MODULI SPACE OF A-INFINITY STRUCTURES AND NONREDUCED CURVES OF GENUS 0

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WEI ZHANG

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Student: Wei Zhang

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This dissertation has been accepted and approved in partial fulfillment of the requirements for the Doctor of Philosophy degree in the Department of Mathematics by:

Alexander Polishchuk	Chair
Nick Addington	Core Member
Ben Elias	Core Member
Dev Sinha	Core Member
Jens Noeckel	Institutional Representative

and

Krista Chronister	Vice Provost for Graduate Studies

Original approval signatures are on file with the University of Oregon Division of Graduate Studies.

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DISSERTATION ABSTRACT

Wei Zhang

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In this thesis, we study A_∞-structures arising from derived categories of certain algebraic curves. More precisely, we consider pairs $(\mathcal{O}_C, \mathcal{O}_D)$, where C is an irreducible projective curve over a field k with $H^0(C, \mathcal{O}_C) = k$ and $H^1(C, \mathcal{O}_C) = 0$, and $D \subset C$ is a Cartier divisor of degree 2, supported at one point. They satisfy certain categorical properties encoded in the notion of an R-pair (of genus 0), (E, F), which we will define. In particular, E is exceptional and F is R-spherical which is a version of the notion of a 1-spherical object defined in the work of Seidel and Thomas. The main result of this thesis is to prove the equivalence between the moduli of the R-pairs and that of certain filtered algebras which permit a simpler description, i.e. given by the quotient stack of a closed subscheme of $\mathbb{A}^3$ for some action of $\mathbb{G}_a$.

CURRICULUM VITAE

NAME OF AUTHOR: Wei Zhang

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:

University of Oregon, Eugene, OR
Capital Normal University, Beijing, China

DEGREES AWARDED:

Doctor of Philosophy, Mathematics, 2022, University of Oregon
Master of Science, Mathematics, 2015, Capital Normal University
Bachelor of Science, Mathematics, 2012, Capital Normal University

AREAS OF SPECIAL INTEREST:

Algebraic Geometry

PROFESSIONAL EXPERIENCE:

Graduate Employee, University of Oregon, 2015 - 2022

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CHAPTER I

INTRODUCTION

In this thesis we study a class of A_∞-algebras arising from derived categories of certain algebraic curves. Derived categories of coherent sheaves can be viewed as refining various cohomological invariants associated with an algebraic variety, such as usual cohomology or Chow groups. While a lot of important studies have been done viewing derived categories as triangulated categories, in many recent applications one has to take into account the natural *enhancement* of this structure, which can be understood as that of a dg-category or of an A_∞-category. Roughly speaking, the dg-enhancement is obtained naturally when calculating morphisms in derived categories using resolutions. An A_∞-enhancement can be obtained from a dg-enhancement using the homological perturbation lemma (see Lemma 2.1.10).

A more specific motivation for us is to study examples in which looking at A_∞-structures one gets some information about the moduli spaces of varieties in question. One can imagine that associating an A_∞-structure to a derived category can be viewed as a kind of algebraic period map. More concretely, we are looking at geometric situations where the derived category $\mathcal{D}$ has some natural generator, i.e., an object G generating $\mathcal{D}$. Then the entire information about the derived category is encoded in an A_∞-enhancement of the endomorphism algebra $\bigoplus_i \operatorname{Hom}^i_{\mathcal{D}}(G,G)$. For example, if G is a coherent sheaf, the latter algebra is the Ext-algebra

$$E(G) = \bigoplus_{i\geq 0} \operatorname{Ext}^i(G,G).$$

This point of view was successfully applied in the works of Polishchuk [7], [8], where he considered reduced projective curves C with marked points $p_1, \dots, p_n$, and picked

$$G = \mathcal{O}_C \oplus \bigoplus_{i=1}^{n} \mathcal{O}_{p_i}$$

as a generator of the perfect derived category of C. Under certain conditions on $(C, p_\bullet)$, the associative algebra $E(G)$ either does not depend on the geometric data, or it depends in a very simple way. This means that the entire geometric information is encoded in the higher products of the A_∞-enhancement on $E(G)$. Furthermore, it was shown in *loc. cit.* that one gets an isomorphism from the relevant moduli space of pointed curves to the moduli space of A_∞-structures on $E(G)$ considered up to gauge equivalence.

In this work we work out the simplest example of a similar picture where we allow the curve C to be nonreduced and replace marked points by (not necessarily reduced) divisors. Namely, we consider pairs (C, D), where C is an irreducible projective curve over a field k with $H^0(C, \mathcal{O}_C) = k$ and $H^1(C, \mathcal{O}_C) = 0$ (so C is of arithmetic genus 0), and $D \subset C$ is a Cartier divisor of degree 2, supported at one point. Thus, if C is smooth (and k is algebraically closed) then $C = \mathbb{P}^1$ and $D = 2p$ for some point p. At the other extreme, C can be a doubled line in $\mathbb{P}^2$ (with the ideal (l^2) where l is a linear form), and $D \subset C$ is the intersection of C with a different line in $\mathbb{P}^2$. We take

$$G = \mathcal{O}_C \oplus \mathcal{O}_D$$

as a generator. Note that in this situation we have

$$\mathrm{Hom}(\mathcal{O}_D, \mathcal{O}_D) = H^0(C, \mathcal{O}_D) \simeq R := k[t]/(t^2),$$

so many relevant spaces become R-modules.

Note that A_∞-algebras are objects of the noncommutative world, so one can expect that studying A_∞-structures arising in some geometric setup one would encounter their noncommutative deformations. This is known to happen in the study of exceptional collections on surfaces (see e.g. [3], [2]). This turns out to be the case for our setup as well.

To study the arising A_∞-algebras, we include our geometric setup into a more abstract homological context. Namely, we axiomatize some properties of the pair $(\mathcal{O}_C, \mathcal{O}_D)$ as above in the following notion of an *R-pair of genus* 0. By such a pair we mean a pair of objects (E, F) in a minimal A_∞-category over k, such that E is an exceptional object, i.e., $\mathrm{Hom}^*(E, E) = k \cdot \mathrm{id}_E$, while F is an *R-spherical object* (a notion to be discussed later), which in particular means that

$$\mathrm{Hom}^0(F, F) = \mathrm{Hom}^1(F, F) = R, \ \mathrm{Hom}^i(F, F) = 0 \text{ for } i \neq 0, 1.$$

Furthermore, we require that $\mathrm{Hom}^i(E, F) = 0$ for $i \neq 0$, $\mathrm{Hom}^j(F, E) = 0$ for $j \neq 1$, and

$$\mathrm{Hom}^0(E, F) = R, \ \mathrm{Hom}^1(F, E) = R,$$

such that all reasonable compositions are given by multiplication in R.

The notion of an R-spherical object is a version of the notion of a 1-spherical object defined in the work of Seidel-Thomas [10]. Similarly to the case of 1-spherical objects, with each R-spherical object F one can associate the spherical twist functor T_F which is an autoequivalence. We also impose a technical condition on our R-pair (E, F): we require an isomorphism of functors

$$T_F^{-1} \simeq \mathcal{S}[-1].$$

where $\mathcal{S}$ is the Serre functor.

Note that in our geometric setup the twist functor associated with the R-spherical object $\mathcal{O}_D$ is the functor $X \mapsto X(D)$. Furthermore, we show that one has an isomorphism

$$\omega_D \simeq \mathcal{O}_C(-D)$$

which implies the above relation between T_F and the Serre functor.

The main idea (borrowed from [8]) is that starting with a pair (E, F) one can construct and study the graded associative algebra

$$\mathcal{R}_{T_F,E} := \bigoplus_{n\geq 0} \operatorname{Hom}(E, T_F^n E),$$

where the associative product uses composition and T_F. We will show that $\mathcal{R}_{T_F,E}$ is the Rees algebra of some filtered algebra $(A, F_\bullet A)$ such that

$$\operatorname{gr}_F A \simeq B := k[u, z]/(z^2), \tag{1.0.1}$$

i.e. $\mathcal{R}_{T_F,E} \simeq \bigoplus_{n\geq 0} F_n A$. The main result is that passage from (E, F) to $\mathcal{R}_{T_F,E}$, or equivalently, to the corresponding filtered algebra $(A, F_\bullet A)$, is an equivalence. Thus, in particular, we can recover the original A_∞-structure on the Ext-algebra of (E, F) from the much simpler data of the filtered algebra A.

Note that for the pair $(E = \mathcal{O}_C, F = \mathcal{O}_D)$ arising from the geometric setup we get the algebra

$$A = \varinjlim \mathrm{H}^0(C, \mathcal{O}(nD))$$

with its natural filtration $F_n A = \cup_{i \le n} \mathrm{H}^0(C, \mathcal{O}(iD))$. As is well known one can recover the curve C (and the divisor D) applying the Proj construction to the Rees algebra of A.

Thus, it is not surprising that in general to go back from a filtered algebra A to an R-pair of genus 0, we use the *noncommutative* Proj construction of [1] for the Rees algebra $\mathcal{R}(A)$. Recall that the latter construction produces an abelian category $\mathrm{qgr} - \mathcal{R}(A)$ (as the quotient of finitely generated graded modules by the subcategory of torsion modules) which is an analog of the category of coherent sheaves. We show that certain natural objects in this category form an R-pair of genus 0.

Finally, we completely resolve the moduli problem corresponding to filtered algebras A such that $\mathrm{gr}_F A \simeq B$. We construct a family of such algebras depending on a parameter in some subscheme $S \subset \mathbb{A}^3$, and show that the corresponding moduli stack is the quotient stack $[S/\mathbb{G}_a]$ for some natural action of the additive group $\mathbb{G}_a$ on S. And we observe that the filtered algebras A in our family are not all commutative, which is in line with the general principle that A_∞-structures may uncover noncommutative deformations.

This thesis is organized as follows. Chapter II contains some preliminaries. First, in Sections 2.1 and 2.2 we review basic results on A_∞-structures and some results from noncommutative projective geometry that are relevant for us. Then in Sections 2.3 we discuss the twist functors associated with R-spherical objects.

Chapter III introduces R-pairs of genus 0. We give the abstract context in Section 3.1 and then discuss the geometric context with curves in Section 3.3. In Section 3.4 we give a sample computation of the A_∞-structure on the Ext-algebra arising from a double point on the smooth curve of genus 0.

We start discussing the connection with moduli of filtered algebras in Chapter IV. We classify all relevant filtered algebras in Section 4.1. Section 4.2 contatins the first main technical result, Theorem 4.2.1, which states the graded algebra $\mathcal{R}_{T_F,E}$ is the Rees algebra of a filtered algebra A satisfying (1.0.1). Next, in Sections 4.3 and 4.4 we study the bimodule over $\mathcal{R} = \mathcal{R}_{T_F,E}$ formed by the Hom^1-spaces between $T_F^i E$ and E. Assuming the compatibility of our R-pair with Serre duality, we identify this bimodule with the restricted dual of $\mathcal{R}$ up to a shift of grading.

In Chapter V we work out the opposite construction via noncommutative projective geometry: we get an R-pair from a filtered algebra satisfying (1.0.1). First, in Section 5.1 we check the AS-Gorenstein property of the Rees algebra associated with a filtered algebra satisfying (1.0.1), which is a technical property needed for the noncommutative projective geometry results. Then in Section 5.2 we give a construction of an R-pair of genus 0 in the noncommutative Proj category of the Rees algebra $\mathcal{R}(A)$.

Finally, we prove the main result on the equivalence of different moduli functors in Chapter VI.

Throughout, we fix a field k of characteristic 0. We denote by ${}_k(e_1, \ldots, e_n)$ the k-linear span of linearly independent vectors $e_1, \ldots, e_n$.

CHAPTER II

PRELIMINARIES

2.1. Basic constructions of A_∞-structures

In this section, we recall several basic definitions and theorems about A_∞-structures. Our main source is [9, sections I.1 and I.3] and we follow the sign convention therein. For an element x in a graded k-vector space, we denote by $|x|$ its degree.

Definition 2.1.1. A (non-unital) A_∞-category $\mathcal{A}$ over k consists of the data:

- a set Obj$\mathcal{A}$ of objects,
- a graded vector space $\hom^{\cdot}_{\mathcal{A}}(X_0, X_1)$ over k for each pair $X_0, X_1 \in \mathrm{Obj}\mathcal{A}$, and
- compositions (higher products):

$$\mu^d_{\mathcal{A}} : \hom^{\cdot}_{\mathcal{A}}(X_{d-1}, X_d) \otimes_k \cdots \otimes_k \hom^{\cdot}_{\mathcal{A}}(X_0, X_1) \longrightarrow \hom^{\cdot}_{\mathcal{A}}(X_0, X_d)[2-d],$$

for all $d \geq 1$ and $X_0, ..., X_d \in \mathrm{Obj}\mathcal{A}$.

The compositions satisfy the quadratic A_∞-associativity equations:

$$\Sigma_{1\leq m\leq d, 0\leq n\leq d-m}(-1)^{\maltese_n}\mu^{d-m+1}_{\mathcal{A}}(a_d \otimes \cdots \otimes a_{n+m+1} \otimes \mu^m_{\mathcal{A}}(a_{n+m} \otimes \cdots \otimes a_{n+1}) \otimes a_n \otimes \cdots \otimes a_1) = 0$$

for all $d \geq 1$ and homogeneous $a_i \in \hom^{\cdot}_{\mathcal{A}}(X_{i-1}, X_i)$ $(1 \leq i \leq d)$, where $\maltese_n := |a_1| + \cdots + |a_n| - n$.

In the above definition, the notation $\maltese_n$ appears quite often involving signs in the context of A_∞-structures. Hence we introduce the following:

Standing Assumption: Suppose $a_1, ..., a_n$ are homogeneous elements in a situation as in Definition 2.1.1 (i.e. they appear in a tail). Then

$$\blacktriangleright_n := |a_1| + \cdots + |a_n| - n.$$

Proposition 2.1.2. *For an A_∞-category $\mathcal{A}$, there is an associated cohomology category* $\mathrm{H}(\mathcal{A})$ *(which is a non-unital linear graded category) that consists of the data:*

- $\mathrm{Obj}\mathrm{H}(\mathcal{A}) := \mathrm{Obj}\mathcal{A}$,
- $\mathrm{Hom}^{\cdot}_{\mathrm{H}(\mathcal{A})}(X_0, X_1) := \mathrm{H}^{\cdot}(\hom^{\cdot}_{\mathcal{A}}(X_0, X_1), \mu^1_{\mathcal{A}})$ *for each pair* $X_0, X_1 \in \mathrm{Obj}\mathrm{H}(\mathcal{A})$, *and*
- *compositions* $[a_2] \cdot [a_1] := (-1)^{|a_1|}[\mu^2_{\mathcal{A}}(a_2 \otimes a_1)]$.

There is a subcategory $\mathrm{H}^0(\mathcal{A}) \subset \mathrm{H}(\mathcal{A})$ *consisting of the degree* 0 *piece of the hom-spaces.*

Definition 2.1.3. (1) A (non-unital) A_∞-functor $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B}$ between A_∞-categories $\mathcal{A}$ and $\mathcal{B}$ consists of the data:

- a map $\mathcal{F} : \mathrm{Obj}\mathcal{A} \longrightarrow \mathrm{Obj}\mathcal{B}$, and
- a homogeneous morphism

$$\hom^{\cdot}_{\mathcal{A}}(X_{d-1}, X_d) \otimes_k \cdots \otimes_k \hom^{\cdot}_{\mathcal{A}}(X_0, X_1) \longrightarrow \hom^{\cdot}_{\mathcal{B}}(\mathcal{F}X_0, \mathcal{F}X_1)[1-d]$$

for all $d \geq 1$ and $X_0, ..., X_d \in \mathrm{Obj}\mathcal{A}$.

They satisfy the equations:

$$\begin{aligned} & \Sigma_{r\geq 1}\Sigma_{s_1+\cdots+s_r=d}\mu^r_{\mathcal{B}}(\mathcal{F}^{s_r}(a_d\otimes\cdots\otimes a_{d-s_r+1})\otimes\cdots\otimes F^{s_1}(a_{s_1}\otimes\cdots\otimes a_1)) \\ = \ & \Sigma_{1\leq m\leq d, 0\leq n\leq d-m}(-1)^{\maltese_n}\mathcal{F}^{d-m+1}(a_d\otimes\cdots\otimes a_{n+m+1}\otimes\mu^m_{\mathcal{A}}(a_{n+m}\otimes\cdots\otimes a_{n+1})\otimes a_n\otimes\cdots\otimes a_1) \end{aligned}$$

for all $d\geq 1$ and homogeneous $a_i\in \hom^{\cdot}_{\mathcal{A}}(X_{i-1},X_i)$ $(1\leq i\leq d)$.

(2) The induced non-unital graded linear functor $\mathrm{H}(\mathcal{F}):\mathrm{H}(\mathcal{A})\longrightarrow\mathrm{H}(\mathcal{B})$ is given by

$$\mathrm{H}(\mathcal{F})([a])=[\mathcal{F}^1(a)]$$

for any $a\in\hom^{\cdot}_{\mathcal{A}}(X_0,X_1)$.

Definition 2.1.4. An A_∞-algebra is an A_∞-category with one object and an A_∞-morphism between A_∞-algebras is an A_∞-functor between them.

Definition 2.1.5. (1) Given an A_∞-category $\mathcal{A}$, a non-unital right A_∞-module $\mathcal{M}$ over $\mathcal{A}$ consists of the data:

- a graded vector space $\mathcal{M}(X)$ over k for all $X\in\mathrm{Obj}(\mathcal{A})$, and
- action maps

$$\mu^d_{\mathcal{M}}:\mathcal{M}(X_{d-1})\otimes_k\hom^{\cdot}_{\mathcal{A}}(X_{d-2},X_{d-1})\otimes_k\cdots\otimes_k\hom^{\cdot}_{\mathcal{A}}(X_0,X_1)\longrightarrow\mathcal{M}(X_0)[2-d]$$

for all $d\geq 1$ and $X_0,...,X_{d-1}\in\mathrm{Obj}\mathcal{A}$.

They satisfy the equations:

$$\Sigma_{1\leq m\leq d,0\leq n\leq d-m}(-1)^{\maltese_n}\mu^{d-m+1}_{\mathcal{M}}(b\otimes a_{d-1}\otimes\cdots\otimes a_{n+m+1}\otimes\mu^m(a_{n+m}\otimes\cdots\otimes a_{n+1})\otimes a_n\otimes\cdots\otimes a_1)=0 \tag{2.1.1}$$

where $\mu^m = \begin{cases} \mu^m_{\mathcal{M}}, & \text{if } n+m=d \\ \mu^m_{\mathcal{A}}, & \text{if } n+m<d \end{cases}$, for all $d \geq 1$ and homogeneous $b \in \mathcal{M}(X_{d-1})$ and $a_i \in \hom^{\cdot}_{\mathcal{A}}(X_{i-1}, X_i)$ $(1 \leq i \leq d-1)$.

(2) There is a non-unital A_∞-category $\mathcal{Q} := \mathrm{nu}-\mathrm{mod}(\mathcal{A}) = \mathrm{mod}_\infty - \mathcal{A}$ of non-unital right A_∞-modules over $\mathcal{A}$ that consists of the data:

- $\mathrm{Obj}(\mathrm{nu}-\mathrm{mod}(\mathcal{A}))$ is the set of non-unital right A_∞-modules over $\mathcal{A}$ (defined as above),
- $\hom^{\cdot}_{\mathcal{Q}}(\mathcal{M}_0, \mathcal{M}_1)$ is the graded vector space over k of pre-module homomorphisms for each pair $\mathcal{M}_0, \mathcal{M}_1 \in \mathrm{Obj}\mathcal{Q}$, where each homogeneous $t \in \hom^{\cdot}_{\mathcal{Q}}(\mathcal{M}_0, \mathcal{M}_1)$ is a sequence $t = (t^1, t^2, ...)$ with

  $$t^d : \mathcal{M}_0(X_{d-1}) \otimes_k \hom^{\cdot}_{\mathcal{A}}(X_{d-2}, X_{d-1}) \otimes_k \cdots \otimes_k \hom^{\cdot}_{\mathcal{A}}(X_0, X_1) \longrightarrow \mathcal{M}_1(X_0)[|t|-d+1]$$

  for $d \geq 1$, and
- compositions (higher products):

$$\begin{aligned}
&\mu^1_{\mathcal{Q}}(t)^d(b \otimes a_{d-1} \otimes \cdots \otimes a_1) = \\
&\qquad \Sigma_{0 \leq n \leq d-1}(-1)^{\#}\mu^{n+1}_{\mathcal{M}_1}(t^{d-n}(b \otimes a_{d-1} \otimes \cdots \otimes a_{n+1}) \otimes a_n \otimes \cdots \otimes a_1) \\
&\qquad + \Sigma_{0 \leq n \leq d-1}(-1)^{\#}t^{n+1}(\mu^{d-n}_{\mathcal{M}_0}(b \otimes a_{d-1} \otimes \cdots \otimes a_{n+1}) \otimes a_n \otimes \cdots \otimes a_1) \\
&\qquad + \Sigma_{m,n}(-1)^{\#}t^{d-m+1}(b \otimes a_{d-1} \otimes \cdots \otimes \mu^m_{\mathcal{A}}(a_{n+m} \otimes \cdots \otimes a_{n+1}) \otimes a_n \otimes \cdots \otimes a_1), \\
&\mu^2_{\mathcal{Q}}(t_2 \otimes t_1)^d(b \otimes a_{d-1} \otimes \cdots \otimes a_1) \\
&\qquad = \Sigma_{0 \leq n \leq d-1}(-1)^{\#}t_2^{n+1}(t_1^{d-n}(b \otimes a_{d-1} \otimes \cdots \otimes a_{n+1}) \otimes a_n \otimes \cdots \otimes a_1), \\
&\mu^l_{\mathcal{Q}} = 0 \quad \text{(for all } l \geq 3),
\end{aligned}$$

where $\# = |a_{n+1}| + \cdots + |a_{d-1}| + |b| - d + n + 1$.

(3) A pre-module homomorphism t is a module homomorphism, if $\mu_Q^1(t) = 0$.

In the following, we define the notion of a twisted complex in an A_∞-category $\mathcal{A}$ in 2 steps: first, the A_∞-category of additive enlargement $\Sigma\mathcal{A}$ of $\mathcal{A}$; then, the A_∞-category Tw$\mathcal{A}$ of twisted complexes.

Theorem 2.1.6. *[9, (3k)] There is an A_∞-category, the additive enlargement $\Sigma\mathcal{A}$ of $\mathcal{A}$, that consists of the data:*

- Obj$\Sigma\mathcal{A}$ *is the set consisting of triples of the form $(I, \{X^i\}, \{V^i\})$, where I is a finite set, $X^i \in \mathrm{Obj}\mathcal{A}$ and V^i is a finite-dimensional graded vector space $(i \in I)$ (write such a triple as a formal direct sum $\oplus_{i\in I} V^i \otimes X^i$).*
- $\hom^{\cdot}_{\Sigma\mathcal{A}}\left(\bigoplus_{i\in I_0} V_0^i \otimes X_0^i, \bigoplus_{j\in I_1} V_1^j \otimes X_1^j\right) \coloneqq \bigoplus_{i,j} \hom^{\cdot}_k(V_0^i, V_1^j) \otimes_k \hom^{\cdot}_{\mathcal{A}}(X_0^i, X_1^j)$ *with the natural grading.*
- *The compositions (higher products) are given as follows: Write an element in*

$$\hom^{\cdot}_{\Sigma\mathcal{A}}\left(\bigoplus_{i\in I_0} V_0^i \otimes X_0^i, \bigoplus_{j\in I_1} V_1^j \otimes X_1^j\right)$$

as a matrix $a = (a^{ji})$ $(i \in I_0, j \in I_1)$ where each a^{ji} is a finite sum $a^{ji} = \Sigma_l \phi^{jil} \otimes x^{jil} \in \hom^{\cdot}_k(V_0^i, V_1^j) \otimes_k \hom^{\cdot}_{\mathcal{A}}(X_0^i, X_1^j)$. We may suppress a^{ji} to be of the form $a^{ji} = \phi^{ji} \otimes x^{ji}$ for simplicity. Then, for $d \geq 1$,

$$\mu^d_{\Sigma\mathcal{A}}(a_d \otimes \cdots \otimes a_1)_{i_d i_0} \coloneqq \Sigma_{i_1,\ldots,i_d}(-1)^{\Delta} \phi^d_{i_d,i_{d-1}} \cdots \phi^1_{i_1,i_0} \otimes \mu^d_{\mathcal{A}}(x^d_{i_d,i_{d-1}} \otimes \cdots \otimes x^1_{i_1,i_0}),$$

where $\Delta = \Sigma_{p<q} |\phi^p_{i_p,i_{p-1}}| \cdot (|x^q_{i_q,i_{q-1}}| - 1)$. Extend $\mu^d_{\Sigma\mathcal{A}}$ for all maps by linearity.

There is an embedding of $\mathcal{A}$ into $\Sigma\mathcal{A}$ by identifying each $X \in \mathrm{Obj}\mathcal{A}$ with the triple $(\{*\}, \{X\}, \{k\}) \in \mathrm{Obj}\Sigma\mathcal{A}$ where the one-dimensional space k in this triple sits in degree 0.

Definition 2.1.7. Let $\mathcal{A}$ be an A_∞-category.

(1) A pre-twisted complex in $\mathcal{A}$ is a pair (X, δ_X) with $X \in \mathrm{Obj}\Sigma\mathcal{A}$ and a differential (or connection) $\delta_X \in \hom^1_{\Sigma\mathcal{A}}(X, X)$; sometimes we write X instead of (X, δ_X).

(2) A sub-complex of a pre-twisted complex $(X = \bigoplus_{i \in I} V^i \otimes X^i, \delta_X = (\delta_X^{ji}))$ with $\delta_X^{ji} = \Sigma_l \phi^{jil} \otimes x^{jil}$ is a pair $(\tilde{X} = \bigoplus_{i\in I} \tilde{V}^i \otimes X^i, \delta_{\tilde{X}} = \delta_{X_{|\tilde{X}}})$, where each $\tilde{V}^i \subset V^i$ is a subspace that is preserved by all ϕ_{jil}.

(3) Given a subcomplex $(\tilde{X} = \bigoplus_{i\in I} \tilde{V}^i \otimes X^i, \delta_{\tilde{X}})$, there induces a quotient complex $\bigoplus_{i\in I} V^i/\tilde{V}^i \otimes X^i$ with the induced differential.

Definition 2.1.8. A twisted complex is a pre-twisted complex (X, δ_X) with two properties:

- δ_X is strictly lower-triangular, i.e. there is a finite decreasing filtration by subcomplexes $X = F^0X \supset F^1X \supset \cdots \supset F^nX = 0$ such that the induced differential on the quotients $F^iX/F^{i+1}X$ is 0.

- δ_X satisfies the Generalized Maurer-Cartan Equation: $\Sigma_{r=1}^{\infty} \mu^r_{\Sigma\mathcal{A}}(\delta_X \otimes \cdots \otimes \delta_X) = 0$.

In the above definition, the Generalized Maurer-Cartan Equation makes sense, i.e. is a finite sum, because δ_X is strictly lower-triangular.

Theorem 2.1.9. *[9, (3l)] There is an A_∞-cateogry* $\mathrm{Tw}\mathcal{A}$ *of twisted complexes in* $\mathcal{A}$ *that consists of the data:*

- $\mathrm{Obj}\mathrm{Tw}\mathcal{A} := \{$*twisted complexes in* $\mathcal{A}\}$,
- $\hom^{\cdot}_{\mathrm{Tw}\mathcal{A}}(X_0, X_1) := \hom^{\cdot}_{\Sigma\mathcal{A}}(X_0, X_1)$ *for* $X_0, X_1 \in \mathrm{Obj}\mathrm{Tw}\mathcal{A}$, *and*
- *compositions (higher products) are given by all the possible deformations by differentials:*

$$\mu^d_{\mathrm{Tw}\mathcal{A}}(a_d \otimes \cdots \otimes a_1) := \Sigma\mu^{d+i_0+\cdots+i_d}_{\Sigma\mathcal{A}} (\delta_{X_d} \otimes \cdots \otimes \delta_{X_d} \otimes a_d \otimes \delta_{X_{d-1}} \otimes \cdots \otimes \delta_{X_{d-1}} \otimes a_{d-1} \otimes \cdots \otimes a_1 \otimes \delta_{X_0} \otimes \cdots \otimes \delta_{X_0})$$

where the sum is over all $i_0, ..., i_d$ *with* i_r *many copies of* δ_{X_r} $(0 \leq r \leq d)$ *in this sum.*

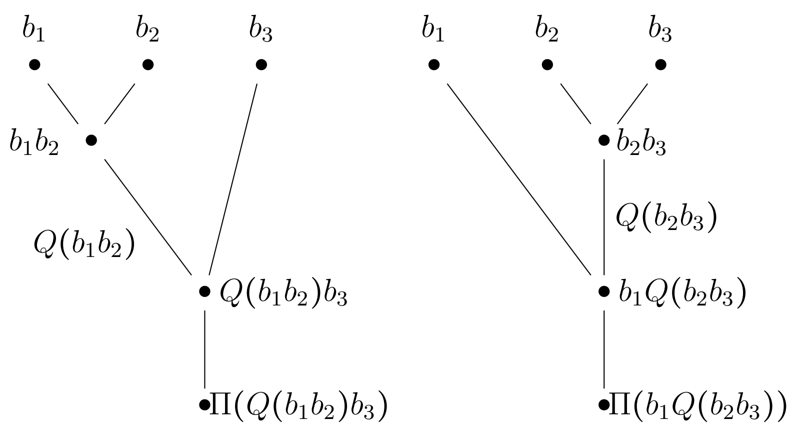
Lemma 2.1.10 (Homological Perturbation)**.** *[6, section 6.4] Let* (A, d) *be a dg-algebra over a field* k*. Let* $\Pi : A \longrightarrow A$ *be an idempotent which commutes with* d*. If there is a homotopy operator* Q *on* A *such that* $\mathrm{id} - \Pi = dQ + Qd$*, then there is an* A_∞*-structure on* $B := \mathrm{im}(\Pi)$ *with higher products given by the formula:*

$$\mu^n_B(b_1 \otimes \cdots \otimes b_n) := \Sigma_T \pm m_T(b_1, ..., b_n)$$

for $b_1, ..., b_n \in B$ $(n \geq 3)$*. Here* T *runs over all oriented planar rooted* 3*-valent trees with* n *leaves (different from the root) marked by* $b_1, ..., b_n$ *and the root marked by* Π *which is the projector. For such a tree, leaves* $b_1, ..., b_n$ *are at the top level from left to right and the root at the bottom level, every inner vertex (i.e. not a leaf or the root) has two edges in-coming from above and one edge out-going below, . Then* $m_T(b_1, ..., b_n)$ *is obtained by going down from leaves to the root, applying the multiplication in* A *at every inner vertex and applying the operator* Q *at every inner* *edge (i.e. its two vertices are inner vertices of the tree), and finally applying the projector* Π.

Remark 2.1.11. The sign for each $m_T(b_1, ..., b_n)$ is determined by the tree T. Here we omit the sign for simplicity by simply writing $\pm$.

Example 2.1.12. Given three elements $b_1, b_2, b_3 \in B$, there are two oriented planar rooted 3-valent trees. For each tree T below, we apply the Homological Perturbation procedure to obtain $m_T(b_1, b_2, b_3)$ at its root:



See Section 3.4 for a calculation of some μ^3 using homological perturbation.

Proposition 2.1.13. *([9, Lemma 3.34]) Let* $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B}$ *be a cohomologically full and faithful* A_∞*-functor. Assume that* $\mathcal{B}$ *is a triangulated* A_∞*-category, and that the objects in the image of* $\mathcal{F}$ *generate it. Then there is a quasi-equivalence* $\tilde{\mathcal{F}} : \mathrm{Tw}\mathcal{A} \longrightarrow \mathcal{B}$ *whose restriction to* $\mathcal{A} \subset \mathrm{Tw}\mathcal{A}$ *is isomorphic to* $\mathcal{F}$ *in* $\mathrm{H}^0(fun(\mathcal{A}, \mathcal{B}))$.

Proposition 2.1.14. *([9, Corollary 4.9]) Let* $\mathcal{B}$ *be a split-closed triangulated* A_∞*-category, and let* $\mathcal{A} \subset \mathcal{B}$ *be a full subcategory which split-generates it. Then there is a quasi-equivalence* $\Pi(\mathrm{Tw}\mathcal{A}) \longrightarrow \mathcal{B}$*, which induces an equivalence of triangulated categories* $\mathrm{H}^0(\mathcal{B}) \simeq D^\pi(\mathcal{A})$.

2.2. Basic constructions of noncommutative projective geometry

In this section, we recall a few definitions on noncommutative projective schemes of [1]. Let S be a noetherian commutative ring and let B be a $\mathbf{Z}_{\geq 0}$-graded noetherian algebra over S. Consider the category $\mathrm{gr} - B$ of graded finitely generated right B-modules. There is a full subcategory tors of $\mathrm{gr} - B$ of torsion modules. Let $\mathrm{qgr} - B$ be the Serre quotient of $\mathrm{gr} - B$ by the subcategory tors, considered as the noncommutative projective scheme.

Notation 2.2.1. (1) Given a graded B-module M, for any $j \in \mathbf{Z}$, $M(j)$ is a graded B-module given by $M(j)_n = M_{j+n}$ $(n \in \mathbf{Z})$.

(2) Denote by $\mathcal{O}(j)$ the image of $B(j)$ in $\mathrm{qgr} - B$.

(3) $\mathrm{H}^i(-) := \mathrm{Ext}^i_{\mathrm{qgr}-R}(\mathcal{O}, -)$; $\Gamma(-) = \oplus_{i \in \mathbf{Z}} \mathrm{H}^i(-)$.

(4) $\underline{\mathrm{Ext}}^{\bullet}_{\mathcal{R}}(M, N) := \oplus_{j \in \mathbb{Z}} \mathrm{Ext}^{\bullet}_{\mathcal{R}-\mathrm{gr}}(M, N(j))$.

2.3. Abstract twist functor

Let $\mathcal{A}$ be any k-linear A_∞-category and let $Y \in \mathrm{Obj}\mathcal{A}$. We will construct an A_∞–functor $T_Y : \mathrm{mod}_\infty{-}\mathcal{A} \longrightarrow \mathrm{mod}_\infty{-}\mathcal{A}$, the abstract twist associated with Y. This functorial abstract twist construction is needed in Section III to define the spherical twist for an R-spherical object F, namely T_F.

Given $\mathcal{M} \in \mathrm{Obj}(\mathrm{mod}_\infty - \mathcal{A})$, define $T_Y(\mathcal{M})$ as follows: for each $X \in \mathrm{Obj}\mathcal{A}$,

$$
\begin{aligned}
\mathcal{T}_Y(\mathcal{M})(X) := \ & \mathcal{M}(X) \\
& \oplus \\
& \mathcal{M}(Y) \otimes_k \hom^{\cdot}(X,Y)[1] \\
& \oplus \\
& \mathcal{M}(Y) \otimes_k \hom^0(Y,Y) \otimes_k \hom^{\cdot}(X,Y)[2] \\
& \oplus \\
& \mathcal{M}(Y) \otimes_k \hom^0(Y,Y) \otimes_k \hom^0(Y,Y) \otimes_k \hom^{\cdot}(X,Y)[3] \\
& \oplus \\
& . \\
& . \\
& .
\end{aligned}
$$

where we label $\mathcal{M}(X)$ as the 1st component, $\mathcal{M}(Y) \otimes_k \hom^{\cdot}(X,Y)[1]$ the 2nd component, so on and so forth; and for each $X_0, X_1, ..., X_{l-1} \in \mathrm{Obj}\mathcal{A}$ ($l \geq 1$), homogeneous maps

$$
\mu^l_{T_Y(\mathcal{M})} : T_Y(\mathcal{M})(X_{l-1}) \otimes_k \hom^{\cdot}(X_{l-2}, X_{l-1}) \otimes_k \cdots \otimes_k \hom^{\cdot}(X_0, X_1) \longrightarrow T_Y(\mathcal{M})(X_0)[2-l]
$$

are given by:

- for $l = 1$,

$$
\begin{aligned}
&\mu^1_{T_Y(\mathcal{M})}(b_d \otimes \cdots \otimes b_1) := \\
&\quad \Sigma_{n+m \leq d} (-1)^{\blacktriangleright_n} b_d \otimes \cdots \otimes b_{n+m+1} \otimes \mu^m(b_{n+m} \otimes \cdots \otimes b_{n+1}) \otimes b_n \otimes \cdots \otimes b_1,
\end{aligned}
$$

where $b_d \otimes \cdots \otimes b_1 \in T_Y(\mathcal{M})(X_0)$ is an element in the d-th component, and

$$\mu^m := \begin{cases} \mu^m_{\mathcal{M}} & \text{if } n+m = d \\ \mu^m_{\mathcal{A}} & \text{if } n+m < d; \end{cases}$$

– for $l > 1$,

$$\mu^l_{T_Y(\mathcal{M})}((b_d \otimes \cdots \otimes b_1) \otimes a_{l-1} \otimes \cdots \otimes a_1) :=$$
$$\Sigma_{1 \le n \le d} b_d \otimes \cdots \otimes b_{n+1} \otimes \mu^{n+l-1}(b_n \otimes \cdots \otimes b_1 \otimes a_{l-1} \otimes \cdots \otimes a_1),$$

where $b_d \otimes \cdots \otimes b_1 \in T_Y(\mathcal{M})(X_{l-1})$ is an element in the d-th component and

$$\mu^{n+l-1} := \begin{cases} \mu^{d+l-1}_{\mathcal{M}} & \text{if } n = d \\ \mu^{n+l-1}_{\mathcal{A}} & \text{if } n < d. \end{cases}$$

Proposition 2.3.1. *$T_Y(\mathcal{M})$ together with $\mu^l_{T_Y(\mathcal{M})}$ $(l \ge 1)$ as above is a right A_∞-module, i.e. equation (2.1.1) is satisfied*

Proof. We need to check that for any $X_0, ..., X_{l-1} \in \mathrm{Obj}\mathcal{A}$ and any

$$b \otimes a_{l-1} \otimes \cdots \otimes a_1 \in T_Y(\mathcal{M})(X_{l-1}) \otimes_k \hom^{\cdot}(X_{l-2}, X_{l-1}) \otimes_k \cdots \otimes_k \hom^{\cdot}(X_0, X_1)$$

with $b = b_d \otimes \cdots \otimes b_1$ an element in the d-th component of $T_Y(\mathcal{M})(X_{l-1})$ $(d \ge 1, l \ge 1)$,

$$\Sigma_{m+n \le l} (-1)^{\maltese_n} \mu^{l-m+1}_{T_Y(\mathcal{M})}(b \otimes a_{l-1} \otimes \cdots \otimes a_{n+m+1} \otimes \mu^m(a_{n+m} \otimes \cdots \otimes a_{n+1}) \otimes a_n \otimes \cdots \otimes a_1) = 0. \quad (2.3.1)$$

Case 1. When $l = 1$, equation (2.3.1) becomes $\mu^1_{T_Y(\mathcal{M})}(\mu^1_{T_Y(\mathcal{M})}(b)) = 0$ which we can write

$$\begin{aligned}
&\mu^1_{T_Y(\mathcal{M})}(\mu^1_{T_Y(\mathcal{M})}(b_d \otimes \cdots \otimes b_1)) \\
&= \Sigma_{n+m\leq d}(-1)^{\blacktriangleright_n}\mu^1_{T_Y(\mathcal{M})}(b_d \otimes \cdots \otimes b_{n+m+1} \otimes \mu^m(b_{n+m} \otimes \cdots \otimes b_{n+1}) \otimes b_n \otimes \cdots \otimes b_1) \\
&= \Sigma_{n+m\leq d}[S_1^{(n,m)} + S_2^{(n,m)} + S_3^{(n,m)}],
\end{aligned}$$

where

$$\begin{aligned}
S_1^{(n,m)} &= \Sigma_{r+s\leq d, r\geq n+m}(-1)^{\blacktriangleright_n + \blacktriangleright_{r-m+1}} b_d \otimes \cdots \otimes b_{r+s+1} \\
&\quad \otimes \mu^s(b_{r+s} \otimes \cdots \otimes b_{r+1}) \otimes b_r \otimes \cdots \otimes b_{n+m+1} \otimes \mu^m(b_{n+m} \otimes \cdots \otimes b_{n+1}) \otimes b_n \otimes \cdots \otimes b_1, \\
S_2^{(n,m)} &= \Sigma_{r+s\leq d-m+1, r\leq n, s\geq n+1-r}(-1)^{\blacktriangleright_n + \blacktriangleright_r} b_d \otimes \cdots \otimes b_{r+s+m} \\
&\quad \otimes \mu^s(b_{r+s+m-1} \otimes \cdots \otimes b_{n+m+1} \otimes \mu^m(b_{n+m} \otimes \cdots \otimes b_{n+1}) \otimes b_n \otimes \cdots \otimes b_{r+1}) \otimes b_r \otimes \cdots \otimes b_1, \\
S_3^{(n,m)} &= \Sigma_{r<n, s<n+1-r}(-1)^{\blacktriangleright_n + \blacktriangleright_r} b_d \otimes \cdots \otimes b_{n+m+1} \\
&\quad \otimes \mu^m(b_{n+m} \otimes \cdots \otimes b_{n+1}) \otimes b_n \otimes \cdots \otimes b_{r+s+1} \otimes \mu^s(b_{r+s} \otimes \cdots \otimes b_{r+1}) \otimes b_r \otimes \cdots \otimes b_1.
\end{aligned}$$

Here, in $S_1^{(n,m)}$,

$$\begin{aligned}
\blacktriangleright_{r-m+1} &= |b_r| + \cdots + |b_{n+m+1}| + |\mu^m(b_{n+m} \otimes \cdots \otimes b_{n+1})| + |b_n| + \cdots + |b_1| - (r - m + 1) \\
&= |b_r| + \cdots + |b_1| - r + 1.
\end{aligned}$$

Note that the terms in $\Sigma_{n+m\leq d}S_1^{(n,m)}$ are in a one-to-one correspondence with the terms in $\Sigma_{n+m\leq d}S_3^{(n,m)}$. For a term in the former given by $n = n_0, m = m_0, r = r_0, s = s_0$ with sign

$$(-1)^{\blacktriangleright_{n_0} + \blacktriangleright_{r_0 - m_0 + 1}} = (-1)^{(|b_{n_0}| + \cdots + |b_1| - n_0) + (|b_{r_0}| + \cdots + |b_1| - r_0 + 1)},$$

the corresponding term in the latter is given by $n = r_0, m = s_0, r = n_0, s = m_0$ with sign

$$(-1)^{\maltese_{r_0}+\maltese_{n_0}} = (-1)^{(|b_{r_0}|+\cdots+|b_1|-r_0)+(|b_{n_0}|+\cdots+|b_1|-n_0)}.$$

Since these two signs cancel out, $\Sigma_{n+m\leq d} S_1^{(n,m)} + \Sigma_{n+m\leq d} S_3^{(n,m)} = 0$. Also, $\Sigma_{n+m\leq d} S_2^{(n,m)} = 0$ by looking at the middle quadratic terms for a fixed segment, using the quadratic equation of either $\mathcal{M}$ or $\mathcal{A}$.

Case 2. When $l > 1$, the left hand side of equation (2.3.1) can be written as

$$\Sigma_{m+n\leq l}(-1)^{\maltese_n}\mu^{l-m+1}_{T_Y(\mathcal{M})}(b\otimes a_{l-1}\otimes\cdots\otimes a_{n+m+1}\otimes\mu^m(a_{n+m}\otimes\cdots\otimes a_{n+1})\otimes a_n\otimes\cdots\otimes a_1) = S_1+S_2,$$

where

$$\begin{aligned}
S_1 &= \Sigma_{n<l}(-1)^{\maltese_n}\mu^{n+1}_{T_Y(\mathcal{M})}(\mu^{l-n}_{T_Y(\mathcal{M})}(b\otimes a_{l-1}\otimes\cdots\otimes a_{n+1})\otimes a_n\otimes\cdots\otimes a_1)\\
&= \mu^1_{T_Y(\mathcal{M})}(\mu^l_{T_Y(\mathcal{M})}(b\otimes a_{l-1}\otimes\cdots\otimes a_1))\\
&\quad + (-1)^{\maltese_{l-1}}\mu^l_{T_Y(\mathcal{M})}(\mu^1_{T_Y(\mathcal{M})}(b)\otimes a_{l-1}\otimes\cdots\otimes a_1)\\
&\quad + \Sigma_{0<n<l-1}(-1)^{\maltese_n}\mu^{n+1}_{T_Y(\mathcal{M})}(\mu^{l-n}_{T_Y(\mathcal{M})}(b\otimes a_{l-1}\otimes\cdots\otimes a_{n+1})\otimes a_n\otimes\cdots\otimes a_1)\\
&= \Sigma_{1\leq n\leq d}\mu^1_{T_Y(\mathcal{M})}(b_d\otimes\cdots\otimes b_{n+1}\otimes\mu^{n+l-1}(b_n\otimes\cdots\otimes b_1\otimes a_{l-1}\otimes\cdots\otimes a_1))\\
&\quad + \Sigma_{s+m\leq d}(-1)^{\maltese_{l-1}+\maltese_s}\\
&\quad\ \mu^l_{T_Y(\mathcal{M})}(b_d\otimes\cdots\otimes b_{s+m+1}\otimes\mu^m(b_{s+m}\otimes\cdots\otimes b_{s+1})\otimes b_s\otimes\cdots\otimes b_1\otimes a_{l-1}\otimes\cdots\otimes a_1)\\
&\quad + \Sigma_{0<n<l-1,1\leq m\leq d}(-1)^{\maltese_n}\\
&\quad\ \mu^{n+1}_{T_Y(\mathcal{M})}(b_d\otimes\cdots\otimes b_{m+1}\otimes\mu^{m+l-1-n}(b_m\otimes\cdots\otimes b_1\otimes a_{l-1}\otimes\cdots\otimes a_{n+1})\otimes a_n\otimes\cdots\otimes a_1)
\end{aligned}$$

and

$$
\begin{aligned}
S_2 &= \Sigma_{m+n<l}(-1)^{\blacktriangleright_n}\mu_{T_Y(\mathcal{M})}^{l-m+1}(b\otimes a_{l-1}\otimes\cdots\otimes a_{n+m+1}\otimes\mu_{\mathcal{A}}^m(a_{n+m}\otimes\cdots\otimes a_{n+1})\otimes a_n\otimes\cdots\otimes a_1)\\
&= \Sigma_{m+n<l,1\leq r\leq d}(-1)^{\blacktriangleright_n}\\
&\quad b_d\otimes\cdots\otimes b_{r+1}\otimes\mu^{l-m+r}(b_r\otimes\cdots\otimes b_1\otimes a_{l-1}\otimes\cdots\otimes a_{n+m+1}\otimes\\
&\qquad\qquad \mu_{\mathcal{A}}^m(a_{n+m}\otimes\cdots\otimes a_{n+1})\otimes a_n\otimes\cdots\otimes a_1).
\end{aligned}
$$

Write $S_1 = S_{11} + S_{12} + S_{13}$ with S_{1j} the j-th sum in the above expression of S_1 ($j = 1,2,3$). We can further write

$$
\begin{cases}
S_{11} = S_{11}^{(1)} + S_{11}^{(2)}\\
S_{12} = S_{12}^{(1)} + S_{12}^{(2)}.
\end{cases}
$$

Here,

$$S_{11}^{(1)} = \Sigma_{1\le n<d}\Sigma_{r+s\le d-n+1, r\ge 1}(-1)^{\blacktriangleright_r}$$
$$b_d \otimes \cdots \otimes \mu^s(b_{n+r+s-1} \otimes \cdots \otimes b_{n+r}) \otimes \cdots \otimes b_{n+1} \otimes \mu^{n+l-1}(b_n \otimes \cdots \otimes b_1 \otimes a_{l-1} \otimes \cdots \otimes a_1),$$

$$S_{11}^{(2)} = \Sigma_{1\le n\le d}\Sigma_{s\le d-n+1}$$
$$b_d \otimes \cdots \otimes \mu^s(b_{n+s-1} \otimes \cdots \otimes b_{n+1} \otimes \mu^{n+l-1}(b_n \otimes \cdots \otimes b_1 \otimes a_{l-1} \otimes \cdots \otimes a_1)),$$

$$S_{12}^{(1)} = \Sigma_{s+m\le d, r\ge s+m}(-1)^{\blacktriangleright_{l-1}+\blacktriangleright_s}$$
$$b_d \otimes \cdots \otimes \mu^{r-m+l}(b_r \otimes \cdots \otimes b_{s+m+1} \otimes \mu^m(b_{s+m} \otimes \cdots \otimes b_{s+1}) \otimes b_s \otimes \cdots \otimes b_1 \otimes a_{l-1} \otimes \cdots \otimes a_1),$$

$$S_{12}^{(2)} = \Sigma_{s+m\le d, 1\le r\le s}(-1)^{\blacktriangleright_{l-1}+\blacktriangleright_s}$$
$$b_d \otimes \cdots \otimes b_{s+m+1} \otimes \mu^m(b_{s+m} \otimes \cdots \otimes b_{s+1}) \otimes \cdots \otimes \mu^{r+l-1}(b_r \otimes \cdots \otimes b_1 \otimes a_{l-1} \otimes \cdots \otimes a_1),$$

$$S_{13} = \Sigma_{0<n<l-1, 1\le m\le d, r\ge m}(-1)^{\blacktriangleright_n}$$
$$b_d \otimes \cdots \otimes \mu^{r-m+n+1}(b_r \otimes \cdots \otimes b_{m+1} \otimes \mu^{m+l-1-n}(b_m \otimes \cdots \otimes b_1 \otimes a_{l-1} \otimes \cdots \otimes a_{n+1})$$
$$\otimes a_n \otimes \cdots \otimes a_1),$$

and the signs are given by

$$\begin{cases} \blacktriangleright_n = |a_n| + \cdots + |a_1| - n, \\ \blacktriangleright_{l-1} = |a_{l-1}| + \cdots + |a_1| - (l-1), \\ \blacktriangleright_s = |b_s| + \cdots + |b_1| - s, \\ \blacktriangleright_r = |b_{n+r-1}| + \cdots + |b_{n+1}| + |\mu^{n+l-1}(b_n \otimes \cdots \otimes b_1 \otimes a_{l-1} \otimes \cdots \otimes a_1)| - r \\ \qquad = |b_{n+r-1}| + \cdots + |b_{n+1}| + |b_n| + \cdots + |b_1| + |a_{l-1}| + \cdots + |a_1| + 2 - (n+l-1) - r \\ \qquad = |b_{n+r-1}| + \cdots + |b_1| + |a_{l-1}| + \cdots + |a_1| + 3 - n - l - r. \end{cases}$$

Now it is easy to verify that $S_{11}^{(2)} + S_{12}^{(1)} + S_{13} + S_2 = 0$ (by looking at terms with a fixed head) and $S_{11}^{(1)} + S_{12}^{(2)} = 0$. □

Let us now define the effect of T_Y on morphisms.

– For $l = 1$, $T_Y^1 : \hom^{\cdot}(\mathcal{M}_0, \mathcal{M}_1) \longrightarrow \hom^{\cdot}(T_Y(\mathcal{M}_0), T_Y(\mathcal{M}_1))$ for A_∞-modules $\mathcal{M}_0$ and $\mathcal{M}_1$ is given by:

  ∗ $d = 1$:

$$(T_Y^1(t))^1 : T_Y(\mathcal{M}_0)(X) \longrightarrow T_Y(\mathcal{M}_0)(X)[|t|] \quad (X \in \mathrm{Obj}\mathcal{A})$$

$$c \otimes b_r \otimes \cdots \otimes b_1 \mapsto \Sigma_{0\le s\le r} t^{r+1-s}(c \otimes b_r \otimes \cdots \otimes b_{s+1}) \otimes b_s \otimes \cdots \otimes b_1$$

  ∗ $d > 1$:

$$(T_Y^1(t))^d : T_Y(\mathcal{M}_0)(X_{d-1}) \otimes_k \hom_{\mathcal{A}}^{\cdot}(X_{d-2}, X_{d-1}) \otimes_k \cdots \otimes_k \hom_{\mathcal{A}}^{\cdot}(X_0, X_1)$$

$$\longrightarrow T_Y(\mathcal{M}_1)(X_0)[|t| - d + 1]$$

$$c \otimes b_r \otimes \cdots \otimes b_1 \otimes a_{d-1} \otimes \cdots \otimes a_1 \mapsto t^{r+d}(c \otimes b_r \otimes \cdots \otimes b_1 \otimes a_{d-1} \otimes \cdots \otimes a_1)$$

– For $l > 1$, $T_Y^l = 0$.

Since $T_Y^{\ge 2}$ vanishes, all that we need to check for T_Y to be an A_∞-functor is

$$\begin{cases} \mu^1\left(T_Y^1(t)\right) = T_Y^1\left(\mu^1(t)\right) \\ \mu^2\left(T_Y^1(t_2) \otimes T_Y^1(t_1)\right) = T_Y^1\left(\mu^2(t_2 \otimes t_1)\right). \end{cases} \tag{2.3.2}$$

This is a straightforward check.

CHAPTER III

R-PAIRS OF GENUS 0

3.1. Definition of R-pairs of genus 0

Let $\mathcal{C}$ be a strictly unital minimal k-linear A_∞-category.

Suppose $F \in \mathrm{Obj}\mathcal{C}$ is an object such that $\mathrm{Hom}^0(F,F) = R \cdot \mathrm{id}_F$ as k-algebras. Since $\mathcal{C}$ is strictly unital and minimal, the μ^2 gives natural R-module structures on $\mathrm{Hom}^*(F,X)$ and $\mathrm{Hom}^*(X,F)$ for any $X \in \mathrm{Obj}\mathcal{C}$. If, in addition, $\mathrm{Hom}^1(F,F) \simeq R$ as R-R-bimodules, then $\mu^2 : \mathrm{Hom}^i(X,F) \otimes_k \mathrm{Hom}^{1-i}(F,X) \longrightarrow \mathrm{Hom}^1(F,F)$ is actually R-bilinear for all i. To see this, let $r \in R, f \in \mathrm{Hom}^i(X,F)$ and $g \in \mathrm{Hom}^{1-i}(F,X)$, then

$$
\begin{aligned}
&\mu^2(\mu^2(r \cdot \mathrm{id}_F \otimes f) \otimes g) \\
&= \mu^2(r \cdot \mathrm{id}_F \otimes \mu^2(f \otimes g)) && \text{(because } \mu^1 = 0\text{)} \\
&= \mu^2(\mu^2(f \otimes g) \otimes r \cdot \mathrm{id}_F) && \text{(because } \mathrm{Hom}^1(F,F) \simeq R \text{ as } R\text{-}R\text{-bimodules)} \\
&= \mu^2(f \otimes \mu^2(g \otimes r \cdot \mathrm{id}_F)). && \text{(because } \mu^1 = 0\text{)}
\end{aligned}
$$

So, there induces a map, also denoted by μ^2,

$$\mathrm{Hom}^i(X,F) \otimes_R \mathrm{Hom}^{1-i}(F,X) \longrightarrow \mathrm{Hom}^1(F,F).$$

Definition 3.1.1. An object $F \in \mathrm{Obj}\mathcal{C}$ is R-1-*spherical* (or simply R-*spherical*), if

- $\mathrm{Hom}^i(F,F) = 0$ for $i \neq 0,1$;
- an isomorphism of k-algebras $\mathrm{Hom}^0(F,F) \simeq R \cdot \mathrm{id}_F$ is fixed;
- $\mathrm{Hom}^1(F,F) \simeq R$ as an R-R-bimodule; and

– for any $X \in \mathrm{Obj}\mathcal{C}$, $\hom(X,F)$ and $\hom(F,X)$ are perfect complexes of R-modules, and the pairings $\mu^2 : \mathrm{Hom}^i(X,F) \otimes_R \mathrm{Hom}^{1-i}(F,X) \longrightarrow \mathrm{Hom}^1(F,F)$ are perfect.

We are interested in pairs of objects (E,F) of special kind such that F is R-spherical. Note that morphism spaces $\mathrm{Hom}^i(E,F)$ and $\mathrm{Hom}^i(F,E)$ have natural R-module structures given by post-composing and precomposing with $\mathrm{End}(F) \simeq R$.

Definition 3.1.2. An object $E \in \mathrm{Obj}\mathcal{C}$ is *exceptional*, if $\mathrm{Hom}^*(E,E) = \mathrm{Hom}^0(E,E) \simeq k$.

Definition 3.1.3. A pair of objects (E,F) in $\mathcal{C}$ is an *R-pair of genus* 0 if E is exceptional, F is R-spherical, and

$$\mathrm{Hom}^*(E,F) = \mathrm{Hom}^0(E,F) \simeq R, \quad \mathrm{Hom}^*(F,E) = \mathrm{Hom}^1(F,E) \simeq R$$

as R-modules.

Let us denote by $\tau : R \to k$ the k-linear map given by $\tau(1) = 0$, $\tau(t) = 1$. Note that the k-linear pairing

$$R \times R \to k : (x,y) \mapsto \tau(xy)$$

is nondegenerate. This implies that for any perfect R-linear pairing

$$b : P \times Q \to R$$

of finitely generated free R-modules, the induced k-linear pairing

$$\tau b : P \times Q \to k$$

is also perfect. This gives a way to identify the dual to a perfect complex of R-modules over R with the dual over k.

Next, we formulate a condition on an R-pair that will be relevant for us.

Compatibility with Serre duality. Assume $\mathcal{S}$ is a Serre functor on the cohomology category $\mathrm{H}(\mathcal{C})$ over k. We say that (E,F) is compatible with Serre duality if an isomorphism

$$T_F(E) \xrightarrow{\alpha} \mathcal{S}^{-1}E[1]$$

is given, such that the composition

$$\hom(E,F) \xrightarrow{T_F} \hom(T_F E, T_F F) \simeq \hom(\mathcal{S}^{-1}E[1], F) \longrightarrow \hom(F, E[1])^*,$$

where the last arrow is given by the Serre duality, coincides with the isomorphism induced by the R-spherical structure on F.

To give an R-pair of genus 0 is equivalent to giving a minimal A_∞-structures (up to gauge equivalence) on the following graded category $\mathcal{C}_R(0)$ over k with two objects E and F (it can be also viewed simply as a graded k-algebra). By definition,

$$\mathrm{Hom}^*(E,E) = \mathrm{Hom}^0(E,E) = k,$$

$$\mathrm{Hom}^0(F,F) = \mathrm{Hom}^1(F,F) = R,$$

$$\mathrm{Hom}^0(E,F) = \mathrm{Hom}^1(F,E) = R,$$

and all interesting compositions (not including $\mathrm{Hom}^0(E,E)$) are given by the multiplication in R.

One can define the corresponding moduli functor in a standard way by considering arbitrary k-algebras S and minimal S-linear A_∞-structures on $\mathcal{C}_R(0) \otimes_k S$ (see [7]). Our goal is to relate this moduli functor (or rather its subfunctor corresponding to R-pairs compatible with Serre duality) with a certain moduli space of filtered algebras, which is much easier to study.

3.2. Computation of the spherical twist by F

Let (E, F) be an R-pair of genus 0. We compute explicitly the twist functor T_F on the twisted complexes

$$\tilde{E}_n := [F \longrightarrow F \longrightarrow \cdots \longrightarrow F \longrightarrow E]$$

with n copies of F ($n \geq 0$) where the differentials are given by $\delta_F : F \longrightarrow F$ and $\delta : F \longrightarrow E$, R-generators of $\mathrm{Hom}^1(F, F)$ and $\mathrm{Hom}^1(F, E)$.

For each A_∞-module $\mathcal{M}$ over R, there is a twisted complex $B(\mathcal{M}, F)$ (also denoted by $\mathcal{M} \otimes_R F$):

$$[\cdots \longrightarrow \mathcal{M} \otimes_k R \otimes_k R \otimes_k F[2] \longrightarrow \mathcal{M} \otimes_k R \otimes_k F[1] \longrightarrow \mathcal{M} \otimes_k F]$$

where the differentials are given by the usual formulas of a bar resolution. For any twisted complex $C^\cdot$ in the A_∞-category $< E, F >$, consider the evaluation map

$$\mathrm{eval} : B\left(\hom^\cdot(F, C^\cdot), F\right) \longrightarrow C^\cdot.$$

Then we have

Proposition 3.2.1. *For each n,* $\mathrm{Cone}\Big(\mathrm{eval} : B\big(\hom^{\cdot}(F, \tilde{E}_n), F\big) \longrightarrow \tilde{E}_n\Big) \simeq T_F(\tilde{E}_n)$.

3.3. R-pairs from curves of arithmetic genus 0

Proposition 3.3.1. *Let C be an irreducible Cohen-Macauley projective curve over k of arithmetic genus 0 (not necessarily reduced) with $h^0(C, \mathcal{O}_C) = 1$. Let $D \subset C$ be an effective Cartier divisor of length 2 supported at one point p. Consider the DG-enhancement of $\mathrm{Perf}(C)$ and view it as an A_∞-category (with vanishing higher products $\mu^{\geq 3} = 0$). Then the pair $(\mathcal{O}_C, \mathcal{O}_D)$ is an R-pair of genus 0.*

Proof. Work locally near p. Since $l(\mathcal{O}_D) = 2$, there is a short exact sequence:

$$0 \longrightarrow \mathcal{O}_p \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{O}_p \longrightarrow 0,$$

where $\mathcal{O}_p$ is the skyscraper sheaf at point p. Note that there is a splitting

$$1 : \mathrm{H}^0(\mathcal{O}_p) \longrightarrow \mathrm{H}^0(\mathcal{O}_D),$$

given by the constant map. So,

$$\mathrm{H}^0(\mathcal{O}_D) \simeq \mathrm{H}^0(\mathcal{O}_p) \oplus \mathrm{H}^0(\mathcal{O}_p) \simeq k \oplus k$$

as k-vector spaces. Let $I = \ker(\mathrm{H}^0(\mathcal{O}_D) \longrightarrow \mathrm{H}^0(\mathcal{O}_p)) \simeq k$. Then $I^2 = 0$. Choose $t \in I$ such that $I = k \cdot t$. Then $\mathrm{H}^0(\mathcal{O}_D) \simeq k[t]/(t^2) = R$. We may assume $D = (f)$ for some function f near p.

The short exact sequence

$$0 \longrightarrow \mathcal{O}_C \xrightarrow{f} \mathcal{O}_C \longrightarrow \mathcal{O}_D \longrightarrow 0,$$

induces a long exact sequence

$$0 \longrightarrow \mathcal{H}\mathrm{om}_{\mathcal{O}_C}(\mathcal{O}_D, \mathcal{O}_C) \longrightarrow \mathcal{H}\mathrm{om}_{\mathcal{O}_C}(\mathcal{O}_C, \mathcal{O}_C) \longrightarrow \mathcal{H}\mathrm{om}_{\mathcal{O}_C}(\mathcal{O}_C, \mathcal{O}_C)$$
$$\longrightarrow \mathcal{E}\mathrm{xt}^1_{\mathcal{O}_C}(\mathcal{O}_D, \mathcal{O}_C) \longrightarrow \mathcal{E}\mathrm{xt}^1_{\mathcal{O}_C}(\mathcal{O}_C, \mathcal{O}_C) = 0.$$

Since $\mathcal{H}\mathrm{om}_{\mathcal{O}_C}(\mathcal{O}_C, \mathcal{O}_C) \simeq \mathcal{O}_C$, we see that $\mathcal{E}\mathrm{xt}^1_{\mathcal{O}_C}(\mathcal{O}_D, \mathcal{O}_C) \simeq \mathrm{coker}(f : \mathcal{O}_C \longrightarrow \mathcal{O}_C) \simeq \mathcal{O}_D$.

There is a 1st-quadrant cohomology spectral sequence

$$E_2^{p,q} \simeq \mathrm{H}^p(C, \mathcal{E}\mathrm{xt}^q_{\mathcal{O}_C}(\mathcal{O}_D, \mathcal{O}_C)) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}_C}(\mathcal{O}_D, \mathcal{O}_C).$$

Since C is a curve, $E_2^{p,q} = 0$ for $p > 1$. Also $\mathcal{H}\mathrm{om}_{\mathcal{O}_C}(\mathcal{O}_D, \mathcal{O}_C) = 0$. So, the E_2-page is

	.	.	.	.	
	.	.	.	.	
2nd row:	$\mathrm{H}^0(C, \mathcal{E}\mathrm{xt}^2_{\mathcal{O}_C}(\mathcal{O}_D, \mathcal{O}_C))$	$\mathrm{H}^1(C, \mathcal{E}\mathrm{xt}^2_{\mathcal{O}_C}(\mathcal{O}_D, \mathcal{O}_C))$	0	0	$\cdots$
1st row:	$\mathrm{H}^0(C, \mathcal{O}_D)$	$\mathrm{H}^1(C, \mathcal{O}_D)$	0	0	$\cdots$
0th row:	0	0	0	0	$\cdots$

Hence the $E_\infty^{p,q} \simeq E_2^{p,q}$ for all p, q. So,

$$\mathrm{Ext}^1_{\mathcal{O}_C}(\mathcal{O}_D, \mathcal{O}_C) \simeq \mathrm{H}^0(C, \mathcal{O}_D) \simeq R.$$

There is a left exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{O}_D, \omega_C \otimes_{\mathcal{O}_C} \mathcal{O}_D) \longrightarrow \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{O}_C, \omega_C \otimes_{\mathcal{O}_C} \mathcal{O}_D)$$
$$\longrightarrow \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{O}(-D), \omega_C \otimes_{\mathcal{O}_C} \mathcal{O}_D).$$

The last arrow induced by f is 0. So, $\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{O}_D, \omega_C \otimes_{\mathcal{O}_C} \mathcal{O}_D) \simeq \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{O}_C, \omega_C \otimes_{\mathcal{O}_C} \mathcal{O}_D)$. Now, by Serre Duality, $\mathrm{Ext}^1(\mathcal{O}_D, \mathcal{O}_D) \simeq \mathrm{Ext}^1(\mathcal{O}_D, \mathcal{O}_C)$. □

We have the following two examples in which conditions of Proposition 3.3.1 are satisfied.

Example 3.3.2. We can take C to be a smooth curve of arithmetic genus 0, i.e., $C = \mathbb{P}^1$ and consider the nonreduced divisor $D = 2p$ (where $p \in C$).

Example 3.3.3. Let $\pi : C := \mathcal{S}\mathrm{pec}\left(\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-1)\right) \longrightarrow \mathbb{P}^1_k$ be the nonreduced curve over $\mathbb{P}^1_k$ given by the obvious embedding $\mathcal{O}_{\mathbb{P}^1_k} \hookrightarrow \mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-1)$ of sheaves of k-algebras, where the algebra structure of $\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-1)$ is $(a,m)\cdot(b,n) = (ab, an+bm)$. Let $p \in \mathbb{P}^1_k$ and set $D = \pi^{-1}(p)$. Then $\mathrm{h}^0(C, \mathcal{O}_C) = 1$, $\mathrm{h}^1(C, \mathcal{O}_C) = 0$, and D is an effective Cartier divisor of length 2 supported at p. So, it gives rise to an R-pair of genus 0, $(\mathcal{O}_C, \mathcal{O}_D)$.

3.4. Computation of the A_∞-structure associated with the double point on a smooth curve of genus 0

The results of this section are not used anywhere else in the text. They are presented here, so as to give an example of explicit computation of the A_∞-structures we are studying that arise in the geometric context.

Specifically, we will use the construction in Lemma 2.1.10 (Homological Perturbation) to compute some products μ^3 associated with the pair $(C, 2p)$, where $C = \mathbb{P}^1$, $p \in C$ is a point. We want to apply the Homological Perturbation to a certain dg-algebra computing $\mathrm{Ext}^*(G, G)$, where $G = \mathcal{O}_C \oplus \mathcal{O}_{2p}$. We follow an approach similar to the one in [7, Sec. 3]: we use an analog of the Cech complex corresponding to the covering of C by the open subset $U := C \setminus \{p\}$ and the formal neighborhood of p in C.

For every coherent sheaf $\mathcal{F}$ on C we can consider the two-term complex $K^\bullet(\mathcal{F}) = K_p^\bullet(\mathcal{F})$ with

$$K^0(\mathcal{F}) = \varprojlim_n H^0\big(C, \mathcal{F}/\mathcal{F}(-np)\big) \oplus H^0(U, \mathcal{F}),$$
$$K^1(\mathcal{F}) = \varinjlim_m \varprojlim_n H^0\big(C, \mathcal{F}(mp)/\mathcal{F}(-np)\big)$$

and the differential

$$d(s_0, s) = \kappa(s) - \iota(s_0),$$

where we use natural maps $\iota : H^0\big(C, \mathcal{F}/\mathcal{F}(-np)\big) \to K^1(\mathcal{F})$ and $\kappa : H^0(U, \mathcal{F}) \to K^1(\mathcal{F})$.

The construction of $K^\bullet(\mathcal{F})$ immediately generalizes to the case when $\mathcal{F}$ is a bounded complex of vector bundles (by taking the total complex of the corresponding bicomplex). Furthermore, if $\mathcal{A}$ is a complex of coherent sheaves equipped with a structure of an $\mathcal{O}$-dg-algebra then we can equip the complex $K^\bullet(\mathcal{A})$ with a structure of a dg-algebra by using the natural componentwise multiplication on $K^0(\mathcal{A})$ and using the multiplications

$$K^0(\mathcal{A}) \otimes K^1(\mathcal{A}) \to K^1(\mathcal{A}) : (s_0, s) \cdot u = \iota(s_0) \cdot u,$$
$$K^1(\mathcal{A}) \otimes K^0(\mathcal{A}) \to K^1(\mathcal{A}) : u \cdot (s_0; s) = u \cdot \kappa(s), \tag{3.4.1}$$

where on the right-hand side we use the natural product on $K^1(\mathcal{A})$.

Since $\mathcal{O}_{2p}$ is not locally free, it is convenient to replace this sheaf by the following resolution:

$$P := [\mathcal{O}(-2p) \longrightarrow \mathcal{O}],$$

where we view P as a complex concentrated in degrees -1 and 0.

We replace our generator $G = \mathcal{O}_C \oplus \mathcal{O}_{2p}$ with the complex $\mathcal{O}_C \oplus P$, and consider the sheaf of dg-algebras

$$\mathcal{A} := \underline{\mathrm{End}}(\mathcal{O}_C \oplus P),$$

so that the hypercohomology algebra $H^*(C, \mathcal{A})$ is identified with $\mathrm{Ext}^*(G, G)$. This hypercohomology is computed as the cohomology of the dg-algebra

$$E^{dg} := K^\bullet(\mathcal{A}).$$

We fix a formal parameter t at p. This choice gives isomorphisms

$$k[[t]] \xrightarrow{\sim} \varprojlim_n H^0\big(C, \mathcal{O}_C/\mathcal{O}_C(-np)\big),$$

$$k((t)) \xrightarrow{\sim} \varinjlim_m \varprojlim_n H^0\big(C, \mathcal{O}_C(mp)/\mathcal{O}_C(-np)\big).$$

Hence, for any integer n we have an identification of $K^1\big(\mathcal{O}_C(np)\big)$ with $k((t))$. For an element $a(t) \in k((t))$ we denote by $[a(t)]$ the corresponding element of $K^1\big(\mathcal{O}_C(np)\big)$.

We have a direct sum decomposition

$$E^{dg} = K_{\mathcal{O}} \oplus K_{\mathcal{O},P} \oplus K_{P,\mathcal{O}} \oplus K_{P,P},$$

where $K_{\mathcal{O}} = K^\bullet(\mathcal{O})$ and $K_{P_1,P_2} = K^\bullet(P_2 \otimes P_1^\vee)$. We denote (local) sections of the 0th term of P by $\mathbf{e} \cdot f$, where $f \in \mathcal{O}$, and local sections of the -1st term of P by $\mathbf{u} \cdot f$, where $f \in \mathcal{O}(-2p)$.

We denote elements of $K_{\mathcal{O}}$ as

$$v + f + [a],$$

where $v \in tk[[t]]$, $f \in \mathcal{O}(U)$, $a \in k((t))$, v and f have degree 0 and $[a]$ has degree 1. The differential on $K_{\mathcal{O},\mathcal{O}}$ is given by

$$d_{\mathcal{O}}(v + f + [a]) = [f - v],$$

where on the right-hand side we use the projection

$$\mathcal{O}(U) \to K^1(\mathcal{O}_C) \simeq k((t))$$

to view f as an element of $k((t))$.

The summand $K_{\mathcal{O},P}$ decomposes as a graded space as

$$\mathbf{u} \cdot \big(t^2 k[[t]] \oplus \mathcal{O}(U)\big)[1] \oplus \mathbf{u} \cdot k((t)) \oplus \mathbf{e} \cdot (k[[t]] \oplus \mathcal{O}(U)) \oplus k((t))[-1].$$

We will write elements of $K_{\mathcal{O},P}$ as formal sums

$$\mathbf{u}\cdot v+\mathbf{u}\cdot f+\mathbf{u}\cdot[a]+\mathbf{e}\cdot w+\mathbf{e}\cdot h+\mathbf{e}\cdot[b],$$

where $v\in t^2k[[t]]$, $w\in k[[t]]$, $a,b\in k((t))$, $f,h\in\mathcal{O}(U)$. Here we treat a,b,v,w,f,h as having degree 0, and use the convention that $\deg(\mathbf{u})=-1$, $\deg(\mathbf{e})=0$ and $\deg([x])=\deg(x)+1$.

Similarly, elements of $K_{P,\mathcal{O}}$ are formal sums

$$v\cdot\mathbf{u}^*+f\cdot\mathbf{u}^*+[a]\cdot\mathbf{u}^*+w\cdot\mathbf{e}^*+h\cdot\mathbf{e}^*+[b]\cdot\mathbf{e}^*,$$

where $\deg(\mathbf{u}^*)=1$, $\deg(\mathbf{e})=0$, $v\in t^{-2}k[[t]]$.

Elements of $K_{P,P}$ are formal sums

$$\mathbf{u}\cdot(v_{uu}+f_{uu}+[a_{uu}])\cdot\mathbf{u}^*+\mathbf{e}\cdot(v_{eu}+f_{eu}+[a_{eu}])\cdot\mathbf{u}^*+\mathbf{u}\cdot(v_{ue}+f_{ue}+[a_{ue}])\cdot\mathbf{e}^*+$$
$$\mathbf{e}\cdot(v_{ee}+f_{ee}+[a_{ee}])\cdot\mathbf{e}^*,$$

where $v_{uu}\in k[[t]]$, $v_{eu}\in t^{-2}k[[t]]$ and $v_{ue}\in t^2k[[t]]$.

The product on $K^0_{\mathcal{O},\mathcal{O}}$ is simply that of the direct sum of rings. The remaining products and the differentials are determined as follows.

Product rules:

$$\mathbf{u}^*\mathbf{u}=1,\quad \mathbf{e}^*\mathbf{e}=1,\quad \mathbf{u}^*\mathbf{e}=\mathbf{e}^*\mathbf{u}=0.$$

$$f\cdot[a]=0,\quad v\cdot[a]=[va],\quad [a]\cdot f=[af],\quad [a]\cdot v=0.$$

Differentials:

$d(\mathbf{u}) = \mathbf{e}, \quad d(\mathbf{e}) = 0, \quad d(\mathbf{e}^*) = -\mathbf{u}^*, \quad d(\mathbf{u}^*) = 0.$

The cohomology algebra of $K^\cdot(\mathcal{E}nd(\mathcal{O}_C \oplus P))$ can be identified with $\text{Ext}^*(\mathcal{O}_C \oplus \mathcal{O}_{2p})$. We have the following

Cohomology representatives:

$K^\cdot_{\mathcal{O},\mathcal{O}}$: $1_{\mathcal{O}} := (1,1) \in K^0_{\mathcal{O},\mathcal{O}}$.

$K^\cdot_{\mathcal{O},P}$: $A_1 := \mathbf{u}[1] + \mathbf{e} \cdot 1 \in K^0_{\mathcal{O},P}$,

$A_t := \mathbf{u}[t] + \mathbf{e} \cdot t \in K^0_{\mathcal{O},P}$.

$K^\cdot_{P,\mathcal{O}}$: $B_{\frac{1}{t}} := [\frac{1}{t}]\mathbf{e}^* + \frac{1}{t}\mathbf{u}^* \in K^1_{P,\mathcal{O}}$,

$B_{\frac{1}{t^2}} := [\frac{1}{t^2}]\mathbf{e}^* + \frac{1}{t^2}\mathbf{u}^* \in K^1_{P,\mathcal{O}}$.

$K^\cdot_{P,P}$: $Y_{\frac{1}{t}} := \mathbf{e}[\frac{1}{t}]\mathbf{e}^* + \mathbf{e} \cdot \frac{1}{t} \cdot \mathbf{u}^* \in K^1_{P,P}$,

$Y_{\frac{1}{t^2}} := \mathbf{e}[\frac{1}{t^2}]\mathbf{e}^* + \mathbf{e} \cdot \frac{1}{t^2} \cdot \mathbf{u}^* \in K^1_{P,P}$,

$e_{2p,1} := \mathbf{u}[1]\mathbf{e}^* + \mathbf{u} \cdot 1 \cdot \mathbf{u}^* + \mathbf{e} \cdot 1 \cdot \mathbf{e}^* \in K^0_{P,P}$,

$e_{2p,t} := \mathbf{u}[t]\mathbf{e}^* + \mathbf{u} \cdot t \cdot \mathbf{u}^* + \mathbf{e} \cdot t \cdot \mathbf{e}^* \in K^0_{P,P}$.

Note: Since $g = h^1(\mathcal{O}) = 0$, the short exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(p) \longrightarrow \mathcal{O}_p \longrightarrow 0$$

gives $h^0(\mathcal{O}(p)) = 2$ and $h^0(\mathcal{O}(np)) = n+1$ $(n \geq 1)$ by induction. We can then choose $f[n] \in \text{H}^0(U, \mathcal{O}_C)$ $(n \geq 1)$ such that

$$\text{H}^0(\mathcal{O}(np)) = < 1, f[1], ..., f[n] >$$

and that, at p, $f[n](t) = \frac{1}{t^n} + (\text{regular part})$.

To run the homological perturbation we use the following

Homotopy Operator Q on $K^{\cdot}(\mathcal{E}nd(\mathcal{O}_C \oplus P))$:

$K^{\cdot}_{\mathcal{O},\mathcal{O}}$: $\operatorname{im} d = K^1_{\mathcal{O},\mathcal{O}}$.

$$Q([v]) = -v \ (v \in k[[t]]),$$

$$Q([\tfrac{1}{t^n}]) = f[n](t)_{\geq 0} \cdot 1 + f[n] \ (n \geq 1).$$

$K^{\cdot}_{\mathcal{O},P}$: $\operatorname{im} d = K^1_{\mathcal{O},P} \oplus \{\mathbf{u}[v] + \mathbf{e} \cdot v \mid v \in t^2 \cdot k[[t]]\} \oplus \{-\mathbf{u} \cdot d(f) + \mathbf{e} f \mid f \in \mathrm{H}^0(U, \mathcal{O})\}$.

$$Q(\mathbf{e}[b]) = \mathbf{u}[b],$$

$$Q(\mathbf{u}[a] + \mathbf{e} \cdot v + \mathbf{e} \cdot f) = \mathbf{u} \cdot v_{\geq 2} + \mathbf{u} \cdot f.$$

$K^{\cdot}_{P,\mathcal{O}}$: $\operatorname{im} d = K^2_{P,\mathcal{O}} \oplus \{[v]\mathbf{e}^* + v \cdot \mathbf{u}^* \mid v \in k[[t]]\} \oplus \{d(f)\mathbf{e}^* - f\mathbf{u}^* \mid f \in \mathrm{H}^0(U, \mathcal{O})\}$.

$$Q([a]\mathbf{u}^*) = [a]\mathbf{e}^*,$$

$$Q([a]\mathbf{e}^* + v\mathbf{u}^* + f\mathbf{u}^*) = -v_{\geq 0}\mathbf{e}^* - f\mathbf{e}^*.$$

$K^{\cdot}_{P,P}$: $\operatorname{im} d =$

$$K^2_{P,P}$$

$\oplus\{\mathbf{e}[a]\mathbf{e}^* + \mathbf{e}v\mathbf{u}^* + \mathbf{e}f\mathbf{u}^* + \mathbf{u}([v] - d(f) - [a])\mathbf{u}^* \mid v \in k[[t]], f \in \mathrm{H}^0(U, \mathcal{O}), [a] \in k((t))\}$

$\oplus\{\mathbf{e}v\mathbf{e}^* + \mathbf{u}[v]\mathbf{e}^* + \mathbf{u}v\mathbf{u}^* + \mathbf{e}f\mathbf{e}^* - \mathbf{u}d(f)\mathbf{e}^* + \mathbf{u}f\mathbf{u}^* \mid v \in t^2 \cdot k[[t]], f \in \mathrm{H}^0(U, \mathcal{O})\}$.

$$Q(\mathbf{e}[a_{eu}]\mathbf{u}^*) = \mathbf{u}[a_{eu}]\mathbf{u}^*,$$

$$Q(\mathbf{e}[a_{ee}]\mathbf{e}^* + \mathbf{u}[a_{uu}]\mathbf{u}^* + \mathbf{e}v_{eu}\mathbf{u}^* + \mathbf{e}f_{eu}\mathbf{u}^*) = \mathbf{u}[a_{ee} - (v_{eu})_{<0}]\mathbf{e}^* + \mathbf{u}(v_{eu})_{\geq 0}\mathbf{u}^* + \mathbf{u}f_{eu}\mathbf{u}^*,$$

$$Q(\mathbf{u}[a_{ue}]\mathbf{e}^* + \mathbf{u}v_{uu}\mathbf{u}^* + \mathbf{u}f_{uu}\mathbf{u}^* + \mathbf{e}v_{ee}\mathbf{e}^* + \mathbf{e}f_{ee}\mathbf{e}^*) = \mathbf{u}(v_{ee})_{\geq 2}\mathbf{e}^* + \mathbf{u}f_{ee}\mathbf{e}^*.$$

Projection via $\Pi := \mathrm{Id} - dQ - Qd$ onto the cohomology representatives:

$K^{\cdot}_{\mathcal{O},\mathcal{O}}$: $\Pi(v) = \Pi([v]) = 0 \ (v \in k[[t]])$,

$\Pi(f[n]) = 0 \ (n \geq 1),$

$\Pi(1_U) = 1_{\mathcal{O}} \ (1_U \in \mathrm{H}^0(U, \mathcal{O})),$

$\Pi([\frac{1}{t^n}]) = 0 \ (n \geq 1).$

$K^{\cdot}_{\mathcal{O},P}$: $\Pi(\mathbf{u}[a] + \mathbf{e}v + \mathbf{e}f) = \mathbf{e}(v - v_{\geq 2}) + \mathbf{u}[v - v_{\geq 2}] = v(0) \cdot A_1 + v'(0) \cdot A_t.$

$K^{\cdot}_{P,\mathcal{O}}$: $\Pi([a]\mathbf{e}^* + v\mathbf{u}^* + f\mathbf{u}^*) = [v - v_{\geq 0}]\mathbf{e}^* + (v - v_{\geq 0})\mathbf{u}^* = \mathrm{Res}^{(-1)}(v) \cdot B_{\frac{1}{t}} + \mathrm{Res}^{(-2)}(v) \cdot B_{\frac{1}{t^2}},$

where $\mathrm{Res}^{(-1)}(v)$ is the coefficient of $\frac{1}{t}$ in v

and $\mathrm{Res}^{(-2)}(v)$ is the coefficient of $\frac{1}{t^2}$ in v.

$K^{\cdot}_{P,P}$: $\Pi(\mathbf{e}[a_{ee}]\mathbf{e}^* + \mathbf{u}[a_{uu}]\mathbf{u}^* + \mathbf{e}v_{eu}\mathbf{u}^* + \mathbf{e}f_{eu}\mathbf{u}^*)$

$= \mathbf{e}(v_{eu} - (v_{eu})_{\geq 0})\mathbf{u}^* + \mathbf{e}[v_{eu} - (v_{eu})_{\geq 0}]\mathbf{e}^*$

$= \mathrm{Res}^{(-1)}(v_{eu}) \cdot Y_{\frac{1}{t}} + \mathrm{Res}^{(-2)}(v_{eu}) \cdot Y_{\frac{1}{t^2}},$

$\Pi(\mathbf{u}[a_{ue}]\mathbf{e}^* + \mathbf{u}v_{uu}\mathbf{u}^* + \mathbf{u}f_{uu}\mathbf{u}^* + \mathbf{e}v_{ee}\mathbf{e}^* + \mathbf{e}f_{ee}\mathbf{e}^*)$

$= \mathbf{u}(v_{ee} - (v_{ee})_{\geq 2})\mathbf{u}^* + \mathbf{e}(v_{ee} - (v_{ee})_{\geq 2})\mathbf{e}^* + \mathbf{u}[v_{ee} - (v_{ee})_{\geq 2}]\mathbf{e}^*$

$= v_{ee}(0) \cdot e_{2p,1} + v'_{ee}(0) \cdot e_{2p,t}.$

Computing μ^3 of the A_∞-structure on $\mathrm{Ext}^*(\mathcal{O}_C \oplus \mathcal{O}_{2p}) \simeq \mathrm{H}^*(K^{\cdot}(\mathcal{E}nd(\mathcal{O}_C \oplus P)))$**:**

In the following, we first apply the homotopy operator Q on the two-term products of the cohomology representatives.

$\mathcal{O} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}$:

$1_{\mathcal{O}} \cdot 1_{\mathcal{O}} = 1_{\mathcal{O}}, \qquad Q(1_{\mathcal{O}} \cdot 1_{\mathcal{O}}) = 0.$

Other products are 0.

$\mathcal{O} \longrightarrow \mathcal{O} \longrightarrow P$:

$$A_1 \cdot 1_{\mathcal{O}} = A_1, \qquad Q(A_1 \cdot 1_{\mathcal{O}}) = 0.$$

$$A_t \cdot 1_{\mathcal{O}} = A_t, \qquad Q(A_t \cdot 1_{\mathcal{O}}) = 0.$$

$\mathcal{O} \longrightarrow P \longrightarrow \mathcal{O}$:

$$B_{\frac{1}{t}} A_1 = [\tfrac{1}{t}], \qquad Q(B_{\frac{1}{t}} A_1) = 0.$$

$$B_{\frac{1}{t^2}} A_1 = [\tfrac{1}{t^2}], \qquad Q(B_{\frac{1}{t^2}} A_1) = 0.$$

$$B_{\frac{1}{t}} A_t = [1], \qquad Q(B_{\frac{1}{t}} A_t) = -1.$$

$$B_{\frac{1}{t^2}} A_t = [\tfrac{1}{t}], \qquad Q(B_{\frac{1}{t^2}} A_t) = 0.$$

$P \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}$:

$$1_{\mathcal{O}} B_{\frac{1}{t}} = B_{\frac{1}{t}}, \qquad Q(1_{\mathcal{O}} B_{\frac{1}{t}}) = 0.$$

$$1_{\mathcal{O}} B_{\frac{1}{t^2}} = B_{\frac{1}{t^2}}, \qquad Q(1_{\mathcal{O}} B_{\frac{1}{t^2}}) = 0.$$

$\mathcal{O} \longrightarrow P \longrightarrow P$:

$$Y_{\frac{1}{t}} A_1 = \mathbf{e}[\tfrac{1}{t}], \qquad Q(Y_{\frac{1}{t}} A_1) = \mathbf{u}[\tfrac{1}{t}].$$

$$Y_{\frac{1}{t^2}} A_1 = \mathbf{e}[\tfrac{1}{t^2}], \qquad Q(Y_{\frac{1}{t^2}} A_1) = \mathbf{u}[\tfrac{1}{t^2}].$$

$$e_{2p,1} A_1 = A_1, \qquad Q(e_{2p,1} A_1) = 0.$$

$$e_{2p,t} A_1 = A_t, \qquad Q(e_{2p,t} A_1) = 0.$$

$$Y_{\frac{1}{t}} A_t = \mathbf{e}[1], \qquad Q(Y_{\frac{1}{t}} A_t) = \mathbf{u}[1].$$

$$Y_{\frac{1}{t^2}} A_t = \mathbf{e}[\tfrac{1}{t}], \qquad Q(Y_{\frac{1}{t^2}} A_t) = \mathbf{u}[\tfrac{1}{t}].$$

$$e_{2p,1} A_t = A_t, \qquad Q(e_{2p,1} A_t) = 0.$$

$$e_{2p,t} A_t = \mathbf{u}[t^2] + \mathbf{e}t^2, \qquad Q(e_{2p,t} A_t) = \mathbf{u}t^2.$$

$P \longrightarrow \mathcal{O} \longrightarrow P$:

$$A_1 B_{\frac{1}{t}} = Y_{\frac{1}{t}}, \qquad Q(A_1 B_{\frac{1}{t}}) = 0.$$
$$A_1 B_{\frac{1}{t^2}} = Y_{\frac{1}{t^2}}, \qquad Q(A_1 B_{\frac{1}{t^2}}) = 0.$$

$$A_t B_{\frac{1}{t}} = \mathbf{e}[1]\mathbf{e}^* + \mathbf{e} \cdot 1 \cdot \mathbf{u}^*, \qquad Q(A_t B_{\frac{1}{t}}) = \mathbf{u}[1]\mathbf{e}^* + \mathbf{u} \cdot 1 \cdot \mathbf{u}^*.$$
$$A_t B_{\frac{1}{t^2}} = Y_{\frac{1}{t}}, \qquad Q(A_t B_{\frac{1}{t^2}}) = 0.$$

$P \longrightarrow P \longrightarrow \mathcal{O}$:

$$B_{\frac{1}{t}} Y_{\frac{1}{t}} = 0, \qquad Q(B_{\frac{1}{t}} Y_{\frac{1}{t}}) = 0.$$
$$B_{\frac{1}{t}} Y_{\frac{1}{t^2}} = 0, \qquad Q(B_{\frac{1}{t}} Y_{\frac{1}{t^2}}) = 0.$$
$$B_{\frac{1}{t}} e_{2p,1} = B_{\frac{1}{t}}, \qquad Q(B_{\frac{1}{t}} e_{2p,1}) = 0.$$
$$B_{\frac{1}{t}} e_{2p,t} = [1]\mathbf{e}^* + 1 \cdot \mathbf{u}^*, \qquad Q(B_{\frac{1}{t}} e_{2p,t}) = -1 \cdot \mathbf{e}^*.$$

$$B_{\frac{1}{t^2}} Y_{\frac{1}{t}} = 0, \qquad Q(B_{\frac{1}{t^2}} Y_{\frac{1}{t}}) = 0.$$
$$B_{\frac{1}{t^2}} Y_{\frac{1}{t^2}} = 0, \qquad Q(B_{\frac{1}{t^2}} Y_{\frac{1}{t^2}}) = 0.$$
$$B_{\frac{1}{t^2}} e_{2p,1} = B_{\frac{1}{t^2}}, \qquad Q(B_{\frac{1}{t^2}} e_{2p,1}) = 0.$$
$$B_{\frac{1}{t^2}} e_{2p,t} = B_{\frac{1}{t}}, \qquad Q(B_{\frac{1}{t^2}} e_{2p,t}) = 0.$$

$P \longrightarrow P \longrightarrow P$:

$$Y_{\frac{1}{t}} Y_{\frac{1}{t}} = 0, \qquad Q(Y_{\frac{1}{t}} Y_{\frac{1}{t}}) = 0.$$
$$Y_{\frac{1}{t}} Y_{\frac{1}{t^2}} = 0, \qquad Q(Y_{\frac{1}{t}} Y_{\frac{1}{t^2}}) = 0.$$
$$Y_{\frac{1}{t}} e_{2p,1} = Y_{\frac{1}{t}}, \qquad Q(Y_{\frac{1}{t}} e_{2p,1}) = 0.$$
$$Y_{\frac{1}{t}} e_{2p,t} = \mathbf{e}[1]\mathbf{e}^* + \mathbf{e}\cdot 1\cdot \mathbf{u}^*, \qquad Q(Y_{\frac{1}{t}} e_{2p,t}) = \mathbf{u}[1]\mathbf{e}^* + \mathbf{u}\cdot 1\cdot \mathbf{u}^*.$$

$$Y_{\frac{1}{t^2}} Y_{\frac{1}{t}} = 0, \qquad Q(Y_{\frac{1}{t^2}} Y_{\frac{1}{t}}) = 0.$$
$$Y_{\frac{1}{t^2}} Y_{\frac{1}{t^2}} = 0, \qquad Q(Y_{\frac{1}{t^2}} Y_{\frac{1}{t^2}}) = 0.$$
$$Y_{\frac{1}{t^2}} e_{2p,1} = Y_{\frac{1}{t^2}}, \qquad Q(Y_{\frac{1}{t^2}} e_{2p,1}) = 0.$$
$$Y_{\frac{1}{t^2}} e_{2p,t} = Y_{\frac{1}{t}}, \qquad Q(Y_{\frac{1}{t^2}} e_{2p,t}) = 0.$$

$$e_{2p,1} Y_{\frac{1}{t}} = Y_{\frac{1}{t}}, \qquad Q(e_{2p,1} Y_{\frac{1}{t}}) = 0.$$
$$e_{2p,1} Y_{\frac{1}{t^2}} = Y_{\frac{1}{t^2}}, \qquad Q(e_{2p,1} Y_{\frac{1}{t^2}}) = 0.$$
$$e_{2p,1} e_{2p,1} = e_{2p,1}, \qquad Q(e_{2p,1} e_{2p,1}) = 0.$$
$$e_{2p,1} e_{2p,t} = e_{2p,t}, \qquad Q(e_{2p,1} e_{2p,t}) = 0.$$

$$e_{2p,t} Y_{\frac{1}{t}} = \mathbf{e}[1]\mathbf{e}^* + \mathbf{e}\cdot 1\cdot \mathbf{u}^*, \qquad Q(e_{2p,t} Y_{\frac{1}{t}}) = \mathbf{u}[1]\mathbf{e}^* + \mathbf{u}\cdot 1\cdot \mathbf{u}^*.$$
$$e_{2p,t} Y_{\frac{1}{t^2}} = Y_{\frac{1}{t}}, \qquad Q(e_{2p,t} Y_{\frac{1}{t^2}}) = 0.$$
$$e_{2p,t} e_{2p,1} = e_{2p,t}, \qquad Q(e_{2p,t} e_{2p,1}) = 0.$$
$$e_{2p,t} e_{2p,t} = \mathbf{u}[t^2]\mathbf{e}^* + \mathbf{u}t^2\mathbf{u}^* + \mathbf{e}t^2\mathbf{e}^*, \qquad Q(e_{2p,t} e_{2p,t}) = \mathbf{u}t^2\mathbf{e}^*.$$

Now, we compute μ^3 on the cohomology using the tree formula

$$\mu^3(x, y, z) = \Pi(\pm Q(xy)z \pm xQ(yz)),$$

where x, y and z are cohomology representatives.

$\mathcal{O} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}$: $\mu^3 = 0$.

$\mathcal{O} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O} \longrightarrow P$: $\mu^3 = 0$.

$\mathcal{O} \longrightarrow \mathcal{O} \longrightarrow P \longrightarrow \mathcal{O}$: $\mu^3 = 0$.

$\mathcal{O} \longrightarrow \mathcal{O} \longrightarrow P \longrightarrow P$: $\mu^3 = 0$.

$\mathcal{O} \longrightarrow P \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}$: $\mu^3 = 0$.

$\mathcal{O} \longrightarrow P \longrightarrow \mathcal{O} \longrightarrow P$: The nontrivial μ^3 are

$$\mu^3(A_1, B_{\frac{1}{t}}, A_t) = \pm(-A_1),$$
$$\mu^3(A_t, B_{\frac{1}{t}}, A_t) = \pm(-A_t).$$

$\mathcal{O} \longrightarrow P \longrightarrow P \longrightarrow \mathcal{O}$: $\mu^3 = 0$.

$\mathcal{O} \longrightarrow P \longrightarrow P \longrightarrow P$: The nontrivial μ^3 are

$$\mu^3(Y_{\frac{1}{t}}, e_{2p,t}, A_t) = \pm A_t,$$
$$\mu^3(Y_{\frac{1}{t^2}}, e_{2p,t}, A_t) = \pm A_1.$$

$P \longrightarrow \mathcal{O} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}$: $\mu^3 = 0$.

$P \longrightarrow \mathcal{O} \longrightarrow \mathcal{O} \longrightarrow P$: $\mu^3 = 0$.

$P \longrightarrow \mathcal{O} \longrightarrow P \longrightarrow \mathcal{O}$: The nontrivial μ^3 are

$$\mu^3(B_{\frac{1}{t}}, A_t, B_{\frac{1}{t}}) = \pm(-B_{\frac{1}{t}}) \pm B_{\frac{1}{t}},$$
$$\mu^3(B_{\frac{1}{t}}, A_t, B_{\frac{1}{t^2}}) = \pm(-B_{\frac{1}{t^2}}),$$
$$\mu^3(B_{\frac{1}{t^2}}, A_t, B_{\frac{1}{t}}) = \pm B_{\frac{1}{t^2}}.$$

$P \longrightarrow \mathcal{O} \longrightarrow P \longrightarrow P$: The nontrivial μ^3 are

$$\mu^3(Y_{\frac{1}{t}}, A_t, B_{\frac{1}{t}}) = \pm Y_{\frac{1}{t}},$$
$$\mu^3(Y_{\frac{1}{t^2}}, A_t, B_{\frac{1}{t}}) = \pm Y_{\frac{1}{t^2}}.$$

$P \longrightarrow P \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}$: $\mu^3 = 0$.

$P \longrightarrow P \longrightarrow \mathcal{O} \longrightarrow P$: The nontrivial μ^3 is

$$\mu^3(A_t, B_{\frac{1}{t}}, e_{2p,t}) = \pm(-e_{2p,t}).$$

$P \longrightarrow P \longrightarrow P \longrightarrow \mathcal{O}$: The nontrivial μ^3 are

$$\mu^3(B_{\frac{1}{t}}, e_{2p,t}, Y_{\frac{1}{t}}) = \pm(-B_{\frac{1}{t}}) \pm B_{\frac{1}{t}},$$
$$\mu^3(B_{\frac{1}{t}}, e_{2p,t}, Y_{\frac{1}{t^2}}) = \pm(-B_{\frac{1}{t^2}}),$$
$$\mu^3(B_{\frac{1}{t}}, Y_{\frac{1}{t}}, e_{2p,t}) = \pm B_{\frac{1}{t}},$$
$$\mu^3(B_{\frac{1}{t^2}}, Y_{\frac{1}{t}}, e_{2p,t}) = \pm B_{\frac{1}{t^2}},$$
$$\mu^3(B_{\frac{1}{t^2}}, e_{2p,t}, Y_{\frac{1}{t}}) = \pm B_{\frac{1}{t^2}}.$$

$P \longrightarrow P \longrightarrow P \longrightarrow P$: The nontrivial μ^3 are

$$\mu^3(Y_{\frac{1}{t}}, e_{2p,t}, Y_{\frac{1}{t}}) = \pm Y_{\frac{1}{t}},$$
$$\mu^3(Y_{\frac{1}{t}}, e_{2p,t}, e_{2p,t}) = \pm e_{2p,t},$$
$$\mu^3(Y_{\frac{1}{t}}, Y_{\frac{1}{t}}, e_{2p,t}) = \pm Y_{\frac{1}{t}},$$
$$\mu^3(Y_{\frac{1}{t^2}}, Y_{\frac{1}{t}}, e_{2p,t}) = \pm Y_{\frac{1}{t^2}},$$
$$\mu^3(Y_{\frac{1}{t^2}}, e_{2p,t}, Y_{\frac{1}{t}}) = \pm Y_{\frac{1}{t^2}},$$
$$\mu^3(Y_{\frac{1}{t^2}}, e_{2p,t}, e_{2p,t}) = \pm e_{2p,1}.$$

CHAPTER IV

FILTERED ALGEBRAS ASSOCIATED WITH R-PAIRS

4.1. Moduli space of filtered algebras

Recall that we denote $R = k[t]/(t^2)$. Let us define the non-negatively graded commutative k-algebra B by setting

$$B_0 = k \quad \text{and } B_n = R \quad \text{for } n \geq 1.$$

The algebra structure for $\oplus_{n\geq 0} B_n$ is given by the rule that all multiplications

$$B_i \otimes_k B_j \to B_{i+j} \quad \text{with } i > 0, j > 0$$

are given by the multiplication in R (the multiplication with $B_0 = k$ is clear). If we take $u = 1, z = t$ as a k-basis of B_1 then B is generated by u and t as an associative k-algebra and has defining relations

$$\begin{cases} uz = zu, \\ z^2 = 0. \end{cases} \tag{4.1.1}$$

In other words, we have an isomorphism $B \simeq k[u,z]/(z^2)$.

We consider the stack $\mathcal{M}_{fa,0}$ of filtered algebras $A = \cup_{n\geq 0} F_n A$ (with an increasing exhaustive filtration such that $F_{-1}A = 0$), together with an isomorphism of graded k-algebras

$$\mathrm{gr}_F A \simeq B. \tag{4.1.2}$$

To define this stack over k, we consider the corresponding functor on commutative k-algebras S, where we consider filtered S-algebras with an isomorphism $\mathrm{gr}_F A \simeq S \otimes_k B$.

Given such an S-algebra A, let us choose generators $\alpha, \beta \in F_1 A$ such that, under the above isomorphism,

$$\alpha \mod F_0 A \mapsto u, \quad \beta \mod F_0 A \mapsto z.$$

We then have relations in A of the form

$$\begin{cases} \alpha\beta - \beta\alpha = a\alpha + b\beta + c, \\ \beta^2 = d\alpha + e\beta + f \end{cases}$$

for some $a, b, c, d, e, f \in S$. Note that $\langle \alpha^n, \beta\alpha^n \mid n \geq 0 \rangle$ form an S-basis of A. By comparing coefficients of the basis components of the identity $(\beta^2)\beta = \beta(\beta^2)$, we get

$$\begin{cases} ad = 0 \\ bd = 0 \\ cd = 0. \end{cases}$$

Similarly, from $(\alpha\beta)\beta = \alpha(\beta^2)$, we get

$$\begin{cases} 2a = 0 \\ a^2 + 2bd = ae \\ ab + 2c + be = 0 \\ ac + 2bf = ce. \end{cases}$$

Since 2 is invertible in S, the above equations become

$$\begin{cases} a = 0 \\ bd = 0 \\ cd = 0 \\ be + 2c = 0 \quad (c = -\frac{be}{2}) \\ 2bf - ce = 0. \end{cases}$$

It is easy to see that the changing of β to $\beta + \lambda$, where $\lambda \in S$, will change the coefficient e to $e + 2\lambda$. Hence, we can eliminate the ambiguity of the choice of β by considering the unique β for which the coefficient e is zero. Note that for this choice the relations in A take form

$$\begin{cases} \alpha\beta - \beta\alpha = b\beta, \\ \beta^2 = d\alpha + f \end{cases}$$

From this we see that there is a natural homomorphism of the additive group $\mathbb{G}_a$ to the group of automorphisms of A as a filtered algebra with a fixed isomorphism $\mathrm{gr}_F A \simeq B$. Namely, for every $c \in S$, we have an automorphism ϕ_c given by

$$\begin{cases} \phi_c(\alpha) = \alpha + bc, \\ \phi_c(\beta) = \beta. \end{cases} \tag{4.1.3}$$

The fact that this is an automorphism follows from $bd = 0$. Note that the definition of ϕ_c does not depend on a choice of α.

There is no canonical way to fix the ambiguity in a choice of α, so instead we will just consider this as a part of the data. We consider the moduli stack $\widetilde{\mathcal{M}}_{fa,0}$ of filtered algebras A with a fixed isomorphism $\mathrm{gr}_F A \simeq B$ and a choice of α. Note that we have a natural action of the additive group $\mathbb{G}_a$ on $\widetilde{\mathcal{M}}_{fa,0}$ corresponding to changing the choice of α to $\alpha + \lambda$ (not to be confused with the above family of automorphisms!). This does not change the coefficients b and d but changes f to $f - \lambda d$.

From the above discussion we get the following result.

Proposition 4.1.1. *The moduli stack $\widetilde{\mathcal{M}}_{fa,0}$ is isomorphic to the closed subscheme $Z_0 \subset \mathbb{A}^3_k$ with coordinates b, d, f, given by the ideal (bd, bf). The natural action of $\mathbb{G}_a$ on $\widetilde{\mathcal{M}}_{fa,0}$ corresponds to the action of $\mathbb{G}_a$ on Z_0 given by automorphisms*

$$\psi_\lambda : (b, d, f) \mapsto (b, d, f - \lambda d),$$

where $\lambda \in S$. The stack $\mathcal{M}_{fa,0}$ is equivalent to the quotient stack $Z_0/\mathbb{G}_a$.

Example 4.1.2. Let C be an irreducible projective curve over k of arithmetic genus 0 (not necessarily reduced) with $h^0(C, \mathcal{O}_C) = 1$, and let $D \subset C$ be an effective Cartier divisor with $H^0(D, \mathcal{O}_D) \simeq k[t]/(t^2)$. Then the filtered algebra

$$A = \varinjlim H^0(C, \mathcal{O}(nD))$$

is an example of such a filtered algebra defined in this section.

4.2. From R-pairs of genus 0 to filtered algebras

Theorem 4.2.1. *Let (E, F) be an R-pair of genus 0 with fixed trivializations*

$$\operatorname{Hom}^1(F, F) \simeq R, \quad \operatorname{Hom}(E, F) \simeq R.$$

Let $T = T_F$ be the spherical twist by F. Let $E_i = T^i(E) \in \mathrm{Tw}(\mathcal{C})$ and let $\mathcal{R} = \mathcal{R}_{T,E} := \oplus_{n\geq 0}\mathrm{Hom}(E, E_n)$. Consider the graded associative algebra structure on $\mathcal{R}$ given by $ab = T^i(a) \circ b$ with $b \in \mathrm{Hom}(E, E_i)$ and $a \in \mathrm{Hom}(E, E_j)$. Then

$$\mathrm{Hom}^*(E, E_n) = \mathrm{Hom}^0(E, E_n) \quad \textit{for } n \geq 0,$$

and $\mathcal{R}$ is canonically isomorphic to the Rees algebra of a filtered algebra $(A, F_\bullet A)$ satisfying $\mathrm{gr}_F A \simeq B$ (see equation 4.1.2).

Proof. Let $L = \mathrm{Hom}^1(F, F)$ and write $ML^i := M \otimes_R L^{\otimes_R i}$ for an R-module M. Let $V = \mathrm{Hom}^0(E, F)$ and set $V^\vee = \mathrm{Hom}_R(V, R)$. By the perfect pairing, we have $\mathrm{Hom}^1(F, E) \simeq V^\vee L$.

Step 1. We first give an explicit twisted complex representing $E_i := T^i(E)$. Let $\tilde{E}_i$ be the following complex:

$$\mathrm{Hom}^1(F, E)L^{i-1} \otimes_R F \longrightarrow \cdots \longrightarrow \mathrm{Hom}^1(F, E) \otimes_R F \longrightarrow E,$$

where the last map is the evaluation map and the other maps are induced by the evaluation maps $L \otimes_R F \longrightarrow F$. These maps all have degree 1. Note that $E_1 = T(E) = \mathrm{Cone}(\hom(F, E) \otimes_R F \longrightarrow E)$ which can be identified with the complex $\mathrm{Hom}^1(F, E) \otimes_R F \longrightarrow E[1]$. So, $E_1 = T(E) \simeq \tilde{E}_1$. We now show that there is a homotopy equivalence

$$\tilde{E}_{i+1} \simeq T(\tilde{E}_i) \simeq T(E_i) = E_{i+1},$$

where the last equivalence is by induction. Since $\hom(F, \tilde{E}_i)$ can be written as

$$\begin{array}{ccccccc}
\mathrm{Hom}^1(F,E)L^{i-1}\otimes_R \mathrm{id}_F & & \mathrm{Hom}^1(F,E)L^{i-2}\otimes_R \mathrm{id}_F & & \cdots & & \mathrm{Hom}^1(F,E)\otimes_R \mathrm{id}_F \\
& \searrow^{\simeq} & & \searrow^{\simeq} & & & \downarrow^{\simeq} \\
\mathrm{Hom}^1(F,E)L^{i} & & \mathrm{Hom}^1(F,E)L^{i-1} & & \cdots & & \mathrm{Hom}^1(F,E)
\end{array}$$

where the first row has degree 0 and the second row has degree 1, we see that the complex $\mathrm{hom}(F,\tilde{E}_i)$ has cohomology $\mathrm{Hom}^1(F,E)L^i$ in degree 1. It is easy to see that the natural embedding $\mathrm{Hom}^1(F,E)L^i[-1] \hookrightarrow \mathrm{hom}(F,\tilde{E}_i)$ and the natural projection $\mathrm{hom}(F,\tilde{E}_i) \longrightarrow \mathrm{Hom}^1(F,E)L^i[-1]$ are homotopy inverses to each other. So,

$$\begin{aligned}
T(\tilde{E}_i) &= \mathrm{Cone}\Big(\mathrm{hom}(F,\tilde{E}_i)\otimes_R F \longrightarrow \tilde{E}_i\Big) \\
&\simeq \mathrm{Cone}\Big(\mathrm{Hom}^1(F,E)L^i \otimes_R F[-1] \longrightarrow \tilde{E}_i\Big) \\
&= \Big(\mathrm{Hom}^1(F,E)L^i \otimes_R F \longrightarrow \tilde{E}_i\Big) \\
&= \tilde{E}_{i+1}.
\end{aligned}$$

Step 2. The complex $\mathrm{hom}(E,E_i)$ can be written as

$$\Big(\oplus_{j=0}^{i-1}\mathrm{Hom}^1(F,E)L^j \otimes_R \mathrm{Hom}^0(E,F)\Big) \oplus \mathrm{Hom}^0(E,E)$$

in degree 0 since E is exceptional. So,

$$\begin{aligned}
\mathrm{Hom}^*(E,E_i) &= \mathrm{Hom}^0(E,E_i) \\
&\simeq \Big(\oplus_{j=1}^{i} V^\vee L^j \otimes_R V\Big) \oplus \mathrm{Hom}^0(E,E) \\
&\simeq \Big(\oplus_{j=1}^{i} \mathrm{End}_R(V)L^j\Big) \oplus \mathrm{Hom}^0(E,E)
\end{aligned}$$

for all $i \geq 0$.

Step 3. For $n = 0$, there is a natural projection (given by identity)

$$\pi_0 : \mathrm{Hom}^0(E, E_0) \twoheadrightarrow \mathrm{Hom}^0(E, E) \simeq k,$$

and for $n \geq 1$, there is a natural projection

$$\pi_n : \mathrm{Hom}^0(E, E_n) \twoheadrightarrow \mathrm{End}_R(V)L^n.$$

The induced map

$$\pi = (\pi_n) : \mathcal{R} = \oplus_{n\geq 0}\mathrm{Hom}^0(E, E_n) \twoheadrightarrow \mathrm{Hom}^0(E, E) \oplus (\oplus_{n\geq 1}\mathrm{End}_R(V)L^n)$$

is a homomorphism of graded algebras.

Step 4. Let $v \in \mathrm{Hom}^0(E, E_1) \simeq \mathrm{End}_R(V)L \oplus \mathrm{Hom}^0(E, E)$ be the element represented by $\mathrm{id}_E \in \mathrm{Hom}^0(E, E)$. Then for each $i \geq 0$, $T^i(v)$ is represented by the following map between complexes:

$$\begin{array}{ccccccccccc}
 & & V^\vee L^i \otimes_R F & \longrightarrow & V^\vee L^{i-1} \otimes_R F & \longrightarrow & \cdots & \longrightarrow & V^\vee L \otimes_R F & \longrightarrow & E \\
 & & \big\downarrow \mathrm{id} & & \big\downarrow \mathrm{id} & & \cdots & & \big\downarrow \mathrm{id} & & \big\downarrow \mathrm{id} \\
V^\vee L^{i+1} \otimes_R F & \longrightarrow & V^\vee L^i \otimes_R F & \longrightarrow & V^\vee L^{i-1} \otimes_R F & \longrightarrow & \cdots & \longrightarrow & V^\vee L \otimes_R F & \longrightarrow & E
\end{array}$$

We then have, for each $i \geq 1$, an exact sequence

$$0 \longrightarrow \mathrm{Hom}^0(E, E_{i-1}) \xrightarrow{v\cdot} \mathrm{Hom}^0(E, E_i) \longrightarrow \mathrm{End}_R(V)L^i \longrightarrow 0.$$

It follows that $\mathcal{R}$ is generated by elements of degree 1.

Step 5. Let $a \in \mathrm{End}_R(V)L \hookrightarrow \mathrm{End}_R(V)L \oplus \mathrm{Hom}^0(E, E) \simeq \mathrm{Hom}^0(E, E_1)$. Thus we can view a as a map $E \longrightarrow E_1$ represented by

$$\begin{array}{ccc} E & & \\ \Big\downarrow a & & \\ V^\vee L \otimes_R F & \xrightarrow{\ \delta_1\ } & E \end{array}$$

where, as we recall, the second row represents E_1. We want to calculate the map

$$T(a) : \mathrm{Cone}\,(\hom(F, E) \otimes_R F \longrightarrow E) \longrightarrow \mathrm{Cone}\,(\hom(F, E_1) \otimes_R F \longrightarrow E_1)\,.$$

To do so, it remains to find the map $\hom(F, E) \longrightarrow \hom(F, E_1)$ induced by a. It suffices to find its effect on cohomology. Let $x \in \mathrm{Hom}^1(F, E)[-1]$. Then x is sent to $\pm\mu^2(a \otimes x) \pm \mu^3(\delta_1 \otimes a \otimes x)$. In this sum, denote the first term by $a_*(x)$ and the second term by $m_a(x)$ Now it is easy to see that $T(a)$ is represented by

$$\begin{array}{ccccc} V^\vee L \otimes_R F & \xrightarrow{\quad \delta_1 \quad} & E & & \\ {\scriptstyle a_* \otimes \mathrm{id}_F}\Big\downarrow & \searrow{\scriptstyle m_a \otimes \mathrm{id}_F} & \Big\downarrow{\scriptstyle a} & & \\ V^\vee L^2 \otimes_R F & \xrightarrow[\quad \delta_2 \quad]{} & V^\vee L \otimes_R F & \xrightarrow{\ \delta_1\ } & E. \end{array}$$

Step 6. We are now ready to check that $v \in \mathcal{R}$ is in fact a central element. Since $\mathcal{R}$ is generated by degree 1 elements, it suffices to show that, for any $a \in \mathrm{End}_R(V)L \hookrightarrow \mathcal{R}_1$, $av = va$. Note that $av = T(a) \circ v$ and $va = T(v) \circ a$ both are represented by

$$\begin{array}{ccccc} & & E & & \\ & & \Big\downarrow a & & \\ V^\vee L^2 \otimes_R F & \xrightarrow{\ \delta_2\ } & V^\vee L \otimes_R F & \xrightarrow{\ \delta_1\ } & E. \end{array}$$

So, $v \in \mathcal{R}_1$ is a central element. It follows that $\mathcal{R}$ is the Rees algebra of the filtered algebra $(A := \cup_{i \geq 0} \mathcal{R}_i, F_\bullet A)$, where each $F_i A = \mathcal{R}_i$, such that $\mathrm{gr}_F A \simeq \mathcal{R}/(v)$ is the desired graded algebra. Since $\mathcal{R}/(v)$ is generated by two elements of degree 1, it follows that $\mathcal{R}$ is generated by three elements of degree 1. $\square$

4.3. Two perfect pairings

Fix trivializations $L := \mathrm{Hom}^1(F,F) = R \cdot \delta_F$ and $W := \mathrm{Hom}^1(F,E) = R \cdot \delta$. Choose $\epsilon \in \mathrm{Hom}^1(F,E)$ s.t. $\mathrm{Hom}^1(F,E) = {}_k(\delta, \epsilon)$. Let $V = \mathrm{Hom}^0(E,F)$. Note that the pairing given by μ^2

$$\mathrm{Hom}^0(E,F) \otimes_k \mathrm{Hom}^1(F,E) \longrightarrow \mathrm{Hom}^1(F,F)/{}_k(\delta_F) \simeq k$$

can be identified with the perfect pairing $R \otimes_k R \longrightarrow R/k \simeq k$. So, there are dual elements $e, f \in \mathrm{Hom}^0(E,F)$ such that

$$\mu^2(e \otimes \delta) = 1, \quad \mu^2(e \otimes \epsilon) = 0, \quad \mu^2(f \otimes \delta) = 0, \quad \mu^2(f \otimes \epsilon) = 1.$$

For each $n \geq 1$, we can write $E_{n-1} = [F \longrightarrow \cdots \longrightarrow F \longrightarrow E]$ with $(n-1)$ many F's where the arrows are given by $\delta_F \in \mathrm{Hom}^1(F,F)$ and $\delta \in \mathrm{Hom}^1(F,E)$. Note that

$$\mathrm{Hom}^*(E, E_{n-1}) = \mathrm{Hom}^0(E, E_{n-1}) = {}_k(\mathrm{id}_E, e_1, f_1, \cdots, e_{n-1}, f_{n-1}),$$

where $e_i = e, f_i = f \in \mathrm{Hom}^0(E,F)$ are maps from E to the i-th F in the complex E_{n-1} (counting from the right). Similarly, we easily compute

$$\mathrm{Hom}^*(E_n, E) = \mathrm{Hom}^1(E_n, E) = {}_k(\epsilon_1, \epsilon_2, \delta_2, \cdots, \epsilon_n, \delta_n),$$

where $\epsilon_1 = \epsilon \in \mathrm{Hom}^1(F, E)$ is the map from the 1st F in E_n (counting from the right) to E, and $\epsilon_i = \epsilon, \delta_i = \delta \in \mathrm{Hom}^1(F, E)$ are maps from the i-th F in E_n to E.

In particular, we have

$$\mathrm{Hom}^*(E_1, E) = \mathrm{Hom}^1(E_1, E) = \mathrm{Hom}^1(F, E)/\mathrm{d}(\mathrm{id}_E) = \mathrm{Hom}^1(F, E)/_k(\delta) \simeq k \cdot \epsilon,$$

where $\mathrm{d} = \mu^1 : \hom^0(E_1, E) \longrightarrow \hom^1(E_1, E)$ is the differential.

Proposition 4.3.1. *For each $n \geq 1$, the pairing*

$$\mathrm{Hom}^0(E, E_{n-1}) \otimes_k \mathrm{Hom}^1(E_n, E) \longrightarrow \mathrm{Hom}^1(E_n, E_{n-1}) \simeq \mathrm{Hom}^1(E_1, E) \simeq k \cdot \epsilon \simeq k$$

is perfect.

Proof. First, let us analyze

$$\mathrm{Hom}^1(E_n, E_{n-1}) = \mathrm{coker}(\mathrm{d} : \hom^0(E_n, E_{n-1}) \longrightarrow \hom^1(E_n, E_{n-1})).$$

For this, we set the following notations:

- $V(j) := V$ is the $\hom^0$-space from E in E_n to the j-th F in E_{n-1}
- $R(m, l) := R \cdot \mathrm{id}_F$ is the $\hom^0$-space from the m-th F in E_n to the l-th F in E_{n-1}
- $W(j) := W$ is the $\hom^1$-space from the j-th F in E_n to E in E_{n-1}
- $L(m, l) := L$ is the $\hom^1$-space from the m-th F in E_n to the l-th F in E_{n-1}.

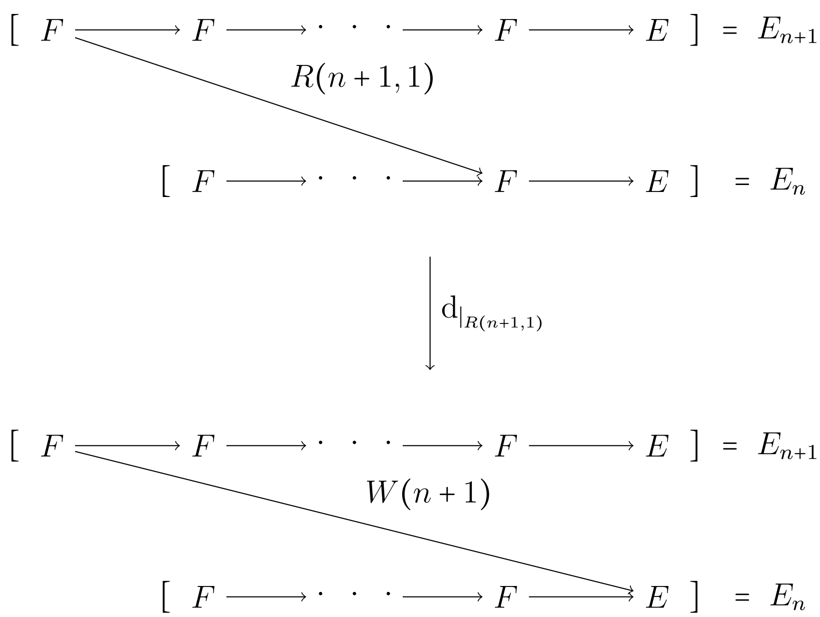
Then

$$\begin{cases} \hom^0(E_n, E_{n-1}) = (\oplus_k V(k)) \oplus (\oplus_{m,l} R(m, l)) \oplus k \cdot \mathrm{id}_E \\ \hom^1(E_n, E_{n-1}) = (\oplus_k W(k)) \oplus (\oplus_{m,l} L(m, l)) \,. \end{cases}$$

For a moment, let us consider the differential

$$\mathrm{d} : \hom^0(E_{n+1}, E_n) \longrightarrow \hom^1(E_{n+1}, E_n).$$

Note that d sends $R(n+1,1)$ isomorphically onto its image $W(n+1)$:



So, $W(n+1) \subseteq \hom^1(E_{n+1}, E_n)$ is in the coboundary.

Since $d_{|R(n+1,2)}$ factors through $L(n+1,1) \oplus W(n+1) \subseteq \hom^1(E_{n+1}, E_n)$ and $W(n+1)$ is already in the coboundary, it is clear that $L(n+1,1)$ is also in the coboundary. Fixing the source of the arrow at the outmost F and moving its target to the left one step at a time, we see that

$$\mathcal{B} := (W(n+1) \oplus L(n+1,1) \oplus \cdots \oplus L(n+1,n-1)) \subseteq \hom^1(E_{n+1}, E_n)$$

is in the coboundary, i.e. $\mathrm{d}^{-1}(B) \longrightarrow B$ is surjective.

We now have the following commutative diagram with exact rows where we denote the last column by $\hom^{\cdot}(E_{n+1}, E_n)/\sim$.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{d}^{-1}(\mathcal{B}) & \longrightarrow & \hom^0(E_{n+1}, E_n) & \longrightarrow & \hom^0(E_{n+1}, E_n)/\mathrm{d}^{-1}(\mathcal{B}) & \longrightarrow & 0 \\
 & & \big\downarrow\!\!\!\!\downarrow & & \big\downarrow \mathrm{d} & & \big\downarrow & & \\
0 & \longrightarrow & \mathcal{B} & \longrightarrow & \hom^1(E_{n+1}, E_n) & \longrightarrow & \hom^1(E_{n+1}, E_n)/\mathcal{B} & \longrightarrow & 0
\end{array}
$$

By the Snake Lemma, $\mathrm{Hom}^1(E_{n+1}, E_n) \simeq \mathrm{H}^1(\hom^{\cdot}(E_{n+1}, E_n)/\sim)$.

There is a natural embedding of chain complexes $\hom^{\cdot}(E_n, E_{n-1}) \hookrightarrow \hom^{\cdot}(E_{n+1}, E_n)/\sim$ and we have the following commutative diagram with exact rows.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \hom^0(E_n, E_{n-1}) & \xrightarrow{\ \mathrm{d}\ } & \hom^1(E_n, E_{n-1}) & \longrightarrow & \mathrm{Hom}^1(E_n, E_{n-1}) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \hom^0(E_{n+1}, E_n)/\mathrm{d}^{-1}(\mathcal{B}) & \xrightarrow{\ \mathrm{d}\ } & \hom^1(E_{n+1}, E_n)/\mathcal{B} & \longrightarrow & \mathrm{H}^1(\hom^{\cdot}(E_{n+1}, E_n)/\sim) & \longrightarrow & 0
\end{array}
$$

Note that

$$\frac{\hom^1(E_{n+1}, E_n)/\mathcal{B}}{\hom^1(E_n, E_{n-1})} = L(n+1, n) \oplus \cdots \oplus L(1, n)$$

and

$$R(n+1, n) \oplus \cdots \oplus R(n+1, 1) \subseteq \mathrm{d}^{-1}(\mathcal{B}).$$

We have the following calculations for the induced map

$$\mathrm{d} : \frac{\hom^0(E_{n+1}, E_n)/\mathrm{d}^{-1}(\mathcal{B})}{\hom^0(E_n, E_{n-1})} \longrightarrow \frac{\hom^1(E_{n+1}, E_n)/\mathcal{B}}{\hom^1(E_n, E_{n-1})}$$

- $$\begin{aligned} R(n,n) &\xrightarrow{\ \mathrm{d}\ } L(n+1,n) \quad [\mathrm{mod}\ \ \hom^1(E_n,E_{n-1})\oplus\mathcal{B}] \\ r_n\cdot \mathrm{id}_F &\longmapsto \pm r_n\cdot\delta_F \qquad (r_n\in R)\end{aligned}$$

- $$\begin{aligned} R(n-1,n) &\xrightarrow{\mathrm{d}} L(n,n)\oplus L(n+1,n) \quad [\mathrm{mod}\ \ \hom^1(E_n,E_{n-1})\oplus\mathcal{B}] \\ r_{n-1}\cdot \mathrm{id}_F &\longmapsto \pm r_{n-1}\cdot\delta_F \pm \mu^3(r_{n-1}\cdot\mathrm{id}_F\otimes\delta_F\otimes\delta_F) \qquad (r_{n-1}\in R)\end{aligned}$$

. .

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- $$\begin{aligned} R(1,n) &\xrightarrow{\ \mathrm{d}\ } L(2,n)\oplus L(3,n)\oplus\cdots\oplus L(n+1,n) \quad [\mathrm{mod}\ \ \hom^1(E_n,E_{n-1})\oplus\mathcal{B}] \\ r_1\cdot \mathrm{id}_F &\longmapsto \pm r_1\cdot\delta_F \pm \mu^3(r_1\cdot\mathrm{id}_F\otimes\delta_F\otimes\delta_F)\pm\cdots\pm\mu^{n+1}(r_1\cdot\mathrm{id}_F\otimes\delta_F\otimes\cdots\otimes\delta_F)\end{aligned}$$

- $$\begin{aligned} V(n) &\xrightarrow{\ \mathrm{d}\ } L(1,n)\oplus L(2,n)\oplus\cdots\oplus L(n+1,n) \quad [\mathrm{mod}\ \ \hom^1(E_n,E_{n-1})\oplus\mathcal{B}] \\ x &\longmapsto \pm\mu^2(x\otimes\delta)\pm\mu^3(r\cdot x\otimes\delta\otimes\delta_F)\pm\cdots\pm\mu^{n+2}(r\cdot x\otimes\delta\otimes\delta_F\otimes\cdots\otimes\delta_F)\end{aligned}$$

Then

$$\frac{\hom^0(E_{n+1},E_n)/\mathrm{d}^{-1}(\mathcal{B})}{\hom^0(E_n,E_{n-1})} = R(n,n)\oplus R(n-1,n)\oplus\cdots\oplus R(1,n)\oplus V(n)$$

and the induced map

$$\frac{\hom^0(E_{n+1},E_n)/\mathrm{d}^{-1}(\mathcal{B})}{\hom^0(E_n,E_{n-1})} \longrightarrow \frac{\hom^1(E_{n+1},E_n)/\mathcal{B}}{\hom^1(E_n,E_{n-1})}$$

is an isomorphism. Therefore,

$$\begin{array}{ccc} \mathrm{Hom}^1(E_n, E_{n-1}) & \xrightarrow{\simeq} & \mathrm{H}^1(\mathrm{hom}^{\cdot}(E_{n+1}, E_n)/\sim) \\ & & \uparrow \simeq \\ & & \mathrm{Hom}^1(E_{n+1}, E_n) \end{array}$$

and the composition gives an isomorphism

$$P : \mathrm{Hom}^1(E_{n+1}, E_n) \simeq \mathrm{Hom}^1(E_n, E_{n-1})$$

for each $n \geq 1$.

In the complex

$$[\mathrm{hom}^{\cdot}(E_{n+1}, E_n)/\sim] = [\overline{\mathrm{d}} : \mathrm{hom}^0(E_{n+1}, E_n)/\mathrm{d}^{-1}(\mathcal{B}) \longrightarrow \mathrm{hom}^1(E_{n+1}, E_n)/\mathcal{B}],$$

we have the following calculations:

- $R(n,1) \xrightarrow{\overline{\mathrm{d}}} W(n)$
- $R(n,2) \xrightarrow{\overline{\mathrm{d}}} L(n,1) \oplus W(n)$

.　　　　.

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- $R(n,n-1) \xrightarrow{\overline{\mathrm{d}}} L(n,n-2) \oplus \cdots \oplus L(n,1) \oplus W(n)$
- $R(n,n) \xrightarrow{\overline{\mathrm{d}}} L(n+1,n) \oplus L(n,n-1) \oplus \cdots \oplus L(n,1) \oplus W(n)$

If $n = 1$, then $\overline{\mathrm{d}}(r \cdot \mathrm{id}_F^{(1,1)}) = \pm r\delta_F^{(2,1)} \pm r\delta^{(1)} \in \mathrm{H}^1(\mathrm{hom}^{\cdot}(E_2, E_1)/\sim)$. So,

$$r\delta_F^{(2,1)} = \pm r\delta^{(1)} \in \mathrm{H}^1(\mathrm{hom}^{\cdot}(E_2, E_1)/\sim).$$

If $n \geq 2$, then $\overline{\mathrm{d}}(r \cdot \mathrm{id}_F^{(n,n)}) = \pm r\delta_F^{(n+1,n)} \pm r\delta_F^{(n,n-1)} \in \mathrm{H}^1(\mathrm{hom}^{\cdot}(E_{n+1}, E_n)/\sim)$. Hence

$$r\delta_F^{(n+1,n)} = \pm r\delta_F^{(n,n-1)} \in \mathrm{H}^1(\mathrm{hom}^{\cdot}(E_{n+1}, E_n)/\sim).$$

Here, $\mathrm{id}_F^{(n,n)}$ is the identity from the n-th F to the n-th F, $\delta_F^{(i+1,i)}$ is the map given by δ_F from the $(i+1)$-st F to the i-th F $(i = n-1, n)$, $\delta^{(1)}$ is the map from the 1-st F in E_2 to E in E_1, and $r \in R$.

The above computations show that the isomorphism P has the effect:

$$P : \mathrm{Hom}^1(E_{n+1}, E_n) \xrightarrow{\simeq} \mathrm{Hom}^1(E_n, E_{n-1})$$

$$r\delta_F^{(n+1,n)} \longmapsto \begin{cases} \pm r\delta^{(1)}, & \text{if } n = 1 \\ \pm r\delta_F^{(n,n-1)}, & \text{if } n \geq 2. \end{cases}$$

Recall that we have

$$\text{basis elements: } \delta_{n+1}, \epsilon_{n+1}, \cdots, \delta_2, \epsilon_2, \epsilon_1 \text{ for } \mathrm{Hom}^1(E_{n+1}, E),$$

and

$$\text{basis elements: } e_n, f_n, \cdots, e_1, f_1, \mathrm{id}_E \text{ for } \mathrm{Hom}^0(E, E_n).$$

Under these bases, the pairing

$$\mathrm{Hom}^0(E, E_n) \otimes_k \mathrm{Hom}^1(E_{n+1}, E) \longrightarrow \mathrm{Hom}^1(E_{n+1}, E_n) \xrightarrow[\simeq]{P^n} \mathrm{Hom}^1(E_1, E) \simeq k$$

is represented by the following upper-triangular $(2n+1)\times(2n+1)$ matrix:

$$\begin{pmatrix} M_2 & * & \cdot & \cdot & \cdot & * & * \\ 0 & M_2 & * & \cdot & \cdot & * & * \\ \cdot & & \cdot & \cdot & \cdot & * & * \\ \cdot & & & \cdot & \cdot & \cdot & * \\ \cdot & & & & 0 & M_2 & * \\ 0 & & \cdot & \cdot & & 0 & \pm 1 \end{pmatrix}$$

where

$$M_2 = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

So, by induction, this pairing is perfect. □

Proposition 4.3.2. *The pairing*

$$\mathrm{Hom}^1(E_n, E) \otimes_k \mathrm{Hom}^0(E_1, E_n) \longrightarrow \mathrm{Hom}^1(E_1, E) \simeq k \cdot \epsilon \simeq k$$

is perfect.

Proof. As before, let us consider basis elements: $e_n, f_n, \cdots, e_1, f_1, \mathrm{id}_E$ for $\mathrm{Hom}^0(E, E_n)$. Under T_F, $\mathrm{Hom}^0(E_1, E_n)$ has basis elements:

$$T_F(e_n), T_F(f_n), \cdots, T_F(e_1), T_F(f_1), T_F(\mathrm{id}_E).$$

Note that $T_F(\mathrm{id}_E)$ can be represented by

$$
\begin{array}{ccccccccc}
 & & & & & & F & \longrightarrow & E \\
 & & & & & & \Big\downarrow \mathrm{id}_F & & \Big\downarrow \mathrm{id}_E \\
F & \longrightarrow & \cdot & \cdot & \cdot & \longrightarrow & F & \longrightarrow & E
\end{array}
$$

and $T_F(e_i)$ or $T_F(f_i)$ can be represented by

$$
\begin{array}{ccccccccccccc}
 & & & & F & \longrightarrow & E & & & & & & \\
 & & & & \Big\downarrow & \searrow & \Big\downarrow e_i \text{ or } f_i & & & & & & \\
F & \longrightarrow & \cdot & \cdot \longrightarrow & F & \longrightarrow & F & \longrightarrow & \cdot & \cdot & \longrightarrow & F & \longrightarrow E
\end{array}
$$

Now, under the bases

$$\delta_n, \epsilon_n, \cdots, \delta_2, \epsilon_2, \epsilon_1 \text{ for } \mathrm{Hom}^1(E_n, E)$$

and

$$T_F(e_n), T_F(f_n), \cdots, T_F(e_1), T_F(f_1), T_F(\mathrm{id}_E) \text{ for } \mathrm{Hom}^0(E_1, E_n),$$

the pairing

$$\mathrm{Hom}^1(E_n, E) \otimes_k \mathrm{Hom}^0(E_1, E_n) \longrightarrow \mathrm{Hom}^1(E_1, E) \simeq k \cdot \epsilon \simeq k$$

is represented by a lower-triangular matrix with diagonal elements given by ±1. □

4.4. The Hom^1-bimodule

In Theorem 4.2.1, we constructed a graded algebra

$$\mathcal{R} := \bigoplus_{n \geq 0} \mathrm{Hom}(E, E_n)$$

with the product $a \cdot b = T^i(a) \circ b$, where $b \in \mathcal{R}_i$. We have also seen that $\mathrm{Hom}^1(E_n, E) = 0$ for $n \le 0$, and we have a canonical isomorphism

$$\tau : \mathrm{Hom}^1(E_1, E) \xrightarrow{\sim} k \tag{4.4.1}$$

Now let us consider the graded space concentrated in negative degrees,

$$\mathcal{M} := \bigoplus_{n \le -1} \mathrm{Hom}^1(E_{-n}, E).$$

We have a natural structure of a graded $\mathcal{R}$-bimodule on $\mathcal{M}$: for $a \in \mathcal{R}_i = \mathrm{Hom}(E, E_i)$ and $m \in \mathcal{M}_j = \mathrm{Hom}^1(E_{-j}, E)$, we set

$$a \cdot m := T^{-i}(a \circ m), \quad m \cdot a = m \circ T^{-i-j}(a).$$

Note that in the first formula we have $a \circ m \in \mathrm{Hom}^1(E_{-j}, E_i)$ and T^{-i} takes it to an element of $\mathrm{Hom}^1(E_{-i-j}, E) = \mathcal{M}_{i+j}$.

It is clear that the full Hom^*-algebra of the collection of objects $(E_i = T^i E)_{i \in \mathbb{Z}}$ is determined by the graded algebra $\mathcal{R}$ together with the graded $\mathcal{R}$-bimodule $\mathcal{M}$.

Proposition 4.4.1. *Assume that the pair (E, F) is compatible with the Serre duality. Then we have a unique isomorphism*

$$\mathcal{M} \simeq \mathcal{R}^*(1),$$

where $\mathcal{R}^* = \bigoplus_i \mathcal{R}^*_{-i}$ is the restricted dual of $\mathcal{R}$, compatible with (4.4.1).

Proof. We showed in Section 4.2.1 that the two pairings

$$\tau_r : \mathcal{M}_{-1-n} \otimes \mathcal{R}_n \to M_{-1} \simeq k : m \otimes a \mapsto \tau(m \cdot a) \text{ and}$$

$$\tau_l : \mathcal{R}_n \otimes M_{-1-n} \to M_{-1} \simeq k : a \otimes m \mapsto \tau(a \cdot m)$$

are nondegenerate. Hence, by [8, Lem. 2.4.3], there exists a unique graded automorphism ϕ of $\mathcal{R}$ such that $\mathcal{M} \simeq \mathcal{R}_\phi^*[1]$, where $\mathcal{R}_\phi$ is the bimodule which is equal to $\mathcal{R}$ as a right $\mathcal{R}$-module, with the left $\mathcal{R}$-module structure given by $a \cdot b = \phi(a)b$. Equivalently, ϕ is determined by the condition

$$\tau_l(\phi(a), m) = \tau_r(m, a).$$

We need to show that $\phi = \text{id}$, the identity automorphism of $\mathcal{R}$. Since $\mathcal{R}$ is generated by $\mathcal{R}_1$, it is enough to show that $\phi(a) = a$ for $a \in \mathcal{R}_1$. Since the pairing $\tau_l : \mathcal{R}_1 \otimes \mathcal{M}_{-2} \to k$ is nondegenerate, $\phi(a)$ is uniquely determined by the functional $\tau_l(\phi(a), ?)$ on $\mathcal{M}_{-2}$. Thus, it is enough to check the identity

$$\tau_l(a, m) = \tau_r(m, a)$$

for $m \in \mathcal{M}_{-2}$ and $a \in \mathcal{R}_1$. Equivalently, we have to check the equality

$$T^{-1}(a \circ m) = m \circ T(a) \tag{4.4.2}$$

in $\text{Hom}(E_1, E)$, for $m \in \mathcal{M}_{-2} = \text{Hom}^1(E_2, E)$ and $a \in \text{Hom}(E, E_1)$.

Now let $\mathcal{S}$ be the Serre functor on the category $\langle E, F\rangle$ (on cohomology level). Then for every object X we have a canonical functional

$$\mathrm{tr}_X : \mathrm{Hom}(X, \mathcal{S}X) \to k$$

which is an isomorphism if $\mathrm{Hom}(X, X) \simeq k$. Furthermore, for a pair of objects X, Y, and morphisms $\alpha : X \to Y$, $\beta : Y \to \mathcal{S}(X)$, we have

$$\mathrm{tr}_X(\beta \circ \alpha) = \mathrm{tr}_Y(\mathcal{S}(\alpha) \circ \beta). \tag{4.4.3}$$

For this and other properties of tr_X used below, see [4].

We want to apply the identity (4.4.3) in our situation. We have $E[1] \simeq \mathcal{S}(E_1)$ and $E_1[1] \simeq \mathcal{S}(E_2)$. Furthermore, since $\mathrm{Hom}(E_1, E_1) \simeq k$, the map tr_{E_1} : $\mathrm{Hom}(E_1, E_0[1]) \to k$ is an isomorphism. Hence, we need to check the equality (4.4.2) after applying tr_{E_1} to both sides. Applying (4.4.3) to $\alpha = m : E_2 \to E[1]$ and $\beta = a : E \to E_1$, we get

$$\mathrm{tr}_{E_2}(a \circ m) = \mathrm{tr}_E(\mathcal{S}(m) \circ a).$$

Note that

$$\mathcal{S}(m) \circ a = T^{-1}(m) \circ a = T^{-1}(m \circ T(a)).$$

It remains to observe that

$$\mathrm{tr}_{E_1}(T^{-1}(x)) = \mathrm{tr}_{E_2}(x)$$

for $x \in \mathrm{Hom}^1(E_2, E_1)$, and

$$\mathrm{tr}_E(T^{-1}(y)) = \mathrm{tr}_{E_1}(y)$$

for $y \in \mathrm{Hom}^1(E_1, E)$. Indeed, both identities are particular case of the general identity

$$\mathrm{tr}_{\mathcal{S}(X)}(\mathcal{S}(x)) = \mathrm{tr}_X(x)$$

for $x \in \mathrm{Hom}(X, \mathcal{S}(X))$. □

As in [8], for a graded $\mathcal{R}$-bimodule $\mathcal{M}$ we consider the bigraded algebra

$$A(\mathcal{R}, \mathcal{M}, 1) := \mathcal{R} \oplus \mathcal{M}[-1], \tag{4.4.4}$$

where $\mathcal{M}[-1]$ is the square zero ideal, and the nonzero products come from the product on $\mathcal{R}$ and the bimodule structure on $\mathcal{M}$. The cohomological grading is $\deg(\mathcal{R}) = 0$, $\deg(\mathcal{M}[-1]) = 0$, and the internal grading comes from the grading of $\mathcal{R}$ and $\mathcal{M}$.

The above proposition shows that we have an isomorphism of algebras

$$\bigoplus_{n \geq 0} \mathrm{Hom}^*(E, E_n) \simeq A(\mathcal{R}, \mathcal{R}^*(1), 1),$$

where $\mathcal{R} = \oplus_{n\geq 0} \mathrm{Hom}^0(E, E_n)$. Thus, from a R-pair (E, F) compatible with Serre duality we get a minimal A_∞-structure on $A(\mathcal{R}, \mathcal{R}^*(1), 1)$.

CHAPTER V

FROM FILTERED ALGEBRAS TO R-PAIRS OF GENUS 0

5.1. AS-Gorenstein property

Let $\mathcal{R}$ be a graded algebra and let M, N be graded $\mathcal{R}$-modules. Recall that (from Notation 2.2.1)

$$\underline{\mathrm{Ext}}^{\bullet}_{\mathcal{R}}(M,N) := \oplus_{j\in\mathbb{Z}} \mathrm{Ext}^{\bullet}_{\mathcal{R}-\mathrm{gr}}(M, N(j)).$$

For a connected graded algebra $\mathcal{R}$ over a field k, there is a notion of left Artin-Schelter Gorenstein (AS-Gorenstein) with parameters (d, m). We will not need the full force of this property, namely we drop the finiteness of injective dimension.

So, we work with this notion of *weak left Artin-Schelter Gorenstein (AS-Gorenstein) with parameters* (d, m):

- $\underline{\mathrm{Ext}}^{\bullet}_{\mathcal{R}}(k, \mathcal{R})$ is 1-dimensional, concentrated in cohomological degree d and internal degree m.

Similarly, we define *weak right AS-Gorenstein with parameters* (d, m).

Recall that $B \simeq k[u, z]/(z^2)$ with $\deg(u) = \deg(z) = 1$.

Proposition 5.1.1. *The commutative algebra* $B[x]$*, with* $\deg(x) = 1$*, is weak left and right AS-Gorenstein with parameters* $(2, -1)$*.*

Proof. Since $B[x]$ is commutative, it suffices to show that it is weak left AS-Gorenstein with parameters $(2, -1)$, i.e. $\underline{\mathrm{Ext}}^{\bullet}_{B[x]}(k, B[x]) = \oplus_{n\in\mathbb{Z}} \mathrm{Ext}^{\bullet}_{B[x]-\mathrm{gr}}(k, B[x](n))$ is 1-dimensional, concentrated in cohomological degree 2 and internal degree -1.

Consider the curve over $\mathbb{P}^1_k$, $p : C := \mathcal{S}pec(\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-1)) \longrightarrow \mathbb{P}^1_k$, given by the embedding into the 1st component $\mathcal{O}_{\mathbb{P}^1_k} \hookrightarrow \mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-1)$ of sheaves of k-algebras, where the latter is equipped with the product $(a,m)\cdot(b,n) = (ab, an+bm)$ for sections (a,m), (b,n) over a common open set. Let $\mathcal{O}_C(n) := (\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-1))(n)$ for $n \in \mathbb{Z}$. Then

$$\mathrm{H}^0(C, \mathcal{O}_C(n)) = \mathrm{H}^0(\mathbb{P}^1_k, p_*\mathcal{O}_C(n)) \simeq \mathrm{H}^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(n)) \oplus \mathrm{H}^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(n-1)).$$

In particular, $\mathrm{H}^0(C, \mathcal{O}_C(n)) = 0$ for $n < 0$, and $\mathrm{H}^0(C, \mathcal{O}_C(1)) \simeq \mathrm{H}^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(1)) \oplus \mathrm{H}^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k})$. Let u, x be a k-basis of the 1st component and z a k-basis of the 2nd component. Then there is an isomorphism of graded k-algebras $\oplus_{n\in\mathbb{Z}}\mathrm{H}^0(C, \mathcal{O}_C(n)) \simeq k[u,z,x]/(z^2) \simeq B[x]$. Note that $\omega_{\mathbb{P}^1_k} \simeq \mathcal{O}_{\mathbb{P}^1_k}(-1-1)$. So,

$$p_*\omega_C \simeq \mathcal{H}om_{\mathbb{P}^1_k}(p_*\mathcal{O}_C, \omega_{\mathbb{P}^1_k}) \simeq \mathcal{H}om_{\mathbb{P}^1_k}(\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-1), \mathcal{O}_{\mathbb{P}^1_k}(-2)) \simeq \mathcal{O}_{\mathbb{P}^1_k}(-2) \oplus \mathcal{O}_{\mathbb{P}^1_k}(-1).$$

Note that here the first isomorphism is the standard fact in the Grothendieck-Serre duality, see [5, Ex. III 7.2]. Therefore, $\omega_C \simeq \mathcal{O}_C(-1)$. By Serre duality on C, we have

$$\mathrm{H}^1(C, \mathcal{O}_C(n-1)) \simeq \mathrm{Hom}_C(\mathcal{O}_C(n-1), \omega_C)^* \simeq \mathrm{H}^0(C, \mathcal{O}_C(-n))^*$$

for all $n \in \mathbb{Z}$. So, $\oplus_{n\in\mathbb{Z}}\mathrm{H}^1(C, \mathcal{O}_C(n)) \simeq \oplus_{n\in\mathbb{Z}}\mathrm{H}^0(C, \mathcal{O}_C(-n-1))^* \simeq B[x]^*(1)$, where $B[x]^* := \oplus_{n\in\mathbb{Z}}\mathrm{Hom}_k(B[x]_n, k)$ is the restricted dual of $B[x]$.

Now, taking the associated sheaf on $\mathbb{P}^1_k$, we get

$$(B[x](m))^{\sim} \simeq \oplus_{n\in\mathbb{Z}}\mathrm{H}^0(\mathbb{P}^1_k, p_*\mathcal{O}_C(m)(n))^{\sim} \simeq p_*\mathcal{O}_C(m)$$

for all $m \in \mathbb{Z}$, see [5, Proposition II 5.15]. Let $\mathfrak{C}_k^\bullet$ be the cochain complex of the bar-resolution of k by free right $B[x]$-modules

$$\cdots \longrightarrow B[x]_+ \otimes_k B[x]_+ \otimes_k B[x] \longrightarrow B[x]_+ \otimes_k B[x] \longrightarrow \underline{B[x]} \longrightarrow 0.$$

Here, the underline in a cochain complex denotes the position in degree 0. Since $\widetilde{k(m)} = 0$ ($m \in \mathbb{Z}$), applying $\widetilde{\ }$ to $\mathfrak{C}_k^\bullet$, we get an exact complex of sheaves of modules on C

$$\cdots \longrightarrow B[x]_+ \otimes_k B[x]_+ \otimes_k \mathcal{O}_C(m) \longrightarrow B[x]_+ \otimes_k \mathcal{O}_C(m) \longrightarrow \underline{\mathcal{O}_C(m)} \longrightarrow 0$$

which we denote by $\widetilde{\mathfrak{C}}(m)^\bullet$.

For each $m \in \mathbb{Z}$, let $\{\mathcal{I}^{-s,t}(m)\}_{s,t\geq 0}$ be a Cartan-Eilenberg resolution of $\widetilde{\mathfrak{C}}(m)^\bullet$ by injectives. There are two cohomology spectral sequences $\{E_{I,r}^{-s,t}(m)\}$ and $\{E_{II,r}^{-s,t}(m)\}$ associated to the 2nd-quadrant double complex $\{\Gamma(C, \mathcal{I}^{-s,t}(m))\}_{s,t\geq 0}$ such that

$E_{I,1}^{-s,t}(m)$ is the vertical cohomology at position t for each column indexed by $-s$;
$E_{II,1}^{-s,t}(m)$ is the horizontal cohomology at position $-s$ for each row indexed by t.

Fix $s \geq 0$. Then

$$\oplus_{m\in\mathbb{Z}} E_{I,1}^{-s,0}(m) \simeq \oplus_{m\in\mathbb{Z}} \mathrm{H}^0(C, (B[x]_+)^{\otimes_k s} \otimes_k \mathcal{O}_C(m)) \simeq (B[x]_+)^{\otimes_k s} \otimes_k B[x];$$
$$\oplus_{m\in\mathbb{Z}} E_{I,1}^{-s,1}(m) \simeq \oplus_{m\in\mathbb{Z}} \mathrm{H}^1(C, (B[x]_+)^{\otimes_k s} \otimes_k \mathcal{O}_C(m)) \simeq (B[x]_+)^{\otimes_k s} \otimes_k B[x]^*(1).$$

So, the $\oplus_{m\in\mathbb{Z}} E_{I,1}(m)$-page has $\mathfrak{C}_k^\bullet$ in the 0th row and $\mathfrak{C}_k^\bullet \otimes_{B[x]} B[x]^*(1)$ in the 1st row, while all the other rows vanish (because C is a curve). Hence the spectral sequence $\{\oplus_{m\in\mathbb{Z}} E_{I,r}^{-s,t}(m)\}$ is biregular and so it strongly converges to the hypercohomology

$$\oplus_{m\in\mathbb{Z}}\mathbb{H}^{\bullet}(\Gamma(C,\mathcal{I}^{\bullet,\bullet}(m))).$$

Since each $\widetilde{\mathfrak{C}}(m)^{\bullet}$ is exact, $E_{II,1}^{-s,t}(m)=0$ for all $s,t\geq 0$. So, $\oplus_{m\in\mathbb{Z}}\mathbb{H}^{\bullet}(\Gamma(C,\mathcal{I}^{\bullet,\bullet}(m)))=0$. It follows that the $\oplus_{m\in\mathbb{Z}}E_{I,2}(m)$-page is

$$\begin{array}{lcccc}
\text{1st row:} & \cdots & \oplus_{m\in\mathbb{Z}}E_{I,2}^{-2,1}(m) & \oplus_{m\in\mathbb{Z}}E_{I,2}^{-1,1}(m) & \underline{\oplus_{m\in\mathbb{Z}}E_{I,2}^{0,1}(m)} \\
 & & & \searrow^{\mathrm{d}_{I,2}^{-2,1}}_{\simeq} & \\
\text{0th row:} & \cdots & 0 & 0 & \underline{k}
\end{array}$$

where $\mathrm{d}_{I,2}^{-2,1}$ is an isomorphism and all the other entries are 0 (due to the convergence to 0 and the fact that the $\oplus_{m\in\mathbb{Z}}E_{I,3}(m)$-page is the E_{∞}-page). In other words, $\mathrm{H}^{\bullet}(\mathfrak{C}_k^{\bullet}\otimes_{B[x]}B[x]^*(1))=\mathrm{H}^{-2}(\mathfrak{C}_k^{\bullet}\otimes_{B[x]}B[x]^*(1))\simeq k$, concentrated in internal degree 0. So,

$$\mathrm{H}^{\bullet}(\mathfrak{C}_k^{\bullet}\otimes_{B[x]}B[x]^*)=\mathrm{H}^{-2}(\mathfrak{C}_k^{\bullet}\otimes_{B[x]}B[x]^*)\simeq k,$$

concentrated in internal degree 1.

Now, let us compute $\underline{\mathrm{Ext}}^{\bullet}_{B[x]}(k,B[x])$. Fix $s\geq 0$. Since each degree $-n$ piece of $\mathfrak{C}_k^{-s}\otimes_{B[x]}B[x]^*$ is computed by

$$\begin{aligned}
&(\mathfrak{C}_k^{-s}\otimes_{B[x]}B[x]^*)_{-n}\\
\simeq\ &\mathrm{coker}[\oplus_{j+r+m=-n}(\mathfrak{C}_k^{-s})_j\otimes_k B[x]_r\otimes_k(B[x]^*)_m\longrightarrow\oplus_{p+q=-n}(\mathfrak{C}_k^{-s})_p\otimes_k(B[x]^*)_q]
\end{aligned}$$

where the map is given by $c \otimes b \otimes b' \mapsto cb \otimes b' - c \otimes bb'$, we obtain

$$
\begin{aligned}
& \mathrm{Hom}_k((\mathfrak{C}_k^{-s} \otimes_{B[x]} B[x]^*)_{-n}, k) \\
\simeq\ & \ker[\Pi_{p+q=-n}\mathrm{Hom}_k((\mathfrak{C}_k^{-s})_p, B[x]_{-q}) \longrightarrow \Pi_{j+r+m=-n}\mathrm{Hom}_k((\mathfrak{C}_k^{-s})_j \otimes_k B[x]_r, B[x]_{-m})] \\
\simeq\ & \mathrm{Hom}_{B[x]^{op}-\mathrm{gr}}(\mathfrak{C}_k^{-s}, B[x](n)) \\
=\ & \mathrm{Hom}_{B[x]-\mathrm{gr}}(\mathfrak{C}_k^{-s}, B[x](n)).
\end{aligned}
$$

So,

$$
\begin{aligned}
\underline{\mathrm{Ext}}^s_{B[x]}(k, B[x]) &\simeq \oplus_{n \in \mathbb{Z}} \mathrm{H}^s(\mathrm{Hom}_{B[x]-\mathrm{gr}}(\mathfrak{C}_k^{\bullet}, B[x](n))) \\
&\simeq \oplus_{n \in \mathbb{Z}} \mathrm{H}^s(\mathrm{Hom}_k((\mathfrak{C}_k^{\bullet} \otimes_{B[x]} B[x]^*)_{-n}, k))
\end{aligned}
$$

is 1-dimensional, precisely when $s = 2$ and $n = -1$; and otherwise vanishing. Hence $B[x]$ is weak left (and right) AS-Gorenstein with parameters $(2, -1)$. □

For the following result, we need to consider the noncommutative projective scheme over k associated with $\mathcal{R}$, i.e. the quotient category $\mathrm{qgr} - \mathcal{R}$ of $\mathrm{gr} - \mathcal{R}$ by the torsion modules. Denote by $\mathcal{O}(j)$ the object in $\mathrm{qgr} - \mathcal{R}$ corresponding to $\mathcal{R}(j)$ and set $\mathrm{H}^i(-) := \mathrm{Ext}^i_{\mathrm{qgr}-\mathcal{R}}(\mathcal{O}, -)$. We also consider $\mathrm{qgr} - \mathcal{R}^{op}$ and set similarly the notation $\mathrm{H}^i(-)$ for this category. (See Section 2.2.)

Proposition 5.1.2. *For any filtered k-algebra $(A, F_\bullet A)$ together with $\mathrm{gr}_F(A) \simeq B$, the Rees algebra $\mathcal{R} := \mathcal{R}(A)$ is weak left and right AS-Gorenstein with parameters $(2, -1)$.*

Proof. Let $v \in \mathcal{R}_1$ be the natural central element of $\mathcal{R}$. Consider $\mathcal{R}_w := \mathcal{R}[w, x]/(v - xw)$ with $\deg(w) = 0$ and $\deg(x) = 1$. Then $\mathcal{R}_w \simeq B[x]$ for $w = 0$ and $\mathcal{R}_w \simeq \mathcal{R}$ as graded $k[w]$-algebras for $w \in k^\times$. Fix $d \geq 0$. Since $\deg(w) = 0$, we have the homogeneous component

$$(\mathcal{R}_w)_d = \oplus_{d_1+d_2=d}\mathcal{R}_{d_1}[w]\cdot x^{d_2}/(v-wx).$$

Note that $\mathcal{R}_{d_1}$ has a k-basis (see the discussion at the beginning of Section 5.2):

$$v^{d_1},\ \ v^{d_1-1}a,\ \ v^{d_1-1}b,\ \ v^{d_1-2}a^2,\ \ v^{d_1-2}ab,\ \ \cdots,\ \ va^{d_1-1},\ \ va^{d_1-2}b,\ \ a^{d_1},\ \ a^{d_1-1}b.$$

So, $(\mathcal{R}_w)_d \simeq \oplus_{d_1+d_2=d}((w\cdot k[w])^{\oplus(2d_1-1)}\cdot x^{d_2+1}\oplus k[w]^{\oplus 2})$ is flat over $k[w]$. Hence, $\mathcal{R}_w$ is a flat family of graded algebras over $\mathbb{A}^1_k \simeq \mathrm{Spec}k[w]$.

Now, we show that $\mathcal{R}$ is weak right AS-Gorenstein with parameters $(2,-1)$, that is, $\underline{\mathrm{Ext}}^\bullet_{\mathcal{R}^{op}}(k,\mathcal{R})$ is 1-dimensional, concentrated in cohomological degree 2 and internal degree -1. Since $\mathcal{R}_w$ is noetherian (see the discussion at the beginning of Section 5.2), we can resolve k by free right graded $\mathcal{R}_w$-modules of finite rank and denote the resolution by $\mathcal{P}^\bullet$. Fix $j\in\mathbb{Z}$. The complex $\mathrm{Hom}_{\mathrm{gr}-\mathcal{R}_w}(\mathcal{P}^\bullet,\mathcal{R}_w(j))$ has at each cohomological degree a direct sum of finite copies of $\mathcal{R}_w(j)$ (hence flat over $\mathbb{A}^1_k$). Let d^s be the differential of this complex starting at position s and let $K^s=\ker(\mathrm{d}^s)$ and $\mathrm{Im}^s=\mathrm{im}(\mathrm{d}^{s-1})$. Let $y\in\mathbb{A}^1_k$. Then

$$\begin{aligned}
&\dim_{k(y)}[\mathrm{Ext}^s_{\mathcal{R}_w^{op}}(k,\mathcal{R}_w(j))\otimes_k k(y)]\\
=\ &\dim_{k(y)}[K^s\otimes_k k(y)]-\dim_{k(y)}[\mathrm{Im}^s\otimes_k k(y)]\\
=\ &\dim_{k(y)}[\mathrm{Hom}_{\mathrm{gr}-\mathcal{R}_w}(\mathcal{P}^{-s},\mathcal{R}_w(j))\otimes_k k(y)]\\
&\qquad -\dim_{k(y)}[\mathrm{Im}^s\otimes_k k(y)]-\dim_{k(y)}[\mathrm{Im}^{s+1}\otimes_k k(y)].
\end{aligned}$$

The first term is constant on $\mathbb{A}^1_k$. Let $r \in \mathbb{Z}$. Then

$$\{y \in \mathbb{A}^1_k \mid \dim_{k(y)}[\mathrm{Im}^s \otimes_k k(y)] < r\} = \{y \in \mathbb{A}^1_k \mid \wedge^r (\mathrm{d}^{s-1} \otimes_k \mathrm{id}_{k(y)}) = 0\} \subseteq \mathbb{A}^1_k$$

is closed. So,

$$\{y \in \mathbb{A}^1_k \mid \dim_{k(y)}[\mathrm{Ext}^s_{\mathcal{R}_w^{op}}(k, \mathcal{R}_w(j)) \otimes_k k(y)] > r\} \subseteq \mathbb{A}^1_k$$

is also closed. Let us now consider the various specializations, i.e.

$$E(s,j)_y := \mathrm{Ext}^s_{\mathcal{R}_w^{op}}(k, \mathcal{R}_w(j)) \otimes_k k(y) \simeq \begin{cases} \mathrm{Ext}^s_{B[x]^{op}}(k, B[x](j)), & \text{if } y = 0 \\ \mathrm{Ext}^s_{\mathcal{R}^{op}}(k, \mathcal{R}(j)), & \text{if } y \in k^\times. \end{cases}$$

For $s = 2$ and $j = -1$, since $\mathrm{Ext}^2_{B[x]^{op}}(k, B[x](-1))$ is 1-dimensional, by the upper-semicontinuity, $\{y \in \mathbb{A}^1_k \mid \dim_{k(y)} E(2,-1)_y \leq 1\}$ is open and contains $0 \in k$ (hence non-empty). So, it also contains some $a \in k^\times$ (assuming $k = \bar{k}$). Then $\mathrm{Ext}^2_{\mathcal{R}^{op}}(k, \mathcal{R}(-1))$ is at most 1-dimensional. For $s \neq 2$ or $j \neq -1$, we similarly deduce that $\mathrm{Ext}^s_{\mathcal{R}^{op}}(k, \mathcal{R}(j)) = 0$. For a non-algebraically closed field k, these dimension results still hold since dimensions don't change after passing to $\bar{k}$.

Below we show that $\underline{\mathrm{Ext}}^\bullet_{\mathcal{R}^{op}}(k, \mathcal{R}) \neq 0$. Suppose $\underline{\mathrm{Ext}}^\bullet_{\mathcal{R}^{op}}(k, \mathcal{R})$ were 0. Since $\mathcal{R}$ is noetherian, we have, by [1, Proposition 7.2(1)(2)], that $\mathrm{H}^0(\mathcal{O}(j)) = \lim\limits_{n\to\infty} \mathrm{Hom}_{\mathrm{gr-}\mathcal{R}}(\mathcal{R}_{\geq n}, \mathcal{R}(j)) = 0$ for all $j \in \mathbb{Z}$ and there is an exact sequence

$$0 \longrightarrow \tau(\mathcal{R}(j))_0 \longrightarrow \mathcal{R}_j \longrightarrow \mathrm{H}^0(\mathcal{O}(j)) \longrightarrow \lim_{n\to\infty} \mathrm{Ext}^1_{\mathrm{gr-}\mathcal{R}}(\mathcal{R}/\mathcal{R}_{\geq n}, \mathcal{R}(j)) \longrightarrow 0,$$

where $\tau(\mathcal{R}(j))$ is the torsion submodule of $\mathcal{R}(j)$. Since v is a non-zero divisor, $\tau(\mathcal{R}(j)) = 0$. By our assumption, $\mathrm{Ext}^1_{\mathrm{gr}-\mathcal{R}}(\mathcal{R}/\mathcal{R}_{\geq n}, \mathcal{R}(j)) = 0$. So, we have an isomorphism $\mathcal{R}_j \simeq \mathrm{H}^0(\mathcal{O}(j))$ for all $j \in \mathbb{Z}$; a contradiction. Hence, $\underline{\mathrm{Ext}}^\bullet_{\mathcal{R}^{op}}(k, \mathcal{R}) \neq 0$.

It is similar to show that $\mathcal{R}$ is also weak left AS-Gorenstein with parameters $(2, -1)$. □

5.2. R-pairs of genus 0 via noncommutative projective geometry

In this section, we work with an arbitrary noetherian commutative ring S. Let A be an increasing filtered algebra with $F_{-1}A = 0$ such that

$$\mathrm{gr}_F(A) = \oplus_{i \geq 0} F_i A / F_{i-1} A \simeq S[u, z]/(z^2) \simeq S \oplus S[t]/(t^2) \oplus S[t]/(t^2) \oplus \cdots,$$

where S is in degree 0 and $\deg(u) = \deg(z) = 1$. So, $F_0(A) \simeq S$ and $F_i(A)/F_{i-1}(A) \simeq S[t]/(t^2)$ concentrated in degree i for $i \geq 1$. Let $\mathcal{R} := \mathcal{R}(A)$ denote the Rees algebra associated with the filtered algebra A. Recall that A has an S-basis $\{\alpha^n, \alpha^n \beta \mid n \geq 0\}$ with $\deg(\alpha) = \deg(\beta) = 1$. So, $\mathcal{R}_0 \simeq \alpha^0 \cdot S$, $\mathcal{R}_1 \simeq (\alpha^0 \cdot S) \oplus (\alpha^1 \cdot S) \oplus (\beta^1 \cdot S)$, etc. Let $v := \alpha^0 \in \mathcal{R}_1$, $a := \alpha^1 \in \mathcal{R}_1$, and $b := \beta^1 \in \mathcal{R}_1$. Then $\mathcal{R}$ is generated by v, a, b as an S-algebra. Note that v is a central element and that $ab - ba \in v \cdot \mathcal{R}_1$. We have a decomposition for $\mathcal{R}$ in terms of basis elements over S:

$$
\begin{array}{ll}
\mathcal{R}_0: & 1 \\
\mathcal{R}_1: & v,\ a,\ b \\
\mathcal{R}_2: & v^2,\ va,\ vb,\ a^2,\ ab \\
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot
\end{array}
$$

It follows that $\mathcal{R} = S \cdot 1 \oplus S \cdot b \oplus (v \cdot \mathcal{R}) \oplus (\oplus_{n \geq 1} S \cdot a^n) \oplus (\oplus_{n \geq 1} S \cdot a^n b)$. Also, since $\mathcal{R}/(v \cdot \mathcal{R}) \simeq S[u,z]/(z^2)$ is commutative noetherian and v is a central element, $\mathcal{R}$ is both left and right noetherian.

Proposition 5.2.1. *Let $\mathcal{R}^*$ be the restricted dual of $\mathcal{R}$. Then:*

(i) $\underline{\mathrm{Ext}}^i_{\mathcal{R}}(S,R) = \underline{\mathrm{Ext}}^i_{\mathcal{R}^{op}}(S,R) = 0$ *for* $i \neq 2$*;* $\underline{\mathrm{Ext}}^2_{\mathcal{R}}(S,R)$ *and* $\underline{\mathrm{Ext}}^2_{\mathcal{R}^{op}}(S,R)$ *are locally free S-modules of rank* 1*, concentrated in internal degree* -1*.*

(ii) $\mathrm{Tor}_i^{\mathcal{R}}(S,\mathcal{R}^*) = \mathrm{Tor}_i^{\mathcal{R}}(\mathcal{R}^*,S) = 0$ *for* $i \neq 2$*;* $\mathrm{Tor}_2^{\mathcal{R}}(S,\mathcal{R}^*)$ *and* $\mathrm{Tor}_2^{\mathcal{R}}(\mathcal{R}^*,S)$ *are locally free S-modules of rank* 1*, concentrated in internal degree* 1*.*

Proof. **Step 1.** $S = k$ is a field.

By Proposition 5.1.2, $\mathcal{R}$ is left and right AS-Gorenstein with parameters $(2,-1)$. Let $\mathfrak{C}^\bullet$ be a cochain complex of free left graded $\mathcal{R}$-modules that resolves k. Then, for $s \geq 0$,

$$\underline{\mathrm{Ext}}^s_{\mathcal{R}}(k,\mathcal{R}) \simeq \oplus_{n \in \mathbb{Z}} \mathrm{H}^s(\mathrm{Hom}_{\mathcal{R}\text{-gr}}(\mathfrak{C}^\bullet, \mathcal{R}(n))) \simeq \oplus_{n \in \mathbb{Z}} \mathrm{H}^s(\mathrm{Hom}_k((\mathcal{R}^* \otimes_{\mathcal{R}} \mathfrak{C}^\bullet)_{-n}, k)).$$

So, $\mathrm{H}^{-s}(\mathcal{R}^* \otimes_{\mathcal{R}} \mathfrak{C}^\bullet)_{-n}$ is 1-dimensional precisely when $s = 2$ and $n = -1$; and otherwise vanishing. Hence $\mathrm{Tor}_\bullet^{\mathcal{R}}(\mathcal{R}^*, k) = \mathrm{Tor}_2^{\mathcal{R}}(\mathcal{R}^*, k) \simeq k$, concentrated in internal degree 1. Similarly, $\mathrm{Tor}_\bullet^{\mathcal{R}}(k, \mathcal{R}^*) = \mathrm{Tor}_2^{\mathcal{R}}(k, \mathcal{R}^*) \simeq k$, concentrated in internal degree 1.

Step 2. S is a local ring with maximal ideal $\mathfrak{m}$.

Set $k := S/\mathfrak{m}$. Since $\mathcal{R}$ is noetherian, there is a resolution $\mathcal{P}^\bullet$ of S by free right graded $\mathcal{R}$-modules of finite rank. Let $\mathcal{R}_k := \mathcal{R} \otimes_S k$. Then $(\mathcal{P}^\bullet \otimes_{\mathcal{R}} \mathcal{R}^*) \otimes_S k \simeq (\mathcal{P}^\bullet \otimes_S k) \otimes_{\mathcal{R}_k} \mathcal{R}_k^*$. Now, resolve k by a complex $\mathcal{Q}^\bullet$ of free S-modules. There is a 3rd-quadrant cohomology spectral sequence

$$E_{I,2}^{-p,-q} \simeq \mathrm{Tor}_p^S(\mathrm{Tor}_q^{\mathcal{R}}(S, \mathcal{R}^*), k) \Rightarrow \mathbb{H}^n((\mathcal{P}^\bullet \otimes_{\mathcal{R}} \mathcal{R}^*) \otimes_S \mathcal{Q}^\bullet),$$

where $\mathrm{d}_{I,r} : E_{I,r}^{-p,-q} \longrightarrow E_{I,r}^{-p+r,-q-r+1}$. There is another spectral sequence

$$E_{II,2}^{-p,-q} \simeq \begin{cases} 0, & \text{if } p > 0 \\ \mathrm{H}^{-q}((\mathcal{P}^\bullet \otimes_{\mathcal{R}} \mathcal{R}^*) \otimes_S k), & \text{if } p = 0 \end{cases} \Rightarrow \mathbb{H}^n((\mathcal{P}^\bullet \otimes_{\mathcal{R}} \mathcal{R}^*) \otimes_S \mathcal{Q}^\bullet).$$

Note that $\mathcal{P}^\bullet \longrightarrow S \longrightarrow 0$ is an exact complex of free S-modules. So, $\mathcal{P}^\bullet \otimes_S k$ is a resolution of k by free right graded $\mathcal{R}_k$-modules. Then, by Step 1, $\mathrm{H}^{-q}((\mathcal{P}^\bullet \otimes_{\mathcal{R}} \mathcal{R}^*) \otimes_S k) \simeq \mathrm{H}^{-q}((\mathcal{P}^\bullet \otimes_S k) \otimes_{\mathcal{R}_k} \mathcal{R}_k^*) \simeq \mathrm{Tor}_q^{\mathcal{R}_k}(k, \mathcal{R}_k^*) \simeq k$ precisely when $q = 2$, concentrated in internal degree 1; and otherwise vanishing. So,

$$\mathbb{H}^n := \mathbb{H}^n((\mathcal{P}^\bullet \otimes_{\mathcal{R}} \mathcal{R}^*) \otimes_S \mathcal{Q}^\bullet) \simeq \begin{cases} 0, & \text{if } n \neq -2 \\ k \;\; \text{(in internal degree 1)}, & \text{if } n = -2. \end{cases}$$

Also, $E_{I,2}$-page looks like (listing only the first few rows)

$$\begin{array}{llll} \text{0th row}: & \cdots & \mathrm{Tor}_1^S(\mathrm{Tor}_0^{\mathcal{R}}(S,\mathcal{R}^*),k) & \mathrm{Tor}_0^{\mathcal{R}}(S,\mathcal{R}^*)\otimes_S k \\ \\ \text{(−1)st row}: & \cdots & \mathrm{Tor}_1^S(\mathrm{Tor}_1^{\mathcal{R}}(S,\mathcal{R}^*),k) & \mathrm{Tor}_1^{\mathcal{R}}(S,\mathcal{R}^*)\otimes_S k \\ \\ \text{(−2)nd row}: & \cdots & \mathrm{Tor}_1^S(\mathrm{Tor}_2^{\mathcal{R}}(S,\mathcal{R}^*),k) & \mathrm{Tor}_2^{\mathcal{R}}(S,\mathcal{R}^*)\otimes_S k \\ \\ \text{(−3)rd row}: & \cdots & \mathrm{Tor}_1^S(\mathrm{Tor}_3^{\mathcal{R}}(S,\mathcal{R}^*),k) & \mathrm{Tor}_3^{\mathcal{R}}(S,\mathcal{R}^*)\otimes_S k \end{array}$$

Since $\mathrm{Tor}_0^{\mathcal{R}}(S,\mathcal{R}^*)\otimes_S k$ survives to the $E_{I,\infty}$-page, it is 0 (because $\mathbb{H}^0 = 0$). By Nakayama Lemma, $\mathrm{Tor}_0^{\mathcal{R}}(S,\mathcal{R}^*) = 0$. So, the 0th row vanishes. Then $\mathrm{Tor}_1^{\mathcal{R}}(S,\mathcal{R}^*)\otimes_S k$ survives to the $E_{I,\infty}$-page and it is 0 (because $\mathbb{H}^{-1} = 0$). By Nakayama Lemma, $\mathrm{Tor}_1^{\mathcal{R}}(S,\mathcal{R}^*) = 0$. So, the (-1)st row vanishes. It then follows that $\mathrm{Tor}_2^{\mathcal{R}}(S,\mathcal{R}^*)\otimes_S k$ survives to the $E_{I,\infty}$-page. Since $\mathbb{H}^{-2}\simeq k$ (sitting in internal degree 1), $\mathrm{Tor}_2^{\mathcal{R}}(S,\mathcal{R}^*)\otimes_S k \simeq k$ in internal degree 1. So, there is a short exact sequence of S-modules:

$$0\longrightarrow \mathfrak{I}\longrightarrow S\longrightarrow \mathrm{Tor}_2^{\mathcal{R}}(S,\mathcal{R}^*)\longrightarrow 0.$$

Also, $\mathrm{Tor}_1^S(\mathrm{Tor}_2^{\mathcal{R}}(S,\mathcal{R}^*),k)$ survives to the $E_{I,\infty}$-page and it is 0 (because $\mathbb{H}^{-3} = 0$). Then $\mathfrak{I}\otimes_S k = 0$. By Nakayama Lemma, $\mathfrak{I} = 0$. So, $\mathrm{Tor}_2^{\mathcal{R}}(S,\mathcal{R}^*)\simeq S$, concentrated in internal degree 1. Hence $\mathrm{Tor}_p^S(\mathrm{Tor}_2^{\mathcal{R}}(S,\mathcal{R}^*),k) = 0$ for $p\geq 1$. For $q\geq 3$, it is similar to show first that $\mathrm{Tor}_q^{\mathcal{R}}(S,\mathcal{R}^*)\otimes_S k$ survives to the $E_{I,\infty}$-page and it is 0 (because $\mathbb{H}^{-q} = 0$ for $q\geq 3$), and then $\mathrm{Tor}_q^{\mathcal{R}}(S,\mathcal{R}^*) = 0$ which renders the $(-q)$th row vanishing on $E_{I,2}$-page. So, $\mathrm{Tor}_\bullet^{\mathcal{R}}(S,\mathcal{R}^*) = \mathrm{Tor}_2^{\mathcal{R}}(S,\mathcal{R}^*)\simeq S$, concentrated in internal degree 1.

Step 3. S is a general noetherian commutative ring.

Localize at every prime ideal of S and apply the same spectral sequence argument as in Step 2. We get that $\mathrm{Tor}^{\mathcal{R}}_{\bullet}(S, \mathcal{R}^*) = \mathrm{Tor}^{\mathcal{R}}_{2}(S, \mathcal{R}^*)$ is locally free, concentrated in internal degree 1. Using $\mathcal{R}^{op}$, we get the similar statement for $\mathrm{Tor}^{\mathcal{R}}_{\bullet}(\mathcal{R}^*, S)$. This proves part (ii). By duality between $\underline{\mathrm{Ext}}^{\bullet}_{\mathcal{R}}(S, \mathcal{R})$ and $\mathrm{Tor}^{\mathcal{R}}_{\bullet}(\mathcal{R}^*, S)$, we get part (i). □

Now, let us consider the noncommutative projective scheme over S associated with $\mathcal{R}$, i.e. the quotient category $\mathrm{qgr} - \mathcal{R}$ of $\mathrm{gr} - \mathcal{R}$ by the torsion modules. Again, denote by $\mathcal{O}(j)$ the object in $\mathrm{qgr} - \mathcal{R}$ corresponding to $\mathcal{R}(j)$ and set $\mathrm{H}^i(-) := \mathrm{Ext}^i_{\mathrm{qgr}-\mathcal{R}}(\mathcal{O}, -)$. (See Section 2.2.)

Proposition 5.2.2. *(i) In the category* $\mathrm{qgr} - \mathcal{R}$*,*

$$\mathrm{H}^i(\mathcal{O}(j)) = 0 \text{ for } i \neq 0, 1,$$
$$\mathrm{H}^1(\mathcal{O}(j)) = 0 \text{ for } j \geq 0,$$

and there is a natural isomorphism of graded algebras $\oplus_{j\geq 0}\mathrm{H}^0(\mathcal{O}(j)) \simeq \mathcal{R}$*.*

(ii) Let $F = S[t]/(t^2)[x]$ *with* $\deg(S[t]/(t^2)) = 0$ *and* $\deg(x) = 1$ *equipped with a right* $\mathcal{R}$*-module structure given by*

$$\pi : \mathcal{R} \twoheadrightarrow \mathcal{R}/(v \cdot \mathcal{R}) \hookrightarrow F, \quad a^n \mapsto x^n, \; a^n b \mapsto x^{n+1} t \quad (n \geq 0).$$

Then the multiplication by x *induces an isomorphism* $F \simeq F(1)$ *and there is a natural short exact sequence in* $\mathrm{qgr} - \mathcal{R}$*:*

$$0 \longrightarrow \mathcal{O}(-1) \xrightarrow{v\cdot} \mathcal{O} \xrightarrow{\pi} F \longrightarrow 0 \tag{5.2.1}$$

Also, we have canonical isomorphisms:

(*a*) $\mathrm{H}^0(F) \simeq S[t]/(t^2),\ \mathrm{H}^{>0}(F) = 0;$

(*b*) $\mathrm{Ext}^1(F, \mathcal{O}) \simeq S[t]/(t^2),\ \mathrm{Ext}^{\neq 1}(F, \mathcal{O}) = 0;$

(*c*) $\mathrm{Hom}(F, F) \simeq S[t]/(t^2) \cdot \mathrm{id}_F,\ \mathrm{Ext}^1(F, F) \simeq S[t]/(t^2),\ \mathrm{Ext}^{\geq 2}(F, F) = 0;$

(*d*) $\mathrm{Hom}(\mathcal{O}, \mathcal{O}) \simeq S \cdot \mathrm{id}_{\mathcal{O}},\ \mathrm{Ext}^{\geq 1}(\mathcal{O}, \mathcal{O}) = 0.$

Moreover, the composition $\mathrm{Hom}(\mathcal{O}, F) \otimes_S \mathrm{Ext}^1(F, \mathcal{O}) \longrightarrow \mathrm{Ext}^1(F, F)$ *is a perfect pairing. Hence, the pair* $(\mathcal{O}, F)$ *is an R-pair of genus* 0*.*

(iii) There are isomorphisms $\mathcal{O}(n+1) \simeq T(\mathcal{O}(n))$ $(n \in \mathbb{Z})$*, where* $T := T_F$ *is the spherical twist associated with* F*. Hence the graded algebra* $\mathcal{R}_{T,\mathcal{O}} := \oplus_{n\geq 0}\mathrm{Hom}(\mathcal{O}, T_F^n(\mathcal{O}))$ *equipped with its natural central element of degree* 1 *is isomorphic to* $(\mathcal{R}, v)$*.*

Proof. (i) Since $\mathcal{R}$ is noetherian, we have, by [1, Proposition 7.2(1)(2)], that

$$\mathrm{H}^i(\mathcal{O}(j)) = \lim_{n\to\infty} \mathrm{Ext}^i_{\mathrm{gr}-\mathcal{R}}(\mathcal{R}_{\geq n}, \mathcal{R}(j)) \simeq \lim_{n\to\infty} \mathrm{Ext}^{i+1}_{\mathrm{gr}-\mathcal{R}}(\mathcal{R}/\mathcal{R}_{\geq n}, \mathcal{R}(j))$$

for $i \geq 1$, and there is an exact sequence

$$0 \longrightarrow \tau(\mathcal{R}(j))_0 \longrightarrow \mathcal{R}_j \longrightarrow \mathrm{H}^0(\mathcal{O}(j)) \longrightarrow \lim_{n\to\infty} \mathrm{Ext}^1_{\mathrm{gr}-\mathcal{R}}(\mathcal{R}/\mathcal{R}_{\geq n}, \mathcal{R}(j)) \longrightarrow 0,$$

where $\tau(\mathcal{R}(j))$ is the torsion submodule of $\mathcal{R}(j)$.

Since $v \in \mathcal{R}$ is a non-zero divisor, $\tau(\mathcal{R}(j)) = 0$. Now, we show that

$$\lim_{n\to\infty} \mathrm{Ext}^1_{\mathrm{gr}-\mathcal{R}}(\mathcal{R}/\mathcal{R}_{\geq n}, \mathcal{R}(j)) = 0.$$

Let $j \geq 0$. By Proposition 5.2.1, $\mathrm{Ext}^1_{\mathrm{gr}-\mathcal{R}}(S(m), R(j)) = 0$ for all $m \in \mathbb{Z}$. Since $\mathcal{R}/\mathcal{R}_{\geq n} = \mathcal{R}_0 \oplus \cdots \oplus \mathcal{R}_{n-1}$, using the short exact sequences

$$0 \longrightarrow \mathcal{R}_{i-1} \longrightarrow \mathcal{R}_i \longrightarrow \mathcal{R}_i/\mathcal{R}_{i-1} \longrightarrow 0$$

and the fact that $\mathcal{R}_0 \simeq S$ and $\mathcal{R}_i/\mathcal{R}_{i-1}$ is $S[t]/(t^2)(-i)$, it is easy to see that

$$\mathrm{Ext}^1_{\mathrm{gr}-\mathcal{R}}(\mathcal{R}/\mathcal{R}_{\geq n}, \mathcal{R}(j)) = 0.$$

So, $\mathcal{R}_j \simeq \mathrm{H}^0(\mathcal{O}(j))$ for $j \geq 0$. These maps assemble to an isomorphism of graded algebras $\mathcal{R} \simeq \oplus_{j\geq 0}\mathrm{H}^0(\mathcal{O}(j))$, see [1, Theorem 4.5(2)].

For $i \neq 0, 1$, $\mathrm{H}^i(\mathcal{O}(j)) \simeq \lim\limits_{n\to\infty} \mathrm{Ext}^{i+1}_{\mathrm{gr}-\mathcal{R}}(\mathcal{R}/\mathcal{R}_{\geq n}, \mathcal{R}(j))$. By Proposition 5.2.1,

$$\mathrm{Ext}^{i+1}_{\mathrm{gr}-\mathcal{R}}(S(m), R(j)) = 0$$

for all $m, j \in \mathbb{Z}$ and for $i \geq 2$. Now, a similar argument as above shows that $\mathrm{H}^i(\mathcal{O}(j)) = 0$ for $i \neq 0, 1$.

Since $\mathrm{H}^1(\mathcal{O}(j)) = \lim\limits_{n\to\infty} \mathrm{Ext}^2_{\mathrm{gr}-\mathcal{R}}(\mathcal{R}/\mathcal{R}_{\geq n}, \mathcal{R}(j))$ and $\mathrm{Ext}^2_{\mathrm{gr}-\mathcal{R}}(S(m), \mathcal{R}(j)) = 0$ when $j - m \neq -1$, it is easy to show that $\mathrm{H}^1(\mathcal{O}(j)) = 0$ for $j \geq 0$.

(ii) Since $F/(x \cdot F) \simeq S[t]/(t^2)$ is torsion, we have $F \simeq F(1)$. The fact that $\mathcal{R}/(v \cdot \mathcal{R}) \hookrightarrow F$ has torsion cokernel gives the short exact sequence (5.2.1)

$$0 \longrightarrow \mathcal{O}(-1) \xrightarrow{v\cdot} \mathcal{O} \xrightarrow{\pi} F \longrightarrow 0$$

in $\mathrm{qgr} - \mathcal{R}$.

(a) Shifting the short exact sequence (5.2.1) and using $F \simeq F(1)$, we get a short exact sequence

$$0 \longrightarrow \mathcal{O} \xrightarrow{v\cdot} \mathcal{O}(1) \xrightarrow{(x^{-1}\cdot)\circ\pi} F \longrightarrow 0 \tag{5.2.2}$$

Applying $\mathrm{H}^*(-)$ and using part (i), we get $\mathrm{H}^0(F) \simeq \mathrm{H}^0(\mathcal{O}(1))/\mathrm{H}^0(\mathcal{O}) \simeq S[t]/(t^2)$ and $\mathrm{H}^{>0}(F) = 0$.

(b) Applying $\mathrm{Ext}^*(-,\mathcal{O})$ to the short exact sequence (5.2.1), we get a long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(F,\mathcal{O}) \longrightarrow \mathrm{Hom}(\mathcal{O},\mathcal{O}) \longrightarrow \mathrm{Hom}(\mathcal{O}(-1),\mathcal{O}) & \\ \longrightarrow \mathrm{Ext}^1(F,\mathcal{O}) \longrightarrow \mathrm{Ext}^1(\mathcal{O},\mathcal{O}) \longrightarrow \mathrm{Ext}^1(\mathcal{O}(-1),\mathcal{O}) & \\ \longrightarrow \mathrm{Ext}^2(F,\mathcal{O}) \longrightarrow \mathrm{Ext}^2(\mathcal{O},\mathcal{O}) \longrightarrow \mathrm{Ext}^2(\mathcal{O}(-1),\mathcal{O}) \longrightarrow \cdots & \end{aligned}$$

By part (i), $\mathrm{Ext}^{\geq 1}(\mathcal{O},\mathcal{O}) = 0$ and $\mathrm{Ext}^{\geq 1}(\mathcal{O}(-1),\mathcal{O}) \simeq \mathrm{Ext}^{\geq 1}(\mathcal{O},\mathcal{O}(1)) = 0$. So,

$$\mathrm{Ext}^1(F,\mathcal{O}) \simeq \mathrm{Hom}(\mathcal{O}(-1),\mathcal{O})/\mathrm{Hom}(\mathcal{O},\mathcal{O}) \simeq \mathrm{H}^0(\mathcal{O}(1))/\mathrm{H}^0(\mathcal{O}) \simeq S[t]/(t^2)$$

and $\mathrm{Ext}^{\neq 1}(F,\mathcal{O}) = 0$.

(c) Applying $\mathrm{Ext}^*(-,F)$ to the short exact sequence (5.2.1), we get a long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(F,F) \longrightarrow \mathrm{Hom}(\mathcal{O},F) \longrightarrow \mathrm{Hom}(\mathcal{O}(-1),F) & \\ \longrightarrow \mathrm{Ext}^1(F,F) \longrightarrow \mathrm{Ext}^1(\mathcal{O},F) \longrightarrow \mathrm{Ext}^1(\mathcal{O}(-1),F) & \\ \longrightarrow \mathrm{Ext}^2(F,F) \longrightarrow \mathrm{Ext}^2(\mathcal{O},F) \longrightarrow \mathrm{Ext}^2(\mathcal{O}(-1),F) \longrightarrow \cdots & \end{aligned}$$

Note that $\mathrm{Hom}(\mathcal{O}, F) \longrightarrow \mathrm{Hom}(\mathcal{O}(-1), F)$ can be identified with the map $\mathrm{H}^0(F) \longrightarrow \mathrm{H}^0(F(1))$ induced by $v\cdot : F \longrightarrow F(1)$ which is the zero map. So,

$$\mathrm{Hom}(F, F) \simeq \mathrm{H}^0(F) \simeq S[t]/(t^2).$$

Also, since $\mathrm{Ext}^1(\mathcal{O}, F) = 0$ by part (ii)(a), we have

$$\mathrm{Ext}^1(F, F) \simeq \mathrm{Hom}(\mathcal{O}, F(1)) \simeq \mathrm{Hom}(\mathcal{O}, F) \simeq S[t]/(t^2).$$

Again, by part (ii)(a), we see that $\mathrm{Ext}^{\geq 2}(F, F) = 0$.

(d) follows from part (i).

It is also clear that the pairing $\mathrm{Hom}(\mathcal{O}, F) \otimes_S \mathrm{Ext}^1(F, \mathcal{O}) \longrightarrow \mathrm{Ext}^1(F, F)$ can be identified with the multiplication $S[t]/(t^2) \otimes_S S[t]/(t^2) \longrightarrow S[t]/(t^2)$, and therefore, it is perfect.

(iii) Shifting the short exact sequence (5.2.2) by $n \in \mathbb{Z}$

$$0 \longrightarrow \mathcal{O}(n) \longrightarrow \mathcal{O}(n+1) \longrightarrow F(n) \longrightarrow 0$$

gives

$$\begin{aligned} \mathcal{O}(n+1)[1] &\simeq \mathrm{Cone}(F(n) \longrightarrow \mathcal{O}(n)[1]) \\ &\simeq \mathrm{Cone}(\mathrm{Ext}^1(F(n), \mathcal{O}(n)) \otimes_R F(n) \longrightarrow \mathcal{O}(n)[1]) \\ &\simeq T_{F(n)}(\mathcal{O}(n))[1], \end{aligned}$$

where $R = S[t]/(t^2)$, since $\mathrm{Ext}^1(F(n), \mathcal{O}(n)) \simeq \mathrm{Ext}^1(F, \mathcal{O}) \simeq R$. So, $\mathcal{O}(n+1) \simeq T_F(\mathcal{O}(n))$ because $F \simeq F(n)$. Then $\mathcal{R}_{T,\mathcal{O}} \simeq \oplus_{n\geq 0}\mathrm{Hom}(\mathcal{O}, \mathcal{O}(n))$ which is also compatible with algebra structures. □

5.3. Normalization of A_∞-structures

Let $\mathcal{O}, F$ and generators v, a, b of $\mathcal{R}_1$ be as in Section 5.2. There is a chain complex in $\mathrm{gr}-\mathcal{R}$:

$$0 \longrightarrow \mathcal{O} \xrightarrow{\psi_1 = \begin{pmatrix} v \\ a \end{pmatrix}} \mathcal{O}(1) \oplus \mathcal{O}(1) \xrightarrow{\psi_2 = (a, -v)} \mathcal{O}(2) \longrightarrow 0 \tag{5.3.1}$$

Proposition 5.3.1. *The chain complex (5.3.1) is exact in* $\mathrm{qgr}-\mathcal{R}$*.*

Proof. Let $C = \mathrm{coker}(\psi_1 : \mathcal{O} \longrightarrow \mathcal{O}(1) \oplus \mathcal{O}(1))$. There induces a map $\psi_2 : C \longrightarrow \mathcal{O}(2)$. We claim that $C \simeq \mathcal{O}(2)$ in $\mathrm{qgr}-\mathcal{R}$ under ψ_2. It suffices to show that $\mathrm{coker}(\psi_2 : C \longrightarrow \mathcal{O}(2))$ is finite dimensional (thus torsion) and that $\psi_2 : C \longrightarrow \mathcal{O}(2)$ is injective.

By the decomposition at the beginning of Section 5.2, $\mathrm{coker}(\psi_2 : C \longrightarrow \mathcal{O}(2)) = S \cdot 1 \oplus S \cdot b$. Now, consider the module $M := \mathrm{Im}(\psi_2 : C \longrightarrow \mathcal{O}(2)) = (v \cdot \mathcal{R}) \oplus (\oplus_{n \geq 1} S \cdot a^n) \oplus (\oplus_{n \geq 1} S \cdot a^n b)$ and the map

$$\begin{aligned} \phi : M &\longrightarrow \mathcal{O}(1) \oplus \mathcal{O}(1) \\ v \cdot y &\mapsto (0, -y) \quad (y \in \mathcal{R}) \\ \lambda \cdot a^n &\mapsto (\lambda \cdot a^{n-1}, 0) \quad (\lambda \in S, \quad n \geq 1) \\ \lambda \cdot a^n b &\mapsto (\lambda \cdot a^{n-1} b, 0) \quad (\lambda \in S, \quad n \geq 1). \end{aligned}$$

Let us compute $\phi \circ \psi_2(p, q)$ for $(p, q) \in \mathcal{O}(1) \oplus \mathcal{O}(1)$. We have $\phi \circ \psi_2(p, q) = \phi(ap - vq) = \phi(ap) + (0, q)$. Again, using the decomposition, we can write

$$p = \lambda_1 \cdot 1 + \lambda_2 \cdot b + v \cdot y + \lambda_3 \cdot a^n + \lambda_4 \cdot a^m b$$

for some $y \in \mathcal{R}$, $n, m \geq 1$ and $\lambda_i \in S$ $(i = 1, 2, 3, 4)$. Then $\phi(ap) = (\lambda_1 \cdot 1 + \lambda_2 \cdot b + \lambda_3 \cdot a^n + \lambda_4 \cdot a^m b, -ay)$. It follows that $\phi(ap) + (vy, ay) = (p, 0)$. Hence, $\phi(ap) = (p, 0) \in C$ and so $\phi \circ \psi_2(p, q) = (p, q) \in C$. Therefore the induced map $\psi_2 : C \longrightarrow \mathcal{O}(2)$ is injective. So, the above chain complex is a short exact sequence in $\mathrm{qgr} - \mathcal{R}$. □

Let $\gamma \in \mathrm{Ext}^1(\mathcal{O}(2), \mathcal{O})$ be the extension corresponding to the short exact sequence (5.3.1). The perfect pairing

$$\mathrm{Ext}^1(\mathcal{O}(2), \mathcal{O}) \otimes_S \mathrm{Hom}(\mathcal{O}(1), \mathcal{O}(2)) \longrightarrow \mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}) \simeq S$$

determines a functional

$$\gamma \circ - : \mathcal{R}_1 \simeq \mathrm{Hom}(\mathcal{O}(1), \mathcal{O}(2)) \longrightarrow \mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}) \tag{5.3.2}$$

Let $\widetilde{\gamma} \in \mathrm{Ext}^1(\mathcal{O}(2), \mathcal{O})$ be the map represented by

$$\begin{array}{ccccl} [F & \xrightarrow{\delta_F} & F & \xrightarrow{\delta} & E] = E_2 = \mathcal{O}(2) \\ & \searrow^{\delta} & & & \\ & & E & & \end{array}$$

Note that $\widetilde{\gamma}$ also determines a functional

$$\widetilde{\gamma} \circ - : \mathcal{R}_1 \simeq \mathrm{Hom}(\mathcal{O}(1), \mathcal{O}(2)) \longrightarrow \mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O})$$

by post-composition.

Proposition 5.3.2. $\widetilde{\gamma} = \gamma$.

Proof. Let us first compute explicitly $\mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O})$. Let $\gamma' \in \mathrm{Ext}^1(F(1), \mathcal{O})$ be the extension corresponding to the short exact sequence (5.2.1)

$$0 \longrightarrow \mathcal{O} \xrightarrow{v\cdot} \mathcal{O}(1) \xrightarrow{\pi} F(1) \longrightarrow 0.$$

Applying $\mathrm{Ext}^*(\mathcal{O}(1), -)$ to the short exact sequence (5.2.1), we get a canonical isomorphism

$$\gamma' : \mathrm{Hom}(\mathcal{O}(1), F(1))/\mathrm{Hom}(\mathcal{O}(1), \mathcal{O}(1)) = \mathrm{Hom}(\mathcal{O}(1), F(1))/(\pi) \xrightarrow{\simeq} \mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}).$$

There is a canonical isomorphism

$$(x^{-1}\cdot) \circ \pi \circ - : \mathrm{Hom}(\mathcal{O}, \mathcal{O}(1))/\mathrm{Hom}(\mathcal{O}, \mathcal{O}) \xrightarrow{\simeq} \mathrm{Hom}(\mathcal{O}, F)$$

by applying $\mathrm{Ext}^*(\mathcal{O}, -)$ to the short exact sequence (5.2.2)

$$0 \longrightarrow \mathcal{O} \xrightarrow{v\cdot} \mathcal{O}(1) \xrightarrow{(x^{-1}\cdot) \circ \pi} F \longrightarrow 0.$$

Then $\mathrm{Hom}(\mathcal{O}, F)$ is generated by $(x^{-1}\cdot) \circ \pi \circ (a\cdot) = (x^{-1}x\cdot) \circ \pi = \pi$ and $(x^{-1}\cdot) \circ \pi \circ (b\cdot) = (x^{-1}x\cdot) \circ (t \cdot \pi) = t \cdot \pi$. So, $\mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}) = (\gamma' \circ (t \cdot \pi))$. Here $t \in S[t]/(t^2)$.

There is a commutative diagram of distinguished triangles

$$\begin{array}{ccccccc}
\mathcal{O} & \xrightarrow{\psi_1 = \begin{pmatrix} v \\ a \end{pmatrix}} & \mathcal{O}(1) \oplus \mathcal{O}(1) & \xrightarrow{\psi_2 = (a, -v)} & \mathcal{O}(2) & \xrightarrow{\gamma} & \mathcal{O}[1] \\
\big\downarrow = & & \big\downarrow \mathrm{pr}_1 & & \big\downarrow (x^{-1}\cdot) \circ \pi & & \big\downarrow = \\
\mathcal{O} & \xrightarrow{v\cdot} & \mathcal{O}(1) & \xrightarrow{\pi} & F(1) & \xrightarrow{\gamma'} & \mathcal{O}[1]
\end{array}$$

where pr_1 is the projection onto the first component. It follows that $\gamma = \gamma' \circ (x^{-1}\cdot) \circ \pi$. For any $r \in \mathrm{Hom}(\mathcal{O}(1), \mathcal{O}(2))$, $\gamma \circ r$ is then represented by

$$\mathcal{O}(1) \xrightarrow{r} \mathcal{O}(2) \xrightarrow{\pi} F(2) \xrightarrow{x^{-1}\cdot} F(1) \xrightarrow{\gamma'} \mathcal{O}[1].$$

So, the functional (5.3.2), $\gamma \circ - : \mathrm{Hom}(\mathcal{O}(1), \mathcal{O}(2)) \longrightarrow \mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O})$, is

$$v \mapsto 0, \quad a \mapsto 0, \quad \text{and} \quad b \mapsto \gamma' \circ (t \cdot \pi) \text{ (a generator)} \tag{5.3.3}$$

Here, we identify $\mathrm{Hom}(\mathcal{O}, \mathcal{O}(1))$ with $\mathrm{Hom}(\mathcal{O}(1), \mathcal{O}(2))$ under T_F and use the generators v, a, b of the former.

In the proof of Proposition 5.2.2, we established a canonical isomorphism

$$\begin{array}{ccc} \mathrm{Ext}^1(F, \mathcal{O}) \simeq \mathrm{Hom}(\mathcal{O}, \mathcal{O}(1))/\mathrm{Hom}(\mathcal{O}, \mathcal{O}) & \simeq & S[t]/(t^2) \\ \gamma' & \longmapsto & 1. \end{array}$$

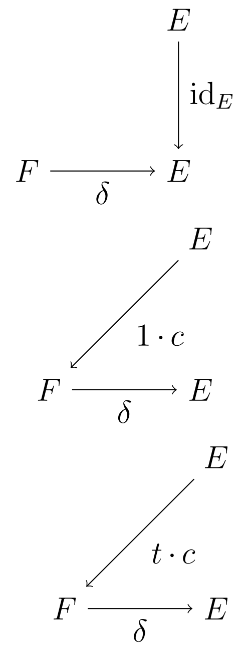
Now, applying $\mathrm{Ext}^*(-, \mathcal{O})$ to the short exact sequence (5.2.1), we get

$$\begin{array}{ccc} \mathrm{Ext}^1(F(1), \mathcal{O}) & \longrightarrow & \mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}) \\ t \cdot \gamma' & \longmapsto & \gamma' \circ (t \cdot \pi). \end{array}$$

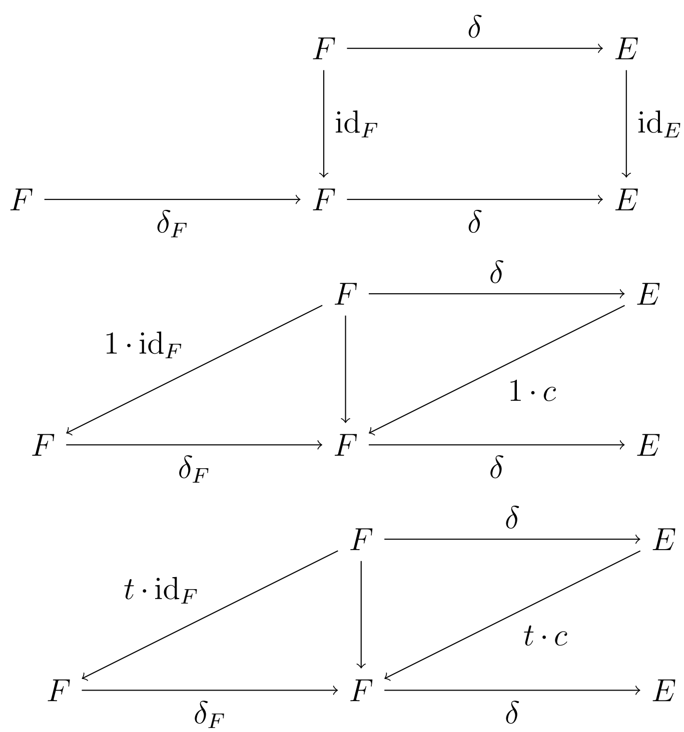
So, the extension $\gamma' \circ (t \cdot \pi)$ corresponds to the following map in the A_∞-category of the R-pair of genus 0, $(E := \mathcal{O}, F)$:

$$\begin{array}{lll} [F & \xrightarrow{\delta} & E] = E_1 = \mathcal{O}(1) \\ & \searrow^{t \cdot \delta} & \\ & & E \end{array}$$

Recall that generators $v, a, b \in \mathcal{R}_1 \simeq \mathrm{Hom}(E, E_1)$ are respectively represented by:



where $c \in \mathrm{Hom}^0(E, F)$ is determined by the perfect pairing $c \circ \delta = \delta_F$. Then $T_F(v), T_F(a), T_F(b)$ are respectively represented by:



In light of (5.3.3), it is now easy to show that γ and $\widetilde{\gamma}$ induce the same functional

$$\mathrm{Hom}(\mathcal{O}(1), \mathcal{O}(2)) \longrightarrow \mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}).$$

Hence $\gamma = \widetilde{\gamma}$. □

Now, it is easy to compute (in the cohomology category of twisted complexes):

- $\mu^3(\gamma \otimes \psi_2 \otimes \psi_1) = \mu^3(\widetilde{\gamma} \otimes \psi_2 \otimes \psi_1) = 0$

- $\mu^2(\gamma \otimes \psi_2) = \mu^2(\widetilde{\gamma} \otimes \psi_2)$ is represented by

$$\begin{array}{ccc} [F & \xrightarrow{\delta} & E] \\ & \searrow^{\delta} & \\ & & E \end{array}$$

which is the image under μ^1 of the map w_1 represented by

$$\begin{array}{ccc} [F & \xrightarrow{\delta} & E] \\ & & \downarrow \mathrm{id}_E \\ & & E \end{array}$$

- $\mu^2(\psi_2 \otimes \psi_1) = \mu^2(-T_F(v) \otimes a) + \mu^2(T_F(a) \otimes v) = 0$

Picking w_1 as above and $w_2 = 0$, we can compute the Massey product (see [9, Remark 1.2]) as

$$< \gamma, \psi_2, \psi_1 >= [\mu^3(\gamma \otimes \psi_2 \otimes \psi_1) + \mu^2(w_1 \otimes \psi_1) - \mu^2(\gamma \otimes w_2)] = [\mathrm{id}_E] \qquad (5.3.4)$$

CHAPTER VI

MAIN RESULT

We consider three moduli functors defined as follows: to each noetherian k-algebra S, assign

(1) the set of equivalence classes of R-pairs (E, F) of genus 0 (or equivalently, of gauge equivalence classes of S-linear strictly unital, minimal A_∞-structures on the category $\mathcal{C}_R(0) \otimes_k S$), compatible with Serre duality;

(2) the set of isomorphism classes of the data $(A, F_\bullet A, \iota, \mu^\bullet)$, where: $(A, F_\bullet A)$ is a filtered algebra (with $F_{-1}A = 0$) equipped with an isomorphism $\iota : \mathrm{gr}^F A \simeq B$, and $\mu^\bullet$ is a minimal A_∞-structure on $A(\mathcal{R}(A), \mathcal{R}(A)^*(1), 1)$ (see (4.4.4)) with given μ^2 and such that $\mu^3(\gamma \otimes \psi_2 \otimes \psi_1) = 1$ (up to gauge equivalence);

(3) the set of isomorphism classes of the data $(A, F_\bullet A, \iota)$ as in (2).

Theorem 6.0.1. *The above three functors are isomorphic.*

Proof. First, let us explain the constructions relating the three functors.

(1) $\longrightarrow$ (2): Consider the twisted complexes $E_n \simeq T_F^n(E)$, for $n \geq 0$. Then Theorem 4.2.1 gives a filtered algebra $(A, F_\bullet A)$ such that $\mathcal{R}(A) \simeq \oplus_{n\geq 0}\mathrm{Hom}^0(E, E_n)$. Moreover, as we have seen in Section 4.4, the algebra $\oplus_{n\geq 0}\mathrm{Hom}^*(E, E_n)$ is isomorphic to $A(\mathcal{R}(A), \mathcal{R}(A)^*(1), 1)$. Now, applying homological perturbation to the A_∞-structure on the subcategory $(E_n)_{n\geq 0}$, we get a minimal A_∞-structure $\mu^\bullet$ on

$A(\mathcal{R}(A), \mathcal{R}(A)^*(1), 1)$. Then $\mu^3(\gamma \otimes \psi_2 \otimes \psi_1) = 1$ (up to gauge equivalence), see equation (5.3.4).

Injectivity: Let $\Pi\mathrm{Tw}(\{E, F\}, \mu^\bullet)$ denote the A_∞ split-closure of the A_∞-category of twisted complexes of $(\{E, F\}, \mu^\bullet)$. Since $\Pi\mathrm{Tw}(\{E, F\}, \mu^\bullet)$ is split-generated by $(\{E, F\}, \mu^\bullet)$ and there is a triangle $E_0 \longrightarrow E_1 \longrightarrow F \longrightarrow E_0[1]$, we see that $\Pi\mathrm{Tw}(\{E, F\}, \mu^\bullet)$ is split-generated by $E_0 = E$ and E_1 as well. So, by Proposition 2.1.14, the inclusion $\{E_i \mid i \geq 0\} \hookrightarrow \Pi\mathrm{Tw}(\{E, F\}, \mu^\bullet)$ induces a quasi-equivalence

$$\Pi\mathrm{Tw}(\{E_i \mid i \geq 0\}) \simeq \Pi\mathrm{Tw}(\{E, F\}, \mu^\bullet).$$

Now, if $(\{E, F\}, \mu^\bullet)$ and $(\{E, F\}, \mu'^\bullet)$ induce gauge-equivalent A_∞-structures on $\{E_i \mid i \geq 0\}$, then the above quasi-equivalence gives an

$$\Phi : \Pi\mathrm{Tw}(\{E, F\}, \mu^\bullet) \simeq \Pi\mathrm{Tw}(\{E, F\}, \mu'^\bullet)$$

such that $\mathrm{H}^*\Phi$ restricts to the identity on $\mathrm{Hom}^*(E_i, E_j)$ $(i, j \geq 0)$.

Note that

- $\mathrm{Hom}^0(E, F)$ is a summand of $\mathrm{Hom}^0(E, E_1)$
- $\mathrm{Hom}^1(F, E)$ is a summand of $\mathrm{Hom}^1(E_2, E)$
  (looking at the Hom^1-space from the left-most F in E_2 to E).

So, $\mathrm{H}^*\Phi$ restricts to identity on $\mathrm{Hom}^0(E, F)$ and $\mathrm{Hom}^1(F, E)$. The composition now shows that $\mathrm{H}^*\Phi$ also induces identity on $\mathrm{Hom}^1(F, F)$.

Let $\alpha : R \longrightarrow R$ be the automorphism induced by $\mathrm{H}^*\Phi$ on $\mathrm{Hom}^0(F, F)$. For convenience, let $\beta := \mathrm{id} : R \longrightarrow R$ be the automorphism induced by $\mathrm{H}^*\Phi$ on $\mathrm{Hom}^0(E, F)$. Then the left action of $\mathrm{Hom}^0(F, F)$ on $\mathrm{Hom}^0(E, F)$ gives

$\beta(r \cdot r') = \alpha(r) \cdot \beta(r')$ for $r, r' \in R$ and so $\alpha(1) = 1$. Hence $\mathrm{H}^*\Phi$ also induces identity on $\mathrm{Hom}^0(F, F)$.

So, $\mathrm{H}^*\Phi$ is identity on $\{E, F\}$.

(2) $\longrightarrow$ (3): The forgetful map is injective due to [8, Lemma 5.2.1].

(3) $\longrightarrow$ (1): Consider the the derived category $D^b(\mathrm{qgr} - \mathcal{R}(A))$ with $(E := \mathcal{O}, F)$ as in Proposition 5.2.2.

Now, we prove that the composition (3)$\longrightarrow$(1)$\longrightarrow$(2)$\longrightarrow$(3) is identity and so each map we have constructed between these data is bijective. Let $(A, F_\bullet A, \iota)$ be given as in (3). Then we construct (E, F) as in (3)$\longrightarrow$(1) and consider the filtered algebra $\cup_{n\geq 0}\mathrm{Hom}^0(E, E_n)$ as in (1)$\longrightarrow$(2). By Proposition 2.1.13, there induces an A_∞-functor

$$\mathrm{Tw}\{E, F\} \longrightarrow D(\mathrm{qgr} - \mathcal{R}(A))$$

which is a quasi-equivalence onto its image. So,

$$\oplus_{n\geq 0}\mathrm{Hom}^0(E, E_n) \simeq \oplus_{n\geq 0}\mathrm{Hom}(\mathcal{O}, T_F^n(\mathcal{O})) \simeq \mathcal{R}(A),$$

where the first isomorphism follows from the quasi-equivalence and the second is Proposition 5.2.2(iii). $\square$

REFERENCES CITED

[1] Michael Artin, James J. Zhang. *Noncommutative projective schemes.* Advances in Mathematics, 109 (1994) 228-287.

[2] Denis Auroux, Ludmil Katzarkov, Dmitri Orlov. *Mirror symmetry for del Pezzo surfaces: vanishing cycles and coherent sheaves.* Invent. Math. 166 (2006), no. 3, 537–582.

[3] Alexei Bondal, Alexander Polishchuk. *Homological properties of associative algebras: the method of helices.* Russian Acad. Sci. Izv. Math. 42 (1994), no. 2, 219–260.

[4] Andrei Caldararu, Simon Willerton. *The Mukai pairing. I. A categorical approach.* New York J. Math. 16 (2010), 61–98.

[5] Robin Hartshorne. *Algebraic Geometry.* Springer, 1997.

[6] Maxim Kontsevich, Yan Soibelman. *Homological mirror symmetry and torus fibrations.* URL https://arxiv.org/abs/math/0011041v2, 2001.

[7] Alexander Polishchuk. *Moduli of curves as moduli of A_∞-structures.* URL https://arxiv.org/abs/1312.4636, 2017.

[8] Alexander Polishchuk. *A_∞-structures associated with pairs of 1-spherical objects and noncommutative orders over curves.* Trans. AMS. 373 (2020), 6029-6093.

[9] Paul Seidel. *Fukaya categories and Picard-Lefschetz theory.* European Mathematical Society, 2008.

[10] Paul Seidel, Richard Thomas. *Braid group actions on derived categories of coherent sheaves.* Duke Math. J. 108 (2001), 37-108.