THE CURTIS-WELLINGTON SPECTRAL SEQUENCE THROUGH COHOMOLOGY

by

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DISSERTATION ABSTRACT

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We study stable homotopy through unstable methods applied to its representing infinite loop space, as pioneered by Curtis and Wellington. Using cohomology instead of homology, we find a width filtration whose subquotients are simple quotients of Dickson algebras. We make initial calculations and determine towers in the resulting width spectral sequence. We also make calculations related to the image of J and conjecture that it is captured exactly by the lowest filtration in the width spectral sequence.

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CHAPTER I

INTRODUCTION

Understanding maps between spheres up to continuous deformation, known as the stable homotopy groups of spheres, is a central question in algebraic topology. These groups also encode questions in geometric topology, including some manifold classification questions. Recent work shows that these structures have an impact even in purely algebraic settings. In this project we study stable homotopy groups through unstable methods, revisiting an approach initiated by Curtis and Wellington. We develop a new filtration on their spectral sequence, and thus stable homotopy groups of spheres themselves, and explore the ramifications of this filtration.

Background

We begin with the set of homotopy classes of based maps between spheres, $f : S^{n+k} \to S^n$, which are denoted $\pi_{n+k}(S^n)$. For $n+k \geq 1$ these sets have a natural group structure, which is abelian for $n+k \geq 2$. These groups behave nicely with respect to the suspension homomorphism $\sigma : \pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$.

Theorem 1.1.1 (Freudenthal Suspension Theorem)**.** *The suspension homomorphism $\sigma : \pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$ is an isomorphism for $k < n-1$ and a surjection for $k = n-1$.*

Corollary 1.1.2. *The group $\pi_{n+k}(S^n)$ depends only on k if $n > k+1$.*

As a result, we can define the k^{th} stable homotopy group (or k-stem), π_k^S, to be $\pi_{n+k}(S^n)$ in the stable range ($n > k+1$). Historically, advances in algebraic topology have been reflected in progress in understanding and calculating these groups. Developments like fiber sequences, cohomology of Eilenberg-Maclane spaces, homological algebra, spectra and stable homotopy, Bott periodicity, chromatic homotopy, and motivic homotopy either originated in or were quickly applied to further understanding of these groups. As such, the question of computing stable homotopy groups of spheres serves as a good indicator of broader progress in the field.

Conspicuously missing from this list is iterated loop space theory. Stable homotopy groups of spheres are isomorphic to the unstable homotopy groups of Q_0S^0, the degree zero component of $\varinjlim \Omega^d S^d$. Building on this connection, unstable approaches have been tried, in particular trying to leverage a deep connection between stable homotopy and symmetric groups. At the level of homology, this was noticed independently and simultaneously by Barratt and Priddy [3] and Quillen. Briefly, one models the classifying space for the nth symmetric group as a colimit over d of space of n disks in $\mathbb{R}^d$. Then given a set of n disks in $\mathbb{R}^d$, associate to it a collapse map from $S^d = \mathbb{R}^d \cup \infty$ to itself which sends the complement of the disks to the base point and each interior of a disk homeomorphically onto the $S^d \backslash \infty$. These maps from the space of disks to $\Omega^d S^d$ can be assembled to a map from the colimit. The Barratt-Priddy-Quillen theorem tells us that a resulting map from the classifying space for the infinite symmetric groups to Q_0S^0 is an isomorphism in homology.

There is then an unstable version of the Adams spectral sequence that can be applied. This was first introduced by Curtis in [10]. He outlined some first calculations, noticing that Adams filtration was lowered and that both the Hopf and Kervaire classes were in filtration zero, leading to the well-known and still open conjecture that these are the only classes to survive on the zero line. But Curtis made some fundamental errors, which Wellington corrected and then went on to establish more global properties, including the classification of Bockstein towers (which in particular preclude any upper vanishing lines).

This unstable version is a much more rarely used tool compared to its stable counterpart since it is often intractable. In particular it has a non-abelian Quillen homology defining its E_2. However, in the case that the cohomology is free, a standard result of Bousfield [5] applies and allows the E_2 term to be equated with Ext in the category of unstable modules over the Steenrod algebra of the desuspended indecomposables.

In their approaches, Wellington and Curtis applied the known structure of the homology of Q_0S^0. This homology is the symmetric algebra on the Kudo-Araki-Dyer-Lashof algebra [8]. While algebraic topologists are comfortable with "homological coalgebra," in this case one runs into difficulties because calculations of the homology coproduct as well as the Steenrod coaction (Nishida relations) require regular applications of Adem relations. Wellington had to filter carefully to make things at all tractable. Thus, any progress on using this connection with iterated loop space theory to progress on homotopy groups of spheres mostly stopped after the early 1980s.

Through non-explicit methods, Nakaoka, [23, 24], had previously shown that the cohomology of the infinite symmetric group and thus Q_0S^0 is polynomial, generated in combinatorially interesting degrees. Finer control of that calculation, and in particular incorporation of the Steenrod algebra action, motivated many authors to study the cohomology of symmetric groups in more detail in the eighties and nineties [1, 2, 12]. Relatively recently, Giusti, Salvatore and Sinha found a new Hopf ring presentation for the cohomology of symmetric groups, as algebras over the Steenrod algebra [13], yielding a "skyline diagram" presentation for the cohomology of $\mathcal{S}_\infty$ in the limit. We take their work as a starting point, and ultimately see that it makes the Curtis-Wellington spectral sequence much more accessible than the previous approach through homology.

Summary of Results

Let $\mathfrak{N}$ denote the algebra indecomposables of $H^*(Q_0S^0)$, which we call the Nakaoka module and which are a module over the Steenrod algebra. Recall that the Dickson algebras are rings of invariants $D_n = \mathbb{F}_2[x_1, \ldots, x_n]^{Gl_n(F_2)}$, calculated by Dickson as polynomial on generators in degrees $2^n - 2^\ell$. The Steenrod algebra action on the ambient polynomial algebras restrict to the Dickson algebras, which provide a rich and still not fully understood collection of (unstable) modules over the Steenrod algebra. Let D_n^o be the quotient of D_n by all perfect squares.

Theorem 1.2.1. *There is a width filtration (see Definition 3.0.2 below) of the Nakaoka module $\mathfrak{N}$ whose subquotients are isomorphic to D_n^o as unstable modules over the Steenrod algebra.*

This width filtration is related to composition length in the Dyer-Lashof algebra. Applying a reduction of Bousfield (see Proposition 2.3.1 below) we have the following.

Corollary 1.2.2. *There is a width spectral sequence with*

$$E_1^{s,t;n} = \mathrm{Ext}_{\mathcal{U}}^{s,t}(\Sigma^{-1}D_n^o, \mathbb{F}_2)$$

and $d_r : \mathrm{Ext}^{s,t}(\Sigma^{-1}D_n^o) \to \mathrm{Ext}^{s+1,t}(\Sigma^{-1}D_{n+r}^o)$ *which converges to the* E_2*-term of the Curtis-Wellington spectral sequence.*

Accordingly, there is a filtration on stable homotopy. While this filtration could be new, preliminary evidence is consistent with this filtration agreeing with the chromatic filtration.

In Section V we present evidence that the first submodule in this filtration detects the image of J. Additionally, the width filtration at the prime two has basic generators in degrees $2^k - 1$ and at odd primes such generators would have degrees $2(p^k - 1)$. In either case – a new filtration on stable homotopy or a new approach to the chromatic filtration – it is compelling to pursue further understanding of the width filtration.

Our calculations rely on a presentation of the cohomology of Q_0S^0 due to Giusti-Salvatore-Sinha. Through that presentation we can already manage by-hand calculations of the Curtis-Wellington spectral sequence more readily than by previous techniques. Moreover, filtration with Dickson algebra subquotients is particularly amenable to computer calculation, for which we thank Hood Chatham - see Appendix A.

It is elementary to eliminate the possibility of many differentials in the width spectral sequence, so these computer calculations provide a good understanding of the E_2-term of the Curtis-Wellington spectral sequence as well. While, disappointingly, the E_2 term of the CWSS is much larger than the classical Adams spectral sequence, we have explored two accessible phenomena.

The first accessible phenomenon is the existence of towers of elements connected by multiplication by h_0. Section IV we make calculations to determine the locations of these towers in the resulting width spectral sequence.

Theorem 1.2.3. *There are infinite h_0-towers in* $\mathrm{Ext}^{s,t}(\Sigma^{-1}D_1^o, \mathbb{F}_2)$ *in degrees* $4a-2$ *for* a *a positive integer.*

Theorem 1.2.4. *Let n be an integer greater than or equal to 2. There are towers in*

$\mathrm{Ext}^{s,t}(\Sigma^{-1}D_n^o, \mathbb{F}_2)$ *corresponding to all integer solutions of*

$$(2^{n-2}-2^{n-3})a_1 + \cdots + (2^{n-2}-1)a_{n-2} + (2^{n-1}-1)b_{n-1} = k$$

where at least one of $a_1, \ldots, a_{n-2}$ are odd in degree $t-s = 4k-1$. And also towers corresponding to all integer solutions of

$$(2^{n-1}-2^{n-2})b_1 + \cdots + (2^{n-1}-1)b_{n-1} + (2^n-1)c_n + (2^n-1) = k$$

in degree $t-s = 4k-2$.

These results agree with the locations of towers identified by Wellington in the E_2 term of the CWSS, thus implying that there are no differentials between the towers internal to the width spectral sequence.

The lowest filtration is also accessible, and related to the image of J. In Section V we make calculations which show preliminarily that the image of J is compatible with the width filtration. In studying the image of J, we first noticed that the unstable Adams Ext chart for $H^*(BO, \mathbb{F}_2)$ is a shifted version of the unstable Adams Ext chart for the first quotient of the width filtration.

Proposition 1.2.5. $\mathrm{Ext}_{\mathcal{U}}^{s,t}(\Sigma^{-1}D_1^o) \cong \mathrm{Ext}_{\mathcal{U}}^{s,t+1}(\Sigma^{-1}\mathrm{Ind}\ H^*(BO))$.

While it seems this would be classical, we haven't found any treatment of this in the literature. This motivated us to study the image of J map on cohomology, and the following.

Theorem 1.2.6. *The map induced by the image of J on cohomology induces a splitting of the Nakaoka module $\mathfrak{N}$.*

Corollary 1.2.7. *The algebraic map induced by the image of J on* Ext *induces a splitting of* $\mathrm{Ext}(\mathfrak{N})$.

We also speculate that the map on the E_2 page of the unstable Adams spectral sequence induced by the J map on cohomology is algebraic – that is, it agrees with the map on Ext induced by the map on cohomology. If this were the case, then the splitting of Theorem 1.2.6 would give rise to the standard splitting of homotopy by the image of J. Such results would invite further study of the compatibility of the width filtration and the chromatic filtration.

Outline

In Section II we review background including homology of Q_0S^0 and Dyer Lashof operations, work of Giusti-Salvatore-Sinha on the cohomology of symmetric groups, the result of Bousfield we referenced above, and work of Hu'ng and Peterson on indecomposables of Dickson algebras.

The unstable Adams spectral sequence is typically intractable, with a non-abelian Quillen homology defining its E_2. But in Section III we apply a standard result of Bousfield [5] in the special case that a cohomology ring is free, as is the case here, equating the E_2 with Ext in the category of unstable modules over the Steenrod algebra of the desuspended indcomposables. Using the skyline diagram presentation of the cohomology of the infinite symmetric group, these indecomposables are manageable.

Indeed, we show that a filtration by skyline diagram width (which corresponds to composition length in the Kudo-Araki-Dyer-Lashof algebra) yields subquotients which are given by the Dickson algebras, modulo perfect squares. The resulting width spectral sequence is relatively tractable, allowing us for example to reveal an error, likely of transcription, in Wellington's Ext charts (at the 11- and 12- stems). We then share computer calculations, which imply many more differentials than in the classical Adams spectral sequence, but regular phenomena as well.

In particular, there are Bockstein (h_0) towers, which we classify in the width spectral sequence in Section IV. These occur in the same dimensions as Wellington identified, with considerably more effort, in the E_2 of the CWSS. Thus, there are no differentials in the width spectral sequence with h_0 inverted, and we conjecture no differentials in the width spectral sequence in general, a purely algebraic question. It would be interesting to understand differentials between these h_0-towers and the special roles the resulting classes in homotopy might play.

Based on a remarkable identification of the unstable Adams E_2 for BO and some preliminary calculations, in Section V we initiate the study of the J-homomorphism. We calculate the induced map on cohomology of the indecomposables and prove that it induces a splitting of the Nakaoka module. The width filtration is algebraic, with its topological meaning an open question. These calculations related to the image of J give evidence for a connection to the chromatic filtration.

CHAPTER II

KEY BACKGROUND

Homology of QS^0 and the Dyer-Lashof algebra

We will begin with a review of the homology of QS^0, which was the basis of the calculations of both Curtis and Wellington and has been understood for fifty years [8]. Recall Dyer-Lashof operations

$$Q^s : H_q(QX) \to H_{q+s}(QX)$$

Let $I = (s_1, \ldots, s_k)$ be a sequence of integers and $Q^I = Q^{s_1} \cdots Q^{s_k}$.

We define the length of I to be $\ell(I) = k$, degree of I to be $d(I) = \sum_{j=1}^{k} s_i$, and the excess of I to be $e(I) = 2s_1 - d(I)$. The sequence I is called admissible if $2s_j \geq s_{j-1}$. These operations are subject to the Adem relations for $r > 2s$

$$Q^r Q^s = \sum_{i>0} \binom{i-s-1}{2i-r} Q^{r+s-i} Q^i.$$

The Dyer Lashof algebra over $\mathbb{F}_2$, is defined in [8] as the quotient of free associative algebra generated by Q^s by the two-sided ideal generated by the Adem relations and the relations $Q^I = 0$ if $e(I) < 0$. Recall as well that the homology of loop spaces, as H-spaces, carry a product structure.

Theorem 2.1.1. *$H_*(QX)$ is the free commutative algebra generated by $Q^I x$, where x ranges over a basis for the reduced homology of X and I ranges over admissible* *sequences with* $e(I) \geq 0$*, modulo the ideal generated by the set*

$$\{Q^s x - x^2 \mid |x| = s\}$$

We are concerned with $X = S^0$, for which $\iota \in H_0$ is the only basis element.

While this presentation gives a clean additive basis for homology, there are a few challenges in using this as input for the unstable Adams spectral sequence.

Cup co-product is given by $\Delta Q^I = \sum_{J+J'=I} Q^J \otimes Q^{J'}$ where J and J' both have excess greater than or equal to 0. But Adem relations are needed. For example, $\Delta Q^{4,2}$ is the sum

$$Q^{3,2} \otimes Q^{1,0} + Q^{2,2} \otimes Q^{2,0} + Q^{3,1} \otimes Q^{1,1} + Q^{2,1} \otimes Q^{2,1} + Q^{1,1} \otimes Q^{3,1} + Q^{2,0} \otimes Q^{2,2} + Q^{1,0} \otimes Q^{3,2}.$$

However, $Q^{1,0}, Q^{2,0}, Q^{3,1}$ are all not admissible and require Adem relations to see that since $Q^{1,0} = 0$, $Q^{2,0} = Q^{1,1}$, and $Q^{3,1} = 0$ the co-product is really

$$\Delta Q^{4,2} = Q^{2,2} \otimes Q^{1,1} + Q^{2,1} \otimes Q^{2,1} + Q^{1,1} \otimes Q^{2,2}.$$

It is thus a combinatorial challenge to even construct primitives, in contrast to our explicit ring generators for cohomology given in Theorem 2.2.2.

Similarly, Steenrod action is given by Nishida relations

$$Sq_*^r Q^s = \sum_i \binom{r-2i}{s-2r+2i} Q^{s-r+i} Sq_*^i$$

where Sq_*^r is the dual to Sq^r. These are combinatorially involved themselves, and moreover, as was the case with cup coproduct, inadmissible terms which arise complicate analysis.

Another challenge in working with homology is that the operations Q^i do not preserve components. For example, start with $\iota \in H_0(\Omega^n_1 S^n)$ and $Q^2\iota \in H_2(\Omega^n_2 S^n)$. To work with the 0-component, define $\bar{\iota}$ as the non-zero class in $H_0(\Omega^n_{-1} S^n)$, so that $Q^2\iota * \bar{\iota} * \bar{\iota} \in H_2(\Omega^n_0 S^n)$. But,

$$Q^1(Q^2\iota * \bar{\iota}^2) = Q^1Q^2\iota * \bar{\iota}^2 + Q^1\bar{\iota}^2 = Q^1Q^2\iota * \bar{\iota}^2$$

which is not the same as $Q^1Q^2\iota * \bar{\iota}^4$. Thus in the most straightforward basis for homology of Q_0S^0, namely monomials in $Q^I\iota$ multiplied by appropriate powers of $\bar{\iota}$, one cannot immediately apply the Nishida relations.

Cohomology

Recall that $H^*(Q_0S^0, \mathbb{F}_2) \cong H^*(BS_\infty, \mathbb{F}_2)$. While cohomology is the formal linear dual of homology, that is not the best perspective to take to actually make calculations. Instead, recent calculations of cohomology of finite symmetric groups by taking all symmetric groups together and considering both cup product and a transfer or induction product gives a concise presentation.

Theorem 2.2.1 (GSS)**.** *As a Hopf ring, $\bigoplus_n H^*(B\mathcal{S}_n; \mathbb{F}_2)$ is generated by classes $\gamma_{\ell[n]} \in H^{n(2^\ell-1)}(B\mathcal{S}_{n2^\ell})$, along with unit classes on each component. The coproduct of $\gamma_{\ell[n]}$ is given by*

$$\Delta\gamma_{\ell[n]} = \sum_{i+j=n} \gamma_{\ell[i]} \otimes \gamma_{\ell[j]}.$$

Relations between transfer products of these generators are given by

$$\gamma_{\ell[n]} \odot \gamma_{\ell[m]} = \binom{n+m}{n}\gamma_{\ell[n+m]}.$$

Relations between cup products of generators are that cup products of generators on different components are zero.

These can be presented graphically, as "skyline diagrams." The generator $\gamma_{\ell[n]}$ is represented by a rectangle of width $n \cdot 2^{\ell-1}$ and total area $n \cdot (2^\ell - 1)$, so that the area of the rectangle corresponds to its degree in cohomology while the width indicates which component it is from. Cup product is indicated by vertical stacking to make columns, whose placement next to each other denotes transfer product. We also draw in vertical dashed lines separating the block into n equal sections, for purposes of the coproduct. An example of such a diagram can be seen in Figure 1.

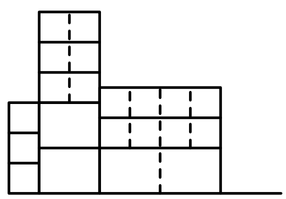


FIGURE 1. Skyline diagram for $\gamma_1^3 \odot \gamma_2 \gamma_{1[2]}^3 \odot \gamma_{2[2]} \gamma_{1[4]}^2 \odot 1_4$

The cohomology of the infinite symmetric group is the inverse limit

$$H^*(BS_\infty) = \varprojlim_n H^*(BS_n).$$

The maps $H^*(BS_n) \to H^*(BS_m)$ for $n > m$ take a diagram in $H^*(BS_n)$ which has a "tail" of width greater than or equal to $\frac{n-m}{2}$ (that is, a $\odot$-product factor of 1_k with $k > \frac{n-m}{2}$) to a diagram in $H^*(BS_m)$ obtained by shortening its tail to make it the appropriate width to be an element of $H^*(BS_m)$. If a class is not such a transfer product with a sufficiently large unit class, it maps to zero. In Figure 1 the diagram has a tail of width 2 and is an element of $H^*(BS_{18})$. Shortening the tail once, it becomes $\gamma_1^3 \odot \gamma_2\gamma_{1[2]}^3 \odot \gamma_{2[2]}\gamma_{1[4]}^2 \odot 1_2 \in H^*(BS_{16})$. Shortening it again produces $\gamma_1^3 \odot \gamma_2\gamma_{1[2]}^3 \odot \gamma_{2[2]}\gamma_{1[4]}^2 \in H^*(BS_{14})$ after which point the tail can no longer be shortened and its image in $H^*(BS_{2k})$ is 0 for $k < 7$.

With restriction maps taking this form, the cohomology of $B\mathcal{S}_\infty$ could be viewed through such diagrams with "infinitely long tails", or in monomial form as "$\odot 1_\infty$". As they confer no additional information, we prefer to omit the tails altogether, only using them implicitly when we calculate through multiplication rules for finite groups. Using this presentation, we next recall another basic result of Giusti-Salvatore-Sinha, refining a classical result of Nakaoka.

Theorem 2.2.2. *The cohomology of BS_∞ is a polynomial algebra. Minimal generators of $H^*(BS_\infty)$ as an algebra under cup-product are represented graphically by single columns with at least one block type appearing an odd number of times.*

These minimal generators form a basis for the indecomposables of the cohomology of BS_∞ (and this also of Q_0S^0), which we call the Nakaoka module $\mathfrak{N}$.

The idea of proof is to consider the product of single-column diagrams which contribute to a skyline diagram. This product results in a sum of diagrams, the widest of which is the original skyline diagram. This shows such products are algebraically independent, and a simple filtration argument shows the polynomial ring they generate exhausts the cohomology. For example, the skyline diagam from Figure 1, is the product of its nontrivial columns, γ_1^3, $\gamma_2^2\gamma_{1[2]}^3$, and $\gamma_{2[2]}\gamma_{1[4]}^2$ plus lower-width terms.

In the same paper, Giusti-Salvatore-Sinha describe the Steenrod algebra action on the cohomology of symmetric groups in terms of the basis elements

$\gamma_{\ell[2^k]}$. Because there are Cartan formulae for both cup and transfer product, this determines Steenrod structure on the whole.

Definition 2.2.3. (i) The algebraic degree of a single column is the total number of Hopf ring generators cup-multiplied to make the single column.

(ii) The height of one of these skyline diagrams is the largest of the algebraic degrees of its constituent columns.

(iii) The effective scale of a single column, composed of $\gamma_{\ell[n]}$ cup-multiplied together, is the largest such ℓ that occurs in the block. The effective scale of a skyline diagram is the minimum of the effective scales of its single columns.

(iv) A monomial is full width as long as it is not a non-trivial transfer product of some monomial with some unit class 1_k.

Theorem 2.2.4 (Theorem 8.3 of [13]). *The Steenrod square $Sq^i\gamma_{\ell[2^k]}$ is the sum of all full-width monomials of total degree $2^k(2^\ell - 1) + i$, height one or two, and effective scale at least ℓ, with height two only allowed if the effective scale $= \ell$.*

For example, Figure 2 illtustrates the three summands of $Sq^3(\gamma_{2[4]})$.

$$Sq^3(\gamma_{2[4]}) = \gamma_{4[1]} + \gamma_{3[1]} \odot \gamma_{2[1]}\gamma_{1[2]} \odot \gamma_{2[1]} + \gamma_{2[1]}^2 \odot \gamma_{2[1]} \odot \gamma_{2[2]}.$$

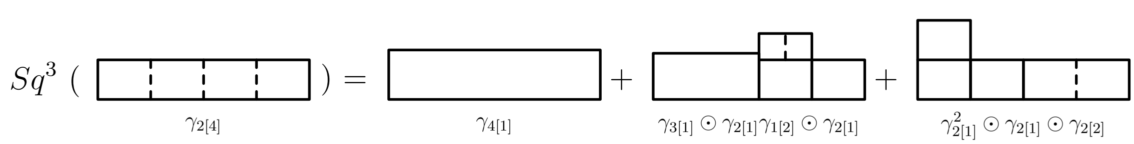


FIGURE 2. Skyline diagrams for the three summands of $Sq^3(\gamma_{2[4]})$

It is straightforward to use Cartan formulae to calculate Steenrod action on $\mathfrak{N}$, the indecomposables. This gives a refinement of Nakaoka's work, which only determined this module additively, and is a much more accessible presentation than of the homology primitives [8, 19]. Indeed, much of Wellington's work on the CWSS is devoted to calculations with these primitives. These calculations are generally simplified or become immaterial through this cohomology approach.

The unstable Adams spectral sequence and a result of Bousfield

We will next review some of the background to the unstable Adams spectral sequence along with a result of Bousfield's that makes the E_2 term actually computable.

Recall that work of [4] and others proves that for simply connected X with $\pi_*(X)$ of finite type, there is an unstable analog to the Adams spectral sequence with $E_2 \cong \mathrm{Ext}^{s,t}_{\mathcal{UA}}(H^*(X), \mathbb{F}_2)$ converging to $\pi_*(X)$.

There have been relatively few computations made of the unstable Adams spectral sequence, with some explicit calculations of Curtis and Mahowald [11, 9] and Miller's proof of the Sullivan Conjecture [22] being spectacular exceptions. A main roadblock is that the Ext groups which occur, which we call $\mathrm{Ext}_{\mathcal{UA}}$ for the category of unstable algebras over the Steenrod algebra, are not Ext groups in the usual sense of derived homomorphisms in an abelian category. While Goerss established that they are a "non-abelian" derived Hom, in the sense of Quillen, in the category of simplicial algebras over $\mathcal{A}$, this has not to our knowledge been used in any way for calculations.

To make calculations, one can hope for equivalent Ext calculations in abelian categories. The category of unstable modules over the Steenrod algebra, $\mathcal{U}$, is abelian and there is a free unstable algebra functor $\mathcal{U} \rightarrow \mathcal{UA}$. However, the cohomology of a space is very rarely in the image of this functor, even if it is free. If the cohomology is free only as an algebra, there is still an alternate form of reduction. Let A be an augmented algebra with $\overline{A}$ its augmentation ideal. We let Ind A denote the algebra indecomposables $\overline{A}/(\overline{A} \cdot \overline{A})$. Note that $\Sigma^{-1}\text{Ind } A$ is naturally in $\mathcal{U}$. The following was originally stated by Bousfield in [5] and follows from the composite functor spectral sequence constructed by Miller [22].

Theorem 2.3.1. *Let P be an unstable algebra over $\mathcal{A}$ that is free as an algebra on* Ind P. *Then*

$$\text{Ext}^{s,t}_{\mathcal{UA}}(P, k) \cong \text{Ext}^{s,t-1}_{\mathcal{U}}(\Sigma^{-1}\text{Ind } P, k)$$

Recall that we refer to Ind $H^*(Q_0S^0, \mathbb{F}_2)$, as the Nakaoka module, $\mathfrak{N}$. Thus, Proposition 2.3.1 applies and will give as an immediate corollary, Proposition 3.0.1 in the next section.

Hung-Peterson's calculations of indecomposables of Dickson algebras

We will now end the background sections with a discussion of Dickson algebras and some of the results of Hung-Peterson on their indecomposables, which will show up on the 0-line of the first page of our spectral sequence.

Let D_n denote the n^{th} Diskson algebra, defined as the invariants $\mathbb{F}_2[x_1, \ldots, x_n]^{GL_n(\mathbb{F}_2)}$. These algebras are polynomial on generators $d_{k,l}$ in dimensions $2^k(2^l - 1)$ where $k + l = n$. Adem and Milgram [2] describe one method to generate these generators.

Theorem 2.4.1 (Theorem 2.4 [2])**.** *Let*

$$D_{n,i}(x_1, \ldots, x_n) = \det \begin{pmatrix} x_1 & \cdots & \hat{x}_1^{2^{i-1}} & \cdots & x_1^{2^n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_n & \cdots & \hat{x}_n^{2^{i-1}} & \cdots & x_n^{2^n} \end{pmatrix}$$

then

$$d_{n-j,j} = D_{n,n+1-j}/D_{n,n+1}$$

are the generators for the n^{th} Dickson algebra $\mathbb{F}_2[x_1, \ldots, x_n]^{GL_n(\mathbb{F}_2)}$.

As an example, consider the second Dickson algebra, D_2. Then,

$$D_{2,1} = \det \begin{pmatrix} x_1^2 & x_1^4 \\ x_2^2 & x_2^4 \end{pmatrix} = x_1^2 x_2^4 + x_1^4 x_2^2,$$

$$D_{2,2} = \det \begin{pmatrix} x_1 & x_1^4 \\ x_2 & x_2^4 \end{pmatrix} = x_1 x_2^4 + x_1^4 x_2,$$

and

$$D_{2,3} = \det \begin{pmatrix} x_1 & x_1^2 \\ x_2 & x_2^2 \end{pmatrix} = x_1 x_2^2 + x_1^2 x_2.$$

We then get our two generators, $d_{1,1}$ in degree $2 = 2^1(2^1 - 1)$ and $d_{0,2}$ in degree $3 = 2^0(2^2 - 1)$ from polynomial long division,

$$d_{1,1} = D_{2,2}/D_{2,3} = \frac{x_1 x_2^4 + x_1^4 x_2}{x_1 x_2^2 + x_1^2 x_2} = x_1^2 + x_1 x_2 + x_2^2$$

and

$$d_{0,2} = D_{2,1}/D_{2,3} = \frac{x_1^2 x_2^4 + x_1^4 x_2^2}{x_1 x_2^2 + x_1^2 x_2} = x_1^2 x_2 + x_1 x_2^2.$$

We show below that one can filter the algebra indecomposables of the cohomology of $Q_0 S^0$ and obtain subquotient modules which are essentially Dickson algebras. Thus, the Steenrod indecomposables of Dickson algebras (as modules) give rise to putative indecomposables in the unstable Adams spectral sequence for $Q_0 S^0$.

Before our work, the Lannes-Zarati homomorphism (which seems not to have been published in full detail) maps from part of the classical Ext algebra of the Steenrod algebra to the indecomposables of a corresponding Dickson algebra, $\mathrm{Ext}_{\mathcal{A}}^{n,n+d}(\mathbb{F}_2, \mathbb{F}_2) \to (\mathbb{F}_2 \otimes_{\mathcal{A}} D_n)_d$. This connection motivated Hung and Peterson to work out a minimal set of generators for D_n as a module over the Steenrod algebra $\mathcal{A}$ through $n = 4$. Through our work below, these can give rise to elements in $\mathrm{Ext}^{0,t}(\mathrm{Ind}\ (H^*(Q_0 S^0)))$ of width n.

For D_1, which is just the cohomology of $\mathbb{R}P^\infty$, the classes ${d_{0,1}}^{2^s-1}$ form a basis as a module over $\mathcal{A}$. For $n = 2$, the classes ${d_{1,1}}^{2^s-1}{d_{0,2}}^0$ form a basis for D_2 as a module over $\mathcal{A}$. The cases for $n = 3$ and $n = 4$ are worked out in [16]. To state these results, we let $I = (i_{n-1}, \ldots, i_0)$ be a sequence of non-negative integers corresponding to

$${d_{n-1,1}}^{i_{n-1}}{d_{n-2,2}}^{i_{n-2}} \cdots {d_{0,n}}^{i_0}$$

Theorem 2.4.2 (Hung-Peterson Theorem 2.6). *The set of monomials corresponding to the sequences I listed below form a monomial basis for $\mathbb{Z}/2 \otimes_{\mathcal{A}} D_3$*

$$\begin{aligned} &(2^s - 1, 0, 0), && s \geq 0 \\ &(2^r - 2^s - 1, 2^s - 1, 1), && r > s > 0. \end{aligned}$$

Theorem 2.4.3 (Hung-Peterson Theorem 2.7). *The set of monomials corresponding to the sequences I listed below form a monomial basis for $\mathbb{Z}/2 \otimes_{\mathcal{A}} D_4$*

$$
\begin{aligned}
(2^s - 1, 0, 0, 0), &\qquad s \geq 0 \\
(2^r - 2^s - 1, 2^s - 1, 1, 0), &\qquad r > s > 0 \\
(2^t - 2^r - 1, 2^r - 2^s - 1, 2^s - 1, 1), &\qquad t > r > s > 1 \\
(2^r - 2^{s+1} - 2^s - 1, 2^s - 1, 2^s - 1, 1), &\qquad r > s + 1 > 2.
\end{aligned}
$$

Below we will discuss these indecomposables further and give a new concrete connection with homotopy groups.

CHAPTER III

WIDTH SPECTRAL SEQUENCE

In this section, we apply Bousfield's result of Proposition 2.3.1 to equate the Curtis-Wellington E_2 with an explicit Ext group in the category of unstable modules over the Steenrod algebra, at which point there is an immediate filtration to develop. The following is an immediate corollary of Proposition 2.3.1.

Proposition 3.0.1. *The E_2 term of the Curtis-Wellington spectral sequence is isomorphic to* $\mathrm{Ext}_{\mathcal{U}}^{s,t}(\Sigma^{-1}\mathfrak{N})$.

Our graphical skyline diagram presentation, in particular of the indecomposables as stated in Theorem 2.2.2, points immediately to a filtration of the Nakaoka module.

Definition 3.0.2. Let F_n be the submodule of $\mathfrak{N}$ of elements of width less than or equal to 2^{n-1}.

These are submodules as unstable Steenrod modules by the formula from Theorem 2.2.4. Moreover, F_n/F_{n-1} will be spanned by single columns of width exactly 2^{n-1} with at least one block type appearing an odd number of times.

Let V_n be the transitive elementary abelian 2-subgroup of S_{2^n}. Because restriction to a subgroup maps to invariants of an action by the normalizer of the subgroup (see [2]), in this case the restriction map is received by rings of Dickson invariants. But Corollary 7.6 of [13], follows an argument of Milgram to show that restriction of $\gamma_{\ell,2^k}$ with $\ell + k = n$ to V_n is the Dickson class $d_{k,l}$. Thus single-column diagrams go to corresponding products of Dickson classes, and these single column skyline diagrams are isomorphic to a quotient of D_n as a module over the Steenrod algebra. In particular, we have the following.

Proposition 3.0.3. *The quotient F_n/F_{n-1} is isomorphic to D_n^o, where D_n^o is the quotient of D_n by all perfect squares.*

This filtration allows us to consider only full width terms in the image of the Steenrod action, substantially simplifying calculations. Assembling the long exact sequences in cohomology associated to the short exact sequences

$$0 \to F_{n-1} \to F_n \to D_n^o \to 0$$

from the filtration described above produces a tri-graded spectral sequence converging to the E_2 term of the Curtis-Wellington spectral sequence for stable homotopy.

Theorem 3.0.4. *The spectral sequence associated to the width filtration has*

$$E_1^{s,t;n} = \mathrm{Ext}_{\mathcal{U}}^{s,t}(\Sigma^{-1} D_n^o, \mathbb{F}_2)$$

and $d_r : \mathrm{Ext}^{s,t}(\Sigma^{-1} D_n^o) \to \mathrm{Ext}^{s+1,t}(\Sigma^{-1} D_{n+r}^o)$. It converges to $\mathrm{Ext}_{\mathcal{U}}^{s,t}(\Sigma^{-1}\mathfrak{N})$, and thus the E_2 of the Curtis-Wellington spectral sequence.

Using the well-known Steenrod structure on Dickson algebras, as presented for example in [15], we have been able to make hand calculations out to the 17 stem. Hood Chatham [7] kindly produced an Ext-chart illustrating the first page of this spectral sequence, out to the the forty-five stem. We share a clip here in Figure 3 and the full chart in the Appendix.

Along the zero-line in this width spectral sequence we see the Steenrod indecomposables of the Dickson algebras, which have been of interest, for example in work of Hung and Peterson [16], and are far from understood. On the E_1 page, we can see the classes listed by Hung and Peterson in Theorems 2.4.2 and 2.4.3. We do not have any calculations about whether these classes will survive to the E_2 page of the Curits Wellington spectral sequence, but hand calculations in low degrees show that they are all still there. These Dickson indecomposables also provide an upper bound on the classes that will survive to the E_∞ page of the CWSS. Curtis conjectured that only Hopf and Kervaire classes will survive, so these results provide a strong connection between the work of Hung and Peterson and that of Curtis. We also note that this zero-line receives the Hurewicz map for Q_0S^0, as studied by Lannes and Zarati [18].

Wellington made similar computations for $\mathrm{Ext}(H_*Q_0S^0)$ at the prime 2, including charts out to the 17 stem. Comparing Wellington's results to ours, they mainly agree, but our calculations reveal an error in the 11 and 12 stem in the Ext chart. The original chart had a class in bidegree (12,4) and a d_2 differential to the class in degree $(11, 6)$. Instead, our hand calculations show that there is a class in degree $(12, 3)$, which appears in the computer calculations in degree $(11, 3)$ after the desuspension. Since we know that the CWSS must converge to stable homotopy, we know that there must be a d_3 differential instead of the d_2 differential.

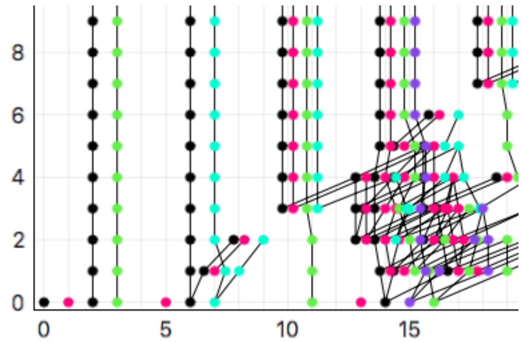
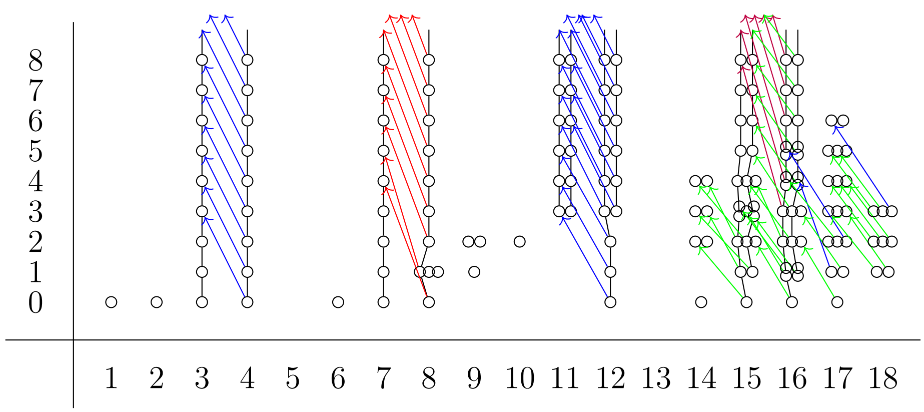
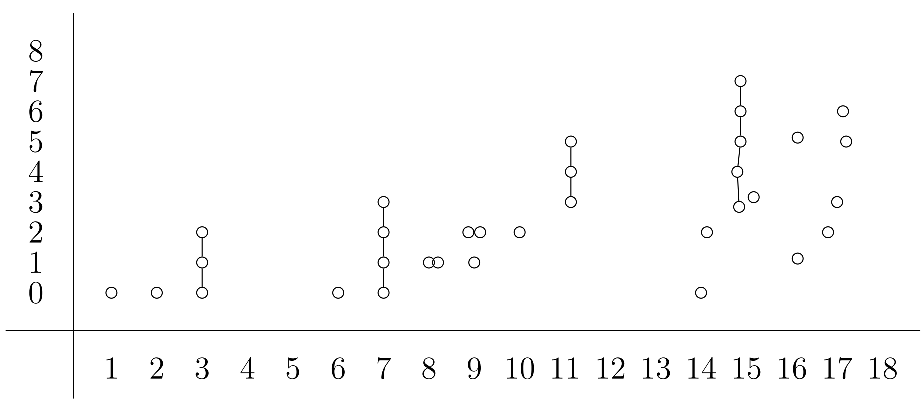


FIGURE 3. The E_1 page of the width spectral sequence, with width filtration encoded by color: black corresponds to D_1, red to D_2, green to D_3, teal to D_4, and purple to D_5.

Part of Table 13 by Wellington [27] (with correction in the 12 stem), depicting the E_2 page and differentials of the CWSS.



E_∞ page of Curtis-Wellington spectral sequence up through the 17 stem.



The differentials of the spectral sequence of Theorem 3.0.4, namely

$$d_r : \mathrm{Ext}^{s,t}(D^o_n) \to \mathrm{Ext}^{s+1,t}(D^o_{n+r}),$$

fix topological degree (t), increase co-bar length by 1 (s), and increase filtration by r. In the charts, they will be moving one unit left, one unit up, and in our representation of the third (width) grading by color move between different colors so that if the source is in D^o_n, the target is in D^o_{n+r}. We can see by hand through at least the 16-stem that there are no possible differentials in the width filtration spectral sequence purely by considering degree and multiplicative strucure.

CHAPTER IV

H_0 TOWERS

Partial action

One of the immediate differences between the Adams spectral sequence and Curtis-Wellington spectral sequence is the presence of infinite h_0 towers. We classify these in the width spectral sequence.

We begin by defining a partial action of $\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$. In the Yoneda approach to Ext, an element of $\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ is an extension of length s from $\mathbb{F}_2$ to $\Sigma^t\mathbb{F}_2$, namely

$$0 \to \Sigma^t\mathbb{F}_2 \to E_1 \to \cdots \to E_s \to \mathbb{F}_2 \to 0,$$

where the E_i are $\mathcal{A}$ modules. An element of $\mathrm{Ext}_{\mathcal{U}}^{p,q}(\Sigma^{-1}D_n^o, \mathbb{F}_2)$ is an extension of length p from $\Sigma^{-1}D_n^o$ to $\Sigma^q\mathbb{F}_2$

$$0 \to \Sigma^q\mathbb{F}_2 \to F_1 \to \cdots \to F_p \to \Sigma^{-1}D_n^o \to 0$$

where the F_i are unstable $\mathcal{A}$ modules and the maps $F_i \to F_{i+1}$ have excess less than or equal to the degree of F_{i+1}.

Definition 4.1.1. Define a partial action of $\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ on $\mathrm{Ext}_{\mathcal{U}}^{p,q}(\Sigma^{-1}D_n^o, \mathbb{F}_2)$, defined when $t - s + 1 \leq q$, by suspending the stable extension q times and concatenating it on the left with the unstable extension to give an extension which defines an element of $\mathrm{Ext}_{\mathcal{U}}^{s+p,t+q}(\Sigma^{-1}D_n^o, \mathbb{F}_2)$.

$$0 \to \Sigma^{t+q}\mathbb{F}_2 \to \Sigma^q E_1 \to \cdots \to \Sigma^q E_s \to \Sigma^q \mathbb{F}_2 \to F_1 \to \cdots \to F_p \to \Sigma^{-1} D_n^o \to 0.$$

Calculations involving the Λ-algebra

We next recall the Λ-algebra [4, 6], which gives an explicit though computationally involved way to compute some Ext groups over the Steenrod algebra. The Λ algebra is the graded associative differential algebra with unit over $\mathbb{F}_2$ with

1. a generator λ_i of degree i for each $i \geq 0$

2. for each i, $k \geq 0$ a relation

$$\lambda_i \lambda_{2i+1+k} = \sum_{j \geq 0} \binom{k-1-j}{j} \lambda_{i+k-j} \lambda_{2i+1+j}$$

3. a differential ∂ given by

$$\partial(\lambda_i) = \sum_{j \geq 1} \binom{i-j}{j} \lambda_{i-j} \lambda_{j-1}.$$

Note that $\Lambda = \bigoplus_{s \geq 0} \Lambda^s$ where Λ^s is genreated by monomials λ_I of length s.

To define complexes using the Λ algebra, we need our action to be on the right. Let $\mathcal{U}_{\mathcal{R}}$ denote the category of unstable right $\mathcal{A}$ modules and continue to denote $\mathcal{U}$ the category of unstable left $\mathcal{A}$ modules. For M of finite type, $\mathrm{Ext}^{s,t}_{\mathcal{U}_{\mathcal{R}}}(\mathbb{F}_2, M) \cong \mathrm{Ext}^{s,t}_{\mathcal{U}}(M^*, \mathbb{F}_2)$.

The Λ algebra gives one method to approach calculation of Ext groups, in particular, from [6],

$$\mathrm{Ext}^{s,t}_{\mathcal{U}_{\mathcal{R}}}(\mathbb{F}_2, M) \cong H^s(V(M))_{t-s}$$

where $V(M)$ is the chain complex

$$M \to M\hat{\otimes}\Lambda^1 \to M\hat{\otimes}\Lambda^2 \to \cdots$$

Theorem 4.2.1 (Theorem 3.3 from [6]). *For $M \in \mathcal{U}_{\mathcal{R}}$ and $s, t \geq 0$ there is a natural isomorpism*

$$\mathrm{Ext}^{s,t}_{\mathcal{U}_{\mathcal{R}}}(\mathbb{F}_2, M) \cong H^s(V(M))_{t-s}$$

Proof. Given a short exact sequence of right $\mathcal{A}$ modules,

$$0 \to A \to B \to C \to 0$$

we get an exact sequence

$$\cdots \to H^{s,t}(V(A)) \to H^{s,t}(V(B)) \to H^{s,t}(V(C)) \to H^{s+1,t}(V(A)) \to \cdots .$$

Now let M be an injective in $\mathcal{U}_{\mathcal{R}}$, then $H^{s,t}(V(M)) = 0$ for $s > 0$. Indeed, let $I(n)$ be the injective unstable $\mathcal{A}$ module on a single n-dimensional generator. For $n \geq 1$, $\Omega I(n) = I(n-1)$ and $\Omega^1 I(n) = 0$ (where Ω is right adjoint to the suspension functor and Ω^1 is its first derived functor). By applying Theorem 3.5 of [6], there is a natural exact sequence

$$\to H^{s,t}(V(\Omega I(n))) \to H^{s,t+1}(V(I(n))) \to H^{s-1,t}(V(\Omega^1 I(n))) \to H^{s+1,t}(V(\Omega I(n))) \to .$$

Thus, $H^{s,t}(V(\Omega I(n))) = H^{s,t}(V(I(n-1))) \cong H^{s,t+1}(V(I(n)))$. Note that for $s > 0$, $H^{s,t}(V(0)) = 0$, so by induction $H^{s,t}(V(n)) = 0$ for $s > 0$. Since any injective right $\mathcal{A}$ module is the retract of a direct sum of injectives $I(n)$ of this form we can conclude $H^{s,t}(V(M)) = 0$ for $s > 0$ for any injective $M \in \mathcal{U}_{\mathcal{R}}$.

Now all that remains to show is that $\mathrm{Ext}^{s,*}_{\mathcal{U}_{\mathcal{R}}}(\mathbb{F}_2, -)$ and $H^s(V(M))_{*-s}$ agree on a point. For M an injective right $\mathcal{A}$ module,

$$\mathrm{Ext}^{0,t}_{\mathcal{U}_{\mathcal{R}}}(\mathbb{F}_2, M) \cong \mathrm{Hom}_{\mathcal{U}_{\mathcal{R}}}(\tilde{H}_*(S^t), M).$$

Next we define a map $\Phi : \mathrm{Hom}_{\mathcal{U}_{\mathcal{R}}}(\tilde{H}_*(S^t), M) \to H^{0,t}(V(M))$ by $f \mapsto f(1)$. We know that $x \in H^{0,t}(V(M))$ if and only if $x \in \ker(\delta) : M\hat{\otimes}\Lambda^0 \to M\hat{\otimes}\Lambda^1$, which means $x \cdot Sq^i = 0$ for all i. This then means the map $f_x : \tilde{H}_*(S^t) \to M$ determined by $1 \mapsto x$ is in $\mathrm{Hom}_{\mathcal{U}_{\mathcal{R}}}(\tilde{H}_*(S^t), M)$ since $f_x(1) \cdot Sq^i = x \cdot Sq^i = f_x(1 \cdot Sq^i) = 0$ for all i. So, we can see that this is an isomorphism.

□

In [5], Bousfield defines the following tower complex as a quotient of the chain complex $V(M)$, as follows.

Definition 4.2.2. Let

$$T^s(M) = \begin{cases} M \otimes (\lambda_0)^s & s = 0, 1 \\ M \otimes (\lambda_0)^s \oplus \sum_{k>0} M_{2k} \otimes \lambda_{2k-1}(\lambda_0)^{s-1} & s > 1, \end{cases}$$

with

$$\delta(x \otimes \lambda_{2k-1}(\lambda_0)^{s-1}) = 0,$$

and

$$\delta(x \otimes (\lambda_0)^s) = \begin{cases} x \cdot Sq^1 \otimes (\lambda_0)^{s+1} + x \cdot Sq^{2k} \otimes \lambda_{2k-1}(\lambda_0)^s & s > 0, x \in M_{4k} \\ x \cdot Sq^1 \otimes (\lambda_0)^{s+1} & \text{otherwise.} \end{cases}$$

Let $O(M)$ be the subcomplex of $V(M)$ generated by $x \otimes \lambda_I \in M \hat{\otimes} \Lambda^s$ such that $\lambda_I = \lambda_{i_1} \cdots \lambda_{i_s}$ is admissible with i_s odd.

Proposition 4.2.3 (Proposition 2.3 of [5]). *For $M \in \mathcal{U}_{\mathcal{R}}$ with $M_0 = 0$ there is a long exact sequence*

$$\cdots \to H^{s-1}(T(M))_{n+1} \to H^s(O(M))_n \to H^s(V(M))_n \to H^s(T(M))_n \to \cdots$$

Proof. Beginning with the exact sequence

$$0 \to \ker(q) \to V(M) \xrightarrow{q} T(M) \to 0$$

we get a long exact sequence in homology

$$\cdots \to H^{s-1}(T(M))_{n+1} \to H^s(\ker(q))_n \to H^s(V(M))_n \to H^s(T(M))_n \to \cdots .$$

Note that the map q is the quotient map that takes anything in $V(M)$ not of the form $M \otimes (\lambda_0)^s$ or $M_{2k} \otimes \lambda_{2k-1}(\lambda_0)^{s-1}$ (for $s > 1$) to 0. In particular since all elements of $O(M)$ are of the form $x \otimes \lambda_I$ for $\lambda_I = \lambda_{i_1} \cdots \lambda_{i_s}$ is admissible with i_s odd, we get natural maps $O(M) \hookrightarrow \ker q$ and $H^s(O(M)) \to H^s(\ker(q))$. Thus, it suffices to prove that this map induces an isomorphism $H^s(\ker(q)) \cong H^s(O(M))$.

The harder direction here is showing surjectivity. Take $x \in \ker q$ such that $\delta(x) = 0$. If x is a boundary, then we are done. Assume x is not a boundary. Then choose y in the same homology class with $y = \sum_{i=0}^{t} y_i \otimes \lambda_{J_i}(\lambda_0)^i$ with t a minimum.

Suppose $t \neq 0$. If the final term, n_s of $J_t = (n_1, \cdots, n_s)$ is even, then $\delta(y_t \otimes \lambda_{J_t}\lambda_0^t)$ will include (among other terms) the term $y_t \otimes \lambda_{n_1,\ldots,n_s-1}\lambda_0^{t+1}$ contradicting the fact that $\delta([x]) = \delta([y]) = 0$. So, n_s is odd, which means that $\delta(y_t \otimes \lambda_{n_1,\cdots,n_s+1}\lambda_0^{t-1})$ contains the term $y_t \otimes \lambda_{n_1,\cdots,n_s}\lambda_0^t)$ plus additional terms where the λ_0 at the end has power $t-1$ or less. Then $y + \delta(y_t \otimes \lambda_{n_1,\cdots,n_s+1}\lambda_0^{t-1})$ is still homologous to x but now has $t-1$ as the maximum degree of the λ_0 portion of the term, which contradicts the minimality of t. Thus, we conclude $t = 0$ (and n_s is still odd) so $[x] = [y \otimes \lambda_{n_1,\ldots,n_s}] \in H^*(O(M))$.

$\square$

Towers on the E_1 page

Remark 2.4 of [5] notes that the towers in $H^*(T(M))$ correspond with those in $H^*(V(M))$ and thus also with the towers in $\mathrm{Ext}_{U_R}^{s,t}(\mathbb{F}_2, M)$. Applying these tower detectors to $\mathrm{Ext}_{\mathcal{U}}^{s,t}(\Sigma^{-1}D_n^o, \mathbb{F}_2)$, we conclude the following two theorems.

Theorem 4.3.1. *There are infinite towers in* $\mathrm{Ext}^{s,t}(\Sigma^{-1}D_1^o, \mathbb{F}_2)$ *in degrees* $t - s = 4a - 2$ *for* a *a positive integer.*

Theorem 4.3.2. *Let* n *be an integer greater than or equal to 2. There are towers in*

$\mathrm{Ext}^{s,t}(\Sigma^{-1}D_n^o, \mathbb{F}_2)$ *corresponding to all integer solutions of*

$$(2^{n-2} - 2^{n-3})a_1 + \cdots + (2^{n-2} - 1)a_{n-2} + (2^{n-1} - 1)b_{n-1} = k$$

where at least one of $a_1, \ldots, a_{n-2}$ *are odd in degree* $t - s = 4k - 1$*. And also towers corresponding to all integer solutions of*

$$(2^{n-1} - 2^{n-2})b_1 + \cdots + (2^{n-1} - 1)b_{n-1} + (2^n - 1)c_n + (2^n - 1) = k$$

in degree $t - s = 4k - 2$*.*

Wellington, in [27] also used these tower detecting complexes, but of course with his homology approach. Recall that the homology of $Q_0 S^0$ is free under the product induced by loop sum on classes Q^I where I is admissible. Wellington proves there are towers in dimensions $4k - 1$ and $4k$ generated by Q^I either in degree $4k$ with excess 0 and some odd index, or Q^I in degree $4k - 1$ with final index odd and all others even. Our calculations agree with Wellington's in that the towers in $\mathrm{Ext}_{\mathcal{U}}^{s,t}(\Sigma^{-1} D_n^o, \mathbb{F}_2)$ for each n correspond with his generated by Q^I with $\ell(I) = n$. Indeed, in degree $(4k - 1) - 1$, we have towers corresponding to each integer solution to

$$(2^{n-1} - 2^{n-2})b_1 + \cdots + (2^{n-1} - 1)b_{n-1} + (2^n - 1)c_n + (2^n - 1) = k.$$

Each of these corresponds to the tower generated by Q^I with $I = (s_1, s_2, \ldots, s_n)$ where $s_n = 2k + 1 + 2c_n$ and the other s_i can be computed inductively from right to left with the formula

$$s_i = \frac{1}{2^{(n-1)-i}} \left(k + 1 + a_n + \sum_{j=i}^{n-1} 2^{(n-1)-j} b_j \right) \text{ for } 2 \leq i \leq n - 1,$$

and finally

$$s_1 = 4k - 1 - \sum_{j=2}^{n} s_j.$$

In degree $(4k)-1$, we found towers corresponding to each integer solution of

$$(2^{n-2}-2^{n-3})a_1+\cdots+(2^{n-2}-1)a_{n-2}+(2^{n-1}-1)b_{n-1}=k.$$

Each of these corresponds to the tower generated by Q^I with $I=(s_1,s_2,\ldots,s_n)$ where we once again find each term in the index working from right to left. First, $s_n=2k$ and $s_{n-1}=k+b_{n-1}$, then inductively we compute

$$s_i=\frac{1}{2^{(n-1)-i}}\left(k+b_{n-1}+\sum_{j=i}^{n-2}2^{(n-2)-j}a_j\right)\ \text{for } 2\leq i\leq n-1,$$

and finally

$$s_1=4k-\sum_{j=2}^{n}s_j.$$

This agreement between our towers in the width spectral sequence and Wellington's imply there are no differentials in the width filtration spectral sequence with h_0 inverted. Between the fact that these results indicate there are no differentials between towers and that some differentials can be eliminated by hand calculations in low degrees, we wonder whether there are any differentials in the width spectral sequence at all, a purely algebraic question.

Proof of Theorem 4.3.1. Let d_1 be the generator of D_1. Then elements of D_1^o are d_1^{2i+1} in degree $2i$ after desuspension. Take $\{x_1,x_3,\cdots,x_{2i+1},\cdots\}$ as a basis for the linear dual, $(\Sigma^{-1}D_1^o)^*$ where x_{2i+1} is dual to d_1^{2i+1}, so each x_{2i+1} is in degree $2i$ in the desuspension. Consider $M=(\Sigma^{-1}D_1^o)^*$ as an unstable right $\mathcal{A}$ module by defining the linear map $x_i\cdot Sq^k$ as

$$(x_i\cdot Sq^k)(y)=x_i(Sq^ky),\ \text{for } y\in D_1^o.$$

We can use Bousfield's tower detector to determine where there are towers in $\mathrm{Ext}^{s,t}_{\mathcal{U}_{\mathcal{R}}}(\mathbb{F}_2, M)$. As defined in as defined in Definition 4.2.2, $T^s(M)$ is constructed so that in degree 2 and above the next degree is constructed from the previous by multiplying by λ_0 on the right and the differential from degree one onward is the same as the previous degree but with an extra factor of λ_0 on the right. Thus, it is sufficient to compute $H^2(T(M))$ to determine where the towers are.

We can see in Definition 4.2.2 that the differential δ only involves Sq^1 and Sq^{2k} and thus in this case is determined by the fact that

$$x_i \cdot Sq^1 = 0 \text{ and } x_{4k+1} \cdot Sq^{2k} = x_{2k+1}.$$

Elements of $H^2(T(M))$ come in three forms. First, all $x_{4k+1} \otimes (\lambda_0)^2$ are not cycles since $\delta(x_{4k+1} \otimes (\lambda_0)^2) = x_{2k+1} \otimes \lambda_{2k-1}(\lambda_0)^2$. Second, all $x_{2k+1} \otimes \lambda_{2k-1}\lambda_0$ are cycles, but also boundaries hit by $x_{4k+1} \otimes \lambda_0$. Finally, all $x_{4k-1} \otimes (\lambda_0)^2$ are cycles and can not be boundaries since the image of $T^1(M)$ is only elements of the form $x_{2k+1} \otimes \lambda_{2k-1}\lambda_0$. Thus, we get a tower for $x_{4k-1} \otimes (\lambda_0)^s$ in degree $4k-2$ for each positive integer k. □

Proof of Theorem 4.3.2. We know that D_n is generated by n elements in degrees $(2^n - 2^i)$ for $0 \leq i \leq n-1$. Let these generators be represented by $d_{2^n-2^i}$. Then an arbitrary basis element of D_n^o is of the form $d_{2^n-2^{n-1}}^{a_1} \cdots d_{2^n-2^0}^{a_n}$ with at least one a_i odd. Let $x_{a_1,\ldots,a_n} \in (D_n^o)^*$ denote the linear dual of $d_{2^n-2^{n-1}}^{a_1} \cdots d_{2^n-2^0}^{a_n}$ in degree $\sum_{i=1}^n a_i(2^n - 2^{n-i})$.

Working now in $(\Sigma^{-1}D_n^o)^*$, $x_{a_1,\ldots,a_n} \in (D_n^o)^*$ will now be in degree $\sum_{i=1}^n a_i(2^n - 2^{n-i}) - 1$. Let $M = (\Sigma^{-1}D_n^o)^*$ and consider the tower detecting complex $T(M)$. As described n the proof of Theorem 4.3.1 it is sufficient to calculate $H^2(T(M))$ to determine the location of the towers.

Since the differential δ as defined in Definition 4.2.2 only uses Sq^1 and Sq^{2k},we need only understand the right action of Sq^1 on an arbitrary $x_{a_1,\ldots,a_n}$ and the right action of Sq^{2k} on $x_{a_1,\ldots,a_n}$ with $\sum_{i=1}^n a_i(2^n - 2^{n-i}) = 4k+1$. These are given by the formulas

$$x_{a_1,\ldots,a_n} \cdot Sq^1 = \begin{cases} x_{a_1,\ldots,a_{n-2},a_{n-1}+1,a_n-1} & \text{if } a_{n-1} \text{ even,} a_n \geq 1 \text{, at least one } a_i \text{ odd} \\ 0 & \text{else} \end{cases}$$

and if $\sum_{i=1}^n a_i(2^n - 2^{n-i}) = 4k+1$

$$x_{a_1,\ldots,a_n} \cdot Sq^{2k} = \begin{cases} x_{\frac{a_1+2}{2},\frac{a_2}{2},\ldots,\frac{a_{n-1}}{2},\frac{a_n-1}{2}} & \text{if } a_n \text{ odd, } a_i \text{ even} \\ x_{\frac{a_1}{2},\ldots,\frac{a_{j-1}}{2},\frac{a_j-1}{2},\frac{a_{j+1}+2}{2},\frac{a_{j+2}}{2}\ldots,\frac{a_{n-1}}{2},\frac{a_n-1}{2}} & \text{if } a_j, a_n \text{ odd,} \\ & \quad a_i \text{ even for } i \neq j,n \\ x_{\frac{a_1}{2},\ldots,\frac{a_{n-2}}{2},\frac{a_{n-1}-1}{2},\frac{a_n+1}{2}} & \text{if } a_{n-1}, a_n \text{ odd,} \\ & \quad a_i \text{ even for } 1 \leq i < n-1 \\ 0 & \text{else,} \end{cases}$$

where throughout i and j are between 1 and $n-1$ inclusively.

With these formulas in hand, we analyze $H^2(T(M))$. Our strategy will be to first characterize all cycles in $H^2(T(M))$ and then go through each type of cycle to determine which are boundaries. All of those that are not boundaries will correspond to our tower generators. There are four types of cycles:

(a) $x_{a_1,\ldots,a_n} \otimes (\lambda_0)^2$ with $\displaystyle\sum_{i=1}^n a_i(2^n - 2^{n-i}) \neq 4k+1$ and a_{n-1} odd

(b) $x_{a_1,\ldots,a_n} \otimes (\lambda_0)^2$ with a_{n-1} even and $a_n = 0$

(c) $x_{a_1,\ldots,a_n} \otimes (\lambda_0)^2$ with $\displaystyle\sum_{i=1}^{n} a_i(2^n - 2^{n-i}) = 4k+1$ and a_n, a_{n-1}, and some other a_i for $1 \leq i < n-1$ odd

(d) $x_{a_1,\ldots,a_n} \otimes \lambda_{2k-1}\lambda_0$ with $\displaystyle\sum_{i=1}^{n} a_i(2^n - 2^{n-i}) = 2k+1$

To see that these are all of the cycles, we look at our formulas for the action of Sq^1 and Sq^{2k}. As long as the degree of $x_{a_i,\ldots,a_n}$ is not a multiple of four, the differential on $x_{a_i,\ldots,a_n} \otimes (\lambda_0)^2$ only involves Sq^1. So, if $\sum_{i=1}^{n} a_i(2^n - 2^{n-i}) - 1 \neq 4k$, $x_{a_i,\ldots,a_n} \otimes (\lambda_0)^2$ is a cycle when $x_{a_i,\ldots,a_n} \cdot Sq^1 = 0$. This means either a_{n-1} is odd and we get cycles of type (a), $a_n = 0$ and a_{n-1} is even and we get cycles of type (b), or all a_i are even, but then $x_{ai,\ldots,a_n}$ is not an element of $(D_1^o)^*$.

If $x_{a_i,\ldots,a_n}$ is in degree $4k$, the differential on $x_{a_i,\ldots,a_n} \otimes (\lambda_0)^2$ involves both Sq^1 and Sq^{2k}. Then $\sum_{i=1}^{n} a_i(2^n - 2^{n-i}) - 1 = 4k$ which means that $a_n > 0$ and a_{n-1} is even. For both Sq^1 and Sq^{2k} to act trivially, a_{n-1} must be odd and some other a_i for $1 \leq i < n-1$ is also odd to give the cycles of type (c).

Finally, we have the elements of the form $x_{a_i,\ldots,a_n} \otimes \lambda_{2k-1}(\lambda_0)^{s-1}$ which are all cycles by definition and give us the cycles in class (d).

Beginning with cycles of class (a), we want to determine which are also boundaries. For cycles in (a), $x_{a_1,\ldots,a_n} \otimes (\lambda_0)^2$ with $\sum_{i=1}^{n} a_i(2^n - 2^{n-i}) \neq 4k+1$ and a_{n-1} odd, we will split into three cases based on the value of $\sum_{i=1}^{n} a_i(2^n - 2^{n-i})$ mod 4.

If $\sum_{i=1}^{n} a_i(2^n - 2^{n-i}) = 4k-1$ and a_{n-1} is odd, then a_n is odd and

$$\delta(x_{a_1,\ldots,a_{n-2},a_{n-1}-1,a_n+1} \otimes \lambda_0) = x_{a_1,\ldots,a_{n-2},a_{n-1},a_n} \otimes (\lambda_0)^2$$

as long as one of $a_1, \ldots, a_{n-2}$ are odd. Thus, we see that the only cycles not hit by boundaries are $x_{a_1,\ldots,a_{n-2},a_{n-1},a_n} \otimes (\lambda_0)^2$ with a_{n-1}, a_n odd and a_i even for $1 \leq i \leq n-2$.

If $\sum_{i=1}^n a_i(2^n - 2^{n-i}) = 4k - 2$ and a_{n-1} is odd , then a_n is even and

$$\delta(x_{a_1,\ldots,a_{n-2},a_{n-1}-1,a_n+1} \otimes \lambda_0) = x_{a_1,\ldots,a_{n-2},a_{n-1},a_n} \otimes (\lambda_0)^2$$

and we see that all these cycles are also boundaries.

If $\sum_{i=1}^n a_i(2^n - 2^{n-i}) = 4k$ and a_{n-1} is odd with $a_n \geq 2$, then $x_{a_1,\ldots,a_{n-2},a_{n-1},a_n} \otimes (\lambda_0)^2$ is the boundary of

$$\begin{cases} \delta(x_{a_1,\ldots,a_{n-2},a_{n-1}-1,a_n+1} \otimes \lambda_0 & \\ \quad +x_{a_1+2,a_2,\ldots,a_{n-2},a_{n-1},a_n-1} \otimes \lambda_0) & a_{n-1} \text{ odd, all other } a_i \text{ even} \\ \delta(x_{a_1,\ldots,a_{n-2},a_{n-1}-1,a_n+1} \otimes \lambda_0 & \\ \quad +x_{a_1,\ldots,a_{j-1},a_j-1,a_{j+1}+2,a_{j+2},\ldots,a_{n-2},a_{n-1},a_n+1} \otimes \lambda_0) & a_j,\ a_{n-1} \text{ odd, all other } a_i \text{ even} \\ \delta(x_{a_1,\ldots,a_{n-2},a_{n-1}-1,a_n+1} \otimes \lambda_0) & a_{n-1} \text{ odd, at least 2 other } a_i \text{ odd} \end{cases}$$

and we see that all these cycles are also boundaries. This now covers all cases for our class (a) cycles.

Turning to our class (b) cycles, $x_{a_1,\ldots,a_n} \otimes (\lambda_0)^2$ with a_{n-1} even and $a_n = 0$ we see that none of these are boundaries. Indeed, they would need to be in the image of some $x_{b_1,\ldots,b_n} \otimes (\lambda_0)$ with $x_{b_1,\ldots,b_n} \cdot Sq^1 = x_{a_1,\ldots,a_n}$. However, Sq^1 changes the parity of each of the last two indices and is only nonzero if b_{n-1} is even, but then its image a_{n-1} must be odd, a contradiction.

Next, all class (c) cycles $x_{a_1,\ldots,a_n} \otimes (\lambda_0)^2$ with $\sum_{i=1}^n a_i(2^n - 2^{n-i}) = 4k+1$ and a_n, a_{n-1}, and some other a_i for $1 \leq i < n-1$ odd are boundaries hit by

$$\delta(x_{a_1,\ldots,a_{n-2},a_{n-1}-1,a_n+1} \otimes \lambda_0) = x_{a_1,\ldots,a_{n-2},a_{n-1},a_n} \otimes (\lambda_0)^2.$$

Thus, all class (c) cycles are boundaries as well.

Finally, for class (d), if $\sum_{i=1}^n a_i(2^n - 2^{n-i}) = 2k+1$

$$\delta(x_{2a_1,\ldots,2a_{n-2},2a_{n-1}+1,2a_n-1} \otimes \lambda_0) = x_{a_1,\ldots,a_n} \otimes \lambda_{2k-1}\lambda_0.$$

So these too are all boundaries.

This leaves us with only those classes of type (a) corresponding to $x_{a_1,\ldots,a_{n-2},a_{n-1},a_n} \otimes (\lambda_0)^2$ with a_{n-1}, a_n odd, a_i even for $1 \leq i \leq n-2$ and all cycles from class (b) with $x_{a_1,\ldots,a_n} \otimes (\lambda_0)^2$ with a_{n-1} even and $a_n = 0$ such that $\sum_{i=1}^n a_i(2^n - 2^{n-i}) = 4k$ in $H^2(T(M))$. These correspond to all integer solutions respectively of the equations:

$$(2^{n-1} - 2^{n-2})b_1 + \cdots + (2^{n-1} - 2)b_{n-2} + (2^{n-1} - 1)b_{n-1} + (2^n - 1)c_n + (2^n - 1) = k.$$

and

$$(2^{n-2} - 2^{n-3})a_1 + \cdots + (2^{n-2} - 1)a_{n-2} + (2^{n-1} - 1)b_{n-1} = k$$

with at least one a_i for $1 \leq i < n-2$ odd. □

While these calculations immediately give towers in the width filtration spectral sequence, we do not perform the analysis to determine which survive to the E_2 of the CWSS, instead citing agreement with Wellington's towers in the E_2 of the CWSS. His argument is substantially more involved than what was required above, so we would like to find a self-contained argument at some point.

It would be interesting to investigate the elements of homotopy which correspond to these towers, which must of course all support or receive differentials. Would the resulting finite towers at E_∞ be exceptional in any way?

CHAPTER V

IMAGE OF J

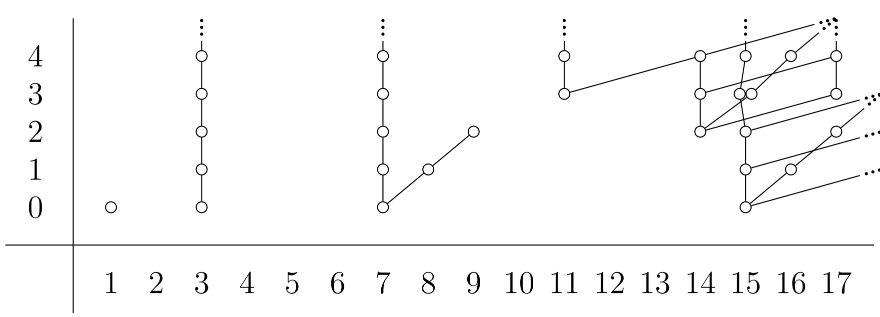
Ext calculations for BO

We start with a fun observation. The Bousfield result [5] equating the E_2 of the unstable Adams spectral sequence with an Ext in unstable modules can be applied to $\mathrm{Ext}^{s,t}_{\mathcal{UA}}(H^*(BO), \mathbb{F}_2)$, as of course $H^*(BO)$ is polynomial on the Stiefel-Whitney classes. We get

$$\mathrm{Ext}^{s,t}_{\mathcal{UA}}(H^*(BO), \mathbb{F}_2) \cong \mathrm{Ext}^{s,t-1}_{\mathcal{U}}(\Sigma^{-1}\mathrm{Ind}\ H^*(BO), \mathbb{F}_2).$$

Up to decomposables, $Sq^i(w_j) = \binom{j-1}{i} w_{j+i}$, where w_j is the jth Stiefel Whitney class [17, 26, 28]. Preliminary calculations for $\mathrm{Ext}_{\mathcal{U}}$ lead to the observation that its Ext chart looked like $\mathrm{Ext}_{\mathcal{U}}(\Sigma^{-1}D_1^o)$, but shifted to the right. Recall that D_1^o is the cohomology of $\mathbb{R}P^\infty$, modulo squares. This then lead us to the following isomorphism between familiar modules, which we found surprising.

Calculations of $\mathrm{Ext}_{\mathcal{U}}(BO, \mathbb{F}_2)$



Proposition 5.1.1. $\mathrm{Ext}^{s,t}_{\mathcal{U}}(\Sigma^{-1}D_1^o) \cong \mathrm{Ext}^{s,t+1}_{\mathcal{U}}(\Sigma^{-1}\mathrm{Ind}\ H^*(BO))$

This hints that the lowest degree in the width filtration yields the image of J, an idea that we expand on more later in this section. Before proving this proposition, we recall notation from [25]. Define $\Phi : \mathcal{U} \to \mathcal{U}$ on $M \in \mathcal{U}$ at the prime 2 to by

$$(\Phi M)^n \cong M^{n/2} \quad Sq^i \Phi x = \Phi Sq^{i/2} x$$

where $M^{n/2}$ is trivial if $n/2$ is not an integer. Let $\lambda_M : \Phi M \to M$ by $\Phi x \mapsto Sq_0 x$. Define the functor Ω and its first (and only nontrivial) left derived function Ω_1 by

$$\ker \lambda_M = \Sigma \Omega_1 M \quad \text{and} \quad \mathrm{coker} \lambda_M = \Sigma \Omega M.$$

Proposition 5.1.2. *If $M \in \mathcal{U}$ and λ_M is injective, then* $\mathrm{Ext}^{s,t}_{\mathcal{U}}(\Omega M, \mathbb{F}_2) \cong \mathrm{Ext}^{s,t+1}(M, \mathbb{F}_2)$.

Proof. We know that

$$0 \to \ker(\lambda_M) \to \Phi(M) \to M \to \mathrm{coker}(\lambda_M) \to 0$$

is an exact sequence where $\ker(\lambda_M) \cong \Sigma\Omega_1 M$ and $\mathrm{coker}(\lambda_M) \cong \Sigma\Omega M$. Since λ_M is injective, $\ker(\lambda_M) = 0$, so $\Omega_i M = 0$ for all $i > 0$. Let $P_\bullet \to M$ be a free resolution of M. Then $\Omega P_\bullet \to \Omega M$ is a free resolution of ΩM. Thus, $\mathrm{Ext}_{\mathcal{U}}(\Omega M, \mathbb{F}_2) \cong H_s(\mathrm{Hom}^t(\Omega P_\bullet, \mathbb{F}_2))$. Since Ω is left adjoint to Σ, we get

$$H_s(\mathrm{Hom}^t(\Omega P_\bullet, \mathbb{F}_2)) \cong H_s(\mathrm{Hom}^t(P_\bullet, \Sigma\mathbb{F}_2)) \cong H_s(\mathrm{Hom}^{t+1}(P_\bullet, \mathbb{F}_2)) \cong \mathrm{Ext}^{s,t+1}_{\mathcal{U}}(M, \mathbb{F}_2)$$

□

Proof of Proposition 5.1.1. The result will follow from Proposition 5.1.2 if we can show that λ_M is injective for $M = \Sigma^{-1}\text{Ind } H^*(BO)$ and $\text{coker}\lambda_M \cong \Sigma\Sigma^{-1}D_1^o$.

For any $i > 0$, $w_i \in M$, w_i has degree $i-1$ and

$$\lambda_M(w_i) = Sq_0(w_i) = Sq^{i-1}(w_i) = \binom{i-1}{i-1} w_{i+i-1} = w_{2i-1}$$

so λ_M is clearly injective. We can also see that the image of λ_M will be w_{2i-1} for $i \geq 1$. Then $\text{coker}\lambda_M$ will be $w_{2i} + im(\lambda_M)$ in degrees $2i-1$. The map $w_{2i} + im(\lambda_M) \mapsto d_1^{2i-1}$ then defines an isomorphism with $D_1^o \cong \Sigma\Sigma^{-1}D_1^o$, so we conclude that $\Sigma\Sigma^{-1}D_1^o \cong \text{coker}\lambda_M$. Thus, applying Proposition 5.1.2, $\text{Ext}^{s,t}(\Sigma^{-1}D_1^o) \cong \text{Ext}^{s,t+1}(\Sigma^{-1}\text{Ind } H^*(BO))$. □

The suggestive calculation above for BO could tell us about the delooping of the image of J. But our calculations in this paper are based on the cohomology of QS^0 and not its delooping. For QS^0 itself, the first calculations above are consistent with the following.

Theorem 5.1.3. *The map induced by the image of J on cohomology induces a splitting of the Nakaoka module $\mathfrak{N}$.*

Corollary 5.1.4. *The algebraic map induced by the image of J on* Ext *induces a splitting of* $\text{Ext}(\mathfrak{N})$.

Naively one would conjecture that J quotient of $\mathfrak{N}$ which sends $\gamma_{1[1]}^{2k+1} \mapsto \beta_{2k+1}$, but this is not a map of unstable modules. To make our calculations of this map and the induced map on Ext, we begin by studying the map on homology.

J_* on homology

We use the basis for the homology of SO presented by Hatcher [14],

$$H_*(SO, \mathbb{Z}) \cong E[e^1, \ldots, e^n, \ldots]$$

and the basis for $H_*(Q_1S^0)$ in Theorem 3.44 of Madsen-Milgram [20],

$$H_*(Q_1S^0, \mathbb{Z}_2) \cong E(g_1, \ldots, g_i, \ldots) \otimes P(g_I)$$

where $I = (i_1, \ldots, i_m)$ and runs over all sequences of integers $0 \leq i_1 \leq \cdots \leq i_m$ and $i_1 = 0$ implies $m = 2$ and $i_2 > 0$. We can translate between this basis and the Kudo-Araki-Dyer-Lashof basis of admissible q_I.

Proposition 5.2.1 (Theorem 3.9 [21])**.** *Let J be the class of the identity map $(S^\infty, *) \to (S^\infty, *)$ in Q_1S^0, let p be any prime, and let I be any admissible sequence of length j. Define*

$$s(Q_I(J)) = Q_I(J) * (-J) * \cdots * (-J)$$

where there are p^j copies of $-J$, the class of a point in $Q_{-1}S^0$. Then

$$H_*(Q_0S^0), \mathbb{Z}_p) = \Lambda\{s(Q_I(J))\}$$

as an algebra.

In our application $p = 2$ and in our notation $J = \iota$, $-J = \bar{\iota}$ and $s(Q_I(J)) = q_I(\iota) * \bar{\iota}^{2^j}$. As discussed in the proof of Theorem C in [21], the product of these generators with ι (shifting to the 1-component) give the homology generators as described by:

$$g_i \otimes 1 \mapsto q_i(\iota) * \bar{\iota}$$
$$1 \otimes g_{0,i} \mapsto q_i^2(\iota) * \bar{\iota}^3$$
$$1 \otimes g_I \mapsto q_I(\iota) * \bar{\iota}^{2^m - 1}.$$

The homology map induced by the J homomorphism follows from Corollary 1.5 of [21], giving that $J_* : H_*(SO) \to H_*(Q_1S^0)$ is defined by $e_i \mapsto g_i \otimes 1$, which is $q_i(\iota) * \bar{\iota}$.

J^* on cohomology indecomposables

We can use this and the pairings between homology and cohomology for both SO and Q_1S^0 to understand the induced map $J^* : \Sigma^{-1}\text{Ind } H^*(Q_1S^0) \to \Sigma^{-1}\text{Ind } H^*(SO)$. For cohomology, use the basis from Hatcher [14]

$$H^*(SO, \mathbb{F}_2) \cong \bigotimes_{i \text{ odd}} \mathbb{F}_2[\beta_i]$$

where β_i is the linear dual to e^i. To work out the pairing between homology and cohomology for SO, we inductively use the Hopf algebra structure.

In general, the induced map on cohomology is given by the following.

Proposition 5.3.1. *The map on cohomology indecomposables induced by the J homomorphism is:*

$$\gamma_{1[1]}^{2k+1} \mapsto \beta_{2k+1}$$

$$\gamma_{1[2]}^{i}\gamma_{2[1]} \mapsto \beta_{2i+3}$$

and all other indecomposables map to 0.

To prove this we make some preliminary calculations in the Lemmas below.

Lemma 5.3.2. *Odd degree elements of $\mathfrak{N}$ pair trivially with any nontrivial product of two or more elements in $H_*(Q_1S^0)$.*

Proof. Let $x \in \mathfrak{N}$ be an indecomposable of odd degree and width 2^k. Any indecomposable of width 2^k in odd degree must contain at least one block of full width, thus $\psi(x) = x \otimes 1 + 1 \otimes x$. Let q_{I_1} and q_{I_2} be nontrivial elements of $H_*(Q_1S^0)$. Then

$$\langle x, q_{I_1} * q_{I_2} \rangle = \langle \psi(x), q_{I_1} \otimes q_{I_2} \rangle = 0$$

using the product and co-product structure of the homology and cohomology. □

Lemma 5.3.3.

$$\beta_{2k+1}^* = e^{2k+1} + \sum_{a+b=2k+1} e^a \wedge e^b$$

Proof. We begin with the identity $\beta_{2k+1} = (e^{2k+1})^*$, since β_{2k+1} is primitive and pairs nontrivially with only e^{2k+1}. Thus, $\langle \beta_{2k+1}, e^{2k+1} + \sum_{a+b=2k+1} e^a \wedge e^b \rangle = 1$.

We would like to show that for any other $x \in H^*(Q_1S^0)$, $\langle x, e^{2k+1} + \sum_{a+b=2k+1} e^a \wedge e^b \rangle = 0$. We begin by writing x as the sum of products of primitives

$$x = \sum_j \prod_i \beta_{2a_{i,j}+1}^{2^{b_{i,j}}}.$$

By linearity and degree arguments it suffices to consider products of the form $x = \prod_{i=1}^{n} \beta_{2a_i+1}^{2^{b_i}}$ where $\deg x = \sum_i (2a_i+1)\cdot 2^{b_i} = 2k+1$. Then $\langle x, e^a \wedge e^b \rangle = 1$ if there are an odd number of ways to split up the n terms in the product into two products of total degrees a and b, otherwise $\langle x, e^a \wedge e^b \rangle = 0$.

The number of ways to split the n terms into two groups is

$$\begin{cases} \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{i} & n \text{ is odd} \\ \sum_{i=1}^{\frac{n}{2}-1} \binom{n}{i} + \binom{n}{\frac{n}{2}} & n \text{ is even.} \end{cases}$$

In either case, this is equivalent to

$$\frac{\sum_{i=0}^{n} \binom{n}{i}}{2} - 1 = \frac{2^n}{2} - 1 = 2^{n-1} - 1,$$

which is odd for $n \geq 2$. Certainly, not all of these pairs will be distinct, but they will cancel each other out in pairs, leaving an odd number of possibilities. Thus, $\langle x, e^a \wedge e^b \rangle = 1$. We can also calculate that $\langle x, e^{2k+1} \rangle = 1$, which means that

$$\langle x, e^{2k+1} + \sum_{a+b=2k+1} e^a \wedge e^b \rangle = 1 + 1 = 0$$

and β_{2k+1} is the only term in $H^*(Q_1 S^0)$ that pairs nontrivially with $e^{2k+1} + \sum_{a+b=2k+1} e^a \wedge e^b$. $\square$

Lemma 5.3.4. $\sum_{i=0}^{m} \binom{n}{i} \equiv_2 \binom{n-1}{m}$

Proof.

$$\begin{aligned}\sum_{i=0}^{m}\binom{n}{i} &= [x^m]\frac{(1+x)^n}{1-x} \\ &\equiv_2 [x^m]\frac{(1+x)^n}{1+x} \\ &= [x^m](1+x)^{n-1} \\ &= \binom{n-1}{m}\end{aligned}$$

$\square$

Lemma 5.3.5.
$$\binom{b-1}{k-b} = [x^{k-b}]F_k(x)$$

where

$$F_k(x) = \frac{C(-x)^{-k} - (-xC(-x))^k}{\sqrt{1-4x}} \equiv_2 C(x)^{-k} + (xC(x))^k$$

and $C(x)$ *is the generating function for the Catalan numbers*

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

Lemma 5.3.6. *Fix* $k > 0$ *for* $\lceil \frac{2k+1}{3} \rceil \leq j \leq k$,

$$\binom{j-1}{2j-k-1} + \sum_{b=\lceil \frac{k+1}{2} \rceil}^{\lfloor \frac{2k-j}{2} \rfloor} \binom{b-1}{2b-k-1}\binom{j-b-1}{2k-j-2b} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{else.} \end{cases}$$

Proof. We know that $\binom{b-1}{2b-k-1} = \binom{b-1}{k-b}$. Applying Lemma 5.3.5 we get

$$\binom{b-1}{k-b} = [x^{k-b}]F_k(x)$$

and

$$\binom{j-b-1}{2k-j-2b} = [x^{2j-(2k+1)+b}]F_{3j-2k-1}(x).$$

Given the range on j, we know that $3j-2k-1 \leq k-1$. Thus $k > 3j-2j-1$, so the sum

$$\begin{aligned}\sum_b \binom{b-1}{2b-k-1}\binom{j-b-1}{2k-j-2b} &= \sum_b([x^{k-b}]F_k(x))([x^{2j-(2k+1)+b}]F_{3j-2k-1}) \\ &= [x^{2j-k-1}]F_k(x) \cdot F_{3j-2k-1}(x)\end{aligned}$$

which is the coefficient of x^{2j-k-1} in $F_k(x) \cdot F_{3j-2k-1}(x)$. Expanding out this product, we get

$$\begin{aligned}&F_k(x) \cdot F_{3j-2k-1}(x) = \\ &\quad = (C(x)^{-k} + x^k C(x)^k)(C(x)^{-3j+2k+1} + x^{3j-2k-1}C(x)^{3j-2k-1}) \\ &\quad = C(x)^{-3j+k+1} + x^k C(x)^{-3j+3k+1} + x^{3j-2k-1}C(x)^{3j-3k-1} + x^{3j-k-1}C(x)^{3j-k-1} \\ &\quad = C(x)^{-(3j-k-1)} + x^{3j-2k-1}(C(x)^{-(3k-3j+1)} + x^{3k-3j+1}C(x)^{3k-3j+1}) + (xC(x))^{3j-k-1} \\ &\quad = F_{3j-k-1} + x^{3j-2k-1}F_{3k-3j+1}.\end{aligned}$$

Then the x^{2j-k-1} coordinate is

$$[x^{2j-k-1}]F_{3j-k-1} + [x^{k-j}]F_{3k-3j+1} = \binom{j-1}{2j-k-1} + \binom{2k-2j}{k-j}.$$

Putting everything together, we have

$$\begin{aligned}\binom{j-1}{2j-k-1} + \sum_{b=\lceil \frac{k+1}{2} \rceil}^{\lfloor \frac{2k-j}{2} \rfloor} \binom{b-1}{2b-k-1}\binom{j-b-1}{2k-j-2b} &= \\ &= \binom{j-1}{2j-k-1} + \binom{j-1}{2j-k-1} + \binom{2k-2j}{k-j} \\ &= \binom{2(k-j)}{k-j},\end{aligned}$$

which is 1 if and only if $j = k$. $\square$

Now we are ready to prove Proposition 5.3.1.

Proof of Proposition 5.3.1. Since $H^*(SO)$ has indecomposables in only odd degrees, we only need to understand the map on odd degree elements in $\mathfrak{N}$ and thus only need to keep track of homology classes that are dual to odd degree indecomposables. Then, by Lemma 5.3.2, we can ignore any products of two or more classes in $H_*(Q_1S^0)$ as we work out the maps. Additionally, for each odd degree there is exactly one indecomposable in $H^*(Q_1S^0)$, namely β_{2k+1}. Then Lemma 5.3.3 tells us β_{2k+1} is a term in $(e^{2k+1})^*$ and $(e^a \wedge e^b)^*$ but not in the linear dual of any products of three or more terms. So, it suffices to look at only products of one and two e^i in $H_*(SO)$ since products of three or more are dual only to decomposables.

The homology map induced by the J homomorphism is $J_* : H_*(SO) \to H_*(Q_1S^0)$ is defined by $e_i \mapsto g_i \otimes 1 = q_i(\iota) * \bar{\iota}$ in the Kudo-Araki-Dyer-Lashof basis. It follows that

$$e^a \wedge e^b \mapsto (q_a(\iota) * \bar{\iota}) \circ (q_b(\iota) * \bar{\iota}) = \sum_{l=\frac{a-b}{2}}^{\frac{a}{2}} \binom{a-l}{a-2l} q_{b+2l-a,a-l} + \text{ products}$$

We consider all $a+b = 2k+1$. Then $(q_{2k+1-2j,j})^* \mapsto \sum (e^a \wedge e^b)^*$ where the sum, runs over all $e^a \wedge e^b$ for which $q_{2k+1-2j,j}$ was one of the terms in the image under J_*. Summing over all $a+b = 2k+1$, the coefficient for admissible $q_{2k+1-2j,j}$ will be

$$\sum_{i=0}^{2j-k-1} \binom{j}{i} + \sum_{b=\lceil \frac{k+1}{2} \rceil}^{\lfloor \frac{2k-j}{2} \rfloor} \left(\sum_{l=0}^{2b-k-1} \binom{b}{l} \right) \binom{j-b-1}{2j-(2k+1)+b},$$

which simplifies by Lemma 5.3.4 mod 2 to

$$\binom{j-1}{2j-k-1} + \sum_{b=\lceil \frac{k+1}{2} \rceil}^{\lfloor \frac{2k-j}{2} \rfloor} \binom{b-1}{2b-k-1} \binom{j-b-1}{2k-j-2b}.$$

By Lemma 5.3.6, we see that $(q_{1,k})^*$ contains an odd number of $(e^a \wedge e^b)^*$ in its image and thus β_{2k+1} is one of the terms in the image. For all other $q_{2k+1-2j,j}$ there are an even number of $(e^a \wedge e^b)^*$ in the image and thus β_{2k+1} does not appear as one of the terms in the linear dual, which leaves only decomposables. So, only $J^*((q_{1,k})^*)$ is nonzero up to decomposables.

Next, we have that for $2a + 3b = 2k+1$,

$$\langle \gamma_{1[2]}^a \gamma_{2[1]}^b, q_{1,k} \rangle = \begin{cases} 1 & a = 1, b = k \\ 0 & \text{else.} \end{cases}$$

Thus, $\gamma_{1[2]}^{k-1}\gamma_{2[1]}$ appears as a term in $(q_{1,k})^*$ and is not a term in any other $(q_{i,j})^*$. So,

$$J^*(\gamma_{1[2]}^{k-1}\gamma_{2[1]}) = J^*((q_{1,k})^*) + \sum J^*((q_{i,j})^*) = \beta_{2k+1} + 0.$$

Any other $\gamma_{1[2]}^{a}\gamma_{2[1]}^{b}$ when expressed as a sum of $(q_{i,j})^*$ will have only terms with $i > 1$ and thus J^* will contain only decomposables. □

For an example of how this works in degree 15 ($k = 7$) see Appendix B.

Theorem 5.3 on the splitting of the Nakaoka module follows from the map on cohomology given in Proposition 5.3.1.

Proof of Theorem 5.1.3. At the level of unstable modules, we have the inclusion map $\Sigma^{-1}D_1^o \to \Sigma^{-1}\mathfrak{N}$ and $J^* : \Sigma^{-1}\mathfrak{N} \to \Sigma^{-1}\mathrm{Ind}\ H^*(SO)$ whose composition induces an isomorphism between the image of the first filtration D_1^o in $\mathfrak{N}$ and the indecomposables of $H^*(SO)$. □

APPENDIX A

THE WIDTH SPECTRAL SEQUENCE

Through degree 16, there are no possible differentials in the width spectral sequence so this gives the E_2 of the CWSS

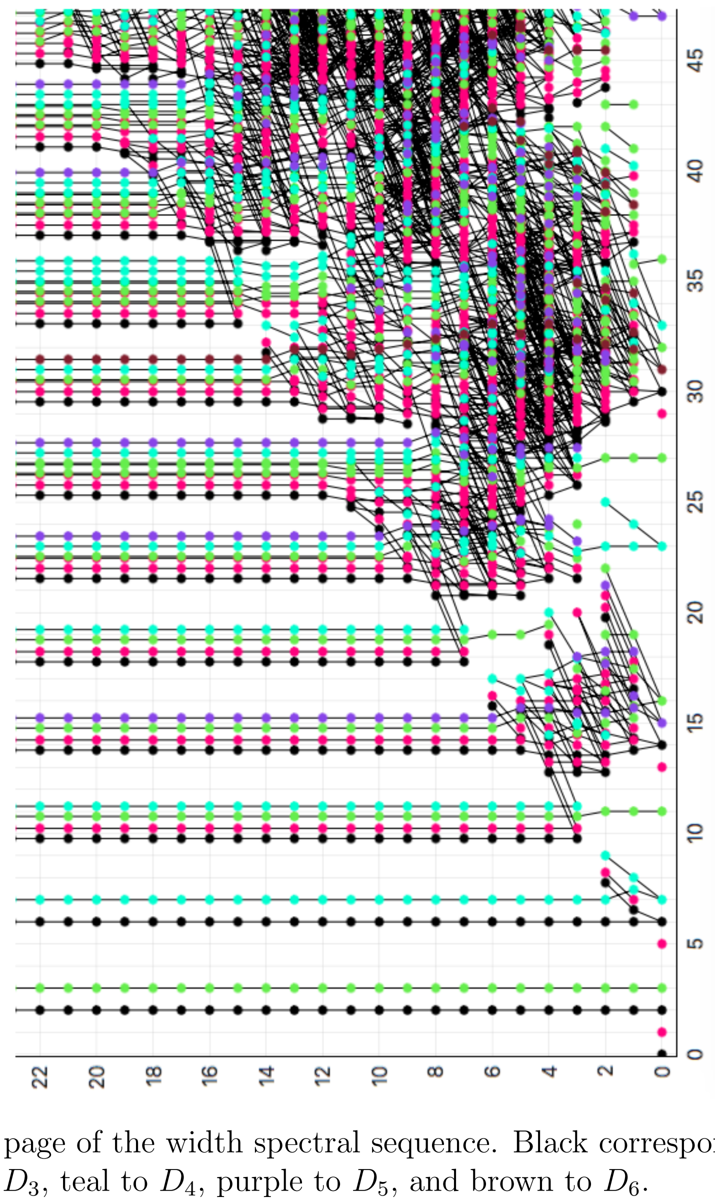


FIGURE 4. E_1 page of the width spectral sequence. Black corresponds to D_1, red to D_2, green to D_3, teal to D_4, purple to D_5, and brown to D_6.

APPENDIX B

J^* CALCULATIONS IN DEGREE 15

We will illustrate what is happening in the proof of Proposition 5.3.1 in degree 15 ($k = 7$). Using Lemma 5.3.2 to ignore nontrivial products in $H_*(Q_1S^0)$ and Lemma 5.3.3 to ignore products of three or more terms in $H_*(SO)$, the relevant homology maps are

$$J_*(e^{15}) = q_{15}$$

and for $a + b = 15$

$$J_*(e^a \wedge e^b) = \sum_{l=\lceil \frac{a-b}{2} \rceil}^{\frac{a}{2}} \binom{a-l}{a-2l} q_{b+2l-a,a-l} + \text{ products.}$$

We conclude that $J^*(q_{15}^*) = J^*(\gamma_{1[1]}^{15} = (e^{15})^* = \beta_{15}$ and $J^*((q_{15-2j,j})^*) = \sum (e^a \wedge e^b)^*$ where the sum, runs over all $e^a \wedge e^b$ for which $q_{15-2j,j}$ was one of the terms in the image under J_*. Thus, summing over the image of J_* for all $a + b = 15$ and looking at the coefficient of $q_{15-2j,j}$ mod-2 will tell us whether $q_{15-2j,j}$ is in the image of an odd or even number of $e^a \wedge e^b$. For $q_{15-2j,j}$ to be admissible, we need $5 \leq j \leq 7$. We then calculate the coefficient for $q_{5,5}$ is

$$\sum_{i=0}^{2} \binom{5}{i} + \sum_{b=4}^{4} \left(\sum_{l=0}^{2b-8} \binom{b}{l} \right) \binom{5-b-1}{-5+b},$$

which by Lemma 5.3.4 is

$$\binom{4}{2} + \sum_{b=4}^{4} \binom{b-1}{2b-8} \binom{5-b-1}{-5+b} = 0.$$

Repeating the same process for $q_{3,6}$, the coefficient is equivalent mod-2 to

$$\binom{6}{4} + \sum_{b=4}^{4} \binom{b-1}{2b-8}\binom{6-b-1}{-3+b} = 1+1 = 0.$$

And finally for $q_{1,7}$, the coefficient is equivalent mod-2 to

$$\binom{6}{6} + \sum_{b=4}^{3} \binom{b-1}{2b-8}\binom{7-b-1}{-1+b} = 1+0 = 1.$$

Thus, $q_{1,7}$ is the only element appearing an even number of times as the image of the $e^a \wedge e^b$ and thus only $J^*((q_{1,7})^*)$ is nonzero up to decomposables.

Next, we know that

$$(q_{1,7})^* = \gamma_{1[2]}^6 \gamma_{2[1]} + \gamma_{1[2]}^3 \gamma_{2[1]}^3 + \gamma_{2[1]}^5,$$

$$(q_{3,6})^* = \gamma_{1[2]}^3 \gamma_{2[1]}^3 + \gamma_{2[1]}^5,$$

and

$$(q_{5,5})^* = \gamma_{2[1]}^5$$

so

$$J^*(\gamma_{2[1]}^5) = J^*((q_{5,5}^*) = 0,$$

$$J^*(\gamma_{1[2]}^3 \gamma_{2[1]}^3) = J^*((q_{3,6})^* + (q_{5,5}^*)) = 0+0,$$

and

$$J^*(\gamma_{1[2]}^6 \gamma_{2[1]}) = J^*((q_{1,7})^* + (q_{3,6})^* + (q_{5,5}^*)) = \beta_{15}$$

all up to decompoasables.

REFERENCES CITED

[1] Adem, A., Maginnis, J., and Milgram, R. J. Symmetric invariants and cohomology of groups. *Math. Ann. 287*, 3 (1990), 391–411.

[2] Adem, A., and Milgram, R. J. *Cohomology of finite groups*, second ed., vol. 309 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2004.

[3] Barratt, M., and Priddy, S. On the homology of non-connected monoids and their associated groups. *Comment. Math. Helv. 47* (1972), 1–14.

[4] Bousfield, A., Curtis, E., Kan, D., Quillen, D., Rector, D., and Schlesinger, J. The mod-p lower central series and the adams spectral sequence. *Topology 5*, 4 (1966), 331 – 342.

[5] Bousfield, A. K. A vanishing theorem for the unstable Adams spectral sequence. *Topology 9* (1970), 337–344.

[6] Bousfield, A. K., and Curtis, E. B. A spectral sequence for the homotopy of nice spaces. *Transactions of the American Mathematical Society 151*, 2 (1970), 457–479.

[7] Chatham, H. Dickson at 2. http://math.mit.edu/ hood/dickson/.

[8] Cohen, F. R., Lada, T. J., and May, J. P. *The homology of iterated loop spaces*. Lecture Notes in Mathematics, Vol. 533. Springer-Verlag, Berlin-New York, 1976.

[9] Curtis, E., and Mahowald, M. *EHP* computations of $E_2(S^n)$. In *Computers in geometry and topology (Chicago, IL, 1986)*, vol. 114 of *Lecture Notes in Pure and Appl. Math.* Dekker, New York, 1989, pp. 105–120.

[10] Curtis, E. B. The Dyer-Lashof algebra and the Λ-algebra. *Illinois J. Math. 19* (1975), 231–246.

[11] Curtis, E. B., and Mahowald, M. The unstable Adams spectral sequence for S^3. In *Algebraic topology (Evanston, IL, 1988)*, vol. 96 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 1989, pp. 125–162.

[12] Feshbach, M. The mod 2 cohomology rings of the symmetric groups and invariants. *Topology 41*, 1 (2002), 57–84.

[13] Giusti, C., Salvatore, P., and Sinha, D. The mod-2 cohomology rings of symmetric groups. *Journal of Topology 5*, 1 (Feb 2012), 169–198.

[14] Hatcher, A. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[15] Hung, N. H. V. The action of the Steenrod squares on the modular invariants of linear groups. *Proc. Amer. Math. Soc. 113*, 4 (1991), 1097–1104.

[16] Hu'ng, N. H. V., and Peterson, F. P. Spherical classes and the Dickson algebra. *Math. Proc. Cambridge Philos. Soc. 124*, 2 (1998), 253–264.

[17] Kochman, S. O. An algebraic filtration of $H_*(MO;\mathbf{Z}_2)$. *Proc. Edinburgh Math. Soc. (2) 26*, 3 (1983), 313–317.

[18] Lannes, J., and Zarati, S. Invariants de Hopf d'ordre supérieur et suite spectrale d'Adams. *C. R. Acad. Sci. Paris Sér. I Math. 296*, 15 (1983), 695–698.

[19] Madsen, I. On the action of the Dyer-Lashof algebra in $H_*(G)$. *Pacific J. Math. 60*, 1 (1975), 235–275.

[20] Madsen, I., and Milgram, R. J. *The classifying spaces for surgery and cobordism of manifolds.* Annals of Mathematics Studies, No. 92. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1979.

[21] Milgram, R. J. The mod 2 spherical characteristic classes. *Ann. of Math. (2) 92* (1970), 238–261.

[22] Miller, H. Correction to: "The Sullivan conjecture on maps from classifying spaces" [Ann. of Math. (2) **120** (1984), no. 1, 39–87; MR0750716 (85i:55012)]. *Ann. of Math. (2) 121*, 3 (1985), 605–609.

[23] Nakaoka, M. Homology of the infinite symmetric group. *Ann. of Math. (2) 73* (1961), 229–257.

[24] Nakaoka, M. Note on cohomology algebras of symmetric groups. *J. Math. Osaka City Univ. 13* (1962), 45–55.

[25] Schwartz, L. *Unstable modules over the Steenrod algebra and Sullivan's fixed point set conjecture.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1994.

[26] Stong, R. E. Determination of $H^*(\mathrm{BO}(k,\cdots,\infty),Z_2)$ and $H^*(\mathrm{BU}(k,\cdots,\infty),Z_2)$. *Trans. Amer. Math. Soc. 107* (1963), 526–544.

[27] Wellington, R. J. The unstable Adams spectral sequence for free iterated loop spaces. *Mem. Amer. Math. Soc. 36*, 258 (1982), viii+225.

[28] Wu, W.-t. Les i-carrés dans une variété grassmannienne. *C. R. Acad. Sci. Paris 230* (1950), 918–920.