FAMILIES OF DIFFERENTIAL OPERATORS ACTING ON OVERCONVERGENT HILBERT MODULAR FORMS

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DISSERTATION ABSTRACT

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We construct differential operators acting on overconvergent Hilbert modular forms. This extends work of Katz in the case of p-adic Hilbert modular forms in [Kat78], and of Harron–Xiao and Liu for overconvergent Siegel modular forms in [HX14, Liu19a]. The result has applications to the construction of p-adic L-functions in the presence of a Damerell-type formula.

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CHAPTER I

INTRODUCTION

In this thesis, we construct differential operators acting on overconvergent Hilbert modular forms. This extends work of Katz in the case of p-adic Hilbert modular forms, and of Harron–Xiao and Liu for overconvergent Siegel modular forms. The result has applications to the construction of p-adic L-functions.

1.1 Motivation

In [Ser72], Serre introduced the idea of using p-adic families of modular forms to p-adically interpolate values of L-functions. In particular, he used the family of Eisenstein series with q-expansion

$$2G_{2k}(q) = \zeta(1-2k) + 2\sum_{n=1}^{\infty} \sigma_{2k-1}(n)q^n.$$

He showed that congruences between the coefficients of q^n for $n \geq 1$ for particular values of k imply the existence of such a congruence between the constant terms $\zeta(1-k)$ as well, and used this to show that (an appropriate normalization of) the Riemann zeta function is p-adically continuous when restricted to inputs of negative odd integers. This is one construction of the Kubota–Leopoldt p-adic zeta function, which interpolates the values of the Riemann zeta function at negative odd integers.

Serre was able to use a well-known family of modular forms whose values at the cusp ∞ are equal to special values of the Riemann zeta function. Though the result was generalized to Dedekind zeta functions for totally real fields in [DR80], these families can be tricky to come up with in more generality. One important addition to the theory was Katz's use of differential operators, in particular those built from the Gauss–Manin connection ∇, as a key ingredient in the construction of these families in [Kat76, Kat78]. Katz's operators give a p-adic analog of the

Maass–Shimura operators which Shimura used to prove algebraicity results in [Shi76, Shi00], and which were adapted by Harris to prove algebraicity results for higher rank groups in [Har81, Har86].

Katz leveraged this idea to p-adically interpolate the zeta function for a CM field K using Hilbert modular forms on its maximal totally real subfield K^+. He started with a holomorphic Eisenstein series and used successive powers of these differential operators to produce a family of not necessarily holomorphic Eisenstein series. By Damerell's formula (in e.g. [GS81] and described in Appendix A), sums of the values of these Eisenstein series at the CM points of the modular curve give the central values $L(\chi, s_0)$ of the L-functions for a specific class of Hecke characters χ, known as characters of type A_0. Viewing the zeta function as a function on the space of characters, this fact reduces the p-adic interpolation of ζ_K to the study of how powers of the Gauss–Manin connection behave p-adically.

As noted in his introduction, Katz's method only succeeds when we can choose an "ordinary" CM type, which is only possible when every prime above p in the totally real field K^+ splits in K/K^+. This is due to the fact that his differential operators are only defined over the ordinary locus, which does not contain the CM points that are supersingular at p. The present work extends these operators to be defined on the overconvergent loci, using methods from [Liu19a] and using the geometry developed in [AIP15, AIP16]. This extension allows Damerell's formula to be used in more general situations, whenever the Eisenstein series is defined at the CM points.

1.2 Description of Results

The present work culminates in the following theorem, which is Theorem 4.6.2 in the body of the dissertation:

Theorem 1.2.1. *Fix a tuple* $\underline{v} = (v_{\mathfrak{p}})_{\mathfrak{p}|p}$ *with each* $v_{\mathfrak{p}} > 0$. *For each embedding* $\sigma\colon F \to K$, *and any* $k \geq 1$, *there is a differential operator* ∇_σ^k *acting on families of nearly* $\underline{v}$*-overconvergent Hilbert modular forms, which raises the weight by* $2k\sigma$ *and the type by* k. *The operators* ∇_σ^k *and* ∇_τ^ℓ *commute for any pair of embeddings* σ *and* τ.

These ∇_σ^k are the p-adic analogs of the Maass-Shimura operators in the Archimedean case, as we prove in Section 3.5. The absence of these "overconvergent" operators in [Kat78] prevented Katz from constructing p-adic L-functions for CM fields in the case that the CM points were not ordinary; i.e., when p is not split in K/K^+. The overconvergent operators for elliptic modular forms have been previously constructed, and are used in [AI19] to construct p-adic L-functions for quadratic imaginary fields; our construction lays the groundwork to generalize that construction to a general CM field.

In defining the differential operators, we work rigid analytically. However, we give a discussion of the integrality of the operator in Section 4.7.

As in [AIP16], we are careful to make a distinction between two possible meanings of the phrase "Hilbert modular form." Given a totally real field F, the term may refer to the automorphic forms on either of the following groups:

$$G = \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}} \mathrm{GL}_2, \quad \text{or} \quad G^* = G \times_{\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}} \mathbb{G}_m} \mathbb{G}_m.$$

For any commutative ring R, the R-points of G are the 2×2 invertible matrices with entries in $\mathcal{O}_F \otimes_{\mathbb{Z}} R$, while the R-points of G^* are those matrices from $G(R)$ with determinant in $R^\times \subset (\mathcal{O}_F \otimes_{\mathbb{Z}} R)^\times$. Each of these groups has an advantage: G has a nicer Hecke theory including a commutative Hecke algebra; while G^* allows us to use geometric tools, since its associated PEL-type moduli problem is representable by a Shimura variety. The inclusion $G^* \subset G$ gives a restriction map from the space of automorphic forms on G to those on G^*, and in fact there is an explicit geometric criterion that picks out the space of automorphic forms for G inside the space of automorphic forms for G^*. In the following, we focus first on G^* so that we may use the geometric tools it affords us, after which we shift our attention to this criterion which allows us to transport our results to the group G. We get the following theorem, Theorem 5.4.2 in the body of this dissertation, which allows these operators to be used in either situation:

Theorem 1.2.2. *The differential operators ∇_σ^k constructed in Theorem 1.2.1 preserve the space of Hilbert modular forms for G inside the space of Hilbert modular forms for G^*.*

In Chapter 2, we give a geometric construction of Hilbert modular forms fort he group G^*. In Section 2.4, we give a comparison theorem that shows that our geometric Hilbert modular forms, when viewed over $\mathbb{C}$, are equivalent to the usual notion of Hilbert modular forms in terms of holomorphic functions on a symmetric space. In Chapter 3, we repeat this for nearly Hilbert modular forms, including a description of a Maass–Shimura differential operator in each setting, and a proof that they correspond. Chapter 4 then uses this as a basis to construct analogous spaces of p-adic Hilbert modular forms, and then overconvergent and nearly overconvergent Hilbert modular forms, as well as a Maass–Shimura differential operator in this setting. Finally, in Chapter 5, we transport our results from G^* to G.

1.3 Relationship to Recent Developments

This dissertation adds to the recent investigations into extending p-adic differential operators past the ordinary locus, or into when the ordinary locus is empty, including those in [dSG14, dSG19, EM20, Urb14, Liu19a, Liu19b]. Shortly after posting the paper on which this dissertation is based on the arXiv, the author learned from Giacomo Graziani that his dissertation [Gra20] (currently in preparation) is closely related. His dissertation uses the VBMS (vector bundles with marked sections) formalism present in e.g. [AI19], while we use $(\mathfrak{g}, Q)$-modules as in [Liu19a].

In [HX14], the authors ask whether or not their construction can be adapted to avoid a choice of the Hodge filtration. Building on the work of [AI19, Liu19a], this work answers that question in the affirmative. Additionally, other works note that in order to use nearly overconvergent Hilbert modular forms to interpolate the values of p-adic L-functions, a "canonical splitting of the Hodge filtration" must be chosen; see for example the introduction to [AI19] for this in the situation of Damerell's formula. Our methods translate this into a need for a canonical choice of frame for the relative de Rham cohomology of the modular curve in the nearly holomorphic case. Seeing as a canonical trivialization (i.e., a frame) for the modular sheaf $\underline{\omega}$ must already be chosen in Serre's setting, this is a natural bit of data to consider. For the nearly overconvergent case, we take preimages of the Andreatta–Iovita–Pilloni torsors for overconvergent Hilbert modular forms in the space of frames for the de Rham cohomology.

This construction works for elliptic modular forms, and seems eminently generalizable to be used for the automorphic forms on more groups, such as Hilbert–Siegel and hermitian modular forms. The author of this dissertation is currently working on a sequel extending his approach to the higher rank setting, in particular Hilbert–Siegel modular forms.

1.4 Notation and Conventions

Fix a totally real field F of degree $[F : \mathbb{Q}] = d > 1$. Its ring of integers will be denoted $\mathcal{O}_F$, and its inverse different will be denoted $\mathfrak{d}^{-1}$. Its Galois closure is F^{Gal}, with ring of integers $\mathcal{O}^{Gal}$.

For a p-adic local field K, let $\mathcal{O}_K$ be its ring of integers, and $\mathfrak{m}$ for the maximal ideal in $\mathcal{O}_K$. We also write $\mathcal{O}_1 = \mathcal{O}_K/\mathfrak{m}$ for the residue field, and $\mathcal{O}_m = \mathcal{O}_K/\mathfrak{m}^m$ for the local ring extending it.

When the local field K contains F^{Gal} as a subfield, we say that K *splits* F. In this case, there are d independent embeddings of F into K. The set of such embeddings is denoted I.

Given a prime ideal $\mathfrak{p}$ of F lying over p in $\mathbb{Q}$, write $F_\mathfrak{p}$ for the completion of F in the $\mathfrak{p}$-adic metric. This is a finite extension of $\mathbb{Q}_p$. Write the local inverse different as $\mathfrak{d}_\mathfrak{p}^{-1}$.

For any integer N, let μ_N denote the group of Nth roots of unity.

CHAPTER II

HILBERT MODULAR FORMS

Fix an integer N. In this section, we give a geometric construction of Hilbert modular forms over F of level $\Gamma_1(N)$, and a comparison to the theory in terms of holomorphic functions. This serves as a base for our geometric construction of nearly Hilbert modular forms in Chapter III.

2.1 Weights

Let K be a local field that splits F, so that $F \otimes_{\mathbb{Q}} K = K^d$; in this section we take $K = \mathbb{R}$, but we state it in generality so that we may use this in Chapter IV. Let I denote the set of embeddings $F \to K$, noting that there are d distinct embeddings. Consider the algebraic group $\mathbb{T} = \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}} \mathbb{G}_m$. The weight space of classical Hilbert modular forms is the space of characters of the split torus $\mathbb{T}_{/K}$.

Each embedding $\sigma \colon F \to K$ determines a character of $\mathbb{T}_{/K}$ via the formula

$$\sigma \colon \mathbb{T}_{/K}(R) \to R^\times, \qquad \sigma(c \otimes r) = \sigma(c) r.$$

All algebraic characters of $\mathbb{T}_{/K}$ can be written in terms of these characters - for any character κ, there exists some tuple of integers $(k_\sigma)_\sigma$ such that

$$\kappa(c \otimes r) = \prod_\sigma (\sigma(c) r)^{k_\sigma}.$$

We often write such a character as a formal sum $\kappa = \sum_{\sigma \in I} k_\sigma \sigma$. In this way, we identify the space of characters of $\mathbb{T}_{/K}$ with $\mathbb{Z}[I]$, the free abelian group on the set I of embeddings $F \to K$.

2.2 Moduli Setup

Let $\mathfrak{c}$ be a fractional ideal of F, p a prime number, and $N \geq 5$ an integer prime to p. A $\mathfrak{c}$-polarized Hilbert–Blumenthal abelian variety (HBAV, we suppress $\mathfrak{c}$ from the notation) defined over a base scheme S is a tuple $(A, \iota, \psi, \lambda)$ where

- $A \to S$ is an semi-abelian scheme[1] of relative dimension d,
- $\iota\colon \mathcal{O}_F \to \mathrm{End}(A)$ is a ring homomorphism known as a *real multiplication on* A,
- $\psi\colon \mu_N \otimes_{\mathbb{Z}} \mathfrak{d}^{-1} \to A$ is a closed embedding known as a *level structure*, and
- $\lambda\colon A \otimes_{\mathcal{O}_F} \mathfrak{c} \to A^\vee$ is a $\mathfrak{c}$-polarization.

If $\mathfrak{c}$ is principal, we say $(A, \iota, \psi, \lambda)$ is principally polarized. Often, we will use A as a shorthand for the entire string. Let S be a scheme over $\mathrm{Spec}\,\mathbb{Z}[N^{-1}]$, $X \to S$ be the moduli space of HBAVs over S, and $\pi\colon \mathcal{A} \to X$ be the universal HBAV over X.

Remark 2.2.1. In the definition of HBAV, we really want A to be an abelian variety, rather than a semi-abelian scheme. If we take this alternate definition, and write Y as the moduli space of these varieties, then X is a projective toroidal compactification of Y. Write $C = X \setminus Y$, and refer to C as the boundary or the cusps. Allowing semi-abelian schemes in the definition of HBAV is similar to talking about the moduli space of generalized elliptic curves and the universal generalized elliptic curve when discussing elliptic modular forms.

The Hodge bundle is

$$\underline{\omega} = \pi_* \Omega^1_{\mathcal{A}/X}.$$

This is locally free of rank d as a sheaf of $\mathcal{O}_X$-modules, with an action of $\mathcal{O}_F$ induced by the real multiplication. There is a largest open subscheme X^R of X, such that $\underline{\omega}$ restricts to a locally free sheaf of $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{X^R}$-modules. If we use an affine scheme $S = \mathrm{Spec}\,\mathcal{O}$ as our base and every prime number that ramifies in

[1]A group scheme which is an extension of an abelian variety by a torus

F is invertible in $\mathcal{O}$, then $X^R = X$. On the other hand, if $\mathcal{O}$ is a finite field of characteristic p, and p ramifies in F, then X^R is of codimension 2. The space X^R is known as the Rapoport locus, and the condition that a HBAV A corresponds to a point of X^R is called the Rapoport condition. A reference for this is [AIP16].

Recall that, in Section 2.1, we fixed a local field K that splits F. Over $X_{/K}$, the bundle $\underline{\omega}$ decomposes as a direct sum $\underline{\omega} = \bigoplus_{\sigma \in I} \underline{\omega}_\sigma$, so that each element $c \in \mathcal{O}_F$ acts on $\underline{\omega}_\sigma$ as multiplication by $\sigma(c) \in K$.

2.3 Hilbert Modular Forms

For a tuple of positive integers $\kappa = (k_\sigma)_\sigma \in \mathbb{Z}_{\geq 1}[I]$, we can form the line bundle over $X^R_{/K}$ by

$$\underline{\omega}_\kappa \coloneqq \underline{\omega}_{\sigma_1}^{\otimes k_{\sigma_1}} \otimes \cdots \otimes \underline{\omega}_{\sigma_d}^{\otimes k_{\sigma_d}}. \tag{2.1}$$

Definition 2.3.1. A Hilbert modular form of level $\Gamma_1(N)$ and weight $\kappa = (k_\sigma)_\sigma$ is a global section of $\underline{\omega}_\kappa$. The K-vector space of Hilbert modular forms of level $\Gamma_1(N)$ and weight κ is thus $H^0(X^R, \underline{\omega}_\kappa)$.

Write $T^\times_{\underline{\omega}} \coloneqq \mathrm{Isom}_{X^R,F}(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{X^R}, \underline{\omega})$ for the X^R-scheme whose points are $\mathcal{O}_F \otimes \mathcal{O}_{X^R}$-linear isomorphisms from $\mathcal{O}_F \otimes \mathcal{O}_{X^R}$ to $\underline{\omega}$. We view this as the frame bundle of $\underline{\omega}$ taking the $\mathcal{O}_F$-module structure into account. Over X^R, $\underline{\omega}$ is a vector bundle whose fiber over an open set U looks like $\mathcal{O}_F \otimes \mathcal{O}_U$. The points of $T^\times_{\underline{\omega}}$ correspond to bases for this free $\mathcal{O}_F \otimes \mathcal{O}_U$-module: for any $\alpha \in T^\times_{\underline{\omega}}$, the element $\alpha(1 \otimes 1)$ gives a basis for $\underline{\omega}$ as an $\mathcal{O}_F \otimes \mathcal{O}_{X^R}$-module. This space $T^\times_{\underline{\omega}}$ has an action of $\mathbb{T}_{/X^R} \cong \mathrm{Aut}_{X^R,F}(\mathcal{O}_F \otimes \mathcal{O}_{X^R})$ by precomposition: for any $g \in \mathbb{T}$ and any $\alpha \in T^\times_{\underline{\omega}}$, $g \cdot \alpha$ is the composition

$$\mathcal{O}_F \otimes \mathcal{O}_{X^R} \xrightarrow{g} \mathcal{O}_F \otimes \mathcal{O}_{X^R} \xrightarrow{\alpha} \underline{\omega}.$$

In fact, $T_{\underline{\omega}}^{\times}$ is a $\mathbb{T}$-torsor. This implies that its sheaf of functions of is graded by the characters of $\mathbb{T}_{/K}$. The graded portion corresponding to the weight $\kappa = (k_\sigma)_\sigma$ is the line bundle $\underline{\omega}_\kappa$ defined in Equation (2.1), and we find that the ring of Hilbert modular forms is the ring of functions on $T_{\underline{\omega}}^{\times}$, $\bigoplus_{\kappa\in\mathbb{Z}[I]} \underline{\omega}_\kappa = \mathcal{O}_{T_{\underline{\omega}}^{\times}}$.

The grading is defined as follows. Each line bundle $\underline{\omega}_\kappa$ can be recovered from $\mathcal{O}_{T_{\underline{\omega}}^{\times}}$ by considering the homogeneous functions with the property that for any $g \in \mathbb{T}$, any HBAV A, and any frame $\alpha\colon \mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{X^R} \xrightarrow{\sim} \underline{\omega}$,

$$f(A, \alpha g) = \left(\prod \sigma(g)^{-k_\sigma}\right) f(A, \alpha) = \kappa(g^{-1}) f(A, \alpha). \tag{2.2}$$

Somewhat less explicitly, we may view these as functions from $T_{\underline{\omega}}^{\times}$ to the rank one representation W_κ of $\mathbb{T}$, defined over K. The action of $\mathbb{T}$ is $g \cdot w = \kappa(g)w$, and the functions are homogeneous in the sense that $f(A, \alpha g) = g^{-1} \cdot f(A, \alpha)$.

Remark 2.3.2. Homogeneous functions on $T_{\underline{\omega}}^{\times}$ which are homogeneous as in Equation (2.2) are identified with sections of $\underline{\omega}_\kappa$ as follows. The pullback of $\underline{\omega}$ from X^R to $T_{\underline{\omega}}^{\times}$ is canonically trivialized by α, giving a trivialization α_σ of $\underline{\omega}_\sigma$ for each σ, and thus a trivialization α_κ of $\underline{\omega}_\kappa$ with $\alpha_\kappa(1) = \bigotimes_\sigma \alpha_\sigma(1)^{\otimes k_\sigma} \in \underline{\omega}_\kappa$. Acting on the trivialization by some $g \in \mathbb{T}(K)$ sends α_σ to $\sigma(g)\alpha_\sigma$, and $(g\alpha)_\kappa = \kappa(g)\alpha_\kappa$.

Take a section v of $\underline{\omega}_\kappa$ defined on all of $X^R_{/K}$, and pull it back to a section over $T_{\underline{\omega}}^{\times}$. Its value $v(A, \alpha)$ at a point corresponding to a HBAV A with a trivialization α of its cotangent bundle is a multiple $f(A, \alpha)$ of the canonical basis $\alpha_\kappa(1)$ for the line bundle $\underline{\omega}_\kappa$. Since v was pulled back from a section defined over X^R, we have that $v(A, \alpha) = v(A, \alpha g)$ for any $g \in \mathbb{T}(K)$. But the canonical trivialization of the fiber of $\underline{\omega}_\kappa$ over $(A, \alpha g)$ is $\kappa(g)$ times that of the canonical trivialization over (A, α). Thus we have

$$f(A, \alpha g) = \kappa(g^{-1}) f(A, \alpha). \tag{2.3}$$

Thus we give the following alternate definition for Hilbert modular forms.

Definition 2.3.3. Let R_0 be a ring. A Hilbert modular form of level $\Gamma_1(N)$ and weight $\kappa = (k_\sigma)_\sigma$, defined over R_0 is an algebraic function $f \in \mathcal{O}_{T_{\underline{\omega}}^\times}$ which satisfies the homogeneity property of Equation (2.3). Write the space of such functions as $\mathcal{O}_{T_{\underline{\omega}}^\times}[-\kappa]$.

Remark 2.3.4. While we defined Hilbert modular forms defined over a ring R_0, we can patch this definition together over affine open sets to give a definition for Hilbert modular forms defined over some scheme S.

Following Katz in [Kat78, Section 1.2], we give the following interpretation of Definition 2.3.3. For any R_0-algebra R, and any HBAV A defined over R equipped with a trivialization α of its cotangent bundle, we should get a number $f(A, \alpha) \in R$ subject to the following conditions:

- the number $f(A, \alpha) \in R$ depends only on the isomorphism class of the pair (A, α);

- f commutes with extension of scalars, in the sense that, for any R_0-algebra morphism $i \colon R \to R'$

$$f(A \times_{\operatorname{Spec} R} \operatorname{Spec} R', \alpha \otimes_R R') = i(f(A, \alpha));$$

- and f satisfies the homogeneity condition from Equation (2.3),

$$f(A, \alpha g) = \kappa(g^{-1}) f(A, \alpha).$$

Many important examples of Hilbert modular forms, such as Eisenstein series, are in fact defined over some finite extension of $\mathbb{Z}\left[\frac{1}{N}\right]$. However, the analytic picture involving holomorphic functions on a symmetric space in fact recovers

Hilbert modular forms defined over $\mathbb{C}$. One should keep this in mind when reading the next section, in which we give this perspective.

2.4 Holomorphic Hilbert Modular Forms

In this section we connect the geometric construction above to the view of Hilbert modular forms as holomorphic functions on a symmetric space.

Following [Kat78], let $\mathcal{O}^{Gal}$ denote the ring of integers in the Galois closure of F. We have d distinct embeddings $\sigma\colon \mathcal{O}_F \to \mathcal{O}^{Gal}$; For any $\mathcal{O}^{Gal}$-algebra A, we get a ring homomorphism

$$\mathcal{O}_F \otimes_{\mathbb{Z}} A \to \prod_{\sigma} A, \qquad n \otimes a \mapsto (\sigma(n)a)_\sigma.$$

When A is a flat $\mathcal{O}^{Gal}$-module, this is an injection. When the discriminant of F is invertible in A, it is a surjection. In particular, when $A = \mathbb{R}$ is the real numbers, we get a ring isomorphism $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{R} \to \prod_\sigma \mathbb{R}$. Note that $\mathbb{R}$ is an $\mathcal{O}^{Gal}$-algebra because F is totally real. Writing $\mathbb{R}$ as an $\mathcal{O}^{Gal}$-algebra is the same as fixing an embedding $\mathcal{O}^{Gal} \to \mathbb{R}$, which then identifies the embeddings $\mathcal{O}_F \to \mathcal{O}^{Gal}$ with the real embeddings of F. Thus we may index the product over the set I of real embeddings of F.

Let $\mathfrak{h} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the complex upper half-plane, with its usual action of the group $\mathrm{GL}_2^+(\mathbb{R})$ of invertible 2×2 matrices with real coefficients and positive determinant given by the formula

$$\gamma \cdot z = \frac{az+b}{cz+d}, \qquad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R}).$$

Then write $\mathfrak{h}_F = \prod_{\sigma \in I} \mathfrak{h}$ for the product of d copies of $\mathfrak{h}$ indexed by the real embeddings of F; it has a similar action of $\mathrm{GL}_2^+(\mathcal{O}_F \otimes \mathbb{R})$ given by the formula

$$\gamma \cdot (z_\sigma)_\sigma = \left(\frac{\sigma(a)z_\sigma + \sigma(b)}{\sigma(c)z_\sigma + \sigma(d)}\right)_\sigma, \qquad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathcal{O}_F \otimes \mathbb{R}).$$

Fix some $N \geq 5$, and let $\Gamma_1(N)$ be the following congruence subgroup.

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F) \,\middle|\, \begin{array}{l} a, d \in 1 + N\mathcal{O}_F \\ b \in \mathfrak{c}^{-1}\mathfrak{d} \\ c \in N\mathfrak{c}\mathfrak{d}^{-1} \end{array} \right\}.$$

Remark 2.4.1. In the case of modular forms ($F = \mathbb{Q}$), congruence subgroups must be subgroups of $\mathrm{SL}_2 \mathbb{Z}$. Note however that $\Gamma_1(N)$ need not be a subgroup of $\mathrm{SL}_2 \mathcal{O}_F$. Rather, we should allow congruence subgroups Γ to be *commensurable* with that group: there is a common subgroup H of both Γ and $\mathrm{SL}_2 \mathcal{O}_F$ which has finite index in both. For the subgroup $\Gamma_1(N)$ above, let $\mathfrak{b}$ be an integral ideal of $\mathcal{O}_F$ contained in both $\mathfrak{c}^{-1}\mathfrak{d}$ and $\mathfrak{c}\mathfrak{d}^{-1}$, and use

$$H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F) \,\middle|\, \begin{array}{l} a, d \in 1 + N\mathcal{O}_F \\ b \in \mathfrak{b} \\ c \in N\mathfrak{b} \end{array} \right\}.$$

The reader may check that this is indeed a a common subgroup of $\mathrm{SL}_2 \mathcal{O}_F$ and $\Gamma_1(N)$, and that it has finite index in both.

To each $\underline{z} = (z_\sigma)_\sigma \in \mathfrak{h}_F$ and each fractional ideal $\mathfrak{c}$, we associate a complex $\mathfrak{c}$-polarized HBAV $(A_{\underline{z}}, \iota_{\underline{z}}, \psi_{\underline{z}}, \lambda_{\underline{z}})$ along with a $\mathcal{O}_F \otimes \mathbb{C}$-basis for its Lie algebra. This will give us a map $\mathfrak{h}_F \to T_{\underline{\omega}}^\times(\mathbb{C})$, so that we can pull back complex modular forms to be functions on $\mathfrak{h}_F$. For a fractional ideal $\mathfrak{c}$ and $\underline{z} \in \mathfrak{h}_F$, let

$$L_{\underline{z}} = \mathfrak{d}^{-1} + \mathfrak{c}^{-1}\underline{z} \subset \mathcal{O}_F \otimes \mathbb{C}.$$

This lattice can be equipped with a $\mathfrak{c}$-polarization associated to the alternating pairing

$$\langle , \rangle \colon L_{\underline{z}} \times L_{\underline{z}} \to \mathfrak{d}^{-1}\mathfrak{c}^{-1}, \qquad \langle a + b\underline{z}, c + d\underline{z} \rangle = ad - bc.$$

This descends to a $\mathfrak{c}$-polarization $\lambda_{\underline{z}}$ on the complex torus $A_{\underline{z}} = (\mathcal{O}_F \otimes \mathbb{C})/2\pi i L_{\underline{z}}$. This torus has a natural $\mathcal{O}_F$-module structure given by acting on the first factor. We can also give a natural level structure. Identify the N-torsion of $A_{\underline{z}}$ with the quotient $\frac{1}{N}L_{\underline{z}}/L_{\underline{z}}$. This has a subgroup $\frac{1}{N}\mathfrak{d}^{-1}/\mathfrak{d}^{-1}$. Then we can use the level structure

$$\mu_N \otimes \mathfrak{d}^{-1} \to \left(\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right) \otimes \mathfrak{d}^{-1} = \frac{1}{N}\mathfrak{d}^{-1}/\mathfrak{d}^{-1} \hookrightarrow \frac{1}{N}L_{\underline{z}}/L_{\underline{z}} = A[N].$$

The first map is the inverse of the exponential isomorphism $\frac{k}{N} + \mathbb{Z} \mapsto e^{2\pi i k/N}$.

Notice that by writing $A_{\underline{z}} = (\mathcal{O}_F \otimes \mathbb{C})/2\pi i L_{\underline{z}}$, we are identifying $\mathrm{Lie}(A_{\underline{z}}) = \mathcal{O}_F \otimes \mathbb{C}$. Thus, in order to trivialize the cotangent bundle, we just have to give a functional $\mathcal{O}_F \otimes \mathbb{C} \to \mathcal{O}_F \otimes \mathbb{C}$. We choose the identity map, and denote it by $\mathrm{d}w$. We think of it as the differential that sends each path in $\mathcal{O}_F \otimes \mathbb{C}$ to its endpoint minus its starting point. Let $\alpha \colon \mathcal{O}_F \otimes \mathbb{C} \to \Omega^1_A$ be the trivialization sending $1 \otimes 1$ to $\mathrm{d}w$.

This gives an embedding $\phi \colon \mathfrak{h}_F \to T^\times_{\underline{\omega}}(\mathbb{C})$. Since we have defined (complex) Hilbert modular forms as functions on $T^\times_{\underline{\omega}}(\mathbb{C})$ in Definition 2.3.3, we can pull back complex Hilbert modular forms to be functions on $\mathfrak{h}_F$. We give the following classical definition of holomorphic Hilbert modular forms, and then prove that all such forms are the pullback of Hilbert modular forms defined over $\mathbb{C}$ in the sense of Definition 2.3.3.

Definition 2.4.2. A holomorphic Hilbert modular form of level $\Gamma_1(N)$ and weight $\kappa = (k_\sigma)_\sigma$ is a holomorphic function $\mathfrak{h}_F \to \mathbb{C}$ satisfying the homogeneity condition

$$f(\gamma \cdot \underline{z}) = \left(\prod_\sigma (\sigma(c)z + \sigma(d))^{k_\sigma}\right) f(\underline{z}), \qquad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N).$$

If $F = \mathbb{Q}$, then we also need a growth condition at the cusps.

Write $M_\kappa(N)$ for the vector space of holomorphic Hilbert modular forms of level $\Gamma_1(N)$ and weight κ, and recall from Definition 2.3.3 that the set of Hilbert modular forms of level $\Gamma_1(N)$ and weight κ can be written $\mathcal{O}_{T^\times_{\underline{\omega}}}[-\kappa]$. Let $\phi^*\colon \mathcal{O}_{T^\times_{\underline{\omega}}}[-\kappa] \to M_\kappa(N)$ be the pullback along the map $\phi\colon \mathfrak{h} \to T^\times_{\underline{\omega}}(\mathbb{C})$ described above.

Theorem 2.4.3. *The pullback ϕ^* gives an isomorphism $\mathcal{O}_{T^\times_{\underline{\omega}}}[-\kappa] \xrightarrow{\sim} M_\kappa(N)$.*

Proof. First we show that the image of ϕ^* actually lands in $M_\kappa(N)$. Fix $f \in \mathcal{O}_{T^\times_{\underline{\omega}}}[-\kappa]$. Certainly $\phi^* f$ is a function $\mathfrak{h}_F \to \mathbb{C}$; we omit the proof that it is holomorphic. Rather, we should focus on the homogeneity property from Definition 2.4.2.

For simplicity, we begin with the assumption that f has level $\Gamma_1(1)$. Pick some arbitrary $\gamma \in \Gamma_1(1)$,

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, d \in \mathcal{O}_F, \quad b \in \mathfrak{c}\mathfrak{d}^{-1}, \quad c \in \mathfrak{c}^{-1}\mathfrak{d}.$$

In particular, $\gamma \cdot \underline{z} = \left(\frac{\sigma(a)z_\sigma + \sigma(d)}{\sigma(c)z_\sigma + \sigma(d)}\right)_\sigma$. Recall that elements of $\mathcal{O}_F$ are included into $\mathcal{O}_F \otimes \mathbb{C} \cong \prod_\sigma \mathbb{C}$ by $n \mapsto (\sigma(n))_\sigma$. Since addition and multiplication happens componentwise, and each σ is a ring homomorphism, we may even write $\gamma \cdot \underline{z} = \frac{a\underline{z}+b}{c\underline{z}+d} = \left(\frac{\sigma(a)z_\sigma + \sigma(d)}{\sigma(c)z_\sigma + \sigma(d)}\right)_\sigma$.

Note that $\gamma \in \Gamma_1(1)$ gives a linear isomorphism from $\mathcal{O}_F \otimes \mathbb{C}$ to itself that sends $L_{\underline{z}}$ into itself; thus

$$L_{\underline{z}} = \underline{z}\mathfrak{c}^{-1} + \mathfrak{d}^{-1} = \gamma L_{\underline{z}} = (a\underline{z}+b)\mathfrak{c}^{-1} + (c\underline{z}+d)\mathfrak{d}^{-1}.$$

Since $L_{\underline{z}} = (a\underline{z}+b)\mathfrak{c}^{-1} + (c\underline{z}+d)\mathfrak{d}^{-1}$, we have that $(c\underline{z}+d)^{-1}L_{\underline{z}} = \left(\frac{a\underline{z}+b}{c\underline{z}+d}\right)\mathfrak{c}^{-1} + \mathfrak{d}^{-1} = L_{\gamma\cdot\underline{z}}$. This gives us a homothety of lattices between $L_{\underline{z}}$ and $L_{\gamma\cdot\underline{z}}$, which descends to an isomorphism $A_{\underline{z}} \cong A_{\gamma\cdot\underline{z}}$. However, the rescaling by $c\underline{z}+d$ means that the pairs $(A_{\underline{z}}, \mathrm{d}w)$ and $(A_{\gamma\cdot\underline{z}}, \mathrm{d}w)$ are not isomorphic. In fact, we get an isomorphism between the pairs $(A_{\underline{z}}, \mathrm{d}w) \cong (A_{\gamma\cdot\underline{z}}, (c\underline{z}+d)\mathrm{d}w)$. Using the homogeneity property, we find

$$f(A_{\underline{z}}, \mathrm{d}w) = f(A_{\gamma\cdot\underline{z}}, (c\underline{z}+d)\mathrm{d}w) = \kappa^{-1}(c\underline{z}+d)f(A_{\gamma\cdot\underline{z}}, \mathrm{d}w).$$

The first term in the equality is $\phi^*(f)(\underline{z})$, and the last is $\kappa^{-1}(c\underline{z}+d)\phi^*(f)(\gamma\cdot\underline{z})$. Unfolding the definition of the character κ, we get the desired transformation property

$$f(\gamma\cdot\underline{z}) = \left(\prod_{\sigma}(\sigma(c)z+\sigma(d))^{k_\sigma}\right)f(\underline{z}), \qquad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(1).$$

Thus, if f is homogeneous in the sense of Definition 2.3.3, then $\phi^* f$ is homogeneous in the sense of Definition 2.4.2, so $\phi^* f \in M_\kappa(N)$.

Now, we need to prove that ϕ^* is a bijection, which we do by giving an inverse map. Every element of $\mathcal{O}_{T^\times_{\underline{\omega}}}[-\kappa]$ is determined by its values on the fibers over the open locus $Y(\mathbb{C}) \subset X(\mathbb{C})$ without the cusps. Now we just have to see that, given the function $\phi^* f$, we can recover f; i.e., given the values $f(A_{\underline{z}}, \mathrm{d}w)$, we can find $f(A,\omega)$ for any complex abelian variety A and any nonvanishing differential ω on A.

Note that the integral homology $H_1(A,\mathbb{Z})$ has an action of $\mathcal{O}_F$ induced by the action of $\mathcal{O}_F$ on A. A choice of differential ω on A gives us the following lattice

$L_\omega \subset \mathcal{O}_F \otimes \mathbb{C}$:

$$L_\omega = \left\{ \int_\eta \omega \mid \eta \in H_1(A, \mathbb{Z}) \right\}.$$

Then A is isomorphic to $A_\omega = (\mathcal{O}_F \otimes \mathbb{C})/2\pi i L_\omega$, and in fact, the pair (A, ω) is isomorphic to the pair $(A_\omega, \mathrm{d}w)$. One can choose an embedding $\mathfrak{d}^{-1} \to L_\omega$ such that $L_\omega/\mathfrak{d}^{-1}$ is torsion-free. So this quotient is projective, and thus is isomorphic to $\mathfrak{c}^{-1}$ by the classification of projective $\mathcal{O}_F$-modules. We can split this projection to write L_ω as a direct sum of two submodules; the first isomorphic to $\mathfrak{d}^{-1}$, and the second isomorphic to $\mathfrak{c}^{-1}$. Write them as $\underline{z}^{(1)}\mathfrak{d}^{-1}$ and $\underline{z}^{(2)}\mathfrak{c}^{-1}$. We can tweak $\underline{z}^{(2)}$ so that $z_\sigma^{(2)}/z_\sigma^{(1)}$ has positive imaginary part by replacing $z_\sigma^{(2)}$ by its negative if necessary. Then $\underline{z} = \frac{\underline{z}^{(1)}}{\underline{z}^{(2)}} \in \mathfrak{h}_F$, and $\underline{z}^{(2)} L_{\underline{z}} = L_\omega$.

This gives us a homothety of lattices between $L_{\underline{z}}$ and L_ω, which descends to an isomorphism $A_{\underline{z}} \cong A$. Taking the differentials into account, we find that the pairs $(A_{\underline{z}}, \underline{z}^{(2)}\mathrm{d}w)$ and (A, ω) are isomorphic. Thus we have $f(A, \omega) = \kappa(\underline{z}^{(2)})^{-1} f(A_{\underline{z}}, \mathrm{d}w) = \kappa(\underline{z}^{(2)})^{-1} \phi^*(f)(\underline{z})$, and f is completely determined by $\phi^*(f)$. Thus ϕ^* is invertible.

For the case when f has level $\Gamma_1(N)$, a similar proof holds. We just have to see that $\Gamma_1(N)$ is exactly the set of elements $g \in G(\mathbb{R})$ such that the action of g preserves both the isomorphism class of the abelian varieties $A_{\underline{z}}$ and the level structure. The level structure is the embedding $\psi\colon \frac{1}{N}\mathfrak{d}^{-1}/\mathfrak{d}^{-1} \to \frac{1}{N} L_{\underline{z}}/L_{\underline{z}}$; this should be thought of as taking the $\mathcal{O}_F$-module isomorphism $\frac{1}{N} L_{\underline{z}}/L_{\underline{z}} \to A_{\underline{z}}[N]$ and restricting it to $\frac{1}{N}\mathfrak{d}^{-1}/\mathfrak{d}^{-1} \subset \frac{1}{N} L_{\underline{z}}/L_{\underline{z}}$. Any $\gamma \in \Gamma_1(1)$ gives an automorphism of $\frac{1}{N} L_{\underline{z}}/L_{\underline{z}}$, and $\gamma \cdot \psi$ is the level structure given by precomposing ψ by this automorphism. Note that $\gamma \in \Gamma_1(N)$ if and only if

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, d \in 1 + N\mathcal{O}_F, \quad b \in \mathfrak{c}\mathfrak{d}^{-1}, \quad c \in N\mathfrak{c}^{-1}\mathfrak{d}.$$

Thus, for any $n \in \mathfrak{d}^{-1}$ and $m \in \mathfrak{c}^{-1}$, $\gamma \cdot \psi$ should send $\frac{n}{N} + \frac{m}{N}\underline{z} + L_{\underline{z}}$ to $\frac{an+cm}{N} + \frac{bn+dm}{N}\underline{z} + L_{\underline{z}}$. But $\frac{an+cm}{N} = \frac{n}{N} + \frac{(a-1)n+cm}{N} \in \frac{n}{N} + \mathfrak{d}^{-1}$, since $(a-1)n$ and cm are both elements of $N\mathfrak{d}^{-1}$. This is enough to see that γ preserves the level structure, since ψ only cares about the restriction of this isomorphism to elements of the form $\frac{n}{N} + \mathfrak{d}^{-1}$ for $n \in \mathfrak{d}^{-1}$. $\square$

Remark 2.4.4. Essentially, the proof that ϕ^* is bijective relies on two facts. First, the image of the composition $\mathfrak{h}_F \xrightarrow{\phi} T_{\underline{\omega}}^{\times} \to X(\mathbb{C})$ is $Y(\mathbb{C})$, and f is determined by its values on the fibers over $Y(\mathbb{C})$. Second, once you know the value of f on one element of each fiber, you can find the values over the entire fiber by using the homogeneity condition.

CHAPTER III

NEARLY HILBERT MODULAR FORMS

In Chapter II, we explained how to view Hilbert modular forms as geometric objects. Here, in we extend that construction to nearly Hilbert modular forms. In Section 3.1 and we discuss an extension of the modular sheaf that is important for this construction. In Sections 3.3 and 3.4, we build the scaffolding a differential operator ∇_σ for each real embedding σ of the field F, using the Gauss–Manin connection as a key ingredient. Finally, in Section 3.6, we shift our focus to holomorphic functions and the classical Maass–Shimura operators δ_κ^σ. The section ends by setting up a "realization functor" ϕ^* that assigns a nearly holomorphic function to each nearly Hilbert modular form as constructed in Section 3.5. Finally, we prove that our operators ∇_σ and δ_κ^σ correspond in the sense that the following square commutes.

$$\begin{tikzcd} N_\kappa^\nu \arrow[r, "\phi^*"] \arrow[d, "\nabla_\sigma"'] & \mathcal{N}_\kappa^\nu \arrow[d, "\delta_\kappa^\sigma"] \\ N_{\kappa+2\sigma}^{\nu+\sigma} \arrow[r, "\phi^*"] & \mathcal{N}_{\kappa+2\sigma}^{\nu+\sigma} \end{tikzcd}$$

Here N_κ^ν is the space of nearly Hilbert modular forms of weight κ and type ν, as constructed in Section 3.5; and $\mathcal{N}_\kappa^\nu$ is the space of nearly holomorphic Hilbert modular forms of weight κ and type ν, as constructed in Section 3.6.

This serves as a motivation for the construction of ∇_σ in Section 4.6, since that will be entirely analogous to the construction of ∇_σ here. For an application of δ_κ^σ to the values of p-adic L-functions for CM fields using Damerell's formula, see the Appendix A.

3.1 The de Rham Sheaf

Another important sheaf is the relative de Rham cohomology $H^1_{dR}(\mathcal{A}/X) = R^1\pi_*\Omega^\bullet_{\mathcal{A}/X}$ of $\mathcal{A} \to X$. This sheaf has a natural subsheaf $\mathcal{H}$ which is locally free of rank 2 as a sheaf of $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{X^R}$-modules;[1] over K, it fits into an exact sequence of sheaves of $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{X^R}$-modules known as the Hodge filtration:

$$0 \to \underline{\omega} \to \mathcal{H} \to \underline{\omega}^\vee \to 0. \tag{3.1}$$

Over K, $\underline{\omega}$ and its dual $\underline{\omega}^\vee$ are locally free of rank 1 as sheaves of $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{X^R}$-modules, so they are projective. Thus the sequence splits, though non-canonically. In addition, $\mathcal{H}$ admits a nondegenerate alternating pairing with respect to which $\underline{\omega}$ is a maximal totally isotropic subspace, which gives the explicit isomorphism between $\mathcal{H}/\underline{\omega}$ and $\underline{\omega}^\vee$.

Finally, we note that $\mathcal{H}$ splits as a direct sum $\mathcal{H} = \bigoplus_\sigma \mathcal{H}_\sigma$ as $\underline{\omega}$ did. This gives a similar exact sequence for each embedding σ:

$$0 \to \underline{\omega}_\sigma \to \mathcal{H}_\sigma \to \underline{\omega}_\sigma^\vee \to 0.$$

This sequence gives a filtration $\underline{\omega}_\sigma \subset \mathcal{H}_\sigma$ of $\mathcal{H}_\sigma$, which induces filtrations on $\operatorname{Sym}^k \mathcal{H}_\sigma$ for each k, with $\operatorname{Fil}^r \operatorname{Sym}^k \mathcal{H}_\sigma = \underline{\omega}_\sigma^{k-r} \otimes \operatorname{Sym}^r \mathcal{H}_\sigma$. We also get a filtration on $\operatorname{Sym}^\kappa \mathcal{H} := \bigotimes_\sigma \operatorname{Sym}^{k_\sigma} \mathcal{H}_\sigma$ indexed by the partially ordered set $\mathbb{Z}_{\geq 0}[I]$, where a pure tensor $\bigotimes_\sigma s_\sigma$ is in $\operatorname{Fil}^\nu \operatorname{Sym}^\kappa \mathcal{H}$ for $\nu = (r_\sigma)_\sigma$ if $s_\sigma \in \operatorname{Fil}^{r_\sigma} \operatorname{Sym}^{k_\sigma} \mathcal{H}_\sigma$ for all σ. In particular, $\operatorname{Fil}^0 \operatorname{Sym}^\kappa \mathcal{H} = \underline{\omega}_\kappa$, where $\nu = 0$ means $\nu = (0)_\sigma$.

3.2 Associated Bundles

So far we have discussed two important bundles $\underline{\omega}$ and $\mathcal{H}$ on X. In addition, we used $\underline{\omega}$ to build the bundles $\underline{\omega}_\kappa$, whose sections are Hilbert modular forms of weight κ; we explicitly related these bundles to spaces of functions on the frame

[1] In fact, $H^1_{dR}(\mathcal{A}/X)$ agrees with $\mathcal{H}$ over Y^R. This $\mathcal{H}$ is a canonical extension of $H^1_{dR}(\mathcal{A}/Y^R)$, a locally free sheaf of $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_F$-modules of rank 2, to such a sheaf defined over all of X^R.

bundle $T_{\underline{\omega}}^{\times}$. In this section, we will give a more general version of this construction of so called "associated bundles" which will be built out of the whole $\mathcal{H}$.

We start with three groups. Write $\mathbb{T} = \mathrm{Aut}_{F,X^R}(\underline{\omega})$ for the automorphism group of $\underline{\omega}$ as an $\mathcal{O}_F \otimes \mathcal{O}_{X^R}$-module. Over an open set where $\underline{\omega}$ is trivial, $\mathbb{T}$ is isomorphic to $\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}} \mathbb{G}_m$. Then let $G = \mathrm{Aut}_{F,X^R}(\mathcal{H})$ be the automorphism group of $\mathcal{H}$ as an $\mathcal{O}_F \otimes \mathcal{O}_{X^R}$-module. Over an open set where $\mathcal{H}$ is trivial, G is isomorphic to $\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}} \mathrm{GL}_2$; this explains its name, since we have been calling that group G since Section 1. Finally, write $Q = \mathrm{Aut}_{F,X^R}(\underline{\omega} \subset \mathcal{H})$ for the subgroup of $G = \mathrm{Aut}_F(\mathcal{H})$ consisting of automorphisms fixing the sub-bundle $\underline{\omega}$. If we restrict to an open set where both $\underline{\omega}$ and $\mathcal{H}$ are trivial, we can choose a basis for $\mathcal{H}$ whose first element is a basis for $\underline{\omega}$; in this basis, the group G is simply a matrix group, and Q consists of the upper triangular matrices.

For each of the three groups, we have an associated torsor. As before, let $T_{\underline{\omega}}^{\times} = \mathrm{Isom}_{X,F}(\mathcal{O}_F \otimes \mathcal{O}_{X^R}, \underline{\omega})$, noting that $T_{\underline{\omega}}^{\times}$ is a $\mathbb{T}$-torsor. We also consider the G-torsor $T_{\mathcal{H}}^{\times} = \mathrm{Isom}_{X,F}((\mathcal{O}_F \otimes \mathcal{O}_X)^{\oplus 2}, \mathcal{H})$ and the Q-torsor $T_{\mathcal{H}}^{\times,+} = \mathrm{Isom}_{X,F}^{+}((\mathcal{O}_F \otimes \mathcal{O}_X)^{\oplus 2}, \mathcal{H})$, where the $+$ superscript denotes the fact that we should only consider isomorphisms that respect the filtrations; i.e., isomorphisms $\alpha\colon (\mathcal{O}_F \otimes \mathcal{O}_X)^{\oplus 2} \to \mathcal{H}$ so that the restriction $\alpha|_{\mathcal{O}_F \otimes \mathcal{O}_X}$ to the first component is an isomorphism $\mathcal{O}_F \otimes \mathcal{O}_X \to \underline{\omega}$. If we think of these isomorphisms as simply giving bases for the fibers of $\mathcal{H}$, we can think of $T_{\mathcal{H}}^{\times,+}$ as the subobject of $T_{\mathcal{H}}^{\times}$ consisting of bases whose first vector is a basis for $\underline{\omega}$, and $T_{\underline{\omega}}^{\times}$ as the quotient of $T_{\mathcal{H}}^{\times,+}$ obtained by forgetting the second basis vector. These maps of torsors correspond to the maps of groups: the first to the inclusion $Q \to G$, and the second to the projection $Q \to \mathbb{T}$ which picks out the top left entry of the upper triangular matrix.

Finally, we explore the concept of "associated bundles." Let $T^{\times} \to S$ be a torsor for some group H, and let V be a representation of H. Then we say that the contracted product $T^{\times} \times_H V$ is an associated bundle, where sections of the contracted product over an affine open $U = \operatorname{Spec} R$ are homogeneous functions

$$(T^{\times} \times_H V)(U) = \{v \colon \mathcal{O}_{T^{\times}}(U) \to V(R) \mid v(\alpha h) = h^{-1} \cdot v(\alpha) \text{ for all } h \in H(R)\}.$$

Example 3.2.1 (Associated Bundles to $\underline{\omega}$)**.** In Remark 2.3.2, we showed that sections of the line bundles $\underline{\omega}_{\kappa}$ are certain homogeneous functions on $T_{\underline{\omega}}^{\times}$. Specifically, write $\kappa = (k_{\sigma})_{\sigma}$. This corresponds to a one-dimensional representation $(\rho_{\kappa}, W_{\kappa})$ of $\mathbb{T}$ given by the formula $\rho_{\kappa}(g) = \prod_{\sigma} \sigma(g)^{k_{\sigma}}$. Then $\underline{\omega}_{\kappa}$ is given as a contracted product by $\underline{\omega}_{\kappa} = T_{\underline{\omega}}^{\times} \times_{\mathbb{T}} W_{\kappa}$.

These bundles can in fact be recovered from $T_{\mathcal{H}}^{\times,+}$ as well. Let W_{κ} also denote the inflation to Q via the map $Q \to \mathbb{T}$. Then $T_{\mathcal{H}}^{\times,+} \times_Q W_{\kappa} \cong T_{\underline{\omega}}^{\times} \times_{\mathbb{T}} W_{\kappa} \cong \underline{\omega}_{\kappa}$.

3.3 The Gauss–Manin Connection

The relative de Rham cohomology admits a Gauss–Manin connection, which extends to a connection on $\mathcal{H}$ with logarithm poles on the boundary C. We denote it by $\nabla \colon \mathcal{H} \to \mathcal{H} \otimes \Omega_X^1(\log C)$. It induces a principal connection on $T_{\mathcal{H}}^{\times}$, and a connection on any associated bundle $T_{\mathcal{H}}^{\times} \times_G V$ for any representation V of G.

We can describe it in terms of the action of the Lie algebra. Specifically, let $U = \operatorname{Spec} R \subset X$ be an affine open for which $\mathcal{H}|_U$ is trivial, and we pick D a derivation $R \to R$, viewing it as an element of $T_X(U)$.[2] Then the covariant derivative in the direction of D is a linear map $\nabla(D) \colon \mathcal{H} \to \mathcal{H}$, which also commutes with the action of $\mathcal{O}_F$. Any frame $\alpha \in T_{\mathcal{H}}^{\times}$ gives a natural basis for the

[2]The exterior derivative $d \colon R \to \Omega_U^1$ gives rise to a directional derivative $d(D) \colon R \to R$ for any $D \in T_X(U)$. The directional derivative in the direction of D is a derivation on R, and every derivation arises this way.

fibers, and $\nabla(D)$ is given in this basis by some matrix $X(D, \alpha) \in M_2(\mathcal{O}_F \otimes_{\mathbb{Z}} R)$ (if U meets the boundary, it will actually be in $M_2(\mathcal{O}_F \otimes_{\mathbb{Z}} \operatorname{Frac} R)$ with logarithm poles over C). This is $\mathfrak{g}(R)$, where $\mathfrak{g}$ is the Lie algebra of G.

The map $\alpha \mapsto X(D, \alpha)$ is G-equivariant for the Ad action of G on $\mathfrak{g}$, in the sense that $X(D, \alpha g) = \operatorname{Ad}(g^{-1})(X(D, \alpha))$ for all $g \in G$. This is the standard condition for ∇ to induce a principal connection on $T_{\mathcal{H}}^{\times}$. One can see this formula at the level of linear algebra by using g as a change of basis matrix, and noting that the Ad action of G on $\mathfrak{g}$ is given by conjugation.

Let V be a representation of G, so that it is also a representation of $\mathfrak{g}$. Then the associated bundle $T_{\mathcal{H}}^{\times} \times_G V$ acquires a connection whose covariant derivative ∇_V can be described as

$$\nabla_V(D)(f)(\alpha) = Df(\alpha) + X(D, \alpha) \cdot f(\alpha). \tag{3.2}$$

The action on pure tensors $f \otimes r \in V \otimes R$ is $D(f \otimes r) = f \otimes D(r)$ and $X(D, \alpha) \cdot (f \otimes r) = (X(D, \alpha) \cdot f) \otimes r$.

Remark 3.3.1. One might think of functions to $V \otimes R$ as R-linear combinations of vectors in V. The $Df(\alpha)$ term in Equation (3.2) differentiates the function $r \in R$ using the exterior derivative, while the $X(D, \alpha) \cdot f(\alpha)$ term differentiates the sections of V. This shows that it is a natural form for a connection that might be easier to describe in other ways.

The Gauss–Manin connection is also used to define the Kodaira–Spencer map. Consider the following composition.

$$\underline{\omega} \otimes \underline{\omega} \to \underline{\omega} \otimes \mathcal{H} \xrightarrow{\mathrm{Id} \otimes \nabla} \underline{\omega} \otimes \mathcal{H} \otimes \Omega_X^1(\log C) \to \underline{\omega} \otimes \underline{\omega}^{\vee} \otimes \Omega_X^1(\log C) \to \Omega_X^1(\log C). \tag{3.3}$$

This morphism is surjective, and over K it identifies $\Omega^1_X(\log C)$ with the summand $\bigoplus_\sigma \underline{\omega}_\sigma^{\otimes 2}$ of $\underline{\omega}^{\otimes 2}$. Write this isomorphism as $\mathrm{KS}\colon \Omega^1_X(\log C) \xrightarrow{\sim} \bigoplus_\sigma \underline{\omega}_\sigma^{\otimes 2}$. We will also use the projection $\mathrm{KS}_\sigma\colon \Omega^1_X(\log C) \to \underline{\omega}_\sigma^{\otimes 2}$.

Remark 3.3.2. There are natural Hecke actions on the space of sections of $\Omega^1_X(\log C)$ and on $\underline{\omega}_\sigma^{\otimes 2}$. The projection KS_σ is *not* equivariant with respect to these actions. While we do not consider Hecke actions in this dissertation, it is important enough to the general theory that this fact should be mentioned.

3.4 $(\mathfrak{g}, Q)$-modules

In the previous section, we described how the Gauss–Manin connection ∇ on $\mathcal{H}$ descends to a covariant derivative on the associated bundles $T^\times_{\mathcal{H}} \times_G V$ for any representation V of G. However, this does not, for example, descend to a connection on the associated bundles for $T^\times_{\underline{\omega}}$ or $T^{\times,+}_{\mathcal{H}}$. Seeing as Example 3.2.1 realized Hilbert modular forms as sections of associated bundles to $T^\times_{\underline{\omega}}$, we need something more to be able define a covariant derivative acting on the space of Hilbert modular forms.

As it turns out, the torsor $T^{\times,+}_{\mathcal{H}}$ will be an important intermediary. Representations of Q will be necessary in order to define the weights of nearly Hilbert modular forms, and a compatible action of the Lie algebra $\mathfrak{g}$ will give us the action of ∇. How exactly these actions should be compatible is codified in the notion of a $(\mathfrak{g}, Q)$-module; to define them, we follow the exposition of [Liu19a].

Remark 3.4.1. The idea of a $(\mathfrak{g}, K)$-module has existed in the theory for much longer, especially in the representation theoretic approach to the subject. These are a fairly natural generalization; the action of the Lie algebra is necessary for a description of differential operators, but rather than describing the weights in terms of representations of the maximal compact subgroup K of the real points of G, we describe it in terms of the group Q.

Let $\mathfrak{g} = \operatorname{Lie} G$ and $\mathfrak{q} = \operatorname{Lie} Q$ be the Lie algebras of G and of its Borel subgroup Q, respectively.

Definition 3.4.2. A $(\mathfrak{g}, Q)$-module defined over an algebra E is an algebraic representation V of both $\mathfrak{g}$ and Q on locally free E-modules, satisfying the following compatibility conditions.

1. The action of $\mathfrak{q} \subset \mathfrak{g}$ is the same as that induced by Q.

2. For any $g \in Q$, $X \in \mathfrak{g}$, and $v \in V$,
$$\operatorname{Ad}(g)(X) \cdot v = g \cdot X \cdot g^{-1} \cdot v.$$

Any representation V of G has a natural action of $\mathfrak{g}$ and of Q. These actions will be compatible in both senses above. In fact, any finite rank $(\mathfrak{g}, Q)$-module arises this way, c.f. [Liu19a, Remark 2.7].

In order to produce a $(\mathfrak{g}, Q)$-module, one can start with a representation V of Q and form a basis of $\mathfrak{g}$ whose first elements are a basis of $\mathfrak{q} \subset \mathfrak{g}$. The elements of $\mathfrak{q}$ act in the way induced by Q, so one only needs to give formulas for the action of the rest in a compatible way.

Remark 3.4.3. Another way to describe a $(\mathfrak{g}, Q)$-module V is to pick some ring R and extend the $Q(R)$-module $V(R)$ to a representation $\widehat{V}(R)$ of an open subgroup $\widehat{Q}(R) \subset G(R)$ containing $Q(R)$. Since $\widehat{Q}(R)$ is open in $G(R)$, its Lie algebra is $\mathfrak{g}$. One then checks that the action of $\mathfrak{g}$ preserves $V(R)$ as a subspace of $\widehat{V}(R)$, and then calculates the formulas for the action, which will automatically be compatible. These formulas will only involve scalars from R, so they will define a $(\mathfrak{g}, Q)$-module over R. If the scalars all come from a subring of R and the basis is defined over that subring, then the formulas will define a $(\mathfrak{g}, Q)$-module over that subring. This is done e.g. in [Liu19a, Section 2.3] ("a more conceptual proof" after Equation (2.5)) for $R = \mathbb{Z}_p$; we will use this method in Section 3.5 as well.

Let V be a $(\mathfrak{g}, Q)$-module. The formula in Equation (3.2) shows that the Gauss–Manin connection on the bundle associated to a G-module is given in terms of the exterior derivative and the action of $\mathfrak{g}$. Thus one might hope that we can define a connection on $\mathcal{V} \coloneqq T_{\mathcal{H}}^{\times,+} \times_Q V$. This is most of the content of the following proposition.

Proposition 3.4.4. *Let V be a $(\mathfrak{g}, Q)$-module. Then there is an integrable connection $\nabla_{\mathcal{V}}$ on the associated bundle $\mathcal{V} \coloneqq T_{\mathcal{H}}^{\times,+} \times_Q V$ given by the formula*

$$\nabla_{\mathcal{V}}(D)(v)(\alpha) = Dv(\alpha) + X(D, \alpha) \cdot v(\alpha).$$

Proof. The formula certainly gives a map $\mathcal{O}_{T_{\mathcal{H}}^{\times,+}} \to \mathcal{O}_{T_{\mathcal{H}}^{\times,+}} \otimes \Omega_X^1(\log C)$. In order to show that it gives a connection on $\mathcal{V}$, we have to show that the homogeneity property holds, $\nabla_{\mathcal{V}}(v)(\alpha g) = g^{-1} \cdot \nabla_{\mathcal{V}}(v)(\alpha)$. In fact, it holds for each term separately, and we treat them separately. First,

$$Dv(\alpha g) = D(g^{-1} \cdot v(\alpha)) = g^{-1} \cdot Dv(\alpha).$$

The first equality is by the homogeneity of v, and the second by the chain rule for the exterior derivative d. Then,

$$X(D, \alpha g) \cdot v(\alpha g) = \mathrm{Ad}(g^{-1})(X(D, \alpha)) \cdot g^{-1} \cdot v(\alpha) = g^{-1} \cdot X(D, \alpha) \cdot g \cdot g^{-1} \cdot v(\alpha).$$

This reduces to $g^{-1} \cdot X(D, \alpha) \cdot v(\alpha)$. Here we have used the second compatibility condition for $(\mathfrak{g}, Q)$-modules in the second equality. Thus we have

$$\begin{aligned} \nabla_{\mathcal{V}}(D)(v)(\alpha g) &= Dv(\alpha g) + X(D, \alpha g) \cdot v(\alpha g) \\ &= g^{-1} \cdot Dv(\alpha) + g^{-1} \cdot X(D, \alpha) \cdot v(\alpha) \\ &= g^{-1} \cdot \nabla_{\mathcal{V}}(v)(\alpha). \end{aligned}$$

This is what we wanted to show. The connection is integrable because d and ∇ both are. $\square$

3.5 Nearly Hilbert Modular Forms

Nearly Hilbert modular forms are sections of certain vector bundles. In particular, there should be finite rank representations V_κ^ν of Q for all $\kappa \in \mathbb{Z}[I]$ and $\nu = (r_\sigma)_\sigma \in \mathbb{Z}_{\geq 0}[I]$ so that a nearly holomorphic Hilbert modular form of weight κ and type ν is a section of $\mathcal{V}_\kappa^\nu = T_{\mathcal{H}}^{\times,+} \times_Q V_\kappa^\nu$. Each individual V_κ^ν is not a $(\mathfrak{g}, Q)$-module, which complicates the definition of the differential operators. However, there is an inclusion $V_\kappa^{\nu_1} \subset V_\kappa^{\nu_2}$ whenever $\nu_i = (r_\sigma^{(i)})_\sigma$ and $r_\sigma^{(2)} \geq r_\sigma^{(1)}$ for all σ, and the union $V_\kappa = \bigcup_\nu V_\kappa^\nu$ is a $(\mathfrak{g}, Q)$-module. It will satisfy $\mathfrak{g}_\sigma V_\kappa^\nu \subset V_\kappa^{\nu+\sigma}$. Thus, by Proposition 3.4.4 and the forthcoming Remark 3.5.4, we will obtain a connection $\nabla_{\mathcal{V}_\kappa}$ on $\mathcal{V}_\kappa = \bigcup_\nu \mathcal{V}_\kappa^\nu$ satisfying

$$\nabla_{\mathcal{V}_\kappa}(\mathcal{V}_\kappa^\nu) \subset \bigoplus_\sigma \mathcal{V}_\kappa^{\nu+\sigma} \otimes \underline{\omega}_\sigma^{\otimes 2} \subset \mathcal{V}_\kappa \otimes \Omega_X^1(\log C)$$

The specifics are laid out in the rest of the section.

For now, we fix an auxiliary prime ℓ which is unramified in $F/\mathbb{Q}$, L an ℓ-adic field which splits F, and $\mathcal{O}_L$ its ring of integers. Let I denote the set of embeddings of F into L.

We fix a weight $\kappa = \sum_\sigma k_\sigma \sigma \in \mathbb{Z}[I]$, with W_κ the corresponding 1-dimensional representation of $\mathbb{T}$ defined over $\mathcal{O}_L$. We build the $(\mathfrak{g}, Q)$-module V_κ following the course laid out in Remark 3.4.3 and [Liu19a].

We begin by giving names to important, non-algebraic subgroups of $G(\mathcal{O}_L)$. We will use the neighborhood $I_G(\mathcal{O}_L) \supset Q(\mathcal{O}_L)$, which contains the lower parabolic subgroup $Q^-_{I_G}(\mathcal{O}_L)$ and the Levi subgroup $H(\mathcal{O}_L)$,

$$I_G(\mathcal{O}_L) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathcal{O}_L) \mid c \in p\mathcal{O}_F \otimes \mathcal{O}_L \right\},$$

$$Q^-_{I_G}(\mathcal{O}_L) = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in G(\mathcal{O}_L) \mid c \in p\mathcal{O}_F \otimes \mathcal{O}_L \right\},$$

$$H(\mathcal{O}_L) = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in G(\mathcal{O}_L) \right\}.$$

Note that $Q^-_{I_G}(\mathcal{O}_L)$ projects onto $\mathbb{T}(\mathcal{O}_L)$ by picking out the top left entry a. We inflate the representation $W_\kappa(L)$ from $\mathbb{T}(\mathcal{O}_L)$ to $Q^-_{I_G}(\mathcal{O}_L)$. We have a unique choice of ℓ-adic topology on the finite dimensional Banach space $W_\kappa(L)$; consider the ℓ-adic analytic induction $\widehat{V}_\kappa(\mathcal{O}_L) = \operatorname{Ind}_{Q^-_{I_G}(\mathcal{O}_L)}^{I_G(\mathcal{O}_L)} W_\kappa(L)$, which is the set of ℓ-adic analytic functions $\phi\colon I_G(\mathcal{O}_L) \to W_\kappa(L)$ which are homogeneous with respect to the action of $Q^-_{I_G}(\mathcal{O}_L)$ in the sense that $\phi(hx) = h \cdot \phi(x)$ for all $x \in I_G(\mathcal{O}_L)$, $h \in Q^-_{I_G}(\mathcal{O}_L)$. It is a representation of $I_G(\mathcal{O}_L)$ by right translation: $(g \cdot \phi)(x) = \phi(xg)$ for all $g, x \in I_G(\mathcal{O}_L)$.

By the Iwahori decomposition, every coset $Q^-_{I_G}(\mathcal{O}_L)x \in Q^-_{I_G}(\mathcal{O}_L)\backslash I_G(\mathcal{O}_L)$ can be written as

$$Q^-_{I_G}(\mathcal{O}_L)x = Q^-_{I_G}(\mathcal{O}_L) \begin{pmatrix} 1 & \underline{Y} \\ 0 & 1 \end{pmatrix}.$$

This is for some unique choice of $\underline{Y} \in \mathcal{O}_F \otimes \mathcal{O}_L$, so that each $\phi \in \widehat{V}_\kappa$ is determined by its values on these matrices. We note that the coset spaces $Q^-_{I_G}(\mathcal{O}_L)\backslash I_G(\mathcal{O}_L)$

and $H(\mathcal{O}_L)\backslash Q(\mathcal{O}_L)$ are the same, down to the choice of representatives, and view $\widehat{V}_\kappa$ as an algebraic representation defined over $\mathcal{O}_L$ which consists of ℓ-adic analytic functions from the coset space $H\backslash Q$ to W_κ. This allows us to write $\widehat{V}_\kappa(R) = W_\kappa \otimes \mathcal{O}^{rig}_{H\backslash Q}$, where $\mathcal{O}^{rig}_{H\backslash Q}$ is the space of ℓ-adic analytic functions on the rigid space $H\backslash Q$.

For any $\mathcal{O}_L$-algebra R, we have natural coordinates on $(H\backslash Q)(R) \cong \mathcal{O}_F \otimes R \cong \prod_\sigma R$ called Y_σ for each embedding $\sigma\colon F \to L$.[3] By the standard theory of rigid spaces (in e.g. [Nic07]), the ring of ℓ-adic analytic functions on $H\backslash Q$ is $\mathcal{O}^{rig}_{H\backslash Q}(\operatorname{Sp} R) = R\langle \underline{Y} \rangle$, the space of power series over R in the variables Y_σ whose coefficients go to 0 ℓ-adically.

One may explicitly compute the action of $I_G(\mathcal{O}_L)$ on $\widehat{V}_\kappa(\mathcal{O}_L)$, obtaining the formula

$$(g \cdot P)(\underline{Y}) = (a + \underline{Y}c) \cdot P((a + \underline{Y}c)^{-1}(b + \underline{Y}d)), \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Here $\underline{Y}$ and the entries of g are viewed as elements of the ring $\mathcal{O}_F \otimes \mathcal{O}_L$. The induced action of $Q(\mathcal{O}_L)$ and $\mathfrak{g}(\mathcal{O}_L)$ may be computed from this formula. For $Q(\mathcal{O}_L)$, this is simple: substitute 0 for c in the formula above. This gives the induced action of $\mathfrak{q}(\mathcal{O}_L)$ on $\widehat{V}_\kappa(\mathcal{O}_L)$. To describe the action of the rest of $\mathfrak{g}(\mathcal{O}_L)$, we need a basis. Since $\mathfrak{g}(\mathcal{O}_L) = \prod_\sigma \mathfrak{gl}_2(\mathcal{O}_L)$, we may specify the actions of $\{\mu_\sigma^- \mid \sigma\colon F \to L\}$, where μ_σ^- is the element of $\prod \mathfrak{gl}_2(\mathcal{O}_L)$ which is $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ in the entry corresponding to σ, and the zero matrix in all other entries. With our fixed choice of coordinates Y_σ, we find that

$$\mu_\sigma^- \cdot P(\underline{Y}) = Y_\sigma \varepsilon_\sigma \cdot P(\underline{Y}) - Y_\sigma^2 \frac{\partial}{\partial Y_\sigma} P(\underline{Y}).$$

[3] This is where we use the fact that our auxilliary prime ℓ is unramified.

Here $\varepsilon_\sigma \in \mathrm{Lie}(\mathbb{T}) = \prod \mathfrak{gl}_1(\mathcal{O}_L)$ is the tuple with a 1 in the entry corresponding to σ and 0 in all other entries, acting naturally on W_κ. Specifically, $\varepsilon_\sigma \cdot w = k_\sigma w$ for any $w \in W_\kappa$.

Notice that the space of polynomials $W_\kappa[\underline{Y}]$ is preserved by the actions of both $Q(\mathcal{O}_L)$ and $\mathfrak{g}(\mathcal{O}_L)$, though it is not preserved by $I_G(\mathcal{O}_L)$. These are the algebraic functions on the scheme $H\backslash Q$. We have thus defined a $(\mathfrak{g}, Q)$-module structure on $V_\kappa = W_\kappa \otimes \mathcal{O}_{H\backslash Q}$, where now $\mathcal{O}_{H\backslash Q}$ is the space of algebraic functions on the coset space $H\backslash Q$. Over $\mathcal{O}_L$, this is the polynomial ring $\mathcal{O}_{H\backslash Q}(R) = \mathcal{O}_L[\underline{Y}] = \mathcal{O}_L[Y_\sigma]_\sigma$.

Now we fix an algebra R over the ring of integers in the Galois closure of F such that the discriminant of F is invertible in R. Over R, we may write $V_\kappa = W_\kappa \otimes \mathcal{O}_{H\backslash Q} = W_\kappa[\underline{Y}]$ where $\underline{Y} = (Y_\sigma)_\sigma$ is our natural set of coordinates, and we can extend any basis for $\mathfrak{q}$ to a basis for $\mathfrak{g}$ by using $\{\mu_\sigma^-\}$. We have actions given by formulas. For Q,

$$(g \cdot P)(\underline{Y}) = a \cdot P(a^{-1}(b + \underline{Y}d)), \qquad g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

For $\mathfrak{g}$, we specify the action of the μ_σ^-.

$$\mu_\sigma^- \cdot P(\underline{Y}) = Y_\sigma \varepsilon_\sigma \cdot P(\underline{Y}) - Y_\sigma^2 \frac{\partial}{\partial Y_\sigma} P(\underline{Y}). \tag{3.4}$$

These formulas give compatible actions since they were compatible on $\widehat{V}_\kappa$.

Remark 3.5.1. The coordinates $\underline{Y}$ are simply indeterminates at this stage. Later, we will turn certain algebraic functions to V_κ into smooth functions on $\mathfrak{h}_F$ by substituting Y_σ with $s_\sigma = \frac{1}{2iy_\sigma}$, where y_σ is the imaginary part of our coordinates z_σ on $\mathfrak{h}_F$. The notation can be confusing at first, since the same letter is used for a quantity and $\frac{1}{2i}$ times its reciprocoal, but it is standard as in e.g. [Liu19a].

Remark 3.5.2. If we replace R by a subring $\mathcal{O} \subset R$, then $\mathfrak{g}(\mathcal{O})$ is a subalgebra of $\mathfrak{g}(R) = \prod_\sigma \mathfrak{gl}_2(R)$. Thus these formulas can still be used to describe the action, but we have to be careful about what elements of the direct product we consider. This is particularly important if p ramifies in F when $R = K$ is our chosen p-adic field and $\mathcal{O} = \mathcal{O}_K$ is its ring of integers.

Not only does the action of Q preserve the degree of the polynomial, it preserves the degree in each variable Y^σ separately. Thus V_κ has an exhaustive filtration indexed by the partially ordered set $\mathbb{Z}_{\geq 0}[I]$, such that V_κ^ν is the space of polynomials of degree at most r_σ in the variable Y_σ. The Lie algebra $\mathfrak{q}$ also preserves the filtration. Write $\mathfrak{g} = \bigoplus_\sigma \mathfrak{g}_\sigma$ over K. Since $\mathfrak{g}_\sigma$ is generated by $\mathfrak{q}_\sigma$ and μ_σ^-, it preserves the degree as a function of Y_τ for each $\tau \neq \sigma$, and raises the degree as a function of Y_σ by at most 1 by the formula in Equation (3.4). We have $\mathfrak{g}_\sigma V_\kappa^\nu \subset V_\kappa^{\nu+\sigma}$. Unlike V_κ, each piece V_κ^ν has finite rank $\prod_\sigma (1 + r_\sigma)$, meaning that its associated bundle $\mathcal{V}_\kappa^\nu = T_{\mathcal{H}}^{\times,+} \times_Q V_\kappa^\nu$ has finite rank as a vector bundle.

Definition 3.5.3. The sheaf of nearly holomorphic Hilbert modular forms of weight κ is $\mathcal{V}_\kappa = T_{\mathcal{H}}^{\times,+} \times_Q V_\kappa$. The sheaf of nearly holomorphic Hilbert modular forms of weight κ and type ν is $\mathcal{V}_\kappa^\nu = T_{\mathcal{H}}^{\times,+} \times_Q V_\kappa^\nu$. The K-vector space of nearly holomorphic Hilbert modular forms of weight κ and type ν is thus $H^0(X, \mathcal{V}_\kappa^\nu)$.

Remark 3.5.4. Using the filtration, we can refine the statement of Proposition 3.4.4. Define the differential operator ∇_σ to be the composition

$$\mathcal{V}_\kappa \xrightarrow{\nabla_{\mathcal{V}_\kappa}} \mathcal{V}_\kappa \otimes \Omega_X^1(\log C) \xrightarrow{1 \otimes KS_\sigma} \mathcal{V}_\kappa \otimes \underline{\omega}_\sigma^{\otimes 2} \cong \mathcal{V}_{\kappa+2\sigma}.$$

Then $\nabla_\sigma(D)$ is given by the action of $X(D_\sigma, \alpha) \in \mathfrak{g}_\sigma$, where D_σ is the projection of $D \in \Omega_X^1(\log C)$ onto the summand $\underline{\omega}_\sigma^{\otimes 2}$. Thus it sends $\mathcal{V}_\kappa^\nu$ into $\mathcal{V}_\kappa^{\nu+\sigma} \otimes \underline{\omega}_\sigma^{\otimes 2} \cong \mathcal{V}_{\kappa+2\sigma}^{\nu+\sigma}$.

Specifically, ∇_σ raises the weight of a nearly holomorphic Hilbert modular form by 2σ and its type by σ.

Recall Remark 3.5.2, which stated that $\mathfrak{g}(\mathcal{O}_K) \subset \prod_\sigma \mathfrak{gl}_2(\mathcal{O}_K)$ may be a strict inclusion if K is a p-adic field and p ramifies in F. In this case, D_σ may not be an integral element of the Lie algebra even if D is. So ∇_σ may not actually act integrally. We discuss this in Section 4.7.

Note that we have written ∇_σ with no reference to κ or ν, since they are clear from context while σ is not. In addition, write $\kappa = \sum_\sigma k_\sigma \sigma$ in order to make better sense of what character $\kappa + 2\sigma$ corresponds to.

It is known (c.f. [Kat78, Lemma 2.1.14]) that ∇_σ and ∇_τ commute for any pair of embeddings σ and τ. This is done by reducing to the case of working over $\mathbb{C}$, where we have the explicit formulas which will be given in the next section. These formulas define commuting operators over $\mathbb{C}$, which are transported to being commuting operators over any base ring. Thus we may unambiguously write $\nabla_{\kappa'}$ for the differential operator that raises weights by $\kappa' = \sum_\sigma 2k'_\sigma \sigma$ and types $(k_\sigma)_\sigma$.

3.6 Nearly Holomorphic Hilbert Modular Forms

In Section 2.4, we described holomorphic Hilbert modular forms over $\mathbb{C}$ in terms of holomorphic functions on the product $\mathfrak{h}_F = \prod_{\sigma \in I} \mathfrak{h}$ of upper half spaces. We then set up a realization functor by giving a map $\phi\colon \mathfrak{h}_F \to T^\times_{\underline{\omega}}$ and pulling back. In this section, we will do the same for nearly holomorphic Hilbert modular forms, with the added bonus that the differential operator described in Remark 3.5.4 will correspond to the classical Maass–Shimura operator.

The complex space $\mathfrak{h}_F$ has coordinates $(z_\sigma)_{\sigma \in I}$; for each $\sigma \in I$, we get a pair of differential operators $\frac{\partial}{\partial z_\sigma}$ and $\frac{\partial}{\partial \bar{z}_\sigma}$. One way to characterize holomorphic functions is to say that a smooth function $f\colon \mathfrak{h}_F \to \mathbb{C}$ is holomorphic if and only if $\frac{\partial f}{\partial \bar{z}_\sigma} = 0$ for all σ. Here we generalize this to the notion of a *nearly holomorphic* function.

Definition 3.6.1. A function f is nearly holomorphic of type $\nu = (r_\sigma)_\sigma$ if $\frac{\partial^{r_\sigma+1} f}{\partial \bar{z}_\sigma^{r_\sigma+1}} = 0$ for each σ. A function f is nearly holomorphic if there exists a tuple of integers $\nu = (r_\sigma)_\sigma$ such that f is nearly holomorphic of type ν.

In particular, a holomorphic function is nearly holomorphic of type $(0)_\sigma$.

Remark 3.6.2. There is another characterization of nearly holomorphic functions. Following the conventions in [Liu19a], write $s_\sigma = \frac{1}{z_\sigma - \overline{z_\sigma}}$. Then a function f is nearly holomorphic of type $\nu = (r_\sigma)_\sigma$ if and only if it can be written as a polynomial in the variables s_σ with degree at most r_σ as a polynomial in s_σ alone, and for which the coefficients are holomorphic functions. There are no algebraic relations between these s_σ's, so this polynomial is uniquely determined by f.

Remark 3.6.3. In [Liu19a], the author's nearly holomorphic functions are polynomials in the variables s_{ij}, where s_{ij} is the ij entry of the matrix $(z - \overline{z})^{-1}$. In contrast, in [Shi00, 13.2], the author uses r_i to denote a set of functions that play this role, noting that we may take them to be the entries of this matrix. We note that in the Hilbert case, $s_\sigma = \frac{1}{z_\sigma - \overline{z_\sigma}} = \frac{1}{2iy_\sigma}$ where y_σ is the imaginary part of z_σ. The theory is often presented in terms of this y_σ; we have chosen to use s_σ to follow [Liu19a] more closely. This y_σ is not related to the coordinate Y_σ that we chose for the coset space $H\backslash Q$.

In the vein of Definition 2.4.2, define nearly holomorphic Hilbert modular forms as follows.

Definition 3.6.4. A nearly holomorphic Hilbert modular form of level $\Gamma_1(N)$, weight $\kappa = (k_\sigma)_\sigma$, and type $(r_\sigma)_\sigma$ is a nearly holomorphic function $\mathfrak{h}_F \to \mathbb{C}$ of type

$(r_\sigma)_\sigma$ satisfying the homogeneity condition

$$f(\gamma \cdot \underline{z}) = \left(\prod_\sigma (\sigma(c)z + \sigma(d))^{k_\sigma} \right) f(\underline{z}), \qquad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N).$$

If $F = \mathbb{Q}$, then we also need a growth condition at the cusps.

Write $N_\kappa^\nu(N)$ for the vector space of holomorphic Hilbert modular forms of level $\Gamma_1(N)$, weight κ and type ν. We also define the classical Maass–Shimura operator as in e.g. [Shi00].

Definition 3.6.5. The weight κ Maass–Shimura operator at σ, denoted δ_κ^σ, acts on functions by the formula

$$(\delta_\kappa^\sigma f)(\underline{z}) = \frac{1}{2\pi i} s_\sigma^{k_\sigma} \frac{\partial}{\partial z_\sigma} s_\sigma^{-k_\sigma} f(\underline{z}). \tag{3.5}$$

If f is a nearly holomorphic Hilbert modular form of weight κ and type $(r_\sigma)_\sigma$, then $\delta_\kappa^\sigma f$ is a nearly holomorphic modular form of weight $\kappa + 2\sigma$ and type $\nu + \sigma$. Write $\delta_\kappa^{n\sigma}$ for the composition

$$\delta_\kappa^{n\sigma} = \delta_{\kappa+2(n-1)\sigma}^\sigma \circ \cdots \circ \delta_\kappa^\sigma.$$

This raises the weight by $2n\sigma$ and the type by $n\sigma$. Since the partial deriviatives commute, and $\frac{\partial s_\sigma}{\partial z_\tau} = 0$ when $\sigma \neq \tau$, we may unambiguously write $\delta_\kappa^{\kappa'}$ for the composition that raises the weight by $2\kappa'$ and the type by κ' for any κ'.

Remark 3.6.6. The factor of $\frac{1}{2\pi i}$ in Equation (3.5) is a normalization factor required to make this an algebraic operation. Our choice of which abelian variety $A_{\underline{z}}$ is associated to each $\underline{z} \in \mathfrak{h}$, as well as the choice to use $\mathrm{d}w$ as the differential, are other normalization factors; our choices here follow [Kat78]. With these choices, we compare δ_κ^σ to ∇_σ directly, rather than requiring another normalization factor.

Our comparison theorem will be built very similarly to Theorem 2.4.3. Specifically, we will construct a function $\mathfrak{h}_F \to T_{\mathcal{H}}^{\times,+}$ such that the composition

$\mathfrak{h}_F \to T_{\mathcal{H}}^{\times,+} \to T_{\underline{\omega}}^{\times}$ is the function ϕ we used in Section 2.4; we will call this lift ϕ as well. Then pulling back along ϕ will give us a function valued in V_κ^ν; setting $\underline{Y} = 0$ will give us a W_κ-valued function. Forgetting the action of $\mathbb{T}$, this is simply a $\mathbb{C}$-valued function. Write, somewhat abusively, ϕ_0^* for the composition of the pullback ϕ^* with the projection $V_\kappa^\nu \to W_\kappa$ that evaluates at $\underline{Y} = 0$.

Once we have established that ϕ_0^* is an isomorphism, we will show that the differential operator ∇_σ described in Remark 3.5.4 corresponds to the Maass–Shimura operator δ_κ^σ by showing that the following square commutes.

$$\begin{tikzcd}
N_\kappa^\nu \arrow[r, "\phi_0^*"] \arrow[d, "\nabla_\sigma"'] & \mathcal{N}_\kappa^\nu \arrow[d, "\delta_\sigma"] \\
N_{\kappa+2\sigma}^{\nu+\sigma} \arrow[r, "\phi_0^*"] & \mathcal{N}_{\kappa+2\sigma}^{\nu+\sigma}
\end{tikzcd}$$

Following [Liu19a], we describe two bases for $H^1_{dR}(A_{\underline{z}})$, and put them together to find a suitable basis that we may use to extend ϕ as previously defined. In this discussion we use the fact that $H^1_{dR}(A)$ is the $\mathbb{C}$-linear dual of $\mathrm{Lie}(A) \otimes_{\mathbb{R}} \mathbb{C}$, and that $\mathrm{Lie}(A_{\underline{z}}) \cong \mathcal{O}_F \otimes \mathbb{C}$.

We have a natural basis for $\mathrm{Lie}(A_{\underline{z}}) \cong \mathcal{O}_F \otimes \mathbb{C}$ as an $\mathcal{O}_F \otimes \mathbb{R}$-module given by 1 and $\underline{z}$. This extends to an $\mathcal{O}_F \otimes \mathbb{C}$-basis of $\mathrm{Lie}(A_{\underline{z}}) \otimes_{\mathbb{R}} \mathbb{C}$; let α and β be the elements of the dual basis, defined by

$$\alpha\,(c_1 + c_2\underline{z}) = c_1, \qquad \beta(c_1 + c_2\underline{z}) = c_2.$$

We may also use the natural $\mathbb{C}$-action on $\mathrm{Lie}(A_{\underline{z}})$ to get a splitting of $\mathrm{Lie}(A_{\underline{z}}) \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Lie}(A_{\underline{z}}) \oplus \overline{\mathrm{Lie}(A_{\underline{z}})}$. The space $\mathrm{Lie}(A_{\underline{z}}) \otimes_{\mathbb{R}} \mathbb{C}$ has two actions of $\mathbb{C}$: one on the first component and one on the second. Write $\mathrm{Lie}(A_{\underline{z}})$ for the subspace on which these actions agree $v \otimes c = cv \otimes 1$; and $\overline{\mathrm{Lie}(A_{\underline{z}})}$ for the subspace on which they differ by complex conjugation, $v \otimes c = \bar{c}v \otimes 1$. The dual space to $\mathrm{Lie}(A_{\underline{z}}) \cong \mathcal{O}_F \otimes \mathbb{C}$ is simply the cotangent bundle $\Omega^1_{A_{\underline{z}}}$; we have a basis $\mathrm{d}w$ for this space, which is the simply the natural identification $\mathrm{Lie}(A_{\underline{z}}) \to \mathcal{O}_F \otimes \mathbb{C}$. The complement of $\Omega^1_{A_{\underline{z}}}$, which we called $\overline{\mathrm{Lie}(A_{\underline{z}})}$, is also naturally identified with $\mathcal{O}_F \otimes \mathbb{C}$, and let $\mathrm{d}\overline{w}$ denote the functional given by this identification.

Neither of these pairs of functionals is quite the basis we want. In order for the chosen basis to be a lift of the function ϕ described in Section 2.4, we need to choose a basis with $\mathrm{d}w$ as the first element. Then, we need another element of $H^1_{dR}(A_{\underline{z}})$ to extend this to a basis of the whole space. We have two useful choices.

First, we might choose a slight rescaling of the second basis, $(\mathrm{d}w, -\mathrm{d}\overline{w}\,\underline{s})$. This rescaling is necessary for the basis to respect the pairing, and in fact this is the basis we choose to use when we define the extension $\phi\colon \mathfrak{h}_F \to T^{\times,+}_{\mathcal{H}}$.

However, we might instead choose the basis $(\mathrm{d}w, \beta)$. This has the advantage that the differential operator is given by an explicit element of the Lie algebra. Specifically, via the identification $\underline{\omega} \otimes_{\mathcal{O}_F} \underline{\omega} \cong \Omega^1_X$, the basis $\mathrm{d}w$ gives a basis for Ω^1_X, and a dual basis $D = (D_\sigma)_\sigma$ for T_X. Writing $\mathrm{d}w_\sigma$ and β_σ for the corresponding basis of $H^1_{dR}(A_{\underline{z}})(\sigma)$ for each $\sigma \in I$, we have that $\nabla(D_\sigma)(\mathrm{d}w_\sigma) = \beta_\sigma$. Thus, in this basis, the differential operator ∇_σ acts via the Lie algebra element μ^-_σ, for which we have an explicit formula. This also has the benefit of being a holomorphic basis: for a nearly Hilbert modular form $f \in \mathcal{O}_{T^{\times,+}_{\mathcal{H}}} \times_Q V_\kappa$, we have that the function $z \mapsto f(A_{\underline{z}}, \mathrm{d}w, \beta)$ can be written as a polynomial $P_f(\underline{Y})$ in the variables Y_σ with holomorphic functions as coefficients.

The final piece of the puzzle, then, is to relate these two bases to each other. The action of Q gives us

$$(\mathrm{d}w, -\mathrm{d}\overline{w}\,\underline{s}) \begin{pmatrix} 1 & \underline{s} \\ 0 & 1 \end{pmatrix} = (\mathrm{d}w, \beta) \qquad \text{or,} \qquad (\mathrm{d}w, -\mathrm{d}\overline{w}\,\underline{s}) = (\mathrm{d}w, \beta) \begin{pmatrix} 1 & -\underline{s} \\ 0 & 1 \end{pmatrix}.$$

In particular, if write $f(A_{\underline{z}}, \mathrm{d}w, \beta) = P_f(Y)$, we have

$$\begin{aligned} f(A_{\underline{z}}, \mathrm{d}w, -\mathrm{d}\overline{w}\,\underline{s}) &= f\left(A_{\underline{z}}, (\mathrm{d}w, \beta)\begin{pmatrix} 1 & -\underline{s} \\ 0 & 1 \end{pmatrix}\right) \\ &= \begin{pmatrix} 1 & \underline{s} \\ 0 & 1 \end{pmatrix} \cdot f(A_{\underline{z}}, \mathrm{d}w, \beta) \\ &= \begin{pmatrix} 1 & \underline{s} \\ 0 & 1 \end{pmatrix} \cdot P_f(\underline{Y}) \\ &= P_f(\underline{s} + \underline{Y}) \end{aligned}$$

Thus $\phi_0^*(f)(\underline{z}) = P_f(\underline{s})$ is a polynomial in the variables s_σ with holomorphic functions of $\underline{z}$ as coefficients. If f has type $\nu = (r_\sigma)_\sigma$ in the sense that $f \in \mathcal{O}_{T_{\mathcal{H}}^{\times,+}} \times_Q V_\kappa^\nu$, then $\phi_0^*(f)$ is nearly holomorphic of type ν by our alternate characterization in Remark 3.6.2.

This process is reversible since $P_f(\underline{Y})$ can be recovered from $P_f(\underline{s})$. Thus, to prove that $\phi_0^*\colon \mathcal{V}_\kappa^\nu \to N_\kappa^\nu(N)$ is a valid realization functor, we just have to show that $\phi_0^*(f)$ satisfies the transformation property, $\phi_0^*(f)(\gamma \cdot \underline{z}) = \kappa(c\underline{z} + d)\phi_0^*(f)(\underline{z})$. We have

$$(A_{\underline{z}}, \mathrm{d}w, -\mathrm{d}\overline{w}\,\underline{s}) \cong (A_{\gamma\cdot\underline{z}}, (c\underline{z}+d)\mathrm{d}w, -(c\underline{z}+d)^{-1}\mathrm{d}\overline{w}\,\underline{s}).$$

Similar to replacing $\mathrm{d}w$ by $(c\underline{z}+d)\mathrm{d}w$, the action of γ replaces $\mathrm{d}\overline{w}$ by $(c\overline{\underline{z}}+d)\mathrm{d}\overline{w}$ and $\underline{s}$ by $\frac{\underline{s}}{|c\underline{z}+d|^2}$, which has the net effect of replacing the product $-\mathrm{d}\overline{w}\,\underline{s}$ by $-(c\underline{z}+d)^{-1}\mathrm{d}\overline{w}\,\underline{s}$. Thus we get the required transformation property by using the formula for the action of Q on V_κ:

$$\phi_0^*(f)(\gamma \cdot \underline{z}) = \begin{pmatrix} (c\underline{z}+d)^{-1} & 0 \\ 0 & (c\underline{z}+d) \end{pmatrix} \cdot \phi_0^*(f)(\underline{z}) = \kappa(c\underline{z}+d)^{-1}\phi_0^*(f)(\underline{z}).$$

We have proved,

Theorem 3.6.7. *The map $\phi_0^* \colon \mathcal{V}_\kappa^\nu \to N_\kappa^\nu(N)$ is an isomorphism.*

We are left to show that this intertwines the differential operators ∇_σ and δ_κ^σ.

Theorem 3.6.8. *The operator ∇_σ is the correct analog of the Maass–Shimura operator δ_κ^σ, in the sense that $\phi_0^* \circ \nabla_\sigma = \delta_\kappa^\sigma \circ \phi_0^*$.*

Proof. We have formulas for each operator, and we compare them. Let $P_f(\underline{Y})$ be the polynomial with coefficients which are holomorphic functions of $\underline{z}$ such that $f(A_{\underline{z}}, \mathrm{d}w, \beta) = P_f(\underline{Y})$. By the formula in Equation (3.4.4), write

$$(\nabla_\sigma \cdot f)(A_{\underline{z}}, \mathrm{d}w, \beta) = \frac{\partial P_f}{\partial z_\sigma}(\underline{Y}) + \mu_\sigma^- P_f(\underline{Y}).$$

Here the operator $\frac{\partial}{\partial z_\sigma}$ differentiates the coefficients of P_f. Using the formula for the action of μ_σ^-, we get

$$(\nabla_\sigma \cdot f)(A_{\underline{z}}, \mathrm{d}w, \beta) = \frac{\partial P_f}{\partial z_\sigma}(\underline{Y}) + Y_\sigma \varepsilon_\sigma P_f(\underline{Y}) - Y_\sigma^2 \frac{\partial}{\partial Y_\sigma} P_f(\underline{Y}).$$

The action of ε_σ is just multiplication by k_σ. Applying the realization functor, we get

$$(\phi_0^* \circ \nabla_\sigma)(f)(\underline{z}) = \frac{\partial P_f}{\partial z_\sigma}(\underline{s}) + k_\sigma s_\sigma P_f(\underline{s}) - s_\sigma^2 \frac{\partial P_f}{\partial Y_\sigma}(\underline{s}). \tag{3.6}$$

Now we should compare this to $(\delta_\kappa^\sigma \circ \phi_0^*)(f)$. We have that $\phi_0^*(f) = P_f(\underline{s})$. Note that $\frac{\partial s_\tau}{\partial z_\sigma} = 0$ if $\sigma \neq \tau$, and

$$\frac{\partial s_\sigma}{\partial z_\sigma} = \frac{\partial}{\partial z_\sigma} \frac{1}{z_\sigma - \overline{z}_\sigma} = -\frac{1}{(z_\sigma - \overline{z}_\sigma)^2} = -s_\sigma^2. \tag{3.7}$$

Using the definition of δ_κ^σ and the product rule, we get

$$(\delta_\kappa^\sigma \cdot \phi_0^*(f))(\underline{z}) = s_\sigma^{k_\sigma} \frac{\partial}{\partial z_\sigma} \left(s_\sigma^{-k_\sigma} P_f(\underline{s}) \right) = \left(s_\sigma^{k_\sigma} \frac{\partial}{\partial z_\sigma} s_\sigma^{-k_\sigma} \right) P_f(\underline{s}) + \frac{\partial}{\partial z_\sigma} \left(P_f(\underline{s}) \right). \quad (3.8)$$

Here, the derivative in $\frac{\partial P_f}{\partial z_\sigma}(\underline{s})$ acts only on the coefficients of P_f; on the other hand, $\frac{\partial}{\partial z_\sigma}(P_f(\underline{s}))$ refers to the derivative of the whole nearly holomorphic function $P_f(\underline{s})$. The product rule gives

$$\frac{\partial}{\partial z_\sigma}(P_f(\underline{s})) \quad = \quad \frac{\partial P_f}{\partial z_\sigma}(\underline{s}) + \frac{\partial P_f}{\partial Y_\sigma}(\underline{s}) \cdot \frac{\partial s_\sigma}{\partial z_\sigma} \quad = \quad \frac{\partial P_f}{\partial z_\sigma}(\underline{s}) - s_\sigma^2 \frac{\partial P_f}{\partial Y_\sigma}(\underline{s}).$$

On the other hand, using Equation (3.7), we have

$$\frac{\partial}{\partial z_\sigma} s_\sigma^{-k_\sigma} = -k_\sigma s_\sigma^{-k_\sigma - 1} \frac{\partial s_\sigma}{\partial z_\sigma} = k_\sigma s_\sigma^{k_\sigma + 1}, \qquad s_\sigma^{k_\sigma} \frac{\partial}{\partial z_\sigma} s_\sigma^{-k_\sigma} = k_\sigma s_\sigma.$$

Plugging the previous two equations into Equation 3.8, we get

$$(\delta_\kappa^\sigma \cdot \phi_0^*(f))(\underline{z}) = k_\sigma s_\sigma P_f(\underline{s}) + \frac{\partial P_f}{\partial z_\sigma}(\underline{s}) - s_\sigma^2 \frac{\partial P_f}{\partial Y_\sigma}(\underline{s}). \quad (3.9)$$

Comparing Equations (3.6) and (3.9), we conclude. □

3.7 Splitting the Hodge Filtration

This section is included to give some intuition for why certain objects might show up in this theory. In it, we attempt to explain why the $(\mathfrak{g}, Q)$-module V_κ is a natural object to consider by giving an interpretation of its relationship with W_κ. It is not necessary in order to follow the results and proofs of this dissertation.

We have not yet talked about one of the most important features of the theory of nearly Hilbert modular forms: the choice of splitting of the Hodge filtration. Recall from Equation (3.1) that the bundles $\underline{\omega}$ and $\mathcal{H}$ fit into the following exact sequence of sheaves of $\mathcal{O}_F \otimes \mathcal{O}_X$-modules known as the Hodge filtration:

$$0 \to \underline{\omega} \to \mathcal{H} \to \underline{\omega}^\vee \to 0$$

We also recall that we defined the group schemes G, H, and Q in terms of these objects. We had G, the group of automorphisms of $\mathcal{H}$ that preserve the pairing; H the group of automorphisms of $\underline{\omega}$; and Q, the group of automorphisms of $\mathcal{H}$ that preserved $\underline{\omega}$ as a subsheaf.

Pick a trivialization $\alpha_0\colon \mathcal{O}_F \otimes \mathcal{O}_X \to \underline{\omega}$, and extend it to a trivialization α of $\mathcal{H}$ such that $\alpha(1,0) = \alpha_0(1)$ and $\langle \alpha(1,0), \alpha(0,1)\rangle = 1$. This gives a *splitting of the Hodge filtration*. Specifically, $\alpha(0,1)$ projects to a basis $\omega' \in \underline{\omega}^\vee$; every element of $\underline{\omega}^\vee$ can be written uniquely as a constant multiple of this element, $c\omega'$. Then we define an inclusion $\eta\colon \underline{\omega}^\vee \to \mathcal{H}$ by sending $c\omega'$ to $c\alpha(0,1)$. This induces a decomposition of $\mathcal{H}$ as an internal direct sum $\mathcal{H} = \operatorname{Span}\{\alpha(1,0)\} \oplus \operatorname{Span}\{\alpha(0,1)\} = \underline{\omega} \oplus \eta\left(\underline{\omega}^\vee\right)$.

In this basis, we can write G as a group of 2×2 matrices, with Q as the subgroup of upper triangular matrices. We also embed $\mathbb{T}$ into Q as a subgroup H of diagonal matrices; write $h \in \mathbb{T}$ as the diagonal matrix $\begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix}$. The group H preserves the direct sum decomposition $\mathcal{H} = \underline{\omega} \oplus \underline{\omega}^\vee$.

On the other hand, while Q does preserve the subspace $\underline{\omega} \subset \mathcal{H}$, it does not preserve the complement $\eta\left(\underline{\omega}^\vee\right)$; in fact, Q acts transitively on the set of such complements, while H is the subset that preserves each decomposition. Thus the set of splittings of the Hodge filtration is a Q-set, and the stabilizer of each splitting is H. So any choice of a splitting gives an isomorphism between the set of splittings of the Hodge filtration and the coset space $H\backslash Q$.

Now recall that the representation V_κ of Q is the set $W_\kappa \otimes \mathcal{O}_{H\backslash Q}$. We should think of this as the space of functions from $H\backslash Q$ to W_κ, or once a choice of a splitting of the Hodge filtration is chosen, perhaps as the space of functions from the set of splittings of the Hodge filtration to W_κ. Given an element of

$T_{\mathcal{H}}^{\times,+}$, we can think of the trivialization of $\mathcal{H}$ as giving us a trivialization of $\underline{\omega}$ along with a choice of splitting of the Hodge filtration. (It also gives a basis for the complement $\eta\,(\underline{\omega}^\vee)$, but for this section we will ignore this fact.) We get the following philosophical result: a nearly Hilbert modular form is a gadget that takes in a splitting of the Hodge filtration and splits out a Hilbert modular form.

Note that a nearly Hilbert modular form f is an honest Hilbert modular form if it doesn't depend on the choice of splitting. On the algebraic side, this happens if and only if it descends to a function on the quotient $T_{\underline{\omega}}^{\times}$ of $T_{\mathcal{H}}^{\times,+}$. On the complex analytic side, this happens if and only if the nearly holomorphic function f is actually holomorphic.

When $F = \mathbb{Q}$, this conversation is more explicit. Working over $\mathbb{C}$, we have a natural splitting of the Hodge filtration given by $\mathcal{H} = \underline{\omega} \oplus \overline{\underline{\omega}}$, where $\underline{\omega}$ is the space of holomorphic differentials on the universal elliptic curve $\mathcal{A}$, and $\overline{\underline{\omega}} \cong \underline{\omega}^\vee$ is the space of anti-holomorphic differentials. The coset representatives of $H\backslash Q$ are matrices of the form

$$\begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix}, \qquad Y \in \operatorname{Hom}(\underline{\omega}^\vee, \underline{\omega}) \cong \underline{\omega}^{\otimes 2}.$$

There are only a few sections of $\underline{\omega}^{\otimes 2}$, and none of them are holomorphic. In fact, they are all multiples of the function $E_2 - \frac{3}{\pi y} = E_2 + \frac{12}{2\pi i}s$, where E_2 is a normalized Eisenstein series of weight 2. As a function on $\mathfrak{h}$, it satisfies the weight 2 modularity condition. Now let $f(z)$ be a holomorphic modular form of weight k, so $\delta_k f = \frac{ks}{2\pi i} f + \frac{1}{2\pi i}\frac{\partial f}{\partial z}$ is a nearly holomorphic modular form of weight $k+2$. Since $\frac{1}{12}(kE_2 + \frac{12ks}{2\pi i})f = \frac{2\pi i k}{12}E_2 f + ksf$ also satisfies the weight $k+2$ modularity condition, their difference $\partial f = \frac{\partial f}{\partial z} - \frac{2\pi i k}{12}E_2 f$ satisfies the weight $k+2$ modularity condition as well, and it is holomorphic. This ∂f is the *holomorphic projection* of $\delta_k f$, and it is an honest *holomorphic* modular form of weight $k+2$ when viewed as a function on $\mathfrak{h}$.

In the sense above, the nearly holomorphic function $\delta_k f$ corresponds to a certain nearly modular form ∇f. This nearly modular form takes a splitting η of the Hodge filtration and spits out a modular form, which corresponds to a function on $\mathfrak{h}$. There is one splitting η_0 of the Hodge filtration for which the function on $\mathfrak{h}$ corresponding to $(\nabla f)(\eta_0)$ is actually the nearly holomorphic function $\delta_k f$, and one splitting η_{holo} of the Hodge filtration for which the function on $\mathfrak{h}$ corresponding to $(\nabla f)(\eta_{\mathrm{holo}})$ is actually holomorphic. All other splittings fall somewhere in between.

CHAPTER IV

P-ADIC THEORY

Let K be a p-adic field that splits F, as in Section 2.1. As established in Section 1.4, $\mathcal{O}_K$ is its ring of integers, and $\mathfrak{m}$ the unique maximal ideal in $\mathcal{O}_K$. For any $m \geq 1$, let $\mathcal{O}_m = \mathcal{O}_K/\mathfrak{m}^m$, so that in particular $\mathcal{O}_1$ is the residue field $\mathcal{O}_K/\mathfrak{m}$, which has characteristic p. In this section we work over the bases $S = \operatorname{Spec} \mathcal{O}_K$ and $S_m = \operatorname{Spec} \mathcal{O}_m$.

4.1 Weights

Recall the discussion of weights in the Archimedean theory, from Section 2.1. In the Archimedean theory, weights were characters of the algebraic group $\mathbb{T} = \operatorname{Res}_{\mathcal{O}_F/\mathbb{Z}} \mathbb{G}_m$. For the p-adic theory, weights are instead characters of the constant group scheme $\mathbb{T}(\mathbb{Z}_p)$.

The weight space $\mathcal{W}$, a rigid analytic space defined over K associated to the algebra $\mathbb{Z}_p[\![\mathbb{T}(\mathbb{Z}_p)]\!]$. The $\mathbb{C}_p$-points of $\mathcal{W}$ parametrize continuous homomorphisms $\mathbb{T}(\mathbb{Z}_p) \to \mathbb{C}_p^\times$, which are the p-adic weights of Hilbert modular forms. As a rigid analytic space, it is isomorphic to a finite disjoint union of open unit balls of dimension d, where each component is labeled by a (finite order) character of the torsion subgroup of $\mathbb{T}(\mathbb{Z}_p)$. A character of $\mathbb{T}(\mathbb{Z}_p)$ is called algebraic if it is in $\mathbb{Z}[I]$, and locally algebraic if it is the product of a finite order character and an algebraic character. Locally algebraic characters are dense in the weight space.

For any value[1] $w \in v(\mathcal{O}_F)$, the rigid analytic group $\mathbb{T}$ has two important subgroups. The first, $\mathbb{T}^0_w$, consists of units congruent to 1 modulo $(p)^w$. The second, $\mathbb{T}_w$, is generated by $\mathbb{T}^0_w$ and $\mathbb{T}(\mathbb{Z}_p)$.

[1] If p ramifies with index e, $w \in \frac{1}{e}\mathbb{Z}$. Writing $(p)^w$ should make sense as an ideal of $\mathcal{O}_F$, and we will sometimes write p^w abusively even if there is no such element.

Let A be an affinoid algebra, and let $\kappa\colon \mathbb{T}(\mathbb{Z}_p) \to A^\times$ be a family of characters parametrized by the affinoid space $\mathcal{U} = \operatorname{Sp} A$. As asserted in [AIP16], such a character is w-analytic for some w, meaning that it factors as a composition as follows for some linear map ψ:

$$\mathbb{T}^0_w(\mathbb{Z}_p) = 1 + p^w(\mathcal{O}_F \otimes \mathbb{Z}_p) \xrightarrow{\log_F} p^w\mathcal{O}_F \otimes \mathbb{Z}_p \xrightarrow{\psi} pA \xrightarrow{\exp} 1 + pA \subset A^\times. \tag{4.1}$$

There is a universal character $\kappa^{un}\colon \mathcal{W} \times \mathbb{T}(\mathbb{Z}_p) \to \mathbb{C}_p^\times$, where $\kappa^{un}(x,t)$ evaluates the character associated to the point x at the input $t \in \mathbb{T}(\mathbb{Z}_p)$, corresponding to the natural character $\mathbb{T}(\mathbb{Z}_p) \to \mathcal{O}_K[\![\mathbb{T}(\mathbb{Z}_p)]\!]$ sending g to g. The universal character is 1-analytic.

For a family of p-adic, or overconvergent, Hilbert modular forms, its weight should be a family of characters; if the family of p-adic Hilbert modular forms is parametrized by a rigid space $\mathcal{U}$, so should be the family of weights. Any family of characters parametrized by a space $\mathcal{U}$ is specified by pulling back the universal character by a map $\mathcal{U} \to \mathcal{W}$.

4.2 Canonical Subgroups for Ordinary HBAVs

In this section we discuss ordinary HBAVs and their properties. Consider an HBAV A_1 defined over the residue field $\mathcal{O}_1$, and a prime ideal $\mathfrak{p}$ of F lying over p. Consider the $\mathfrak{p}^n$-torsion of A_1,

$$A_1[\mathfrak{p}^n] = \{a \in A_1 \mid x \cdot a = 0 \text{ for all } x \in \mathfrak{p}^n\}.$$

For our fixed A_1, there are two possibilities for the sizes of these subgroups. Either $\#A_1[\mathfrak{p}^n] = \operatorname{Nm}(\mathfrak{p})^n$ for all n, or $\#A_1[\mathfrak{p}^n] = 1$ for all n. In the first case, we say that A_1 is *ordinary*, while in the second we say that A_1 is *supersingular*.

For an HBAV A defined over $\mathcal{O}_K$, we say that A is ordinary or supersingular depending on its base change $A_1 = A \times_{\mathcal{O}_K} \mathcal{O}_1$ to the residue field. If A_1 is ordinary, we say A is ordinary, and if A_1 is supersingular, we say that A is supersingular.

There is a test for whether or not A_1 is ordinary depending on the value of a modular form, known as the Hasse invariant. Let $q = \mathrm{Nm}(\mathfrak{p})$, Fr be the relative q-power Frobenius map $A_1 \to A_1^{(q)}$, defined by the following diagram:

$$\begin{tikzcd}
A_1 \arrow[dr, dashed, "\mathrm{Fr}"] \arrow[drr, bend left, "\mathrm{Fr}_{\mathrm{abs}}"] \arrow[ddr, bend right] & & \\
& A_1^{(q)} \arrow[r] \arrow[d] & A_1 \arrow[d] \\
& \operatorname{Spec}\mathcal{O}_1 \arrow[r, "\mathrm{Fr}_{\mathrm{abs}}"] & \operatorname{Spec}\mathcal{O}_1
\end{tikzcd}$$

In this diagram, both maps $A_1 \to \operatorname{Spec}\mathcal{O}_1$ are the structure maps. The maps $\operatorname{Spec}\mathcal{O}_1 \to \operatorname{Spec}\mathcal{O}_1$ and $A_1 \to A_1$ are both the *absolute Frobenius* maps, which are the identity on points, but pulling back functions raises them to the qth power. The bottom right square is a pullback diagram which defines $A_1^{(q)}$, and the relative Frobenius Fr fits in as the dashed arrow by the universal property of the pullback.

Remark 4.2.1. In fact, $A_1^{(q)} \cong A_1$, which is easiest to see in terms of equations. Since A_1 is an abelian variety, it is projective. Embed it into some projective space; it is the zero set of some collection of homogeneous polynomials $\{f_i\}_{i \in I}$, and since it is defined over $\mathcal{O}_1$, the coefficients of these polynomials are elements of $\mathcal{O}_1$. Then $A_1^{(q)}$ is the zero set of the polynomials $\{f_i^{(q)}\}_{i \in I}$, where $f_i^{(q)}$ is the polynomial obtained from f_i by raising all of its coefficients to the q power. Since the coefficients are in $\mathcal{O}_1$, raising them to the q power is the identity, and $A_1^{(q)}$ is the same subvariety of the same projective space as A_1 was. This is one advantage of using the relative q-power Frobenius map instead of p-power. For elliptic curves, this is spelled out in e.g. [Sil09, Example II.4.6]. We will not distinguish between the two for the rest of this section.

The dual isogeny to the relative Frobenius, $V\colon \hat{A}_1 \to \hat{A}_1$ is known as verschiebung. Vershiebung is either separable, if A_1 is ordinary, or inseparable, if A_1 is supersingular at $\mathfrak{p}$ for some $\mathfrak{p}$ lying over p.

Fix an HBAV A_1 defined over an $\mathcal{O}_1$-algebra R_1 such that $\Omega^1_{A_1}$ is a free $\mathcal{O}_F \otimes R_1$-module of rank 1 (i.e., A_1 satisfies the Rapoport condition). Then choose a trivialiation $\alpha\colon \mathcal{O}_F \otimes R_1 \to \Omega^1_{A_1}$, and let $\omega = \alpha(1)$. Using the decomposition $\Omega^1_{A_1} \cong \bigoplus_\sigma \Omega^1_{A_1}(\sigma)$, we get an R_1-basis $\{\omega_\sigma\}_\sigma$ for $\Omega^1_{A_1}$, where ω_σ is the image of ω after projecting to $\Omega^1_{A_1}(\sigma)$.

The polarization λ induces a perfect, $\mathcal{O}_F$-linear pairing $\Omega^1_{A_1} \times T_{\hat{A}_1} \to R_1$. Let $\{\eta_\sigma\}_\sigma$ be the dual basis to $\{\omega_\sigma\}_\sigma$, and notice that pullback by Fr gives a skew-linear map $T_{\hat{A}_1} \to T_{\hat{A}_1}$, such that

$$\mathrm{Fr}(c\eta) = c^q \,\mathrm{Fr}(\eta). \tag{4.2}$$

Notice that, since $\sigma(c^q) = \sigma(c)^q$, this map is $\mathcal{O}_F$-linear. Thus, if η is dual to an element of $\Omega^1_{A_1}(\sigma)$, so is $\mathrm{Fr}(\eta)$. Define the function Ha_σ to be the coefficient

$$\mathrm{Fr}(\eta_\sigma) \mapsto \mathrm{Ha}_\sigma(A_1, \alpha)\eta_\sigma.$$

We note that this satisfies a homogeneity property: for any $c \in \mathcal{O}_F \otimes R_1$, $c \cdot \alpha$ gives the new dual basis $\{\sigma(c)^{-1}\eta_\sigma\}_\sigma$. Using Equation (4.2), we see that

$$\mathrm{Fr}(\sigma(c)^{-1}\eta_\sigma) = \sigma(c)^{-q}\,\mathrm{Fr}(\eta_\sigma).$$

Writing each side of the equation above in terms of the Hasse invariant, we get

$$\mathrm{Ha}_\sigma(A_1, c\alpha)\sigma(c)^{-1}\eta_\sigma = \sigma(c)^{-q}\,\mathrm{Ha}_\sigma(A_1, \alpha)\eta_\sigma.$$

Multiplying both sides by $\sigma(c)$ and comparing coefficients, we find

$$\mathrm{Ha}_\sigma(A_1, c\alpha) = \sigma(c)^{1-q}\,\mathrm{Ha}_\sigma(A_1, \alpha). \tag{4.3}$$

Definition 4.2.2. For any abelian variety A_1 as above and any trivialization α of its cotangent bundle, define the *Hasse invariant at* σ, Ha_σ, to be coefficient

$$\mathrm{Fr}(\eta_\sigma) \mapsto \mathrm{Ha}_\sigma(A_1, \alpha)\eta_\sigma$$

By Equation (4.3), we see that Ha_σ is a Hilbert modular form of weight $(q-1)\sigma$.

For every prime $\mathfrak{p}$ lying over p, let $I_\mathfrak{p}$ be the set of embeddings $F \to K$ which "belong to $\mathfrak{p}$." Specifically,

$$I_\mathfrak{p} = \{\sigma \in I \mid \mathcal{O}_F \xrightarrow{\sigma} \mathcal{O}_K \to \mathcal{O}_1 \text{ factors through the reduction } \mathcal{O}_F \to \mathcal{O}_F/\mathfrak{p}\}.$$

Then define the *Hasse invariant at* $\mathfrak{p}$, $\mathrm{Ha}_\mathfrak{p}$, to be the product

$$\mathrm{Ha}_\mathfrak{p}(A_1, \alpha) = \prod_{\sigma \in I_\mathfrak{p}} \mathrm{Ha}_\sigma(A_1, \alpha).$$

These are Hilbert modular forms of weight $\sum_{\sigma \in I_\mathfrak{p}}(q-1)\sigma$; we also call them *partial Hasse invariants.* Finally, define the full *Hasse invariant* Ha to be the product

$$\mathrm{Ha}(A_1, \alpha) = \prod_{\sigma \in I} \mathrm{Ha}_\sigma(A_1, \alpha) = \prod_{\mathfrak{p} \mid p} \mathrm{Ha}_\mathfrak{p}(A_1, \alpha).$$

This is a Hilbert modular form of parallel weight $q-1$.

Remark 4.2.3. The partial Hasse invariants Ha_σ are only defined over the Rapoport locus. However, as noted in [AIP16, Remark 3.1, page 11], the products $\mathrm{Ha}_\mathfrak{p}$ extend to be defined over the whole modular curve.

An abelian variety A_1 over $\mathcal{O}_1$ will be ordinary at p, i.e., V will be separable, if and only if $\mathrm{Ha}(A_1, \alpha) \neq 0$ for some trivialization α of $\Omega^1_{A_1}$. The homogeneity property shows that one may check this condition for any choice of α. We say A_1 is ordinary at $\mathfrak{p}$ if $\mathrm{Ha}_\mathfrak{p}(A_1, \alpha) \neq 0$ for some trivialization α. The product formula

$\mathrm{Ha} = \prod_{\mathfrak{p}} \mathrm{Ha}_{\mathfrak{p}}$ shows that A_1 is ordinary at p if and only if it is ordinary at $\mathfrak{p}$ for all $\mathfrak{p}$ dividing p.

Remark 4.2.4. When $F = \mathbb{Q}$, this story simplifies somewhat, since an HBAV is just an elliptic curve. Here an elliptic curve is canonically polarized; let E be an elliptic curve, $\mathrm{Fr}\colon E \to E$ be the relative p-power Frobenius, and note that we can view its dual isogeny as a map $V\colon E \to E$. Since Fr has degree p, we can factor the multiplication by p map as $[p] = V \circ \mathrm{Fr}$, and since Fr is (totally) inseparable, it has trivial kernel, giving that the p-torsion is $E[p] = \ker[p] = \ker V$. This shows that $\#E[p] = \#\ker V$, which is 1 if V is inseparable and p if V is separable.

So we should find a way to determine whether V is separable or inseparable. In general, an isogeny $f\colon E \to E'$ is separable if and only if the pullback on cotangent bundles $f^*\colon \Omega^1_{E'} \to \Omega^1_E$ is nonzero, see e.g. [Sil09, III.4.2c]. Since the cotangent bundles are rank 1, we only have to check whether V^* is nonero, or that $V^*(\omega) \neq 0$ for a single generator $\omega \in \Omega^1_E$. This is essentially what the Hasse invariant does for us; more specifically, Ha checks whether or not the dual to this map is nonzero.

Remark 4.2.5. Though we have taken the view that Hilbert modular forms should be thought of as functions on $T^\times_{\underline{\omega}}$, here it is useful to realize that the Hasse invariant is a section of a line bundle on X, the moduli space of HBAVs. Specifically, this view allows us to see that its zero locus, consisting of points that correspond to supersingular HBAVs, is a closed subscheme $X^{ss} \subset X$. On the other hand, the complement of X^{ss}, which consists of points that correspond to ordinary HBAVs, is a dense open subscheme $X^{ord} \subset X$. We call X^{ss} the *supersingular locus*, and X^{ord} is the *ordinary locus*.

Let $\mathfrak{d}_{\mathfrak{p}}^{-1}$ be the local inverse different in the completion $F_{\mathfrak{p}}$. If A is ordinary at $\mathfrak{p}$, then as an algebraic group, we have $A[\mathfrak{p}^n] \cong \mathfrak{d}_{\mathfrak{p}}^{-1} \otimes \mu_{p^n} \times \mathcal{O}_F/\mathfrak{p}^n$. The maximal connected subgroup $\mathfrak{d}_{\mathfrak{p}}^{-1} \otimes \mu_{p^n}$ is called the *canonical subgroup of level* $\mathfrak{p}^n$ of A, denoted $C_{\mathfrak{p}^n} = C_{\mathfrak{p}^n}(A)$. For $n_1 \leq n_2$, we have $C_{\mathfrak{p}^{n_1}} \subset C_{\mathfrak{p}^{n_2}}$, and the union over all n gives the *canonical subgroup of level* $\mathfrak{p}^\infty$ of A, denoted $C_{\mathfrak{p}^\infty}$. For $n \leq \infty$, let C_{p^n} be the maximal connected subgroup of $A[p^n]$. This is the canonical subgroup of level p^n, and it is precisely the subgroup of $A[p^n]$ generated by all the groups $C_{\mathfrak{p}^n}$ for $\mathfrak{p}$ dividing p.

Remark 4.2.6. For all $n \leq \infty$, the only point of $\mathfrak{d}_{\mathfrak{p}}^{-1} \otimes \mu_{p^n}$ over a field of characteristic p is the identity; this accounts for the fact that $A[\mathfrak{p}^n]$ and $\mathcal{O}_F/\mathfrak{p}^n$ both have size $\mathrm{Nm}(\mathfrak{p})^n$ over the special point S_1 of S, even though $\#A[\mathfrak{p}^n] = \mathrm{Nm}(p)^{2n}$ at the generic point $\operatorname{Spec} K$. In this sense, we can think of the canonical subgroup as the "kernel of the reduction map".

For any $n < \infty$ and $m < n$, we have the following as in [CEF+16, Equation (9)]. It is valid for any ordinary abelian variety A.

$$\Omega^1_{A/\mathcal{O}_m} = (\mathrm{Lie}(A))^D = (\mathrm{Lie}(A)[p^n])^D = (\mathrm{Lie}(A)[p^n]^\circ)^D = (\mathrm{Lie}\, C_n(A))^D$$

Here $A[p^n]^\circ$ is the connected component of the identity, which is the canonical subgroup C_n. We are allowed to consider the p^n-torsion because we are working over $\mathcal{O}_m$ for $m < n$. Then, since the Tate module is isomorphic to the Lie algebra of a p^n-torsion group,

$$(\mathrm{Lie}\, C_n(A))^D \cong (T_p C_n(A))^D \cong C_n^D(A) \otimes \mathcal{O}_m.$$

This shows that the cotangent bundle $\Omega^1_{A/\mathcal{O}_m}$ is isomorphic to the dual of the canonical subgroup $C_n^D(A) \cong \mathcal{O}_F/p^n\mathcal{O}_F$. This is also true in families; write

$X_m^{ord} := X^{ord} \times_{\mathcal{O}_K} \mathcal{O}_m$, and $\mathcal{A}_m^{ord} = \mathcal{A}^{ord} \times_{\mathcal{O}_K} \mathcal{O}_m$. Then

$$\Omega^1_{\mathcal{A}_m^{ord}/\mathcal{O}_m} \cong C_n^D(\mathcal{A}_m^{ord}). \tag{4.4}$$

Remark 4.2.7. Let $A \to T$ be a family of HBAVs parametrized by some S-scheme T. We view this as an HBAV defined over T. Then if A is ordinary, its canonical subgroup is also a family $C_{\mathfrak{p}^n} \to T$. For any $T' \to T$, we can define the fiber product $A \times_T T'$ which will also be ordinary; its canonical subgroup will be the fiber product $C_{\mathfrak{p}^n} \times_T T'$.

4.3 p-adic Hilbert Modular Forms

In Chapter II, we described Hilbert modular forms as functions on the torsor $T^\times_{\underline{\omega}}$, the frame bundle of the pushforward of the relative cotangent bundle $\underline{\omega} = \pi_*\Omega^1_{\mathcal{A}/X}$ of the universal HBAV $\pi\colon \mathcal{A} \to X$. In the p-adic setting, however, the cotangent bundle is not as well-behaved as one might hope. In Section 4.2 we described canonical subgroups and the Hodge–Tate map, which connected these canonical subgroups C_{p^n} to the cotangent bundle Ω^1_A. We will use canonical subgroups as a replacement for the cotangent bundle in the p-adic setting.

Recall that the ordinary locus X^{ord} is an open subscheme of X, and let $\mathcal{A}^{ord} = \mathcal{A} \times_X X^{ord}$ be the universal ordinary HBAV. Since $\mathcal{A}^{ord}$ is ordinary, it has a canonical subgroup C_{p^∞}, and the Hodge–Tate map gives an isomorphism $\Omega^1_{\mathcal{A}^{ord}} \to C_{p^\infty}$ defined over $X^{ord} \times_{\mathcal{O}_K} \mathcal{O}_m$ for any m. This gives rise to a morphism (not an isomorphism) $\Omega^1_{\mathcal{A}^{ord}} \to C_{p^\infty}$ over $\mathcal{O}_K$.

For all $n < \infty$, let $T^\times_{C_{p^n}} = \mathrm{Isom}_{X^{ord}}(C_{p^n}, \mathfrak{d}^{-1} \otimes \mu_{p^n})$ be the set of $\mathcal{O}_F$-linear isomorphisms from the canonical subgroup C_{p^n} of the universal ordinary HBAV $\mathcal{A}^{ord}$ over the ordinary locus X^{ord} to $\mathfrak{d}^{-1} \otimes \mu_{p^n}$. This is a torsor for the group of $\mathcal{O}_F$-linear automorphisms of $\mathfrak{d}^{-1} \otimes \mu_{p^n}$; when $n < \infty$, this automorphism group is isomorphic to $\mathrm{Aut}_F(\mathfrak{d}^{-1} \otimes \mu_{p^n}) \cong (\mathcal{O}_F/p^n\mathcal{O}_F)^\times = \mathbb{T}(\mathbb{Z}/p^n\mathbb{Z})$.

When $n = \infty$, consider instead the formal scheme $\mathfrak{T}^{\times}_{\mathfrak{C}_{p^\infty}} = \varprojlim_n T^{\times}_{C_{p^n}}$. This is a formal torsor for the group $\varprojlim_n \mathbb{T}(\mathbb{Z}/p^n\mathbb{Z}) \cong \mathbb{T}(\mathbb{Z}_p)$, known as the Igusa tower. A function on $\mathfrak{T}^{\times}_{\mathfrak{C}_p^\infty}$ takes in a pair (A, u) of an HBAV A defined over an $\mathcal{O}_K$-algebra R and a trivialization $u\colon C_{p^n} \to \mathfrak{d}^{-1} \otimes \mu_{p^n}$ and returns an element of R.

Definition 4.3.1. Let R_0 be a complete and separated $\mathcal{O}_K$-algebra, and κ a character of $\mathbb{T}(\mathbb{Z}_p)$. A p-adic Hilbert modular form of level $\Gamma_1(N)$ and weight κ defined over R_0 is an algebraic function $f \in \mathcal{O}_{\mathfrak{T}^{\times}_{C_{p^\infty}}}$ which satisfies the homogeneity property

$$f(A, g^t u) = \kappa(g^{-1}) f(A, u). \tag{4.5}$$

Write the space of such functions as $\mathcal{O}_{\mathfrak{T}^{\times}_{C_{p^\infty}}}[-\kappa]$.

Remark 4.3.2. Let R_∞ be a normal $\mathcal{O}_K$-algebra which is p-adically complete, separated, and topologically of finite type. Let $R_m = R_\infty \otimes_{\mathcal{O}_K} \mathcal{O}_m$. Given a classical Hilbert modular form f, i.e., a function on $T^{\times}_{\underline{\omega}}$, defined over R_∞, we now describe how to view it as a p-adic Hilbert modular form.

Each isomorphism $u\colon C_{p^\infty} \to \mathfrak{d}^{-1} \otimes \mu_{p^\infty}$ restricts to an isomorphism on the p^n-torsion, $u_n\colon C_{p^\infty} \to \mathfrak{d}^{-1} \otimes \mu_{p^\infty}$. We dualize, obtaining isomorphisms $u_n^D\colon (\mathfrak{d}^{-1} \otimes \mu_{p^n})^D \to C^D_{p^n}$ for each n. Now fix some $m \geq 1$. When $m < n$, composing with the isomorphism of Equation (4.4) we obtain, for any pair (A, u) of an ordinary HBAV A and a trivialization $u\colon C_{p^\infty}(A) \to \mathfrak{d}^{-1} \otimes \mu_{p^\infty}$, a trivialization $\alpha_n\colon (\mathfrak{d}^{-1} \otimes \mu_{p^n})^D \to \Omega^1_{A/\mathcal{O}_m}$. In fact, for a fixed m, this trivialization α_n is independent of n (as long as $m < n$). This gives us an isomorphism $T^{\times}_{C_{p^n}} \cong T^{\times}_{\underline{\omega}} \times_{X^R} X^{ord}$. We also have that $(\mathfrak{d}^{-1} \otimes \mu_{p^n})^D \cong \mathcal{O}_F \otimes R_m$. Using this viewpoint, one sees that, in an appropriate category of locally ringed spaces,

$$\mathfrak{T}^{\times}_{C_{p^\infty}} = \varprojlim_n T^{\times}_{\underline{\omega}} \times_{X^R} X^{ord}_m. \tag{4.6}$$

Thus, a p-adic Hilbert modular form f defined over R_∞ is a coherent system of classical Hilbert modular forms $(f_m)_{m=1}^\infty$ defined over R_m with the caveat that we only care about their values over the ordinary locus, rather than over the whole Rapoport locus X^R.

Notice that the homogeneity property in Equation (4.5) includes a transpose, as well as putting the group element on the opposite side of the trivialization compared to Equation (2.3). This is because of the step in the process above where we have to dualize the automorphism – the dual of $g^t u$ becomes $u^D g$, which looks much closer to the form taken by the latter equation. In fact, this is how we check the homogeneity condition.

Since $\mathbb{T}$ is a group of 1×1 matrices, the transpose map is just the identity; we could have left it out. However, in the case of higher rank groups, this is a crucial piece, so we include it here.

The ring of p-adic Hilbert modular forms of level $\Gamma_1(N)$ is $\mathcal{O}_{\mathfrak{T}^\times_{C_{p^\infty}}}$. By the discussion in Remark 4.3.2, the ring $\mathcal{O}_{T^\times_{\underline{\omega}}}$ of classical Hilbert modular forms maps to the ring of p-adic Hilbert modular forms, simply by reducing the classical Hilbert modular form modulo $\mathfrak{m}^m$ for each m. This map is injective with dense image (see [Hid04]), which justifies saying that the ring of p-adic Hilbert modular forms is the completion of the ring of Hilbert modular forms; this is a good thing, as it was the initial motivation behind Serre's introduction of p-adic modular forms in [Ser72].

4.4 Overconvergent Loci and Partial Canonical Subgroups

For each of the g primes $\mathfrak{p}$ of F lying over p, we have a partial Hasse invariant $\mathrm{Ha}_\mathfrak{p}$ defined in Definition 4.2.2. This is a Hilbert modular form defined over the finite field $\mathcal{O}_1$. We would like to choose a lift of this to a Hilbert modular form defined over $\mathcal{O}_K$, i.e., a Hilbert modular form $\mathrm{Ha}_{\mathfrak{p}}^{\mathrm{lift}}$ defined over $\mathcal{O}_K$ such that

$$\mathrm{Ha}_{\mathfrak{p}}^{\mathrm{lift}}(A,\alpha) \equiv \mathrm{Ha}_{\mathfrak{p}}(A \times_{\mathcal{O}_K} \mathcal{O}_1, \alpha \times_{\mathcal{O}_K} \mathcal{O}_1) \pmod{\mathfrak{m}}.$$

Unfortunately, this is not always possible. However, we are always able to produce such a lift of a power of the Hasse invariant $\mathrm{Ha}_{\mathfrak{p}}^{w}$ for some positive integer w. We will refer to this lift as $\mathrm{Ha}_{\mathfrak{p}}^{w,\mathrm{lift}}$. Note that an HBAV A defined over $\mathcal{O}_K$ is ordinary at $\mathfrak{p}$ if and only if $\mathrm{Ha}_{\mathfrak{p}}^{w,\mathrm{lift}}(A,\alpha)$ is a unit for some choice of α. In addition, because of the transformation property that comes with $\mathrm{Ha}_{\mathfrak{p}}^{w,\mathrm{lift}}$ being a Hilbert modular form of weight $\sum_{\sigma\in I_{\mathfrak{p}}} w(q-1)\sigma$, the $\mathfrak{p}$-adic valuation of $\mathrm{Ha}_{\mathfrak{p}}^{w,\mathrm{lift}}(A,\alpha)$ does not depend on the choice of α.

Define the *Hodge height* $\mathrm{Hdg}_{\mathfrak{p}}$ for each prime $\mathfrak{p}$,

$$\mathrm{Hdg}_{\mathfrak{p}}(A) = \min\left\{\frac{1}{w} v_p(\mathrm{Ha}_{\mathfrak{p}}^{w,\mathrm{lift}}(A,\alpha)), 1\right\}.$$

The condition that A is ordinary at $\mathfrak{p}$ if and only if $\mathrm{Ha}_{\mathfrak{p}}^{w,\mathrm{lift}}(A,\alpha)$ is a unit translates to the condition that A is ordinary at $\mathfrak{p}$ if and only if $\mathrm{Hdg}_{\mathfrak{p}}(A) = 0$.

Consider a tuple $\underline{v} = (v_{\mathfrak{p}})_{\mathfrak{p}}$ of rational numbers $v_{\mathfrak{p}} \in (0,1)\cap\mathbb{Q}$, indexed by the primes $\mathfrak{p}$. For each such tuple, we construct the $\underline{v}$-overconvergent locus $\mathfrak{X}(\underline{v})$, which is a normal formal scheme that classifies HBAVs A with $\mathrm{Hdg}_{\mathfrak{p}}(A) < v_{\mathfrak{p}}$ for all $\mathfrak{p}$. This description gives a well-defined subset of the rigid analytic fiber $\mathcal{X}(\underline{v}) \subset \mathcal{X}$; we take a formal model $\mathfrak{X}'(\underline{v})$ for it by taking admissible formal blowups as in [AIP16, Section 3.2], and we normalize the resulting formal scheme in $\mathcal{X}(\underline{v})$ to obtain the formal overconvergent locus $\mathfrak{X}(\underline{v})$. We also consider the formal completion of the ordinary locus $\mathfrak{X}^{ord}$ inside $\mathfrak{X}$; since an ordinary HBAV has $\mathrm{Hdg}_{\mathfrak{p}}(A) = 0$ for all $\mathfrak{p}$, we have an inclusion $\mathfrak{X}^{ord} \to \mathfrak{X}(\underline{v})$ for any tuple $\underline{v}$.

For each tuple $\underline{v}$ with $v_{\mathfrak{p}} < \frac{1}{p^n}$ for all $\mathfrak{p}$, the canonical subgroup of level p^n on $\mathfrak{X}^{ord}$ extends to a canonical subgroup $\mathfrak{C}_n \to \mathfrak{X}(\underline{v})$ over the $\underline{v}$-overconvergent locus. Its fiber over the ordinary locus is locally isomorphic to the formal group scheme $\mathfrak{d}^{-1} \otimes \mu_{p^n}$. Its fiber over the complement $\mathfrak{X}(\underline{v}) \setminus \mathfrak{X}^{ord}$ is isomorphic to a different formal group scheme, but both become isomorphic to each other and to the constant group scheme $\mathcal{O}_F/p^n\mathcal{O}_F$ at the rigid fiber.

We move to the rigid fiber $\mathcal{C}_n \to \mathcal{X}(\underline{v})$ in order to define $\mathcal{I}_n = \mathrm{Isom}_{\mathcal{X}(\underline{v}),F}(\mathfrak{d}^{-1} \otimes \mu_{p^n}, \mathcal{C}_n)$, the group of $\mathcal{O}_F$-linear isomorphisms between $\mathfrak{d}^{-1} \otimes \mu_{p^n}$ and $\mathcal{C}_n$. The forgetful map $h_n \colon \mathcal{I}_n \to \mathcal{X}(\underline{v})$ is a finite étale cover with Galois group $\mathrm{Aut}_F(\mathfrak{d}^{-1} \otimes \mu_{p^n}) = \mathbb{T}(\mathbb{Z}/p^n\mathbb{Z})$. We then let $\mathfrak{I}_n$ be the formal model of $\mathcal{I}_n$ obtained by normalizing $\mathfrak{X}(\underline{v})$ inside $\mathcal{I}_n$, and let h_n also refer to the covering $\mathfrak{I}_n \to \mathfrak{X}(\underline{v})$. We call $\mathcal{I}_n$ and $\mathfrak{I}_n$ "partial Igusa towers".

Remark 4.4.1. Here we are really using the fact that the fibers of the rigid space $\mathcal{C}_{p^n}$ are all isomorphic, even though the fibers of the formal model $\mathfrak{C}_{p^n}$ are not. This partial Igusa tower $\mathfrak{I}_n$ is supposed to be the overconvergent version of the full Igusa tower $\mathfrak{T}^{\times}_{\mathfrak{C}_{p^\infty}}$ from Section 4.3, classifying trivializations of the canonical subgroup. Over the rigid fiber $\mathcal{X}(\underline{v})$, $\mathcal{I}_n$ literally does this. However, this becomes more complicated over the special point when the fibers of $\mathfrak{C}_{p^n}$ are no longer all isomorphic to each other.

The canonical subgroups help us pick out an important subsheaf $\mathcal{F} \subset \underline{\omega}$, which is described in [AIP16, Proposition 3.4].

Proposition 4.4.2. *There is a unique subsheaf $\mathcal{F}$ of $\underline{\omega}$ which is locally free of rank* 1 *as a $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{\mathfrak{I}_n}$-module and contains $p^{\frac{\sup_{\mathfrak{p}}\{v_{\mathfrak{p}}\}}{p-1}}\underline{\omega}$. Moreover, for all $0 < w < n - \sup_{\mathfrak{p}}\{v_{\mathfrak{p}}\}\frac{p^n}{p-1}$, we have $\mathcal{O}_F$-linear maps*

$$HT_w \colon \mathfrak{C}^D_{p^n} \to h_n^*(\mathcal{F})/p^w h_n^*(\mathcal{F}).$$

These induce isomorphisms

$$HT_w \otimes 1\colon \mathfrak{C}^D_{p^n} \otimes (\mathcal{O}_{\mathfrak{I}_n}/p^w\mathcal{O}_{\mathfrak{I}_n}) \to h_n^*(\mathcal{F})/p^w h_n^*(\mathcal{F}). \tag{4.7}$$

Sketch. We sketch the proof, referring to [AIP16, Proposition 3.4], and [AIP15, Proposition 4.3.1], for a full proof.

The maps HT_w and the sheaf $\mathcal{F}$ are constructed at the same time, and then we check that HT_w has the required properties of being $\mathcal{O}_F$-linear and inducing the isomorphism in Equation (4.7). Let $A(\underline{v}) = \mathcal{A} \times_{\mathfrak{X}} \mathfrak{X}(\underline{v})$ be the universal object over $\mathfrak{X}(\underline{v})$. Pick any basis $e_1^0, \ldots, e_d^0$ of the rigid group $\mathfrak{d}^{-1} \otimes \mu_{p^n}$, and for each point (A, u) of $\mathcal{I}_n$ we get a basis $e_1, \ldots, e_d$ for $\Omega^1_{\mathcal{C}_{p^n}}$ by $e_j = u^D(e_j^0) \in \mathcal{C}^D_{p^n} \cong \Omega^1_{\mathcal{C}^D_{p^n}}$. Recall that the inclusion $\mathfrak{C}_{p^n} \to A(\underline{v})$ induces a pullback $\Omega^1_{A(\underline{v})} \to \Omega^1_{\mathfrak{C}_{p^n}}$. Lift each e_j to an element $\tilde{e}_j \in \Omega^1_{A(\underline{v})}$ such that $\tilde{e}_j$ maps to e_j for each j. The sheaf $\mathcal{F}$ is the subsheaf of $\Omega^1_{A(\underline{v})}$ generated by $\tilde{e}_1, \ldots, \tilde{e}_d$. The map HT_w is determined by sending e_j to $\tilde{e}_j$ for each j. In *loc. cit.* the authors give the details of the construction. $\square$

Remark 4.4.3. The fact that $\underline{\omega} \subset \mathcal{F} \subset p^{\frac{\sup_{\mathfrak{p}}\{v_{\mathfrak{p}}\}}{p-1}} \underline{\omega}$ shows that $\mathcal{F} \otimes_{\mathcal{O}_K} K = \underline{\omega} \otimes_{\mathcal{O}_K} K$ as families of modules. Their rigid fibers give different integral structures to the rigid analytification of $\underline{\omega}$, and only $\mathcal{F}$ gives a locally free sheaf of $\mathcal{O}_F$-modules. Specifically, the rigid fiber of $\underline{\omega}$ and the rigid fiber of $\mathcal{F}$ should be thought of as two versions of "the ball of radius 1" inside the analytification of $\underline{\omega}$ pulled back to $\mathcal{I}_n$.

4.5 Overconvergent Hilbert Modular Forms

Fix a tuple $\underline{v}$ such that $v_{\mathfrak{p}} < \frac{1}{p^n}$ for all i. We have our canonical subgroup $\mathfrak{C}_{p^n} \to \mathfrak{X}(\underline{v})$ of level p^n and a partial Igusa tower $\mathfrak{I}_n$.

Write $\mathrm{ev}_1^D\colon \mathcal{I}_n \to \mathcal{C}_n^D$ for the map sending u to $u^D(1)$. This descends to a map of formal schemes $\mathrm{ev}_1^D\colon \mathfrak{I}_n \to \mathfrak{C}_n^D$ by the universal property of the relative normalization. Following [AIP16], define the formal affine morphism $\gamma_w\colon \mathfrak{T}^\times_{\mathcal{F},w} \to$

$\mathfrak{I}_n$, for any $n-1 < w \leq n - \sup\{v_{\mathfrak{p}}\}\frac{p^n}{p-1}$, as follows. For every normal p-adically complete and separated, flat $\mathcal{O}_K$-algebra R and for every morphism $\gamma\colon \mathrm{Spf}(R) \to \mathfrak{I}_n$, its R-valued points over γ classify frames $\alpha\colon \gamma^*(\mathcal{F}) \to \mathcal{O}_F \otimes_{\mathbb{Z}} R$, such that $HT_w(\mathrm{ev}_1^D(u)) \equiv \alpha^{-1}(1) \pmod{p^w}$. The reference cited above writes this as needing to send 1 to 1 in the composite

$$\mathcal{O}_F/p^n\mathcal{O}_F \xrightarrow{u^D} \mathfrak{C}_n^D(R) \xrightarrow{HT_w} \gamma^*\mathcal{F}/p^w\gamma^*(\mathcal{F}) \xrightarrow{\alpha} \mathcal{O}_F \otimes_{\mathbb{Z}} R/p^wR. \tag{4.8}$$

The map $\gamma_w\colon \mathfrak{T}^\times_{\mathcal{F},w} \to \mathfrak{I}_n$ is a formal torsor for the group $\mathbb{T}_w^0$. Composing with the projection $\mathfrak{I}_n \to \mathfrak{X}(\underline{v})$ gives a formal torsor $\mathrm{pr}_w\colon \mathfrak{T}^\times_{\mathcal{F},w} \to \mathfrak{X}(\underline{v})$ for the group $\mathbb{T}_w$. The rigid fiber is denoted $\mathcal{T}^\times_{\mathcal{F},w}$.

Remark 4.5.1. In [AIP16], this is called $\mathfrak{IW}_w^+$. In other references, such as [AIP15, Liu19a], there are two related formal torsors: one is called $\mathfrak{IW}_w^+$, while the other is $\mathfrak{T}^\times_{\mathcal{F},w}$. In the case of Hilbert modular forms, the two coincide – we use the second name here so that our discussion in Section 4.6 better parallels that of [Liu19a, Section 3.4].

For any representation (κ, W_κ) of $\mathbb{T}_w$, we may construct a corresponding sheaf $\underline{\omega}_{\kappa,w}$ on $\mathcal{X}(\underline{v})$. Over an affinoid open $U = \mathrm{Sp}\, R \subset \mathcal{X}(\underline{v})$ when $\underline{\omega}|_U$ is trivial, sections of $\underline{\omega}_{\kappa,w}$ correspond to functions

$$H^0(\mathcal{X}(\underline{v}), \underline{\omega}_{\kappa,w}) = \left\{ f\colon \mathcal{T}^\times_{\mathcal{F},w}(U) \to W_\kappa \otimes R \;\middle|\; f(A, \alpha g) = g^{-1} \cdot f(A,\alpha) \right\}.$$

Definition 4.5.2. Let κ be a w-analytic weight. A $\underline{v}$-overconvergent Hilbert modular form of level $\Gamma_1(N)$ and weight κ is a section of $\underline{\omega}_{\kappa,w}$ defined over $\mathcal{X}(\underline{v})$. The K-vector space of $\underline{v}$-overconvergent Hilbert modular forms of level $\Gamma_1(N)$ and weight κ is thus $H^0(\mathcal{X}(\underline{v}), \underline{\omega}_{\kappa,w})$.

An overconvergent Hilbert modular form is a $\underline{v}$-overconvergent Hilbert modular form for some tuple $\underline{v}$ with each $v_{\mathfrak{p}} > 0$. The K-vector space of overconvergent Hilbert modular forms is the union $\bigcup_{\underline{v}} H^0(\mathcal{X}(\underline{v}), \underline{\omega}_{\kappa,w})$.

Remark 4.5.3. In Remark 4.3.2, we saw how to view classical Hilbert modular forms as p-adic Hilbert modular forms. The idea is that, over the ordinary locus, there is a tight connection between the canonical subgroup of the universal ordinary HBAV $C_{p^\infty}(\mathcal{A}^{ord})$ and the Hodge bundle $\underline{\omega}$. Since overconvergent Hilbert modular forms are p-adic modular forms, we would like to use the canonical subgroup when describing them. However, the full canonical subgroup C_{p^∞} does not extend past the ordinary locus at all – if we want to use the canonical subgroup, we have to extend one of the finite level canonical subgroups C_{p^n}. The sheaf $\mathcal{F}$ fills in the extra information that we would otherwise lose by not having the full canonical subgroup of infinite level. The compatibility condition, requiring that 1 be sent to 1 in Equation (4.8), essentially means that the trivialization u of $\mathfrak{C}_{p^n}$ controls as much as possible about the trivialization α of $\mathcal{F}$.

4.6 Nearly Overconvergent Hilbert Modular Forms and the Main Construction

There is a natural homomorphism $Q \to \mathbb{T}$ that picks out the top left entry. Let Q_w^0 be the rigid analytic group which is the preimage of $\mathbb{T}_w^0$ under this projection, and Q_w the preimage of $\mathbb{T}_w$.

We have previously considered the $\mathcal{O}_K$-schemes $T_{\underline{\omega}}^\times = \mathrm{Isom}_X(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_X, \underline{\omega})$ and $T_{\mathcal{H}}^{\times,+} = \mathrm{Isom}_X^+([\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_X]^{\oplus 2}, \mathcal{H})$, where the $+$ superscript means that the isomorphisms should respect the filtrations. We write $\mathcal{T}_{\underline{\omega},an}^\times$ and $\mathcal{T}_{\mathcal{H},an}^{\times,+}$ for their rigid analytifications, and $\mathcal{T}_{\underline{\omega},an}^\times(\underline{v})$ and $\mathcal{T}_{\mathcal{H},an}^{\times,+}(\underline{v})$ for their base changes to $\mathcal{X}(\underline{v})$. We have a map $\mathcal{T}^{\times,+}_{\mathcal{H},an}(\underline{v}) \to \mathcal{T}^{\times}_{\underline{\omega},an}(\underline{v})$ given by forgetting everything but the isomorphism on the submodules $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{X}(\underline{v})}$ and $\underline{\omega}$.

Then let $\mathcal{T}^{\times}_{\mathcal{F},w} \to \mathcal{I}_w \to \mathcal{X}(\underline{v})$ be the rigid fiber of the chain of morphisms defined in Section 4.4. This $\mathcal{T}^{\times}_{\mathcal{F},w}$ maps to $\mathcal{T}^{\times}_{\underline{\omega},an}$ as well.[2] We thus form the fiber product

$$\mathcal{T}^{\times,+}_{\mathcal{H},w}(\underline{v}) \coloneqq \mathcal{T}^{\times,+}_{\mathcal{H},an}(\underline{v}) \times_{\mathcal{T}^{\times}_{\underline{\omega},an}(\underline{v})} \mathcal{T}^{\times}_{\mathcal{F},w}(\underline{v}).$$

The map $\pi^+_w \colon \mathcal{T}^{\times,+}_{\mathcal{H},w}(\underline{v}) \to \mathcal{X}(\underline{v})$ is a torsor for the group Q_w.

Let κ be a w-analytic weight. There is an associated rigid analytic representation $W_{\kappa,w}$ of $\mathbb{T}_w$, which we inflate to Q_w. In similar fashion to the holomorphic case, we define the analytic $(\mathfrak{g}, Q_w)$-module $V_{\kappa,w}$ by

$$V_{\kappa,w} = W_{\kappa,w} \otimes \mathcal{O}_{H\backslash Q}$$

We give $\mathfrak{g}(K)$ the same basis as before; since Q_w is open in Q, its Lie algebra is still $\mathfrak{q} \subset \mathfrak{g}$. We define the actions of $\mathfrak{g}(K)$ and $Q_w(K)$ by the same formulas,

$$(g \cdot P)(\underline{Y}) = a \cdot P(a^{-1}(b + \underline{Y}d)) \quad \text{for all } g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in Q_w(R),$$

$$(\mu^-_\sigma \cdot P)(\underline{Y}) = Y_\sigma \varepsilon_\sigma \cdot P(\underline{Y}) - Y_\sigma^2 \frac{\partial}{\partial Y_\sigma} P(\underline{Y}).$$

Compatibility is checked in the same way. Since the action of Q_w preserves the degree of each Y_σ, there is a Q_w-submodule $V^{\nu}_{\kappa,w} \subset V_{\kappa,w}$ consisting of polynomials with degree at most r_σ in the variable Y_σ for each σ. Since $\mathfrak{g}_\sigma$

[2] It does *not* map to the rigid fiber of the formal completion of $T^{\times}_{\underline{\omega}}$, as the points of either space should be $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_K$-bases for the respective sheaves, which do not match up. The points of $\mathcal{T}^{\times}_{\underline{\omega},an}$ are instead $\mathcal{O}_F \otimes K$-bases for $\underline{\omega}$.

preserves the degree as a function of Y_τ for each $\tau \neq \sigma$, and raises the degree as a function of Y_σ by at most 1, we have $\mathfrak{g}_\sigma V^\nu_{\kappa,w} \subset V^{\nu+\sigma}_{\kappa,w}$.

Definition 4.6.1. The sheaf of nearly $\underline{v}$-overconvergent Hilbert modular forms of level $\Gamma_1(N)$, w-analytic weight κ, and type ν is $\mathcal{V}^\nu_{\kappa,w}(\underline{v}) = \mathcal{T}^{\times,+}_{\mathcal{H},w}(\underline{v}) \times_{Q_w} V^\nu_{\kappa,w}$. A nearly overconvergent Hilbert modular form of level $\Gamma_1(N)$, w-analytic weight κ, and type ν is a section of $\mathcal{V}^\nu_{\kappa,w}(\underline{v})$ for some $\underline{v}$ with each $v_\mathfrak{p}$ satisfying $0 < v_\mathfrak{p} < \frac{1}{p^w}$.

Proposition 3.4.4 gives us a connection $\nabla_{\kappa,w} \colon \mathcal{V}_{\kappa,w} \to \mathcal{V}_{\kappa,w} \otimes \Omega^1_{\mathcal{X}(\underline{v})}(\log C)$, and the Kodaira–Spencer morphism gives us our differential operators,

$$\nabla_{\sigma,w} \colon \mathcal{V}_{\kappa,w} \to \mathcal{V}_{\kappa+2\sigma,w}$$

The discussion of Remark 3.5.4 applies, giving

$$\nabla_{\sigma,w} \colon \mathcal{V}^\nu_{\kappa,w} \to \mathcal{V}^{\nu+\sigma}_{\kappa+2\sigma,w}.$$

Each $\nabla_{\sigma,w}$ commutes with $\nabla_{\tau,w}$ for any pair of embeddings $\sigma, \tau \in I$. Thus we may unambiguously write $\nabla_{\kappa',w}$ for the differential operator that raises weights by $\kappa' = \sum_\sigma 2k'_\sigma \sigma$, and types by $\nu' = (k'_\sigma)_\sigma$. We can also base change to $\mathcal{X}(\underline{v}')$ for any $\underline{v}' = (v'_\mathfrak{p})$ with $0 < v'_\mathfrak{p} < v_\mathfrak{p}$ for each $\mathfrak{p} \mid p$.

This construction works just as well for a single weight κ as it does for the universal w-analytic weight κ^{un}_w, and pulling back by the inclusion of an affinoid open $\mathcal{U} \subset \mathcal{W}_w$ allows us to apply these differential operators to families parametrized by $\mathcal{U}$.

We summarize these results in the following theorem.

Theorem 4.6.2. *Fix a tuple $\underline{v} = (v_\mathfrak{p})$ with $0 < v_\mathfrak{p} < \frac{1}{p^w}$ for all i. For each embedding $\sigma \colon F \to K$, and any $k \geq 1$, there is a differential operator ∇^k_σ acting on families of nearly $\underline{v}$-overconvergent Hilbert modular forms of w-analytic* *weight, which raises the weight by $2k\sigma$ and the type by k. The operators ∇_σ^k and ∇_τ^ℓ commute for any pair of embeddings σ and τ.*

4.7 Integrality

As constructed, this differential operator is defined over K. However, when p is unramified in F, it is defined over $\mathcal{O}_K$. Each module V_κ^ν is defined over $\mathbb{Z}$, but our description and the basis we chose are only defined over K a priori. It is true that the connection ∇ is defined over $\mathcal{O}_K$ in general; however, there is a problem to defining ∇_σ when p is ramified in F.

Over K, we can decompose $\mathfrak{g} = \bigoplus_\sigma \mathfrak{g}_\sigma$. Any trivialization $\alpha \in T_{\mathcal{H}}^{\times,+}$ over an affine open $U = \operatorname{Spec} R$ defined over K determines a trivialization of $\underline{\omega}$, and thus of

$$\bigoplus_\sigma \underline{\omega}_\sigma^{\otimes 2} \cong \Omega_X^1(\log C).$$

We let $D \in T_X(U)$ be the direction dual to that trivialization, and $X(D, \alpha) \in \mathfrak{g}(R)$ such that $\nabla(D)(v)(\alpha) = (X(D,\alpha)\cdot v)(\alpha)$. In fact, we get a basis $D_\sigma^\vee$ for each $\underline{\omega}_\sigma^{\otimes 2}$; we embed that sheaf as a summand of $\Omega_X^1(\log C)$ and view the set $\{D_\sigma^\vee\}$ as a basis of the latter sheaf. Let $D_\sigma \in T_X(U)$ be dual to $D_\sigma^\vee$. We have $X(D_\sigma, \alpha) \in \mathfrak{g}_\sigma(R)$, and $\sum_\sigma D_\sigma = D$ giving $\sum_\sigma X(D_\sigma, \alpha) = X(D, \alpha)$. Specifically, $X(D,\alpha) \in \mathfrak{g}(R)$ corresponds to the tuple $(X(D_\sigma, \alpha))_\sigma \in \bigoplus_\sigma \mathfrak{g}_\sigma(R)$. Under this formalism, we have

$$\nabla_\sigma(v)(\alpha) = d(v)(D_\sigma)(\alpha) + (X(D_\sigma,\alpha)\cdot v)(\alpha).$$

With this in mind, we turn to integrality. When p is unramified in F, the decomposition $\mathfrak{g} = \bigoplus_\sigma \mathfrak{g}_\sigma$ manifests over $\mathcal{O}_K$, and ∇_σ acts integrally. However, when p ramifies, there is no guarantee that $X(D_\sigma, \alpha) \in \mathfrak{g}(\mathcal{O}_K)$ under the assumption that $X(D,\alpha) \in \mathfrak{g}(\mathcal{O}_K)$. However, we can quantify by how much this fails, and we do so using the following lemma, formulated and proved by the author for the discussion in this section.

Lemma 4.7.1. *Let $n = (n_\sigma)_\sigma \in \prod_\sigma \mathcal{O}_K$. There exists some ℓ not depending on n such that $p^\ell n \in \mathcal{O}_F \otimes \mathcal{O}_K \subset \prod_\sigma \mathcal{O}_K$.*

Proof. Pick a $\mathbb{Z}$-basis $\{e_1, \ldots, e_d\}$ for $\mathcal{O}_F$, and order the embeddings of F into K, $\{\sigma_1, \ldots, \sigma_d\}$. The $\mathcal{O}_K$-linear map $\mathcal{O}_F \otimes \mathcal{O}_K \to \prod_\sigma \mathcal{O}_K$ is given in this basis by the matrix

$$M_{\underline{\sigma}} = \begin{pmatrix} \sigma_1(e_1) & \ldots & \sigma_d(e_1) \\ \vdots & \ddots & \vdots \\ \sigma_1(e_d) & \ldots & \sigma_d(e_d) \end{pmatrix}$$

It is classical that $\det M_{\underline{\sigma}}^2 = \Delta$ is the discriminant of $F/\mathbb{Q}$. Further, the index of the image of this $\mathcal{O}_K$-linear map is $\#\mathcal{O}_K/(\det M_{\underline{\sigma}})$. Let $\mathfrak{p}$ be the maximal ideal in $\mathcal{O}_K$ and write $(\det M_\sigma) = \mathfrak{p}^\ell$ for some ℓ. We have $\#\mathcal{O}_K/\mathfrak{p}^\ell = p^{f\ell}$, where f is the residue field degree. Thus $p^{f\ell}(n_\sigma)_\sigma \in \mathcal{O}_F \otimes \mathcal{O}_K$ whenever $(n_\sigma)_\sigma \in \prod_\sigma \mathcal{O}_K$. This exponent does not depend on the choice of n, so the lemma is proven. □

Remark 4.7.2. In fact, we can say more. Since $\left(\prod_\sigma \mathcal{O}_K\right)/(\mathcal{O}_F \otimes \mathcal{O}_K)$ is a $\mathcal{O}_K$-module of order $p^{f\ell}$, it is killed by p^ℓ - the worst case scenario is that the quotient is indecomposable, and thus isomorphic to the indecomposable module $\mathcal{O}_K/\mathfrak{p}^\ell$. This module has size $p^{f\ell}$, but is killed by p^ℓ.

We use this to prove more about the Lie algebra $\mathfrak{g}$.

Corollary 4.7.3. *Let $X = (X_\sigma)_\sigma \in \mathfrak{g}(\mathcal{O}_K) \subset \prod_\sigma \mathfrak{gl}_2(\mathcal{O}_K) \subset \mathfrak{g}(K)$. View X_σ as the tuple $(0, \ldots, X_\sigma, \ldots, 0)$ in $\prod_\sigma \mathfrak{gl}_2(\mathcal{O}_K) \subset \mathfrak{g}(K)$. Then there exists some ℓ independent of X such that $p^\ell X_\sigma \in \mathfrak{g}(\mathcal{O}_K)$.*

Proof. Pick ℓ as in Lemma 4.7.1. Note that the elements of $\mathfrak{g}(\mathcal{O}_K)$ should be viewed as 2×2 matrices with entries in $\mathcal{O}_F \otimes \mathcal{O}_K$, while X_σ is a priori a tuple of 2×2 matrices with entries in $\mathcal{O}_K$. We may instead view X_σ as a 2×2 matrix with entries in $\prod_\sigma \mathcal{O}_K$. Scaling X_σ by p^ℓ simply scales its entries, so by our choice of ℓ the entries of $p^\ell X_\sigma$ are in $\mathcal{O}_F \otimes \mathcal{O}_K$, and $p^\ell X_\sigma \in \mathfrak{g}(\mathcal{O}_K)$. $\square$

By the corollary, we have $p^\ell X(D_\sigma, \alpha) \in \mathfrak{g}(\mathcal{O}_K)$. Thus $p^\ell X(D_\sigma, \alpha)$ acts integrally.

$$X(D_\sigma, \alpha) \cdot (V_\kappa^\nu(\mathcal{O}_K)) \subset \frac{1}{p^\ell} V_\kappa^{\nu+\sigma}(\mathcal{O}_K) \subset V_\kappa^{\nu+\sigma}(K).$$

On the level of sheaves, the above translates to

$$\nabla_\sigma(\mathcal{V}_\kappa^\nu(\mathcal{O}_K)) \subset \frac{1}{p^\ell} \mathcal{V}_{\kappa+2\sigma}^{\nu+\sigma}(\mathcal{O}_K) \subset \mathcal{V}_{\kappa+2\sigma}^{\nu+\sigma}(K).$$

This allows us to control denominators.

CHAPTER V

DESCENT TO G

5.1 Two Groups

The phrase "Hilbert modular form" is ambiguous in that it can refer to the space of automorphic forms on one of two groups. The group $G = \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}} \mathrm{GL}_2$ is one. For any commutative ring R, its R-points are

$$G(R) = \mathrm{GL}_2(\mathcal{O}_F \otimes_{\mathbb{Z}} R).$$

Determinants of such matrices live in $\mathbb{T} = \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}} \mathbb{G}_m$. Recall that over K, $\mathbb{T}$ is a split torus; its diagonal subgroup is isomorphic to $\mathbb{G}_m$. The second group, denoted G^*, is defined as the fiber product $G^* = G \times_{\mathbb{T}} \mathbb{G}_m$. It is a subgroup of G, with R-points

$$G^*(R) = \{g \in G(R) \mid \det(g) \in 1 \otimes R^\times \subset (\mathcal{O}_F \otimes R)^\times\}.$$

Each group has advantages. The Hecke theory for G is canonical, while the theory for G^* is not. In fact, unlike G, G^* has a noncommutative Hecke algebra. On the other hand, while both are the groups of interest in a PEL type moduli problem, only the moduli problem associated to G^* is representable by a scheme. This is the reason why the previous sections were only concerned with the automorphic forms on G^*.

Since G^* is a subgroup of G, we may restrict the automorphic forms on G to the group G^*, viewing them as automorphic forms on the latter group. This allows us to use the geometric viewpoint of the previous sections to study automorphic forms on G. In the next section, we describe the weights of such form, and following [AIP16] we record the geometric condition that distinguishes these restricted forms from other automorphic forms on G^*.

5.2 Weights

In Section 2.1, we introduced the weight space for overconvergent modular forms on G^*; here we continue with the weight space for G, including the weight of a restricted modular form.

The space of algebraic weights for G is $\{\text{characters of } \mathbb{T}_{/K}\} \times \mathbb{Z}$, with a map to the algebraic weights for G^* given by $\rho\colon (\theta, w) \mapsto 2\theta + w\,\mathrm{Nm}_{F/\mathbb{Q}}$. In particular, if (θ, w) is a weight for G, $\rho(\theta, w)(g) = \theta(g)^2 \cdot (\mathrm{Nm}_{F/\mathbb{Q}}(g))^w$ for any $g \in \mathbb{T}(\mathbb{Z}_p)$. If we have a modular form of weight (θ, w) on G, its restriction to G^* will have weight $\rho(\theta, w)$.

The space of p-adic weights for G is $\{\text{characters of } \mathbb{T}(\mathbb{Z}_p)\} \times \mathbb{Z}_p$. The map ρ sending a p-adic weight for G to a p-adic weight for G^* uses the same formula as above.

Let f be an automorphic form on G^* of weight $\theta^2 \cdot \mathrm{Nm}_{F/\mathbb{Q}}^w$. It can be extended to an automorphic form on G if, for all $\epsilon \in \mathcal{O}_F^{\times,+}$, we have

$$f(A, \iota, \psi, \epsilon\lambda, \omega) = \theta(\epsilon) f(A, \iota, \psi, \lambda, \omega). \tag{5.1}$$

We will avoid using this criterion. It is included to stress the fact that whether or not an automorphic form on G^* extends to G can be detected geometrically. We will opt instead to use a criterion involving the symmetric space. This will be enough, as Proposition 5.4.1 can be stated in terms of classical Hilbert modular forms and easily transported to the space of overconvergent ones.

5.3 Descent of Modular Forms

We can quantify the difference between automorphic forms on G^* and automorphic forms on G using the adèlic viewpoint. Let K_∞^+ be a maximal compact subgroup of the connected component of the identity $G(\mathbb{R})^+$ in $G(\mathbb{R})$, and K_∞^* the same for $G^*(\mathbb{R})$. Let $K_0(N)$ and $K_0^*(N)$ be the natural choices of compact open subsets of the finite adèles, consisting of matrices which are upper triangular modulo N. The inclusion $G^* \to G$ induces a map on the symmetric spaces $G^*(\mathbb{R})^+/K^*_\infty Z(G^*(\mathbb{R})) \to G(\mathbb{R})^+/K^+_\infty Z(G(\mathbb{R}))$ which ends up being a bijection. Thus automorphic forms on G and automorphic forms on G^* are functions on the same space. The only difference comes from the congruence subgroups

$$\begin{aligned}\Gamma_0^*(N) = G^*(\mathbb{Q}) \cap K_0^*(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_F) \mid c \equiv 0 \pmod{N} \right\}, \\ \Gamma_0^G(N) = G(\mathbb{Q}) \cap K_0(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathcal{O}_F) \mid c \equiv 0 \pmod{N} \right\}.\end{aligned} \tag{5.2}$$

For any $\epsilon \in \mathcal{O}_F^{\times,+}$, let $g_\epsilon = \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}$. Equation (5.1) is simply a translation of the fact that $g_\epsilon \in \Gamma_0^G(N)$, while $g_\epsilon \notin \Gamma_0^*(N)$. In a strict sense, this is the only discrepancy, as $\Gamma_0^*(N)$ and these matrices g_ϵ together generate $\Gamma_0^G(N)$.

We define an action of $\mathcal{O}_F^{\times,+}$ on the space of automorphic forms on G^* of weight $\rho(\nu, w)$ to be

$$\epsilon \cdot f = f|g_\epsilon. \tag{5.3}$$

The discussion above implies that f is a modular form for G if and only if $\mathcal{O}_F^\times$ acts on f via its nebentypus.

5.4 Descent of Operators

At the moment, the differential operators ∇_σ are defined as maps that send Hilbert modular forms on G^* to Hilbert modular forms on G^*. A priori, if f is a nearly overconvergent Hilbert modular form for G, $\nabla_\sigma f$ is only a nearly overconvergent Hilbert modular form for G^*. We will argue that it extends to G.

Proposition 5.4.1. *The action of $\mathcal{O}_F^{\times,+}$ defined in Equation (5.3) commutes with the Gauss–Manin connection $\nabla \colon \mathcal{H} \to \mathcal{H} \otimes \Omega_X^1$.*

Proof. This is classical, and seen e.g. as a special case of [Shi00, Proposition 12.10(2)].

For another proof, we note that the slash operator is defined in such a way that it commutes with the exterior derivative d defined for differential forms. Since ∇ arises as a differential in the spectral sequence associated to a filtered de Rham complex whose differentials are this d, it commutes with the slash operator as well. □

This leads to our second Main Theorem.

Theorem 5.4.2. *The differential operators ∇_σ constructed in Theorem 4.6.2 preserve the space of Hilbert modular forms for G inside the space of Hilbert modular forms for G^*.*

Proof. This is essentially a special case of Proposition 3.4.4. The fiber of $T_{\mathcal{H}}^{\times,+}$ over some point x corresponding to a HBAV $(A, \iota, \psi, \lambda)$ only depends on the substring (A, ι), so we may identify the fibers over any x with the fibers over $g_\epsilon \cdot x$. Under this identification, the slash operator is just the natural action of $Q \ni g_\epsilon$ on $\mathcal{V}_\kappa^\nu$, and we proved that this action commutes with $\nabla = \bigoplus_\sigma \nabla_\sigma$ in the previously cited proposition. Thus the fact descends to each ∇_σ. □

APPENDIX

DAMERELL'S FORMULA

This text is adapted from a note published on the author's website. It is independent from the rest of the dissertation, but is included to show the arithmetic application that motivated the construction of the differential operators in the main body of the text.

A.1 Introduction

In [Ser72], Serre gave a construction of the Kubota–Leopoldt p-adic zeta function using the theory of p-adic of modular forms and the fact that the values of holomorphic Eisentein series at the cusps are related to the values of the zeta function. This construction was generalized in [DR80] to construct the p-adic zeta functions of totally real fields.

These two "interpolation" results were preceded by "algebraicity" results: up to a renormalization, the values of these zeta functions at certain inputs are in fact algebraic numbers, so that it makes sense to ask about congruences between them. For quadratic imaginary fields, this study was initiated by Damerell, and completed using the Maass–Shimura operators and the theory of nearly holomorphic modular forms. From here, p-adic interpolation of the zeta functions of quadratic imaginary fields, and then for CM fields,[1] were given by Katz in [Kat76] and [Kat78] respectively, though with conditions on p.

Here, we make explicit the relationship between values of Eisenstein series at CM points and values of zeta functions for CM fields. We begin by describing the L-functions to be interpolated in Section 2, introducing Hilbert modular forms

[1] Recall that a CM field is a totally imaginary field K which is a degree 2 extension of a totally real field F.

in Section 3. Once we have both tools, we relate them in Section 4. Section 5 gives some reasons one might care about the result.

An excellent technical reference on the theory of nearly holomorphic automorphic forms is [Shi00], which summarizes and builds on earlier results of Shimura and Maass. Katz's construction of p-adic L-functions has been extended in the case of quadratic imaginary fields by Andreatta and Iovita in [AI19]; the author of the present note is working to extend this to the case of CM fields.

A.1.1 Notation, Conventions, and Assumed Knowledge.

We assume the reader has taken a first course in algebraic number theory (e.g. out of [Mil20]) and has a basic understanding of complex analysis.

A.2 *L*-functions

For this section, we fix a totally real field F of degree d, and a CM extension $K = F(\alpha)$. For simplicity, we assume that the ring of integers is $\mathcal{O}_K = \mathcal{O}_F + \alpha\mathcal{O}_F$.[2] We also fix a CM type of K; for each real embedding $\sigma\colon F \to \mathbb{R}$, we choose a preferred embedding of K into $\mathbb{C}$ which agress with σ when restricted to F, which we also call σ. Thus the set of complex embeddings of K is the set of σ's and all $\overline{\sigma}$'s as σ runs over the real embeddings of F. Write I for the set of real embeddings of F.

A.2.1 The 1-variable *L*-function.

The standard, 1-variable Dedekind zeta function of K can be written, for $\mathrm{Re}(s) > 1$, as the sum over all nonzero ideals of $\mathcal{O}_K$.

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{\mathrm{Nm}(\mathfrak{a})^s}. \tag{A.1}$$

[2] If F is a PID, this is possible because K/F is quadratic. If not, this decomposition should be $\mathcal{O}_K = \mathcal{O}_F + \alpha\mathfrak{a}$ for a fractional ideal $\mathfrak{a}$ of F.

We may rewrite this in order to make most of the summation happen with elements of the field itself. Pick representatives $\mathfrak{a}_1, \dots, \mathfrak{a}_h$ for the class group of $\mathcal{O}_K$, and note that every integral ideal $\mathfrak{a}$ can be written as $\mathfrak{a} = \mathfrak{a}_i^{-1}(\alpha)$ for some i and some $\alpha \in \mathfrak{a}_i$.

Writing $\mathfrak{a}$ as $\mathfrak{a}_i^{-1}(\alpha)$ for some i and some $\alpha \in \mathfrak{a}_i$ is unique up to rescaling α by a unit in $\mathcal{O}_K^\times$. In the quadratic imaginary case, we can sum over all elements of $\mathfrak{a}_i$, and divide by $\#\mathcal{O}_K^\times$ to account for repetition. When K is not quadratic imaginary, its unit group is infinite, and we can no longer do this. We should instead view $\mathfrak{a}_i$ as a $\mathcal{O}_K^\times$-set, where $u \cdot \alpha = u\alpha$ for any unit u and $\alpha \in \mathfrak{a}_i$. This action is useful because each orbit $\alpha\mathcal{O}_K^\times \in \mathfrak{a}_i/\mathcal{O}_K^\times$ corresponds to a unique integral ideal $\mathfrak{a}_i^{-1}(\alpha)$, and each integral ideal $\mathfrak{a}_i^{-1}(\alpha)$ corresponds to a unique orbit $\alpha\mathcal{O}_K^\times$.

In fact, for reasons that will arise later, we will look at orbits for the restricted action of $\mathcal{O}_F^{\times,+} \subset \mathcal{O}_K^\times$ consisting of *totally positive units* u with $\sigma(u) > 0$ for every real embedding σ of F. The association $\alpha\mathcal{O}_F^{\times,+} \mapsto \mathfrak{a}_i^{-1}(\alpha)$ is then $[\mathcal{O}_K^\times : \mathcal{O}_F^{\times,+}]$-to-one, where $[\mathcal{O}_K^\times : \mathcal{O}_F^{\times,+}]$ is the index of the subgroup $\mathcal{O}_F^{\times,+}$ in $\mathcal{O}_K^\times$.[3] Thus we write the following, where the innermost sum is over cosets $0 \neq \alpha\mathcal{O}_F^{\times,+} \in \mathfrak{a}_i/\mathcal{O}_F^{\times,+}$.

$$\zeta_K(s) = \frac{1}{[\mathcal{O}_K^\times : \mathcal{O}_F^{\times,+}]} \sum_{i=1}^{h} \sum_{\alpha\mathcal{O}_F^{\times,+}} \frac{\mathrm{Nm}(\mathfrak{a}_i)^s}{\mathrm{Nm}(\alpha)^s}. \tag{A.2}$$

A.2.2 Hecke Characters. We will give a definition in terms of Hecke characters in order to write down the $(d+1)$-variable zeta function in all cases. However, in order to avoid delving into the idèlic theory, we give the following ideal-theoretic definition of a Hecke character, instead of the standard one.

[3]This index is finite. In particular, $[\mathcal{O}_F^\times : \mathcal{O}_F^{\times,+}] \leq 2^d$, so that $\mathcal{O}_F^{\times,+}$ is a finitely generated Abelian group of the same rank as $\mathcal{O}_F^\times$. Since $\mathcal{O}_K^\times$ also has the same rank as $\mathcal{O}_F^\times$, so $\mathcal{O}_F^{\times,+} \subset \mathcal{O}_K^\times$ is an inclusion of finitely generated Abelian groups of the same rank. The quotient is finitely generated of rank 0, hence finite; its size is the index we're looking for.

Definition A.2.1. An *unramified Hecke character* is a homomorphism $\chi\colon \mathcal{I}_K \to \mathbb{C}^\times$ from the group of fractional ideals of K to the complex numbers. An unramified Hecke character χ is *unitary* if $|\chi(\mathfrak{a})| = 1$ for all ideals $\mathfrak{a} \in \mathcal{I}_K$. We say that χ has *infinity type* $(k_{1,\sigma}, k_{2,\sigma})_\sigma$ if χ can be written as a product of characters $\chi = \chi_u\chi_\infty$, where χ_u is unitary and $\chi_\infty((\alpha)) = \prod_\sigma \sigma(\alpha)^{k_{1,\sigma}}\overline{\sigma}(\alpha)^{k_{2,\sigma}}$ for all principal ideals (α).

Remark A.2.2. Unramified, unitary Hecke characters χ have the property that $\chi((\alpha)) = 1$ for all principal ideals (α).

We note that ramified Hecke characters exist. For simplicity, we only work with unramified Hecke characters, but some of what we say will be true for all Hecke characters. We distinguish these statements and constructions by writing "(unramified) Hecke character" when the unramified hypothesis is not needed.

In particular, we note that the map $\mathfrak{a} \mapsto \mathrm{Nm}(\mathfrak{a})^s$ is an unramified Hecke character of infinity type $(s,s)_\sigma$. For any (unramified) Hecke character χ, write:

$$L(\chi, s) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{\mathrm{Nm}(\mathfrak{a})^s}, \qquad L(\chi^{-1}, 0) = \sum_{\mathfrak{a}} \frac{1}{\chi(\mathfrak{a})}.$$

When $\chi = \mathrm{Nm}^s$, we see that $L(\chi^{-1}, 0) = \zeta_K(s)$. More generally, $L(\chi\,\mathrm{Nm}^k, s) = L(\chi, s-k)$ for any (unramified) Hecke character χ and any two complex numbers k and s.

A.2.3 The $(d+1)$-variable L-function.

We build the $(d+1)$-variable L-function following Equation (A.2). Again, the innermost sum is over cosets $0 \neq \alpha\mathcal{O}_F^\times \in \mathfrak{a}_i/\mathcal{O}_F^\times$.

$$L(\chi, (s_\sigma, t_\sigma)_\sigma) = \frac{1}{[\mathcal{O}_K^\times : \mathcal{O}_F^\times]} \sum_{i=1}^{h} \sum_{\alpha\mathcal{O}_F^\times} \frac{\chi(\mathfrak{a}_i)}{\chi(\alpha)\prod_\sigma \sigma(\alpha)^{s_\sigma}\overline{\sigma}(\alpha)^{t_\sigma}}. \tag{A.3}$$

Note that $L(\chi, s) = L(\chi, (s, s)_\sigma)$, and changing the CM type at σ (i.e., replacing the preferred choice of complex embedding σ by $\overline{\sigma}$) interchanges the variables s_σ and t_σ.

Remark A.2.3. For this to be well-defined, we need the sum $s_\sigma + t_\sigma$ to be independent of σ, since α is only defined up to a totally positive unit in F. Write S_0 for this common value. If we replace α by $u\alpha$ for some $u \in \mathcal{O}_F^{\times,+}$, $\chi(u\alpha) = \chi(u)\chi(\alpha)$ does not change; since χ is unramified and unitary, we have $\chi(u) = 1$. However, we must also check that

$$\prod_\sigma \sigma(u\alpha)^{s_\sigma}\overline{\sigma}(u\alpha)^{t_\sigma} = \prod_\sigma \sigma(\alpha)^{s_\sigma}\overline{\sigma}(\alpha)^{t_\sigma}. \tag{A.4}$$

Since $\sigma(u) = \overline{\sigma}(u)$, the left hand side differs from the right hand side by a factor of

$$\prod_\sigma \sigma(u)^{s_\sigma+t_\sigma} = \prod_\sigma \sigma(u)^{S_0} = \left(\prod_\sigma \sigma(u)\right)^{S_0} = \mathrm{Nm}(u)^{S_0} = 1.$$

Thus we have verified the equality in Equation (A.4), and so we see that the function defined in Equation (A.3) is independent of the choices of the representatives $\alpha \in \alpha\mathcal{O}_F^{\times,+}$.

This is why we describe it as a $(d+1)$-variable L-function when it looks like there are $2d$ variables - once the common sum $S_0 = s_\sigma + t_\sigma$ is chosen, the choice of s_σ forces the choice of t_σ, and vice-versa. Thus we are left with $d+1$ variables; one way to choose these variables is to vary S_0 and the s_σ's, though there is no preferred way to choose.

A.3 Hilbert Modular Forms

For now, we focus on the totally real field F of degree $[F : \mathbb{Q}] = d$. There are many ways of viewing Hilbert modular forms over F. In this section we describe three.

(a) Hilbert modular forms as functions on the space of lattices,

(b) Hilbert modular forms as holomorphic functions on a symmetric space, and

(c) Hilbert modular forms as sections of a line bundle on a moduli space of Abelian varieties.

The first description will be important for relating values of certain Hilbert modular forms to the values of the $(d+1)$-variable L-function. The second is where the Maass–Shimura operators will be described. The third will be important for applications to algebraicity.

A.3.1 Setup. Fix a set of representatives $\mathfrak{a}_1, \ldots, \mathfrak{a}_h$ for the class group of F. Note that this also serves as a complete set of isomorphism classes of locally free $\mathcal{O}_F$-modules of rank 1.

Each fractional ideal $\mathfrak{a}_i$ lives naturally in F. Since $\mathbb{R}$ is a flat Abelian group, we may also view $\mathfrak{a}_i$ as a subset of $F \otimes_{\mathbb{Z}} \mathbb{R} \cong \prod_\sigma \mathbb{R}$, where the isomorphism sends a pure tensor $n \otimes t \in F \otimes \mathbb{R}$ to the tuple $(\sigma(n)t)_\sigma \in \prod_\sigma \mathbb{R}$. Each $\mathfrak{a}_i \subset F \otimes \mathbb{R}$ is a discrete subgroup; we assign a volume to the quotient $(F \otimes \mathbb{R})/\mathfrak{a}_i$ by choosing a fundamental domain D[4], and defining the volume of the quotient to be the volume of this subset of $\mathbb{F} \otimes_{\mathbb{Z}} \mathbb{R}$. We also refer to this as the *covolume* of the lattice $\mathfrak{a}_i$.

We scale the representatives $\mathfrak{a}_i$ for the class group. Let V_i denote the covolume of $\mathfrak{a}_i$ in $F \otimes \mathbb{R}$, and Δ the discriminant of F. Replace $\mathfrak{a}_i$ by the lattice $\frac{\sqrt[2d]{|\Delta|}}{\sqrt[d]{V_i}}\mathfrak{a}_i \subset F \otimes \mathbb{R}$; this need not be a fractional ideal of F, but it is a lattice in $F \otimes \mathbb{R}$ with covolume $\sqrt{|\Delta|}$. In particular, the choice of $\mathfrak{a}_i$ is unique, and the representative of the class consisting of principal ideals is $\mathcal{O}_F$.

A.3.2 Lattices. We describe Hilbert modular forms as functions on a space of lattices. First, we should define what a lattice is.

[4] D is a connected subset of $F \otimes \mathbb{R}$ such that, for all $x \in F \otimes \mathbb{R}$, exactly one element $d \in x + \mathfrak{a}_i$ is in D.

Definition A.3.1. Consider the vector space $F \otimes_{\mathbb{Z}} \mathbb{C}$. It has the structure of a $\mathcal{O}_F$-module by acting on the first component, and a real vector space by acting on the second component. A *lattice*, or more precisely an $\mathcal{O}_F$-lattice in $F \otimes \mathbb{C}$, is a discrete $\mathcal{O}_F$-submodule L of $F \otimes \mathbb{C}$ which is locally free of rank 2 as an $\mathcal{O}_F$-module, and which spans $F \otimes \mathbb{C}$ over $\mathbb{R}$.

Since a lattice L is locally free of rank 2 as an $\mathcal{O}_F$-module, it is isomorphic to $\mathfrak{a}_{i_1} \oplus \mathfrak{a}_{i_2}$ for two indices i_1 and i_2, where $\mathfrak{a}_i$ refers to one of the representatives of the class group of F chosen above. Picking an isomorphism, we may write $L = \omega_1\mathfrak{a}_{i_1} \oplus \omega_2\mathfrak{a}_{i_2}$ for some $\omega_1, \omega_2 \in (F \otimes \mathbb{C})^\times$.

We say two lattices L and L' are *homothetic* if there is some $\lambda \in (F \otimes \mathbb{C})^\times$ for which $\lambda L = \{\lambda\ell \in F \otimes \mathbb{C} \mid \ell \in L\} = L'$. By scaling $L = \omega_1\mathfrak{a}_{i_1} + \omega_2\mathfrak{a}_{i_2}$, we see that any lattice is homothetic to a lattice of the form $\mathfrak{a}_{i_1} + \tau\mathfrak{a}_{i_2}$ for some $\tau \in (F \otimes \mathbb{C})^\times$; e.g., for $\tau = \frac{\omega_2}{\omega_1}$.

We now define Hilbert modular forms in terms of functions on lattices.

Definition A.3.2. Let $\mathcal{L}$ be the set of $\mathcal{O}_F$-lattices in $F \otimes \mathbb{C}$, and let $\underline{k} = (k_\sigma)_\sigma$ be a tuple of integers indexed by the real embeddings of F. A *Hilbert modular form* of weight $\underline{k}$ is a function $f\colon \mathcal{L} \to \mathbb{C}$ satisfying some analytic conditions (to be specified in the next section) and the homogeneity property

$$f(\lambda L) = \left(\prod_\sigma \sigma(\lambda)^{-k_\sigma}\right) f(L) \qquad \text{for all } \lambda \in (F \otimes \mathbb{C})^\times.$$

Here, if $\lambda = n \otimes z \in F \otimes \mathbb{C}$ is a pure tensor, we write $\sigma(\lambda) = \sigma(n)z$. It is nonzero for all σ if $\lambda \in (F \otimes \mathbb{C})^\times$.

Notice that f is determined by its values on lattices of the form $\mathfrak{a}_{i_1} \oplus \tau\mathfrak{a}_{i_2}$.

Remark A.3.3. In Section A.4.1, we will give our first example of a Hilbert modular form using this viewpoint. This will be the easiest way for us to relate its values to the values of the L-functions.

A.3.3 Holomorphic Functions.

Hilbert modular forms can also be viewed as holomorphic functions on a symmetric space. Let $\mathfrak{h} \subset \mathbb{C}$ be the upper half-plane consisting of complex numbers with positive imaginary part. Then define $\mathfrak{h}_F$ to be the product of d copies of $\mathfrak{h}$, indexed by the real embeddings of F. We write $\underline{z} = (z_\sigma)_\sigma$ for elements of $\mathfrak{h}_F$.

We have an action of the group $\mathrm{SL}_2 F$ on $\mathfrak{h}_F$ given by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z_\sigma)_\sigma = \left(\frac{\sigma(a)z_\sigma + \sigma(b)}{\sigma(c)z_\sigma + \sigma(d)}\right)_\sigma .$$

This action gives us the following definition.

Definition A.3.4. A *holomorphic Hilbert modular form* of weight $\underline{k} = (k_\sigma)_\sigma$ is a holomorphic function $f\colon \mathfrak{h}_F \to \mathbb{C}$ such that $|f(\underline{z})|$ is bounded as every $\mathrm{Im}(z_\sigma)$ goes to ∞ at once[5] and which satisfies the transformation property

$$f(\gamma \cdot \underline{z}) = \left(\prod_\sigma (\sigma(c)z_\sigma + \sigma(d))^{k_\sigma}\right) f(\underline{z}) \qquad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2 \mathcal{O}_F.$$

To connect this with the previous definition, we have to associate a lattice to each $\underline{z} \in \mathfrak{h}_F$. In fact, the space of lattices is disconnected, while $\mathfrak{h}_F$ is connected – a modular form in the sense of Definition A.3.2 is actually a collection of Hilbert modular forms in the sense of Definition A.3.4, one for each connected component. Thus we should actually associate multiple lattices to each element of $\mathfrak{h}_F$.

View $\underline{z} \in \mathfrak{h}_F$ as an element of $F \otimes \mathbb{C} \supset \mathfrak{h}_F$. To every triple $(\mathfrak{a}_{i_1}, \mathfrak{a}_{i_2}, \underline{z})$, we associate a lattice $\mathfrak{a}_{i_1} + \underline{z}\mathfrak{a}_{i_2} = \{a_1 + a_2\underline{z} \in F \otimes \mathbb{C} \mid a_j \in \mathfrak{a}_{i_j}\}$. This allows us to

[5] This condition is important for $F = \mathbb{Q}$ and automatic for $d > 1$.

translate between Definitions A.3.2 and A.3.4. Note that the same lattice may be written in different ways.

For a Hilbert modular form f_L viewed as a function on the space of lattices, we build a function $f_h(\mathfrak{a}_{i_1}, \mathfrak{a}_{i_2}, -)\colon \mathfrak{h}_F \to \mathbb{C}$ for each orered pair $(\mathfrak{a}_{i_1}, \mathfrak{a}_{i_2})$ by the formula $f_h(\mathfrak{a}_{i_1}, \mathfrak{a}_{i_2}, \underline{z}) = f_L(\mathfrak{a}_{i_1} + \underline{z}\mathfrak{a}_{i_2})$. One may check that this function satisfies the requisite transformation property. The analytic conditions mentioned in Definition A.3.2 correspond to the function $f_h(\mathfrak{a}_{i_1}, \mathfrak{a}_{i_2}, -)$ being holomorphic and bounded at the cusps.

Now, for each ordered pair $(\mathfrak{a}_{i_1}, \mathfrak{a}_{i_2})$, fix a holomorphic modular form $f_h(\mathfrak{a}_{i_1}, \mathfrak{a}_{i_2}, -)\colon \mathfrak{h}_F \to \mathbb{C}$. Every lattice $\omega_1\mathfrak{a}_{i_1} + \omega_2\mathfrak{a}_{i_2}$ is homothetic to $\mathfrak{a}_{i_1} + \underline{z}\mathfrak{a}_{i_2}$ for some $\underline{z} \in \mathfrak{h}_F$. Specifying that $f_L(\mathfrak{a}_{i_1} + \underline{z}\mathfrak{a}_{i_2}) = f_h(\mathfrak{a}_{i_1}, \mathfrak{a}_{i_2}, \underline{z})$, the homogeneity property then determines f_L on all lattices.

Remark A.3.5. Viewing Hilbert modular forms as functions on the space of $\mathcal{O}_F$-lattices in $F \otimes \mathbb{C}$ will give a tight connection to the relevant values of L-functions. However, one of the important pieces of the theory, the Maass–Shimura operator, is best understood in the realm of holomorphic functions. We give both viewpoints here, and a rough translation between them, in order to take advantage of both sides of the theory. The third viewpoint, in terms of algebraic geometry, is given for the purpose of one of our applications. The next section introduces this, and ties it back to the viewpoint involving lattices.

A.3.4 Geometry.

In order to motivate the following discussion, we give a preliminary definition of classical modular forms; they may be viewed as a special case of Hilbert modular forms over the totally real field $F = \mathbb{Q}$.

Definition A.3.6. Fix a base ring R. A *modular form* defined over R is an algebraic function f that assigns to every pair (E, ω) of an elliptic curve E defined over some R-algebra S and a basis ω for $\Omega^1_{E/S}$ as an S-module, an element of S. A modular form of weight k is such a function which satisfies the homogeneity property that for any $c \in S^\times$, $f(E, c\omega) = c^{-k} f(E, \omega)$.

Remark A.3.7. Since $\Omega^1_{E/S}$ is a free S-module of rank 1, any two bases ω_1, ω_2 for $\Omega^1_{E/S}$ are related by $\omega_1 = c\omega_2$ for a unique $c \in S^\times$. Thus the value of $f(E, \omega)$ for any ω is determined by the value of $f(E, \omega_0)$ for some fixed ω_0.

Elliptic curves are one-dimensional examples of Abelian varieties. In order to generalize the definition above, one may look at more general classes of Abelian varieties of higher dimension, or endowed with extra structure. For Hilbert modular forms, we consider Hilbert–Blumenthal Abelian varieties.

Definition A.3.8. Fix a ring R, and let F be a totally real field of dimension d. A *Hilbert–Blumnenthal Abelian variety*, or HBAV, is a tuple $(A, \iota, \psi, \lambda)$ consisting of

– an Abelian variety A of relative dimension d over R,

– a *real multiplication* $\iota\colon \mathcal{O}_F \to \mathrm{End}(A)$,

– a *level structure* ψ, and

– a *polarization* λ.

For our purposes, the level structure ψ is trivial, and we suppress it from the notation.

Remark A.3.9. In general, the moduli space of HBAVs is represented by a stack, rather than a scheme. However, with enough level structure, the moduli space is "rigidified" to be represented by a scheme. When we do not fix a polarization as part of the data, this never happens. For our purposes, we will have no level structure, which will mean that we have to deal with stacks; we will ignore the issues that come up.

The real multiplication ι gives $\Omega^1_{A/R}$ the structure of an $\mathcal{O}_F \otimes R$-module. For a dense open subspace of the moduli space of HBAVs, this is a free module of rank 1. In this case, we can form its frame bundle consisting of bases ω for $\Omega^1_{A/R}$ which generate it as an $\mathcal{O}_F \otimes R$-module. Any two such bases ω_1, ω_2 are related by $\omega_1 = c\omega_2$ for a unique $c \in (\mathcal{O}_F \otimes R)^\times$.

Definition A.3.10. Fix a base ring R. A *Hilbert modular form* defined over R is an algebraic function f that assigns to every tuple $(A, \iota, \lambda, \omega)$ of an HBAV (A, ι, λ) defined over some R-algebra S and a basis ω for $\Omega^1_{E/S}$ as an $\mathcal{O}_F \otimes S$-module, an element of S. A Hilbert modular form of weight $(k_\sigma)_\sigma$ is such a function which satisfies the homogeneity property that for any $c \in (\mathcal{O}_F \otimes S)^\times$, $f(A, \iota, \lambda, c\omega) = \left(\prod_\sigma \sigma(c)^{-k_\sigma}\right) f(A, \iota, \lambda, \omega)$. Here σ acts on pure tensors $n \otimes s \in (\mathcal{O}_F \otimes S)^\times$ by $\sigma(n \otimes s) = \sigma(n) \otimes s$.

The algebraic geometry involved gives a way to investigate algebraicity, and in fact the integrality, of the values of Hilbert modular forms. The modular forms we consider will be defined over $\mathbb{Z}$. Thus the algebraicity of their values will depend on the algebraicity of the inputs at which we evaluate them; the integrality will depend on those inputs and on the structure of the moduli space of HBAVs viewed as a stack, which will contribute some predictable denoninators.

We connect this section back to the theory of lattices. Let (A, ι, λ) be an HBAV defined $\mathbb{C}$, and let $\omega \in \Omega^1_{A/R}$ be a basis. The choice of ω gives an isomorphism between Lie algebra $\mathrm{Lie}(A) = H_1(A, \mathbb{R})$ and $F \otimes \mathbb{C}$, by sending a path $\gamma \in H_1(A, \mathbb{R})$ to $\int_\gamma \omega$. To $(A, \iota, \lambda, \omega)$, we associate the lattice $\{\int_\gamma \omega \mid \gamma \in H_1(A, \mathbb{Z}) \subset H_1(A, \mathbb{R})\}$.

On the other hand, to any lattice L, we associate the complex torus $A = (F \otimes \mathbb{C})/L$. Since $\mathcal{O}_F$ acts on $(F \otimes \mathbb{C})$ through the first component, and L is stable under this action, we may associate an endomorphism structure ι. Further, there is a natural alternating pairing on $F \otimes \mathbb{C}$ which induces a polarization on A, assuring us that it is the set of complex points of an Abelian variety, given by the formula

$$((z_\sigma)_\sigma, (w_\sigma)_\sigma) \mapsto \sum_\sigma \operatorname{Im}(z_\sigma \overline{w}_\sigma).$$

Write $\mathrm{d}\tau$ for the natural differential on $\mathbb{C}$ with coordinate τ, viewed as an $F \otimes \mathbb{C}$-basis for $F \otimes \mathbb{C}$. The lattice L can be recovered from A by considering the integrals $\left\{\int_\gamma \frac{\mathrm{d}\tau}{\pi} \mid \gamma \in H_1(A, \mathbb{Z})\right\}$.

We now have three separate ways to view Hilbert modular forms. Our next step is to give some examples of Hilbert modular forms, and use all three viewpoints to use them to learn about L-values.

A.4 Damerell's Formula

Damerell's formula relates the values of the zeta functions described above to the values of certain Hilbert modular forms, known as Eisenstein series. We give a description of the pieces that go into the construction.

A.4.1 Holomorphic Eisenstein Series. We give examples of Hilbert modular forms, in terms of Definition A.3.2. First, note that we have an action of $\mathcal{O}_F^{\times,+}$ on any lattice L, induced by the action of $\mathcal{O}_F$. Denote the set of orbits of this action by $L/\mathcal{O}_F^{\times,+}$, and write the orbit of $\alpha \in L$ by $\alpha\mathcal{O}_F^{\times,+} \in L/\mathcal{O}_F^{\times,+}$.

Definition A.4.1. Let $k > 2$ be an integer. The *holomorphic Eisenstein series of parallel weight k* is the function on the space of lattices

$$G_k(L) = \sum_{0 \neq \alpha\mathcal{O}_F^{\times,+} \in L/\mathcal{O}_F^{\times,+}} \frac{1}{\prod_\sigma \sigma(\alpha)^k}.$$

Here the sum is over nonzero orbits of the action of $\mathcal{O}_F^{\times,+}$ on L. This is independent of the choice of representatives α since $\prod_\sigma \sigma(u)^k = \mathrm{Nm}(u)^k = 1$ for any totally positive unit u and any integer k.

The holomorphic avatar has a simple formula on the connected component corresponding to lattices of the form $\mathcal{O}_F + \underline{z}\mathcal{O}_F$. We write it

$$G_k(\underline{z}) = \sum_{(c,d)} \frac{1}{\prod_\sigma (\sigma(c)z_\sigma + \sigma(d))^k}. \tag{A.5}$$

The sum is over nonzero representatives (c, d) of the diagonal action of $\mathcal{O}_F^{\times,+}$ on $\mathcal{O}_F \oplus \mathcal{O}_F$. i.e., we choose exactly one pair (c, d) from the elements (cu, du) as u runs over the totally positive units $u \in \mathcal{O}_F^{\times,+}$, but we exclude $(0, 0)$ from the sum. It converges absolutely using the fact that $\mathrm{Im}(z_\sigma) > 0$ for all σ.

We conclude this section with a proposition relating the values of this Eisenstein series at certain lattices to certain values of the $(d + 1)$-variable L-function.

Proposition A.4.2. *Let K be a CM field with totally real subfield F, and fix $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ a set of representatives for the class group of K. Fix a CM type for K and an unramified, unitary Hecke character χ. Then*

$$L(\chi, (k, 0)_\sigma) = \frac{1}{[\mathcal{O}_K^\times : \mathcal{O}_F^{\times,+}]} \sum_{i=1}^r \chi(\mathfrak{a}_i) G_k(\mathfrak{a}_i).$$

Proof. Recall that

$$L(\chi, (s_\sigma, t_\sigma)_\sigma) = \frac{1}{[\mathcal{O}_K^\times : \mathcal{O}_F^{\times,+}]} \sum_{i=1}^r \sum_{\alpha\mathcal{O}_F^\times} \frac{\chi(\mathfrak{a}_i)}{\chi(\alpha) \prod_\sigma \sigma(\alpha)^{s_\sigma} \overline{\sigma}(\alpha)^{t_\sigma}}.$$

The innermost sum is over representatives α of nonzero orbits of the $\mathcal{O}_F^{\times,+}$ action on $\mathfrak{a}_i$. We can pull out a factor of $\chi(\mathfrak{a}_i)$ from the innermost sum, and set $\chi(\alpha) = 1$

since χ is unramified and unitary. We are left to verify that

$$G_k(\mathfrak{a}_i) = \sum_{\alpha\mathcal{O}_F^\times} \frac{1}{\prod_\sigma \sigma(\alpha)^k}.$$

This is exactly the sum from Definition A.4.1. $\square$

A.4.2 Maass–Shimura Operators.

An important player in the story is the Maass–Shimura operator.

Definition A.4.3. Let f be a Hilbert modular form of parallel weight $\underline{k}$. Write $\underline{z} = (z_\sigma)_\sigma = (x_\sigma + iy_\sigma)_\sigma$ for the coordinate on $\mathfrak{h}_F$, and $s_\sigma = (z_\sigma - \overline{z}_\sigma)^{-1} = \frac{1}{2iy_\sigma}$. The *weight $\underline{k}$ Maass–Shimura operator at σ $\delta_{\underline{k}}^\sigma$* acts on f by the formula

$$\delta_{\underline{k}}^\sigma(f) = s_\sigma^{k_\sigma} \frac{\partial}{\partial z_\sigma} s_\sigma^{-k_\sigma} f = k_\sigma s_\sigma f + \frac{\partial f}{\partial z_\sigma}.$$

The resulting function is a *nearly holomorphic modular form* of weight $k + 2\sigma$ and type σ.[6] We iterate the Maass–Shimura operator by the formula

$$\delta_{\underline{k}}^{j\sigma}(f) = \left(\delta_{\underline{k}+2j\sigma-2\sigma}^\sigma \circ \cdots \circ \delta_{\underline{k}+2\sigma}^\sigma \circ \delta_{\underline{k}}^\sigma\right)(f).$$

The resulting function is a nearly holomorphic modular form of weight $k + 2j\sigma$ and type $j\sigma$. Using the fact that the partial derivatives $\frac{\partial}{\partial z_\sigma}$ commute for different σ, one may show that a similar definition gives a well-defined operator $\delta_{\underline{k}}^{\sum j_\sigma \sigma}$, which sends a holomorphic Hilbert modular form of weight $\underline{k}$ to a nearly holomorphic Hilbert modular form of weight $k + \sum 2j_\sigma\sigma$ and type $\sum j_\sigma \sigma$.

Remark A.4.4. Note that we do not define what it means for a function $\mathfrak{h}_F \to \mathbb{C}$ to be a nearly holomorphic Hilbert modular form, or what its type is. The reader may take it to be the definition that

[6] Weight $k+2\sigma$ refers to the weight $\underline{k} = (k_\tau)_\tau$ where $k_\tau = k$ for $\tau \neq \sigma$ and $k_\sigma = k+2$. In general, weight $k + \sum 2j_\sigma\sigma$ refers to the weight $\underline{k} = (k_\tau)_\tau$ where $k_\sigma = k + 2j_\sigma$ for all σ.

(a) a nearly holomorphic modular form of type 0 is simply a holomorphic modular form,

(b) the vector space nearly holomorphic modular forms of type $\sum j_\sigma \sigma$ includes the vector space of nearly holomorphic modular forms of type $\sum_\sigma j'_\sigma \sigma$ where $0 \leq j'_\sigma \leq j_\sigma$ for all σ, and

(c) if f is a nearly holomorphic modular form of weight $\underline{k}$ and type $\sum_\tau j_\tau \tau$, then $\delta_{\underline{k}}^\sigma f$ is a nearly holomorphic modular form of weight $\underline{k}+2\sigma$ and type $\sigma + \sum_\tau j_\tau \tau$.

When $2j_\sigma < k_\sigma$ for all σ, this process produces all nearly holomorphic modular forms of weight $\underline{k}$ and type $\sum j_\sigma \sigma$.

A.4.3 Nearly Holomorphic Eisenstein Series.

Write $G_{k,\underline{j}} := \delta_{\underline{k}}^{\sum_\sigma j_\sigma \sigma} G_k$ for $\underline{j} = (j_\sigma)_\sigma$. We find a formula for this in the following proposition.

Proposition A.4.5. *Let $k > 2$ be an integer, and $\underline{j} = (j_\sigma)_\sigma$ a tuple of non-negative integers. Then a formula for $G_{k,\underline{j}}$ is*

$$G_{k,\underline{j}}(\underline{z}) = \left(\prod_\sigma k(k+1)\dots(k+j-1) s_\sigma^{j_\sigma} \right) \sum_{(c,d)} \frac{\prod_\sigma (\sigma(c)\overline{z}_\sigma + \sigma(d))^{j_\sigma}}{\prod_\sigma (\sigma(c) z_\sigma + \sigma(d))^{k+j_\sigma}}.$$

Proof. We will prove this by induction, noting that the base case $\underline{j} = (0)_\sigma$ we may compare with Equation (A.5). For the induction step, we want to verify that $G_{k,\underline{j}+\sigma} = \delta_{k+\underline{j}}^\sigma G_{k,\underline{j}}$. Since $\delta_{k+\underline{j}}^\sigma f = (k+2j_\sigma) s_\sigma f + \frac{\partial f}{\partial z_\sigma}$, we may factor out the terms involving variables that do not depend on z_σ (including $\overline{z}_\sigma$, but not including s_σ) and simply verify, using that formula, that

$$\delta_{k+2\underline{j}}^\sigma \left(s_\sigma^{j_\sigma} \sum \frac{1}{(\sigma(c) z_\sigma + \sigma(d))^{k+j_\sigma}} \right) = (k_\sigma + j_\sigma) s_\sigma^{j_\sigma+1} \sum \frac{(\sigma(c)\overline{z}_\sigma + \sigma(d))}{(\sigma(c) z_\sigma + \sigma(d))^{k+j_\sigma+1}}. \tag{A.6}$$

Write $f = s_\sigma^{j_\sigma} \sum \frac{1}{(\sigma(c)z_\sigma + \sigma(d))^{k+j_\sigma}}$. For the first term of $\delta_{k+2\underline{j}}^\sigma f$, we have

$$(k+2j_\sigma) s_\sigma \left(s_\sigma^{j_\sigma} \sum \frac{1}{(\sigma(c) z_\sigma + \sigma(d))^{k+j_\sigma}} \right).$$

The term $\frac{\partial f}{\partial z_\sigma}$ splits into two terms by the product rule.

$$\frac{\partial f}{\partial z_\sigma} = \frac{\partial s_\sigma^{j_\sigma}}{\partial z_\sigma} \sum \frac{1}{(\sigma(c)z_\sigma + \sigma(d))^{k+j_\sigma}} + s_\sigma^{j_\sigma} \frac{\partial}{\partial z_\sigma} \sum \frac{1}{(\sigma(c)z_\sigma + \sigma(d))^{k+j_\sigma}}$$

Call these terms 2a and 2b respectively. Term 2a simplifies using the formula $\frac{\partial s_\sigma^{j_\sigma}}{\partial z_\sigma} = -j_\sigma s_\sigma^{j_\sigma+1}$, and combines with the $(k+2j_\sigma)s_\sigma f$ term to produce

$$[(k+2j_\sigma) - j_\sigma]\, s_\sigma^{j_\sigma+1} \sum \frac{1}{(\sigma(c)z_\sigma + \sigma(d))^{k+j_\sigma}} = (k+j_\sigma) s_\sigma^{j_\sigma+1} \sum \frac{(\sigma(c)z_\sigma + \sigma(d))}{(\sigma(c)z_\sigma + \sigma(d))^{k+j_\sigma+1}} \tag{A.7}$$

We calculate term 2b.

$$s_\sigma^{j_\sigma} \frac{\partial}{\partial z_\sigma} \sum \frac{1}{(\sigma(c)z_\sigma + \sigma(d))^{k+j_\sigma}} = (k+j_\sigma) s_\sigma^{j_\sigma+1} \sum \frac{-\sigma(c)(z_\sigma - \overline{z}_\sigma)}{(\sigma(c)z_\sigma + \sigma(d))^{k+j_\sigma+1}} \tag{A.8}$$

Note that we have added an extra factor of s_σ before the sum, in exchange for adding a factor of $s_\sigma^{-1} = z_\sigma - \overline{z}_\sigma$ to the numerator of every term inside the sum. We now add the final result from Equation (A.7) to that from Equation (A.8) to obtain $\delta_{k+2\underline{j}}^\sigma f$. We have arranged for the factors in front to match, as well as the denominators for each term in the sum, so that we only have to add the numerators.

$$\delta_{k+\underline{j}}^\sigma f = (k+j_\sigma) s^{j_\sigma+1} \sum \frac{\sigma(c)z_\sigma + d_\sigma - \sigma(c)(z_\sigma - \overline{z}_\sigma)}{(\sigma(c)z_\sigma + \sigma(d))^{k+j_\sigma+1}}.$$

Use the fact that $\sigma(c)z_\sigma + d_\sigma - \sigma(c)(z_\sigma - \overline{z}_\sigma) = \sigma(c)\overline{z} + \sigma(d)$ to compare with Equation (A.6). □

We now have values for the $G_{k,\underline{j}}$ that will be used to relate their values at CM points to the values of the L-functions from the previous section.

A.4.4 Damerell's Formula. Fix representatives $\mathfrak{a}_1, \ldots, \mathfrak{a}_h$ for the class group of K, and let χ be an unramified, unitary Hecke character. We write the $(d+1)$-variable L-function, recalling that the inner sum is over nonzero cosets

$0 \neq \alpha\mathcal{O}_F^{\times,+} \in \mathfrak{a}_i/\mathcal{O}_F^{\times,+}$.

$$L(\chi, (s_\sigma, t_\sigma)_\sigma) = \frac{1}{[\mathcal{O}_K^\times : \mathcal{O}_F^{\times,+}]} \sum_{i=1}^{h} \sum_{\alpha\mathcal{O}_F^{\times,+}} \frac{\chi(\mathfrak{a}_i)}{\prod_\sigma \sigma(\alpha)^{s_\sigma}\overline{\sigma}(\alpha)^{t_\sigma}}.$$

We have left our a factor of $\chi(\alpha)$ from the denominator; these are forced to be 1 by the assumptions that χ be unramified and unitary.

In the previous section, we gave the formula for $G_{k,\underline{j}}$ as a holomorphic function on $\mathfrak{h}_F$. As a function on lattices, we have

$$G_{k,\underline{j}}(L) = \left(\prod_\sigma k(k+1)\dots(k+j_\sigma-1)s_\sigma^{j_\sigma}\right) \sum_{\alpha\mathcal{O}_F^{\times,+}} \frac{\prod_\sigma \overline{\sigma}(\alpha)^{j_\sigma}}{\prod_\sigma \sigma(\alpha)^{k+j_\sigma}}. \tag{A.9}$$

Here the sum is over nonzero orbits $0 \neq \alpha\mathcal{O}_F^{\times,+} \in L/\mathcal{O}_F^{\times,+}$. Compare to Equation (A.6) to see that this is the correct formula, at least for lattices $L = \mathcal{O}_F + \underline{z}\mathcal{O}_F$. We might describe $y_\sigma = \frac{1}{2is_\sigma}$ as the "covolume of L at σ", noting that $\prod_\sigma \frac{1}{2is_\sigma}$ is the covolume of L.

Theorem A.4.6 (Damerell's Formula)**.** *Fix an integer $k > 2$, a tuple $\underline{j} = (j_\sigma)_\sigma$ of positive integers indexed by the real embeddings of F, and an unramified unitary Hecke character χ. Then we may relate special values of the $(d+1)$-variable L-function with the values of various Eisenstein series at lattices corresponding to fractional ideals of K as follows:*

$$L(\chi, (k+j_\sigma, -j_\sigma)_\sigma) = \frac{\sum_{i=1}^h \chi(\mathfrak{a}_i) G_{k,\underline{j}}(\mathfrak{a}_i)}{[\mathcal{O}_K^\times, \mathcal{O}_F^{\times,+}] \left(\prod_\sigma k(k+1)\dots(k+j_\sigma-1)s_\sigma^{j_\sigma}\right)}.$$

Proof. Compare Equations (A.9) and (A.3). □

A.5 Takeaways

In this section we describe some applications for Damerell's formula as discussed in the previous two sections. Specifically, we carry over the setup from Section 3: fix a totally real field F of degree d, and a CM extension $K = F(\alpha)$ with $\mathcal{O}_K = \mathcal{O}_F + \alpha\mathcal{O}_F$. We also fix a CM type of K, i.e., a preferred extension of each real embedding $\sigma\colon F \to \mathbb{R}$ to a complex embedding $K \to \mathbb{C}$ which we also call σ. Thus the set of complex embeddings of K is the set of all σ's and all $\overline{\sigma}$'s as σ runs over the real embeddings of F.

A.5.1 Algebraicity of L-Values: Damerell's Theorem. In order to talk about algebraicity, we recall the algebraic definition from Section A.3.4. For simplicity, we focus on the quadratic imaginary case.

Definition A.5.1 (Definition A.3.6)**.** Fix a base ring R. A *modular form* defined over R is an algebraic function f that assigns to every pair (E, ω) of an elliptic curve E defined over some R-algebra S and a basis ω for $\Omega^1_{E/S}$ as an S-module, an element of S. A modular form of weight k is such a function which satisfies the homogeneity property that for any $c \in S^\times$, $f(E, c\omega) = c^{-k} f(E, \omega)$.

In particular, given a modular form f defined over a number field H, an elliptic curve E defined over H, and a generator $\omega \in \Omega^1_{E/H}$, the value $f(E, \omega) \in H$ is algebraic. A similar definition, and a similar statement about algebraicity, can be made for nearly modular forms.

To prove the algebraicity of certain values of L-functions, we can use our previous results relating these values to the values of Eisenstein series. Eisenstein series are in fact defined over $\mathbb{Z}$, so it will be enough to show that evaluating their complex avatars at the CM points of the modular curve corresponds to evaluating their algebraic avatars at a pair consisting of an elliptic curve defined over a number field H and a generator $\mathrm{d}w \in \Omega^1_{E/\mathbb{C}}$ which is explicitly related to an algebraic generator $\omega \in \Omega^1_{E/H}$ by a constant $\mathrm{d}w = c \cdot \omega$.

In fact, each point of the upper half plane $\tau \in \mathfrak{h}$ corresponds to a specific complex elliptic curve E_τ and a chosen basis $\mathrm{d}w \in \Omega^1_{E/\mathbb{C}}$. Any complex torus is an elliptic curve; we let L_τ denote the lattice $\mathbb{Z} + \tau\mathbb{Z} \subset \mathbb{C}$, and write $E_\tau = \mathbb{C}/L_\tau$.

The projection $\mathbb{C} \to E_\tau$ gives an isomorphism between the cotangent bundle of E_τ and the cotangent bundle of $\mathbb{C}$. Writing w for the coordinate on $\mathbb{C}$, we get a generator $\mathrm{d}w$ of $\Omega^1_{E_\tau/\mathbb{C}}$. Evaluating the holomorphic function at $\tau \in \mathfrak{h}$ corresponds to evaluating the algebraic function at the pair $(E_\tau, \mathrm{d}w)$.

When $\tau = \alpha$ is a CM point, the corresponding elliptic curve E_α has complex multiplication by the quadratic imaginary field $K = \mathbb{Q}(\alpha)$. A celebrated result in explicit class field theory gives that this E_α is defined over the Hilbert class field H of K. We also have that $\pi \mathrm{d}w$ is defined over H, so that

$$G_k\left(E_\alpha, \pi \mathrm{d}w\right) = \pi^{-k} G_k(E_\alpha, \mathrm{d}w) = \pi^{-k} G_k(\alpha) \in H.$$

Thus by Damerell's Formula (Theorem A.4.6), we have

$$\frac{L(\chi, k+j, -j)}{\pi^k} = \frac{\sum_{i=1}^h \chi(\mathfrak{a}_i) G_{k,j}(\mathfrak{a}_i)/s(\mathfrak{a}_i)^j}{k(k+1)\dots(k+j-1)(\#\mathcal{O}_K^\times)\pi^k} \in H(\mu_h).$$

Here we write $s(\mathfrak{a}_i)$ for the covolume of $\mathfrak{a}_i$ in $F \otimes_{\mathbb{Z}} \mathbb{C}$. Notice that since χ is a character of a group of order h, it takes values in the hth roots of unity μ_h. This is a standard algebraicity result showing that, up to a "period" π^k, the value $L(\chi, k+j, -j)$ lives in a particular number field. Above we wrote $H(\mu_h)$, but many authors would bound it more precisely by writing $H(\text{the values of } \chi)$. In addition to algebraicity, there are integrality results, bounding the denominators of $\pi^{-k} L(\chi, k+j, -j) \notin \mathcal{O}_H(\mu_h)$.

A similar construction can be carried out for a general CM field K, using Hilbert modular forms on its maximal totally real subfield and Hilbert Blumenthal Abelian varieties in place of elliptic curves. We omit it here, as it is nearly the same once we have the geometric desctiption of Hilbert modular forms from Section A.3.4.

A.5.2 p-adic Interpolation. One application that requires the algebraicity and integrality results described in the previous section is the construction of the p-adic L-function for a CM field. These are laid out carefully in [Kat76] and [AI19] for quadratic imaginary K (respectively for p split in K and p nonsplit in K), and in [Kat78] for a general CM field K (under the assumption that all primes of the maximal totally real subfield K^+ of K which divide p are split in K/K^+). The case when primes above p are nonsplit in K/K^+ is not settled. Very roughly, the construction goes like this.

First, we relate the values of the L-function at certain inputs to the values of certain modular forms. We then modify it to write

$$L_p(\chi, (k+j_\sigma, -j_\sigma)_\sigma) = \left(\prod_{\mathfrak{p}|p} 1 - \chi(\mathfrak{p}) \operatorname{Nm}(\mathfrak{p})^k\right) L(\chi, (k+j_\sigma, -j_\sigma)_\sigma) =$$

$$= [\text{predictable constants}] \sum_{i=1}^h \chi(\mathfrak{a}_i) G_{k,\underline{j}}^{[p]}(\mathfrak{a}_i).$$

Here L_p is the function to be interpolated, and it is modified from L by removing the Euler factor at p. We also have $G_{k,\underline{j}}^{[p]}$, the p-depletion of the Eisenstein series $G_{k,\underline{j}}$. When K is quadratic imaginary, one might describe $G_{k,\underline{j}}^{[p]}$ in terms of its q-expansion as the form whose nth Fourier coefficient is 0 whenever n is a power of p, whose nth Fourier coefficient is the same as that for $G_{k,\underline{j}}$ when n is prime to p.

Second, we prove congruences modulo p^n between the values $G_{k,0}(\mathfrak{a}_i)$ and $G_{k',0}(\mathfrak{a}_i)$ whenever $k \equiv k' \pmod{(p-1)p^{n-1}}$. For K quadratic imaginary, this is established using a q-expansion principle in [Ser72]. Serre's intended use is to p-adically interpolate the standard Riemann zeta function, but the result is useful here as well.

Finally, we prove congruences modulo p^n between $\delta_k^{\sum j_\sigma \sigma} f$ and $\delta_k^{\sum j'_\sigma \sigma} f$ whenever f is a p-depleted form and each $j_\sigma \equiv j'_\sigma \pmod{(p-1)p^{n-1}}$. From here we have the neccessary congruences to claim that $L_p(\chi, -)$ is p-adically continuous as a function of the $d+1$ variables. Thus by the p-adic density of the set of characters of infinity type $(k + j_\sigma, -j_\sigma)_\sigma$ in the set of all p-adic characters, we get a unique extension of L_p to a continuous function on the set of all p-adic characters, which is the p-adic L-function.

Remark A.5.2. Note that this is a very rough outline of the arguments. In particular, we made no reference to formal or rigid geometry which is a key tool in defining Hilbert modular forms in a p-adic setting, or to the Frobenius and U_p operators that give an algebraic way to p-deplete forms. We also made no reference to how the behavior of p takes this result from one proven in the 1970's when p splits to one proven in the 2010's when p is nonsplit; we note here that it has to do with whether or not the HBAVs with CM by K are ordinary at p.

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