HOMOLOGICAL ALGEBRA FOR POLYNOMIAL MACKEY RINGS OVER PRIME CYCLIC GROUPS.

by

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A DISSERTATION

Presented to the Department of Mathematics
and the Division of Graduate Studies of the University of Oregon
in partial fulfillment of the requirements
for the degree of
Doctor of Philosphy

September 2022

DISSERTATION APPROVAL PAGE

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Title: Homological Algebra for Polynomial Mackey Rings over Prime Cyclic Groups

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Degree awarded September 2022

DISSERTATION ABSTRACT

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Department of Mathematics

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Let C_l denote the cyclic group of prime order l and let k be a field. We define a Mackey $\underline{k}$-algebra $\underline{k}[x_\theta]$ which is constructed by adjoining a free commutative variable to the free side of the constant Mackey functor $\underline{k}$. When $char(k)$ is relatively prime to l we show that there is a an equivalence of categories between $\underline{k}[x_\theta] - \underline{Mod}$ and the category of modules over a certain twisted group ring. We calculate the free side of a certain Ext object $\underline{Ext}^*_{\underline{k}[x_\theta]}(\underline{k}, \underline{k})$ in the two cases when $char(k)$ is relatively prime to l and when $char(k) = l = 2$.

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ACKNOWLEDGEMENTS

I would like to thank Dan Dugger for his amazing patience, insight and guidance. I would also like to thank Xã for her support, understanding and love.

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CHAPTER I

INTRODUCTION

Introduction

Let G be a finite group. Mackey functors over G serve as the natural coefficients for G-equivariant cohomology in the same way that abelian groups do for singular cohomology [May96]. Mackey functors and their homological algebra are much less understood than that of abelian groups. We will investigate some examples of homological algebra with Mackey functors in this thesis. Considering the special case $G = C_l$, the cyclic group of prime order l, we focus on defining a type of polynomial ring in the category of Mackey functors and computing the internal Ext object for the residue field Mackey functor.

In this thesis we will focus on Mackey functors over the cyclic groups of prime order l with a fixed field k. We will define a Mackey functor $\underline{k}[x_\theta]$ which arises by adjoining a free commutative variable to the free side of the constant coefficient Mackey ring $\underline{k}$. For a field k with characteristic relatively prime to l, we will prove that the category of Mackey modules over a Mackey $\underline{k}$-algebra $\mathcal{R}$ is equivalent to the category of ordinary modules over a certain "twisted group ring". When k has characteristic l, the category is more complicated. We end by calculating the Mackey Ext ring $\underline{\mathrm{Ext}}^*_{\mathcal{R}}(\underline{k}, \underline{k})$ both when $\mathrm{char}(k) \neq l$ and when $\mathrm{char}(k) = l = 2$. In both cases, we investigate resolutions of the residue field and the associated Ext groups, establishing a component of Koszul duality in some cases.

Before stating the results in more detail, we begin with some brief background information. Let G be a finite group. The structure of a Mackey functor $\mathcal{F}$ includes an abelian group $\mathcal{F}(G/H)$ for each subgroup H of G, along with various restriction and transfer maps between these values of $\mathcal{F}$. In the case $G = C_l$, this takes a simple form: a Mackey functor $\mathcal{F}$ over C_l is the data of two abelian groups $\mathcal{F}_\theta := \mathcal{F}(C_l/e)$ and $\mathcal{F}_\bullet := \mathcal{F}(C_l/C_l)$, with maps of abelian groups $p_*\colon \mathcal{F}_\theta \to \mathcal{F}_\bullet$, $p^*\colon \mathcal{F}_\bullet \to \mathcal{F}_\theta$, and an automorphism $t\colon \mathcal{F}_\theta \to \mathcal{F}_\theta$. These maps satisfy the following relations:

$$\begin{aligned} &p^* \circ p_* = \sum_{i=0}^{l-1} t^i, \qquad p_* \circ t^n = p_* \quad \text{for all } n, \\ &t^n \circ p^* = p^* \quad \text{for all } n, \qquad \text{and} \qquad t^l = \mathrm{id}_{\mathcal{F}_\theta}. \end{aligned}$$

We will draw these Mackey functors as
$$\begin{tikzcd} F_\theta \arrow[loop left, "t"] \arrow[r, bend left, "p_*"] & F_\bullet \arrow[l, bend left, "p^*"] \end{tikzcd}$$
Mackey functors have been studied extensively and can be read about in [Dre73], [Dre71], [Gre71], and [Web00].

The category of Mackey functors over C_l is equipped with a tensor product called the box product $-\Box-$. Mackey rings are defined to be monoids in the monoidal category of Mackey functors with the box product. Unravelling the definitions, a Mackey ring $\mathcal{R}$ is a Mackey functor where both $\mathcal{R}_\theta$ and $\mathcal{R}_\bullet$ are rings, p^* and t are ring maps, and p_* is a map of $\mathcal{R}_\bullet$-modules. If k is a ring, an important example of a Mackey ring is $\underline{k}$, the "constant coefficient" Mackey ring:

$$\begin{tikzcd} k \arrow[loop left, "\mathrm{id}_k"] \arrow[r, bend left, "l\mathrm{id}_k"] & k \arrow[l, bend left, "\mathrm{id}_k"] \end{tikzcd}$$

We can define a left Mackey module over a Mackey ring $\mathcal{R}$ to be a Mackey functor $\mathcal{M}$ with a unital and associative structure map $\mu_{\mathcal{M}}\colon \mathcal{R}\Box\mathcal{M} \to \mathcal{M}$. It turns out that a Mackey functor $\mathcal{M}$ is a $\underline{k}$-module if and only if $\mathcal{M}_\theta$ and $\mathcal{M}_\bullet$ are k-modules, p_*, p^*, and t are k-linear, and $p_* \circ p^* = l \cdot \mathrm{id}_{\mathcal{M}_\bullet}$.

The main object of interest in this paper is the commutative Mackey $\underline{k}$-algebra $\underline{k}[x_\theta]$. This Mackey functor has $\underline{k}[x_\theta]_\theta = k[x_1, \ldots, x_l]$ with $t(x_i) = x_{i+1 \bmod l}$ and comes equipped with the following universal property: for any commutative Mackey $\underline{k}$-algebra $\mathcal{S}$ and any element $y \in \mathcal{S}_\theta$, there is a unique map of Mackey $\underline{k}$-algebras $f \colon \underline{k}[x_\theta] \to \mathcal{S}$ for which $f_\theta(x_1) = y$. This property is similar to the universal property of polynomial algebras over a field k, which was our motivation in defining and studying this object.

The category of $\mathcal{R}$-modules for a Mackey ring $\mathcal{R}$ is abelian with enough projectives and injectives, so the usual machinery of homological algebra applies. In particular, we can talk about $\mathrm{Ext}^*_{\mathcal{R}}(\mathcal{M}, \mathcal{N})$. If $\mathcal{R}$ is a Mackey ring and $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{R}$-modules, then $\mathrm{Ext}^*_{\mathcal{R}}(\mathcal{M}, \mathcal{N})$ is the $\bullet$-side of an internal Ext object denoted $\underline{\mathrm{Ext}}^*_{\mathcal{R}}(\mathcal{M}, \mathcal{N})$. That is, $\underline{\mathrm{Ext}}^*_{\mathcal{R}}(\mathcal{M}, \mathcal{N})$ is a Mackey functor and $\underline{\mathrm{Ext}}^*_{\mathcal{R}}(\mathcal{M}, \mathcal{N})_\bullet = \mathrm{Ext}^*_{\mathcal{R}}(\mathcal{M}, \mathcal{N})$. When $\mathcal{M} = \mathcal{N}$ this is a Mackey ring via the Yoneda product [Wei94]. Our aim in this paper is to investigate the case $\underline{\mathrm{Ext}}^*_{\underline{k}[x_\theta]}(\underline{k}, \underline{k})$. We will show the following results:

Theorem 1.1.1. *$\underline{Ext}^*_{\underline{k}[x_\theta]}(\underline{k}, \underline{k})_\theta$ is an exterior k-algebra on l generators in two cases:*

1. *When char(k) is relatively prime to l, and*
2. *when char$(k) = l = 2$.*

In the case when char(k) is relatively prime to l, the theory simplifies somewhat. In this case, for any Mackey $\underline{k}$-algebra $\mathcal{R}$ the structure of an $\mathcal{R}$-module

$\mathcal{M}$ is determined solely by the $\mathcal{R}_\theta$-module structure of $\mathcal{M}_\theta$ and the action of t on $\mathcal{M}_\theta$. Here one can use a Koszul resolution [Mac63] to compute the Ext Mackey functors of $\underline{k}$ as a $\underline{k}[x_\theta]$-module, since the this resolution is exact.

However, when $\mathrm{char}(k) = l$ the category is more complicated. When $\mathrm{char}(k)$ is relatively prime to l we can give $\underline{k}$ a finite projective resolution, but when $\mathrm{char}(k) = l = 2$ any resolution for $\underline{k}$ as a $\underline{k}[x_\theta]$-module must be infinite. For the case $\mathrm{char}(k) = l = 2$, we exhibit a short exact sequence of $\underline{k}[x_\theta]$-modules ending with a certain module $\mathcal{M}$ and beginning with $\bigoplus_{i=0}^{\infty} \mathcal{M}$. We then stitch this short exact sequence with itself (infinitely many times) to get a projective resolution of $\underline{k}$ as a $\underline{k}[x_\theta]$-module. From there, we compute $\underline{\mathrm{Ext}}^*_{\underline{k}[x_\theta]}(\underline{k}, \underline{k})$ and show that its θ-side is an exterior k-algebra on 2 generators. It is remarkable that, despite the resolution being infinite and complicated, the Ext groups are themselves very simple. We end by calculating a portion of the $\bullet$-side ring structure.

One difficulty in the case $\mathrm{char}(k) = l$ is that while $\underline{k}[x_\theta]_\theta = k[x_1, \ldots, x_l]$, a nice polynomial ring, $\underline{k}[x_\theta]_\bullet$ is a more complicated ring requiring infinitely many ring generators. Even for $\mathrm{char}(k) = l = 2$, $\underline{k}[x_\theta]_\bullet$ is a ring with two seperate infinite families of generators, along with many relations. To manage this, we rely heavily on a decomposition of $k[a_0, a_1, \ldots, b_1, b_2, \ldots]/\sim$ as a $k[a_0]$-module, over which it is the sum of an infinite rank free module and an infinite rank sum of k's. The complexity of this ring is the main obstacle in extending the results to the case where $\mathrm{char}(k) = l > 2$.

CHAPTER II

BACKGROUND

We now develop the necessary background information on Mackey functors.

Definition 2.0.1. *Let C_l be a finite cyclic group of prime order l. A Mackey functor $\mathcal{F}$ over C_l consists of abelian groups $\mathcal{F}_\theta$ and $\mathcal{F}_\bullet$ and maps of abelian groups $t\colon \mathcal{F}_\theta \to \mathcal{F}_\theta$, $p_*\colon \mathcal{F}_\theta \to \mathcal{F}_\bullet$, and $p^*\colon \mathcal{F}_\bullet \to \mathcal{F}_\theta$ which satisfy the following identities:*

$$\begin{aligned} t^l &= id_{\mathcal{F}_\theta} & p_* \circ t &= p_* \\ t \circ p^* &= p^* & p^* \circ p_* &= \sum_{i=0}^{l-1} t^i. \end{aligned}$$

A map of Mackey functors $f\colon \mathcal{F} \to \mathcal{G}$ consists of two maps of abelian groups, $f_\theta\colon \mathcal{F}_\theta \to \mathcal{G}_\theta$ and $f_\bullet\colon \mathcal{F}_\bullet \to \mathcal{G}_\bullet$, which satisfy the following identities:

$$t_{\mathcal{G}} \circ f_\theta = f_\theta \circ t_{\mathcal{F}} \qquad f_\bullet \circ p_{*,\mathcal{F}} = p_{*,\mathcal{G}} \circ f_\theta \qquad f_\theta \circ p^*_{\mathcal{F}} = p^*_{\mathcal{G}} \circ f_\bullet.$$

There are two particularly important Mackey functors, the free functors $\mathcal{F}_\theta(\mathbb{Z})$ and $\mathcal{F}_\bullet(\mathbb{Z})$:

$$\begin{tikzcd} \mathcal{F}_\theta(\mathbb{Z})\colon & \mathbb{Z}^l \arrow[loop left, "t"] \arrow[r, bend left, "\nabla"] & \mathbb{Z} \arrow[l, bend left, "\Delta"] & & \mathcal{F}_\bullet(\mathbb{Z})\colon & \mathbb{Z} \arrow[loop left, "\mathrm{id}_{\mathbb{Z}}"] \arrow[r, bend left, "p_*"] & \mathbb{Z}^2 \arrow[l, bend left, "p^*"] \end{tikzcd}$$

The maps in $\mathcal{F}_\theta(\mathbb{Z})$ are $\nabla\colon (a_1, \ldots, a_l) \mapsto a_1 + \cdots + a_l$ and $\Delta\colon a \mapsto (a, a, \ldots, a)$, and t acts as cyclic permutation on $\mathbb{Z}^l$. The maps in $\mathcal{F}_\bullet(\mathbb{Z})$ are $p_*\colon a \mapsto (0, a)$ and $p^*\colon (a, b) \mapsto a + bl$. We will present $\mathcal{F}_\theta(\mathbb{Z})$ where $\mathcal{F}_\theta(\mathbb{Z})_\theta$ is generated by the element

$g = (1, 0, 0, \dots, 0)$ as a $\mathbb{Z}[C_l]$-module. We also identify the element $1_\bullet = (1, 0) \in \mathcal{F}_\bullet(\mathbb{Z})$ and its image under p^*, denoted $1_\theta \in \mathcal{F}_\theta(\mathbb{Z})$. The images of these elements determine all maps out of $\mathcal{F}_\theta(\mathbb{Z})$ and $\mathcal{F}_\bullet(\mathbb{Z})$, respectively. This is stated precisely in the following proposition.

Proposition 2.0.2. *For any Mackey functor $\mathcal{G}$,*

$$Hom(\mathcal{F}_\bullet(\mathbb{Z}), \mathcal{G}) \cong \mathcal{G}_\bullet \qquad Hom(\mathcal{F}_\theta(\mathbb{Z}), \mathcal{G}) \cong \mathcal{G}_\theta$$

where the isomorphisms are $f \mapsto f_\bullet(1_\bullet)$ and $h \mapsto h_\theta(g)$. [RAE19]

Definition 2.0.3. *Let $\mathcal{F}$ and $\mathcal{G}$ be Mackey functors. The box product $\mathcal{F}\Box\mathcal{G}$ is the Mackey functor with*

$$(\mathcal{F}\Box\mathcal{G})_\theta = \mathcal{F}_\theta \otimes \mathcal{G}_\theta$$
$$(\mathcal{F}\Box\mathcal{G})_\bullet = ((\mathcal{F}_\theta \otimes \mathcal{G}_\theta) \oplus (\mathcal{F}_\bullet \otimes \mathcal{G}_\bullet))/\sim$$

where $\sim$ is defined as

$$a_\theta \otimes p^*(b_\bullet) \sim p_*(a_\theta) \otimes b_\bullet$$
$$p^*(a_\bullet) \otimes b_\theta \sim a_\bullet \otimes p_*(b_\theta)$$
$$t(a_\theta) \otimes t(b_\theta) \sim a_\theta \otimes b_\theta$$

for any $a_\theta \in \mathcal{F}_\theta$, $b_\theta \in \mathcal{G}_\theta$, $a_\bullet \in \mathcal{F}_\bullet$, *and* $b_\bullet \in \mathcal{G}_\bullet$. *The map* t *is induced by the diagonal action* $t(a \otimes b) = t(a) \otimes t(b)$. *The map* p_* *is induced by the inclusion* $\mathcal{F}_\theta \otimes \mathcal{G}_\theta \to (\mathcal{F}_\theta \otimes \mathcal{G}_\theta) \oplus (\mathcal{F}_\bullet \otimes \mathcal{G}_\bullet)$. *The map* p^* *is induced by the map* $a_\theta \otimes b_\theta \mapsto \sum_{i=1}^{l} t^i(a_\theta) \otimes t^i(b_\theta)$ *and* $a_\bullet \otimes b_\bullet \mapsto p^*(a_\bullet) \otimes p^*(b_\bullet)$. *The box product is symmetric monoidal with unit* $\mathcal{F}_\bullet(\mathbb{Z})$.

Using the box product we can define a ring object in the category of Mackey functors over C_l.

Definition 2.0.4. *Let* $\mathcal{R}$ *be a Mackey functor. We say that* $\mathcal{R}$ *is a Mackey ring if there are maps* $\iota\colon \mathcal{F}_\bullet(\mathbb{Z}) \to \mathcal{R}$ *and* $\mu_{\mathcal{R}}\colon \mathcal{R}\Box\mathcal{R} \to \mathcal{R}$ *such that* $(\iota\Box id_{\mathcal{R}}) \circ \mu_{\mathcal{R}} = id_{\mathcal{R}}$ *and* $\mu_{\mathcal{R}} \circ (\mu_{\mathcal{R}}\Box id_{\mathcal{R}}) = \mu_{\mathcal{R}} \circ (id_{\mathcal{R}}\Box\mu_{\mathcal{R}})$.

Proposition 2.0.5. *[Rae19, Theorem 2.2.2] Let* $\mathcal{R}$ *be a Mackey functor. Then* $\mathcal{R}$ *is a Mackey ring if and only if* $\mathcal{R}_\theta$ *and* $\mathcal{R}_\bullet$ *are rings,* p^* *and* t *are ring maps, and* p_* *is a map of left* $\mathcal{R}_\bullet$*-modules (with* $\mathcal{R}_\theta$ *as a left* $\mathcal{R}_\bullet$*-module induced from* p^**). A commutative Mackey ring is a Mackey ring where* $\mathcal{R}_\theta$ *and* $\mathcal{R}_\bullet$ *are commutative rings.*

Definition 2.0.6. *Let* $\mathcal{R}$ *be a Mackey ring. A Mackey functor* $\mathcal{N}$ *is a left* $\mathcal{R}$*-module if there is a map* $\mu_{\mathcal{M}}\colon \mathcal{R}\Box\mathcal{M} \to \mathcal{M}$ *which is unital and associative. A map of left* $\mathcal{R}$*-modules* $f\colon \mathcal{M} \to \mathcal{N}$ *is a map of Mackey functors such that* $f \circ \mu_{\mathcal{M}} = \mu_{\mathcal{N}} \circ (id_{\mathcal{R}}\Box f)$. *Right* $\mathcal{R}$*-modules are defined similarly.*

Remark 2.0.7. Let $\mathcal{M}$ be a Mackey functor and $\mathcal{R}$ be a Mackey ring. Then $\mathcal{M}$ is an $\mathcal{R}$-module if $\mathcal{M}_\theta$ is an $\mathcal{R}_\theta$-module, $\mathcal{M}_\bullet$ is an $\mathcal{R}_\bullet$-module, p^* and p_* are $\mathcal{R}_\bullet$-module maps and $t(r_\theta m_\theta) = t(r_\theta)t(m_\theta)$ for any $r_\theta \in \mathcal{R}_\theta$ and $m_\theta \in \mathcal{M}_\theta$. A map of Mackey functors f is a map of $\mathcal{R}$-modules if f_θ is a map of $\mathcal{R}_\theta$-modules and $f_\bullet$ is a map of $\mathcal{R}_\bullet$-modules.

The Mackey rings we will consider in this paper are all commutative, so the distinction between left and right Mackey modules is unimportant for us.

The category of $\mathcal{R}$-modules is also monoidal with its own product $\Box_{\mathcal{R}}$ and unit $\mathcal{R}$.

Definition 2.0.8. *Let $\mathcal{R}$ be a commutative Mackey ring and $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{R}$-modules. Define $\mathcal{M}\Box_{\mathcal{R}}\mathcal{N} := coeq(\mathcal{M}\Box\mathcal{R}\Box\mathcal{N} \rightrightarrows \mathcal{M}\Box\mathcal{N})$ where the two maps are $\mu_{\mathcal{M}}\Box id_{\mathcal{N}}$ and $id_{\mathcal{M}}\Box\mu_{\mathcal{N}}$.*

Similar to the roles that $\mathcal{F}_\theta(\mathbb{Z})$ and $\mathcal{F}_\bullet(\mathbb{Z})$ play as free Mackey functors in the category of Mackey functors, the two main examples of free functors in the category of $\mathcal{R}$-modules are $\mathcal{F}_\bullet(\mathcal{R}) = \mathcal{R}\Box\mathcal{F}_\bullet(\mathbb{Z}) \cong \mathcal{R}$ and $\mathcal{F}_\theta(\mathcal{R}) = \mathcal{R}\Box\mathcal{F}_\theta(\mathbb{Z})$. We will denote $\mathcal{F}_\theta(\mathcal{R})$ as $\mathcal{F}_\theta(\mathcal{R}_\theta)$ in the future.

There are also distinguished elements $1_\bullet \in \mathcal{R}_\bullet$ and $1_\theta \in \mathcal{R}_\theta$, the ring units. We denote the element $1_\theta \otimes g \in \mathcal{F}_\theta(R_\theta)_\theta$ also by g.

Proposition 2.0.9. *Let $\mathcal{R}$ be a Mackey ring and let $\mathcal{M}$ be a $\mathcal{R}$-module. The map $f \mapsto f_\bullet(1_\bullet)$ is an isomorphism $Hom_{\mathcal{R}}(\mathcal{R},\mathcal{M}) \xrightarrow{\cong} \mathcal{M}_\bullet$ and the map $h \mapsto h_\theta(g)$ is an isomorphism $Hom_{\mathcal{R}}(\mathcal{F}_\theta(\mathcal{R}_\theta),\mathcal{M}) \xrightarrow{\cong} \mathcal{M}_\theta$.*

Proof. Routine. □

We can present $\mathcal{F}_\theta(\mathcal{R}_\theta)$ in several ways. One way is with $\mathcal{F}_\theta(\mathcal{R}_\theta)_\theta = \mathcal{R}_\theta^l$, $\mathcal{F}_\theta(\mathcal{R}_\theta)_\bullet = \mathcal{R}_\theta$ with t acting as cyclic permutation on $\mathcal{R}_\theta^l$, $p_*\colon (a_0,\ldots,a_{l-1}) \mapsto \sum_{i=0}^{l-1} a_i$ for $a_0,\ldots,a_{l-1} \in \mathcal{R}_\theta$ and $p^*\colon a \mapsto (a,\ldots,a)$ for $a \in \mathcal{R}_\theta$. In this presentation, $\mathcal{F}_\theta(\mathcal{R}_\theta)$ has the $\mathcal{R}$-module structure where $r(a_0,\ldots,a_{l-1}) =$

$(ra_0, (t^{-1}r)a_1, (t^{-2}r)a_2, \dots)$ is the $\mathcal{R}_\theta$ action on $\mathcal{F}_\theta(R_\theta)_\theta$ and $\mathcal{F}_\theta(R_\theta)_\bullet$ has the induced $\mathcal{R}_\bullet$-action since $\mathcal{R}_\theta$ is an $\mathcal{R}_\bullet$-module from the Mackey ring structure of $\mathcal{R}$.

We say that an $\mathcal{R}$-module is free if it is a direct sum of copies of $\mathcal{R}$ and $\mathcal{F}_\theta(\mathcal{R}_\theta)$.

Proposition 2.0.10. *[Rae19, Theorem 2.2.2] Free $\mathcal{R}$-modules are projective. In particular, $\mathcal{R}$ and $\mathcal{F}_\theta(\mathcal{R}_\theta)$ are projective. [Rae19]*

There is another presentation of $\mathcal{F}_\theta(R_\theta)$ which is more useful in our calculations. $\mathcal{F}_\theta(\mathcal{R}_\theta)\colon$

$$\begin{tikzcd} \mathcal{R}_\theta^l \arrow[loop left, "t"] \arrow[r, bend left, "p_*"] & \mathcal{R}_\theta \arrow[l, bend left, "p^*"] \end{tikzcd}$$

Here, we view $\mathcal{F}_\theta(R_\theta)_\theta$ as the free $\mathcal{R}_\theta$-module $\mathcal{R}_\theta\langle g, tg, \dots, t^{l-1}g\rangle$, where t acts by $t(ut^ig) = t(u)t^{i+1}g$, where $u \in R_\theta$ and $t(u) \in \mathcal{R}_\theta$. We identify $u \in \mathcal{R}_\theta = \mathcal{F}_\theta(\mathcal{R}_\theta)_\bullet$ with $p_*(ug)$. We define p_* as $\sum_{i=0}^{l-1} u_i t^i g \mapsto \sum_{i=0}^{l-1} t^{l-i}(u_i)p_*(g)$ and p^* as $p_*(ug) \mapsto \sum_{i=0}^{l-1} t^i(u)t^i g$. Here, $\mathcal{R}_\theta$ acts diagonally on $\mathcal{F}_\theta(R_\theta)$. We will temporarily call this Mackey functor $\mathcal{F}_\theta(R_\theta)^{conc}$ for sake of convenience.

Proposition 2.0.11. $\mathcal{F}_\theta(R_\theta) \cong \mathcal{F}_\theta(R_\theta)^{conc}$.

Proof. The isomorphism is $f\colon \mathcal{F}_\theta(R_\theta)_\theta \to \mathcal{F}_\theta(R_\theta)_\theta^{conc}$, $(a_0, \dots, a_{l-1}) \mapsto \sum_{i=0}^{l-1}(t^i a_i)t^i g$ and $u \mapsto p_*(ug)$. $\square$

Definition 2.0.12. *[Rae19] Let $\mathcal{R}$ be a commutative Mackey ring and $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{R}$-modules. The internal Hom object in the category of $\mathcal{R}$-modules is $\underline{Hom}_{\mathcal{R}}(\mathcal{M}, \mathcal{N})$ which is the Mackey functor*

$$\begin{tikzcd}
\mathit{Hom}_{\mathcal{R}}(\mathcal{F}_\theta(\mathcal{R})\Box\mathcal{M},\mathcal{N}) \arrow[loop left, "t"] \arrow[r, bend left, "p_*"] & \mathit{Hom}_{\mathcal{R}}(\mathcal{M},\mathcal{N}). \arrow[l, bend left, "p^*"]
\end{tikzcd}$$

The map t is induced by $s\Box id_{\mathcal{M}}$, where s is the map $s\colon \mathcal{F}_\theta \to \mathcal{F}_\theta$, $g \mapsto tg$. The map p_* is induced by $r_*\Box id_{\mathcal{M}}$, where r_* is the map $r_*\colon \mathcal{R} \to \mathcal{F}_\theta(R_\theta)$, $1_\bullet \mapsto p_*(g)$. The map p^* is induced by $r^*\Box id_{\mathcal{M}}$, where r^* is the map $r^*\colon \mathcal{F}_\theta(\mathcal{R}_\theta) \to \mathcal{R}$, $g \mapsto p^*(1_\bullet)$.

Definition 2.0.13. *Let $\mathcal{R}$ be a Mackey ring and $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{R}$-modules. The internal Ext object in the category of $\mathcal{R}$-modules is $\underline{Ext}^*_{\mathcal{R}}(\mathcal{M},\mathcal{N})$ which is the Mackey functor*

$$\begin{tikzcd}
\mathit{Ext}_{\mathcal{R}}(\mathcal{F}_\theta(\mathcal{R})\Box\mathcal{M},\mathcal{N}) \arrow[loop left] \arrow[r, bend left] & \mathit{Ext}_{\mathcal{R}}(\mathcal{M},\mathcal{N}) \arrow[l, bend left]
\end{tikzcd}$$

Our main computational interest in this thesis is the above Ext object, specifically $\underline{\mathrm{Ext}}^*_{\underline{k}[x_\theta]}(\underline{k},\underline{k})_\theta$. It should be noted that $\underline{\mathrm{Ext}}^*_{\mathcal{R}}(\mathcal{M},\mathcal{N})_\bullet = \mathrm{Ext}^*_{\mathcal{R}}(\mathcal{M},\mathcal{N})$ for a Mackey ring $\mathcal{R}$ and $\mathcal{R}$-modules $\mathcal{M}$ and $\mathcal{N}$.

Finally, it will be important for us to understand box products of free $\mathcal{R}$-modules. Since $\mathcal{F}_\bullet(\mathcal{R})$ is the unit for $\Box_{\mathcal{R}}$, we only need to determine $\mathcal{F}_\theta(\mathcal{R}_\theta)\Box_{\mathcal{R}}\mathcal{F}_\theta(\mathcal{R}_\theta)$. This is $\mathcal{R}\Box\mathcal{F}_\theta(\mathbb{Z})\Box\mathcal{F}_\theta(\mathbb{Z})$, and so is determined by the following result.

Lemma 2.0.14. *[Rae19] Let C_l be a finite cyclic group of prime order and let $\mathcal{F}_\theta(\mathbb{Z})$ be the free Mackey functor on the θ-side in the category of Mackey functors over C_l. Then $\mathcal{F}_\theta(\mathbb{Z})\Box\mathcal{F}_\theta(\mathbb{Z}) \cong \bigoplus_{i=1}^{l}\mathcal{F}_\theta(\mathbb{Z})$, where the map is $g_i \mapsto g\Box t^i g$.*

This last lemma is useful in our computations of $\underline{\mathrm{Ext}}^*_{\underline{k}[x_\theta]}(\underline{k},\underline{k})_\theta$.

CHAPTER III

CONSTRUCTION AND PROPERTIES OF POLYNOMIAL MACKEY RINGS.

We will be working in the category of Mackey functors over the group C_l, the cyclic group of order l for a fixed prime l. Moreover, we will fix a field k of characteristic q.

Definition 3.0.1. *For $l = 2$, define $\underline{k}[x_\theta]$ to be the Mackey functor with*

$$\underline{k}[x_\theta]_\theta = k[x, y], \qquad \underline{k}[x_\theta]_\bullet = k[a_0, a_1, \ldots, b_1, b_2, \ldots]/I$$

where I is the ideal $(a_n a_m - a_0 a_{n+m}, a_n b_m - 2a_{m+n}, b_n b_m - 2b_{n+m})$. The ring maps t and p^* are defined by

$$t\colon x \mapsto y, \qquad p^*\colon a_n \mapsto (x+y)(xy)^n, \qquad p^*\colon b_n \mapsto 2(xy)^n.$$

The map p_* is a k-linear map defined by the relation $p_* \circ t = p_*$ and

$$p_*\colon x^{n+1}y^n \mapsto a_n, \qquad p_*\colon x^n y^n \mapsto b_n.$$

The value of p_* is extended to other monomials by induction

$$p_*(x^{n+m}y^n) = a_0 p_*(x^{n+m-1}y^n) - p_*(x^{n+m-1}y^{n+1})$$

and similarly for $p_*(x^n y^{n+m})$.

For $l = 2$ we give a very concrete definition of $\underline{k}[x_\theta]$, but for $\mathrm{char}(k) \neq l$ we give a less explicit definition.

Definition 3.0.2. *For $l > 2$ and char$(k) \neq l$, define $\underline{k}[x_\theta]$ to be the Mackey functor with*

$$\underline{k}[x_\theta]_\theta = k[x_0, \ldots, x_{l-1}] \qquad \underline{k}[x_\theta]_\bullet = k[x_0, \ldots, x_{l-1}]^{C_l}.$$

The ring maps t and p^* are defined by

$$t\colon x_i \mapsto x_{i+1 \mod (l)}, \qquad p^*\colon f(x_0, \ldots, x_{l-1}) \mapsto \sum_{i=0}^{l-1} t^i f(x_0, \ldots, x_{l-1}).$$

The map p_* is defined by $f(x_0, \ldots, x_{l-1}) \mapsto \frac{1}{l}\sum_{i=0}^{l-1} t^i f(x_0, \ldots, x_{l-1})$, which is well defined because $\sum_{i=0}^{l-1} t^i f(x_0, \ldots, x_{l-1})$ is C_l-invariant.

We will prove in this section that when $l = 2$ and char$(k) \neq 2$ these definitions agree. For the rest of this chapter we will assume $l = 2$. We will return to the case $l > 2$ in the next chapter.

Proposition 3.0.3. *$\underline{k}[x_\theta]$ is a Mackey algebra over $\underline{k}$.*

Proof. We begin by checking that $\underline{k}[x_\theta]$ is a well-defined Mackey ring. Note that the relations $p_* \circ t = p_*$ and $t \circ p^* = p^*$ follow by definition of p_* and p^*. We begin by showing that $p^* \circ p_* = \mathrm{id} + t$.

We show that $p^* \circ p_* = \mathrm{id} + t$ by induction. First, note that

$$p^*(p_*(x^n y^n)) = p^*(b_n) = 2x^n y^n = x^n y^n + x^n y^n = (\mathrm{id} + t)(x^n y^n), \text{ and}$$

$$p^*(p_*(x^{n+1} y^n)) = p^*(a_n) = x^{n+1} y^n + x^n y^{n+1} = (\mathrm{id} + t)(x^{n+1} y^n).$$

Now, suppose that $p^*(p_*(x^{n+i}y^n)) = (\mathrm{id} + t)(x^{n+i}y^n)$ for all $i \leq m$. Then

$$\begin{aligned}
p^*(p_*(x^{n+m+1}y^n)) &= a_0 p_*(x^{n+m}y^n) - p_*(x^{n+m}y^{n+1}) \\
&= p^*(p_*(x^{n+m}y^n)p_*(x) - p_*(x^{n+m}y^{n+1})) \\
&= p^*(p_*(x^{n+m}y^n))p^*(p_*(x)) - p^*(p_*(x^{n+m}y^{n+1})) \\
&= (\mathrm{id} + t)(x^{n+m}y^n)(\mathrm{id} + t)(x) - (\mathrm{id} + t)(x^{n+m}y^{n+1}) \\
&= (x^{n+m}y^n + x^n y^{n+m})(x + y) - (x^{n+m}y^{n+1} + x^{n+1}y^{n+m}) \\
&= x^{n+m+1}y^n + x^{n+1}y^{n+m} + x^{n+m}y^{n+1} + x^n y^{n+m+1} \\
&\qquad - (x^{n+m}y^{n+1} + x^{n+1}y^{n+m}) \\
&= x^{n+m+1}y^n + x^n y^{n+m+1} = (\mathrm{id} + t)(x^{n+m+1}y^n)
\end{aligned}$$

as desired. By induction, $p^* \circ p_* = \mathrm{id} + t$ on all of $k[x, y]$.

Next, we need to show that $p_*(u)v = p_*(up^*(v))$ for any $u \in k[x, y]$ and $v \in k[a_0, \ldots, b_1, \ldots]/I$. We will first show that this relation holds when u is any monomial $x^{n+k}y^n \in k[x, y]$ and v is a generator a_n or b_n in $k[a_0, \ldots, b_1, \ldots]/I$.

We check the four following base cases for monomials $x^{n+1}y^n$ and x^ny^n:

Case 1 :

$$\begin{aligned} p_*(x^{n+1}y^np^*(a_m)) &= p_*(x^{n+1}x^n(x^{m+1}y^m + x^my^{m+1})) \\ &= p_*(x^{n+m+2}y^{n+m} + x^{n+m+1}y^{n+m+1}) \\ &= p_*(x^{n+m+1}y^{n+m})p_*(x) - p_*(x^{n+m+1}y^{n+m+1}) \\ &\qquad + p_*(x^{n+m+1}y^{n+m+1}) \\ &= a_{n+m}a_0 = a_na_m = p_*(x^{n+1}y^n)a_m. \end{aligned}$$

Case 2 :

$$\begin{aligned} p_*(x^{n+1}y^np^*(b_m)) &= p_*(x^{n+1}y^n(2x^my^m)) = 2p_*(x^{n+m+1}y^{n+m}) \\ &= 2a_{n+m} = a_nb_m = p_*(x^{n+1}y^n)b_m. \end{aligned}$$

Case 3 :

$$\begin{aligned} p_*(x^ny^np^*(a_m)) &= p_*(x^ny^n(x^{m+1}y^m + x^my^{m+1})) \\ &= p_*(x^{n+m+1}y^{n+m} + x^{n+m}y^{n+m+1}) \\ &= p_*((\mathrm{id} + t)(x^{n+m+1}y^{n+m})) = p_*(2x^{n+m+1}y^{n+m}) \\ &= 2p_*(x^{n+m+1}x^{n+m}) = 2a_{n+m} = b_na_m \\ &= p_*(x^ny^n)a_m. \end{aligned}$$

Case 4 :

$$\begin{aligned} p_*(x^n y^n p^*(b_m)) &= p_*(x^n y^n (2x^m y^m)) \\ &= 2p_*(x^{n+m} y^{n+m}) = 2b_{n+m} = b_n b_m \\ &= p_*(x^n y^n) b_m. \end{aligned}$$

Next, we show that $p_*(x^{n+m}y^n)a_0 = p_*(x^{n+m}y^n p^*(a_0))$ by the definition $p_*(x^{n+m}y^n) = p_*(x^{n+m-1}y^n)a_0 - p_*(x^{n+m-1}y^{n+1})$:

$$\begin{aligned} p_*(x^{n+m}y^n)a_0 &= p_*(x^{n+m+1}y^n) + p_*(x^{n+m}y^{n+1}) = p_*(x^{n+m}y^n(x+y)) \\ &= p_*(x^{n+m}y^n p^*(a_0)). \end{aligned}$$

Therefore, since $p_* \circ t = p_*$ and p_* is k-linear by definition, we have $p_*(u)a_0 = p_*(up^*(a_0))$ for any $u \in R_\theta$.

Now, we want to show that $p_*(u)c = p_*(up^*(c))$ for $c = a_i, b_i$. To do so, we induct on m. Let c be any of the generators a_j or b_j for $\underline{k}[x_\theta]_\bullet$. Suppose that $p_*(x^{n+i}y^n)c = p_*(x^{n+i}y^n p^*(c))$ for all $i < m$. Then we have

$$\begin{aligned} p_*(x^{n+m}y^n)c &= (p_*(x^{n+m-1}y^n)a_0 - p_*(x^{n+m-1}y^{n+1}))c \\ &= p_*(x^{n+m-1}y^n p^*(c))a_0 - p_*(x^{n+m-1}y^{n+1}p^*(c)) \\ &= p_*(x^{n+m-1}y^n p^*(c)p^*(a_0)) - p_*(x^{n+m-1}m^{n+1}p^*(c)) \\ &= p_*(x^{n+m-1}y^n(x+y)p^*(c)) - p_*(x^{n+m-1}m^{n+1}p^*(c)) \\ &= p_*(x^{n+m}y^n p^*(c) + x^{n+m-1}y^{n+1}p^*(c)) - p_*(x^{n+m-1}y^{n+1}p^*(c)) \\ &= p_*(x^{n+m}y^n p^*(c)). \end{aligned}$$

Therefore, by induction we have $p_*(x^{n+m}y^n)c = p_*(x^{n+m}y^n p^*(c))$ for all n, m, and where c is any of the generators a_j or b_j.

Next, we show that $p_*(x^{n+m}y^n)c = p_*(x^{n+m}y^n p^*(c))$ where c is a_ia_j, a_ib_j, or b_ib_j. We can use the relations $a_ib_j = 2a_{i+j}$ and $b_ib_j = 2b_{i+j}$ to reduce to the previous case where c is a single generator and the case $c = a_ia_j$ is as follows:

$$\begin{aligned}
p_*(x^{n+m}y^n)a_ia_j &= p_*(x^{n+m}y^n p^*(a_i))a_j = p_*(x^{n+m}y^n(x+y)x^iy^i)a_j \\
&= p_*(x^{i+n+m+1}y^{i+n} + x^{i+n+m}y^{i+n+1})a_j \\
&= p_*(x^{i+n+m+1}y^{i+n})a_j + p_*(x^{i+n+m}y^{i+n+1})a_j \\
&= p_*(x^{i+n+m+1}y^{i+n}p^*(a_j)) + p_*(x^{i+n+m}y^{i+n+1}p^*(a_j)) \\
&= p_*((x^{i+n+m+1}y^{i+n} + x^{i+n+m}y^{i+n})p^*(a_j)) \\
&= p_*(x^{n+m}y^n(x+y)x^iy^ip^*(a_j)) = p_*(x^{n+m}y^np^*(a_i)p^*(a_j)) \\
&= p_*(x^{n+m}y^np^*(a_ia_j)).
\end{aligned}$$

Since $p_*(x^{n+m}y^n)c = p_*(x^{n+m}y^np^*(c))$ for any n, m, and any product $c = a_ia_j, a_ib_j$, or b_ib_j, then the relation $p_*(x^{n+m}y^n)c = p_*(x^{n+m}y^np^*(c))$ holds for any monomial c by induction. Since $p_* \circ t = p_*$ and p_* is a k-linear map we can conclude that $p_*(v)u = p_*(vp^*(u))$ for any $v \in R_\theta$ and any $u \in R_\bullet$.

These relations show that $\underline{k}[x_\theta]$ is a well-defined Mackey ring. That $\underline{k}[x_\theta]$ is a Mackey algebra over $\underline{k}$ follows from the map $\underline{k} \to \underline{k}[x_\theta]$. $\square$

Proposition 3.0.4. *Let $\mathcal{G}$ be a commutative Mackey algebra over $\underline{k}$ and $z \in \mathcal{G}_\theta$. Then there is a unique map of Mackey algebras $f \colon \underline{k}[x_\theta] \to \mathcal{G}$ such that $f_\theta \colon x \mapsto z$.*

Proof. By the universal property of k-algebras there is a unique map of k-algebras $f_\theta \colon k[x,y] \to \mathcal{G}_\theta$ such that $x \mapsto z$ and $y \mapsto tz$. Now, define $f_\bullet \colon \underline{k}[x_\theta]_\bullet \to \mathcal{G}_\bullet$ by

$$f_\bullet \colon a_n \mapsto p_*(z^{n+1}tz^n), \qquad f_\bullet \colon b_n \mapsto p_*(z^n tz^n).$$

It remains to check that $f_\bullet$ is well-defined and that all the squares commute.

To show that $f_\bullet$ is well-defined we must show that $f_\bullet \colon I \to 0$. In particular, we must show that

$$f_\bullet(a_n)f_\bullet(a_m) = f_\bullet(a_{n+m})f_\bullet(a_0),$$

$$f_\bullet(a_n)f_\bullet(b_m) = 2f_\bullet(a_{n+m}), \qquad \text{and}$$

$$f_\bullet(b_n)f_\bullet(b_m) = 2f_\bullet(b_{n+m}).$$

We begin by showing the first relation holds.

$$\begin{aligned}
f_\bullet(a_n)f_\bullet(a_m) &= p_*(z^{n+1}tz^n)p_*(z^{m+1}tz^m) \\
&= p_*(z^{n+1}tz^n p^*(z^{m+1}tz^m)) = p_*(z^{n+1}tz^n(z^{m+1}tz^m + z^m tz^{m+1})) \\
&= p_*(z^{n+m+2}tz^{n+m} + z^{n+m+1}tz^{n+m+1}) \\
&= p_*(z^{n+m+1}tz^{n+m})p_*(z) - p_*(z^{n+m+1}tz^{n+m+1}) + p_*(z^{n+m+1}tz^{n+m+1}) \\
&= p_*(z^{n+m+1}tz^{n+m})p_*(z) = f_\bullet(a_{n+m})f_\bullet(a_0).
\end{aligned}$$

Next, we show that the second relation holds.

$$\begin{aligned} f_\bullet(a_n)f_\bullet(b_m) &= p_*(z^{n+1}tz^n)p_*(z^mtz^m) \\ &= p_*(z^{n+1}tz^n p^*(p_*(z^mtz^m)) = p_*(z^{n+1}tz^n(2z^mtz^m)) \\ &= 2p_*(z^{n+m+1}tz^{n+m}) = 2f_\bullet(a_{n+m}). \end{aligned}$$

Finally, we show that the third relation holds.

$$\begin{aligned} f_\bullet(b_n)f_\bullet(b_m) &= p_*(z^ntz^n)p_*(z^mtz^m) \\ &= p_*(z^ntz^n p^*(p_*(z^mtz^m))) = p_*(z^ntz^n(2z^mtz^m)) \\ &= 2p_*(z^{n+m}tz^{n+m}) = 2f_\bullet(b_{n+m}). \end{aligned}$$

Therefore, $f_\bullet$ is a well-defined map of rings.

Next, we wish to show that the pair f_θ, $f_\bullet$ constitute a well-defined map of $\underline{k}$-Mackey algebras. This requires checking that the appropriate squares commute, namely that

$$p_* \circ f_\theta = f_\bullet \circ p_*,$$

$$f_\theta \circ p^* = p^* \circ f_\bullet,$$

$$t \circ f_\theta = f_\theta \circ t.$$

The latter two relations are all between ring maps, so it suffices to check that these hold on the ring generators. In particular, we see that $t \circ f_\theta = f_\theta \circ t$ because

$$t(f_\theta(x)) = tz = f_\theta(y) = t(f_\theta(y)) = t(tz) = z = f_\theta(x) = f_\theta(t(y)).$$

Similarly, we see that $f_\theta \circ p^* = p^* \circ f_\bullet$ because

$$\begin{aligned} f_\theta(p^*(a_n)) &= f_\theta(x^{n+1}y^n + x^n y^{n+1}) = z^{n+1}tz^n + z^n tz^{n+1} \\ &= (\mathrm{id} + t)(z^{n+1}tz^n) = p^*(p_*(z^{n+1}tz^n)) = p^*(f_\bullet(a_n)), \end{aligned}$$

and

$$\begin{aligned} f_\theta(p^*(b_n)) &= f_\theta(2x^n y^n) = 2z^n tz^n \\ &= (\mathrm{id} + t)(z^n tz^n) = p^*(p_*(z^n tz^n)) = p^*(f_\bullet(b_n)). \end{aligned}$$

We will next show that the last relation holds by induction. In particular, we will induct on m for monomials $x^{n+m}y^n$. First, note that by definition we have

$$f_\bullet(p_*(x^n y^n)) = f_\bullet(b_n) = p_*(z^n tz^n) = p_*(f_\theta(x^n y^n)),$$

$$f_\bullet(p_*(x^{n+1}y^n)) = f_\bullet(a_n) = p_*(z^{n+1}tz^n) = p_*(f_\theta(x^{n+1}y^n)).$$

Now, suppose that $f_\bullet \circ p_* = p_* \circ f_\theta$ for all monomials of the form $x^{n+i}y^n$ where $i < m$. Then we have

$$\begin{aligned}
f_\bullet(p_*(x^{n+m}y^n)) &= f_\bullet(p_*(x^{n+m-1}y^n)p_*(x) - p_*(x^{n+m-1}y^{n+1})) \\
&= f_\bullet(p_*(x^{n+m-1}y^n))f_\bullet(p_*(x)) - f_\bullet(p_*(x^{n+m-1}y^{n+1})) \\
&= p_*(f_\theta(x^{n+m-1}y^n))p_*(f_\theta(x)) - p_*(f_\theta(x^{n+m-1}y^{n+1})) \\
&= p_*(f_\theta(x^{n+m-1}y^n)p^*p_*(f_\theta(x)) - f_\theta(x^{n+m-1}y^{n+1})) \\
&= p_*(f_\theta(x^{n+m-1}y^n)(f_\theta(x) + tf_\theta(x)) - f_\theta(x^{n+m-1}y^{n+1}) \\
&= p_*(f_\theta(x^{n+m}y^n + x^{n+m-1}y^{n+1} - x^{n+m-1}y^{n+1})) \\
&= p_*(f_\theta(x^{n+m}y^n))
\end{aligned}$$

as desired. Therefore, we are guaranteed a map of $\underline{k}$-Mackey algebras sending $x \mapsto z$ for any $z \in \mathcal{G}_\theta$. It remains to see that f is uniquely determined by the choice of z. But this is clear from the definition of $\underline{k}[x_\theta]$. □

Proposition 3.0.5. *If* 2 *is a unit in* k*, then* $\underline{k}[x_\theta]_\bullet \cong k[z,w]$.

Proof. We will show the map of rings $f\colon \underline{k}[x_\theta]_\bullet \to k[z,w]$,

$$a_n \mapsto \frac{zw^n}{2^n}, \qquad b_n \mapsto \frac{w^n}{2^{n-1}}$$

is an isomorphism. The above formulas give a map $k[\underline{a},\underline{b}] \to k[z,w]$ and we need to check it sends the ideal $\mathcal{I} := (a_na_m - a_{n+m}a_0, a_nb_m - 2a_{n+m}, b_nb_m - 2b_{n+m})$ to 0. It suffices to check this on the generators a_n, b_m, which we do below:

$$\begin{aligned}
1. \quad \hat{f}(a_n a_m - a_{n+m} a_0) &= \left(\frac{zw^n}{2^n}\right)\left(\frac{zw^m}{2^m}\right) - \left(\frac{zw^{m+n}}{2^{m+n}}\right) z \\
&= \frac{z^2 w^{m+n}}{2^{m+n}} - \frac{z^2 w^{m+n}}{2^{m+n}} = 0. \\
2. \quad \hat{f}(a_n b_m - 2a_{n+m}) &= \left(\frac{zw^n}{2^n}\right)\left(\frac{w^m}{2^{m-1}}\right) - 2\left(\frac{zw^{m+n}}{2^{m+n}}\right) \\
&= \frac{zw^{m+n}}{2^{m+n-1}} - \frac{2zw^{m+n}}{2^{m+n}} = 0. \\
3. \quad \hat{f}(b_n b_m - 2b_{m+n}) &= \left(\frac{w^n}{2^{n-1}}\right)\left(\frac{w^m}{2^{m-1}}\right) - 2\left(\frac{w^{m+n}}{2^{m+n-1}}\right) \\
&= \frac{w^{m+n}}{2^{m+n-2}} - \frac{2w^{m+n}}{2^{m+n-1}} = 0.
\end{aligned}$$

Therefore, $\hat{f}$ sends $\mathcal{I} \to 0$, so $\hat{f}$ induces the map f described above.

Now, let $\iota\colon k[z,w] \to \underline{k}[x_\theta]_\bullet$ be the map $z \mapsto a_0$ and $w \mapsto b_1$. We will now show that f and ι are inverses. It is trivial that $f \circ \iota = \mathrm{id}$, and

$$\begin{aligned}
(\iota \circ f)(a_n) &= \iota\left(\frac{zw^n}{2^n}\right) = \frac{a_0 b_1^n}{2^n} = a_n \\
(\iota \circ f)(b_n) &= \iota\left(\frac{w^n}{2^{n-1}}\right) = \frac{b_1^n}{2^{n-1}} = b_n.
\end{aligned}$$

Therefore, $\iota \circ f = \mathrm{id}$ and $f \circ \iota = \mathrm{id}$, so f is an isomorphism of rings. □

Corollary 3.0.6. *When $l = 2$ and char$(k) \neq 2$, the definitions of 3.0.1 and 3.0.2 agree.*

Proposition 3.0.7. *Let $A = k[a_0, a_1, \dots]/(a_n a_m + a_0 a_{n+m})$ be the k-subalgebra of $\underline{k}[x_\theta]_\bullet$ generated by the a_i, and $B = k\langle b_m | m \geq 1\rangle$. Then $\underline{k}[x_\theta]_\bullet = A \oplus B$ as a $k[a_0]$-module. Also, A is a free $k[a_0]$-module on the basis $\{1, a_i | i > 0\}$ and*

$B \cong k[a_0]/(a_0)\langle b_m | m > 0\rangle$.

Proof. We can take the set $\{a_0^n a_m, b_i | n \geq 0, m, i > 0\}$ as a k-basis for $\underline{k}[x_\theta]_\bullet$, derived from the monomials in the a_n and the b_m using the relations $a_n a_m = a_0 a_{n+m}$ and $a_n b_m = 0$. The result follows routinely from this. $\square$

Proposition 3.0.8. *$\underline{k}[x_\theta]$ is generated by the elements $x^n y^n$ and $x^{n+1}y^n$ as a $k[a_0]$-module, and thus also as an $\underline{k}[x_\theta]_\bullet$-module.*

Proof. It suffices to show that every monomial in $k[x, y]$ is in the $k[a_0]$-span of $\{x^n y^n, x^{n+1}y^n | n \geq 0\}$. We will prove this by induction. First, notice that

$$x^{n+2}y^n = x^{n+1}y^n(x + y) - x^{n+1}y^{n+1} = x^{n+1}y^n p^*(a_0) - x^{n+1}y^{n+1}$$

and for $m \geq 2$

$$x^{n+m}y^n = x^{n+m-1}y^n(x + y) - x^{n+m-1}y^{n+1} = x^{n+m-1}y^n p^*(a_0) - x^{(n+1)+(m-2)}y^{n+1}.$$

Therefore, by induction on m we deduce that all monomials of the form $x^{n+m}y^n$ are in the $k[a_0]$-span of $\{x^{n+1}y^n, x^n y^y | n \geq 0\}$. We also have $x^n y^{n+1} = x^n y^n(x + y) - x^{n+1}y^n = x^n y^n p^*(a_0) - x^{n+1}y^n$, so for all $n \geq 0$ we have $x^n y^{n+1}$ is in the $k[a_0]$-span of $\{x^{m+1}y^m, x^m y^m | m \geq 0\}$. A similar argument to above shows that all $x^n y^{n+m}$ are also in the $k[a_0]$-span of $\{x^{n+1}y^n, x^n y^n | n \geq 0\}$. $\square$

Lemma 3.0.9. *$k[x_\theta]_\theta$ is free as a $k[a_0]$-module via p^*, on the basis $\{x^{n+1}y^n, x^n y^n \mid n \geq 0\}$.*

Proof. Note that this is indeed a generating set for $\underline{k}[x_\theta]_\theta$ even over $k[a_0]$, but it remains to find linear independence. It is a classical result that $k[x,y]$ is a free $k[x+y,xy]$-module with the basis $\{1,x\}$. Furthermore, since $k[x+y,xy] \cong k[x+y][xy]$ as k-algebras, then $k[x+y,xy] = k[x+y]\langle 1,(xy)^n\rangle$ as $k[x+y]$-modules. Therefore, $k[x_\theta]_\theta$ is free as a $k[a_0]$-module over the basis $\{1(xy)^n, x(xy)^n\}$. $\square$

In this section we have expanded on the free commutative k-algebra generated by one element on the θ-side. For good measure, we point out that one can also consider the free commutative k-algebra generated by one element on the $\bullet$-side. But this is much simpler:

Proposition 3.0.10. *Let $\underline{k}[x_\bullet]$ be the Mackey algebra over $\underline{k}$ with*

$$\underline{k}[x_\bullet]_\theta = \underline{k}[x_\bullet]_\bullet = k[x]$$

and maps

$$t = p^* = id_{k[x]}, \qquad p_* = 2.$$

Let $\mathcal{G}$ be a $\underline{k}$-algebra and $z \in \mathcal{G}_\bullet$. Then there is a unique map of $\underline{k}$-algebras $f\colon \underline{k}[x_\bullet] \to \mathcal{G}$ such that $f_\bullet(x) = z$.

Proof. Routine. $\square$

CHAPTER IV

$\underline{K}[X_\theta]$-MODULES WHEN $L^{-1} \in K$

We begin this section by fixing a prime order cyclic group C_l and a field k with $\mathrm{char}(k) \neq l$. Recall the definition of $\underline{k}[x_\theta]$ from Definition 3.0.2. We will now investigate $\underline{k}[x_\theta]$ and its modules in this setting. Under the assumption that l is invertible in k, it turns out that $\underline{k}[x_\theta]$ is a nicely behaved object essentially determined by everything on the θ-side. We will prove a result generalizing this to more general Mackey $\underline{k}$-algebras with $l^{-1} \in k$ which tells us that $\underline{k}[x_\theta]$-modules are determined by their underlying $\underline{k}[x_\theta]_\theta$-modules along with the action of C_l.

Lemma 4.0.1. *Let $\mathcal{R}$ be a Mackey $\underline{k}$-algebra and $\mathcal{M}$ be an $\mathcal{R}$-module. Let $j\colon \mathcal{M}_\theta^{C_l} \to \mathcal{M}_\theta$ be the inclusion map. The maps*

$$(1/l)p_* \circ j\colon \mathcal{M}_\theta^{C_l} \to \mathcal{M}_\bullet \qquad \textit{and } \tilde{p}^*\colon \mathcal{M}_\bullet \to \mathcal{M}_\theta^{C_l}$$

are inverse $\mathcal{R}_\bullet$-module maps, where $j \circ \tilde{p}^* = p^*$.

Proof. First, note that $(1/l)p^*(p_*(z)) = z$ for all $z \in \mathcal{M}_\theta^{C_l}$. This is because

$$(1/l)p^*(p_*(z)) = (1/l)\left(\sum t^n\right)(z) = (1/l)\sum_{C_l} z = z,$$

so $\tilde{p}^* \circ (1/l)p_* \circ j = \mathrm{id}_{\mathcal{M}_\theta^{C_l}}$. Finally, since $p_* \circ p^* = l\mathrm{id}_{\mathcal{M}_\bullet}$ and $\mathrm{im}\, p^* \subset \mathcal{M}_\theta^{C_l}$ then $p_* \circ j \circ \tilde{p}^* = l\mathrm{id}_{\mathcal{M}_\bullet}$ as well. □

Corollary 4.0.2. *If $\mathcal{M}$ is a $\underline{k}$-module then $\mathcal{M}$ is isomorphic to the $\underline{k}$-module*

$$\begin{tikzcd} \mathcal{M}_\theta \arrow[loop left] \arrow[r, bend left] & \mathcal{M}_\theta^{C_l} \arrow[l, bend left] \end{tikzcd}.$$

Note that Proposition 3.0.5 is a special case of Lemma 4.0.1 for $l = 2$.

Definition 4.0.3. *Let R be a commutative k-algebra with an action of C_l on R. Let $R[C_l]_{tw}$ be the ring which is $R\langle C_l\rangle$ as an R-module and*

$$t\left(\sum_{C_l} a_i t^i\right) = \sum_{C_l} t(a_i)t^{i+1}$$

for a generator $t \in C_l$. We call $R[C_l]_{tw}$ the "twisted group ring".

Note that an $R[C_l]_{tw}$-module M is the same as an R-module M together with an additive map $t\colon M \to M$ such that $t^l = \mathrm{id}_M$ and $t(rm) = t(r)t(m)$ for all $r \in R$ and $m \in M$.

Remark 4.0.4. For $t \in C_l$ and $r \in \mathcal{R}_\theta$ we have the basic relation $t \cdot r = t(r) \cdot t$ in $(\mathcal{R}[C_l]_{tw})_\theta$.

Theorem 4.0.5. *Let $\mathcal{R}$ be a $\underline{k}$-algebra as Mackey functors over C_l, where l is a prime and k is a field with $l^{-1} \in k$. There is an equivalence of categories between $\mathcal{R}$-Mod and $R_\theta[C_l]_{tw}$-mod.*

Proof. Let $\mathcal{R}$ be a $\underline{k}$ Mackey algebra. Let $\mathcal{U}$ be the forgetful functor from $\mathcal{R}$-mod to $R[C_l]_{tw}$-Mod sending $\mathcal{M}$ to $\mathcal{M}_\theta$ and a map f to f_θ, and let $\mathcal{G}$ be the functor from $R[C_l]_{tw}$-Mod to $\mathcal{R}$-mod such that $\mathcal{G}\colon M \mapsto \mathcal{M}$, where

$$\mathcal{M}_\theta = M, \qquad \mathcal{M}_\bullet = M^{C_l},$$

$$p_* = \sum t^m, \qquad \text{and } p^* = i \text{ is the canonical inclusion map.}$$

M_θ inherits an action of C_l by nature of being a $R[C_l]_{tw}$-module, via the action of $t \in R[C_l]_{tw}$ on M_θ. Let $\mathcal{G}(f)$ be the map of $\mathcal{R}$ Mackey modules with $\mathcal{G}(f)_\theta = f$ and $\mathcal{G}(f)_\bullet = \tilde{f}$, where $\tilde{f}$ is the unique map $\tilde{f}\colon M^{C_l} \to N^{C_l}$, $y \mapsto f(y)$. Note that this is well-defined since if $y \in M^{C_l}$ then $ty = y$ for all $t \in C_l$, hence $tf(y) = f(ty) = f(y)$, so $f(M^{C_l}) \subseteq N^{C_l}$. Since f is a map of $R_\theta[C_l]_{tw}$-modules, we have $f(rx) = rf(x)$ for any $x \in M_\theta$ and any $r \in R_\theta$ and $f(t^n(x)) = t^n f(x)$, hence $\mathcal{G}(f)_\theta$ is a map of R_θ-modules. Since R_θ is also an $R_\bullet$-module, this means that f, and hence $\tilde{f}$, is also a map of $R_\bullet$-modules. Therefore, $\mathcal{G}(f)$ is a map of $\mathcal{R}$-modules.

To show that $\mathcal{U}$ and $\mathcal{G}$ constitute an equivalence of categories we need to find a unit and counit, namely natural isomorphisms $\epsilon\colon \mathcal{UG} \to \mathbf{Id}_{\mathcal{R}-Mod}$ and $\eta\colon \mathcal{GU} \leftarrow \mathbf{Id}_{R[C_l]_{tw}-Mod}$. We can take ϵ to be the identity.

Next, consider the map $\eta = (\eta_{\mathcal{M}})$ where $\eta_{\mathcal{M}}$ is the map of Mackey $\mathcal{R}$ modules with $\eta_{\mathcal{M},\theta} = \mathrm{id}_{M_\theta}$ and $\eta_{\mathcal{M},\bullet} = p^*_{\mathcal{M}}$. This is well-defined since im $p^* \subseteq M^{C_l}$, so this makes sense. We next need to see that $\eta_{\mathcal{M}}$ is a map of $\mathcal{R}$-modules. It suffices to show that $\eta_{\mathcal{M}}$ is a map of Mackey functors such that η_θ is a map of R_θ-modules and $\eta_\bullet$ is a map of $R_\bullet$-modules. We can see that $\eta_{\mathcal{M}}$ is a map of Mackey functors by inspection and η_θ is a map of R_θ-modules and $\eta_\bullet$ is a map of $R_\bullet$ modules because $\mathcal{M}$ is an $\mathcal{R}$-module.

Finally, since $i_{\mathcal{N}_\theta^{C_l}}$ is injective and hence a monomorphism, we can conclude that

$$\eta_{\mathcal{N},\bullet} \circ f_\bullet = \tilde{p}^*_{\mathcal{N}} \circ f_\bullet = \mathcal{G}(f_\theta) \circ \tilde{p}^*_{\mathcal{M}} = \mathcal{G}(f_\theta) \circ \eta_{\mathcal{M},\bullet}.$$

Therefore, η is natural. Moreover, both $\eta_{\mathcal{N},\theta}$ and $\eta_{\mathcal{N},\bullet}$ are isomorphisms ($\eta_{\mathcal{N},\bullet}$ is an isomorphism since $p^*_{\mathcal{M}}$ surjects onto its image and is injective since $p_* \circ p^* = l \cdot \mathrm{id}_{\mathcal{M}_\bullet}$ is an isomorphism), so each $\eta_{\mathcal{M}}$ is an isomorphism and thus η is a natural isomorphism. Therefore, $\mathcal{G}$ and $\mathcal{U}$ constitute an equivalence of categories. □

Applying the above results to $\underline{k}[x_\theta]$ leads us to want to understand $\underline{k}[x_\theta]_\theta[C_l]_{tw}$. The following result calculates this ring:

Proposition 4.0.6. *The ring map*

$$(\underline{k}[x_\theta])_\theta[C_l]_{tw} \leftarrow k\langle x, t\rangle/(t^l = 1, x(t^n x t^{-n}) = (t^n x t^{-n})x)_{0\leq n\leq l-1}$$

defined by $x \mapsto x_1$*,* $t \mapsto t$*, is an isomorphism. Furthermore,* $(\underline{k}[x_\theta])_\theta[C_l]_{tw} \cong k[x_1, \ldots, x_l]\langle 1, t, \ldots, t^{l-1}\rangle$ *as a* $k[x_1, \ldots, x_l]$*-module.*

Proof. This can be seen by checking the vector space isomorphism. □

Let $\mathcal{G}\colon \mathcal{R}_\theta[C_l]_{tw} - \mathrm{Mod} \to \mathcal{R} - \mathrm{Mod}$ be the functor defined in the proof of Theorem 4.0.5. Recall $\mathcal{G}(M)_\theta = M$ and $\mathcal{G}(M)_\bullet = M^{C_l}$.

Corollary 4.0.7. *Let* $\mathcal{R}$ *be a Mackey* $\underline{k}$*-algebra and let* M *and* N *be* $R_\theta[C_l]_{tw}$*-modules. Then* $\mathcal{G}(M)\Box_{\mathcal{R}}\mathcal{G}(N) \cong \mathcal{G}(M \otimes_{R_\theta} N)$*.*

Proof. By Theorem 4.0.5 it is enough to check the isomorphism after applying $\mathcal{U}$ to both sides, and then it is obvious. □

Koszul Complex for Mackey functors over C_l

We now use the above machinery to investigate the homological algebra of the Mackey ring $\underline{k}[x_\theta]$. It turns out the there is a Mackey functor analog of the Koszul complex for k as a $k[x_1, \ldots, x_l]$-module which is exact as well. Constructing this relies on the equivalence of categories above.

Proposition 4.1.1. *$k[x_1, \ldots, x_l]$ is a projective $\underline{k}[x_\theta]_\theta[C_l]_{tw}$-module, where $rt^i \cdot a = rt^i(a)$ for $r, a \in k[x_1, \ldots, x_l]$.*

Proof. First, this definition of $k[x_1, \ldots, x_l]$ as a $\underline{k}[x_\theta]_\theta[C_l]_{tw}$-module is well-defined. To see this, notice that by construction $t^l = \mathrm{id}_{k[x_1,\ldots,x_l]}$, so

$$t^l \cdot a = t^l(a) = \mathrm{id}_{k[x_1,\ldots,x_l]}a = a = 1 \cdot a.$$

Moreover, for $r, a, b \in k[x_1, \ldots, x_l]$ we have

$$(rt^i) \cdot (ab) = rt^i(ab) = rt^i(a)t^i(b) = rt^i(a)(t^i \cdot b) = (rt^i(a)t^i) \cdot b = (rt^i \cdot a) \cdot b,$$

so $k[x_1, \ldots, x_l]$ is a well-defined $\underline{k}[x_\theta]_\theta[C_l]_{tw}$.

Now, consider the $\underline{k}[x_\theta]_\theta[C_l]_{tw}$-module map $f\colon \underline{k}[x_\theta]_\theta[C_l]_{tw} \to k[x_1, \ldots, x_l]$, where $f\colon 1 \mapsto 1$ and $t \mapsto 1$ and the $k[x_1, \ldots, x_l]$-module map $g\colon k[x_1, \ldots, x_l] \to \underline{k}[x_\theta]_\theta[C_l]_{tw}$, where $g\colon a \mapsto \frac{1}{l}\sum_{i=0}^{l-1} at^i$. Note that g is also a $\underline{k}[x_\theta]_\theta[C_l]_{tw}$-module map, since

$$g((at) \cdot b) = g(at(b)) = \frac{1}{l}\sum_{i=0}^{l-1} at(b)t^i = at\left(\frac{1}{l}\sum_{i=0}^{l-1} t^{i-1}\right) = atg(b)$$

for any $a, b \in k[x_1, \ldots, x_l]$. Moreover, g is a splitting for f, since

$$(f \circ g)(1) = f\left(\frac{1}{l}\sum_{i=0}^{l-1} t^i\right) = \frac{1}{l}\sum_{i=0}^{l-1} 1 = 1.$$

Therefore, $k[x_1, \ldots, x_l]$ is a summand of $\underline{k}[x_\theta]_\theta[C_l]_{tw}$ and hence is projective. □

Proposition 4.1.2. *Let* $N := k[x_1, \ldots, x_l]\langle e_1, \ldots, e_l\rangle$ *be the free* $k[x_1, \ldots, x_l]$*-module on the basis* $e_1, \ldots, e_l$*. Let*

$$P_\bullet := k[x_1, \ldots, x_l] \to \bigwedge^{l-1} N \to \cdots \to \bigwedge^{2} N \to \bigwedge^{1} N \to k[x_1, \ldots, x_l] \xrightarrow{\epsilon} k$$

be the usual Koszul complex for $k[x_1, \ldots, x_l]$ *with differential*

$d\colon e_{i_1} \wedge \cdots \wedge e_{i_n} \mapsto \sum_j (-1)^{j+1} x_{i_j} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_n}$*. This is also a projective resolution of* $k[x_\theta]_\theta[C_l]_{tw}$*-modules, where the* C_l *action is given by* $t(\alpha e_{i_1} \wedge \ldots e_{i_n}) = t(\alpha) e_{i_1+1} \wedge \cdots \wedge e_{i_n+1}$ *for* $t \in C_l$*,* $\alpha \in k[x_1, \ldots, x_l]$ *and* $e_{i+l} := e_i$*.*

Proof. By standard theory, the Koszul complex resolving k as a $k[x_1, \ldots, x_l]$-module is exact. The following calculation shows that $d(t\omega) = td(\omega)$.

$$\begin{aligned}
d(t(e_{i_1} \wedge \cdots \wedge e_{i_n})) &= d(e_{i_1+1} \wedge \cdots \wedge e_{i_l+1}) \\
&= \sum_j (-1)^j x_{i_j+1} e_{i_1+1} \wedge \cdots \wedge \widehat{e_{i_j+1}} \wedge \cdots \wedge e_{i_n+1} \\
&= \sum_j (-1)^j t(x_{i_j}) t(e_{i_1}) \wedge \cdots \wedge t(\widehat{e_{i_j}}) \wedge \cdots \wedge t(e_{i_n}) \\
&= t\left(\sum_j (-1)^j x_{i_j} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_n}\right) = td(e_{i_1} \wedge \cdots \wedge e_{i_n}).
\end{aligned}$$

It remains to see that the modules in the resolution are projective. For $0 < n < l$, the action of C_l on n-element subsets of $\{1, \ldots, l\}$ is free (since l is prime). Let $S = \{S_1, \ldots, S_l\}$ be a collection of n-element subsets of $\{1, \ldots, l\}$ which is fixed under the action of C_l. Denote $e_{S_i} := e_{i_1} \wedge \cdots \wedge e_{i_n}$, where $S_i = \{i_1, \ldots, i_n\}$, with

$i_1 < i_2 < \cdots < i_n$. Then we have that $k[x_1, \ldots, x_l]\langle e_{S_1}, ..., e_{S_l}\rangle$ is a direct summand of $\bigwedge^n N$ as a $k[x_1, \ldots, x_l]$-module. The other summands correspond to different choices of S. For each choice of S, $k[x_1, \ldots, x_l]\langle e_{S_1}, \ldots, e_{S_l}\rangle$ is also closed under the C_l action by construction, so it is moreover a $\underline{k}[x_\theta]_\theta[C_l]_{tw}$-submodule. In fact, it is a free $\underline{k}[x_\theta]_\theta[C_l]_{tw}$-module of rank 1 and hence $\bigwedge^n N \cong \bigoplus_{\binom{l}{n}/l} \underline{k}[x_\theta]_\theta[C_l]_{tw}$ as a $\underline{k}[x_\theta]_\theta[C_l]_{tw}$-module for $0 < n < l$. By 4.1.1, $k[x_1, \ldots, x_l]$ is a projective $\underline{k}[x_\theta]_\theta[C_l]_{tw}$-module as well. Therefore, the sequence $P_\bullet$ is also an exact sequence of projective $k[x_\theta][C_l]_{tw}$-modules.

□

Proposition 4.1.3. *Let $\mathcal{G}$ be the functor $(\underline{k}[x_\theta])_\theta[C_l]_{tw} - Mod \to \underline{k}[x_\theta] - \underline{Mod}$ in Theorem 4.0.5. Then $\mathcal{G}(k) \cong \underline{k}$, $\mathcal{G}(k[x_\theta][C_l]_{tw}) \cong \mathcal{F}_\theta(k[x_1, \ldots, x_l])$, and $\mathcal{G}(k[x_1, \ldots, x_l]) \cong \underline{k}[x_\theta]$.*

Proof. The isomorphisms $\mathcal{G}(k) \cong \underline{k}$, $\mathcal{G}(k[x_\theta][C_l]_{tw}) \cong \mathcal{F}_\theta(k[x_1, \ldots, x_l])$, and $\mathcal{G}(k[x_1, \ldots, x_l]) \cong \underline{k}[x_\theta]$ follow from the fact that $\underline{k}_\theta = k$ where $t = \text{id}$,

$\underline{k}[x_\theta]_\theta = k[x_1, \ldots, x_l]$ where t is the map $x_i \mapsto x_{i+1}$, and

$\mathcal{F}_\theta(k[x_1, \ldots, x_l]) \cong k[x_1, \ldots, x_l]\langle g, tg, \ldots, t^{l-1}g\rangle$ where t is the map $t^n g \mapsto t^{n+1}g$. □

Theorem 4.1.4. *$\underline{Ext}^*_{\underline{k}[x_\theta]}(\underline{k}, \underline{k})_\theta$ is isomorphic to the exterior k-algebra on l generators.*

Proof. Let $P_\bullet$ be the resolution of k as a $k[x_\theta][C_l]$-module from Proposition 4.1.2 (above). Then $P_\bullet$ is a projective resolution of k as a $k[x_\theta][C_l]$-module, so we have

$$\begin{aligned}
\underline{\mathrm{Ext}}^*_{\underline{k}[x_\theta]}(\underline{k},\underline{k})_\theta &= H^*(\mathrm{Hom}_{\underline{k}[x_\theta]}(\mathcal{F}_\theta(\underline{k}[x_\theta])\square_{\underline{k}[x_\theta]}\mathcal{G}(P_\bullet),\underline{k}))\\
&\cong H^*(\mathrm{Hom}_{\underline{k}[x_\theta]}(\mathcal{G}(k[x_\theta][C_l]_{tw})\square_{\underline{k}[x_\theta]}\mathcal{G}(P_\bullet),\underline{k}))\\
&\cong H^*(\mathrm{Hom}_{\underline{k}[x_\theta]}(\mathcal{G}(k[x_\theta][C_l]_{tw}\otimes_{k[x_1,\ldots,x_l]} P_\bullet),\underline{k}))\\
&\cong H^*(\mathrm{Hom}_{k[x_\theta][C_l]_{tw}}(k[x_\theta][C_l]_{tw}\otimes_{k[x_1,\ldots,x_l]} P_\bullet,k))\\
&\cong H^*(\mathrm{Hom}_{k[x_1,\ldots,x_l]}(P_\bullet,k))\cong \mathrm{Ext}^*_{k[x_1,\ldots,x_l]}(k,k).
\end{aligned}$$

Finally, by a classical result $\mathrm{Ext}^*_{k[x_1,\ldots,x_l]}(k,k)$ is an exterior k-algebra on l generators, giving the desired result. □

Remark 4.1.5.

$$\begin{aligned}
\underline{\mathrm{Ext}}^i_{\underline{k}[x_\theta]}(\underline{k},\underline{k})_\bullet &= H^i(\mathrm{Hom}_{\underline{k}[x_\theta]}(\mathcal{G}(P_\bullet),\underline{k})\\
&= H^i(\mathrm{Hom}_{(\underline{k}[x_\theta])_\theta[C_l]_{tw}}(P_\bullet,k))\\
&\cong \mathrm{Hom}_{(\underline{k}[x_\theta])_\theta[C_l]_{tw}}\left(\bigwedge^i N,k\right) \quad = \begin{cases} k & i=0,1,l-1,l\\ \bigoplus_{\binom{l}{i}/l} k & 2\le i\le l-2\end{cases}
\end{aligned}$$

The multiplication structure for $\underline{\mathrm{Ext}}^*_{(\underline{k}[x_\theta])_\theta[C_l]_{tw}}(\underline{k},\underline{k})_\bullet$ is complicated.

CHAPTER V

$\underline{K}[X_\theta]$-MODULES OVER C_2 WHEN $\mathrm{char}(K) = 2$

We now begin analyzing $\underline{k}[x_\theta]$-modules over the group C_2 when $\mathrm{char}(k) = 2$. Unlike the case when $l^{-1} \in k$, the case $\mathrm{char}(k) = l$ is much more difficult. We will explore this case for $l = \mathrm{char}(k) = 2$. In this section, we will exhibit a short exact sequence of $\underline{k}[x_\theta]$-modules which we stitch together to form a projective resolution of $\underline{k}$. This in turn gives us a construction for $\underline{\mathrm{Ext}}^*_{\mathcal{R}}(\underline{k}, \underline{k})$, the internal Ext object, from which we compute the additive and multiplicative structures.

One of the surprises in this case is that we need an infinite resolution of $\underline{k}$, though this resolution turns out to have a periodicity to it. We build this resolution by finding a four-term exact sequence ending in a submodule of $\mathcal{F}_\theta(\underline{k}[x_\theta])$. This submodule appears naturally as the kernel of a Koszul-like complex. We find that $\underline{\mathrm{Ext}}^*_{\underline{k}[x_\theta]}(\underline{k}, \underline{k})_\theta$ is nonzero in only finitely many degrees, though, while $\underline{\mathrm{Ext}}^*_{\underline{k}[x_\theta]}(\underline{k}, \underline{k})_\bullet$ is nonzero in infintely many degrees.

From now on, we will refer to $\underline{k}[x_\theta]$ as $\mathcal{R}$.

Constructing the free resolution of $\underline{k}$

Definition 5.1.1. *Let $\mathcal{M}$ be the $\mathcal{R}$-submodule of $\mathcal{F}_\theta(R_\theta)$ given by*

$$\mathcal{M}_\theta = R_\theta\langle g + tg\rangle$$

$$\mathcal{M}_\bullet = k\langle a_0^r p_*(x^n y^n g)\rangle_{n,r\geq 0}$$

The map p^* *is defined by* $p^*\colon p_*(x^n y^n g) \mapsto x^n y^n(g + tg)$. *The map* p_* *is defined by* $p_*\colon x^n y^n(g + tg) \mapsto 0$ *and* $p_*\colon x^{n+m} y^n(g + tg) \mapsto a_0^m p_*(x^n y^n g)$. *The map* t *is defined by* $t\colon x^n y^m(g + tg) \mapsto x^m y^n(g + tg)$.

Proposition 5.1.2. $\mathcal{M}$ *is a well-defined* $\mathcal{R}$*-submodule of* $\mathcal{F}_\theta(R_\theta)$.

Proof. It suffices to show that $\mathcal{M}$ is closed under the maps p_*, p^*, and t, since $t_{\mathcal{M}}$, $p_{*,\mathcal{M}}$, and $p^*_{\mathcal{M}}$ are restrictions of the corresponding maps for $\mathcal{F}_\theta(\mathcal{R}_\theta)$, and that $\mathcal{M}_\bullet$ and $\mathcal{M}_\theta$ are well-defined $\mathcal{R}_\bullet$ and $\mathcal{R}_\theta$-submodules of $\mathcal{F}_\theta(\mathcal{R}_\theta)_\bullet$ and $\mathcal{F}_\theta(\mathcal{R}_\theta)_\theta$, respectively. These are all routine verifications. □

Lemma 5.1.3. $0 \to \mathcal{M} \xrightarrow{\alpha} \mathcal{F}_\theta(\mathcal{R}_\theta) \xrightarrow{\beta} \mathcal{F}_\theta(\mathcal{R}_\theta)$ *is exact, where* α *is the inclusion map and* β *is determined by* $\beta_\theta\colon g \mapsto g + tg$.

Proof. First, notice that

$$\beta_\theta(g + tg) = (g + tg) + t(g + tg) = 0,$$

so $\beta_\theta \circ \alpha_\theta = 0$. Also notice that

$$\begin{aligned}\beta_\bullet(p_*(x^n y^n g)) &= p_*(x^n y^n \beta_\theta(g)) = p_*(x^n y^n(g + tg)) = p_*(x^n y^n g + x^n y^n tg)\\ &= p_*(x^n y^n g + t(x^n y^n g)) = p_*(2x^n y^n g) = 0,\end{aligned}$$

so $\beta_\bullet \circ \alpha_\bullet = 0$ as well.

It remains to find the kernel of β. First, notice that $\ker \beta_\theta = \langle g + tg \rangle \subseteq \operatorname{im} \alpha_\theta$. We wish to show that $\ker \beta_\bullet \subseteq \operatorname{im} \alpha_\bullet$. Since $\mathcal{F}_\theta(\mathbb{R}_\theta)_\bullet$ is spanned over $\mathcal{R}_\bullet$ by elements $p_*(x^n y^n g), p_*(x^{n+1} y^n g)$ for $n \geq 0$ and we know that $p_*(x^n y^n g) \in \ker \beta_\bullet$, it suffices to show that elements in $\mathcal{R}_\bullet \langle p_*(x^{n+1} y^n g) \rangle \cap \ker \beta_\bullet$ are zero.

To that end, first notice that

$$\begin{aligned} a_0^i a_m p_*(x^{n+1} y^n g) &= a_0^i p_*(p^*(a_m) x^{n+1} y^n g) = a_0^i p_*((x+y) x^{n+m+1} y^{n+m} g) \\ &= a_0^i p_*(p^*(a_0) x^{n+m+1} y^{n+m} g) = a_0^{i+1} p_*(x^{n+m+1} y^{n+m} g), \end{aligned}$$

and the b_i's kill $p_*(x^{n+1} y^n g)$, so we can write any arbitrary element of $R_\bullet \langle p_*(x^{n+1} y^n g) \rangle \subset \mathcal{F}_\theta(R_\theta)_\bullet$ as $\sum_j c_j a_0^{m_j} p_*(x^{n_j+1} y^{n_j} g)$ where each $c_j \in k$.

Then we have

$$\begin{aligned} 0 &= \beta_\bullet \left(\sum c_j a_0^{m_j} p_*(x^{n_j+1} y^{n_j} g) \right) = \sum c_j a_0^{m_j} p_*(x^{n_j+1} y^{n_j} (g + tg)) \\ &= \sum c_j a_0^{m_j} p_*(x^{n_j+1} y^{n_j} g + t(x^{n_j+1} y^{n_j} tg)) \\ &= \sum c_j a_0^{m_j} p_*(x^{n_j+1} y^{n_j} g + y^{n_j+1} x^{n_j} g) \\ &= \sum c_j a_0^{m_j} p_*((x+y) x^{n_j} y^{n_j} g) = \sum c_j a_0^{m_j} p_*(p^*(a_{n_j}) g) \\ &= \sum c_j a_0^{m_j} a_{n_j} p_*(g). \end{aligned}$$

Therefore, $\sum c_j a_0^{m_j} a_{n_j} = 0$, and since the a_n are linearly independent over $k[a_0]$ we can conclude that $\sum_{j|n_j=n} c_j a_0^{m_j} = 0$ for each choice of n. Therefore,

$$\begin{aligned}\sum_j c_j a_0^{m_j} p_*(x^{n_j+1}y^{n_j}g) &= \sum_n \sum_{j|n_j=n} c_j a_0^{m_j} p_*(x^{n+1}y^n g) = \sum_n \left(\sum_{j|n_j=n} c_j a_0^{m_j} \right) p_*(x^{n+1}y^n g) \\ &= \sum_n 0 p_*(x^{n+1}y^n g) = 0.\end{aligned}$$

Therefore, $\ker \beta_\bullet \subseteq \text{im } \alpha_\bullet$ and thus $\ker \beta = \text{im } \alpha$ as desired. □

Remark 5.1.4. Let $\mathcal{Q}$ be the cokernel of the inclusion $\iota\colon \mathcal{M} \to \mathcal{F}_\theta(R_\theta)$. The short exact sequence

$$0 \to \mathcal{M} \to \mathcal{F}_\theta(R_\theta) \xrightarrow{q} \mathcal{Q} \to 0$$

is not split.

Proof. Notice that both M_θ and Q_θ are fixed by t. Therefore, any splitting would imply that $\mathcal{F}_\theta(R_\theta)$ were also fixed by t, which is not the case.

□

Theorem 5.1.5. *There is an exact sequence of $\mathcal{R}$-modules:*

$$0 \to \bigoplus_{i\geq 1} \mathcal{M} \xrightarrow{\oplus\alpha_i} \bigoplus_{i\geq 1} \mathcal{F}_\theta(R_\theta) \xrightarrow{\oplus\beta_i} \bigoplus_{i\geq 1} \mathcal{F}_\theta(R_\theta) \xrightarrow{\gamma} \bigoplus_{i\geq 0} \mathcal{R} \xrightarrow{\delta} \mathcal{M} \to 0.$$

The map $\alpha_i\colon \mathcal{M} \to \mathcal{F}_\theta(R_\theta)$ is the inclusion map.

The map $\oplus\beta_i\colon \bigoplus_{i\geq 1} \mathcal{F}_\theta(R_\theta) \to \bigoplus_{i\geq 1} \mathcal{F}_\theta(R_\theta)$ is defined by $g_i \mapsto g_i + tg_i$.

The map $\gamma\colon \bigoplus_{i\geq 1} \mathcal{F}_\theta(R_\theta) \to \bigoplus_{i\geq 0} \mathcal{R}$ *is defined by* $g_i \mapsto x^i y^i 1_{\theta,0} + 1_{\theta,i}$.

The map $\delta\colon \bigoplus_{i\geq 0} \mathcal{R} \to \mathcal{M}$ *is defined by* $1_{\bullet,i} \mapsto p_*(x^i y^i g)$.

Proof. The exactness of the (α, β) spot follows from Lemma 5.1.3. We will now show that $\ker \gamma = \text{im } \oplus \beta_i$. To see that $\gamma_\theta \circ \oplus\beta_{i,\theta} = 0$, notice that

$$(\gamma_\theta \circ \oplus\beta_{i,\theta})(g_i) = \gamma_\theta(g_i + tg_i) = x^i y^i 1_{0,\theta} + 1_{i,\theta} + t(x^i y^i 1_{0,\theta} + 1_{i,\theta}) = 0.$$

This shows that $\text{im } \oplus \beta_i \subseteq \ker \gamma$.

Next, we will see that $\ker \gamma \subseteq \text{im } \oplus \beta_i$. Let $\sum_j u_j g_j + v_j t g_j \in \ker \gamma_\theta$, where $u_j, v_j \in R_\theta$. Let $\pi_i\colon \bigoplus_{i\geq 0} \mathcal{R}_\theta \to \mathcal{R}_\theta$ be the projection onto the ith summand. Then we have

$$(u_j + v_j)1_{i,\theta} = \pi_j\left(\gamma_\theta\left(\sum_j u_j g_j + v_j t g_j\right)\right) = \pi_i(0) = 0,$$

so $u_j + v_j = 0$ since the annihilator of $1_{i,\theta}$ is 0 for all i. Therefore,

$$\begin{aligned}\sum_j u_j g_j + v_j t g_j &= \sum_j u_j g_j + u_j t g_j = \sum_j u_j(g_j + tg_j) = \sum_j \beta_{j,\theta}(u_j g_j)\\ &= \beta_\theta\left(\sum_j u_j g_j\right) \in \text{im } \beta_\theta.\end{aligned}$$

Therefore, $\ker \gamma_\theta = (g_i + tg_i)_{i\geq 1} = \text{im } \oplus \beta_\theta$ as desired.

Next, we investigate $\ker \gamma_\bullet$. Let $A = k[a_0, a_1, \ldots]/(a_n a_m + a_0 a_{n+m})_{n,m}$ be the

k-subalgebra of $\mathcal{R}_\bullet$ generated by the a_i, and $B = k\langle b_m | m > 0\rangle$. Notice that $\mathcal{R}_\bullet = A \oplus B$ as a $k[a_0]$-module. Furthermore, since

$$\begin{aligned}\gamma_\bullet(p_*(x^{n+1}y^n g_i)) &= p_*(x^{n+i+1}y^{n+i}1_{0,\theta} + x^{n+1}y^n 1_{i,\theta}) \\ &= a_{n+i}1_{0,\bullet} + a_n 1_{i,\bullet} \in \bigoplus A \subset \bigoplus \mathcal{R}_\bullet\end{aligned}$$

and

$$\begin{aligned}\gamma_\bullet(p_*(x^n y^n g_i)) &= p_*(x^{n+i}y^{n+i}1_{0,\theta} + x^n y^n 1_{i,\theta}) \\ &= b_{n+i}1_{0,\bullet} + b_n 1_{i,\bullet} \in \bigoplus B \subset \bigoplus \mathcal{R}_\bullet\end{aligned}$$

we can see that $\ker\gamma_\bullet = \ker(\oplus\pi_A \circ \gamma_\bullet) \oplus \ker(\oplus\pi_B \circ \gamma_\bullet)$, where π_A and π_B are the projections of $k[a_0]$-modules $\mathcal{R}_\bullet \to \mathcal{R}_\bullet/B \cong A$ and $\mathcal{R}_\bullet \to \mathcal{R}_\bullet/A \cong B$.

First, note that $\gamma_\bullet(p_*(x^n y^n g_i)) = b_{n+i}1_{0,\bullet} + b_n 1_{i,\bullet} \neq 0$ for all n and i, so $p_*(x^n y^n g_i) \notin \ker\gamma_\bullet$ for all n and i. However,

$$\gamma_\bullet(a_m p_*(x^n y^n g_i)) = a_m b_{n+i}1_{0,\bullet} + a_m b_n 1_{i,\bullet} = 01_{0,\bullet} + 01_{i,\bullet} = 0,$$

so $a_m p_*(x^n y^n g_i) \in \ker\gamma_\bullet$ for all n, m, and i. Since each b_m annihilates $\mathcal{F}_\theta(R_\theta)_\bullet = R_\theta$, it remains to determine which sums of the form $\sum c_{i,m}p_*(x^n y^n g_i)$ are in $\ker\gamma_\bullet$. Let $\pi_{i,n}$ be the $k[a_0]$-module projection $\bigoplus \mathcal{R}_\bullet \to B$, which picks out the $k[a_0]$ component of $\bigoplus \mathcal{R}_\bullet$ spanned by $b_n 1_{i,\bullet}$. Then we can see that

$$c_{n,i}b_n 1_{i,\bullet} = \pi_{n,i}\left(\gamma_\bullet\left(\sum c_{n,i}p_*(x^n y^n g_i)\right)\right) = \pi_{n,i}(0) = 0$$

therefore $c_{n,i} = 0$ for each n and i. Thus $\sum c_{n,i} p_*(x^n y^n g_i) = 0$. Therefore, $\ker(\oplus \pi_B \circ \gamma_\bullet) = \langle a_m p_*(x^n y^n g_i) | m, n, i \geq 0 \rangle$.

Next, we classify which elements are in $\ker(\oplus \pi_A \circ \gamma_\bullet)$. Let $\sum_j c_j a_0^{m_j} p_*(x^{n_j+1} y^{n_j} g_{i_j}) \in \ker \gamma_\bullet$ with $c_j \in k$. Let $\varpi_{n,i}$ be the $k[a_0]$-module projection $\bigoplus \mathcal{R}_\bullet \to k[a_0]\langle a_n 1_{i,\bullet} \rangle$. Then we have

$$\begin{aligned}\sum_{j | n_j = n, i_j = i} c_j a_0^{m_j} a_n 1_{i,\bullet} &= \varpi_{n,i} \left(\gamma_\bullet \left(\sum c_j a_0^{m_j} p_*(x^{n_j+1} y^{n_j} g_{i_j}) \right) \right) \\ &= \varpi_{n,i}(0) = 0,\end{aligned}$$

so $\sum_{j | n_j = n, i_j = i} c_j a_0^{m_j} = 0$. Therefore,

$$\begin{aligned}\sum c_j a_0^{m_j} p_*(x^{n_j+1} y^{n_j} g_i) &= \sum_n \sum_i \sum_{j | n_j = n, i_j = i} c_j a_0^{m_j} p_*(x^{n+1} y^n g_i) \\ &= \sum_n \sum_i \left(\sum_{j | n_j = n, i_j = i} c_j a_0^{m_j} \right) p_*(x^{n+1} y^n g_i) \\ &= \sum_n \sum_i 0 = 0,\end{aligned}$$

thus $\ker(\oplus \pi_A \circ \gamma_\bullet) = 0$. Thus $\ker \gamma_\bullet = R_\bullet \langle a_m p_*(x^n y^n g_i) | i, n, m \geq 0 \rangle$.

Lastly, we can see that $R_\bullet \langle a_m p_*(x^n y^n g_i) | i, n, m \geq 0 \rangle \subseteq \text{im } \beta_\bullet$ since

$$\begin{aligned}a_0 p_*(x^n y^n g_i) &= p_*((x + y) x^n y^n g_i) = p_*(x^{n+1} y^n g_i + x^n y^{n+1} g_i) \\ &= p_*(x^{n+1} y^n g_i + x^{n+1} y^n t g_i) = p_*(x^{n+1} y^n (g_i + t g_i)) \\ &= \oplus \beta_{i,\bullet}(p_*(x^{n+1} y^n g_i)).\end{aligned}$$

Therefore, $\ker \gamma_\bullet = \text{im } \beta_\bullet$ and thus $\ker \gamma = \text{im } \beta$ as desired.

Finally, we will show that im $\gamma = \ker \delta$. First, notice that

$$\begin{aligned}(\delta_\theta \circ \gamma_\theta)(g_i) &= \delta_\theta(x^i y^i 1_{0,\theta} + 1_{i,\theta}) = x^i y^i \delta_\theta(1_{0,\theta}) + \delta_\theta(1_{i,\theta}) \\ &= x^i y^i p^*(p_*(g)) + p^*(p_*(x^i y^i g)) = x^i y^i (g + tg) + (x^i y^i g + t(x^i y^i g)) = 0.\end{aligned}$$

Therefore, $\delta \circ \gamma = 0$.

It remains to see that $\ker \delta \subseteq$ im γ. First, note that since b_m annihilates $\mathcal{F}_\theta(R_\theta)_\bullet$ and hence it annihilates $\mathcal{M}_\bullet$ as well, $b_m 1_{i,\bullet} \in \ker \delta$ for all i, m. Let $\sum_j c_j a_0^{n_j} a_{m_j} 1_{i_j,\bullet} \in \ker \delta_\bullet$ where $c_j \in k$. Then

$$\begin{aligned}0 = \delta_\bullet \left(a_0^{n_j} a_{m_j} 1_{i_j,\bullet}\right) &= \sum c_j a_0^{n_j} a_{m_j} p_*(x^{i_j} y^{i_j} g) = \sum c_j a_0^{n_j} p_*(p^*(a_{m_j}) x^{i_j} y^{i_j} g) \\ &= \sum c_j a_0^{n_j} p_*((x+y) x^{i_j+m_j} y^{i_j+m_j} g) = \sum c_j a_0^{n_j} p_*(p^*(a_{i_j+m_j}) g) \\ &= \sum c_j a_0^{n_j} a_{i_j+m_j} p_*(g).\end{aligned}$$

Therefore, we conclude that $\sum c_j a_0^{n_j} a_{i_j+m_j} = 0$. Since the elements a_n are linearly independent over $k[a_0]$, this means that $\sum_{j|i_j+m_j=N} c_j a_0^{n_j} = 0$. Therefore,

$$\begin{aligned}\sum_j c_j a_0^{n_j} a_{m_j} 1_{i_j,\bullet} &= \sum_N \sum_{j|i_j+m_j=N} c_j a_0^{n_j} a_{m_j} 1_{N-i_j,\bullet} = \sum_N \left(\sum_{j|i_j+m_j=N} c_j a_0^{n_j} \right) a_N p_*(g) \\ &= \sum_N 0 a_N p_*(g) = 0.\end{aligned}$$

Therefore, $\ker \delta_\bullet = \langle b_m 1_{n,\bullet} | n \geq 0, m \geq 1 \rangle$. Lastly, note that

$$\begin{aligned}
&\gamma_\bullet(p_*(g_{m+n}) + p_*(x^m y^m g_n)) \\
&\qquad\qquad = p_*(x^{m+n}y^{m+n}1_{0,\theta} + 1_{m+n,\theta}) + p_*(x^{m+n}y^{m+n}1_{0,\theta} + x^m y^m 1_{n,\theta}) \\
&\qquad\qquad = b_m 1_{n,\bullet},
\end{aligned}$$

so $\ker \delta_\bullet \subseteq \text{im } \gamma_\bullet$ as desired.

We now finally show that $\ker \delta_\theta \subseteq \text{im } \gamma_\theta$. Let $\sum c_i 1_{i,\theta} \in \ker \delta$. Then we have

$$0 = \delta\left(\sum c_i 1_{i,\theta}\right) = \sum c_i x^i y^i (g + tg) = \left(\sum c_i x^i y^i\right)(g + tg)$$

hence $\sum c_i x^i y^i = 0$. Therefore, we have

$$\sum c_i 1_{i,\theta} = \sum c_i 1_{i,\theta} + \left(\sum c_i x^i y^i\right) 1_{0,\theta} = \sum c_i \left(1_{i,\theta} + x^i y^i 1_{0,\theta}\right),$$

so $\ker \delta \subseteq \langle 1_{i,\theta} + x^i y^i 1_{0,\theta} \rangle$. Importantly,

$$\begin{aligned}
x^i y^i 1_{\theta,j} + 1_{\theta,i+j} &= x^i y^i 1_{\theta,j} + x^{i+j} x^{i+j} 1_{\theta,0} + x^{i+j} y^{i+j} 1_{\theta,0} + 1_{\theta,i+j} \\
&= x^i y^i (1_{\theta,j} + x^j y^j 1_{\theta,0}) + x^{i+j} y^{i+j} 1_{\theta,0} + 1_{\theta,i+j} \\
&= x^i y^i \gamma_\theta(g_j) + \gamma_\theta(g_{i+j}) = \gamma_\theta(x^i y^i g_j + g_{i+j}).
\end{aligned}$$

Therefore, $\ker \delta_\theta = \text{im } \gamma_\theta$ and thus $\ker \delta = \text{im } \gamma$ as desired. $\square$

Proposition 5.1.6. *There is an infinite length free resolution $\mathcal{P}_* \to \underline{k}$ of the form*

$$\mathcal{P}_* = \ldots \xrightarrow{f_4} \bigoplus_{i\geq 0} \mathcal{F}_\theta(R_\theta) \xrightarrow{f_3} \bigoplus_{i\geq 0} \mathcal{R} \xrightarrow{f_2} \mathcal{F}_\theta(R_\theta) \xrightarrow{f_1} \mathcal{F}_\theta(R_\theta) \xrightarrow{f_0} \mathcal{R} \xrightarrow{\epsilon} \underline{k} \to 0.$$

For $n \geq 1$ the modules in the resolution are

$$\mathcal{P}_{3n} = \bigoplus_{i_1,\ldots,i_{n-1}\geq 1, \quad i_n\geq 0} \mathcal{R}, \mathcal{P}_{3n+1} = \bigoplus_{i_1,\ldots,i_n\geq 1} \mathcal{F}_\theta(R_\theta), \quad \text{and} \quad \mathcal{P}_{3n+2} = \bigoplus_{i_1,\ldots,i_n\geq 1} \mathcal{F}_\theta(R_\theta).$$

1. *The map $f_0 \colon \mathcal{F}_\theta(R_\theta) \to \mathcal{R}$ is defined by $f_{0,\theta} \colon g \mapsto x1_\theta$ and the map $f_1 \colon \mathcal{F}_\theta(R_\theta) \to \mathcal{F}_\theta(R_\theta)$ is defined by $f_{1,\theta} \colon g \mapsto yg + xtg$.*
2. *The map $f_2 \colon \bigoplus_{i\geq 0} \mathcal{R} \to \mathcal{F}_\theta(R_\theta)$ is defined by $f_{2,\bullet} \colon 1_{i,\bullet} \mapsto p_*(x^i y^i g)$.*
3. *For $n \geq 1$, the maps $f_{3n} \colon \bigoplus_{i_1,\ldots,i_n\geq 1} \mathcal{F}_\theta(R_\theta) \to \bigoplus_{i_1,\ldots,i_{n-1}\geq 1, i_n\geq 0} \mathcal{R}$ are $f_{3n} = \oplus\gamma$, $g_I \mapsto x^{i_n} y^{i_n} 1_{(i_1,\ldots,i_{n-1},0),\theta} + 1_{I,\theta}$.*
4. *For $n \geq 1$, the maps $f_{3n+1} \colon \bigoplus_{i_1,\ldots,i_n\geq 1} \mathcal{F}_\theta(R_\theta) \to \bigoplus_{i_1,\ldots,i_n\geq 1} \mathcal{F}_\theta(R_\theta)$ are $f_{3n+1} = \oplus\beta$, $g_I \mapsto g_I + tg_I$.*
5. *For $n \geq 1$, the maps $f_{3k+2} \colon \bigoplus_{i_1,\ldots,i_{n-1}\geq 1, i_n\geq 0} \mathcal{R} \to \bigoplus_{i_1,\ldots,i_{n-1}\geq 1} \mathcal{F}_\theta(R_\theta)$ are $f_{3n+2} = \oplus_I \delta$, $1_{I,\bullet} \mapsto p_*(x^{i_n} y^{i_n} g_{(i_1,\ldots,i_{n-1})})$.*

Proof. We begin by showing that $\ker\epsilon = \text{im } f_0$. First, because

$$(\epsilon_\theta \circ f_{0,\theta})(g) = \epsilon_\theta(x1_\theta) = 0$$

we see that im $f_0 \subseteq \ker \epsilon$. Also, since $f_{0,\theta}(g) = x$ and $f_{0,\theta}(tg) = y$ then $\ker \epsilon_\theta = (x, y) \subseteq$ im $f_{0,\theta}$. Furthermore, since

$$f_{0,\bullet}(p_*(x^n y^n g)) = p_*(x^n y^n f_{0,\theta}(g)) = p_*(x^n y^n (x 1_\theta)) = a_n$$

and

$$f_{0,\bullet}(p_*(x^{n-1} y^n g)) = p_*(x^{n-1} y^n f_{0,\theta}(g)) = p_*(x^n y^n 1_\theta) = b_n$$

then $\ker \epsilon_\bullet = (a_n, b_n) \subseteq$ im $f_{0,\bullet}$, thus $\ker \epsilon =$ im f_0.

Next, we will show that $\ker f_0 =$ im f_1. First, because

$$(f_{0,\theta} \circ f_{1,\theta})(g) = f_{0,\theta}(yg + xtg) = y(xg) + x(yg) = 0$$

we have im $f_1 \subseteq \ker f_0$.

Let $ug + vtg \in \ker f_{0,\theta}$ for $u, v \in \mathcal{R}_\theta$. Then we have

$$0 = f_{0,\theta}(ug + vtg) = ux + vy,$$

so $ux = vy$. Since (x, y) is a regular sequence in $\mathcal{R}_\theta$, this means there is some $w \in R_\theta$ such that $xyw = ux = vy$, and consequently $ug + vtg = wyg + wxtg$. Therefore,

$$f_{1,\theta}(wg) = w(yg + xtg) = ug + vtg,$$

so $\ker f_{0,\theta} \subseteq$ im $f_{1,\theta}$. Finally, we will show that $\ker f_{0,\bullet} \subseteq$ im $f_{1,\bullet}$. First, recall that

$$p_*(x^{s_j+1} y^{s_j} g) = a_0 p_*(x^{s_j} y^{s_j} g) + p_*(x^{s_j} y^{s_j+1} g)$$

so we may write any element in $\ker f_{0,\bullet}$ as $\sum_j c_j a_0^{m_j} p_*(x^{n_j} y^{n_j} g) \; + \; d_j a_0^{r_j} p_*(x^{s_j} y^{s_j+1} g) \in \ker f_{0,\bullet}$ where $c_j, d_j \in k$. First, note that

$$f_{0,\bullet}(a_0 p_*(x^n y^{n+1} g)) = a_0(p_*(x^{n+1} y^{n+1} 1_\theta)) = a_0 b_{n+1} = 0$$

and that

$$\begin{aligned} f_{1,\bullet}(p_*(x^{n+1} y^n g)) &= p_*(x^{n+1} y^n f_{1,\theta}(g)) = p_*(x^{n+1} y^n (yg + xtg)) \\ &= p_*(x^{n+1} y^{n+1} g + x^{n+2} y^n tg) = p_*(x^{n+1} y^{n+1} g) + p_*(t(x^{n+2} y^n tg)) \\ &= p_*(x^{n+1} y^{n+1} g) + p_*(x^n y^{n+2} g) \\ &= p_*(x^{n+1} y^{n+1} g) + a_0 p_*(x^n y^{n+1} g) + p_*(x^{n+1} y^{n+1} g) \\ &= a_0 p_*(x^n y^{n+1} g). \end{aligned}$$

Therefore, we will now show that elements of the form $\sum_j c_j a_0^{m_j} p_*(x^{n_j} y^{n_j} g) \; + \; d_j p_*(x^{s_j} y^{s_j+1} g)$ in $\ker f_{0,\bullet}$ are in im $f_{1,\bullet}$. To that end, notice that

$$\begin{aligned} 0 &= f_{0,\bullet}\left(\sum_j c_j a_0^{m_j} p_*(x^{n_j} y^{n_j} g) + d_j p_*(x^{s_j} y^{s_j+1} g)\right) \\ &= \sum_j c_j a_0^{m_j} p_*(x^{n_j} y^{n_j} f_{0,\theta}(g)) + d_j p_*(x^{s_j} y^{s_j+1} f_{0,\theta}(g)) \\ &= \sum_j c_j a_0^{m_j} p_*(x^{n_j} y^{n_j} (x 1_\theta)) + d_j p_*(x^{s_j} y^{s_j+1} (x 1_\theta)) \\ &= \sum_j c_j a_0^{m_j} p_*(x^{n_j+1} y^{n_j} 1_\theta) + d_j p_*(x^{s_j+1} y^{s_j+1} 1_\theta) \\ &= \sum_j c_j a_0^{m_j} a_{n_j} + d_j b_{s_j+1}. \end{aligned}$$

This implies that $\sum_{j|n_j=n} c_j a_0^{m_j} = 0$ for all $n \geq 0$ and that $\sum_{j|s_j=s} d_j b_{s+1} = 0$ for all $s \geq 0$. Therefore,

$$\begin{aligned}
&\sum_j c_j a_0^{m_j} p_*(x^{n_j} y^{n_j} g) + d_j p_*(x^{s_j} y^{s_j+1} g) \\
&\qquad = \sum_n \sum_{j|n_j=n} c_j a_0^{m_j} p_*(x^n y^n g) + \sum_s \sum_{j|s_j=s} d_j p_*(x^s y^{s+1} g) \\
&\qquad = \sum_n \left(\sum_{j|n_j=n} c_j a_0^{m_j} \right) p_*(x^n y^n g) + \sum_s \left(\sum_{j|s_j=s} d_j \right) p_*(x^s y^{s+1} g) \\
&\qquad = \sum_n 0 p_*(x^n y^n g) + \sum_s 0 p_*(x^s y^{s+1} g) = 0.
\end{aligned}$$

Therefore, $\ker f_{0,\bullet} = R_\bullet \langle a_0 p_*(x^n y^{n+1}) \rangle \subseteq \text{im } f_{1,\bullet}$, so $\ker f_0 = \text{im } f_1$.

Now, we will show that $\ker f_1 = \mathcal{M} = \text{im } f_2$. First, note that

$$\begin{aligned}
(f_{1,\bullet} \circ f_{2,\bullet})(1_{i,\bullet}) &= f_{1,\bullet}(p_*(x^i y^i g)) = p_*(x^i y^i f_{1,\theta}(g)) = p_*(x^i y^i (yg + xtg)) \\
&= p_*(x^i y^{i+1} g + x^{i+1} y^i tg) = p_*(x^i y^{i+1} g + t(x^{i+1} y^i tg)) \\
&= p_*(2x^i yi + 1g) = 0,
\end{aligned}$$

so $f_1 \circ f_2 = 0$. Next, let $ug + vtg \in \ker f_{1,\theta}$ for some $u, v \in R_\theta$. Then we have

$$0 = f_{1,\theta}(ug + vtg) = u(yg + xtg) + vt(yg + xtg) = uyg + uxtg + vxtg + vyg,$$

so $uy + vy = 0$ giving $u = v$. Thus $ug + vtg = u(g + tg) \in \mathcal{M}_\theta$, so $\ker f_{1,\theta} = \text{im } f_{2,\theta}$. Lastly, we will show that $\ker f_{1,\bullet} \subseteq \mathcal{M}_\bullet$. To that end, from the computation above we can see that $f_{1,\bullet}(p_*(x^n y^n g)) = 0$ so $R_\theta \langle p_*(x^n y^n g) \rangle \subseteq \ker f_{1,\bullet}$. Let

$\sum_j c_j a_0^{m_j} p_*(x^{n_j+1}y^{n_j}g) \in \ker f_{1,\bullet}$. Then we have

$$\begin{aligned}
0 &= f_{1,\bullet}\left(\sum_j c_j a_0^{m_j} p_*(x^{n_j+1}y^{n_j}g)\right) = \sum_j c_j a_0^{m_j} p_*(x^{n_j+1}y^{n_j}f_{1,\theta}(g)) \\
&= \sum_j c_j a_0^{m_j} p_*(x^{n_j+1}y^{n_j}(yg+xtg)) = \sum_j c_j a_0^{m_j} p_*(x^{n_j+1}y^{n_j+1}g + x^{n_j+2}y^{n_j}tg) \\
&= \sum_j c_j a_0^{m_j}(p_*(x^{n_j+1}y^{n_j+1}g) + p_*(x^{n_j+2}y^{n_j}tg)) \\
&= \sum_j c_j a_0^{m_j}(p_*(x^{n_j+1}y^{n_j+1}g) + p_*(x^{n_j}y^{n_j+2}g)) \\
&= \sum_j c_j a_0^{m_j}(p_*(x^{n_j+1}y^{n_j+1}g) + a_0 p_*(x^{n_j}y^{n_j+1}g) + p_*(x^{n_j+1}y^{n_j+1}g)) \\
&= \sum_j c_j a_0^{m_j+1} p_*(x^{n_j}y^{n_j+1}g).
\end{aligned}$$

By the linear independence of the $p_*(x^{n_j}y^{n_j+1}g)$ over $k[a_0]$ we deduce that $\sum_{j|n_j=n} c_j a_0^{m_j+1} p_*(x^n y^{n+1}g) = 0$ and thus $\sum_{j|n_j=n} c_j a_0^{m_j} = 0$. Therefore, we have

$$\sum_j c_j a_0^{m_j} p_*(x^{n_j+1}y^{n_j}g) = \sum_n \sum_{j|n_j=n} c_j a_0^{m_j} p_*(x^{n+1}y^n g) = \sum_n 0 p_*(x^{n+1}y^n g) = 0.$$

Therefore, $\ker f_{1,\bullet} = R_\theta\langle p_*(x^n y^n g)\rangle \subseteq \mathcal{M}_\bullet = \text{im } f_{2,\bullet}$. Thus $\ker f_1 = \text{im } f_2$ as desired.

Finally, the fact that $\ker f_2 = \text{im } f_3$, $\ker f_{3n} = \text{im } f_{3n+1}$, $\ker f_{3n+1} = \text{im } f_{3n+2}$, and $\ker f_{3n+2} = \text{im } f_{3(n+1)}$ follows from the previous theorem. Therefore, the sequence is exact, as desired. $\square$

Computation of the $\underline{\mathbf{Ext}}^*_{\underline{k}[x_\theta]}(\underline{k},\underline{k})$ Mackey functors

Proposition 5.2.1. *Let* $\psi \in Hom_{\mathcal{R}}(\mathcal{F}_\theta(R_\theta)\Box\mathcal{F}_\theta(R_\theta),\underline{k})$ *be the map defined by*

$$\psi\colon g\Box g \mapsto 1_\theta, \qquad \textit{and} \qquad \psi\colon tg\Box g \mapsto 0.$$

The map $\mathcal{F}_\theta(k) \to \underline{Hom}_{\mathcal{R}}(\mathcal{F}_\theta(R_\theta),\underline{k})$, $g \mapsto \psi$ *is an isomorphism of* $\mathcal{R}$*-modules.*

Proof. Let $f\colon \mathcal{F}_\theta(k) \to \underline{\mathrm{Hom}}_{\mathcal{R}}(\mathcal{F}_\theta(R_\theta),\underline{k})$ be the map $g \mapsto \psi$. First, we will show that f_θ is surjective. Let $\alpha \in \mathrm{Hom}_{\mathcal{R}}(\mathcal{F}_\theta(R_\theta)\Box\mathcal{F}_\theta(R_\theta),\underline{k})$. Consider $\alpha(g\Box g)g + \alpha(tg\Box g)tg \in \mathcal{F}_\theta(R_\theta)$. Then we have

$$\begin{aligned} f_\theta(\alpha(g\Box g)g + \alpha(tg\Box g)tg)\colon g\Box g &\mapsto \alpha(g\Box g)\psi(g\Box g) + \alpha(tg\Box g)t\psi(g\Box g) \\ &= \alpha(g\Box g)1_\theta + \alpha(tg\Box g)0 = \alpha(g\Box g), \end{aligned}$$

and also

$$\begin{aligned} f_\theta(\alpha(g\Box g)g + \alpha(tg\Box g)tg)\colon tg\Box g &\mapsto \alpha(g\Box g)\psi(tg\Box g) + \alpha(tg\Box g)t\psi(tg\Box g) \\ &= \alpha(g\Box g)0 + \alpha(tg\Box g)1_\theta = \alpha(tg\Box g). \end{aligned}$$

Furthermore, since $t_{\underline{k}} = \mathrm{id}_k$, we have

$$\begin{aligned} f_\theta(\alpha(g\Box g)g + \alpha(tg\Box g)tg)(tg\Box tg) &= tf_\theta(\alpha(g\Box g)g + \alpha(tg\Box g))(g\Box g) \\ &= t\alpha(g\Box g) = \alpha(tg\Box tg), \end{aligned}$$

and

$$\begin{aligned} f_\theta(\alpha(g\Box g)g + \alpha(tg\Box g)tg)(g\Box tg) &= tf_\theta(\alpha(g\Box g)g + \alpha(tg\Box g))(tg\Box g) \\ &= t\alpha(tg\Box g) = \alpha(g\Box tg). \end{aligned}$$

Therefore, $\alpha = f(\alpha(g\Box g)g + \alpha(tg\Box g)tg)$, so f_θ is surjective.

Next, we will show that $\ker f_\theta = 0$. Let $ag + btg \in \ker f_\theta$. Then we have $a\psi + bt\psi = 0$, so $a\psi + bt\psi\colon g\Box g \mapsto 0$ and $a\psi + bt\psi\colon tg\Box g \mapsto 0$. But then

$$a = a1_\theta = a\psi(g\Box g) + bt\psi(g\Box g) = 0$$

and

$$b = b1_\theta = a\psi(tg\Box g) + bt\psi(tg\Box g) = 0,$$

so $ag + btg = 0$. Therefore, f_θ is injective.

Next, we will see that $f_\bullet$ is an isomorphism of $R_\bullet$-modules. First, let $\alpha \in \mathrm{Hom}_{\mathcal{R}}(\mathcal{F}_\theta(R_\theta), \underline{k})$. Consider the element $\alpha(g)p_*(g) \in \mathcal{F}_\theta(R_\theta)_\theta$. Then

$$\begin{aligned} f_\bullet(\alpha(g)p_*(g))(g) &= \alpha(g)f_\bullet(p_*(g))(g) = \alpha(g)p_*(f_\theta(g))(g) \\ &= \alpha(g)p_*(\psi)(g) = \alpha(g)1_\theta = \alpha(g). \end{aligned}$$

Therefore, $f_\bullet$ is surjective.

It remains to see that $f_\bullet$ is injective. Let $ap_*(g) \in \ker f_\bullet$. Then we have

$$a = a1_\theta = ap_*(\psi)(g) = ap_*(f_\theta(g))(g) = af_\bullet(p_*(g))(g) = f_\bullet(ap_*(g))(g) = 0,$$

so $f_\bullet$ is also injective. This shows that f is an isomorphism, as desired. ☐

Lemma 5.2.2. *Let $\phi \in Hom_{\mathcal{R}}(\mathcal{R}, \underline{k})$ be the map $1_\bullet \mapsto 1_\bullet$. The map defined by*

$$\begin{aligned} f\colon \underline{k} &\to \underline{Hom}_{\mathcal{R}}(\mathcal{R}, \underline{k}) \\ 1_\bullet &\mapsto \phi \end{aligned}$$

is an isomorphism of $\mathcal{R}$-modules.

Proof. Similar to above. ☐

Proposition 5.2.3. *The map $f_0^*\colon \underline{Hom}_{\mathcal{R}}(\mathcal{P}_0, \underline{k}) \to \underline{Hom}_{\mathcal{R}}(\mathcal{P}_1, \underline{k})$ is 0.*

The map $f_1^*\colon \underline{Hom}_{\mathcal{R}}(\mathcal{P}_1, \underline{k}) \to \underline{Hom}_{\mathcal{R}}(\mathcal{P}_2, \underline{k})$ is 0.

The map $f_2^*\colon \underline{Hom}_{\mathcal{R}}(\mathcal{P}_2, \underline{k}) \to \underline{Hom}_{\mathcal{R}}(\mathcal{P}_3, \underline{k})$ is defined by $\psi \mapsto p^*(\phi_0)$.

The maps $f_{3n}^*\colon \underline{Hom}_{\mathcal{R}}(\mathcal{P}_{3n}, \underline{k}) \to \underline{Hom}_{\mathcal{R}}(\mathcal{P}_{3n+1}, \underline{k})$ are defined by $\phi_I \mapsto p_*(\psi_I)$.

The maps $f_{3n+1}^*\colon \underline{Hom}_{\mathcal{R}}(\mathcal{P}_{3n+1}, \underline{k}) \to \underline{Hom}_{\mathcal{R}}(\mathcal{P}_{3n+2}, \underline{k})$ are defined by $\psi_I \mapsto \psi_I + t\psi_I$.

The maps $f_{3n+2}^*\colon \underline{Hom}_{\mathcal{R}}(\mathcal{P}_{3n+2}, \underline{k}) \to \underline{Hom}_{\mathcal{R}}(\mathcal{P}_{3(n+1)}, \underline{k})$ are defined by $\psi_I \mapsto p^*(\phi_{i_1,\ldots,i_n,0})$.

Proof. We begin by proving that $f_0^* = 0$. Let ϕ be the generator of $\underline{\mathrm{Hom}}_{\mathcal{R}}(\mathcal{P}_0, \underline{k}) \cong \underline{k}$. Then we have

$$f_{0,\bullet}^*(\phi)_\theta\colon g \mapsto \phi_\theta(x1_\theta) = x\phi(1_\theta) = 0$$

in $\underline{k}_\bullet$. Therefore, $f_0^* = 0$.

Next, we prove that $f_1^* = 0$. Let ψ be the generator of $\underline{\mathrm{Hom}}_{\mathcal{R}}(\mathcal{P}_1, \underline{k}) \cong \mathcal{F}_\theta(k)$. Then we have

$$\begin{aligned} f_{1,\theta}^*(\psi)_\theta \colon g\Box g \mapsto \psi(f_{1,\theta}(g)\Box g) = \psi((yg + xtg)\Box g) = y\psi(g\Box g) + x\psi(tg\Box g) = y1_\theta + x0 \\ = 0 \end{aligned}$$

and similarly

$$\begin{aligned} f_{1,\theta}^*(\psi)_\theta \colon tg\Box g \mapsto \psi(f_{1,\theta}(g)\Box tg) = \psi((yg + xtg)\Box tg) = \psi(yg\Box tg + xtg\Box tg) \\ = 0. \end{aligned}$$

Therefore, $f_1^* = 0$.

Now, we describe f_2^*. Let ψ be the generator of $\underline{\mathrm{Hom}}_{\mathcal{R}}(\mathcal{P}_2, \underline{k}) \cong \mathcal{F}_\theta(k)$ and let ϕ_i be the generators of $\underline{\mathrm{Hom}}_{\mathcal{R}}(\mathcal{P}_3, \underline{k}) \cong \bigoplus_{i\geq 0} \underline{k}$. Then we have

$$\begin{aligned} f_{2,\theta}^*(\psi)_\theta \colon 1_{i,\theta}\Box g \mapsto \psi_\theta(f_{2,\theta}(1_{i,\theta})\Box g) = \psi_\theta(x^i y^i (g + tg)\Box g) = x^i y^i \psi_\theta((g + tg)\Box g) \\ = x^i y^i \left(\psi_\theta(g\Box g) + \psi_\theta(g\Box tg)\right) = x^i y^i (1_\theta + 0) = x^i y^i 1_\theta \end{aligned}$$

and similarly

$$\begin{aligned} f_{2,\theta}^*(\psi)_\theta \colon 1_{i,\theta}\Box tg \mapsto \psi_\theta(f_{2,\theta}(1_{i,\theta})\Box tg) = \psi_\theta(x^i y^i (g + tg)\Box tg) = x^i y^i \psi_\theta((g + tg)\Box tg) \\ = x^i y^i \left(\psi_\theta(g\Box tg) + \psi_\theta(tg\Box tg)\right) = x^i y^i (0 + 1_\theta) = x^i y^i 1_\theta \end{aligned}$$

Since $x^i y^i 1_\theta = 1_\theta$ in $\underline{k}$ precisely when $i = 0$, otherwise $x^i y^i 1_\theta = 0$, then $f_{2,\theta}^* \colon \psi \mapsto p^*(\phi_0)$.

Next, we describe the maps f^*_{3n}. Let ϕ_J be the generators of $\underline{\text{Hom}}_{\mathcal{R}}(\mathcal{P}_{3n}, \underline{k}) \cong \bigoplus_{i_1,\ldots,i_{n-1}>0, i_n \geq 0} \underline{k}$. Then we have

$$\begin{aligned} f^*_{3n,\bullet}(\phi_J)_\theta \colon g_I &\mapsto \phi_{J,\theta}(x^{i_n}y^{i_n}1_{i_1,\ldots,i_{n-1},0,\theta} + 1_{I,\theta}) \\ &= x^{i_n}y^{i_n}\phi_{J,\theta}(1_{i_1,\ldots,i_{n-1},0,\theta}) + \phi_{J,\theta}(1_{I,\theta}). \end{aligned}$$

Note that if $i_n = 0$, then $i_0, \ldots, i_{n-1}, 0 = I$, so $f^*_{3n,\bullet}(\phi_J)_\theta(g_{i_1,\ldots,i_{n-1},0}) = 0$.

Otherwise, if $i_n > 0$, then $x^{i_n}y^{i_n}\phi_{J,\theta}(1_{i_1,\ldots,i_{n-1},0,\theta}) + \phi_{J,\theta}(1_{I,\theta}) = \phi_{J,\theta}(1_{I,\theta})$ in $\underline{k}_\theta$. Since $\phi_{J,\theta}(1_{I,\theta}) = 1_\theta$ exactly when $J = I$, otherwise $\phi_{J,\theta}(1_{I,\theta}) = 0$, then we have $f^*_{3k,\bullet} \colon \phi_I \mapsto p_*(\psi_I)$ and $f^*_{3k,\bullet} \colon \phi_{i_1,\ldots,i_{n-1},0} \mapsto 0$.

Next, we describe the maps f^*_{3n+1}. Let ψ_J be the generators of $\underline{\text{Hom}}_{\mathcal{R}}(\mathcal{P}_{3n+1}, \underline{k}) \cong \bigoplus_{i_1,\ldots,i_n>0} \mathcal{F}_\theta(k)$. Then we have

$$f^*_{3n+1}(\psi_J)_\theta \colon g_I \Box g \mapsto \psi_{J,\theta}((g_I + tg_I)\Box g) = \psi_{J,\theta}(g_I\Box g + tg_I \Box g) = \psi_{J,\theta}(g_I \Box g)$$

and similarly

$$\begin{aligned} f^*_{3n+1}(\psi_J)_\theta \colon g_I \Box g &\mapsto \psi_{J,\theta}((g_I + tg_I)\Box tg) = \psi_{J,\theta}(g_I\Box tg + tg_I \Box tg) = \psi_{J,\theta}(tg_I \Box tg) \\ &= t\psi_{J,\theta}(g_I \Box g). \end{aligned}$$

Since $\psi_{I,\theta}(g_J \Box g) = 1_\theta$ exactly when $J = I$ and $t1_\theta = 1_\theta$, then $f^*_{3n+1,\theta}(\psi_J) = \psi_J + t\psi_J$.

Finally, we describe the maps f^*_{3n+2}. Let ψ_J be the generators of

$\underline{\mathrm{Hom}}_{\mathcal{R}}(\mathcal{P}_{3n+2}, \underline{k}) \cong \bigoplus_I \mathcal{F}_\theta(k)$. Then we have

$$\begin{aligned} f^*_{3n+2}(\psi_J)_\theta \colon 1_{I,\theta}\Box g \mapsto{}& \psi_{J,\theta}(x^{i_{n+1}}y^{i_{n+1}}(g_{i_1,\ldots,i_n} + tg_{i_1,\ldots,i_n}))\Box g) \\ &= \psi_{J,\theta}(x^{i_{n+1}}y^{i_{n+1}}(g_{i_1,\ldots,i_n}\Box g + tg_{i_1,\ldots,i_n}\Box g)) = (x^{i_{n+1}}y^{i_{n+1}}\psi_{J,\theta}(g_{i_1,\ldots,i_n}\Box g)) \\ &= x^{i_{n+1}}y^{i_{n+1}}1_\theta. \end{aligned}$$

Note that this is 0 exactly $i_{n+1} \neq 0$, otherwise it is 1_θ. Therefore, $f^*_{3n+2,\theta}(\psi_J) = p^*(\phi_{i_1,\ldots,i_n,0})$. $\Box$

Proposition 5.2.4. *1.* $\underline{Ext}^0_{\mathcal{R}}(\underline{k}, \underline{k}) = \underline{Hom}_{\mathcal{R}}(\underline{\mathcal{R}}, \underline{k})/\underline{0} \cong \underline{k}$

2. $\underline{Ext}^1_{\mathcal{R}}(\underline{k}, \underline{k}) = \underline{Hom}_{\mathcal{R}}(\mathcal{F}_\theta(R_\theta), \underline{k})/\underline{0} \cong \mathcal{F}_\theta(\underline{k})$

3. $\underline{Ext}^2_{\mathcal{R}}(\underline{k}, \underline{k}) =$
$$\begin{tikzcd} k\langle\psi + t\psi\rangle \arrow[loop left, "1"] \arrow[r, bend left, "0"] & k\langle p_*(\psi)\rangle \arrow[l, bend left] \end{tikzcd} \cong \underline{k}$$

4. $\underline{Ext}^3_{\mathcal{R}}(\underline{k}, \underline{k}) =$
$$\begin{tikzcd} 0 \arrow[loop left, "0"] \arrow[r, bend left, "0"] & k\langle\phi_0\rangle \arrow[l, bend left, "0"] \end{tikzcd}$$

5. For $n > 1$, $\underline{Ext}^{3n}_{\mathcal{R}}(\underline{k}, \underline{k}) =$
$$\begin{tikzcd} 0 \arrow[loop left, "0"] \arrow[r, bend left, "0"] & k\langle\phi_I \mid i_n = 0\rangle \arrow[l, bend left, "0"] \end{tikzcd}$$

6. For $n > 1$, $\underline{Ext}^{3n+1}_{\mathcal{R}}(\underline{k}, \underline{k}) = \underline{0}$

7. For $n > 1$, $\underline{Ext}^{3n+2}_{\mathcal{R}}(\underline{k}, \underline{k}) =$
$$\begin{tikzcd} 0 \arrow[loop left, "0"] \arrow[r, bend left, "0"] & k\langle p_*(\psi_I)\rangle \arrow[l, bend left, "0"] \end{tikzcd}$$

Proof. Since $f_0^* = f_1^* = 0$ then $\underline{\mathrm{Ext}}^1_{\mathcal{R}}$ and $\underline{\mathrm{Ext}}^0$ are canonically isomorphic to $\underline{\mathrm{Hom}}_{\mathcal{R}}(\mathcal{P}_1, \underline{k})$ and $\underline{\mathrm{Hom}}_{\mathcal{R}}(\mathcal{P}_0, \underline{k})$, respectively. We begin by finding $\ker f_2^*$. Let $a\psi + bt\psi \in \ker f_{2,\theta}^*$. Then we have

$$f_{2,\theta}^*(a\psi + bt\psi) = ap^*(\phi_0) + btp^*(\phi_0) = (a+b)p^*(\phi_0),$$

so $a + b = 0$. Therefore, $a = b$ and so $\ker f_{2,\theta}^* = k\langle \psi + t\psi \rangle$. Also, notice that

$$f_{2,\bullet}^*(p_*(\psi)) = p_*(f_{2,\theta}^*(\psi)) = p_*(p^*(\phi_0)) = 2\phi_0 = 0,$$

so $\ker f_{2,\bullet}^* = k\langle p_*(\psi)\rangle$. Therefore, we see that

$$\underline{\mathrm{Ext}}^2_{\mathcal{R}}(\underline{k}, \underline{k}) = \begin{tikzcd} k\langle \psi + t\psi \rangle \arrow[loop left, "1"] \arrow[r, bend left, "0"] & k\langle p_*(\psi) \rangle \arrow[l, bend left] \end{tikzcd} \quad \text{as desired.}$$

Next, we calculate $\underline{\mathrm{Ext}}^3_{\mathcal{R}}(\underline{k}, \underline{k})$. First, note that $f_{3,\bullet}^*(\phi_0) = 0$ and therefore $f_{3,\theta}^*(p^*(\phi_0)) = 0$ as well. Let $\sum_{i=1}^n c_i\phi_i \in \ker f_{3,\bullet}^*$. Then we have

$$0 = f_{3,\bullet}^*\left(\sum_{i=0}^n c_i\phi_i\right) = \sum_{i=0}^n c_i p_*(\psi_i),$$

but the $p_*(\psi_i)$ are linearly independent, so we must have each $c_i = 0$. Thus, $\ker f_{3,\bullet}^* = k\langle \phi_0 \rangle$. Now, let $\sum_{i=1}^m c_i p^*(\phi_i) \in \ker f_{3,\theta}^*$. Then we have

$$0 = f_{3,\theta}^*\left(\sum_{i=1}^m c_i p^*(\phi_i)\right) = p^*\left(\sum_{i=1}^m c_i f_{3,\bullet}^*(\phi_i)\right) = p^*\left(\sum_{i=1}^m c_i p_*(\psi_i)\right) = \sum_{i=1}^m c_i(\psi_i + t\psi_i),$$

so by linear independence of the elements $\psi_i + t\psi_i$ over k we must have all $c_i = 0$. Therefore, $\ker f_{3,\theta}^* = k\langle p^*(\phi_0)\rangle$. Finally, since $f_{2,\theta}^*(\psi) = p^*(\phi_0)$, then $f_2^*(t\psi) =$

$tp^*(\phi_0) = p^*(\phi_0)$ and $f_2^*(p_*(\psi)) = p_*(p^*(\phi_0)) = 0$. Therefore, im $f^*_{2,\theta} = k\langle p^*(\phi_0)\rangle$ and im $f^*_{2,\bullet} = 0$. Therefore, we see that

$$\underline{\mathrm{Ext}}^3_{\mathcal{R}}(\underline{k}, \underline{k}) = \begin{tikzcd} 0 \arrow[loop left, "0"] \arrow[r, bend left, "0"] & k\langle \phi_0 \rangle \arrow[l, bend left, "0"] \end{tikzcd}$$

as desired.

Next, we compute $\underline{\mathrm{Ext}}^{3n}_{\mathcal{R}}(\underline{k}, \underline{k})$ for $n > 1$. By a similar calculation to above, $\ker f^*_{3,\bullet} = k\langle \phi_{i_1,\ldots,i_{n-1},0}\rangle$ and $\ker f^*_{3,\theta} = k\langle p^*(\phi_{i_1,\ldots,i_{n-1},0})\rangle$. Since $f^*_{3(n-1)+2,\theta}(\psi_I) = p^*(\phi_{i_1,\ldots,i_{n-1},0})$, then we also have $f^*_{3(n-1)+2,\theta}(t\psi_I) = tp^*(\phi_{i_1,\ldots,i_{n-1},0}) = p^*(\phi_{i_1,\ldots,i_{n-1},0})$ and $f^*_{3(n-1)+2,\bullet}(p_*(\psi_I)) = p_*(p^*(\phi_{i_1,\ldots,i_{n-1},0})) = 2p^*(\phi_{i_1,\ldots,i_{n-1},0}) = 0$. Therefore, im $f^*_{3(n-1)+2,\theta} = k\langle p^*(\phi_{i_1,\ldots,i_{n-1},0})\rangle$ and im $f^*_{3(n-1)+2,\bullet} = 0$. Therefore, we see that

$$\underline{\mathrm{Ext}}^{3n}_{\mathcal{R}}(\underline{k}, \underline{k}) = \begin{tikzcd} 0 \arrow[loop left, "0"] \arrow[r, bend left, "0"] & k\langle \phi_I | i_n = 0 \rangle \arrow[l, bend left, "0"] \end{tikzcd}$$

as desired.

Next, we compute $\underline{\mathrm{Ext}}^{3n+1}_{\mathcal{R}}(\underline{k}, \underline{k})$ for $n > 0$. Since $f^*_{3n+1,\theta}(\psi_I) = \psi_I + t\psi_I$, then $\ker f_{3n+1,\theta} = k\langle \psi_I + t\psi_I\rangle$. Also, $\ker f^*_{3n+1} = k\langle p_*(\psi_I)\rangle$ since $f^*_{3n+1,\bullet}(p_*(\psi_I)) = p_*(f^*_{3n+1,\theta}(\psi_I)) = p_*(\psi_I + t\psi_I) = p_*(2\psi_I) = 0$. Now, since $f^*_{3n,\bullet}(\phi_I) = p_*(\psi_I)$ and thus $f^*_{3n,\theta}(p^*(\phi_I)) = p^*(p_*(\psi_I)) = \psi_I + t\psi_I$, then im $f^*_{3n} = \ker f^*_{3n+1}$. Therefore, $\underline{\mathrm{Ext}}^{3n+1}_{\mathcal{R}}(\underline{k}, \underline{k}) = \underline{0}$ as desired.

Finally, we compute $\underline{\mathrm{Ext}}^{3n+2}_{\mathcal{R}}(\underline{k}, \underline{k})$ for $n > 0$. Since $f^*_{3n+2,\theta}(\psi_*) = p^*(\phi_{i_1,\ldots,i_n,0})$ then $f^*_{3n+2,\theta}(t\psi_I) = tp^*(\phi_{i_1,\ldots,i_n,0}) = p^*(\phi_{i_1,\ldots,i_n,0})$, so $\psi_I + t\psi_I \in \ker f^*_{3n+2,\theta}$. Let $\sum_{l=1}^n c_l\psi_{J_l} + d_i t\psi_{J_l} \in \ker f^*_{3n+2,\theta}$. Then we have

$$\begin{aligned} 0 = f^*_{3n+2,\theta}\left(\sum_{l=1}^m c_l\psi_{J_l} + d_l t\psi_{J_l}\right) &= \sum_{l=1}^m c_l p^*(\phi_{i_{l_1},\ldots,i_{l_n},0}) + d_l tp^*(\phi_{i_{l_1},\ldots,i_{l_n},0}) \\ = \sum_{l=1}^m (c_l + d_l)p^*(\phi_{i_{l_1},\ldots,i_{l_n},0}). \end{aligned}$$

Since the $p^*(\phi_{i_{l_1},\ldots,i_{l_n},0})$ are linearly independent over k, this means $c_l + d_l = 0$ and thus $c_l = d_l$. Therefore, $\ker f^*_{3n+2,\theta} = k\langle \psi_I + t\psi_I\rangle$. Also, since $f^*_{3n+2,\bullet}(p_*(\psi_I)) = p_*(f^*_{3n+2,\theta}(\psi_I)) = p_*(p^*(\phi_{i_1,\ldots,i_n,0})) = 2\phi_{i_1,\ldots,i_n,0} = 0$, then $\ker f^*_{3n+2,\bullet} = k\langle p_*(\psi_I)\rangle$. It remains to find im f^*_{3n+1}. Since $f^*_{3n+1,\theta}(\psi_I) = \psi_I + t\psi_I$ and thus $f^*_{3n+1,\bullet}(p_*(\psi_I)) = p_*(f^*_{3n+1,\theta}(\psi_I)) = p_*(\psi_I + t\psi_I) = p_*(2\psi_I) = 0$, then im $f^*_{3n+1,\theta} = k\langle \psi_I + t\psi_I\rangle$ and im $f^*_{3n+1,\bullet} = 0$. Therefore, we see that $\underline{\mathrm{Ext}}^{3n+2}_{\mathcal{R}}(\underline{k},\underline{k}) =$

$$\begin{tikzcd} 0 \arrow[loop left, "0"] \arrow[r, bend left, "0"] & k\langle p_*(\psi_I)\rangle \arrow[l, bend left, "0"] \end{tikzcd}$$

as desired. □

Computation of products in $\underline{\mathbf{Ext}}^*_{\underline{k}[x_\theta]}(\underline{k},\underline{k})$

Proposition 5.3.1. *The map* $\omega\colon \mathcal{F}_\theta(R_\theta) \to \mathcal{F}_\theta(R_\theta)\Box\mathcal{F}_\theta(R_\theta)$, *where* $g \mapsto g\Box g$, *is the unique counital, coassociative map.*

Proof. This is a straightforward calculation checking the necessary equations. □

Remark 5.3.2. To multiply two classes in $\underline{\mathrm{Ext}}^*_{\mathcal{R}}(\underline{k},\underline{k})_\theta$ represented by cocycles $u\colon \mathcal{P}_n\Box\mathcal{F}_\theta(\mathcal{R}_\theta) \to \underline{k}$ and $v\colon \mathcal{P}_m\Box\mathcal{F}_\theta(\mathcal{R}_\theta) \to \underline{k}$ in $\underline{\mathrm{Ext}}^*_{\mathcal{R}}(\underline{k},\underline{k})$, we take the product $[u][v]$ to be the class represented by the cocycle

$$\mathcal{P}_{n+m}\Box\mathcal{F}_\theta(R_\theta) \xrightarrow{\mathrm{id}_{\mathcal{P}_{n+m}}\Box\omega} \mathcal{P}_{n+m}\Box\mathcal{F}_\theta(R_\theta)\Box\mathcal{F}_\theta(R_\theta) \xrightarrow{\tilde{u}_m\Box\mathrm{id}_{\mathcal{F}_\theta(R_\theta)}} \mathcal{P}_m\Box\mathcal{F}_\theta(R_\theta) \xrightarrow{v} \underline{k},$$

where $\tilde{u}\colon \mathcal{P}_*\Box\mathcal{F}_\theta(\mathcal{R}_\theta) \to \mathcal{P}_{*-n}$ is a lifting of u.

Theorem 5.3.3. *Let* $\alpha_x, \alpha_y \in \underline{Hom}_{\mathcal{R}}(\mathcal{F}_\theta(R_\theta)\Box\mathcal{F}_\theta(R_\theta),\underline{k})$ *be the maps*

$$\alpha_x\colon g\Box g \mapsto 1_\theta,\quad tg\Box g \mapsto 0,\quad \textit{and}\quad \alpha_y\colon g\Box g \mapsto 0,\quad tg\Box g \mapsto 1_\theta.$$

Then $\underline{Ext}^*_{\mathcal{R}}(\underline{k},\underline{k})_\theta$ *is an exterior algebra on the classes* $[\alpha_x]$ *and* $[\alpha_y]$ *in* $\underline{Ext}^1_{\mathcal{R}}(\underline{k},\underline{k})_\theta$.

Proof. We begin by computing lifts of α_x. In particular, let $\tilde{\alpha}_{x,0}\colon \mathcal{F}_\theta(\mathcal{R}_\theta)\Box\mathcal{F}_\theta(\mathcal{R}_\theta) \to \mathcal{R}$ and $\tilde{\alpha}_{x,1}\colon \mathcal{F}_\theta(\mathcal{R}_\theta)\Box\mathcal{F}_\theta(\mathcal{R}_\theta) \to \mathcal{F}_\theta(\mathcal{R}_\theta)$ be the maps

$$\tilde{\alpha}_{x,0,\theta}\colon g\Box g \mapsto 1_\theta, \quad tg\Box g \mapsto 0 \quad \text{and} \quad \tilde{\alpha}_{x,1,\theta}\colon g\Box g \mapsto tg, \quad tg\Box g \mapsto tg.$$

First, note that

$$(\epsilon_\theta \circ \tilde{\alpha}_{x,0,\theta})(g\Box g) = \epsilon_\theta(1_\theta) = 1_\theta = \alpha_{x,\theta}(g\Box g)$$

and

$$(\epsilon_\theta \circ \tilde{\alpha}_{x,0,\theta})(tg\Box g) = \epsilon_\theta(0) = 0 = \alpha_{x,\theta}(tg\Box g),$$

so $\tilde{\alpha}_{x,0}$ is a lift of α_x.

Next, we will see that $\tilde{\alpha}_{x,1}$ is a lift of $\tilde{\alpha}_{x,0}$. This is because

$$\begin{aligned}(f_{0,\theta} \circ \tilde{\alpha}_{x,1,\theta})(g\Box g) &= f_{0,\theta}(tg) = y1_\theta = \tilde{\alpha}_{x,0,\theta}(yg\Box g)\tilde{\alpha}_{x,0,\theta}((yg + xtg)\Box g)\\ &= \tilde{\alpha}_{x,0,\theta}(f_{1,\theta}\Box\mathrm{id}_{\mathcal{F}_\theta(R_\theta)}(g\Box g))\end{aligned}$$

and

$$\begin{aligned}(f_{0,\theta} \circ \tilde{\alpha}_{x,1,\theta})(tg\Box g) &= f_{0,\theta}(tg) = y1_\theta = \tilde{\alpha}_{x,0,\theta}(yg\Box g)\tilde{\alpha}_{x,0,\theta}((yg + xtg)\Box g)\\ &= \tilde{\alpha}_{x,0,\theta}(f_{1,\theta}\Box\mathrm{id}_{\mathcal{F}_\theta(R_\theta)}(tg\Box g)).\end{aligned}$$

Next, we will calculate lifts of α_y. In particular, similarly to α_x, let $\tilde{\alpha}_{y,0} \colon \mathcal{F}_\theta(R_\theta \Box \mathcal{F}_\theta(R_\theta) \to \mathcal{R}$ and $\tilde{\alpha}_{y,1} \colon \mathcal{F}_\theta(R_\theta) \Box \mathcal{F}_\theta(R_\theta) \to \mathcal{F}_\theta(R_\theta)$ be the maps

$$\tilde{\alpha}_{y,0,\theta} \colon g\Box g \mapsto 0, \quad tg\Box g \mapsto 1_\theta, \quad \text{and} \quad \tilde{\alpha}_{y,1,\theta} \colon g\Box g \mapsto g, \quad tg\Box g \mapsto g.$$

First, note that

$$(\epsilon_\theta \circ \tilde{\alpha}_{y,0,\theta})(g\Box g) = \epsilon_\theta(0) = 0 = \alpha_{y,\theta}(g\Box g)$$

and

$$(\epsilon_\theta \circ \tilde{\alpha}_{y,0,\theta})(tg\Box g) = \epsilon_\theta(1_\theta) = 1_\theta = \alpha_{y,\theta}(tg\Box g),$$

so $\tilde{\alpha}_{y,0}$ is a lift of α_x.

Next, we will see that $\tilde{\alpha}_{y,1}$ is a lift of $\tilde{\alpha}_{y,0}$. This is because

$$\begin{aligned}(f_{0,\theta} \circ \tilde{\alpha}_{y,1,\theta})(g\Box g) = f_{0,\theta}(g) = x1_\theta = \tilde{\alpha}_{y,0,\theta}(xtg\Box g)\tilde{\alpha}_{y,0,\theta}((yg+xtg)\Box g) \\ = \tilde{\alpha}_{y,0,\theta}(f_{1,\theta}\Box \mathrm{id}_{\mathcal{F}_\theta(R_\theta)}(g\Box g))\end{aligned}$$

and

$$\begin{aligned}(f_{0,\theta} \circ \tilde{\alpha}_{y,1,\theta})(tg\Box g) = f_{0,\theta}(g) = x1_\theta = \tilde{\alpha}_{y,0,\theta}(xtg\Box g)\tilde{\alpha}_{y,0,\theta}((yg+xtg)\Box g) \\ = \tilde{\alpha}_{y,0,\theta}(f_{1,\theta}\Box \mathrm{id}_{\mathcal{F}_\theta(R_\theta)}(tg\Box g)).\end{aligned}$$

It remains to compute $[\alpha_x][\alpha_y]$ and $[\alpha_y][\alpha_x]$, since $\underline{\mathrm{Ext}}^2_{\mathcal{R}}(\underline{k}, \underline{k})_\theta \cong k$ and $\underline{\mathrm{Ext}}^n_{\mathcal{R}}(\underline{k}, \underline{k}) = 0$ for $n > 2$. The following computations show that

$[\alpha_x][\alpha_y] = [\alpha_y][\alpha_x] = [\psi + t\psi] \in \underline{\mathrm{Ext}}^2_{\mathcal{R}}(\underline{k}, \underline{k})_\theta$:

$$\begin{aligned}(\alpha_{y,\theta} \circ \tilde{\alpha}_{x,1,\theta} \Box \mathrm{id}_{\mathcal{F}_\theta(R_\theta)} \circ \mathrm{id}_{\mathcal{F}_\theta(R_\theta)})(g \Box g) &= \alpha_{y,\theta}(\tilde{\alpha}_{x,1,\theta} \Box \mathrm{id}_{\mathcal{F}_\theta(R_\theta)}(g \Box g \Box g)) \\ &= \alpha_{y,\theta}(tg \Box g) = 1_\theta\end{aligned}$$

$$\begin{aligned}(\alpha_{y,\theta} \circ \tilde{\alpha}_{x,1,\theta} \Box \mathrm{id}_{\mathcal{F}_\theta(R_\theta)} \circ \mathrm{id}_{\mathcal{F}_\theta(R_\theta)})(tg \Box g) &= \alpha_{y,\theta}(\tilde{\alpha}_{x,1,\theta} \Box \mathrm{id}_{\mathcal{F}_\theta(R_\theta)}(tg \Box g \Box g)) \\ &= \alpha_{y,\theta}(tg \Box g) = 1_\theta\end{aligned}$$

$$\begin{aligned}(\alpha_{x,\theta} \circ \tilde{\alpha}_{y,1,\theta} \Box \mathrm{id}_{\mathcal{F}_\theta(R_\theta)} \circ \mathrm{id}_{\mathcal{F}_\theta(R_\theta)})(g \Box g) &= \alpha_{x,\theta}(\tilde{\alpha}_{y,1,\theta} \Box \mathrm{id}_{\mathcal{F}_\theta(R_\theta)}(g \Box g \Box g)) \\ &= \alpha_{x,\theta}(g \Box g) = 1_\theta\end{aligned}$$

$$\begin{aligned}(\alpha_{x,\theta} \circ \tilde{\alpha}_{y,1,\theta} \Box \mathrm{id}_{\mathcal{F}_\theta(R_\theta)} \circ \mathrm{id}_{\mathcal{F}_\theta(R_\theta)})(tg \Box g) &= \alpha_{x,\theta}(\tilde{\alpha}_{y,1,\theta} \Box \mathrm{id}_{\mathcal{F}_\theta(R_\theta)}(tg \Box g \Box g)) \\ &= \alpha_{x,\theta}(tg \Box g) = 1_\theta.\end{aligned}$$

Furthermore, from the following calculations we see that $[\alpha_x]^2 = [\alpha_y]^2 = 0$:

$$\begin{aligned}(\alpha_{x,\theta} \circ \tilde{\alpha}_{x,1,\theta} \Box \mathrm{id}_{\mathcal{F}_\theta(R_\theta)} \circ \mathrm{id}_{\mathcal{F}_\theta(R_\theta)})(g \Box g) &= \alpha_{x,\theta}(\tilde{\alpha}_{x,1,\theta} \Box \mathrm{id}_{\mathcal{F}_\theta(R_\theta)}(g \Box g \Box g)) \\ &= \alpha_{y,\theta}(tg \Box g) = 0\end{aligned}$$

$$\begin{aligned}(\alpha_{x,\theta} \circ \tilde{\alpha}_{x,1,\theta} \square \mathrm{id}_{\mathcal{F}_\theta(R_\theta)} \circ \mathrm{id}_{\mathcal{F}_\theta(R_\theta)})(tg \square g) &= \alpha_{x,\theta}(\tilde{\alpha}_{x,1,\theta} \square \mathrm{id}_{\mathcal{F}_\theta(R_\theta)}(tg \square g \square g)) \\ &= \alpha_{x,\theta}(tg \square g) = 0\end{aligned}$$

$$\begin{aligned}(\alpha_{y,\theta} \circ \tilde{\alpha}_{y,1,\theta} \square \mathrm{id}_{\mathcal{F}_\theta(R_\theta)} \circ \mathrm{id}_{\mathcal{F}_\theta(R_\theta)})(g \square g) &= \alpha_{y,\theta}(\tilde{\alpha}_{y,1,\theta} \square \mathrm{id}_{\mathcal{F}_\theta(R_\theta)}(g \square g \square g)) \\ &= \alpha_{y,\theta}(g \square g) = 0\end{aligned}$$

$$\begin{aligned}(\alpha_{y,\theta} \circ \tilde{\alpha}_{y,1,\theta} \square \mathrm{id}_{\mathcal{F}_\theta(R_\theta)} \circ \mathrm{id}_{\mathcal{F}_\theta(R_\theta)})(tg \square g) &= \alpha_{y,\theta}(\tilde{\alpha}_{y,1,\theta} \square \mathrm{id}_{\mathcal{F}_\theta(R_\theta)}(tg \square g \square g)) \\ &= \alpha_{y,\theta}(g \square g) = 0.\end{aligned}$$

By previous computation, we know that $\underline{\mathrm{Ext}}^n_{\mathcal{R}}(\underline{k}, \underline{k})_\theta = 0$ for $n > 2$, so this determines $\underline{\mathrm{Ext}}^*_{\mathcal{R}}(\underline{k}, \underline{k})_\theta$ as a k-algebra. □

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