \wedge \gamma_J^A.$$

Note, we require even forms (possibly either γ_j^A or $\eta_{j,k}^A$) so they commute. For $1 \leq j \leq r$, L_j is even, and γ_j^A is an even L_j-form. For the L_j which are odd, $\eta_{j,k}^A$ combines minors of odd $L_j \times L_j$ dimensions with minors of odd $L_k \times L_k$ dimensions to produce an even $(L_j + L_k)$-form. In case 1, the requirement that N be even means there are an even number of odd L_j to be paired down into $(J-r)/2$ pairs. In case 2, there are an odd number of odd L_j, so γ_J^A remains as an odd L_J-form. Though this extra γ_J^A is an odd form, it commutes with all the even forms.

In our applications, it is necessary to extend the ε_j basis for $\mathbb{R}^N$ to a basis for $\mathbb{R}^{N+k}$ and extend the odd γ_J^A form by these new basis vectors to create another even form. In general, we can write

$$\varepsilon_{\mathrm{vol}_k} = \varepsilon_{\mathrm{vol}} \wedge \xi_k = \varepsilon_{\mathrm{vol}} \wedge \varepsilon_{N+1} \wedge \varepsilon_{N+2} \wedge \cdots \wedge \varepsilon_{N+k}.$$

Then for any $\omega \in \bigwedge(\mathbb{R}^N) \leq \bigwedge(\mathbb{R}^{N+k})$, we have

$$\int \omega \, d\varepsilon_{\mathrm{vol}} = \int \omega \wedge \varepsilon_{N+1} \wedge \cdots \wedge \varepsilon_{N+k} \; d\varepsilon_{\mathrm{vol}} \, d\varepsilon_{N+1} \cdots d\varepsilon_{N+k} = \int \omega \wedge \xi_k \; d\varepsilon_{\mathrm{vol}_k}.$$

Thus, we can embed any Berezin integral computation in a higher dimension if desired.

Recall, we assume the functions which make up $A(\vec{x})$ are suitably integrable so that all integrals which appear in γ_j^A and $\eta_{j,k}^A$ are finite. However, we do not assume any resemblance between the L_j many columns in x_j and the L_k many columns in x_k. Assuming some additional consistency, we obtain a Hyperpfaffian analogue of the de Bruijn integral identities.

Corollary. *Let $\xi_k = \varepsilon_{N+1} \wedge \varepsilon_{N+2} \wedge \cdots \wedge \varepsilon_{N+k}$. Suppose $L_1 = \cdots = L_J = L$. Under the additional assumption that $\gamma_j^A = \gamma$ for all j, and $\eta_{j,k}^A = \eta$ for all j, k (typically because the entries of $A(\vec{x})$ in one variable x_j are the same as the entries in any other variable x_k),*

$$\int_{-\infty < x_1 < \cdots < x_J < \infty} \det A(\vec{x}) \, dx_1 \cdots dx_J = \mathrm{BE}_{\mathrm{vol}_k}(\omega) = \mathrm{PF}(\omega),$$

where ω and k depend on M and L.

1. *If L is even, then $\omega = \gamma$ and $\mathrm{BE}_{\mathrm{vol}_k} = \mathrm{BE}_{\mathrm{vol}}$.*
2. *If L is odd and M is even, then $\omega = \eta$ and $\mathrm{BE}_{\mathrm{vol}_k} = \mathrm{BE}_{\mathrm{vol}}$.*
3. *If L is odd and M is odd, then $\omega = \eta + \gamma \wedge \xi_L$ and $\mathrm{BE}_{\mathrm{vol}_k} = \mathrm{BE}_{\mathrm{vol}_L}$.*

Note, we extend γ by ξ_L instead of just $\xi_1 = \varepsilon_{N+1}$ in case 3 only so that $\gamma \wedge \xi_L$ is a $2L$-form and therefore ω is homogeneous (for which the Hyperpfaffian is

defined). Every choice of k produces a different but equally valid Berezin integral expression. We obtain the (Pfaffian) de Bruijn integral identities for classical $\beta = 1$ and $\beta = 4$ when $L = 1$ and $L = 2$, respectively.

In chapter V, the confluent Vandermonde structure of section 2.7 allows us to write the relevant density functions as a determinant to which Theorem 3.1 will apply. Likewise, in chapter IV, the proto-confluent Vandermonde structure of section 2.8 allows us to do the same for those density functions. This chapter builds toward a proof of the more general algebraic framework outlined in Theorem 3.1.

Even further generalization in Theorem 3.1 is still possible, if desired. Suppose instead the first L_1 columns of $A(\vec{x})$ are made up of functions, not necessarily single variable, of variables $x_1, \ldots, x_a$, and the next L_2 columns of $A(\vec{x})$ are made up of functions of variables $x_{a+1}, \ldots, x_b$. If L_1 is even, then we replace γ_1^A with

$$\gamma_1^A = \sum_{\mathfrak{t}:\underline{L_1} \nearrow \underline{N}} \int_{-\infty<x_1<\cdots<x_a<\infty} \det A_{\mathfrak{t}}(x_1, \ldots, x_a)\, dx_1 \cdots dx_a\, \varepsilon_{\mathfrak{t}},$$

which now features iterated integrals of the multivariate minors. If L_1 and L_2 are odd, then we replace $\eta_{1,2}^A$ with

$$\eta_{1,2}^A = \sum_{\mathfrak{t}:\underline{L_1} \nearrow \underline{N}} \sum_{\mathfrak{s}:\underline{L_2} \nearrow \underline{N}} \int_{-\infty<x_1<\cdots<x_b<\infty} \Big[\det A_{\mathfrak{t}}(x_1, \ldots, x_a) \times \det A_{\mathfrak{s}}(x_{a+1}, \ldots, x_b) \Big]\, dx_1 \cdots dx_b\, \varepsilon_{\mathfrak{t}} \wedge \varepsilon_{\mathfrak{s}}.$$

In general, we integrate the minors with respect to whichever variables appear, in the same total order on the domain induced by the original integral of $\det A(\vec{x})$.

3.1. Additional Conventions

For any injection $\mathfrak{t} : \underline{K} \to \underline{N}$, let $Q_{\mathfrak{t}}$ denote $Q_{\mathfrak{t}(1),\ldots,\mathfrak{t}(K)}$ whenever it is clear from context Q admits K many indices, and let $Q_{\mathfrak{t}}$ denote $\{Q_{\mathfrak{t}(k)}\}_{k=1}^{K}$ whenever it is clear from context Q admits only one index. For any permutation $\sigma \in S_K$, we can view $\sigma : \underline{K} \to \underline{K}$ as a bijection and then write $Q_{\mathfrak{t}\circ\sigma}$ to denote $Q_{\mathfrak{t}\circ\sigma(1),\ldots,\mathfrak{t}\circ\sigma(K)}$ or $\{Q_{\mathfrak{t}\circ\sigma(k)}\}_{k=1}^{K}$ as appropriate in context.

For example, when V is a matrix, $V_{\mathfrak{t}}$ is a submatrix. We should think of $V_{\mathfrak{t}}$ as a single object with K many indices (which indicate a choice of K many rows $\mathfrak{t}(1), \ldots, \mathfrak{t}(K)$ from which our submatrix is constructed). Similarly, if $\omega \in \bigwedge^K(\mathbb{R}^N)$, then $A_{\mathfrak{t}}$ might denote the coefficient of $\varepsilon_{\mathfrak{t}}$ (equivalently, an entry in a K-dimensional hyper array). In contrast, if $\vec{f} = \{f_k\}_{k=1}^{N}$ is a family of functions, then $\vec{f}_{\mathfrak{t}} = \{f_{\mathfrak{t}(k)}\}_{k=1}^{K}$ is a subfamily of K functions, each indexed by a single integer. In the statement of our main results, we use these increasing function subscripts in both ways, but it is clear from context how these subscripts should be applied differently to different objects.

Let $\mathfrak{t} : \underline{K} \nearrow \underline{N}$ denote a strictly increasing function from $\underline{K}$ to $\underline{N}$. Note, every injection $\mathfrak{s} : \underline{K} \to \underline{N}$ can be written uniquely as $\mathfrak{s} = \mathfrak{t} \circ \sigma$ for some $\mathfrak{t} : \underline{K} \nearrow \underline{N}$ and $\sigma \in S_K$. For any $\mathfrak{t} : \underline{K} \nearrow \underline{N}$, there exists a unique complementary $\mathfrak{t}' : \underline{N - K} \nearrow \underline{N}$ with $\mathfrak{t}(\underline{K}) \cup \mathfrak{t}'(\underline{N - K}) = \underline{N}$. Define $\operatorname{sgn}(\mathfrak{t})$ to be the signature of the permutation $\sigma \in S_N$ given by

$$\sigma(k) = \begin{cases} \mathfrak{t}(k) & \text{if } k \in \underline{K} \\ \mathfrak{t}'(k - K) & \text{if } k \in \underline{N} \setminus \underline{K} \end{cases}.$$

Equivalently,

$$\operatorname{sgn}(\mathfrak{t}) = \int \varepsilon_{\mathfrak{t}} \wedge \varepsilon_{\mathfrak{t}'} \, d\varepsilon_{\mathrm{vol}}.$$

For any $\Lambda = (\lambda_1, \ldots, \lambda_K)$ which partitions N and $\mathfrak{t} : \underline{N} \to \underline{M}$, write $\mathfrak{t} = (\mathfrak{t}_1 | \cdots | \mathfrak{t}_K)$ to indicate a decomposition of $\mathfrak{t}$ in which $\mathfrak{t}_1$ is the restriction of $\mathfrak{t}$ to the first λ_1 positive integers, and each $\mathfrak{t}_k$ is the restriction of $\mathfrak{t}$ to the next λ_k positive integers. For convenience, we will treat each $\mathfrak{t}_k$ as having domain $\underline{\lambda_k}$ instead of the appropriate subset of $\underline{N}$ of size λ_k.

Conversely, for any $\mathfrak{t}_1 : \underline{\lambda_1} \to \underline{M}, \ldots, \mathfrak{t}_K : \underline{\lambda_K} \to \underline{M}$, we can construct $(\mathfrak{t}_1 | \cdots | \mathfrak{t}_K) = \mathfrak{t} : \underline{N} \to \underline{M}$ by defining for each $n \in \underline{N}$, $\mathfrak{t}(n) = \mathfrak{t}_k \left(n - \sum_{j=1}^{k-1} \lambda_j \right)$ where this k, which depends on n, is the largest k for which the difference inside the parentheses is positive. As before, it will be convenient to identify the restrictions of this new $\mathfrak{t}$ with the original $\mathfrak{t}_1, \ldots, \mathfrak{t}_K$ even though the domains are not exactly the same.

In general, this is just a bookkeeping device which gives us a convenient notation for a choice of indices from the codomain $\underline{M}$. For example, when $\sigma : \underline{N} \to \underline{N}$ is a permutation, we can think of σ as sorting N many possible indices into K blocks of different sizes $\lambda_1, \ldots, \lambda_K$ as specified by the images of the σ_k.

3.2. Decomposition of the Symmetric Group

For any $\Lambda = (\lambda_1, \ldots, \lambda_K)$ which partitions N, let $\mathrm{H}(\Lambda) \subseteq S_N$ denote the *Young subgroup*, meaning $\mathrm{H}(\Lambda) \cong S_{\lambda_1} \times \cdots \times S_{\lambda_K}$. Viewing σ as a function from $\underline{N}$ to $\underline{N}$, we can write the decomposition (with respect to Λ) as $\sigma = (\sigma_1 | \cdots | \sigma_K)$. Then we should think of these σ_k belonging to the appropriate S_{λ_k}.

Let $\mathrm{Sh}(\Lambda) \subseteq S_N$ denote the subset of *shuffle permutations*. These are permutations which satisfy $\sigma(i) < \sigma(j)$ whenever

$$\lambda_1 + \cdots + \lambda_k < i < j \leq \lambda_1 + \cdots + \lambda_{k+1}.$$

Each shuffle permutation represents a way to iteratively riffle shuffle K stacks of $\lambda_1, \ldots, \lambda_K$ many cards into a single pile of N cards while preserving the original ordering within each of the K stacks. Equivalently, the shuffle permutations are the $\sigma \in S_N$ for which each σ_k is a strictly increasing function from $\underline{\lambda_k}$ to $\underline{N}$.

Let $\mathrm{Sh}^\circ(\Lambda) \subseteq \mathrm{Sh}(\Lambda)$ denote the subset of *ordered* shuffle permutations. These are shuffle permutations which also satisfy

$$\sigma(1) < \sigma(\lambda_1 + 1) < \sigma(\lambda_1 + \lambda_2 + 1) < \cdots < \sigma(\lambda_1 + \cdots + \lambda_{K-1} + 1).$$

Using the decomposition $\sigma = (\sigma_1 | \cdots | \sigma_K)$, we can conveniently rewrite the above condition as

$$\sigma_1(1) < \sigma_2(1) < \cdots < \sigma_K(1).$$

Let $\mathrm{Bl}(\Lambda) \subseteq S_N$ denote the subset of *block permutations*. A block permutation represents shuffling a deck of cards by first separating the deck into a pile of the first λ_1 cards, a pile of the second λ_2 cards, and so on, then reassembling the deck without interlacing the piles or shuffling within any of the piles. Using the decomposition $\sigma = (\sigma_1 | \cdots | \sigma_K)$, block permutations are permutations for which each $\sigma_k : \underline{\lambda_k} \to \underline{N}$ acts by $\sigma_k(j) = (j-1) + \sigma_k(1)$.

Clearly, any block permutation is determined entirely by the action on the first element of each block (of λ_k elements), of which there are K many. Given a block permutation σ, define $\theta_\sigma \in S_K$ to be the unique permutation for which $\theta_\sigma(i) < \theta_\sigma(j)$ if and only if $\sigma_i(1) < \sigma_j(1)$. Heuristically, σ moves blocks of λ_k consecutive elements together. θ_σ is the unique action on the K many blocks determined by σ. The map $\sigma \mapsto \theta_\sigma$ is bijective, so $\mathrm{Bl}(\Lambda) \cong S_K$.

Note, because the blocks have (possibly) different sizes λ_j, block permutations do not (in general) preserve partitions. Define

$$\Lambda^\sigma = (\lambda_{\theta_\sigma^{-1}(1)}, \ldots, \lambda_{\theta_\sigma^{-1}(K)}),$$

obtained from Λ by reordering its entries according to θ_σ.

In section 3.5, we make use of the following Lemma which decomposes permutations in the symmetric group as products of ordered shuffles, block permutations, and young permutations (composed in the opposite order).

Lemma 3.1. *Let $\Lambda = (\lambda_1, \ldots, \lambda_K)$ be a partition of N. Given any $\varphi \in S_N$, there exists unique permutations $\tau \in \mathrm{H}(\Lambda)$, $\pi \in \mathrm{Bl}(\Lambda)$, and $\sigma \in \mathrm{Sh}^\circ(\Lambda^\pi)$ so that $\varphi = \sigma \circ \pi \circ \tau$.*

For completeness, we include a proof of this lemma in section 3.3.

3.3. Proof of Decomposition Lemma

Let $\Lambda = (\lambda_1, \ldots, \lambda_K)$ be a partition of N. Recall from section 3.2 the definitions of the *Young subgroup* $\mathrm{H}(\Lambda) \subseteq S_N$, the subset of *block permutations* $\mathrm{Bl}(\Lambda) \subset S_N$, the subset of *shuffle permutations* $\mathrm{Sh}(\Lambda)$, and the subset of *ordered shuffle permutations* $\mathrm{Sh}^\circ(\Lambda)$.

The following proof of Lemma 3.1 is due to Wells [35]. Conducive to this proof, it will be convenient to give alternate definitions for the different subsets of permutations. First, define $s_j = \sum_{k=1}^{j-1} \lambda_k$ to be the partial sums of the λ_k, up to but not including λ_j so that $s_1 = 0$, $s_2 = \lambda_1$, $s_3 = \lambda_1 + \lambda_2$, and so on. We

alternatively define the Young subgroup $\mathrm{H}(\Lambda)$ to be the $\sigma \in S_N$ such that

$$s_k + 1 \leq \sigma(s_k + j) \leq s_{k+1}$$

for all $1 \leq k \leq K$ and $1 \leq j \leq \lambda_k$. We define the block permutations $\mathrm{Bl}(\Lambda)$ to be the $\sigma \in S_N$ such that

$$\sigma(s_k + j) + 1 = \sigma(s_k + j + 1)$$

for all $1 \leq k \leq K$ and $1 \leq j \leq \lambda_k$. Recall the definition of $\theta_\sigma \in S_K$ from section 3.2. This is the unique permutation such that

$$\sigma(s_{\theta_\sigma^{-1}(1)} + 1) < \sigma(s_{\theta_\sigma^{-1}(2)} + 1) < \cdots < \sigma(s_{\theta_\sigma^{-1}(K)} + 1).$$

The shuffle permutations $\mathrm{Sh}(\Lambda)$ are the $\sigma \in S_N$ such that

$$\sigma(s_k + i) < \sigma(s_k + j)$$

for all $1 \leq k \leq K$ and $1 \leq i < j \leq \lambda_k$. The ordered shuffle permutations $\mathrm{Sh}^\circ(\Lambda)$ additionally satisfy

$$\sigma(s_j + 1) < \sigma(s_k + 1)$$

for all $1 \leq j < k \leq K$.

Demonstrably, $|\mathrm{H}(\Lambda)| = \prod_{k=1}^{K} |S_{\lambda_k}| = \lambda_1! \cdots \lambda_K!$, and $|\mathrm{Bl}(\Lambda)| = |S_K| = K!$. Heuristically, a shuffle permutation $\sigma \in \mathrm{Sh}(\Lambda)$ is constructed by choosing from N positions the location of the first λ_1 elements, then the next λ_2 elements, and so on until all elements are exhausted. It is straightforward to see $|\mathrm{Sh}(\Lambda)|$ is the

multinomial coefficient

$$|\mathrm{Sh}(\Lambda)| = \binom{N}{\lambda_1, \ldots, \lambda_K} = \frac{N!}{\lambda_1! \cdots \lambda_K!}.$$

We prove Lemma 3.1 in two steps. First, we show the decomposition of an arbitrary permutation into a product of a shuffle permutation after a permutation belonging to the Young subgroup. Second, we show this shuffle permutation can be further decomposed into a product of an ordered shuffle permutation after a block permutation.

Lemma 3.2. *Given any $\varphi \in S_N$, there exists unique $\tau \in \mathrm{H}(\Lambda)$ and $\rho \in \mathrm{Sh}(\Lambda)$ so that $\varphi = \rho \circ \tau$.*

Proof. Consider the collection of right cosets $S_N/\mathrm{H}(\Lambda)$. For each $1 \leq k \leq K$ and $1 \leq j \leq \lambda_k$, let a_j^k be the j^{th} smallest element of the set $\{\varphi(s_k + 1), \varphi(s_k + 2), \ldots, \varphi(s_k + \lambda_k)\}$, and define a permutation $\tau \in S_N$ by

$$\tau(s_k + j) = \varphi^{-1}(a_j^k)$$

for $1 \leq k \leq K$ and $1 \leq j \leq \lambda_k$. Then $\tau \in \mathrm{H}(\Lambda)$, and

$$\varphi \circ \tau(s_k + i) < \varphi \circ \tau(s_k + j)$$

whenever $1 \leq i < j \leq \lambda_k$. Thus, every coset $T \in S_N/\mathrm{H}(\Lambda)$ contains at least one shuffle permutation. Note,

$$|S_N/\mathrm{H}(\Lambda)| = \frac{N!}{\lambda_1! \cdots \lambda_K!} = |\mathrm{Sh}(\Lambda)|.$$

Thus, each coset contains a unique shuffle permutation. Define $\rho \in \mathrm{Sh}(\Lambda)$ to be this unique shuffle permutation in the coset to which φ belongs. Then $\varphi = \rho \circ \tau$ as desired. $\square$

Lemma 3.3. *Given any $\rho \in \mathrm{Sh}(\Lambda)$, there exists unique permutations $\pi \in \mathrm{Bl}(\Lambda)$ and $\sigma \in \mathrm{Sh}^{\circ}(\Lambda^{\pi})$ so that $\rho = \sigma \circ \pi$.*

Proof. For each $1 \leq k \leq K$, define b_k to be the k^{th} smallest element of the set

$$\{\rho(s_1 + 1), \rho(s_2 + 1), \ldots, \rho(s_K + 1)\}.$$

Let $\alpha \in S_K$ be the permutation satisfying

$$\rho(s_k + 1) = b_{\alpha(k)}$$

for $1 \leq k \leq K$. Define $\pi \in \mathrm{Bl}(\Lambda)$ by

$$\pi(s_k + j) = \rho^{-1}(b_{\alpha(k)}) + j - 1$$

for $1 \leq k \leq K$ and $1 \leq j \leq \lambda_k$. Then $\alpha = \theta_{\pi}$. Let $\sigma = \rho \circ \pi^{-1}$. We want to show $\sigma \in \mathrm{Sh}^{\circ}(\Lambda^{\pi})$.

Let $\mu = \Lambda^\pi$, and let $t_j = \sum_{k=1}^{j-1} \mu_k$. Suppose $1 \leq k \leq K$ and $1 \leq j \leq \mu_k$. Observe that

$$\begin{aligned}\sigma(t_k + j) &= \rho \circ \pi^{-1}(t_k + 1 + j - 1) \\ &= \rho \circ \pi^{-1}(\pi(s_{\alpha^{-1}(k)} + 1) + j - 1) \\ &= \rho \circ \pi^{-1}(\pi(s_{\alpha^{-1}(k)} + j)) \\ &= \rho(s_{\alpha^{-1}(k)} + j).\end{aligned}$$

Thus,

$$\sigma(t_k + j) = \rho(s_{\alpha^{-1}(k)} + j).$$

If $1 \leq i < j \leq \mu_k$, then $\rho(s_{\alpha^{-1}(k)} + i) < \rho(s_{\alpha^{-1}(k)} + j)$ since $\rho \in \mathrm{Sh}(\Lambda)$. Thus, $\sigma(t_k + i) < \sigma(t_k + j)$, and $\sigma \in \mathrm{Sh}(\Lambda^\pi)$.

Next, if $i < j$, then

$$\rho(s_{\alpha^{-1}(i)} + 1) = b_i < b_j = \rho(s_{\alpha^{-1}(j)} + 1).$$

Thus, $\sigma(t_i + 1) < \sigma(t_j + 1)$, and $\sigma \in \mathrm{Sh}^\circ(\Lambda^\pi)$.

It remains to show that this decomposition $\rho = \sigma \circ \pi$ is unique. Suppose $\rho = \sigma' \circ \pi'$ for some $\pi' \in \mathrm{Bl}(\Lambda)$ and $\sigma' \in \mathrm{Sh}^\circ(\Lambda^{\pi'})$. As before, let $\alpha \in S_K$ be the permutation satisfying

$$\rho(s_k + 1) = b_{\alpha(k)}$$

for $1 \leq k \leq K$. Define c_k to be the k^{th} smallest element of the set $\{\pi'(s_1+1), \pi'(s_2+1), \ldots, \pi'(s_K+1)\}$. Since $\sigma' \in \mathrm{Sh}^\circ(\Lambda^{\pi'})$, we have $\sigma'(c_k) = b_k$ for each $1 \leq k \leq K$, and thus $\theta_{\pi'} = \alpha = \theta_\pi$. Since π is completely determined by θ_π, we have $\pi = \pi'$. Thus, $\sigma' = \pi^{-1} \circ \rho = \sigma$ as desired. $\square$

Finally, Lemma 3.1 follows immediately from applying Lemma 3.3 after Lemma 3.2.

3.4. Chen's Lemma

For a set X and a ring R, let $R\langle X\rangle$ denote the free unital algebra on X over R. Given $u = u_1 \cdots u_k$ and $u' = u_{k+1} \cdots u_n$, define an operation $\sqcup\!\sqcup$ on $R\langle X\rangle$ as follows:

$$u \sqcup\!\sqcup u' = \sum_{\sigma \in \mathrm{Sh}(k,n-k)} u_{\sigma^{-1}(1)} \cdots u_{\sigma^{-1}(n)}$$

and by $e \sqcup\!\sqcup e = e$ for the empty word $e \in R\langle X\rangle$. Denote by $R\langle X\rangle_{\sqcup\!\sqcup}$ the algebra $R\langle X\rangle$ with (shuffle) multiplication $\sqcup\!\sqcup$, which is called the *shuffle algebra* on X. This shuffle product was first introduced by Eilenberg and Mac Lane in [13].

Let $\mathcal{H}$ be the Hilbert space $L^2(\mathbb{R})$ of square integrable functions with respect to Lebesgue measure on $\mathbb{R}$, and suppose H is a finite-dimensional subspace of $\mathcal{H}$ with basis X. We assume H is large enough to contain all functions of interest to us.

For any $\sigma \in S_k$, let $\Delta_k(\sigma) \subset (a,b)^k$ denote the region where $a < x_{\sigma^{-1}(1)} < \cdots < x_{\sigma^{-1}(k)} < b$. For each non-negative integer k, define a linear functional $\langle\cdot\rangle_k$ on the k^{th} graded component of $\mathbb{R}\langle X\rangle \cong T(V)$ by

$$\langle f_1 \otimes \cdots \otimes f_k\rangle_k = \int_{\Delta_k(\mathrm{id})} f_1(x_1) \cdots f_K(x_k)\, dx_1 \cdots dx_k.$$

Note, $T(V)$ denotes the tensor algebra to which $\mathbb{R}\langle X\rangle$ is isomorphic. Though not strictly necessary, we write $f_1 \otimes \cdots \otimes f_k \in T(V)$ instead of $f_1 \cdots f_k \in \mathbb{R}\langle X\rangle$ to avoid confusing concatenation with function multiplication.

This collection $\{\langle\cdot\rangle_k\}_{k=0}^{\infty}$ defines a functional $\langle\cdot\rangle$ on $\mathbb{R}\langle X\rangle$ whereby $\langle\cdot\rangle$ acts as $\langle\cdot\rangle_k$ on the k^{th} graded component of a non-homogeneous tensor. The following lemma, due to Chen [8], asserts that this operator $\langle\cdot\rangle$ is actually an algebra homomorphism from $T(V)_{\sqcup\!\sqcup}$ to $\mathbb{R}$:

Lemma 3.4 (Chen)**.** *If $f, g \in \mathbb{R}\langle X\rangle$, then $\langle f \sqcup\!\sqcup g\rangle = \langle f\rangle\langle g\rangle$.*

A major hurdle in computing the partition function Z_N is the absolute value inside the defining integral. We will remove the absolute value by decomposing the domain of integration into these totally ordered subsets $\Delta_k(\sigma)$. However, when we change the domain of integration, we lose the ability to use Fubini's Theorem. Chen's Lemma serves the role of Fubini's Theorem, provided we can demonstrate the integrand to have the appropriate form.

Proof. We can assume $f = f_1 \otimes \cdots \otimes f_k$ is a pure tensor of length k and $g = f_{k+1} \otimes \cdots \otimes f_n$ is a pure tensor of length $n-k$ (because $\langle\cdot\rangle$ is linear and $\sqcup\!\sqcup$ distributes over addition). Then $\langle f \sqcup\!\sqcup g\rangle = \langle f \sqcup\!\sqcup g\rangle_n$ is an n-fold iterated integral over $\Delta_n(\mathrm{id})$. Likewise, $\langle f\rangle\langle g\rangle = \langle f\rangle_k\langle g\rangle_{n-k}$ is the product of a k-fold iterated integral over $\Delta_k(\mathrm{id})$ and an $(n-k)$-fold iterated integral over $\Delta_{n-k}(\mathrm{id})$. Under the integrability criteria on X, Fubini's Theorem tells us this is the same as an n-fold iterated integral over the product space $\Delta_k(\mathrm{id}) \times \Delta_{n-k}(\mathrm{id})$.

Key to the proof of Chen's Lemma is the observation that

$$\Delta_k(\mathrm{id}) \times \Delta_{n-k}(\mathrm{id}) = Z \cup \bigcup_{\sigma\in \mathrm{Sh}(k,n-k)} \Delta_n(\sigma),$$

where $Z \subset (a,b)^n$ has measure 0. To this end, let Y be the set of points in $(a,b)^n$ whose coordinates are all distinct, and then let $Z = ((a,b)^n \setminus Y) \cap (\Delta_k(\mathrm{id}) \times \Delta_{n-k}(\mathrm{id}))$. If $\vec{x} \in (\Delta_k(\mathrm{id}) \times \Delta_{n-k}(\mathrm{id})) \setminus Z \subset Y$, then its coordinates are all

distinct and can be ordered according to some permutation, meaning $\vec{x} \in \Delta_n(\sigma)$ for a unique $\sigma \in S_n$. Since $\vec{x} \in \Delta_k(\mathrm{id}) \times \Delta_{n-k}(\mathrm{id})$, this permutation is one which separately preserves the relative order of the first k coordinates and the relative order of the remaining $n-k$ coordinates, meaning $\sigma \in \mathrm{Sh}(k, n-k)$. Thus,

$$\begin{aligned}\langle f \rangle \langle g \rangle &= \int_{\Delta_k(\mathrm{id}) \times \Delta_{n-k}(\mathrm{id})} f_1(x_1) \cdots f_n(x_n)\, dx_1 \cdots dx_n \\ &= \sum_{\sigma \in \mathrm{Sh}(k,n-k)} \int_{\Delta_n(\sigma)} f_1(x_1) \cdots f_n(x_n)\, dx_1 \cdots dx_n.\end{aligned}$$

By relabeling the variables as $x_j = y_{\sigma(j)}$, we can rewrite this sum as

$$\begin{aligned}&= \sum_{\sigma \in \mathrm{Sh}(k,n-k)} \int_{\Delta_n(\mathrm{id})} f_1(y_{\sigma(1)}) \cdots f_n(y_{\sigma(n)})\, dy_1 \cdots dy_n \\ &= \sum_{\sigma \in \mathrm{Sh}(k,n-k)} \int_{\Delta_n(\mathrm{id})} f_{\sigma^{-1}(1)}(y_1) \cdots f_{\sigma^{-1}(n)}(y_n)\, dy_1 \cdots dy_n \\ &= \langle f \sqcup\!\sqcup g \rangle.\end{aligned}$$

□

Until chapter VI, we will only need $(a, b) = (-\infty, \infty) = \mathbb{R}$, but we have shown Chen's Lemma to hold for more general intervals. Also, we do not need our functions to be real-valued. We just need the codomain to be associative and commutative, provided our functions are appropriately integrable. In particular, we will use complex-valued functions on real domain $[0, 2\pi)$ for the circular ensembles in chapter VI.

3.5. Exterior Shuffle Algebra

Let $\{Q^j_{\mathfrak{t}} \mid j \in \underline{J}, \mathfrak{t} : \underline{L_j} \to \underline{N}\}$ be a subset of an alphabet I, and let F be a field of characteristic 0. We use single integer superscripts to emphasize there are different Q's which admit different numbers of integer subscripts, chosen by $\mathfrak{t}$'s of different sizes. For each $\mathfrak{t} : \underline{L_j} \to \underline{N}$, define $A^j_{\mathfrak{t}} \in F\langle I \rangle$ by

$$A^j_{\mathfrak{t}} = \sum_{\tau \in S_{L_j}} \operatorname{sgn}(\tau) Q^j_{\mathfrak{t} \circ \tau}.$$

We call this the *antisymmetrization* of $Q^j_{\mathfrak{t}}$. We should think of this as analogous to taking a pure tensor in the tensor algebra and then adding to it all possible orderings of the basis vectors with signs. Let V be a rank N free module over $R = F\langle I \rangle_{\sqcup\!\sqcup}$. Define $\alpha_j \in \bigwedge_R V$ by

$$\alpha_j = \sum_{\mathfrak{t}:\underline{L_j} \to \underline{N}} Q^j_{\mathfrak{t}}\, \varepsilon_{\mathfrak{t}} = \sum_{\mathfrak{t}:\underline{L_j} \nearrow \underline{N}} A^j_{\mathfrak{t}}\, \varepsilon_{\mathfrak{t}}.$$

We call this an *antisymmetrized* L_j-form.

Let $\vec{M} = (M_1, \ldots, M_J)$, and let $\vec{L} = (L_1, \ldots, L_J)$ such that each L_j is even. Let $N = \vec{L} \cdot \vec{M}$, and let $\Lambda = (L_1, \ldots, L_1, L_2, \ldots, L_2, \ldots, L_J, \ldots, L_J) = (\lambda_1, \lambda_2, \ldots, \lambda_K)$ with each L_j appearing M_j times. Let $K = \sum_{j=1}^{J} M_j$. For $\sigma \in S_N$, write $\sigma = (\sigma_1 | \cdots | \sigma_K)$ so that each σ_k is the restriction of $\sigma : \underline{N} \to \underline{N}$ to a subset of size $\lambda_k = L_j$ of which there are M_j many. In particular, when $\sigma \in \mathrm{Sh}(\Lambda)$, we have $\sigma_k : \underline{L_j} \nearrow \underline{N}$.

Lemma 3.5. *Under the above assumptions (particularly requiring each L_j to be even) and definitions (such as how to obtain α_j from the $Q^j_{\mathfrak{t}}$), we have*

$$\sum_{\sigma\in S_N} \operatorname{sgn}(\sigma) Q^1_{\sigma_1}\cdots Q^J_{\sigma_K} = \int \frac{\alpha_1^{\wedge_{\sqcup\!\sqcup} M_1}\wedge_{\sqcup\!\sqcup}\cdots\wedge_{\sqcup\!\sqcup}\alpha_J^{\wedge_{\sqcup\!\sqcup} M_J}}{K!}\, d\varepsilon_{\mathrm{vol}}.$$

In particular, when $J = 1$, the right hand side is $\mathrm{PF}_{\sqcup\!\sqcup}(\alpha_1)$, where the subscript $\sqcup\!\sqcup$ is added to emphasize the coefficients $A^j_{\mathfrak{t}}$ are in $F\langle I\rangle_{\sqcup\!\sqcup}$ in which multiplication of coefficients is done by $\sqcup\!\sqcup$.

Though this lemma holds for a more general collection of $Q^j_{\mathfrak{t}}$, we should think of the left hand side as being an $N \times N$ determinant. A single $Q^j_{\sigma_k}$ is a product of L_j many entries from the matrix. $A^j_{\sigma_k}$, the antisymmetrization of $Q^j_{\sigma_k}$, is the determinant of an $L_j \times L_j$ submatrix selected by σ_k. In summary, this lemma transforms any determinantal integrand into one for which (Chen's) Lemma 3.4 will apply. This is functionally similar to the Laplace expansion of the determinant (over complimentary $L_j \times L_j$ minors) which Sinclair uses in his proofs. A version of Lemma 3.5 (with proof) appears in Wells' 2019 dissertation [34] with the simplification that all block sizes L_j be the same L.

Proof. Starting with the right hand side, note each factor α_j is a sum of $A^j_{\mathfrak{t}}\varepsilon_{\mathfrak{t}}$'s. If we expand the product of the sums, each summand will be a product of some $A^j_{\mathfrak{t}}\varepsilon_{\mathfrak{t}}$'s. Any time $\mathfrak{t} : \underline{L_j} \nearrow \underline{N}$ and $\mathfrak{s} : \underline{L_k} \nearrow \underline{N}$ have overlapping ranges, $\varepsilon_{\mathfrak{t}} \wedge \varepsilon_{\mathfrak{s}} = 0$. Thus, each nonzero summand corresponds to a permutation $(\mathfrak{t}_1|\cdots|\mathfrak{t}_K) = \sigma \in S_N$. Since each $\mathfrak{t}_k$ is an increasing function, we have $\sigma \in \mathrm{Sh}(\Lambda)$. We use the Berezin integral $\int d\varepsilon_{\mathrm{vol}}$ because it sends $\varepsilon_\sigma = \varepsilon_{\mathfrak{t}_1}\wedge\cdots\wedge\varepsilon_{\mathfrak{t}_K}$ to $\operatorname{sgn}(\sigma)$ and picks up the

coefficients $A^1_{\sigma_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup A^J_{\sigma_K}$. We can rewrite the right hand side as

$$\text{RHS} = \frac{1}{K!} \sum_{\sigma \in \text{Sh}(\Lambda)} \text{sgn}(\sigma) A^1_{\sigma_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup A^J_{\sigma_K}.$$

We eliminate the factorial denominator by requiring $\sigma_1(1) < \sigma_2(1) < \cdots < \sigma_K(1)$.

$$\text{RHS} = \sum_{\sigma \in \text{Sh}^\circ(\Lambda)} \text{sgn}(\sigma) A^1_{\sigma_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup A^J_{\sigma_K}.$$

Expanding each $A^j_{\sigma_k}$ according to the definition, we get

$$\text{RHS} = \sum_{\sigma \in \text{Sh}^\circ(\Lambda)} \text{sgn}(\sigma) \left(\sum_{\tau_1 \in S_{L_1}} \text{sgn}(\tau_1) Q^1_{\sigma_1 \circ \tau_1} \right) \sqcup\!\sqcup \cdots \sqcup\!\sqcup \left(\sum_{\tau_K \in S_{L_J}} \text{sgn}(\tau_K) Q^J_{\sigma_K \circ \tau_K} \right).$$

Collect the τ_k's as a single element of $\text{H}(\Lambda) \cong (S_{L_1})^{M_1} \times \cdots \times (S_{L_J})^{M_J}$ with $\text{sgn}(\tau) = \text{sgn}(\tau_1) \cdots \text{sgn}(\tau_K)$, so that

$$\text{RHS} = \sum_{\sigma \in \text{Sh}^\circ(\Lambda)} \text{sgn}(\sigma) \sum_{\tau \in \text{H}(\Lambda)} \text{sgn}(\tau) Q^1_{\sigma_1 \circ \tau} \sqcup\!\sqcup \cdots \sqcup\!\sqcup Q^J_{\sigma_K \circ \tau}.$$

Next, we apply an identity of the $\sqcup\!\sqcup$ operation. Note, we are shuffling individual letters together, not strings of letters, so the sum is over $\pi \in S_K = \text{Sh}(1, \ldots, 1)$. The action on the subscripts can also be viewed as a permutation of the K many "blocks" of N as prescribed by the partition Λ. Thus,

$$\begin{aligned} \text{RHS} &= \sum_{\sigma \in \text{Sh}^\circ(\Lambda)} \text{sgn}(\sigma) \sum_{\tau \in \text{H}(\Lambda)} \text{sgn}(\tau) \sum_{\pi \in S_K} Q^1_{\sigma_{\pi(1)} \circ \tau} \cdots Q^J_{\sigma_{\pi(K)} \circ \tau} \\ &= \sum_{\varphi \in S_N} \text{sgn}(\varphi) Q^1_{\varphi_1} \cdots Q^J_{\varphi_K}. \end{aligned}$$

Note, the equality in the last line is not an obvious one but follows from Lemma 3.1. In the context of Lemma 3.5, all of the block sizes L_j are even, so $\operatorname{sgn}(\pi) = 1$ for all $\pi \in \mathrm{Bl}(\Lambda)$. Thus,

$$\operatorname{sgn}(\varphi) = \operatorname{sgn}(\sigma \circ \pi \circ \tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)$$

□

3.6. Proof of de Bruijn Identities

Recall, Theorem 3.1 applies to an $N \times N$ matrix $A(\vec{x})$ whose entries are functions of the variables $\vec{x}$. In this section, we will first prove the result for the confluent alternant $V^{\vec{L}}(\vec{x})$ (see section 2.7) whose entries are derivatives. Then, if additional generality is desired, one can replace the particulars of the confluent alternant with that of more general $A(\vec{x})$.

This proof reduces to applying (Chen's) Lemma 3.4 after Lemma 3.5. Recall, in Lemma 3.5, we required all of the block sizes L_j to be even. As mentioned at the beginning of this chapter, the $\eta_{j,k}$ forms of Theorem 3.1 are even $(L_j + L_k)$-forms constructed from combining an odd L_j-form with an odd L_k-form. In this section, we verify that this construction is valid and compatible with the "antisymmetrized forms" of Lemma 3.5. Wells does this for $L_1 = \cdots = L_J = L$ in his 2019 dissertation [34].

When All L_j Are Even

First, suppose all L_j are already even. For $\pi = (\pi_1|\cdots|\pi_J) \in S_N$ and family of functions $\vec{f} = \{f_n\}_{n=1}^N$, define

$$Q^j_{\pi_j}(x) = \prod_{l=1}^{L_j} D^{l-1} f_{\pi_j(l)}(x),$$

so that we can write

$$\det V^{\vec{L}}(\vec{x}) = \sum_{\pi \in S_N} \operatorname{sgn}(\pi) Q^1_{\pi_1}(x_1) \cdots Q^J_{\pi_J}(x_J).$$

Note, each Q_{π_j} is a product of L_j many matrix entries which we have grouped together. These entries are taken from the L_j many derivative columns for the one variable x_j. Writing the determinant this way allows us to invoke Lemma 3.5 which gives us

$$\det V^{\vec{L}}(\vec{x}) = \int \frac{\alpha_1 \wedge_{\sqcup\!\sqcup} \cdots \wedge_{\sqcup\!\sqcup} \alpha_J}{J!}\, d\varepsilon_{\mathrm{vol}}.$$

where each α_k is defined by

$$\alpha_j = \sum_{\mathfrak{t}:\underline{L_j} \nearrow \underline{N}} A^j_{\mathfrak{t}}(x)\, \varepsilon_{\mathfrak{t}} = \sum_{\mathfrak{t}:\underline{L_j} \nearrow \underline{N}} \sum_{\tau \in S_{L_j}} \operatorname{sgn}(\tau) Q^j_{\mathfrak{t}\circ\tau}(x)\, \varepsilon_{\mathfrak{t}} = \sum_{\mathfrak{t}:\underline{L_j} \nearrow \underline{N}} \operatorname{Wr}(\vec{f}_{\mathfrak{t}}, x)\, \varepsilon_{\mathfrak{t}}.$$

As mentioned in section 2.7, these Wronskians are merely the univariate $L_j \times L_j$ minors of the confluent Vandermonde matrix. Recall from the beginning of this chapter,

$$\gamma_j = \sum_{\mathfrak{t}:\underline{L_j} \nearrow \underline{N}} \int_{\mathbb{R}} \det V_{\mathfrak{t}}(x_j)\, dx_j\, \varepsilon_{\mathfrak{t}} = \sum_{\mathfrak{t}:\underline{L_j} \nearrow \underline{N}} \int_{\mathbb{R}} \operatorname{Wr}(\vec{f}_{\mathfrak{t}}, x_j)\, dx_j\, \varepsilon_{\mathfrak{t}}.$$

Applying (Chen's) Lemma 3.4 sends integration of shuffle products to ordinary products of integrals. Thus,

$$\int_{-\infty<x_1<\cdots<x_N<\infty} \det V^{\vec{L}}(\vec{x})\, dx_1 \cdots dx_J = \left\langle \int \frac{\alpha_1 \wedge_{ш} \cdots \wedge_{ш} \alpha_J}{J!}\, d\varepsilon_{\text{vol}} \right\rangle$$
$$= \int \frac{\gamma_1 \wedge \cdots \wedge \gamma_J}{J!}\, d\varepsilon_{\text{vol}}.$$

Even Number of Odd L_j

Next, suppose all $L_1, \ldots, L_J$ are odd but J is even. Recall from the beginning of this chapter,

$$\eta_{j,k} = \sum_{\mathfrak{t}:\underline{L_j}\nearrow\underline{N}} \sum_{\mathfrak{s}:\underline{L_k}\nearrow\underline{N}} \int\int_{x_j<x_k} \det V_{\mathfrak{t}}(x_j) \det V_{\mathfrak{s}}(x_k)\, dx_j\, dx_k\, \varepsilon_{\mathfrak{t}} \wedge \varepsilon_{\mathfrak{s}}.$$

Following the same argument as in the "all even" case, we can apply our Lemma 3.5 provided there exists antisymmetrized *even* forms $\alpha_{j,k}$ such that each $\eta_{j,k}$ is obtained from $\alpha_{j,k}$ by applying our functional $\langle\cdot\rangle$ (see section 3.4) to the coefficients of $\alpha_{j,k}$. To this end, define $\alpha_{j,k}$ by

$$\alpha_{j,k} = \sum_{\mathfrak{t}:\underline{L_j+L_k}\nearrow\underline{N}} A^{j,k}_{\mathfrak{t}}\, \varepsilon_{\mathfrak{t}} = \sum_{\mathfrak{t}:\underline{L_j+L_k}\nearrow\underline{N}} \sum_{\tau\in S_{L_j+L_k}} \operatorname{sgn}(\tau) Q^{j,k}_{\mathfrak{t}\circ\tau}\, \varepsilon_{\mathfrak{t}},$$

where

$$Q^{j,k}_{\mathfrak{t}\circ\tau}(x,y) = \prod_{l=1}^{L_j} D^{l-1} f_{\mathfrak{t}\circ\tau(l)}(x) \prod_{l=1}^{L_k} D^{l-1} f_{\mathfrak{t}\circ\tau(L_j+l)}(y),$$

then $\alpha_{j,k}$ is an antisymmetrized $(L_j + L_k)$-form by construction. As before, we should think of $Q^{j,k}_{\mathfrak{t}}$ as taking one entry from each of the L_j many derivative columns for one variable x and then one entry from each of the L_k derivative columns for the next variable y. Pairing an odd number of entries with another

odd number of entries produces an even form. It remains to be shown that the antisymmetrizations $A_{\mathfrak{t}}^{j,k}$ are composed of complementary Wronskian minors.

For $\mathfrak{t} : \underline{L_j + L_k} \nearrow \underline{N}$, let $V_{\mathfrak{t}}^{\vec{L}}(x_j, x_k)$ denote the $(L_j+L_k)\times(L_j+L_k)$ submatrix of $V^{\vec{L}}(\vec{x})$ comprised of rows $\mathfrak{t}(1), \ldots, \mathfrak{t}(L_j + L_k)$ taken from the L_j columns in x_j and the L_k columns in x_k. When viewed as a two variable function, $A_{\mathfrak{t}}^{j,k}$ is the determinant of this submatrix. Explicitly,

$$A_{\mathfrak{t}}^{j,k}(x_j, x_k) = \det V_{\mathfrak{t}}^{\vec{L}}(x_j, x_k).$$

For $\mathfrak{t}_1 : \underline{L_j} \nearrow \underline{L_j + L_k}$, let $V_{\mathfrak{t}_1}^{\vec{L}}(x_j)$ denote the $L_j \times L_j$ submatrix of $V_{\mathfrak{t}}^{\vec{L}}(x_j, x_k)$ comprised of rows $\mathfrak{t}_1(1), \ldots, \mathfrak{t}_1(L_j)$ taken from the L_j columns in x_j. Similarly, for $\mathfrak{t}_2 : \underline{L_k} \nearrow \underline{L_j + L_k}$, let $V_{\mathfrak{t}_2}^{\vec{L}}(x_k)$ denote a $L_k \times L_k$ submatrix in x_k. By the Laplace expansion of the determinant,

$$\begin{aligned}
\det V_{\mathfrak{t}}^{\vec{L}}(x_j, x_k)\, \varepsilon_{\mathfrak{t}} &= \sum_{\mathfrak{t}_1 : \underline{L_j} \nearrow \underline{L_j+L_k}} \; \sum_{\mathfrak{t}_2 : \underline{L_k} \nearrow \underline{L_j+L_k}} \det V_{\mathfrak{t}_1}^{\vec{L}}(x_j) \det V_{\mathfrak{t}_2}^{\vec{L}}(x_k) \mathrm{sgn}(\mathfrak{t}_1, \mathfrak{t}_2)\, \varepsilon_{\mathfrak{t}} \\
&= \sum_{\mathfrak{t}_1 : \underline{L_j} \nearrow \underline{L_j+L_k}} \; \sum_{\mathfrak{t}_2 : \underline{L_k} \nearrow \underline{L_j+L_k}} \det V_{\mathfrak{t}_1}^{\vec{L}}(x_j) \det V_{\mathfrak{t}_2}^{\vec{L}}(x_k)\, \varepsilon_{\mathfrak{t}_1} \wedge \varepsilon_{\mathfrak{t}_2} \\
&= \sum_{\mathfrak{t}_1 : \underline{L_j} \nearrow \underline{L_j+L_k}} \; \sum_{\mathfrak{t}_2 : \underline{L_k} \nearrow \underline{L_j+L_k}} \mathrm{Wr}(\vec{f}_{\mathfrak{t}_1}, x_j) \mathrm{Wr}(\vec{f}_{\mathfrak{t}_2}, x_k)\, \varepsilon_{\mathfrak{t}_1} \wedge \varepsilon_{\mathfrak{t}_2}.
\end{aligned}$$

Thus, $\eta_{j,k}$ is the result of applying $\langle\cdot\rangle_2$ to the two-variable coefficients of $\alpha_{j,k}$ as desired. Proceeding as we did in the "all even" case, applying (Chen's) Lemma 3.4 gives us

$$\begin{aligned}
\int_{-\infty<x_1<\cdots<x_N<\infty} \det V^{\vec{L}}(\vec{x})\, dx_1 \cdots dx_J &= \left\langle \int \frac{\alpha_{1,2} \wedge_{\sqcup\!\sqcup} \cdots \wedge_{\sqcup\!\sqcup} \alpha_{J-1,J}}{(J/2)!}\, d\varepsilon_{\mathrm{vol}} \right\rangle \\
&= \int \frac{\eta_{1,2} \wedge \cdots \wedge \eta_{J-1,J}}{(J/2)!}\, d\varepsilon_{\mathrm{vol}}.
\end{aligned}$$

Odd Number of Odd L_j

We still consider all $L_1, \ldots, L_J$ odd but assume J is odd, too. When J was even, we constructed even forms $\alpha_{j,k}$ and subsequent $\eta_{j,k}$ by pairing an L_j form with an L_k form. We also showed taking determinants of appropriate submatrices produces an antisymmetrized form $\alpha_{j,k}$ (for which our Lemma 3.5 will apply). We will do this pairing again for the first $J-1$ variables, which makes $(J-1)/2$ pairs. Explicitly, simply define $Q_{\mathfrak{t}}^{j,k}$, $A_{\mathfrak{t}}^{j,k}$, $\alpha_{j,k}$, and $\eta_{j,k}$ as before.

Next, consider the following modification of $V^{\vec{L}}$:

$$V^{\vec{L},1}(\vec{x}) = \begin{bmatrix} V^{\vec{L}}(\vec{x}) & 0 \\ 0 & 1 \end{bmatrix}.$$

Construct the final α_J by taking $(L_J+1) \times (L_J+1)$ submatrices from the last L_J+1 columns of $V^{\vec{L},1}(\vec{x})$. These submatrices have non-zero determinant only when the last row is chosen. Thus, valid submatrices are entirely determined by a choice of only L_J many other rows, and

$$\alpha_J = \sum_{\mathfrak{t}:\underline{L_J} \nearrow \underline{N}} \sum_{\tau \in S_{L_J}} \operatorname{sgn}(\tau) Q^J_{\mathfrak{t} \circ \tau}\, \varepsilon_{\mathfrak{t}} \wedge \varepsilon_{N+1}$$

is an antisymmetrized (L_J+1)-form when

$$Q^J_{\mathfrak{t}\circ\tau}(x) = \prod_{l=1}^{L_J} D^{l-1} f_{\mathfrak{t}\circ\tau(l)}(x).$$

As before, applying $\langle \cdot \rangle_1$ to α_J produces $\gamma_J \wedge \varepsilon_{N+1}$.

This completes the proof of our main Theorem 3.1. For $L_1, \ldots, L_r$ even, we take $L_j \times L_j$ minors of $V^{\vec{L}}(\vec{x})$ and get a γ_j factor for each. For $L_{r+1}, \ldots, L_J$ odd, we

take $(L_{r+2m-1}+L_{r+2m})\times(L_{r+2m-1}+L_{r+2m})$ minors of $V^{\vec{L}}(\vec{x})$ and get an $\eta_{r+2m-1,r+2m}$ factor for each. When the number of odd L_j is odd, we get an extra γ_J for that last odd L_J.

CHAPTER IV

CONSTELLATION ENSEMBLES

Recall the (one-dimensional) log-gas setup of section 1.3. Suppose this system is copied onto a parallel line (translated vertically in the complex plane). In addition to the internal interactions between particles on the same line, particles from different lines are also able to interact with each other, with the strength of this interaction depending on the distance between the lines. This is an example of what we will call a *linear constellation ensemble*. We will consider several variations on this setup:

1. *The (K-fold) First Constellation Ensemble*, in which charge $L = 1$ particles are copied onto K many parallel lines, subject to $\beta = 1$.

2. *The (K-fold) Monocharge Constellation Ensemble*, in which particles of the same integer charge L are copied onto K many lines.

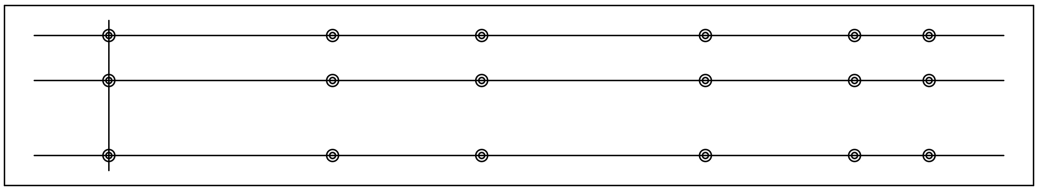
3. *The (K-fold) Homogeneous Constellation Ensemble*, in which particles on the same line have the same integer charge L_k, but particles on different lines may have different charges.

4. *The (K-fold) Multicomponent Constellation Ensemble*, in which the original line may have particles of different charges, but all the parallel lines are copies, featuring the same charges in the same positions.

The first is a special case of the second, which is a special case of either the third or the fourth. Rather than start with the case which is most general (and therefore convoluted), we will work our way up through the different levels of complexity,

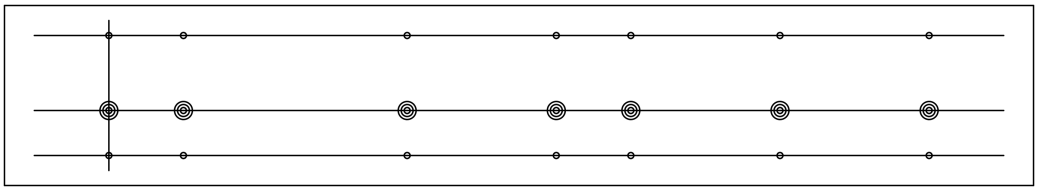
introducing various tools along the way only as necessary. For each of these ensembles, we will also consider *circular* constellation ensembles of concentric circles in the complex plane (see chapter VI).

FIGURE 1. A Monocharge (Linear) Constellation Ensemble



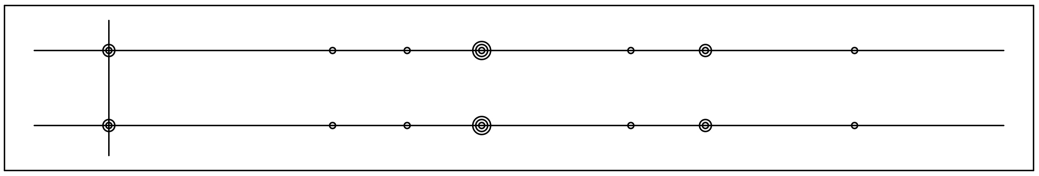
In Figure 1, there are $K = 3$ parallel lines (not necessarily equidistant) on which charge $L = 2$ particles have been placed, represented in this figure by pairs of concentric circles. Note, each horizontal line is a copy of the others, so they have the same number of particles at the same (horizontal) locations. Particles which land on the same vertical line are called a *constellation*. In this example, each constellation is made up of $K = 3$ particles of the same charge $L = 2$. In general, constellation ensembles are ensembles of constellations, of which there are $M = 6$ in this configuration.

FIGURE 2. A Homogeneous (Linear) Constellation Ensemble



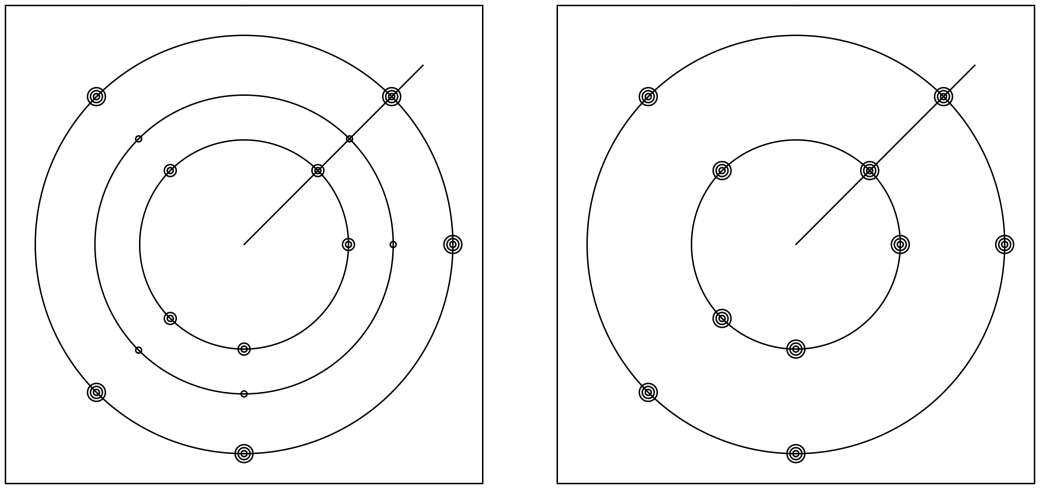
In Figure 2, there are still $K = 3$ parallel lines, but now there are both charge $L_1, L_3 = 1$ particles and charge $L_2 = 3$ particles. Note, the top line features only particles of charge $L_3 = 1$, while the middle line features only particles of charge $L_2 = 3$. Each constellation (of which there are $M = 7$) is made up of one particle of charge 3 and two particles of charge 1, for a total charge of $R_1 = 5$.

FIGURE 3. A Multicomponent (Linear) Constellation Ensemble



In Figure 3, each horizontal line features a mix of charge 1, charge 2, and charge 3 particles. However, particles which land on the same vertical line have the same charge. On the left, we have marked a constellation of charge 2 particles. This example is a *multicomponent* ensemble because it is made up of different *species* of constellations, namely $M_1 = 4$ constellations of charge 1 particles, $M_2 = 2$ constellations of charge 2 particles, and $M_3 = 1$ constellation of charge 3 particles.

FIGURE 4. A Homogeneous Circular Constellation Ensemble



On the left side of Figure 4, there are $K = 3$ concentric circles. Note, each constellation (of which there are $M = 5$) is made up of particles on the same ray. One such constellation (of three particles) has been marked. The box on the right depicts the result of reducing the radius of the second circle to the radius of the innermost circle. Each charge 1 particle merges with a charge 2 particle to form a charge $1 + 2 = 3$ particle.

Though these particle arrangements are somewhat contrived physically, the resulting joint probability density functions give us insight into limiting ensembles which we can interpolate between (by adjusting the distances between the parallel lines or circles). For example, taking the limit of the first constellation ensemble as the distance between the lines (or circles) goes to zero (and correcting for the singularities as particles collapse onto each other) produces a one-dimensional $\beta = K^2$ ensemble. On the other end, taking the limit as the distance between the lines (or circles) goes to infinity produces the equivalent of a one-dimensional $\beta = K$ ensemble. The case of the 2-fold first constellation ensemble (of charge $L = 1$ particles on $K = 2$ parallel lines) was explored by Shum in his 2013 dissertation [29], resulting in an interpolation between classical $\beta = 2$ and $\beta = 4$ ensembles.

Recall, Theorem 3.1 of chapter III is a generalization of the de Bruijn integral identities in which the iterated integral of a determinant is now expressed as the Hyperpfaffian or Berezin integral of an appropriate alternating tensor (also *form*). As the first application, we substitute the particulars for the partition function of the monocharge constellation ensemble in section 4.2. In section 4.4, we extend this to homogeneous constellation ensembles, the most general classification (in this volume) which still produces homogeneous forms (and therefore Hyperpfaffian partition functions). In contract, in section 5.6, we consider multicomponent constellation ensembles which produce non-homogeneous forms instead. Finally, in chapter VI, we consider (circular) ensembles of concentric circles in instead of parallel lines. In all cases, the generalized de Bruijn identities are used, further demonstrating the versatility in the methods established in this volume.

4.1. The Monocharge Setup

Let $\vec{x} \in \mathbb{R}^M$, and let $\vec{y} \in \mathbb{R}^K$ such that $0 \leq y_1 < \cdots < y_K$. We call $\vec{y}$ the *translation vector* of the system, giving the locations of the K many lines $\mathbb{R} + iy_k$ in the complex plane. Consider M many charge $L \in \mathbb{Z}_{>0}$ particles on each line $\mathbb{R} + iy_k$ having the same real parts, meaning for each location $x_m \in \mathbb{R}$, and $1 \leq k \leq K$, there is a charge L particle at location $x_m + iy_k$. Denote the (total KM) particle locations by

$$\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^M) \in \mathbb{C}^{KM},$$

where $\mathbf{x}^m = x_m + i\vec{y} = (x_m + iy_1, x_m + iy_2, \ldots, x_m + iy_K) \in \mathbb{C}^K$. We call $\mathbf{x}$ the *location vector* of the system, in which each $x_k^m \in \mathbb{C}$ gives the location of a particle. We call $\mathbf{x}^m$ the location vector of the *constellation* of K many particles which all share the same real part x_m. We call $\vec{x} = (x_1, \ldots, x_M)$ the location vector of the real parts which generate each constellation. As an example, see Figure 1 at the beginning of this chapter.

The particles are assumed to interact logarithmically so that the contribution of energy to the system by two (charge L) particles at locations $x_m + iy_k$ and $x_n + iy_j$ is given by $-L^2 \log |(x_m + iy_k) - (x_n + iy_j)|$. Let $U : \mathbb{R} \to \mathbb{R}$ be a potential on the real axis. Let $\overline{U} : \mathbb{C} \to \mathbb{R}$ be a extension of this potential to the entire complex plane such that $\overline{U}(z) = U(\mathrm{Re}(z))$. Without loss of generality, we can assume $x_1 < \cdots < x_M$. Then at inverse temperature β, the total potential energy of

the system is given by

$$\begin{aligned} E(\vec{x},\vec{y}) = \beta L \sum_{k=1}^{K}\sum_{m=1}^{M} \overline{U}(x_m+iy_k) - \beta L^2 \sum_{k=1}^{K}\sum_{n<m}^{M} \log|(x_m+iy_k)-(x_n+iy_k)| \\ - \beta L^2 \sum_{j<k}^{K}\sum_{m=1}^{M} \log|(x_m+iy_k)-(x_m+iy_j)| \\ - \beta L^2 \sum_{j<k}^{K}\sum_{n<m}^{M} \log|(x_m+iy_k)-(x_n+iy_j)| \\ - \beta L^2 \sum_{j<k}^{K}\sum_{n<m}^{M} \log|(x_m+iy_j)-(x_n+iy_k)|. \end{aligned}$$

The first iterated sum in the first line accounts for the potential $\overline{U}$. We can substitute $\overline{U}(x_m+iy_k) = U(x_m)$ of which there are K many for each m. The second iterated sum in the first line accounts for interactions between particles which share a line. Note, the differences in that iterated sum are all positive by assumption on the ordering of the x_m, and the differences are the same for all $1 \leq k \leq K$. The iterated sum in the second line accounts for interactions between particles of the same constellation, meaning same real part x_m. The differences in that iterated sum are the same for $1 \leq m \leq M$. The iterated sums in the third and fourth lines account for the remaining interactions between particles. For each quadruple (m,k,n,j), we get four points which make up a rectangle in the complex plane. The four sides of this rectangle are already accounted for by the other interactions. The product of the lengths of the two diagonals is the sum of the squares of the

lengths of the sides. Thus, the potential energy simplifies to

$$E(\vec{x},\vec{y}) = \beta LK \sum_{m=1}^{M} U(x_m) - \beta L^2 K \sum_{n<m}^{M} \log(x_m - x_n) - \beta L^2 M \sum_{j<k}^{K} \log|i(y_k - y_j)|$$
$$- \beta L^2 \sum_{j<k}^{K} \sum_{n<m}^{M} \log\left((x_m - x_n)^2 + (y_k - y_j)^2\right).$$

With this setup, the relative density of states (corresponding to varying location vectors $\vec{x}$ and translation vectors $\vec{y}$) is given by the Boltzmann factor

$$\begin{aligned}\Omega(\vec{x},\vec{y}) &= \exp(-E(\vec{x},\vec{y})) \\ &= |\Delta(\mathbf{x})|^{\beta L^2} \prod_{m=1}^{M} e^{-\beta LKU(x_m)} \\ &= \Delta(\mathbf{x})^{\beta L^2} \prod_{m=1}^{M} \left((-i)^{L(K-1)/2} e^{-U(x_m)}\right)^{\beta LK},\end{aligned}$$

where $\Delta(\mathbf{x})$ denotes the Vandermonde determinant (see section 2.7), evaluated at the variables $\mathbf{x}$. Note, the last equality comes from $|i| = (i)(-i)$, of which there are $\beta L^2 M\binom{K}{2}$ many instances. Thus, the probability of finding the system in a state corresponding to a location vector $\vec{x}$ and fixed translation vector $\vec{y}$ is given by the joint probability density function

$$\rho(\vec{x},\vec{y}) = \frac{\Omega(\vec{x},\vec{y})}{Z_M(\vec{y})M!},$$

where the *partition function* (of the K-fold monocharge constellation ensemble) $Z_M(\vec{y})$ is the normalization constant given by

$$\begin{aligned} Z_M(\vec{y}) &= \frac{1}{M!}\int_{\mathbb{R}^M} \Omega(\vec{x},\vec{y})\, dx_1 \cdots dx_M \\ &= \int_{-\infty<x_1<\cdots<x_M<\infty} \Delta(\mathbf{x})^{\beta L^2}\, d\mu(x_1)\cdots d\mu(x_M), \end{aligned}$$

in which $d\mu(x) = \left((-i)^{L(K-1)/2}e^{-U(x)}\right)^{\beta LK} dx$. As in the one-dimensional log-gas setup of section 1.3, it is necessary to assume the potential U is one for which $Z_M(\vec{y})$ is finite.

Recall, unit charges (meaning $L = 1$) at inverse temperature $\beta = b^2$ have the same Boltzmann factor (and resulting density function) as charge $L = b$ particles at inverse temperature $\beta = 1$ (subject to different but related potentials $U(x)$). In general, replacing β with $\beta' = \beta/b^2$ and replacing L with $L' = bL$ leaves $\Delta(\mathbf{x})^{\beta L^2}$ unchanged. Then replacing U with $U' = bU$ leaves $\Omega(\vec{x},\vec{y})$ unchanged. Thus, for computational purposes, we can change to $\beta = 1$ (provided $\sqrt{\beta}L \in \mathbb{Z}$ for the original β).

In addition to the dependence on inverse temperature β, potential U, and charges L, the partition function $Z_M(\vec{y})$ of a constellation ensemble is also a function of the translation vector $\vec{y}$ which determines how the parallel lines are spaced in the complex plane. As $y_k - y_j \to 0$, the corresponding interaction terms shrink, and the potential energy grows. Conversely, as $y_k - y_j \to \infty$, the corresponding interaction terms grow, and the potential energy shrinks.

Note, this $Z_M(\vec{y})$ is an iterated integral in M many variables. As mentioned in section 1.3, our goal in this chapter is not to compute these integrals for any particular choice of several parameters. Instead, we demonstrate, in general, how to

write $Z_M(\vec{y})$ as a Hyperpfaffian (or Berezin integral in the multicomponent case) of a form whose coefficients are only single or double integrals of Wronskians and/or proto-Wronskians (see section 2.6).

4.2. Monocharge Partition Functions

In all Constellation Ensembles,

$$Z_M(\vec{y}) = \mathrm{BE}_{\mathrm{vol}}(\omega(\vec{y})),$$

for some appropriately defined $\omega(\vec{y})$. Any time $\omega(\vec{y})$ is homogeneous, we also get

$$Z_M(\vec{y}) = \mathrm{PF}(\omega(\vec{y})).$$

Recall (from section 4.1), in the monocharge constellation ensemble, L is the charge of each particle, K is the number of parallel lines, and M is the number of particles on each line. Let $\vec{p}$ be a complete N-family of monic polynomials, where $N = LKM$. Define

$$\gamma_L(\vec{y}) = \sum_{\mathfrak{t}:\underline{LK}\nearrow\underline{N}} \int_{\mathbb{R}} \mathrm{Wr} \otimes \mathrm{Pr}_{\vec{y}}(\vec{p}_{\mathfrak{t}}, x)\, d\mu(x)\, \varepsilon_{\mathfrak{t}},$$

and define

$$\eta_L(\vec{y}) = \sum_{\mathfrak{t}:\underline{LK}\nearrow\underline{N}} \sum_{\mathfrak{s}:\underline{LK}\nearrow\underline{N}} \int\int_{x_1<x_2} \Big[\mathrm{Wr} \otimes \mathrm{Pr}_{\vec{y}}(\vec{p}_{\mathfrak{t}}, x_1) \times \mathrm{Wr} \otimes \mathrm{Pr}_{\vec{y}}(\vec{p}_{\mathfrak{s}}, x_2)\Big]\, d\mu(x_1)\, d\mu(x_2)\, \varepsilon_{\mathfrak{t}} \wedge \varepsilon_{\mathfrak{s}}.$$

Provided we can write the Boltzmann factor integrand $\Omega(\vec{x}, \vec{y})$ as a determinant of an $N \times N$ matrix with univariate minors of the form $\mathrm{Wr} \otimes \mathrm{Pr}_{\vec{y}}(\vec{p}_{\mathfrak{t}}, x)$, Theorem 3.1 immediately gives us the desired Hyperpfaffian expression for the partition function $Z_M(\vec{y})$.

Theorem 4.1 (K-fold Monocharge Partition Function)**.**

$$Z_M(\vec{y}) = \mathrm{PF}(\omega(\vec{y})),$$

where $\omega(\vec{y})$ is defined by:

1. *If LK is even, then $\omega(\vec{y}) = \gamma_L(\vec{y})$.*
2. *If LK is odd, but M is even, then $\omega(\vec{y}) = \eta_L(\vec{y})$.*
3. *If LKM is odd, then $\omega(\vec{y}) = \eta_L(\vec{y}) + \gamma_L(\vec{y}) \wedge \xi_{LK}$.*

As in the corollary to Theorem 3.1, ξ_{LK} upgrades $\gamma_L(\vec{y})$ from an LK-form to a $2LK$-form and makes $\omega(\vec{y})$ homogeneous so that the Hyperpfaffian $\mathrm{PF}(\omega(\vec{y}))$ is well-defined. Alternatively, $Z_M(\vec{y}) = \mathrm{BE}_{\mathrm{vol}}(\omega(\vec{y}))$ in cases 1 and 2, while $Z_M(\vec{y}) = \mathrm{BE}_{\mathrm{vol}_{LK}}(\omega(\vec{y}))$ in case 3.

Recall also, the first constellation ensemble is the special case in which $L = 1$. In that case, the $\mathrm{Wr} \otimes \mathrm{Pr}_{\vec{y}}(\vec{p}_{\mathfrak{t}}, x)$ minors are actually $\mathrm{Pr}_{\vec{y}}(\vec{p}_{\mathfrak{t}}, x)$.

Corollary (K-fold First Constellation Partition Function)**.** *When $L = 1$, the partition function $Z_M(\vec{y})$ is given as in Theorem 4.1 with the following modifications to $\gamma_1(\vec{y})$ and $\eta_1(\vec{y})$:*

$$\gamma_1(\vec{y}) = \sum_{\mathfrak{t}:\underline{K} \nearrow \underline{N}} \int_{\mathbb{R}} \mathrm{Pr}_{\vec{y}}(\vec{p}_{\mathfrak{t}}, x)\, d\mu(x)\, \varepsilon_{\mathfrak{t}},$$

and

$$\eta_1(\vec{y}) = \sum_{\mathfrak{t}:\underline{K}\nearrow\underline{N}} \sum_{\mathfrak{s}:\underline{K}\nearrow\underline{N}} \int\int_{x_1<x_2} \mathrm{Pr}_{\vec{y}}(\vec{p}_{\mathfrak{t}}, x_1)\mathrm{Pr}_{\vec{y}}(\vec{p}_{\mathfrak{s}}, x_2)\, d\mu(x_1)\, d\mu(x_2)\, \varepsilon_{\mathfrak{t}} \wedge \varepsilon_{\mathfrak{s}}.$$

Alternatively, any one-dimensional log-gas with a single species is a special case of a constellation ensemble in which $K = 1$ (meaning only one line). Theorem 4.1 agrees with Sinclair's Hyperpfaffian [31] and Berezin integral [32] expressions for the partition functions of β-ensembles and one-dimensional multicomponent log-gases, respectively. In particular, our $\mathrm{Wr} \otimes \mathrm{Pr}_{\vec{y}}(\vec{p}_{\mathfrak{t}}, x)$ minors become his $\mathrm{Wr}(\vec{p}_{\mathfrak{t}}, x)$ minors when $K = 1$.

To prove Theorem 4.1 (and its analogues which appear in sections 4.4, 5.6, and 6.1), we need to write the Boltzmann factor integrand $\Omega(\vec{x}, \vec{y})$ as a determinant with the appropriate structure so that Theorem 3.1 can be applied. In section 2.7, we noted a confluent alternant has Wronskian minors. Similarly, a proto-confluent alternant has proto-Wronskian minors. Moreover, mixing these structures by feeding a translated $\mathbf{x}$ into an already confluent alternant produces the "mixed" minors at the end of section 2.6. Explicitly, for $\mathfrak{t} : \underline{L_m K} \nearrow \underline{N}$,

$$\det V_{\mathfrak{t}}^{L_m}(x_m) = \mathrm{Wr} \otimes \mathrm{Pr}_{\vec{y}}(\vec{f}_{\mathfrak{t}}, x_m)$$

is an $L_m K \times L_m K$ minor of $V^{\vec{L}}(\mathbf{x})$ in the single variable x_m.

Define $H^L(\mathbf{x})$ from $V^L(\mathbf{x})$ by multiplying each entry by the appropriate $(-i)^{L(K-1)/2} e^{-U(x_m)}$, the LK^{th} root of the Radon-Nikodym derivative of μ. Note, there are LK many columns for each variable x_m, so this multiplies the determinant by the LK^{th} power of the additional factors. Using the confluent

Vandermonde determinant identity,

$$\det H^L(\mathbf{x})\, dx_1 \cdots dx_m = \Delta(\mathbf{x})^{L^2}\, d\mu(x_1) \cdots d\mu(x_m) = \Omega(\vec{x}, \vec{y})\, dx_1 \cdots dx_m.$$

Thus, we have shown the joint probability density function $\Omega(\vec{x}, \vec{y})$ to be the determinant of a matrix with the appropriate minors, completing the proof of Theorem 4.1.

4.3. Homogeneous Constellation Ensembles

Let $\vec{L} \in (\mathbb{Z}_{>0})^K$ be a vector of positive integers which we will call the *charge vector* of the system. Modify the setup in section 4.1 by changing the charge of each $x_m + iy_k$ particle from L to L_k. Note, the M many particles on each line $\mathbb{R} + iy_k$ all have the same charge L_k. As an example, see Figure 2 at the beginning of this chapter. The contribution of energy to the system by a charge L_k particle at location $x_m + iy_k$ and a charge L_j particle at location $x_n + iy_j$ is given by $-L_k L_j \log|(x_m + iy_k) - (x_n + iy_j)|$. Assuming without loss of generality $\beta = 1$, the total potential energy of this new system is given by

$$\begin{aligned} E(\vec{x}, \vec{y}) = &\sum_{k=1}^{K} \sum_{m=1}^{M} L_k U(x_m) - \sum_{k=1}^{K} \sum_{n<m}^{M} L_k^2 \log(x_m - x_n) - M \sum_{j<k}^{K} L_j L_k \log|i(y_k - y_j)| \\ &- \sum_{j<k}^{K} \sum_{n<m}^{M} L_j L_k \log\left((x_m - x_n)^2 + (y_k - y_j)^2\right). \end{aligned}$$

Let $\mathbf{L} = (\vec{L}, \ldots, \vec{L}) \in (\mathbb{Z}_{>0})^{KM}$, let $R_1 = \sum_{k=1}^{K} L_k$, and let $R_2 = \sum_{j<k}^{K} L_j L_k$. With this setup, the relative density of states (corresponding to varying location vectors

$\vec{x}$ and translation vectors $\vec{y}$) is given by the Boltzmann factor

$$\begin{aligned}\Omega(\vec{x},\vec{y}) &= \exp(-E(\vec{x},\vec{y})) \\ &= \left|\Delta^{\mathbf{L}}(\mathbf{x})\right| \prod_{m=1}^{M} e^{-R_1 U(x_m)} \\ &= \Delta^{\mathbf{L}}(\mathbf{x}) \prod_{m=1}^{M} (-i)^{R_2} e^{-R_1 U(x_m)} \\ &= \det H^{\mathbf{L}}(\mathbf{x}).\end{aligned}$$

Recall, $H^{\mathbf{L}}(\mathbf{x})$ was defined from $V^{\mathbf{L}}(\mathbf{x})$ (in section 4.2) by multiplying the entries by the Radon-Nikodym derivative of μ, divided evenly over the columns. In this case, we define $d\mu(x) = (-i)^{R_2} e^{-R_1 U(x)}\, dx$. Thus, with another determinantal Boltzmann factor $\Omega(\vec{x},\vec{y})$, we can already apply Theorem 3.1 to $A(\vec{x}) = H^{\mathbf{L}}(\mathbf{x})$.

4.4. Homogeneous Partition Functions

$H^{\mathbf{L}}(\mathbf{x})$ (which corresponds to shape $\mathbf{L} = (\vec{L},\ldots,\vec{L})$) is the matrix which has L_1 many columns of derivatives evaluated at $x_1 + iy_1$, L_2 many columns of derivatives evaluated at $x_1 + iy_2$, and then so on up through L_K many columns of derivatives evaluated at $x_1 + iy_K$, starting over at L_1 many columns for $x_2 + iy_1$. In general, there are L_k many columns for $x_m + iy_k$, and the total $R_1 = \sum_{k=1}^{K} L_k$ many columns corresponding to x_m are consecutive. An $R_1 \times R_1$ minor in x_m resembles $\mathrm{Wr} \otimes \mathrm{Pr}_{\vec{y}}(\vec{p}_{\mathfrak{t}}, x)$ (as in the Monocharge case) but has different numbers of derivatives for each y_k. Define

$$\mathrm{Wr}^{\vec{L}} \otimes \mathrm{Pr}_{\vec{y}}(\vec{f}, x) = \det\left[\left[D^{l-1} f_n(x + iy_k)\right]_{l=1}^{L_k}\right]_{n,k=1}^{R_1,K}.$$

The first column is R_1 many functions evaluated at $x + iy_1$. The second column is the first derivatives of those functions evaluated at the same $x + iy_1$, and so on until the first L_1 many columns have been exhausted. The next L_2 many columns are L_2 many derivatives of the same functions evaluated at $x + iy_2$, and so on until all y_k have been exhausted. The resulting $R_1 \times R_1$ matrix will have $L_k \times L_k$ Wronskian blocks evaluated at one of the K many $x + iy_k$. Note,

$$\lim_{\vec{y}\to 0} \frac{\mathrm{Wr}^{\vec{L}} \otimes \mathrm{Pr}_{\vec{y}}(\vec{f}, x)}{\Delta^{\vec{L}}(i\vec{y})} = \mathrm{Wr}(\vec{f}, x).$$

Let $\vec{p}$ be a complete N-family of monic polynomials, where $N = R_1 M$. Define

$$\gamma_{\vec{L}}(\vec{y}) = \sum_{\mathfrak{t}:\underline{R_1} \nearrow \underline{N}} \int_{\mathbb{R}} \mathrm{Wr}^{\vec{L}} \otimes \mathrm{Pr}_{\vec{y}}(\vec{p}_{\mathfrak{t}}, x)\, d\mu(x)\, \varepsilon_{\mathfrak{t}},$$

and define

$$\eta_{\vec{L}}(\vec{y}) = \sum_{\mathfrak{t}:\underline{R_1} \nearrow \underline{N}} \sum_{\mathfrak{s}:\underline{R_1} \nearrow \underline{N}} \int\int_{x_1<x_2} \Big[\mathrm{Wr}^{\vec{L}} \otimes \mathrm{Pr}_{\vec{y}}(\vec{p}_{\mathfrak{t}}, x_1) \times \mathrm{Wr}^{\vec{L}} \otimes \mathrm{Pr}_{\vec{y}}(\vec{p}_{\mathfrak{s}}, x_2) \Big]\, d\mu(x_1)\, d\mu(x_2)\, \varepsilon_{\mathfrak{t}} \wedge \varepsilon_{\mathfrak{s}}.$$

Applying Theorem 3.1 in this context produces the following generalization of Theorem 4.1:

Theorem 4.2 (K-fold Homogeneous Partition Function)**.**

$$Z_M(\vec{y}) = \int_{-\infty<x_1<\cdots<x_M<\infty} \Omega(\vec{x}, \vec{y})\, dx_1 \cdots dx_M = \mathrm{PF}(\omega(\vec{y})),$$

where $\omega(\vec{y})$ is defined by:

1. *If R_1 is even, then $\omega(\vec{y}) = \gamma_{\vec{L}}(\vec{y})$.*

2. *If R_1 is odd, but M is even, then $\omega(\vec{y}) = \eta_{\vec{L}}(\vec{y})$.*

3. *If $R_1 M$ is odd, then $\omega(\vec{y}) = \eta_{\vec{L}}(\vec{y}) + \gamma_{\vec{L}}(\vec{y}) \wedge \xi_{R_1}$.*

The three cases are the same as those appearing in Theorem 4.1, replacing all instances of LK with $R_1 = \sum_{k=1}^{K} L_k$. As before, the ξ_{R_1} in case 3 is a pure tensor which upgrades $\gamma_{\vec{L}}(\vec{y})$ from an odd R_1-form to an even $2R_1$-form so that the Hyperpfaffian is well-defined. As mentioned at the beginning of this chapter, homogeneous constellation ensembles are the most general classification (in this volume) for which the partition functions are Hyperpfaffians (because of homogeneous $\omega(\vec{y})$).

4.5. Limits of Linear Constellations

Starting with a homogeneous constellation ensemble, taking the limit as $\vec{y} \to 0$ produces infinite potential energy, so the resulting Boltzmann factor $\Omega_M(\vec{x}, 0)$ is identically zero. In our physical interpretation, collapsing the parallel lines onto each other forces particles with the same real parts (who want to repel each other) onto each other. This is represented by the interaction terms with $L_j L_k \log|i(y_k - y_j)|$. To obtain meaningful limits, we remove these singularities by removing the appropriate interaction terms. Taking the limit inside the integral, it is easy to see

$$\lim_{\vec{y}\to 0} \frac{\Delta^{\mathbf{L}}(\mathbf{x})}{\left(\Delta^{\vec{L}}(i\vec{y})\right)^{M}} = \Delta(\vec{x})^{R_1^2}.$$

Thus, the limiting Boltzmann factor corresponds to a one-dimensional ensemble of particles with charge $R_1 = \sum_{k=1}^{K} L_k$. In terms of confluent matrices,

$$\lim_{\vec{y}\to 0} \frac{V^{\mathbf{L}}(\mathbf{x})}{\left(\Delta^{\vec{L}}(i\vec{y})\right)^M} = V^{R_1}(\vec{x}).$$

This limit turns all proto-confluent translation columns into further derivative columns, a total of R_1 many for each variable x_m. In terms of the partition function,

$$\begin{aligned}
\lim_{\vec{y}\to 0} \frac{Z_M(\vec{y})}{\left(\Delta^{\vec{L}}(i\vec{y})\right)^M} &= \lim_{\vec{y}\to 0} \int \frac{\gamma_{\vec{L}}(\vec{y})^{\wedge M}}{M!\left(\Delta^{\vec{L}}(i\vec{y})\right)^M}\,\varepsilon_{\mathrm{vol}} \\
&= \lim_{\vec{y}\to 0} \frac{1}{M!} \int \left[\sum_{\mathfrak{t}:\underline{R_1}\nearrow\underline{N}} \frac{1}{\Delta^{\vec{L}}(i\vec{y})} \int_{\mathbb{R}} \mathrm{Wr}^{\vec{L}} \otimes \mathrm{Pr}_{\vec{y}}(\vec{p}_{\mathfrak{t}}, x)\, d\mu(x)\, \varepsilon_{\mathfrak{t}}\right]^{\wedge M} \varepsilon_{\mathrm{vol}} \\
&= \frac{1}{M!} \int \left[\sum_{\mathfrak{t}:\underline{R_1}\nearrow\underline{N}} \int_{\mathbb{R}} \mathrm{Wr}(\vec{p}_{\mathfrak{t}}, x)\, d\mu(x)\, \varepsilon_{\mathfrak{t}}\right]^{\wedge M} \varepsilon_{\mathrm{vol}},
\end{aligned}$$

in the case $R_1 K$ is even. The limit of the Hyperpfaffian is again Hyperpfaffian. Also, the Wronskian minors which appear in this Hyperpfaffian are the minors of the confluent limit of the proto-confluent matrix $V^{\mathbf{L}}(\mathbf{x})$. An analogous result holds for $R_1 K$ odd, attaching a $\Delta^{\vec{L}}(i\vec{y})$ denominator to each of two Wronskian-like integrands at a time.

It is not necessary that all y_k go to zero. We could instead take limits as some $y_j \to y_k$. Physically, this means collapsing some lines together but not all. If we did not already have the confluent Vandermonde technology, we could produce any homogeneous constellation ensemble as a limit of first linear constellation ensembles (in which case all the particles have charge 1 and only the ordinary Vandermonde

determinant is needed). As a special case of this, collapsing K many lines produces a one-dimensional $\beta = K^2$ ensemble (equivalently, charge K particles).

4.6. Limits at Infinity

Next, we consider limits as the distances between our lines increase without bound. Not only do we want $y_k \to \infty$, but also $(y_k - y_j) \to \infty$. For simplicity, we start by setting $y_k = (k-1)h$ and then consider limits as $h \to \infty$. This limit produces negatively infinite potential energy, so the resulting Boltzmann factor is positively infinite. This comes from interaction terms with $L_j L_k \log\left((x_m - x_n)^2 + (y_k - y_j)^2\right)$. Denote

$$G_M^{\vec{L}}(h) = \left[\prod_{j<k} \left(1 + ((k-j)h)^2\right)^{L_j L_k}\right]^{\binom{M}{2}}.$$

Note, $\lim_{h\to 0} G_M^{\vec{L}}(h) = 1$, so we can add $G_M^{\vec{L}}(h)$ to the denominators in section 4.5 without changing the limits (as $h \to 0$). On the other hand, it is straightforward to check

$$\lim_{h\to\infty} \frac{\Delta^{\mathbf{L}}(\mathbf{x})}{\left(\Delta^{\vec{L}}(ih\underline{K})\right)^M G_M^{\vec{L}}(h)} = \Delta(\vec{x})^{L_1^2+\cdots+L_K^2}.$$

Thus, in terms of the Boltzmann factor, the limit produces a one-dimensional $\beta = \sum_k L_k^2$ ensemble (of charge $\sqrt{\sum_k L_k^2}$). As a special case of this, if we take the limit of the first linear constellation, the result is a one-dimensional $\beta = K$ ensemble corresponding to possibly non-integer charge $\sqrt{K}$. Physically, moving our lines away from each other without bound breaks the interactions between particles from different lines. The remaining energy contributions from internal interactions within each line are additive. A pair of charge 1 particles repel another pair of

charge 1 particles with a force greater than that between just two charge 1 particles but weaker than that of two charge 2 particles. Together with section 4.5, we now have an interpolation between one-dimensional $\beta = K^2$ and $\beta = K$ ensembles.

In terms of confluent matrices, our existing methods do not allow us to produce square-free powers of the ordinary Vandermonde determinant. Additionally, it is unclear how to bring the limit inside $V^{\mathbf{L}}(\mathbf{x})$ in hopes of producing an entirely new determinantal expression for $\Delta(\vec{x})^{L_1^2+\cdots+L_K^2}$, which, as stated, is a power of a determinant, not a lone determinant. Equivalently, it is unclear how to distribute the denominator of the limit over the Wronskian-like minors of the confluent determinant (which would have allowed us to bring the limit inside the Hyperpfaffian expression for the partition functions). Without this, the limit of the Hyperpfaffian partition function cannot simply be written as a Hyperpfaffian using the methods demonstrated thus far (from Theorem 3.1). However, for each h (or $\vec{y}$) fixed along the way, the partition function is Hyperpfaffian as stated in Theorem 4.2.

Recall (from section 1.2), Shum considered the 2-fold first constellation ensembles in his 2013 dissertation. First, he demonstrated the partition function is Pfaffian (instead of Hyperpfaffian, because $K = 2$). Using this, he gave the kernel of which the correlation functions are the Pfaffian. When computing the limits (as $h \to 0$ and $h \to \infty$), he worked directly with the kernel, producing the expected kernels of the limiting ensembles in both directions (classical $\beta = 4$ as $h \to 0$ and classical $\beta = 2$ as $h \to \infty$). In this way, the limiting ensembles were demonstrated to be solvable Pfaffian point processes without needing to explicitly express the limiting partition functions as Pfaffians. Analogously, square-free $\beta = K$ ensembles

may still have Hyperpfaffian correlation functions even though the methods of this volume do not produce an explicitly Hyperpfaffian partition function in the limit.

CHAPTER V

MULTICOMPONENT ENSEMBLES

This chapter contains unpublished coauthored material. In particular, all sections except section 5.6 appear largely as is in [35].

Recall the (single-component) log-gas setup of section 1.3. In a multicomponent ensemble, we allow the particles to have possibly different charges, provided the charges are all integers of the same sign. We assume any two particles of the same charge, which we will call same *species*, are indistinguishable.

We consider two ensembles:

1. *The Canonical Ensemble*, in which the number of particles of each species is fixed; in this case, we say fixed population.

2. *The Isocharge Grand Canonical Ensemble*, in which the sum of the charges of the particles is fixed, but the number of particles of each species is allowed to vary; in this case, we say the total charge of the system is fixed.

In contrast, the *grand canonical ensemble* traditionally refers to the ensemble in which the total number of particles is not fixed. For computational purposes, it is beneficial to group configurations which share the same total charge. The true grand canonical ensemble is then a disjoint union (over all possible sums of charges) of our isocharge ensembles.

In 2012, Sinclair [32] provided a closed form of the partition function for both ensembles in terms of Berezin integrals of alternating tensors, but only for certain β and only for ensembles with at most one species of odd charge. In this chapter, we provide an alternative framework which allows us to generalize the result to

ensembles with an arbitrary mix of odd and even charges, albeit with the same limitations on β. As mentioned in chapter III, we continue to demonstrate the versatility of Theorem 3.1 whenever appropriate.

By first conditioning on the number of particles of each species, the partition function for the isocharge grand canonical ensemble is built up from the partition functions of the canonical type, revealing the former to be a generating function of the latter as a function of the *fugacities* of each species (roughly, the probability of the occurrence of any one particle of a given charge). In section 6.4, we produce analogous results for charged particles placed on the unit circle in the complex plane.

5.1. The Multicomponent Setup

Let $J \in \mathbb{Z}_{>0}$ be a positive integer, the maximum number of distinct charges in the system. Let $\vec{L} = (L_1, L_2, \ldots, L_J) \in (\mathbb{Z}_{>0})^J$ be a vector of distinct positive integers which we will call the *charge vector* of the system. Let $\vec{M} \in (\mathbb{Z}_{\geq 0})^J$ be a vector of non-negative integers which we will call the *population vector* of the system. Each M_j gives the number (possibly zero) of indistinguishable particles of charge L_j. Let

$$\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^J) \in \mathbb{R}^{M_1} \times \mathbb{R}^{M_2} \times \cdots \times \mathbb{R}^{M_J}$$

so that $\mathbf{x}^j = (x_1^j, x_2^j, \ldots, x_{M_j}^j) \in \mathbb{R}^{M_j}$ for each j. We call $\mathbf{x}$ the *location vector* of the system in which each $x_m^j \in \mathbb{R}$ gives the location of a particle of charge L_j. We call $\mathbf{x}^j$ the *location vector* for the *species* with charge L_j. If some $M_j = 0$, then we take $\mathbf{x}^j$ to be the empty vector.

The particles are assumed to interact logarithmically on an infinite wire so that the contribution of energy to the system by two particles of charge L_j and L_k at locations x_m^j and x_n^k respectively is given by $-L_jL_k \log|x_n^k - x_m^j|$. If U is the potential on the system, then at inverse temperature β, the total potential energy of the system is given by

$$\begin{aligned} E_{\vec{M}}(\mathbf{x}) = \beta \sum_{j=1}^{J} L_j \sum_{m=1}^{M_j} U(x_m^j) - \beta \sum_{j=1}^{J} L_j^2 \sum_{m<n} \log|x_n^j - x_m^j| \\ - \beta \sum_{j<k} L_jL_k \sum_{m=1}^{M_j} \sum_{n=1}^{M_k} \log|x_n^k - x_m^j|. \end{aligned}$$

The first type of iterated sum accounts for the potential U, the second type of iterated sum accounts for interactions between particles of the same charge L_j, and the third type of iterated sum accounts for the interactions between particles of distinct charges L_j and L_k.

With this setup, the relative density of states (corresponding to varying location vectors $\mathbf{x}$) is given by the Boltzmann factor

$$\begin{aligned} \Omega_{\vec{M}}(\mathbf{x}) &= \exp(-E_{\vec{M}}(\mathbf{x})) \\ &= \prod_{j=1}^{J} \prod_{m=1}^{M_j} \exp\left(-\beta L_j U(x_m^j)\right) \times \prod_{j=1}^{J} \prod_{m<n} \left|x_n^j - x_m^j\right|^{\beta L_j^2} \\ &\quad \times \prod_{j<k} \prod_{m=1}^{M_j} \prod_{n=1}^{M_k} \left|x_n^k - x_m^j\right|^{\beta L_jL_k}. \end{aligned}$$

Later, it will be convenient to write $W_{\vec{M}}(\mathbf{x})$ in place of the first of the three iterated products above. In the case when $\sqrt{\beta}L_j \in \mathbb{Z}$ for all j, we will also write $\left|\det V^{\vec{L},\vec{M}}(\mathbf{x})\right|$ in place of the product of the remaining two iterated products. In section 5.3, we explicitly construct the (confluent Vandermonde) matrix $V^{\vec{L},\vec{M}}(\mathbf{x})$

of which this is the determinant. Then the probability of finding the system in a state corresponding to a location vector $\mathbf{x}$ is given by the joint probability density function

$$\rho_{\vec{M}}(\mathbf{x}) = \frac{\Omega_{\vec{M}}(\mathbf{x})}{Z_{\vec{M}} M_1! M_2! \cdots M_J!} = \frac{W_{\vec{M}}(\mathbf{x}) \left|\det V^{\vec{L},\vec{M}}(\mathbf{x})\right|}{Z_{\vec{M}} M_1! M_2! \cdots M_J!},$$

where the partition function (of the canonical ensemble) $Z_{\vec{M}}$ is the normalization constant given by

$$\begin{aligned} Z_{\vec{M}} &= \frac{1}{M_1! M_2! \cdots M_J!} \int_{\mathbb{R}^{M_1}} \cdots \int_{\mathbb{R}^{M_J}} \Omega_{\vec{M}}(\mathbf{x})\, d\nu^{M_1}(\mathbf{x}^1)\, d\nu^{M_2}(\mathbf{x}^2) \cdots d\nu^{M_J}(\mathbf{x}^J) \\ &= \frac{1}{M_1! M_2! \cdots M_J!} \int_{\mathbb{R}^{M_1}} \cdots \int_{\mathbb{R}^{M_J}} \left|\det V^{\vec{L},\vec{M}}(\mathbf{x})\right|\, d\mu_1^{M_1}(\mathbf{x}^1)\, d\mu_2^{M_2}(\mathbf{x}^2) \cdots d\mu_J^{M_J}(\mathbf{x}^J), \end{aligned}$$

with Lebesgue measure ν^{M_j} on $\mathbb{R}^{M_j}$ and $d\mu_j(x) = w_j(x)\, dx = \exp\left(-\beta L_j U(x)\right) dx$.

Note, the factorial denominators appear since particles of the same charge are indistinguishable, giving many different representatives for each state. In particular, the integrand is invariant under permutation of $\{x_1^j, x_2^j, \ldots, x_{M_j}^j\}$ for any j fixed. As in other log-gas models, it is necessary to assume the potential U is one for which $Z_{\vec{M}}$ is finite. Also, replacing β with $\beta' = \beta/b^2$ and replacing each L_j with $L_j' = bL_j$ leaves $\left|\det V^{\vec{L},\vec{M}}(\mathbf{x})\right|$ unchanged. Then replacing U with $U' = bU$ leaves $W_{\vec{M}}(\mathbf{x})$ unchanged. Thus, for computational purposes, we can always assume $\beta = 1$.

Next, allowing the number of particles of each species to vary, let $P(\vec{M})$ be the probability of finding the system with population vector $\vec{M}$. Let $\vec{z} = (z_1, \ldots, z_J) \in (\mathbb{R}_{>0})^J$ be a vector of positive real numbers called the *fugacity vector*. Classically, the probability $P(\vec{M})$ is given by

$$P(\vec{M}) = z_1^{M_1} z_2^{M_2} \cdots z_J^{M_J} \frac{Z_{\vec{M}}}{Z_N},$$

where Z_N is the partition function of the isocharge grand canonical ensemble (corresponding to fixed total charge N) given by

$$Z_N = \sum_{\vec{L}\cdot\vec{M}=N} z_1^{M_1} z_2^{M_2} \cdots z_J^{M_J} Z_{\vec{M}}.$$

In the above expression, the vector $\vec{L}$ of allowed charges is fixed, so we are summing over allowed population vectors $\vec{M}$. A population vector is valid only when the sum of the charges $\sum_{j=1}^{J} L_j M_j$ is equal to the prescribed total charge N.

As before, Z_N is the primary object of interest in this chapter, and it varies with charge vector $\vec{L}$, potential U, and inverse temperature β. Unique to isocharge ensembles is the dependence on the fixed total charge N. Additionally, taking the fugacity vector $\vec{z}$ to be a vector of indeterminants, Z_N is a polynomial in these indeterminants which generates the partition functions of the canonical ensembles.

As mentioned at the beginning of the chapter, Sinclair [32] already produced the desired Berezin integral formulae for ensembles with at most one L_j odd. Even in the case when exactly one L_j is, say L_1, he required the additional restriction that the total charge N be even so that M_1 would be even. In this chapter, we will show his expression can be extended to arbitrary $\vec{L}$ (for which any number of the L_j may be odd), and in the case when N is odd, we give an analogous Berezin integral expression with respect to the volume form on $\mathbb{R}^{N+1}$ (instead of $\mathbb{R}^N$).

The major obstruction remains writing the integrand $\left|\det V^{\vec{L},\vec{M}}(\mathbf{x})\right|$ as an honest determinant without absolute value (for which Theorem 3.1 will apply). When all charges L_j are even, all of the $\beta L_j L_k$ exponents in the integrand are even, so the absolute value on the differences $|x_n^k - x_m^j|$ can be ignored. Alternatively, in chapter V, we had only a single species of indistinguishable particles (or

constellations), so we could (partially) resolve the absolute value by integrating over the domain in which $-\infty < x_1 < \cdots < x_M < \infty$. In the multicomponent case, more work is required, as detailed in section 5.3.

5.2. Multicomponent Partition Functions

Recall the setup from section 5.1. Let $\vec{p}$ be a complete N-family of monic polynomials (see section 2.6). Define

$$\gamma_j = \sum_{\mathfrak{t}:\underline{L_j}\nearrow\underline{N}} \int_{\mathbb{R}} \mathrm{Wr}(\vec{p}_{\mathfrak{t}}, x)\, d\mu_j(x)\ \varepsilon_{\mathfrak{t}},$$

and define

$$\eta_{j,k} = \sum_{\mathfrak{t}:\underline{L_j}\nearrow\underline{N}} \sum_{\mathfrak{s}:\underline{L_k}\nearrow\underline{N}} \int\int_{x<y} \mathrm{Wr}(\vec{p}_{\mathfrak{t}}, x)\mathrm{Wr}(\vec{p}_{\mathfrak{s}}, y)\, d\mu_j(x) d\mu_k(y)\ \varepsilon_{\mathfrak{t}} \wedge \varepsilon_{\mathfrak{s}}.$$

Theorem 5.1. *If all L_j are even, then*

$$Z_N = \mathrm{BE}_{\mathrm{vol}} \left(\sum_{j=1}^{J} z_j \gamma_j \right).$$

The above theorem is Sinclair's (2012) for which we give a different proof and then the following generalization:

Theorem 5.2. *If the first r many L_j are even and N is even, then*

$$Z_N = \mathrm{BE}_{\mathrm{vol}} \left(\sum_{j=1}^{r} z_j \gamma_j + \sum_{j=r+1}^{J} \sum_{k=r+1}^{J} z_j z_k \eta_{j,k} \right).$$

Theorem 5.3. *If the first r many L_j are even and N is odd, then*

$$Z_N = \mathrm{BE}_{\mathrm{vol}_1}\left(\sum_{j=1}^{r} z_j\gamma_j + \sum_{j=r+1}^{J}\sum_{k=r+1}^{J} z_j z_k \eta_{j,k} + \sum_{j=r+1}^{J} z_j\gamma_j \wedge \varepsilon_{N+1}\right),$$

where $\mathrm{BE}_{\mathrm{vol}_1}$ *includes the Berezin integral with respect to* $\varepsilon_{\mathrm{vol}_1} = \varepsilon_{\mathrm{vol}} \wedge \varepsilon_{N+1} \in \bigwedge^{N+1}(\mathbb{R}^{N+1})$.

As mentioned in section 3.6, it is necessary to extend the basis by ε_{N+1} so that the new volume form $\varepsilon_{\mathrm{vol}_1}$ has even length $N+1$. More generally, we can write

$$\varepsilon_{\mathrm{vol}_k} = \varepsilon_{\mathrm{vol}} \wedge \xi_k = \varepsilon_{\mathrm{vol}} \wedge \varepsilon_{N+1} \wedge \varepsilon_{N+2} \wedge \cdots \wedge \varepsilon_{N+k}.$$

Corollary. *In the single species case (N indistinguishable particles of charge L), we get the known Hyperpfaffian expression (Sinclair 2011):*

$$Z_N = \mathrm{BE}_{\mathrm{vol}_k}(\omega) = \mathrm{PF}(\omega),$$

where ω and k depends on N and L.

1. *If L is even, then $\omega = \gamma_1$ and $\mathrm{BE}_{\mathrm{vol}_k} = \mathrm{BE}_{\mathrm{vol}}$.*
2. *If L is odd and N is even, then $\omega = \eta_{1,1}$ and $\mathrm{BE}_{\mathrm{vol}_k} = \mathrm{BE}_{\mathrm{vol}}$.*
3. *If L is odd and N is odd, then $\omega = \eta_{1,1} + \gamma_1 \wedge \xi_L$ and $\mathrm{BE}_{\mathrm{vol}_k} = \mathrm{BE}_{\mathrm{vol}_L}$.*

Note, we extend by ξ_L instead of just $\xi_1 = \varepsilon_{N+1}$ in case 3 only so that $\gamma_1 \wedge \xi_L$ is a $2L$-form and therefore ω is homogeneous. Every choice of k produces a different but equally valid Berezin integral expression. We obtain the (Pfaffian) de Bruijn integral identities for classical $\beta = 1$ and $\beta = 4$ when $L = 1$ and $L = 2$, respectively.

As Theorem 3.1, we have already given general methods for manipulating iterated integrals of determinantal integrands. In section 5.4, we apply these identities first to $Z_{\vec{M}}$, the partition function of the canonical ensemble with arbitrary but fixed population $\vec{M}$. In section 5.5, we sum over all possible population vectors $\vec{M}$ to obtain Z_N, the partition function of the isocharge grand canonical ensemble.

5.3. Absolute Value of Determinants

In section 2.7, we defined $V^{\vec{L}}(\vec{x})$ to be the confluent Vandermonde matrix with L_j derivative columns for each variable x_j. For multicomponent ensembles, we expand the variables to $\mathbf{x} = (\mathbf{x}^1, \ldots, \mathbf{x}^J)$ and allow M_j many variables $\mathbf{x}^j = (x_1^j, \ldots, x_{M_j}^j)$ to share the same charge type L_j. Explicitly, fix charge vector $\vec{L}$, population vector $\vec{M}$, and location vector $\mathbf{x}$ as in section 5.1. Recall $N = \sum_{j=1}^{J} M_j L_j$. Let $\vec{f} = \{f_n\}_{n=1}^{N}$ be a family of $\max(L_1, \ldots, L_J) - 1$ times differentiable functions. For each j, define the $N \times L_j$ matrix

$$V^{L_j}(x) = \left[D^{l-1} f_n(x)\right]_{n,l=1}^{N,L_j}.$$

For each $\mathbf{x}^j \in \mathbb{R}^{M_j}$, define the $N \times M_j L_j$ matrix

$$V^{L_j,M_j}(\mathbf{x}^j) = \begin{bmatrix} V^{L_j}(x_1^j) & V^{L_j}(x_2^j) & \cdots & V^{L_j}(x_{M_j}^j) \end{bmatrix}.$$

Finally, define the $N \times N$ matrix

$$V^{\vec{L},\vec{M}}(\mathbf{x}) = \begin{bmatrix} V^{L_1,M_1}(\mathbf{x}^1) & V^{L_2,M_2}(\mathbf{x}^2) & \cdots & V^{L_J,M_J}(\mathbf{x}^J) \end{bmatrix},$$

in which each variable x_m^j appears in L_j many consecutive columns, generated from $\vec{f}$ by taking derivatives. As before, the Wronskians which appear in section 5.2 are merely the determinants of the univariate $L_j \times L_j$ minors of this matrix.

With the additional restriction that $\vec{f}$ be a complete N-family of monic polynomials, we call $V^{\vec{L},\vec{M}}(\mathbf{x})$ the confluent Vandermonde matrix (with respect to shape $\vec{L}, \vec{M}$) in variables $\mathbf{x}$. Using the same confluent Vandermonde determinant identity as in [25] and section 2.7, we get

$$\det V^{\vec{L},\vec{M}}(\mathbf{x}) = \prod_{j=1}^{J} \prod_{m<n} \left(x_n^j - x_m^j\right)^{L_j^2} \times \prod_{j<k} \prod_{m=1}^{M_j} \prod_{n=1}^{M_k} \left(x_n^k - x_m^j\right)^{L_j L_k}.$$

Note, we can only construct whole numbers of columns for each variable. This is where our restrictions on $\vec{L}$ and β come from. On the physical side, we only consider whole number charges for our particles. Using the above confluent Vandermonde determinant identity, we get

$$\Omega_{\vec{M}}(\mathbf{x}) = W_{\vec{M}}(\mathbf{x}) \left|\det V^{\vec{L},\vec{M}}(\mathbf{x})\right|.$$

Moreover, if we define the $N \times L_j$ matrices

$$H^{L_j}(x) = \exp(-U(x))V^{L_j}(x) = \left[\exp(-U(x))\, D^{l-1} f_n(x)\right]_{n,l=1}^{N,L_j}$$

and the combined $N \times N$ matrix $H^{\vec{L},\vec{M}}(\mathbf{x})$, then

$$\left|\det H^{\vec{L},\vec{M}}(\mathbf{x})\right| = \prod_{j=1}^{J} \prod_{m=1}^{M_j} \exp\left(-L_j U(x_m^j)\right) \left|\det V^{\vec{L},\vec{M}}(\mathbf{x})\right| = W_{\vec{M}}(\mathbf{x}) \left|\det V^{\vec{L},\vec{M}}(\mathbf{x})\right|.$$

Thus, integration of $V^{\vec{L},\vec{M}}(\mathbf{x})$ with respect to the $d\mu_j$'s is equivalent to integration of $H^{\vec{L},\vec{M}}(\mathbf{x})$ with respect to Lebesgue measure. For us to use Theorem 3.1, it is important the entire integrand $\Omega_{\vec{M}}(\mathbf{x})$ be determinantal, with the extra weight functions $W_{\vec{M}}(\mathbf{x})$ incorporated into $H^{\vec{L},\vec{M}}(\mathbf{x})$.

As an example, consider one charge 2 particle, one charge 3 particle, and three charge 1 particles with potential $U(x) = x^2$. This gives us $\vec{L} = (2, 3, 1)$, $\vec{M} = (1, 1, 3)$, and $N = 8$. For simplicity, we will use the variables $\mathbf{x} = (a, b, c_1, c_2, c_3)$. Let $\vec{f} = \{x^{n-1}\}_{n=1}^{N}$. The 8×8 confluent Vandermonde matrix is

$$V^{\vec{L},\vec{M}}(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ a & 1 & b & 1 & 0 & c_1 & c_2 & c_3 \\ a^2 & 2a & b^2 & 2b & 1 & c_1^2 & c_2^2 & c_3^2 \\ a^3 & 3a^2 & b^3 & 3b^2 & 3b & c_1^3 & c_2^3 & c_3^3 \\ \vdots & & & & & & & \vdots \\ a^7 & 7a^6 & b^7 & 7b^6 & 21b^5 & c_1^7 & c_2^7 & c_3^7 \end{bmatrix}.$$

We obtain $H^{\vec{L},\vec{M}}(\mathbf{x})$ by multiplying the first two columns by $\exp(-a^2)$, the next three columns by $\exp(-b^2)$, and the last three columns by the appropriate $\exp(-c_j^2)$. This changes the determinant by

$$W_{\vec{M}}(\mathbf{x}) = \exp(-2a^2)\exp(-3b^2)\exp(-c_1^2)\exp(-c_2^2)\exp(-c_3^2).$$

Absolute Value

Though our Boltzmann factor integrand $\Omega_{\vec{M}}(\mathbf{x})$ is now recognizably determinantal, we still need to remove the absolute value before we can apply Theorem 3.1. This can be done by decomposing the domain of integration into

subsets over which the sign of the determinant is constant. Namely, we use totally ordered subsets $\Delta_N(\sigma)$ over which the differences in the confluent Vandermonde determinant never change signs. Recall from section 3.4, $\Delta_N(\sigma)$ is the subset of $\mathbb{R}^N$ in which the N many variables are ordered according to σ. These smaller domains of integration are exactly the ones which allow us to apply (Chen's) Lemma 3.4.

As in section 5.2, suppose L_j is even for $1 \leq j \leq r$. Let $K_e = \sum_{j=1}^{r} M_j$ be the total number of particles with even charge, and let $K_o = \sum_{j=r+1}^{J} M_j$ be the total number of particles with odd charge. Relabel

$$y_1 = x_1^1, \quad y_2 = x_2^1, \quad \cdots \quad y_{M_1} = x_{M_1}^1, \quad y_{M_1+1} = x_1^2 \quad \cdots \quad y_{K_e} = x_{M_r}^r$$

so that $\vec{y} = (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^r)$ gives the locations of the particles of even charge. Similarly, relabel

$$w_1 = x_1^{r+1}, \quad \cdots \quad w_{M_{r+1}} = x_{M_{r+1}}^{r+1}, \quad w_{M_{r+1}+1} = x_1^{r+2} \quad \cdots \quad w_{K_e} = x_{M_J}^J$$

so that $\vec{w} = (\mathbf{x}^{r+1}, \mathbf{x}^{r+2}, \ldots, \mathbf{x}^J)$ gives the locations of the particles of odd charge. Define λ_j^e to be the L_k which corresponds to y_j so that

$$\Lambda^e = (\lambda_1^e, \ldots, \lambda_{K_e}^e) = (L_1, \ldots, L_1, L_2, \ldots, L_2, \ldots, L_r, \ldots, L_r)$$

gives the list of even charges, with each L_j appearing M_j times. Similarly define λ_j^o for corresponding w_j so that

$$\Lambda^o = (\lambda_1^o, \ldots, \lambda_{K_o}^o) = (L_{r+1}, \ldots, L_{r+1}, L_{r+2}, \ldots, L_{r+2}, \ldots, L_J, \ldots, L_J)$$

gives the list of the odd charges. We will treat all charges λ_j^e and λ_j^o as distinct until it is relevant to recall which charges are repeated (and how many times each).

For each $(\sigma, \tau) \in S_{K_e} \times S_{K_o}$, define $V_{\sigma\tau}^{\vec{L},\vec{M}}(\mathbf{x})$ to be the matrix

$$V_{\sigma,\tau}^{\vec{L},\vec{M}}(\mathbf{x}) = \Big[V^{\lambda^e_{\sigma^{-1}(1)}}(y_{\sigma^{-1}(1)}) \quad \cdots \quad V^{\lambda^e_{\sigma^{-1}(K_e)}}(y_{\sigma^{-1}(K_e)}) \quad V^{\lambda^o_{\tau^{-1}(1)}}(w_{\tau^{-1}(1)}) \quad \cdots \quad V^{\lambda^o_{\tau^{-1}(K_o)}}(w_{\tau^{-1}(K_o)}) \Big]$$

obtained from $V^{\vec{L},\vec{M}}(\mathbf{x})$ by permuting the columns so that the columns with $y_{\sigma^{-1}(1)}$ come first (of which there are $\lambda^e_{\sigma^{-1}(1)}$ many), then all of the columns with $y_{\sigma^{-1}(2)}$ come next (of which there are $\lambda^e_{\sigma^{-1}(2)}$ many), and so on until the y_j are exhausted, doing the same for the w_j.

Using the confluent Vandermonde determinant identity once more, we get

$$\begin{aligned} \det V_{\sigma,\tau}^{\vec{L},\vec{M}}(\mathbf{x}) = &\prod_{j<k} (y_{\sigma^{-1}(k)} - y_{\sigma^{-1}(j)})^{\lambda^e_{\sigma^{-1}(k)}\lambda^e_{\sigma^{-1}(j)}} \\ &\times \prod_{j=1}^{K_e} \prod_{k=1}^{K_o} (w_{\sigma^{-1}(k)} - y_{\tau^{-1}(j)})^{\lambda^o_{\sigma^{-1}(k)}\lambda^e_{\tau^{-1}(j)}} \\ &\times \prod_{j<k} (w_{\tau^{-1}(k)} - w_{\tau^{-1}(j)})^{\lambda^o_{\tau^{-1}(k)}\lambda^o_{\tau^{-1}(j)}}. \end{aligned}$$

Next, consider $\mathbf{x} = (\vec{y}, \vec{w}) \in \Delta_{K_e}(\sigma) \times \Delta_{K_o}(\tau)$ in which the even charged particles (located by $\vec{y}$) are ordered according to σ and the odd charged particles (located by $\vec{w}$) are ordered according to τ. In particular, $w_{\tau^{-1}(k)} > w_{\tau^{-1}(j)}$ whenever $j < k$. Thus, all differences in the third product are positive. Additionally, each difference in the first and second products have even exponents λ_j^e. Thus,

$$\left| \det V_{\sigma,\tau}^{\vec{L},\vec{M}}(\mathbf{x}) \right| = \det V_{\sigma,\tau}^{\vec{L},\vec{M}}(\mathbf{x})$$

on the domain $\Delta_{K_e}(\sigma) \times \Delta_{K_o}(\tau) \subset \mathbb{R}^{K_e} \times \mathbb{R}^{K_o}$.

Note, permuting the variables $\vec{y}$ involves permuting blocks of even numbers of columns at a time, leaving the determinant of $V^{\vec{L},\vec{M}}(\mathbf{x})$ unchanged. In contrast, permuting variables $\vec{w}$ involves permuting blocks of odd numbers of columns at a time, changing the determinant by $\text{sgn}(\tau)$. Thus,

$$\left|\det V^{\vec{L},\vec{M}}(\mathbf{x})\right| = \left|\text{sgn}(\tau) \det V^{\vec{L},\vec{M}}_{\sigma,\tau}(\mathbf{x})\right| = \det V^{\vec{L},\vec{M}}_{\sigma,\tau}(\mathbf{x})$$

on the domain $\Delta_{K_e}(\sigma) \times \Delta_{K_o}(\tau)$. Analogous results hold if we replace $V^{\vec{L},\vec{M}}(\mathbf{x})$ with $H^{\vec{L},\vec{M}}(\mathbf{x})$ (to account for the weight functions).

Recall the example which preceded this subsection, in which we have a single particle of even charge 2. Then $K_e = 1$, $\vec{y} = (a)$, and $\Lambda^e = (2)$. There is one particle of odd charge 3, and there are three particles of odd charge 1. Then $K_o = 4$, $\vec{w} = (b, c_1, c_2, c_3)$, and $\Lambda^o = (3, 1, 1, 1)$. Let τ be the permutation which swaps b with c_1. The new matrix (with permuted columns) is given by

$$V^{\vec{L},\vec{M}}_{\text{id},\tau}(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ a & 1 & c_1 & b & 1 & 0 & c_2 & c_3 \\ a^2 & 2a & c_1^2 & b^2 & 2b & 1 & c_2^2 & c_3^2 \\ a^3 & 3a^2 & c_1^3 & b^3 & 3b^2 & 3b & c_2^3 & c_3^3 \\ \vdots & & & & & & & \vdots \\ a^7 & 7a^6 & c_1^7 & b^7 & 7b^6 & 21b^5 & c_2^7 & c_3^7 \end{bmatrix}$$

Note, swapping two variables requires more than just swapping two columns. We swap the entire three-column block $V^3(b)$ with the one-column block $V^1(c_1)$.

5.4. Canonical Ensembles

With the modification to the integrand outlined in section 5.3, we can now decompose $Z_{\vec{M}}$ into integrals without absolute value, provided we divide the domain of integration appropriately. Explicitly,

$$\begin{aligned} Z_{\vec{M}} &= \frac{1}{M_1!M_2!\cdots M_J!}\int_{\mathbb{R}^{K_e}}\int_{\mathbb{R}^{K_o}} \Omega_{\vec{M}}(\mathbf{x})\, dy_1\cdots dy_{K_e}\, dw_1\cdots dw_{K_o} \\ &= \frac{1}{M_1!M_2!\cdots M_J!}\sum_{\sigma\in S_{K_e}}\sum_{\tau\in S_{K_o}}\int_{\Delta_{K_e}(\sigma)}\int_{\Delta_{K_o}(\tau)} \left|\det V^{\vec{L},\vec{M}}(\mathbf{x})\right| W_{\vec{M}}(\mathbf{x})\, dy_1\cdots dw_{K_o} \\ &= \frac{1}{M_1!M_2!\cdots M_J!}\sum_{\sigma\in S_{K_e}}\sum_{\tau\in S_{K_o}}\int_{\Delta_{K_e}(\sigma)}\int_{\Delta_{K_o}(\tau)} \det H^{\vec{L},\vec{M}}_{\sigma,\tau}(\mathbf{x})\, dy_1\cdots dw_{K_o}, \end{aligned}$$

summing over all totally ordered subsets $\Delta_{K_e}(\sigma)\subset\mathbb{R}^{K_e}$ and $\Delta_{K_o}(\tau)\subset\mathbb{R}^{K_o}$.

When All L_j Are Even

Starting from

$$Z_{\vec{M}} = \frac{1}{M_1!M_2!\cdots M_J!}\sum_{\sigma\in S_{K_e}}\int_{\Delta_{K_e}(\sigma)} \det H^{\vec{L},\vec{M}}_{\sigma}(\vec{y})\, dy_1\cdots dy_{K_e},$$

relabeling the variables $x_j = y_{\sigma^{-1}(j)}$ produces

$$Z_{\vec{M}} = \frac{1}{M_1!M_2!\cdots M_J!}\sum_{\sigma\in S_{K_e}}\int_{\Delta_{K_e}(\mathrm{id})} \det H^{\vec{L},\vec{M}}_{\sigma}(\vec{x})\, dx_1\cdots dx_{K_e}.$$

As in section 3.6, applying Theorem 3.1 yields

$$Z_{\vec{M}} = \frac{1}{M_1!M_2!\cdots M_J!}\sum_{\sigma\in S_{K_e}}\int \frac{\gamma_1^{\wedge M_1}\wedge\cdots\wedge\gamma_J^{\wedge M_J}}{K_e!}\, d\varepsilon_{\mathrm{vol}},$$

where each γ_j is as defined in section 5.2 by

$$\gamma_j = \sum_{\mathfrak{t}:\underline{L_j}\nearrow\underline{N}} \int_{\mathbb{R}} \mathrm{Wr}(\vec{p}_{\mathfrak{t}}, x)\, d\mu_j(x)\ \varepsilon_{\mathfrak{t}},$$

with $d\mu_j(x) = \exp(-L_j U(x))\, dx$. Note, there exist M_j many k for which $\lambda_k^e = L_j$, so each factor γ_j appears M_j many times. This happens independent of $\sigma \in S_{K_e}$, of which there are $|S_{K_e}| = K_e!$ many. Thus,

$$Z_{\vec{M}} = \frac{1}{M_1! M_2! \cdots M_J!} \int \gamma_1^{\wedge M_1} \wedge \cdots \wedge \gamma_J^{\wedge M_J}\, d\varepsilon_{\mathrm{vol}}.$$

We have now proven the following lemma:

Lemma 5.1. *If all L_j are even, then*

$$Z_{\vec{M}} = \int \frac{\gamma_1^{\wedge M_1}}{M_1!} \wedge \cdots \wedge \frac{\gamma_J^{\wedge M_J}}{M_J!}\, d\varepsilon_{\mathrm{vol}}.$$

When All L_j Are Odd

In this subsection, we first assume all L_j are odd, but the total number of particles $K_o = \sum_{j=1}^{J} M_j = 2K$ is even. This happens, for example, when total charge N is even. Recall from section 5.2,

$$\eta_{j,k} = \sum_{\mathfrak{t}:\underline{L_j}\nearrow\underline{N}} \sum_{\mathfrak{s}:\underline{L_k}\nearrow\underline{N}} \int \int_{x<y} \mathrm{Wr}(\vec{p}_{\mathfrak{t}}, x)\mathrm{Wr}(\vec{p}_{\mathfrak{s}}, y)\, d\mu_j(x) d\mu_k(y)\, \varepsilon_{\mathfrak{t}} \wedge \varepsilon_{\mathfrak{s}}.$$

Again, proceeding as we did in then "all even" case, applying Theorem 3.1 to $Z_{\vec{M}}$ produces

$$\begin{aligned} Z_{\vec{M}} &= \frac{1}{M_1!M_2!\cdots M_J!} \sum_{\tau\in S_{K_o}} \int_{\Delta_{K_o}(\tau)} \det H_\tau^{\vec{L},\vec{M}}(\vec{w})\, dw_1\cdots dw_{K_o} \\ &= \frac{1}{M_1!M_2!\cdots M_J!} \sum_{\tau\in S_{K_o}} \int_{\Delta_{K_o}(\mathrm{id})} \det H_\tau^{\vec{L},\vec{M}}(\vec{x})\, dx_1\cdots dx_{K_e} \\ &= \frac{1}{M_1!M_2!\cdots M_J!} \sum_{\tau\in S_{K_o}} \int \frac{1}{K!} \bigwedge_{j=1}^{J}\bigwedge_{k=1}^{J} \eta_{j,k}^{\wedge M_\tau^{j,k}}\, d\varepsilon_{\mathrm{vol}}, \end{aligned}$$

where $M_\tau^{j,k}$ is the number of times $\lambda^o_{\tau^{-1}(2n-1)} = L_j$ while $\lambda^o_{\tau^{-1}(2n)} = L_k$, and $K = K_o/2 = \sum_{j,k} M_\tau^{j,k}$ is the total number of factors in the wedge product. Note, these $M_\tau^{j,k}$ exponents depend on τ, so we are not able to drop the sum. However, the M_j many particles with the same charge L_j are indistinguishable. Restricting to shuffle permutations removes the redundancy in permuting variables which have the same L_j. We have now proven another lemma:

Lemma 5.2. *If all L_j are odd, but total charge N is even, then*

$$Z_{\vec{M}} = \sum_{\tau\in \mathrm{Sh}(M_1,\ldots,M_J)} \int \frac{1}{K!} \bigwedge_{j=1}^{J}\bigwedge_{k=1}^{J} \eta_{j,k}^{\wedge M_\tau^{j,k}}\, d\varepsilon_{\mathrm{vol}}.$$

Recall the example from section 5.3. Modify this example by replacing the even charge 2 particle with an odd charge 3 particle, leaving the other particle of odd charge 3 and three particles of odd charge 1. The three columns in variables a and b produce 3×3 Wronskian minors, while the remaining columns in the variables c_1, c_2, c_3 produce 1×1 Wronskian minors.

Under the identity permutation, we pair the three columns in variable a with the three columns in variable b to produce $\eta_{3,3}$. Pairing the one column in variable c_1 with the one column in variable c_2 produces $\eta_{1,1}$.

Under the permutation τ which previously swapped the three columns in variable b with the one column in variable c_1, we pair variable a (charge 3) with c_1 (charge 1), and we pair b (charge 3) with c_2 (charge 1). After integrating out all the variables, the result is two copies of $\eta_{3,1}$.

The permutation which swaps a with b produces the same $\eta_{3,3}$ as the identity permutation. To avoid this redundancy, we consider only shuffle permutations. The permutation which moves c_1 to the front (ordering the variables as c_1, a, b, c_2, c_3) produces $\eta_{1,3}$ followed by the distinct $\eta_{3,1}$.

Note, in this example as stated, the last variable c_3 is unpaired because we have an odd number of variables. As demonstrated in section 3.6, amending an extra column to the confluent Vandermonde matrix allows us to pair the last variable with a placeholder. Once integrated, this last "pair" produces the single Wronskian form γ_j instead of the double Wronskian form $\eta_{j,k}$.

In general, if all L_j are odd, but the total number of particles $K_o = \sum_{j=1}^{J} M_j = 2K - 1$ is odd (when total charge N is odd, for example), we get a variant of Lemma 5.2:

Lemma 5.3. *If all L_j are odd, and the total charge N is odd, then*

$$Z_{\vec{M}} = \sum_{\tau \in \mathrm{Sh}(M_1,\ldots,M_J)} \int \frac{1}{K!} \bigwedge_{j=1}^{J} \bigwedge_{k=1}^{J} \eta_{j,k}^{\wedge M_\tau^{j,k}} \wedge \gamma_{\tau^{-1}(K_o)}\, d\varepsilon_{\mathrm{vol}},$$

where $K = (K_o + 1)/2 = 1 + \sum_{j,k} M_\tau^{j,k}$ is the total number of factors in the wedge product.

Recall the example with two particles of charge 3 and three particles of charge 1. Under the identity permutation, we pair a charge 3 with a charge 3, pair a charge 1 with a charge 1, and leave a charge 1 unpaired. This produces $\eta_{3,3} \wedge \eta_{1,1} \wedge \gamma_1$. Under the permutation τ which swapped the second charge 3 with the first charge 1, we got $\eta_{3,1} \wedge \eta_{3,1} \wedge \gamma_1$.

Consider instead the permutation which puts all of the charge 1 particles before the charge 3 particles. We pair a charge 1 with a charge 1, pair the last charge 1 with a charge 3, and leave a charge 3 unpaired. This produces $\eta_{1,1} \wedge \eta_{1,3} \wedge \gamma_3$.

Arbitrary Charge Vector

Finally, we allow any mix of odd and even charges. Recall (from the beginning of section 5.4),

$$Z_{\vec{M}} = \frac{1}{M_1! M_2! \cdots M_J!} \sum_{\sigma \in S_{K_e}} \sum_{\tau \in S_{K_o}} \int_{\Delta_{K_e}(\sigma)} \int_{\Delta_{K_o}(\tau)} \det H^{\vec{L},\vec{M}}_{\sigma,\tau}(\mathbf{x})\, dy_1 \cdots dw_{K_o}.$$

Let $N_e = \sum_{j=1}^{r} L_j M_j$ be the total charge of the even charges, and let $N_o = \sum_{j=r+1}^{J} L_j M_j$ be the total charge of the odd charges. By the Laplace expansion of the determinant,

$$\det H^{\vec{L},\vec{M}}_{\sigma,\tau}(\vec{y},\vec{w})\, \varepsilon_{\text{vol}} = \sum_{\mathfrak{t}:\underline{N_e} \nearrow \underline{N}} \sum_{\mathfrak{s}:\underline{N_o} \nearrow \underline{N}} \det H^{\vec{L},\vec{M}}_{\sigma,\mathfrak{t}}(\vec{y}) \det H^{\vec{L},\vec{M}}_{\tau,\mathfrak{s}}(\vec{w})\, \varepsilon_{\mathfrak{t}} \wedge \varepsilon_{\mathfrak{s}},$$

where $\det H^{\vec{L},\vec{M}}_{\sigma,\mathfrak{t}}(\vec{y})$ is an $N_e \times N_e$ minor taken only from columns in the (even charge) variables $\vec{y}$, and $\det H^{\vec{L},\vec{M}}_{\sigma,\mathfrak{s}}(\vec{w})$ is an $N_o \times N_o$ minor taken only from columns

in the (odd charge) variables $\vec{w}$. Note, $\varepsilon_{\mathfrak{t}} \wedge \varepsilon_{\mathfrak{s}} = 0$ whenever these minors are not complimentary.

With the variables separated in this way, we can apply Lemma 5.1 to the determinant in the even charges, and we can apply either Lemma 5.2 or Lemma 5.3 to the determinant in the odd charges. For the even charges, we have

$$\frac{1}{M_1! \cdots M_r!} \sum_{\sigma \in S_{K_e}} \int_{\Delta_{K_e}(\sigma)} \det H^{\vec{L},\vec{M}}_{\sigma,\mathfrak{t}}(\vec{y})\, dy_1 \cdots dy_{K_e}\, \varepsilon_{\mathfrak{t}} = \frac{\gamma_1^{\mathfrak{t}\wedge M_1}}{M_1!} \wedge \cdots \wedge \frac{\gamma_r^{\mathfrak{t}\wedge M_r}}{M_r!},$$

where $\gamma_j^{\mathfrak{t}}$ is subtly different from γ_j because $H^{\vec{L},\vec{M}}_{\sigma,\mathfrak{t}}(\vec{y})$ is already an $N_e \times N_e$ minor chosen by $\mathfrak{t}$. Explicitly,

$$\gamma_j^{\mathfrak{t}} = \sum_{\mathfrak{t}_j : \underline{L_j} \nearrow \underline{N_e}} \int_{\mathbb{R}} \mathrm{Wr}(\vec{p}_{\mathfrak{t}\circ\mathfrak{t}_j}, x)\, d\mu_j(x)\, \varepsilon_{\mathfrak{t}_j}.$$

Taking the sum over all $\mathfrak{t} : \underline{N_e} \nearrow \underline{N}$ gives us back the original γ_j forms

$$\sum_{\mathfrak{t}:\underline{N_e}\nearrow\underline{N}} \frac{\gamma_1^{\mathfrak{t}\wedge M_1}}{M_1!} \wedge \cdots \wedge \frac{\gamma_r^{\mathfrak{t}\wedge M_r}}{M_r!} = \frac{\gamma_1^{\wedge M_1}}{M_1!} \wedge \cdots \wedge \frac{\gamma_r^{\wedge M_r}}{M_r!}.$$

It is straightforward to check an analogous result holds for the determinant in the odd charges. The following lemma then supersedes Lemmas 5.1, 5.2, and 5.3:

Lemma 5.4. *Suppose L_j is even for $1 \le j \le r$, then when N is even,*

$$Z_{\vec{M}} = \int \frac{\gamma_1^{\wedge M_1}}{M_1!} \wedge \cdots \wedge \frac{\gamma_r^{\wedge M_r}}{M_r!} \wedge \sum_{\tau \in \mathrm{Sh}(M_{r+1},\ldots,M_J)} \frac{1}{K!} \bigwedge_{j=1}^{J} \bigwedge_{k=1}^{J} \eta_{j,k}^{\wedge M_\tau^{j,k}}\, d\varepsilon_{\mathrm{vol}},$$

and when N is odd,

$$Z_{\vec{M}} = \int \frac{\gamma_1^{\wedge M_1}}{M_1!} \wedge \cdots \wedge \frac{\gamma_r^{\wedge M_r}}{M_r!} \wedge \sum_{\tau \in \mathrm{Sh}(M_{r+1},\ldots,M_J)} \frac{1}{K!} \bigwedge_{j=1}^{J} \bigwedge_{k=1}^{J} \eta_{j,k}^{\wedge M_\tau^{j,k}} \wedge \gamma_{\tau^{-1}(K_o)} \, d\varepsilon_{\mathrm{vol}}.$$

5.5. Isocharge Grand Canonical Ensemble

Recall from section 5.1, we want to compute

$$Z_N = \sum_{\vec{L}\cdot\vec{M}=N} z_1^{M_1} z_2^{M_2} \cdots z_J^{M_J} Z_{\vec{M}}.$$

When All L_j Are Even

Starting from Lemma 5.1, we have

$$Z_N = \sum_{\vec{L}\cdot\vec{M}=N} z_1^{M_1} \cdots z_J^{M_J} \int \frac{\gamma_1^{\wedge M_1}}{M_1!} \wedge \cdots \wedge \frac{\gamma_J^{\wedge M_J}}{M_J!} \, d\varepsilon_{\mathrm{vol}}.$$

Recall from section 2.4, the Berezin integral is a projection onto the highest exterior power $\bigwedge^N(\mathbb{R}^N)$. If each γ_j is an L_j-form, then the wedge product above is an $\vec{L}\cdot\vec{M}$-form. If we extend the sum over all $\vec{M}$, the Berezin integral will eliminate any summands for which $\vec{L}\cdot\vec{M} \neq N$. Thus,

$$\begin{aligned}
Z_N &= \int \sum_{M_1=0}^{\infty} \cdots \sum_{M_J=0}^{\infty} \frac{(z_1\gamma_1)^{\wedge M_1}}{M_1!} \wedge \cdots \wedge \frac{(z_J\gamma_J)^{\wedge M_J}}{M_J!} \, d\varepsilon_{\mathrm{vol}} \\
&= \int \bigwedge_{j=1}^{J} \sum_{M=1}^{\infty} \frac{(z_j\gamma_j)^{\wedge M}}{M!} \, d\varepsilon_{\mathrm{vol}} \\
&= \int \exp(z_1\gamma_1) \wedge \cdots \wedge \exp(z_J\gamma_J) \, d\varepsilon_{\mathrm{vol}} \\
&= \mathrm{BE}_{\mathrm{vol}}(z_1\gamma_1 + \cdots + z_J\gamma_J).
\end{aligned}$$

In the last line, we replace the product of these exponentials with the exponential of the sum, which we can do because our forms are even and therefore commute. This completes the proof of Theorem 5.1.

When All L_j Are Odd

Let us start by assuming there are no even species, and the total charge N is even. Recall Lemma 5.2 which gives us

$$Z_N = \sum_{\vec{L}\cdot\vec{M}=N} z_1^{M_1}\cdots z_J^{M_J} \sum_{\tau\in \mathrm{Sh}(M_1,\ldots,M_J)} \int \frac{1}{K!} \bigwedge_{j=1}^{J}\bigwedge_{k=1}^{J} \eta_{j,k}^{\wedge M_\tau^{j,k}}\, d\varepsilon_{\mathrm{vol}}.$$

For $\vec{M}$ fixed and $\tau \in \mathrm{Sh}(M_1,\ldots,M_J)$, the number of other permutations which produce the same pairs (j,k) is

$$K!\prod_{j=1}^{J}\prod_{k=1}^{J}\frac{1}{M_\tau^{j,k}!}.$$

This is just a multinomial coefficient, recalling K is the sum of the $M_\tau^{j,k}$. Of these permutations, there exists a unique representative which orders the pairs (j,k) lexicographically. Let $\mathcal{L}_{\vec{M}}$ be the set of these representatives, then

$$Z_N = \int \sum_{M_1=0}^{\infty}\cdots\sum_{M_J=0}^{\infty}\sum_{\tau\in\mathcal{L}_{\vec{M}}}\bigwedge_{j=1}^{J}\bigwedge_{k=1}^{J}\frac{(z_j z_k\eta_{j,k})^{\wedge M_\tau^{j,k}}}{M_\tau^{j,k}!}\, d\varepsilon_{\mathrm{vol}}.$$

Next, we condition on population vectors $\vec{M}$ which produce the same K, the number of (j,k) pairs, and so

$$Z_N = \int \sum_{K=0}^{\infty}\sum_{M_1+\cdots+M_J=2K}\sum_{\tau\in\mathcal{L}_{\vec{M}}}\bigwedge_{j=1}^{J}\bigwedge_{k=1}^{J}\frac{(z_j z_k\eta_{j,k})^{\wedge M_\tau^{j,k}}}{M_\tau^{j,k}!}\, d\varepsilon_{\mathrm{vol}}.$$

Collecting these together, we get the K^{th} power of the sum over all possible pairs (j,k). Thus,

$$\begin{aligned} Z_N &= \int \sum_{K=0}^{\infty} \frac{1}{K!} \left(\sum_{j=1}^{J} \sum_{k=1}^{J} z_j z_k \eta_{j,k} \right)^{\wedge K} d\varepsilon_{\text{vol}} \\ &= \text{BE}_{\text{vol}} \left(\sum_{j=1}^{J} \sum_{k=1}^{J} z_j z_k \eta_{j,k} \right). \end{aligned}$$

In the case with total charge N odd, we can go through the same steps starting from Lemma 5.3. This produces

$$Z_N = \text{BE}_{\text{vol}_1} \left(\sum_{j=1}^{J} \sum_{k=1}^{J} z_j z_k \eta_{j,k} + \sum_{j=1}^{J} z_j \gamma_j \wedge \varepsilon_{N+1} \right).$$

Note, when N is odd, every $Z_{\vec{M}}$ has exactly one γ_j for each τ (see Lemma 5.3). The ε_{N+1} shown attached to each of the γ_j above ensures this is the case when we exponentiate and take the Berezin integral. First, $\varepsilon_{N+1} \wedge \varepsilon_{N+1} = 0$, so $\gamma_j \wedge \varepsilon_{N+1} \wedge \gamma_k \wedge \varepsilon_{N+1} = 0$. Thus, terms in the expansion of the exponential with more than one γ_j are annihilated. Because the Berezin integral with respect to $d\varepsilon_{\text{vol}_1}$ projects onto the highest exterior power $\bigwedge^{N+1}(\mathbb{R}^{N+1})$, terms in the expansion of the exponential with no γ_j are missing the basis vector ε_{N+1} and are annihilated by the Berezin integral (with respect to $d\varepsilon_{\text{vol}_1}$). Thus, we only get summands (in the expansion of the exponential) with exactly one γ_j, as in Lemma 5.3.

Recall Lemma 5.4, in which the $\gamma_1, \ldots, \gamma_r$ corresponding to the even charges are already factored out. Summing over all possible $M_1, \ldots, M_r$, we can factor out an $\exp(z_1\gamma_1 + \cdots z_r\gamma_r)$ as in the "all evens" case. From what remains, we obtain the exponential of the sum of the $\eta_{j,k}$, possibly with an extra set of $\gamma_j \wedge \varepsilon_{N+1}$ forms. For

N even,

$$\begin{aligned} Z_N &= \int \left[\bigwedge_{j=1}^{r} \sum_{M=1}^{\infty} \frac{(z_j \gamma_j)^{\wedge M}}{M!} \right] \wedge \sum_{K=0}^{\infty} \frac{1}{K!} \left(\sum_{j=1}^{J} \sum_{k=1}^{J} z_j z_k \eta_{j,k} \right)^{\wedge K} d\varepsilon_{\text{vol}} \\ &= \int \exp\left(\sum_{j=1}^{r} z_j \gamma_j \right) \wedge \exp\left(\sum_{j=r+1}^{J} \sum_{k=r+1}^{J} z_j z_k \eta_{j,k} \right) \varepsilon_{\text{vol}} \\ &= \text{BE}_{\text{vol}} \left(\sum_{j=1}^{r} z_j \gamma_j + \sum_{j=r+1}^{J} \sum_{k=r+1}^{J} z_j z_k \eta_{j,k} \right). \end{aligned}$$

This concludes the proof of Theorem 5.2 and, with only a slight modification, Theorem 5.3.

5.6. Multicomponent Constellation Ensembles

Recall from section 4.1 how a constellation ensemble is created from a one-dimensional log-gas. Starting with a *multicomponent* configuration on the line $\mathbb{R} + iy_1$, copy the configuration onto the other lines $\mathbb{R} + iy_k$ for K many total copies of the same one-dimensional configuration. As an example, see Figure 3 at the beginning of chapter IV. With this setup, we can take the lemmas and theorems of this chapter (particularly Lemma 5.4, Theorem 5.2 and Theorem 5.3) entirely as written with only slight modification to how γ_j and $\eta_{j,k}$ are defined (to account for the added translation vector $\vec{y}$).

In section 4.1, we demonstrated the Boltzmann factor of the monocharge constellation is the same as the Boltzmann factor of the single-species β-ensemble with $\beta = L^2$ but with the KM many translated variables $\mathbf{x}$ substituted in. In both cases, the Boltzmann factors are determinantal. The Wronskian minors of the former resemble the minors of the latter except made proto-confluent (see section 2.8) by the addition of the translation vector $\vec{y}$. Likewise, the forms which

give the partition functions for multicomponent constellation ensembles are simply the proto-confluent versions of the forms which give the partition functions for one-dimensional multicomponent log-gases.

Let $\mathbf{x} = (\mathbf{x}^1, \ldots, \mathbf{x}^J) \in \mathbb{R}^{M_1} \times \cdots \times \mathbb{R}^{M_J}$ so that $\mathbf{x}^j = (x_1^j, \ldots, x_{M_j}^j) \in \mathbb{R}^{M_j}$ gives the real parts of all particles of charge L_j. Define $\mathbf{x}_{\vec{y}} = (\mathbf{x}_{\vec{y}}^1, \ldots, \mathbf{x}_{\vec{y}}^J) \in \mathbb{C}^{KM_1} \times \cdots \times \mathbb{C}^{KM_J}$ so that

$$\mathbf{x}_{\vec{y}}^j = (x_1^j + i\vec{y}, \ldots, x_{M_j}^j + i\vec{y}) \in \mathbb{C}^{KM_j},$$

with $x_m^j + i\vec{y} = (x_m^j + iy_1, \ldots, x_m^j + iy_k) \in \mathbb{C}^K$. As a list, $\mathbf{x}_{\vec{y}}$ is generated from $\mathbf{x}$ by replacing each real location x_m^j with $x_m^j + i\vec{y}$, the list of its K many translations. Let $N = K(\vec{L} \cdot \vec{M})$ be the total charge of this expanded system.

Without writing out the full Boltzmann factor for the interactions between these particles, it is straightforward to verify all instances of i vanish (as in section 4.1 and the analogous start of section 4.3) except for the interactions between two particles which share a real part. To obtain the absolute value of these factors, we factored out powers of $-i$ and included them in the (complex) measure μ (which otherwise comes just from the potential U). Dealing with one real part at a time, we can use what we know from the monocharge case to get the correct combinatorial exponent on $-i$.

Explicitly, for any real part x_m^j (corresponding to a constellation of K many charge L_j particles), the energy contribution from the potential is $L_j U(x_m^j)$ times the number of translations K. We get one factor of $i^{L_j^2}$ for each pair of particles in the constellation $x_m^j + i\vec{y}$ of which there are $\binom{K}{2}$ many. Thus, we set

$$d\mu_j(x) = \left((-i)^{L_j(K-1)/2} e^{-U(x)}\right)^{L_j K} dx.$$

With any homogeneous constellation ensemble, we could assume the real parts were ordered $x_1 < \cdots < x_M$ because all particles on the same line were indistinguishable (same charge). This was necessary to drop the absolute value from the Boltzmann factor. In particular, whenever the real parts are labeled with the same order as the domain of integration, all differences in the confluent Vandermonde determinant are positive. In the case of differently-charged particles, the order in which they occur is relevant, and some additional tools are needed, as already demonstrated in section 5.3. Nothing unique to constellation ensembles occurs here.

Comparing to the analogous forms of section 5.2, define the $\vec{y}$-modified $\gamma_j(\vec{y})$ and $\gamma_{j,k}(\vec{y})$ by

$$\gamma_j(\vec{y}) = \sum_{\mathfrak{t}:\underline{L_jK}\nearrow\underline{N}} \int_{\mathbb{R}} \mathrm{Wr} \otimes \mathrm{Pr}_{\vec{y}}(\vec{p}_{\mathfrak{t}}, x)\, d\mu(x)\, \varepsilon_{\mathfrak{t}}$$

and

$$\eta_{j,k}(\vec{y}) = \sum_{\mathfrak{t}:\underline{L_jK}\nearrow\underline{N}} \sum_{\mathfrak{s}:\underline{L_kK}\nearrow\underline{N}} \int\int_{x_1<x_2} \Big[\mathrm{Wr} \otimes \mathrm{Pr}_{\vec{y}}(\vec{p}_{\mathfrak{t}}, x_1) \times \mathrm{Wr} \otimes \mathrm{Pr}_{\vec{y}}(\vec{p}_{\mathfrak{s}}, x_2)\Big]\, d\mu(x_1)\, d\mu(x_2)\, \varepsilon_{\mathfrak{t}} \wedge \varepsilon_{\mathfrak{s}}.$$

When K is odd, we are done (because the parity of L_jK is determined by the L_j). However, when K is even, all of our minors have even dimensions L_jK, and the total charge $N = K(\vec{L}\cdot\vec{M})$ is even as well. Thus, for K even (and no additional restrictions on L_j), we get the "all even" versions of our Berezin integral expressions:

Theorem 5.4. *When K is even,*

$$Z_{\vec{M}}(\vec{y}) = \int \frac{\gamma_1(\vec{y})^{\wedge M_1}}{M_1!} \wedge \cdots \wedge \frac{\gamma_J(\vec{y})^{\wedge M_J}}{M_J!}\, d\varepsilon_{\mathrm{vol}},$$

and

$$Z_N(\vec{y}) = \mathrm{BE}_{\mathrm{vol}}\left(\sum_{j=1}^{J} z_j \gamma_j(\vec{y})\right).$$

Note, the fugacity parameters z_j are no longer the probabilities of individual charge L_j particles appearing. Instead, z_j is the probability of a constellation of K many points all having the same charge L_j.

CHAPTER VI

CIRCULAR ENSEMBLES

This chapter contains unpublished coauthored material. In particular, sections 6.4-6.6 appear largely as is in [35].

We will begin with homogeneous circular constellation ensembles of which monocharge circular constellation ensembles are a special case. As an example, see Figure 4 at the beginning of chapter IV. Consider K concentric circles in the complex plane with radii $\vec{y} \in (\mathbb{R}_{>0})^K$. Define $\vec{L}$ and $\mathbf{L} = (\vec{L}, \ldots, \vec{L}) \in (\mathbb{Z}_{>0})^{KM}$ as in section 4.3. Replace $\mathbb{R}$ in the definition of $\vec{x}$ by $[0, 2\pi)$. For each angle $x_m \in [0, 2\pi)$, and $1 \leq k \leq K$, place a charge L_k particle at location $y_k e^{ix_m}$. Denote the (total KM) particle locations by

$$\mathbf{z} = (\mathbf{z}^1, \mathbf{z}^2, \ldots, \mathbf{z}^M) \in \mathbb{C}^{KM}$$

where $\mathbf{z}^m = \vec{y}e^{ix_m} = (y_1 e^{ix_m}, y_2 e^{ix_m}, \ldots, y_K e^{ix_m}) \in \mathbb{C}^K$. Assuming logarithmic interaction between the particles, the total potential energy of this system is given by

$$\begin{aligned} E(\vec{x}, \vec{y}) = &- \sum_{k=1}^{K} \sum_{n<m}^{M} L_k^2 \log \left| y_k e^{ix_m} - y_k e^{ix_n} \right| - \sum_{j<k}^{K} \sum_{m=1}^{M} L_j L_k \log \left| y_k e^{ix_m} - y_j e^{ix_m} \right| \\ &- \sum_{n<m}^{M} \sum_{j<k}^{K} L_j L_k \log \left| y_k e^{ix_m} - y_j e^{ix_n} \right| + L_j L_k \log \left| y_j e^{ix_m} - y_k e^{ix_n} \right|. \end{aligned}$$

As observed in [24], we can express the Boltzmann factor $\Omega_M(\vec{x},\vec{y}) = e^{-E(\vec{x},\vec{y})}$ without absolute values using the following identities:

$$\left|y_k e^{ix_m} - y_k e^{ix_n}\right| = -ie^{-i(x_m+x_n)/2}\left(y_k e^{ix_m} - y_k e^{ix_n}\right)\operatorname{sgn}(x_m - x_n).$$

$$\left|y_k e^{ix_m} - y_j e^{ix_m}\right| = e^{-ix_m}\left(y_k e^{ix_m} - y_j e^{ix_m}\right).$$

$$\left|y_k e^{ix_m} - y_j e^{ix_n}\right|\left|y_j e^{ix_m} - y_k e^{ix_n}\right| = -e^{-i(x_m+x_n)}\left(y_k e^{ix_m} - y_j e^{ix_n}\right)\left(y_j e^{ix_m} - y_k e^{ix_n}\right).$$

As in section 4.1 and section 4.3, we can assume without loss of generality $0 \le x_1 < \cdots < x_M < 2\pi$. Then $\operatorname{sgn}(x_m - x_n) > 0$ for all $n < m$. Thus, the relative density of states (corresponding to varying location vectors $\vec{x}$ and translation vectors $\vec{y}$) is given by the Boltzmann factor

$$\Omega(\vec{x},\vec{y}) = e^{-E(\vec{x},\vec{y})} = \left|\Delta^{\mathbf{L}}(\mathbf{z})\right| = \Delta^{\mathbf{L}}(\mathbf{z})\prod_{m=1}^{M}(-ie^{-ix_m})^{R_3(M-1)/2}(e^{-ix_m})^{R_2} = \det H^{\mathbf{L}}(\mathbf{z}),$$

where $d\mu(x) = (-ie^{-ix})^{R_3(M-1)/2}(e^{-ix})^{R_2}\,dx$, $R_2 = \sum_{j<k}^{K} L_j L_k$, and $R_3 = \sum_{j,k=1}^{K} L_j L_k$. Also, this $\Delta^{\mathbf{L}}(\mathbf{z})$ is once again the confluent Vandermonde determinant of section 2.7.

6.1. Circular Partition Functions

Recall from section 4.4, $R_1 = \sum_{k=1}^{K} L_k$. For circular constellation ensembles, instead of L_k columns for each of the $x_m + iy_k$ (in the linear case), $H^{\mathbf{L}}(\mathbf{z})$ has L_k columns for each of the $y_k e^{ix_m}$. Define

$$\operatorname{Cr}_{\vec{y}}(\vec{f},x) = \det[f_n(y_k e^{ix})]_{n,k=1}^{K},$$

and define

$$\mathrm{Wr}^{\vec{L}} \otimes \mathrm{Cr}_{\vec{y}}(\vec{f}, x) = \det \left[\left[D^{l-1} f_n(y_k e^{ix}) \right]_{l=1}^{L_k} \right]_{n,k=1}^{R_1,K}.$$

This is analogous to the definition of $\mathrm{Wr}^{\vec{L}} \otimes \mathrm{Pr}_{\vec{y}}(\vec{f}, x)$ with linear translations $x + iy_k$ replaced with circular translations $y_k e^{ix}$. These are the $R_1 \times R_1$ minors of $H^{\mathbf{L}}(\mathbf{z})$ which correspond to a single position x_m.

Proceeding as in section 4.4, define

$$\gamma_{\vec{L}}(\vec{y}) = \sum_{\mathfrak{t}:\underline{R_1} \nearrow \underline{N}} \left[\int_0^{2\pi} \mathrm{Wr}^{\vec{L}} \otimes \mathrm{Cr}_{\vec{y}}(\vec{p}_{\mathfrak{t}}, x)\, d\mu(x) \right] \varepsilon_{\mathfrak{t}},$$

and define

$$\eta_{\vec{L}}(\vec{y}) = \sum_{\mathfrak{t}:\underline{R_1} \nearrow \underline{N}} \sum_{\mathfrak{s}:\underline{R_1} \nearrow \underline{N}} \int \int_{0<x_1<x_2<2\pi} \Big[\mathrm{Wr}^{\vec{L}} \otimes \mathrm{Cr}_{\vec{y}}(\vec{p}_{\mathfrak{t}}, x_1) \times \mathrm{Wr}^{\vec{L}} \otimes \mathrm{Cr}_{\vec{y}}(\vec{p}_{\mathfrak{s}}, x_2) \Big] d\mu(x_1)\, d\mu(x_2)\, \varepsilon_{\mathfrak{t}} \wedge \varepsilon_{\mathfrak{s}}.$$

By Theorem 3.1, the expressions for partition functions of homogeneous circular ensembles are the same as in Theorem 4.2 using these new (circular) $\gamma_{\vec{L}}(\vec{y})$ and $\eta_{\vec{L}}(\vec{y})$. For monocharge constellation ensembles, we can specialize to the expressions given in Theorem 4.1.

6.2. Computational Techniques

Recall from section 2.7, $\det V_{\vec{p}}^{\vec{L}}(\vec{x}) = \Delta^{\vec{L}}(\vec{x})$ for any choice of complete N-family of monic polynomials $\vec{p}$. When actually computing integrals of Wronskians, some choices are better than others. In some cases (such as the circular case), there exist polynomials for which the integrals of Wronskians are often zero. This depends on the measures μ which come from the potential U.

For example, consider $\vec{g} = \{x^{n-1}\}_{n=1}^{N}$. It is straightforward to verify that the Wronskian of a collection of monomials will again be a monomial. In particular, for any $\mathfrak{t} : \underline{K} \nearrow \underline{N}$, we have

$$\mathrm{Wr}(\vec{g}_{\mathfrak{t}}, x) = x^{\sum_k \mathfrak{t}(k)-k} \frac{\Delta(\mathfrak{t}(\underline{K}))}{\Delta(\underline{K})}.$$

Thus, for $R \in \mathbb{Z}$,

$$\int_0^{2\pi} \mathrm{Wr}(\vec{g}_{\mathfrak{t}}, re^{ix})(e^{-ix})^R \, dx = \int_0^{2\pi} r^{\sum_k \mathfrak{t}(k)-k} \frac{\Delta(\mathfrak{t}(\underline{K}))}{\Delta(\underline{K})} (e^{ix})^{-R+\sum_k \mathfrak{t}(k)-k} \, dx = 0,$$

unless $-R + \sum_k \mathfrak{t}(k) - k = 0$. This gives a sum condition which all $\mathfrak{t}$ of the same size must satisfy. Likewise,

$$\mathrm{Cr}_{\vec{y}}(\vec{g}_{\mathfrak{t}}, x) = \det \left[y_k^{\mathfrak{t}(j)-1} \right]_{j,k=1}^{K} (e^{ix})^{-K+\sum_k \mathfrak{t}(k)}.$$

Thus,

$$\int_0^{2\pi} \mathrm{Cr}_{\vec{y}}(\vec{g}_{\mathfrak{t}}, x)(e^{-ix})^R \, dx = \det \left[y_k^{\mathfrak{t}(j)-1} \right]_{j,k=1}^{K} \int_0^{2\pi} (e^{ix})^{-R-K+\sum_k \mathfrak{t}(k)} \, dx = 0$$

unless $-R-K+\sum_k \mathfrak{t}(k) = 0$. This condition is actually quite strong and makes our $\gamma(\vec{y})$ forms quite sparse. For example, when $K = 2$, knowing $\mathfrak{t}(1) \in \underline{N}$ determines $\mathfrak{t}(2) = \mathfrak{t}(1) + R + 2$, no matter how big N is.

Historically, being able to "diagonalize" the form γ by a clever choice of (potentially orthogonal or skew orthogonal) polynomials is incredibly useful in obtaining Pfaffian correlation functions from the Pfaffian partition functions. We expect this to be the case with Hyperpfaffian partition functions and correlation functions as well, though this is admittedly still speculation.

6.3. Limits of Circular Constellations

As in section 4.5, we first consider limits (of homogeneous constellation ensembles) as the distances between the circles shrinks to zero. The interaction terms which would give us singularities are the ones with $L_j L_k \log|y_k e^{ix_m} - y_j e^{ix_m}| = L_j L_k \log(y_k - y_j)$, coming from particles which share an angle x_m. Thus, the correct denominator which accounts for these singularities is $\Delta^{\vec{L}}(\vec{y})$ so that

$$\lim_{\vec{y}\to\vec{1}} \frac{\Delta^{\mathbf{L}}(\mathbf{z})}{\left(\Delta^{\vec{L}}(\vec{y})\right)^M} = \Delta(e^{i\vec{x}})^{R_1^2},$$

in which we take the limit as $y_1 = \cdots = y_K = 1$ to represent all the circles collapsing onto the unit circle. As before, the limiting Boltzmann factor corresponds to a one-dimensional ensemble of particles with charge $R_1 = \sum_{k=1}^{K} L_k$. In terms of confluent matrices,

$$\lim_{\vec{y}\to\vec{1}} \frac{V^{\mathbf{L}}(\mathbf{z})}{\left(\Delta^{\vec{L}}(\vec{y})\right)^M} = V^{R_1}(e^{i\vec{x}}),$$

we get the same result as the linear case, with the location vector of real points replaced by a location vector of points on the unit circle. Additionally, in terms of the partition function, we get the same result as the linear case with $\mathrm{Cr}(\vec{p}_t, x)$ in place of $\mathrm{Wr}(\vec{p}_t, x)$. Explicitly,

$$\lim_{\vec{y}\to\vec{1}} \frac{Z_M(\mathbf{z})}{\left(\Delta^{\vec{L}}(\vec{y})\right)^M} = \frac{1}{M!} \int \left[\sum_{\mathfrak{t}:\underline{R_1} \nearrow \underline{N}} \int_{\mathbb{R}} \mathrm{Cr}(\vec{p}_{\mathfrak{t}}, x)\, d\mu(x)\, \varepsilon_{\mathfrak{t}} \right]^{\wedge M} \varepsilon_{\mathrm{vol}}$$

when R_1 is even (and the analogous double-Wronskian expression holds when R_1 is odd).

Proceeding as we did in the linear case, we next consider limits as the distances between our circles increase without bound. For simplicity, we start by setting $y_k = 1 + hk$ (so that $\vec{y} \to \vec{1}$ as $h \to 0$) and then consider limits as $h \to \infty$. Recall (from the beginning of this chapter), there are three types of interaction terms in the Boltzmann factor. First, particles which share an angle x_m produce an interaction factor of $(h(k-j))^{L_j L_k}$. These interactions are already accounted for by the $\left(\Delta^{\vec{L}}(\vec{y})\right)^M$ denominator.

Next, particles on the same circle of radius $y_k = 1 + hk$ produce an interaction factor of $\left((1+hk)\left(e^{ix_m} - e^{ix_n}\right)\right)^{L_k^2}$, which grows on the order of $(1+hk)^{L_k^2}$. There are $\binom{M}{2}$ many of these for each $1 \leq k \leq K$. Note, this is unique to the circular case, in which the particles drift apart as the radius of the circle grows without bound. Finally, particles at different angles on different circles produce an interaction factor of $\left((1+hk)e^{ix_m} - (1+hj)e^{ix_n}\right)^{L_j L_k}$, which grows on the order of $(h(k-j))^{L_j L_k}$. There are $\binom{M}{2}\binom{K}{2}$ many of these. Thus, if we set

$$P_M^{\vec{L}}(h) = \left[\prod_{j \neq k}(1 + h(k-j))^{L_j L_k} \prod_{k=1}^{K}(1+hk)^{L_k^2}\right]^{\binom{M}{2}},$$

then $\lim_{h \to 0} P_M^{\vec{L}}(h) = 1$, and it is straightforward to check

$$\lim_{h \to \infty} \frac{\Delta^{\mathbf{L}}(\mathbf{z})}{\left(\Delta^{\vec{L}}(h\underline{K})\right)^M P_M^{\vec{L}}(h)} = \Delta(e^{i\vec{x}})^{L_1^2 + \cdots + L_K^2}.$$

However, the limitations of the linear case also apply in the circular case. In particular, the limiting partition function (as $h \to \infty$) is still a limit of Hyperpfaffians rather than an honest Hyperpfaffian in its own right.

6.4. Multicomponent Circular Ensembles

Returning to the multicomponent setup in section 5.1, consider instead charged particles on the unit circle. Substitute all instances of $\mathbb{R}$ with $[0, 2\pi)$ so that

$$\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^J) \in [0, 2\pi)^{M_1} \times [0, 2\pi)^{M_2} \times \cdots \times [0, 2\pi)^{M_J}$$

gives the locations of the particles around the unit circle with each $x_n^j \in [0, 2\pi)$ corresponding to an angle. Assuming logarithmic interaction between the particles, the energy contributed by interaction between two particles of charge L_j and L_k at angles x_n^j and x_m^k respectively is given by $-L_j L_k \log \left| e^{ix_n^k} - e^{ix_m^j} \right|$. Thus, at inverse temperature β, the total potential energy of the system is given by

$$E_{\vec{M}}(\mathbf{x}) = -\beta \sum_{j=1}^{J} L_j^2 \sum_{m<n} \log \left| e^{ix_n^j} - e^{ix_m^j} \right| - \beta \sum_{j<k} L_j L_k \sum_{m=1}^{M_j} \sum_{n=1}^{M_k} \log \left| e^{ix_n^k} - e^{ix_m^j} \right|,$$

with Boltzmann factor

$$\Omega_{\vec{M}}(\mathbf{x}) = \exp(-E_{\vec{M}}(\mathbf{x})) = \prod_{j=1}^{J} \prod_{m<n} \left| e^{ix_n^j} - e^{ix_m^j} \right|^{\beta L_j^2} \times \prod_{j<k} \prod_{m=1}^{M_j} \prod_{n=1}^{M_k} \left| e^{ix_n^k} - e^{ix_m^j} \right|^{\beta L_j L_k}.$$

We will write $\mathbf{y} = \exp(i\mathbf{x})$ to mean the vector with entries of the form $e^{ix_m^j}$. Then the probability of finding the system in a state corresponding to a location vector $\mathbf{x}$ is given by the joint probability density function

$$\rho_{\vec{M}}(\mathbf{x}) = \frac{\Omega_{\vec{M}}(\mathbf{x})}{Z_{\vec{M}} M_1! M_2! \cdots M_J!} = \frac{\left| \det V^{\vec{L},\vec{M}}(\mathbf{y}) \right|}{Z_{\vec{M}} M_1! M_2! \cdots M_J!},$$

with partition function

$$Z_{\vec{M}} = \frac{1}{M_1!M_2!\cdots M_J!}\int_{[0,2\pi)^{M_1}}\cdots\int_{[0,2\pi)^{M_J}}\left|\det V^{\vec{L},\vec{M}}(\mathbf{y})\right|\, d\nu^{M_1}(\mathbf{x}^1)\cdots d\nu^{M_J}(\mathbf{x}^J),$$

where ν^{M_j} is Lebesque measure on $[0,2\pi)^{M_j}$. Using the same modifications as before in the linear case, we can assume $\beta = 1$ for computational purposes. Note, the same confluent Vandermonde determinant gives us the same product of differences (with exponents) as before, even with the new complex variables $\mathbf{y}$ in place of the real variables $\mathbf{x}$.

6.5. Complex Modulus

We will be able to apply our same Theorem 3.1 to our determinantal integrand once we resolve the absolute value. As observed in [24], each absolute difference can be decomposed as

$$\begin{aligned}\left|e^{ix_n^k} - e^{ix_m^j}\right| &= -ie^{-i\left(x_n^k+x_m^j\right)/2}\left(e^{ix_n^k} - e^{ix_m^j}\right)\operatorname{sgn}(x_n^k - x_m^j)\\ &= -ie^{-i\left(x_n^k+x_m^j\right)/2}\left(e^{ix_n^k} - e^{ix_m^j}\right)\frac{\left(x_n^k - x_m^j\right)}{\left|x_n^k - x_m^j\right|}.\end{aligned}$$

Next, we define

$$d\mu_j(x) = \left(-ie^{-ix}\right)^{L_jT/2}\, dx,$$

where

$$T = -L_j + \sum_{k=1}^{J} L_kM_k,$$

so that we can bring the (complex valued) weight functions $(-ie^{-ix_m^j})^{T/2}$ inside the matrix $V^{\vec{L},\vec{M}}(\mathbf{y})$ the same way we did in section 5.3. Explicitly, construct $H^{\vec{L},\vec{H}}(\mathbf{y})$

by multiplying each column with the variable x_m^j by $(-ie^{-ix_m^j})^{T/2}$. Note, there will be L_j many columns for each x_m^j. Finally, we can write the $(x_n^k - x_m^j)^{L_j L_k}/|x_n^k - x_m^j|^{L_j L_k}$ factors as separate confluent Vandermonde determinants in $\mathbf{x}$.

We can use the same procedure of separating the odd species from the even species and decomposing the integral over ordered subsets as in section 5.3. Thus,

$$\begin{aligned} Z_{\vec{M}} &= \frac{1}{M_1!M_2!\cdots M_J!}\int_{[0,2\pi)^{K_e}}\int_{[0,2\pi)^{K_o}}\left|\det V^{\vec{L},\vec{M}}(\mathbf{y})\right| dy_1\cdots dy_{K_e}\, dw_1\cdots dw_{K_o} \\ &= \frac{1}{M_1!M_2!\cdots M_J!}\sum_{\sigma\in S_{K_e}}\sum_{\tau\in S_{K_o}}\int_{\Delta_{K_e}(\sigma)}\int_{\Delta_{K_o}(\tau)}\det H^{\vec{L},\vec{M}}(\mathbf{y})\frac{\det V^{\vec{L},\vec{M}}(\mathbf{x})}{\left|\det V^{\vec{L},\vec{M}}(\mathbf{x})\right|}dy_1\cdots dw_{K_o}. \end{aligned}$$

Observe

$$\begin{aligned} \det H^{\vec{L},\vec{M}}(\mathbf{y})\frac{\det V^{\vec{L},\vec{M}}(\mathbf{x})}{\left|\det V^{\vec{L},\vec{M}}(\mathbf{x})\right|} &= \operatorname{sgn}(\tau)\det H^{\vec{L},\vec{M}}_{\sigma,\tau}(\mathbf{y})\frac{\operatorname{sgn}(\tau)\det V^{\vec{L},\vec{M}}_{\sigma,\tau}(\mathbf{x})}{\left|\operatorname{sgn}(\tau)\det V^{\vec{L},\vec{M}}_{\sigma,\tau}(\mathbf{x})\right|} \\ &= \operatorname{sgn}(\tau)^2\det H^{\vec{L},\vec{M}}_{\sigma,\tau}(\mathbf{y}) \end{aligned}$$

for $\mathbf{y}\in\Delta_{K_e}(\sigma)\times\Delta_{K_o}(\tau)$. Thus,

$$Z_{\vec{M}} = \frac{1}{M_1!M_2!\cdots M_J!}\sum_{\sigma\in S_{K_e}}\sum_{\tau\in S_{K_o}}\int_{\Delta_{K_e}(\sigma)}\int_{\Delta_{K_o}(\tau)}\det H^{\vec{L},\vec{M}}_{\sigma,\tau}(\mathbf{y})\, dy_1\cdots dw_{K_o}.$$

6.6. Multicomponent Partition Functions

Proceeding through the same methods presented in section 5.4 and section 5.5, we get the same theorems from section 5.2 with slight modification to our forms γ_j and $\eta_{j,k}$. Given a complete N-family of monic polynomials, define

$$\gamma_j = \sum_{\mathfrak{t}:\underline{L_j}\nearrow\underline{N}}\left[\int_0^{2\pi}\operatorname{Wr}(\vec{p}_{\mathfrak{t}},e^{ix})\,d\mu_j(x)\right]\varepsilon_{\mathfrak{t}},$$

and define

$$\eta_{j,k} = \sum_{\mathfrak{t}:\underline{L_j}\nearrow\underline{N}} \sum_{\mathfrak{s}:\underline{L_k}\nearrow\underline{N}} \left[\int\int_{0<x<y<2\pi} \mathrm{Wr}(\vec{p}_{\mathfrak{t}}, e^{ix})\mathrm{Wr}(\vec{p}_{\mathfrak{s}}, e^{iy})\, d\mu_j(x) d\mu_k(y) \right] \varepsilon_{\mathfrak{t}} \wedge \varepsilon_{\mathfrak{s}},$$

where $d\mu_j(x) = (-ie^{-ix})^{L_j T/2}\, dx$.

6.7. Multicomponent Circular Constellations

We should think of multicomponent circular constellation ensembles as being variations on the multicomponent linear case (see section 5.6) in which we plug in variables $\mathbf{z}$ instead of $\mathbf{x}$. At the beginning of this chapter and again in section 6.5, we demonstrated we can account for the absolute value (complex modulus) by factoring out the sign corrections and then grouping them in place of the potential U. Alternatively, we can view multicomponent constellation ensembles as being the constellation variant of the appropriate one-dimensional ensemble as in section 6.4.

Following the setup from the beginning of this chapter, let $\mathbf{x}$ be the collection of angles $x_m^j \in [0, 2\pi)$. Define $\mathbf{x}_{\vec{y}}$ as before with all instances of $x_m^j + iy_k$ replaced with $y_k e^{ix_m^j}$. For particles $y_k e^{ix_m^j}$ and $y_k e^{ix_n^l}$ on the same circle,

$$\left| y_k e^{ix_m^j} - y_k e^{ix_n^l} \right| = -ie^{-i(x_m^j + x_n^l)/2} \left(y_k e^{ix_m^j} - y_k e^{ix_n^l} \right) \mathrm{sgn}(x_m^j - x_n^l).$$

Using what we know from the one-dimensional case, the sign correction factors in x_m^j are

$$\left(-ie^{-ix_m^j} \right)^{KL_jT/2},$$

where

$$T = -L_j + \sum_{k=1}^{J} L_k M_k.$$

Next, for particles $y_k e^{ix_m^j}$ and $y_l e^{ix_m^j}$ which share an angle x_m^j,

$$\left| y_k e^{ix_m^j} - y_l e^{ix_m^j} \right| = e^{-ix_m^j} \left(y_k e^{ix_m^j} - y_l e^{ix_m^j} \right),$$

giving us the sign correction factor

$$\left(e^{-ix_m^j} \right)^{L_j^2 \binom{K}{2}}.$$

Finally, for particles $y_k e^{ix_m^j}$ and $y_h e^{ix_n^l}$, which share neither an angle nor a radius,

$$\left| y_k e^{ix_m^j} - y_h e^{ix_n^l} \right| \left| y_h e^{ix_m^j} - y_k e^{ix_n^l} \right| = -e^{-i(x_m^j + x_n^l)} \left(y_k e^{ix_m^j} - y_h e^{ix_n^l} \right) \left(y_h e^{ix_m^j} - y_k e^{ix_n^l} \right),$$

giving us the last sign correction factor

$$\left(-ie^{-ix_m^j} \right)^{\binom{K}{2} L_j T}.$$

Thus,

$$d\mu_j(x) = \left(-ie^{-ix} \right)^{K^2 L_j T/2} \left(e^{-ix} \right)^{L_j^2 \binom{K}{2}} dx.$$

Finally, we obtain the same Berezin integral expressions for the partition functions as the linear case (Lemma 5.4 for the canonical, Theorems 5.2 and 5.3 for the isocharge grand canonical) with new $\gamma_j(\vec{y})$ and $\eta_{j,k}(\vec{y})$ defined by

$$\gamma_j(\vec{y}) = \sum_{\mathfrak{t}: \underline{L_j K} \nearrow \underline{N}} \int_0^{2\pi} \mathrm{Wr} \otimes \mathrm{Cr}_{\vec{y}}(\vec{p}_{\mathfrak{t}}, x)\, d\mu(x) \varepsilon_{\mathfrak{t}},$$

and

$$\eta_{j,k}(\vec{y}) = \sum_{\mathfrak{t}:\underline{L_j K}\nearrow\underline{N}} \sum_{\mathfrak{s}:\underline{L_k K}\nearrow\underline{N}} \int_0^{2\pi}\int_0^{2\pi} \Big[\mathrm{Wr}\otimes \mathrm{Cr}_{\vec{y}}(\vec{p}_{\mathfrak{t}}, x_1) \times \mathrm{Wr}\otimes \mathrm{Cr}_{\vec{y}}(\vec{p}_{\mathfrak{s}}, x_2)\Big]\, d\mu(x_1)\, d\mu(x_2)\, \varepsilon_{\mathfrak{t}} \wedge \varepsilon_{\mathfrak{s}}.$$

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