FIXED-POINT-FREE INVOLUTION WORD DIAGRAMS

by

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DISSERTATION ABSTRACT

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We define a useful diagram for studying fixed-point-free involution words. Following the example of Little, we define a specific and a general bumping algorithm on these diagrams. These algorithms serve as the basis for bijective proofs of corresponding transition equations for fixed-point-free involution Stanley symmetric functions. We also use the diagrams to show deletion and exchange properties of FPF involution words.

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CHAPTER I

INTRODUCTION

This work is a development of the theory of the transition equation for fixed-point-free involutions (See [2]). In particular, the main new contribution is a diagrammatic model for involution words, together with a description of a bumping algorithm that uses the properties of these words. For the benefit of the reader unfamiliar with these concepts, we begin with a brief overview of the corresponding theory in Type A, the theory of reduced words and the transition equation for the symmetric group (see [3]).

Let $\mathbb{P}$ be the set of (strictly) positive integers. Denote the set of permutations of $\mathbb{P}$ which fix all but finitely many elements of $\mathbb{P}$ by $S_{\mathbb{P}}$. For any permutation $\pi \in S_{\mathbb{P}}$, we can factor $\pi = s_{i_1} s_{i_2} \cdots s_{i_k}$ into a product of adjacent transpositions. The list $i_1 i_2 \cdots i_k$ is called a *word* (for π). A descent in this word is an index j so that $i_j > i_{j+1}$. We can then consider a *wire diagram* (see the definition of "line diagram" in [4]) corresponding to this word. A word (or wire diagram) is called *reduced* if no two wires cross each other more than once. The set of reduced words for π is denoted $\mathcal{R}(\pi)$.

In [5], Stanley introduced the generating functions F_π, which we will call *Stanley Symmetric Functions*, defined as

$$F_\pi = \sum_{\substack{w \in \mathcal{R}(\pi) \\ w = i_1 i_2 \cdots i_k}} \left(\sum_{\substack{t_1 \le t_2 \le \cdots \le t_k \\ i_j > i_{j+1} \implies t_j < t_{j+1}}} x_{t_1} x_{t_2} \cdots x_{t_k} \right).$$

If π has at most one descent, π is called Grassmannian and has an associated partition $\lambda(\pi)$. Denote the Schur function in the variables x_i for the partition λ by S_λ. For a Grassmannian permutation, π, we have $F_\pi = S_{\lambda(\pi)}$.

In [4], Little defines a bumping algorithm on wire diagrams for reduced wire diagrams. He then goes on to show the algorithm produces a bijection from the set of wire diagrams for π to the set of wire diagrams for all children of π in the Lascoux-Schützenberger (L-S) tree (see [3]). Iterating this algorithm yields a bijection, φ, from the set of wire diagrams for π to the set of wire diagrams for all Grassmannian descendants of π in the L-S tree. Because each application of the bumping algorithm preserves the indices of descents of the word, this gives us a bijective proof that

$$F_\pi = \sum_\sigma S_{\lambda(\sigma)},$$

where the sum is over the Grassmannian descendants of π in the L-S tree. This implies that, for any $\pi \in S_{\mathbb{P}}$, F_π has only positive coefficients when written as a linear combination of Schur functions (originally proved non-bijectively in [3]).

We now focus our attention on fixed-point-free (FPF) involutions of the set $\mathbb{P}$. See Section 2.1 for our definition of these FPF involutions. Denote the set of these

FPF involutions by $\mathsf{FPF}_{\mathbb{P}}$. The set FPF is not a subgroup of $S_{\mathbb{P}}$. However, similar to an identity element, we have a "simplest" FPF involution $\theta : \mathbb{P} \to \mathbb{P}$ defined by $\theta(i) = i - (-1)^i$. The map $\psi : S_{\mathbb{P}} \to \mathsf{FPF}_{\mathbb{P}}$ defined by $\sigma \mapsto \sigma^{-1}\theta\sigma$ is surjective. An *atom* of $\pi \in \mathsf{FPF}_{\mathbb{P}}$ is a permutation in the preimage $\mathcal{A}(\pi) := \psi^{-1}(\pi)$. We define the FPF involution Stanley symmetric function indexed by $\pi \in \mathsf{FPF}_{\mathbb{P}}$ to be

$$\hat{F}_\pi = \sum_{\sigma \in \mathcal{A}(\pi)} F_\sigma.$$

In [1], the authors give an analogue of Little's bumping algorithm for FPF involutions. This FPF bumping algorithm is described symbolically using reduced (and nearly reduced) words for atoms. This algorithm, just like the one we define in Section 2.6, is a bijection from the reduced words for atoms of π to the reduced words for atoms of children of π in an FPF version of the L-S tree. It is therefore a bijective proof that

$$\hat{F}_\pi = \sum_{\sigma} \hat{F}_\sigma,$$

where the sum is over children of π in the FPF L-S tree.

The next steps are done by the same authors in [2]. Here, instead of Schur functions, we use Schur P functions, P_λ, indexed by strict partitions λ. Similar to before, if $\pi \in \mathsf{FPF}_{\mathbb{P}}$ is FPF-Grassmannian (see [2] for definition), then there is a strict partition $\nu(\pi)$ so that $\hat{F}_\pi = P_{\nu(\pi)}$. We may then iterate the FPF bumping algorithm to produce a bijection from the reduced words for atoms of π to the reduced words for atoms of FPF-Grassmannian descendants of π in the FPF L-S tree. This provides us with a bijective proof that

$$\hat{F}_\pi = \sum_{\sigma} P_{\nu(\sigma)},$$

where the sum is over the FPF-Grassmannian descendants of π in the L-S tree. This implies for any $\pi \in \mathsf{FPF}_{\mathbb{P}}$, $\hat{F}_\pi$ has only positive coefficients when written as a linear combination of Schur P functions.

In this paper, we define FPF involution word diagrams (see Section 2.1), which are an analogue of the wire diagram used by Little. Using these diagrams, we define a new FPF bumping map (see Definition 2.6.4) in a visual way, which more closely resembles Little's description of his algorithm. Along the way, several useful properties of these diagrams are shown, making it (hopefully) clear that FPF involution word diagrams are a useful tool for studying FPF involutions. We include a clear visual description of what it means for a diagram to be reduced and we classify the types of defects that may arise in non-reduced diagrams. We also use the diagrams to prove deletion and exchange properties for FPF involution words.

Conjecturally, the bumping algorithm in this paper, when thought of as a map from and to reduced involution words, is the same as the one defined in [1].

The relationship of the bumping maps to their corresponding L-S trees depends on fixing a convention to start bumping at a particular crossing. However, without such a convention, we still obtain a useful transition equation. In [4], we have a generalized bumping algorithm, which provides a method for proving the bijectivity of the specific (convention-following) bumping map. This generalized bumping algorithm also provides us with the general transition equation

$$\sum_{\sigma \in \Psi^+(\nu,r)} F_\sigma = \sum_{\sigma \in \Psi^-(\nu,r)} F_\sigma.$$

In Section 2.5, we present a general bumping algorithm, which provides a bijective proof of the FPF-version of the general transition equation

$$\sum_{\sigma\in\hat{\Psi}^{+}(\nu,b)} \hat{F}_{\sigma} = \sum_{\sigma\in\hat{\Psi}^{-}(\nu,a)} \hat{F}_{\sigma}.$$

Similar to [4], we show that if we follow the convention of starting at the lexicographically largest crossing, one of our index sets is a singleton: $\hat{\Psi}^{+}(\nu, r) = \{\pi\}$ (See Proposition 2.6.5 for a more precise statement).

CHAPTER II

FIXED-POINT-FREE INVOLUTIONS

2.1. Definitions

Definition 2.1.1. *Let $\theta : \mathbb{P} \to \mathbb{P}$ be the function defined by*

$$i \mapsto i - (-1)^i = \begin{cases} i+1 & i \text{ odd} \\ i-1 & i \text{ even} \end{cases}.$$

Definition 2.1.2. *A fixed point free (FPF) involution is an element $\pi \in S_{\mathbb{P}}$ such that*

- *π^2 is the identity*
- *for all $i \in \mathbb{P}$, $\pi(i) \neq i$*
- *for all but finitely many $i \in \mathbb{P}$, $\pi(i) = \theta(i)$.*

The set of all FPF-involutions will be denoted $\mathsf{FPF}_{\mathbb{P}}$.

Definition 2.1.3. *Let $\pi \in \mathsf{FPF}_{\mathbb{P}} \setminus \{\theta\}$. The <u>size</u> of π, denoted $\mathsf{size}(\pi)$, is the largest value $m \in \mathbb{P}$ such that $\pi(m) \neq \theta(m)$. We also have $\mathsf{size}(\theta) = 0$.*

Definition 2.1.4. *For $i \in \mathbb{P}$, let $s_i : \mathbb{P} \to \mathbb{P}$ be the i^{th} transposition, i.e.*

$$s_i(j) = \begin{cases} i+1 & j = i \\ i & j = i+1 \\ j & \text{otherwise} \end{cases}$$

Definition 2.1.5. *Let $\pi : \mathbb{P} \to \mathbb{P}$ be any FPF involution. The the list $w = w_1 w_2 \cdots w_k$ is an FPF involution word for π if $\pi = s_{w_k} \cdots s_{w_1} \theta s_{w_1} \cdots s_{w_k}$. The length of w, denoted $\hat{\ell}_{\mathsf{FPF}}(w)$, is k.*

Definition 2.1.6. *An FPF involution word $w_1 w_2 \cdots w_k$ for π is called reduced if there does not exist another FPF involution word for π which has length smaller than k.*

Definition 2.1.7. *Let $\pi \in \mathsf{FPF}_{\mathbb{P}}$. The length of π, denoted $\hat{\ell}_{\mathsf{FPF}}(\pi)$, is the length of a reduced word for π.*

FPF involutions will be written in one-line notation, i.e. for each $\pi \in \mathsf{FPF}_{\mathbb{P}}$, $\pi = \pi(1)\pi(2)\ldots\pi(n)$, where $n \geq \mathsf{size}(\pi)$.

The following definition is based on a suggestion from Zachary Hamaker.

Definition 2.1.8. *Let π be an FPF involution. Let $w = w_1 \cdots w_k$ be an FPF involution word for π. An involution word diagram for π corresponding to w is a diagram (see example below) which has a rectangular grid of q rows ($q \geq k$) and countably infinitely many columns (indexed by $\mathbb{P}$), where:*

- *Rows $1 \leq \rho_1 < \ldots < \rho_k \leq q$ are chosen to contain a crossing.*
- *For each $i \in \{1, \ldots, k\}$, a cross $\left(\times \right)$ is in column w_i of row ρ_i.*
- *For each pair of consecutive grid positions in the same row, if neither contain a cross, a vertical line segment is drawn between them. A vertical line segment is also drawn on the left side of any position in column 1 which does not have a crossing.*
- *Above every odd numbered column, there is an arc connecting the left side of the top of the column to the right side.*

Remark. *The locations of the crosses are all in the first* $\mathsf{size}(\pi) - 1$ *columns. Each diagram ends with an ellipsis (normally omitted) signifying infinitely many columns to the right, none of which contain any crosses.*

Remark. *An FPF involution word diagram* D *is equivalent to the set of locations of the crosses*

$$D = \{(\rho_i, \kappa_i) \mid 1 \leq i \leq k\}$$

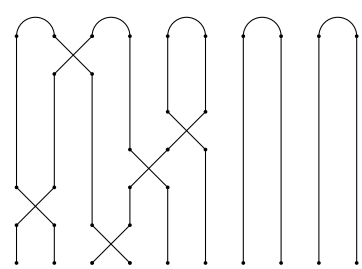
where for each i, ρ_i *is the row number and* κ_i *is the column number.*

Definition 2.1.9. *Let* D *be an involution word diagram. The FPF involution which* D *is for is denoted* $\pi(D)$*. We say* D *is reduced if there does not exists a smaller diagram for* $\pi(D)$*. In other words,* D *is reduced if there is not an involution word diagram* E *so that* $|E| < |D|$ *and* $\pi(D) = \pi(E)$*.*

Note: The cardinality $|D|$ is the length of the corresponding word w. Thus $\hat{\ell}_{\mathsf{FPF}}(\pi) = \min\{|D| : \pi(D) = \pi\}$.

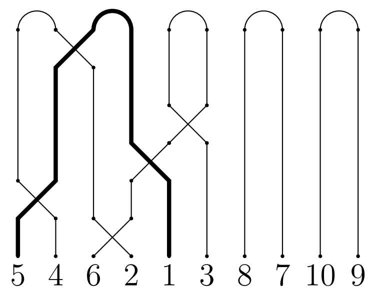
Example 2.1.10. Let $\pi = 5\,4\,6\,2\,1\,3 \in \mathsf{FPF}_{\mathbb{P}}$. Then $w = 2\,5\,4\,1\,3$ is an FPF involution word for π. The FPF involution word diagram for π corresponding to w is shown below.

$D = \{(1,2), (3,5), (4,4), (5,1), (6,3)\}$ =



$\cdots$

Note: As we will see in the next proposition, the FPF involution π can be recovered from any of its FPF involution word diagrams by following the wires. For $k \in \mathbb{P}$, wire position k is the location of the wire which is left of column k. In the following figure, the wire position k, specifically at the bottom of the diagram, is labeled as $\pi(k)$. In this example, we can see that $\pi(1) = 5$ (or, equivalently, $\pi(5) = 1$) by tracing the bold wire below, which connects the first and fifth wire positions at the bottom of the diagram. This wire will be denoted either W_1 or W_5. In general, we have a wire $W_i = W_j$ for every pair i, j such that $\pi(i) = j$.



Proposition 2.1.11. *Let $\pi \in \mathsf{FPF}_{\mathbb{P}}$. Let $w = w_1 w_2 \ldots w_k$ be an FPF involution word for π. Let D be an FPF involution word diagram corresponding to w. Then for $k \in \mathbb{P}$, the wire which is in wire position k at the bottom of D can be traced up, over the arc at the top, and back down. When reaching the bottom of D again, we will be at wire position $\pi(k)$.*

Proof. Let q be the number of rows in D. We have $D = \{(\rho_i, w_i) \mid 1 \leq i \leq k\}$. For $i \in \{0, 1, \ldots, q\}$, let $D_i = \{(\rho, \kappa) \in D \mid \rho \leq i\}$. The involution word diagram $D_0 = D_{\rho_1 - 1}$ has no crossings. Thus for each $\ell \in \mathbb{P}$, the wire in wire position ℓ can be traced up to the top of D_0, then around the arc and back down to wire position $\theta(\ell)$.

Let $\rho_0 = 0$. We induct on $i \in \{0, 1, \ldots, k\}$ to show the diagram D_{ρ_i} has the property that for any $\ell \in \mathbb{P}$, the wire in wire position ℓ can be traced up to the top of D_{ρ_i}, then around the arc and back down to wire position $(s_{w_i} \ldots s_{w_1} \theta s_{w_1} \ldots s_{w_i})(\ell)$. The case when $i = 0$ is done.

We now fix $i \in \{0, 1, \ldots, k-1\}$ and assume that D_{ρ_i} has the required property. Note that $D_{\rho_{i+1}-1} = D_{\rho_i}$. For any wire position ℓ at the bottom of $D_{\rho_{i+1}}$, we trace wire W_ℓ straight up to wire position ℓ at the bottom of row ρ_{i+1}. Then the crossing in row ρ_{i+1} will take us to wire position $s_{w_{i+1}}(\ell)$ at the top of row ρ_{i+1}. Note that if wire W_ℓ is not involved in this crossing, then $\ell \notin \{w_{i+1}, w_{i+1}+1\}$, which means $s_{w_{i+1}}(\ell) = \ell$ From here, we use our induction assumption. Because the portion of $D_{\rho_{i+1}}$ above row ρ_{i+1} is the same as in D_{ρ_i}, wire W_ℓ can continue to be traced over the top of the diagram and back down to wire position $(s_{w_i} \ldots s_{w_1} \theta s_{w_1} \ldots s_{w_i})(s_{w_{i+1}}(\ell))$ at the top of row ρ_{i+1}. Then the crossing in row ρ_{i+1} will apply a final transposition $s_{w_{i+1}}$, to bring us to wire position $(s_{w_{i+1}} s_{w_i} \ldots s_{w_1} \theta s_{w_1} \ldots s_{w_i} s_{w_{i+1}})(\ell)$, which we stay in until we reach the bottom of $D_{\rho_{i+1}}$, as desired.

Our induction says, in particular that, when $i = k$, the wire W_ℓ in D can be traced from wire position ℓ at the bottom all the way around the top and back down to wire position $(s_{w_k} \ldots s_{w_1} \theta s_{w_1} \ldots s_{w_k})(\ell) = \pi(\ell)$. □

2.2. Top and Bottom Labels

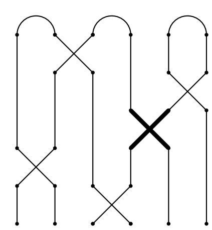
Fix $\pi \in \mathsf{FPF}_{\mathbb{P}}$ and let D be an FPF involution word diagram for π. Let q be the number of rows of D. It will be useful to label grid locations (whether or not they are crossings in D) in a manner which indicates which wires are involved. We do this in two ways.

For each position $T = (\rho, \kappa) \in \mathbb{P} \times \mathbb{P}$ with $\rho \leq q$, we trace the wires coming out of the bottom left and bottom right of the position T until they reach the bottom of the diagram and note those wire positions. If the wire meeting the bottom left of T reaches wire position i and the wire meeting the bottom right of T reaches wire position j, then we say T has <u>bottom label</u> (i, j). Note that if D is reduced and $T \in D$, then $i < j$ (see Corollary 2.2.10).

The top label of T is defined similarly, except that when we trace the wires coming out of the top left and top right of the position, the wires will arc at the top before descending to their final wire position at the bottom of the diagram. The location of the arc is irrelevant for these labels. If the wire meeting the top left of T reaches wire position k and the wire meeting the top of T reaches wire position ℓ, then we say T has <u>top label</u> (k, ℓ). Note that if $T \in D$, we have $k = \pi(j)$, $\ell = \pi(i)$, and furthermore, if the diagram is reduced, we also have $k < \ell$ (see Corollary 2.2.10).

A position $X \in \mathbb{P} \times \mathbb{P}$ is said to have label (a, b) (in D) if (a, b) is either a top or bottom label of X (in D). To avoid ambiguity, we will sometimes specify the involution word diagram by calling (a, b) a D-label.

Example 2.2.1. The bold crossing in the diagram below has bottom label $(3, 5)$ and top label $(1, 6)$.



Definition 2.2.2. *Given an FPF involution word diagram D and a particular crossing $X \in D$, a new involution word diagram <u>$D \setminus \{X\}$</u> is obtained by deleting the crossing X.*

Definition 2.2.3. *Let D be an FPF involution word diagram. Let $\rho \in \mathbb{P}$ be any row in which D has no crossings, i.e. for all $c \in \mathbb{P}$, $(\rho, c) \notin D$. Let $\kappa \in \mathbb{P}$ be any column* *and let* $X = (\rho, \kappa)$*. Then a new involution word diagram* $\underline{D \cup \{X\}}$ *is obtained by inserting a crossing at the given location.*

Note that applying deletion or insertion to a reduced diagram may produce a diagram which is not reduced.

Let $\pi \in \mathsf{FPF}_{\mathbb{P}}$ and let D be an involution word diagram for π. Let $X \in D$. Let (a, b) be the top label and (c, d) be the bottom label of X. Then $\pi(a) = d$ and $\pi(b) = c$. Suppose that the crossing X happens in column i in D. Then we can factor $\pi = \lambda s_i \eta\theta\eta^{-1} s_i \lambda^{-1}$. In this factorization, the central

$$\eta\theta\eta^{-1} = \pi\big(\{(\rho, \kappa) \in D \mid \rho < \text{the row that } X \text{ is in}\}\big)$$

is the permutation for the truncated diagram consisting of just the crossing which are above the crossing X. We conjugate by s_i to account for the crossing X and then by λ to account for the crossings below X. Note that since (c, d) is the bottom label of X, $\lambda(i) = c$ and $\lambda(i + 1) = d$. Let $E = D \setminus \{X\}$. We now want to look at the new fixed-point-free involution $\sigma = \pi(E) = \lambda\eta\theta\eta^{-1}\lambda^{-1}$.

First, we can look at the case where $d = b$. Then X involves only one wire crossing itself in D. In other words, $(\eta\theta\eta^{-1})(i) = i + 1$. We can see in Figures 1 and 2 that uncrossing these wires does not change the permutation.

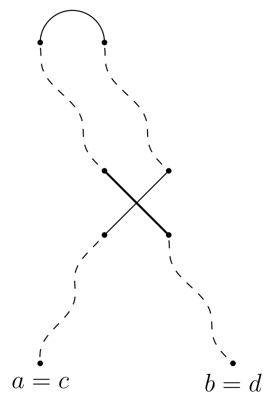


FIGURE 1.

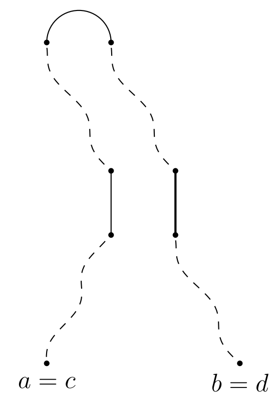


FIGURE 2.

We can confirm this algebraically by checking a couple of things. Firstly,

$$\begin{aligned}\sigma(a) &= (\lambda\eta\theta\eta^{-1}\lambda^{-1})(a) \\ &= (\lambda\eta\theta\eta^{-1})(i) \\ &= \lambda(i+1) \\ &= b \\ &= \pi(a)\end{aligned}$$

Secondly, let $x \in \mathbb{P} \setminus \{a, b\}$. We know that

$$\lambda^{-1}(x) \notin \left\{\lambda^{-1}(c), \lambda^{-1}(d)\right\} = \{i, i+1\} \quad \text{and} \tag{2.2.1}$$

$$(\eta\theta\eta^{-1}s_i\lambda^{-1})(x) \notin \left\{(\eta\theta\eta^{-1}s_i\lambda^{-1})(a), (\eta\theta\eta^{-1}s_i\lambda^{-1})(b)\right\} = \left\{(\lambda^{-1}\pi)(a), (\lambda^{-1}\pi)(b)\right\} = \{i, i+1\}. \tag{2.2.2}$$

Also, by (2.2.1), we know $(s_i\lambda^{-1})(x) = s_i(\lambda^{-1}(x)) = \lambda^{-1}(x)$, and by (2.2.2), we know

$$(s_i\eta\theta\eta^{-1}\lambda^{-1})(x) = s_i(\eta\theta\eta^{-1}\lambda^{-1}(x)) = (\eta\theta\eta^{-1}\lambda^{-1})(x).$$

Therefore,

$$
\begin{aligned}
\sigma(x) &= (\lambda\eta\theta\eta^{-1}\lambda^{-1})(x) \\
&= \lambda\big((\eta\theta\eta^{-1}\lambda^{-1})(x)\big) \\
&= \lambda\big((s_i\eta\theta\eta^{-1}\lambda^{-1})(x)\big) \\
&= (\lambda s_i\eta\theta\eta^{-1})\big(\lambda^{-1}(x)\big) \\
&= (\lambda s_i\eta\theta\eta^{-1})\big(s_i\lambda^{-1}(x)\big) \\
&= (\lambda s_i\eta\theta\eta^{-1}s_i\lambda^{-1})(x) \\
&= \pi(x).
\end{aligned}
$$

In this case, we can conclude that $\sigma = \pi$.

Now we will look at the case where $d \neq b$. This means that X involves two distinct wires in D crossing each other. In other words,

$$(\eta\theta\eta^{-1})(i) \neq i+1. \tag{2.2.3}$$

Since (a, b) is the top label of X in D, $(\lambda s_i\eta\theta\eta^{-1})(i) = a$ and $(\lambda s_i\eta\theta\eta^{-1})(i+1) = b$. Here, we want to show $\sigma = t_{a,b}\pi(D)t_{a,b}$. Let $j = (\eta\theta\eta^{-1})(i)$ and $k = (\eta\theta\eta^{-1})(i+1)$. By (2.2.3), we know $j, k \notin \{i, i+1\}$.

We can now say

$$
\begin{aligned}
\sigma(c) &= (\lambda\eta\theta\eta^{-1}\lambda^{-1})(c) \\
&= (\lambda\eta\theta\eta^{-1})(i) \\
&= \lambda(j) \\
&= \lambda(s_i j) && \text{(because } j \notin \{i, i+1\}\text{)} \\
&= (\lambda s_i \eta\theta\eta^{-1})(i) \\
&= (\lambda s_i \eta\theta\eta^{-1} s_i)(i+1) \\
&= (\lambda s_i \eta\theta\eta^{-1} s_i \lambda^{-1})(d) \\
&= \pi(d) \\
&= a.
\end{aligned}
$$

Similarly, $\sigma(d) = b$. For $x \in \mathbb{P} \setminus \{a, b, c, d\}$, we may again conclude (2.2.1) and (2.2.2), which implies $\sigma(x) = \pi(x)$.

All of this can be summed up into the following lemma.

Lemma 2.2.4. *Let D be an involution word diagram and let $X \in D$ with label (a, b). Then $\pi(D \setminus \{X\}) = t_{a,b}\pi(D)t_{a,b}$.*

Proof. For $\pi \in \mathsf{FPF}_{\mathbb{P}}$ with $\pi(a) = b$, we have $t_{a,b}\pi t_{a,b} = t_{\pi(a),\pi(b)}\pi t_{\pi(a),\pi(b)}(x) = \pi$. And for $\pi \in \mathsf{FPF}_{\mathbb{P}}$ with $\pi(a) \neq b$, we have

$$t_{a,b}\pi t_{a,b}(x) = t_{\pi(a),\pi(b)}\pi t_{\pi(a),\pi(b)}(x) = \begin{cases} \pi(b) & x = a \\ \pi(a) & x = b \\ a & x = \pi(b) \\ b & x = \pi(a) \\ \pi(x) & \text{else.} \end{cases}$$

In each of these cases, we check above that $\sigma = \pi(E)$ is the same as $t_{a,b}\pi t_{a,b}$. $\square$

Corollary 2.2.5. *Let D be an involution word diagram and let X be a position on some row of D which has no crossing. Let (a, b) be either the top or bottom label of X. Then $\pi(D \cup \{X\}) = t_{a,b}\pi(D)t_{a,b}$.*

Proof. Let $\pi = \pi(D \cup \{X\})$. Let $\sigma = \pi(D)$. Let (c, d) be the bottom label of X in $D \cup \{X\}$. Since $D = (D \cup \{X\}) \setminus \{X\}$, Lemma 2.2.4 tells us that $\sigma = t_{c,d}\pi t_{c,d}$. If (a, b) is the bottom label of X in D, then $(a, b) = (c, d)$ and we are done. If (a, b) is the top label, we have two cases.

First, if $\sigma(a) = b$, then as we saw before, the bottom label of X in D is also (a, b).

Second, if $\sigma(a) \neq b$, then the bottom label of X in D is $(\sigma(b), \sigma(a))$. Thus the bottom label of X in $D \cup \{X\}$ is also $(\sigma(b), \sigma(a))$. Now $t_{c,d}\pi t_{c,d} = t_{\sigma(b),\sigma(a)}\pi t_{\sigma(b),\sigma(a)} = t_{a,b}\pi t_{a,b}$. $\square$

Lemma 2.2.6. *Let $\pi \in \mathsf{FPF}_{\mathbb{P}}$. Let D be an involution word diagram for π. Let $a, b \in \mathbb{P}$ and assume that wires W_a and W_b cross at least three times in D. Then D is not reduced.*

Proof. We start by noting that if $\pi(a) = b$, then any of the three crossings may be deleted to yield an involution word diagram for π with fewer crossings. Thus D is not reduced. Now assume $\pi(a) \neq b$. Any crossing of wires W_a and W_b must have one of the following 8 bottom labels, which we divide into two categories:

1) (a, b), (b, a), $(\pi(a), \pi(b))$, $(\pi(b), \pi(a))$

2) $(a, \pi(b))$, $(b, \pi(a))$, $(\pi(a), b)$, $(\pi(b), a)$

Note that the permutation resulting from deleting a crossing depends only on the category of the bottom label. This means that if there are 3 crossings of W_a and W_b, 2 of them must have bottom labels from the same category. Let X and Y be these two crossings with X in the smaller number row. Let (i, j) be the bottom label of X and (k, ℓ) be the bottom label of Y in D. Note that the bottom label of Y (in any involution word diagram) only depends on the crossings below Y, so the bottom label of Y in $D \setminus \{X\}$ is also (k, ℓ). Let $E = D \setminus \{X, Y\}$. Then $\pi(E) = t_{k,\ell} t_{i,j} \pi t_{i,j} t_{k,\ell}$. The goal here is to say that $\pi(E) = \pi$. We can do this by checking the 32 cases, one for each combination of labels for X and Y. Half of these are completely trivial because, for example, $t_{a,b} t_{b,a}$ is the identity. We will look at one of the other 16 cases:

Assume $(i, j) = (a, b)$ and $(k, \ell) = (\pi(a), \pi(b))$. Let $\sigma = \pi(E)$. Then $\sigma = t_{\pi(a),\pi(b)} t_{a,b} \pi t_{a,b} t_{\pi(a),\pi(b)}$. Now for $x \notin \{a, b, \pi(a), \pi(b)\}$, we also have

$\pi(x) \notin \{a, b, \pi(a), \pi(b)\}$. Hence

$$\begin{aligned}
\sigma(x) &= (t_{\pi(a),\pi(b)} t_{a,b} \pi t_{a,b} t_{\pi(a),\pi(b)})(x) \\
&= (t_{\pi(a),\pi(b)} t_{a,b})(\pi(x)) \\
&= \pi(x).
\end{aligned}$$

Also,

$$\begin{aligned}
\sigma(a) &= (t_{\pi(a),\pi(b)} t_{a,b} \pi t_{a,b} t_{\pi(a),\pi(b)})(a) \\
&= (t_{\pi(a),\pi(b)} t_{a,b} \pi t_{a,b})(a) \\
&= (t_{\pi(a),\pi(b)} t_{a,b} \pi)(b) \\
&= (t_{\pi(a),\pi(b)} t_{a,b})(\pi(b)) \\
&= (t_{\pi(a),\pi(b)})(\pi(b)) \\
&= \pi(a).
\end{aligned}$$

Similarly, $\sigma(b) = \pi(b)$. The remaining cases follow in a similar fashion.

Therefore E is an involution word diagram for π with fewer crossings than D, which implies D is not reduced. □

Let $X_D(a, b) \subset D$ denote the set of crossings of wires W_a and W_b in the diagram D.

Lemma 2.2.7. *Let $\pi \in \mathsf{FPF}_{\mathbb{P}}$. Let D be any reduced involution word diagram for π. Let $a, b \in \mathbb{P}$ with $a < b < \pi(b)$ and $a < \pi(a)$. The relative order of $a, \pi(a), b$, and $\pi(b)$ is determined by $|X_D(a, b)|$ according to:*

- *If $|X_D(a, b)| = 0$, then $a < \pi(a) < b < \pi(b)$.*

– *If* $|X_D(a,b)| = 1$, *then* $a < b < \pi(a) < \pi(b)$.

– *If* $|X_D(a,b)| = 2$, *then* $a < b < \pi(b) < \pi(a)$.

Furthermore, $|X_D(a,a)| = 0$.

Proof. Let q be the number of rows in D. For $0 \leq i \leq q$, we define truncated diagrams $D_i = \{(\rho, \kappa) \in D \mid \rho \leq i\}$, which contain the just the crossings in the top i rows of D. Let $\pi_i = \pi(D_i)$.

Suppose first that for some i, D_i is not reduced. This means that there is an involution word diagram E for π_i which has few crossings than D_i. Without loss of generality, E also has fewer rows than D_i. Then $E \cup (D \setminus D_i)$ is an involution word diagram for π which is shorter than D. This contradicts the fact that D is reduced. Thus we can conclude that D_i is in fact reduced for each i.

We now proceed by induction on i. For the base of our induction, we note that for any a and b, $|X_{D_0}(a,b)| = 0$. Since $\pi_0 = \theta$, $a < \pi(a)$ and $b < \pi(b)$ implies a and b are odd. Also, $\pi(a) = a + 1$ and $\pi(b) = b + 1$. The relative order $a < a + 1 < b < b + 1$ is satisfied.

Now fix $i \in \{1, \ldots, q\}$. Assume that D_{i-1} satisfies the relative order requirements for each pair of wires. If there is no crossing in row i, then the number of crossings as well as the relative orders remain the same for every pair of wires, so there is nothing to show. Thus we may assume that there is a crossing $(\kappa_i, i) \in D$. Since this crossing is at the bottom of D_i, we know that it crosses wires W_{κ_i} and W_{κ_i+1} in D_i. We have $\pi_i = s_{\kappa_i}\pi_{i-1}s_{\kappa_i}$. For $x \in \mathbb{P} \setminus \{\kappa_i, \pi_{i-1}(\kappa_i), \kappa_i + 1, \pi_{i-1}(\kappa_i + 1)\}$, $\pi_i(x) = (s_{\kappa_i}\pi_{i-1}s_{\kappa_i})(x) = \pi_{i-1}(x)$.

Suppose that $\pi_{i-1}(\kappa_i) = \kappa_i + 1$. Then $\pi_i(\kappa_i) = (s_{\kappa_i}\pi_{i-1}s_{\kappa_i})(\kappa_i) = (s_{\kappa_i}\pi_{i-1})(\kappa_i + 1) = s_{\kappa_i}(\kappa_i) = \kappa_i + 1 = \pi_{i-1}(\kappa_i)$. This would mean that $\pi_i = \pi_{i-1}$, which implies D_i

is not reduced. Thus wires W_{κ_i} and W_{κ_i+1} are indeed distinct in D_{i-1}. Hence

$$\pi_i(\kappa_i) = (s_{\kappa_i}\pi_{i-1}s_{\kappa_i})(\kappa_i) = (s_{\kappa_i}\pi_{i-1})(\kappa_i+1) = \pi_{i-1}(\kappa_i+1), \tag{2.2.4}$$

$$\text{and} \quad \pi_i(\kappa_i+1) = (s_{\kappa_i}\pi_{i-1}s_{\kappa_i})(\kappa_i+1) = (s_{\kappa_i}\pi_{i-1})(\kappa_i) = \pi_{i-1}(\kappa_i). \tag{2.2.5}$$

There are several things to check. Let's start by looking at the three cases for the value of $|X_{D_{i-1}}(\kappa_i, \kappa_i+1)|$.

- Assume $|X_{D_{i-1}}(\kappa_i, \kappa_i+1)| = 0$. Then $\pi_{i-1}(\kappa_i) < \kappa_i < \kappa_i + 1 < \pi_{i-1}(\kappa_i+1)$. Using equations (2.2.4) and (2.2.5), we get $\pi_i(\kappa_i+1) < \kappa_i < \kappa_i+1 < \pi_i(\kappa_i)$. This correlates with the value $|X_{D_i}(\kappa_i, \kappa_i+1)| = 1$, as desired.
- Assume $|X_{D_{i-1}}(\kappa_i, \kappa_i+1)| = 1$. Then $\kappa_i < \kappa_i+1 < \pi_{i-1}(\kappa_i) < \pi_{i-1}(\kappa_i+1)$. Using equations (2.2.4) and (2.2.5), we get $\kappa_i < \kappa_i+1 < \pi_i(\kappa_i+1) < \pi_i(\kappa_i)$. This correlates with the value $|X_{D_i}(\kappa_i, \kappa_i+1)| = 2$, as desired.
- Assume $|X_{D_{i-1}}(\kappa_i, \kappa_i+1)| = 2$. Then $|X_{D_i}(\kappa_i, \kappa_i+1)| = 3$. By Lemma 2.2.6, D_i is not reduced. Therefore this case is not possible.

Now we will check that nothing unexpected happens with the other wires in the diagram. Let $x, y \in \mathbb{P} \setminus \{\kappa_i, \pi_i(\kappa_i), \kappa_i+1, \pi_i(\kappa_i+1)\}$ with $y \neq x < \pi_i(x) \neq y$. Since W_x is not involved in the new crossing, we have $|X_{D_i}(x,y)| = |X_{D_{i-1}}(x,y)|$. Therefore we want to say that the relative order of $x, y, \pi_i(x)$, and $\pi_i(y)$ is the same as the relative order of $x, y, \pi_{i-1}(x)$, and $\pi_{i-1}(y)$, respectively. Since $\pi_i(x) = \pi_{i-1}(x)$, $\pi_i(y) = \pi_{i-1}(y)$, this is indeed the case.

The crossings of W_x with W_{κ_i} in D_{i-1} occur at the same locations as crossings of W_x with $W_{\kappa_{i+1}}$ in D_i. Again, since W_x is not involved in the new crossing, we have $|X_{D_i}(x, \kappa_i+1)| = |X_{D_{i-1}}(x, \kappa_i)|$. We now want to say that the relative order of $x, \pi_{i-1}(x), \kappa_i$, and $\pi_{i-1}(\kappa_i)$ coincides with the relative order of $x, \pi_i(x), \kappa_i + 1$, and $\pi_i(\kappa_i + 1)$. Three of these values do not change: $x = x$, $\pi_i(x) = \pi_{i-1}(x)$, and $\pi_i(\kappa_i + 1) = \pi_{i-1}(\kappa_i)$. The remaining value, κ_i can be replaced by $\kappa_i + 1$ without changing the relative order because the two wire positions are adjacent.

Similarly, $|X_{D_i}(x, \kappa_i)| = |X_{D_{i-1}}(x, \kappa_i + 1)|$. And for the same reason, the relative order of $x, \pi_{i-1}(x), \kappa_i + 1$, and $\pi_{i-1}(\kappa_i + 1)$ coincides with the relative order of $x, \pi_i(x), \kappa_i$, and $\pi_i(\kappa_i)$.

Therefore D_i satisfies the relative order requirements for each pair of wires. This concludes the induction. Thus for all pairs of distinct wires, W_a and W_b, the relative order of $a, \pi(a), b$, and $\pi(b)$ is determined by the value $|X_{D_i}(a, b)|$.

Finally, we note that a wire cannot cross itself in a reduced involution word diagram. By Lemma 2.2.4, deleting such a crossing does not change the resulting permutation. Thus a diagram which includes one of these crossings cannot be reduced. Therefore $|X_D(a, a)| = 0$. $\square$

Lemma 2.2.8. *Let $\pi \in \mathsf{FPF}_{\mathbb{P}}$. Let $a, b \in \mathbb{P}$ with $a < b$, $a < \pi(a)$, and $b < \pi(b)$. Let D be any involution word diagram for π in which neither wire W_a nor W_b crosses itself. Assume wires W_a and W_b cross each other in D. Let X be the top-most crossing of W_a and W_b. Then X has label $(a, \pi(b))$.*

Proof. Let q be the number of rows in D. Let $(r, j) = X$. As in the proof of Lemma 2.2.7, we will consider truncated diagrams $D_i = \big\{(\rho, \kappa) \in D \mid \rho \leq i\big\}$. For each i, let $\pi_i = \pi(D_i)$.

We start by factoring $\pi = \lambda s_j \eta \theta \eta^{-1} s_j \lambda^{-1}$. Just as in the exposition before Lemma 2.2.4, the central $\eta\theta\eta^{-1} = \pi_{r-1}$ corresponds to the portion of D above X. We conjugate by s_j to account for X (in column j) and then by λ to account for the crossings below X.

We can factor $\pi_i = \eta_i \theta \eta_i^{-1}$ where η_0 is the identity (of S_n) and for $i > 0$,

$$\eta_i = \begin{cases} s_\kappa \eta_{i-1} & \text{if } (i, \kappa) \in D_i \\ \eta_{i-1} & \text{if } D_i = D_{i-1}\,. \end{cases}$$

To keep track of all of the η's floating around, Figure 3 shows the names used for each of the relevant wire positions on the wires W_j and W_{j+1} in D_r.

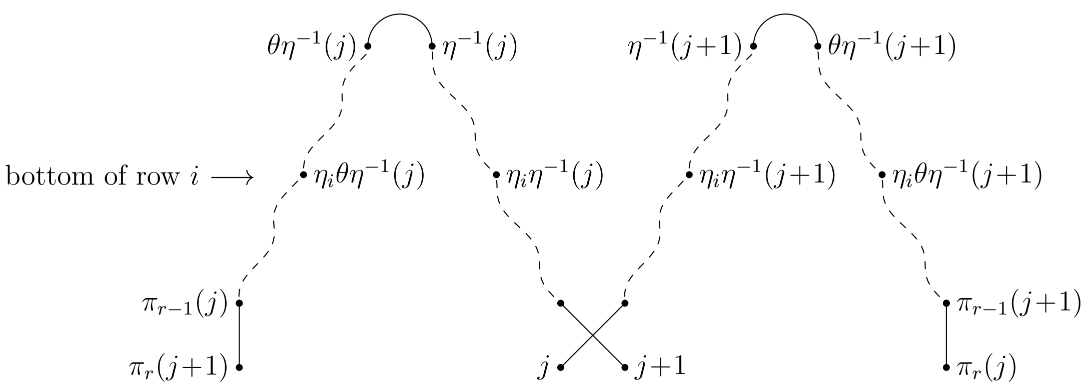


FIGURE 3.

We will now induct on i to show that for $0 \le i < r$,

$$(\eta_i \theta \eta^{-1})(j) < (\eta_i \eta^{-1})(j) < (\eta_i \eta^{-1})(j+1) < (\eta_i \theta \eta^{-1})(j+1).$$

Since $|X_{D_{r-1}}(j, j+1)| = 0$, we also have $|X_{D_i}((\eta_i \eta^{-1})(j), (\eta_i \eta^{-1})(j+1))| = 0$ for $i < r-1$.

For the base of the induction, when $i = 0$, since D_0 is reduced, the desired relative order comes from the $|X_D(a, b)| = 0$ case of Lemma 2.2.7.

Now fix $i \ge 1$ and assume

$$(\eta_{i-1} \theta \eta^{-1})(j) < (\eta_{i-1} \eta^{-1})(j) < (\eta_{i-1} \eta^{-1})(j+1) < (\eta_{i-1} \theta \eta^{-1})(j+1). \qquad (2.2.6)$$

The crossing in row i (if there is one) may involve one of the two wires, but does not involve both of them, otherwise, X would not be the top-most crossing of the two wires in D. Also, the crossing in row i does not involve crossing either of the wires with themselves. Thus one of the 4 values in (2.2.6) may increment or decrement by 1 when switching from $i-1$ to i, but it is not possible for them to change their relative order. Therefore,

$$(\eta_i \theta \eta^{-1})(j) < (\eta_i \eta^{-1})(j) < (\eta_i \eta^{-1})(j+1) < (\eta_i \theta \eta^{-1})(j+1).$$

Hence by induction,

$$(\eta_{r-1} \theta \eta^{-1})(j) < (\eta_{r-1} \eta^{-1})(j) < (\eta_{r-1} \eta^{-1})(j+1) < (\eta_{r-1} \theta \eta^{-1})(j+1),$$

which we can rewrite as

$$\pi_{r-1}(j) < j < j+1 < \pi_{r-1}(j+1).$$

Now, using the fact that $\pi_r = s_j \pi_{r-1} s_j$, we get

$$\pi_r(j+1) < j < j+1 < \pi_r(j).$$

The top label of X in D_r is $(\pi_r(j+1), \pi_r(j))$, which matches the format $(a, \pi(b))$, as desired, in the case where $D = D_r$.

We will show that X has the desired label in D by inducting on i. Our base case, where $i = r$, is already done.

Now fix $i > r$ and assume that X has label $(a, \pi_{i-1}(b))$ in D_{i-1} where $a < b < \pi_{i-1}(b)$ and $a < \pi_{i-1}(a)$.

We will assume there is a crossing $(i, \kappa) \in D_i$ in row i, because otherwise there is nothing to show. The label of X in D_i can be determined by tracing the paths of the wires coming out (of either the top or the bottom) of X. Since $(a, \pi_{i-1}(b))$ was a label of X in D_{i-1}, tracing the left wire to the bottom of row $i-1$ in D_i will get us to wire position a. Then continuing to the bottom of row i will take us to wire position $s_\kappa(a)$. Similarly, the right wire can be traced to wire position $s_\kappa(\pi_{i-1}(b))$. Thus $(s_\kappa(a), (s_\kappa\pi_{i-1})(b))$ is a label of X in D_i.

If the new crossing involves neither W_a nor W_b, then the labels of X remain the same in D_i. Also, the relative order of the endpoints of these two wires is preserved.

If the new crossing involves exactly one of these two wires, then we have a few cases. If (i) $a \in \{\kappa, \kappa+1\}$, then $\pi_{i-1}(a) \notin \{\kappa, \kappa+1\}$. The label of X changes from $(a, \pi_{i-1}(b))$ to $(s_\kappa(a), \pi_{i-1}(b))$. Since $\pi_i(s_\kappa(a)) = (s_\kappa\pi_{i-1})(a) = \pi_{i-1}(a) > s_\kappa(a)$, the relative order of $a, b, \pi_{i-1}(a)$, and $\pi_{i-1}(b)$ is the same as the relative order of $s_\kappa(a), b, \pi_i(s_\kappa(a))$, and $\pi_i(b)$. The other cases (ii) $\pi_{i-1}(a) \in \{\kappa, \kappa+1\}$, (iii) $b \in \{\kappa, \kappa+1\}$, and (iv) $\pi_{i-1}(b) \in \{\kappa, \kappa+1\}$ work similarly.

We will now look at what happens when the new crossing involves both of the wires. Since $a < b < \pi_{i-1}(b)$, $a < \pi_{i-1}(a)$, and only adjacent wires can cross, there are only three cases to look at:

- If $\{\kappa, \kappa+1\} = \{a, b\}$, then $b = a+1$. We have $\pi_i(a) = (s_\kappa\pi_{i-1}s_\kappa)(a) = \pi_{i-1}(b)$ and $\pi_i(b) = \pi_{i-1}(a)$. Since $b = a+1$ and $a < \pi_{i-1}(a) = \pi_i(b)$, we must also have $b < \pi_i(b)$. Because $(a, \pi_{i-1}(b))$ is a label of X in D_{i-1}, so is $(b, \pi_{i-1}(a))$. Hence $(s_\kappa(b), (s_\kappa\pi_{i-1})(a)) = (a, \pi_i(b))$ is a label of X in D_i as desired.

- If $\{\kappa, \kappa+1\} = \{b, \pi_{i-1}(a)\}$, then $b = \pi_{i-1}(a) \pm 1$. We have $\pi_i(a) = (s_\kappa\pi_{i-1}s_\kappa)(a) = s_\kappa(\pi_{i-1}(a)) = b$ and $\pi_i(\pi_{i-1}(a)) = \pi_{i-1}(b)$. Since $b < \pi_{i-1}(b)$

and $b = \pi_{i-1}(a) \pm 1$, we also have $\pi_{i-1}(a) < \pi_{i-1}(b) = \pi_i(\pi_{i-1}(a))$. Now $(s_\kappa(a), (s_\kappa\pi_{i-1})(b)) = (a, \pi_{i-1}(b)) = (a, \pi_i(\pi_{i-1}(a)))$ is our desired label of X in D_i.

– If $\{\kappa, \kappa+1\} = \{\pi_{i-1}(a), \pi_{i-1}(b)\}$, then $\pi_{i-1}(b) = \pi_{i-1}(a) \pm 1$. We have $\pi_i(a) = (s_\kappa\pi_{i-1}s_\kappa)(a) = s_\kappa(\pi_{i-1}(a)) = \pi_{i-1}(b)$ and $\pi_i(b) = \pi_{i-1}(a)$. Since $b < \pi_{i-1}(b)$ and $\pi_{i-1}(b) = \pi_{i-1}(a) \pm 1$, we also have $b < \pi_{i-1}(a) = \pi_i(b)$. Now $(s_\kappa(a), (s_\kappa\pi_{i-1})(b)) = (a, \pi_{i-1}(a)) = (a, \pi_i(\pi_i(b)))$ is our desired label of X in D_i.

These are all of the possible cases for how the new crossing interacts with wires W_a and W_b. Thus, by induction, for all $i \in \{r, \ldots, q\}$, the label of X in D_i is of the form $(a, \pi_i(b))$ for some $a, b \in \mathbb{P}$ satisfying $a < b < \pi_i(b)$ and $a < \pi_i(a)$. In particular, the label of X in $D = D_q$ is of the form $(a, \pi(b))$ for some $a, b \in \mathbb{P}$ satisfying $a < b < \pi(b)$ and $a < \pi(a)$. $\square$

Corollary 2.2.9. *Let D be a reduced involution word diagram for π. Let $a, b \in \mathbb{P}$ with $a < b$, $a < \pi(a)$, and $b < \pi(b)$. Then wires W_a and W_b cross according to these rules:*

(i) If $a < b < \pi(a) < \pi(b)$, then D has a unique crossing, X, with label $(a, \pi(b))$

(ii) If $a < b < \pi(b) < \pi(a)$, then D has exactly two crossings, X and Y, where X has label $(a, \pi(b))$ and Y has label (a, b). Furthermore, X is in a smaller value row than Y, i.e. X appears higher up in D than Y.

(iii) If $a < \pi(a) < b < \pi(b)$, then wires W_a and W_b do not cross in D.

Proof. The number of crossings in each case follows from Lemma 2.2.7 and the fact that these are all of the possible orderings of $a, b, \pi(a)$, and $\pi(b)$ subject to the given constraints. The label in part (i) is the result of the first induction in the proof of Lemma 2.2.8. The label of X in part (ii) follows from Lemma 2.2.8.

The label of Y in part 2 requires a little bit of work. First, we note that since D is reduced and X has a label in the second category from the proof of Lemma 2, Y must have a label which is in the first category. Since we don't care to distinguish top from bottom labels, this leaves us with 2 possibilities. Either (a, b) is a label of Y or (b, a) is a label of Y.

If (b, a) is the bottom label of Y, then because $a < b$, the wires must cross again below Y in D. Since the only other crossing of these wires is X, which happens above Y, (b, a) cannot be the bottom label of Y.

Now if (b, a) is the top label of Y, then $(\pi(a), \pi(b))$ is the bottom label of Y and we know that (in this case) $\pi(b) < \pi(a)$. So again, there would need to be another crossing of these two wires below Y. Thus (b, a) cannot be the top label of Y. Therefore (a, b) is a label of Y. □

Corollary 2.2.10. *Let D be a reduced involution word diagram. Let (x, y) be a label of any crossing in D. Then $x < y$.*

Proof. We check each of the crosses in the cases presented in Corollary 2.2.9.

For case (i). The unique crossing has labels $(a, \pi(b))$ and $(b, \pi(a))$. In this case, we know that $a < \pi(b)$ and $b < \pi(a)$.

For case (ii). The two crossings have labels $(a, \pi(b))$, $(b, \pi(a))$, (a, b), and $(\pi(b), \pi(a))$. In this case, we know that $a < \pi(b)$, $b < \pi(a)$, $a < b$, and $\pi(b) < \pi(a)$. □

2.3. Deletion and Exchange Properties

Lemma 2.3.1. *Let $\pi \in \mathsf{FPF}_{\mathbb{P}}$. Let D be an involution word diagram for π. Then*

$$\hat{\ell}_{\mathsf{FPF}}(\pi) \leq \sum_{W_a, W_b} |X_D(a,b)|$$

where the sum is over all pairs of wires. Furthermore, D is reduced if and only if for every $a \in \mathbb{P}$, $|X_D(a,a)| = 0$, and for every pair of wires, W_a and W_b, $|X_D(a,b)| \leq 2$ and coincides with the relative orders of $a, \pi(a), b$ and $\pi(b)$ in accordance with Lemma 2.2.7.

Proof. The length $\hat{\ell}_{\mathsf{FPF}}(\pi)$ is the smallest number of crossings in any involution word diagram for π. Since D is an involution word diagram for π and the sum on the right side is number of crossing in D, we get the desired inequality.

If we assume D is reduced, then the inequality $|X_D(a,b)| \leq 2$ comes from Lemma 2.2.6. The desired relative orders for the endpoints of the wire pairs as well as $|X_D(a,a)| = 0$ (for every a) come from Lemma 2.2.7.

Now we instead assume that for every $a \in \mathbb{P}$, $|X_D(a,a)| = 0$ and for every pair of wires, W_a and W_b, $|X_D(a,b)| \leq 2$ and the relative order of the endpoints agrees with Lemma 2.2.7. Suppose that D is not reduced. Then there must a shorter involution word diagram E such that $\pi(E) = \pi$. Without loss of generality, we may assume E is reduced. Since

$$\sum_{W_a, W_b} |X_E(a,b)| = \hat{\ell}_{\mathsf{FPF}}(\pi) < \sum_{W_a, W_b} |X_D(a,b)|,$$

there must be a pair of wires $W_a \neq W_b$ so that $|X_E(a,b)| < |X_D(a,b)|$. Since both $|X_E(a,b)|$ and $|X_D(a,b)|$ agree with Lemma 2.2.7 on the relative order of a, b, $\pi(a)$

and $\pi(b)$, we must have $|X_E(a,b)| = |X_D(a,b)|$, which is a contradiction. Thus D is reduced. $\square$

Definition 2.3.2. *Let $\pi \in \mathsf{FPF}_{\mathbb{P}}$. The set of cycles of π is*

$$Cyc(\pi) = \big\{(i,j) \in \mathbb{P} \times \mathbb{P} \;\big|\; i < j = \pi(i)\big\}.$$

The set of inversions of π is

$$Inv(\pi) = \big\{(i,j) \in \mathbb{P} \times \mathbb{P} \;\big|\; i < j \text{ and } \pi(i) > \pi(j)\big\}.$$

The set of FPF-inversions of π is

$$Inv_{\mathsf{FPF}}(\pi) = Inv(\pi) \setminus Cyc(\pi) = \big\{(i,j) \in \mathbb{P} \times \mathbb{P} \;\big|\; i < j \neq \pi(i) \text{ and } \pi(i) > \pi(j)\big\}.$$

Corollary 2.3.3. *Let $\pi \in \mathsf{FPF}_{\mathbb{P}}$. Then*

$$\hat{\ell}_{\mathsf{FPF}}(\pi) = \frac{1}{2} \cdot \Big| Inv_{\mathsf{FPF}}(\pi) \Big|.$$

Proof. Let D be a reduced FPF involution word diagram for π. Then $\hat{\ell}_{\mathsf{FPF}}(\pi) = \sum_{W_a, W_b} |X_D(a,b)|$ where the sum is over all pairs of wires. For each $a, b \in \mathbb{P}$, let $P(a,b) = \{a, b, \pi(a), \pi(b)\}^2$ be the set of pairs among the endpoints of wires W_a and W_b. For $i \in \{0,1,2\}$, let $C_i = \{(a,b) \in \mathbb{P} \times \mathbb{P} : a < b < \pi(b),\ a < \pi(a), \text{ and } |X_D(a,b)| = i\}$.

If $(a,b) \in C_0$, then $a < \pi(a) < b < \pi(b)$. The cycles $(a, \pi(a))$ and $(b, \pi(b))$ are the only inversions in $P(a,b)$. Thus there are no FPF-inversions.

If $(a,b) \in C_1$, then $a < b < \pi(a) < \pi(b)$. So $(b, \pi(a))$ and $(a, \pi(b))$ are the only FPF-inversions in $P(a,b)$.

If $(a,b) \in C_2$, then $a < b < \pi(b) < \pi(a)$. So (a,b), $(b,\pi(a))$, $(a,\pi(b))$, and $(\pi(a),\pi(b))$ are the only FPF-inversions in $P(a,b)$.

In each case, we see that if $(a,b) \in C_i$, then there are exactly $2i$ FPF-inversions in $P(a,b)$.

Now

$$\begin{aligned}
\hat{\ell}_{\mathsf{FPF}}(\pi) &= \sum_{W_a, W_b} |X_D(a,b)| \\
&= \sum_{i=0}^{2} \sum_{(a,b)\in C_i} |X_D(a,b)| \\
&= \sum_{i=0}^{2} \sum_{(a,b)\in C_i} i \\
&= \frac{1}{2} \cdot \sum_{i=0}^{2} \sum_{(a,b)\in C_i} 2i \\
&= \frac{1}{2} \cdot \sum_{i=0}^{2} \sum_{(a,b)\in C_i} \Big|\mathrm{Inv}_{\mathsf{FPF}}(\pi) \cap P(a,b)\Big| \\
&= \frac{1}{2} \cdot \Big|\mathrm{Inv}_{\mathsf{FPF}}(\pi)\Big|.
\end{aligned}$$

Note that each of the summands only has finitely many non-zero terms. $\square$

Corollary 2.3.4. *Let $\pi \in \mathsf{FPF}_{\mathbb{P}}$. Let D be a reduced involution word diagram for π. Let $x, y \in \mathbb{P}$. Then D has a crossing with label (x,y) if and only if $(x,y) \in$ Inv$_{\mathsf{FPF}}(\pi)$.*

Proof. Let $A = \{(a,b) \in \mathbb{P} \times \mathbb{P} \mid (a,b)$ is a label of some crossing $X \in D\}$. We want to show $A = \mathrm{Inv}_{\mathsf{FPF}}(\pi)$. First assume $(x,y) \in A$, i.e. there is a crossing $X \in D$ with bottom label (x, y). Then the other label of X is $(\pi(y), \pi(x))$. By Corollary 2.2.10, $x < y$ and $\pi(y) < \pi(x)$. Hence $(x, y) \in \mathrm{Inv}_{\mathsf{FPF}}(\pi)$. Therefore $A \subset \mathrm{Inv}_{\mathsf{FPF}}(\pi)$.

By Corollary 2.3.3, we have

$$|A| = 2 \cdot |D| = \hat{\ell}_{\mathsf{FPF}}(\pi) = |\mathrm{Inv}_{\mathsf{FPF}}(\pi)|,$$

which implies $A = \mathrm{Inv}_{\mathsf{FPF}}(\pi)$. □

Corollary 2.3.5. *Let $\pi \in \mathsf{FPF}_{\mathbb{P}}$. Let D be a reduced involution word diagram for π. Let $x, y, z \in \mathbb{P}$ with $x < y < z$ and $\{x, y, z\} \cap \{\pi(x), \pi(y), \pi(z)\} = \varnothing$. If D does not have any crossings with label (x, y) or (y, z), then D does not have a crossing with label (x, z).*

Proof. Assume D does not have any crossings with label (x, y) or (y, z). By Corollary 2.3.4, $(x, y), (y, z) \notin \mathrm{Inv}_{\mathsf{FPF}}(\pi)$. Hence $\pi(y) > \pi(x)$ and $\pi(z) > \pi(y)$. This means that $\pi(z) > \pi(x)$, which implies $(x, z) \notin \mathrm{Inv}_{\mathsf{FPF}}(\pi)$. By Corollary 2.3.4, D does not have a crossing with label (x, z). □

Proposition 2.3.6. *(Exchange Property) Let $\pi \in \mathsf{FPF}_{\mathbb{P}}$. Let $q = \hat{\ell}_{\mathsf{FPF}}(\pi)$. Let $D = \{(\rho_i, \kappa_i) \mid 0 \leq i < q\}$ be a reduced involution word diagram for π. Let $\rho_q, \kappa_q \in \mathbb{P}$ with $\kappa_q > \kappa_{q-1}$. Suppose that the diagram $E = D \cup \{(\rho_q, \kappa_q)\}$ is not reduced. Let $\sigma = \pi(E)$. Then one of the following is true:*

(i) $\sigma = \pi(D \setminus \{(\rho_i, \kappa_i)\})$ for some $i \in \{0, \ldots, q-1\}$.

(ii) $\sigma = \pi$

Proof. The crossing (ρ_q, κ_q) crosses wires W_{κ_q} and W_{κ_q+1} in E. If $\pi(\kappa_q) = \kappa_q + 1$, then $\sigma = t_{\kappa_q, \kappa_q+1} \pi t_{\kappa_q, \kappa_q+1} = \pi$. Note that $\pi(\kappa_q) = \kappa_q + 1$ implies that (ρ_q, κ_q) is a crossing of a single wire with itself.

Now assume $\pi(\kappa_q) \neq \kappa_q + 1$. By Lemma 2.3.1, since E is not reduced, there is a pair of wires, W_a and W_b, so that $|X_E(a,b)|$ does not coincide with the relative orders of $a, \pi(a), b$ and $\pi(b)$ in accordance with Lemma 2.2.7. For every pair of wires besides W_{κ_q} and W_{κ_q+1}, both $|X_D(a,b)|$ and the relative order of $a, \pi(a), b$ and $\pi(b)$ stay the same. Hence the pair W_{κ_q} and W_{κ_q+1} is the only pair which does not agree with Lemma 2.2.7.

The new crossing has bottom label $(\kappa_q, \kappa_q + 1)$. Therefore

$$\sigma(\kappa_q) = (s_{\kappa_q} \pi s_{\kappa_q})(\kappa_q) = \pi(\kappa_q + 1), \qquad \text{and} \tag{2.3.7}$$

$$\sigma(\kappa_q + 1) = (s_{\kappa_q} \pi s_{\kappa_q})(\kappa_q + 1) = \pi(\kappa_q). \tag{2.3.8}$$

We now look at the 3 cases for the number of crossings of W_{κ_q} and W_{κ_q+1} in D. Let $a = \min\{\kappa_q, \pi(\kappa_q), \kappa_q + 1, \pi(\kappa_q + 1)\}$, and let $b = \min\big(\{\kappa_q, \pi(\kappa_q), \kappa_q + 1, \pi(\kappa_q + 1)\} \setminus \{a, \pi(a)\}\big)$. Notice that $a < \pi(a)$, $b < \pi(b)$, and $a < b$, just as in Lemma 2.2.7.

- Assume $|X_D(\kappa_q, \kappa_q+1)| = 0$. Then $a < \pi(a) < b < \pi(b)$ and $|X_E(\kappa_q, \kappa_q+1)| = 1$. Since $\pi(\kappa_q) \neq \kappa_q + 1$, we know that $\pi(a) = \kappa_q$ and $b = \kappa_q + 1$. By (2.3.7) and (2.3.8), we get $\sigma(\kappa_q+1) < \kappa_q < \kappa_q+1 < \sigma(\kappa_q)$. Since $|X_E(\kappa_q, \kappa_q+1)| = 1$, this agrees with Lemma 2.2.7. By Lemma 2.3.1, E is reduced. In other words, this case is not possible.

- Now assume $|X_D(\kappa_q, \kappa_q + 1)| = 1$. Then $a < b < \pi(a) < \pi(b)$ and $|X_E(\kappa_q, \kappa_q + 1)| = 2$.

  If $a = \kappa_q$, then $b = \kappa_q + 1$ and by (2.3.7) and (2.3.8), we get $\kappa_q < \kappa_q + 1 < \sigma(\kappa_q + 1) < \sigma(\kappa_q)$. This agrees with Lemma 2.2.7.

If instead, $\pi(a) = \kappa_q$, then $\pi(b) = \kappa_q + 1$. By (2.3.7) and (2.3.8), we get $\sigma(\kappa_q + 1) < \sigma(\kappa_q) < \kappa_q < \kappa_q + 1$. This also agrees with Lemma 2.2.7.

Now if $b = \kappa_q$, then $\pi(a) = \kappa_q + 1$. Let $X \in D$ be the unique crossing of W_{κ_q} and W_{κ_q+1} in D. Let $F = D \setminus \{X\}$ and $\nu = \pi(F)$. By Lemma 2.2.8, $(a, \pi(b))$ is a label of X in D. Thus, by Lemma 2.2.4,

$$\nu = t_{a,\pi(b)}\pi t_{a,\pi(b)} = t_{b,\pi(a)}\pi t_{b,\pi(a)} = s_{\kappa_q}\pi s_{\kappa_q} = \sigma.$$

This is means that in this case, statement (i) holds.

- Finally, assume $|X_D(\kappa_q, \kappa_q + 1)| = 2$. Then $a < b < \pi(b) < \pi(a)$ and $|X_E(\kappa_q, \kappa_q + 1)| = 3$. Since $\pi(\kappa_q) \neq \kappa_q + 1$, we know that $b \neq \kappa_q$. Let $Y \in D$ be the bottom-most crossing of W_{κ_q} and W_{κ_q+1} in D. Let $F = D \setminus \{Y\}$ and $\nu = \pi(F)$. Then by Lemma 2.2.9, (a, b) is a label of Y in D. Thus, by Lemma 2.2.4,

$$\nu = t_{a,b}\pi t_{a,b} = t_{\pi(b),\pi(a)}\pi t_{\pi(b),\pi(a)}.$$

If $a = \kappa_q$, then $b = \kappa_q + 1$, which means

$$\sigma = s_{\kappa_q}\pi s_{\kappa_q} = t_{a,b}\pi t_{a,b} = \nu.$$

If $\pi(b) = \kappa_q$, then $\pi(a) = \kappa_q + 1$, which means

$$\sigma = s_{\kappa_q}\pi s_{\kappa_q} = t_{\pi(b),\pi(a)}\pi t_{\pi(b),\pi(a)} = \nu.$$

Thus in both of these cases, statement (i) holds.

□

Corollary 2.3.7. *(Deletion Property) Let* $\pi \in \mathsf{FPF}_{\mathbb{P}}$*. Let* $D = \{(\rho_i, \kappa_i) \mid 0 \leq i < q\}$ *be an involution word diagram for* π *which is not reduced. Then at least one of the following is true:*

(i) There exist distinct crossings $X, Y \in D$ *so that deleting both crossings yields a shorter diagram for* π*, i.e.* $\pi(D \setminus \{X, Y\}) = \pi$*.*

(ii) There is a single crossing $X \in D$ *so that deleting* X *yields a shorter diagram for* π*, i.e.* $D \setminus \{X\} = \pi$*.*

Proof. Let q be the number of rows in D. For $i \in \{0, 1, \ldots, q\}$, let $D_i = \{(\rho_j, \kappa_j) \mid 0 < j \leq i\}$ be the i^{th} truncated diagram. Since D_0 is reduced and $D_q = D$ is not reduced, there must exist a smallest index $j \in \{1, \ldots, q\}$ so that D_j is not reduced.

By Proposition 2.3.6, either (i) $\pi_j = \pi(D_j \setminus \{(\rho_t, \kappa_t)\})$ for some $t < j$, or (ii) $\pi_j = \pi_{j-1}$. In case (i), deleting both $X := (\rho_j, \kappa_j)$ and $Y := (\rho_t, \kappa_t)$ from D_j yields a shorter diagram for π_j. In case (ii), deleting just $X := (\rho_j, \kappa_j)$ from D_j yields a shorter diagram for π_j.

Note that case (ii), just as in the proof of Proposition 2.3.6, only comes about by having a single wire cross itself.

In either of these cases, we get a shorter diagram for π by deleting the same crossing(s) from D. □

2.4. Defects

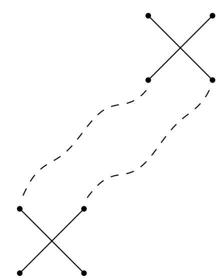
Definition 2.4.1. *An FPF involution word diagram is* <u>*reduced*</u> *if the word corresponding to it is reduced.*

Definition 2.4.2. *An FPF involution word diagram may have "defects." A* <u>*defect*</u> *is a pair of crossings in the FPF involution word diagram. Given one crossing from this pair, we refer to the other crossing as the defect counterpart.*

There are 3 types of defects in FPF involution diagrams:

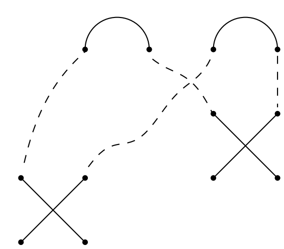
- *Type 1: the "normal" type.*

  This is where two different wires cross each other twice without either wire reaching the top between these two crossings.

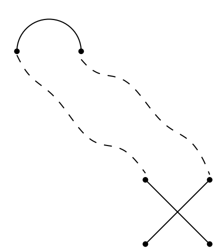
- *Type 2.*

  This is where two different wires cross each other twice with both wires reaching the top between these two crossings.

- *Type 3.*

  This is where a single wire crosses itself. In this case, the defect counterpart of the crossing is itself.

Remark. *It is possible for a crossing to be in more than one defect pair. We will not be concerned with classifying these situations as they never arise in the* *algorithms presented. Any time a pair of crossings is said to be of a particular type, we also assume neither of the crossings is involved in another type of defect.*

Claim 2.4.3. *An FPF involution word diagram is reduced if and only if it has none of these defects.*

Proof. Let $\pi \in \mathsf{FPF}_{\mathbb{P}}$. Let D be an involution word diagram for π which has none of the defects described in Definition 2.4.2. Suppose that D is not reduced.

Corollary 2.3.7 tells us that there are two distinct wires, W_a and W_b, and a pair of crossings X and Y of those wires so that $\sigma := \pi(D \setminus \{X, Y\}) = \pi$. Note that we may eliminate conclusion (ii) from Corollary 2.3.7 as a possibility because this only comes about by having a type 3 defect.

Without loss of generality, assume Y is lower in the diagram (larger row number) than X. Suppose now that (x, y) is the top label of Y in D. Since $W_a \neq W_b$, we must have $\pi(x) \neq y$. When following the two wires coming out of the top of Y in D, both must eventually reach the crossing X. Note that X cannot be reached by following either of the wires coming from the bottom of Y because X is in a higher row than Y.

Since D does not have any defects of type 1 or 2, of the two wires coming out of the top of Y exactly one must reach the top of the diagram. Without loss of generality, W_y is the one that reaches the top between X and Y. This means that continuing to follow W_y, over the top of the diagram and down through X, will eventually lead to wire position y at the bottom of D. In other words, y is part of a bottom label of X. Since W_x does not reach the top of the diagram between X and Y, following W_x out of the top of Y will lead up through X and continue up to the top of the diagram before descending to wire position x. In other words, x is part of the top label of X, which implies $\pi(x)$ is part of the bottom label of X.

We now know that the bottom label of X is either $(\pi(x), y)$ or $(y, \pi(x))$. Hence $\sigma = t_{\pi(x),y} t_{x,y} \pi t_{x,y} t_{\pi(x),y}$. We can see that σ and π are not equal because $\sigma(x) = y \neq \pi(x)$. This contradicts the conclusion we drew from Corollary 2.3.7. Therefore D must actually be reduced.

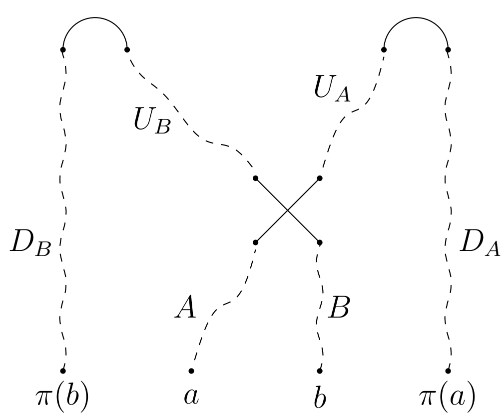
It is left to the reader to verify the reverse implication, that each of the defects indeed provide one or more crossings which may be deleted to obtain a shorter diagram for the same permutation. □

Definition 2.4.4. *Let D be an involution word diagram. Let $(\rho, \kappa) \in D$. We say D is* *<u>nearly reduced at κ</u>* *(or at the crossing (ρ, κ)) if the diagram $D \setminus \{(\rho, \kappa)\}$ is reduced.*

Lemma 2.4.5. *Let D be an involution word diagram and $X \in D$. If D is not reduced, but is nearly reduced at X, then X has a defect counterpart.*

Proof. Let $(\rho, \kappa) = X$. Let (a, b) be the bottom label of $X \in D$. Let $E = D \setminus \{X\}$. We will be assuming that E is reduced and D is not reduced. Let $\sigma = \pi(E)$ and $\pi = \pi(D) = t_{a,b} \sigma t_{a,b}$.

For convenience, we will segment wires W_a and W_b into several parts. These parts are dashed and labeled in the following figure. None of them involve the crossing X nor the arcs at the top of D.



Segments A and B are the parts of wires W_a and W_b, respectively, which come out of the bottom of X and reach wire positions a and b, respectively, at the bottom of D. Segments U_A and U_B are the parts of wires W_a and W_b, respectively, which come out of the top of X and end just before arcing at the top of D. Segments D_A and D_B are the parts of wires W_a and W_b, respectively, which come down from after arcing at the top of D to reach wire positions $\pi(b)$ and $\pi(a)$, respectively.

Our goal is to show that X has a defect counterpart in D. Suppose that X does not have a defect counterpart in D.

If segments A and B cross, then that crossing is a defect counterpart for X. Thus we may assume that A and B do not cross. Hence A is left of B in each row, including at the bottom of D. This means that $a < b$.

Similarly, we know that U_A does not cross U_B. Thus the top of U_B is left of the top of U_A. Since the arc (applying the function θ) does not cross any wires, we must have the top of D_B to the left of the top of D_A. If D_A crosses D_B anywhere, then that crossing is a (type 2) defect counterpart of X. Thus D_B is left of D_B in all rows, including at the bottom of the diagram. Hence $\pi(b) < \pi(a)$.

We now consider whether D_B is to the left or right of X in row ρ.

First, assume that D_B is to the left of X. Both D_B and A are both part of wire W_a in E. Since E is reduced, we know that D_B cannot cross A. This means that if D_B is to the left of X in row ρ, then $\pi(b) < a$.

Now assume D_B is to the right of X. Then D_B is to the right of U_A at the bottom of row $\rho - 1$, which implies D_B and U_A cross. If D_B crosses both U_A and B, then these two crossings would be a type 1 defect pair in E. Thus D_B does not also cross B. This means that if D_B is to the right of X in row ρ, then $b < \pi(b)$.

Combining these two inequalities (along with $a < b$), we can say that $\pi(b)$ is not between a and b. A similar argument using D_A shows that $\pi(a)$ is not between a and b.

Since E is reduced, $\hat{\ell}_{\mathsf{FPF}}(\sigma) = |E|$. Since D is not reduced, the deletion property tells us that $\hat{\ell}_{\mathsf{FPF}}(\pi) < |D| = |E| + 1 = \hat{\ell}_{\mathsf{FPF}}(\sigma) + 1$. In other words, $\hat{\ell}_{\mathsf{FPF}}(\pi) \leq \hat{\ell}_{\mathsf{FPF}}(\sigma)$.

Below, we consider cases for the relative order of $a, b, \pi(a)$, and $\pi(b)$. In each case, we will arrive at the contradiction by counting FPF-inversions and noting that π has at least one more than σ. This means that $\hat{\ell}_{\mathsf{FPF}}(\pi) > \hat{\ell}_{\mathsf{FPF}}(\sigma)$, which contradicts the inequality in the previous paragraph.

For each of these cases, we will count FPF-inversions among $a, b, \pi(a)$, and $\pi(b)$ for both π and σ. We also have to consider FPF-inversions involving a third cycle $(c, \pi(c)) = (c, \sigma(c)) \in \mathrm{Cyc}(\pi) \cap \mathrm{Cyc}(\sigma)$ for an arbitrary $c \notin \{a, b, \pi(a), \pi(b)\}$. We will not need to consider pairs of cycles involving neither a nor b as it is not possible for conjugation by $t_{a,b}$ to change their relative order.

Note that (x, y) is an FPF-inversion of π if and only if $(\pi(y), \pi(x))$ is. To reduce the length of our lists, we will only include one from each of these FPF-inversion pairs. For convenience, we will only include FPF-inversions (x, y) for which $x < \pi(x)$ and $x < \pi(y)$. In other words, we only include the inversion (x, y) where $x = \min\{x, y, \pi(x), \pi(y)\}$. Denote the set of (half of the) FPF-inversions of π involving only $x, y, \pi(x)$, or $\pi(y)$ by $I_{x,y}(\pi)$.

We have $a < b$, $\pi(b) < \pi(a)$ and a and b are adjacent. This leaves us with 3 possible relative orders. Let $c \in \mathbb{P} \setminus \{a, b, \pi(a), \pi(b)\}$ so that $c < \pi(c)$.

(i) Assume $\pi(b) < a < b < \pi(a)$. Thus $\sigma(a) < a < b < \sigma(b)$.

Then $I_{a,b}(\pi) = \{(\pi(b), \pi(a))\}$ and $I_{a,b}(\sigma) = \varnothing$. Hence $|I_{a,b}(\pi)| = |I_{a,b}(\sigma)| + 1$.

We now look at the 15 cases for the relative or of $a, b, c, \pi(a), \pi(b)$, and $\pi(c)$. In each case, we want to show $|I_{a,c}(\pi)| + |I_{b,c}(\pi)| \geq |I_{a,c}(\sigma)| + |I_{b,c}(\sigma)|$.

* If $c < \pi(c) < \pi(b) < a < b < \pi(a)$, then $c < \sigma(c) < \sigma(a) < a < b < \sigma(b)$. So

  $|I_{a,c}(\pi)| = |\varnothing| = 0,$

  $|I_{b,c}(\pi)| = |\varnothing| = 0,$

  $|I_{a,c}(\sigma)| = |\varnothing| = 0,$ and

  $|I_{b,c}(\sigma)| = |\varnothing| = 0.$

* If $c < \pi(b) < \pi(c) < a < b < \pi(a)$, then $c < \sigma(a) < \sigma(c) < a < b < \sigma(b)$. So

  $|I_{a,c}(\pi)| = |\varnothing| = 0,$

  $|I_{b,c}(\pi)| = |\{(c, b)\}| = 1,$

  $|I_{a,c}(\sigma)| = |\{(c, a)\}| = 1,$ and

  $|I_{b,c}(\sigma)| = |\varnothing| = 0.$

* If $c < \pi(b) < a < \pi(c) < b < \pi(a)$, then $c < \sigma(a) < a < \sigma(c) < b < \sigma(b)$. So

  $|I_{a,c}(\pi)| = |\{(c, \pi(a))\}| = 1,$

  $|I_{b,c}(\pi)| = |\{(c, b)\}| = 1,$

  $|I_{a,c}(\sigma)| = |\{(c, a), (c, \sigma(a))\}| = 2,$ and

  $|I_{b,c}(\sigma)| = |\varnothing| = 0.$

* If $c < \pi(b) < a < b < \pi(c) < \pi(a)$, then $c < \sigma(a) < a < b < \sigma(c) < \sigma(b)$. So

  $|I_{a,c}(\pi)| = |\{(c, \pi(a))\}| = 1,$

$$|I_{b,c}(\pi)| = |\{(c,b),(c,\pi(b))\}| = 2,$$

$$|I_{a,c}(\sigma)| = |\{(c,a),(c,\sigma(a))\}| = 2, \text{ and}$$

$$|I_{b,c}(\sigma)| = |\{(c,\sigma(b))\}| = 1.$$

* If $c < \pi(b) < a < b < \pi(a) < \pi(c)$, then $c < \sigma(a) < a < b < \sigma(b) < \sigma(c)$. So

$$|I_{a,c}(\pi)| = |\{(c,\pi(a)),(c,a)\}| = 2,$$

$$|I_{b,c}(\pi)| = |\{(c,b),(c,\pi(b))\}| = 2,$$

$$|I_{a,c}(\sigma)| = |\{(c,a),(c,\sigma(a))\}| = 2, \text{ and}$$

$$|I_{b,c}(\sigma)| = |\{(c,\sigma(b)),(c,b)\}| = 2.$$

* If $\pi(b) < c < \pi(c) < a < b < \pi(a)$, then $\sigma(a) < c < \sigma(c) < a < b < \sigma(b)$. So

$$|I_{a,c}(\pi)| = |\varnothing| = 0,$$

$$|I_{b,c}(\pi)| = |\{(c,b),(\pi(b),c)\}| = 2,$$

$$|I_{a,c}(\sigma)| = |\{(c,a),(\sigma(a),c)\}| = 2, \text{ and}$$

$$|I_{b,c}(\sigma)| = |\varnothing| = 0.$$

* If $\pi(b) < c < a < \pi(c) < b < \pi(a)$, then $\sigma(a) < c < a < \sigma(c) < b < \sigma(b)$. So

$$|I_{a,c}(\pi)| = |\{(c,\pi(a))\}| = 1,$$

$$|I_{b,c}(\pi)| = |\{(c,b),(\pi(b),c)\}| = 2,$$

$$|I_{a,c}(\sigma)| = |\{(c,a)\}| = 1, \text{ and}$$

$$|I_{b,c}(\sigma)| = |\varnothing| = 0.$$

* If $\pi(b) < c < a < b < \pi(c) < \pi(a)$, then $\sigma(a) < c < a < b < \sigma(c) < \sigma(b)$. So

$|I_{a,c}(\pi)| = |\{(c, \pi(a))\}| = 1,$

$|I_{b,c}(\pi)| = |\{(c, b)\}| = 1,$

$|I_{a,c}(\sigma)| = |\{(c, a)\}| = 1$, and

$|I_{b,c}(\sigma)| = |\{(c, \sigma(b))\}| = 1.$

* If $\pi(b) < c < a < b < \pi(a) < \pi(c)$, then $\sigma(a) < c < a < b < \sigma(b) < \sigma(c)$.

  So

  $|I_{a,c}(\pi)| = |\{(c, \pi(a)), (c, a)\}| = 2,$

  $|I_{b,c}(\pi)| = |\{(c, b)\}| = 1,$

  $|I_{a,c}(\sigma)| = |\{(c, a)\}| = 1$, and

  $|I_{b,c}(\sigma)| = |\{(c, \sigma(b)), (c, b)\}| = 2.$

* If $\pi(b) < a < c < \pi(c) < b < \pi(a)$, then $\sigma(a) < a < c < \sigma(c) < b < \sigma(b)$.

  So

  $|I_{a,c}(\pi)| = |\{(a, c), (a, \pi(c))\}| = 2,$

  $|I_{b,c}(\pi)| = |\{(c, b), (\pi(b), c)\}| = 2,$

  $|I_{a,c}(\sigma)| = |\varnothing| = 0$, and

  $|I_{b,c}(\sigma)| = |\varnothing| = 0.$

* If $\pi(b) < a < c < b < \pi(c) < \pi(a)$, then $\sigma(a) < a < c < b < \sigma(c) < \sigma(b)$.

  So

  $|I_{a,c}(\pi)| = |\{(a, c), (a, \pi(c))\}| = 1,$

  $|I_{b,c}(\pi)| = |\{(c, b)\}| = 1,$

  $|I_{a,c}(\sigma)| = |\varnothing| = 0$, and

  $|I_{b,c}(\sigma)| = |\{(c, \sigma(b))\}| = 1.$

* If $\pi(b) < a < c < b < \pi(a) < \pi(c)$, then $\sigma(a) < a < c < b < \sigma(b) < \sigma(c)$.

  So

$$|I_{a,c}(\pi)| = |\{(a, \pi(c))\}| = 1,$$

$$|I_{b,c}(\pi)| = |\{(c, b)\}| = 1,$$

$$|I_{a,c}(\sigma)| = |\varnothing| = 0, \text{ and}$$

$$|I_{b,c}(\sigma)| = |\{(c, \sigma(b)), (c, b)\}| = 2.$$

* If $\pi(b) < a < b < c < \pi(c) < \pi(a)$, then $\sigma(a) < a < b < c < \sigma(c) < \sigma(b)$. So

$$|I_{a,c}(\pi)| = |\{(a, c), (a, \pi(c))\}| = 2,$$

$$|I_{b,c}(\pi)| = |\varnothing| = 0,$$

$$|I_{a,c}(\sigma)| = |\varnothing| = 0, \text{ and}$$

$$|I_{b,c}(\sigma)| = |\{(b, \sigma(c)), (b, c)\}| = 2.$$

* If $\pi(b) < a < b < c < \pi(a) < \pi(c)$, then $\sigma(a) < a < b < c < \sigma(b) < \sigma(c)$. So

$$|I_{a,c}(\pi)| = |\{(a, \pi(c))\}| = 1,$$

$$|I_{b,c}(\pi)| = |\varnothing| = 0,$$

$$|I_{a,c}(\sigma)| = |\varnothing| = 0, \text{ and}$$

$$|I_{b,c}(\sigma)| = |\{(b, \sigma(c))\}| = 1.$$

* If $\pi(b) < a < b < \pi(a) < c < \pi(c)$, then $\sigma(a) < a < b < \sigma(b) < c < \sigma(c)$. So

$$|I_{a,c}(\pi)| = |\varnothing| = 0,$$

$$|I_{b,c}(\pi)| = |\varnothing| = 0,$$

$$|I_{a,c}(\sigma)| = |\varnothing| = 0, \text{ and}$$

$$|I_{b,c}(\sigma)| = |\varnothing| = 0.$$

(ii) Assume $a < b < \pi(b) < \pi(a)$. Thus $a < b < \sigma(a) < \sigma(b)$. ...

(iii) Assume $\pi(b) < \pi(a) < a < b$. Thus $\sigma(a) < \sigma(b) < a < b$. ...

In the second and third cases, we omit the details as they are extremely similar to those from the first case. □

Lemma 2.4.6. *Let D be an involution word diagram and $X \in D$. If D is not reduced, but is nearly reduced at X, then X has a unique defect counterpart.*

Proof. We already know that there is a least one defect counterpart of X by Lemma 2.4.5.

Let $\pi = \pi(D)$. Suppose that both Y and Z are defect counterparts of X in D. We want to show that $Y = Z$. Let (a, b) be the bottom label of X. If $\pi(a) = b$, then X is a type 3 defect, which means $Y = X = Z$.

We may now assume $\pi(a) \neq b$, i.e. $W_a \neq W_b$. Then Y and Z must also cross wires W_a and W_b. More specifically, Y and Z must have one of the following 8 bottom labels, which we divide into two categories (as in the proof of Lemma 2.2.6):

1) $(a, b),\quad (b, a),\quad (\pi(a), \pi(b)),\quad (\pi(b), \pi(a))$

2) $(a, \pi(b)),\quad (b, \pi(a)),\quad (\pi(a), b),\quad (\pi(b), a)$

Since the bottom label of X is in category 1, the bottom label of Y is as well. Otherwise, we can check that $\pi(D \setminus \{X, Y\}) \neq \pi$, which would mean that Y is not a defect counterpart of X. We check for one example here, and leave the rest to the reader. Assume the bottom label of Y is $(\pi(a), b)$. Then we consider the case where Y is in a smaller number row than X. Since X is lower down in the diagram, its bottom label is preserved after deleting Y. Thus $\pi(D \setminus \{X, Y\}) = t_{a,b} t_{\pi(a),b} \pi t_{\pi(a),b} t_{a,b}$.

Now

$$\begin{aligned}
(t_{a,b}t_{\pi(a),b}\pi t_{\pi(a),b}t_{a,b})(a) &= (t_{a,b}t_{\pi(a),b}\pi t_{\pi(a),b})(b) \\
&= (t_{a,b}t_{\pi(a),b}\pi)(\pi(a)) \\
&= (t_{a,b}t_{\pi(a),b})(a) \\
&= t_{a,b}(a) \\
&= b \\
&\neq \pi(a)
\end{aligned}$$

Therefore $\pi(D \setminus \{X, Y\}) \neq \pi$. Similarly, if Y is in a larger number row than X, then the bottom label of X is preserved after deleting Y, so $\pi(D \setminus \{X, Y\}) = t_{\pi(a),b}t_{a,b}\pi t_{a,b}t_{\pi(a),b}$. Since $(t_{\pi(a),b}t_{a,b}\pi t_{a,b}t_{\pi(a),b})(a) = \pi(b)$, we know $\pi(D \setminus \{X, Y\}) \neq \pi$.

Similarly, we may conclude that the bottom label of Z is also in category 1.

Suppose that $Y \neq Z$. We will now show that D is not nearly reduced at X. Let $E = D \setminus \{X\}$. Then $\pi(E) = t_{a,b}\pi t_{a,b} = t_{\pi(a),\pi(b)}\pi t_{\pi(a),\pi(b)}$. Since $t_{a,b} = t_{b,a}$ and $t_{\pi(a),\pi(b)} = t_{\pi(b),\pi(a)}$, it suffices to consider only labels (a, b), and $(\pi(a), \pi(b))$.

We will now delete all 3 of the crossings from D. Let $F = D \setminus \{X, Y, Z\}$. Since the resulting diagram, F, is the same regardless of the order in which we delete X, Y, and Z, we will calculate $\pi(F)$ by deleting crossings from the top down (the crossing with the smallest numbered row to the largest numbered row). This way, it is clear that after deleting each crossing, the bottom label of the remaining crossings is unaffected. This means that $\pi(F) = t_3t_2t_1\pi t_1t_2t_3$ where for each i, $t_i \in \{t_{a,b}\,, t_{\pi(a),\pi(b)}\}$. But $t_{a,b}$ and $t_{\pi(a),\pi(b)}$ commute and each have order 2, so we may cancel to arrive at $\pi(F) = t\pi t$ for some $t \in \{t_{a,b}\,, t_{\pi(a),\pi(b)}\}$. Hence $\pi(F) = \pi(E)$.

Since $\pi(F) = \pi(E)$ and $|F| < |E|$, E is not reduced, which means that D is not nearly reduced at X. This means that if we assume that D is nearly reduced at X, we may conclude that $Y = Z$, as desired.

$\square$

Lemma 2.4.7. *Let D be an FPF involution word diagram with defect pair $X, Y \in D$. If D is nearly reduced at X, then D is also nearly reduced at Y.*

Proof. Assume D is nearly reduced at X. Just as in the proof of Lemma 2.4.6, we know that the bottom label of Y is in the same label category as X. Therefore $\pi(D \setminus \{X\}) = \pi(D \setminus \{Y\})$.

We have $\hat{\ell}_{\mathsf{FPF}}(\pi(D \setminus \{Y\})) = \hat{\ell}_{\mathsf{FPF}}(\pi(D \setminus \{X\})) = |D \setminus \{X\}| = |D \setminus \{Y\}|$. Therefore $D \setminus \{Y\}$ is reduced, which implies D is nearly reduced at Y.

$\square$

Lemma 2.4.8. *Let D be an involution word diagram. Let $X \in D$. Let (a, b) be the top label and (c, d) be the bottom label of X in D. Let $E = D \setminus \{X\}$. The E-labels, $top_E(X)$ and $bot_E(X)$, of X are described by:*

- *If X is a type 3 defect in D, then $top_E(X) = (b, a) = (d, c)$ and $bot_E(X) = (a, b) = (c, d)$.*
- *If X is not a type 3 defect in D, then $top_E(X) = (a, b)$ and $bot_E(X) = (c, d)$.*

Proof. If we trace the wires coming out of the bottom left and bottom right of position X, the wires take the exact same path in both D and E because none of the crossings below X have been affected. Therefore the bottom label of X is the same in D as it is in E.

Let $\pi = \pi(D)$ and $\sigma = \pi(E)$. Since (c, d) is a label of X in D, $\sigma = t_{c,d}\pi t_{c,d}$. In both D and in E, the top label of X is determined by the bottom label. Since X

is a crossing in D, and (c, d) is the bottom label, the top label must be $(\pi(d), \pi(c))$. Hence $a = \pi(d)$ and $b = \pi(c)$.

Since X is not a crossing in E, and (c, d) is the bottom label, the top label must be $(\sigma(c), \sigma(d))$.

If X is a type 3 defect of D, then $c = \pi(d)$ and $d = \pi(c) = b$, which means $\sigma(c) = (t_{c,d}\pi t_{c,d})(c) = t_{c,d}(\pi(d)) = t_{c,d}(c) = d = b$. Similarly, if X is a type 3 defect of D, $\sigma(d) = a$. Hence the top label of X in E is $(b, a) = (d, c)$.

Now in the case where X is not a type 3 defect of D, we have $c \neq \pi(d)$, which means $\sigma(c) = (t_{c,d}\pi t_{c,d})(c) = t_{c,d}(\pi(d)) = \pi(d) = a$ and $\sigma(d) = (t_{c,d}\pi t_{c,d})(d) = t_{c,d}(\pi(c)) = \pi(c) = b$. Hence the top label of X in E is (a, b). $\square$

Lemma 2.4.9. *Let E be an involution word diagram which has no crossings in row $\rho \in \mathbb{P}$. Let $\kappa \in \mathbb{P}$ and $X = (\rho, \kappa)$. Let (a, b) be the top label and (c, d) be the bottom label of X in E. Let $D = E \cup \{X\}$. The D-labels, $top_D(X)$ and $bot_D(X)$, of X are described by:*

- *If X is a type 3 defect in D, then $top_D(X) = (b, a) = (c, d) = bot_D(X)$.*

- *If X is not a type 3 defect in D, then $top_D(X) = (a, b)$ and $bot_D(X) = (c, d)$.*

Proof. Just as in the proof of Lemma 2.4.8, the bottom label of X in D must be (c, d). Let (x, y) be the top label of X in D. Since $E = D \setminus \{X\}$, we may apply Lemma 2.4.8 to say:

- If X is a type 3 defect in D, then the top label of X in E is $(y, x) = (d, c)$. Thus $(y, x) = (a, b)$, which means the top label of X in D is $(b, a) = (c, d)$.

- If X is not a type 3 defect in D, then the top label of X in E is (x, y). Thus $(x, y) = (a, b)$, which means the top label of X in D is (a, b).

□

Lemma 2.4.10. *Let D be an involution word diagram which is nearly reduced at $X \in D$. Let $Y \in D$ be a defect counterpart of X. Let (a, b) be the top (resp. bottom) label of X. The labels of Y are then determined by the defect type as follows:*

- *If the defect is of type 1, then (b, a) is the top (resp. bottom) label of Y.*
- *If the defect is of type 2, then (b, a) is the bottom (resp. top) label of Y.*
- *If the defect is of type 3, then (a, b) is both the bottom and top label of Y.*

Proof. Since D is nearly reduced at X, by Lemma 2.4.6, Y is the only defect counterpart of X. By Lemma 2.4.7, D is nearly reduced at Y. For convenience, we will refer to the label (x, y) as the "swap" of label (y, x). Note that the top labels of X and Y are swaps of each other if and only if the bottom labels are swaps of each other.

We start with the case where X and Y are a type 1 defect pair. Let $T \in \{X, Y\}$ be the crossing in the smallest row (furthest up in the diagram). Let $B \in \{X, Y\} \setminus \{T\}$ be the other crossing. Let (x, y) be the bottom label of T. Tracing wires W_x and W_y down from the bottom of T, the wires cross exactly once at B, otherwise, another crossing would be a second defect counterpart to X. Hence wire W_x is on the left side of wire W_y at the top of B. At the bottom of B, W_x is now right of W_y. This means that the bottom label of B is (y, x), which is the swap of the bottom label of T, as desired.

Now we look at the case where X and Y are a type 2 defect pair. Let (x, y) be the bottom label of X. Then wire W_x is to the right of W_y at the top of X. If by tracing wires W_x and W_y to the top of the diagram the wires cross (without either one arcing over first), that cross would be a type 1 defect counterpart of X. Therefore W_x is to the right of W_y at the top of the diagram (before arcing). Since the arcs do not cross any wires and $W_x \neq W_y$, W_x is still to the right of W_y after arcing over. If we continue to trace the wires down after arcing over, W_x cannot cross W_y before reaching Y, else that crossing would be a second type 2 counterpart of X. Therefore W_x is to the right of W_y at the top of Y. Thus (y, x) is the top label of Y. So, indeed, the top label of Y is the swap of the bottom label of X. Moreover, this means the bottom label of Y is the swap of the top label of X, as desired.

Finally, we assume $X = Y$ is a type 3 defect. Let (x, y) be the bottom label of X. Then $(\pi(D)(y), \pi(D)(x)) = (x, y)$ is also the top label of $X = Y$.

$\square$

Lemma 2.4.11. *Let $\pi \in \mathsf{FPF}_{\mathbb{P}}$. Let $a, b \in \mathbb{P}$ with $a < \pi(a) = b$. Let D be a involution word diagram for π. Let ρ be a row of D. For each $i \in \mathbb{P}$, let (x_i, y_i) be the top label of (ρ, i). Then the following statements hold:*

(i) For all i, $y_i = x_{i+1}$.

(ii) For any $k \in \mathbb{P}$, there exists a unique i so that $x_i = k$.

(iii) If D is reduced and $x_i = a$ and $x_j = b$, then $i > j$.

Proof. Statement (i) follows from the definition of top label as for each i, the quantities y_i and x_{i+1} are obtained by tracing the exact same wire in the same direction.

We start by factoring $\pi = \lambda\eta\theta\eta^{-1}\lambda^{-1}$. Similar to the exposition before Lemma 2.2.4, the central $\eta\theta\eta^{-1}$ corresponds to the portion of D strictly above row ρ. Then for $k \in \mathbb{P}$, let $i_k = \eta\theta\eta^{-1}\lambda^{-1}(k)$. Then $x_{i_k} = k$. Also, for $k, k' \in \mathbb{P}$, if

$i_k = i_{k'}$, then $\eta\theta\eta^{-1}\lambda^{-1}(k) = \eta\theta\eta^{-1}\lambda^{-1}(k')$ implies $k = k'$ because η, θ and λ are invertible. Alternatively, we can trace the wire in wire position k (at the bottom of the diagram) up and over the arc and back down to the top of row ρ. The resulting wire position is i_k. Thus statement (ii) holds.

Now assume D is reduced and let $i, j \in \mathbb{P}$ so that $x_i = a$ and $x_j = b$. Then if we trace the wire in wire position x_i (at the top of row ρ) down instead of up, we will reach wire position $b = \pi(a)$ at the bottom of the diagram. Similarly, tracing the wire in position x_j down will reach wire position $b > a$. Since wire $W_a = W_b$ does not cross itself, between the top of row ρ and the bottom of the diagram, we must have $j > i$. Hence statement (iii) holds.

□

2.5. The FPF Involution Bumping Algorithm

The general bumping algorithm defined in this section is inspired by Little's general bumping algorithm (see Algorithm 2 in [4]). The suggestion to try such an algorithm on involution word diagrams was given to the author by Zachary Hamaker. See also [1] for a similar bumping algorithm which does not use involution word diagrams.

Definition 2.5.1. *Given an FPF involution word diagram D, and a particular crossing $X = (\rho, \kappa) \in D$ with $\kappa > 1$, we can perform a <u>bump</u> at X by doing the following:*

1. *Delete the crossing X.*

2. *Insert a new crossing in the same row and 1 column left of where the crossing X was.*

In other words, the new diagram obtained by bumping the crossing is

$$\mathfrak{B}\big(D, (\rho, \kappa)\big) = (D \setminus \{(\rho, \kappa)\}) \cup \{(\rho, \kappa - 1)\}.$$

Definition 2.5.2. *Given an FPF involution word diagram D, a new involution word diagram $\underline{\mathfrak{p}\big(D\big)}$ is obtained by prepending a wire which does not cross any other wires onto the left side of D, then relabel so that we maintain the underlying set, $\mathbb{P}$. This is equivalent to shifting all of the crossings of D to the right by two. In other words,*

$$\mathfrak{p}\big(D\big) = \{(\rho, \kappa + 2) \mid (\rho, \kappa) \in D\}.$$

Lemma 2.5.3. *Let D be an involution word diagram. Let $(\rho, \kappa) \in D$ with $\kappa > 1$. Then*

$$\mathfrak{p}\Big(\mathfrak{B}(D, (\rho, \kappa))\Big) = \mathfrak{B}\big(\mathfrak{p}(D), (\rho, \kappa + 2)\big).$$

Proof. We have

$$\begin{aligned}
\mathfrak{p}\Big(\mathfrak{B}(D, (\rho, \kappa))\Big) &= \mathfrak{p}\Big(\big(D \setminus \{(\rho, \kappa)\}\big) \cup \{\rho, \kappa - 1\}\Big) \\
&= \Big\{(i, j + 2) \;\Big|\; (i, j) \in \big(D \setminus \{(\rho, \kappa)\}\big) \cup \{\rho, \kappa - 1\}\Big\} \\
&= \Big(\big\{(i, j + 2) \mid (i, j) \in D\big\} \setminus \{(\rho, \kappa + 2)\}\Big) \cup \{\rho, \kappa + 1\} \\
&= \mathfrak{B}\big(\mathfrak{p}(D), (\rho, \kappa + 2)\big).
\end{aligned}$$

$\square$

Lemma 2.5.4. *Let $\pi \in \mathsf{FPF}_{\mathbb{P}}$ and D be an involution word diagram for π which is nearly reduced at $(\rho, \kappa) \in D$. If (ρ, κ) is a type 3 defect, then $\mathfrak{B}(D, (\rho, \kappa))$ does not have a type 3 defect.*

Proof. Let $E = D \setminus \{X\}$. Let $(a, \pi(a))$ be the top label of (ρ, κ) in D. Then by Lemma 2.4.8, $(\pi(a), a)$ is the top label of (ρ, κ) in E. Then wire W_a is in wire positions κ and $\kappa+1$ at the top of row ρ in E, which means that wire position $\kappa-1$ is occupied by a different wire, $W_b \neq W_a$. Thus the new crossing at $(\rho, \kappa-1)$ crosses wires W_a and W_b and is not a type 3 defect.

□

Lemma 2.5.5. *Let $\nu \in \mathsf{FPF}_{\mathbb{P}}$. Let $(a, b) \in Cyc(\nu)$. Let $c \in \mathbb{P}$ with $c > b$. Let $\pi = t_{b,c}\nu t_{b,c}$. Let D be a reduced involution word diagram for π. If $\hat{\ell}_{\mathsf{FPF}}(\pi) = \hat{\ell}_{\mathsf{FPF}}(\nu) + 1$, then D has a crossing with label (b, c). Moreover, D is nearly reduced at this crossing.*

Proof. Let $d = \nu(c)$. Notice $\pi(a) = (t_{b,c}\nu t_{b,c})(a) = t_{b,c}(\nu(a)) = t_{b,c}(b) = c$. Similarly, $\pi(b) = d$. We know $a < b < c$. This leaves 4 possibilities for the relative value of d.

Case 1 ($d < a < b < c$): Let E be a reduced involution word diagram for ν. By Corollary 2.2.9 (ii), E has a crossing with label (b, c). Deleting this crossing yields an involution word diagram for $t_{b,c}\nu t_{b,c} = \pi$ which has fewer crossings than E. Therefore $\hat{\ell}_{\mathsf{FPF}}(\nu) > \hat{\ell}_{\mathsf{FPF}}(\pi)$, which means this case cannot occur.

Case 2 ($a < d < b < c$): A crossing with label (b, c) is guaranteed by Corollary 2.2.9 (ii).

Case 3 ($a < b < d < c$): By Corollary 2.2.9 (ii), wires W_a and W_b cross twice in D. Let $X \in D$ be the crossing with label (b, c) and Y be the other crossing. Deleting X yields an involution word diagram, $D \setminus \{X\}$, for $t_{b,c}\pi t_{b,c} = \nu$. Since X is in a higher row than Y in D, the bottom label of Y is the same in $D \setminus \{X\}$ as it is in D. According to Corollary 2.2.9 (ii), this bottom label of Y is either (a, b) or (d, c). Since $\nu(a) = b$ and $\nu(d) = c$, Y must be a type 3 defect of $D \setminus \{X\}$. Hence

$D \setminus \{X\}$ is not reduced. Therefore $\hat{\ell}_{\mathsf{FPF}}(\nu) < \hat{\ell}_{\mathsf{FPF}}(\pi) - 1$, which means this case cannot occur.

Case 4 ($a < b < c < d$): A crossing with label (b, c) is guaranteed by Corollary 2.2.9 (i).

In cases 2 and 4, D must be nearly reduced at the crossing with label (b, c) because deleting this crossing yields an involution word diagram, F, for $t_{b,c}\pi t_{b,c} = \nu$ which has one fewer crossing than D. Since $\hat{\ell}_{\mathsf{FPF}}(\nu) = \hat{\ell}_{\mathsf{FPF}}(\pi) - 1$, F has exactly $\hat{\ell}_{\mathsf{FPF}}(\nu)$ crossings. Therfore F is reduced. $\square$

Definition 2.5.6. *Let D be an involution word diagram corresponding to $w = w_1 w_2 \cdots w_k$. Fix an even number $m \geq \mathsf{size}(\pi(D))$. Define the m^{th} complement of a grid position (ρ, κ) by $(\rho, \kappa)^{\mathsf{c}} = (\rho, m - \kappa)$. Also, define the m^{th} complement of D by $D^{\mathsf{c}} = \{(\rho, \kappa)^{\mathsf{c}} \mid (\rho, \kappa) \in D\}$. Then D^{c} is an involution word diagram for $w^{\mathsf{c}} = (m - w_1)(m - w_2) \cdots (m - w_k)$. Finally, define the m^{th} complement of a pair $\big(D, (\rho, \kappa)\big)$ by $\big(D, (\rho, \kappa)\big)^{\mathsf{c}} = \big(D^{\mathsf{c}}, (\rho, \kappa)^{\mathsf{c}}\big)$.*

Note that the m here is not needed in the notation as each time we use it, the value of m will either be specified ahead of time or taken to be $\mathsf{size}(\pi(D))$ by default.

Remark. *Taking a complement amounts to flipping the part of the involution word diagram that we care about across a vertical line. Since each defect type is preserved under this flip, the property of being reduced is also preserved. Since labels of crossings are determined by tracing wires, labels of corresponding crossings behave as expected: If X has label (a, b) in D, then X^{c} has label $(\mathsf{size}(\pi(D)) - b, \mathsf{size}(\pi(D)) - a)$ in D^{c}.*

Corollary 2.5.7. *Let $\nu \in \mathsf{FPF}_{\mathbb{P}}$. Let $(a,b) \in$ Cyc(ν). Let $c \in \mathbb{P}$ with $c < a$. Let $\pi = t_{c,a}\nu t_{c,a}$. Let D be a reduced involution word diagram for π. If $\hat{\ell}_{\mathsf{FPF}}(\pi) = \hat{\ell}_{\mathsf{FPF}}(\nu) + 1$, then D has a crossing with label (c,a). Moreover, D is nearly reduced at this crossing.*

Proof. Let $m = \mathsf{size}(\pi(D))$. The diagram D^{c} is a reduced involution word diagram for some permutation ν^{c}. Since $(a,b) \in \mathrm{Cyc}(\nu)$, we also have $(m-b, m-a) \in \mathrm{Cyc}(\nu^{\mathsf{c}})$. Then $\pi^{\mathsf{c}} = \left(t_{c,a}\nu t_{c,a}\right)^{\mathsf{c}} = t_{m-c,m-a}\nu^{\mathsf{c}} t_{m-c,m-a}$ and $\hat{\ell}_{\mathsf{FPF}}(\pi^{\mathsf{c}}) = \hat{\ell}_{\mathsf{FPF}}(\pi) = \hat{\ell}_{\mathsf{FPF}}(\nu) + 1 = \hat{\ell}_{\mathsf{FPF}}(\nu^{\mathsf{c}}) + 1$. By Lemma 2.5.5, D^{c} is nearly reduced at some crossing $X \in D^{\mathsf{c}}$ with label $(m-a, m-c)$. Therefore D is nearly reduced at $X^{\mathsf{c}} \in D$ with label (c,a). $\square$

Definition 2.5.8. *Let $\nu \in \mathsf{FPF}_{\mathbb{P}}$. Let $q \in \mathbb{P}$. Then we define the following:*

$$\Psi^{+}(\nu,q) = \left\{\eta \in \mathsf{FPF}_{\mathbb{P}} \;\middle|\; \hat{\ell}_{\mathsf{FPF}}(\eta) = \hat{\ell}_{\mathsf{FPF}}(\nu)+1 \textit{ and } z = t_{q,j}\cdot\nu\cdot t_{q,j} \textit{ for some } j > q\right\}$$
$$\hat{\Psi}^{-}_{0}(\nu,q) = \left\{\eta \in \mathsf{FPF}_{\mathbb{P}} \;\middle|\; \hat{\ell}_{\mathsf{FPF}}(\eta) = \hat{\ell}_{\mathsf{FPF}}(\nu)+1 \textit{ and } \eta = t_{i,q}\cdot\nu\cdot t_{i,q} \textit{ for some } i < q\right\}$$
$$\hat{\Psi}^{-}(\nu,q) = \begin{cases} \hat{\Psi}^{-}_{0}(\nu,q) & \textit{if } \ \hat{\Psi}^{-}_{0}(\nu,q) \neq \varnothing \\ \hat{\Psi}^{-}_{0}(21 \otimes \nu, q+2) & \textit{otherwise} \end{cases}.$$

Definition 2.5.9. *<u>The generalized bumping algorithm</u>:*

Let $\nu \in \mathsf{FPF}_{\mathbb{P}}$. Let $(a,b) \in$ Cyc(ν). Let $\pi \in \hat{\Psi}^{+}(\nu,b)$. Let D be a reduced FPF involution word diagram for π. The output to this algorithm will be a pair $\big(E,(\rho,\kappa)\big)$, where E is a reduced involution word diagram and $(\rho,\kappa) \in E$ is a crossing.

Since $\pi \in \hat{\Psi}^{+}(\nu,b)$, there is $c > b$ such that $\pi = t_{b,c}\nu t_{b,c}$ and $\hat{\ell}_{\mathsf{FPF}}(\pi) = \hat{\ell}_{\mathsf{FPF}}(\nu) + 1$. Therefore, D has a crossing with label (b,c) by Lemma 2.5.5.

Set $D_0 := D$. Let (ρ_0,κ_0) be the crossing with label (b,c).

If $\kappa_0 = 1$*, stop. Output:* $\mathsf{GBA}\big(D, (\rho_0, \kappa_0)\big) = \Big(\mathfrak{B}\big(\mathfrak{p}(D_0), (\rho_0, 3)\big), (\rho_0, 2)\Big)$

Now set $D_1 := \mathfrak{B}\big(D_0, (\rho_0, \kappa_0)\big)$.

After having found D_i*, we do the following:*

If D_i *is reduced, stop. Output:* $\mathsf{GBA}\big(D, (\rho_0, \kappa_0)\big) = \Big(D_i, (\rho_{i-1}, \kappa_{i-1} - 1)\Big)$.

Find the defect counterpart (ρ_i, κ_i) *of the most recently bumped crossing,* $(\rho_{i-1}, \kappa_{i-1} - 1)$.

If $\kappa_i = 1$*, stop. Output:* $\mathsf{GBA}\big(D, (\rho_0, \kappa_0)\big) = \Big(\mathfrak{B}\big(\mathfrak{p}(D_i), (\rho_i, 3)\big), (\rho_i, 2)\Big)$

Set $D_{i+1} := \mathfrak{B}\big(D_i, (\rho_i, \kappa_i)\big)$.

Remark. *The necessary defect counterparts exist and are unique due to Lemmas 2.4.5 and 2.4.6 and the fact that for each* i*,* D_i *is nearly reduced at the relevant location. Deleting the crossing yields a reduced involution word diagram for* ν *as seen in the proof of Claim 2.5.15.*

Remark. *The input to this algorithm is equivalent to being given a reduced involution word diagram* D *which is nearly reduced at a given starting crossing,* (ρ_0, κ_0).

Definition 2.5.10. *For convenience, we will use the notation* $\mathsf{GBA}_*\big(D, (\rho_0, \kappa_0)\big)$ *to be just the diagram part of the output (ignoring the final crossing).*

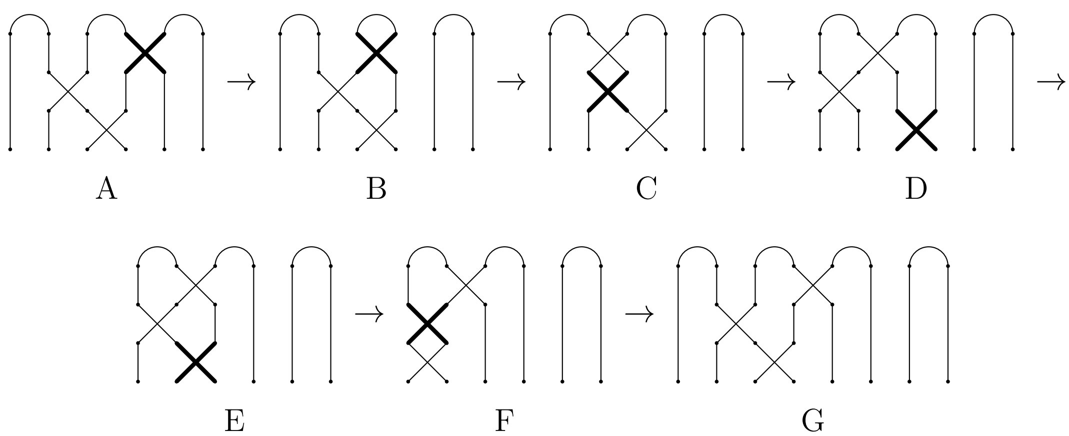
Proposition 2.5.11. *This algorithm terminates.*

Proof. A crossing which starts in column κ can be bumped at most $\kappa - 1$ times. This is because the κ^{th} bump of this crossing would correspond to $\kappa_i = 1$, which ends the algorithm immediately. Since there are finitely many crossings, each only able to be bumped finitely many times, the algorithm must terminate. □

Example 2.5.12. To illustrate the generalized bumping algorithm, we will start with $\nu = 4321 \in \mathsf{FPF}_{\mathbb{P}}$. Then $(2, 3) \in \mathrm{Cyc}(\nu)$. Also, $\pi = 456123 \in \hat{\Psi}^+(\nu, 3)$ because

$\pi = t_{3,5}\, \nu\, t_{3,5}$. The involution word diagram show in part A of the following figure is a reduced FPF involution word diagram for π. We proceed by bumping the bold crossing, which has bottom label $(3, 5)$.

Subsequent bumps are show in the other parts of the figure. In each part, the crossing to be bumped next has been emboldened. The bold crossings in parts B, C, D, E, and F are defect counterparts (of the most recently bumped crossing) of types 3, 1, 2, 3, and 1, respectively. Notice that if we delete the bold crossing in any part, we get a reduced involution word diagram for ν.



Since the bold crossing in part F is in column 1, we prepend a wire to the left side before bumping one final time. Note that the output diagram shown in part G is the same as the input diagram from part A.

Definition 2.5.13. *For $\pi \in \mathsf{FPF}_{\mathbb{P}}$, let $\hat{\mathcal{R}}(\pi)$ be the set of reduced FPF involution word diagrams for π in which every row has a crossing.*

Remark. *For any $\pi \in \mathsf{FPF}_{\mathbb{P}}$, the set $\hat{\mathcal{R}}(\pi)$ is in one to one correspondence with reduced involution words for π.*

Proposition 2.5.14. *Let $\nu \in \mathsf{FPF}_{\mathbb{P}}$. Let $(a,b) \in Cyc(\nu)$. The generalized bumping algorithm yields a map between these two sets:*

$$\bigcup_{\sigma \in \hat{\Psi}^+(\nu,b)} \hat{\mathcal{R}}(\sigma) \xrightarrow{GBA_*} \bigcup_{\sigma \in \hat{\Psi}^-(\nu,a)} \hat{\mathcal{R}}(\sigma).$$

Proof.

Let $\pi \in \hat{\Psi}^+(\nu,b)$. Then $\pi = t_{b,c}\nu t_{b,c}$ for some $c > b$. Let D be a reduced involution word diagram for π. For each i, define D_i, ρ_i, and κ_i as they are in the algorithm. The result, $E = GBA_*\big(D, (\rho_0, \kappa_0)\big)$, is an involution word diagram for an involution $\sigma := \pi(E) \in \mathsf{FPF}_{\mathbb{P}}$. Our goal is to show $\sigma \in \hat{\Psi}^-(\nu,a)$.

We start by assuming for all i, $\kappa_i \neq 1$. Let n be the number of bumps taking place in the algorithm. Then $E = D_n$. Since E is reduced and has the same number of crosses as D, σ has the desired length: $\hat{\ell}_{\mathsf{FPF}}(\sigma) = \hat{\ell}_{\mathsf{FPF}}(\pi) = \hat{\ell}_{\mathsf{FPF}}(\nu) + 1$. We will delay the proof of the following claim.

Claim 2.5.15. *Using the above notation, for each $i \in \{1, \ldots, n\}$,*

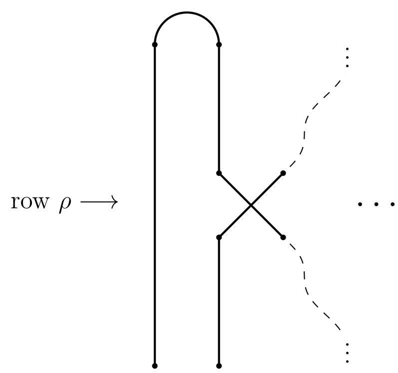
(i) $D_{i-1} \setminus \{(\rho_{i-1}, \kappa_{i-1})\}$ is a reduced involution word diagram for ν, and

(ii) if $(\rho_{i-1}, \kappa_{i-1} - 1)$ is not a type 3 defect of D_i, then there exists $x_i \neq a$ such that $(\rho_{i-1}, \kappa_{i-1} - 1)$ has label (x_i, a).

Let $(\rho, \kappa) = (\rho_{n-1}, \kappa_{n-1} - 1)$. We know $(\rho, \kappa) \in E = D_n$ is not a type 3 defect of E because E is reduced. By Claim 2.5.15 (ii), there exists $x_n \in \mathbb{P}$ such that (ρ, κ) has label (x_n, a) in E. Since E is reduced, we know $x_n < a$ by Corollary 2.2.10. By Claim 2.5.15 (i), $E \setminus \{(\rho, \kappa)\} = D_{n-1} \setminus \{(\rho_{n-1}, \kappa_{n-1})\}$ is a reduced involution word diagram for ν. Therefore $\sigma = t_{x_n,a}\nu t_{x_n,a}$. Hence $\sigma \in \hat{\Psi}_0^-(\nu,a) = \hat{\Psi}^-(\nu,a)$.

Now we instead assume that there is an m so that $\kappa_m = 1$. Using $m+1$ applications of Lemma 2.5.3, the output of the algorithm, $E = \mathfrak{B}\big(\mathfrak{p}(D_m), (\rho_m, 3)\big)$ and $(\rho, \kappa) = (\rho_m, 2)$, is the same as the result of applying the first $m+1$ bumps of the bumping algorithm to $\mathfrak{p}(D)$ with starting crossing $(\rho_0, \kappa_0 + 2)$.

Suppose that E is not reduced. We know that E is nearly reduced at (ρ, κ). Then Lemma 2.4.5 tells us that (ρ, κ) has a defect counterpart Y in E. We know (ρ, κ) is the only crossing in column 2 and there are no crossings in column 1. This is shown in the following figure.



The first two columns of E

Since (ρ, κ) involves wires W_1 and $W_2 \neq W_1$, we know (ρ, κ) is not a type 3 defect. Thus our defect pair is of type 1 or 2. We know that the non-dashed segments (from the above figure) are not involved in any crossings except at (ρ, κ). So Y must only involve the dashed segments. Since the dashed segment coming out the bottom right of (ρ, κ) does not go above row ρ, we know Y must appear below row ρ. Therefore, between (ρ, κ) and Y, wire W_2 arcs over the top of the diagram, but wire W_1 does not. Hence Y is neither a type 1 defect, nor a type 2 defect. In other words, no such defect counterpart Y exists, which implies E is in fact reduced.

Since E is reduced, we know that GBA applied to $\mathfrak{p}(D)$ with starting crossing $(\rho_0, \kappa_0 + 2)$ has exactly $m + 1$ bumps and also outputs the involution word diagram E. When bumping $\mathfrak{p}(D)$, the column values are each 2 more than they are in the original version, which means the smallest "new κ_i" value is 3 and the result which uses the assumption $\kappa_i \neq 1$ for all i can be used. Hence $\sigma \in \hat{\Psi}_0^-(21 \otimes \nu, a + 2)$. We are done as soon as we show $\hat{\Psi}_0^-(21 \otimes \nu, a + 2) = \hat{\Psi}^-(\nu, a)$, which is equivalent to $\hat{\Psi}_0^-(\nu, a) = \varnothing$.

Suppose $\hat{\Psi}_0^-(\nu, a) \neq \varnothing$ and let $\eta \in \hat{\Psi}_0^-(\nu, a)$. Then $\hat{\ell}_{\mathsf{FPF}}(\eta) = \hat{\ell}_{\mathsf{FPF}}(\nu) + 1$ and $\eta = t_{y,a} \nu t_{y,a}$ for some $y < a$. Applying Claim 2.5.15 (ii), there exists $x_m \neq a$ such that $(\rho_{m-1}, \kappa_{m-1} - 1)$ has label (x_m, a) in D_m. This means the crossing $(\rho_m, 1) \in D_m$ either has label (x_m, a) or (a, x_m). Let $F = D_m \setminus \{(\rho_m, 1)\}$. Then F is a reduced involution word diagram for ν. By Lemma 2.4.8, $(\rho_m, 1)$ either has label (x_m, a) or (a, x_m) in F.

Suppose (y, a) is the bottom label of $(\rho_m, 1)$ in F. Then $D_m = F \cup \{(\rho_m, 1)\}$ is an involution word diagram for $t_{y,a} \nu t_{y,a} = \eta$. Also, D_m has one more crossing than F. Since F is reduced and $\hat{\ell}_{\mathsf{FPF}}(\eta) = \hat{\ell}_{\mathsf{FPF}}(\nu) + 1$, D_m must also be reduced. This implies the algorithm would have stopped one step sooner than it actually does, which is a contradiction. Therefore (y, a) is not the bottom label of $(\rho_m, 1)$ in F.

Now instead suppose (x, a) is the bottom label of $(\rho_m, 1)$ in F for some $x \neq y$. By Lemma 2.4.11 (ii), there exist $i, d \in \mathbb{P}$ so that $(\nu(y), d)$ is the top label of (ρ_m, i) in F. Because F has no crossings in row ρ_m, $(y, \nu(d))$ is the bottom label of (ρ_m, i). Since $x \neq y$, we know $i \neq 1$. By Lemma 2.4.11 (i), since $a \neq y$, we know $i \neq 2$. Hence $i > 2$. Since y appears as part of a bottom label to the right of a in row ρ_m and $y < a$, wires W_a and W_y must cross at some $X \in F$ between row ρ_m and the bottom of F. Tracing the wires on the bottom left and bottom right of X

lead to wire positions y and a, respectively. Therefore X has bottom label (y, a). However, there cannot be a crossing in F with label (y, a) because deleting this crossing would yield an involution word diagram for $t_{y,a}\nu t_{y,a} = \eta$ which has fewer crossings than F. This contradicts the fact that $\hat{\ell}_{\mathsf{FPF}}(\eta) = \hat{\ell}_{\mathsf{FPF}}(\nu) + 1$. Thus, for any $x \in \mathbb{P}$, (x, a) is not the bottom label of $(\rho_m, 1)$ in F.

Suppose now that (a, z) is the bottom label of $(\rho_m, 1)$ in F for some $z \in \mathbb{P}$. Then we are again in the situation where y appears as a bottom label to the right of a in row ρ_m, which cannot happen. Thus (a, z) cannot be the bottom label of $(\rho_m, 1)$ in F for any z.

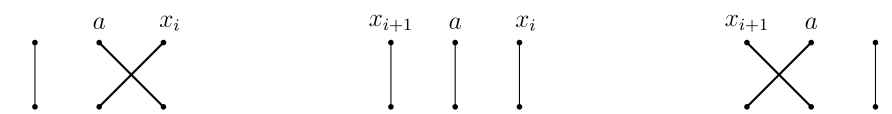
By Lemma 2.4.11 (iii), b must appear to the left of a as a top label in row ρ_m. Thus (a, x_m) cannot be the top label of $(\rho_m, 1)$. Moreover, if (x_m, a) is the top label of $(\rho_m, 1)$, then $x_m = b$, which would mean (b, a) is also the bottom label of $(\rho_m, 1)$. We have already seen that this cannot happen.

In every case, we have reached a contradiction. Therefore, no such η can exist and $\hat{\Psi}_0^-(\nu, a)$ is indeed empty.

□

Proof. (of Claim 2.5.15) For each i, let $\pi_i = \pi(D_i)$. We proceed by induction on i. After the deletion step of the first bump, the result, $D_0 \setminus \{(\rho_0, \kappa_0)\}$, is an involution word diagram for $t_{b,c}\pi t_{b,c} = t_{b,c}(t_{b,c}\nu t_{b,c})t_{b,c} = \nu$. Now fix $i \in \{1, \ldots, n-1\}$ such that D_i does not have a defect of type 3. Assume statements (i) and (ii) hold for this value i. Since $i < n$, D_i is not reduced. Since $D_i \setminus \{(\rho_{i-1}, \kappa_{i-1} - 1)\} = D_{i-1} \setminus \{(\rho_{i-1}, \kappa_{i-1})\}$ is a reduced involution word diagram for ν, D_i is nearly reduced at $(\rho_{i-1}, \kappa_{i-1} - 1)$. By Lemmas 2.4.5 and 2.4.6, $(\rho_{i-1}, \kappa_{i-1} - 1)$ has a unique defect counterpart $(\rho_i, \kappa_i) \in D_i$. By Lemma 2.4.10, (ρ_i, κ_i) has label (a, x_i) in D_i. Since $(\rho_{i-1}, \kappa_{i-1} - 1) \in D_i$ has label (x_i, a), $D_i \setminus \{(\rho_{i-1}, \kappa_{i-1} - 1)\}$ is an involution word diagram for $\nu = t_{x_i,a}\pi_i t_{x_i,a}$, which implies $\pi_i = t_{x_i,a}\nu t_{x_i,a}$. Therefore $D_i \setminus \{(\rho_i, \kappa_i)\}$ is an involution word diagram for $t_{x_i,a}\pi_i t_{x_i,a} = t_{x_i,a}(t_{x_i,a}\nu t_{x_i,a})t_{x_i,a} = \nu$. Also, $D_i \setminus \{(\rho_i, \kappa_i)\}$ is reduced since it has the same number of crossings as $D_{i-1} \setminus \{(\rho_{i-1}, \kappa_{i-1})\}$. There are 2 cases. Case 1: Assume D_{i+1} does not have a defect of type 3.

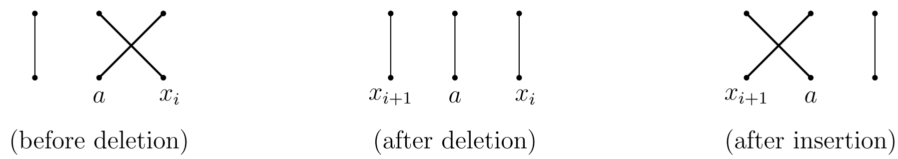
We will now focus on the (up to) 3 wires in row ρ_i of D_i and D_{i+1} which are involved in the bump. By Lemmas 2.4.8 and 2.4.9, deletion and insertion at a position do not change the labels of that position. We have the following figure depicting the relevant columns of row ρ_i before, during, and after the bump. In this figure, and the other figures to follow, labels we are interested in are displayed above or below the relevant wire positions.



(before deletion) (after deletion) (after insertion)

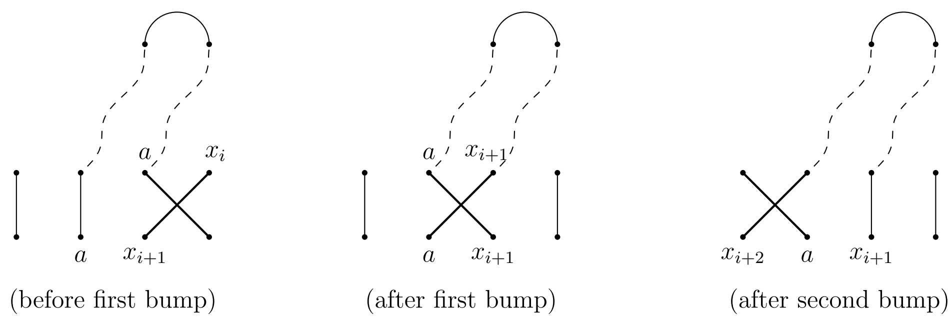
Whatever wire ends up as the left side of the label when the new crossing is inserted becomes x_{i+1}, which means $(\rho_i, \kappa_i - 1)$ has label (x_{i+1}, a) in D_{i+1}. Note that (x_{i+1}, a) is not necessarily a label of $(\rho_i, \kappa_i - 1)$ in D_i (before the deletion). Lemmas 2.4.8 and 2.4.9 do not guarantee that other labels will not change. What is important here is that a is the right side of the label (after the insertion). This must be the case because a is part of the label for both the insertion and deletion. By the uniqueness part of Lemma 2.4.11 (ii), $x_{i+1} \neq a$.

We may instead have bottom labels instead of top labels. The same argument holds.



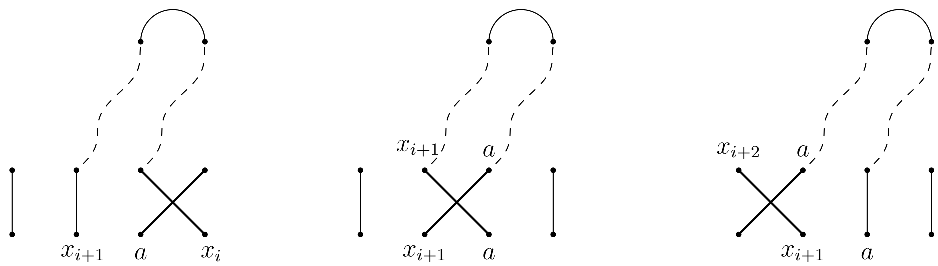
Case 2: We now assume $(\rho_i, \kappa_i - 1)$ is a type 3 defect of D_{i+1}.

By Lemma 2.5.4, (ρ_i, κ_i) is not a type 3 defect of D_i. Hence $(\rho_{i-1}, \kappa_{i-1} - 1)$ is not a type 3 defect of D_i either. We will be doing two consecutive bumps to get past the type 3 defect. Again, using Lemmas 2.4.8 and 2.4.9, if (a, x_i) is the top label of (ρ_i, κ_i) in D_i, then our wires look like this:



Again, we have new values of x_{i+1} and x_{i+2} determined by the labels of the crosses as shown above. The nature of the involvement of the wire W_a is the only thing of consequence here. Note that the existence of x_{i+1} is not guaranteed by our induction assumption, but we also don't need it to obtain a valid value of x_{i+2}, so it only serves as a placeholder in the figures for Case 2.

Just as in case 1, we may have to start with bottom labels instead of top labels.



(before first bump) (after first bump) (after second bump)

To prove the base of the induction for statement (ii), set $i = 0$ and $x_0 = \pi(b)$ and use case 1 or 2 as determined by whether or not D_1 has a defect of type 3.

$\square$

Definition 2.5.16. *Let* $\nu \in \mathsf{FPF}_{\mathbb{P}}$*. Let* $(a, b) \in$ *Cyc*(ν)*. Let* $\sigma \in \hat{\Psi}^{-}(\nu, a)$*. Let* E *be a reduced FPF involution word diagram for* σ*. By Corollary 2.5.7, there must be some* $(\rho, \kappa) \in E$ *so that* $E \setminus \{(\rho, \kappa)\}$ *is an involution word diagram for* ν*. Let* $m = \mathsf{size}(\pi(E)) + 2$*. We now use* m^{th} *complements from Definition 2.5.6. Let*

$$\mathsf{GBA}^{-1}\big(E, (\rho, \kappa)\big) = \Big(\mathsf{GBA}\big(E^{\mathsf{c}}, (\rho, \kappa)^{\mathsf{c}}\big)\Big)^{\mathsf{c}}.$$

Effectively, we can think of GBA^{-1} as the map which comes from unbumping each of the individual bump steps by bumping crossings right instead of left. When "bumping right," we never need to prepend because we always have a wire which does not cross any wires to the right of all of our crossings. In other words, the prepend is done preemptively via using $m = \mathsf{size}(\pi(E)) + 2$ instead of $m = \mathsf{size}(\pi(E))$.

Therefore, if GBA applied to D starting at (ρ, κ) does not require a prepend (at the end), then

$$\mathsf{GBA}^{-1}\big(\mathsf{GBA}\big(D, (\rho, \kappa)\big)\big) = \big(D, (\rho, \kappa)\big),$$

and if it does require a prepend, then

$$\mathsf{GBA}^{-1}\big(\mathsf{GBA}\big(D,(\rho,\kappa)\big)\big) = \big(\mathfrak{p}(D),(\rho,\kappa+2)\big).$$

Proposition 2.5.17. *Let $\nu \in \mathsf{FPF}_{\mathbb{P}}$. Let $(a,b) \in Cyc(\nu)$. The map*

$$\bigcup_{\sigma\in\hat{\Psi}^{+}(\nu,b)} \hat{\mathcal{R}}(\sigma) \xrightarrow{GBA_*} \bigcup_{\sigma\in\hat{\Psi}^{-}(\nu,a)} \hat{\mathcal{R}}(\sigma).$$

is bijective.

Proof. We start by showing injectivity. Let $\pi, \pi' \in \hat{\Psi}^{+}(\nu, b)$. Let D and D' be reduced involution word diagrams for π and π', respectively. Define ρ_0 and κ_0 as they are in the algorithm. Similarly, define the starting crossing $(\rho_0', \kappa_0') \in D'$. Assume the generalized bumping algorithm yields the same involution word diagram $E := \mathsf{GBA}_*\big(D,(\rho_0,\kappa_0)\big) = \mathsf{GBA}_*\big(D,(\rho_0',\kappa_0')\big)$ for an involution $\sigma \in \mathsf{FPF}_{\mathbb{P}}$. Let $(\rho,\kappa),(\rho',\kappa') \in E$ be the ending crossings for the two bumping algorithm implementations. Then (ρ,κ) has label (x,a) for some $x < a$. Also, (ρ',κ') has label (y,a) for some $y < a$. As in the proof of Proposition 2.5.14, deleting either of the ending crossings (ρ,κ) or (ρ',κ') yields a reduced involution word diagram for ν. Therefore, $\nu = t_{x,a}\sigma t_{x,a} = t_{y,a}\sigma t_{y,a}$. Since E is reduced, the ending crossings cannot be type 3 defects. Thus, we have $x \neq \sigma(a)$ and $y \neq \sigma(a)$. Now,

$$\nu(x) = (t_{x,a}\sigma t_{x,a})(x) = t_{x,a}(\sigma(t_{x,a}(x))) = t_{x,a}(\sigma(a)) = \sigma(a).$$

Similarly, $\nu(y) = (t_{y,a}\sigma t_{y,a})(y) = \sigma(a)$. But $\nu(x) = \nu(y)$ implies $x = y$. Since there can only be one crossing with label $(x,a) = (y,a)$, we must have $(\rho,\kappa) = (\rho',\kappa')$.

There are two cases:

Case 1: Assume the GBA applied to D does not involve a prepend (at the last step). Then $\hat{\Psi}_0^-(\nu, a) \neq \varnothing$, which means the GBA applied to D' also does not involve a prepend. Thus applying GBA^{-1} to E starting at $(\rho, \kappa) = (\rho', \kappa')$ will output $\big(D, (\rho_0, \kappa_0)\big)$, which tells us that $D = D'$ (also, $\pi = \pi'$ and $(\rho_0, \kappa_0) = (\rho_0', \kappa_0')$).

Case 2: Assume the GBA applied to D does involve a prepend (at the last step). Then $\hat{\Psi}_0^-(\nu, a) = \varnothing$ and $\hat{\Psi}^-(\nu, a) = \hat{\Psi}_0^-(21 \otimes \nu, a + 2)$, which means the GBA applied to D' also involves a prepend. Therefore, applying the GBA to $\mathfrak{p}(D)$ and $\mathfrak{p}(D')$ yield the outputs $\big(E, (\rho, \kappa)\big)$ and $\big(E, (\rho, \kappa)\big)$, respectively, without using a prepend (at the end of the algorithm). By Case 1, we have $\mathfrak{p}(D) = \mathfrak{p}(D')$, which implies $D = D'$.

Hence GBA_* is injective.

To show surjectivity, we start with any $\sigma \in \hat{\Psi}^-(\nu, a)$ and any involution word diagram, E, for σ. By Corollary 2.5.7, there must be some $(\rho, \kappa) \in E$ so that $E \setminus \{(\rho, \kappa)\}$ is an involution word diagram for ν. There are two cases.

Case 1: Assume $\hat{\Psi}^-(\nu, a) = \hat{\Psi}_0^-(\nu, a)$. We apply our inverse bumping map to E to arrive at $\big(D, (\rho_0, \kappa_0)\big) := \mathsf{GBA}^{-1}\big(E, (\rho, \kappa)\big)$. By an argument similar to (almost) the entire proof of Proposition 2.5.14, D is an involution word diagram for some $\pi = t_{b,c} \nu t_{b,c} \in \hat{\Psi}^+(\nu, b)$ and (ρ_0, κ_0) has label (b, c) for some $c > b$. Hence $\mathsf{GBA}_*\big(D, (\rho_0, \kappa_0)\big) = E$.

Case 2: Assume $\hat{\Psi}^-(\nu, a) = \hat{\Psi}_0^-(21 \otimes \nu, a + 2)$. Hence $\hat{\Psi}_0^-(\nu, a) = \varnothing$. We apply our inverse bumping map to E to arrive at $\big(D, (\rho_0, \kappa_0)\big) := \mathsf{GBA}^{-1}\big(E, (\rho, \kappa)\big)$. Since $E \setminus \{(\rho, \kappa)\}$ is a reduced involution word diagram for $21 \otimes \nu$, Corollary 2.2.9 tells us that $W_1 = W_2$ does not cross any wire in D. Hence we know that $E \setminus \{(\rho, \kappa)\}$ has no crossings in the first two columns. Thus E has at most one crossings in the first two columns. Since E is reduced, we know that the crossing (ρ, κ) is not a type 3 defect of E, so it cannot be in column 1. If (ρ, κ) is in column 2, since (ρ, κ) is the first crossing to be bumped "right" (see comments after Definition 2.5.16), the output D of the inverse bumping map will not have any crossings in the first two columns. In the case where (ρ, κ) is not in column 2 of E, then E has no crossings in the first two columns, which means neither does D. Hence we know that $D = \mathfrak{p}(F)$ for some reduced involution word diagram F. We know D is a reduced involution word diagram for some $\pi = t_{b+2,c}(21 \otimes \nu)t_{b+2,c} \in \hat{\Psi}^+(21 \otimes \nu, b+2)$ and $(\rho_0, \kappa_0) \in D$ has label $(b+2, c)$ for some $c > b+2$. Therefore F is a reduced involution word diagram for $\eta = t_{b,c-2}\nu t_{b,c-2} \in \hat{\Psi}^+(\nu, b)$, $(\rho_0, \kappa_0 - 2) \in F$ has label $(b, c-2)$, and $c - 2 > b$. Moreover, since $\hat{\Psi}_0^-(\nu, a) = \varnothing$, applying GBA to F starting at $(\rho_0, \kappa_0 - 2)$ will end in a prepend. Hence $\mathsf{GBA}_*\big(F, (\rho_0, \kappa_0 - 2)\big) = \mathsf{GBA}_*\big(D, (\rho_0, \kappa_0)\big) = E$.

Thus the map GBA_* is surjective. □

2.6. The Lexicographically Largest Crossing

The specific bumping algorithm defined in this section is inspired by Little's algorithm (see Algorithm 1 in [4]). The lemma and proposition in this section reflect ideas from the same article.

Definition 2.6.1. *Fix $\pi \in \mathsf{FPF}_{\mathbb{P}} \setminus \{\theta\}$ and a reduced involution word diagram D for π. The lexicographically largest crossing is the unique crossing in D with label (r, s) where:*

- $s = \mathsf{size}(\pi)$.

– *r is the maximal index so that $r < s$, $\pi(r) > \pi(s)$, and $r \neq \pi(s)$.*

Proof. (r is well defined)

First, we show s is even. Suppose not, i.e. suppose s is odd. We then have $\theta(s) = s - (-1)^s = s + 1$. Since $s + 1 > s$, we must have $\pi(s+1) = \theta(s+1) = s+1-(-1)^{s+1} = s+1-1 = s$. Since π is an involution, this also yields $\pi(s) = s+1$. This contradicts the fact that $\pi(s) \neq \theta(s)$. Therefore s must indeed be even.

Second, we show $\pi(s) \neq s - 1$. Since s is even, $\theta(s) = s - (-1)^s = s - 1$. We know from the definition of $\mathsf{size}(\pi)$ that $\pi(s) \neq \theta(s)$.

Third, we show $\pi(s) < s$. Suppose not, i.e. suppose $\pi(s) > s$. From the definition of $\mathsf{size}(\pi)$, we have $\theta(\pi(s)) = \pi(\pi(s)) = s$. But $\theta(s) = s - (-1)^s = s - 1 \neq \pi(s)$. This contradicts the fact that θ is an involution. Therefore $\pi(s) < s$.

Fourth, we show $\pi(s-1) \leq s$. Again, we suppose not, i.e. suppose $\pi(s-1) > s$. We then have $\theta(\pi(s-1)) = \pi(\pi(s-1)) = s - 1$. But $\theta(s-1) = s - 1 - (-1)^{s-1} = s \neq s - 1$. This contradicts the fact that θ is an involution. Therefore $\pi(s) < s$.

Hence $\pi(s) < s - 1$ and $\pi(s-1) < s$.

Finally, let $A = \{x \in \mathbb{P} \mid x < s,\ \pi(x) = s - 1 > \pi(s),\ \text{and } x \neq \pi(s)\}$. We know that $\pi(s-1) \in A$ because $\pi(s-1) < s$, $\pi(\pi(s-1)) = s - 1 > \pi(s)$, and $\pi(s-1) \neq \pi(s)$. Thus $A \neq \varnothing$. Because it is bounded above by s, A is finite. Therefore A has a maximal element and r is well defined. $\square$

Proof. (there is a unique crossing with label (r, s))

We start by showing $\pi(r) < s$. Suppose not, i.e. suppose $\pi(r) \geq s$. We know $\pi(r) \neq s$ because $r \neq \pi(s)$. Hence $\pi(r) > s$, which means $\theta(\pi(r)) = \pi(\pi(r))$. We then have

$$s - 1 \leq s - (-1)^{\pi(r)} < \pi(r) - (-1)^{\pi(r)} = \theta(\pi(r)) = \pi(\pi(r)) = r < s.$$

This means that r is an integer strictly between $s-1$ and s. This is a contradiction. Therefore $\pi(r) < s$.

Let D be any reduced involution word diagram for π. Given the inequalities $r < s$, $\pi(s) < s$, and $\pi(r) < s$, there are three possible orderings for r, $\pi(r)$, s, and $\pi(s)$.

- Case 1 ($\pi(s) < r < \pi(r) < s$)

  This case is not possible. We have $\pi(r) < s$, $\pi(\pi(r)) = r > \pi(s)$ and $\pi(r) \neq \pi(s)$. Since $\pi(r) > r$, this contradicts the maximality of r.

- Case 2 ($\pi(s) < \pi(r) < r < s$)

  By Corollary 2.2.9, there are exactly 2 crossings of wires W_r and W_s, One has labels $(\pi(r), s)$ and $(\pi(s), r)$. The other crossing has labels $(\pi(s), \pi(r))$ and (r, s).

- Case 3 ($r < \pi(s) < \pi(r) < s$)

  By Corollary 2.2.9, there is exactly 1 crossing of wires W_r and W_s, which has label (r, s).

Therefore (r, s) is necessarily a label for exactly 1 crossing in D. □

Claim 2.6.2. *This choice of r and s is lexicographically largest in the sense that: If you list the top and bottom label pairs (i, j) for all crossings $X \in D$, then s will be the largest such j on the list and r will be the largest i among those paired with s.*

Proof. (of Claim 2.6.2)

Let $X \in D$ be any crossing and let (t, u) be a label of X. Our first goal here is to show $u \leq s$. Suppose not, i.e. suppose $u > s$. Since D is reduced, $W_t \neq W_u$, which implies $\pi(t) \neq u$. By Corollary 2.2.10, $t < u$.

Since $u > s$, we have $\pi(u) = \theta(u) = u - (-1)^u \geq u - 1 \geq s$. Since $\pi(s) < s$, we must have $\pi(u) \neq s$. Thus $\pi(u) > s$. Since $X \in X_D(t, u)$, we know $X_D(t, u) \neq \varnothing$. By Lemma 2.2.7, since $t < u$ and $\pi(u) = u \pm 1$, we can conclude $|X_D(t, u)| = 2$ and $\pi(t) > u$. Hence $\pi(t) > s$. Thus $u < \pi(t) = \theta(t) = t - (-1)^t \leq t + 1 \leq u$ is a contradiction. Therefore $u \leq s$.

We now restrict our attention to the case where $u = s$. Our goal here is to show $t \leq r$. Since (t, s) is a label of X and D is reduced, $t < s$ by Corollary 2.2.10. Since $(\pi(s), \pi(t))$ is the other label of X, $\pi(s) < \pi(t)$ for the same reason. Since X is not a type 3 defect, $t \neq \pi(s)$. Thus t satisfies the three inequalities which define r. By maximality of r, $t \leq r$. □

Lemma 2.6.3. *Let $\pi \in \mathsf{FPF}_{\mathbb{P}}$ and D be a reduced FPF involution word diagram for π. Let $X \in D$ be the lexicographically largest crossing. Then D is nearly reduced at X.*

Proof. Let r and s be defined as they are in Definition 2.6.1. Let $E = D \setminus \{X\}$ and $\sigma = \pi(E)$.

Then D is nearly reduced at X if and only if E is reduced. Also, E is reduced if and only if $\hat{\ell}_{\mathsf{FPF}}(\sigma) = \hat{\ell}_{\mathsf{FPF}}(\pi) - 1$.

We will use the notation $I_{x,y}(\pi)$ from the proof of Lemma 2.4.5 to denote the set of (half of the) FPF-inversions of π involving only $x, y, \pi(x)$, or $\pi(y)$. We know that by deleting the crossing X, wires W_r and W_s cross one fewer time, which means $|I_{r,s}(\sigma)| = |I_{r,s}(\pi)| + 1$.

Also, for two distinct wires $W_a, W_b \notin \{W_r, W_s\}$, the relative order of $a, b, \pi(a) = \sigma(a)$, and $\pi(b) = \sigma(b)$ is unchanged by deleting X.

This leaves us to check that the number of inversions which involve a third wire is the same in π as in σ. Let $x \in \mathbb{P} \setminus \{r, s, \pi(r), \pi(s)\}$. All we need to do is verify that for our arbitrary x, $|I_{x,r}(\pi)| + |I_{x,s}(\pi)| = |I_{x,r}(\sigma)| + |I_{x,s}(\sigma)|$.

We first use Lemma 2.2.7 to restrict our attention to relative orders of $r, s, \pi(r)$, and $\pi(s)$ which yield a crossing in D.

Next we note that if $\pi(s) < \pi(x)$, then we must have $x < r$ in order to not violate the maximality of r. Similarly, if $\pi(s) < x$, then $\pi(x) < r$. This leaves us with twelve cases for the relative orders of $x, r, s, \pi(x), \pi(r)$, and $\pi(s)$. Each of these 12 cases is checked in the following table, which concludes the proof.

#	$*$	Relative Order	$\lvert I_{x,r}(*)\rvert$	$\lvert I_{x,s}(*)\rvert$	$\lvert I_{x,r}(*)\rvert + \lvert I_{x,s}(*)\rvert$
1	π	$x < \pi(x) < r < \pi(s) < \pi(r) < s$	0	0	0
	σ	$x < \sigma(x) < r < \sigma(r) < \sigma(s) < s$	0	0	
2	π	$x < r < \pi(x) < \pi(s) < \pi(r) < s$	1	0	1
	σ	$x < r < \sigma(x) < \sigma(r) < \sigma(s) < s$	1	0	
3	π	$x < r < \pi(s) < \pi(x) < \pi(r) < s$	1	1	2
	σ	$x < r < \sigma(r) < \sigma(x) < \sigma(s) < s$	2	0	
4	π	$x < r < \pi(s) < \pi(r) < \pi(x) < s$	2	1	3
	σ	$x < r < \sigma(r) < \sigma(s) < \sigma(x) < s$	2	1	
5	π	$r < x < \pi(x) < \pi(s) < \pi(r) < s$	2	0	2
	σ	$r < x < \sigma(x) < \sigma(r) < \sigma(s) < s$	2	0	
6	π	$x < \pi(x) < \pi(s) < \pi(r) < r < s$	0	0	0
	σ	$x < \sigma(x) < \sigma(r) < \sigma(s) < r < s$	0	0	
7	π	$x < \pi(s) < \pi(x) < \pi(r) < r < s$	0	1	1
	σ	$x < \sigma(r) < \sigma(x) < \sigma(s) < r < s$	1	0	
8	π	$x < \pi(s) < \pi(r) < \pi(x) < r < s$	1	1	2
	σ	$x < \sigma(r) < \sigma(s) < \sigma(x) < r < s$	1	1	
9	π	$x < \pi(s) < \pi(r) < r < \pi(x) < s$	2	1	3
	σ	$x < \sigma(r) < \sigma(s) < r < \sigma(x) < s$	2	1	
10	π	$\pi(s) < x < \pi(x) < \pi(r) < r < s$	0	2	2
	σ	$\sigma(r) < x < \sigma(x) < \sigma(s) < r < s$	2	0	
11	π	$\pi(s) < x < \pi(r) < \pi(x) < r < s$	1	2	3
	σ	$\sigma(r) < x < \sigma(s) < \sigma(x) < r < s$	2	1	
12	π	$\pi(s) < \pi(r) < x < \pi(x) < r < s$	2	2	4
	σ	$\sigma(r) < \sigma(s) < x < \sigma(x) < r < s$	2	2	

$\square$

Definition 2.6.4. *The bumping algorithm:*

Let $\pi \in \mathsf{FPF}_{\mathbb{P}}$. Let D be a reduced FPF involution word diagram for π. The output to this algorithm will be a pair $\big(E, (\rho, \kappa)\big)$, where E is a reduced involution word diagram and $(\rho, \kappa) \in E$ is a crossing.

Set $D_0 := D$.

Let (ρ_0, κ_0) be the crossing with label (r, s), which is guaranteed by Definition 2.6.1.

If $\kappa_0 = 1$, stop. Output: $\mathsf{GBA}\big(D, (\rho_0, \kappa_0)\big) = \Big(\mathfrak{B}\big(\mathfrak{p}(D_0), (\rho_0, 3)\big), (\rho_0, 2)\Big)$

Now set $D_1 := \mathfrak{B}\big(D_0, (\rho_0, \kappa_0)\big)$.

After having found D_i, we do the following:

If D_i is reduced, stop. Output: $\mathsf{GBA}\big(D, (\rho_0, \kappa_0)\big) = \Big(D_i, (\rho_{i-1}, \kappa_{i-1} - 1)\Big)$.

Find the defect counterpart (ρ_i, κ_i) of the most recently bumped crossing, $(\rho_{i-1}, \kappa_{i-1} - 1)$.

If $\kappa_i = 1$, stop. Output: $\mathsf{GBA}\big(D, (\rho_0, \kappa_0)\big) = \Big(\mathfrak{B}\big(\mathfrak{p}(D_i), (\rho_i, 3)\big), (\rho_i, 2)\Big)$

Set $D_{i+1} := \mathfrak{B}\big(D_i, (\rho_i, \kappa_i)\big)$.

In Example 2.5.12, the starting crossing for the generalized bumping algorithm had top label $(r, s) = (2, 6)$. Hence this was actually an example of the specific bumping algorithm as well.

Proposition 2.6.5. *Let $\pi \in \mathsf{FPF}_{\mathbb{P}}$. Let r and s be defined as they are in Definition 2.6.1. Let $\nu = t_{r,s}\pi t_{r,s}$.*

$$\text{If } \pi(s) < r, \text{ then } \quad \hat{\Psi}^{+}(\nu, r) = \{\pi\}.$$

$$\text{If } r < \pi(s), \text{ then } \quad \hat{\Psi}^{+}(\nu, \pi(s)) = \{\pi\}.$$

Proof. We have $\pi = t_{r,s}\nu t_{r,s}$. By Lemma 2.6.3, any reduced involution word diagram for π is nearly reduced at the crossing with label (r,s). Therefore $\ell(\pi) = \ell(\nu)+1$. Let D be a reduced involution word diagram for π and let $X \in D$ be the crossing with label (r,s). Let $D_X = D \setminus \{X\}$. Then D_X is a reduced involution word diagram for ν.

We first assume $\pi(s) < r$. We know that $\pi \in \hat{\Psi}^+(\nu, r)$ because $s > r$.

Suppose there is an involution $\eta \in \hat{\Psi}^+(\nu, r) \setminus \{\pi\}$. Then $\eta = t_{r,x}\nu t_{r,x}$ and $\ell(\eta) = \ell(\nu)+1$ for some $x > r$ with $x \neq s$. We know $\pi(s) < \pi(r) < r$ by the proof of (the uniqueness of the crossing in) Definition 2.6.1. Thus $x \notin \{\pi(r), \pi(s)\}$. Let E be a reduced involution word diagram for η. Lemma 2.5.5 tells us there is a crossing $Y \in E$ with label (r,x). Let $E_Y = E \setminus \{Y\}$. Then E_Y is a reduced involution word diagram for ν.

Since D_X is a reduced involution word diagram for ν and every crossing is left of column s, we know $\mathsf{size}(\nu) \leq s$. Hence E_Y does not have any crossings to the right of column $s-1$. Let κ be the column containing Y.

Suppose $\kappa \geq s$. Then since there are no other crossings in column κ, the bottom label of Y is $(\kappa, \kappa+1)$. Since $r < s$, we know (r,x) cannot be the bottom label of Y. Also, since $\nu(r) = \pi(s) < s$, we know $(\eta(x), \eta(r)) = (\nu(r), \nu(x))$ cannot be the bottom label of Y, which means (r,x) cannot be the top label of Y. Thus, we can conclude $\kappa < s$, which implies $x \leq s$. Since $x \neq s$, we have $x < s$.

By the maximality of r in Definition 2.6.1, we know that $\pi(x) < \pi(s)$, which means $(x,s) \notin \mathrm{Inv}_{\mathsf{FPF}}(\pi)$. Thus by Corollary 2.3.4, there cannot be a crossing in D with label (x,s). Therefore we have $\pi(x) < \pi(s) < \pi(r) < r < x < s$. By Corollary 2.2.9, there is a crossing $Z \in D$ with label (r,x).

We now want to find the label of Z in D_X. When we delete the crossing X from D, we must preserve either the top or bottom label of Z. This is because the two parts of the wire W_r that we follow to determine that labels of Z are (r, x) and $(\pi(x), \pi(r))$ cannot both be involved in the crossing X, so one of those parts must take the exact same path to the exact same wire position at the bottom of the diagram. We have two cases:

Case 1: Assume the label (r, x) is preserved, i.e. (r, x) is a label of Z in D_X. Then by Corollary 2.3.4, $(r, x) \in \mathrm{Inv}_{\mathsf{FPF}}(\nu)$. Again, by Corollary 2.3.4, (r, x) is the label of some crossing $Z' \in E_Y$. Deleting this crossing gives us a diagram $E_Y \setminus \{Z'\}$ for $t_{r,x}\nu t_{r,x} = \eta$ with fewer crossings than E. Since E is reduced, this is a contradiction. Thus (r, x) cannot be a label of Z in D_X.

Case 2: Assume the other label, $(\pi(x), \pi(r))$, is preserved, i.e. $(\pi(x), \pi(r))$ is a label of Z in D_X. Since $\pi(r) = \nu(s)$, the other label of Z in D_X is (s, x). Since $s > x$ and D_X is reduced, this contradicts Corollary 2.2.10.

In both cases, we have reached a contradiction. Therefore no such η exists and $\hat{\Psi}^+(\nu, r) = \{\pi\}$.

We now instead assume $r < \pi(s)$. Then $r < \pi(s) < \pi(r) < s$ by the proof of (the uniqueness of the crossing in) Definition 2.6.1. We know that $\pi \in \hat{\Psi}^+(\nu, \pi(s))$ because $s > \pi(s)$.

Suppose there is an involution $\eta \in \hat{\Psi}^+(\nu, \pi(s)) \setminus \{\pi\}$. Then $\eta = t_{\pi(s),x}\nu t_{\pi(s),x}$ and $\ell(\eta) = \ell(\nu) + 1$ for some $x > \pi(s)$. Since $\eta \neq \pi$, we know $x \neq \pi(r)$. Let E be a reduced involution word diagram for η. Lemma 2.5.5 tells us there is a crossing $Y \in E$ with label $(\pi(s), x)$. Let $E_Y = E \setminus \{Y\}$. Then E_Y is a reduced involution word diagram for ν.

By the same argument as in the case where $\pi(s) < r$, we know $x \leq s$. However, it is no longer obvious that $x \neq s$.

Suppose $x = s$. Then $\eta = t_{\nu(r),s}\nu t_{\nu(r),s}$ implies $\eta(r) = s$ and $\eta(\nu(r)) = \nu(s)$. Now our ordering $r < \nu(r) < \nu(s) < s$ tells us that wires W_r and $W_{\nu(r)}$ cross twice in E. By Corollary 2.2.9, Y is the top-most crossing of these two wires. Let Z be the other crossing. Then Z has labels $(r, \nu(r))$ and $(\nu(s), s)$. We know that deleting Y will preserve one of these labels (whichever is the bottom label). Thus the crossing Z either has label $(r, \nu(r)$ or $(\nu(s), s)$ in E_Y. Either of these labels imply that Z is a type 3 defect in E_Y, contradicting the fact that E_Y is reduced. Therefore $x \neq s$, which implies $x < s$.

Since $\pi(x)$ satisfies $\pi(x) < s$, $\pi(\pi(x)) > \pi(s)$ and $\pi(x) \neq \pi(s)$, in order to not violate the maximality of r in Definition 2.6.1, we must have $\pi(x) < r$. Hence $\pi(x) < \pi(s)$, which means $(x, s) \notin \mathrm{Inv}_{\mathsf{FPF}}(\pi)$. Thus by Corollary 2.3.4, there cannot be a crossing in D with label (x, s). Therefore we have $\pi(x) < r < \pi(s) < x < s$. By Corollary 2.2.9, there is a crossing $Z \in D$ with label $(\pi(s), x)$.

We now want to find the label of Z in D_X. When we delete the crossing X from D, we must preserve either the top or bottom label of Z. Just as before have two cases:

Case 1: Assume the label $(\pi(s), x)$ is preserved, i.e. $(\pi(s), x)$ is a label of Z in D_X. Then by Corollary 2.3.4, $(\pi(s), x) \in \mathrm{Inv}_{\mathsf{FPF}}(\nu)$. Again, by Corollary 2.3.4, $(\pi(s), x)$ is the label of some crossing $Z' \in E_Y$. Deleting this crossing gives us a diagram $E_Y \setminus \{Z'\}$ for $t_{\pi(s),x}\nu t_{\pi(s),x} = \eta$ with fewer crossings than E. Since E is reduced, this is a contradiction. Thus $(\pi(s), x)$ cannot be a label of Z in D_X.

Case 2: Assume the other label, $(\pi(x), s)$, is preserved, i.e. $(\pi(x), s)$ is a label of Z in D_X. Since $\pi(x) = \nu(x)$, the other label of Z in D_X is $(\nu(s), x)$. Since D_X

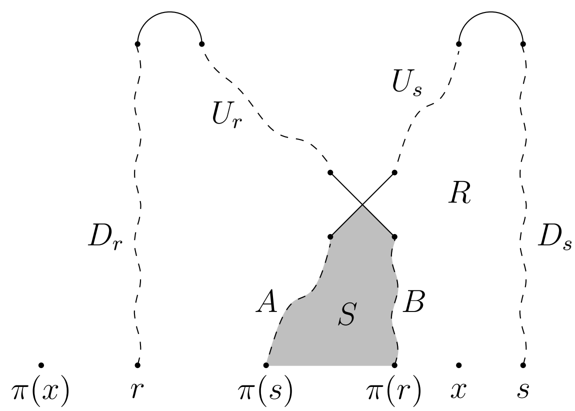


FIGURE 4.

is reduced, Corollary 2.2.10 tells us that $x > \nu(s) = \pi(r)$. We now segment wires W_r and W_s in D (see Figure 4). Let A be the part of W_s which comes out of the left side of the bottom of X and descends to wire position $\pi(s)$ at the bottom of D. Let B be the part of W_r which comes out of the right side of the bottom of X and descends to wire position $\pi(r)$ at the bottom of D. Similarly, U_A, D_A, U_B, and D_B are the other dashed segments labeled in Figure 4. Let S be the region enclosed by W_r, W_s, and the bottom of the diagram below the crossing X (as shown). Let R be the (unshaded) region to the right of S.

Now, we will follow the path of W_x starting at wire position x at the bottom of D. At the bottom of the diagram, W_x is in region R. The point where W_x arcs over the top of the diagram is not in R. Thus we must have W_x leave R by crossing B, U_s or D_s on the way up to the top of D. If we cross D_s, that crossing would have bottom label (x, s), which is not possible.

Suppose now that we cross B. Then we are in region S and must leave. We may not cross B again as this would create a type 1 defect with the crossing used to enter S. Thus we must cross A. But a crossing with A would have bottom label $(\pi(s), x)$. Since this crossing is below X, its bottom label would be preserved when deleting X. This situation has already been covered by Case 1. Thus we may assume that we do not cross B on the way up.

Therefore the first of these wires which we cross is U_s. Since $\pi(x) < \pi(s) < x < s$, by Corollary 2.2.9, there is only one crossing of wires W_x and W_s. Therefore we cannot reenter the region $S \cup R$ enclosed by the wire W_s. This means that on our way back down after arcing over at the top of D, we still cannot cross B.

We know that $\pi(x) < r < \pi(r) < x$. By Corollary 2.2.9, there is a crossing $Z' \in D$ with label $(\pi(x), r)$. Since Z' does not involve segment B of W_r, we know that the label $(\pi(x), r)$ is preserved when deleting X. In other words, Z' has label $(\pi(x), r)$ in D_X because the part of the wire W_r which leads to wire position r is the same as it is in D. Since $\pi(s) = \nu(r)$, the other label of Z' in D_X is $(\pi(s), x)$. Just as in Case 1, we know this is not possible.

Again, in both cases, we have reached a contradiction. Therefore no such η exists and $\hat{\Psi}^+(\nu, \pi(s)) = \{\pi\}$. □

Theorem 2.6.6. *Let $\pi \in \mathsf{FPF}_{\mathbb{P}}$. Let r and s be defined as they are in Definition 2.6.1. Let $\nu = t_{r,s}\pi t_{r,s}$.*

If $\pi(s) < r$, then the bumping algorithm yields a bijection

$$\hat{\mathcal{R}}(\pi) \longrightarrow \bigcup_{\sigma \in \hat{\Psi}^-(\nu, \pi(s))} \hat{\mathcal{R}}(\sigma).$$

If $r < \pi(s)$, then the bumping algorithm yields a bijection

$$\hat{\mathcal{R}}(\pi) \longrightarrow \bigcup_{\sigma \in \hat{\Psi}^-(\nu, r)} \hat{\mathcal{R}}(\sigma).$$

Proof. If $\pi(s) < r$, then $(\pi(s), r) \in \mathrm{Cyc}(\nu)$. If $r < \pi(s)$, then $(r, \pi(s)) \in \mathrm{Cyc}(\nu)$. The result follows directly from combining Propositions 2.5.17 and 2.6.5. $\square$

REFERENCES CITED

[1] Zachary Hamaker, Eric Marberg, and Brendan Pawlowski. Transition formulas for involution Schubert polynomials. Sel. Math. New Ser., 24:2991–3025, 2018.

[2] Zachary Hamaker, Eric Marberg, and Brendan Pawlowski. Fixed-point-free involutions and Schur P-positivity. J. Combin., 11:65–110, 2020.

[3] A. Lascoux and M.P. Schützenberger. Schubert polynomials and the Littlewood-Richardson rule. Lett. Math. Phys., 10 (2-3):111–124, 1985.

[4] D. P. Little. Combinatorial aspects of the Lascoux-Schützenberger tree. Advances in Mathematics, 174:236–253, 2003.

[5] R. P. Stanley. on the number of reduced decompositions of elements of Coxeter groups. Europ. J. Combinatorics, 5:359–372, 1984.