DIAGRAMMATIC REPRESENTATION THEORY OF THE RANK TWO SYMPLECTIC GROUP

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DISSERTATION ABSTRACT

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Title: Diagrammatic Representation Theory of the Rank Two Symplectic Group

We study the diagrammatic representation theory of the group Sp_4 and the quantum group $U_q(\mathfrak{sp}_4)$, expanding on the previous results of Kuperberg about type $B_2 = C_2$ webs. In particular, we construct a basis for an integral form of Kuperberg's web category. Using this basis we prove that the Karoubi envelope of the C_2 web category is equivalent to the category of tilting modules $\mathbf{Tilt}(U_q(\mathfrak{sp}_4))$. We also use the basis to give recursive formulas for the idempotent projecting to a top summand in a tensor product of fundamental representations. Finally, using our result about the equivalence between Kuperberg's web category and $\mathbf{Tilt}(U_q(\mathfrak{sp}_4))$, we prove that when $[3] = 0$ or $[4] = 0$, the semisimple quotient of $U_q(\mathfrak{sp}_4)$ is equivalent to $\mathbf{Rep}(O(2))$.

This dissertation contains previously published material.

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CHAPTER I

INTRODUCTION

This chapter contains previously published material. The material in Section 1.1. appeared in [11]. The material in Section 1.2. appeared in [12]. Also, Section 1.4. contains material from [11, 12].

1.1. Diagrammatics for Tilting Modules

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\mathbf{Rep}(\mathfrak{g})$ denote the category of finite dimensional modules for $\mathfrak{g}$. By Weyl's theorem on complete reducibility $\mathbf{Rep}(\mathfrak{g})$ is a semisimple category, so as an abelian category $\mathbf{Rep}(\mathfrak{g})$ is determined by the number of its simple objects. Since isomorphism classes of finite dimensional irreducible $\mathfrak{g}$-modules are in bijection with the countably infinite set of dominant integral weights X_+, $\mathbf{Rep}(\mathfrak{g}) \cong \mathbf{Rep}(\mathfrak{g}')$ as abelian categories, for any two semisimple Lie algebras.

A Lie algebra acts on the tensor product of two representations, so $\mathbf{Rep}(\mathfrak{g})$ is a monoidal category. Viewing $\mathbf{Rep}(\mathfrak{g})$ as a monoidal semisimple category, we capture much more information about $\mathfrak{g}$ (the amount of information can be made precise through Tannaka–Krein duality). One then may ask for a presentation by generators and relations of the monoidal category $(\mathbf{Rep}(\mathfrak{g}), \otimes)$. A modern point of view on this problem is to find a combinatorial replacement for $\mathbf{Rep}(\mathfrak{g})$ and then use planar diagrammatics to describe the combinatorial replacement by generators and relations.

By combinatorial replacement, we mean a full subcategory of $\mathbf{Rep}(\mathfrak{g})$ monoidally generated by finitely many objects, such that all objects in $\mathbf{Rep}(\mathfrak{g})$

are direct sums of summands of objects in the subcategory. We will focus on the combinatorial replacement $\mathbf{Fund}(\mathfrak{g})$, which is the full subcategory of $\mathbf{Rep}(\mathfrak{g})$ monoidally generated by the irreducible modules $V(\varpi)$ of highest weight ϖ for all fundamental weights ϖ. Note that $\mathbf{Fund}(\mathfrak{g})$ is not an additive category.

We use the terminology $\mathfrak{g}$-webs to refer to a diagrammatic category equivalent to $\mathbf{Fund}(\mathfrak{g})$. The history of $\mathfrak{g}$-webs begins with the Temperley–Lieb algebra [40, 48] for $\mathfrak{sl}_2$ and Kuperberg's "rank two spiders" [31] for $\mathfrak{sl}_3$, $\mathfrak{sp}_4 \cong \mathfrak{so}_5$, and $\mathfrak{g}_2$. D. Kim gave a conjectural presentation for $\mathfrak{sl}_4$-webs [29], and then Morrison gave a conjectural description of $\mathfrak{sl}_n$-webs [36]. Proving that the diagrammatic category was equivalent to $\mathbf{Fund}(\mathfrak{sl}_n)$ proved difficult, but was eventually carried out by Cautis, Kamnitzer, and Morrison using skew Howe duality [16]. Recently a conjectural description of $\mathfrak{sp}_6$-webs has appeared in a preprint by Rose and Tatham [39].

The Lie algebras $\mathfrak{g}$ for which there are $\mathfrak{g}$-web categories which are known to be equivalent to $\mathbf{Fund}(\mathfrak{g})$ are

$$\mathfrak{g} \in \{\mathfrak{sl}_n, \mathfrak{gl}_n, \mathfrak{sp}_4 \cong \mathfrak{so}_5, \mathfrak{g}_2\}.$$

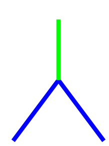
Each of these $\mathfrak{g}$-web categories has a q-deformed integral form, which we denote by $\mathcal{D}_{\mathfrak{g}}$, over $\mathbb{Z}[q, q^{-1}]$ (or some localization). On the representation theory side we have Lusztig's divided powers form of the quantum group, denoted $U_q^{\mathbb{Z}}(\mathfrak{g})$. This algebra has modules $V^{\mathbb{Z}}(\varpi)$, which are lattices inside $V(\varpi)$, for each fundamental weight. One should keep in mind that these lattices may not be irreducible after scalar extension to a field. The full subcategory monoidally generated by the modules $V^{\mathbb{Z}}(\varpi)$ will be denoted $\mathbf{Fund}(U_q^{\mathbb{Z}}(\mathfrak{g}))$.

Let $\mathbf{k}$ be a field and let $q \in \mathbf{k}^\times$. We can specialize the integral versions of both the diagrammatic category and the combinatorial replacement category to $\mathbf{k}$. It is natural to ask if these two categories are equivalent [5, 5A.4]. Taking all sums of summands of objects in $\mathbf{Fund}(\mathbf{k} \otimes U_q^{\mathbb{Z}}(\mathfrak{g}))$, one obtains the category of tilting modules $\mathbf{Tilt}(\mathbf{k} \otimes U_q^{\mathbb{Z}}(\mathfrak{g}))$. So a positive answer to this question means we have found generators and relations for the monoidal category of tilting modules.

For $\mathfrak{g} = \mathfrak{gl}_n$ an answer to this question appears in a paper of Elias [7]. Using ideas from Libedinsky's work [34] on constructing bases for maps between Soergel bimodules, Elias constructs a set of diagrams, denoted $\mathbb{LL}$ and referred to as double ladders, in the $\mathbb{Z}[q, q^{-1}]$-linear category $\mathcal{D}_{\mathfrak{gl}_n}$. There are two main arguments in [7]. First, a diagrammatic argument shows that $\mathbb{LL}$ spans the category over $\mathbb{Z}[q, q^{-1}]$. Second, Elias describes a functor $\Gamma : \mathcal{D}_{\mathfrak{gl}_n} \to \mathbf{Fund}(U_q^{\mathbb{Z}}(\mathfrak{gl}_n))$ and proves that $\Gamma(\mathbb{LL})$ is linearly independent. After observing that the ranks of homomorphism spaces in $\mathbf{Fund}(\mathbf{k} \otimes U_q^{\mathbb{Z}}(\mathfrak{gl}_n))$ are equal to $\#\mathbb{LL}$ [19], it follows that the diagrams $\mathbf{k} \otimes \mathbb{LL}$ are a basis for $\mathbf{k} \otimes \mathcal{D}_{\mathfrak{gl}_n}$ and the functor $\mathbf{k} \otimes \Gamma$ is an equivalence.

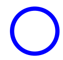
The category we focus on in is the $\mathfrak{sp}_4$ web category, which we denote by $\mathcal{D}_{\mathfrak{sp}_4}$. We recall the definition here using the convention that $[n]_v := \dfrac{v^n - v^{-n}}{v - v^{-1}}$ and $[n] := [n]_q \in \mathbb{Z}[q^{\pm 1}]$.

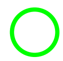
Definition 1.1..1. The category $\mathcal{D}_{\mathfrak{sp}_4}$ is the strict pivotal $\mathbb{Z}[q, q^{-1}, [2]^{-1}]$-linear category generated by two self dual objects, with morphisms generated by

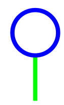


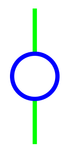
(1.1..1)

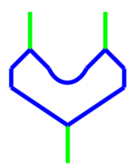
subject to the following relations.

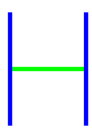
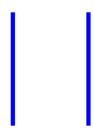
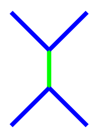
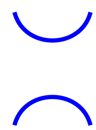
 $= -\frac{[6][2]}{[3]}$ (1.1..2)

 $= \frac{[6][5]}{[3][2]}$ (1.1..3)

 $= 0$ (1.1..4)

 $= -[2]$ (1.1..5)

 $= 0$ (1.1..6)

 $= \frac{1}{[2]}$  $+$  $- \frac{1}{[2]}$  (1.1..7)

We will refer to these relations as the circle relations, the monogon, bigon, and trigon relation, and the $H \equiv I$ relation.

Kuperberg proved [31] there is a monoidal equivalence $\mathbf{k} \otimes \mathcal{D}_{\mathfrak{sp}_4} \to \mathbf{Fund}(\mathbf{k} \otimes U_q(\mathfrak{sp}_4))$, when $\mathbf{k} = \mathbb{C}(q)$ and when $\mathbf{k} = \mathbb{C}$ and $q = 1$. Our goal is to prove this equivalence with as few restrictions on $\mathbf{k}$ and q as possible.

The arguments in Chapter II are completely indebted to Elias's approach, and the basis we construct for Kuperberg's $\mathcal{D}_{\mathfrak{sp}_4}$ webs is the analogue of Elias's light ladder basis for $\mathfrak{sl}_n$-webs in [7]. However, our arguments take less effort, since we can use Kuperberg's result [31] that non-elliptic webs span $\mathcal{D}_{\mathfrak{sp}_4}$ over $\mathbb{Z}[q, q^{-1}]$, and are a basis for $\mathcal{D}_{\mathfrak{sp}_4}$ over $\mathbb{C}$, when $q = 1$. Most of our work is to carefully construct an explicit functor $\Xi : \mathcal{D}_{\mathfrak{sp}_4} \to U_q^{\mathbb{Z}}(\mathfrak{sp}_4) - \text{mod}$.

The following theorem is our first main result in Chapter II.

Theorem 1.1..2. *If* $\mathbf{k}$ *is a field and* $q \in \mathbf{k}^\times$ *is such that* $q + q^{-1} \neq 0$*, then the functor*

$$\Xi : \mathbf{k} \otimes \mathcal{D}_{\mathfrak{sp}_4} \longrightarrow \mathbf{Fund}(\mathbf{k} \otimes U_q^{\mathbb{Z}}(\mathfrak{sp}_4)).$$

is a monoidal equivalence, and therefore induces a monoidal equivalence between the Karoubi envelope of $\mathbf{k} \otimes \mathcal{D}_{\mathfrak{sp}_4}$ *and the category* ***Tilt***$(\mathbf{k} \otimes U_q^{\mathbb{Z}}(\mathfrak{sp}_4))$*.*

Remark 1.1..3. The reader who is already well acquainted with [31] may wonder why we are talking about type C_2 and $\mathfrak{sp}_4$, instead of type B_2 and $\mathfrak{so}_5$. This certainly makes no difference classically, since $\mathfrak{sp}_4(\mathbb{C}) \cong \mathfrak{so}_5(\mathbb{C})$. For the purposes of this paper there is no difference over other fields either. Under our hypothesis that $q + q^{-1} \neq 0$ (note that this includes the possibility that $q = 1$ and $\mathbf{k}$ is not characteristic two), there is an isomorphism $\mathbf{k} \otimes U_q^{\mathbb{Z}}(\mathfrak{sp}_4) \cong \mathbf{k} \otimes U_q^{\mathbb{Z}}(\mathfrak{so}_5)$, as well as an equivalence between $\mathbf{k} \otimes \mathcal{D}_{\mathfrak{sp}_4}$ and the base change from $\mathbb{Z}[q, q^{-1}]$ to $\mathbf{k}$ of Kuperberg's B_2 spider category.

We chose C_2 over B_2 hoping it would prevent confusion, since the defining relations in $\mathcal{D}_{\mathfrak{sp}_4}$ are slightly different than the relations in Kuperberg's B_2 spider.

The following result is a consequence of Theorem (1.1..2), and is new even if $\mathbf{k} = \mathbb{C}$ and $q = 1$ or if $\mathbf{k} = \mathbb{C}(q)$.

Theorem 1.1..4. *Let* $\mathbf{k}$ *be a field and let* $q \in \mathbf{k}^\times$ *so that* $q + q^{-1} \neq 0$*. The double ladder diagrams defined in Section 2.6. form a basis for the morphism spaces in* $\mathbf{k} \otimes \mathcal{D}_{\mathfrak{sp}_4}$*.*

Remark 1.1..5. As we have already mentioned, Kuperberg's B_2 web category is spanned by the same non-elliptic diagrams over $\mathbb{Z}[q, q^{-1}]$. The work of Sikora–Westbury [43] proves that these diagrams are linearly independent whenever

$q + q^{-1} \neq 0$. Although their techniques are quite different than ours and certainly are worth studying, their result is a consequence of ours.

Suppose that one could show that either double ladder diagrams span or are linearly independent. Since the number of double ladders is equal to the number of non-elliptic webs, the result from [43] would imply that the double ladder diagrams are a basis.

However, it is not possible to obtain Theorem (1.1..2) with just their result. Even though their paper and some basic representation theory imply the dimensions of homomorphism spaces in $\mathbf{k} \otimes \mathcal{D}_{\mathfrak{sp}_4}$ and $\mathbf{Fund}(\mathbf{k} \otimes U_q^{\mathbb{Z}}(\mathfrak{sp}_4))$ are equal, it is not enough to deduce that $\mathbf{k} \otimes \Xi$ is an equivalence. The difficulty is best illustrated via analogy: the lattice $\mathbb{Z}$ becomes a one-dimensional vector space after base change to any field, but the map $\mathbb{Z} \xrightarrow{x \mapsto 2x} \mathbb{Z}$ is not an isomorphism after tensoring with a field of characteristic two. We really need to know that the map $\mathbf{k} \otimes \Xi$ is an isomorphism and to do this we must explicitly construct and analyze the functor Ξ.

1.2. Triple Clasp Formulas

Let $\mathfrak{g}$ be a semisimple Lie algebra. Fix a dominant integral weight λ. We can uniquely write $\lambda = \sum_{\varpi} \lambda_{\varpi} \varpi$, where the ϖ are fundamental weights. The irreducible representation with highest weight λ, denoted $V(\lambda)$, occurs with multiplicity one as a direct summand in the tensor product

$$\bigotimes_{\varpi} V(\varpi)^{\otimes \lambda_{\varpi}}, \tag{1.2..1}$$

and all other irreducible summands are isomorphic to $V(\mu)$, with $\mu < \lambda$ in the dominance order. Moreover, the isomorphism class of the tensor product is unaffected by the order of the tensor factors in Equation (1.2..1). For any sequence of fundamental weights: $\varpi_{i_1}, \varpi_{i_2}, \ldots, \varpi_{i_d}$, we refer to

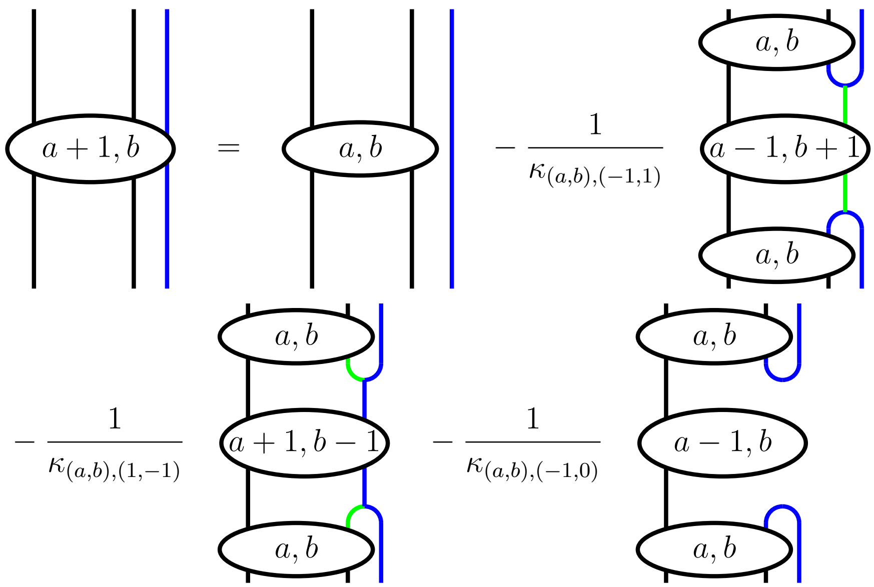
$$V(\varpi_{i_1} + \varpi_{i_2} + \ldots + \varpi_{i_d}) \subset V(\varpi_{i_1}) \otimes V(\varpi_{i_2}) \otimes \ldots \otimes V(\varpi_{i_d}) \tag{1.2..2}$$

as the *top summand*.

We will use Kuperberg's type $B_2 = C_2$ webs to give a recursive description of the idempotent projecting to the top summand in an arbitrary tensor product of fundamental representations of $\mathfrak{sp}_4$. Our formulas are the $\mathfrak{sp}_4$ analogue of Elias's conjectural recursive formulas describing the idempotent projecting to the top summand for $\mathfrak{sl}_n$ [7].

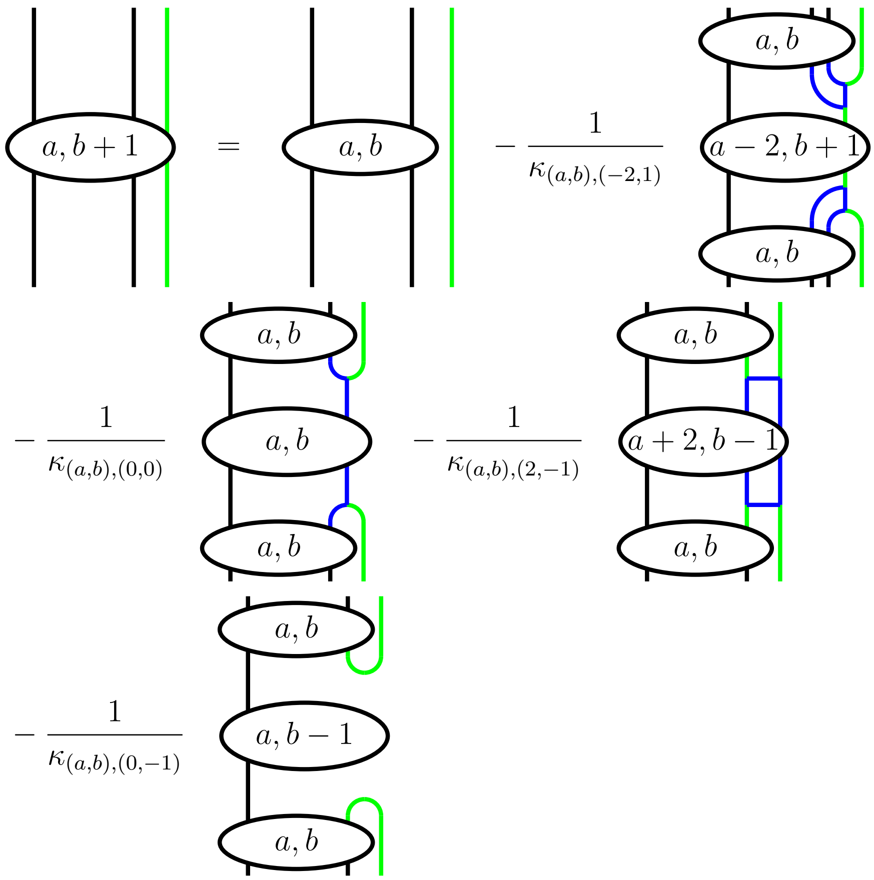
Given a tensor product of fundamental representations so that $V(\varpi_1)$ occurs a times and $V(\varpi_2)$ occurs b times, we will write $V(a, b)$ to denote the top summand. In [30], Kim gives formulas for the $\mathfrak{sp}_4$ clasps projecting to $V(a, 0)$ and $V(0, b)$. The main result of Chapter III is a recursive triple clasp formula for the idempotent projecting to $V(a, b)$.

Theorem 1.2..1. *Let an oval with (a, b) label denote the idempotent with image $V(a, b)$, then*



(1.2..3)

and



(1.2..4)

where

$$\kappa_{(a,b),(-1,1)} = -\frac{[a+1]}{[a]} \tag{1.2..5}$$

$$\kappa_{(a,b),(1,-1)} = \frac{[a+2b+3][2b+2]}{[a+2b+2][2b]} \tag{1.2..6}$$

$$\kappa_{(a,b),(-1,0)} = -\frac{[2a+2b+4][a+2b+3][a+1]}{[2a+2b+2][a+2b+2][a]} \tag{1.2..7}$$

and

$$\kappa_{(a,b),(-2,1)} = -\frac{[a+1][2a+2b+4]}{[a-1][2a+2b+2]} \tag{1.2..8}$$

$$\kappa_{(a,b),(0,0)} = \frac{[a+2][a+2b+4]}{[2][a][a+2b+2]} \tag{1.2..9}$$

$$\kappa_{(a,b),(2,-1)} = -\frac{[2b+2]}{[2b]} \tag{1.2..10}$$

$$\kappa_{(a,b),(0,-1)} = \frac{[2a+2b+4][a+2b+3][2b+2]}{[2a+2b+2][a+2b+1][2b]} \tag{1.2..11}$$

Using the double ladder basis for homomorphism spaces in $\mathcal{D}_{\mathfrak{sp}_4}$ [11], and the ideas from Elias's work on clasps for type A webs [7], we can argue that such a recursive formula exists without knowing the κ's explicitly. The recursive nature of the clasp formula implies recursive relations among the κ's. Our theorem then follows from showing that these relations force the κ's to be the values specified in Theorem (1.2..1).

1.3. Semisimplification of the Category of Tilting Modules

Suppose that $\mathbf{k} = \mathbb{C}$ and $q = e^{i\pi/\ell}$ for some integer $\ell > 4$. There is a well known construction of a $\mathbb{C}$-linear fusion category as the quotient of $\mathbf{Tilt}(U_q^{\mathbf{k}}(\mathfrak{sp}_4))$ by the ideal of negligible morphisms [1]. One may ask for a generators and relations presentation of these quotient categories. Since we have a presentation of the category of tilting modules, it remains to find relations which generate the ideal of negligible morphisms, denoted $\mathcal{N}$. In general the ideal of negligible morphisms is not the monoidal ideal generated by the identity morphisms of all negligible objects. However, the hypothesis $\ell > 4$ guarantees that the non-negligible tilting modules are such that $\dim \operatorname{Hom}(T^{\mathbf{k}}(\lambda), T^{\mathbf{k}}(\mu)) = \delta_{\lambda,\mu}$. It follows that the quotient by the ideal generated by the negligible objects is a semisimple category. Since the ideal of negligible morphisms is the only monoidal ideal with semisimple quotient, the ideal of negligible morphisms coincides with the monoidal ideal generated by the negligible objects.

There is a ρ shifted and ℓ dilated affine Weyl group action on the weight lattice X. Extending this action to $\mathbb{R} \otimes X$ allows us to partition the dominant weights by their relationship to alcoves in $\mathbb{R} \otimes X$. The weights in the interior of the lowest alcove in the cone $-\rho + \mathbb{R}_{\geq 0} \cdot X_+$ are exactly the highest weights of the indecomposable non-negligible tilting modules. In particular, the indecomposable tilting modules with highest weight on the upper closure of the lowest alcove is negligible. Furthermore, each indecomposable negligible tilting module is a direct summand of a tensor product of some tilting module and an indecomposable tilting module with highest weight on the upper closure of the lowest alcove. Thus, the indecomposable negligible tilting modules are contained in the monoidal ideal generated by the indecomposable tilting modules on the upper closure of the lowest alcove.

The tilting modules on the upper closure of the lowest alcove are

$$T^{\mathbf{k}}\left(2k, \frac{\ell-3}{2} - k\right), \text{ for } k = 0, \ldots, \ell - 3 \tag{1.3..1}$$

when ℓ is odd, and

$$T^{\mathbf{k}}\left(k, \frac{\ell-4}{2} - k\right), \text{ for } k = 0, \ldots, \frac{\ell-4}{2} \tag{1.3..2}$$

when ℓ is even. Let $\mathcal{K}$ denote the monoidal ideal in $\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}$ generated by the λ clasps for all λ on the upper closure of the lowest alcove. We may conclude that there is a monoidal functor

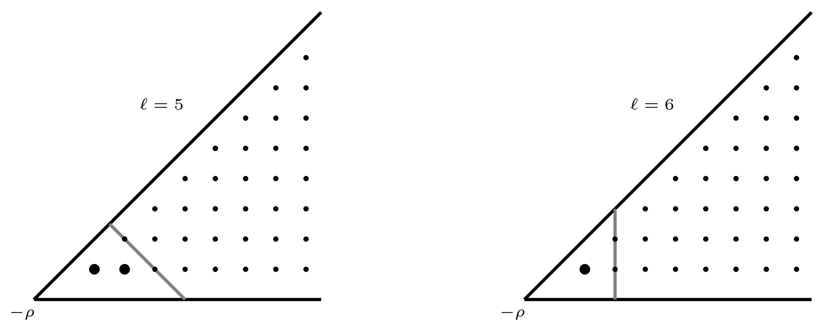
$$\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4} \longrightarrow \mathbf{Tilt}(U_q^{\mathbf{k}}(\mathfrak{sp}_4))/\mathcal{N}, \tag{1.3..3}$$

with kernel $\mathcal{K}$. Furthermore, the induced functor

$$\mathbf{Kar}(\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}/\mathcal{K}) \longrightarrow \mathbf{Tilt}(U_q^{\mathbf{k}}(\mathfrak{sp}_4))/\mathcal{N} \tag{1.3..4}$$

is an equivalence.

Example 1.3..1. Let $\mathfrak{g} = \mathfrak{sp}_4$. In the following diagrams the dots represent the dominant integral weights and the gray line represents the upper wall in the closure of the fundamental Weyl alcove. The bold dots then are the dominant integral weights λ so that the indecomposable tilting module of highest weight λ is non-zero in the negligible quotient.



If $\ell = 5$, then the ideal of negligible morphisms is generated by $\mathrm{id}_{T^{\mathbf{k}}(0,1)}$ and $\mathrm{id}_{T^{\mathbf{k}}(2,0)}$. The quotient category is equivalent to $\mathbf{Rep}(\mathbb{Z}/2)$, where the sign representation corresponds to $T^{\mathbf{k}}(1,0)$.

If $\ell = 6$, then the ideal of negligible morphisms is generated by $\mathrm{id}_{T^{\mathbf{k}}(1,0)}$ and $\mathrm{id}_{T^{\mathbf{k}}(0,1)}$. Thus, the quotient category is equivalent to $\mathbf{Vec}_{\mathbb{C}}$.

Example 1.3..2. It is a pleasant exercise to use the diagrammatic category $\mathcal{TL}$ to show that if $\ell = 8$, the negligible quotient of $\mathbf{Tilt}(U_q^{\mathbf{k}}(\mathfrak{sp}_4))$ is equivalent to

$\mathbf{Tilt}(\mathbb{C}(e^{i\pi/4}) \otimes U_q^{\mathbb{Z}}(\mathfrak{sl}_2))/\mathcal{N}$. Note that if $q = e^{i\pi/8}$, then

$$-\frac{[6]_q[2]_q}{[3]_q} = -[2]_{q^2}. \tag{1.3..5}$$

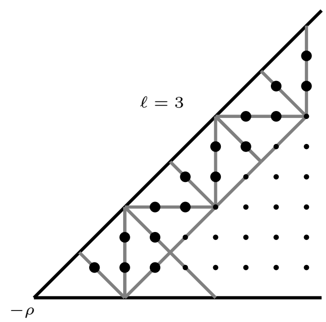
The case when the size of the negligible quotient is controlled by the weights in the fundamental Weyl alcove is particularly well studied since it is related to the construction of modular tensor categories [42]. However, the negligible quotient of tilting modules makes sense for all ℓ. The case of the reductive algebraic group GL_n in characteristic p is completely understood [14]. It also seems likely that similar techniques could handle the case of the quantum group for $\mathfrak{gl}_n$ for all ℓ. However, it seems that very little is known for general $\mathfrak{g}$ when ℓ is small. We will focus on $\mathfrak{sp}_{2n}$, for which the cases of interest are when $\ell' \leq 2n$.

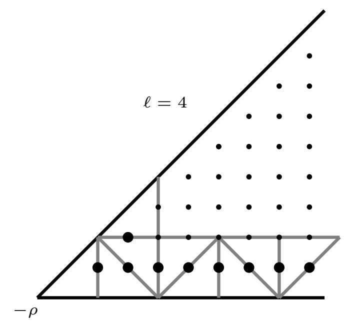
The main result in Chapter IV is the following.

Theorem 1.3..3. *The semisimplification of the category of tilting modules for (quantum) $\mathfrak{sp}_4$ is equivalent to $\mathbf{Rep}(O(2))$ when $q = \zeta_{2\cdot 2}$ and when $q = \zeta_{2\cdot(2+1)}$.*

The proof of this theorem uses Kuperberg's C_2 webs [31], Deligne's category $\underline{\mathbf{Rep}}(O(T))$ [17], and our results connecting C_2 webs with tilting modules for quantum $\mathfrak{sp}_4$ [11].

Remark 1.3..4. Theorem (1.3..3) implies that when $n = 2$, the bold dots are exactly corresponding to the indecomposable tilting modules which do not have dimension zero.





We recall the notion of a principal graph of a self dual object V in a semisimple rigid tensor category $\mathcal{C}$. For simplicity we assume that all irreducible objects in $\mathcal{C}$ appear as a summand of some tensor power of V. The principal graph of V, denoted Γ_V is the graph with vertices the isomorphism classes of irreducibles appearing in some tensor power of V, and the number of edges between $[S_1]$ and $[S_2]$ is equal to the multiplicity of S_2 in $S_1 \otimes V$, which by the self-duality of V is a symmetric relation.

Let $G_u := \mathbb{C}^\times \rtimes \mathbb{Z}/2$, where the $\mathbb{Z}/2$ action on $\mathbb{C}^\times$ is $x \mapsto x^{-1}$. Also, let $H := \mathbb{C}^\times$, and $V := \mathrm{Ind}_H^{G_u}(\mathbb{C})$. Victor Ostrik made two observations about the bold dots in the diagrams above (personal communication, December 2019). This first is that the bold dots appear in the same configuration as the principal graph Γ_V, making it appear that principal graph $\Gamma_{\overline{T(\varpi_1)}}$ is the same as Γ_V. The second is that the alcoves in this picture are the alcoves in the two highest antispherical Kazhdan-Lusztig cells. The lowest of these two cells is the unique reduced expression cell. Moreover, under Lusztig's bijection between antispherical cells and nilpotent orbits [35, Theorem 4.8] the nilpotent orbit corresponding to the unique reduced expression cell has centralizer with maximal reductive part isomorphic to $\mathbb{C}^\times \rtimes \mathbb{Z}/2$. These two pieces of evidence led to the conjecture that the negligible quotient of the category of tilting modules is equivalent to $\mathbf{Rep}(G_u)$.

Note that $O(2) \cong G_u$[1]. The McKay subgroup of $SU(2)$ of type D_∞ is the unique subgroup (up to conjugacy) in $SU(2)$ with principal graph of the restriction of the natural module for $SU(2)$ given by the D_∞ Coxeter graph. The D_∞ McKay group is a non-split extension of $\mathbb{C}^\times$ with $\mathbb{Z}/2$ quotient and is not isomorphic to the group $O(2)$. However, $O(2)$ is isomorphic to the image of D_∞ in the (non-faithful) representation $S^2(\mathbb{C}^2)$.

Remark 1.3..5. We claim that a correct generalization of Ostrik's conjecture for all n is the following.

Conjecture 1.3..6. *The semisimplification of the category of tilting modules for (quantum)* $\mathfrak{sp}_{2n}$ *is equivalent to* $\mathbf{Rep}(O(2))$ *when* $q = \zeta_{2\cdot 2n}$ *and when* $q = \zeta_{2\cdot(n+1)}$.

In order to study these semisimple quotient categories for small roots of unity it will be useful to be able to understand the category of tilting modules via generators and relations. The category of tilting modules is generated as a monoidal category by the fundamental Weyl modules as long as ℓ is not too small. Unfortunately the literature is not clear for exactly which ℓ this is the case, but for the analogous case of reductive algebraic groups in characteristic p the answer is given in [27]. Note that for type C_n, they observe that the fundamental Weyl modules are tilting modules whenever $p > n$. Thus, we expect that for $\mathfrak{sp}_{2n}$ the most accessible cases which are still interesting are when $n + 1 \leq \ell' \leq 2n$.

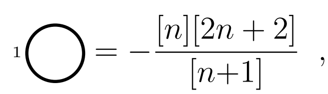
The natural generalization of our proof of Theorem (1.3..3) to a proof of Conjecture (1.3..6) would replace C_2 webs with the following diagrammatic category.

[1] We expect that the reductive part of the centralizer of the subregular nilpotent orbit is isomorphic to $O(2)$, but cannot prove this.

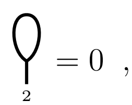
Definition 1.3..7. [9, Definition 1.1] Let $\mathbf{Web}(\mathfrak{sp}_{2n})$ be the $\mathbb{C}(q)$-linear pivotal category defined by the following presentation. The objects are generated monoidally by self-dual objects $\{1, \ldots, n\}$. In addition to the cap/cup unit/counit morphisms implicit in the pivotal structure, the morphisms are generated by

(1.3..6)

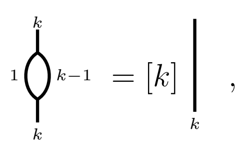
for $k \in \{1, \ldots, n-1\}$. One then takes the quotient by the tensor ideal generated by the following (local) relations:



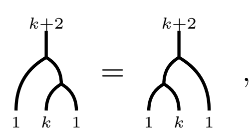
(1.3..7a)



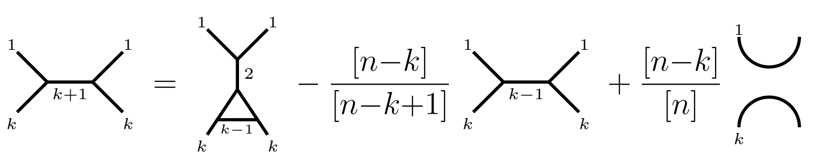
(1.3..7b)



(1.3..7c)



(1.3..7d)



(1.3..7e)

The category $\mathbf{Kar}\,\mathbf{Web}(\mathfrak{sp}_{2n})$ is known to be equivalent to the category of finite dimensional type 1 representations of $U_q(\mathfrak{sp}_{2n})$ [9, Theorem 1.4]. It is conjectured that when $\ell > n$, the category $\mathbf{Kar}\,\mathbb{C} \otimes_{q=e^{i\pi/\ell}} \mathbf{Web}(\mathfrak{sp}_{2n})$ is equivalent to the category of tilting modules for $\mathbb{C} \otimes_{q=e^{i\pi/\ell}} U_q^{\mathbb{Z}}(\mathfrak{sp}_{2n})$ [9, Remark 3.8]. We give sketches of arguments below that will prove Conjecture (1.3..6) once the conjectural relationship between $\mathbf{Web}(\mathfrak{sp}_{2n})$ and tilting modules is established.

1.4. Potential Applications

Remark 1.4..1. If we take $\mathbf{k}$ to be an algebraically closed field of characteristic p and let $q = 1$, then $\mathbf{Tilt}(\mathbf{k} \otimes U_q^{\mathbb{Z}}(\mathfrak{sp}_4))$ is equivalent to the category of tilting modules for the reductive algebraic group $\mathrm{Sp}_4(\mathbf{k})$ [26, H.6]. Very little is known about tilting modules for reductive groups in characteristic $p > 0$, and our results apply in this setting as well for all $p > 2$.

Remark 1.4..2. If $q + q^{-1} = 0$, then the fundamental representation $\mathbf{k} \otimes V^{\mathbb{Z}}(\varpi_2)$ is not tilting. So if one is interested in tilting objects the category $\mathbf{Fund}(\mathfrak{g})$ is not the correct category to study. Also, the category $\mathcal{D}_{\mathfrak{sp}_4}$ is not defined when $q + q^{-1} = 0$, because some relations have coefficients with $q + q^{-1}$ in the denominator. One could clear denominators in the relations and obtain a category which is defined when $q + q^{-1} = 0$. However, we do not know what this diagrammatic category would describe.

Let $\mathbf{k} = \mathbb{C}$ and let $q = e^{\pi i/\ell}$. Soergel conjectured [45] and then proved [46] a formula for the character of a tilting module for $\mathbf{k} \otimes U_q^{\mathbb{Z}}(\mathfrak{g})$ when $\ell > h$, where h is the Coxeter number of $\mathfrak{g}$.

In Section 2.24..4, we will prove that the category $\mathcal{D}_{\mathfrak{sp}_4}$ is a strictly object adapted cellular category [21]. Thus, the discussion in [22, 11.5] allows one to adapt the algorithm in [32] from the context of Soergel bimodules to $\mathfrak{sp}_4$-webs. Using this algorithm, which we outline in Section 2.25., one can compute tilting characters for the quantum group at a root of unity as long as $\ell \geq 3$ (the $\ell = 2$ case is ruled out by the assumption in our theorem that $q + q^{-1} \neq 0$). The Coxeter number of $\mathfrak{sp}_4$ is $h = 4$. This means that when $\ell = 3$, Soergel's conjecture for tilting characters does not apply but the diagrammatic category $\mathbf{k} \otimes \mathcal{D}_{\mathfrak{sp}_4}$ does still describe tilting modules.

There may be a conjecture for the characters of tilting modules of quantum groups that includes $\ell \leq h$, along the lines of [8, Section 8.1] and [44, Theorem 1.6]. Ideally, the conjecture would relate tilting characters for the quantum group at a root of unity to singular, antispherical Kazhdan-Lusztig polynomials. One could use $\mathfrak{sp}_4$ webs to check such a conjecture for small weights.

With applications to modular representation theory in mind, there has been some work on writing the idempotents projecting to all indecomposable tilting modules in terms of the double ladder basis for $SL_2(\overline{\mathbb{F}}_p)$ [15] [50]. Such idempotents have been referred to as p-Jones-Wenzl projectors. Since nobody knows the characters of tilting modules for rank two groups in positive characteristic, this question is not appropriate in our setting. However, the tilting characters are known for the quantum group at a root of unity [46, Section 8]. A key first step in determining the formulas for the p-Jones-Wenzl projectors is to argue that if the characteristic p tilting module is simple, then the characteristic zero clasp can be reduced modulo a maximal ideal to obtain the projector in characteristic p. We are careful to point out how this works for the case of $\mathcal{D}_{\mathfrak{sp}_4}$ in Section 3.5..9, but do not explore the topic further in this dissertation.

It remains an open problem to adapt the arguments in [7] to prove that double ladder diagrams span $\mathcal{D}_{\mathfrak{sp}_4}$ without using Kuperberg's results about non-elliptic webs. The first steps in this adaptation would be to rewrite every composition of elementary light ladder diagrams of the form $L_\mu \circ (\mathrm{id} \otimes L_\nu)$ as a linear combination of double ladder diagrams. This is an easy exercise which may convince the reader that such an adaptation is possible. The second step is to prove that any diagram of the form $(\mathrm{id} \otimes L_\mu \otimes \mathrm{id}) \circ N \circ (\mathrm{id} \otimes \mathbb{D}(L_\nu) \otimes \mathrm{id})$, where N is an arbitrary neutral diagram, is a linear combination of double ladder diagrams. The case when N is the identity is another easy exercise, and considering the case of arbitrary N may help convince the reader that writing a complete adaptation of [7] would be non-trivial.

It is work in progress of Victor Ostrik and Noah Snyder to find the precise relationship between Kuperberg's G_2 webs and tilting modules.

Work in progress of Ben Elias and Geordie Williamson uses $\mathcal{D}_{\mathfrak{sp}_4}$ to extend the quantum algebraic Satake equivalence [20] to type B_2/C_2. As a consequence our results may have implications in geometric representation theory.

Lastly, we mention that in joint work with Haihan Wu, we solved the problem of finding triple clasp formulas for $\mathfrak{g}_2$ [13].

CHAPTER II

DIAGRAMMATICS FOR TILTING MODULES

This chapter contains previously published material. The material in this chapter originally appeared in [11].

2.1. Outline

Section 2: We discuss how to decompose tensor products of representations for $\mathfrak{sp}_4$. Then use the plethysm patterns to describe an algorithm for light ladder diagrams. Finally we define the double ladder diagrams. Section 3: We define an evaluation functor from the diagrammatic category to the representation theoretic category. After reviewing some of the theory of tilting modules for quantum groups/reductive algebraic groups, we interpret the image of the evaluation functor as an integral form of the category of tilting modules. Then we argue that the main theorem follows from linear independence of the image of the double ladder diagrams. Section 4: We argue that the double ladder diagrams are linearly independent. Then we deduce that $\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}$ is an object adapted cellular category, and describe an algorithm to compute tilting characters.

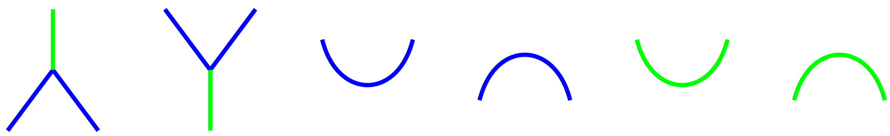
2.2. C_2-Webs

We use the convention that the quantum integers in $\mathbb{Z}[q, q^{-1}]$ are defined as

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad \text{for } n \in \mathbb{Z}. \tag{2.2..1}$$

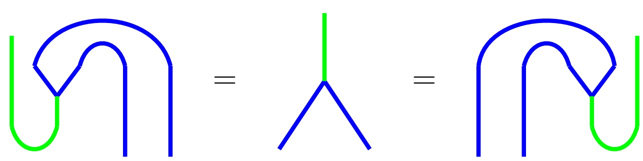
Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}, [2]_q^{-1}]$, the ring $\mathbb{Z}[q, q^{-1}]$ localized at $[2]_q$.

Definition 2.2..1. Let $\mathcal{D}$ be the $\mathcal{A}$-linear monoidal category defined by generators and relations. The generating objects are 1 and 2, the generating morphisms are the following diagrams.

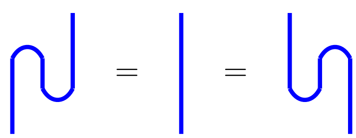


(2.2..2)

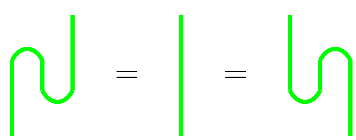
The relations are the following local relations on diagrams.



(2.2..3)



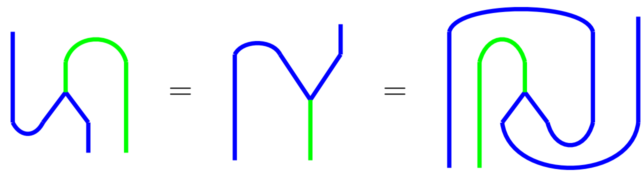
(2.2..4)



(2.2..5)

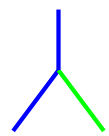
Remark 2.2..2. Our convention is that diagrams are read as morphisms from the bottom boundary to the top boundary. Composition of morphisms is vertical stacking. The monoidal structure on objects is concatenation of words and the monoidal unit is the empty word. The monoidal product on morphisms is horizontal concatenation of diagrams, and the identity morphism of the empty word is the empty diagram.

Notation 2.2..3. The defining relations in $\mathcal{D}$ imply the following equalities of morphisms in $\mathrm{Hom}_{\mathcal{D}}(12, 1)$.



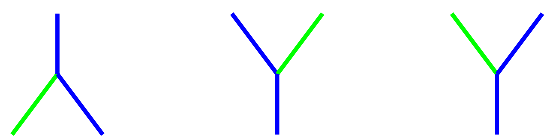
$$= \quad = \tag{2.2..6}$$

We will denote any one of these morphisms by the following trivalent vertex diagram in $\mathrm{Hom}_{\mathcal{D}}(12, 1)$.



(2.2..7)

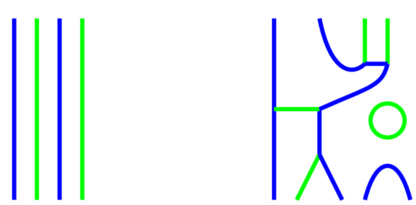
There are similar equalities for every possible vertical and horizontal reflection, and we will write the corresponding trivalent morphisms as follows.



(2.2..8)

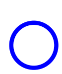
Thanks to this notation, we may now view morphisms in $\mathcal{D}$ as $\mathcal{A}$-linear combinations of isotopy classes of trivalent graphs.

Example 2.2..4. The identity morphism of 1211 and a morphism from 12111 to 1122 are drawn as follows.

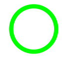


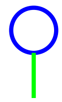
(2.2..9)

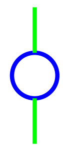
Definition 2.2..5. The $\mathcal{A}$-linear monoidal category $\mathcal{D}_{\mathfrak{sp}_4}$ is the quotient of $\mathcal{D}$ by the following local relations.

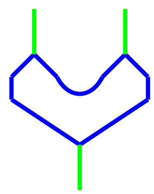


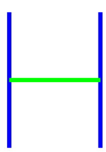
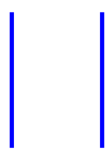
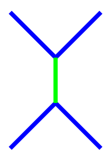
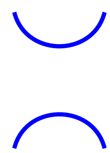
$$= -\frac{[6]_q[2]_q}{[3]_q} \tag{2.2..10}$$

 $= \frac{[6]_q[5]_q}{[3]_q[2]_q}$ (2.2..11)

 $= 0$ (2.2..12)

 $= -[2]_q$ (2.2..13)

 $= 0$ (2.2..14)

 $= \frac{1}{[2]_q}$  $+$  $- \frac{1}{[2]_q}$  (2.2..15)

Notation 2.2..6. When $\mathbf{k}$ is an $\mathcal{A}$-algebra, we can base change the category $\mathcal{D}_{\mathfrak{sp}_4}$ to $\mathbf{k}$, denoted $\mathbf{k} \otimes \mathcal{D}_{\mathfrak{sp}_4}$. The category $\mathbf{k} \otimes \mathcal{D}_{\mathfrak{sp}_4}$ has the same objects as $\mathcal{D}_{\mathfrak{sp}_4}$ and we apply $\mathbf{k} \otimes_{\mathcal{A}} (-)$ to homomorphism spaces. We may also write $\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4} := \mathbf{k} \otimes \mathcal{D}_{\mathfrak{sp}_4}$ for short.

Remark 2.2..7. The coefficients in the circle relations are written as fractions but are actually elements of $\mathcal{A}$, as can be observed in the following quantum number calculations.

$$[5]_q - [1]_q = \frac{([5]_q - [1]_q)\,[3]_q}{[3]_q} = \frac{[7]_q + [5]_q + [3]_q - [3]_q}{[3]_q} = \frac{[6]_q[2]_q}{[3]_q}. \qquad (2.2..16)$$

$$[7]_q - [5]_q + [3]_q = \frac{[8]_q + [2]_q}{[2]_q} = \frac{[10]_q + [8]_q + [6]_q + [4]_q + [2]_q}{[3]_q[2]_q} = \frac{[6]_q[5]_q}{[3]_q[2]_q}. \qquad (2.2..17)$$

Remark 2.2..8. The category $\mathcal{D}_{\mathfrak{sp}_4}$ is almost the B_2 spider category in [31]. But we replaced q with q^2 and rescaled the trivalent vertex by $[2]_q^{-1/2}$. The trivalent vertex in $\mathcal{D}_{\mathfrak{sp}_4}$ may seem less natural since the relations now require us to insist $[2]_q$ is invertible, but when we connect the diagrammatic category to representation theory the rescaled trivalent vertex in $\mathcal{D}_{\mathfrak{sp}_4}$ will be more natural.

2.3. Decomposing Tensor Products in $\mathbf{Rep}(\mathfrak{sp}_4(\mathbb{C}))$

We now recall some basic facts about $\mathfrak{sp}_4(\mathbb{C})$ and its representation theory. Some of this is worked out in detail in [24, Lecture 16]. Then we will record some formulas describing the decomposition of certain tensor products in $\mathbf{Rep}(\mathfrak{sp}_4)$.

Let $X = \mathbb{Z}\epsilon_1 \oplus \mathbb{Z}\epsilon_2$ be the weight lattice for $\mathfrak{sp}_4(\mathbb{C})$. The weights $\varpi_1 = \epsilon_1$ and $\varpi_2 = \epsilon_1 + \epsilon_2$ are called the fundamental weights, and $X_+ = \mathbb{Z}_{\geq 0}\varpi_1 \oplus \mathbb{Z}_{\geq 0}\varpi_2$ is the set of dominant weights.

Let $\mathbf{Fund}(\mathfrak{sp}_4(\mathbb{C}))$ be the full monoidal subcategory of $\mathbf{Rep}(\mathfrak{sp}_4(\mathbb{C}))$ generated by $V(\varpi_1)$ and $V(\varpi_2)$. The decomposition

$$V(\varpi_1) \otimes V(\varpi_1) \cong V(2\varpi_1) \oplus V(\varpi_2) \oplus V(0). \tag{2.3..1}$$

implies there is a one-dimensional space of maps between $V(\varpi_1) \otimes V(\varpi_1)$ and $V(\varpi_2)$. We will later prove that there is a choice for this map so that sending the trivalent vertex to the chosen map gives a well-defined monoidal functor from $\mathbb{C} \otimes \mathcal{D}_{\mathfrak{sp}_4}$ to $\mathbf{Fund}(\mathfrak{sp}_4(\mathbb{C}))$. We will then show that this functor is full and faithful.

For now we will take the equivalence on faith, and use it to guide our intuition for constructing a basis for Hom spaces in $\mathcal{D}_{\mathfrak{sp}_4}$. Let λ and μ be dominant integral weights. There is a direct sum decomposition

$$V(\lambda) \otimes V(\mu) \cong \bigoplus_{\nu \in X(\lambda,\mu) \subset \mathrm{wt}(V(\mu))} V(\lambda + \nu), \tag{2.3..2}$$

where $\mathrm{wt}(V(\mu))$ is the multiset of weights in $V(\mu)$ and $X(\lambda, \mu)$ is a submultiset. Our goal is to determine the set $X(\lambda, \mu)$.

To simplify notation, we may write $V(a, b)$ in place of $V(a\varpi_1 + b\varpi_2)$. The following formulas are easy to work out using classical theory. For example, one can use [38, 2.16].

$$V(a,b) \otimes V(1,0) \cong \begin{cases} V(1,0), \text{if } a = b = 0 \\ V(a+1,0) \oplus V(a-1,1) \oplus V(a-1,0), \text{if } a \geq 1, b = 0 \\ V(1,b) \oplus V(1,b-1), \text{if } a = 0, b \geq 1 \\ V(a+1,b) \oplus V(a-1,b+1) \oplus V(a-1,b) \oplus V(a+1,b-1), \\ \qquad \text{if } a \geq 1, b \geq 1 \end{cases} \tag{2.3..3a}$$

$$V(a,b) \otimes V(0,1) \cong \begin{cases} V(0,1), \text{if } a = b = 0 \\ V(0,b+1) \oplus V(2,b-1) \oplus V(0,b-1), \text{if } a = 0, b \geq 1 \\ V(1,1) \oplus V(1,0), \text{if } a = 1, b = 0 \\ V(1,b+1) \oplus V(1,b) \oplus V(3,b-1) \oplus V(1,b-1), \text{ if } a = 1, b \geq 1 \\ V(a,1) \oplus V(a,0) \oplus V(a-2,1), \text{ if } a \geq 2, b = 0 \\ V(a,b+1) \oplus V(a+2,b-1) \oplus V(a,b-1) \oplus V(a,b) \oplus V(a-2,b+1), \\ \qquad \text{if } a \geq 2, b \geq 1 \end{cases} \tag{2.3..3b}$$

Notation 2.3..1. We will write $V(1) = V(\varpi_1) = V(1,0)$ and $V(2) = V(\varpi_2) = V(0,1)$ as well as $\mathrm{wt}\, 1 = \varpi_1$ and $\mathrm{wt}\, 2 = \varpi_2$. Also, for a sequence $\underline{w} = (w_1, \ldots, w_n)$,

$w_i \in \{1, 2\}$ we will write $V(\underline{w}) = V(w_1) \otimes \ldots \otimes V(w_n)$, $\operatorname{wt} \underline{w} = \operatorname{wt} w_1 + \operatorname{wt} w_2 + \ldots \operatorname{wt} w_n$, and $\underline{w}_{\leq k} = (w_1, w_2, \ldots, w_k)$.

Definition 2.3..2. Let $\underline{w} = (w_1, \ldots, w_n)$ with $w_i \in \{1, 2\}$. A sequence $(\mu_1, \ldots, \mu_n)$ where $\mu_i \in \operatorname{wt}(V(w_i))$ is a **dominant weight subsequence** of $\underline{w}$ if:

1. μ_1 is dominant;
2. $V(\mu_1 + \ldots + \mu_{i-1} + \mu_i)$ is a summand of $V(\mu_1 + \ldots + \mu_{i-1}) \otimes V(w_i)$.

We write $E(\underline{w})$ for the set of all dominant weight subsequences of $\underline{w}$ and

$$E(\underline{w}, \lambda) := \{(\mu_1, \ldots, \mu_n) \in E(\underline{w}) \ : \ \mu_1 + \ldots + \mu_n = \lambda\} \tag{2.3..4}$$

for all $\lambda \in X_+$.

Lemma 2.3..3. *Let $\underline{w} = (w_1, \ldots, w_n)$, $w_i \in \{1, 2\}$, then*

$$V(\underline{w}) \cong \bigoplus_{(\mu_1, \ldots, \mu_n) \in E(\underline{w})} V(\mu_1 + \ldots + \mu_n). \tag{2.3..5}$$

Moreover, if we denote the multiplicity of $V(\lambda)$ as a summand of $V(\underline{w})$ by $[V(\underline{w}) : V(\lambda)]$, then

$$[V(\underline{w}) : V(\lambda)] = \# E(\underline{w}, \lambda). \tag{2.3..6}$$

Proof. If we begin with $V(\emptyset) = \mathbb{C}$ and tensor with $V(w_1)$, there is only one irreducible summand. This summand corresponds to the dominant weight in $\operatorname{wt} V(w_1)$, which we record as μ_1. Then we tensor $V(w_1)$ by $V(w_2)$ and note that $V(w_1) \otimes V(w_2)$ contains $V(\mu_1) \otimes V(w_2)$ as a summand. Choose a summand of $V(\mu_1) \otimes V(w_2)$. The chosen summand is isomorphic to $V(\mu_1 + \mu_2)$ for some weight $\mu_2 \in \operatorname{wt} V(w_2)$, and we record this choice of summand by the weight $\mu_2 \in \operatorname{wt} V(w_2)$.

Next, we tensor $V(w_1) \otimes V(w_2)$ by $V(w_3)$, observe that $V(w_1) \otimes V(w_2) \otimes V(w_3)$ contains a summand isomorphic to $V(\mu_1 + \mu_2) \otimes V(w_3)$, and choose a summand of this summand. The chosen summand is isomorphic to $V(\mu_1 + \mu_2 + \mu_3)$ and we record the choice by the weight $\mu_3 \in \operatorname{wt} V(w_3)$. Iterating this procedure, we end up with a sequence of weights $(\mu_1, \ldots, \mu_n)$, which is a dominant weight subsequence of $\underline{w}$, and a summand in $V(\underline{w})$ isomorphic to $V(\mu_1 + \ldots + \mu_n)$. Furthermore, all summands of $V(\underline{w})$ can be realized uniquely as the end result of the process we just described. $\square$

Lemma 2.3..4. *Let* $\underline{u} = (u_1, \ldots, u_n)$ *be a sequence with* $u_i \in \{1, 2\}$, *then*

$$\dim \operatorname{Hom}_{\mathfrak{sp}_4(\mathbb{C})}(V(\underline{w}), V(\underline{u})) = \sum_{\lambda \in X_+} [V(\underline{w}) : V(\lambda)][V(\underline{u}) : V(\lambda)]. \tag{2.3..7}$$

Proof. Thanks to Lemma (2.3..3), this is consequence of Schur's lemma. $\square$

2.4. Motivating the Light Ladder Algorithm

We outline a well-known construction of a basis of homomorphism spaces in the category $\mathbf{Fund}(\mathfrak{sp}_4(\mathbb{C}))$.

Suppose that $(\mu_1, \ldots, \mu_m) \in E(\underline{w}, \lambda)$. For $i = 1, \ldots, m$ there is a projection map $P_{(\mu_1, \ldots, \mu_i)} : V(w_1) \otimes \ldots \otimes V(w_i) \longrightarrow V(\mu_1 + \ldots + \mu_i)$. The map $P_{(\mu_1, \ldots, \mu_i)}$ is the projection $P_{(\mu_1, \ldots, \mu_{i-1})} : V(w_1) \otimes \ldots \otimes V(w_{i-1}) \longrightarrow V(\mu_1 + \ldots + \mu_{i-1})$ postcomposed with the projection $p_{\mu_i} : V(\mu_1 + \ldots + \mu_{i-1}) \otimes V(w_i) \longrightarrow V(\mu_1 + \ldots + \mu_i)$.

Let $(\nu_1, \ldots, \nu_n) \in E(\underline{v}, \lambda)$. Now, for $i = 1, \ldots, n$ there are inclusion maps $I^{(\nu_1, \ldots, \nu_i)} : V(\nu_1 + \ldots + \nu_i) \longrightarrow V(u_1) \otimes \ldots \otimes V(u_i)$. Composing the projection with the inclusion we get a map $I^{(\nu_1, \ldots, \nu_n)} \circ P_{(\mu_1, \ldots, \mu_m)} : V(\underline{w}) \longrightarrow V(\underline{u})$, factoring through $V(\lambda)$.

Since $[V(\lambda) : V(\underline{w})] = E(\underline{w}, \lambda)$ and $[V(\lambda) : V(\underline{u})] = E(\underline{u}, \lambda)$, the maps

$$\bigcup_{\substack{\lambda \in X_+ \\ (\mu_1,\ldots,\mu_m) \in E(\underline{w},\lambda) \\ (\nu_1,\ldots,\nu_n) \in E(\underline{u},\lambda)}} \{I^{(\nu_1,\ldots,\nu_n)} \circ P_{(\mu_1,\ldots,\mu_m)}\} \tag{2.4..1}$$

form a basis in $\mathrm{Hom}_{\mathfrak{sp}_4(\mathbb{C})}(V(\underline{w}), V(\underline{u}))$.

The maps $P_{(\mu_1,\ldots,\mu_n)}$ are built inductively out of the p_{μ_i}'s in a way that is analogous to how we will define light ladder diagrams in terms of elementary light ladder diagrams. The inclusion map $I^{(\nu_1,\ldots,\nu_n)} : V(\lambda) \longrightarrow V(\underline{u})$ is analogous to what we will call upside down light ladder diagrams. We will define double ladder diagrams as the composition of a light ladder diagram and an upside down light ladder diagram, in analogy with the $I \circ P$'s. Then our work will be to argue that double ladder diagrams are a basis.

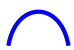
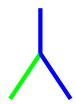
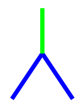
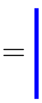
Remark 2.4..1. The projection and inclusion maps we discuss here are not the image of the light ladder diagrams under a functor $\mathcal{D}_{\mathfrak{sp}_4} \longrightarrow \mathbf{Fund}(\mathfrak{sp}_4(\mathbb{C}))$. There are at least two reasons for this. The first being that the object $V(\lambda)$ is not in the category $\mathbf{Fund}(\mathfrak{sp}_4(\mathbb{C}))$, so we have to construct light ladder maps not from $V(\underline{w})$ to $V(\lambda)$, but from $V(\underline{w})$ to $V(\underline{x})$ where $\mathrm{wt}\,\underline{x} = \lambda$.

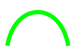
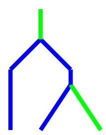
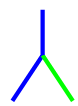
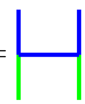
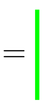
The second reason is that we want to construct a basis for the diagrammatic category which descends to a basis in **Fund** for fields other than $\mathbb{C}$. Over other fields the representation theory is no longer semisimple so $V(\lambda)$ may not be a summand of $V(\underline{w})$. There will still be the same number of maps from $V(\underline{w})$ to a suitable variant of $V(\lambda)$ but they may not be inclusions and projections.

2.5. Light Ladder Algorithm

Now we define some morphisms in the diagrammatic category.

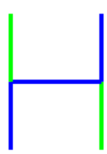
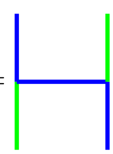
Definition 2.5..1. An **elementary light ladder diagram** is one of the following diagrams in $\mathcal{D}_{\mathfrak{sp}_4}$. We will say that L_μ is the elementary light ladder diagram of weight μ.

$L_{(-1,0)} =$  $L_{(1,-1)} =$  $L_{(-1,1)} =$  $L_{(1,0)} =$  (2.5..1)

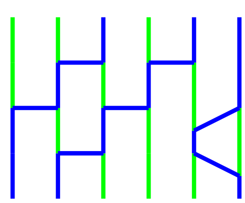
$L_{(0,-1)} =$  $L_{(-2,1)} =$  $L_{(0,0)} =$  $L_{(2,-1)} =$  $L_{(0,1)} =$  (2.5..2)

Remark 2.5..2. If $L_\mu : \underline{u}* \to \underline{w}$, for $* \in \{1, 2\}$, then $\mu \in \operatorname{wt} V(*)$ and $\operatorname{wt} \underline{w} = \operatorname{wt} \underline{u} + \mu$.

Definition 2.5..3. A **neutral diagram** is any diagram which is the horizontal and/or vertical composition of identity maps and the following **basic neutral diagrams**.

$N_{12}^{21} =$  $N_{21}^{12} =$  (2.5..3)

Example 2.5..4. A neutral diagram from 112221 to 221211.



(2.5..4)

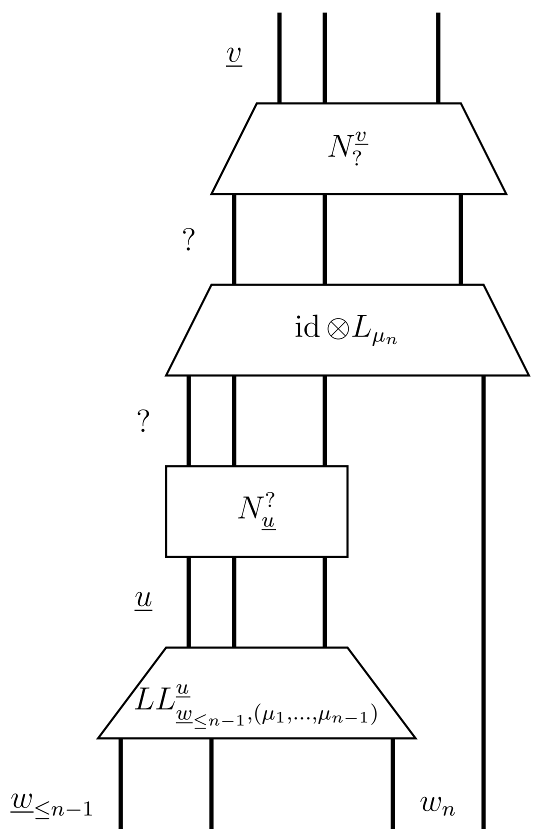
Definition 2.5..5. Fix an object $\underline{w} = (w_1, \ldots, w_n)$ in $\mathcal{D}_{\mathfrak{sp}_4}$, a dominant weight subsequence $\vec{\mu} = (\mu_1, \ldots, \mu_n) \in E(\underline{w})$, and an object $\underline{v} = (v_1, \ldots v_m)$ in $\mathcal{D}_{\mathfrak{sp}_4}$ such that $\operatorname{wt} \underline{v} = \mu_1 + \ldots + \mu_n$. We will describe an algorithm, which we will refer to as the **light ladder algorithm**, to construct a diagram in $\mathcal{D}_{\mathfrak{sp}_4}$ with source $\underline{w}$ and target $\underline{v}$. This diagram will be denoted $LL^{\underline{v}}_{\underline{w},\vec{\mu}}$ and we will call it a **light ladder diagram**.

We define the diagrams inductively, starting by defining $LL^{\emptyset}_{\emptyset,(\emptyset)}$ to be the empty diagram. Suppose we have constructed $LL^{\underline{u}}_{\underline{w}_{\leq n-1},(\mu_1,\ldots,\mu_{n-1})}$, where $\mathrm{wt}(\underline{u}) = \mu_1 + \ldots + \mu_{n-1}$. Then we define

$$LL^{\underline{v}}_{\underline{w},(\mu_1,\ldots,\mu_n)} = N^{\underline{v}}_{?} \circ (\mathrm{id} \otimes L_{\mu_n}) \circ \left(N^{?}_{\underline{u}} \otimes \mathrm{id}\right) \circ \left(LL^{\underline{u}}_{\underline{w}_{\leq n-1},(\mu_1,\ldots,\mu_{n-1})} \otimes \mathrm{id}_{w_n}\right) \quad (2.5..5)$$

where $N^{?}_{?}$ is a neutral diagram with appropriate source (subscript) and target (superscript).

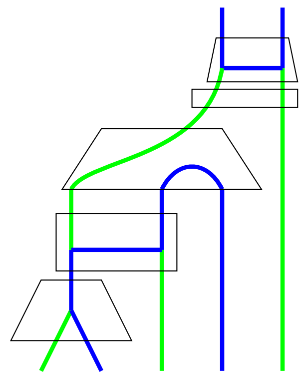
Example 2.5..6. A schematic for the inductive definition of a light ladder diagram $LL_{\underline{w},(\mu_1,\ldots,\mu_n)}$.



(2.5..6)

To further aid the readers understanding of the light ladder construction we give an example and some clarifying comments.

Example 2.5..7. A light ladder diagram $LL^{11}_{21212,((0,1),(1,-1),(0,1),(-1,0),(2,-1))}$

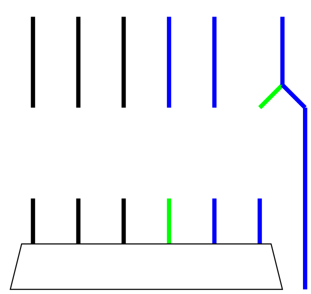


(2.5..7)

Our convention of rectangles and trapezoids is to indicate whether a diagram is a neutral diagram or a diagram of the form id $\otimes$ elementary light ladder diagram. We omitted the first and third steps corresponding to $\mu = (0, 1)$.

The elementary light ladder diagrams have fixed source and target. As a result one can construct $LL^{\underline{u}}_{\underline{w}_{\leq n-1},(\mu_1,\ldots,\mu_{n-1})}$, and then see that $V(\mu_n)$ is a summand of $V(\mu_1 + \ldots + \mu_{n-1}) \otimes V(w_n)$. The reader may wonder if there really is an object $\underline{y}$ in $\mathcal{D}_{\mathfrak{sp}_4}$ such that $\underline{y}w_n$ is the source of id$\otimes L_{\mu_n}$.

Example 2.5..8. An example of what can go wrong without neutral diagrams.



(2.5..8)

Basic neutral diagrams encode isomorphisms $12 \to 21$ and $21 \to 12$, while arbitrary neutral diagrams encode isomorphisms $\underline{w} \to \underline{w}'$.

Remark 2.5..9. The reason we use basic neutral diagrams instead of the braiding is the latter is a non-trivial linear combination of diagrams in $\mathcal{D}_{\mathfrak{sp}_4}$, while the former is a single diagram in $\mathcal{D}_{\mathfrak{sp}_4}$.

Lemma 2.5..10. *Given two sequences $\underline{w}$ and $\underline{w}'$ such that $\operatorname{wt}\underline{w} = \operatorname{wt}\underline{w}'$, there is a neutral diagram connecting $\underline{w}$ to $\underline{w}'$.*

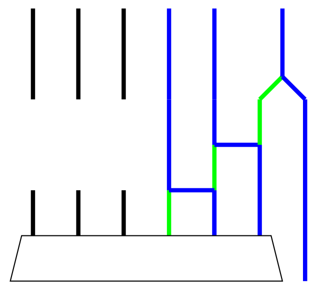
Proof. Suppose that $\operatorname{wt}\underline{w} = a\varpi_1 + b\varpi_2 = \operatorname{wt}\underline{w}'$. Connect both $\underline{w}$ and $\underline{w}'$ via colored neutral diagrams to the standard sequence $1^{\otimes a} \otimes 2^{\otimes b}$ and then compose the neutral diagram from $\underline{w}$ to the standard sequence with the vertical flip of the neutral diagram from the standard diagram to $\underline{w}'$. $\square$

The following lemma uses this observation to fix the problem, in the light ladder algorithm, of elementary diagrams having fixed source and target.

Lemma 2.5..11. *Let $(\mu_1, \ldots, \mu_n) \in E(\underline{w})$ (in particular, $V(\mu_n)$ is a summand of $V(\mu_1 + \ldots + \mu_{n-1}) \otimes V(w_n)$). Suppose we have constructed $LL^{\underline{u}}_{\underline{w}_{\leq n-1},(\mu_1,\ldots,\mu_{n-1})}$. There is an object $\underline{y}$ in $\mathcal{D}_{\mathfrak{sp}_4}$ and a neutral map $N^{\underline{y}}_{\underline{w}_{\leq n-1}}$ such that $\underline{y} \otimes w_n$ is the source of $\operatorname{id} \otimes L_{\mu_n}$.*

Proof. We will argue this for the elementary diagram $L_{(1,-1)}$, so $\mu_n = (1,-1)$ and $w_n = 1$. The arguments for the rest of the cases follow the same pattern. From the tensor product decomposition formulas (2.3..3) we see that $V(1,-1)$ being a summand of $V(\mu_1 + \ldots + \mu_{n-1})$ implies that, if $\mu_1 + \ldots + \mu_{n-1} = a\varpi_1 + b\varpi_2$, then $b \geq 1$. Thus, in the sequence $\underline{u} = (u_1, \ldots, u_k)$ there is some k such that $u_k = 2$. By Lemma (2.5..10) there is a neutral diagram from the sequence $\underline{u}$ to a sequence which ends in 2. The target of this neutral diagram will be an object $\underline{y}$ such that $\underline{y} \otimes 1$ is the source of $\mathrm{id} \otimes L_{(1,-1)}$. □

Example 2.5..12. Using a neutral map to fix the problem.



(2.5..9)

Comparing the tensor product decompositions in (2.3..3) with the elementary light ladder diagrams it is apparent that dominant weight subsequences always produce a light ladder diagram. However, neutral diagrams from one word to another are not unique. The choice of neutral diagram could result in several different light ladder diagrams for a given dominant weight subsequence.

Remark 2.5..13. For any $\underline{w}$ and $\underline{u}$ such that $\mathrm{wt}\, \underline{w} = \mathrm{wt}\, \underline{u}$, there is a distinguished choice of neutral diagram corresponding to the minimal coset representative in the symmetric group realizing the shuffle from one sequence to the other. However, we do not require that we choose particular elements as our neutral diagrams in the light ladder algorithm.

2.6. Double Ladders

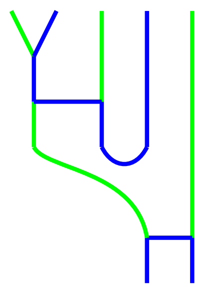
We define a contravariant endofunctor $\mathbb{D}$ on the category $\mathcal{D}_{\mathfrak{sp}_4}$ by requiring that $\mathbb{D}$ fixes objects and turns diagrams upside down. Note that $\mathbb{D}^2 = \mathrm{id}_{\mathcal{D}_{\mathfrak{sp}_4}}$, so $\mathbb{D}$ is a duality (i.e. a contravariant functor $\mathcal{D}_{\mathfrak{sp}_4} \longrightarrow \mathcal{D}_{\mathfrak{sp}_4}$, which is an equivalence and squares to id) on the category.

Definition 2.6..1. Let $LL^{v}_{\underline{w},\vec{\mu}}$ be a light ladder diagram. The associated **upside down light ladder diagram** is defined to be

$$\mathbb{D}(LL^{v}_{\underline{w},\vec{\mu}}). \tag{2.6..1}$$

Example 2.6..2. An upside down light ladder diagram

$$\mathbb{D}(LL^{121}_{112121,((1,0),(1,0),(-2,1),(1,0),(2,-1),(-1,1))}).$$



(2.6..2)

For each dominant weight λ fix a word $\underline{x}_\lambda$ in the alphabet $\{1, 2\}$ corresponding to a sequence of fundamental weights which sum to λ. For all words $\underline{w}$ and for each dominant weight subsequence $\vec{\mu} \in E(\underline{w}, \lambda)$, we choose one light ladder diagram from $\underline{w}$ to $\underline{x}_\lambda$. If $\underline{w} = \underline{x}_\lambda$ and each μ_i is dominant, then we choose the identity diagram. From now on we denote this chosen light ladder diagram by $L_{\underline{w},\vec{\mu}}$.

Remark 2.6..3. The choice of $LL_{\underline{x}_\lambda,\vec{\lambda}} = \mathrm{id}_{\underline{x}_\lambda}$ when the λ_i are all dominant is not essential for our arguments, but does ensure our construction is aligned with other conventions. For example this is required in the definition of an object adapted cellular category in [21].

Definition 2.6..4. If $\underline{w}$ and $\underline{u}$ are fixed words in $\{1, 2\}$ and λ is a dominant weight, then for $\vec{\mu} \in E(\underline{w}, \lambda)$ and $\vec{\nu} \in E(\underline{u}, \lambda)$ we obtain a **double ladder diagram**

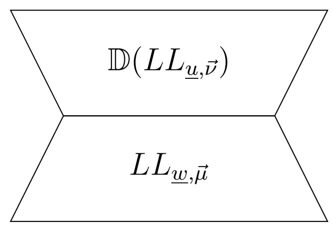
(associated to our choices of $\underline{x}_\lambda$'s and our choices of light ladder diagrams)

$$\mathbb{LL}_{\underline{w},\vec{\mu}}^{\underline{u},\vec{\nu}} = \mathbb{D}(LL_{\underline{u},\vec{\nu}}) \circ LL_{\underline{w},\vec{\mu}}. \tag{2.6..3}$$

Remark 2.6..5. One reason for fixing an $\underline{x}_\lambda$ for all λ is so the composition on the right hand side of (2.6..3) is well defined.

Example 2.6..6. A schematic for a double ladder diagram:

$$\mathbb{LL}_{\underline{w},\vec{\mu}}^{\underline{u},\vec{\nu}} =$$



(2.6..4)

Remark 2.6..7. Note that light ladder diagrams ending in $\underline{x}_\lambda$ are double ladder diagrams, where the upside-down light ladder happens to be the identity diagram.

Definition 2.6..8. We define the set of all double ladder diagrams from $\underline{w}$ to $\underline{u}$ factoring through λ (associated to our choice of $\underline{x}_\lambda$'s and light ladder diagrams) to be

$$\mathbb{LL}_{\underline{w}}^{\underline{u}}(\lambda) = \left\{ \mathbb{LL}_{\underline{w},\vec{\mu}}^{\underline{u},\vec{\nu}} \ : \ \vec{\mu} \in E(\underline{w}, \lambda), \ \vec{\nu} \in E(\underline{u}, \lambda) \right\}, \tag{2.6..5}$$

and define the set of all double ladder diagrams from $\underline{w}$ to $\underline{u}$ (associated to our choice of $\underline{x}_\lambda$'s and light ladder diagrams) to be

$$\mathbb{LL}_{\underline{w}}^{\underline{u}} = \bigcup_{\lambda \in X_+} \mathbb{LL}_{\underline{w}}^{\underline{u}}(\lambda) \tag{2.6..6}$$

Remark 2.6..9. Anytime we write $\mathbb{LL}_{\underline{w},\vec{\mu}}^{\underline{u},\vec{\nu}}$ or $\mathbb{LL}_{\underline{w}}^{\underline{u}}$, we have already fixed choices of $\underline{x}_\lambda$'s and choices of light ladder diagrams. The notation does not account for these choices, but we will not be comparing double ladders for different choices so the notation should not lead to confusion.

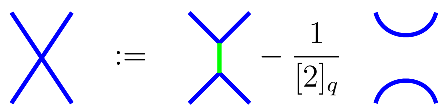
2.7. Relating Non-Elliptic Webs to Double Ladders

Our next goal is to define an evaluation functor from $\mathcal{D}_{\mathfrak{sp}_4}$ to the category $\mathbf{Fund}(\mathfrak{sp}_4)$, and then to prove that the functor is an equivalence. That the functor is an equivalence will follow from showing that double ladder diagrams span the category $\mathcal{D}_{\mathfrak{sp}_4}$, and map to a set of linearly independent morphisms in $\mathbf{Fund}(\mathfrak{sp}_4)$. This approach is modeled on the work on type A webs in [7], where most of the work goes into showing that double ladder diagrams span the diagrammatic category. Checking linear independence is comparatively easy once you know the functor explicitly. But for $\mathcal{D}_{\mathfrak{sp}_4}$, the extra work to show double ladders span can be circumvented by bootstrapping known results about B_2 webs which we recall below.

Kuperberg's paper [31, pp. 14-15] introduces a tetravalent vertex in the B_2 web category which can be used to remove all internal double edges. Let $\mathbf{B}$ be the set of B_2 diagrams with no internal double edges and with no faces having one, two, or three adjacent edges. These diagrams are called non-elliptic in [31]. There are local relations in the B_2 category (now including the tetravalent vertex) which can be used to reduce triangular faces, bigons, monogons, and circles to sums of diagrams with fewer crossings (i.e. $\mathbf{B}$ is the set of irreducible webs with respect to the relations). It follows that the set $\mathbf{B}$ spans the B_2 category over $\mathbb{Z}[q, q^{-1}]$. Let $\mathbf{B}_{\underline{w}}$ be the set of diagrams in $\mathbf{B}$ with $\underline{w}$ on the boundary. One of the main results of [31] is that

$$\#\mathbf{B}_{\underline{w}} = \dim V(\underline{w})^{\mathfrak{sp}_4(\mathbb{C})}. \tag{2.7..1}$$

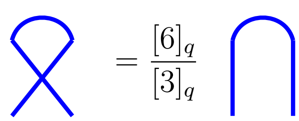
Notation 2.7..1. If we work in the $\mathcal{A}$-linear category $\mathcal{D}_{\mathfrak{sp}_4}$, there is an analogous 90 degree rotation invariant morphism, which we will call the tetravalent vertex, in $\mathrm{End}_{\mathcal{D}_{\mathfrak{sp}_4}}(11)$.



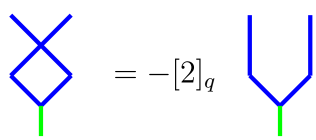
(2.7..2)

There is an augmented graphical calculus in which the generating diagrams are the cups, caps, and trivalent vertices in the definition of $\mathcal{D}_{\mathfrak{sp}_4}$ along with the tetravalent vertex (2.7..2). For the remainder of this section when we say a diagram in $\mathcal{D}_{\mathfrak{sp}_4}$ we mean a diagram in the augmented graphical calculus.

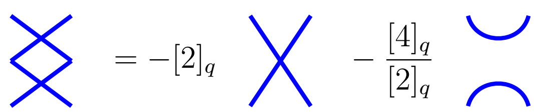
Since $[2]_q$ is invertible in our ground ring, we can use this tetravalent vertex to remove all internal green labelled edges in any diagram in $\mathcal{D}_{\mathfrak{sp}_4}$. The tetravalent vertex satisfies the following relations in $\mathcal{D}_{\mathfrak{sp}_4}$



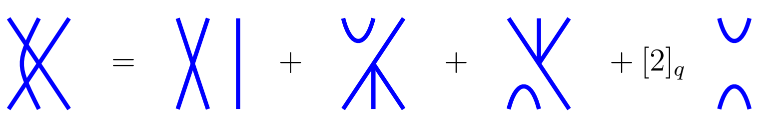
(2.7..3)



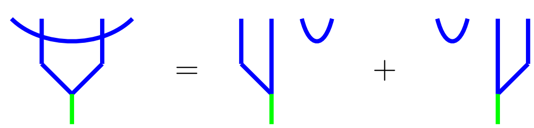
(2.7..4)



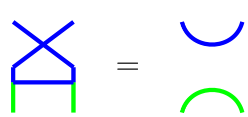
(2.7..5)



(2.7..6)



(2.7..7)



(2.7..8)

Remark 2.7..2. Due to the identity $[2n]_q/[n]_q = [n+1]_q - [n-1]_q$, the coefficients in these relations all lie in the ring $\mathbb{Z}[q, q^{-1}]$.

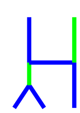
Definition 2.7..3. A **closed component** of a diagram in $\mathcal{D}_{\mathfrak{sp}_4}$ is a connected component of the diagram which does not touch the boundary. A **circle component** in a diagram in $\mathcal{D}_{\mathfrak{sp}_4}$ is a closed component which contains no vertices.

Definition 2.7..4. A **face** of a diagram in $\mathcal{D}_{\mathfrak{sp}_4}$ is a simply connected component of the complement of the diagram, which does not touch the boundary.

Definition 2.7..5. A **non-elliptic diagram** in $\mathcal{D}_{\mathfrak{sp}_4}$ is a diagram such that all faces have more than three sides (i.e a diagram with no triangular faces, bigons, monogons, or circles).

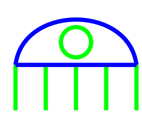
Definition 2.7..6. An **internal 2 edge** of a diagram in $\mathcal{D}_{\mathfrak{sp}_4}$ is a 2 edge in the diagram which does not connect to the boundary.

Example 2.7..7. An example of a non-elliptic web with internal 2 edges in $\mathcal{D}_{\mathfrak{sp}_4}$. This diagram is a light ladder for the dominant weight sequence $((1,0),(-1,1),(1,0))$.



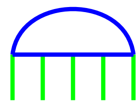
(2.7..9)

Example 2.7..8. An example of an elliptic web with internal 2 edges in $\mathcal{D}_{\mathfrak{sp}_4}$. The only face is the interior of the 2 circle.



(2.7..10)

Example 2.7..9. An example of a non-elliptic web with no internal 2 edges in $\mathcal{D}_{\mathfrak{sp}_4}$. There is only one face and it has five sides.



(2.7..11)

Definition 2.7..10. The set $\mathbf{D}$ is the collection of all non-elliptic diagrams in $\mathcal{D}_{\mathfrak{sp}_4}$ with no internal 2 edges, and the set $\mathbf{D}_{\underline{w}}^{\underline{u}}$ is the set of diagrams in $\mathbf{D} \cap \mathrm{Hom}_{\mathcal{D}_{\mathfrak{sp}_4}}(\underline{w}, \underline{u})$.

Lemma 2.7..11. *Let D be a diagram in $\mathcal{D}_{\mathfrak{sp}_4}$ with no internal 2 edges. Then D is in the span of $\boldsymbol{D}$.*

Proof. Given a diagram X with no internal 2 edges, write $|X|_v$ to denote the total number of 3-valent and 4-valent vertices in X, and write $|X|_c$ to denote the total number of circle components.

We leave it as an exercise to classify the possible faces with less than four sides in a diagram with no internal 2 edges, verifying the following claims. Any face with one edge will be one of the left hand side of Equation (2.2..10), (2.2..12), or (2.7..3). Any face with two edges will be one of the left hand side of Equation (2.2..13), (2.7..4), or (2.7..5). Any face with three edges will be one of the left hand side of Equation (2.2..14), (2.7..6), (2.7..7), or (2.7..8). Note that applying these relations to a diagram with no internal 2 edges will result in a linear combination of diagrams with no internal 2 edges.

Given two diagrams X and Y with no internal 2 edges, we write $X \to Y$ if there is a face with less than or equal to three sides in X, and applying one of the above relations to remove the face from X results in a linear combination of diagrams, one of which is Y. Observe that if $X \to Y$, then $|X|_v \geq |Y|_v$, with equality if and only if we removed a circle using Equation (2.2..10), but in that case $|X|_c > |Y|_c$.

Let $S(k,m)$ be the statement: "If A is a diagram in $\mathcal{D}_{\mathfrak{sp}_4}$ with no internal 2 edges, at most k vertices, and at most m circle components, then A is in the span of $\mathbf{D}$."

We claim that $S(k,m)$ implies $S(k,m+1)$. To see this, assume $S(k,m)$ is true and let A be a diagram with $|A|_v \leq k$ and $|A|_c \leq m+1$. If $|A|_c < m+1$, then we are done. Suppose $|A|_c = m+1$. Our assumptions imply that there is at least one circle, denoted O, in A. Let S be the subdiagram of A which is contained in the region bounded by O. Note that $|S|_v \leq |A|_v \leq k$ and $|S|_c < |A|_c = m+1$. Since we assumed $S(k,m)$ is true, it follows that S is in the span of $\mathbf{D}$. Also, S is contained in the region bounded by O so S has empty boundary. Since the only diagram in $\mathbf{D}$ with empty boundary is the empty diagram, it follows that S is equal to a scalar multiple of the empty diagram. Thus, A is a scalar multiple of a diagram B, such that B is the same outside of O, but now has O bounding a face. We can apply Equation (2.2..10) to remove O from B and obtain a diagram C such that B is a scalar multiple of C. Thus, A is a scalar multiple of C, and $|C|_v = |B|_v \leq |A|_v = k$ and $|C|_c = |B|_c - 1 \leq |A|_c - 1 = m$. By our inductive hypothesis, C is in the span of $\mathbf{D}$, and therefore A is in the span of $\mathbf{D}$.

We claim that $S(k,m)$ for all m implies $S(k+1,0)$. Assume that $S(k,m)$ is true for all m, and let A be a diagram with no internal 2 edges, such that $|A|_v = k+1$ and $|A|_c = 0$. If A all of A's faces have more than three sides (this includes the case of A having no faces), then $A \in \mathbf{D}$ and we are done. Suppose that A has a face with less than or equal to three sides. Since $|A|_c = 0$, it must be possible to remove this face using one of the relations discussed in the first paragraph other than Equation (2.2..10) i.e. "circle removal". Therefore, A is a linear combination of diagrams: $A = \sum \xi_i B_i$, for some scalars ξ_i. For each B_i, since $A \to B_i$, and since we did not use Equation (2.2..10), we have $|B_i|_v < |A|_v = k+1$ (although it is possible that $|B_i|_c > 0$). Since we assume $S(k,m)$ for all m, each B_i may be expressed as a linear combination of diagrams in $\mathbf{D}$. Thus, A is in the span of $\mathbf{D}$.

To see that $S(0,0)$ is true, observe that a diagram with no vertices or circles must necessarily have no faces (in fact this is just the empty diagram). By induction it follows that $S(k,m)$ is true for all m and k. Thus, we may conclude that D can be expressed as a linear combination of diagrams in $\mathbf{D}$. $\square$

Lemma 2.7..12. *The set $\boldsymbol{D}$ spans $\mathcal{D}_{\mathfrak{sp}_4}$ over $\mathcal{A}$.*

Proof. Let D be an arbitrary diagram in $\mathcal{D}_{\mathfrak{sp}_4}$. We will argue that D is a linear combination of non-elliptic webs with no internal 2 edges. If a 2 edge does not connect to a trivalent vertex, then you can use the bigon relation to introduce one. Thus, every 2 edge either connects to the boundary of D, or connects two trivalent vertices. Using the tetravalent vertex to remove all pairs of trivalent vertices, we can rewrite D as a linear combination of diagrams with no internal 2 edges. Thus, we may assume that D is a diagram with no internal 2 edges, and the claim follows from Lemma (2.7..11). $\square$

Remark 2.7..13. In order to introduce a trivalent vertex, we used the bigon relation backwards, which required $[2]_q^{-1} \in \mathcal{A}$.

Lemma 2.7..14. *Let $\mathbf{k}$ be a field and let $q \in \mathbf{k}^\times$ be such that $q + q^{-1} \neq 0$. Then*

$$\dim \operatorname{Hom}_{\mathbf{k}\otimes\mathcal{D}_{\mathfrak{sp}_4}}(\underline{w}, \underline{u}) \leq \dim \operatorname{Hom}_{\mathfrak{sp}_4(\mathbb{C})}(V(\underline{w}), V(\underline{u})). \tag{2.7..12}$$

Proof. There is an obvious bijection between the set **B** and the set **D**. The result then follows from Equation (2.7..1). $\square$

Remark 2.7..15. We sketch a more direct argument to deduce the inequality (2.7..12). The dimension of the $\mathfrak{sp}_4(\mathbb{C})$ invariants in $V(1)^{\otimes 2n}$ is known to be equal to the number of matchings of $2n$ points on the boundary of a disc such that there is no 6-point star in the matching [47][31, 8.4]. One can argue that the local condition of being non-elliptic implies the global condition of having no six point star. Then, noting that non-elliptic diagrams have a unique representative up to isotopy (there are no potential Reidemeister moves), it follows that there is a bijection between non-elliptic diagrams and matchings without a 6-point star. This proves that the inequality (2.7..12) holds when $\underline{w} = 1^{\otimes a}$ and $\underline{u} = 1^{\otimes b}$ for some $a, b \in \mathbb{Z}_{\geq 0}$. Since 2 is a direct summand of 11 it follows that (2.7..12) holds for any words $\underline{w}$ and $\underline{u}$ in the alphabet $\{1, 2\}$.

We have defined a set $\mathbb{LL}_{\underline{w}}^{\underline{u}}$ of double ladders in $\mathcal{D}_{\mathfrak{sp}_4}$. It follows from the construction of $\mathbb{LL}_{\underline{w}}^{\underline{u}}$ and (2.3..7) that

$$\#\mathbb{LL}_{\underline{w}}^{\underline{u}} = \sum_{\lambda \in X_+} \#E(\underline{w}, \lambda)\#E(\underline{u}, \lambda) = \dim \mathrm{Hom}_{\mathfrak{sp}_4(\mathbb{C})}(V(\underline{w}), V(\underline{u})). \tag{2.7..13}$$

We want to show linear independence of the set of double ladders, or equivalently that the inequality of dimensions in (2.7..12) is in fact an equality, for a general choice of base ring $\mathbf{k}$. To this end we will define an evaluation functor from the diagrammatic category $\mathcal{D}_{\mathfrak{sp}_4}^{\mathbf{k}}$ to the representation theoretic category $\mathbf{Fund}(\mathbf{k} \otimes U_q^{\mathcal{A}}(\mathfrak{sp}_4))$, and interpret the image of the evaluation functor in terms of tilting modules. If we can show that the image of the double ladder diagrams under the evaluation functor is a linearly independent set, then the double ladder diagrams must be linearly independent in $\mathcal{D}_{\mathfrak{sp}_4}$ (a dependence relation in $\mathcal{D}_{\mathfrak{sp}_4}$ would map to a dependence relation). This implies that the inequality in (2.7..12) is an equality, and it follows that the evaluation functor maps bases to bases, so is fully faithful.

Remark 2.7..16. Since $\mathbf{D}$ spans $\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}$ and is in a non-canonical bijection with the set of double ladder diagrams (for fixed choices of $\underline{x}_\lambda$ and fixed choices of light ladders), linear independence of the double ladder diagrams over $\mathbf{k}$ implies that both sets are bases.

Note that double ladders have many internal 2 label edges while the diagrams in $\mathbf{D}$ will have none. On the other hand, sometimes the double ladder diagrams will be non-elliptic webs with no internal 2 edges. A good exercise for the reader is to rewrite the diagram in Example (2.7..9) as a double ladder diagram. A hint is that a double ladder diagram in $\mathrm{Hom}_{\mathcal{D}_{\mathfrak{sp}_4}}(2^{\otimes 5}, \emptyset)$ will just be a light ladder diagram $LL^{\emptyset}_{2^{\otimes 5},?}$.

2.8. Defining the Evaluation Functor on Objects

We are now going to be more precise about what category of representations associated to $\mathfrak{sp}_4$ we are considering. The discussion below is well-known, but we reproduce it here to help the reader follow certain calculations which come later.

Our main reference for quantum groups is Jantzen's book [25]. Recall that $\mathfrak{sp}_4(\mathbb{C})$ gives rise to a root system Φ and a Weyl group W. We choose simple roots $\Delta = \{\alpha_s = \epsilon_1 - \epsilon_2, \alpha_t = 2\epsilon_2\}$. There is a unique W invariant symmetric form $(-,-)$ on the root lattice $\mathbb{Z}\Phi$ such that the short roots pair with themselves to be 2. This is the form $(\epsilon_i, \epsilon_j) = \delta_{ij}$, restricted to the root lattice. For $\alpha \in \Phi$ we define the coroot $\alpha^\vee = 2\alpha/(\alpha,\alpha)$, in particular $\alpha_s^\vee = \alpha_s$ and $\alpha_t^\vee = \alpha_t/2$ and the Cartan matrix

$((\alpha_i^\vee, \alpha_j))$ is

$$\begin{pmatrix} \alpha_s^\vee(\alpha_s) & \alpha_s^\vee(\alpha_t) \\ \alpha_t^\vee(\alpha_s) & \alpha_t^\vee(\alpha_t) \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}.$$

Define the algebra $U_q(\mathfrak{sp}_4)$ as the $\mathbb{Q}(q)$ algebra given by generators

$$F_s, F_t, K_s^{\pm 1}, K_t^{\pm 1} E_s, E_t$$

and relations

- $K_s K_s^{-1} = 1 = K_s K_s^{-1}, K_t K_t^{-1} = 1 = K_t^{-1} K_t, K_s K_t = K_t K_s$
- $K_t E_t = q^4 E_t K_t, K_t E_s = q^{-2} E_s K_t$
- $K_s E_t = q^{-2} E_t K_s, K_s E_s = q^2 E_s K_s$
- $K_t F_t = q^{-4} F_t K_t, K_t F_s = q^2 F_s K_t$
- $K_s F_t = q^2 F_t K_s, K_s F_s = q^{-2} F_s K_s$
- $E_t F_s = F_s E_t, E_s F_t = F_t E_s$
- $E_t F_t = F_t E_t + \dfrac{K_t - K_t^{-1}}{q^2 - q^{-2}}$
- $E_s F_s = F_s E_s + \dfrac{K_s - K_s^{-1}}{q - q^{-1}}$
- $E_t^2 E_s - \dfrac{[4]_q}{[2]_q} E_t E_s E_t + E_s E_t^2 = 0$
- $E_s^3 E_t - [3]_q E_s^2 E_t E_s + [3]_q E_s E_t E_s^2 - E_s E_t^3$

Our convention is $[n]_q := \dfrac{q^n - q^{-n}}{q - q^{-1}}$ and $[n]_q! = [n]_q [n-1]_q \dots [2]_q [1]_q$.

Recall that $\mathcal{A} = \mathbb{Z}[q, q^{-1}, [2]_q^{-1}]$. Let $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$ be the unital $\mathcal{A}$-subalgebra of $U_q(\mathfrak{sp}_4)$ spanned by $K_s^{\pm 1}, K_t^{\pm 1}$, and the divided powers

$$E_s^{(n)} = \frac{E_s^n}{[n]_q!}, F_s^{(n)} = \frac{F_s^n}{[n]_q!}, E_t^{(n)} = \frac{E_t}{[n]_{q^2}!}, F_t^{(n)} = \frac{F_t}{[n]_{q^2}!}$$

for all $n \in \mathbb{Z}_{\geq 1}$. So $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$ is Lusztig's divided powers quantum group [3].

Let $V^{\mathcal{A}}(\varpi_1)$ denote the free $\mathcal{A}$ module with basis

$$v_{(1,0)}, v_{(-1,1)}, v_{(1,-1)}, v_{(-1,0)}, \tag{2.8..1}$$

and action of $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$ given by:

$$v_{(-1,0)} \underset{F_s=1}{\overset{E_s=1}{\rightleftarrows}} v_{(1,-1)} \underset{F_t=1}{\overset{E_t=1}{\rightleftarrows}} v_{(-1,1)} \underset{F_s=1}{\overset{E_s=1}{\rightleftarrows}} v_{(1,0)}. \tag{2.8..2}$$

Also, let $V^{\mathcal{A}}(\varpi_2)$ denote the free $\mathcal{A}$ module with basis

$$v_{(0,1)}, v_{(2,-1)}, v_{(0,0)}, v_{(-2,1)}, v_{(0,-1)}, \tag{2.8..3}$$

and action of $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$ given by:

$$v_{(0,-1)} \underset{F_t=1}{\overset{E_t=1}{\rightleftarrows}} v_{(-2,1)} \underset{F_s\,=\,[2]_q}{\overset{E_s=1}{\rightleftarrows}} v_{(0,0)} \underset{F_s=1}{\overset{E_s=[2]_q}{\rightleftarrows}} v_{(2,-1)} \underset{F_t=1}{\overset{E_t=1}{\rightleftarrows}} v_{(0,1)}. \tag{2.8..4}$$

The elements K_α act on the basis vectors by

$$K_s \cdot v_{(i,j)} = q^i v_{(i,j)} \quad \text{and} \quad K_t \cdot v_{(i,j)} = q^{2j} v_{(i,j)}. \tag{2.8..5}$$

Our convention is that whenever we do not indicate the action of E_α or F_α they act by zero. The action of higher divided powers on these modules can be extrapolated from the given data. For example, $F_s^{(2)} v_{(2,-1)} = v_{(-2,1)}$.

Remark 2.8..1. Why are we using $\mathcal{A}$ instead of $\mathbb{Z}[q, q^{-1}]$? When $[2]_q = 0$, the Weyl module $\mathbf{k} \otimes V^{\mathcal{A}}(\varpi_2)$ is not irreducible and the correct choice of combinatorial category seems to be the $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-linear monoidal category generated by V_q and $\Lambda^2(V_q)$. The module $\Lambda^2(V_q)$ has the $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-basis

$$v_{(0,1)}, v_{(2,-1)}, X_0, Y_0, v_{(-2,1)}, v_{(0,-1)}, \tag{2.8..6}$$

and action of $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$ given by:

$$v_{(0,-1)} \underset{F_t=1}{\overset{E_t=1}{\rightleftarrows}} v_{(-2,1)} \underset{F_s}{\overset{E_s}{\rightleftarrows}} X_0 \oplus Y_0 \underset{F_s}{\overset{E_s}{\rightleftarrows}} v_{(2,-1)} \underset{F_t=1}{\overset{E_t=1}{\rightleftarrows}} v_{(0,1)}. \tag{2.8..7}$$

where

$$\begin{gathered} E_s \cdot Y_0 = q^{-1/2} v_{(2,-1)} \qquad E_s \cdot X_0 = q^{1/2} v_{(2,-1)} \\ E_s \cdot v_{(-2,1)} = q^{1/2} X_0 + q^{-1/2} Y_0 \\ F_s \cdot Y_0 = q^{-1/2} v_{(-2,1)} \qquad F_s \cdot X_0 = q^{1/2} v_{(-2,1)} \\ F_s \cdot v_{(2,-1)} = q^{1/2} X_0 + q^{-1/2} Y_0. \end{gathered} \tag{2.8..8}$$

The module $V^{\mathcal{A}}(\varpi_2)$ can be defined over $\mathbb{Z}[q^{1/2}, q^{-1/2}]$. There is a map from $V^{\mathcal{A}}(\varpi_2)$ into $\Lambda^2(V_q)$, such that $v_{(0,0)} \mapsto q^{1/2} X_0 + q^{-1/2} Y_0$. Moreover, the cokernel of this inclusion map will be isomorphic to the trivial module. Thus, $\Lambda^2(V_q)$ is filtered by Weyl modules, and the filtration splits when $[2]_q \neq 0$. If $[2]_q = 0$, then $\Lambda^2(V_q)$ is indecomposable with socle and head isomorphic to the trivial module, and middle subquotient isomorphic to the irreducible module of highest weight ϖ_2.

The algebra $U_q(\mathfrak{sp}_4)$ is a Hopf algebra with structure maps (Δ, S, ϵ) defined on generators by

- $\Delta(E_\alpha) = E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha, \Delta(F_\alpha) = F_\alpha \otimes K_\alpha^{-1} + 1 \otimes F_\alpha, \Delta(K_\alpha) = K_\alpha \otimes K_\alpha$
- $S(E_\alpha) = -K_\alpha^{-1} E_\alpha, S(F_\alpha) = -F_\alpha K_\alpha, S(K_\alpha) = K_\alpha^{-1}$
- $\epsilon(E_\alpha) = 0, \epsilon(F_\alpha) = 0, \epsilon(K_\alpha) = 1.$

Furthermore, the algebra $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$ is a sub-Hopf-algebra of $U_q(\mathfrak{sp}_4)$ [3]. Therefore, $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$ will act on the tensor product of representations through the coproduct Δ.

Using the antipode S, we can define an action of $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$ on

$$V^{\mathcal{A}}(\varpi_1)^* = \mathrm{Hom}_{\mathcal{A}}(V^{\mathcal{A}}(\varpi_1), \mathcal{A}) \tag{2.8..9}$$

by

$$-q^4 v^*_{(1,0)} \underset{F_s=1}{\overset{E_s=1}{\rightleftarrows}} q^3 v^*_{(-1,1)} \underset{F_t=1}{\overset{E_t=1}{\rightleftarrows}} -q v^*_{(1,-1)} \underset{F_s=1}{\overset{E_s=1}{\rightleftarrows}} v^*_{(-1,0)}, \tag{2.8..10}$$

and on

$$V^{\mathcal{A}}(\varpi_2)^* = \mathrm{Hom}_{\mathcal{A}}(V^{\mathcal{A}}(\varpi_2), \mathcal{A}) \tag{2.8..11}$$

by

$$q^6 v^*_{(0,-1)} \underset{F_t=1}{\overset{E_t=1}{\rightleftarrows}} -q^4 v^*_{(-2,1)} \underset{F_s\,=\,[2]_q}{\overset{E_s=1}{\rightleftarrows}} q^2[2]_q v^*_{(0,0)} \underset{F_s=1}{\overset{E_s=[2]_q}{\rightleftarrows}} -q^2 v^*_{(2,-1)} \underset{F_t=1}{\overset{E_t=1}{\rightleftarrows}} v^*_{(0,1)}. \tag{2.8..12}$$

Comparing (2.8..2) and (2.8..10) we see there is an isomorphism of $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$ modules

$$\varphi_1 : V^{\mathcal{A}}(\varpi_1) \to V^{\mathcal{A}}(\varpi_1)^* \tag{2.8..13}$$

such that basis elements in (2.8..2) are sent to the basis elements in (2.8..10). By comparing (2.8..4) and (2.8..12) we similarly obtain an isomorphism

$$\varphi_2 : V^{\mathcal{A}}(\varpi_2) \to V^{\mathcal{A}}(\varpi_2)^* \tag{2.8..14}$$

sending basis elements in (2.8..4) to the basis elements in (2.8..12).

In Section 2.11. we will define a monoidal functor from $\mathcal{D}_{\mathfrak{sp}_4}$ to $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$−mod. The functor will send 1 to $V^{\mathcal{A}}(\varpi_1)$ and 2 to $V^{\mathcal{A}}(\varpi_2)$. The dual modules $V^{\mathcal{A}}(\varpi_1)^*$ and $V^{\mathcal{A}}(\varpi_2)^*$ will not be in the image of the functor Ξ. However, the maps φ_1 and φ_2 are fixed isomorphisms of these dual modules with modules which are in the image of the functor.

2.9. Caps and Cups

Lemma 2.9..1. *If V is any finite rank $\mathcal{A}$ lattice with basis e_i, define maps:*

$$\mathcal{A} \xrightarrow{u} V \otimes \mathrm{Hom}_{\mathcal{A}}(V, \mathcal{A}) \xrightarrow{c} \mathcal{A} \tag{2.9..1}$$

$$\mathcal{A} \xrightarrow{u'} \mathrm{Hom}_{\mathcal{A}}(V, \mathcal{A}) \otimes V \xrightarrow{c'} \mathcal{A} \tag{2.9..2}$$

where $u(1) = \sum e_i \otimes e_i^*$, $u'(1) = \sum e_i^* \otimes e_i$, $c(v \otimes f) = f(v)$, and $c'(f \otimes v) = f(v)$. Then

$$(\mathrm{id}_V \otimes c') \circ (u \otimes \mathrm{id}_V) = \mathrm{id}_V = (c \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes u') \tag{2.9..3}$$

and

$$(\mathrm{id}_{V^*} \otimes c) \circ (u' \otimes \mathrm{id}_{V^*}) = \mathrm{id}_{V^*} = (c' \otimes \mathrm{id}_{V^*}) \circ (\mathrm{id}_{V^*} \otimes u). \tag{2.9..4}$$

Proof. We will show that

$$(\mathrm{id}_V \otimes c') \circ (u \otimes \mathrm{id}_V) = \mathrm{id}_V$$

the arguments to establish the other three equalities in (2.9..3) and (2.9..4) are similar.

Let $v \in V$. Since e_i is a basis for V we can write $v = \sum v_i e_i$ for some $v_i \in \mathcal{A}$. Thus,

$$(\mathrm{id}_V \otimes c') \circ (u \otimes \mathrm{id}_V)(v) = (\mathrm{id}_V \otimes c')(\sum e_i \otimes e_i^* \otimes v) = \sum e_i \cdot e_i^*(v) = \sum v_i e_i = v.$$

□

Lemma 2.9..2. *Fix an isomorphism $\varphi : V \to V^*$ and write $\boldsymbol{cap} = c' \circ (\varphi \otimes \mathrm{id})$ and $\boldsymbol{cup} = (\mathrm{id} \otimes \varphi^{-1}) \circ u$. Then*

$$(\mathrm{id}_V \otimes \boldsymbol{cap}) \circ (\boldsymbol{cup} \otimes \mathrm{id}_V) = \mathrm{id}_V = (\boldsymbol{cap} \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes \boldsymbol{cup}). \tag{2.9..5}$$

Proof. Using $\varphi \circ \varphi^{-1} = \mathrm{id} = \varphi^{-1} \circ \varphi$, (2.9..5) follows easily from (2.9..3) and (2.9..4). □

The $\mathcal{A}$-linear maps

$$\mathcal{A} \xrightarrow{\mathbf{cup}_i := (\mathrm{id} \otimes \varphi_i^{-1}) \circ u_i} V^{\mathcal{A}}(\varpi_i) \otimes V^{\mathcal{A}}(\varpi_i) \xrightarrow{\mathbf{cap}_i := c_i' \circ (\varphi_i \otimes \mathrm{id})} \mathcal{A}, \qquad \text{for } i = 1, 2, \tag{2.9..6}$$

are actually maps of $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$ modules, where $\mathcal{A}$ is the trivial module. The functor Ξ will send the cups and caps from the diagrammatic category to the maps $\mathbf{cup}_i$ and $\mathbf{cap}_i$.

The module $V^{\mathcal{A}}(\varpi_1)$ has basis

$$\{v_{(1,0)}, v_{(-1,1)} = F_s v_{(1,0)}, v_{(1,-1)} = F_t F_s v_{(1,0)}, v_{(-1,0)} = F_s F_t F_s v_{(1,0)}\}, \tag{2.9..7}$$

and the module $V^{\mathcal{A}}(\varpi_2)$ has basis

$$\{v_{(0,1)}, v_{(2,-1)} = F_t v_{(0,1)}, v_{(0,0)} = F_s F_t v_{(0,1)}, v_{(-2,1)} = F_s^{(2)} F_t v_{(0,1)}, v_{(0,-1)} = F_t F_s^{(2)} F_t v_{(0,1)}\}. \tag{2.9..8}$$

With respect to these bases, we can write $\mathbf{cup}_1 : \mathcal{A} \to V^{\mathcal{A}}(\varpi_1) \otimes V^{\mathcal{A}}(\varpi_1)$ as

$$1 \mapsto -q^{-4} v_{(1,0)} \otimes v_{(-1,0)} + q^{-3} v_{(-1,1)} \otimes v_{(1,-1)} - q^{-1} v_{(1,-1)} \otimes v_{(-1,1)} + v_{(-1,0)} \otimes v_{(1,0)}, \tag{2.9..9}$$

and $\mathbf{cup}_2 : \mathcal{A} \to V^{\mathcal{A}}(\varpi_2) \otimes V^{\mathcal{A}}(\varpi_2)$ as

$$1 \mapsto q^{-6} v_{(0,1)} \otimes v_{(0,-1)} - q^{-4} v_{(2,-1)} \otimes v_{(-2,1)} + \frac{q^{-2}}{[2]_q} v_{(0,0)} \otimes v_{(0,0)} - q^{-2} v_{(-2,1)} \otimes v_{(2,-1)} + v_{(0,-1)} \otimes v_{(0,1)}. \tag{2.9..10}$$

To record the maps $\mathbf{cap}_i$ in our basis we use the matrices

$$\mathbf{cap}_1(v_i \otimes v_j) = \begin{array}{c} \\ v_{(-1,0)} \\ v_{(1,-1)} \\ v_{(-1,1)} \\ v_{(1,0)} \end{array} \begin{array}{c} \begin{array}{cccc} v_{(-1,0)} & v_{(1,-1)} & v_{(-1,1)} & v_{(1,0)} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & -q^4 \\ 0 & 0 & q^3 & 0 \\ 0 & -q & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \tag{2.9..11}$$

and

$$\mathbf{cap}_2(v_i \otimes v_j) = \begin{array}{c|c} & \begin{matrix} v_{(0,-1)} & v_{(-2,1)} & v_{(0,0)} & v_{(2,-1)} & v_{(0,1)} \end{matrix} \\ \begin{matrix} v_{(0,-1)} \\ v_{(-2,1)} \\ v_{(0,0)} \\ v_{(2,-1)} \\ v_{(0,1)} \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & q^6 \\ 0 & 0 & 0 & -q^4 & 0 \\ 0 & 0 & q^2[2]_q & 0 & 0 \\ 0 & -q^2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \qquad (2.9..12)$$

.

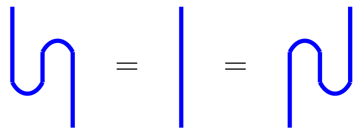
Example 2.9..3. We give two calculations to clarify how we arrived at these formulas:

$$\mathbf{cap}_2(v_{(0,0)} \otimes v_{(0,0)}) = c_2' \circ (\varphi_2 \otimes \mathrm{id})(v_{(0,0)} \otimes v_{(0,0)}) = c_2'(q^2[2]_q v^*_{(0,0)} \otimes v_{(0,0)}) = q^2[2]_q$$

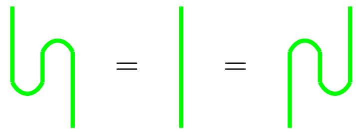
and

$$\begin{aligned} \mathbf{cup}_1(1) &= (\mathrm{id} \otimes \varphi_1^{-1}) \circ u_1(1) \\ &= v_{(-1,0)} \otimes \varphi_1^{-1}(v^*_{(-1,0)}) + v_{(1,-1)} \otimes \varphi_1^{-1}(v^*_{(1,-1)}) \\ &\qquad + v_{(-1,1)} \otimes \varphi_1^{-1}(v^*_{(-1,1)}) + v_{(1,0)} \otimes \varphi_1^{-1}(v_{(1,0)}) \\ &= -q^{-4} v_{(1,0)} \otimes v_{(-1,0)} + q^{-3} v_{(-1,1)} \otimes v_{(1,-1)} - q^{-1} v_{(1,-1)} \otimes v_{(-1,1)} + v_{(-1,0)} \otimes v_{(1,0)}. \end{aligned}$$

The maps $\mathbf{cup}_i$ and $\mathbf{cap}_i$ in $U_q^{\mathcal{A}}(\mathfrak{sp}_4) - \text{mod}$ are going to correspond to the colored cap and cup maps in $\mathcal{D}_{\mathfrak{sp}_4}$. In which case, the equation (2.9..5) corresponds to the isotopy relations.



(2.9..13)



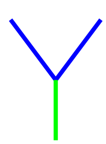
(2.9..14)

2.10. Trivalent Vertices

Consider the module $V^{\mathcal{A}}(\varpi_1) \otimes V^{\mathcal{A}}(\varpi_1)$. We observe that the vector $q^{-1}v_{(1,0)} \otimes v_{(0,1)} - v_{(0,1)} \otimes v_{(1,0)}$ is annihilated by E_s and E_t. The action of K_s scales this vector by 1 and the action of K_t scales the vector by q^2. There is an $\mathcal{A}$-linear map

$$\begin{aligned}
\mathbf{i} : V^{\mathcal{A}}(\varpi_2) &\to V^{\mathcal{A}}(\varpi_1) \otimes V^{\mathcal{A}}(\varpi_1) \\
v_{(0,1)} &\mapsto q^{-1}v_{(1,0)} \otimes v_{(-1,1)} - v_{(-1,1)} \otimes v_{(1,0)} \\
v_{(2,-1)} &\mapsto q^{-1}v_{(1,0)} \otimes v_{(1,-1)} - v_{(1,-1)} \otimes v_{(1,0)} \\
v_{(0,0)} &\mapsto q^{-1}v_{(1,0)} \otimes v_{(-1,0)} + q^{-2}v_{(-1,1)} \otimes v_{(1,-1)} \\
&\qquad - v_{(1,-1)} \otimes v_{(-1,1)} - q^{-1}v_{(-1,0)} \otimes v_{(1,0)} \\
v_{(-2,1)} &\mapsto q^{-1}v_{(-1,1)} \otimes v_{(-1,0)} - v_{(-1,0)} \otimes v_{(-1,1)} \\
v_{(0,-1)} &\mapsto q^{-1}v_{(1,-1)} \otimes v_{(-1,0)} - v_{(-1,0)} \otimes v_{(1,-1)}.
\end{aligned} \tag{2.10..1}$$

Using the explicit description of $V^{\mathcal{A}}(\varpi_2)$ in (2.8..4), one checks that $\mathbf{i}$ is a map of $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$-modules by computing the action of the generators of $U_q^{\mathcal{A}}(\mathfrak{sp}_4)$ on the vectors appearing on the right hand side of (2.10..1). The morphism $\mathbf{i}$ will correspond to the following diagram.



(2.10..2)

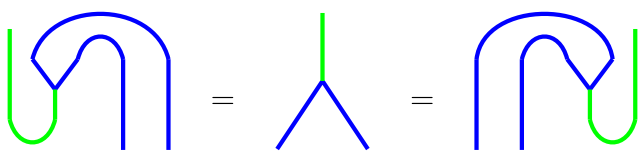
One can also check the equality of the following two elements of $\mathrm{Hom}_{U_q^{\mathcal{A}}(\mathfrak{sp}_4)}(V^{\mathcal{A}}(\varpi_1) \otimes V^{\mathcal{A}}(\varpi_1), V^{\mathcal{A}}(\varpi_2))$:

$$(\mathrm{id} \otimes \mathbf{cap}_1) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \mathbf{cap}_1 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathbf{i} \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathbf{cup}_2 \otimes \mathrm{id} \otimes \mathrm{id}) \qquad (2.10..3)$$

and

$$(\mathbf{cap}_1 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathbf{cap}_1 \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \mathbf{i} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \mathbf{cup}_2). \qquad (2.10..4)$$

Then we will unambiguously denote both maps by $\mathbf{p}$. In the graphical calculus this corresponds to the following.



(2.10..3) $\qquad$ $\mathbf{p}$ $\qquad$ (2.10..4) $\qquad$ (2.10..5)

The equality of (2.10..3) and (2.10..4) follows from verifying that both maps act on a basis as follows.

$$\mathbf{p} : V^{\mathcal{A}}(\varpi_1) \otimes V^{\mathcal{A}}(\varpi_1) \to V^{\mathcal{A}}(\varpi_2). \qquad (2.10..6)$$

$$v_{(1,0)} \otimes v_{(1,0)} \mapsto 0 \qquad v_{(-1,1)} \otimes v_{(1,0)} \mapsto q v_{(0,1)}$$

$$v_{(1,0)} \otimes v_{(-1,1)} \mapsto -v_{(0,1)} \qquad v_{(-1,1)} \otimes v_{(-1,1)} \mapsto 0$$

$$v_{(1,0)} \otimes v_{(1,-1)} \mapsto -v_{(2,-1)} \qquad v_{(-1,1)} \otimes v_{(1,-1)} \mapsto \frac{-1}{[2]_q} v_{(0,0)}$$

$$v_{(1,0)} \otimes v_{(-1,0)} \mapsto \frac{-q}{[2]_q} v_{(0,0)} \qquad v_{(-1,1)} \otimes v_{(-1,0)} \mapsto -v_{(-2,1)}$$

$$v_{(1,-1)} \otimes v_{(1,0)} \mapsto qv_{(2,-1)} \qquad v_{(-1,0)} \otimes v_{(1,0)} \mapsto \frac{q}{[2]_q} v_{(0,0)}$$

$$v_{(1,-1)} \otimes v_{(-1,1)} \mapsto \frac{q^2}{[2]_q} v_{(0,0)} \qquad v_{(-1,0)} \otimes v_{(-1,1)} \mapsto qv_{(-2,1)}$$

$$v_{(1,-1)} \otimes v_{(1,-1)} \mapsto 0 \qquad v_{(-1,0)} \otimes v_{(1,-1)} \mapsto qv_{(0,-1)}$$

$$v_{(1,-1)} \otimes v_{(-1,0)} \mapsto -v_{(0,-1)} \qquad v_{(-1,0)} \otimes v_{(-1,0)} \mapsto 0$$

Remark 2.10..1. We sketch a method to compute (2.10..3) evaluated on $v_{(-1,1)} \otimes v_{(1,-1)}$, the other calculations follow the same pattern. The $\mathbf{cap}_1$'s in the definition of (2.10..3) are only non-zero on basis vectors of the form $v_\mu \otimes v_{-\mu}$. Also, in the formula for $\mathbf{i}$ (2.10..1) the only basis vector with a tensor of the form $v_{(-1,1)} \otimes v_{(1,-1)}$ is $v_{(0,0)}$. Therefore, (2.10..3) acts as

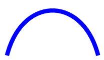
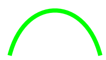
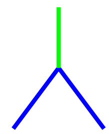
$$\begin{aligned} v_{(-1,1)} \otimes v_{(1,-1)} \mapsto & (\mathrm{id} \otimes \mathbf{cap}_1) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \mathbf{cap}_1 \otimes \mathrm{id}) \circ \\ & \left(\frac{q^{-2}}{[2]_q} v_{(0,0)} \otimes \mathbf{i}(v_{(0,0)}) \otimes v_{(-1,1)} \otimes v_{(1,-1)} \right) \\ & = q^{-2}\, \mathbf{cap}_1(v_{(1,-1)} \otimes v_{(-1,1)})\, \mathbf{cap}_1(v_{(-1,1)} \otimes v_{(1,-1)}) \frac{q^{-2}}{[2]_q} v_{(0,0)} \\ & = q^{-2} q^3 (-q) \frac{q^{-2}}{[2]_q} v_{(0,0)} \\ & = \frac{-1}{[2]_q} v_{(0,0)}. \end{aligned} \tag{2.10..7}$$

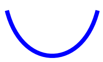
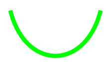
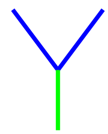
2.11. The Definition of the Evaluation Functor

Theorem 2.11..1. *There is a monoidal functor*

$$\Xi : \mathcal{D}_{\mathfrak{sp}_4} \to U_q^{\mathcal{A}}(\mathfrak{sp}_4) - mod.$$

defined on objects by defining $\Xi(\textcolor{blue}{1}) = V^{\mathcal{A}}(\varpi_1)$ *and* $\Xi(\textcolor{green}{2}) = V^{\mathcal{A}}(\varpi_2)$ *and then extending monoidally. The functor* Ξ *is defined on morphisms by first defining*

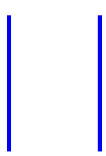
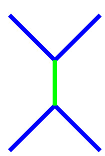
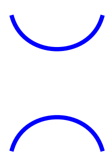
 $\mapsto \boldsymbol{cap}_1$  $\mapsto \boldsymbol{cap}_2$  $\mapsto \boldsymbol{p}$ (2.11..1)

 $\mapsto \boldsymbol{cup}_1$  $\mapsto \boldsymbol{cup}_2$  $\mapsto \boldsymbol{i}$ (2.11..2)

and then extending $\mathcal{A}$*-linearly so that horizontal concatenation of diagrams corresponds to tensor product of morphisms in* $U_q^{\mathcal{A}}(\mathfrak{sp}_4) -$ *mod and vertical composition of diagrams corresponds to composition of morphisms in* $U_q^{\mathcal{A}}(\mathfrak{sp}_4)-$*mod.*

Example 2.11..2. We illustrate how Ξ is defined on objects and on morphisms:

$$\Xi(\textcolor{blue}{1}\textcolor{green}{22}) = V^{\mathcal{A}}(\textcolor{blue}{1}\textcolor{green}{22}) = V^{\mathcal{A}}(\varpi_1) \otimes V^{\mathcal{A}}(\varpi_1) \otimes V^{\mathcal{A}}(\varpi_2) \tag{2.11..3}$$

$\frac{1}{[2]_q}$  $+$  $-\frac{1}{[2]_q}$  $\mapsto \frac{1}{[2]_q}\,\mathrm{id} \otimes \mathrm{id} - \mathbf{i} \circ \mathbf{p} + \frac{1}{[2]_q}\,\mathbf{cup}_1 \circ \mathbf{cap}_1\,.$

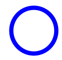
(2.11..4)

2.12. Checking Relations

Since $\mathcal{D}_{\mathfrak{sp}_4}$ is defined by generators and relations, in order to verify the theorem we must check that the diagrammatic relations hold in $U_q^{\mathcal{A}}(\mathfrak{sp}_4) -$ mod.

Proof of Theorem 2.11..1. The isotopy relations follow from (2.9..5) and the equality of (2.10..3) and (2.10..4).

To verify the relation



$$= -\frac{[6]_q[2]_q}{[3]_q} \tag{2.12..1}$$

it suffices to show that

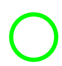
$$\mathbf{cap}_1 \circ \mathbf{cup}_1(1) = -\frac{[6]_q[2]_q}{[3]_q}. \tag{2.12..2}$$

Using (2.9..9) and (2.9..11) we find

$$\mathbf{cap}_1 \circ \mathbf{cup}_1(1) = -q^{-4}\cdot 1 + q^{-3}\cdot(-q) - q^{-1}\cdot q^3 + 1\cdot(-q^4) = -\left([5]_q - [1]_q\right). \tag{2.12..3}$$

The desired equality (2.12..2) comes from the quantum number calculation in (2.2..16).

One can similarly argue that the relation



$$= \frac{[6]_q[5]_q}{[3]_q[2]_q} \tag{2.12..4}$$

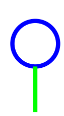
is satisfied. Use (2.9..10) and (2.9..12) to compute

$$\mathbf{cap}_2 \circ \mathbf{cup}_2(1) = q^{-6} + q^{-2} + 1 + q^2 + q^6 = [7]_q - [5]_q + [3]_q, \tag{2.12..5}$$

then use (2.2..17) to deduce

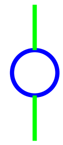
$$\mathbf{cap}_2 \circ \mathbf{cup}_2(1) = \frac{[6]_q[5]_q}{[3]_q[2]_q}. \tag{2.12..6}$$

To check the monogon relation



$$= 0 \tag{2.12..7}$$

and the bigon relation

 $= -[2]_q$ (2.12..8)

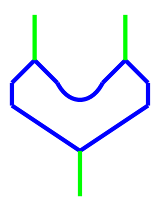
we need to show $\mathbf{cap}_1 \circ \mathbf{i} = 0$ and $\mathbf{p} \circ \mathbf{i} = -[2]_q \, \mathrm{id}$ respectively. Since the module $V^{\mathcal{A}}(\varpi_2)$ is generated by the highest weight vector $v_{(0,1)}$ it suffices to show that $\mathbf{cap}_1 \circ \mathbf{i}(v_{(0,1)}) = 0$ and $\mathbf{p} \circ \mathbf{i}(v_{(0,1)}) = -[2]_q v_{(0,1)}$. The calculations go as follows:

$$\begin{aligned} \mathbf{cap}_1 \circ \mathbf{i}(v_{(0,1)}) &\overset{(2.10..1)}{=} \mathbf{cap}_1(q^{-1} v_{(1,0)} \otimes v_{(-1,1)} - v_{(-1,1)} \otimes v_{(1,0)}) \\ &\overset{(2.9..9)}{=} 0. \end{aligned} \tag{2.12..9}$$

and

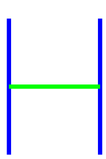
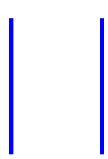
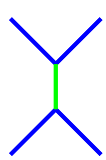
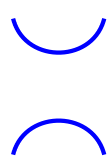
$$\begin{aligned} \mathbf{p} \circ \mathbf{i}(v_{(0,1)}) &\overset{(2.10..1)}{=} \mathbf{p}(q^{-1} v_{(1,0)} \otimes v_{(-1,1)} - v_{(-1,1)} \otimes v_{(1,0)}) \\ &\overset{(2.10..6)}{=} -q^{-1} v_{(0,1)} + q v_{(0,1)} \\ &= -[2]_q v_{(0,1)}. \end{aligned} \tag{2.12..10}$$

Verifying the trigon relation

 $= 0$ (2.12..11)

is left as an exercise (Hint: apply $(\mathbf{p} \otimes \mathbf{p}) \circ (\mathrm{id} \otimes \mathbf{cup}_1 \otimes \mathrm{id}) \circ \mathbf{i}$ to the vector $v_{(0,1)}$ and use (2.10..1) and (2.9..9) and (2.10..6)).

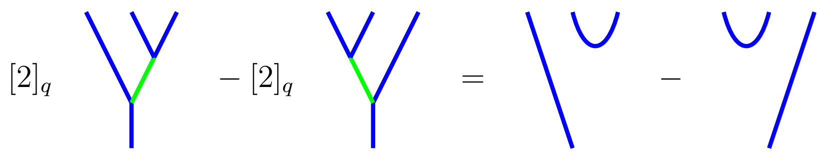
Now we endeavor to check the $H = I$ relation.

 $= \dfrac{1}{[2]_q}$  $+$  $- \dfrac{1}{[2]_q}$  (2.12..12)

Precomposing with $\mathrm{id} \otimes \mathbf{cup}_1$ is an $\mathcal{A}$-linear map

$$\mathrm{Hom}_{U_q^{\mathcal{A}}(\mathfrak{sp}_4)}(V^{\mathcal{A}}(\varpi_1)^{\otimes 2}, V^{\mathcal{A}}(\varpi_1)^{\otimes 2}) \longrightarrow \mathrm{Hom}_{U_q^{\mathcal{A}}(\mathfrak{sp}_4)}(V^{\mathcal{A}}(\varpi_1), V^{\mathcal{A}}(\varpi_1)^{\otimes 3}), \quad (2.12..13)$$

while postcomposing with $\mathrm{id} \otimes \mathrm{id} \otimes \mathbf{cap}_1$ is an $\mathcal{A}$-linear map in the other direction. From (2.9..5) it follows that the two maps are mutually inverse isomorphisms of $\mathcal{A}$-modules, so we can instead check the following relation.



(2.12..14)

From the discussion in Remark (2.2..3) it follows that we need to show

$$[2]_q(\mathrm{id} \otimes \mathbf{i}) \circ (\mathrm{id} \otimes \mathbf{p}) \circ (\mathbf{cup}_1 \otimes \mathrm{id}) - [2]_q(\mathbf{i} \otimes \mathrm{id})(\mathbf{p} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathbf{cup}_1) \quad (2.12..15)$$

is equal to

$$\mathrm{id} \otimes \mathbf{cup}_1 - \mathbf{cup}_1 \otimes \mathrm{id}\,. \quad (2.12..16)$$

Since $V^{\mathcal{A}}(\varpi_1)$ is generated by the vector $v_{(1,0)}$ it suffices to check that (2.12..15) and (2.12..16) send $v_{(1,0)}$ to the same vector in $V^{\mathcal{A}}(\varpi_1) \otimes V^{\mathcal{A}}(\varpi_1) \otimes V^{\mathcal{A}}(\varpi_1)$.

From (2.9..9), (2.10..6), and (2.10..1) it follows that

$$[2]_q(\mathrm{id} \otimes \mathbf{i}) \circ (\mathrm{id} \otimes \mathbf{p}) \circ (\mathbf{cup}_1 \otimes \mathrm{id})(v_{(1,0)}) - [2]_q(\mathbf{i} \otimes \mathrm{id})(\mathbf{p} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathbf{cup}_1)(v_{(1,0)}) \quad (2.12..17)$$

is equal to

$$-q^{-3}v_{(1,0)}\otimes \mathbf{i}(v_{(0,0)}) + q^{-2}[2]_q v_{(-1,1)} \otimes \mathbf{i}(v_{(2,-1)}) - [2]_q v_{(1,-1)} \otimes \mathbf{i}(v_{(0,1)})$$
$$+ q^{-3}[2]_q\, \mathbf{i}(v_{(0,1)}) \otimes v_{(1,-1)} - q^{-1}[2]_q\, \mathbf{i}(v_{(2,-1)}) \otimes v_{(-1,1)} + q\, \mathbf{i}(v_{(0,0)}) \otimes v_{(1,0)} \tag{2.12..18}$$

Using (2.9..9), we also find that

$$\mathrm{id} \otimes \mathbf{cup}_1(v_{(1,0)}) - \mathbf{cup}_1 \otimes \mathrm{id}(v_{(1,0)}) \tag{2.12..19}$$

is equal to

$$\begin{aligned} & v_{(1,0)} \otimes (-q^{-4}v_{(1,0)} \otimes v_{(-1,0)} + q^{-3}v_{(-1,1)} \otimes v_{(1,-1)} - \\ & q^{-1}v_{(1,-1)} \otimes v_{(-1,1)} + v_{(-1,0)} \otimes v_{(1,0)}) \\ & - (q^{-4}v_{(1,0)} \otimes v_{(-1,0)} - q^{-3}v_{(-1,1)} \otimes v_{(1,-1)} \\ & + q^{-1}v_{(1,-1)} \otimes v_{(-1,1)} - v_{(-1,0)} \otimes v_{(1,0)}) \otimes v_{(1,0)} \end{aligned} \tag{2.12..20}$$

Using (2.10..1) to show that (2.12..18) = (2.12..20) is left as an exercise. □

2.13. Background on Tilting Modules

Let $\mathbf{k}$ be a field and let $q \in \mathbf{k}^\times$ be such that $q + q^{-1} \neq 0$. We will write $U_q^{\mathbf{k}}(\mathfrak{sp}_4) = \mathbf{k} \otimes U_q^{\mathcal{A}}(\mathfrak{sp}_4)$, and $U_q^{\mathbf{k}}(\mathfrak{sp}_4) - \mathrm{mod}$ for the category of finite dimensional $U_q^{\mathbf{k}}(\mathfrak{sp}_4)$ modules which are direct sums of their weight spaces and such that K_α acts on the μ weight space as $q^{(\mu,\alpha^\vee)}$.

Everything we say in this section is well-known to experts, but the results are essential for our arguments so we include some discussion for completeness. Two excellent references are Jantzen's book [26] (only the second edition contains the appendix on representations of quantum groups and the appendix on tilting modules) and the eprint [4]. To deal with specializations when q is an even root of unity we will also need some results from [41] and [28].

For each $\lambda \in X_+$ there is a **dual Weyl module** of highest weight λ, denoted $\nabla^{\mathbf{k}}(\lambda)$, which is defined as an induced module [26, H.11]. The dual Weyl modules are a direct sum of their weight spaces and therefore have formal characters. Recall that we wrote $V(\lambda)$ for the irreducible module $\mathfrak{sp}_4(\mathbb{C})$ module of highest weight λ. We will write $[V(\lambda)]$ for the formal character of $V(\lambda)$ in $\mathbb{Z}[X]$, the group algebra of the weight lattice. It is known that a q-analogue of Kempf's vanishing holds for any $\mathbf{k}$ [41]. This implies that dual Weyl modules have formal character $[V(\lambda)]$ [3, Theorem 5.12].

The dual Weyl module always has a unique simple submodule with highest weight λ. We will denote this module by $L^{\mathbf{k}}(\lambda)$. The module $L^{\mathbf{k}}(\lambda)$ should not be thought of as a base change of $V(\lambda)$. In fact quite often the two modules will have distinct formal characters.

Since $U_q^{\mathbf{k}}(\mathfrak{sp}_4)$ is a Hopf-algebra, it acts on the dual vector space of any finite dimensional representation. Then we define the Weyl module of highest weight λ by $V^{\mathbf{k}}(\lambda) = \nabla^{\mathbf{k}}(-w_0\lambda)^*$ [26, H.15]. The dual Weyl module $V^{\mathbf{k}}(\lambda)$ has the same formal character as $\nabla^{\mathbf{k}}(\lambda)$, i.e. $[V(\lambda)]$, and $V^{\mathbf{k}}(\lambda)$ has a unique simple quotient isomorphic to $L^{\mathbf{k}}(\lambda)$.

Remark 2.13..1. In type C_2 the longest element w_0 acts on the weight lattice as -1. Therefore $V^{\mathbf{k}}(\lambda) = \nabla^{\mathbf{k}}(\lambda)^*$.

Definition 2.13..2. A **tilting module** is a module which has a (finite) filtration by Weyl modules, and a (finite) filtration by dual Weyl modules. The category of tilting modules, denoted $\mathbf{Tilt}(U_q^{\mathbf{k}}(\mathfrak{sp}_4))$, is the full subcategory of $U_q^{\mathbf{k}}(\mathfrak{sp}_4) - \mathrm{mod}$ where the objects are tilting modules.

Proposition 2.13..3. *The tensor product of two Weyl modules*

$$V^{\mathbf{k}}(\lambda_1) \otimes V^{\mathbf{k}}(\lambda_2)$$

has a filtration by Weyl modules.

Proof. That this holds over $\mathbf{k}$ follows from [28] where the result is shown to hold integrally using the theory of crystal bases. □

Corollary 2.13..4. *The tensor product of two tilting modules is a tilting module.*

Proof. Since $(-)^*$ is exact, it follows from Proposition (2.13..3) that the tensor product of dual Weyl modules

$$V^{\mathbf{k}}(\lambda_1)^* \otimes V^{\mathbf{k}}(\lambda_2)^*$$

has a filtration by dual Weyl modules. Thus the tensor product of two tilting modules will have a Weyl filtration and a dual Weyl filtration and is therefore a tilting module. □

Proposition 2.13..5. *Let $\lambda, \mu \in X_+$. Then $\dim_{\mathbf{k}} \mathrm{Ext}^i(V^{\mathbf{k}}(\lambda), \nabla^{\mathbf{k}}(\mu)) = \delta_{i,0}\delta_{\lambda,\mu}$ for all $i \geq 0$.*

Proof. A standard argument [4, Proof of Claim 3.1] shows that the vanishing of higher extension groups follows from Kempf's vanishing [41]. □

Proposition 2.13..6. *The category $\boldsymbol{Tilt}(U_q^{\mathbf{k}}(\mathfrak{sp}_4))$ is closed under direct sums, direct summands, and tensor products. The isomorphism classes of indecomposable* *objects in the category are in bijection with X_+. We will write $T^{\mathbf{k}}(\lambda)$ for the indecomposable tilting module corresponding to the dominant integral weight λ. The module $T^{\mathbf{k}}(\lambda)$ is characterized as the unique indecomposable tilting module with a one dimensional λ highest weight space.*

Proof. [26, E.3-E.6]. □

Lemma 2.13..7. *Weyl modules or dual Weyl modules give a basis for the Grothendieck group of the category $U_q^{\mathbf{k}}(\mathfrak{sp}_4) - mod$.*

Proof. Both $V^{\mathbf{k}}(\lambda)$ and $\nabla^{\mathbf{k}}(\lambda)$ have the same formal character: $[V(\lambda)]$. In particular $V^{\mathbf{k}}(\lambda)$ and $\nabla^{\mathbf{k}}(\lambda)$ both have one dimensional λ weight spaces. □

For a tilting module T, we will write $(T : V^{\mathbf{k}}(\lambda))$ to denote the filtration multiplicity. Formal character considerations also imply that $(T : V^{\mathbf{k}}(\lambda)) = (T : V^{\mathbf{k}}(\lambda)^*)$ [26, E.10].

Lemma 2.13..8. *The following are equivalent.*

1. *The Weyl module $V^{\mathbf{k}}(\lambda)$ is simple.*

2. *$V^{\mathbf{k}}(\lambda) \cong \nabla^{\mathbf{k}}(\lambda)$*

3. *The Weyl module $V^{\mathbf{k}}(\lambda)$ is a tilting module.*

Proof. It is not hard to see (1) implies (2) implies (3) [26, E.1]. That (3) implies (2) follows from Lemma (2.13..7), along with the equality of formal characters $[V^{\mathbf{k}}(\lambda)] = [\nabla^{\mathbf{k}}(\lambda)]$. To see that (2) implies (1), observe that the composition

$$L^{\mathbf{k}}(\lambda) \to \nabla^{\mathbf{k}}(\lambda) \xrightarrow{\sim} V^{\mathbf{k}}(\lambda) \to L^{\mathbf{k}}(\lambda)$$

is non-zero on the λ weight space. So the composition is a non-zero endomorphism of a simple module and therefore is an isomorphism. Thus, $L^{\mathbf{k}}(\lambda)$ is a direct summand of $\nabla^{\mathbf{k}}(\lambda)$. Since $\nabla^{\mathbf{k}}(\lambda)$ has a simple socle, we may conclude that $\nabla^{\mathbf{k}}(\lambda) \cong L^{\mathbf{k}}(\lambda)$. $\square$

Lemma 2.13..9. *1. If X has a filtration by Weyl modules, then for all $\lambda \in X_+$*

$$dim\,\mathrm{Hom}_{U_q^{\mathbf{k}}(\mathfrak{sp}_4)}(X, \nabla^{\mathbf{k}}(\lambda)) = (X : V^{\mathbf{k}}(\lambda)).$$

2. If Y has a filtration by dual Weyl modules, then for all $\lambda \in X_+$

$$dim\,\mathrm{Hom}_{U_q^{\mathbf{k}}(\mathfrak{sp}_4)}(V^{\mathbf{k}}(\lambda), Y) = (Y : \nabla^{\mathbf{k}}(\lambda)).$$

Proof. Both claims follow from Proposition (2.13..5) and a long exact sequence argument. $\square$

Proposition 2.13..10. *If T and T' are tilting modules, then*

$$\dim \mathrm{Hom}_{U_q^{\mathbf{k}}(\mathfrak{sp}_4)}(T, T') = \sum_{\lambda \in X_+} (T : V^{\mathbf{k}}(\lambda))(T' : V^{\mathbf{k}}(\lambda)). \tag{2.13..1}$$

Proof. Since T has both Weyl and dual Weyl filtrations, this follows from 2.13..9 and the fact that $(T' : \nabla^{\mathbf{k}}(\lambda)) = (T' : V^{\mathbf{k}}(\lambda))$. $\square$

2.14. The Image of the Evaluation Functor and Tilting Modules

We continue with our assumption that $\mathbf{k}$ is a field and $q \in \mathbf{k}^\times$ such that $q + q^{-1} \neq 0$.

Definition 2.14..1. The category $\mathbf{Fund}(U_q^{\mathbf{k}}(\mathfrak{sp}_4))$ is the full subcategory of $\mathrm{Rep}(U_q^{\mathbf{k}}(\mathfrak{sp}_4))$ with objects

$$V^{\mathbf{k}}(\underline{w}) = V^{\mathbf{k}}(w_1) \otimes V^{\mathbf{k}}(w_2) \otimes \ldots \otimes V^{\mathbf{k}}(w_n),$$

where $\underline{w} = w_1 w_2 \ldots w_n$ and $w_i \in \{1, 2\}$.

After changing coefficients to $\mathbf{k}$, the functor from Theorem (2.11..1) becomes

$$\mathbf{k} \otimes \Xi : \mathcal{D}_{\mathfrak{sp}_4}^{\mathbf{k}} \longrightarrow \mathbf{Fund}(U_q^{\mathbf{k}}(\mathfrak{sp}_4)) \tag{2.14..1}$$

We will abuse notation and write Ξ for $\mathbf{k} \otimes \Xi$.

Lemma 2.14..2. *The modules $V^{\mathbf{k}}(\underline{w})$ are tilting modules.*

Proof. From the description of the integral forms of the modules in (2.8..2) and (2.8..4), it is easy to see that $V^{\mathbf{k}}(\varpi_1)$ and $V^{\mathbf{k}}(\varpi_2)$ are irreducible with highest weight ϖ_1 and ϖ_2. They also have the same formal character as $[V(\varpi_1)]$ and $[V(\varpi_2)]$ respectively. So (2.13..8) implies that $V^{\mathbf{k}}(\underline{w})$ is a tensor product of tilting modules and therefore is a tilting module. $\square$

Remark 2.14..3. If $q + q^{-1} = 0$, then the Weyl module $V^{\mathbf{k}}(\varpi_1)$ is still simple and therefore tilting but the Weyl module $V^{\mathbf{k}}(\varpi_2)$ is not. In particular, $V^{\mathbf{k}}(\varpi_2)$ has two Jordan–Hölder factors, a simple socle isomorphic to $L^{\mathbf{k}}(0)$ and the simple quotient $L^{\mathbf{k}}(\varpi_2)$.

Lemma 2.14..4. *For all $\underline{w}$ and $\underline{u}$*

$$\dim_{\mathbf{k}} \mathrm{Hom}_{U_q^{\mathbf{k}}(\mathfrak{sp}_4)}(V^{\mathbf{k}}(\underline{w}), V^{\mathbf{k}}(\underline{u})) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{sp}_4(\mathbb{C})}(V(\underline{w}), V(\underline{u})). \tag{2.14..2}$$

Proof. Suppose that

$$V(\underline{w}) \cong \bigoplus_{\lambda} V(\lambda)^{m_\lambda}, \tag{2.14..3}$$

so we have an equality of formal characters $[V(\underline{w})] = \sum m_\lambda [V(\lambda)]$. Since $[V^{\mathbf{k}}(\underline{w})] = [V(\underline{w})]$ and $[V^{\mathbf{k}}(\lambda)] = [V(\lambda)]$ it follows that $(V^{\mathbf{k}}(\underline{w}) : V^{\mathbf{k}}(\lambda)) = m_\lambda$. The claim then follows from Proposition (2.13..10) and (2.3..7) □

Theorem 2.14..5. *The functor*

$$\Xi : \mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4} \longrightarrow \mathbf{Fund}(U^{\mathbf{k}}_q(\mathfrak{sp}_4)).$$

is a monoidal equivalence.

Proof. The functor Ξ is monoidal and essentially surjective, so it suffices to prove Ξ is full and faithful.

Let $\underline{w}$ and $\underline{u}$ be objects in $\mathcal{D}_{\mathfrak{sp}_4}$. In the next section we will prove that $\Xi(\mathbb{LL}^{\underline{u}}_{\underline{w}})$ is a linearly independent set of homomorphisms in $\mathbf{Fund}(U^{\mathbf{k}}_q(\mathfrak{sp}_4))$.

Since

$$\#\mathbb{LL}^{\underline{u}}_{\underline{w}} = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{sp}_4(\mathbb{C})}(V(\underline{w}), V(\underline{u})) = \dim_{\mathbf{k}} \mathrm{Hom}_{U^{\mathbf{k}}_q(\mathfrak{sp}_4)}(V^{\mathbf{k}}(\underline{w}), V^{\mathbf{k}}(\underline{u})), \tag{2.14..4}$$

the linear independence of $\Xi(\mathbb{LL}^{\underline{u}}_{\underline{w}})$ implies that Ξ maps $\mathbb{LL}^{\underline{u}}_{\underline{w}}$ to a basis in $\mathbf{Fund}(U^{\mathbf{k}}_q(\mathfrak{sp}_4))$. These observations imply that $\mathbb{LL}^{\underline{u}}_{\underline{w}}$ is a linearly independent set of homomorphisms in $\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}$. From the inequality in Lemma (2.7..14) we deduce that $\mathbb{LL}^{\underline{u}}_{\underline{w}}$ is a basis. So Ξ maps a basis to a basis and $\mathrm{Hom}_{\mathcal{D}_{\mathfrak{sp}_4}}(\underline{w}, \underline{u}) \xrightarrow{\Xi} \mathrm{Hom}_{U^{\mathbf{k}}_q(\mathfrak{sp}_4)}(V^{\mathbf{k}}(\underline{w}), V^{\mathbf{k}}(\underline{u}))$ is an isomorphism. □

Corollary 2.14..6. *The functor Ξ induces a monoidal equivalence between the Karoubi envelope of $\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}$ and the category* ***Tilt***$(U^{\mathbf{k}}_q(\mathfrak{sp}_4))$.

Proof. Tensor products and direct summands of tilting modules are tilting modules. Therefore, Lemma (2.14..2) implies that every direct summand of $V^{\mathbf{k}}(\underline{w})$ is a tilting module.

Let $\lambda \in X_+$, so $\lambda = a\varpi_1 + b\varpi_2$ for $a, b \in \mathbb{Z}_{\geq 0}$. The module $V^{\mathbf{k}}(1^{\otimes a} \otimes 2^{\otimes b})$ has a one dimensional λ highest weight space and all other non-zero weight spaces in X_+ are less than λ. From (2.13..6) we deduce that $V^{\mathbf{k}}(1^{\otimes a} \otimes 2^{\otimes b})$ must contain $T^{\mathbf{k}}(\lambda)$ as a direct summand. Therefore every indecomposable tilting module is a direct summand of some $V^{\mathbf{k}}(\underline{w})$. $\square$

2.15. Outline of Proof that Double Ladders are Linearly Independent

In this section we will finish the proof of Theorem (2.14..5) by arguing that the set $\Xi(\mathbb{LL}_{\underline{w}}^{\underline{u}})$ is linearly independent for all words $\underline{w}$ and $\underline{u}$.

The idea of the proof is best illustrated as follows. Suppose we just wanted to prove that the image of light ladder diagrams from $\underline{w}$ to $\emptyset$ are linearly independent. Recall that $E(\underline{w}, 0)$ is the set of dominant weight subsequences $\vec{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$, such that $\sum \mu_i = 0$. Assume that for each dominant weight subsequence in $E(\underline{w}, 0)$, we have fixed a choice of light ladder $LL_{\vec{\mu}}$ and a vector $v_{\vec{\mu}} \in V^{\mathbf{k}}(\underline{w})$. Since $\vec{\mu} \in E(\underline{w}, 0)$ is such that $\sum \mu_i = 0$, the light ladder $L_{\vec{\mu}}$ will map under Ξ to a homomorphism $V^{\mathbf{k}}(\underline{w}) \to V^{\mathbf{k}}(\emptyset) = \mathbf{k}$. If the following matrix of elements in $\mathbf{k}$

$$(\Xi(LL_{\vec{\mu}})(v_{\vec{\nu}}))_{\vec{\mu}, \vec{\nu} \in E(\underline{w}, 0)} \tag{2.15..1}$$

is upper triangular with invertible elements on the diagonal, then a non-trivial linear dependence among the maps $\Xi(LL_{\vec{\mu}})$ will give rise to a non-zero vector in the kernel of the matrix (2.15..1).

In the following subsections we will fix a choice of vectors associated to dominant weight subsequences. Then, since we want to argue double ladder diagrams are linearly independent, we must consider the image of the dominant weight subsequence vectors under both light ladders and upside down light ladders. The inductive construction of light ladders allows us to reduce these calculations to elementary light ladders, neutral ladders, and upside down elementary light ladders. In the end we still deduce linear independence of double ladder diagrams from an upper triangularity argument.

2.16. Subsequence Basis

Recall that the modules $V^{\mathbf{k}}(1)$ (2.9..7) and $V^{\mathbf{k}}(2)$ (2.9..8) both have a fixed basis of weight vectors v_ν for $\nu \in \operatorname{wt} V^{\mathbf{k}}(1) \cup \operatorname{wt} V^{\mathbf{k}}(2)$.

Definition 2.16..1. Fix $\underline{w} = (w_1, \ldots, w_n)$, a word in the alphabet $\{1, 2\}$, and let

$$S(\underline{w}) := \{(\nu_1, \ldots \nu_n) \ : \ \nu_i \in \operatorname{wt} V^{\mathbf{k}}(w_i)\}. \tag{2.16..1}$$

We set

$$v_{\underline{w},+} := v_{w_1} \otimes v_{w_2} \otimes \ldots \otimes v_{w_n} \tag{2.16..2}$$

where $v_1 = v_{(1,0)}$ and $v_2 = v_{(0,1)}$. Also, for any sequence of weights $\vec{\nu} = (\nu_1, ..., \nu_n) \in S(\underline{w})$, we define

$$v_{\underline{w},\vec{\nu}} := v_{\nu_1} \otimes \ldots \otimes v_{\nu_n} \in V^{\mathbf{k}}(\underline{w}). \tag{2.16..3}$$

The **subsequence basis** of $V^{\mathbf{k}}(\underline{w})$ is the set

$$\{v_{\vec{\nu}} \ : \ \vec{\nu} \in S(\underline{w})\}. \tag{2.16..4}$$

Lemma 2.16..2. *The subsequence basis of $V^{\mathbf{k}}(\underline{w})$ is a basis of $V^{\mathbf{k}}(\underline{w})$.*

Proof. This is clear. $\square$

Definition 2.16..3. Let $\chi \in X_+$. The χ **weight space** of $V^{\mathbf{k}}(\underline{w})$, denoted $V^{\mathbf{k}}(\underline{w})[\chi]$, is the $\mathbf{k}$-span of the subsequence basis vectors $v_{\vec{\nu}}$ such that $\sum \nu_i = \chi$.

Note that $E(\underline{w}) \subset S(\underline{w})$. In particular, for each $\vec{\nu} \in E(\underline{w})$ we get a subsequence basis vector $v_{\underline{w},\vec{\nu}}$. In the special case that the dominant weight subsequence is such that $\nu_i = \operatorname{wt} w_i$ for all i, then $v_{\underline{w},\vec{\nu}} = v_{\underline{w},+}$. Also, there is a partition of the set of dominant weight subsequences of $\underline{w}$:

$$E(\underline{w}) = \bigcup_{\lambda \in X_+} E(\underline{w}, \lambda), \tag{2.16..5}$$

where $\vec{\nu} \in E(\underline{w})$ is in $E(\underline{w}, \lambda)$ whenever $\sum \nu_i = \lambda$ or equivalently $v_{\underline{w},\vec{\nu}} \in V^{\mathbf{k}}(\underline{w})[\lambda]$.

Definition 2.16..4. Recall that our choice of simple roots was $\Delta = \{\alpha_s, \alpha_t\}$. There is a partial order on the set of weights defined by $\mu \leq \nu$ if $\nu - \mu \in \mathbb{Z}_{\geq 0} \cdot \Delta$. If we restrict this partial order to the set $\operatorname{wt} V^{\mathcal{A}}(1) \cup \operatorname{wt} V^{\mathcal{A}}(2)$, the resulting order is:

$$(-1, 0) < (1, -1) < (-1, 1) < (1, 0) \tag{2.16..6}$$

$$(0, -1) < (-2, 1) < (0, 0) < (2, -1) < (0, 1). \tag{2.16..7}$$

The lexicographic order gives a total order on the set $S(\underline{w})$. We will transport this total order to give a total order on the subsequence basis.

Example 2.16..5. In the image of $E(2121, (2,0)) \longrightarrow V^{\mathbf{k}}(2121)[(2,0)]$ we have,

$$v_{((0,1),(1,0),(2,-1),(-1,0))} > v_{((0,1),(1,0),(0,-1),(1,0))} > v_{((0,1),(1,-1),(0,0),(1,0))}.$$

Lemma 2.16..6. *If* $\operatorname{wt}\underline{w} \ngeq \chi$*, then* $V^{\mathbf{k}}(\underline{w})[\chi] = 0$*.*

Proof. If $\vec{\nu} \in S(\underline{w})$ is such that $\nu_i \in \operatorname{wt} V^{\mathbf{k}}(w_i)$, then $\sum \nu_i \leq \operatorname{wt}\underline{w}$. The subsequence basis spans $V^{\mathbf{k}}(\underline{w})$, so whenever $V^{\mathbf{k}}(\underline{w})[\chi] \neq 0$, we must have $\chi \leq \operatorname{wt}\underline{w}$. □

2.17. The Evaluation Functor and Elementary Diagrams

Notation 2.17..1. In the remainder of the section, we will use the same notation for diagrammatic morphisms and their image under the functor Ξ. But instead of saying diagram we will say map, for example the image of a light ladder diagram under Ξ will be referred to as a light ladder map.

To further simplify some of the statements below, our convention is that $\underline{w}$ and $\underline{u}$ are words in the alphabet $\{1, 2\}$ and ξ represents an invertible element of $\mathbf{k}$.

Recall that for each weight $\mu \in \operatorname{wt} V^{\mathbf{k}}(1) \cup \operatorname{wt} V^{\mathbf{k}}(2)$ there is an elementary light ladder diagram. The images of the elementary light ladder diagrams under the evaluation functor are the following elementary light ladder maps:

$$L_{(1,0)} = \mathrm{id} : V^{\mathbf{k}}(1) \to V^{\mathbf{k}}(1)$$

$$L_{(-1,1)} = \mathbf{p} : V^{\mathbf{k}}(11) \to V^{\mathbf{k}}(2)$$

$$L_{(1,-1)} = (\mathrm{id} \otimes \mathbf{cap}_1) \circ (\mathbf{i} \otimes \mathrm{id}) : V^{\mathbf{k}}(21) \to V^{\mathbf{k}}(1)$$

$$L_{(-1,0)} = \mathbf{cap}_1 : V^{\mathbf{k}}(11) \to \mathbf{k}$$

$$L_{(0,1)} = \mathrm{id} : V^{\mathbf{k}}(2) \to V^{\mathbf{k}}(2)$$

$$L_{(2,-1)} = (\mathrm{id} \otimes \mathbf{cap}_2 \otimes \mathrm{id}) \circ (\mathbf{i} \otimes \mathbf{i}) : V^{\mathbf{k}}(22) \to V^{\mathbf{k}}(11)$$

$$L_{(0,0)} = (\mathbf{cap}_1 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathbf{i}) : V^{\mathbf{k}}(12) \to V^{\mathbf{k}}(1)$$

$$L_{(-2,1)} = \mathbf{p} \circ (\mathrm{id} \otimes \mathbf{cap}_1 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \mathbf{i}) : V^{\mathbf{k}}(112) \to V^{\mathbf{k}}(2)$$

$$L_{(0,-1)} = \mathbf{cap}_2 : V^{\mathbf{k}}(22) \to \mathbf{k}.$$

There are two simple neutral diagrams, and their images under the evaluation functor are the simple neutral maps:

$$N_{12}^{21} = (\mathbf{p} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathbf{i}) : V^{\mathbf{k}}(12) \to V^{\mathbf{k}}(21)$$
$$N_{21}^{12} = (\mathrm{id} \otimes \mathbf{p}) \circ (\mathbf{i} \otimes \mathrm{id}) : V^{\mathbf{k}}(21) \to V^{\mathbf{k}}(12).$$

Lemma 2.17..2. *If $f : V^{\mathbf{k}}(\underline{w}) \longrightarrow V^{\mathbf{k}}(\underline{u})$ is a morphism which is in the image of the functor Ξ, then $f : V^{\mathbf{k}}(\underline{w})[\chi] \longrightarrow V^{\mathbf{k}}(\underline{u})[\chi]$, for all $\chi \in X$.*

Proof. It is well known that every $U_q^{\mathbf{k}}(\mathfrak{sp}_4)$ module homomorphism between finite dimensional modules will preserve weight spaces. But we could also deduce this from observing that the maps id, $\mathbf{i}$, and the cap and cup maps all preserve weight spaces and that any map in the image of Ξ is a linear combination of vertical and horizontal compositions of these basic maps. □

Recall that to construct light ladder diagrams and double ladder diagrams we need to fix a word $\underline{x}_\lambda$ in 1 and 2 for all $\lambda \in X_+$, and we need to make choices of neutral diagrams in the algorithmic construction. We now fix an $\underline{x}_\lambda$ for all $\lambda \in X_+$ and fix a light ladder diagram $LL_{\underline{w},(\mu_1,\ldots,\mu_m)}$ for all $\underline{w}$ and all $(\mu_1, \ldots, \mu_m) \in E(\underline{w})$. This allows us to construct double ladder diagrams. The double ladder maps are the image of these double ladder diagrams under the evaluation functor.

Remark 2.17..3. The form of the arguments below do not depend on our choice of light ladder maps.

2.18. Pairing Vectors and Neutral Maps

Lemma 2.18..1. *If $N : V^{\mathbf{k}}(\underline{w}) \to V^{\mathbf{k}}(\underline{u})$ is a neutral map, then $N(v_{\underline{w},+}) = \xi \cdot v_{\underline{u},+}$. Furthermore, if $(\mu_1, \ldots, \mu_n)$ is a sequence of weights such that $\mu_i \in \mathrm{wt}\, V^{\mathbf{k}}(w_i)$,* *and $N(v_{\underline{w},(\mu_1,\ldots,\mu_n)})$ has a non-zero coefficient for $v_{\underline{u},+}$ after being written in the subsequence basis, then $v_{\underline{w},(\mu_1,\ldots,\mu_n)} = v_{\underline{w},+}$.*

Proof. Neutral maps are vertical and horizontal compositions of identity maps, and the basic neutral maps N_{12}^{21} and N_{12}^{21}. The lemma will follow from verifying its validity for the two basic neutral maps.

The following maps factor through $V^{\mathbf{k}}(1)$:

$$I_{12}^{21} := \mathbb{D}(L_{(1,-1)}) \circ L_{(0,0)} \qquad \text{and} \qquad I_{21}^{12} := \mathbb{D}(L_{(0,0)}) \circ L_{(1,-1)}. \tag{2.18..1}$$

Since $V^{\mathbf{k}}(1)$ contains no vectors of weight $\varpi_1 + \varpi_2$, it follows that

$$I_{12}^{21}(v_{(1,0)} \otimes v_{(0,1)}) = 0 \qquad \text{and} \qquad I_{21}^{12}(v_{(0,1)} \otimes v_{(1,0)}) = 0. \tag{2.18..2}$$

It is easy to use the diagrammatic relations to compute that the maps

$$b_{12}^{21} = qN_{12}^{21} + q^{-1}I_{12}^{21} \tag{2.18..3}$$

and

$$b_{21}^{12} = q^{-1}N_{21}^{12} + qI_{21}^{12} \tag{2.18..4}$$

are mutual inverses.

Both b_{12}^{21} and b_{21}^{12} are isomorphisms so they restrict to isomorphisms of weight spaces. Since the $\varpi_1 + \varpi_2$ weight spaces of $V^{\mathbf{k}}(12)$ and $V^{\mathbf{k}}(21)$ are one dimensional, it follows that N_{12}^{21} sends the vector $v_{(1,0)} \otimes v_{(0,1)}$ to a non-zero scalar multiple of $v_{(0,1)} \otimes v_{(1,0)}$ and N_{12}^{21} sends $v_{(0,1)} \otimes v_{(1,0)}$ to a non-zero multiple of $v_{(1,0)} \otimes v_{(0,1)}$. Furthermore, the only subsequence basis vector which N_{12}^{21} sends to a non-zero multiple of $v_{(0,1)} \otimes v_{(1,0)}$ is $v_{(1,0)} \otimes v_{(0,1)}$, and the only subsequence basis vector which N_{21}^{12} sends to a non-zero multiple of $v_{(1,0)} \otimes v_{(0,1)}$ is $v_{(0,1)} \otimes v_{(1,0)}$. $\square$

2.19. Pairing Vectors and Light Ladders

Lemma 2.19..1. *Let* $* \in \{1, 2\}$ *and* $\mu \in \mathrm{wt}\, V^{\mathbf{k}}(*)$. *Then the map* $\mathrm{id} \otimes L_\mu : V^{\mathbf{k}}(\underline{w}) \otimes V^{\mathbf{k}}(*) \to V^{\mathbf{k}}(\underline{u})$, *is such that for all* $\nu \in \mathrm{wt}(V^{\mathbf{k}}(*))$,

$$\mathrm{id} \otimes L_\mu(v_{\underline{w},+} \otimes v_\nu) = \begin{cases} 0 & \textit{if } \nu > \mu \\ \xi \cdot v_{\underline{u},+} & \textit{if } \nu = \mu. \end{cases} \tag{2.19..1}$$

Proof. It suffices to check the claim for L_μ and not all $\mathrm{id} \otimes L_\mu$. The claim is obvious for $L_{(1,0)}$ and $L_{(0,1)}$. For the rest of the cases, the claim follows from the calculation in Section 2.22.. Note that in the L_μ step of the calculation, the first non-zero entry is $v_\mu \mapsto \xi \cdot v_{\underline{u},+}$. $\square$

Let $\vec{\mu} = (\mu_1, \dots, \mu_n) \in E(\underline{w}, \lambda)$. The light ladder map $LL_{\underline{w},\vec{\mu}} : V^{\mathbf{k}}(\underline{w}) \to V^{\mathbf{k}}(\underline{x}_\lambda)$ restricts to a map

$$LL_{\underline{w},\vec{\mu}} : V^{\mathbf{k}}(\underline{w})[\lambda] \longrightarrow V^{\mathbf{k}}(\underline{x}_\lambda)[\lambda]. \tag{2.19..2}$$

Moreover, $V^{\mathbf{k}}(\underline{x}_\lambda)[\lambda] = \mathbf{k} \cdot v_{\underline{x}_\lambda,+}$. There is also a totally ordered set of linearly independent vectors in $V^{\mathbf{k}}(\underline{w})[\lambda]$, namely $v_{\underline{w},\vec{\nu}}$ for all $\vec{\nu} = (\nu_1, \dots, \nu_n) \in E(\underline{w}, \lambda)$.

Proposition 2.19..2.

$$LL_{\underline{w},\vec{\mu}}(v_{\underline{w},\vec{\nu}}) = \begin{cases} 0 & \textit{if } \vec{\nu} > \vec{\mu} \\ \xi \cdot v_{\underline{x}_\lambda,+} & \textit{if } \vec{\nu} = \vec{\mu}. \end{cases} \tag{2.19..3}$$

Proof. By the inductive definition of the light ladder map $LL_{\underline{w},\vec{\mu}}$ and of the vector $v_{\underline{w},\vec{\nu}}$, this proposition follows from repeated use of Lemmas (2.19..1) and (3.4..6).

$\square$

2.20. Pairing Vectors and Upside Down Light Ladders

In the results of the previous subsection we found the lexicographic order on sequences of weights was adapted to light ladders. There is another order on weights which is convenient for upside down light ladders.

Definition 2.20..1. Fix $\underline{w}$ and let $\vec{\mu} = (\mu_1, \ldots, \mu_n)$ and $\vec{\nu} = (\nu_1, \ldots, \nu_n)$ be sequences of weights such that $\mu_i, \nu_i \in \operatorname{wt} V^{\mathbf{k}}(w_i)$. Define a total order $<^{\mathbb{D}}$ on weight sequences by setting $\vec{\nu} <^{\mathbb{D}} \vec{\mu}$ if $(\nu_n, \ldots, \nu_1) < (\mu_n, \ldots, \mu_1)$ in the lexicographic order. We may also transport this order to give a total order on the subsequence basis.

Lemma 2.20..2. *Let* $* \in \{1, 2\}$ *and* $\mu \in \operatorname{wt} V^{\mathbf{k}}(*)$. *Then the map* $\mathrm{id} \otimes \mathbb{D}(L_\mu) : V^{\mathbf{k}}(\underline{w}) \to V^{\mathbf{k}}(\underline{u}) \otimes V^{\mathbf{k}}(*)$ *is such that*

$$\mathrm{id} \otimes \mathbb{D}(L_\mu)(v_{\underline{w},+}) = \xi \cdot v_{\underline{u},+} \otimes v_\mu + \sum c_{\vec{\tau}} \cdot v_{\underline{u},\vec{\tau}} \otimes v_\nu, \qquad c_{\vec{\tau}} \in \mathbf{k}, \tag{2.20..1}$$

where $v_{\underline{u},\vec{\tau}} \otimes v_\nu$ *is a subsequence basis vector,* $v_\nu > v_\mu$, *and* $v_{\underline{u},\vec{\tau}} < v_{\underline{u},+}$.

Proof. It suffices to check the claim for $\mathbb{D}(L_\mu)$ and not all $\mathrm{id} \otimes \mathbb{D}(L_\mu)$. The claim is obvious for $\mathbb{D}(L_{(1,0)})$ and $\mathbb{D}(L_{(0,1)})$. The rest of the cases follow from the calculation in Section 2.23.. Note that the first line in the $\mathbb{D}(L_\mu)$ calculation is $v_{\underline{w},+} \mapsto \xi \cdot v_{\underline{u},+} \otimes v_\mu$, while the remaining terms are of the form $v_{\underline{u},\vec{\tau}} \otimes v_\nu$ where $\nu > \mu$. $\square$

Let $\vec{\mu} = (\mu_1, \dots, \mu_n) \in E(\underline{w}, \lambda)$. The associated upside down light ladder map $\mathbb{D}(LL_{\underline{w},\vec{\mu}}) : V^{\mathbf{k}}(\underline{x}_\lambda) \longrightarrow V^{\mathbf{k}}(\underline{w})$ restricts to a map

$$\mathbb{D}(LL_{\underline{w},\vec{\mu}}) : V^{\mathbf{k}}(\underline{x}_\lambda)[\lambda] \longrightarrow V^{\mathbf{k}}(\underline{w})[\lambda]. \tag{2.20..2}$$

Proposition 2.20..3.

$$\mathbb{D}(LL_{\underline{w},\vec{\mu}})(v_{\underline{x}_\lambda,+}) = \xi \cdot v_{\underline{w},\vec{\mu}} + \sum c_{\vec{\tau}} \cdot v_{\underline{w},\vec{\tau}}, \qquad c_{\vec{\tau}} \in \mathbf{k}, \tag{2.20..3}$$

where $v_{\underline{w},\vec{\mu}} <^{\mathbb{D}} v_{\underline{w},\vec{\tau}}$.

Proof. By the inductive definition of the light ladder map $LL_{\underline{w},(\mu_1,\dots,\mu_n)}$, this proposition follows from repeated use of Lemmas (2.20..2) and (3.4..6). □

2.21. Proof of Linear Independence

Theorem 2.21..1. *The set*

$$\mathbb{LL}_{\underline{w}}^{\underline{u}} = \bigcup_{\lambda \in X_+} \mathbb{LL}_{\underline{w}}^{\underline{u}}(\lambda) \tag{2.21..1}$$

is a linearly independent subset of $\mathrm{Hom}_{U_q^{\mathbf{k}}(\mathfrak{sp}_4)}(V^{\mathbf{k}}(\underline{w}), V^{\mathbf{k}}(\underline{u}))$.

Proof. Let

$$\sum_{\lambda} \sum_{\substack{\vec{\mu} \in E(\underline{w},\lambda) \\ \vec{\nu} \in E(\underline{u},\lambda)}} {}^{\lambda}c_{\vec{\mu}}^{\vec{\nu}} \cdot \mathbb{LL}_{\underline{w},\vec{\mu}}^{\underline{u},\vec{\nu}} = 0, \qquad {}^{\lambda}c_{\vec{\mu}}^{\vec{\nu}} \in \mathbf{k} \tag{2.21..2}$$

be a non-trivial linear relation. There is at least one $\lambda_0 \in X_+$ with ${}^{\lambda_0}c_{\vec{\mu}}^{\vec{\nu}} \neq 0$ such that if ${}^{\lambda}c_{\vec{\mu}}^{\vec{\nu}} \neq 0$ then $\lambda \not> \lambda_0$. Lemma (2.16..6) implies that for all $\lambda \neq \lambda_0$ with ${}^{\lambda}c_{\vec{\mu}}^{\vec{\nu}} \neq 0$, $V^{\mathbf{k}}(\underline{x}_\lambda)[\lambda_0] = 0$. If $v_0 \in V^{\mathbf{k}}(\underline{w})[\lambda_0]$, then since light ladder maps preserve the weight of a vector (2.17..2)

$$0 = \sum_{\lambda} \sum_{\vec{\mu},\vec{\nu}} {}^{\lambda}c_{\vec{\mu}}^{\vec{\nu}} \cdot \mathbb{LL}_{\underline{w},\vec{\mu}}^{\underline{u},\vec{\nu}}(v_0) = \sum_{\vec{\mu},\vec{\nu}} {}^{\lambda_0}c_{\vec{\mu}}^{\vec{\nu}} \cdot \mathbb{LL}_{\underline{w},\vec{\mu}}^{\underline{u},\vec{\nu}}(v_0). \tag{2.21..3}$$

Note that for $\vec{\mu} \in E(\underline{w}, \lambda_0)$, $v_{\underline{w},\vec{\mu}} \in V^{\mathbf{k}}(\underline{w})[\lambda_0]$.

Let $\vec{\mu_0}$ be the largest $\vec{\mu}$, in the lexicographic order, such that ${}^{\lambda_0}c_{\vec{\mu}}^{\vec{\nu}} \neq 0$. Taking $v_0 = v_{\underline{w},\vec{\mu_0}}$ in (2.21..3) results in

$$0 = \sum_{\vec{\mu},\vec{\nu}} {}^{\lambda_0}c_{\vec{\mu}}^{\vec{\nu}} \cdot \mathbb{LL}_{\underline{w},\vec{\mu}}^{\underline{u},\vec{\nu}}(v_{\underline{w},\vec{\mu_0}}) = \sum_{\vec{\mu},\vec{\nu}} {}^{\lambda_0}c_{\vec{\mu}}^{\vec{\nu}} \cdot \mathbb{D}(LL_{\underline{u},\vec{\nu}}) \circ LL_{\underline{w},\vec{\mu}}(v_{\underline{w},\vec{\mu_0}}). \tag{2.21..4}$$

Proposition (2.19..2) implies

$$0 = \sum_{\vec{\nu}} {}^{\lambda_0}c_{\vec{\mu_0}}^{\vec{\nu}} \cdot \mathbb{D}(LL_{\underline{u},\vec{\nu}}) \circ LL_{\underline{w},\vec{\mu_0}}(v_{\underline{w},\vec{\mu_0}}) = \sum_{\vec{\nu}} {}^{\lambda_0}c_{\vec{\mu_0}}^{\vec{\nu}} \xi \cdot \mathbb{D}(LL_{\underline{u},\vec{\nu}})(v_{\underline{x}_\lambda,+}). \tag{2.21..5}$$

Let $\vec{\nu_0}$ be the smallest $\vec{\nu}$, in the $<^{\mathbb{D}}$ order, such that ${}^{\lambda_0}c_{\vec{\mu_0}}^{\vec{\nu}} \neq 0$. Proposition (2.20..3) implies

$$\begin{aligned} 0 &= {}^{\lambda}c_{\vec{\mu_0}}^{\vec{\nu_0}} \xi \cdot \mathbb{D}(LL_{\underline{u},\vec{\nu_0}})(v_{\underline{x}_\lambda,+}) + \sum_{\vec{\nu_0} <^{\mathbb{D}} \vec{\nu}} {}^{\lambda_0}c_{\vec{\mu_0}}^{\vec{\nu}} \xi \cdot \mathbb{D}(LL_{\underline{u},\vec{\nu}})(v_{\underline{x}_\lambda,+}) \\ &= {}^{\lambda_0}c_{\vec{\mu_0}}^{\vec{\nu_0}} \xi \cdot v_{\underline{u},\vec{\nu_0}} + \text{“higher terms”}, \end{aligned} \tag{2.21..6}$$

where “higher terms” is a linear combination of subsequence basis vectors all of which are greater than $v_{\underline{u},\vec{\nu_0}}$ in the $<^{\mathbb{D}}$ order. Since the subsequence basis vectors are linearly independent, we must have ${}^{\lambda_0}c_{\vec{\mu_0}}^{\vec{\nu_0}} \xi = 0$, which is a contradiction. $\square$

2.22. Elementary Light Ladder Calculations

$$L_{(-1,1)}(v_{(1,0)} \otimes (-)) : \begin{cases} v_{(1,0)} & \mapsto 0 \\ v_{(-1,1)} & \mapsto -v_{(0,1)} \\ v_{(1,-1)} & \mapsto -v_{(2,-1)} \\ v_{(-1,0)} & \mapsto \dfrac{-q}{[2]_q} v_{(0,0)}, \end{cases} \tag{2.22..1}$$

$$L_{(1,-1)}(v_{(0,1)} \otimes (-)) : \begin{cases} v_{(1,0)} & \mapsto 0 \\ v_{(-1,1)} & \mapsto 0 \\ v_{(1,-1)} & \mapsto -v_{(1,0)} \\ v_{(-1,0)} & \mapsto -v_{(-1,1)}, \end{cases} \tag{2.22..2}$$

$$L_{(-1,0)}(v_{(1,0)} \otimes (-)) : \begin{cases} v_{(1,0)} & \mapsto 0 \\ v_{(-1,1)} & \mapsto 0 \\ v_{(1,-1)} & \mapsto 0 \\ v_{(-1,0)} & \mapsto 1, \end{cases} \tag{2.22..3}$$

$$L_{(2,-1)}(v_{(0,1)} \otimes (-)) : \begin{cases} v_{(0,1)} & \mapsto 0 \\ v_{(2,-1)} & \mapsto v_{(1,0)} \otimes v_{(1,0)} \\ v_{(0,0)} & \mapsto v_{(1,0)} \otimes v_{(-1,1)} + q^{-1} v_{(-1,1)} \otimes v_{(1,0)} \\ v_{(-2,1)} & \mapsto v_{(-1,1)} \otimes v_{(-1,1)} \\ v_{(0,-1)} & \mapsto -v_{(1,0)} \otimes v_{(-1,0)} + v_{(-1,1)} \otimes v_{(1,-1)}, \end{cases} \tag{2.22..4}$$

$$L_{(0,0)}(v_{(1,0)} \otimes (-)) : \begin{cases} v_{(0,1)} & \mapsto 0 \\ v_{(2,-1)} & \mapsto 0 \\ v_{(0,0)} & \mapsto -q^{-1}v_{(1,0)} \\ v_{(-2,1)} & \mapsto -v_{(-1,1)} \\ v_{(0,-1)} & \mapsto -v_{(1,-1)}, \end{cases} \tag{2.22..5}$$

$$L_{(-2,1)}(v_{(1,0)} \otimes v_{(1,0)} \otimes (-)) : \begin{cases} v_{(0,1)} & \mapsto 0 \\ v_{(2,-1)} & \mapsto 0 \\ v_{(0,0)} & \mapsto 0 \\ v_{(-2,1)} & \mapsto v_{(0,1)} \\ v_{(0,-1)} & \mapsto v_{(2,-1)}, \end{cases} \tag{2.22..6}$$

$$L_{(0,-1)}(v_{(0,1)} \otimes (-)) : \begin{cases} v_{(0,1)} & \mapsto 0 \\ v_{(2,-1)} & \mapsto 0 \\ v_{(0,0)} & \mapsto 0 \\ v_{(-2,1)} & \mapsto 0 \\ v_{(0,-1)} & \mapsto 1. \end{cases} \tag{2.22..7}$$

2.23. Upside Down Elementary Light Ladder Calculations

$$\begin{aligned} \mathbb{D}(L_{(-1,1)}) : v_{(0,1)} \mapsto\ & q^{-1}v_{(1,0)} \otimes v_{(-1,1)} \\ & -v_{(-1,1)} \otimes v_{(1,0)} \end{aligned} \tag{2.23..1}$$

$$\begin{aligned}\mathbb{D}(L_{(1,-1)}) : v_{(1,0)} \mapsto & -q^{-3} v_{(0,1)} \otimes v_{(1,-1)} \\ & + q^{-1} v_{(2,-1)} \otimes v_{(-1,1)} \\ & - \frac{q}{[2]_q} v_{(0,0)} \otimes v_{(1,0)}\end{aligned} \tag{2.23..2}$$

$$\begin{aligned}\mathbb{D}(L_{(-1,0)}) : 1 \mapsto & -q^{-4} v_{(1,0)} \otimes v_{(-1,0)} \\ & + q^{-3} v_{(-1,1)} \otimes v_{(1,-1)} \\ & - q^{-1} v_{(1,-1)} \otimes v_{(-1,1)} \\ & + v_{(-1,0)} \otimes v_{(1,0)},\end{aligned} \tag{2.23..3}$$

$$\begin{aligned}\mathbb{D}(L_{(2,-1)}) : v_{(1,0)} \otimes v_{(1,0)} \mapsto & -q^{-2} v_{(0,1)} \otimes v_{(2,-1)} \\ & + v_{(2,-1)} \otimes v_{(0,1)}\end{aligned} \tag{2.23..4}$$

$$\begin{aligned}\mathbb{D}(L_{(0,0)}) : v_{(1,0)} \mapsto & \frac{-q^{-3}}{[2]_q} v_{(1,0)} \otimes v_{(0,0)} \\ & + q^{-2} v_{(-1,1)} \otimes v_{(2,-1)} \\ & - v_{(1,-1)} \otimes v_{(0,1)}\end{aligned} \tag{2.23..5}$$

$$
\begin{aligned}
\mathbb{D}(L_{(-2,1)}) : v_{(0,1)} \mapsto & -q^{-4} v_{(1,0)} \otimes v_{(1,0)} \otimes v_{(-2,1)} \\
& + \frac{q^{-2}}{[2]_q} v_{(1,0)} \otimes v_{(-1,1)} \otimes v_{(0,0)} + \frac{q^{-3}}{[2]_q} v_{(-1,1)} \otimes v_{(1,0)} \otimes v_{(0,0)} \\
& - q^{-2} v_{(-1,1)} \otimes v_{(-1,1)} \otimes v_{(2,-1)} \\
& - q^{-1} v_{(1,0)} \otimes v_{(-1,0)} \otimes v_{(0,1)} + v_{(-1,1)} \otimes v_{(1,-1)} \otimes v_{(0,1)}
\end{aligned}
\tag{2.23..6}
$$

and

$$
\begin{aligned}
\mathbb{D}(L_{(0,-1)}) : 1 \mapsto & \, q^{-6} v_{(0,1)} \otimes v_{(0,-1)} \\
& - q^{-4} v_{(2,-1)} \otimes v_{(-2,1)} \\
& + \frac{q^{-2}}{[2]_q} v_{(0,0)} \otimes v_{(0,0)} \\
& - q^{-2} v_{(-2,1)} \otimes v_{(2,-1)} \\
& + v_{(0,-1)} \otimes v_{(0,1)}.
\end{aligned}
\tag{2.23..7}
$$

2.24. Object Adapted Cellular Category Structure

We refer to [21, Definition 2.4] for the definition of a strictly object adapted cellular category or SOACC.

Let $\mathbf{k}$ be a field and let $q \in \mathbf{k}^\times$ such that $q + q^{-1} \neq 0$. In this section we will show that $\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}$ is an SOACC. It follows that the endomorphism algebras in $\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}$ are cellular algebras. Since we proved that $\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}$ is equivalent to $\mathbf{Fund}(U_q^{\mathbf{k}}(\mathfrak{sp}_4))$, the result about cellular algebras also follows from [5] and the result about $\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}$ being an SOACC follows from [2, Proposition 2.4]. For more discussion about the relation between our work and [5] we recommend [7, p. 6] (but replace $\mathfrak{sl}_n$ webs with $\mathcal{D}_{\mathfrak{sp}_4}$).

For each $\lambda \in X_+$, choose an object $\underline{x}_\lambda$ in $\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}$ such that $\operatorname{wt} \underline{x}_\lambda = \lambda$. The set $\Lambda = \{\underline{x}_\lambda\}_{\lambda \in X_+}$ is in bijection with X_+, and we define a partial order on Λ by setting $\underline{x}_\lambda \leq \underline{x}_\mu$ whenever $\lambda \leq \mu$ i.e. $\mu - \lambda \in \mathbb{Z}_{\geq 0}\Phi_+$.

For any object $\underline{w}$ in $\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}$ and for all $\vec{\mu} \in E(\underline{w}, \lambda)$ we fix a light ladder diagram $LL_{\vec{\mu}} := LL_{\underline{w},\vec{\nu}} \in \operatorname{Hom}_{\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}}(\underline{w}, \underline{x}_\lambda)$ and an upside down light ladder diagram $\mathbb{D}(LL_{\vec{\nu}}) := \mathbb{D}(LL_{\underline{w},\vec{\nu}}) \in \operatorname{Hom}_{\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}}(\underline{x}_\lambda, \underline{w})$.

If $\underline{x}_\lambda = x_1 x_2 \ldots x_n$ where $x_i \in \{1, 2\}$, then the set $E(\underline{x}_\lambda, \lambda)$ contains a single element, $\vec{\lambda} = (\operatorname{wt} x_1, \operatorname{wt} x_2 \ldots \operatorname{wt} x_n)$. Recall that in our definition of double ladder diagrams we choose $LL_{\vec{\lambda}} = \operatorname{id}_{\underline{x}_\lambda} = \mathbb{D}(LL_{\vec{\lambda}})$.

For $\vec{\mu} \in E(\underline{w}, \lambda)$ and $\vec{\nu} \in E(\underline{u}, \lambda)$ we set

$$\mathbb{LL}^{\lambda}_{\vec{\mu},\vec{\nu}} := \mathbb{D}(LL_{\vec{\nu}}) \circ LL_{\vec{\mu}} \in \operatorname{Hom}_{\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}}(\underline{w}, \underline{u}). \tag{2.24..1}$$

It follows from our main theorem that $\{\mathbb{LL}^{\lambda}_{\vec{\mu},\vec{\nu}}\}_{\lambda \in X_+}$ forms a basis for $\operatorname{Hom}_{\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}}(\underline{w}, \underline{u})$.

Remark 2.24..1. In the definition of an SOACC, one fixes the data of two sets, $E(\underline{w}, \lambda)$ and $M(\underline{w}, \lambda)$, which are in a fixed bijection. We are choosing to ignore the set $M(\underline{w}, \lambda)$.

Definition 2.24..2. Fix $\lambda \in X_+$. Let $(\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4})_{<\lambda}$ be the $\mathbf{k}$-linear subcategory whose morphisms are spanned by $\mathbb{LL}^{\chi}_{\vec{\mu},\vec{\nu}}$ with $\chi < \lambda$.

Lemma 2.24..3. *Let* $f \in \operatorname{Hom}_{\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}}(\underline{w}, \underline{u})$ *and let* $\vec{\mu} \in E(\underline{u}, \lambda)$. *Then*

$$LL_{\vec{\mu}} \circ f \equiv \sum_{\vec{\nu} \in E(\underline{w},\lambda)} * \cdot LL_{\vec{\nu}} \qquad \textit{modulo } (\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4})_{<\lambda}, \tag{2.24..2}$$

where $*$ *represents an element of* $\mathbf{k}$.

Proof. Writing $LL_{\vec{\mu}} \circ f$ in the double ladder basis, we find that

$$
\begin{aligned}
LL_{\vec{\mu}} \circ f &= \sum_{\substack{\chi \in X_+ \\ \vec{\nu} \in E(\underline{w}, \chi) \\ \vec{\tau} \in E(\underline{x}_\lambda, \chi)}} * \cdot \mathbb{LL}^{\chi}_{\vec{\nu}, \vec{\tau}} \\
&\equiv \sum_{\substack{\vec{\mu} \in E(\underline{w}, \lambda) \\ \vec{\tau} \in E(\underline{x}_\lambda, \lambda)}} * \cdot \mathbb{LL}^{\lambda}_{\vec{\nu}, \vec{\tau}} \qquad \text{modulo } (\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4})_{<\lambda} \qquad (2.24..3) \\
&\equiv \sum_{\vec{\nu} \in E(\underline{w}, \lambda)} * \cdot LL_{\vec{\nu}} \qquad \text{modulo } (\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4})_{<\lambda}
\end{aligned}
$$

The second equality follows from the observation that if $\chi \in X_+$ and $E(\underline{x}_\lambda, \chi) \neq \emptyset$, then $\chi \leq \lambda$. The third equality follows from recalling that $E(\underline{x}_\lambda, \lambda) = \{\vec{\lambda}\}$ and $LL_{\vec{\lambda}} = \mathrm{id}_{\underline{x}_\lambda}$. □

Corollary 2.24..4. *The category $\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}$ with fixed choices of $\underline{x}_\lambda$ and light ladder diagrams is an SOACC.*

2.25. Tilting Character Algorithm

We will describe a way to compute the filtration multiplicities $(T^{\mathbf{k}}(\lambda), V^{\mathbf{k}}(\mu))$ for all $\lambda, \mu \in X_+$ using the light ladder diagrams in $\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}$. The ideas in this section are standard, and we follow [22, 32]. The reader may also wish to consult [5, Appendix 4B] and compare our discussion with the theory of cell modules for cellular algebras.

Lemma 2.25..1. *The indecomposable tilting module $T^{\mathbf{k}}(\lambda)$ has a local endomorphism ring, and if J is the Jacobson radical of the ring $\mathrm{End}_{U_q^{\mathbf{k}}(\mathfrak{sp}_4)}(T^{\mathbf{k}}(\lambda))$, then*

$$\mathrm{End}_{U_q^{\mathbf{k}}(\mathfrak{sp}_4)}(T^{\mathbf{k}}(\lambda))/J \xrightarrow{\sim} \mathbf{k} \cdot \mathrm{id}, \qquad (2.25..1)$$

where $\varphi + J = c_\varphi \operatorname{id} + J \mapsto c_\varphi \operatorname{id}$.

Proof. Restriction to $T^{\mathbf{k}}(\lambda)[\lambda]$ is a $\mathbf{k}$-linear ring homomorphism

$$\operatorname{End}_{U_q^{\mathbf{k}}(\mathfrak{sp}_4)}(T^{\mathbf{k}}(\lambda)) \to \operatorname{End}_{\mathbf{k}}(T^{\mathbf{k}}(\lambda)[\lambda]) = \mathbf{k} \cdot \operatorname{id}. \tag{2.25..2}$$

Since id acts on the λ weight space as multiplication by 1, this ring homomorphism is surjective. Also, $T^{\mathbf{k}}(\lambda)$ is indecomposable so its endomorphism ring is local, and therefore the kernel of the ring homomorphism in (2.25..2) is J. □

Lemma 2.25..2. *Let $\underline{w}$ be an object in $\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}$ such that $\operatorname{wt}\underline{w} = \lambda$. Then*

$$V^{\mathbf{k}}(\underline{w}) = T^{\mathbf{k}}(\lambda) \oplus \bigoplus_{\mu<\lambda} T^{\mathbf{k}}(\mu)^{r_{\underline{w},\mu}} \tag{2.25..3}$$

and $r_{\underline{w},\mu}$ is the rank of the pairing

$$\begin{aligned}\kappa_{\underline{w},\mu} : \operatorname{Hom}_{U_q^{\mathbf{k}}(\mathfrak{sp}_4)}(V^{\mathbf{k}}(\underline{w}), T^{\mathbf{k}}(\mu)) \times \operatorname{Hom}_{U_q^{\mathbf{k}}(\mathfrak{sp}_4)}(T^{\mathbf{k}}(\mu), V^{\mathbf{k}}(\underline{w})) &\to \mathbf{k} \cdot \operatorname{id} \\ (f, g) &\mapsto c_{f\circ g} \cdot \operatorname{id}.\end{aligned}$$

Proof. The claim about the decomposition of $V^{\mathbf{k}}(\underline{w})$ follows from character considerations. The second claim about computing multiplicities using the rank of the composition pairing is standard [22, Lemma 11.65]. □

Remark 2.25..3. The pairing $\kappa_{\underline{w},\operatorname{wt}\underline{w}}$ will always have rank 1.

Lemma 2.25..4. *The light ladder diagrams $\{LL_{\vec{\nu}}\}_{\vec{\nu}\in E(\underline{w},\mu)}$ form a basis for*

$$\operatorname{Hom}_{\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}}(\underline{w}, \underline{x}_\mu)/(\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4})_{<\mu},$$

and the upside down light ladder diagrams $\{\mathbb{D}(LL_{\vec{\nu}})\}_{\vec{\nu}\in E(\underline{w},\mu)}$ *form a basis for*

$$\mathrm{Hom}_{\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}}(\underline{x}_\mu, \underline{w})/(\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4})_{<\mu}.$$

For all pairs $\vec{\chi}, \vec{\nu} \in E(\underline{w}, \mu)$ *there is a scalar* $c^{\vec{\nu}}_{\vec{\chi}} \in \mathbf{k}$ *such that*

$$LL_{\vec{\nu}} \circ \mathbb{D}(LL_{\vec{\chi}}) = c^{\vec{\nu}}_{\vec{\chi}} \,\mathrm{id}_{\underline{x}_\mu} + (\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4})_{<\mu},$$

which is computed as the coefficient of the identity in the double ladder basis. The rank of the matrix $(c^{\vec{\nu}}_{\vec{\chi}})_{\vec{\chi},\vec{\nu}\in E(\underline{w},\mu)}$ *is equal to the rank of the pairing* $\kappa_{\underline{w},\mu}$.

Proof. Since $\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}$ is an object adapted cellular category, this follows from the discussion in [22, Appendix 11.5]. □

Proposition 2.25..5. *The character of the indecomposable tilting module with highest weight* $\lambda \in X_+$ *is*

$$[T^{\mathbf{k}}(\lambda)] = [V^{\mathbf{k}}(\lambda)] + \sum_{\mu<\lambda} \#E(\underline{x}_\lambda, \mu)[V^{\mathbf{k}}(\mu)] - \sum_{\mu<\lambda} \mathrm{rk}_{\mathbf{k}}(c^{\vec{\nu}}_{\vec{\chi}})_{\vec{\chi},\vec{\nu}\in E(\underline{x}_\lambda,\mu)}[T^{\mathbf{k}}(\mu)]. \quad (2.25..4)$$

Proof. Since $(V^{\mathbf{k}}(\underline{x}_\lambda) : V^{\mathbf{k}}(\mu)) = \#E(\underline{x}_\lambda, \mu)$, the claim follows from Lemma (2.25..2) and Lemma (2.25..4). □

Remark 2.25..6. Since the sums on the right hand side of (2.25..4) are indexed over $\mu < \lambda$, and the partially ordered set $(X_+, <)$ has the descending chain condition, one can determine $[T^{\mathbf{k}}(\lambda)]$ by computing $\#E(\underline{x}_\chi, \nu)$ and $r_{\underline{x}_\chi,\nu}$ for all $0 \le \nu \le \chi \le \lambda$.

Remark 2.25..7. Calculations of tilting module characters can be made completely within the diagrammatic category $\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}$. The quantity $\#E(\underline{x}_\lambda, \mu)$ is equal to the number of light ladder diagrams from $\underline{x}_\lambda$ to $\underline{x}_\mu$, and $r_{\underline{x}_\lambda,\mu}$ is equal to the rank of the matrix $(c_{\vec{\chi}}^{\vec{\nu}})_{\vec{\chi},\vec{\nu}\in E(\underline{w},\mu)}$. Moreover, these matrices can be computed in $\mathcal{D}_{\mathfrak{sp}_4}$. If $M \subset \mathcal{A}$ is a maximal ideal and $\mathbf{k} = \mathcal{A}/M$, then the rank of the mod M reduction of the matrix $(c_{\vec{\chi}}^{\vec{\nu}})_{\vec{\chi},\vec{\nu}\in E(\underline{w},\mu)}$ is equal to $(V^{\mathbf{k}}(\underline{w}) : T^{\mathbf{k}}(\lambda))$.

CHAPTER III

TRIPLE CLASP FORMULAS

This chapter contains previously published material. The material in this chapter originally appeared in [12].

3.1. Outline

Elias's formulas in type A and our formulas in type B_2 were inspired by Wenzl's recursive formula for $\mathfrak{sl}_2$. The $\mathfrak{sl}_2$ case so well illustrates the arguments used to derive our main Theorem (1.2..1) that we recall Wenzl's recursion below. Section 2: We recall some facts about the double ladders basis for $\mathfrak{sp}_4$ webs and deduce the triple clasp formula. Section 3: The recursive formulas for the local intersection forms are stated, and then derived via diagrammatic calculations. We prove the main theorem by showing the conjectured formulas satisfy the recursion. Lastly, we explain how to generalize Elias's clasp conjecture and show that our main theorem verifies this conjecture in type C_2.

3.2. Wenzl's Triple Clasp Formula for A_1 Webs

Let $\mathbb{C}^2$ be the two dimensional defining representation of $\mathfrak{sl}_2(\mathbb{C})$. The tensor powers $(\mathbb{C}^2)^{\otimes d}$ carry an action of $\mathfrak{sl}_2(\mathbb{C})$ and the module $S^d(\mathbb{C}^2)$ is an irreducible quotient of $(\mathbb{C}^2)^{\otimes d}$. The finite dimensional irreducible representations of $\mathfrak{sl}_2$ are in bijection with $\mathbb{Z}_{\geq 0}$ via $d \mapsto S^d(\mathbb{C}^2)$ and the composition factors of the kernel of the map $(\mathbb{C}^2)^{\otimes d} \longrightarrow S^d(\mathbb{C}^2)$ are all of the form $S^k(\mathbb{C}^2)$ for $k < d$.

Let $\mathbf{Rep}(\mathfrak{sl}_2(\mathbb{C}))$ denote the abelian monoidal category of finite dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$. Since $\mathbf{Rep}(\mathfrak{sl}_2(\mathbb{C}))$ is semisimple and has simple objects in bijection with the nonnegative integers, it is equivalent to $\oplus_{\mathbb{Z}_{\geq 0}} \mathbf{Vec}_{\mathbb{C}}$, and is therefore uninteresting as an abelian category. However, a semisimple monoidal category contains much more information than just the number of simple objects. For example, by the general theory of Tannakian reconstruction [18] one can recover the group $SL_2(\mathbb{C})$ as the automorphisms of the monoidal functor $\mathbf{Fund}(\mathfrak{sl}_2(\mathbb{C})) \longrightarrow \mathbf{Vec}_{\mathbb{C}}$.

Define $\mathbf{Fund}(\mathfrak{sl}_2(\mathbb{C}))$ to be the full monoidal subcategory of $\mathbf{Rep}(\mathfrak{sl}_2(\mathbb{C}))$ with objects arbitrary tensor products of $\mathbb{C}^2$. Since each irreducible finite dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$ is a direct summand of a tensor power of the defining representation, there is an equivalence of monoidal categories:

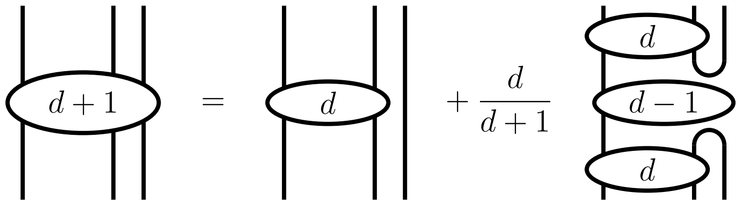
$$\mathbf{Kar}(\mathbf{Fund}(\mathfrak{sl}_2(\mathbb{C}))) \cong \mathbf{Rep}(\mathfrak{sl}_2(\mathbb{C})).^1$$

Threfore, the study of $\mathbf{Rep}(\mathfrak{sl}_2(\mathbb{C}))$ as a monoidal category is reduced to the study of idempotents in $\mathbf{Fund}(\mathfrak{sl}_2(\mathbb{C}))$.

Let $\mathcal{TL}$ be the strict, pivotal, and $\mathbb{C}$-linear category generated by one self dual object of dimension -2. It is well known that $\mathcal{TL}$ is equivalent to the monoidal category $\mathbf{Fund}(\mathfrak{sl}_2(\mathbb{C}))$ [40, 48]. Thus, we are led to consider the problem of using the category $\mathcal{TL}$ to describe the idempotent in $\mathrm{End}_{\mathfrak{sl}_2(\mathbb{C})}((\mathbb{C}^2)^{\otimes d})$ which projects to $S^d(\mathbb{C}^2)$. Wenzl found a recursive description of these idempotents [51]. Using the usual graphical calculus for $\mathcal{TL}$ and using a d labelled oval to represent

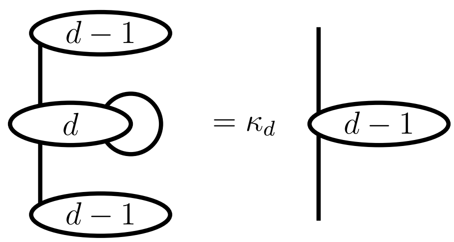
[1] The Karoubi envelope of a category $\mathcal{C}$, denoted $\mathbf{Kar}(\mathcal{C})$ is the category with objects pairs: (X, e), where X is an object in $\mathcal{C}$ and $e \in \mathrm{End}_{\mathcal{C}}(X)$ is an idempotent. The morphisms $(X, e) \longrightarrow (Y, f)$ in $\mathbf{Kar}(\mathcal{C})$ are all morphisms of the form $f \circ \varphi \circ e$, where $\varphi : X \longrightarrow Y$ in $\mathcal{C}$. When $\mathcal{C}$ is a monoidal category, $\mathbf{Kar}(\mathcal{C})$ is also naturally a monoidal category. Moreover, if $\mathcal{A}$ is a semisimple monoidal category, $\mathcal{C} \subset \mathcal{A}$ is a full monoidal subcategory, and every irreducible object in $\mathcal{A}$ is a direct summand of an object in $\mathcal{C}$, then there is a monoidal equivalence $\mathbf{Kar}(\mathcal{C}) \longrightarrow \mathcal{A}$.

the idempotent with image $S^d(\mathbb{C}^2)$, the Wenzl recursion becomes the following.



(3.2..1)

One way to think about Wenzl's recursion, which we learned from [7], is to first note that by Schur's lemma there is some $\kappa_d \in \mathbb{C}$ such that the following equality holds.

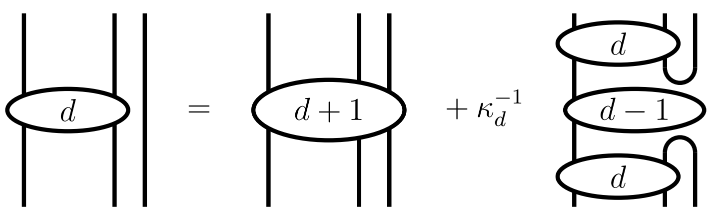


(3.2..2)

Since

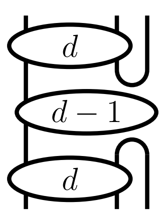
$$S^d(\mathbb{C}^2) \otimes \mathbb{C}^2 \cong S^{d+1}(\mathbb{C}^2) \oplus S^{d-1}(\mathbb{C}^2),$$

we also observe that there is a relation in $\mathrm{End}_{\mathfrak{sl}_2(\mathbb{C}))}(\mathbb{C}^2)$ of the following form.

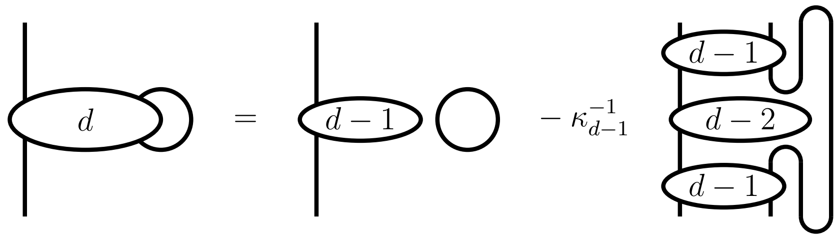


(3.2..3)

Where κ_d^{-1} is the coefficient needed to make the quasi-idempotent



into an idempotent. Using the $d \mapsto d-1$ version of the relation in Equation (3.2..3) to rewrite the middle clasp labelled d on the left hand side of Equation (3.2..2), we find the following.



(3.2..4)

Using the relations in $\mathcal{TL}$, along with the fact that post-composing the ovals with any cap map results in zero, we can simplify the right hand side of Equation (3.2..4) and find $\kappa_d = -2 - 1/\kappa_{d-1}$. From the initial condition $\kappa_0^{-1} = 0$, it is easy to verify that $\kappa_d = -(d+1)/d$.

Notation 3.2..1. The terminology of *clasp* was introduced in [31] to refer to idempotents projecting to the top summand expressed in terms of the graphical calculus. Following [7], we will refer to a recursive formula for clasps, in which the terms on the right hand side of the recursion are three clasps linked together by diagrammatic morphisms, as a *triple clasp formula*. We will refer to the diagrammatic morphisms in the clasp formula as *elementary light ladder diagrams*.

Remark 3.2..2. Equation (3.2..1) is still true without the middle clasp labelled $d-1$. However, the middle clasp in the triple clasp keeps track of which summand the morphism is factoring through and therefore has representation theoretic meaning. Moreover, in higher rank examples, like the one considered in this chapter, removing the middle clasp will not result in a valid identity. One would have to also change the coefficients and we do not expect these new coefficients to be as nice as those occurring in the triple clasp formula.

Remark 3.2..3. The category $\mathbf{Rep}(\mathfrak{sl}_2(\mathbb{C}))$ has a $\mathbb{C}(q)$-linear analogue, the category of finite dimensional type 1 representations of $U_q(\mathfrak{sl}_2(\mathbb{C}))$. We denote this category by $\mathbf{Rep}(U_q(\mathfrak{sl}_2(\mathbb{C})))$. Inside this category is the full monoidal subcategory generated by the q analogue of $\mathbb{C}^2$, which we call $\mathbf{Fund}(U_q(\mathfrak{sl}_2(\mathbb{C})))$. Finally, there is also a q analogue of $\mathcal{TL}$, denoted $\mathcal{TL}_q$ which is the strict pivotal $\mathbb{C}(q)$-linear category generated by a self dual object of dimension $-q - q^{-1}$. Again, we have $\mathcal{TL}_q \cong \mathbf{Fund}(U_q(\mathfrak{sl}_2(\mathbb{C})))$ and as long as q is not a root of unity we also have $\mathbf{Kar}(\mathbf{Fund}(U_q(\mathfrak{sl}_2(\mathbb{C})))) \cong \mathbf{Rep}(U_q(\mathfrak{sl}_2(\mathbb{C})))$.

3.3. Recollection of Double Ladder Basis

In Section 2.5. and 2.6. we defined *elementary light ladders*, *neutral ladders*, *light ladders*, *upside down light ladders*, and *double ladders*. Let us recall the important aspects of these constructions below.

Start by associating an elementary light ladder diagram in $\mathcal{D}_{\mathfrak{sp}_4}$ to each weight in a fundamental representation: $\mu \mapsto L_\mu$. For a dominant weight subsequence $\vec{\mu} \in E(\underline{w})$, there are light ladder diagrams $LL_{\underline{w},\vec{\mu}}$. The elementary light ladder diagrams for μ_i are the building blocks of the light ladder diagram $LL_{\underline{w},\vec{\mu}}$, but in order to make light ladder diagrams out of elementary light ladders we also require neutral diagrams which are used to shuffle words in 1 and 2. The ability to freely choose neutral diagrams means that for a given dominant weight subsequence there may be many choices of light ladder diagrams.

There is a duality $\mathbb{D}$ on the diagrammatic category, which takes a diagram and flips it upside down. We define upside down light ladders as the image, under $\mathbb{D}$, of usual light ladders.

When defining the double ladder basis, we first fix, for each dominant weight λ, a choice of object $\underline{x}_\lambda \in \mathcal{D}_{\mathfrak{sp}_4}$ satisfying $\mathrm{wt}(\underline{x}_\lambda) = \lambda$. Then for each object $\underline{w}$ and each $\vec{\mu} \in E(\underline{w})$ we fix a choice of light ladder $LL_{\vec{\mu}}$ such that if $\mathrm{wt}\,\vec{\mu} = \lambda$, then the target of the diagram is $\underline{x}_\lambda$. Moreover, for each λ there is a unique $\vec{\mu} \in E(\underline{x}_\lambda)$ such that $\mathrm{wt}\,\vec{\mu} = \lambda$ and we insist the chosen light ladder is the identity. The double ladder basis $\mathbb{LL}$ is then constructed by composing all light ladders with all upside down light ladders.

We refer to the image of these various diagrams under Ξ as maps, e.g. Ξ applied to a neutral ladder is a neutral map.

Suppose we have fixed a choice of $\underline{x}_\lambda$ for all $\lambda \in X_+$ and then fixed a choice of light ladders for all $\underline{w}$ and $\vec{\mu} \in E(\underline{w})$. We write $\mathbb{LL}$ to denote the associated double ladder basis, $\mathbb{LL}_{\underline{w}}^{\underline{x}}$ to denote $\mathbb{LL} \cap \mathrm{Hom}_{\mathcal{D}_{\mathfrak{sp}_4}}(\underline{w}, \underline{x})$, and $\mathbb{LL}_{\underline{w}}^{\underline{x}}(\lambda)$ for the collection of double ladders of the form $\mathbb{D}(LL_{\underline{x},\vec{\nu}}) \circ LL_{\underline{w},\vec{\mu}}$, where $\vec{\nu} \in E(\underline{x}, \lambda)$ and $\vec{\mu} \in E(\underline{w}, \lambda)$. Thus,

$$\mathbb{LL}_{\underline{w}}^{\underline{x}} = \bigcup_{\lambda \in X_+} \mathbb{LL}_{\underline{w}}^{\underline{x}}(\lambda). \tag{3.3..1}$$

Note that the middle of a diagram in $\mathbb{LL}_{\underline{w}}^{\underline{x}}(\lambda)$ is the identity of $\underline{x}_\lambda$. We say that such diagrams *factor through* λ.

Definition 3.3..1. Fix $\lambda \in X_+$. Let $(\mathcal{D}_{\mathfrak{sp}_4}^{\mathbf{k}})_{<\lambda}$ be the $\mathbf{k}$-linear subcategory whose morphisms are spanned by all double ladders in $\mathbb{LL}_{\underline{u}}^{\underline{v}}(\chi)$ for all $\underline{u}$ and $\underline{v}$ and all $\chi < \lambda$.

Lemma 3.3..2. *Fix $\lambda \in X_+$. The subcategory $(\mathcal{D}_{\mathfrak{sp}_4}^{\mathbf{k}})_{<\lambda}$ is an ideal, i.e. if $D \in (\mathcal{D}_{\mathfrak{sp}_4}^{\mathbf{k}})_{<\lambda}$, then $g \circ D \circ f \in (\mathcal{D}_{\mathfrak{sp}_4}^{\mathbf{k}})_{<\lambda}$ for all f, g.*

Proof. See [21, Lemma 2.8]. Note that $\mathbb{D}$ induces a bijection on homomorphisms spaces in $\mathcal{D}_{\mathfrak{sp}_4}$ and preserves the basis $\mathbb{LL}$. Thus, the claim follows from Lemma (2.24..3). $\square$

3.4. Downward Diagrams and Neutral Coefficients

Definition 3.4..1. Let $D \in \mathrm{Hom}_{\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}}(\underline{w}, \underline{x})$ be an arbitrary diagram. Suppose that there is a horizontal cross section of D which intersects D in the word $\underline{y}$. If $\mathrm{wt}\, \underline{y} < \mathrm{wt}\, \underline{w}$, then we say D is a **downward diagram**. Suppose that there is a horizontal cross section of D which intersects D in the word $\underline{y}$. If $\mathrm{wt}\, \underline{y} < \mathrm{wt}\, \underline{x}$, then we say D is an **upward diagram**.

Example 3.4..2. Any elementary light ladder L_ν, for $\nu \notin \{\varpi_1, \varpi_2\}$, is a downward diagram. Any light ladder $LL_{\underline{w},\vec{\mu}}$ is a downward diagram, unless $\mu_i = \mathrm{wt}\, w_i$ for all i.

Remark 3.4..3. The duality $\mathbb{D}$ induces a bijection between upward diagrams and downward diagrams.

Lemma 3.4..4. *If* $D \in \mathrm{Hom}_{\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}}(\underline{w}, \underline{x})$ *is a downward diagram then* $D \in (\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4})_{<\mathrm{wt}\, \underline{w}}$.

Proof. Suppose that D is a downward diagram, in particular there is some $\underline{y}$ with $\mathrm{wt}\, \underline{y} < \mathrm{wt}\, \underline{w}$ such that $D = A \circ B$ for $A \in \mathrm{Hom}_{\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}}(\underline{y}, \underline{x})$ and $B \in \mathrm{Hom}_{\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}}(\underline{w}, \underline{y})$. If we write A in terms of the double ladder basis, then it is easy to see that A is a linear combination of diagrams in $(\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4})_{<\mathrm{wt}\, \underline{w}}$. Then from Lemma (3.3..2) it follows that $D \in (\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4})_{<\mathrm{wt}\, \underline{w}}$. $\square$

Lemma 3.4..5. *Any downward map from* $\underline{w}$ *to* $\underline{x}$ *will send* $v_{\underline{w},+}$ *to zero.*

Proof. Since homomorphisms of $U_q^{\mathbf{k}}(\mathfrak{sp}_4)$ modules preserve weight spaces, this follows from the observation that if $\mathrm{wt}\, \underline{y} < \mathrm{wt}\, \underline{w}$, then $V^{\mathbf{k}}(\underline{y})[\mathrm{wt}\, \underline{w}] = 0$. $\square$

Lemma 3.4..6. *If $N : V^{\mathbf{k}}(\underline{w}) \to V^{\mathbf{k}}(\underline{u})$ is a neutral map, then $N(v_{\underline{w},+}) = v_{\underline{u},+}$. Furthermore, if $v_{\underline{w},\vec{\mu}}$ is a subsequence basis element, and $N(v_{\underline{w},\vec{\mu}})$ has a non-zero coefficient for $v_{\underline{u},+}$ after being written in the subsequence basis, then $v_{\underline{w},\vec{\mu}} = v_{\underline{w},+}$.*

Proof. Since $V^{\mathbf{k}}(\underline{w})[\mathrm{wt}\,\underline{w}] = \mathbf{k} \cdot v_{\underline{w},+}$, the second claim follows from the fact that maps of $U_q^{\mathbf{k}}(\mathfrak{sp}_4)$ modules preserve weight spaces. Neutral maps are compositions and tensor products of identity maps, N_{12}^{21}, and N_{21}^{12}. So to to verify the first claim we just need to check that $N_{12}^{21}(v_{12,+}) = v_{21,+}$ and $N_{21}^{12}(v_{21,+}) = v_{12,+}$. From the calculations in Sections 2.22. and 2.23., we find

$$N_{12}^{21}(v_{12,+}) = L_{(-1,1)} \otimes \mathrm{id}(v_{1,+} \otimes \mathbb{D}(L_{(-1,1)})(v_{2,+})) = v_{21,+} \tag{3.4..1}$$

and

$$N_{21}^{12}(v_{21,+}) = \mathrm{id} \otimes L_{(-1,1)}(\mathbb{D}(L_{(-1,1)})(v_{2,+}) \otimes v_{1,+}) = v_{12,+}. \tag{3.4..2}$$

□

Remark 3.4..7. We rescaled the generating trivalent vertex in the B_2 spider from [31]. Explicitly our trivalent vertex is equal to $\dfrac{1}{\sqrt{[2]}}$ times Kuperberg's trivalent generator. One reason our choice may be preferable to the original, is that using Kuperberg's trivalent vertex the neutral maps have $\xi = [2]$ instead of $\xi = 1$.

Let $\lambda \in X_+$ and let $\underline{w}$ and $\underline{x}$ be such that $\mathrm{wt}(\underline{w}) = \lambda = \mathrm{wt}(\underline{x})$. Then $\mathbb{LL}_{\underline{w}}^{\underline{x}}(\lambda)$ contains a single diagram. Denote this diagram by $I_{\underline{w}}^{\underline{x}}$. After applying Ξ, it follows from Lemma (3.4..6) that $I_{\underline{w}}^{\underline{x}}$ sends $v_{\underline{w},+}$ to $v_{\underline{x},+}$.

Suppose we made another choice of $\underline{x}'_\lambda$ for each dominant weight λ (along with choices of all the necessary neutral maps) and then constructed a double ladder basis $\mathbb{LL}'$. Again, there is a unique double ladder diagram $I'^{\underline{x}}_{\underline{w}} \in$

$\mathrm{Hom}_{\mathcal{D}_{\mathfrak{sp}_4}}(\underline{w}, \underline{x})$ which maps $v_{\underline{w},+}$ to $v_{\underline{x},+}$. Since $\mathbb{LL}'$ is a basis, we can express $I_{\underline{w}}^{\underline{x}}$ as a linear combination of diagrams in $\mathbb{LL}'$

$$I_{\underline{w}}^{\underline{x}} = c \cdot I'^{\underline{x}}_{\underline{w}} + (\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4})_{<\mathrm{wt}\,\underline{w}}. \tag{3.4..3}$$

Looking at how both sides of (3.4..3) act on $v_{\underline{w},+}$ we deduce that $c = 1$.

Definition 3.4..8. Let $f \in \mathrm{Hom}_{\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}}(\underline{w}, \underline{x})$. We define the *neutral coefficient* of f to be the coefficient of $I_{\underline{w}}^{\underline{x}}$ when f is expressed in the basis $\mathbb{LL}$.

Lemma 3.4..9. *If* $\mathrm{wt}\,\underline{w} = \mathrm{wt}\,\underline{x}$*, then the neutral coefficient of* $f \in \mathrm{Hom}_{\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}}(\underline{w}, \underline{x})$ *is* c *if and only if* $f(v_{\underline{w},+}) = c \cdot v_{\underline{x},+}$.

Proof. Follows from Lemma (3.4..5) and Lemma (3.4..6). □

Remark 3.4..10. The discussion given above ensures that the neutral coefficient is independent of any choices that are made in the light ladder algorithm.

3.5. Definition and Basic Properties of Clasps

Our exposition is based on [7] and [22, Chapter 11].

Definition 3.5..1. We say that a morphism in $\mathrm{Hom}_{\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}}(\underline{w}, \underline{x})$ is a *clasp*, if it is killed by postcomposition with any downward diagram and has neutral coefficient 1. If $\mathrm{wt}(\underline{w}) = \lambda = \mathrm{wt}(\underline{x})$, then we may call such a map a λ-*clasp*.

Lemma 3.5..2. *Let* $C \in \mathrm{Hom}_{\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}}(\underline{w}, \underline{x})$ *have neutral coefficient equal to* 1*. Then the following are equivalent:*

1. *C is a clasp,*

2. *C is killed by postcomposition with any diagram in* $(\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4})_{<\mathrm{wt}\,\underline{w}}$*,*

3. *C is killed by postcomposition with any diagram of the form* $(\mathrm{id} \otimes L_\mu \otimes \mathrm{id}) \circ N$ *where N is a neutral diagram and L_μ is an elementary light ladder diagram for $\mu \notin \{\varpi_1, \varpi_2\}$.*

Proof. Since a diagram in $(\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4})_{<\mathrm{wt}\,\underline{w}}$ is a linear combination of downward diagrams, (1) implies (2). The diagram $(\mathrm{id} \otimes L_\mu \otimes \mathrm{id}) \circ N$ in (3) is a downward diagram, thanks to the assumption on μ, so (1) implies (3). From Lemma (3.4..4) we deduce that (2) implies (1).

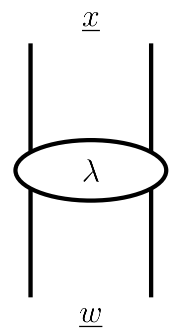
By the definition of double ladders as the composition of light ladders and upside down light ladders, and since light ladders are in particular double ladders where the upside down double ladder is the identity, we see that C is killed by postcomposition with $(\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4})_{<\mathrm{wt}\,\underline{w}}$ if and only if C is killed by postcomposition with any light ladder of the form $LL_{\underline{w},\vec{\mu}}$ where $\mathrm{wt}\,\vec{\mu} < \mathrm{wt}\,\underline{w}$. Then from the inductive definition of light ladders, we see (3) implies (2). □

Proposition 3.5..3. *If a clasp exists then it is unique, and it is also characterized as the map with neutral coefficient 1 which is killed by precomposition with any upward diagram. The composition of a clasp with a neutral ladder is a clasp (so if any λ-clasp exists, then all λ-clasps exist), the composition of two clasps is a clasp, and clasps are preserved by $\mathbb{D}$.*

Proof. We leave it as an exercise to adapt the proof in [7, Proposition 3.2] to our setting. □

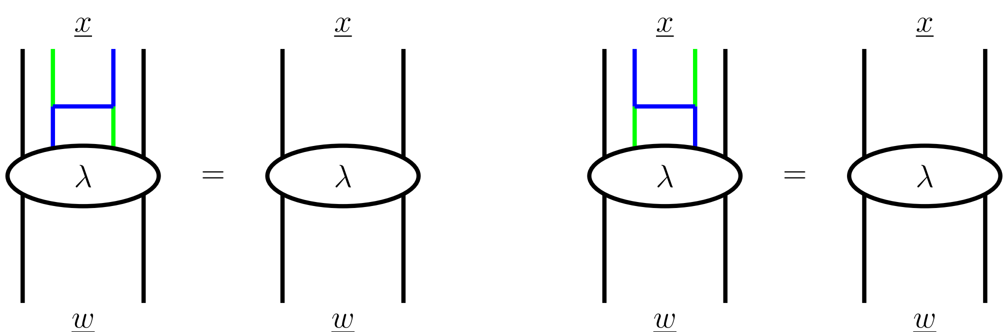
Graphically we will depict λ-clasps as ovals labelled by λ with source $\underline{w}$ and target $\underline{x}$. In writing we will denote it by C_λ.

C_λ =



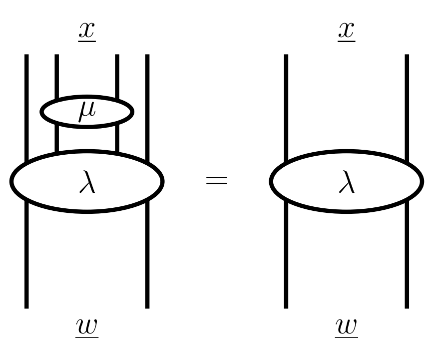
(3.5..1)

Proposition (3.5..3) says that the composition of a clasp with a neutral ladder is a clasp, we will refer to this as *neutral absorption*, depicted diagrammatically as follows.



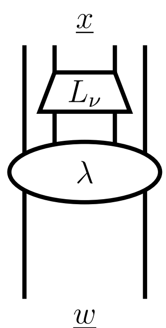
(3.5..2)

We also observe that Proposition (3.5..3) says the composition of two clasps is a clasp, which is what we will call *clasp absorption*. This is expressed diagrammatically as follows.



(3.5..3)

Finally, note that postcomposing a clasp with a non-identity elementary light ladder results in zero. We will refer to this phenomenon as *clasp orthogonality*, and it can be expressed diagrammatically as follows.



$$= 0 \tag{3.5..4}$$

Remark 3.5..4. The λ clasps give a *compatible system of idempotents* [7, Definition 3.3], and therefore represent an object in the Karoubi envelope of $\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}$. This object is a common summand of the objects $\underline{w}$ such that $\mathrm{wt}(\underline{w}) = \lambda$.

Notation 3.5..5. Given an idempotent $e \in \mathrm{End}_{\mathbf{k}\otimes\mathcal{D}_{\mathfrak{sp}_4}}(\underline{w})$ we get an object $(\underline{w}, e)$ in $\mathrm{Kar}(\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4})$. The object in the Karoubi envelope which corresponds to $(\underline{w}, C_\lambda)$ will be denoted by λ, for all $\underline{w}$ such that $\mathrm{wt}\,\underline{w} = \lambda$.

Recall that in the Karoubi envelope we have

$$\mathrm{Hom}_{\mathrm{Kar}(\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4})}((\underline{w}, e), (\underline{x}, f)) = f\,\mathrm{Hom}_{\mathbf{k}\otimes\mathcal{D}_{\mathfrak{sp}_4}}(\underline{w}, \underline{x})e. \tag{3.5..5}$$

Corollary 3.5..6. *Suppose the λ and χ clasps both exist. Then* $\mathrm{Hom}_{Kar(\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4})}(\lambda, \chi)$ *is spanned by the identity if $\lambda = \chi$ and is zero otherwise.*

Proof. Compare with [7, Corollary 3.6]. Let $D \in \mathbb{LL}^{\underline{x}_\chi}_{\underline{x}_\lambda}$. From clasp orthogonality it follows that $C_\chi \circ D \circ C_\lambda = 0$ unless $D = I^{\underline{x}_\chi}_{\underline{x}_\lambda}$. If $\lambda \neq \chi$, then every diagram in $\mathbb{LL}^{\underline{x}_\chi}_{\underline{x}_\lambda}$ is strictly lower, so $\mathrm{Hom}_{\mathrm{Kar}(\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4})}(\lambda, \chi) = 0$. Thanks to neutral absorption (3.5..2) we have $C_\chi \circ I_{\underline{x}_\lambda}^{\underline{x}_\chi} \circ C_\lambda = C_\chi \circ \mathrm{id} \circ C_\lambda$. Then from clasp absorption (3.5..3) it follows that $C_\chi = C_\chi \circ \mathrm{id} \circ C_\lambda = C_\lambda$. ☐

Lemma 3.5..7. *The λ clasp exists in $\mathcal{D}_{\mathfrak{sp}_4}^{\mathbf{k}}$ if and only if $V^{\mathbf{k}}(\lambda)$ is a direct summand of $V^{\mathbf{k}}(\underline{x}_\lambda)$. Moreover, when the λ clasp exists we have $C_\lambda = \Xi^{-1}(e_\lambda)$ where e_λ is the endomorphism of $V^{\mathbf{k}}(\underline{x}_\lambda)$ projecting to $V^{\mathbf{k}}(\lambda)$.*

Proof. Suppose that the λ clasp does exist. Consider the idempotent $e_\lambda \in \mathrm{End}(V^{\mathbf{k}}(\underline{x}_\lambda))$ which is the image under Ξ of the λ clasp in $\mathrm{End}_{\mathcal{D}_{\mathfrak{sp}_4}^{\mathbf{k}}}(\underline{x}_\lambda)$. The map e_λ projects to a direct summand of $V^{\mathbf{k}}(\underline{x}_\lambda)$, and by Corollary (3.5..6) the summand has endomorphism ring $\mathbf{k} \cdot \mathrm{id}$. Since $V^{\mathbf{k}}(\underline{x}_\lambda)[\lambda] = \mathbf{k} \cdot v_{\underline{x}_\lambda,+}$ and the lambda clasp preserves the λ weight vector $v_{\underline{x}_\lambda,+}$, the object $\mathrm{im}(e_\lambda)$ has a one dimensional lambda weight space. An object with a one-dimensional λ weight space and a local endomorphism ring must be the indecomposable tilting module of highest weight λ. Since the endomorphism ring of a tilting module is $\mathbf{k} \cdot \mathrm{id}$ if and only if the indecomposable tilting module is an irreducible Weyl module, it follows that $\mathrm{im}(e_\lambda) \cong V^{\mathbf{k}}(\lambda)$.

Suppose $V^{\mathbf{k}}(\lambda)$ is a summand of $V^{\mathbf{k}}(\underline{x}_\lambda)$, so there is an idempotent $e_\lambda \in \mathrm{End}_{U_q^{\mathbf{k}}(\mathfrak{sp}_4)}(V^{\mathbf{k}}(\underline{x}_\lambda))$ which projects to $V^{\mathbf{k}}(\lambda)$. Since $V^{\mathbf{k}}(\lambda)[\lambda]$ is one dimensional, it follows that $\mathrm{im}(e_\lambda)[\lambda] = V^{\mathbf{k}}(\underline{x}_\lambda)[\lambda] = \mathbf{k} \cdot v_{\underline{x}_\lambda,+}$. Restricting e_λ to the λ weight space induces an isomorphism. Hence, $e_\lambda(v_{\underline{x}_\lambda,+}) = \xi v_{\underline{x}_\lambda,+}$ for some $\xi \in \mathbf{k}^\times$. Since e_λ is idempotent, $\xi = 1$, so e_λ has neutral coefficient one. By Lemma (3.4..5), postcomposing e_λ with a downward map induces a map $V^{\mathbf{k}}(\lambda) \to V^{\mathbf{k}}(\underline{y})$ which has $V^{\mathbf{k}}(\lambda)[\lambda]$ in its kernel, and therefore is zero. We conclude that e_λ is a clasp. ☐

Remark 3.5..8. Since the finite dimensional representations of $\mathfrak{sp}_4(\mathbb{C})$ are completely reducible, it follows that when $\mathbf{k} = \mathbb{C}$ and $q = 1$, λ clasps exist for all $\lambda \in X_+$. If K is any field and $q \in K$ is transcendental, then λ clasps exist over K for all $\lambda \in X_+$.

Remark 3.5..9. We argue that if $V^{\mathbf{k}}(\lambda)$ is a direct summand of $V^{\mathbf{k}}(\underline{x}_\lambda)$, i.e. the λ clasp exists over $\mathbf{k}$, then the characteristic zero clasp can be used to compute the $\mathbf{k}$ clasp.

Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}, [2]_q^{-1}]$. When we say "all fields" we mean all pairs $\mathbf{k}$ and $q \in \mathbf{k}$ such that $q + q^{-1} \neq 0$. Any quotient of $\mathcal{A}$ by a maximal ideal will give such a pair.

From [31] we know that the set $\mathbf{D}$ of non-elliptic webs spans $\mathcal{D}_{\mathfrak{sp}_4}$ over $\mathcal{A}$, and we know from Theorem (2.14..5) that $\mathbb{LL}$ is linearly independent over all fields $\mathbf{k}$. It follows that the coefficients of a linear dependence among double ladders over $\mathcal{A}$ must all be contained in every maximal ideal of $\mathcal{A}$. But the Jacobson radical of $\mathcal{A}$ is zero. So $\mathbb{LL}$ is linearly independent over $\mathcal{A}$. Furthermore, from Theorem (2.14..5) it follows that the sets $\mathbf{D}$ and $\mathbb{LL}$ both give bases of $\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}$ for all fields $\mathbf{k}$.

Fix objects $\underline{w}, \underline{u} \in \mathcal{D}_{\mathfrak{sp}_4}$. Consider the matrix which expresses a double ladder in terms of the spanning set $\mathbf{D}$

$$A_{\underline{w}}^{\underline{u}} : \mathcal{A}\mathbb{LL}_{\underline{w}}^{\underline{u}} \longrightarrow \mathcal{A}B_{\underline{w}}^{\underline{u}} = \operatorname{Hom}_{\mathcal{D}_{\mathfrak{sp}_4}}(\underline{w}, \underline{u}). \tag{3.5..6}$$

This matrix is an isomorphism over $\mathbf{k}$, for all fields $\mathbf{k}$, so the determinant is not contained in any maximal ideal in $\mathcal{A}$. Thus, $\det A_{\underline{w}}^{\underline{u}}$ is a unit in $\mathcal{A}$, and $A_{\underline{w}}^{\underline{u}}$ is invertible over $\mathcal{A}$. Hence, $\mathbb{LL}$ spans $\mathcal{D}_{\mathfrak{sp}_4}$ over $\mathcal{A}$.

Let $\mathcal{O}$ be a complete discrete valuation ring, which is an $\mathcal{A}$ algebra, and such that $\mathcal{O}/\mathfrak{m} = \mathbf{k}$. Assume that the field $K = \operatorname{Frac}(\mathcal{O})$ is characteristic zero and $q \in K$ is transcendental. Suppose that $V^{\mathbf{k}}(\lambda)$ is a summand of $V^{\mathbf{k}}(\underline{x}_\lambda)$, i.e. there is a clasp $e_\lambda^{\mathbf{k}} \in \operatorname{End}_{\mathbf{k}\otimes\mathcal{D}_{\mathfrak{sp}_4}}(\underline{x}_\lambda)$. The endomorphism $e_\lambda^{\mathbf{k}}$ is an idempotent so it can be lifted to $e_\lambda^{\mathcal{O}} \in \operatorname{End}_{\mathcal{O}\otimes\mathcal{D}_{\mathfrak{sp}_4}}(\underline{x}_\lambda)$. Since $\mathbb{LL}$ is a basis of $\mathcal{D}_{\mathfrak{sp}_4}$ over $\mathcal{A}$, it follows that $\mathbb{LL}$ is a basis of the category $\mathcal{O} \otimes \mathcal{D}_{\mathfrak{sp}_4}$. Therefore,

$$e_\lambda^{\mathcal{O}} = x \operatorname{id} + (\mathcal{O} \otimes \mathcal{D}_{\mathfrak{sp}_4})_{<\lambda}.$$

Since $e_\lambda^{\mathcal{O}}$ specializes to $e_\lambda^{\mathbf{k}}$, which in turn sends $v_{\underline{x}_\lambda,+}$ to $v_{\underline{x}_\lambda,+}$, there is some $m \in \mathfrak{m}$ such that $x = 1 - m$. Also, $e_\lambda^{\mathcal{O}}$ is an idempotent and $(\mathcal{O} \otimes \mathcal{D}_{\mathfrak{sp}_4})_{<\lambda}$ is an ideal in $\mathcal{O} \otimes \mathcal{D}_{\mathfrak{sp}_4}$ so

$$x \operatorname{id} + (\mathcal{O} \otimes \mathcal{D}_{\mathfrak{sp}_4})_{<\lambda} = \left(x \operatorname{id} + (\mathcal{O} \otimes \mathcal{D}_{\mathfrak{sp}_4})_{<\lambda}\right)^2 = x^2 \operatorname{id} + (\mathcal{O} \otimes \mathcal{D}_{\mathfrak{sp}_4})_{<\lambda}.$$

Comparing neutral coefficients, we find $x = x^2$. It follows that $1 - m = (1-m)^2 = 1 - 2m + m^2$, which implies $m^2 = m$. Since $m \in \mathfrak{m}$, we may conclude that $m = 0$.

From the fact that $\mathbb{LL}$ is a basis over $\mathcal{O}$ it follows that the homomorphism spaces in $\mathcal{O} \otimes \mathcal{D}_{\mathfrak{sp}_4}$ are free and finitely generated $\mathcal{O}$-modules. Thus, the $\mathcal{O}$ module

$$e_\lambda^{\mathcal{O}} \operatorname{Hom}_{\mathcal{O} \otimes \mathcal{D}_{\mathfrak{sp}_4}}(\underline{x}_\lambda) e_\lambda^{\mathcal{O}} \tag{3.5..7}$$

is a finitely generated projective $\mathcal{O}$ module. Since $\mathcal{O}$ is local, one can use Nakayama's lemma to show that projective and finitely generated implies free of finite rank. A consequence is the equality

$$\operatorname{rk}_{\mathcal{O}} e_\lambda^{\mathcal{O}} \operatorname{End}_{\mathcal{O} \otimes \mathcal{D}_{\mathfrak{sp}_4}}(\underline{x}_\lambda) e_\lambda^{\mathcal{O}} = \dim_{\mathbf{k}} e_\lambda^{\mathbf{k}} \operatorname{End}_{\mathbf{k} \otimes \mathcal{D}_{\mathfrak{sp}_4}}(\underline{x}_\lambda) e_\lambda^{\mathbf{k}}. \tag{3.5..8}$$

We know $e_\lambda^{\mathbf{k}} \operatorname{End}_{\mathbf{k} \otimes \mathcal{D}_{\mathfrak{sp}_4}}(\underline{x}_\lambda) e_\lambda^{\mathbf{k}} = \mathbf{k} \cdot e_\lambda^{\mathbf{k}}$, so we may deduce that $e_\lambda^{\mathcal{O}} \operatorname{Hom}_{\mathcal{O} \otimes \mathcal{D}_{\mathfrak{sp}_4}}(\underline{x}_\lambda) e_\lambda^{\mathcal{O}} = \mathcal{O} \cdot e_\lambda^{\mathcal{O}}$.

On the other hand, we know there is a characteristic zero clasp, $e_\lambda^K \in \operatorname{End}_{K \otimes \mathcal{D}_{\mathfrak{sp}_4}}(\underline{x}_\lambda)$. Using that e_λ^K has neutral coefficient one and is orthogonal to all downward diagrams, we may conclude that $e_\lambda^{\mathcal{O}} e_\lambda^K e_\lambda^{\mathcal{O}} = e_\lambda^K$, so $e_\lambda^K \in e_\lambda^{\mathcal{O}} \operatorname{End}_{K\otimes\mathcal{D}_{\mathfrak{sp}_4}}(\underline{x}_\lambda) e_\lambda^{\mathcal{O}} = K e_\lambda^{\mathcal{O}}$. By comparing neutral coefficients we see that $e_\lambda^K = e_\lambda^{\mathcal{O}}$.

Over K the λ clasp exists for all $\lambda \in X_+$, so to compute e_λ^K we are free to use the recursion from our main theorem to expand this clasp in terms of the basis $\mathbb{LL}$. The argument we just sketched implies that the coefficients, of the double ladders, in the expanded clasp actually lie in $\mathcal{O}$. So we can reduce e_λ^K modulo a maximal ideal to obtain $e_\lambda^{\mathbf{k}}$.

3.6. Intersection Forms and Triple Clasp Formulas

Let $a \in \{1, 2\}$. If $\mathbf{k} = \mathbb{C}$ and $q = 1$, we know that $\Xi(\lambda \otimes a) = V(\lambda) \otimes V(\varpi_a)$ decomposes as described Equation (2.3..3).

Definition 3.6..1. Let $a \in \{1, 2\}$ and let $\lambda \in X_+$. Define the set $S_{\lambda,a}$ to be the collection of weights $\mu \in \operatorname{wt} V(a)$ such that $V(\lambda + \mu)$ is a direct summand of $V(\lambda) \otimes V(a)$. Since each weight in $\operatorname{wt} V(a)$ is multiplicity one,

$$V(\lambda) \otimes V(a) \cong \bigoplus_{\mu \in S_{\lambda,a}} V(\lambda + \mu). \tag{3.6..1}$$

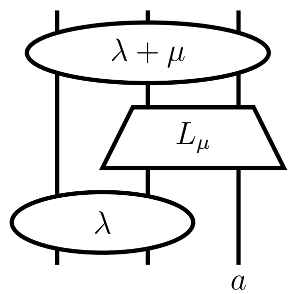
Lemma 3.6..2. *$V^{\mathbf{k}}(\lambda) \otimes V^{\mathbf{k}}(a)$ has a filtration by the Weyl modules $V^{\mathbf{k}}(\lambda + \mu)$ for $\mu \in S_{\lambda,a}$.*

Proof. The tensor product of Weyl filtered modules has a Weyl module filtration. Since $V^{\mathbf{k}}(\chi)$ has the same character as $V(\chi)$, the filtration multiplicities are determined by the character of the Weyl filtered module. Therefore, the claim follows from Equation (3.6..1). □

Definition 3.6..3. Let $\mu \in S_{\lambda,a}$. Suppose the λ and $\lambda+\mu$ clasps exist over $\mathbf{k}$. There is an elementary light ladder L_μ which induces a map

$$C_{\lambda+\mu} \circ (\mathrm{id} \otimes L_\mu) \circ (C_\lambda \otimes \mathrm{id}_a) : \lambda \otimes a \to \lambda + \mu.$$

We denote this map by $E_{\lambda,\mu}$ and depict it diagrammatically by

$E_{\lambda,\mu} =$ 

(3.6..2)

Proposition 3.6..4. *Suppose the λ clasp exists and that the $\lambda+\mu$ clasp exists. Then $\{E_{\lambda,\mu}\}$ is a basis for* $\mathrm{Hom}_{Kar\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}}(\lambda \otimes a, \lambda+\mu)$.

Proof. Since double ladders are a basis, it follows that after postcomposing with $C_{\lambda+\mu}$ and precomposing with $C_\lambda \otimes \mathrm{id}_a$ the double ladders $\mathbb{LL}^{\underline{x}_{\lambda+\mu}}_{\underline{x}_\lambda a}$ will span $\mathrm{Hom}_{\mathrm{Kar}\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}}(\lambda \otimes a, \mu)$.

Let $D \in \mathbb{LL}^{\underline{x}_{\lambda+\mu}}_{\underline{x}_\lambda a}$. By the definition of double ladders, there are dominant weight sequences $\vec{\nu} \in E(\underline{x}_{\lambda+\mu})$ and $\vec{\chi} = (\chi_1, \dots \chi_n) \in E(\underline{x}_\lambda a)$ such that $D = \mathbb{D}(LL_{\underline{x}_{\lambda+\mu},\vec{\nu}}) \circ LL_{\underline{x}_\lambda a,\vec{\chi}}$. Due to clasp orthogonality (3.5..4), $C_{\lambda+\mu} \circ D \circ (C_\lambda \otimes \mathrm{id}_a) = 0$ unless $\vec{\nu} \in E(\underline{x}_{\lambda+\mu}, \lambda+\mu)$ and $(\chi_1, \dots \chi_{n-1}) \in E(\underline{x}_\lambda, \lambda)$. Using the neutral absorption property of clasps (3.5..2), we now see that

$$E_{\lambda,\mu} = C_{\lambda+\mu} \circ \mathbb{LL}^{\underline{x}_{\lambda+\mu}}_{\underline{x}_\lambda a} \circ (C_\lambda \otimes \mathrm{id}_a).$$

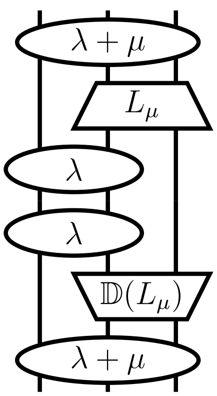
Therefore, $E_{\lambda,\mu}$ spans $\mathrm{Hom}_{\mathrm{Kar}\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}}(\lambda \otimes a, \lambda+\mu)$.

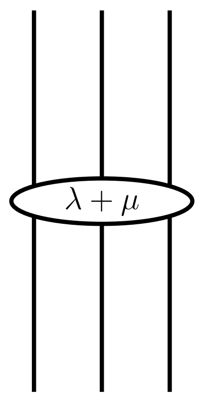
The $\lambda + \mu$ clasp exists, so $V^{\mathbf{k}}(\lambda+\mu)$ is an irreducible Weyl module, and therefore $V^{\mathbf{k}}(\lambda+\mu) \cong V^{\mathbf{k}}(\lambda+\mu)^*$. Since $V^{\mathbf{k}}(\lambda+\mu)$ occurs exactly once in the Weyl filtration of $V^{\mathbf{k}}(\lambda) \otimes V^{\mathbf{k}}(a)$, it follows from Lemma (2.13..9) that

$$\dim \operatorname{Hom}_{\operatorname{Kar}\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}}(\lambda \otimes a, \lambda+\mu) = \dim \operatorname{Hom}_{U_q^{\mathbf{k}}(\mathfrak{sp}_4)}(V^{\mathbf{k}}(\lambda) \otimes V^{\mathbf{k}}(a), V^{\mathbf{k}}(\lambda+\mu)^*) = 1.$$

Thus $E_{\lambda,\mu}$ spans a one dimensional vector space and therefore is a basis. $\square$

Definition 3.6..5. The map $K_{\lambda,\mu} := E_{\lambda,\mu} \circ \mathbb{D}E_{\lambda,\mu}$ is an endomorphism of $\lambda+\mu$, and this endomorphism space is spanned by the identity map. We define the *local intersection form* $\kappa_{\lambda,\mu}$ to be the neutral coefficient of $E_{\lambda,\mu} \circ \mathbb{D}E_{\lambda,\mu}$.

$K_{\lambda,\mu} =$ 

$= \kappa_{\lambda,\mu}$ 

(3.6..3)

Lemma 3.6..6. *Suppose that both the λ clasp and the $\lambda+\mu$ clasp exist. If the local intersection form $\kappa_{\lambda,\mu}$ is nonzero, then $\frac{1}{\kappa_{\lambda,\mu}}\mathbb{D}E_{\lambda,\mu} \circ E_{\lambda,\mu}$ is an idempotent in* $\operatorname{End}_{Kar(\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4})}(\lambda \otimes a)$ *which projects to $V^{\mathbf{k}}(\lambda+\mu)$. If the local intersection form is zero, then $V^{\mathbf{k}}(\lambda+\mu)$ is not a summand of $V^{\mathbf{k}}(\lambda) \otimes V^{\mathbf{k}}(a)$.*

Proof. If $\kappa_{\lambda,\mu} \neq 0$, then $\frac{1}{\kappa_{\lambda,\mu}}\mathbb{D}E_{\lambda,\mu} \circ E_{\lambda,\mu}$ is a non-zero idempotent factoring through $V^{\mathbf{k}}(\lambda+\mu)$. Since the $\lambda+\mu$ clasp exists, the module $V^{\mathbf{k}}(\lambda+\mu)$ is irreducible. Thus, the idempotent has image isomorphic to $V^{\mathbf{k}}(\lambda+\mu)$.

Since $E_{\lambda,\mu}$ is a basis for $\mathrm{Hom}_{\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}}(\lambda \otimes a, \lambda + \mu)$ it follows that $\mathbb{D}(E_{\lambda,\mu})$ is a basis for $\mathrm{Hom}_{\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4}}(\lambda + \mu, \lambda \otimes a)$. If $\kappa_{\lambda,\mu} = 0$, then every pair of projection $V^{\mathbf{k}}(\lambda) \otimes V^{\mathbf{k}}(a) \to V^{\mathbf{k}}(\lambda + \mu)$ and inclusion $V^{\mathbf{k}}(\lambda + \mu) \to V^{\mathbf{k}}(\lambda) \otimes V^{\mathbf{k}}(a)$ compose to be $0 \in \mathrm{End}_{U_q^{\mathbf{k}}(\mathfrak{sp}_4)}(V^{\mathbf{k}}(\lambda + \mu))$. Thus, $V^{\mathbf{k}}(\lambda + \mu)$ is not a direct summand of $V^{\mathbf{k}}(\lambda) \otimes V^{\mathbf{k}}(a)$. □

Remark 3.6..7. By working modulo the ideals $(\mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4})_{<\lambda}$ instead of with clasps, one can show that the indecomposable tilting module $T^{\mathbf{k}}(\lambda + \mu)$ is a direct summand of $V^{\mathbf{k}}(\lambda) \otimes V^{\mathbf{k}}(a)$ if and only if $\kappa_{\lambda,\mu} \neq 0$.

Proposition 3.6..8. *Suppose that the λ clasp exists. Also assume the $\lambda + \mu$ clasps exist and the $\kappa_{\lambda,\mu}$ are invertible in $\mathbf{k}$, for all $\mu \in S_{\lambda,a} - \{\varpi_a\}$. Then the $\lambda + \varpi_a$ clasp exists and*

$$C_{\lambda+\varpi_a} = C_\lambda \otimes \mathrm{id}_a - \sum \kappa_{\lambda,\mu}^{-1} \mathbb{D}E_{\lambda,\mu} \circ E_{\lambda,\mu}, \tag{3.6..4}$$

where the sum is over all $\mu \in S_{\lambda,a} - \{\varpi_a\}$.

Proof. Lemma (3.6..6) implies that $V^{\mathbf{k}}(\lambda+\mu)$ is a direct summand of $V^{\mathbf{k}}(\lambda) \otimes V^{\mathbf{k}}(a)$ for all $\mu \in S_{\lambda,a} - \{\varpi_a\}$. So

$$V^{\mathbf{k}}(\lambda) \otimes V^{\mathbf{k}}(a) = X \bigoplus_{\mu \in S_{\lambda,a} - \{\varpi_a\}} V^{\mathbf{k}}(\lambda + \mu), \tag{3.6..5}$$

where X is a direct summand with highest weight $\lambda + \varpi_a$. Direct summands of Weyl filtered modules have a Weyl filtration, so X is filtered by Weyl modules. Comparing characters of both sides of (3.6..5) implies that $X \cong V^{\mathbf{k}}(\lambda + \varpi_a)$. The decomposition in (3.6..5) gives rise to the following equality in the endomorphism algebra of $V^{\mathbf{k}}(\lambda) \otimes V^{\mathbf{k}}(a)$:

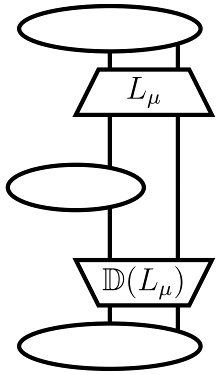
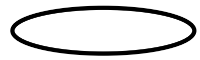
$$\mathrm{id}_{V^{\mathbf{k}}(\lambda)} \otimes \mathrm{id}_{V^{\mathbf{k}}(a)} = \sum_{\mu \in S_{\lambda,a}} e_{\lambda+\mu}, \tag{3.6..6}$$

where $e_{\lambda+\mu}$ is the projection to the summand isomorphic to $V^{\mathbf{k}}(\lambda+\mu)$. The module $V^{\mathbf{k}}(\lambda)$ is a direct summand of $V^{\mathbf{k}}(\underline{x}_\lambda)$, the image of C_λ, so $V^{\mathbf{k}}(\lambda) \otimes V^{\mathbf{k}}(a)$ is a direct summand of $V^{\mathbf{k}}(\underline{x}_\lambda a)$, the image of $C_\lambda \otimes \mathrm{id}_a$. Precomposing and postcomposing Equation (3.6..6) with $C_\lambda \otimes \mathrm{id}_a$ yields the desired equality from Equation (3.6..4), in the endomorphism algebra of $V^{\mathbf{k}}(\underline{x}_\lambda a)$. □

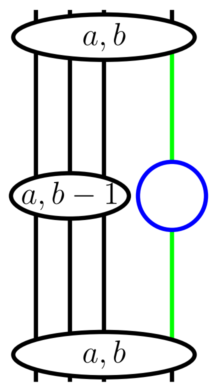
3.7. Deriving the Recursive Formula for Clasp Coefficients

We will compute recursive formulas for the local intersection forms $\kappa_{\lambda,\mu}$ using the graphical calculus for $\mathcal{D}_{\mathfrak{sp}_4}$.

Notation 3.7..1. To simplify notation, we will write (a, b) for $a\varpi_1 + b\varpi_2$. We will also leave off labels of clasps when the highest weight is understood. Furthermore, we will often leave off extra strands below (above) clasps which are on the bottom (top) of the diagram, as well as strands to the left of a diagram which has a clasp at the top or bottom. This is justified because all clasps with the same highest weight are transformed to one another by applying neutral diagrams on the top and bottom (in other contexts this could be nontrivial to verify, but it is easy to see that any two words in 1 and 2 of the same weight differ by a neutral diagram). We also freely use clasp absorption (3.5..3) to simplify formulas. For example (3.6..3) becomes

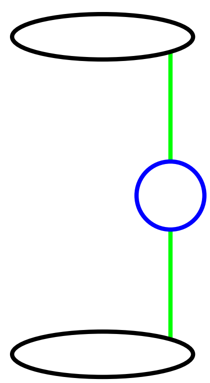
$K_{\lambda,\mu} =$  $= \kappa_{\lambda,\mu}$  (3.7..1)

and



(3.7..2)

becomes the following.



(3.7..3)

We define $\kappa_{\lambda,\varpi_a} = 1$ for all $\lambda \in X_+$ and $a \in \{1, 2\}$. Whenever $\mu \notin S_{(a,b),\varpi}$, we set $\kappa^{-1}_{(a,b),\mu}$ equal to zero. This results in the following initial conditions for our recursion:

$$\kappa^{-1}_{(a,b),(1,-1)} = 0 \qquad \text{when} \qquad b = 0, \tag{3.7..4}$$

$$\kappa^{-1}_{(a,b),(-1,0)} = \kappa^{-1}_{(a,b),(-1,1)} = 0 \qquad \text{when} \qquad a = 0, \tag{3.7..5}$$

$$\kappa^{-1}_{(a,b),(0,0)} = 0 \qquad \text{when} \qquad a = 0, \tag{3.7..6}$$

$$\kappa^{-1}_{(a,b),(-2,1)} = 0 \qquad \text{when} \qquad a = 0 \text{ or } 1, \tag{3.7..7}$$

and

$$\kappa^{-1}_{(a,b),(0,-1)} = \kappa^{-1}_{(a,b),(2,-1)} = 0 \qquad \text{when} \qquad b = 0. \tag{3.7..8}$$

Proposition 3.7..2. *The $\kappa_{\lambda,\mu}$'s satisfy the following relations.*

$$\kappa_{(a,b),(1,0)} = 1 \tag{3.7..9}$$

$$\kappa_{(a,b),(0,1)} = 1 \tag{3.7..10}$$

$$\kappa_{(a,b),(-1,1)} = -[2] - \frac{1}{\kappa_{(a-1,b),(-1,1)}} \tag{3.7..11}$$

$$\kappa_{(a,b),(2,-1)} = -\frac{[4]}{[2]} - \frac{1}{\kappa_{(a,b-1),(2,-1)}} \tag{3.7..12}$$

$$\kappa_{(a,b),(0,0)} = \frac{[5]}{[2]} - \frac{\kappa_{(a-2,b+1),(2,-1)}}{\kappa^{-1}_{(a-1,b),(-1,1)}} - \frac{1}{\kappa_{(a-1,b),(1,-1)}} \tag{3.7..13}$$

$$\kappa_{(a,b),(1,-1)} = \frac{[5]}{[2]} - \frac{\kappa_{(a+2,b-2),(-1,1)}}{\kappa_{(a,b-1),(2,-1)}} - \frac{1}{[2]^2\kappa_{(a,b-1),(0,0)}} \tag{3.7..14}$$

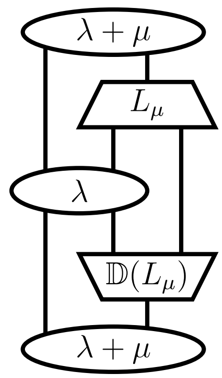
$$\begin{aligned}\kappa_{(a,b),(-2,1)} = \frac{[5]}{[2]}\kappa_{(a-1,b),(-1,1)} - (-[2] - \frac{1}{\kappa_{(a-2,b),(-1,1)}})\frac{\kappa_{(a-1,b),(-1,1)}}{\kappa_{(a-1,b),(-1,0)}} \\ - \frac{\kappa_{(a-2,b+1),(0,0)}}{\kappa^2_{(a-2,b),(-1,1)}\kappa_{(a-1,b),(-1,1)}}\end{aligned} \tag{3.7..15}$$

$$\kappa_{(a,b),(-1,0)} = -\frac{[6][2]}{[3]} - \frac{1}{\kappa_{(a-1,b),(-1,0)}} - \frac{\kappa_{(a-2,b+1),(1,-1)}}{\kappa_{(a-1,b),(-1,1)}} - \frac{\kappa_{(a,b-1),(-1,1)}}{\kappa_{(a-1,b),(1,-1)}} \tag{3.7..16}$$

$$\kappa_{(a,b),(0,-1)} = \frac{[6][5]}{[3][2]} - \frac{1}{\kappa_{(a,b-1),(0,-1)}} - \frac{\kappa_{(a+2,b-2),(-2,1)}}{\kappa_{(a,b-1),(2,-1)}} - \frac{\kappa_{(a,b-1),(0,0)}}{\kappa_{(a,b-1),(0,0)}} - \frac{\kappa_{(a-2,b),(2,-1)}}{\kappa_{(a,b-1),(-2,1)}} \tag{3.7..17}$$

Proof. We will use the established properties of clasps to derive the recursion relations. Recall that $\kappa_{\lambda,\mu}$ is the coefficient of the neutral map in $K_{\lambda,\mu}$.

$$K_{\lambda,\mu} = \qquad (3.7..18)$$

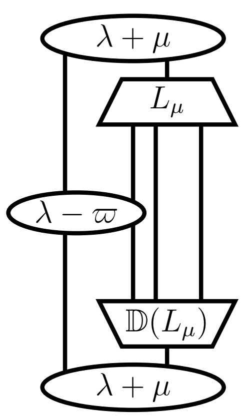


Using the equation

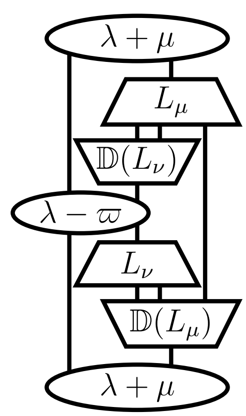
$$\lambda = (\lambda - \varpi) \otimes \varpi - \sum_{\nu \in \mathrm{wt}\, V(\varpi) - \{\varpi\}} \kappa^{-1}_{\lambda-\varpi,\nu} \mathbb{D}E_{\lambda-\varpi,\nu} \circ E_{\lambda-\varpi,\nu} \qquad (3.7..19)$$

to rewrite the λ clasp, and then using clasp absorption (3.5..3), we can rewrite $K_{\lambda,\mu}$ as the following.

$$K_{\lambda,\mu} =$$

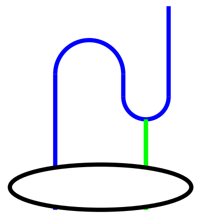


$$- \sum_{\nu \in \mathrm{wt}\, V(\varpi) - \{\varpi\}} \kappa^{-1}_{\lambda-\varpi,\nu} \qquad (3.7..20)$$



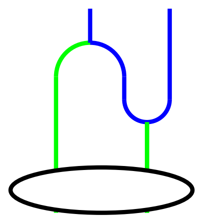
Having established the general pattern one follows to derive these recursive formulas, we proceed to apply it for each $\kappa_{(a,b),\mu}$.

To compute $\kappa_{(a,b),(-1,1)}$, we resolve the (a,b) clasp in $K_{(a,b),(-1,1)}$ as in (3.7..20). Since strictly lower diagrams are orthogonal to clasps (3.5..4), we find



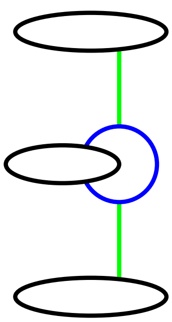
$$= 0 \tag{3.7..21}$$

and

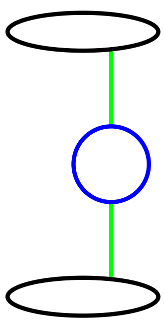


$$= 0. \tag{3.7..22}$$

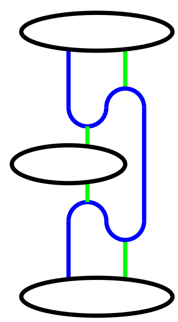
So after expanding the (a,b) clasp in $K_{(a,b),(-1,1)}$ the only $\nu \in \operatorname{wt} V(1)$ which contributes to the sum in (3.7..20) is $(-1,1)$. This means we can rewrite $K_{(a,b),(-1,1)}$ as follows.



$=$



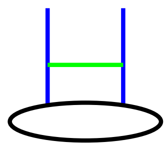
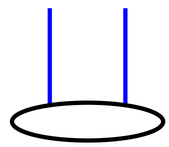
$- \kappa^{-1}_{(a-1,b),(-1,1)}$



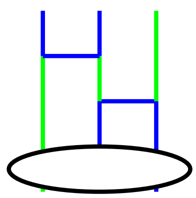
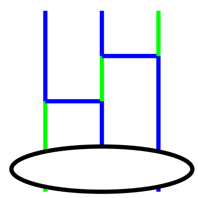
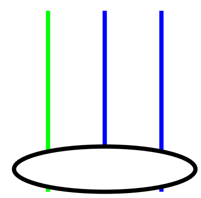
(3.7..23)

Using neutral ladder absorption (3.5..2) and clasp absorption (3.5..3), we deduce $\kappa_{(a,b),(-1,1)} = -[2] - \kappa^{-1}_{(a-1,b),(-1,1)}$.

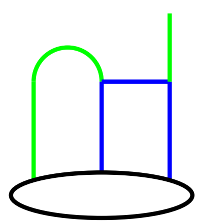
We proceed similarly with the remaining $K_{\lambda,\mu}$. It is useful to note that from the $H \equiv I$ relation (1.1..7) we have the following "clasped" relation.

 $= \frac{1}{[2]}$  (3.7..24)

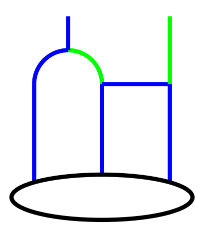
Similarly, using the $H \equiv I$ relation (1.1..7), orthogonality to clasps (3.5..4), and neutral map absorption (3.5..2) we can also deduce the following.

 $=$  $=$  (3.7..25)

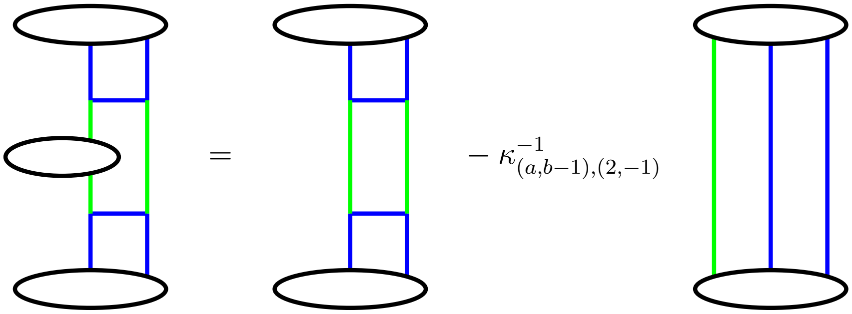
For $\kappa_{(a,b),(2,-1)}$, we begin by observing that by clasp orthogonality (3.5..4)

 $= 0,$ (3.7..26)

and by (3.7..24) and clasp orthogonality (3.5..4),

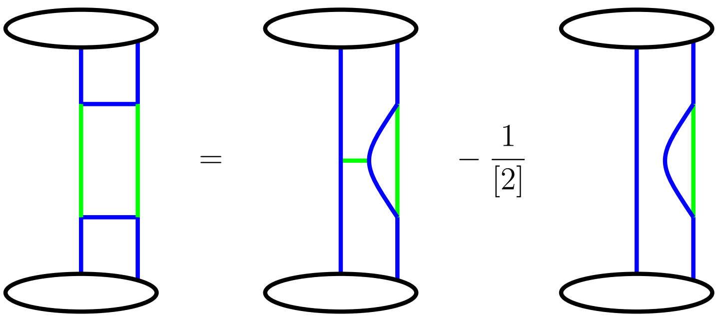
 $= 0.$ (3.7..27)

Using these observations and (3.7..25), $K_{(a,b),(2,-1)}$ can be resolved as follows.



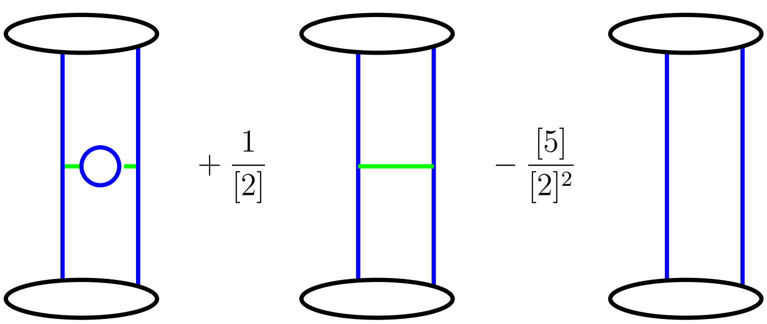
$$= \qquad - \kappa^{-1}_{(a,b-1),(2,-1)} \qquad (3.7..28)$$

Then we apply the $H \equiv I$ relation (1.1..7) to find



$$= \qquad - \frac{1}{[2]} \qquad (3.7..29)$$

Here the vanishing of the third term is due to clasp orthogonality (3.5..4). If we apply the $H \equiv I$ relation (1.1..7) again, then by the monogon relation (1.1..4) and clasp orthogonality (3.5..4) we can rewrite the right hand side as follows.



$$+ \frac{1}{[2]} \qquad - \frac{[5]}{[2]^2} \qquad (3.7..30)$$

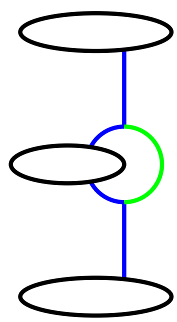
Finally, using the bigon relation and the clasped $H \equiv I$ relation (1.1..7) we can rewrite (3.7..30) as

$$-[2]\frac{1}{[2]}+\frac{1}{[2]^2}-\frac{[5]}{[2]^2}=-\frac{[4]}{[2]} \tag{3.7..31}$$

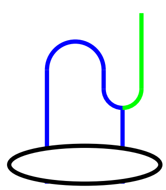
times the clasp, and then conclude that

$$\kappa_{(a,b),(2,-1)}=-\frac{[4]}{[2]}-\kappa^{-1}_{(a,b-1),(2,-1)}. \tag{3.7..32}$$

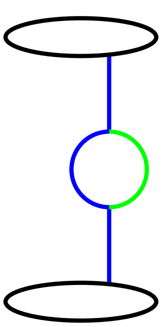
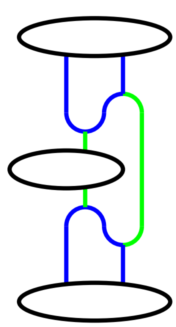
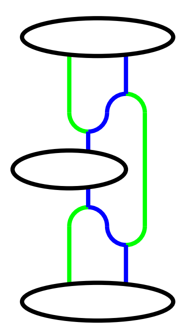
To compute $\kappa_{(a,b),(0,0)}$ we will expand the middle clasp in $K_{(a,b),(0,0)}$.

$K_{(a,b),(0,0)} =$  (3.7..33)

Since

 $= 0,$ (3.7..34)

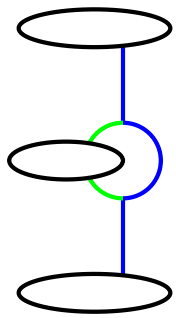
we can rewrite $K_{(a,b),(0,0)}$ as follows.

 $-\kappa^{-1}_{(a-1,b),(-1,1)}$  $-\kappa^{-1}_{(a-1,b),(1,-1)}$  (3.7..35)

Observing that the second term in (3.7..35) is $K_{(a-2,b+1),(2,-1)}$, and then using neutral map absorption (3.5..2) and clasp absorption (3.5..3) for the third term in (3.7..35), we deduce that

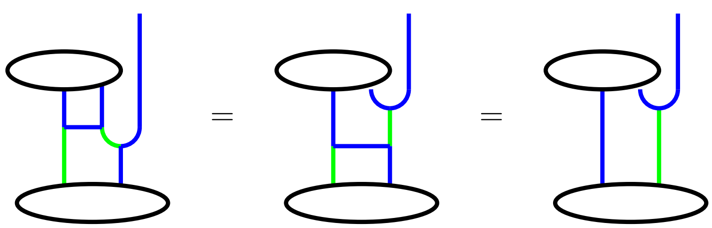
$$\kappa_{(a,b),(0,0)} = \frac{[5]}{[2]} - \kappa^{-1}_{(a-1,b),(-1,1)}\kappa_{(a-2,b+1),(2,-1)} - \kappa^{-1}_{(a-1,b),(1,-1)}. \tag{3.7..36}$$

To compute $\kappa_{(a,b),(1,-1)}$ we expand the middle clasp in $K_{(a,b),(1,-1)}$.



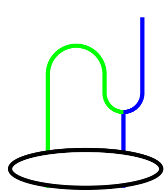
(3.7..37)

We begin by using the $H \equiv I$ relation (1.1..7) and clasp orthogonality (3.5..4), followed by neutral absorption (3.5..2), to calculate the following.



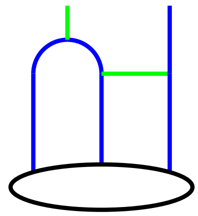
(3.7..38)

Clasp orthogonality (3.5..4) implies



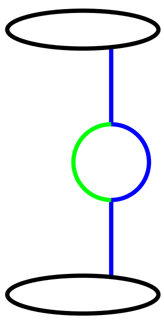
$= 0,$ (3.7..39)

and the $H \equiv I$ relation followed by clasp orthogonality (3.5..4) implies that

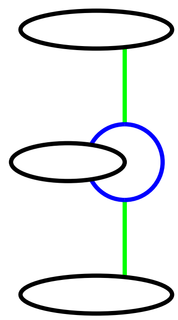


$$= 0. \tag{3.7..40}$$

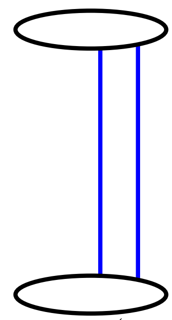
Therefore, we can rewrite $K_{(a,b),(1,-1)}$ as follows.



$$- \kappa^{-1}_{(a,b-1),(2,-1)}$$



$$- \kappa^{-1}_{(a,b-1),(0,0)} \frac{1}{[2]^2}$$



(3.7..41)

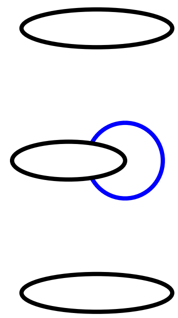
Identifying the second term in (3.7..41) as $K_{(a+2,b-2),(-1,1)}$, we deduce that

$$\kappa_{(a,b),(1,-1)} = \frac{[5]}{[2]} - \kappa^{-1}_{(a,b-1),(2,-1)} \kappa_{(a+2,b-2),(-1,1)} - \frac{1}{[2]^2} \kappa^{-1}_{(a,b-1),(0,0)}. \tag{3.7..42}$$

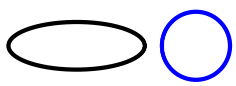
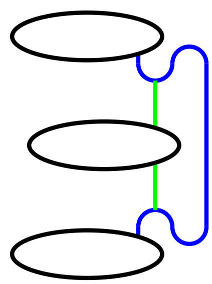
Remark 3.7..3. Note that at this point we could start solving these recursive relations, as the local intersection forms for the weights $(-1,1)$ and $(2,-1)$ are linked only to themselves in their recursion relation. While the local intersection forms for the weights $(0,0)$ and $(1,-1)$ have recursions which link them to themselves, each other, and the local intersection forms for the weights $(-1,1)$ and $(2,-1)$.

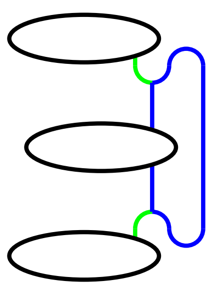
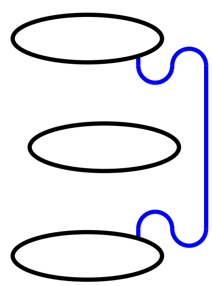
Continuing with our derivation of recursive relations for local intersection forms, we expand the middle clasp in

$$K_{(a,b),(-1,0)} = \tag{3.7..43}$$



and apply clasp absorption (3.5..3) to deduce the following.

$K_{(a,b),(-1,0)} =$  $- \kappa^{-1}_{(a-1,b),(-1,1)}$ 

$- \kappa^{-1}_{(a-1,b),(1,-1)}$  $- \kappa^{-1}_{(a-1,b),(-1,0)}$ 

(3.7..44)

By identifying the second and third terms on the right hand side of (3.7..44) as $K_{(a-2,b+1),(1,-1)}$ and $K_{(a,b-1),(-1,1)}$ respectively, we find

$$\kappa_{(a,b),(-1,0)} = -\frac{[6][2]}{[3]} - \frac{\kappa_{(a-2,b+1),(1,-1)}}{\kappa_{(a-1,b),(-1,1)}} - \frac{\kappa_{(a,b-1),(-1,1)}}{\kappa_{(a-1,b),(1,-1)}} - \frac{1}{\kappa_{(a-1,b),(-1,0)}}. \tag{3.7..45}$$

Remark 3.7..4. Again, we could stop here and solve the recursive relations since the local intersection form for the weight $(-1,0)$ involves the weights $(-1,0)$ and the weights we have computed recursions for previously.

The antidominant weight in $V(\varpi_2)$ is $(0,-1)$. A calculation similar to the derivation of the recursion for $\kappa_{(a,b),(-1,0)}$ results in

$$\kappa_{(a,b),(0,-1)} = \frac{[6][5]}{[3][2]} - -\frac{\kappa_{(a-2,b),(2,-1)}}{\kappa_{(a,b-1),(-2,1)}} - \frac{\kappa_{(a,b-1),(0,0)}}{\kappa_{(a,b-1),(0,0)}} - \frac{\kappa_{(a+2,b-2),(-2,1)}}{\kappa_{(a,b-1),(2,-1)}} - \frac{1}{\kappa_{(a,b-1),(0,-1)}}. \tag{3.7..46}$$

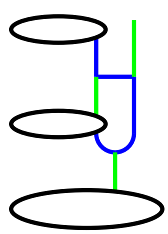
The last local intersection form to resolve is $K_{(a,b),(-2,1)}$. Recall that

$$\mathbf{k} \otimes \Xi : \mathcal{D}^{\mathbf{k}}_{\mathfrak{sp}_4} \longrightarrow \mathbf{Fund}(U_q^{\mathbf{k}}(\mathfrak{sp}_4)) \tag{3.7..47}$$

is an equivalence. Also, we know that if a Weyl module $\mathbf{k} \otimes V^{\mathbf{k}}(\lambda)$ is simple, then we can compute the dimension of homomorphism spaces involving that Weyl module in characteristic zero. Thus, from Equation (2.3..3) we see that

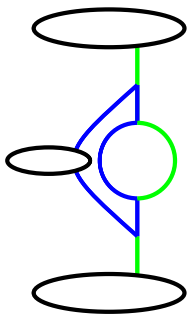
$$\dim \mathrm{Hom}_{\mathbf{k}\otimes\mathcal{D}_{\mathfrak{sp}_4}}((a-2,b+1),(a,b-1)\otimes 2) = \\ \dim \mathrm{Hom}_{\mathfrak{sp}_4(\mathbb{C})}(V(a-2,b+1), V(a,b-1)\otimes V(\varpi_2)) = 0,$$

and it follows that

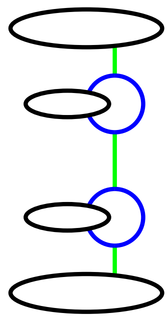


$$= 0. \tag{3.7..48}$$

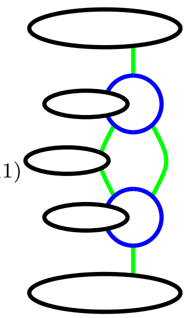
When we expand the (a, b) clasp in $K_{(a,b),(-2,1)}$, one of the terms has (3.7..48) as a sub-diagram and therefore is zero, so we get the following three terms.



$- \kappa^{-1}_{(a-1,b),(-1,0)}$



$- \kappa^{-1}_{(a-1,b),(-1,1)}$

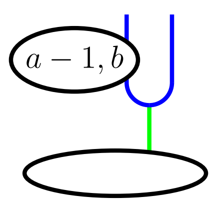


(3.7..49)

The first term in (3.7..49) simplifies to

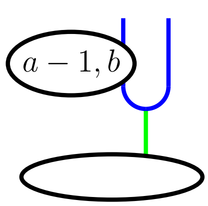
$$\frac{[5]}{[2]}\kappa_{(a-1,b),(-1,1)}. \tag{3.7..50}$$

We need to resolve the second and third terms on the right hand side of (3.7..49). Both terms contain the following sub-diagram.



(3.7..51)

Expanding the $(a-1, b)$ clasp and using clasp orthogonality (3.5..4) and neutral absorption (3.5..2), we can rewrite (3.7..51) as follows.



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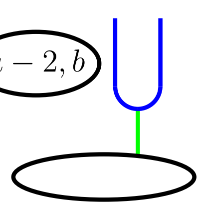
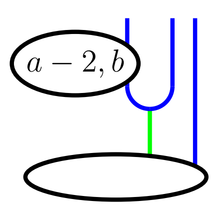
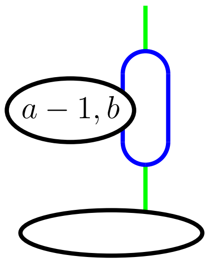


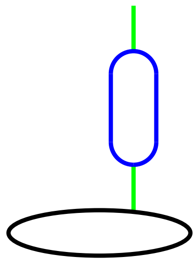
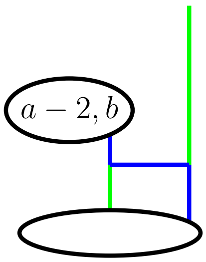
$- \kappa^{-1}_{(a-2,b),(-1,1)}$



(3.7..52)

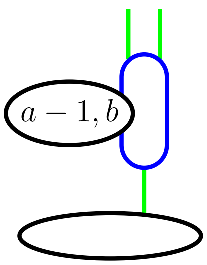
Then, from the relation (3.7..52), we have

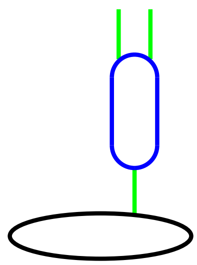
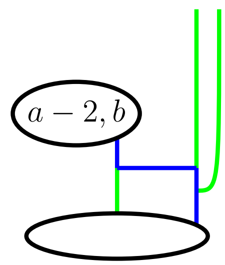


$=$

$-\,\kappa^{-1}_{(a-2,b),(-1,1)}$


(3.7..53)

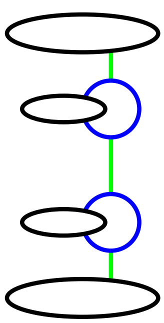
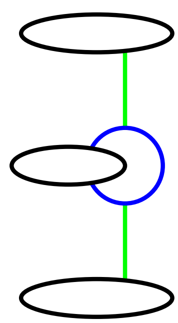
and

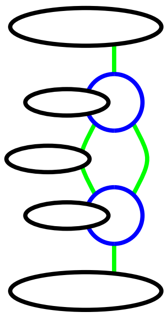


$=$

$-\,\kappa^{-1}_{(a-2,b),(-1,1)}$


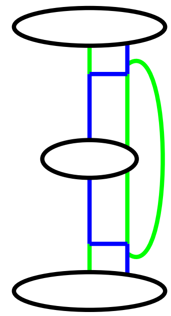
(3.7..54)

Applying these local relations to the second and third term on the right hand side of (3.7..49), and simplifying diagrams using the defining relations of $\mathcal{D}_{\mathfrak{sp}_4}$, we obtain the next two equations.


$= -\left([2] + \kappa^{-1}_{(a-2,b),(-1,1)}\right)$

(3.7..55)



$$= \kappa^{-2}_{(a-2,b),(-1,1)}$$



(3.7..56)

Note that the diagram on the right hand side of (3.7..55) is $K_{(a-1,b),(-1,1)}$. Moreover, after applying neutral absorption (3.5..2) the diagram on the right hand side of (3.7..56) is $K_{(a-2,b+1),(0,0)}$. Therefore, we can use (3.7..50), (3.7..55), and (3.7..56) to rewrite (3.7..49), then deduce that

$$\begin{aligned}\kappa_{(a,b),(-2,1)} = \frac{[5]}{[2]}\kappa_{(a-1,b),(-1,1)} - (-[2] - \kappa^{-1}_{(a-2,b),(-1,1)})\frac{\kappa_{(a-1,b),(-1,1)}}{\kappa_{(a-1,b),(-1,0)}} \\ - \frac{\kappa_{(a-2,b+1),(0,0)}}{\kappa^2_{(a-2,b),(-1,1)}\kappa_{(a-1,b),(-1,1)}}.\end{aligned} \tag{3.7..57}$$

□

3.8. Solving the Recursion

Proposition 3.8..1. *The recursive relations in Proposition* (3.7..2) *together with the initial conditions in Equations* (3.7..4), (3.7..5), (3.7..6), (3.7..7), *and* (3.7..8) *are uniquely solved by*

$$\kappa_{(a,b),(1,0)} = 1 \tag{3.8..1}$$

$$\kappa_{(a,b),(0,1)} = 1 \tag{3.8..2}$$

$$\kappa_{(a,b),(-1,1)} = -\frac{[a+1]}{[a]} \tag{3.8..3}$$

$$\kappa_{(a,b),(2,-1)} = -\frac{[2b+2]}{[2b]}. \tag{3.8..4}$$

$$\kappa_{(a,b),(0,0)} = \frac{[a+2][a+2b+4]}{[2][a][a+2b+2]}. \tag{3.8..5}$$

$$\kappa_{(a,b),(1,-1)} = \frac{[a+2b+3][2b+2]}{[a+2b+2][2b]}. \tag{3.8..6}$$

$$\kappa_{(a,b),(-2,1)} = -\frac{[a+1][2a+2b+4]}{[a-1][2a+2b+2]}. \tag{3.8..7}$$

$$\kappa_{(a,b),(-1,0)} = -\frac{[2a+2b+4][a+2b+3][a+1]}{[2a+2b+2][a+2b+2][a]}. \tag{3.8..8}$$

$$\kappa_{(a,b),(0,-1)} = \frac{[2a+2b+4][a+2b+3][2b+2]}{[2a+2b+2][a+2b+1][2b]} \tag{3.8..9}$$

Proof. There is a recursive relation for each non-dominant weight in a fundamental representation. We say that the right hand side of a relation *involves* the weight μ if $\kappa_{?,\mu}$ appears in the right hand side of the recursion.

That relation (3.7..11) (with the specified initial conditions) is solved by (3.8..3) is easily seen to be equivalent to showing that

$$-[a+1] = -[2][a] + [a-1]. \tag{3.8..10}$$

This is a well known identity for quantum numbers, but we will describe a different way to derive (3.8..10). First, we multiply equation (3.8..10) by $(q-q^{-1})$, resulting in

$$-\left(q^{a+1} - q^{-(a+1)}\right) = -(q+q^{-1})\left(q^a - q^{-a}\right) + \left(q^{(a-1)} - q^{(-(a-1))}\right). \tag{3.8..11}$$

Second, we temporarily replace q^a with the variable A, so (3.8..11) becomes the following.

$$Aq - A^{-1}q^{-1} = -(q+q^{-1})(A-A^{-1}) + (Aq^{-1} - A^{-1}q) \tag{3.8..12}$$

Equation (3.8..12) is easily seen to be true in $\mathbb{Z}[A^{\pm 1}, q^{\pm 1}]$, then specializing A to q^a we find that (3.8..11) holds as well. To see that relation (3.7..12) is solved by (3.8..3) is similar, and we leave it as an exercise.

By using

$$\kappa_{(a,b),(-1,1)} = -\frac{[a+1]}{[a]} \qquad \text{and} \qquad \kappa_{(a,b),(2-1)} = -\frac{[2b+2]}{[2b]}, \tag{3.8..13}$$

we can simplify (3.7..13) and (3.7..14) as follows.

$$\kappa_{(a,b),(0,0)} = \frac{[5]}{[2]} - \frac{[a-1]}{[a]}\frac{[2b+4]}{[2b+2]} - \kappa^{-1}_{(a-1,b),(1,-1)} \tag{3.8..14}$$

$$\kappa_{(a,b),(1,-1)} = \frac{[5]}{[2]} - \frac{[2b-2]}{[2b]}\frac{[a+3]}{[a+2]} - \frac{1}{[2]^2}\kappa^{-1}_{(a,b-1),(0,0)} \tag{3.8..15}$$

Then by induction our claim that (3.7..13) is solved by (3.8..5) and (3.7..14) is solved by (3.8..6) follows from verifying the following two equalities.

$$\frac{[a+2][a+2b+4]}{[2][a][a+2b+2]} = \frac{[5]}{[2]} - \frac{[a-1][2b+4]}{[a][2b+2]} - \frac{[a+2b+1][2b]}{[a+2b+2][2b+2]} \tag{3.8..16}$$

$$\frac{[a+2b+3][2b+2]}{[a+2b+2][2b]} = \frac{[5]}{[2]} - \frac{[a+3][2b-2]}{[a+2][2b]} - \frac{1}{[2]^2}\frac{[2][a][a+2b]}{[a+2][a+2b+2]} \tag{3.8..17}$$

We focus on the quantum number calculation needed to verify the first of these two equalities. After clearing denominators the desired equality (3.8..16) will follow from the following identity.

$$\begin{aligned}[a+2][2b+2][a+2b+4] &= [5][a][2b+2][a+2b+2] \\ &\quad - [2][a-1][2b+4][a+2b+2] - [2][a][2b][a+2b+1].\end{aligned} \tag{3.8..18}$$

Multiplying through by $(q-q^{-1})^3$ and replacing q^a with A and q^b with B, we find the desired quantum number identity is a consequence of the following identity in $\mathbb{Z}[A^{\pm 1}, B^{\pm 1}, q^{\pm 1}]$.

$$\begin{aligned}(Aq^2 - A^{-1}q^{-2})(B^2q^2 - B^{-2}q^{-2})(AB^2q^4 - A^{-1}B^{-2}q^{-4}) = \\ (q^4+q^2+1+q^{-2}+q^{-4})(A-A^{-1})(B^2q^2-B^{-2}q^{-2})(AB^2q^2-A^{-1}B^{-2}q^{-2}) \\ -(q+q^{-1})(Aq^{-1}-A^{-1}q)(B^2q^4-B^{-2}q^{-4})(AB^2q^2-A^{-1}B^{-2}q^{-2}) \\ -(q+q^{-1})(A-A^{-1})(B^2-B^{-2})(AB^2q-A^{-1}B^{-2}q^{-1})\end{aligned} \tag{3.8..19}$$

We leave the details of checking (3.8..19) by hand as an exercise for the reader. Then replacing A with q^a and B with q^b we may deduce the equality (3.8..18).

The calculations needed to verify (3.8..17) are omitted, as are the rest of the details of the quantum number calculations. We simply outline the remainder of the proof below.

Once the first four relations are solved, we can simplify the fifth relation so the simplified recursion only involves the weight $(-1,0)$. By using induction we reduce proving the recursion relation (3.7..15) is solved by (3.8..7) to a quantum number calculation. The sixth recursion relation only involves the previous five weights, so we can use these solutions to simplify the right hand side of (3.7..16). A quantum number calculation will verify that the right hand side is in fact equal to (3.8..8). After using the first six solutions to simplify the last recursion, (3.7..17) only involves the weight $(0,-1)$ and so can be solved by induction and a quantum number calculation. $\square$

Remark 3.8..2. The point of writing (3.8..10) in the form (3.8..12) and (3.8..16) in the form (3.8..19) is that it makes it possible (and in fact easy) to have a computer verify the desired quantum number identities [10].

3.9. Relation to Elias's Clasp Conjecture

In the following, we will reinterpret Elias's type A clasp conjecture [7] in type C_2. We then discuss how we expect Elias's conjecture generalizes to a type independent statement.

Recall that the Weyl group for the C_2 root system, which we denote simply by W, acts on the weight lattice X by

$$s(\varpi_1) = -\varpi_1 + \varpi_2 \qquad \text{and} \qquad s(\varpi_2) = \varpi_2 \tag{3.9..1}$$

while

$$t(\varpi_1) = \varpi_1 \qquad \text{and} \qquad t(\varpi_2) = 2\varpi_1 - \varpi_2. \tag{3.9..2}$$

For a weight μ, we will denote by d_μ the minimal length element in W which when takes μ to a dominant weight. Thus,

$$d_{\varpi_1} = 1 = d_{\varpi_2} \tag{3.9..3}$$

and

$$d_{-\varpi_1+\varpi_2} = s, d_{2\varpi_1-\varpi_2} = t, d_{\varpi_1-\varpi_2} = st, d_{-2\varpi_1+\varpi_2} = ts, d_{-\varpi_1} = sts, \text{and } d_{-\varpi_2} = tst. \tag{3.9..4}$$

We then define the set $\Phi_\mu = \{\alpha \in \Phi_+ \mid d_\mu(\alpha) \in \Phi_-\}$. Thus,

$$\Phi_{-\varpi_1+\varpi_2} = \{\alpha_s\} \tag{3.9..5}$$

$$\Phi_{2\varpi_1-\varpi_2} = \{\alpha_t\} \tag{3.9..6}$$

$$\Phi_{\varpi_1-\varpi_2} = \{\alpha_t, t(\alpha_s)\} \tag{3.9..7}$$

$$\Phi_{-2\varpi_1+\varpi_2} = \{\alpha_s, s(\alpha_t)\} \tag{3.9..8}$$

$$\Phi_{-\varpi_1} = \{\alpha_s, s(\alpha_t), st(\alpha_s)\} \tag{3.9..9}$$

$$\Phi_{-\varpi_2} = \{\alpha_t, t(\alpha_s), ts(\alpha_t)\} \tag{3.9..10}$$

Let $(-,-)$ be the standard inner product on X so the ϵ_i are an orthonormal basis. Recall that $\alpha^\vee = 2\alpha/(\alpha,\alpha)$, and that ρ is the sum of the fundamental weights. We define $q_\alpha = q$ when α is a short root and $q_\alpha = q^2$ when α is a long root.

Corollary 3.9..1. *In type C_2, if μ is an (extremal) weight in a fundamental representation and $\lambda \in X_+$, then*

$$\kappa_{\lambda,\mu} = \pm \prod_{\alpha\in\Phi_\mu} \frac{[(\alpha^\vee, \lambda+\rho)]_{q_\alpha}}{[(\alpha^\vee, \lambda+\mu+\rho)]_{q_\alpha}} \tag{3.9..11}$$

Proof. Using the formula $\dfrac{[2n]_v}{[2]_v} = [n]_{v^2}$ it is an easy exercise to use (3.8..1) and our description of Φ_μ to check the corollary. □

It is natural to expect Elias's clasp conjecture to generalize as follows. Let Φ be an irreducible root system with associated simple Lie algebra $\mathfrak{g}$. Let $(-,-)$ be the W invariant bilinear form on Φ so that $(\alpha,\alpha) = 2$ for all short roots $\alpha \in$

Φ. Fix a fundamental weight ϖ and a weight $\mu \in \operatorname{wt} V(\varpi)$ which is in the Weyl group orbit of ϖ. Let d_μ be the minimal length element in the Weyl group such that $d_\mu(\mu) = \varpi$. Set

$$\Phi_+(\mu) = \{\alpha \mid d_\mu(\alpha) \in \Phi_-\}.$$

Conjecture 3.9..2. *There is an elementary light ladder map L_μ, which is a morphism of $U_q(\mathfrak{g})$ modules, and for each dominant weight λ a map $E_{\lambda,\mu}$ (which may be zero) which is a composition of the clasps C_λ and $C_{\lambda+\mu}$ with L_μ as in Equation* (3.6..2)*. Moreover, there is a duality $\mathbb{D}$ on $\mathbf{Fund}(\mathfrak{g})$ which, interpreted in the graphical calculus, is flipping a diagram upside down. Finally, we expect that*

$$E_{\lambda,\mu} \circ \mathbb{D}(E_{\lambda,\mu}) = \prod_{\alpha \in \Phi_+(\mu)} \frac{[(\alpha^\vee, \lambda + \rho)]_{q_\alpha}}{[(\alpha^\vee, \lambda + \rho + \mu)]_{q_\alpha}} C_\lambda, \tag{3.9..12}$$

where $q_\alpha = q^{(\alpha,\alpha)/2}$.

We can already conjecture the general form of one of the recursive relations satisfied by the local intersection forms. The local intersection form calculations in type C_2 show that every weight in $V(\varpi)$ appears in this recursion for the local intersection form of $\kappa_{\lambda,-\varpi}$. In arbitrary type, $-\varpi \in \operatorname{wt} V(-w_0(\varpi)) \cong V(\varpi)^*$.

Conjecture 3.9..3. *There is an elementary light ladder map*

$$L_{-\varpi} : V(\varpi) \otimes V(-w_0(\varpi)) \to \mathbb{C}(q).$$

Moreover, if

$$E_{\lambda,-\varpi} \circ \mathbb{D}(E_{\lambda,-\varpi}) = \kappa_{\lambda,-\varpi} \cdot C_\lambda,$$

then

$$\kappa_{\lambda,-\varpi} = \dim_q V(\varpi) - \sum_{\mu \in \mathrm{wt}\, V(\varpi)} \frac{\kappa_{\lambda-\varpi+\mu,-\mu}}{\kappa_{\lambda-\varpi,\mu}}. \tag{3.9..13}$$

One might hope to prove Conjecture (3.9..2) by finding a combinatorial description of the recursive formulas themselves, then proving these recursions are both given by calculations with webs and solved uniquely by Equation (3.9..12).

Remark 3.9..4. The conjecture in type A only deals with $\kappa_{\lambda,\mu}$ when μ is in the Weyl group orbit of a dominant fundamental weight. In type C_2, we cannot currently explain the local intersection form for the weight $(0,0)$ in a way that suggests any generalization. However, we do expect there is a general formula which, for any simple Lie algebra $\mathfrak{g}$ and any fundamental weight ϖ, computes $\kappa_{\lambda,\mu}$ for all $\mu \in \mathrm{wt}\, V(\varpi)$ in terms of the root system Φ.

CHAPTER IV

SEMISIMPLIFICATION OF TILTING MODULE CATEGORY

4.1. Outline

We give details about semisimplifying diagrammatic categories. Then we recall Deligne's diagrammatic description of $\mathbf{Rep}(O(n))$ and the classification of braidings on $\mathbf{Rep}(O(2))$. We then also recall Kuperberg's type C_2 webs and state precisely the relationship with tilting modules. We show that when q^2 has order 4 there is a full and essentially surjective functor from $\underline{\mathbf{Rep}}(O(2))$ to the category of tilting modules for $U_q^{\mathbb{Z}}(\mathfrak{sp}_4)$, and we show that when q^2 has order 3 there is a full and essentially surjective functor from the category of tilting modules to $\mathbf{Rep}(O(2))$. We then deduce the desired equivalences. Lastly, we generalize these arguments to the case of $\mathfrak{sp}_{2n}$, modulo the conjectural relationship between C_n webs and tilting modules.

4.2. Diagrammatics for Spherical Categories

Notation 4.2..1. Let $\mathbf{k}$ be a field. We write $\mathbf{k}\langle\langle O \mid M \mid R\rangle\rangle$ to denote the $\mathbf{k}$-linear monoidal category generated by objects O, morphisms M, modulo the relations R.

We always will assume that this category is spherical. In particular, the category is rigid so there is an involution on the set of generating objects O, denoted $o \mapsto o^*$ and extended to tensor products such that $o_1 \otimes o_2 \mapsto o_1^* \otimes o_2^*$. Rigidity also implies that there is an involution on the set M such that if $m \in \mathrm{Hom}(A, B)$, then $m^* \in \mathrm{Hom}(B^*, A^*)$. Moreover, for all $o \in O$, there are morphisms in M:

$$\mathrm{cap}_o^R : o \otimes o^* \to \mathbf{k}, \mathrm{cap}_o^L : o^* \otimes o \to \mathbf{k},$$

$$\mathrm{cup}_o^L : \mathbf{k} \to o \otimes o^*, \text{ and } \mathrm{cup}_o^R : \mathbf{k} \to o^* \otimes o$$

which satisfy the following relations.

$$(\mathrm{cap}_o^R \otimes \mathrm{id}_o) \circ (\mathrm{id} \otimes \mathrm{cup}_o^R) = \mathrm{id}_o = (\mathrm{id}_o \otimes \mathrm{cap}_o^R) \circ (\mathrm{cup}_o^R \otimes \mathrm{id}_o)$$

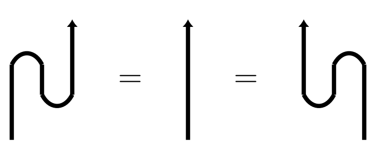
$$(\mathrm{id}_o \otimes \mathrm{cap}_o^L) \circ (\mathrm{cup}_o^L \otimes \mathrm{id}_o) = \mathrm{id}_o = (\mathrm{cap}_o^L \otimes \mathrm{id}_o) \circ (\mathrm{id}_o \otimes \mathrm{cup}_o^L)$$

These cup and cap maps also give rise to traces on the category:

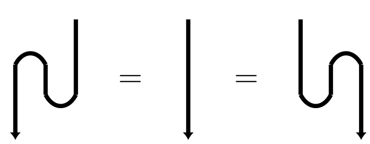
$$Tr^? : \mathrm{Hom}(B, A) \times \mathrm{Hom}(A, B) \to \mathrm{End}(\mathbf{k}) \qquad (f, g) \mapsto \mathrm{cap}_A^? \circ (f \circ g \otimes \mathrm{id}_{A^*}) \circ \mathrm{cup}_A^?, \tag{4.2..1}$$

for $? \in \{L, R\}$. The assumption that our category is spherical implies that these two traces agree and therefore we are justified in simply writing Tr. We will also omit from our notation the superscript L or R on the cup and cap maps when computing this trace.

In this paper, all generators and relations monoidal categories will be described using planar diagrams. In this language, the cup and cap maps will be drawn as cups and caps, and the relations above are the "zig-zag" relations.



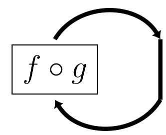
(4.2..2)



(4.2..3)

One can think of Tr in the diagrammatic notation as follows.

$$Tr(f,g) = \tag{4.2..4}$$



If $\mathrm{End}(\mathbf{k}) = \mathbf{k} \cdot \mathrm{id}_{\mathbf{k}}$, then we define the *dimension* of an object X to be $Tr(\mathrm{id}_X) \in \mathbf{k}$.

Definition 4.2..2. A category $\mathcal{C}$ is *Karoubian* if for all objects X and for all idempotent endomorphisms $e \in \mathrm{End}_{\mathcal{C}}(X)$, there is an object $Y \in \mathcal{C}$ and projection and inclusion morphisms $\pi : X \to Y$ and $\iota : Y \to X$ such that $\pi \circ \iota = \mathrm{id}_Y$ and $\iota \circ \pi = e$.

Definition 4.2..3. Let $\mathcal{C}$ be a category. We define the *Karoubi envelope* of $\mathcal{C}$, denoted $\mathbf{Kar}\,\mathcal{C}$ as the category with objects: pairs (X, e) where X is an object in $\mathcal{C}$ and $e \in \mathrm{End}_{\mathcal{C}}(X)$ is an idempotent, and morphisms:

$$\mathrm{Hom}_{\mathbf{Kar}(\mathcal{C})}((X,e),(Y,f)) = f \circ \mathrm{Hom}_{\mathcal{C}}(X,Y) \circ e.$$

In the case that $\mathcal{C}$ is additive, $\mathbf{k}$-linear, and monoidal then $\mathbf{Kar}(\mathcal{C})$ is as well, with tensor product of objects defined as $(X,e) \otimes (Y,f) := (X \otimes Y, e \otimes f)$.

The Karoubi envelope is a Karoubian category. Moreover, every functor from $\mathcal{C}$ to a Karoubian category factors through $\mathbf{Kar}(\mathcal{C})$.

4.3. Ideal of Negligible Morphisms in Spherical Categories

We recall some well known-results which appear in [14], [17], [6], and [23].

Let $\mathbf{k}$ be an algebraically closed field.

Definition 4.3..1. Let $\mathcal{C}$ be an $\mathbf{k}$-linear, spherical tensor category, with $\mathrm{End}_{\mathcal{C}}(\mathbf{k}) = \mathbf{k} \cdot \mathrm{id}_{\mathbf{k}}$. The *negligible ideal* in $\mathcal{C}$ is the subcategory $\mathcal{N}(\mathcal{C}) \subset \mathcal{C}$ with objects the same as $\mathcal{C}$ and morphisms

$$\mathrm{Hom}_{\mathcal{N}(\mathcal{C})}(X,Y) = \{f \in \mathrm{Hom}_{\mathcal{C}}(X,Y) \mid Tr(f \circ g) = 0 \text{ for all } g : Y \to X\}$$

Lemma 4.3..2. *[6, Theorem 2.9] The subcategory $\mathcal{N}(\mathcal{C})$ forms a monoidal ideal in $\mathcal{C}$.*

Notation 4.3..3. Since $\mathcal{N}(\mathcal{C})$ is a monoidal ideal, it follows that $\overline{\mathcal{C}} := \mathcal{C}/\mathcal{N}(\mathcal{C})$ is a monoidal category and $\pi_{\mathcal{C}} : \mathcal{C} \to \overline{\mathcal{C}}$ is a monoidal functor.

The objects in $\overline{\mathcal{C}}$ are the same as the objects in $\mathcal{C}$, but we will denote them by $\overline{o}$ when considered as objects in $\overline{\mathcal{C}}$. We also write $\overline{f}$ for the image in $\overline{\mathcal{C}}$ of a morphism $f \in \mathcal{C}$.

Definition 4.3..4. A category is *Krull-Schmidt* if every object in the category decomposes into a finite direct sum of indecomposable objects, and every indecomposable object has a local endomorphism ring.

Lemma 4.3..5. *A $\mathbf{k}$-linear Karoubian category with finite dimensional homomorphism spaces is Krull-Schmidt.*

Proof. See [22, 11.4.2]. □

Definition 4.3..6. A category $\mathcal{C}$ is *semisimple* if it is abelian and every object is isomorphic to a finite direct sum of simple objects.

Lemma 4.3..7. *[14, Lemma 2.4] Let $\mathcal{C}$ be an $\mathbf{k}$-linear, Karoubian, spherical tensor category, with finite dimensional homomorphism spaces, and such that $\mathrm{End}_{\mathcal{C}}(\mathbf{k}) = \mathbf{k} \cdot \mathrm{id}_{\mathbf{k}}$. Assume that every nilpotent endomorphism in $\mathcal{C}$ has trace zero.*

Let X be an indecomposable object in $\mathcal{C}$.

1. *If* $\dim X \neq 0$*, then the negligible endomorphisms of* X *coincide with the Jacbobson radical of* $\mathrm{End}_{\mathcal{C}}(X)$*.*

2. *If* $\dim X = 0$*, then all endomorphisms of* X *are negligible.*

3. *If* Y *is an indecomposable object in* $\mathcal{C}$ *and* Y *is not isomorphic to* X*, then all homomorphisms from* X *to* Y *are negligible.*

Proof. See [14], proof of Lemma 2.4. □

Lemma 4.3..8. *[14, Theorem 2.5] Let* $\mathcal{C}$ *be a* $\mathbf{k}$*-linear, Karoubian, spherical tensor category, with finite dimensional homomorphism spaces, and such that* $\mathrm{End}_{\mathcal{C}}(\mathbf{k}) = \mathbf{k} \cdot \mathrm{id}_{\mathbf{k}}$.

1. *If every nilpotent endomorphism in* $\mathcal{C}$ *has trace zero, then* $\overline{\mathcal{C}}$ *is semisimple.*

2. *The irreducible objects in* $\overline{\mathcal{C}}$ *are the (image under* $\pi_{\mathcal{C}}$ *of) indecomposable objects in* $\mathcal{C}$ *of non-zero dimension.*

3. *Two irreducible objects in* $\overline{\mathcal{C}}$ *are isomorphic if and only if the corresponding indecomposable objects of non-zero dimension are isomorphic.*

Proof. See [14], proof of Theorem 2.5. □

Lemma 4.3..9. *[17, Section 6] Let* $\mathcal{C}$ *be a* $\mathbf{k}$*-linear, spherical tensor category. If* $\mathcal{C}$ *is semisimple, has a simple unit object, and the endomorphism rings of all simple objects are spanned over* $\mathbf{k}$ *by the identity, then* $\mathcal{C}$ *has no nonzero negligible morphisms.*

Proof. We give a proof sketch based on the discussion in [17] Proposition 5.7 and Section 6.

Let $f : X \to Y$ be negligible. Since $\mathcal{C}$ is semisimple, we can fix decompositions $X \cong \oplus X_i$ and $Y \cong \oplus Y_j$ where each X_i and Y_j are simple, and thus can view f as $\oplus f_{ij} : \oplus X_i \to \oplus Y_j$. For each $X_i \cong Y_j$ fix an isomorphism $\varphi^i_j : Y_j \to X_i$. Note that if no X_i is isomorphic to Y_j, then $f = 0$. Write $g(ij) : Y \to Y_j \xrightarrow{\varphi^i_j} X_i \to X$. Then since f is negligible $0 = Tr(f \circ g(ij)) = Tr(f_{ij} \circ g(ij)) = c_{ij} \dim(X_i)$ for some nonzero scalar $c_{ij} \in \mathbf{k}$. Thus, to show that $f = 0$ it suffices to show that the dimensions of simple objects in $\mathcal{C}$ are nonzero.

Let S be a simple object in $\mathcal{C}$, we will show that $\dim S \neq 0$. Since $\mathcal{C}$ is rigid it follows that $\dim \mathrm{Hom}_{\mathcal{C}}(\mathbf{k}, S \otimes S^*) = 1$. Since $\mathcal{C}$ is semisimple and $\mathbf{k}$ is simple, this implies that $\mathbf{k}$ is a direct summand of $S \otimes S^*$. Then Equation (4.2..2), along with $S \neq 0$ implies that the coevaluation map, cup_S, must be a nonzero element of, and hence a basis for, $\mathrm{Hom}_{\mathcal{C}}(\mathbf{k}, S \otimes S^*)$. Similarly, the evaluation map, cap_S, is a basis for $\mathrm{Hom}_{\mathcal{C}}(S \otimes S^*, \mathbf{k})$. Thus, if $0 = \dim S = Tr(\mathrm{id}_S) = \mathrm{cap}_S \circ \mathrm{cup}_S$, then $\mathbf{k}$ cannot be a summand of $S \otimes S^*$. □

Remark 4.3..10. Since we assume that $\mathbf{k}$ is algebraically closed throughout, it will always be the case that endomorphisms of simple objects in semisimple categories are spanned over $\mathbf{k}$ by the identity.

4.4. Semisimplification of Diagrammatic Spherical Categories

Definition 4.4..1. The $\mathbf{k}$-linear category $\mathbf{k}\langle\langle O \mid M \mid R\rangle\rangle$ is *semisimplifiable* if the space of endomorphisms of the empty word (the monoidal unit in the category) is spanned, over $\mathbf{k}$, by the empty diagram, all homomorphism spaces are finite dimensional, and the trace of any nilpotent endomorphism is zero.

Lemma 4.4..2. *Let* $\mathcal{C} = \mathbf{Kar}\,\mathbf{k}\langle\langle O \mid M \mid R\rangle\rangle$ *and let* $\mathcal{C}' = \mathbf{Kar}\,\mathbf{k}\langle\langle O' \mid M' \mid R'\rangle\rangle$. *Suppose that there are objects* $F(o) \in \mathcal{C}'$, *for all* $o \in O$, *and morphisms* $F(m) \in \mathcal{C}'$, *for all $m \in M$, such that modulo the ideal $\mathcal{N}(\mathcal{C}')$ the morphisms $F(m)$ satisfy all the relations in R. Then there is a monoidal functor $F : \mathcal{C} \to \overline{\mathcal{C}'}$, extending the mapping F.*

Proof. The fact that there is a monoidal functor $F : \mathcal{C} \to \overline{\mathcal{C}'}$ such that $o \mapsto \overline{o}$ and $m \mapsto \overline{m}$ follows from a generators and relations check using our hypotheses, along with the universal property of **Kar**. $\square$

Lemma 4.4..3. *Let $\mathcal{C} = \mathbf{Kar}\,\mathbf{k}\langle\langle O \mid M \mid R\rangle\rangle$ and let $\mathcal{C}' = \mathbf{Kar}\,\mathbf{k}\langle\langle O' \mid M' \mid R'\rangle\rangle$. Assume that $\mathbf{k}\langle\langle O \mid M \mid R\rangle\rangle$ and $\mathbf{k}\langle\langle O' \mid M' \mid R'\rangle\rangle$ are semisimplifiable* (4.4..1). *Let $F : \mathcal{C} \to \overline{\mathcal{C}'}$ be a monoidal functor. If F is full, then $\ker F = \mathcal{N}(\mathcal{C})$.*

Proof. We will first establish that $\ker F \subset \mathcal{N}(\mathcal{C})$. Let $f : X \to Y \in \ker(F)$ and let $g : Y \to X$. Since F is monoidal, F sends the unit object to the unit object, A^* to $F(A)^*$, and cap_A to $\mathrm{cap}_{F(A)}$. Since F is $\mathbf{k}$-linear we have

$$F(c \cdot \mathrm{id}_{\mathbf{k}_{\mathcal{C}}}) = c \cdot \mathrm{id}_{\mathbf{k}_{\mathcal{C}'}}\,.$$

Thus,

$$\begin{aligned} Tr(f \circ g) \cdot \mathrm{id}_{\mathbf{k}_{\mathcal{C}'}} &= F(Tr(f \circ g) \cdot \mathrm{id}_{\mathbf{k}_{\mathcal{C}}}) \\ &= F(\mathrm{cap}_Y \circ (f \circ g \otimes \mathrm{id}_{Y^*}) \circ \mathrm{cup}_Y) \\ &= \mathrm{cap}_{F(Y)} \circ (F(f) \circ F(g) \otimes \mathrm{id}_{F(Y)^*}) \circ \mathrm{cap}_{F(Y)} \\ &= 0, \end{aligned}$$

where the last equality follows from knowing $F(f) = 0$. We assumed that $\overline{\mathcal{C}'}$ is semisimple, so in particular is nonzero. It follows that we can deduce that $Tr(f \circ g) = 0$. Hence, $f \in \mathcal{N}(\mathcal{C})$.

It remains to show that $\mathcal{N}(\mathcal{C}) \subset \ker F$. Suppose that $f : X \to Y$ is a negligible morphism in $\mathcal{C}$. If we can show that $F(f) \in \mathcal{N}(\mathcal{C}')$, then since $\overline{\mathcal{C}'}$ is semisimple it follows from Lemma (4.3..9) that $F(f) = 0$.

To see that $F(f)$ is negligible, let $g : F(Y) \to F(X)$. Since F is full, there is $g' : Y \to X$ such that $F(g') = g$. This allows us to compute

$$\begin{aligned}
Tr(F(f) \circ g) \cdot \mathrm{id}_{\mathbf{k}_{\mathcal{C}'}} &= Tr(F(f) \circ F(g')) \cdot \mathrm{id}_{\mathbf{k}_{\mathcal{C}'}} \\
&= \mathrm{cap}_{F(Y)} \circ (F(f) \circ F(g) \otimes \mathrm{id}_{F(Y)^*}) \circ \mathrm{cup}_{F(Y)} \\
&= F(\mathrm{cap}_Y \circ (f \circ g \otimes \mathrm{id}_{Y^*}) \circ \mathrm{cup}_Y) \\
&= F(Tr(f \circ g)) \\
&= F(0) \\
&= 0,
\end{aligned}$$

where the second to last equality is a consequence of f being negligible. □

Proposition 4.4..4. *Let* $\mathcal{C} = \mathbf{Kar}\,\mathbf{k}\langle\langle O \mid M \mid R\rangle\rangle$ *and let* $\mathcal{C}' = \mathbf{Kar}\,\mathbf{k}\langle\langle O' \mid M' \mid R'\rangle\rangle$*. Assume that* $\mathbf{k}\langle\langle O \mid M \mid R\rangle\rangle$ *and* $\mathbf{k}\langle\langle O' \mid M' \mid R'\rangle\rangle$ *are semisimplifiable* (4.4..1)*.*

If there is a full and essentially surjective monoidal functor $F : \mathcal{C} \to \overline{\mathcal{C}'}$*, then there is a monoidal equivalence* $\overline{F} : \overline{\mathcal{C}} \to \overline{\mathcal{C}'}$ *such that* $\overline{F} \circ \pi_{\mathcal{C}} = F$*.*

Proof. Since F is full, Lemma (4.4..3) implies that $\ker F = \mathcal{N}(\mathcal{C})$. Thus, there is an induced functor $\overline{F} : \overline{\mathcal{C}} \longrightarrow \overline{\mathcal{C}'}$ such that $\overline{F} \circ \pi_{\mathcal{C}} = F$. Since F is essentially surjective, so is $\overline{F}$, and therefore $\overline{F}$ is a monoidal equivalence. □

Lemma 4.4..5. *Let $\mathcal{C}$ and $\mathcal{A}$ be additive, rigid, monoidal categories with* $\mathrm{End}(\mathbf{\mathit{1}}) = \mathbf{k} \cdot \mathrm{id}$. *Suppose that $\mathcal{A}$ is an abelian category and that the monoidal product is exact. If there is an additive, monoidal functor $F : \mathcal{C} \longrightarrow \mathcal{A}$, then the trace of a nilpotent endomorphism in $\mathcal{C}$ is zero.*

Proof. Let $n \in \mathrm{End}_{\mathcal{C}}(X)$ be such that $n^k = 0$, for some $k \geq 0$. Thus, $F(n) \in \mathrm{End}_{\mathcal{A}}(F(X))$ also satisfies $F(n)^k = F(n^k) = F(0) = 0$. Since $\mathcal{A}$ is abelian, it follows from [17, Lemma 3.5, Corollary 3.6] that $Tr(F(n)) = 0$. Thus,

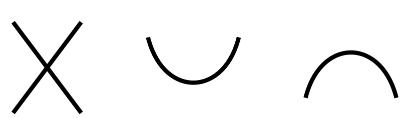
$$\begin{aligned}
Tr(n) \cdot \mathrm{id}_{\mathbf{k}_{\mathcal{A}}} &= Tr(n) \cdot \mathrm{id}_{F(\mathbf{k}_{\mathcal{C}})} \\
&= Tr(n) \cdot F(\mathrm{id}_{\mathbf{k}_{\mathcal{C}}}) \\
&= F(Tr(n) \cdot \mathrm{id}_{\mathbf{k}_{\mathcal{C}}}) \\
&= F(\mathrm{cap} \circ (n \otimes \mathrm{id}_X) \circ \mathrm{cup}) \\
&= F(\mathrm{cap}) \circ (F(n) \otimes \mathrm{id}_{F(x)}) \circ F(\mathrm{cup}) \\
&= Tr(F(n)) \cdot \mathrm{id}_{\mathbf{k}_{\mathcal{A}}} \\
&= 0,
\end{aligned}$$

so $Tr(n) = 0$. $\square$

4.5. Deligne's Description of Rep(O(t)).

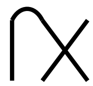
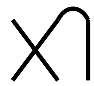
Definition 4.5..1. [17, Definition 9.2]

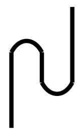
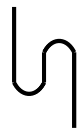
Let $\mathcal{R}$ be the $\mathbb{C}(T)$-linear monoidal category with generating object $\bullet$ and the following generating morphisms.



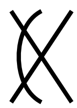
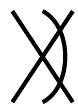
(4.5..1)

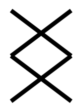
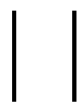
The relations are the following local relations on diagrams.

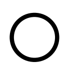
 $=$  (4.5..2)

 $=$ $=$  (4.5..3)

 $=$  (4.5..4)

 $=$  (4.5..5)

 $=$  (4.5..6)

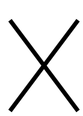
 $= T$ (4.5..7)

Definition 4.5..2. Define $\underline{\mathbf{Rep}}(O(T))$ to be the Karoubi envelope of the category $\mathcal{R}$. If $t \in \mathbb{C}$, then we also define

$$\underline{\mathbf{Rep}}(O(t)) := \mathbf{Kar}(\mathbb{C} \otimes_{T=t} \mathcal{R}).$$

Let V be an n dimensional vector space equipped with a non-degenerate symmetric bilinear form B. If e_i and f_i are dual bases for V with respect to B, then element $\Delta = \sum e_i \otimes f_i \in V \otimes V$ is independent of choice of basis.

Proposition 4.5..3. *[17, Theorem 9.6] The assignment $\bullet \mapsto V$ along with*

 $\mapsto (v \otimes w \mapsto w \otimes v)$ (4.5..8)

$$\cup \quad \mapsto (1 \mapsto \Delta) \tag{4.5..9}$$

$$\cap \quad \mapsto (v \otimes w \mapsto B(v, w)) \tag{4.5..10}$$

defines a full functor from $\underline{\mathbf{Rep}}(O(n))$ *to* $\mathbf{Rep}(O(V, B))$*. Moreover, the kernel of this functor is the ideal of negligible morphisms in* $\underline{\mathbf{Rep}}(O(n))$*.*

Remark 4.5..4. The category $\mathcal{R}$ does not actually appear in [17]. Instead, Deligne worked with a category with objects $\{0, 1, 2, \ldots\}$ and morphism spaces having a basis of Brauer diagrams. Then [33, Theorem 2.6] showed that Deligne's category of Brauer diagrams [33, Definition 2.3], denoted $\mathcal{B}(T)$ by Lehrer and Zhang in loc. cit., is equivalent to the generators and relations category $\mathcal{R}$.

Lemma 4.5..5. *Let* $n \in \mathbb{Z}_{\geq 0}$*. The category* $\underline{\mathbf{Rep}}(O(n)) = \mathbf{Kar}\, \mathbb{C} \otimes_{T=n} \mathcal{R}$ *is semisimplifiable, see Definition* (4.4..1)*.*

Proof. That the endomorphisms of the unit are spanned by the empty diagram and all homomorphism spaces are finite dimensional is an immediate consequence of the homomorphism spaces in $\underline{\mathbf{Rep}}(O(n))$ having a basis of Brauer diagrams (4.5..4). By considering the composition of monoidal functors

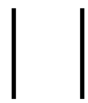
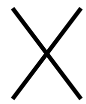
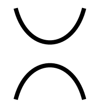
$$\underline{\mathbf{Rep}}(O(n)) \longrightarrow \mathbf{Rep}(O(n)) \longrightarrow \mathbb{C} - \mathrm{mod},$$

one can apply Lemma (4.4..5) and deduce that a nilpotent endomorphism in $\underline{\mathbf{Rep}}(O(n))$ must have trace zero. $\square$

4.6. Braidings on $\mathbf{Rep}(O(2))$

We want to determine all braidings on $\mathbf{Rep}(O(2))$. This has already been accomplished in [49], but we repeat the calculations here for completeness.

In this section, for ease of notation, we write the following.

$I :=$  $X :=$  $Q :=$  (4.6..1)

Let $\beta_{(-),(-)}$ be a *potential* braiding on $\underline{\mathbf{Rep}}(O(T))$. Since $\{I, X, Q\}$ is a basis of $\mathrm{End}_{\mathcal{R}}(\bullet \otimes \bullet)$, it follows that there are scalars $a, b, c \in \mathbb{C}$ such that

$$\beta_{\bullet,\bullet} = aI + bX + cQ. \tag{4.6..2}$$

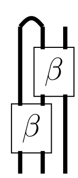
The *hexagon* equation implies that

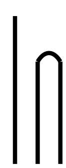
$$\beta_{\bullet,\bullet\otimes\bullet} = (\beta_{\bullet,\bullet} \otimes \mathrm{id}_\bullet) \circ (\mathrm{id}_\bullet \otimes \beta_{\bullet,\bullet}).$$

Also,

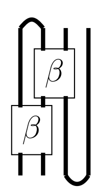
$$\beta_{\bullet,\mathbf{k}} = \mathrm{id}_\bullet = \beta_{\mathbf{k},\bullet}.$$

Therefore, naturality of β implies



$=$  (4.6..3)

which in turn implies that



$=$  (4.6..4)

In other words, using cups and caps to *rotate* β we obtain β^{-1}. Expanding β in Equation (4.6..4) we find that

$$(b^2 + ac)I + (ab + bc)X + (a^2 + ab + tac + bc + c^2)Q = I, \tag{4.6..5}$$

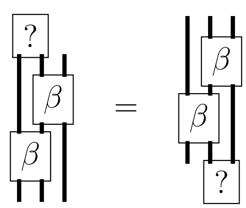
so in particular, $b(a + c) = 0$. Thus, we have the following:

$$\text{If } a + c = 0, \text{ then } b^2 = a^2 + 1, \text{ and } a^2(2 - t) = 0, \tag{4.6..6}$$

and

$$\text{If } b = 0,\ c = a^{-1} \neq 0, \text{ and } t = -(a^2 + a^{-2}). \tag{4.6..7}$$

Since $\beta_{\bullet,\bullet}$ is a linear combination of I, X, and Q we can deduce that β satisfies the *braid relation* if and only if



(4.6..8)

for $? \in \{I, X, Q\}$. The case of $? = I$ is trivial, and the case of $? = Q$ follows from Equation (4.6..3). A calculation shows that for the case of $? = X$ to hold one must have

$$\begin{aligned} &a^2 X \otimes I + acQ \otimes I + ac(X \otimes I) \circ (I \otimes Q) + c^2 (X \otimes I) \circ (I \otimes Q) \circ (Q \otimes I) \\ &= \\ &a^2 I \otimes X + acI \otimes Q + ac(Q \otimes I) \circ (I \otimes X) + c^2 (I \otimes Q) \circ (Q \otimes I) \circ (I \otimes X). \end{aligned} \tag{4.6..9}$$

Remark 4.6..1. Since partition diagrams form a basis for homomorphisms spaces in $\underline{\mathbf{Rep}}(O(T))$, this calculation shows that a braidings on $\underline{\mathbf{Rep}}(O(T))$ would have $a = c = 0$, and therefore by Equation (4.6..6) $b = \pm 1$.

In order to see which β give rise to a braiding on $\mathbf{Rep}(O(2))$ we need to specialize T to 2 and consider the image of β under a functor $\underline{\mathbf{Rep}}(O(2)) \to \mathbf{Rep}(O(2))$.

Let $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ and let $B(-,-)$ be the symmetric bilinear form on V determined by

$$B(e_1, e_1) = 0 = B(e_2, e_2) \text{ and } B(e_1, e_2) = 1 = B(e_2, e_1). \tag{4.6..10}$$

By Proposition (4.5..3), there is a monoidal functor

$$F_B : \underline{\mathbf{Rep}}(O(2)) \longrightarrow \mathbf{Rep}(O(V, B)). \tag{4.6..11}$$

Subtracting the right hand side from the left hand side in Equation (4.6..9), applying F_B, then writing the matrix with respect to the basis

$$\{e_1 \otimes e_1 \otimes e_1, e_1 \otimes e_1 \otimes e_2, e_1 \otimes e_2 \otimes e_1, e_2 \otimes e_1 \otimes e_1,$$
$$e_1 \otimes e_2 \otimes e_2, e_2 \otimes e_1 \otimes e_2, e_2 \otimes e_2 \otimes e_1, e_2 \otimes e_2 \otimes e_2\}$$

results in

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \\ 0 & a^2-c^2 & c^2-a^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -(a+c)^2 & 0 & a^2-c^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & (a+c)^2 & c^2-a^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c^2-a^2 & (a+c)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & a^2-c^2 & 0 & -(a+c)^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & c^2-a^2 & a^2-c^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This matrix is identically zero if and only if $c = -a$.

Proposition 4.6..2. *There is a bijection between the set* $\mathbb{C} \times \langle \pm 1 \rangle$ *and braidings on* $\mathbf{Rep}(O(2))$

$$(a, \epsilon) \mapsto {}^{a,\epsilon}\beta$$

such that

$${}^{a,\epsilon}\beta_{\bullet,\bullet} = aI + \epsilon\sqrt{a^2+1}X - aQ.$$

Proof. The preceding calculations show that the morphisms $\beta_{\bullet,\bullet} = aI + bX + cQ$ (interpreted as $O(2)$ equivariant endomorphisms of $V \otimes V$) satisfy the braid relation and naturality if and only if $c = -a$ and $b = \pm 1$.

The Karoubi envelope of a braided category is braided. Since $Rep(O(2))$ is equivalent to the Karoubi envelope of the full monoidal subcategory generated by V, it suffices to describe a braiding on this subcategory. Define ${}^{a,\epsilon}\beta_{\bullet^{\otimes i},\bullet^{\otimes k}}$ to be the morphism corresponding the the positive braid lift of the minimal length coset representative for the double coset $S_i \times S_k \cdot w_0 \cdot S_i \times S_k$, where $w_0 \in S_{i+k}$ is the longest word, and to each crossing we assign ${}^{a,\epsilon}\beta_{\bullet,\bullet}$. We leave it to the reader to verify that this determines a braiding on the monoidal subcategory generated by V. $\square$

Remark 4.6..3. For a more detailed discussion about this instance of a braiding on a subcategory inducing a braiding on the Karoubi envelope see [49, Section 4].

4.7. Webs and Tilting Modules

Notation 4.7..1. We write $[n]_q$, the *quantum integer* n, to denote the element $\frac{q^n - q^{-n}}{q - q^{-1}}$ in the ring $\mathbb{Z}[q, q^{-1}]$.

Let $\mathbf{k}$ be a field and let $\xi \in \mathbf{k}$. Then we can specialize a quantum integer $[n]_q$ to the element $[n]_\xi = \frac{\xi^n + \xi^{-n}}{\xi + \xi^{-1}}$.

Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}, [2]_q^{-1}]$. If $\mathbf{k}$ is a field and $\xi \in \mathbf{k}$ is such that $\xi + \xi^{-1} \neq 0$, then we can specialize the quantum numbers $\frac{[m]}{[2]^k} \in \mathcal{A}$ to $\mathbf{k}$ in exactly the same way.

When the context makes it clear we will drop the subscript of q or ξ and just write $[n]$.

Example 4.7..2. For the present purposes we are most interested in when $\mathbf{k} = \mathbb{C}(\zeta_{2n})$, where $\zeta_{2n} = e^{i\pi/n}$, for some $n \in \mathbb{Z}_{\geq 0}$. In this case we have

$$[k]_{\zeta_{2n}} = [n-k]_{\zeta_{2n}}$$

for all $k \in \mathbb{Z}$.

By combining the usual tricks for working with quantum numbers, like the identity $[2][n] = [n+1] + [n-1]$, with the identity $[k] = [n-k]$ we can explicitly determine $[k] \in \mathbb{C}(\zeta_{2 \cdot n})$. To illustrate this, consider the case of $\mathbb{C}(\zeta_{2 \cdot 4})$. Since

$$[1] = [4-1] = [3] \text{ and } [2]^2 = [3] + [1].$$

we deduce that

$$[2]^2 = 2.$$

Since we chose $\zeta_{2\cdot 4} = e^{i\pi/4}$, we know that $[2] > 0$, hence $[2] = \sqrt{2}$.

Notation 4.7..3. Let $\zeta_n = e^{i\pi/n}$. If $n \geq 3$, then we can consider $\mathbb{C}(\zeta_n) \otimes_{\mathcal{A}} \mathcal{D}_{\mathfrak{sp}_4}$, and write

$$\mathcal{T}_n := \mathbf{Kar}(\mathbb{C}(\zeta_n) \otimes_{\mathcal{A}} \mathcal{D}_{\mathfrak{sp}_4})$$

We recall the main result of Chapter II.

Theorem 4.7..4. *There is an equivalence of monoidal categories*

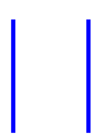
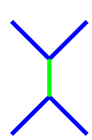
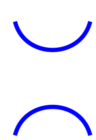
$$\mathcal{T}_n \longrightarrow \mathbf{Tilt}(\mathbb{C}(\zeta_n) \otimes_{\mathcal{A}} U_q^{\mathcal{A}}(\mathfrak{sp}_4))$$

such that $1 \mapsto V(\varpi_1)$, $2 \mapsto V(\varpi_2)$, *and the trivalent vertex maps to a specified intertwiner in Equation* (2.10..1).

Lemma 4.7..5. *The category* $\overline{\mathcal{T}_n}$ *is semisimplifiable* (4.4..1).

Proof. The category $\mathcal{T}_n$ is equivalent to the category of tilting modules for $\mathbb{C}(\zeta_n) \otimes_{\mathcal{A}} U_q^{\mathcal{A}}(\mathfrak{sp}_4)$. This is an additive and monoidal subcategory of the category of finite dimensional representations of $\mathbb{C}(\zeta_n) \otimes_{\mathcal{A}} U_q^{\mathcal{A}}(\mathfrak{sp}_4)$, and it follows that $\mathbf{Tilt}(\mathbb{C}(\zeta_n) \otimes_{\mathcal{A}} U_q^{\mathcal{A}}(\mathfrak{sp}_4))$ has finite dimensional homomorphisms, the endomorphisms of the monoidal unit are all scalar multiples of the identity, and that nilpotent endomorphisms have trace zero. $\square$

Remark 4.7..6. [31] The category $\mathcal{D}_{\mathfrak{sp}_4}$ is braided monoidal with

$\beta_{1,1} = q$  $+$  $+ \frac{q^{-2}}{[2]}$  (4.7..1) and the equivalence in Theorem (4.7..4) is braided monoidal with $\beta_{1,1}$ mapping to the usual quantum group braiding. For more details see [9, Porism 5.4].

4.8. Outline of Proof of Main Theorem

We will define a full and essentially surjective monoidal functor $\underline{\mathbf{Rep}}(O(2)) \longrightarrow \overline{\mathcal{T}_{2\cdot 4}}$. Then, we argue that the kernel of this functor is the ideal of negligible morphisms in $\underline{\mathbf{Rep}}(O(2))$. Thus we have monoidal equivalences

$$\mathbf{Rep}(O(2)) \longleftarrow \overline{\underline{\mathbf{Rep}}(O(2))} \longrightarrow \overline{\mathcal{T}_{2\cdot 4}} \longrightarrow \overline{\mathbf{Tilt}(\mathbb{C}(\zeta_{2\cdot 4}) \otimes U_q^{\mathcal{A}}(\mathfrak{sp}_4))}.$$

From this, we may deduce the desired equivalence between $Rep(O(2))$ and the negligible quotient of tilting modules. Finally, we compute that the usual braiding on $\mathbf{Tilt}(\mathbb{C}(\zeta_{2\cdot 4}) \otimes U_q^{\mathcal{A}}(\mathfrak{sp}_4))/\mathcal{N}$ corresponds to an unusual braiding on $\mathbf{Rep}(O(2))$.

Next, we argue that there is a full, essentially surjective, monoidal functor $\mathcal{T}_{2\cdot 3} \to \mathbf{Rep}(O(2))$. We then show that the kernel of this functor is the ideal of negligible morphisms, and deduce monoidal equivalences

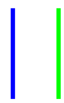
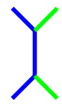
$$\mathbf{Rep}(O(2)) \longleftarrow \overline{\underline{\mathbf{Rep}}(O(2))} \longleftarrow \overline{\mathcal{T}_{2\cdot 3}} \longrightarrow \overline{\mathbf{Tilt}(\mathbb{C}(\zeta_{2\cdot 3}) \otimes U_q^{\mathcal{A}}(\mathfrak{sp}_4))}.$$

Again, we observe that the braiding on the quantum group category does not correspond to the usual symmetric braiding on $\mathbf{Rep}(O(2))$.

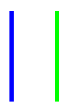
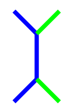
4.9. A Functor $\underline{\mathbf{Rep}}(O(2)) \longrightarrow \overline{\mathcal{T}_{2\cdot 4}}$

We first derive a relation which holds in $\overline{\mathcal{T}_{2\cdot 4}}$, but does not hold in $\mathcal{T}_{2\cdot 4}$

Lemma 4.9..1. *The following relation holds in $\overline{\mathcal{T}_{2\cdot 4}}$.*

 $= -[2]$  (4.9..1)

Proof. If X is indecomposable and $\dim X = 0$, then Lemma (4.3..7) implies that $\mathrm{id}_X \in \mathcal{N}$. Moreover, the results of Chapter II imply that

$I =$  and $E =$  (4.9..2)

is a basis for $\mathrm{End}_{\mathbb{C}(\zeta_{2\cdot 4})\otimes \mathcal{D}_{\mathfrak{sp}_4}{}^{\mathcal{A}}}(1 \otimes 2)$. Note that $E^2 = \frac{[5]}{[2]} I$. Since $[5]/[2] \neq 0$ when $q = \zeta_{2\cdot 4}$, there is an idempotent projecting to $V(\varpi_1 + \varpi_2)$:

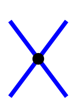
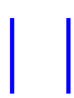
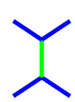
$$\pi_{V(\varpi_1+\varpi_2)} = I - \frac{[2]}{[5]} E. \tag{4.9..3}$$

Moreover, the trace of this idempotent is

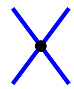
$$-\frac{[8][6]}{[3]},$$

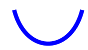
which is 0 when $q = \zeta_{2\cdot 4}$. It follows that, $\pi_{V(\varpi_1+\varpi_2)}$ is in $\mathcal{N}$. The relation in the statement of the lemma then follows from observing that $[5] = -1$ if $q = \zeta_{2\cdot 4}$. $\square$

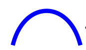
Notation 4.9..2. We will write a dotted crossing to represent the following linear combination in $\mathrm{End}_{\mathcal{D}_{\mathfrak{sp}_4}}(1 \otimes 1)$.

 $:=$  $+ [2]$  (4.9..4)

Lemma 4.9..3. *The assignment* $\bullet \mapsto 1$ *and*

 $\mapsto$  (4.9..5)

 $\mapsto$  (4.9..6)

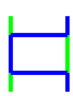
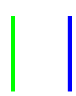
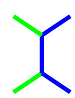
 $\mapsto$ . (4.9..7)

extends to a monoidal functor

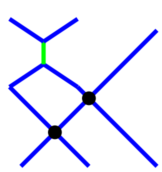
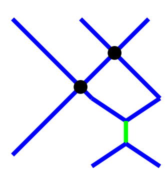
$$F : \underline{\mathbf{Rep}}(O(2)) \longrightarrow \overline{\mathcal{T}_{2\cdot 4}}.$$

Proof. We need to verify that the relations for $\underline{\mathbf{Rep}}(O(2))$ are satisfied by the cups, caps, and the morphism (4.9..4) in $\overline{\mathcal{T}_{2\cdot 4}}$. Note that $[2] = \sqrt{2}$ and $-[6][2]/[3] = 2$ in $\mathbb{C}(\zeta_{2\cdot 4})$. By only using the relations for $\mathcal{D}_{\mathfrak{sp}_4}$ it is easy to verify the relations for $\mathbb{C} \otimes_{T=2} \mathcal{R}$ in Equations (4.5..2), (4.5..3), (4.5..4), (4.5..6), and (4.5..7).

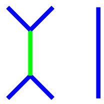
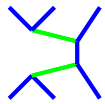
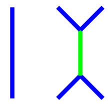
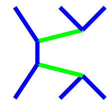
It remains to verify the braid relation in Equation (4.5..5). Using Equation (1.1..7) we deduce the following.

 $=$  $+ [2]$  (4.9..8)

If we use Equation (4.9..4) to expand the dotted crossings in the following diagrams

 $-$  (4.9..9)

and then use (4.9..8) to simplify, we obtain

 $+[2]$  $-$  $-[2]$  $\overset{\text{Lemma (4.9..1)}}{=} 0.$

(4.9..10)

The braid relation in Equation (4.5..5) is an immediate consequence of the expression in (4.9..9) being zero.

Therefore, there is a monoidal functor

$$\mathbb{C} \otimes_{T=2} \mathcal{R} \longrightarrow \overline{\mathcal{T}_{2\cdot 4}}.$$

Since the target category is Karoubian, this functor extends to a monoidal functor

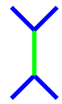
$$F : \mathbf{Kar}(\mathbb{C} \otimes_{T=2} \mathcal{R}) \longrightarrow \overline{\mathcal{T}_{2\cdot 4}}$$

with the desired action on generating objects and morphisms. $\square$

Lemma 4.9..4. *The functor F is full and essentially surjective.*

Proof. Let $\langle 1 \rangle_{\otimes}$ denote the full monoidal subcategory of $\mathcal{D}_{\mathfrak{sp}_4}$ with objects tensor products of 1. Since 2 is a direct summand of $1 \otimes 1$ and the objects in $\mathcal{D}_{\mathfrak{sp}_4}$ are tensor products of 1 and 2, it follows that the functor $\langle 1 \rangle_{\otimes} \to \mathcal{T}_{2\cdot n}$ induces a monoidal equivalence $\mathbf{Kar}\langle 1 \rangle_{\otimes} \to \mathcal{T}_{2\cdot n}$.

The functor F has image in $\overline{\mathbf{Kar}\langle 1 \rangle_{\otimes}}$, so it suffices to show that F is full. To this end, we consider an arbitrary morphism in $\langle 1 \rangle_{\otimes}$. This is a linear combination of diagrams with no 2's on the boundary. We also know from [31] that we can assume that there are no closed subdiagrams. Thus, any occurrence of 2 in a given diagram must locally be of the form



(4.9..11)

Then by repeatedly applying Equation (4.9..4) we can write any morphism as a linear combination of diagrams which are in the image of F. $\square$

Theorem 4.9..5. *There is a monoidal equivalence of categories*

$$\overline{F} : \overline{\underline{\mathbf{Rep}}(O(2))} \longrightarrow \overline{\mathcal{T}_{2\cdot 4}}$$

Proof. This follows from Lemma (4.7..5), Lemma (4.5..5), and Proposition (4.4..4). $\square$

Corollary 4.9..6. *Under the equivalence in Theorem* (4.9..5) *the braiding on* $\overline{\mathcal{T}_{2\cdot 4}}$ *induces the braiding* ${}^{a,\epsilon}\beta$ *on* $\mathbf{Rep}(O(2))$ *from Proposition* (4.6..2)*, for* $a = \sqrt{i}/\sqrt{2}$ *and* $\epsilon = 1$.

Proof. First, use Equation (4.9..2) to write $\beta_{1,1}$ in terms of the image under F of the generators of $\underline{\mathbf{Rep}}(O(2))$. Then, note that $\zeta_{2\cdot 4}^2 = \sqrt{i}$ and $\zeta_{2\cdot 4} + \zeta_{2\cdot 4}^{-1} = \sqrt{2}$ to see $a = \sqrt{i}/\sqrt{2}$. $\square$

4.10. A Functor $\mathcal{T}_{2\cdot 3} \longrightarrow \mathbf{Rep}(O(2))$

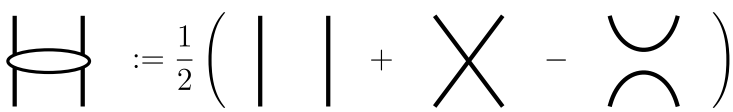
Let $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2$, let B be the symmetric bilinear form in Equation (4.6..10), and let F_B be the full and essentially surjective functor $\mathbf{Kar}\,\mathbb{C} \otimes_{T=2} \mathcal{R} \to \mathbf{Rep}(O(V, B))$ in Equation (4.6..11). Note that the group $O(V, B)$ is generated by the matrices

$$s := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad t_x := \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \quad x \in \mathbb{C}. \tag{4.10..1}$$

This makes it apparent that the module $V \otimes V$ decomposes into a direct sum of three irreducible $O(V, B)$ modules

$$V \otimes V = \mathbb{C}\{e_1 \otimes e_1, e_2 \otimes e_2\} \oplus \mathbb{C}\{e_1 \otimes e_2 - e_2 \otimes e_1\} \oplus \mathbb{C}\{e_1 \otimes e_2 + e_2 \otimes e_1\}. \tag{4.10..2}$$

Lemma 4.10..1. *The endomorphism*



(4.10..3)

maps under F_B, see Equation (4.6..11)*, to the idempotent endomorphism which, under the decomposition in Equation* (4.10..2) *projects to $\mathbb{C}\{e_1 \otimes e_1, e_2 \otimes e_2\}$.*

Proof. The image of (4.10..3) under F_B acts as

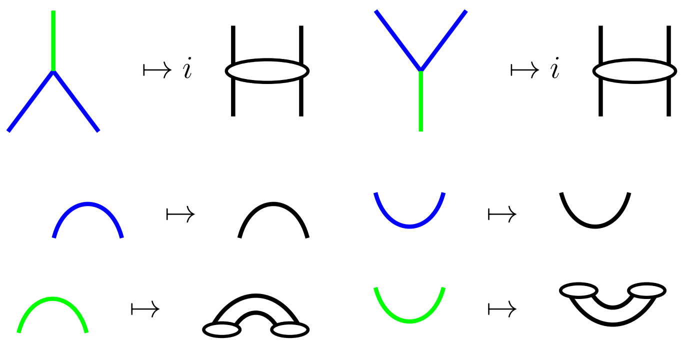
$$\begin{aligned}
e_1 \otimes e_1 &\mapsto \frac{1}{2}\left(e_1 \otimes e_1 + e_1 \otimes e_1 + 0\right) = e_1 \otimes e_1 \\
e_1 \otimes e_2 &\mapsto \frac{1}{2}\left(e_1 \otimes e_2 + e_2 \otimes e_1 - (e_1 \otimes e_2 + e_2 \otimes e_1)\right) = 0 \\
e_2 \otimes e_1 &\mapsto \frac{1}{2}\left(e_2 \otimes e_1 + e_1 \otimes e_2 - (e_1 \otimes e_2 + e_2 \otimes e_1)\right) = 0 \\
e_2 \otimes e_2 &\mapsto \frac{1}{2}\left(e_2 \otimes e_2 + e_2 \otimes e_2 + 0\right) = e_2 \otimes e_2.
\end{aligned}$$

$\square$

Notation 4.10..2. In this section we will continue to use the notation I, X, and Q from Equation (4.6..1) to denote the three basis elements in $\mathrm{End}(\bullet \otimes \bullet)$. We will also write the idempotent in Lemma (4.10..1) as

$$E := \frac{1}{2}\left(I + X - Q\right).$$

Lemma 4.10..3. *The assignment* $s \mapsto (\bullet, \mathrm{id})$, $t \mapsto (\bullet \otimes \bullet, E)$, *and*



(4.10..4)

(4.10..5)

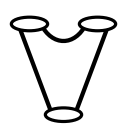
(4.10..6)

determines a monoidal functor

$$G : \mathcal{T}_{2\cdot 3} \longrightarrow \overline{\underline{\mathbf{Rep}}(O(2))}. \tag{4.10..7}$$

Proof. This is a another generators and relations check. Note that if $q = \zeta_{2\cdot 3}$, then $[2] = 1$ and $-[6][2]/[3] = 2 = [6][5]/[3][2]$. It is immediate from the definition of E that all relations hold, except the relation in Equation (1.1..6).

The relation in Equation (1.1..6) is not satisfied in $\underline{\mathbf{Rep}}(O(2))$, but does hold in the negligible quotient. The simplest way to verify the relation in the negligible quotient is to use Lemma (4.10..1) to check that applying the functor $F_B : \underline{\mathbf{Rep}}(O(2)) \longrightarrow \mathbf{Rep}(O(V, B))$ to the diagram



(4.10..8)

results in a linear map which sends each basis vector in $\mathbb{C}\{e_1 \otimes e_1, e_2 \otimes e_2\} \subset V \otimes V$ to 0. $\square$

Lemma 4.10..4. *The functor G is full and essentially surjective.*

Proof. Since $1 \mapsto \bullet$, G is essentially surjective. The identity maps of the objects $\bullet^{\otimes m}$, the idempotent E, and the cup and cap maps are in the image of G. Since X can be expressed in terms of $I = \mathrm{id}_{\bullet\otimes\bullet}$, E and Q, it follows that the symmetric crossing is also in the image of G. Thus, G is full. $\square$

Theorem 4.10..5. *There is a monoidal equivalence*

$$\overline{G} : \overline{\mathcal{T}_{2\cdot 3}} \longrightarrow \overline{\underline{\mathbf{Rep}}(O(2))}.$$

Proof. This follows from Lemma (4.5..5), Lemma (4.7..5), and Proposition (4.4..4). $\square$

Corollary 4.10..6. *Under the equivalence in Theorem , the braiding on $\overline{\mathcal{T}_{2\cdot 3}}$ induces the braiding ${}^{a,\epsilon}\beta$ on $\mathbf{Rep}(O(2))$ from Proposition (4.6..2) for $a = \zeta_{2\cdot 3} - \dfrac{1}{2}$ and $\epsilon = -1$.*

Proof. We leave this as an exercise to the reader. $\square$

4.11. Sketch of Proof of Conjecture (1.3..6)

Notation 4.11..1. Let $n \in \mathbb{Z}_{\geq 2}$. Write $\mathcal{A} := \mathbb{Z}[q, q^{-1}, [2]^{-1}, [3]^{-1}, \ldots, [n]^{-1}]$. Let $\mathbf{Web}^{\mathcal{A}}(\mathfrak{sp}_{2n})$ be the obvious $\mathcal{A}$ form of $\mathbf{Web}(\mathfrak{sp}_{2n})$. Let $U_q^{\mathcal{A}}(\mathfrak{sp}_{2n})$ be Lusztig's divided powers quantum group, viewed as an $\mathcal{A}$ algebra. Fix $\ell > n$. Write $\mathcal{T}_\ell(\mathfrak{sp}_{2n}) := \mathbf{Kar}\left(\mathbb{C} \otimes_{q=e^{2\pi i/\ell}} \mathbf{Web}^{\mathcal{A}}(\mathfrak{sp}_{2n})\right)$.

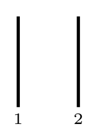
Conjecture 4.11..2. *[9, Remark 3.8] There is a monoidal equivalence* $\mathcal{T}_\ell(\mathfrak{sp}_{2n}) \longrightarrow \mathbf{Tilt}(\mathbb{C} \otimes_{q=e^{2\pi i/\ell}} U_q^{\mathcal{A}}(\mathfrak{sp}_{2n}))$.

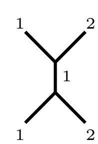
We will assume this conjecture is true for the rest of this section.

Proof Sketch of Conjecture (1.3..6). Suppose that $q = \zeta_{2\cdot 2n}$. Since $2n > n$, the fundamental Weyl modules for $\mathfrak{sp}_{2n}$ are tilting modules. Moreover, using [9, Proposition 2.2] we find that

$$\dim_q V(\varpi_1) = 2, \dim_q V(\varpi_2) = 1, \text{ and } \dim_q V(\varpi_k) = 0 \text{ for } k = 3, 4, \dots n.$$

Therefore, the only diagrams in $\mathbb{C} \otimes_{q=e^{i\pi/2n}} \mathbf{Web}^{\mathcal{A}}(\mathfrak{sp}_{2n})$ which survive in the negligible quotient are those with labels 1 and 2. Using that webs with label 3 are zero, one can argue similarly to the proof of Lemma (4.9..1) that the following relation holds in the negligible quotient.

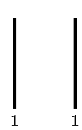


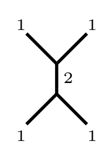
$= \frac{2}{[2]}$


(4.11..1)

Note that $[n+1] = [2n - (n+1)] = [n-1]$, so $[2][n] = [n+1] + [n-1] = 2[n+1]$.

Minor variations of the arguments given in the proofs of Lemma (4.9..3) and Lemma (4.9..4) for $\mathfrak{sp}_4$ will then show that the assignment


$\mapsto$


$- \frac{2}{[2]}$


(4.11..2)

determines a full and essentially surjective monoidal functor $\underline{\mathbf{Rep}}(O(2)) \longrightarrow \overline{\mathcal{T}_{2\cdot 2n}(\mathfrak{sp}_{2n})}$. The desired result then follows from Proposition (4.4..4).

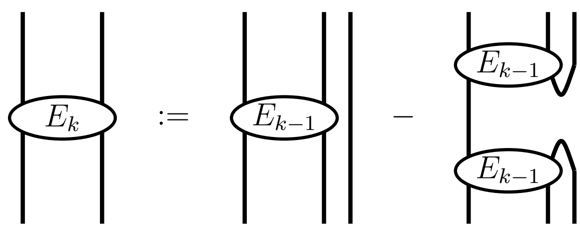
Let $q = \zeta_{2\cdot(n+1)}$. Since $n+1 > n$, the fundamental Weyl modules for $\mathbb{C}_{q=zeta_{2(n+1)}} U_q^{\mathcal{A}}(\mathfrak{sp}_{2n})$ are tilting modules and from [9, Proposition 2.2] we also find that

$$\dim_q V(\varpi_k) = 2 \text{ for } k = 1, 2, \ldots n.$$

We will define idempotents $E_k \in \mathrm{End}_{\overline{\underline{\mathbf{Rep}}(O(2))}}(\bullet^{\otimes k})$ for all $n \geq 2$. Set

$$E_1 := \mathrm{id}, E_2 := E = \frac{1}{2}\left(I + X - Q\right),$$

and for $k \geq 3$



Recall, from Equation (4.6..11), the monoidal functor F_B from $\underline{\mathbf{Rep}}(O(2))$ to $\mathbf{Rep}(O(V))$, and from Equation (4.10..1), the group $O(V)$ is generated by elements s and t_x, for all $x \in \mathbb{C}$. Note that the $O(V)$ module $V^{\otimes k}$ has a submodule $\mathbb{C}\{e_1^{\otimes k}, e_2^{\otimes k}\}$ and

$$s(e_1^{\otimes k}) = e_2^{\otimes k}, s(e_2^{\otimes k}) = e_1^{\otimes k}, t_x(e_1^{\otimes k}) = x^k e_1^{\otimes k}, \text{ and } t_x(e_2^{\otimes k}) = x^{-k} e_2^{\otimes k}.$$

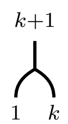
Since all the other basis elements in $V^{\otimes k}$ are acted on by t_x as x^i for $i < k$, we see that the submodule $\mathbb{C}\{e_1^{\otimes k}, e_2^{\otimes k}\}$ has multiplicity one in $V^{\otimes k}$. Thus, there is a unique endomorphism of $V^{\otimes k}$ which projects to this isotypic component and is the identity on $e_i^{\otimes k}$. One can check that the image of E_k under F_B corresponds to this idempotent.

It is easy to see that $E_k \circ (E_{k-1} \otimes \mathrm{id}_\bullet) = E_k$. Using that $F_B(E_k)$ is projection to $\mathbb{C}\{e_1^{\otimes k}, e_2^{\otimes k}\}$, and that $\overline{F_B} : \overline{\underline{\mathbf{Rep}}(O(2))} \longrightarrow \mathbf{Rep}(O(V))$ is an equivalence, we can also deduce that $E_k \circ (\mathrm{id}_\bullet \otimes E_{k-1}) = E_k$.

Let $\lambda_k \in \mathbb{C}$ be such that $\lambda_k^2 = [k]_{e^{i\pi/(n+1)}}$ for $k = 1, 2, \ldots, n$. We claim that there is a functor

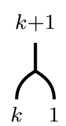
$$\mathbb{C} \otimes_{q=e^{i\pi/(n+1)}} \mathbf{Web}^{\mathcal{A}}(\mathfrak{sp}_{2n}) \longrightarrow \overline{\underline{\mathbf{Rep}}(O(2))}$$

such that $k \mapsto (\bullet^{\otimes k}, E_k)$,



$$\mapsto \lambda_k E_{k+1} \circ (\mathrm{id}_\bullet \otimes E_k) : \bullet \otimes (\bullet^{\otimes k}, E_k) \longrightarrow (\bullet^{\otimes k+1}, E_{k+1}),$$

and



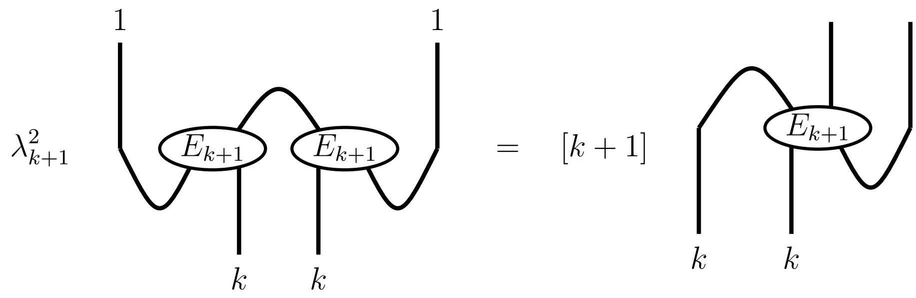
$$\mapsto \lambda_k E_{k+1} \circ (E_k \otimes \mathrm{id}_\bullet) : (\bullet^{\otimes k}, E_k) \otimes \bullet \longrightarrow (\bullet^{\otimes k+1}, E_{k+1}).$$

If there is such a functor, it is clearly full and essentially surjective, so we are reduced to a generators and relations check. The most interesting relation is Relation (1.3..7e).

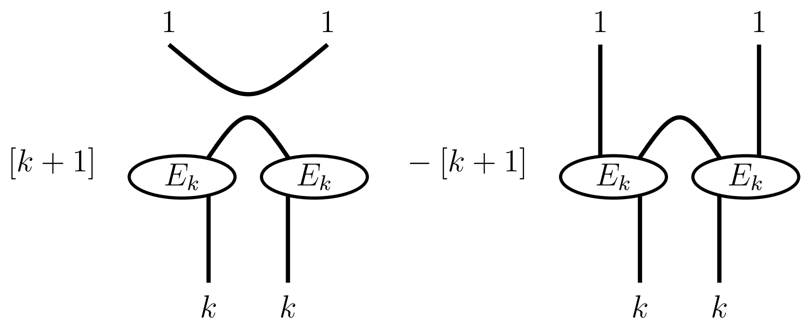
Note that since $q^{n+1} = -1$ we have

$$[n-k] = [k+1], [n-k+1] = [k], \text{and } [n] = 1.$$

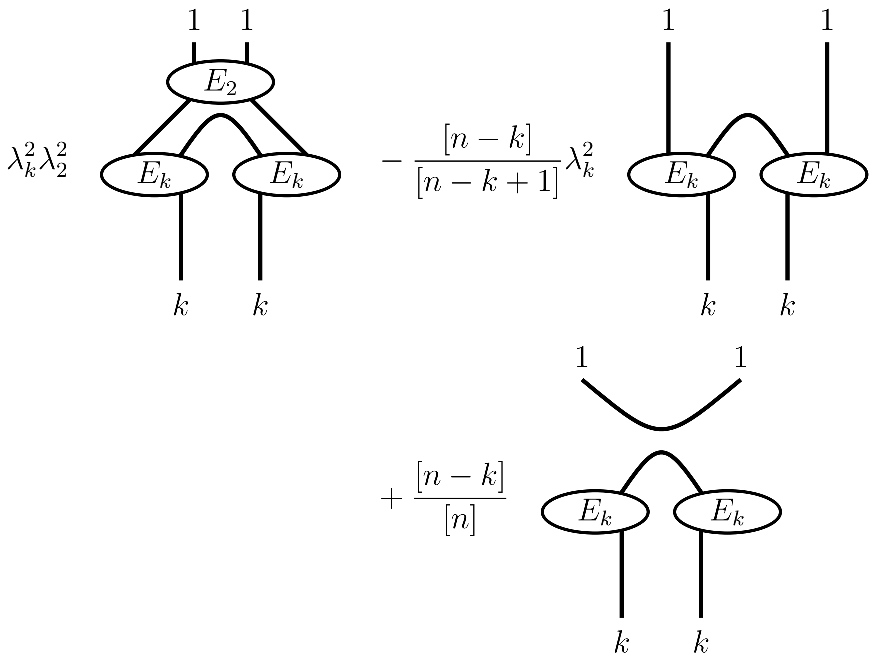
Let $k \geq 3$. The left hand side of Relation (1.3..7e) maps to



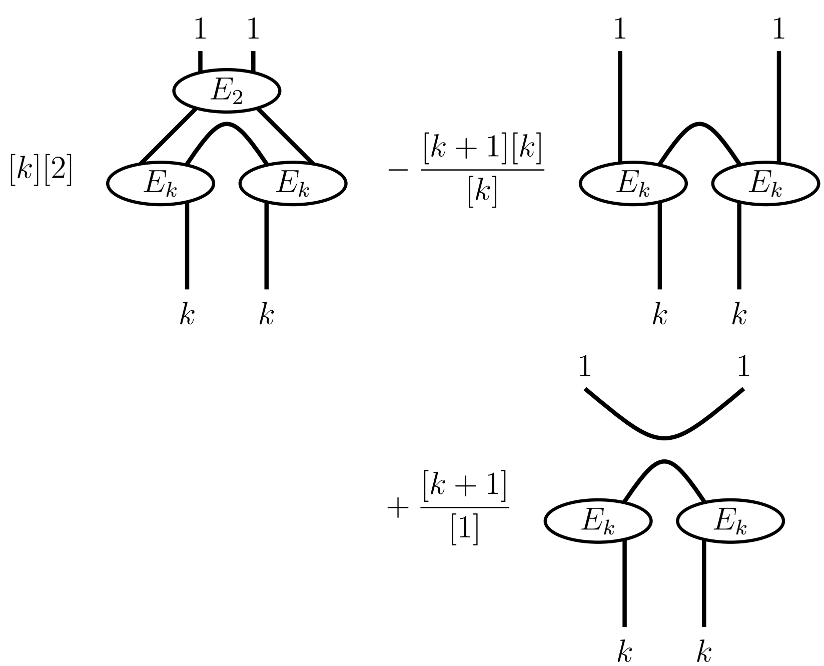
which after expanding E_{k+1} using our inductive definition we get



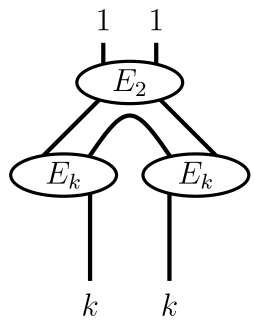
The right hand side of Relation (1.3..7e) maps to



which after simplifying the coefficients becomes



Thus, the result will follow if we can show that



$= 0.$

When $k \geq 2$ this is a consequence of the fact that $\mathbb{C}\{e_1^{\otimes k}, e_2^{\otimes k}\} \otimes \mathbb{C}\{e_1^{\otimes k}, e_2^{\otimes k}\}$ does not have any basis vectors such that t_x acts by x^2, so

$$\mathrm{Hom}_{O(V)}(\mathbb{C}\{e_1^{\otimes k}, e_2^{\otimes k}\} \otimes \mathbb{C}\{e_1^{\otimes k}, e_2^{\otimes k}\}, \mathbb{C}\{e_1^{\otimes 2}, e_2^{\otimes 2}\}) = 0.$$

$\square$

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