A STUDY IN THE THEORY OF ECONOMIC GROWTH

AND INCOME DISTRIBUTION

By

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CHAPTER I

INTRODUCTION

Section I: The Scope of the Present Study

In this work a study of the theory of economic growth under three different technological assumptions is attempted. The first of them is the famous Harrod's[1] model of economic growth which is based on the assumption of fixed coefficients of production. The second is the neoclassical model that assumes that a given amount of goods can be produced by varying combinations of different factors of production. The third model, which is based on that of Johansen,[2] attempts a synthesis of the other two by assuming that factor substitution is possible only at the time when the old capital is replaced by the new or when new investment is made, whereas for the old plant which is still operating, the coefficients of production retain the values planned at the time the plant is newly built.

The above three models, taken together, have many interesting properties that could be studied with reference to the actual trends

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[1] R. F. Harrod, "An Essay on Dynamic Theory," Economic Journal, XLIX (1939), 14-33. See also his Towards a Dynamic Economics (London: Macmillan and Co., Ltd., 1943).

[2] L. Johansen, "Substitution versus Fixed Production Coefficients in the Theory of Economic Growth: A Synthesis," Econometrica, XXVII, No. 2 (1959), 157-76.

of the economy over time. The purpose here is to study those properties of the models that are basic from the point of view of economic analysis. The basic properties are those which characterize the equilibrium pattern of an economy and also reveal the behavior of the economy when out of equilibrium.

In the study of equilibrium and stability consideration of the market forces which act through prices of goods and services becomes necessary. In a multisectoral model the prices of goods could be studied with some simplification. The present study is concerned with a one-commodity model except on a few occasions when breakdown of the economy into sectors becomes necessary. In some places mention will be made of the effect of changing prices on dynamic equilibrium even in a one-commodity model.

The study of factor prices will be prominent in this discussion as far as the determination of equilibrium trend of the economy is concerned. Economic literature abounds in controversies as to whether growth equilibrium exists, is stable, and whether market forces have stabilizing or destabilizing influences on the economy envisaged by a given model. Some of the controversies regarding stability in connection with Harrod's model will be considered, which may, however, be considered as a detour. The influence of market forces as revealed by the pricing of factor services consistent with equilibrium growth will be the main subject matter of this work.

In this discussion all interpretations of distribution will be based on the economic models studied. It can be shown that each model has its more or less definite implications about prices in general which can be derived very simply in some cases, whereas others may require much more elaborate process, though the basic principles may be the same. For example, in the models assuming substitutability of factors the application of the marginal productivity theory is simple enough to determine income distribution. But in the fixed-coefficient models this is not so. Thus a more general theoretical scheme has to be followed to explain income determination for the models concerned.

Distribution theory is, at present, a controversial subject. Since the consideration of this in the context of growth equilibrium is a significant part of the problem, it is thought worthwhile to dwell on this subject alone by way of clarifying the approach to be taken. In the next section the basic idea that underlies almost all theories of distribution will be studied. In later sections some existing static and dynamic models will be considered which are not studied as the main subject matter of this work, but which provide an insight into the state in which the theory of distribution is at present together with the theory of equilibrium in general.

The second chapter will be devoted to a study of Harrod's model of economic growth and neoclassical criticisms of the model. The neoclassical model of growth is to be found in the course of the arguments. In studying Harrod's model Kaldor's Keynesian theory of income distribution will be considered. The latter theory will be shown to be inadequate as an explanation of equilibrium.

In the same chapter the constancy of production coefficients in Harrod's model will be discussed, giving consideration to Samuelson's substitution theorem. The theorem will be studied with the assumption of profit maximization on the part of the entrepreneurs and the result will be used to show that if factor prices are flexible, the coefficients of production will be such that Harrod's equilibrium rate of growth will equal the natural rate of growth determined by the rate of technological progress and the rate of growth of the labor force. Since this is the neoclassical line of reasoning, objections to it will be considered in the same chapter.

The third chapter will be concerned with Johansen's model of economic growth. This model will be studied with various assumptions regarding the durability or depreciation of capital goods. Though they will not differ from Johansen's assumptions, the determination of equilibrium growth of the economy or even the stationary equilibrium in cases in which they exist will be studied with additional constraints derived from the valuation aspect of the economy. Also to be considered is a controversy whether Johansen's assumption that fixed coefficients of production rule for capital goods already existent in a world where productivity of factors is rising is legitimate or not. A synthesis will be attempted between Johansen's approach and the neoclassical approach by using two types of production function in Johansen's model. The two production functions are those relevant for short and long periods, respectively, which are assumed to be characterized by two distinct sets of parameters. The fourth chapter will present conclusions.

Section II: Distribution Theory in its Basic Form

As was already indicated, the study of income distribution is a part of the study of equilibrium growth. That it is an integral part of the discussions will be apparent. But here it should be mentioned that any inference made about factor prices in this analysis is in conformity with the classical marginal analysis. In this section the object is to see the basic ideas underlying most of the distribution theories and particularly to defend the idea that all distribution theories are variants of marginal productivity ideas. A defense of this basic notion is the central theme of the rest of this chapter.

Perhaps it would not be too bold an assertion to say that almost all theoretical models dealing with distribution of income imply or explicitly say that pricing of productive services follows the same rule as that of commodities in general, namely, that all agents of production are valued according to their supply and demand. In the usual textbook approach the demand for a factor is determined by its marginal revenue product. The supply is determined by either purely exogenous factors, by institutional factors, by the choice of the factor-owners between income received from employment of the factor and satisfaction from avoiding the exertion involved in such employment, or by any combination of these conditions.

That demand, as determined by tastes, and supply, as derived from scarcity of means, are the determinants of prices, and thus of distributive shares, is implicit in nearly all theories and are generally derived by marginal analysis. For instance, the wages fund theory employed by the classical economists, although it says that wage is determined by the amount of wages fund, also implies that the wage rate in fact tends to equal the marginal product of labor. However, only in the stationary state do supply and demand of labor reach equality and marginal product of labor becomes equal to the equilibrium wage rate. Regarding profit, the arguments are not quite clear. Even many modern writers have no definite theory of profit. If it is regarded as entrepreneurial income, namely, the return on the skill of the entrepreneur and his undertaking of risk in investing capital, the term profit would include a number of elements. But if the minimum risk premium and the payment for the skill of management are deducted, the rate of interest remains to be determined.

In a determination of the rate of interest the marginal analysis holds its own if it is realized that the supply of capital is the result of the allocation process of income by its recipients and that demand is determined by the addition which it makes to the total product of the enterprise. But the idea of marginal productivity has been a matter of controversy among economists in some instances. Capital, as an important element of the production process, may or may not be regarded as a "factor" of production, depending on how one conceives of a factor of production in economics. Capital consists of a large variety of intermediate goods which are the result of the productive activities. If capital is thus conceived as a set of intermediate goods[1] waiting to be finished, the idea that it leads to a surplus value by leading to an increase in the total product of the enterprise requires explanation. Bohm-Bawerk's criticism[2] of the marginal productivity theory of interest springs mainly from this characteristic of capital. In his theory Bohm-Bawerk introduces the concept of period of production as a measure of capital, which is used as a device to evolve an analytic framework in which rate of interest is determined independently of the marginal productivity theory. But since the more roundabout method of production involving more use of capital can be used, according to this theory, only if this results in higher productivity of the production process, one is again led to explain the rate of interest along the marginal productivity lines. The achievement of the Austrian theory, therefore, lies not in repudiating the marginal productivity theory of interest, but in evolving a new approach of analyzing the production process of the economy in which capital is closely entangled with "time."

Other important attempts at explaining rate of interest are Fisher's time-preference theory, loanable fund theory, and liquidity preference theory.[3] The first of these is related to marginal productivity on the demand side of the determinant of the rate of interest.

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[1] See E. von Bohm-Bawerk, Positive Theory of Capital, trans. W. Smart (New York: G. E. Stechert and Co., 1923), I, 17-23.

[2] For Bohm-Bawerk's criticism of productivity theories, see J. W. Conard, An Introduction to the Theory of Interest (University of California Press, 1959), pp. 28-30.

[3] For a discussion of various theories of interest, see J. W. Conard, ibid.

The latter two theories, the loanable fund and the liquidity preference, are not so directly connected with marginal analysis, but various authors have attempted to show that within the framework of a more general Walrasian type of model the rate of interest explained according to one theory is the same as that explained by the other. Without going into detail, it may be stated that these theories are consistent with the proposition given above.

So far, the other elements which enter into profit, viz., risk and skill of management have been neglected. Skill of management does not present any serious difficulty. With regard to the former, risk, the uncertainty theory of profit which says that profit is the return on risk-bearing in an uncertain world should be considered. If revenue is interpreted over and above the costs over a given period as the return due to the risk undertaken by the investor, it is sure to lead to some ambiguities. When profit is earned, what sufficient criterion is there to distinguish between the earnings due to the efficiency of the firm, risk-bearing of the entrepreneur, and other complexes of events going on in the real world? In the opinion of the writer, the notion of profit as a phenomenon of uncertainty is misleading. In an uncertain world, a business enterprise can be regarded as a player in a game in which there are mixed strategies. The expenditure made by the business is the cost of playing the game. The knowledge about the probability distribution of events such as the price of goods and services, wage rates, interest rates, and so on, is presumably derived from empirical study before an investment decision is made. On this basis the course of action which maximizes expected gain or minimizes expected loss in some relevant sense is chosen. This is not to say that there is pure objectivity in calculating chances of gain and loss. But whatever subjectivity exists, it has an empirical foundation in the experience of the players. When the enterprise is started the gains or losses are outcomes of chance. The expected return is the stochastic return on the plant and is estimated on the basis of available information. A profit higher or lower than the expected level is liable to affect the expectation of the entrepreneurs and lead to a change in the decisions to invest. Hence, profit is the return on capital in the long run, even in an uncertain world.

Before concluding this section, how the proposition outlined at the beginning of this section applies to the case of models with continuous substitutability of factors and to the one with fixed coefficients of production will be studied. In the case in which substitution of factors is allowed, there are three distinct possibilities regarding the return to scale of output. If it is assumed that there is constant return to scale, the famous Euler's theorem holds and the payment to the factors according to their marginal product exhausts the total product. The return to various factors, their level of employment, or the level of production of each good and also prices of goods and the coefficients of production are simultaneously determined by the supply and demand. The prices of factors determine the proportion in which they are employed. Level of employment is determined by the demand for goods.

When increasing returns to scale prevail, the marginal productivity theory cannot be applied as in the previous case. This is one important reason why it is necessary to consider demand for goods as playing a significant role in the determination of factor prices in order to obtain a more general theory of distribution. The scarcity of factors is also important because of its bearing on prices. The present work, however, will be restricted to the case of constant returns to scale to simplify arguments.

In a fixed coefficient model the factors are assumed to be employed in fixed proportions which cannot be altered. This makes it impossible to evaluate the marginal product of any factor as in the previous case by comparing factor-proportions, because one factor cannot be increased by keeping others constant in order to increase output. But this absence of substitution possibility of factors is no hindrance to the application of the marginal theory of distribution. The substitution by consumers among various items through choice becomes the basis on which the theory of distribution can be built.

In a multi-commodity case it can be assumed that there are n commodities, each of which is produced by a given technique. In the hierarchy of commodities existing at any time, various amounts of them are produced and sold at various prices. In equilibrium the price per unit of any good is equal to its unit cost, which in turn is determined by the prices of factors employed and the coefficients of production which characterize the technique of producing that commodity. Price of any particular factor is homogeneous in the economy. In competitive equilibrium the demand conditions are such that the marginal unit of output of each good produced finds a purchaser with a price which just covers its cost of production. The importance of demand becomes obvious if the consequence of a change in a particular factor price can be seen. Assume that equilibrium is disturbed due to the rise in wage rate. The consequence of this change is to render the existing level of activities in all sectors unprofitable. In the labor intensive sector prices will rise more than in others. Thus a reallocation is necessary in order to restore the equilibrium. Allowing consumer substitution employment will rise in the capital intensive sector and fall in the labor intensive sector. Thus in the present model the significant part of the problem is the determination of allocation of factors in different sectors. Such allocation is guided by valuation of consumers. The existence of valuation with a common purchasing power implies substitution of some type.

The above discussion reveals that there is similarity between the model with the fixed coefficient of production and the one with variable factor proportion. But in the former case the marginal revenue productivity is determined with reference to consumer pricing only. This point is illustrated by the following example.

Suppose that there are two goods, A and B. It is assumed that one unit of A is produced with one unit of capital and two units of labor. Similarly it is assumed that the production of one unit of B requires one unit of labor and one unit of capital. It is further assumed that their demand functions are:

$$X_a = 100 - p_a \quad \text{and}$$

$$X_b = 50 - p_b$$

where X_a and X_b are quantities of A and B demanded and p_a and p_b their respective prices. Suppose that the initial equilibrium values were $p_a = 25$ and $p_b = 15$, with $X_a = 75$ and $X_b = 35$, wage rate = 10 and rate of profit = 5. It follows that the total labor employed is 185 and capital is 110. If one unit of capital and two units of labor are transferred to sector A from sector B the result is $X_a = 76$, which implies that $p_a = 24$ and in the other sector $X_b = 33$ and $p_b = 17$. The resulting new wage rate of 7 is the marginal product of labor, and the marginal product of capital or the rate of profit now becomes 10. This explains how new price structure leads to new levels of output of the goods A and B which become profitable.

However, there are some complications in the above illustration. For instance, one may ask about the missing one unit of capital in the reallocation process. This would not present any serious problem in a more general model which considers the allocation of income between current consumption and saving. The above example was designed simply to explain the role of demand in the marginal productivity interpretation of distribution in the type of model studied here. But one can easily visualize that the cause of reallocation in the above example is to be found in the decrease in the supply of capital relative to supply of labor. Thus a new equilibrium with no relative scarcity or superfluity of capital is established with new factor-incomes. The effect of superfluity of capital arising from, say, a change in the time preference of the people may similarly be considered as leading to more capital intensive process.

The theory outlined above can be easily extended to the dynamic models. There will be occasion later on to consider the distribution problem relating to the fixed coefficient models. Here it is worthwhile to make some remarks about the nature of problems that will be confronted in considering a dynamic model.

In a dynamic model changes are allowed in technology, supply of resources like labor and capital, in taste of the people, and so on, which are regarded as given in a static model. It is very difficult to develop a general dynamic theory capable of explaining all varieties of change and their implications. But the effect of changing supply of factors of production on distribution can be explained. In a model based on the assumption of substitutability of factors it is quite clear that changing supply of factors presents no difficulty in finding how the economy reacts to the change. The factor-proportions can be varied according to the availability of various factors of production. But in a fixed coefficient model it causes some difficulty. The difficulty is only apparent if consideration is given to a factor such as capital that can adapt to the supply of other factors and to the demand for commodities by the consumers. If it does not, zero returns are implied for some factors and this would have drastic consequences! The precise way in which capital adapts to the supply of other factors in a fixed coefficient world depends upon a variety of circumstances. Consider, for example, an economy in which only labor and capital are the factors of production. Suppose the supply of labor is outrunning the supply of capital. In such a circumstance it is definite that return on capital becomes high. Labor can be unemployed and its marginal product reduced to zero. But such an awkward situation is unlikely to occur because the fall in wage means reduction in consumption and increase in the rate of accumulation. Equilibrium is achieved if the rate of accretion of the value of invested capital equals the rate of growth. If it is higher, accumulation will increase in rate and vice versa. Another possibility is that labor intensive or capital saving technology will develop and this will restore the equilibrium between the supply and demand of factors.

It should be noted that the net increment of the value of the product resulting from the investment of an additional unit of capital is the marginal revenue product of capital. The wage rate at which the additional labor finds employment as a result of the new addition to the stock of capital is the marginal product of labor. In other words, marginal analysis of distribution is possible with fixed coefficients.

The fundamental premise on which the above analysis rests is that the resources controlled by an economic unit are scarce and bring a positive net return. One particular aspect of this premise is that whenever marginal productivity analysis is used to explain factor prices with the assumption of factor substitutability the marginal productivity of capital is always positive, similar to that of labor. The realism of this assumption is questionable. But in equilibrium of a non-controlled competitive economy it is doubtful that any other line of reasoning could provide an adequate analysis of distribution.

It may also be argued that the theory of distribution developed in this section can be applied only in the case of micro-economics. For the aggregative models it may be considered inadequate. Perhaps the dissatisfaction is due to the presence of institutional factors, among other complications, and also is due to the interpretation of the marginal productivity theory in its simplest form based on factor substitution. The purpose of the so-called macro-economic theories of distribution has been to avoid all the complications introduced by the details of market forces within the institutional framework of the economy and also the difficulties arising from the production function approach to this problem and within these confines to evolve a consistent theory of distribution. One version, that of Kaldor,[1] will be considered in the next chapter. It will show that it only gives an answer to the question as to what part of the total income should go to labor and what part to capital under full employment conditions in order to satisfy equilibrium conditions, and that this is done on the basis of some arbitrary assumptions which may not be true in general. It will be shown that the question of factor price determination is not answered by this model. It may be said that the marginal analysis explained in this section allows no alternative as a general principle, however much its precise meaning may vary as assumed conditions vary.

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[1]See N. Kaldor, "Alternative Theories of Distribution," Review of Economic Studies, XXIII, No. 61, 94-100.

The next sections will be devoted to a survey of ideas about static or dynamic equilibrium with their implications about income distribution as they exist today.

Section III: Some Important Economic Models

It is thought desirable to note some basic features of some of the important economic models that exist at present and see how the observations about distribution theory apply to them. In order to do this some scheme of classification is desirable. Although Davis's[1] classification of the theories into (a) "Marginal Productivity Theories," (b) "Macro-economic Theories," and (c) "Institutional Theories" or "Group Dynamics" could be maintained, a different procedure is intended here. The purpose here is to study the theory of distribution in the context of economic growth, growth equilibrium, and its stability. The classification is then based on the general theoretical scheme developed by various authors in their works. These are as follows:

1. Static growth models maintaining the assumption of substitutability of factors. Neoclassical theories are the best representatives of this class.

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[1]R. M. Davis, "Recent Development in the Theory of Income Distribution," Proceedings of the Thirty-fourth Annual Conference of the Western Economic Association, 1959, p. 12.

Bohm-Bawerk's[1] theory of capital, including its modern versions,[2] also belongs to this class. In the sphere of capital accumulation and optimal resource allocation over time, Ramsey's[3] model remains a classic example.

As far as the treatment of multiplicity of technique is concerned Robinson's[4] model may be classed here. This is true if multiplicity of technique means variability of the ratio of factors used in producing one unit of output. Her discussions are predominantly characterized by the assumptions of fixed coefficients of production, however.

If it is thought that the alternative processes of production assumed by von Neumann[5] in his general equilibrium model are points in a production function, this, also, can be classed here.

2. Fixed coefficient models without growth. In this class two

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[1]Bohm-Bawerk, op. cit.

[2]C. Blyth, "The Theory of Capital and its Time Measures," Econometrica, XXIV (1956), 467-79. See also his article "Towards a More General Theory of Capital," Economica, XXVII (1960), 120-36.

[3]F. P. Ramsey, "A Mathematical Theory of Saving," Economic Journal, XXXVIII (1927).

[4]Joan Robinson, Accumulation of Capital (Irwin, 1956). See the same author's "Production Function and the Theory of Capital," Review of Economic Studies, XXI, No. 55 (1953-1954), 81-106.

[5]J. von Neumann, "A Model of General Equilibrium," The Review of Economic Studies, XIII, No. 33 (1945-1946), 10-18.

important models, namely the static Leontief model[1] and the Walrasian model of general equilibrium,[2] may be taken.

3. Fixed coefficient models with growth. Harrod-Domar[3] models belong to this class. They deal with aggregate quantities; hence they may be regarded as one-commodity models. The study of Harrod's model is the purpose of the second chapter. The multisectoral generalization of this class of models is to be found in the dynamic Leontief model.[4]

4. Models dealing with unemployment or income-variation condition. In this class Keynesian[5] theory is prominent. This is predominantly a short-run theory. However, some models dealing with economic growth may be regarded as an extension of Keynesian theory. This is true of the Harrod model. In the static level the Keynesian model is efficiently designed to explain the effect of money demand or prices on output. The behavior of entrepreneurs with respect to investment and the whole economy with respect to the purchase of the output of consumer goods and the disposition of money is the most significant aspect of the study in this class.

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[1]W. Leontief, The Structure of American Economy, 1919-1939 (2d ed.; New York: Oxford University Press).

[2]L. Walras, Elements of Pure Economics, trans. W. Jaffe (Irwin, Illinois, 1954).

[3]Harrod, op. cit.; E. D. Domar, "Capital Expansion, Rate of Growth and Employment," Econometrica, XIV (1946), 137-47.

[4]W. Leontief, et al., Studies in the Structure of the American Economy (New York: Oxford University Press, 1953), Chap. III.

[5]J. M. Keynes, The General Theory of Employment, Interest and Money (New York, 1936).

Here a subclass of economic theory can be noted which is based on the earlier ideas of Keynes, mentioned by him in the Treatise on Money[1] about the determination of profit by an entrepreneur's decision to consume and the totality of the decisions of consumers and financial institutions regarding saving, expenditures, lending policies, and so on. Kaldor[2] and Robinson[3] belong in this category. One might even include Kalecki[4] in this subclass for similar reasons. The next section gives a brief idea of their work, although Chapter II considers Kaldor in detail.

6. Lastly, mention is made of the model of Johansen[5] which is in the area of economic growth and shares the assumptions of classes 1 and 3 mentioned above. It has already been mentioned that the properties of this model will be studied in the third chapter.

The purpose of this classification is to provide a way to look at how different theories have sought to explain the forces at work in the economy with different conceptual tools. The models listed in the classification are, it is believed, the ones that are predominant in the economic analysis. In some form or other their implication

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[1] J. M. Keynes, A Treatise on Money, I (London: Macmillan and Co., 1930).

[2] N. Kaldor, "Alternative Theories of Distribution," Review of Economic Studies, XXIII, No. 3, 83-100, and also "A Model of Economic Growth," Economic Journal, LXVII, No. 268 (1957), 591-629.

[3] Joan Robinson, Accumulation of Capital (Irwin, Illinois, 1956).

[4] M. Kalecki, Theory of Economic Dynamics (London: Allen and Unwin, 1954).

[5] Johansen, loc. cit.

influences the ideas which would otherwise look novel. In all these models the analysis of equilibrium attempts to study how equilibrium output, prices of goods, and incomes of factors are determined, to mention only a few. The determination of all these are closely interrelated.

It is not claimed that the above classification is exhaustive. Because of the vast area they cover it is not possible to study carefully all the existing theories in a work like this. Even among the works included in the classification some are not the principal subject matter of this work. This is true of all the static theories. But for the present, some remarks about them will be valuable.

Section IV: Some Observations on Static Models

In this section a brief remark on some existing static theories is attempted without aiming at an exhaustive treatment of the fields they cover. The gist of the remark is that the system of equilibrium envisaged by all such models is consistent with or dependent on the theory of distribution outlined in Section II, above. It will be found that many of the arguments will be repeated, but not, it is hoped, unnecessarily.

A static theory is distinguished from a dynamic theory by its assumption of fixity in the supply of factors of production, consumer taste, and technology. In this framework a variety of theories are developed, some assuming production function with variability of factor proportions and some assuming fixed coefficients, classes 1 and 2 of the last section. The former is characteristic of the neoclassical school whereas the latter is characteristic of the Walraw-Leontief models. Keynesian theory may be considered as a static theory. Kaldor-Kalecki-Robinson theories are designed for dynamics, but their theories of distribution may be applied in a static system too. Before going on to these, the first two will be considered.

In a neoclassical system with factor substitutability, as already observed in Section II, the existence of equilibrium coincides with the determination of factor-prices by marginal productivity. It is the given theory that an economy with larger capital-labor ratio has lower marginal productivity of capital than in the opposite case. This has an important consequence regarding the stability of equilibrium, because it implies that excess of any factor relative to the other is absorbed in the economy by the change in the price of the factors. Thus the economy is perfectly stable.

The given neoclassical theory is challenged by authors who do not believe in the working of stabilizing forces. It should be emphasized that the marginal productivity theory is an integral part of neoclassical theory. Robinson, while not objecting to marginal productivity ideas, finds that the simple comparative statics employed in explaining the two different equilibrium situations is unacceptable. Without going into the complications of her arguments, the writer notes some of the obvious facts which are implied by the variation of factor-proportions in a simple neoclassical model. Consider two economies, A and B, with equal aggregate amounts of output as a whole and equal durability of capital goods. Of these two, A with larger capital-labor ratio has a larger proportion of capital goods in the total output. Thus, in this economy the aggregate consumption is lower than in B having a lower capital-labor ratio. This has some important implications. In the economy A with low rate of consumption the propensity to save is higher at the same level of output than in the other economy. Suppose that in A a low rate of discount of future satisfaction prevails and the rate of interest is low, though savings are high. Assuming zero rate of profit for equilibrium under perfect competition, the wage rate is higher in A. Much confusion will arise if the high wage rate is taken to mean high aggregate consumption. This may not be true theoretically. Aggregate consumption need not rise or it may even fall and still the real wage rate of labor may be higher. If constant return to scale is assumed, as has been done here implicitly, and if both economies are assumed to have the same type of production function, the proportions of income going to labor and capitalists may be invariant under changes in capital-labor ratio.[1] Thus any difference in capital-labor ratio in two economies with the same aggregate output may be explicable only in terms of time preference or something of this sort.

Further, it may be argued that a higher wage rate in the economy with the higher capital-labor ratio is liable to lead to higher cost of producing capital. But this does not present any serious difficulty if it is realized that in capital goods production the new process is more capitalistic because of higher wage rate. Cost in real terms

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[1]This is true if there is unit elasticity of substitution between factors.

is the same. Only the breakdown of the total unit cost between amount of labor and amount of service of capital has changed, while the services weighted by their price may remain invariant too.

Above, the cost of capital was intended to mean the cost of real capital assumed to be measured in physical units in which the goods in general are measured. This may be objected to. If the measurement fairly represents the productive capacity of a plant this cannot be said to be unjustifiable. Even if it is admitted that this measure is not justifiable and the measurement given by Bohm-Bawerk[1] is accepted in terms of average period of production or in terms of the cumulants of the distribution of inputs and outputs in time given by Blyth,[2] no significantly different conclusion would be reached. In such theories capital assumes the character of process of production. The decision of entrepreneurs in choosing a particular time-shape of input and output depends on factor prices and technical conditions. A wage rate higher than the previous one will necessitate expenditure in fixed capital and it naturally happens that more expenditure is concentrated in the beginning and less expenditure is distributed over the lifetime of a process of production.

The dynamic part of this analysis involves the study of the process of adjustment and, as Robinson rightly says, it is not easy.

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[1]See Bohm-Bawerk, loc. cit.

[2]See Blyth, "Towards a More General Theory of Capital," Economica, XXVII (1960), 20-36.

From the comparative statics above it is known that in the model considered here it is always possible that for any arbitrary amounts of various factors of production there can be a full employment stationary equilibrium, and factors earn positive rate of return however small it may be for some of them that are plentiful in supply. But if a fixed-coefficient model is taken, there is difficulty in obtaining an equilibrium with full utilization of factors for any arbitrary amounts of them.

In a Walrasian model, for example, with given amounts of factors, one can determine equilibrium, in general, characterized by excess capacity. The equations relating outputs of goods to the given factors cannot have solutions with positive outputs of all goods except by chance. Hence with given production functions and demand functions for all the goods, equilibrium can exist in which some factors are in excess supply and can be said to have their marginal productivity equal to zero. They naturally become free goods.[1] If this state of affairs is the result of the persistent long-run tendency in the behavior of the economy as a whole perhaps such equilibrium is to be regarded as a genuine long-run equilibrium. Otherwise one cannot accept such a solution as anything more than market equilibrium. In the long run the supply of factors should be allowed to vary. Thus with varying supply of factors it is necessary to consider the effect of the rate of return on the factors. The writer therefore believes that even in the case of

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[1]See Dorfman, Samuelson, and Solow, Linear Programming and Economic Analysis (New York: McGraw-Hill Book Co., Inc., 1958), pp. 351-75.

fixed coefficient of production in the long run each factor remains scarce.

This analysis in the dynamic setting will be pursued in the next chapter. It may be observed that from a purely theoretical point of view the difficulty of obtaining a long-run full employment equilibrium solution of a fixed coefficient model is very often exaggerated.

Some aspects of the Keynesian model which is a static model as far as it is concerned with the short period in which factor supply is fixed, there is no change in the technique of production and the taste of consumers also is fixed, may now be considered. The role of investment in this model is one of generating income rather than that of augmenting the capacity of enterprises. Thus new investment in the Keynesian model does not increase factor supply.

As far as the determination of factor income is concerned Keynes does not depart from the marginal productivity theory though he disagrees with the idea that full employment equilibrium is always possible as a necessary state of affairs. The role of income variation as a determinant of change in aggregate demand and, therefore, of the supply of goods is the crucial part of Keynesian theory.[1] At each level of income with savings equaling planned investment there is equilibrium with corresponding level of employment. This is true for the classical model as much as it is true for Keynes. But for Keynes when there is unemployment a reduction in money wage rate does not raise employment in

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[1]Keynes, General Theory.

general because this leads to a fall in demand thereby reducing the profit. This argument is not in conflict with the distribution theory outlined in Section II, above. In accordance with the present analysis it is through demand and price the marginal products of factors are realized and even explained in terms of them. Fall in demand in Keynesian theory lowers the marginal productivity of plant and equipment. In the short run when the stock of capital is given, a low level of demand is associated with excess of capital compared to the level of production. But this situation cannot be regarded as normal and enduring. The writer believes, with Kaldor,[1] that the unemployment equilibrium is unstable from the point of view of the long-run development of the economy. This will become clear in the analysis in the next chapter.

When demand increases, the level of production rises as long as there is possibility of increasing employment. In the short run, however, increasing employment leads to diminishing marginal productivity of labor. Keynes says that up to the point when surplus labor is absorbed "the decreasing return from applying more labour to a given capital equipment has been offset by the acquiescence of labour in a diminishing real wage."[2] After this point a rise in employment necessitates a rise in wage by the equivalent amount the product increases,

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[1]See N. Kaldor, "A Model of Economic Growth," Economic Journal, LXVII, No. 268, 593-94.

[2]See Keynes, General Theory, p. 289.

"whereas the yield from applying a further unit would be diminished quantity of the product. The conditions of strict equilibrium require, therefore, that wages and prices, and consequently profits also, should all rise in the same proportions as expenditure, the 'real' position, . . . being left unchanged."[1]

The above suggests that the marginal productivity theory is used by Keynes to explain distribution. It is true that labor does not always clamour for its marginal product. But in equilibrium at full employment wage rate is in line with the marginal product. It may be asked, however, whether marginal product is the only limit to which wage rate can rise. This is the attitude held by Kaldor.[2]

If marginal product of labor is regarded as the upper bound of wage rate it follows that under constant returns to scale the marginal product of capital is the lower bound for the rate of profit. In the next chapter it will be seen that with some additional restrictions on these boundaries, Kaldor's theory of distribution says that the profit rate and wage rate should lie between these boundaries. The Keynesian argument above shows that in full employment the marginal product of labor is also the lower bound for wage. Hence it follows that wage rate must be equal to marginal product of labor. Further it should be observed that in a competitive world a rate of profit higher than the marginal product of capital would induce further investment until it equals the

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[1] Ibid.

[2] See N. Kaldor, "A Rejoinder to Mr. Atsumi and Professor Tobin," Review of Economic Studies, XXVII, p. 119.

latter. Thus in full employment the marginal product of capital is the upper bound for profit because a return higher than this is liable to be competed away.

The reason for developing a theory of distribution without taking marginal productivity into account was in the fact that profit and wage, according to Kaldor, could be anywhere within the limits referred to above, depending upon the expenditure habits of capitalists and labor. Kaldor develops a theory of distribution which is based on the idea given by Keynes in his Treatise on Money that the rate of profit depends on the consumption decision of labor and enterpreneurs, the decision of the latter to invest and the policy of the banks regarding credits.[1] In that same work Keynes mentioned the widow's cruse doctrine according to which the profits of entrepreneurs increases with the increase in the consumer expenditure of enterpreneurs. As a casual interpretation the validity of such a theory is dubious. But as will be seen in the next chapter, Kaldor builds up a systematic theory of distribution out of these ideas. The same is true of the theory of Robinson, though quite consistently she does not give up the concept of marginal productivity factors.

Within this same category Kalecki's theory can be considered. First in his theory of profit, he equates, in his model,[2] gross profit to the gross investment plus capitalists' consumption assuming that

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[1]Keynes, A Treatise on Money, I, Chap. VI.

[2]Kalecki, Theory of Economic Dynamics, pp. 45-52.

workers consume all their income. He says that the significance of the model lies in the fact that it reveals a way to interpret which of the terms of the equation is under the influence of capitalists' decisions. He says, "Now, it is clear that capitalists may decide to consume and to invest more in a given period than in the preceding one, but they cannot decide to earn more. It is, therefore, their investment and consumption decisions which determine profits and not vice versa."[1] For explaining economic behavior this idea seems to have little value, because the more acceptable line of thought would say that it is the expected profit that determines the investment activities of the entrepreneurs and their consumption expenditures. Without going into further criticism of this theory mention might be made of another part of Kalecki's theory in which the level of wage is explained with the help of the degree of monopoly arguments.

One formulation of this theory is the measurement of the degree of monopoly by the ratio of gross profit to the gross proceeds of the economy. This leads to the conclusion that rise in the degree of monopoly leads to the fall in wage rate. This is another aspect of the theory of profit mentioned. It can be observed that for a given output a rise in profit means a rise in the degree of monopoly and a fall in wage level. This follows from the identity in which total income is equated to wage and profit.

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[1] Ibid., p. 46.

Another formulation of this theory is by defining the degree of monopoly as the ratio of aggregate proceeds to wage plus material-cost. This does not add anything new to the theory except that a new term, material-cost, is introduced and the terms of the equation are manipulated in a different way. It is obvious that the share of wage in the total income declines as degree of monopoly increases and/or the ratio of material cost to wage in the prime cost rises. As expost identities these ideas are obvious. But as an explanation of the determination of factor prices they are inadequate. Further discussion on this point is postponed until the next chapter. It should be noted that Kaldor's theory and Kalecki's theory are similar.

The above discussion may be summed up by saying that the so-called macro-theory of distribution does not become meaningful unless the terms of the equation are represented as functions, behavioral or otherwise. Demands for labor or investment cannot be explained except on marginal productivity lines. It is doubtful whether any other approach can take its place. Consumption demand is essential in distribution theory, not because of what Kalecki or Kaldor think to be its role, but because this determines the price of consumer goods and, therefore, the derived demand for factors.

In passing it should be noted that the Keynesian "general" theory has many virtues of a dynamic theory. The income variation by change in the level of investment is a dynamic concept. But the full elaboration of the dynamics requires the consideration of the effect of investment on the productive capacity of the economy. Thus a continuous investment activity does not lead to generation of income by the multiplier process alone. It also leads to increasing income over time by augmenting the supply of capital. In the next chapter the Harrod model will be found to be one important generalization of Keynesian theory to cover the dynamics absent in Keynes' work.

The next section will consider in brief some dynamic models which exist today as an introduction to the present work.

Section V: Some Dynamic Models

The principal purpose of the study of some dynamic models in this section is to discuss some important features of the dynamic theory with reference to income distribution and the concept of equilibrium growth. It was already noted that a static theory was concerned with the study of forces acting in an economy with fixed supply of factors and constant taste and technology. In a dynamic model, however, these are allowed to vary. But the manner of variation of these factors is either to be treated as being influenced by economic events or in some ways that are not directly related to the process of economic development such that the economist assumes certain regularity or irregularity about them. The study of economic dynamics with the closest approximation to reality would have to sacrifice much of analysis and logic and become a mere history of growth. For analysis sufficient simplifying assumptions are helpful in deriving logical conclusions. Comparisons of conclusions derived under different sets of assumptions reveal many interesting properties of a developing economy.

At the outset one observation about the dynamic model is necessary. The preoccupation in such models being to study the overall development of an economy, the study of price theory in its details disappears. In multisectoral models one may have price theory for determining relative prices of goods and services in general. But a general theory of determination of absolute prices becomes difficult even in such models. In aggregate models the theory of price becomes a little awkward, not to speak of relative prices.

The pricing of factor services, however, does not disappear from the growth models. In fact this assumes more importance. This is one basic question around which the analysis of equilibrium growth is to be developed. The classical model of economic growth is perhaps the best evidence of the value of the growth model for studying income distribution. In the Ricardian model, for instance, the problem of capital accumulation and distribution are one. In this theory the whole process of capitalist development is a closed historic process except for technological progress which does not have an obvious connection with the rest of the events.

But modern dynamics differs from the classical in that the assumptions made by the classical economists, especially by Ricardo, about, for instance, diminishing returns to land and the Malthusian theory of population, are no longer considered in the models as basic facts. However, there are some properties of modern dynamic analysis that resemble the classical.

It should be emphasized that in modern economic dynamics the influence of the Keynesian idea of underemployment equilibrium is visible. In some of the theories this idea is accepted and given a more elaborate treatment. This is true of the Harrod model. In this model unemployment is not only a possibility, but it is regarded as consistent with long run equilibrium. Full employment is, in this model, not necessarily equivalent to equilibrium. This idea is held by Robinson also.

At the other extreme is Kaldor's analysis[1] which regards unemployment as an unstable and temporary situation. His model disregards entirely the possibility of unemployment. He evidently attempts to distinguish between the theory of long run economic development and the theory of cyclical fluctuations.

Still another variety of theoretical system exists which concludes that the economic system is perfectly stable. This is the neoclassical theory. Except for the basic premises on which his analysis rests Kaldor would belong to the neoclassical school.

Some further observations on these models are desirable at this point. According to the neoclassical theory, in growth equilibrium it should be noted that the time pattern of income of all output should satisfy the decision making units in the economy--the entrepreneurs, factor-owners, and consumers. The technological assumption in neoclassical models was already mentioned in the last section. Under such

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[1]See Kaldor, "A Model of Economic Growth," pp. 593-94.

conditions for all time patterns of supply of factors and technological advance and even change in consumer preference, the model leads to the conclusion that equilibrium exists with full employment. The reason is that distribution of income between factor-owners has a stabilizing influence. If only the growth of labor supply is taken it will be shown in the next chapter that a stationary solution for growth rate can be obtained such that any deviation from this will set forces at work toward the restoration of equilibrium. This happens because the earnings of the entrepreneurs become maximum at the equilibrium. A decline in the growth rate of labor, for example, leads to a higher wage rate and increased capital-labor ratio.

This fact was noted in the last section on the basis of comparative statics. In dynamics the process of adjustment is to be explained and it should be admitted that it is hard to visualize clearly. But some points that are clear should be noted. Assuming that capital and labor are the only factors of production, a sharp reduction in the growth rate of labor relative to the growth rate of capital leads to a higher wage rate. If equilibrium is to continue capital must replace labor to some desirable extent. But such a tendency cannot stop unless the growth rate of capital falls to the same level as that of labor. However, this is quite possible. With a low rate of growth of labor, output cannot increase at the previous rate. Given constant marginal propensity to save, the rate of accumulation declines. But this decline does not lead immediately to the new equilibrium rate. In each period the economy experiences a growth rate that is lower than in the previous period. In mathematical language this leads to an infinite sequence of decreasing growth rate of output and capital which steadily converges to the growth rate of labor. The necessary condition for this to be realized is every time entrepreneurs should move along the production function in the direction dictated by the supply of factors of production. How this happens is a problem which is too complicated to visualize. The intricate problems as to various possible reactions of the economy during the adjustment are avoided in this work.

In the fixed coefficient models there are difficulties mentioned in the previous section in connection with the statics. There is the possibility of superfluity of some factors because of the difference in the growth rates. The question of the existence of full employment equilibrium in such a case will be studied in the next chapter. Here it should be noted that the pricing mechanism which is implied in such models can be a guide for study.

Kaldor[1] has, by assuming variability of aggregate saving as a proportion of income, rising capital output with the increased availability of capital and profit, and also by making the growth rate of labor depend in some manner on the growth rate of income, made the system more flexible.

There is one more economic model which is worth mentioning. That is von Neumann's model of general equilibrium.[2] This model has

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[1] *Ibid*.

[2] Von Neumann, *op. cit*.

some interesting properties which require attention. In most models the possibility of joint products is ruled out. But in this model no difficulty arises by allowing joint products. Unlike other models this one has no primary factors of production like land and labor and no final consumption. All are included within a systematic input-output scheme so that all goods and factor services are inputs and outputs. Consumption is an input for the production of labor as its output. Thus this model may be regarded as a pure production model.

The neglecting of the fixed factors of production by von Neumann in his model is mainly for the purpose of studying a growing economy.[1] The assumption about technology is that there are alternative processes of producing a commodity. The choice of any particular process is determined under equilibrium. Each process or activity is of the fixed coefficient variety which implies that constant returns to scale prevail. Each process has a unit time duration, and if any one has more than a unit time duration, intermediate stages can be introduced and the resulting processes would last one unit interval each.

Similarly for services of durable capital goods he devises a method which says that "wear and tear of capital goods are to be described by introducing different stages of wear as different goods using a separate process for each of these."[2] The non-depreciating capital good might appear as both input and output or one of many

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[1]Such assumption would be unnecessary if the idea of constancy of growth rate were given up or technological progress considered.

[2]Von Neumann, op. cit., p. 2.

possible joint outputs.

This model shows the outputs of one period becoming inputs for the next period. The amount of production at any time is limited by the available output at that time. All the relationships appear as inequalities which are characteristic of the linear programming problems. The solution should rule out the possibility of negative outputs and prices. Some goods, however, may be in excess of the amount required as current input. In that case their prices will be zero. The process of production chosen should bring the maximum amount of return on capital invested. Otherwise such a process is not chosen. Regarding profit rate, the model is based on the assumption of competitive conditions ruling in the economy, and the profit rate is zero.

One feature of the model is that it takes care of both production and consumption at the same time. This, however, is subject to criticism, to be discussed later.

The solution of the model yields an economy expanding at a constant balanced rate of growth in which the processes adopted yield zero rate of profit. This property, as shall be seen later, is shared by most of the dynamic models. If the equilibrium rate of growth is r then the model implies that there is at least one good which grows at the rate r. All goods growing faster than at the rate r are free goods. Since the ratio of the value of output to the value of input gives the rate of growth and the same ratio gives the rate of interest in equilibrium, it follows that the two rates are equal. The proof of von Neumann shows that the two rates are equal at only one value. Thus growth rate and interest rate are uniquely determined. But for prices of goods and their amounts there is more than one solution.

It should be noted that the determination of prices in this model does not differ from the one in the Walrasian model. There also it was observed that all excess resources become free. But the difference lies in the explicit introduction of the demand functions in the Walrasian model which is not true for the von Neumann model. Though demand is implied in the latter it is by way of much idealization.

In the assumption that consumption is an input to labor it resembles the Malthusian model. But it may be noted that in the von Neumann model wage need not be at the subsistence level though it will remain at a constant rate because of the constant coefficient assumption. Further, the model has close similarity to the classical ideas in that the total output of any time becomes input for another period, if it is realized that the total output consists of fixed capital, considered as intermediate goods in von Neumann's model, working capital or goods-in-process, and wages-fund represented by the consumer goods assuming that the capitalists do not consume.

The model can be criticized on the ground that it presents an extremely "idealized" picture of economic development.[1] The idea of equilibrium should in this model cover a very wide range of phenomena. The determination of economic goods and free goods, the determination

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[1]See Champernowne, "A Note on J. v. Neumann's Article on 'A Model of Economic Equilibrium,'" Review of Economic Studies, XXIII, No. 33, pp. 10-19 for an elaborate explanation of the model and also some criticisms.

of economically useful processes and processes worth rejecting are some of them. In the usual theoretical analysis one is concerned directly with usable processes and economic goods, so that any solution that leads to zero value of goods becomes more likely a disequilibrium situation.

Again it is too bold to use a closed model in which factor supply is explained by the simple economic factors considered in the model. In case of labor supply there may be some difficulty in accepting the solution. Moreover the solution becomes entirely unacceptable when it happens that the supply of labor grows at a rate faster than the equilibrium rate of growth. Thus it is necessary to exclude some of the factors from the model, which are considered here as being determined within the model, and consider them separately. Labor supply could be studied separately and the pricing of labor might be studied along the line of the more acceptable theories. The role of interest could be studied by reference to the behavior of consumers regarding the disposition of their income between consumption and saving. Some attempts have been made in recent years to bring the von Neumann model in line with the usual approach by allowing labor to grow at some finite given rate, by allowing workers' consumption to depend both on their real income and prices, and so on.[1]

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[1]J. G. Kemeny, et al., "A Generalization of the von Neumann Model of an Expanding Economy," Econometrica, XXIV (1956), 115-135. Also M. Morishima, "Economic Expansion and the Interest Rate in Generalized von Neumann Models," Econometrica, XXVIII (1960), 352-363.

With proper changes in assumptions in the directions followed by authors like Morishima, one can have a much better approach for studying equilibrium growth. It should be noted in this connection that in order to be realistic it is necessary to allow some exogenous trend in the economy and therefore to introduce a sufficient degree of freedom.

In the above discussion about the von Neumann model it was noted that in equilibrium certain processes of production are determined as selected by the economic units. This means that although there are alternative processes the equilibrium process for any one good exists. This is similar to the neoclassical conclusion. Later, in the next chapter it will be observed that under certain circumstances the coefficients of production become uniquely determined. This is the content of the so-called substitution theorem. Interpreted within the framework of pricing of factors this theorem will be shown to have some valuable implications for the growth models to be studied.

Before concluding this section it should be observed that in the line of multisectoral growth Johansen has also made a valuable contribution.[1] The assumptions about technology are neoclassical. However the more interesting work at present is the one previously mentioned in which there is an attempt to synthesize the fixed coefficient models with the substitution models. A multisectoral extension of his work is most difficult. But satisfaction can be obtained by finding out

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[1]L. Johansen, A Multisectoral Study of Economic Growth (Netherlands: North Holland Publishing Co., 1960).

various implications of that model with the use of the marginal productivity theory of distribution.

There may be many other growth models but they will, it is believed, fall into one class or another discussed in this section. In conclusion it may be said that there are, strictly speaking, two broad categories of dynamic theory, one applying Harrod's assumption about technology, another using the assumption of the neoclassical theories. Controversies exist about the question whether in the long run full-employment equilibrium is possible with the working of economic forces that are often emphasized as characteristic of the freely working capitalistic economy.

Section VI: Conclusion

A reflection of the arguments and discussions in the preceding pages will show that the theory of economic growth even in all its various forms is subject to much controversy. But the complexity of the real world is responsible for all this. However, for the economist the value of any particular theory lies not in answering all questions with which one is confronted in reality, but in deriving conclusions about the things to which the theory is relevant; and a theory may not be relevant for all the questions asked. In this work the aim is neither to develop a realistic theory nor to answer questions about any practical issue. The object is to find some of the important logical implications of the models mentioned in the first section of this chapter. The foremost among them is to find out how distribution of income works in determining equilibrium growth.

The author's own prejudice is in favor of Johansen's model. The reason is that it is not only more realistic than others from the point of view of technological assumptions, but it also gives explicit recognition to the fact that durability of capital or a production process, if one may call it, has its role in determining equilibrium, and it is an important determinant of output. This is not to be taken to mean that this will resolve the complicated issues about the proper analysis of the theory of capital with due regard to its complexity. The most that is hoped at present is that some simple properties that are not considered explicitly in other models will find explanation in the one referred to here.

CHAPTER II

HARROD'S MODEL OF ECONOMIC GROWTH AND ITS CRITICISM

Section I: Introduction

The purpose of this chapter is to investigate the properties of Harrod's model of economic growth. The reasons for the choice of this model as the center of discussion are: (1) This model has been the subject of many interesting controversies and attacks by many writers; hence, almost all alternative formulations of the theory of economic growth can be studied on the basis of the criticisms advanced against it. (2) This model has acquired much prominence in the discussion among economists who are concerned with the problem of development of underdeveloped areas. (3) The problem of economic instability may be attacked with this type of model or with some modification or extension of it. (4) This model has its peculiar implications about prices of goods and factors in general, whose relationship with the problem of full employment versus equilibrium growth rate is the crucial point to be discussed.

In the next section Harrod's model will be developed. The third section is devoted to the concept of instability which is one significant part of the model. This is related to the idea of divergence of actual from equilibrium growth rate which provides, it is held, a basis for studying cyclical fluctuations. More significant from the point of view of the secular trend of the economy is the impact of resources and technological progress on growth. The model has to suffer much criticism on the point that equilibrium growth determined by the usual income equation and the equation relating saving and investment is unrelated to the potential growth determined by the increasing supply of labor, for instance, and the rate of progress of know-how, though the difference between the two may influence the actual development of the economy. In the fourth section attention is diverted to Solow's criticism of the model and to consideration of his contribution to analysis of growth. The effect of divergence between equilibrium and actual rates of growth on the distribution of income and its role in restabilizing the economy in the long run acquires importance in the discussion. In regard to income distribution Harrod's model suffers from lack of determinacy because it is based on the assumption of a fixed coefficient of production. The alternative theory of distribution given by Kaldor seeks to resolve the problem of income distribution and the problem of adjustment of equilibrium rate of growth to the maximum achievable rate of growth at one stroke. In Section V this theory of distribution will be studied which will be found inadequate to explain distribution and also it will be shown that it assumes income distribution which is then to be explained by some other theory. Besides, it will be found that the independence of investment-income ratio assumed by Kaldor in his distribution theory is arbitrary. One might use the marginal productivity theory to explain distribution in Harrod's model. But this does not become precise enough for one commodity model of

Harrod. In order to keep the exposition simple a three commodity model is built up instead of a general multisectoral one for explaining distribution in Section VI, which is the main thesis of the present chapter. Two of the commodities will be consumer goods and the last one will be capital good. Allowing sufficient flexibility of prices of goods and factors in the market it will be shown that balanced growth with full employment can result. This implies that fixed coefficient of production is no hindrance to the realization of full employment growth rate.

In the seventh section consideration is given to the important possibility of capital-deepening under certain conditions, which is not denied by Harrod. Before showing this, consideration is given to the substitution theorem developed by Samuelson and Morishima which shows that under constant returns to scale with certain given assumptions it is possible that a unique ratio between factors becomes observable in the economy despite the fact that the possibility of substitution exists. The implication of this theorem for Harrod's model will be studied, and using a modified dual of this problem it will be observed that this fact is implied in Harrod's theory. However, a discussion will be presented of the arguments about whether full employment growth is guaranteed once this idea is introduced.

In the last section conclusions of the chapter are presented with a view to the subsequent analysis of Johansen's model.

Section II: The Structure of the Model

The fundamental character of Harrod's model is that it is predominantly Keynesian. It is distinguished from the latter in that investment is more specifically a dynamic element of the economy. Harrod gives investment a two-fold character, namely, that of creating income via the multiplier and that of creating additional capacity.

It is assumed that investment in any time is planned by entrepreneurs in such a way as to maintain a certain constant relationship between change in income, dY/dt, at that time and investment, $I(t)$. That is, a constant acceleration coefficient is assumed. Thus with $Y(t)$ as income at t and $I(t)$ as investment (both regarded as functions of time), and C_r as the acceleration coefficient:[1]

II-1 $$I(t) = C_r \frac{dY}{dt}.$$

The above equation gives the ex-ante investment of the entrepreneurs. It is meant to be behavioral in nature expressing the inducement to investment proportionate to change in output. It is also a technological assumption, however, saying that there is a constant proportion of capital, K, employed per unit of output Y. It is not necessary that investment expost should satisfy the above relationship except in equilibrium.

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[1]The formulation of Harrod's model here is based on R. G. D. Allen's Mathematical Economics (London: Macmillan and Co., Ltd., 1957), pp. 64-69. For similar versions see D. Hamberg, Economic Growth and Instability (New York: W. W. Norton and Co., Inc., 1956), Chap. III.

Another significant assumption is about the constancy of the marginal and average propensity to save. If s is the savings coefficient and if S(t) is saving at time t,

II-2 $$S(t) = sY(t)$$

This is the equation for the planned saving for the community as a whole. All the plans for saving out of current income are assumed to be realized.

Equilibrium requires that ex-ante saving should equal ex-ante investment at all levels of income. That is, with a given savings coefficient there should be at any time a rate of investment which will insure a sufficient level of demand "to leave producers content with what they have done." The equilibrium condition is:

II-3 $$sY(t) = C_r \cdot \frac{dY}{dt}$$

This is a simple differential equation whose solution is:

II-4 $$Y(t) = ke^{s/C_r \cdot t}$$

where k is determined by initial conditions and $s/C_r = G_w$ is called the warranted rate of growth. It shows that income has to grow at the exponential rate of s/C_r. This gives the equilibrium time path for output and capital as well. The rate s/C_r is an equilibrium rate because "if it is realized it will leave all parties satisfied that they have produced neither more nor less than the right amount. Or, to state the matter otherwise," to quote Harrod, "it will put them in a frame of mind which will cause them to give such orders as will maintain the same rate of growth."[1]

In actuality, however, the equilibrium so defined may not exist and the rate of growth may not equal the warranted rate of growth. The precise explanation of this statement will occupy the next section. But in Harrod's line of reasoning the difference between the warranted rate and actual rate of growth results from the fact that actual capital-output ratio and the desired capital-output ratio may be different. The totality of events in the economy may result in a capital-output ratio, C, in the community whereas the required ratio is C_r. If $C = C_r$ equilibrium exists and is self-perpetuating. If there is inequality between the two the actual growth rate, G, is different from G_w. The equations for G and G_w are then given as

II-5 $G = s/C$

II-6 $G_w = s/C_r$

From the above equations it can be seen that if G has a higher value than G_w, with s constant, C will have a value below C_r, which means that on balance producers and traders find the goods in the pipe-line or the equipment insufficient to sustain existing turnover.[2] Regarding this as an extraordinarily simple and notable demonstration of an advancing system Harrod argues that "around the line of advance, which if adhered to would alone give satisfaction, centrifugal forces

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[1]Harrod, "An Essay in Dynamic Theory," Economic Journal, 1939, p. 16.

[2]Harrod, Towards a Dynamic Economics (London, 1948), p. 85.

are at work, causing the system to depart further and further from the required line of advance."[1]

Section III: Instability

At the outset of this discussion the explanation of the term C_r is considered worth repeating, because of its relevance to the explanation of instability. Considering C_r as a term in the production function meaning that one unit increment of output requires C_r unit increment of capital it is clear that with less than C_r units of increment of capital the production of an additional unit of output cannot be achieved. Take the following two relationships

II-7 $$\frac{dY}{dt} \cdot \frac{1}{I(t)} = C$$ and

II-8 $$\frac{dY}{dt} \cdot \frac{1}{\underline{I}(t)} = C_r$$

where $\underline{I}(t)$ is the ex-ante investment and $I(t)$ is the expost investment. It is hard to distinguish between I and $\underline{I}$, along the line in which the accelerator is defined. By introducing lag, however, one can explain the divergence between warranted rate and actual rate. The following arguments are subdivided into special cases depending on the assumptions made in explaining instability. In cases A and B the arguments of other writers are given and in C some other possible explanations are discussed.

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[1] Ibid., p. 86.

Case A. Here are summarized the arguments of R. G. D. Allen.[1] If all the equations are put in the period form,

II-9 $\underline{I}(t) = C_r \Delta Y(t) = C_r [Y(t) - Y(t-1)]$

This investment $\underline{I}(t)$ may not be realized. If saving has one period lag and $S(t) = sY(t-1)$, because $S(t)$ is exactly realized expost,

II-10 $sY(t-1) = I(t) =$ investment expost.

The difference between $\underline{I}(t)$ and $I(t)$ has an impact on business conditions by generating greater or less pressure on demand than would be in equilibrium when the two were equal. If the difference is written as,

II-11 $D(t) = \underline{I}(t) - I(t) = C_r \{Y(t)-Y(t-1)\} - sY(t-1).$

A positive value of $D(t)$ means that there is an excess demand for investment goods in period t, the magnitude of this excess being determined by the value of $\underline{I}(t)$ and $I(t)$. If $D(t)$ is negative there is surplus of saving or excess of investment supply.

If it is assumed that in period t+1 output is planned in such a way as to fill up the deficiency or remove the surplus,

II-12 $Y(t+1) = Y(t) + D(t) = Y(t) + C_r \{Y(t)-Y(t-1)\} - sY(t-1)$

$= Y(t)\ (1+C_r) - Y(t-1)\ (C_r + s)$

In this formulation it should be noted that output in t+1 was meant just to fill up the deficiencies of the past income or remove the surplus, if there were any, and if $D(t)$ were zero the output at t+1 would be naturally at the stationary level in which it was in t. The

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[1] Allen, op. cit., pp. 74-78.

fallacy of this argument is obvious. However, the result may be seen. II-12 is a second order difference equation whose characteristic roots depend on the values of C_r and s. Allen[1] suggests that for likely values of C_r between 1-s and $1+2\sqrt{s}$ the course of Y(t+1) given by II-12 is an explosive oscillation.

Another model formulated as an improvement of the one given above is built up on the assumption that output at t+1 grows at the warranted rate. Thus the model takes into consideration that whereas output grows at the warranted rate any discrepancy of the previous period is also sought to be removed. Thus,

$$\text{II-13} \qquad Y(t+1) = (1+G_w)\ Y(t) + D(t).$$

$$= (1+G_w)\ Y(t) + C_r\,[Y(t)-Y(t-1)] - sY(t-1)$$

$$= (1+G_w+C_r)Y(t) - (C_r+s)Y(t-1)$$

This is another second order equation whose solutions are $1+G_w$ and C_r. Again it is said that for likely values of s and C_r the path of Y(t) is non-oscillatory and explosive.[2]

This leads to the conclusion arrived at by Harrod, though by a different route, that the dynamic system is unstable.[3] Hence a slight departure from equilibrium is likely to be explosive. In such a case

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[1] Ibid., p. 76.

[2] Ibid., p. 77; the details of the solution and the discussion about the various possibilities of the behavior of Y(t) need not detain one here. In most cases C_r is sufficiently high to warrant the conclusion reached above.

[3] For similar proof of instability see D. Hamberg, op. cit., pp. 202-204.

the only explanation about the stabilization of an economy should turn on the possibility of the limiting forces acting on the parameters of the system. Thus an economy may be thrown out of one regime--one regime being that in which one particular set of values of parameters rule--to another. This possibility cannot be accepted for a model which is based on the assumption of fixed values of the parameters.

Case B. Another interpretation of instability may be considered. In this case the concept of equilibrium requires some explanation. It is held that the very nature of warranted rate of growth is that this is the resultant of all forces that drive the various components of the economy into equilibrium relationships with one another and describe a time path for them that the relationships hold forever unless some external forces disturb them. In such overall development of the economy an external disturbance that drives the economy out of equilibrium path may generate forces that lead the economy away from equilibrium. But as time passes such forces may be swamped by the equilibrium tendency of the economy. In such case the economy may be perfectly stable. As an important representative of this class of instability analysis the arguments of Jorgenson may be considered.[1] The substance of Jorgenson's argument is as follows:

When investment expost is less than ex-ante investment there is excess demand. In such a case the entrepreneurs react to excess demand

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[1] D. W. Jorgenson, "On Stability in the Sense of Harrod," Economica, XXVII, No. 107 (1960), 243-52.

by ordering more investment goods and at the same time they maintain the warranted level of output. If D(t-1) is defined as excess demand at time t-1, it can be conceived that it may grow in the next period. This is in line with the previous reasoning. However, the growth of excess demand depends on the nature of the reaction of the entrepreneurs. Assume that excess demand is growing at the rate k. Then,

II-14 Excess demand at $t = D(t) - (1+k)D(t-1)$; D is as previously defined. k is assumed to be greater than zero. Using II-11,

II-15 $$Y(t) = C_r Y(t-1) + (s/C_r)Y(t-1) + D(t)/C_r,$$
$$= Y(t-1)\,(1+s/C_r) + (1+k)/C_r\, D(t-1);$$
$$= Y(t-1)\,(1+G_w) + (1+k)/C_r\, D(t-1).$$

Thus,

II-16 $$D(t-1) = C_r\left\{Y(t-1) - Y(t-2)\right\} - sY(t-2)$$

Inserting II-16 in II-15 and multiplying both sides of II-15 by C_r there is by rearrangement of terms,

II-17 $$C_r\left[\left\{Y(t) - Y(t-1)\right\} - sY(t-1)\right] = (1+k)\,\left[C_r\, Y(t-1)-Y(t-2) - sY(t-2)\right]$$

This equation can be further rearranged to yield:

II-18 $$Y(t) = Y(t-1)\left\{(1+G_w)+(1+k)\right\} - Y(t-2)(1+k)(1+G_w)$$

This is a second order difference equation with roots $1+G_w$ and $1+k$. The solution can be written in the form:

II-19 $$Y(t) = A_1\,(1+G_w)^t + A_2\,(1+k)^t$$

where A_1 and A_2 are determined by initial conditions. The course of Y(t) is naturally determined by the dominant root. If $G_w > k$ warranted rate of growth will be arrived at; this can be regarded as a sufficient condition for stability of Harrod's dynamics. It can be conjectured that G_w is always likely to exceed k. Thus Harrod's conclusion need not hold for all situations where warranted rate is different from the actual rate.

Perhaps the significant point in the above argument lies in the fact that for a small departure of the actual growth rate from the warranted rate of growth the latter remains dominant in the system and the stress of excess demand or excess supply slowly withers away. In order to have explosive movement the deviation of the actual rate from the warranted rate should be such as to disrupt all balance altogether between the various sectors of the economy.

One might also interpret the role of k in another special way. In a disequilibrium situation it may not retain one constant value. As time advances it may become smaller and smaller. This implies that the mistakes of the decision-makers will decrease over time. In the neighborhood of equilibrium it may be so. But whether it can be so in all situations, or, in other words, whether G_w remains bigger in all conditions, is a question which cannot be answered without going to the analysis of market mechanism in all phases of the development of the economy in detail. Among other things, it should be observed, although Jorgenson does not argue in this line, that the inability of entrepreneurs to accumulate stocks or to decumulate them determines the value of k.

Although in the above discussion the concern is with the situation when there is a departure from the equilibrium, the same argument can be applied for the condition of restoration of equilibrium of the economy when it is not in equilibrium to begin with.

There are yet other possibilities for describing the course of events resulting from the divergence of actual from the warranted rate of growth. In what follows are presented three different models.[1]

<u>Case C</u>. If it is assumed that increment in output at time t results from the investment in period t as before,

II-20 $$Y(t)-Y(t-1) = \frac{1}{C_r} I(t).$$

Assume that in period t-1 the capital-output ratio was $C \neq C_r$, i.e.,

II-21 $$C\{Y(t-1)-Y(t-2)\} = I(t)$$

The difference in the lags given by the two equations above show the difference between the desired ratio and the actual ratio of capital to output in the margin.

II-22 $$Y(t) = Y(t-1)(1+\frac{C}{C_r}) - C/C_r\ Y(t-2)$$

The solution of the above yields two roots 1 and C/C_r. If $C/C_r > 1$, the economy will grow at that rate; if not it will tend to be stationary. Another way to depict the situation is to assume that investment in period t leads to increment in output at t+1. Thus, instead of II-20,

II-23 $$I(t) = C_r\{Y(t+1) - Y(t)\}$$

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[1]The following line of reasoning is suggested to the author by Professor Paul Simpson.

Equating this to II-21,

II-24 $$Y(t+1) - Y(t) - C/C_r\{Y(t-1)-Y(t-2)\} = 0$$

The above is a third order difference equation which has three roots, $+\sqrt{C/C_r}$, $-\sqrt{C/C_r}$, and 1. The solution can be written in the form:

II-25 $$Y(t) = A_1\ (\sqrt{C/C_r})^t + A_2\ (-\sqrt{C/C_r})^t + A_3$$

In the above solution it may be observed that output will converge to a stationary level if $C/C_r < 1$. In the opposite case it may grow if $A_1 + A_2 > 0$, and $A_1 > A_2$. However, if $A_1 = A_2$ there is oscillation and Y(t) becomes explosive for large even value of t.

In the above models it is easily seen that if $C = C_r$ output in each period would rise by the same amount as it did in the previous period.[1] The role of saving is not quite obvious, because what part of income is saved by the community has not been considered nor its effects. If a more complicated system is built with equations involving savings, this would naturally be a more complete model, but the gain in precision is doubtful. Moreover the above analysis leads to an entirely different notion which is not consistent with Harrod's analysis. It seeks to establish a different growth pattern which would be an equilibrium one with different initial conditions. It is possible that if the assumptions underlying the model are not so different as to distort Harrod's model, the instability would mean merely passage from one equilibrium to another. This interpretation obviously implies a special meaning

[1]The series of output describes an arithmetic progression.

of stability.

The savings function may now be considered, keeping the assumption of Harrod that savings decision is always fulfilled. Using the saving-investment identity the following equation is obtained:

II-26 $$sY(t) - sY(t-1) = I(t) - I(t-1)$$

Further, let I(t) and I(t-1) be given by the equations

II-27 $$I(t-1) = C_r \{Y(t-1)-Y(t-2)\}$$

and $$I(t) = C \{Y(t) - Y(t-1)\}$$

Combining II-26 and II-27 and rearranging the terms the following is obtained:

II-28 $$(s-C)Y(t)-(s-C-C_r)Y(t-1)-C_rY(t-2) = 0$$

This model is straightforward enough to require no explanation in particular. The roots of II-28 are 1 and $C_r/(C-2)$. The solution can be written as

II-29 $$Y(t) = A_1+A_2 \{C_r/(C-s)\}^t$$

where A_1 and A_2 are determined by initial conditions. This solution resembles another growth equilibrium. The term, $C_r/(C-s)$, can be compared to the warranted rate of growth in Harrod's model, if $C_r = C$. In the present case, however, if $C-s > C_r$ and $C \neq s$ there will be a long run tendency for the income to converge to a stationary level. If $C_r > C-s$ and if $C \neq s$ there may be a growth rate that is different from the warranted rate of growth. This possibility is not without significance. It is not impossible to imagine a situation in which production took place in the previous period with full utilization of capacity and later on excess capacity was allowed as a matter of business policy, or the change in capital-output ratio was accepted by the entrepreneurs.

But the disequilibrium interpretation of the model is important for Harrod's case. In a case where $C > C_r$ excess capacity rules in the economy and this is undesirable considering the production technology. Although for $C_r < C < C_r + s$ there still is growth in the above model the entrepreneurs are not to be thought of as satisfied with the new capital-output ratio which is the result, for instance, of the undesirable piling of the goods or otherwise. The removal of this from the point of view of the entrepreneur can be effected by reducing output which leads to further rise in C. Thus output will conceivably fall at an accelerated rate. Similarly, one can argue about the case when $C < C_r$ which leads to higher and higher output, than under the warranted rate of growth.

In this kind of analysis of disequilibrium the model of Harrod does not give any precise notion of how the values of the parameters react during the process of departure from the equilibrium. It is not always true to assume that savings decisions are realized, for example. Perhaps Harrod is not to be blamed for that, because the main object in formulating the model that has been studied here was only to establish the properties of equilibrium rate of growth. The events that occur in disequilibrium were naturally left for trade cycle theory. The notion of inherent instability that was emphasized and which implied that the movement away from equilibrium would be explosive is a subject that requires study in the framework of a model which takes adequate account of entrepreneurial expectation and the reactions of the economy as a whole.

The above discussions about instability are, however, very significant. This is so because the equilibrium defined by Harrod may be realized in reality only in exceptional circumstances. If warranted rate is different from the actual rate one is naturally inclined to inquire whether there can be any other form of equilibrium path taken by the economy. In the short run there are many possibilities for the economy. But if the warranted rate is to be regarded as long-run trend of the economy the problem of instability should revolve around the question of whether equilibrium so determined and experienced is consistent with the long-run trend in the availability of factors, markets of goods, and technological progress, and so on. When Harrod speaks of "warranted rate of growth" it is a sort of quasi-long-run equilibrium. It does not satisfy the rule of consistency with the other long-run tendencies of the economy necessarily. This is one reason for the instability of the equilibrium growth. Moreover if warranted rate is a genuine long-period equilibrium concept the stability can be expected.

To sum up, Harrod's argument about instability says that any shock in the economy leading to actual capital-output ratio different from the capital-coefficient or an actual growth rate in capacity different from warranted growth rate sets up a tendency for disequilibrium, there being no forces to reconcile the two rates of growth. However, it is possible in a long-run balanced growth situation for the economy to be stable, as Jorgenson's argument shows. Because there is possibility that the force that emerges to drive the economy further from equilibrium withers away. There is further the possibility for the economy to settle down to neutral equilibrium provided the conditions are favorable for this to happen, as was seen in the last model discussed above. Harrod's model is not quite conclusive on the point of instability and also not very clear. In order to understand this part of his theory it is necessary to study thoroughly Harrod's picture of the long-run tendencies of the economy.

Section IV: Warranted Rate and Natural Rate of Growth; and Solow's Criticism of Harrod's Model

Warranted rate of growth in Harrod's model is the resultant of the basic equations relating saving and investment assuming the behaviors of savers and investors. This growth rate is sufficient to justify the actions of producers. But at the same time the economy may not be in full-employment equilibrium.

The rate of growth which the increase of population and the technological improvements allow is called the "natural rate of growth." He says that natural rate of growth or "G_n represents the line of output at each point of which producers of all kinds will be satisfied that they are making a correct balance between work and leisure."[1] It is the maximum rate at which income can grow at an economy. Looked at from the point of view of factor supply it may be regarded as a full employment rate of growth.

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[1]Harrod, Towards a Dynamic Economics, p. 87.

In Harrod's model full-employment rate of growth and warranted rate of growth can be equal only by chance. But if warranted rate of growth is higher than natural rate the actual rate of growth tends to be lower than the warranted rate most of the time and vice versa. With the help of this peculiar relationship between G_n, G_w, and G, Harrod seeks to explain how secular stagnation or secular inflation are possible. Just in terms of probability it can be seen that the chance of equality between all the growth rates is negligible!

The independent behavior of G_w from G_n is rather doubtful, because economic equilibrium is liable to be affected by G_n. But for Harrod the influence of G_n lies in determining the direction in which the economy in disequilibrium should move. The direction of movement is not toward the achievement of full-employment equilibrium but for the contrary purpose, namely, to intensify disequilibrium.

The difficulty which this kind of analysis presents is to be found in the lack of its adequate explanation about how market mechanism affects the value of the parameters in the model, especially if the economy does not tend to equilibrium. But the defect of Harrod's model most often emphasized is the assumption of a production function with fixed proportionality of labor and capital or of inputs in general. The assumption is valid at best in short periods only. Hence Harrod's model has utilized, in fact, a short period tool for dealing with long period problems. In the short period when there are fixed types of capital equipment available in fixed quantity the analysis with Harrod's fixed coefficient assumption is realistic. But in the long run new capital goods can take different character. So it becomes more realistic to assume that there are a multiplicity of combinations in which factors can be used to produce a given amount of output. Thus it may be a better approach to assume a production function which prescribes various proportions of factors that can be used per unit of output under a given technological environment. If there is a change in know-how or change in the technology it can be represented by change in the shape and/or position of the function.

If one assumes variability of proportion of factors it is not hard to see how one can derive a system under which the growth rates will converge to the natural rate of growth, determined by the rate of growth of factor supply and technological progress. The argument underlying this possibility is subject to many assumptions, of course. But the possibility, traditionally used by economists, remains that relative prices of goods and factors act as forces to drive the economy to full employment rate of growth. It is said that the lack of possibility of factor-substitution produces the "knife-edge" situation characteristic of Harrod's model.

In fact, it is possible to choose a number of production functions which fulfill the assumption of factor-substitution and it is not hard to get one which insures equality between natural and warranted rate of growth with stable properties. Solow[1] in an important article

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[1]R. M. Solow, "A Contribution to the Theory of Economic Growth," Quarterly Journal of Economics, LXX, No. 1 (1956), 65-99. For a similar discussion by T. W. Swan see "Economic Growth and Capital Accumulation," Economic Order, XXXII, No. 63 (1956), 334-361.

has not only shown such possibility but also has made explicit the defects underlying Harrod's model. Hence it is considered worthwhile to consider his arguments at this point at some length.

The production function may be assumed, according to Solow, as

II-30 $$Y = F(K,L)$$

where Y is the total output at time t, K, the total stock of capital and L the total supply of labor at time t. Capital goods are measured in physical units. It is assumed that the production given above is continuous throughout.

Assume that s is the constant marginal and average propensity to save. At time t the total saving is

II-31 $$sY(t) = sF(K,L)$$

Net investment at t is given by $\frac{dK}{dt}$. The following equilibrium condition results:

II-32 $$\frac{dK}{dt} = sF(K,L)$$

Assume that labor supply is growing at a constant rate n per unit of time. Though the introduction of the term for neutral technological progress does not complicate this model, it is omitted for the present. With L(0) as the initial supply of labor total supply of labor at time t is given by the following expression:

II-33 $$L(t) = L(0)e^{nt}$$

Inserting II-33 in II-32

II-34 $$\frac{dK}{dt} = sF\{K,L(0)e^{nt}\}$$

With Solow one can assume constant returns to scale; this means that II-30 is homogeneous of degree one in K and L. One can write this equation in the following form in which $K/L = r$.

II-35 $$Y(t) = L(0)e^{nt}F(r,1)$$

Since $K = Lr = L(0)e^{nt}r$,

II-36 $$\frac{dK}{dt} = \frac{dr}{dt} L(0)e^{nt} \quad nrL(0)e^{nt} = L(0)e^{nt} (\frac{dr}{dt} + nr)$$

Combining II-32, II-35, and II-36

II-37 $$L(0)e^{nt}(\frac{dr}{dt} + nr) = sL(0)e^{nt}F(r,1)$$

Cancelling the common factor $L(0)e^{nt}$ the basic equation of this model is

II-38 $$\frac{dr}{dt} + nr = sF(r,1).$$

or

II-38a $$\frac{dr}{dt} = sF(r,1) - nr$$

The rate of change of capital-output ratio at each point of time is $\frac{dr}{dt}$. This rate of change has its course determined by the terms on the right hand side of the equation II-38a. The solution may take any kind of shape depending on the nature of the production function. If there is a production function such that for some value of r, $\frac{dr}{dt}$ in II-38a becomes zero r remains constant at that value. Then capital will grow at the same rate as labor. This means that warranted rate of growth equals natural rate of growth. Even though a function of this type may be found it is not certain whether the value at which $\frac{dr}{dt} = 0$ is unique. Further, there is the question of stability of such point. If the function F is of the Cobb-Douglas type the solution of II-38a can be illustrated in Figure 1 by the thick curve.

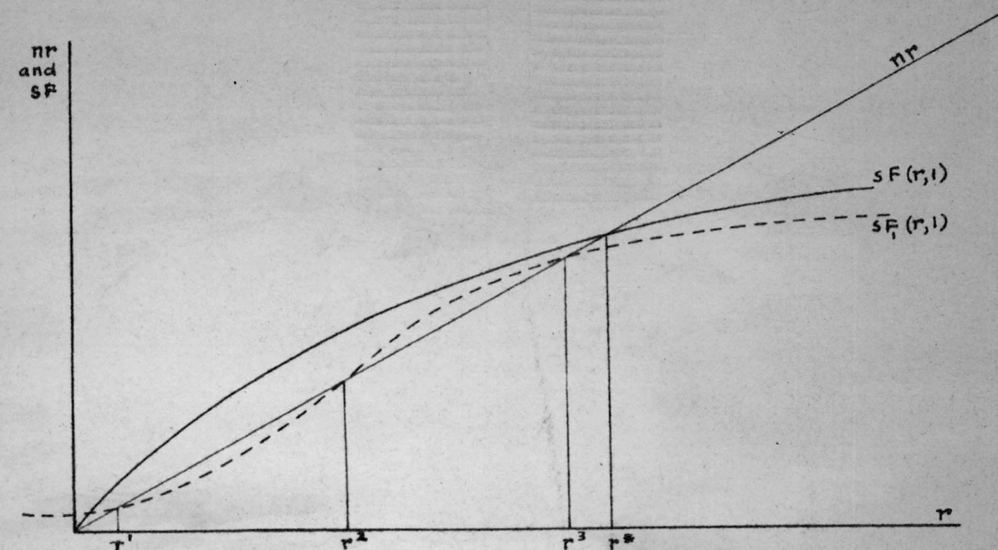

Figure 1.

In the figure above r^* gives the solution with $\frac{dr}{dt} = 0$ for the function F. This solution is stable. For the global stability the condition $r > 0$ is required because $r = 0$ is another solution though unstable. It is at the value $r = r^*$ the ratio of capital to labor remains constant such that the warranted rate of growth of Harrod is realized and maintained. At any point $r > r^*$ or $r < r^*$, sF(r,1) is either greater or less than nr. This sets up tendencies which ultimately lead to r^*. Any point aside from the origin could be chosen as an initial condition and the forces will operate to stabilize the dynamic system at the equilibrium ratio r^*.

The dotted curve in Figure 1 obtained by assuming different form of the function $F = F_1$ does not possess the property of having any unique point at which $\frac{dr}{dt}$ becomes zero. There are three points at which r becomes constant, namely r^1, r^2, r^3. At r^1 there is stability within some range of r. Point r^2 is unstable. Point r^3 is stable for the values of $r > r^2$.

The existence of a production function which allows continuous variability of factor-proportion does not necessarily produce or guarantee a unique and globally stable system. One can have a production function which leads to a constant solution for r that might be unstable even locally. This obviously happens in the case where the sF curve cuts nr from below. If this possibility is ruled out there are yet other situations in which no equilibrium with constant r may exist at all. In Figure 2 the two functions $F = F_2$ and $F = F_3$ illustrate this. The curves are adopted from Solow.[1] The dotted lines used by the author illustrate the case of linear homogeneous function in which there is the possibility that if the coefficient of L is zero in the equation of the type $Y = aK+bL$ the two sides of II-38a will be equal to zero for all r if $sa = n$. For f^*, however, there is a stable unique point, r^*, at which r remains constant.

If $F = F_2$, $sF(r,1)-nr$ does not tend to equal zero. There is an ever increasing capital-labor ratio aside from the point $r = 0$. In case of function $F = F_3$ there is the reverse situation of an indefinitely

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[1]Solow, <u>op. cit</u>.

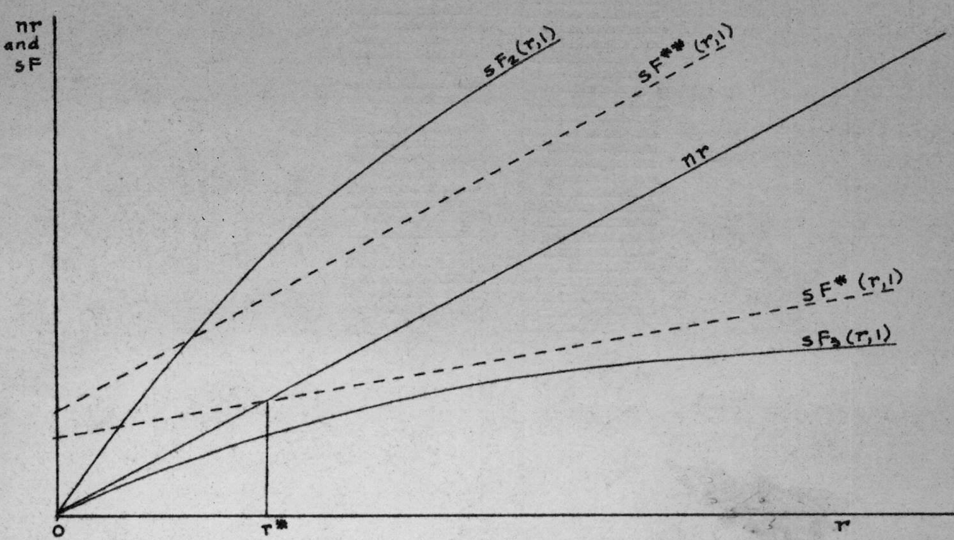

Figure 2.

falling capital-labor ratio. A similar argument holds for the line F^1. In such cases warranted and natural rates of growth cannot be equal.[1]

The gain from the above arguments in demonstrating the possibility of warranted growth is dubious. The "knife-edge" situation in which G_n equalled G_w under the assumption of fixed proportionality of factors was not very plausible. But in the continuous variability assumption about factor proportions, too, a way out was not found except by choosing a specific kind of production function. If one could prove that a production function of this type fairly approximates

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[1]It may be remarked that warranted rate of Harrod's definition may not be realized either.

reality a case for the inevitability of equality between the two rates could be justified. Then this would justify that Harrod's warranted rate would exist. But in most cases the Cobb-Douglas type of production function has been assumed rather than proved reliable. The use of this function has been made to explain income distribution or in finding the nature of technological progress.[1]

For all varieties of production function implied by the illustrations above the movements of the ratio of capital to labor should be considered as taking place in the framework of market. Choice of any ratio depends among other things on profitability. Even in a constant factor-ratio case of Harrod's model if it could be shown that market mechanisms lead warranted rate of growth toward natural rate of growth this model would have a stable property. But the contention of Solow is that in Harrod's model the forces of market as a mechanism for adjustment of warranted rate of growth to natural rate is in general non-existent and moreover factor-prices cannot be meaningfully determined.

To see this a return is made to Harrod's model, again neglecting technological progress. Since fixed coefficient of production is assumed the isoquant representing the ratio in which the factors can be

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[1]See A. Smithies, "Productivity, Real Wages and Economic Growth," Quarterly Journal of Economics, LXIV (1960), 189-205; R. M. Solow, "Technological Progress and Aggregate Production Function," The Review of Economics and Statistics, XXXIX, No. 3, 312-20. In his book, A Multisectoral Study of Economic Growth (Amsterdam, 1960), Leif Johansen uses Cobb-Douglas production function. The reasons for this are perhaps the absence of any other convincing and convenient functions.

used for a given amount of output becomes a right-angled corner as in Figure 3. Here labor and capital are considered as the only factors of production.

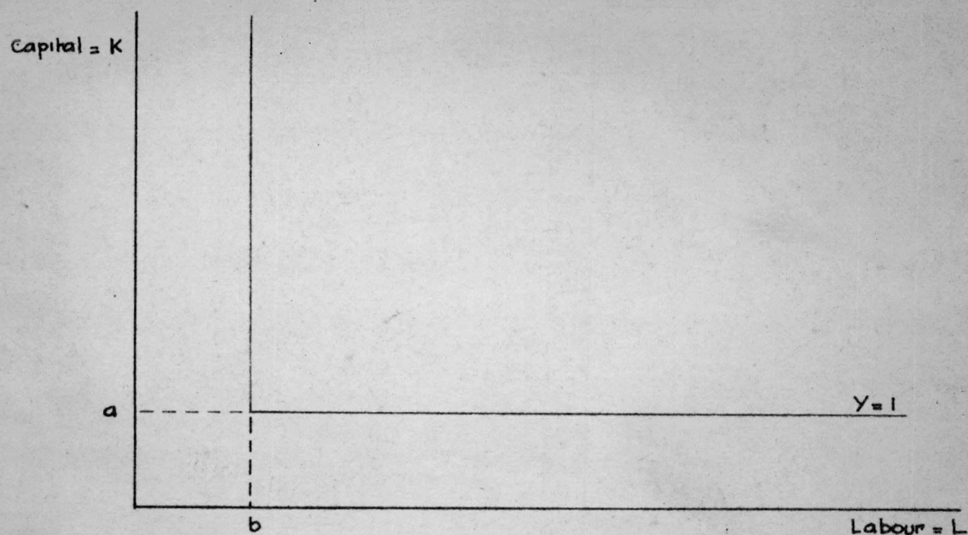

Figure 3.

If the amount of output produced were 1, then in Figure 3 a units of capital and b units of labor become necessary. If available capital were more than this the production would not increase, because labor has become a bottleneck. Hence the production function can be written, following Solow, as follows:

II-39 $$Y = F(K,L) = \min(\frac{K}{a},\frac{L}{b})$$

where a and b are positive constants. The meaning of the right hand term in the above equation is simply that the amount of output is determined by the scarce of the two factors assuming the respective coefficients a and b. If, for instance, labor is relatively scarce then this would limit the amount of output produced despite the profuseness of the supply of capital in the economy.

Following the previous procedure the following analogue, according to Solow, of equation II-38a for equation II-39 can be written as follows:

II-40 $$\frac{dr}{dt} = s \min(\frac{r}{a}, \frac{1}{b}) - nr$$

Since the scarce of the two factors becomes a bottleneck, if $\frac{r}{a} < \frac{1}{b}$, the following equation will hold:

II-41 $$\frac{dr}{dt} = s(\frac{r}{a}) - nr$$

For $\frac{dr}{dt} = 0$, which is true for Harrod's model, it gives $\frac{sr}{a} = nr$, or $n = s/a$. The warranted rate of growth in Harrod's model is s/a, the term a being equal to C_r. If $r/a > 1/b$, then $r > a/b$; in which case the relevant equation should be:

II-42 $$\frac{dr}{dt} = \frac{s}{b} - nr$$

For $\frac{dr}{dt} = 0$, $s/b = nr$, or $r = \frac{s}{nb}$.

The situations described above and the properties of Harrod's model can be graphically illustrated with the device used by Solow which is reproduced for reference as Figure 4.

In Figure 4, $s\text{Min}(\frac{r}{a}, \frac{1}{b})$ has slope s/a from origin onwards until $r = a/b$ is reached. At $r = a/b$ none of the factors are bottleneck. For values of r greater than a/b the graph of $s\text{Min}(\frac{r}{a}, \frac{1}{b})$ has its function

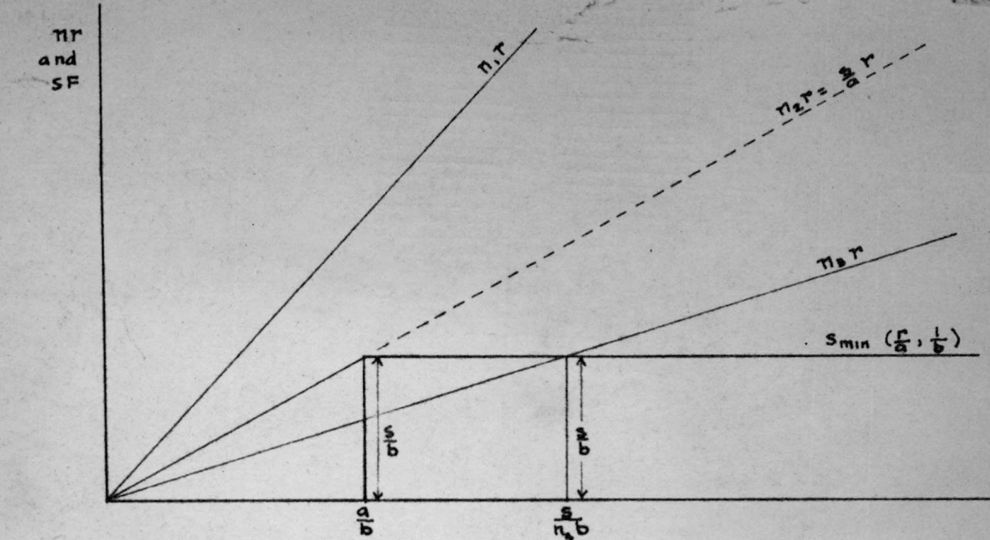

Figure 4.

value s/b throughout. It is easily seen that only for some values of r the natural rate is equal to the warranted rate of growth. But if $n \neq s/a$, is there any possibility of adjustment which would establish the equality? Solow argues as follows.

If $n = n_1 > s/a$, n_1r is always greater than $s\text{Min}(\frac{r}{a},\frac{1}{b})$ which implies that r will always have the tendency to fall as labor supply grows. Assume an initial value of r as $r_o > a/b$. In that case the equation II-42 will hold, with n replaced by n_1. The solution of this equation is $r = (r_o - \frac{s}{n_1b})e^{-nt} \frac{s}{n_1b}$. As t increases r will tend to the value $\frac{s}{n_1b}$. $\frac{s}{n_1b}$ is not shown in the figure. It lies to the left of a/b. The tendency to move toward s/n_1b persists because when $r = a/b$ is reached, movement to the left of a/b occurs and capital becomes bottleneck and equation II-41 begins to hold. The solution of this equation is $r = \frac{a}{b}e^{(s/a - n_1)(t - t_1)}$. Since $s/a < n$, r will tend to zero as t increases indefinitely.

But if $n = n_2 = s/a$, warranted rate is equal to the natural rate. With an initial value of $r = r_o > a/b$, r decreases to a/b because labor has become bottleneck. However, it would not fall below a/b because this will create bottleneck in capital. If $r = r_o < a/b$ initially there is a superfluity of labor. Capital is accumulating at the warranted rate s/a. Thus r remains at r_o over time "in a sort of neutral equilibrium."[1] Any superfluity of labor that existed initially will remain. This illustrates a peculiar situation of equality of warranted rate to natural rate with unemployment or a situation which Kahn would call a "Bastard Golden Age Economy."[2]

If $n = n_3 < s/a$ warranted rate is higher than the natural rate of growth. In this case there is value of $r = \frac{s}{n_3 b}$ at which the economy is in stable equilibrium. Here the marginal productivity of capital has fallen to zero. The stability of equilibrium at this value of r is obvious if the initial condition is assumed to be $r_o > a/b$. If $r_o < a/b$, since $s/a > n_3$, r increases exponentially at the rate $s/a - n_3$; when $r = a/b$ is reached it tends steadily to $s/(n_3 b)$.

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[1]Solow, "A Contribution," p. 75.

[2]R. F. Kahn, "Exercises in the Analysis of Growth," Oxford Economic Papers, XI, No. 2 (1959), 150.

This equilibrium is not to be interpreted as one characterized by excess capacity, strictly speaking. As a matter of fact superfluity of capital under equilibrium conditions means that capital has become a free good. Similarly any equilibrium point along the thick segment of the line n_2r in Figure 4, other than $r = a/b$, is characterized by superfluity of labor, and hence labor is a free good. In such cases, Solow argues that the marginal product of the scarce factor is the whole product and that of the superfluous factor is zero. Consequently the distribution aspect of Harrod's theory interpreted in this fashion seems too simple to provide an insight into the working of the market forces. It is alleged that this is the consequence of the assumption of fixed proportionality of factors.[1]

Moreover, if in Harrod's model represented as above warranted and natural rates were equal with $r = a/b$ in Figure 4, the distribution of income between labor and capital remains indeterminate. If ρ is the return per unit of capital and w is the wage rate per unit of labor, the total cost of producing one unit of output becomes $a\rho + wb$, because a units of capital and b units of labor have been used for producing one unit of output. Since the total wage and total return on capital exhaust the product, the equation $a\rho + bw = 1$ is obtained. Thus any positive value satisfying this equation is consistent with equilibrium.

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[1]Samuelson has made a similar observation in "Wages and Interest; A Modern Dissection of Marxian Models," American Economic Review, XLVII, No. 6 (1957), 884-920. Note especially the short passage: "The case of a single fixed-coefficient is a very peculiar one indeed. Increase labor by epsilon and its share of the product may go from 100 per cent to zero!" p. 906.

If any production problem, static or dynamic, is considered, it is generally accepted that it has two aspects. One is directly concerned with the finding of various possible outputs under the technological conditions given, and furthermore, with the finding of the condition under which equilibrium of some sort is attained--usually the maximum. The other is the dual aspect of the problem, namely, finding the value implications of the various production possibilities. This was the line adopted precisely in the above analysis. In a model with variable factor-proportions the same type of reasoning leads, with the traditional approach of the neoclassical economists of course, to the simple marginal productivity theory of distribution.

If the statement that the excess supply of any factor in Harrod's model leads to its zero marginal productivity is reflected on, one is naturally inclined to ask, "What is the marginal product of each factor when none of the factors are in excess supply?" If there were an answer to this question indeterminacy of factor prices would disappear. If there is no answer to this question there seems little justification for using the marginal productivity concept in describing distribution in any type of equilibrium situation that is worth the name in Harrod's model. If capital and labor were free goods they would disappear from the production function too--the whole economic problem would then cease to exist. If either becomes free, there is possibly something wrong with the analysis. Capital is not only a factor to be allocated but it is also a consequence of the allocation process and it has a real cost involved in producing it. If it is abundant to the degree of superfluity the economy is in disequilibrium. Similarly if labor is getting something less than what can keep it alive and working there is disequilibrium.

As will be apparent later, the marginal productivity theory of distribution is significant for Harrod's model as for other famous economic models. Solow's criticism, though it reveals some important properties of the model, is not valid as far as distribution is concerned. Even with the fixed coefficient assumption adjustment of the warranted to natural rate could be explained as being possible just as in the neoclassical model if desired. This will be returned to later.

Distribution according to the marginal productivity theory may not be quite apparent for Harrod's model at first sight. There are some writers who are skeptical about the application of this theory in micro-economics. One of the most prominent is Kaldor. His alternative to the marginal productivity theory is preeminently designed to explain distribution in aggregate economic analysis and also is used by him in his own dynamic model.[1] In the next section his theory will be considered.

Section V: Kaldor's Theory of Distribution[2] and Its Criticism

Given two classes of income-recipients, owners of capital and

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[1]N. Kaldor, "A Model of Economic Growth," Economic Journal, LXXVII, No. 268 (1957), 591-624.

[2]N. Kaldor, "Alternative Theories of Distribution," Review of Economic Studies, XXIII, No. 6, 83-100. See especially pp. 94-100.

laborers, the pattern of disposition of their respective income between consumption and saving is significant in the distribution of income between labor and capital. This is basic among the important premises on which Kaldor's theory of distribution is developed. In the Keynesian model, given the marginal propensity of the community to save, total saving and investment are equalized by the variation of income and employment. If full employment is assumed and if the level of output is given, in Kaldor's model saving and investment are equal through variation in wage and profit if the saving propensities of the two classes are given. He says that "the principle of the multiplier . . . could be alternatively applied to a determination of the relation between prices and wages, if the employment and output is given"[1] and vice versa.

The expost saving-investment identity is $S = I$. The total income is divided into wage W and profit P. Thus there is another equation relating income to distributive shares.

II-43 $$Y = W + P$$

Wage earners and profit earners have their respective marginal and average propensities to save equal to s_w and s_p. From this the aggregate saving equation becomes

II-44 $$S = s_w W + s_p P$$

It follows that $I = s_p P + s_w W = s_p P + s_w (Y-P)$; which results in the basic equation of Kaldor's model:

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[1] Ibid., p. 94.

II-45 $$\frac{I}{Y} = (s_p - s_w)\frac{P}{Y} + s_w$$

and

II-46 $$\frac{P}{Y} = \frac{1}{s_p - s_w}\,\frac{I}{Y} - \frac{s_w}{s_p - s_w}\ .\ s_p \neq s_w$$

For interpreting II-46 for the purpose of the distribution theory one important assumption is necessary. It is that the term I/Y should be independent of savings decisions. If this assumption is made the share of profit in total income varies with I/Y. If $s_p > s_w$, P/Y varies directly with I/Y, and in the reverse case the relation is inverse. For $s_p > s_w$, it can be seen that P/Y can be negative if $s_w > I/Y$. Thus, if labor does all the saving and also finances the consumption of the capitalists, labor will be exploiting the capitalists! On such points more will be said later. Here it should first be noted that for stability, according to Kaldor, $s_p > s_w$, though the precise meaning of stability is not obvious. If the condition was supposed to rule out the possibility of negative profit and implied that level of profit should be taken into consideration in determination of I/Y, then a separate theory of profit is required. But this is not done.

Another assumption that is required for this theory to be valid is that there should be full-employment; because under conditions of unemployment there are other forces influencing wage rate and profit rate.

In the equation II-46 if $s_w = 0$, one special case of distribution arises. In this case $P = I/s_p$. This says that profit increases if entrepreneurs' propensity to save is low. This is described as the

"Widow's Cruse" theory of distribution.[1] With $s_w = 0$ profit being determined by s_p and I wage becomes a residue.

The above theory is internally consistent only under certain conditions, as Kaldor explains.[2] Firstly, the theory holds only when a minimum wage condition is satisfied. This implies that profit can grow only within some defined range given the level of output and the number of workers in the economy.[3] Secondly, the profit rate cannot be below a certain minimum, determined by uncertainties, or the degree of monopoly, and so forth. Lastly, profit should not be related to capital-output ratio. That is, capital coefficient should not be sensitive to P/Y.

If these assumptions are satisfied then according to this theory Harrod's model acquires some important properties. The first to

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[1]This theory was first mentioned by J. M. Keynes in his book, A Treatise on Money (London: Macmillan Co., 1930), I, 139. For other references see above, Chapter I, p.

[2]Kaldor, "Alternative Theories," pp. 97-99.

[3]This is clearly recognized by Joan Robinson whose theory of distribution is basically similar to the one being discussed here. The following quotation exemplifies this. "A higher proportion of investment-wages to consumption-wages entails a higher ratio of quasi-rent to wages bill in the sales of commodities and a higher share of quasi-rent is likely to give rise to a higher level of consumption expenditure out of profits, which, in turn, entails a higher share of quasi-rent . . . the more the entrepreneurs and rentiers (taken as a whole) spend on investment and consumption, the more they get as quasi-rent.

"But there is a limit to the possible maximum proportion of quasi-rent to wages, which is set by what we may call the inflation barrier. . . . There is a limit to the level to which real-wage rates can fall. . . ." Joan Robinson, The Accumulation of Capital (Irwin, Illinois, 1956), p. 48.

be noted is that the income shares are not indeterminate in spite of the fact that fixed coefficient of production is assumed. The other property which is important in the light of previous discussions about the relationship between natural and warranted rate of growth is that from the present theory it follows that the two rates can be equal. In fact, Kaldor argues that the warranted rate of growth and the natural rate of growth "are not independent of one another; if profit margins are flexible the former will adjust itself to the latter through a consequential change in P/Y."[1]

In Harrod's model

II-47 $$s = G_w C_r = \frac{dY}{Y} \cdot \frac{I}{dY} = I/Y$$

If the required savings-ratio s_r for natural rate of growth is $(I/Y)^*$ which is not equal to actual $I/Y = s$, equality between s_r and s can be brought by variation in P/Y. Difficulty arises in this model because it does not indicate precisely which way the causation acts. Consider that $G_w < G_n$ implying $s < s_r$. Consequently P/Y is lower than it should be if natural rate of growth were equal to the warranted rate. Does it mean that P/Y will be expected to rise? The model cannot answer this question because this requires an explanation of how the market behaves when $G_w < G_n$; it is not capable of showing that the actual rate of growth must rise of necessity. One might be inclined to justify Kaldor's argument and say that when $G_w < G_n$, G will have a

[1]Kaldor, "Alternative Theories," p. 97.

tendency to rise above G_w, in accordance with what Harrod has said. In Harrod's case it happens because demand remains high and investment is profitable. In Kaldor's case, however, one may interpret that natural rate and warranted are not independent because in the long run exogenous forces like technology and the growth of the labor force completely dictate the flow of output. But in that case the problem of determining the adjustment mechanism by which the equilibrium growth rate becomes full employment growth rate evaporates. This interpretation has sufficient ground if we see that the treatment of I/Y as an independent variable in the equation II-46 has justification in case that it is a product of two factors, $G_n = \frac{dY}{Y}$ and $C_r = \frac{I}{dY}$, as above; that is, the first term of the product is determined exogenously by growth of labor force and so on and the second is technologically given.[1]

As an alternative to other distribution theories Kaldor's theory attempts obviously to avoid using market adjustments mechanism as explaining distribution. However, I/Y is regarded, by Kaldor, not necessarily in the long run sense of peculiar character mentioned above. There is apparently some confusion which arises in treating P/Y as dependent and I/Y as independent variables. To explain stability of G_w at G_n one has to explain the course of events when $G_w \neq G_n$. When G_w is less than G_n if the argument of previous paragraphs is brought in it is possible that the actual rate may rise above G_w. But this need not change the proportion of investment to total income.

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[1] Davis, op. cit., p. 15.

Production of consumption goods might rise as fast as investment. However it should be noted that this theory is not supposed to be valid for unemployment conditions, which is true when $G_w < G_n$ and the economy realizes the rate G_w.

When $G_w > G_n$ the theory will have least applicability, because this presents a case of unemployment more genuine than the former.

Even in a full employment situation the theory as an explanation of distribution is dubious. The idea that P/Y changes with I/Y in full employment implies a definite behavior pattern of prices of consumer goods and capital goods. The relative proportion of profit and wage in national income depends upon the relative change in prices of consumer goods and capital goods when employment is increased in the investment goods sector and decreased in the consumer goods sector. This means that for Kaldor's theory to hold the prices should behave in a fashion which will raise profit-income ratio with rising investment-output ratio.

According to Kaldor's assumption, in a full employment situation there is a given level of output which can be split into output of consumer goods and investment goods in proportions that can vary within a definite range, such that his condition for rising profit is satisfied. Thus the set of wage rate and profit rate that is acceptable is determined. In order to show that the whole set of investment-output ratio and the corresponding profit-output ratio is an equilibrium set, one has to show that the passage from one such ratio of investment-output ratio to another means behavior of prices in a compatible way. But this may not be the case. Under conditions of full employment a variation in the ratio of investment to output may lead to a rise in the prices of consumer goods quite out of proportion to the rise in price of investment goods. In such a case one sector of the economy may be running into losses while the other may be earning huge profits. Moreover a fall in the ratio of investment to output might lead not to another full-employment situation but to unemployment.

It is clear that prices are important determinants of the ratio of profit to income. Unless the prices are known, income distribution in Kaldor's model is indeterminate. But the ratios in which various goods are produced in the economy are independent of prices. It is the characteristic of the economy that the demand conditions and production functions determine the amount of various goods produces and their respective prices simultaneously with the prices of factors employed. Kaldor's approach, however, leaves the determination of prices out of the scene. He does not show that the acceptable distribution pattern is always guaranteed by price movements that will occur in desired direction. If this were possible it would be perhaps worth pointing out that equilibrium distribution would exist at a point where the wage rate will be at subsistence level because that would guarantee the maximum profit. If it is argued that the achievement of maximum profit is to be ruled as being beyond the capacity of an entrepreneur it is also not possible for entrepreneurs to vary the investment-output ratio in any way independently of other developments in the economy. Kaldor does not like to insert any more explanation as to why wage rate and prices of consumer goods should behave in the way he assumes. If this were done there would be a more genuine theory of distribution.

Kaldor's basic notion that multiplier mechanism works towards raising the level of output and employment if there is unemployment, but when there is full employment it determines distribution of income through price variation is confusing. Because even under unemployment conditions multiplier operates through price. At full employment there can be change in the level of investment relative to the total output and correspondingly change in the prices toward equilibrium only if the particular pattern existing before were not the equilibrium one.

Moreover, the theory lacks generality because of the assumptions that $s_p \neq s_w$ and $s_p > s_w$. To have a more general theory one might formulate the theory in a slightly different form which would not only do away with the above assumption but also assume away the independence of I/Y and obtain a determinate solution for P and W. In other words, a distribution theory proper is required to see that the saving-investment equation in Kaldor's model defined in terms of entrepreneurial and non-entrepreneurial consumption and saving behavior is satisfied with unique values of P and W.

If Kaldor's savings function for labor and entrepreneurs is taken, as before, then at any time for a level investment I_o equilibrium requires that $s_w W + s_p P = I_o$. For this value of investment it is not precise what values P and W will have. If it is said that with the investment I_o there will be a total income Y_o which will determine W and P, one is back to Kaldor's model. What has to be shown is that with the investment I_o there is a given total income Y_o and wage and profit such that equilibrium is determined. One way to deal with this problem is to use Schneider's argument[1] which is summarized as follows.

It is assumed that entrepreneurs plan non-entrepreneurial income by contract. Thus their profit is planned indirectly. In this case profit becomes a residue which the entrepreneurs seek to maximize. This assumption makes profit dependent on the total income and wages. One can define the relationship between wages in some way. For example, if P^* is the profit having equilibrium relationship with wage W, one may write in functional form:

II-48 $$P^* = f(W) = \alpha W; \quad \alpha > 0$$

The above equation says that profit varies directly with wage. Now taking the equation, $s_p P^* + s_w W = I_o$, by substituting II-48 in it, the following is obtained:

II-49 $$s_p P^* + s_w W = s_p P^* + (s_w/\alpha)P^* = P^* \frac{(\alpha s_p + s_w)}{\alpha} = I_o$$

From II-49 the following results:

II-50 $$P^* = \left(\frac{\alpha}{\alpha s_p + s_w}\right) I_o$$

and $$W = \left(\frac{I_o}{\alpha s_p + s_w}\right)$$

The above formulation removes much of the difficulty of Kaldor's theory, because this provides a solution for all values of s_p and s_w and also I/Y does not have to be assumed independent. In Kaldor's model

[1]E. Schneider, "Income and Income Distribution in Macro Economic Theory," trans. E. Henderson, in International Economic Papers, No. 8, pp. 111-121.

the classes of income earners are distinguished by their having different savings functions.[1] In the present case this is not so. Moreover, in the present case entrepreneurial behavior is clearly brought to the fore.

Despite the fact that the assumption of a functional form in which W and P can be related fills up the deficiency of the model of distribution studied at present the question as to what determines the functional form arises. In using the functional form given above it was accepted that whenever there is a rise in wage profit rises. If this is based on empirical observation it is questionable. It is desirable to have a theoretical basis for this. A thorough theoretical treatment of the problem will reveal various forms that f in II-48 would take.

To see this there should be a more elaborate analysis of the process by which the relationship between P and W is established. It is true that when a production process starts some factor shares are fixed by contract. However, it is not to be denied that contractual incomes are influenced by market forces, as the other residual incomes are. In the long run contractual incomes vary with variation in market conditions. Here only wage is considered as the contractual income and profit as the residue. The contractual income is influenced by market conditions.

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[1]It should be noted that if savings coefficients of labor and capitalists are equal in Kaldor's model there is only one class.

When an entrepreneur makes a decision to invest and employs more labor, he takes the wage rate determined by the market in a perfectly competitive system, which is assumed here. The plan for a given amount of employment involves a planned total expenditure. The entrepreneur has an expectation of the total income which he would receive at the end of the process by the sale of the output. The difference between the sale proceeds and the total expenditure, which may be called quasi-rent or expected profit, may or may not be realized in actuality. But the motivation of the entrepreneur lies in maximizing this residue.

Let Y^* be the real output which the entrepreneur expects at the end of the period. Assuming labor as the only input the total cost consists of the wage bill in the enterprise. Let N be the amount of labor employed in the economy and W be the wage level, or w the wage rate. Then for expected profit P^* there is the following relationship:

II-51 $$P^* = Y^* - wN$$

But Y^* depends upon the amount of labor employed. Hence:

II-52 $$Y^* = \phi(N)$$

In planning employment of labor the entrepreneur is assumed to maximize P^*. Differentiating II-51 with respect to N:

$$dP^*/dN = \phi'(N) - w$$

Setting this derivative equal to zero the following relationship is obtained:

II-53 $$\phi'(N) = w$$

Here it has been taken for granted that the maximum exists. This is in line with Keynesian reasoning. Equation II-53 says that wage rate should equal the marginal product of labor.

Combining the three equations, II-51, II-52, and II-53:

II-54 $$P^* = \phi(N) - \phi'(N)N$$

Equation II-54 gives the relationship between P^* and W. This can be illustrated by assuming some particular shape of ϕ. In Figure 5 is illustrated a function ϕ_2 which is concave for all values of N, and a function ϕ_1 which is convex for small values of N and concave for all larger values of N. Both terms of the right hand side of II-54 are plotted in the same figure separately by deriving the respective derivatives of the functions and multiplying them by N. In Figure 6 the information given by Figure 5 is used to illustrate the relationship between P^* and N. It should be noted that the figures presented below give only a rough approximation of the shapes of functions assumed. In Figure 6 if all terms are thought of in wage units then the graphs will

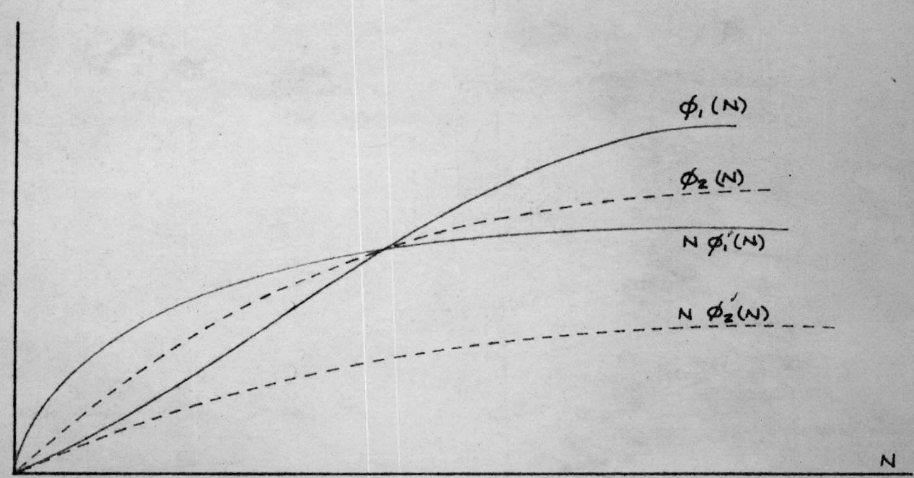

Figure 5.

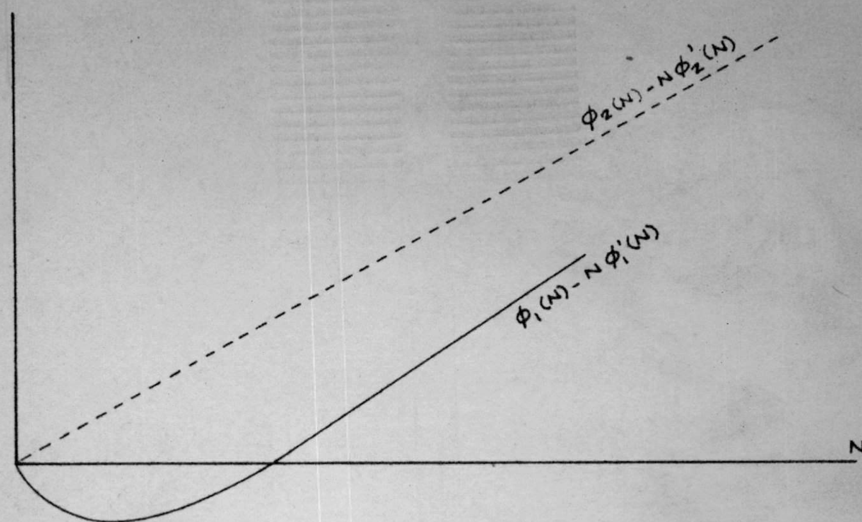

Figure 6.

denote the relationship between wage and profit.[1] The above analysis provides the basis for Schneider's assumption that P^* is functionally related to W.

The above illustrations and arguments show that marginal productivity determine wage rate. This is in line with Keynesian argument. But the question arises whether this argument is sufficient for income distribution or whether some more conditions are required. The arguments of those who intend to give a determinate solution to Kaldor's model do not stop here. Their next step is to find precisely what the

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[1]For this line of argument see H. Atsumi, "Mr. Kaldor's Theory of Income Distribution," Review of Economic Studies, XXVII, No. 73 (1960), 109-18.

levels of wage and profit will be in the economy given that marginal productivity determines wage. In order to know this one should know what is the level of investment in the economy and what are the propensities of saving of the two respective classes. The saving-investment equation lays down the condition for equality between supply and demand. With this the total wages and total profit are determined uniquely. All that is required is to solve the simultaneous equations as in II-50 for a given investment I_o.

If distribution is explained by marginal productivity, the use of multiplier analysis seems unnecessary. However, the value of the latter lies in determining the level of output at which equilibrium will be established. Thus a complete description of macro-economic equilibrium, according to the above analysis, is possible by bringing in two theories together, one for determining distribution of income and another for determining level of output. Hence as a theory of distribution multiplier ceases to have any importance.

It should be remarked that in the theory above, I/Y is not an independent variable. If Schneider's wage-profit relationship, $P^* = \alpha W$, is taken, the term for I/Y becomes $\frac{\alpha s_p + s_w}{\alpha + 1}$ which is obtained from equation II-50. This ratio may be constant or variable depending on whether α is a constant or a variable, assuming that other parameters are constant. In any case, I/Y besides being dependent on α is also dependent on s_p and s_w. This appears more acceptable than Kaldor's assumption.

It is now necessary to make some observations about the marginal productivity approach outlined above. It is not known what theory of distribution has or will ever be valid for explaining distribution of income. But on the basis of the fact that the marginal productivity theory has a micro-economic foundation it is hard to replace it by any other theory which lacks this and confuses the chain of causation in economic life by assumptions which have a weak foundation. However, there are limitations to the theory which have been often emphasized by its critics. The obvious one is that the theory is valid only in case the law of diminishing returns is operating.[1] It may be argued that in the neighborhood of full-employment this law is more likely to operate. In an unemployment condition neither Kaldor's theory nor this particular theory will hold.

Even if it were true that in the real world at all levels of output and employment increasing returns would hold there is no reason why marginal productivity should not explain distribution. If only the production side is looked at and the output evaluated in terms of the physical units it will be found that additional product due to an increment of labor employed would be rising more than proportionately. Whether this is true or not will not be discussed here. But the writer's argument is that it is not the physical unit added but the value of the increment of output that determines the price of labor.

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[1]See N. Kaldor, "A Rejoinder to Mr. Atsumi and Professor Tobin," Review of Economic Studies, XXVII, No. 73 (1960), 121-22.

Valuation of goods is the only legitimate guide in determining marginal productivity of any agent of production. That pricing of goods determines allocation of factors and the factor-prices is the basis of much of our theory.

In brief, one has not a theory of distribution in the so-called Keynesian theory of distribution, but a theory of how in an economy with a given level of output and employment there are various profit-wage combinations corresponding to different levels of investment given the propensities to save of different classes of income earners. For Harrod's model this theory has little use, because, as already seen, this theory is built up on the assumption that warranted rate is equal to the natural rate of growth, whereas the problem here is to see the reconciliation of the two growth rates by the adjustment of prices provided this is possible.

Hence what is required is a study of Harrod's model in the framework in which the forces determining prices are made explicit. In the next section the rigid assumption of the fixed proportionality of factors is kept and the model will be considered with more than one commodity.

Section VI: Equilibrium in a Fixed-Coefficient Model and its Price Implication

It was remarked in the last section that Kaldor's theory of income distribution clearly asserted that in the long run the equilibrium rate of growth is not independent of the natural rate of growth. It was also remarked that in the long run such interdependence was assured by the independent behavior of I/Y. Thus Kaldor implied that there must be some process of adjustment of the warranted rate to the natural rate when he said that with a flexible profit margin this is possible. It was not, however, explained how the price mechanism should work for establishing the equality between the warranted rate of growth and the natural rate of growth. In this section an explanation will be attempted of the relationship between the two rates taking into consideration the demand and supply functions for goods and factors.

The production function of the type given by Solow[1] for Harrod's model is used. In this section the study is based on the assumption that there is more than one sector in the economy. This would give rise to a model of the variety which is called the Leontief dynamic model.[2] To keep the exposition simple, only two sectors are assumed to produce consumer goods and one sector to produce capital goods. All natural resources are assumed to be free and they are not included in the production functions. Labor is the only factor that has a supply exogenously determined in this model.

Let x_1 and x_2 be the quantity of consumer goods 1 and 2 produced, and x_3 be the quantity of capital goods. L is the quantity of labor supply in the economy of which x_4 is employed. By x_{ij} is denoted the amount of ith good used in the production of jth good. α_i is used to denote the amount of ith good that is produced per unit of capital used

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[1]See Section IV above.

[2]W. Leontief (ed.), Studies in the Structure of the American Economy (Oxford: Oxford University Press, 1953), Chapters II and III.

in that sector. Similarly, β_i is used to denote the amount of ith good produced per unit of labor employed in that sector. For simplification it is assumed that $\alpha_3 = 0$. All the α's and β's are assumed to be constant positive numbers.

There are then the following production functions:

II-51 (a) $x_1 = \alpha_1 x_{31}$; $x_1 = \beta_1 x_{41}$; for good 1

(b) $x_2 = \alpha_2 x_{32}$; $x_2 = \beta_2 x_{42}$; for good 2

(c) $x_3 = \beta_3 x_{43}$; for capital goods

For the total amount of labor employed there is the expression:

II-52′ $$x_4 = x_{41} + x_{42} + x_{43} = \frac{1}{\beta_1}x_1 + \frac{1}{\beta_2}x_2 + \frac{1}{\beta_3}x_3 \leq L$$

where L is the amount of labor given exogenously. All the variables are to be treated as functions of time.

For a full employment situation with L inelastic, II-52′ holds with strict equality. One of the problems of this section is to find the implication of an equilibrium situation where full employment exists.

Since consumer goods are produced with labor and capital in fixed proportions and since capital can be expressed in terms of labor, because of II-51 (c), the production possibility schedule for the two consumer goods can be written in terms of labor alone. The result, for instance, is that the total requirement of labor per unit of consumer good 1 is A_1 and that per unit of consumer good 2 is A_2, respectively, such that $A_1 = \frac{1}{\alpha_1\beta_3} + \frac{1}{\beta_1}$ and $A_2 = \frac{1}{\alpha_2\beta_3} + \frac{1}{\beta_2}$. With these coefficients as the only parameters the production possibility schedule can be derived which is written as:

$$\text{II-53}' \qquad A_1x_1 + A_2x_2 = x_4 \leq L$$

Equation II-53$'$ is graphically represented in Figure 7. This equation gives a number of combinations of x_1 and $x_2 \geq 0$ that could be achieved for any particular value of x_4. In fact at any time t with actual labor force L(t) a production set satisfying (II-53$'$) with $x_i \geq 0$ $(i = 1,2)$ can be found to consist of an infinite number of points inside and on the boundary of the triangle OMN in Figure 7. If the preference of the community is considered as between consuming 1 and 2, with the indifference curves defined, as usual, to be convex, a number of points of the set OMN satisfy the preference pattern of the community. But there may be a unique point Q that assures optimum for the economy. Point Q represents the Pareto-optimum at which the productive resources of the economy are fully utilized and the satisfaction of the people as a whole is maximized. Price ratio of the two goods

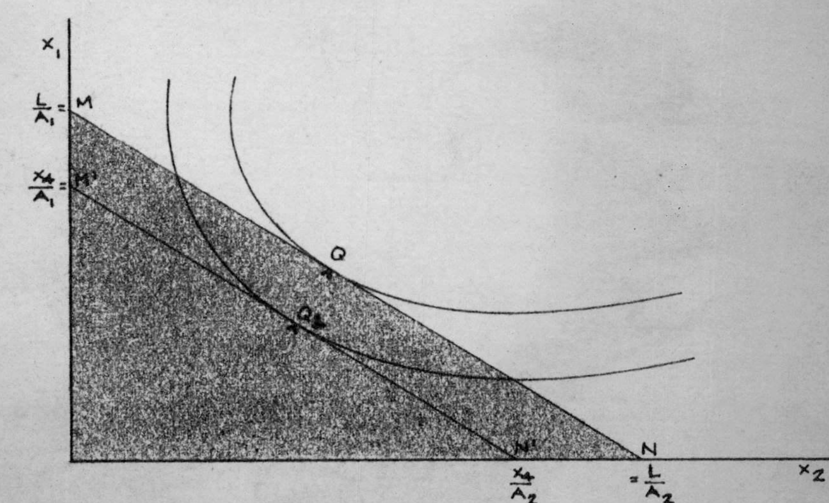

Figure 7.

is determined by the coefficients of production, or more specifically by the slope of the line MN.

The above argument further shows that a part of the labor force is directed to the production of capital goods and this remains a constant proportion of the total output of goods for any specific combination of the two goods 1 and 2. If the labor force is growing at any given rate the line MN shifts to the right, in a parallel fashion. Similar effect is produced if neutral technological progress is taking place. If in such a dynamic situation the preference pattern does not change the proportion of additional labor allocated to the production of capital goods to the total labor employed in producing all goods remains unchanged. In other words, a fixed proportion of income is saved in each period. Each point in the line MN may be regarded as consistent with Harrod's natural rate of growth. This rate of growth is realized in the economy for all sectors if the strict assumption about the preference of the people between present and future consumption is satisfied. The assumption is that the preference function includes consumption at present and that in the future as two different things and the function may be assumed homogeneous of degree one. This means that with rising output the total income saved remains a constant fraction of the aggregate income so that satisfaction increases proportionately with balanced increase of all outputs. If this is not satisfied the curve MN in the future should shift in such a fashion that there is a change in its slope which should assure a changed proportion in the employment of labor in various sectors of the economy. From this observation one corollary immediately follows that even if savings coefficient were constant flexibility of choice between various goods consumed at present which might assure flexibility in curve PP' can assure output along the line MN in Figure 7.

The above discussion has one characteristic which requires some comments. This is that capital goods have been converted into labor and, hence, they are not dealt with in a way that would bring out their peculiarities as a factor of production. There is much validity in this objection because by the present procedure the productive contribution of capital taken by itself is concealed. Introducing durable capital does not basically alter the nature of the problem. It was said, however, that the consumer allocation of income between future and present consumption is necessarily the determinant of the amount of capital being created in the economy at any given time. This presents the supply side of the problem of capital accumulation. On the demand side it calls for explicitness on the point that demand for consumer goods affects the amount of capital produced.

Another important problem which has to be noted in connection with the above analysis is about equilibrium, assuming the type of technology noted here. It is undeniable that with the preference function assumed the point Q is the welfare optimum. But this need not be the unique equilibrium point for the economy.[1] There may, in fact, be

[1]If the equilibrium conditions are not satisfied the point like Q_1 cannot be called an optimum. The point is that an equilibrium may not be an optimum though the converse of this statement is false.

many possible equilibrium positions. The point Q_1 in Figure 7 may be, for example, another equilibrium situation though characterized by unemployment. In dynamics the economy may be on the left side of the ever-shifting MN line. In Harrod's terminology the warranted rate of growth may be different from the natural rate.

It is also conceivable that equilibrium might exist with an excess of capital goods. Apparently one may have a high level of employment of labor but the production of capital goods might outrun the amount required. However, this situation, if it leads to ever-increasing excess capacity, cannot be stable. But if a slight excess capacity exists and continues at that level it may not be inconsistent with stable equilibrium.

In order to have a more detailed picture of the economy being studied here the demand functions are considered for goods and factors with which a better understanding may be obtained of how equilibrium is determined. The demand functions are as follows:

II-54 (a) $x_1^d = f^1(p_1, p_2, p_3, w, r, \pi)$

(b) $x_2^d = f^2(p_1, p_2, p_3, w, r, \pi)$

(c) $x_{3i}^d = f^{3i}(p_1, p_2, p_3, w, r, \pi)$ $(i = 1, 2,)$

(d) $x_{4j}^d = f^{4j}(p\ ,\ p\ ,\ p\ , w, r, \pi)$ $(j = 1, 2, 3)$

In the above equations x^d stands for quantity demanded. The amount of capital (resp. labor) demanded by the ith (resp. jth) industry is denoted by x_{3i}^d (resp. x_{4j}^d). Reference to the price of ith good is p_i. Other elements beside the cost of direct inputs may be included in p_3. No other items constituting price of capital are introduced and it is assumed that they are included in p_3. The rate of interest, wage rate, and profit rate are r, w, and π. For profit per unit of goods sold there are the following equations:

$$\text{II-55} \qquad \pi = p_1 - w/\beta_1 - rp_3/\alpha_1$$
$$\pi = p_2 - w/\beta_2 - rp_3/\alpha_2$$
$$\pi = p_3 - w/\beta_3 - rw/\beta_3$$

Here it has been assumed that profit rate, wage and interest are homogeneous for all sectors of the economy. In a perfectly competitive economy profit rate may be zero, in which case the equilibrium price of a unit of a good equals its unit cost. This is not inconsistent with the assumption that producers maximize their profit. It may also be assumed that consumers' attempts to maximize their satisfaction underlies their demand functions.

The equilibrium conditions may now be written as follows:

$$\text{II-56 (a)} \qquad p_1x_1 = wx_{41} + rp_3x_{31}$$
$$p_2x_2 = wx_{42} + rp_3x_{32}$$
$$p_3x_3 = w(1+r)x_{43}$$

The constraints are given by:

$$\text{II-56 (b)} \qquad p_3(x_{31} + x_{32}) = p_3x_3$$
$$w(x_{41} + x_{42} + x_{43}) = wx_4 \leq wL$$

There are now a complete set of equations which have to be satisfied for equilibrium to exist. But some more remarks are necessary to see how the solutions are obtained, such that they become economically meaningful. In the type of model here some problems of bottleneck and negative values might arise. Such possibilities have to be ruled out by assuming that the x's and p's and other variables r and w cannot fall below zero. If this fact and the conditions that output cannot exceed the capacity imposed by the availability of the factors are considered the inequality signs have to be used instead of the equality in the relevant places of the above system of equations. For the production functions given above, the change indicated here leads to the following set of inequalities in which a_{11} is the inverse of $_1$, a_{12} the inverse of $_2$ and similarly for the is substituted a_{21}, a_{22}, and so forth.

II-57 $$a_{11}x_1 + a_{12}x_2 - x_3 \leq 0$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \leq L$$

For other changes it is enough to assume that π should be greater than or equal to zero.

From the theory of linear programming it is known that the two inequalities represent two linear half-spaces. The intersection of these two half-spaces with the positive orthant of the cartesian coordinate space results in a convex set, which in this case has the shape of a prism as illustrated in Figure 8 below.

The interior and the boundary of the prism represent the feasible set of output.

The space toward the direction of the arrow on the plane LQM represents the second of the inequality II-57 and the space upward left of the plane denoted by QPR represents the first. It should be remarked that the plane QRP shifts upward as L increases. It appears from the illustration that the maximum should lie along the boundary QRP

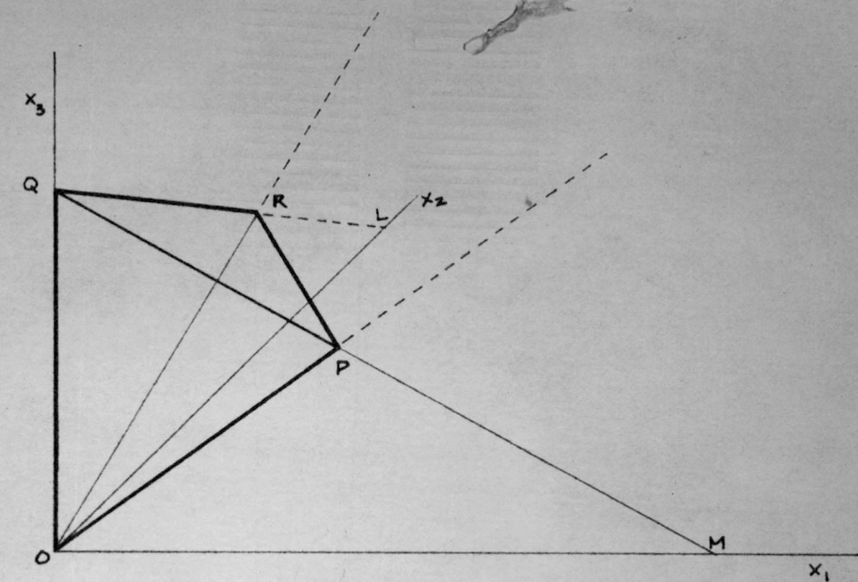

Figure 8.

of the set. For any positive set of prices of goods it appears that the economy maximizes the value of goods produced by moving to some point on QRP. The line RP is the only set of goods produced which utilizes capital and labor fully. Along QRP all points to the left of RP denote excess capacity.

From the above model it seems that, provided maximum means equilibrium, full employment equilibrium is always possible. The determination of prices of goods is the dual of the above problem. It is not intended here to enter into the detailed analysis of the process by which the existence of equilibrium is to be proved with the help of all the conditions given above put together. But it is enough to indicate that in finding every feasible output and then the optimal output the demand conditions are put to work. If a solution, which is an equilibrium solution, is obtained the prices are easily known with the help of equation II-55 with any two of the six variables assumed to be determined outside the system so that the number of unknowns is equated to the number of equations, assuming that the rate of profit is zero, such that the competitive condition is satisfied. If one of the prices is chosen as a numeraire one has only either wage or interest to be determined outside.

Although this section was begun with the observation that the analysis was to explain the tendency of the economy that is growing, the approach to the problem of determining equilibrium was static. But it was designed for simplicity of analysis by which the purpose of understanding the nature of equilibrium under the assumed technological conditions could be achieved. It was found not only that there can be full employment in this type of economy but also that the prices of goods and factors are determinate. From equations II-55 the factor prices can be easily determined with the prices of goods. Full employment requires that if the rate of interest is given there should be a certain given set of prices of goods and wage rate which will assure the employment of capital and labor available. Similarly, if the wage rate is given other values can be so determined as to assure full employment equilibrium.

The question arises whether the equilibrium so determined implies that if the economy is outside it there are inherent forces which drive it towards it. Stability of an equilibrium requires that any deviation from it should affect costs and prices in such a way as to motivate a reallocation of factors and income of the community which would lead to equilibrium. In the present model equilibrium, although it may be a full employment one, need not be unique. One may introduce a number of assumptions to have a unique equilibrium. It is not of interest here. It is held here that if prices of goods and factors behave as they should there can be full-employment equilibrium and this is stable whether it is unique or not. If full employment does not exist, output has to be low. This affects the rate of investment in the economy. One might conceive that a low level of employment might as well be consistent with the equilibrium conditions. The author is inclined to support this idea. But there is one qualification necessary for this. A rise in the level of output will set a tendency to raise employment to its maximum and any movement from a point of unemployment equilibrium is indeterminate. Thus any unemployment equilibrium is to be regarded as unstable and temporary.

In a consideration of the implication of excess capital stock there are two main conditions of the economy that have to be considered in which this situation might occur. One is that the economy might be in a process of decline with growing unemployment and falling output. Another is the situation in which employment might have been already at its maximum. In the former situation the excess will be possibly depleted as time goes by due to the low rate of saving at low level of output and a quasi-equilibrium of the kind discussed above might arise. In the latter case there might be increased demand for labor which might raise wage and prices and cause redistribution of income leading to a fall in the rate of saving. In this case there is another possibility too. The excess stock of capital might lead to a fall in demand for capital goods and then to a fall in employment. The case of low rate of capital production similarly implies a low level of output. In such a case full employment depends upon the effect of wage on the cost structure of the economy. At a low level of employment if there is a downward pressure on wage rate the labor intensive sector might benefit from it, other things remaining the same. This might raise the level of employment.

The above are remarks on some of the likely situations from a static view. From the point of view of growth the stability of full-employment equilibrium has to be studied with reference to the development in time of the relationships between the different variables in the model. The arguments do not differ much in this case from the main points indicated above. In the following paragraphs some remarks about Harrod's model are made.

It is known that Harrod's warranted rate of growth will arise if equilibrium conditions are satisfied, such that a fixed proportion of total output belongs to the investment goods category, and consumer goods are in constant proportion to each other. This implies, as has already been seen, that the preference map of the community does not change. If the preference map changed with increased income it is possible that the economy would still develop along the equilibrium path but not at a given constant exponential rate s/C_r. In such a case the savings-coefficient of a community might change with changed preference as between present consumption and consumption in the future; capital-coefficient also might change for the economy as a whole because of changing proportion in which various goods are consumed. Although historical data are often brought to support the idea that both savings-coefficient and capital-coefficient have remained fairly constant this does not rule out the possibility of a changing output composition that has taken place in the economy. The explanation of constancies should take account of many other factors besides the given static technology, which itself might have been in a series of change. However, the rigidity of the present model is no bar to an understanding of the logic of the development process.

To recapitulate the author's idea of the warranted rate of growth on the basis of the present model: Suppose that the equilibrium rate were not equal to the full-employment rate. This case presents a difficulty, because if equilibrium is characterized with excess supply of labor, wage rate has to be zero. In the present model assume that wage rate is exogenously determined. This assumption is necessary because it is impossible to have zero wage rate. With this assumption equilibrium outputs and prices and interest can still be derived. But real wage in an unemployment situation does not have any definite determinant, except that in critical conditions it may be said to have been determined by the subsistence minimum. However, it may be said here that income distribution is not a serious problem when there is unemployment provided there is equilibrium. The relationship between

G_w and G_n may now be seen.

Natural and warranted rate of growth are not independent of each other if it is understood that the latter has to adjust itself to the former, if it is to be realized at all. If the natural rate is determined by purely exogenous factors and if the warranted rate were different from it either labor or capital will be superfluous. This renders one of the factors more expensive than the other. Assuming consumer substitution of goods the price mechanism may work toward either changing the proportion of various goods produced which affects the capital-coefficient on the average and/or the savings-coefficient. If the natural rate is constant warranted rate should approach this constant rate. If the former is changing then the latter should change in the same fashion if equilibrium is to be maintained. Although it is not the intention of the author to carry the argument too far it appears to him that the influence of the preference pattern of the consumers may have much effect on natural rate as well.

Kaldor's argument throws some light on the issue of equality between the two rates by showing the effect of income redistribution on the aggregate saving though his assumption of independent investment-income ratio is not acceptable. By the redistribution of income from one class to another the investment-income ratio can be affected if the saving propensity of the two classes is different. This redistribution can absorb excess saving or create more saving for the economy and restore the equality between the two rates of growth.

The above observations are made on the basis of the simplified model assumed here. It is not impossible to have a more general model covering a larger number of sectors in the economy. This is exemplified by the dynamic Leontief model,[1] which is studied by several authors in its various aspects. As an exact analogue of Harrod's model it deserves some mention. In this aspect the closed Leontief model is to be considered. Labor might be included in the general input-output equation to be given below. But the argument does not change significantly by excluding it. The problem of income distribution has been already dealt with. All that is required is to note the general model and its implication of prices over time. Regarding relative prices the above arguments can be used.

For the aggregate output of n goods at time t there is a vector X(t) whose elements are $x_i(t)$; (i = 1, 2 . . . n). Differentiating each $x_i(t)$ with respect to time there is the increment in output of ith good at time t. This is represented in vector form as $\dot{X}(t)$ in which the dot denotes differentiation with respect to time. The result is

$$\text{II-58} \qquad X(t) = AX(t) + B\dot{X}(t)$$

where A is an n$\times$n matrix of coefficients a_{ij} which means the amount of ith good used per unit of goods on the jth sector, household being considered as one sector; B is the n$\times$n matrix of capital coefficients b_{ij}, which indicates the amount of commodity i held as the stock per unit of commodity j. In one commodity case the equation becomes

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[1]Leontief, loc. cit.

exactly like Harrod's $y(t) = (1-s)y(t) + C_r \frac{dy}{dt}$. II-58 can be written as:

$$\text{II-59} \qquad (I-A)X(t) = B\dot{X}(t)$$

Assuming B as non-singular

$$\text{II-60} \qquad B^{-1}(I-A)X(t) = \dot{X}(t)$$

Assume the vector of initial outputs to be X^* and λ as the rate of growth such that $x_i^* e^{\lambda t}$ is the solution of the ith equation:

$$\text{II-61} \qquad B^{-1}(I-A)X(t) = \frac{dX^* e^{\lambda t}}{dt}$$

from which is obtained $B^{-1}(I-A)X^* e^{\lambda t} = \lambda X^* e^{\lambda t}$, in which $e^{\lambda t}$ may be cancelled. This results in:

$$\text{II-62} \qquad \left\{B^{-1}(I-A) - I\lambda\right\} X^* = 0$$

where I is the unit vector and 0 is the zero vector. In equation II-62 $B^{-1}(I-A)$ is the inverse of the non-negative matrix $B(I-A)^{-1}$. The interesting property of such a matrix is that it has a positive characteristic root $\bar{\lambda}$ with corresponding eigenvectors having positive elements, $\bar{x}_1$, $\bar{x}_2$. . . $\bar{x}_n$, as the solution of x_1, and so on.[1] Thus it is seen that the initial conditions should be determined by the eigenvectors that are consistent with the solution desired.

The dual interpretation of the above solution gives what prices of goods and factors will exist. It is sufficient to note that prices may or may not stay at the stationary level depending upon the nature

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[1]There are many works in which the properties of positive matrices are discussed. Mentioned for reference here is S. Karlin, Mathematical Methods and Theory in Games, Programming and Economics, I (Addison-Wesley, 1959), 246-56.

of entrepreneurial expectation,[1] especially. If prices are expected to change, putting the argument in period form, differential equations are obtained similar to the differential equations encountered above, but in this case relating prices of goods in period t to that in period t-1 for including the appreciation or depreciation in the value of stock due to price change. Price is to include the expenditure made in input flow and also change in the value of stock of capital. Thus the following equation:

$$\text{II-63} \qquad p(t) = p(t)\,A - \left[p(t) - p(t-1)\right] B$$

where A and B are as above and p(t) is the price vector. Interest and wage are excluded from this discussion. The solution is obtained in a similar way by writing II-63 as:

$$\text{II-64} \qquad p(t)\,(I-A+B) = p(t-1)B$$

or

$$p(t) = p(t-1)B(I-A+B)^{-1}$$

provided I-A+B has inverse. Whether prices tend to be stationary or keep increasing or decreasing, and so forth, depends on the characteristic roots of the above.[2]

But it should be mentioned here that this solution does not include the influence of interest and profit which may change. If they are allowed to have their influence on II-64 it may as well be the case that their influence will shift the economy from one initial condition

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[1]See R. M. Solow, "Competitive Valuation in a Dynamic Input-Output System," Econometrica, XXVII, No. 1 (1959), 30-53.

[2]See Solow, ibid., for the discussions.

to another. The way the model is set up it says that change in capital value fully affects the price development. This is not quite clear, because the effect of capital gain or loss on price depends on the entrepreneurs' expectations about the future and not the other way around in many circumstances. It is not to deny that prices can change and sometimes may be oscillatory or even explosive. But such changes are hard to explain with equations of the simple kind used here. In the next section there will be occasion to mention the equilibrium relationship of price level, interest rate, and growth rate. Before passing to the next section the results of this section are summarized.

This section studied a multi-sectoral extension of Harrod's model in its most rigid form. In the model it was found that adjustment mechanisms are not absent for equalizing G_n and G_w of Harrod's model. It was found that price mechanisms can be effective if they have a sufficient degree of flexibility. Allocation of factors is governed by consumer choice and this also determines the income of factors together with prices of goods. It was also argued that in the long run full-employment equilibrium is stable. Equilibrium with unemployment as a possibility was not disputed above. It was observed that this kind of situation is unstable in the long run. Also mentioned were some of the properties of Harrod's warranted rate of growth which follow from assuming constant capital-coefficient and constant savings-coefficient. Those arguments need not be repeated here. In brief, the fixed coefficient model is found to be not as rigid as some writers with this notion seek to impress.

However, the approach of assuming a fixed coefficient of production with extreme rigidity may not do much justice to Harrod, because he did not maintain that a given capital-coefficient is true for all rates of interest. In his theory capital-output ratio can vary with variation in the rate of interest. Thus the idea that with a given rate of interest that capital-coefficient is chosen which maximizes the profit of entrepreneurs seems to be present in the theory. This implies that Harrod's technology also allows substitution. The problem then arises that if such a possibility exists the interpretation should be that the failure of market mechanism to induce change in capital-output ratio leads to disequilibrium.

Similarly in the input-output models the fixed-coefficients may actually have been the results of market behavior rather than something given.

This brings up an important issue, namely, whether the fixed-coefficient assumed in Harrod's model is cause or effect of the market phenomena in general. Since Harrod's position is not quite obvious, in the next section the neoclassical substitution model will be studied in order to see under what conditions which fixed coefficients are established. Then Harrod's model will be considered from the new point of view.

Section VII: Substitutability of Factors and Harrod's Model

It is already known that the assumption of fixed proportionality of factors in the Harrod-Domar model and in the Leontief models is regarded as unrealistic by their critics. A more realistic way to deal with the problem of production would be to assume variable proportions of factors as the opinions say. However, it might have been the case that the above models might have assumed such variability of factor-proportions, but the working of the economic system would be such that some given coefficients would necessarily rule. In this section the problem will be discussed as to what conditions can lead to this type of situation. First this is discussed in the framework of a static model and then the method is extended to dynamics.

As in all analyses of competitive equilibrium the arguments here are based on the postulate that the economic problem is one of maximization of output or minimization of cost. This has a close resemblance to the programming problems in which the vital question one faces is about the choice of factor or process combinations which would result in minimum real cost for a given amount of output or maximum output per unit of real cost. The choice is made from a number of possibilities, which may be finite. If some assumptions about the nature of techniques of production are made one can easily see that such optimal coefficients of production exist irrespective of the level of output of each good produced. The most significant assumption that should be noted here is that there are constant returns to scale.

One could apply the linear programming technique to study this problem, as Arrow and Koopmans have done.[1] This technique leads to similar conclusions regarding the simultaneous determination of prices of goods and factors, and for all levels of output of goods one has the optimum coefficients of production. However, the argument will be restricted to the simple version of Samuelson's[2] approach.

The static model is now considered, assuming a continuous production function which is homogeneous of degree one in the inputs employed. Labor is regarded as the only primary factor of production. All other inputs are producible in the system whereas labor is not. Since labor is the only primary factor, wage is the only cost involved for the economy as a whole in producing various goods. For the sake of simplicity only two goods are assumed, namely, consumer good and capital good denoted by the indices 1 and 2. The respective amounts of these goods produced are represented by x_1 and x_2. Labor is regarded as input denoted by index 3, the total supply or rather the employment of which is x_3. The symbol used to denote the amount of j used in the production of i is x_{ij} $(i = 1,2;\ j = 1,2,3)$. The following production functions for 1 and 2 are written as:

II-65
$$X_1 = F(X_{12}, X_{13})$$
$$X_2 = G(X_{22}, X_{23})$$

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[1]See T. C. Koopmans (ed.), Activity Analysis of Production and Allocation (New York: Wiley, 1951), Chapters VIII and IX.

[2]*Ibid.*, Chapter VII.

Also:

$$\text{II-66} \qquad X_{12} + X_{22} = X_2$$

$$X_{13} + X_{23} = X_3$$

If commodity 1 is selected arbitrarily and its amount maximized using the above equation, the function F is maximized to do this using the second equation in II-65 and the conditions II-66 as constraints. The following Lagrangean expression may be formed:

$$\text{II-67} \qquad L = \lambda_1 F(X_{12}, X_{13}) + \lambda_2 \left\{ G(X_{22}, X_{23}) - X_{12} - X_{22} \right\} + \lambda_3 \left\{ X_3 - X_{13} - X_{23} \right\}$$

where λ_i $(i = 1,2,3)$ are the Lagrangean multipliers. From II-67 there are the following maximum conditions:

$$\text{II-68} \qquad \frac{\partial L}{\partial X_{12}} = \lambda_1 \frac{\partial F}{\partial X_{12}} - \lambda_2 = 0$$

$$\frac{\partial L}{\partial X_{13}} = \lambda_1 \frac{\partial F}{\partial X_{13}} - \lambda_3 = 0$$

$$\frac{\partial L}{\partial X_{22}} = \lambda_2 \frac{\partial G}{\partial X_{22}} - \lambda_2 = 0$$

$$\frac{\partial L}{\partial X_{23}} = \lambda_2 \frac{\partial G}{\partial X_{23}} - \lambda_3 = 0$$

On eliminating the λ's:

$$\text{II-69} \qquad \frac{\partial G}{\partial X_{22}} = 1$$

$$\frac{\partial G}{\partial X_{23}} \cdot \frac{\partial F}{\partial X_{12}} - \frac{\partial F}{\partial X_{13}} = 0$$

The solution of these equations gives the ratio in which capital and labor should be used at any level of output. The amount of output does not affect the ratio.

The above is the substance of the substitution theorem which says that under the conditions assumed above regarding technology there is a unique coefficient of production which makes the output maximum or which minimizes the cost of production. The dual aspect of the theorem is obvious. It shows that the marginal productivity determines the share of each factor employed.

This has so far been a static point of view. However, it has been a starting point for the dynamic. To be considered now is the question whether the substitution theorem holds for the dynamic. If in the production functions given above all the variables were treated as functions of time and if saving and investment were assumed to be going on as leading to increment of the stock of capital there would be a growing economy. It might appear that the theorem would hold even for this case. In order to see the validity of the theorem the problem would be approached from the dual side. That is, the pricing problem would be explicitly formulated to see whether the theorem that under constant returns to scale there is only one coefficient of production regardless of the level of output produced at each point in time. To do this a one commodity model can be taken. If the model given in Section IV is taken in which the amount of one commodity X is a function of K, as capital, and L, as labor, it may be written using the present assumption regarding the return to scale as:

II-70 $$F(a,b) = 1, \text{ where } a = \frac{K}{X}, \text{ and } b = \frac{L}{X}$$

X, K, and L are time functions.

One can study the problem from the point of view of entrepreneurial behavior; because it is in the interest of entrepreneurs as to what ratio of factors should be chosen so that the allocation of resources in the enterprise is efficient. If it is assumed that the enterprises maximize profit the following price equation would be a better guide for seeing the effect of profit maximization on the various relationships of the variables:

II-71 $$p = (arqp + wb)\ (1 + \pi)$$

Or $$\pi(arqp + wb) + arqp + wb - p = 0$$

where π is the rate of profit, p the price, w the wage rate, r the rate of interest, and q includes those elements in the value of capital which are taken into consideration in fixing price and which do not appear in its cost, interpreted in terms of material or input cost. To maximize π the following expression is used:

II-72 $$\pi(arpq + wb) + arpq + wb - p - \lambda\{F(a,b)-1\} = 0$$

where λ is the Lagrangean multiplier.

Setting the partial derivatives of with respect to a and b equal to zero:

II-73 $$\frac{\lambda F_a(a,b) - \pi rpq - rpq}{arpq + wb} = 0$$

$$\frac{\lambda F_b(a,b) - \pi w - w}{arpq + wb} = 0$$

where F_a and F_b are partial derivatives of F with respect to a and b respectively. From the above equations the conditions for maximum profit may be written as follows:

$$\text{II-74} \qquad rpq(1+\pi) = \lambda F_a(a,b)$$
$$w(1+\pi) = \lambda F_b(a,b)$$

The equilibrium conditions in II-74 say that under competitive conditions the marginal productivity ratio of the factors employed should equal their price ratio. If perfect competition is assumed, π has to be zero. But for the present argument the positive value of π does not introduce any complication. In II-74 there is also the condition that λ has the value 1 because the two incomes, namely the return on capital and the return to labor, exhaust the total product. If the present argument is combined with the one given previously there is a complete picture of how the factor ratios and their earning ratios are invariant under any change in the scale of output.

This evidently is not too convincing an argument, because in II-74 it is seen that the ratio of rpq to w determined a and b. In order to establish the conclusion about the invariance of the coefficients of production, it is necessary to prove that any change in such a ratio is impossible if equilibrium is to exist. One should be able to show that any change in a factor price is compensated by an appropriate change in other price or prices in such a way that the coefficient of production is left unchanged.

Before entering into further details of this argument some observations will be made about the determination of factor prices according to the equilibrium conditions II-74. While the rate of profit does not affect the coefficients of production in II-74 it is clear from II-71 that it affects price. Hence to have a determinate price one should know the rate of profit. Moreover r and p taken together may be regarded as constituting one term or separate terms. This latter alternative is valid. Taking the price items there is then one equation, II-71, in four unknowns taking q as given. Thus if profit rate is positive there are three degrees of freedom in determining prices. If competitive profit is assumed to be zero there are two degrees of freedom, or in case p = 1, there is only one degree of freedom.[1] Thus in the former case if any three prices are known and a and b are known the remaining two can be determined. In the case that profit is zero the one price has to be known as if it is determined wholly outside the system. If, however, all prices are determined the equations II-71 can be solved for the coefficients of production, a and b, which satisfy II-74.

With these observations, if the theory that the constancy of a and b must obtain at unique values for all variations in output is considered, there are several remaining assumptions to choose among. But all such assumptions should lead to the same conclusion that the relationship between w and rpq should be constant. To see what assumptions about the behavior of the economy are necessary it can be noted what would happen if equilibrium is disturbed by change in any of the prices in equation II-74. Assume that there is a change in the time preference of

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[1]In this case note the similar argument by M. Morishima in "Prices, Interest and Profits in a Dynamic Leontief System," Econometrica, XXVI, No. 3 (1958), 358-370. His treatment of the substitution theorem for a general multisectoral growth model is of especial interest.

the people so that the rate of interest is lowered. This might be considered as leading to an increased proportion of income saved. Further it may be argued that this leads to increased investment and increased wage rate. If this happens then, according to II-74, there will be a change in the coefficient of production if price remains at the old level which may be held to be true. There can be a restoration of the coefficients a and b to the older values if and only if there is a compensatory rise in price which has been assumed away for the present. If a fall in the rate of interest is compensated only by a rise in price there is the possibility of constancy of the coefficients with wage rate at the old level. Similarly one might find numerous conditions which should hold for the substitution theorem to be true. But the logic of this theorem should indicate that the coefficients rule over the prices and not the other way around. The disturbance of equilibrium is, according to this theorem, accompanied by changes in prices in order to establish the values of a and b.

The above may appear to have slightly distorted the content of the theorem by ignoring, for example, the explicit discussion of Morishima[1] who proves this by saying that if prices are given in the economy the coefficients are determined by them. This is perhaps to be regarded as a convincing argument. In this case the coefficients of production would be regarded as not independent of factor-price ratios. But the variation of the latter leads to the variation of the former. This interpretation gives the conclusion that the optimal coefficient

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[1] *Ibid.*

of production for an economy is determined by the condition of prices of goods and factors, given the production function. If a community has achieved excessive capital accumulation resulting in low rate of interest the ratio b, other things remaining the same, must increase in order that the equilibrium conditions be satisfied. Similarly if wage rate is very low due to excessive supply of labor and if high rate of interest prevails, the coefficient, a, must increase. This is so because at new factor prices the costs of production change and they can be minimum only with a different combination of factors.

In the above dynamics the price was assumed to be constant for all time. If price is assumed to change, which may be imagined in terms of money, there will be a situation in which the determination of the coefficients of production by reference to factor prices may be hard. It is worthwhile to see what implications can be derived from a model that assumes money prices changing.[1]

Assume that price at time t is equal to the rate of return on capital (using price at time t-1 for evaluating return for t-1), whose amount per unit of output is a, corrected for capital loss due to change in price at time t, plus the wage bill per unit of output. One may assume that all the elements of value of capital are included in the capital-coefficient. One may also assume real wage is lagging behind the change in price by one period. This assumption will be

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[1]This argument is to some extent based on Solow's model, "Competitive Valuation in Dynamic Input-Output System," Econometrica, XXVII, No. 1 (1959), 30-53.

made here. Then for price at t:

$$\text{II-75} \qquad p_t = a(1+r)p_{t-1} - ap_t + \overline{w}p_{t-1}b$$

where $\overline{w}$ is the real wage. II-75 has the following solution:

$$\text{II-76} \qquad p_t = p_o \left[\frac{a(1+r) + \overline{w}b}{1 + a}\right]^t$$

where p_o is price at time 0.

Let the case of stationary price be taken first. For this $a(1+r) + \overline{w}b = 1 + a$, or $ar + \overline{w}b = 1$ is required. Thus in a stationary condition of prices it is seen that the theory that marginal productivity determines prices of sectors holds. If, however, prices are changing there are two possibilities. One is that $ar + \overline{w}b < 1$ in which case price tends to fall to zero. The other is that there is the possibility of an explosive rise in prices if $ar + \overline{w}b > 1$.[1] In the two possibilities given above it seems that under the technological assumptions made here changing prices may be an impossibility.

If the condition is neglected that money wage rate is changing with one period lag and the equation II-75 is made non-homogeneous by substituting wb for $\overline{w}p_{t-1}b$ the condition may be obtained for stationary prices which will be in line with the argument of the previous pages. In fact the solution of such an equation becomes:

$$\text{II-77} \quad p_t = \left(p_o - \frac{wb}{1+a-a(1+r)}\right)\left(\frac{a(1+r)}{1+a}\right)^t + \frac{wb}{1+a-a(1+r)}$$

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[1]It should be noted that the arguments given here are the author's derivation from the model being discussed and the model, although similar in some essential respects to that of Solow (see previous footnote), the arguments may differ.

This is the solution of equation II-75 in which $\bar{w}bp_{t-1}$ was replaced by wb. Stability of price requires that either p_t should converge to the last term on the right hand side of II-77 or $a(1+r) = (1+a)$. The former leads to the price equation of the earlier discussion,[1] except for the terms q and π. These arguments indicate that the substitution theorem may hold only under the assumptions of stable price. But it should be remarked that there is no reason why price should be unstable in the above model. Instability for II-77 requires that $ar \geqslant 1$. It is doubtful if there can ever be a circumstance in which this situation may ever occur except for an exceptionally short interval. The author would rather hold that price is stable in the long run.

It is not denied, however, that there might be the possibility of price instability. But to deal with this problem the above device would be inadequate. In the above model it is assumed that while prices would change, the rate of interest does not react to it in any way. This does not seem justifiable. If one may consider all interrelations of prices and interest it is through a different type of approach which should also bring into the picture the monetary aspect of the economy. This is not the scope of the present work. Thus in the long run price will be assumed to be stable and all rising tendencies that might occur or should have occurred would be assumed to be explained by exogenous factors. Hence it is assumed here that the substitution theorem in Morishima's version holds without difficulty

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[1]See p. 115, equation II-71.

for the above model.

The implication of the above analysis for Harrod's model is worth considering now. It is first noted that Harrod says that a given capital coefficient rules in the economy, if there is neutral technological progress and if a given rate of interest exists. If under the conditions given by Harrod all the conditions discussed above hold, it is apparent that Harrod's argument is proved. But it is not quite clear how a given rate of interest should be uniquely related to other prices. If price offsets the variation of the rate of interest, different rates of interest may exist with the same capital coefficient according to the author's argument. Thus Harrod's proposition seems to have ruled out the effect of prices. Or, it should imply that price change cannot offset the effect of change in the rate of interest.

The argument about price will be discussed a little later. Here the grounds for the condition of constant capital-coefficient given by Harrod may be made precise. In this connection an argument given by Green[1] will be considered. According to this argument the capital-coefficient and the rate of interest are functionally related as the following analysis will show. Assume a production function of the Cobb-Douglas type as below:

II-78 $$X(t) = K^a L^b \qquad a,b > 0 \qquad a+b = 1$$

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[1]The arguments used here are to be found in H. A. J. Green, "Growth Models, Capital and Stability," Economic Journal, LXX, No. 277 (1960), 57-73. See especially pp. 57-59.

where K and L are capital and labor which are time functions, and X is output. Differentiating the logarithm of both sides of II-78 with respect to time and using dot on the top of the variable to indicate its time derivative:

II-79 $$\frac{\dot{X}}{X} = a\frac{\dot{K}}{K} + b\frac{\dot{L}}{L}$$

Assuming rate of growth of labor to be n and using the savings-investment identity $sX(t) = \dot{K}$

II-80 $$\frac{\dot{X}}{X} = a\frac{sX(t)}{K(t)} + bn$$

Introduce one further condition in the model, namely that entrepreneurs equate the present value of the expected income stream from an investment to the cost of capital. Assume that income expected from the investment is continuous over the infinite life of capital. Also assume that an investment is made and the cost incurred is one dollar. Then the following relation will hold:

II-81 $$\int_0^\infty \frac{\partial X}{\partial K} e^{-rt} dt = 1$$

where the partial derivative of X with respect to K means the marginal productivity of capital, which is the return on capital. It should be noted that this return is not expected to vary. Even if it varies due to obsolescence which might be expected to take place in a given manner the argument will not be affected to any significant extent. Thus if the condition II-81 holds, the equilibrium relation between the capital-coefficient and interest rate can be determined as follows.

On solving II-81, $1/r . \frac{\partial X}{\partial K} = 1$. From II-78 it is known that $\frac{\partial X}{\partial K} = \frac{aX}{K}$. Therefore, $K/(aX) = 1/r$. But $K/X = C_r$ in Harrod's model.

II-82 $$C_r = \frac{a}{r}$$

Equation II-82 shows the relationship explicitly between the rate of interest and the capital coefficient. If this relationship is used the warranted rate of growth becomes:

II-83 $$G_w = \frac{sr}{a}$$

The substance of this part of the argument is that II-82 lays down the rule of substitution and II-83 shows how equilibrium growth is related to the rate of interest. Using them in II-80 the relationship is obtained between growth rate of output with the equilibrium growth rate and natural rate of growth, n, which may be written as:

II-84 $$\frac{\dot{X}}{X} = a\frac{sr}{a} + bn$$

It follows that the warranted rate and natural rate of growth tend to equality by change in the rate of interest. Harrod argues that the equality between the two rates can be achieved in an economy with higher warranted rate compared to the natural rate by progressive lowering of the rate of interest which would lead to the deepening of capital. This is in line with the neoclassical thinking. But Harrod suspects that the substitution principle might not work. In this connection the critics of the neoclassical model hold that the assumption underlying the principle of substitution is unrealistic. They hold that the assumption of substitutability of factors implies that the rate of profit remains positive however large the ratio of capital to labor is made.[1] They hold that for a sufficiently large capital-labor ratio the rate of profit can be zero. Hence no change in the rate of interest could be of any avail toward raising capital-labor ratio.

Another argument used to show the invalidity of the neoclassical approach is that even if the rate of profit could remain always positive for all capital-labor ratios the rate of interest may not fall below a certain minimum. If the rate of interest is above the marginal productivity of capital there is no scope of raising the capital-coefficient.

Before commenting on all these arguments one more contention should be noted--that the rate of discount used in evaluating the present value of capital may not be related to the market rate of interest. But this argument is inadequate if it means that it is true even under equilibrium. Further, this argument is of little use because there is no other way to explain investment behavior except by relating it to the demand condition in general and the rate of interest. The fact that the warranted rate is equilibrium rate and the interest

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[1]In this connection one may note Eisner's argument about whether the fixed-coefficient assumption of Harrod's model is more realistic than the implicit assumption of the existence of positive profit however large the capital-labor ratio in the neoclassical type of models. He says: "Critics of growth models deal with functions which implicitly or explicitly (like the Cobb-Douglas illustration which Solow employs) imply the assumption that marginal net product of capital is always positive, regardless of how high the capital-labor ratio rises; barring demand problems it must always pay to invest. This indeed, is one of the crucial assumptions on which growth model critiques rest. It embodies again the optimistic notion of unlimited investment opportunities." "On Growth Models and Neoclassical Resurgence," Economic Journal, LXVIII, No. 272 (1958), 713-14.

rate is such that this equilibrium is satisfied makes this argument unjustifiable.

The reader is now in a position to see the main points that constitute the argument of the criticis of the neoclassical growth model. One is that the production function that assures positive rate of return on capital is unrealistic. The second is that the rate of interest as an adjustment channel is ineffective. Concerning point one it might be said that there is nothing in a production function of any type that can explain income distribution in an economy irrespective of the conditions of demand. In the real world the possibility of varying proportions of labor and capital exists and such variations follow the rule of profitability. But the precise manner in which this works is hard to understand. With the Cobb-Douglas function one does not necessarily reach the conclusion that the economy is perfectly stable and the profit rate will always remain positive. Whether profit is to go up or go down depends upon the behavior of prices. In the unspecified function which was used for the purposes of this section variability of factor-proportions was assumed and assigned the property that is characteristic of the Cobb-Douglas production function, namely, that of homogeneity of degree one. The author, however, did not elaborate upon the point that even in such a case there is the possibility of price instability and the profit rate may assume any value. But such possibilities are regarded as not lasting and in the long run there is stable price and profit is similarly stable.

The second point mentioned above is important. In the second essay on dynamic theory[1] Harrod says that C_r is not responsive to the changes in the rate of interest. In the light of the argument outlined in this section the irresponsiveness of the rate of interest could be explained as follows. The fall in interest could lead to rise in price and also rise in wage. The total effect might then be the change in the absolute level of these variables. But relatively they might remain invariant. This would leave the capital coefficient invariant. However, if this kind of argument holds at all it would hold only under the conditions of full-employment. It would rather seem more justifiable to explain the ineffectiveness of the rate of interest by means of profit expectations. Harrod uses the term natural rate of interest as determining the capital coefficient and also the natural rate of growth. Without following Harrod, low natural rate of interest may be taken to mean low profit expectation. In such a case lowering the rate of interest might not induce change in C_r because of the low prospect of earning. This would imply that the effect of lower interest rate is offset by other changes.

Harrod, however, considers that there is a rate appropriate to the natural rate of growth which may be regarded as the natural rate of interest. He says that G_n is itself determined to a significant extent by this rate of interest. Its precise relationship to the latter is not unique. In summary, the argument appears to be that since there is a

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[1]R. F. Harrod, "Second Essay in Dynamic Theory," Economic Journal, LXX, No. 278 (1960), 277-93.

maximum achievable rate of growth considering the resources and technology of an economy there is a rate of interest that will help the economy achieve it, or rather be consistent with the behavior of the consumers of the economy while at that growth rate. A change in the rate of interest implies that the growth rate should vary assuming the preference of the individuals in the society. If G_n is dependent on interest and consumer utility maximization is also an important element in determining G_n or interest it might appear that market forces should affect G_n. But this is not true because all these are given relations and the market may not establish them. The value of C_r is determined by the technology, the utility condition is given, the rate of interest and corresponding G_n is given at any time. C_r is not liable to be affected in the market as long as G_n is given. Under these conditions the achievement of equilibrium at G_n requires that the propensity to save should be such that G_w should equal G_n. This means that with the type of consumer behavior there is a natural rate of interest which is appropriate to G_n which requires a given propensity to save which may not be realized under the existing conditions. Then if there is a difference between the required and actual saving propensity, G_w and G_n differ with all the consequences of inflation or stagnation following.

Although there are a number of inconsistencies in the above arguments only the point concerning the divergence between the required and actual saving coefficient will be considered here. It is necessary to note that the propensity to save of the community as a whole is dependent upon a number of psychological and institutional factors. To assume a constant coefficient without going into the details as to why it should be so is at best a simplification of the analysis which helps an analyst as to what effect of changing value of this parameter on equilibrium income may be expected. If it is recognized that difference in the pattern of income distribution can affect the aggregate propensity to save chronic inflation or secular stagnation naturally affect it. On this point the author is on the side of Kaldor. In Harrod's model the instability of G_w is due to the fact that entrepreneurs pursue a policy that is just the opposite of what is required for attaining equilibrium. This is true because equilibrium determined at less than full-employment or with excessive capital is unstable, though Harrod's explanation is different. Thus the natural cure for such instability is the establishment of equilibrium at natural rate of growth.

In concluding this section the following points may be noted. Although Harrod seems to allow substitution possibility in his model he strongly adheres to the assumption of fixed coefficient of production. In his argument there is obviously a prejudice for showing disequilibrium without showing adequately the reason for entrepreneurial choice of coefficient of production and the community's choice of any particular savings-coefficient in the framework of a market economy. It would be much more appropriate to regard G_w as short run equilibrium and G_n as the long period equilibrium. If the two are treated as independent, all growth rates of Harrod's model will be actual but not equilibrium because in that case there is no equality between supply of factors to demand in the long run sense nor does Harrod's explanation show any inherent tendency towards long-run equilibrium. Further, if G_n is to be regarded as something dependent on a number of forces that in all their complexity determine the potentialities of a society from one phase of its history to another its value for economic analysis has to await exploitation until much comparatively easier things have been known.

Section VIII: Conclusion

In this section the long controversy about Harrod's model is concluded with the following summary of this writer's arguments. The discussions above were mainly concerned with the usually emphasized properties of Harrod's model. The most important among them was the so-called inherent instability of the dynamic model which Harrod seeks to establish and the related questions of independence of equilibrium growth from the natural rate of growth. As far as instability is concerned, it was observed that the argument of Harrod is not sufficient to prove this. Jorgenson's argument was used to show that the model might be perfectly stable. Other arguments were used from the present writer's own formulation of lagged models using Harrod's technological assumptions and showed that there are possibilities of neutral equilibrium in the model.

In the long-run context Harrod's model was found to exhibit the fact that the warranted rate of growth would in no way approach the natural rate of growth. Solow's criticism was considered that this conclusion in Harrod's model was due to the assumption of fixed proportionality of factors. Solow's argument was studied that with the neoclassical type of technological assumption one can have a perfectly stable warranted rate of growth that equals the natural rate of growth as a natural course of events. The distributive mechanism implied in Solow's model contributed to the explanation of stable equilibrium growth rate at full-employment level. It was also studied that in Harrod's model income distribution exhibited an unacceptable character or it was found that there is no way to explain distribution in Harrod's model which could explain the property of equilibrium with the help of the pricing mechanism in general. However, recourse could be taken to Kaldor's theory of distribution in this connection.

Two important points were noted in relation to Kaldor's theory. One is that the theory can explain distribution if it is assumed that natural rate of growth is established. The other point is that if the previous argument is not accepted, Kaldor's theory is inadequate to explain income distribution. In fact, Kaldor's theory assumes distribution rather than explaining it as other prices are determined. It is observed here that some other theory is required to explain distribution in Kaldor's model. For this the arguments of Schneider and Atsumi were brought in. This writer holds that multiplier is the determinant of total income and marginal productivity may be used to explain distribution. Thus was undertaken the task of explaining distribution for a fixed coefficient model with the help of supply and demand conditions. The conclusion was arrived at that the assumption of fixed coefficient of production was no hindrance to the application of a modified version of marginal productivity theory. A more important conclusion on the basis of the theory of multisectoral equilibrium was that there was a genuine equilibrium which would be realized only at full-employment. Thus not only the question of indeterminacy of factor shares was answered but also it was found that under competitive conditions equilibrium required that warranted rate should equal natural rate although the constancy of the rate of growth would require assumptions which need not be of concern here. It was also observed that the multisectoral model of the type discussed would tend to grow at a constant relative rate if the assumptions required for such constancy are satisfied. In such a case the model would be similar to Leontief's closed dynamic model which was chosen to compare with Harrod's model.

In this last section are considered the statement of Harrod that a given fixed coefficient of production is valid only at a given rate of interest and if the technological progress is of a neutral kind. Since at present there is more logical basis for such assertion made by the substitution theorems the simplified version of them was considered with Samuelson's approach. In order to see the impact of prices of goods and factors on coefficients the dual of the theorem was considered in which case the author's model was a simplified version of that of Morishima. The conditions under which such a theorem could hold were noted. The substitution theorem tells nothing more than what a neo-classical theory does. For the theorem to hold it is necessary that changing factor prices or prices of goods should not lead to instability in price.

The theorem in Harrod's model was applied and also derived on the basis of Green's argument of an explicit relationship between rate of interest and capital coefficient. Although the relationship was in line with the substitution principle the arguments about the unrealistic nature of the production function used in the neoclassical approach was directed toward proving the ineffectiveness of change in the rate of interest in changing capital coefficient. Thus in spite of the long argument it was found that Harrod did not believe that the coefficient of production can change.

Finally, some arguments were noted regarding the absence of any tendency for G_w to equal G_n. It was noted that G_w might be regarded as a temporary equilibrium if it is different from G_n. This argument is clear from the observations made in Section VI.

The conclusion here is not different in essentials from those of Kaldor,[1] who argues that the underemployment equilibrium is unstable. Moreover, Kaldor makes the growth rate of population depend on the actual trend of the economy. This is in contrast with the idea of Harrod. However, it is not easy to relate the growth of population in any simple way to the economic trend because this would presuppose a knowledge of the relationship of all other social factors determining the change in population with the trend of the economy. Moreover this assumption is not essential for the case here. Another important assumption which appears more realistic than in the case of Harrod's

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[1]See N. Kaldor, "A Model of Economic Growth," Economic Journal, LXVII, No. 268 (1957), 591-624.

model is that about change in the capital-labor ratio with growing capital. In this case, however, Kaldor thinks that rise in capital accumulation leads to the increase of capital per unit of labor and rules out the assumption of neutral technological progress.[1] The following figure[2] taken from his article illustrates the relationship he assumes between rate of growth and the rate of change in the capital-labor ratio. The curve shows more like a production function. Whatever argument one may have against this shape of curve relating innovation to capital accumulation the basic argument appears more justifiable than that of Harrod. On the whole, Kaldor's model establishes similar conclusions as the neoclassical model not actually by using more basic causal analysis but rather by assuming the conclusions.

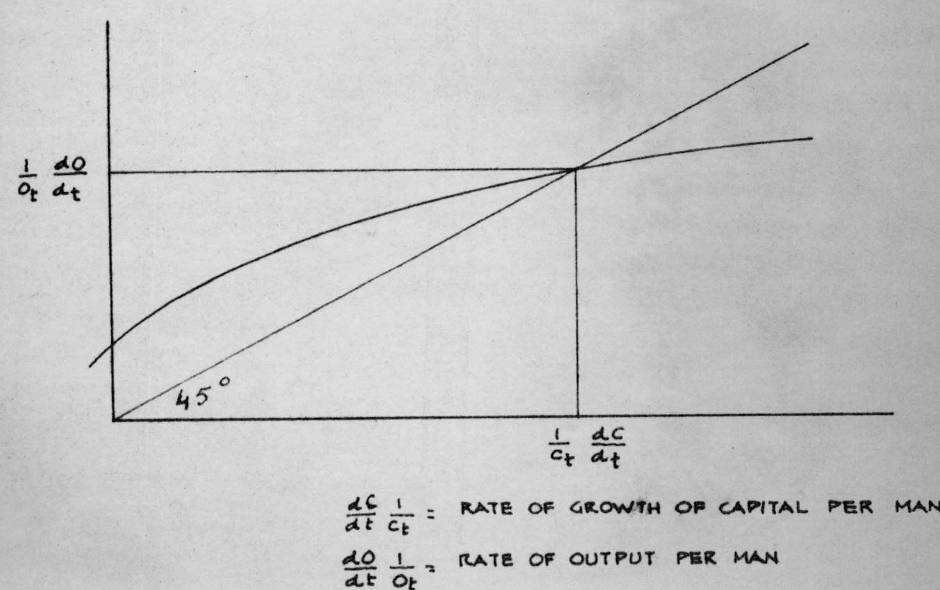

Figure 9.

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[1] Ibid.

[2] Ibid., p. 597.

The above observations complete the discussion regarding the characteristics of the fixed coefficient model of Harrod and also the neoclassical criticism. From the point of view of realism both these models are extreme. A fixed coefficient model is inadequate to explain growth because the assumption is not valid for long period problems. Similarly the substitution model is inadequate because it is true only in the long run. In the next chapter Johansen's model which attempts a synthesis of the two will be studied.

CHAPTER III

JOHANSEN'S MODEL OF ECONOMIC GROWTH

Section I: Introduction

In the last chapter the problems of equilibrium growth were studied under two different technological assumptions underlying the production functions on which the models were based. Harrod's model was considered as belonging to the class of growth theories in which fixed proportionality of factors of production is assumed. The class of models based on the assumptions of substitutability of factors was regarded as neoclassical. The assumptions of both these classes of theories are extreme and unrealistic. The former of the two theories mentioned above uses an assumption that is valid in the short period only, because only in the short period the existing capital stocks do not permit any substitution between factors. This fact, as noted earlier, makes the theory of less significance for dealing with long run growth equilibrium. In the long run the neoclassical assumption appears to be valid by assuming that there is no restraint imposed on the entrepreneur in substituting one factor for another.

The real world, however, presents a different picture. It is not true that an entrepreneur is limited by his technological knowledge or prejudice to any particular proportion of factor inputs. Nor is he able to change the proportions at any time, whatever justification he may have for the change. Thus it is necessary that in order to be realistic these facts be clearly recognized. In this chapter the analysis of economic growth is based on the assumption that when a capital good is installed the proportion of factors is fixed at the value which is planned when the decision for such installation is made. However, when the new capital replaces the old the factor-ratios might change at the discretion of the entrepreneur. There is, thus, fixed coefficient ex post and substitution possibility ex ante.

On this assumption a general model of economic growth and solutions of the model for some particular cases were presented by Johansen recently.[1] It will be seen later that the conclusions emerging from the model do not differ significantly from those observed in the case of Harrod's model or the neoclassical model. The role of durability of capital goods in the economy is made more explicit, although Johansen has represented it more as a parameter than a variable. This is a major accomplishment, but one with limitations. To assume that durability of capital is determined exogenously and is unalterable after the installation of capital is to assume away one important aspect of the theory of capital. However, to introduce it as any other variable of the economic system makes the problem extremely involved. An example of the parameter treatment of durability of capital concerns factor-ratios. The change in factor-ratios may be related in some

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[1]L. Johansen, "Substitution versus Fixed Production Coefficients in the Theory of Economic Growth: A Synthesis," Econometrica, XXVII, No. 2 (1959), 157-76.

unknown way to the durability of capital, thus becoming an involved variable in entrepreneurial decisions. Similarly one may suspect that many innovations taking place in the economy affect the durability of capital goods which already exist in the economy and of those that are being produced. The other confusion this leads to is the appropriate definition of the production function. A complete description of the production process is not accomplished by simply saying that so much of output is produced with so much of capital and so much of labor and other inputs. Unless it is specified that the durability of capital has such and such effect, in a precise way, the production function gives but an incomplete and possibly a misleading picture of the economic structure. Further, the role of market forces in determining durability of capital is extremely important. These are some of the problems introduced by considering the new variable in the model of economic growth. Johansen's treatment deals with these matters only in a limited way. Without going into the intricacies of the problems the object of the writer in the present chapter is to restrict himself to some particular simple cases which are designed to illustrate the relationships that exist among the rate of growth of output, choice of production coefficients, life of capital goods, or the rate of depreciation, the rate of interest, and factor prices.

It has been emphasized in various places that the present object is to study some properties of equilibrium growth. The purpose of the present chapter will be to derive some relations between the variables mentioned above on the basis of some simplifying arbitrary assumptions regarding the behavior of employment, the rate of depreciation of capital, the propensity to save, and the production function and technology. The term variable is used above rather loosely to include even the parameters when desired. This will be clear later when the relevant problems are met.

In the second section of this chapter a simple general model is outlined without specifying the nature of functions involved in the various structures that may be conceived. In the third section the case of infinite life of capital goods is studied, where the term durability of capital is used in a physical sense different from the one that is economically meaningful. It will be said that a capital good is infinitely durable if it can produce a given amount of goods per unit of time with an appropriate combination of other factors for ever. This can be taken to mean the physical durability of capital. The reason for this choice of definition of durability is to see the effect of other variables on the economic life of capital given the physical durability exogenously. In the section mentioned the various equilibrium solutions assuming fixed population or labor force will be elaborated. The solution for the case of a growing labor force will be mentioned, however.

In the fourth section a constant finite life of capital is assumed. The solutions will be obtained for various cases considered with different assumptions regarding the supply of labor and the progress of technology.

In Section V the case of exponentially depreciating capital goods will be considered. Other assumptions will be similar to the ones used in Section IV.

In Section VI the main conclusions and criticisms will be given. The object of the author in the final analysis is to suggest a further synthesis of the models of economic growth--that of Johansen with the neoclassical model.

Section II: A Simple General Model[1]

In this section a general model based on some simplifying assumptions is developed. This chapter will be concerned with a one commodity case throughout. The single commodity may be considered as an aggregate or otherwise. All the index number problems are avoided by considering one commodity in the economy which is used both for current consumption and as a stock used for further production. The commodity and the stock of capital are both assumed to be measured in common physical units.

In the production process it is assumed that there are only two types of factor inputs, namely, labor and capital. Since the durability of capital is introduced in the present model the production process has to be analyzed with proper recognition of the effect of this variable. This, however, leads to much complication if a realistic description of the production structure is desired. In any particular

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[1]The basic models in all cases are similar to Johansen's but the details of variation in models and equilibrium conditions are all the author's own.

enterprise there may exist a large variety of ages of capital assets. The same is true about the economy as a whole. In such a case it is rather difficult to obtain a production function of the types studied in the last chapter. The reason is that the capital installed in each time period might have embodied in it a different technology and also a different ratio of labor and capital might have been planned with the installation of each new capital equipment. This fact introduces the difficulty. But this is one of the problems which is to be considered in detail. In fact this study has to unfold the technological developments at the margin. In other words, this discussion has to provide an answer to the question, "What technology will be adopted and what direction of movement along a production function will be chosen when a new investment is made, given the progress in the knowledge of new techniques and production functions associated with them, together with the prices of factors of production?" An exhaustive study of this question requires a more comprehensive theory of investment.

It is assumed that the gross investment at any time which consists of the replacement of the wornout capital goods and the new addition to the stock of total existing capital reflects a choice of an appropriate, possibly new, technology and also an appropriate combination of factors. Let $k(t)$ represent the gross capital formation in the economy at time t. This investment might lead to employment of a certain amount of labor. The amount of labor employed with the capital $k(t)$, measured in units of goods, is represented by $n(t)$. If it is assumed that one labor unit utilizes one capital equipment, then $n(t)$

would measure the number of capital equipments installed at time t. For time t, one may represent the production technology in terms of the ratio of k(t) to n(t), this being true for only the capital installed at t.

Since the possibility of change of the shape of the production function over time with the progress in technology is being considered here, it is assumed that for time t, F_t will denote the production function. The function F_t transforms k(t) and n(t) to the gross increment of output at t which is represented by y(t). The production function is then written as:

$$\text{III-1} \qquad y(t) = F_t \left\{ k(t) \quad , \quad n(t) \right\}$$

At any time there is a certain age distribution of equipment. It is assumed that only the oldest of the existing capital is replaced when new is being installed. In general, it is assumed that there is a certain function a(t) which represents the time of installation of the oldest capital good operating at time t. If all capital goods installed at interval $\{t, a(t)\}$ are assumed to be operating at time t and if the contributions made by the gross investments and corresponding allocation of labor, with appropriate production function, at each point of time in the interval $\{t, a(t)\}$ are considered, the following integral would represent the total output at time t.

$$\text{III-2} \qquad X(t) = \int_{a(t)}^{t} F_u \left\{ k(u), \ n(u) \right\} du$$

where X(t) is the total output at t. Similarly, if N(t) and K(t) are the total volume of employment of labor and the total stock of capital existing at t, the following equations will hold:

$$\text{III-3} \qquad N(t) = \int_{a(t)}^{t} n(u)\, du$$

$$\text{III-4} \qquad K(t) = \int_{a(t)}^{t} k(u)\, du$$

In the succeeding analysis it is assumed that the production functions are homogeneous of degree one in n and k.[1] One production function differs from another in its embodiment of new technology. If there is no technological change the function remains constant. This does not mean that F becomes constant for all n and k. The meaning of constancy in this case is that the shape of the function does not change nor does the function value shift in time. In case of neutral technological progress the function f_t would be better represented as a product of two functions, g(t) and f where g(t) would be solely a function of time and f would be the usual production function studied in static models. The function g(t) may be interpreted as a measure of technological progress as affecting the increment of output. Very often this function is represented by the form e^{gt} in which g is called the rate of technological progress. This is neutral in the sense that it does not affect the relative productivity of the factors employed. When such progress is taking place the productivity of all factors will be rising at an equal pace.

If the technological progress is non-neutral the function F_t becomes more complicated to describe symbolically. In a static

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[1]"Production functions" is used in the plural because there are many production functions between the time interval t and a(t).

production function there are some parameters involved in it. The constancy of such parameters over time would imply the absence of progress in technique, if neutral progress of the kind discussed in the last paragraph is ruled out. It is, however, hard to find out how such parameters change in time. In most of the discussion here only neutral technological progress is considered. Yet on some occasions observations will be made about the situations in which non-neutral technological progress could be analyzed.

One further point which deserves mention regarding the production technology is that the equation III-2 implies that a capital good produced at time a(t) yields a uniform rate of output until time t. This may not be true, because as time goes on the cost of utilization of old equipment may rise and the producers might operate the old equipment at lower intensity. This would result in decline of output from that equipment. This, however, will not be considered too important by itself. In all the models some exogenously determined life of capital will be assumed. But later it will be necessary to introduce new assumptions regarding the behavior of firms when productivity is changing continuously in time. This will lead to the consideration of the obsolescence resulting from the endogenous forces of the economy.

The model presented above describes mainly the production aspect of the economy. Equilibrium conditions will now be considered. One of these is the equality between saving and investment. As before, a constant marginal and average propensity to save of the economy out of current gross income is assumed. One may apply this coefficient to income after deducting the depreciation allowances or to gross income. Most of the time it will be assumed that the gross saving is a constant proportion of gross income. In this case the following equation holds:

$$\text{III-5} \qquad sX(t) = k(t)$$

In cases where depreciation formulas are considered the alternative procedure may be relevant too.

It is assumed that the supply and demand for labor is always equal. Full employment of labor is assumed. Labor supply is assumed to be constant in some cases and increasing at exponential rate in some cases. The economy is perfectly competitive, there is no uncertainty and the factors are paid according to their marginal productivity.

This completes a brief general outline of this model. Any modification or elaboration that may be required will be postponed until the appropriate contexts.

Section III: Infinite Life of Capital Goods

In this section the case of an economy in which capital goods are assumed to have infinite life is studied. As an approximation to the real world, it is apparent that this model will not lead one far enough. It may, however, reveal how an economy of this type utilizes the possibilities of substitution between various factors that are open to it.

Before proceeding with the study of such an economy it is desirable to clarify the meaning of infinite durability of capital. It is first necessary to understand what Johansen means by it. In order to do this Johansen's model may be summarized for the present case.[1]

In Johansen's model the term infinite durability is used to indicate that a capital good installed at any time is not only capable of yielding a uniform rate of output for ever according to the technology embodied in it, but it is also actually operated at full capacity by the entrepreneurs for all time. This implies that if there is net accumulation going on, the effect of changes in economic circumstances, if any, will be borne by the new plants only, the old plants being apparently immune from such effects.

It is assumed that there is no effect of scale of output on the total amount produced. It is further assumed that there is neutral technological progress going on in the economy. One may, therefore, write the function F_t in the form $e^{gt}f(k,n)$ where f is assumed to be of the Cobb-Douglas type, although Johansen does not assume that the exponents of k and n add up to 1. Since the life of capital is assumed to be infinite the function a(t) in the model becomes minus infinity. Further, labor force is assumed to be increasing at an exponential rate λ, such that with N_o as the initial supply of labor the total labor available at time t is $N_o e^{\lambda t}$. Full employment of factors is assumed. Under these assumptions the following equations are obtained:

III-6 $$X(t) = \int_{-\infty}^{t} e^{gu}\{k(u)\}^{\beta}\{n(u)\}^{1-\beta}\,du$$

III-7 $$\int_{-\infty}^{t} n(u)\,du = N_o e^{\lambda t}$$

Differentiation of III-7 with respect to t yields:

III-8 $$\lambda N_o e^{\lambda t} = n(t)$$

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[1]Johansen, op. cit., pp. 165-67.

Using III-5 and III-8 in III-6 the solution is found to be:

$$\text{III-9} \qquad X(t) = \left\{ \frac{(1-\beta)s^{\beta}\ (N_0\lambda)^{1-\beta}}{g+\ \lambda(1-\beta)}\ e^{gt+\lambda(1-\beta)t} + C \right\}^{1/1-\beta}$$

In the above solution the possibility of zero rate of technological progress and constant labor force should be ruled out. Zero rate of technological progress would mean static technique, and in such a case one may accept the solution. If there is a constant labor force, first of all, the equation III-6 is undefined because n(t) becomes zero and the integral is zero. However, considering   as a variable and letting it approach zero as the limit in the solution III-9 output may be found to be constant. This implies that constancy of labor force stops innovation of all kinds. It should be remembered that when output is constant due to constant labor force saving may not be zero. In such a case what would happen to the amount of saving coming each period? Johansen's model does not provide an answer. Moreover, his model does not provide the reason why old capital remains in operation in spite of the fact that new plants are more productive than the old. By assuming that old equipment and new equipment alike have infinite life, all the complications are avoided.

The economic reason to be given for the possibility, in theory at least, of infinite life of capital is to be sought in the nature of technological progress and its rate in the economy and in the behavior of the entrepreneurs assuming that each capital equipment would produce a uniform rate of output according to its original efficiency for an infinite length of time. Johansen assumes the economic life of capital to be infinite without going into the reasons that are responsible for such a possibility. To assume that technological progress is taking place and the life of capital goods is infinite is to assume that entrepreneurs cannot distinguish between more productive and less productive capital goods. When technological progress is taking place labor and capital in new plants will be earning a larger rate of return and old plants will obviously be running into losses. Yet Johansen's model shows that old plants live their full life. The purpose of this argument is to indicate that Johansen's model is valid only under static technique and constant relative prices of factors, in general. If constant technique is assumed and if the supply of labor is fixed infinite durability of capital should imply zero rate of saving.

In what follows some simple analyses are attempted assuming that the durability of capital is infinite only in a physical sense. Its economic life is determined within the economic structure by the market forces and the technological developments. Further argument on these points is based on the assumption of a static technology and the effect of factor prices and interest are noted. Later some observations on the effect of technological progress on the life of capital goods will be made. To keep the analysis simple fixed supply of labor is assumed.

Static Technology and Fixed Supply of Labor. Let the amount of labor available be a constant number N. Since the assumption is made that there is a fixed proportion of total income saved in each period and since it is also assumed that the factors of production are fully employed it is required that some older capital has to be scrapped.

Since a constant technology is also assumed the cause of scrapping is to be discovered in the variation of factor prices mainly. Let it be assumed that as a result of scrapping a fixed number of labor, n, is made available for employment with the new capital each period. From equation III-3 one has:

$$\text{III-10} \qquad N = \int_{a(t)}^{t} n \, du$$

From III-10 a(t) is equal to $t-\frac{N}{n}$. Using the Cobb-Douglas type function one obtains:

$$\text{III-11} \qquad X(t) = \int_{t-\frac{N}{n}}^{t} n^{1-\beta} \left\{sX(u)\right\}^{\beta} du \qquad 0<\beta<1; \quad 0<s<1.$$

The above equation is rather too difficult to solve. However, one may easily find some important properties that are economically significant. From the point of view of long run development of the economy one may be interested to see how the total output should behave given an arbitrary initial condition. If only the growth aspect of the problem is taken it should be noted that for increasing output the derivative of III-11 with respect to time should be positive, which implies that:

$$\text{III-12} \qquad n^{1-\beta} s^{\beta} \left\{X^{\beta}(t) - X^{\beta}(t-\tfrac{N}{n})\right\} > 0$$

The condition for the monotonic increasing function X(t) can be found easily with the help of the following diagram. In Figure 10 the horizontal axis measures time with 0 taken as the arbitrary origin. The curve with positive slope represents the function X(t). Along the horizontal axis two points $t-\frac{N}{n}$ and t which represent the time interval given by the equation III-11 are marked. The distance between the two points is N/n. It is clear from the equation that the height of each point along the curve X(t) is obtained by the summation of the betath power of the heights of such points times the coefficients $n^{1-\beta}s^{\beta}$ over the interval N/n left to the point being considered. Since the function X (t) is increasing the height at t that is raised to the power beta and multiplied by the relevant coefficients and summed over the interval N/n should exceed that particular height. More precisely one has the following condition:

$$\text{III-13} \qquad X(t) \leq N/n\ n^{1-\beta}s^{\beta}X^{\beta}(t)$$

In the above equation the relation $\leq$ is used in order to show that a stationary maximum level of output is also possible, in case the

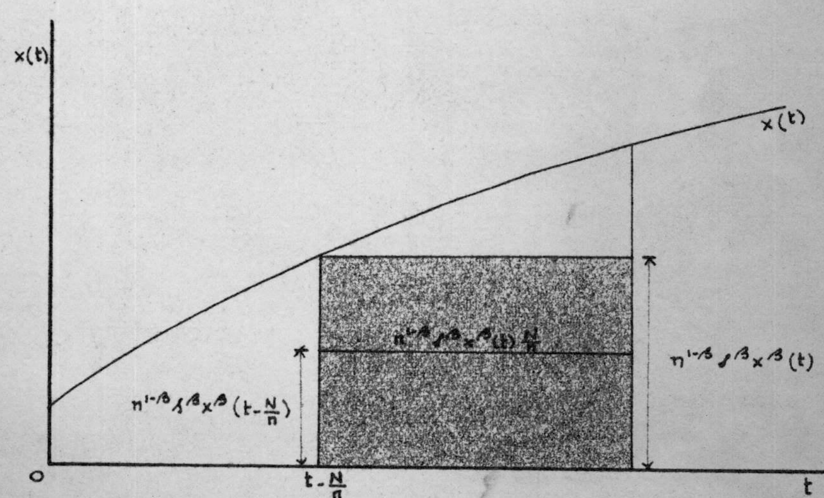

Figure 10.

equality holds. By further rearrangement of the terms of III-13 the following condition may be written:

III-14 $$X(t) \leq \left\{ N/n \; n^{1-\beta} \; s^{\beta} \right\}^{1/(1-\beta)}$$

From III-14 some interesting properties of the economy being considered may be derived. The condition says that the output in an economy having static technology and constant finite life of capital, which in this model is necessitated by the need to employ the available saving and, therefore, new capital having the same productivity as the old, is bounded above. The output cannot exceed the limit set by the right hand side of the inequality III-14. One might say that the limit is asymptotically approached by the economy.

It follows that the function X(t) becomes concave as time increases. This is because of the boundedness of the function X(t). The conclusion regarding the concavity of the function may not hold if one allows the fact that the equation III-11 need not have only a solution yielding increasing X(t). But whatever be the nature of the solution it is doubtful that output will exceed the limit. It is perhaps fair enough to approximate X(t) by a concave function. It may also be maintained that for a variety of initial conditions the function need not be concave throughout the interval $(0, \infty)$. This is the reason why it should be said that X(t) becomes concave for the larger value of t.

It is known that the boundary to which output could rise is determined by the constant terms of the equation III-1. If the boundary is regarded as the stationary equilibrium one may apply comparative statics to see the effect of change in the value of the parameters on the output. It is easily seen that high value of s raises the asymptotic output. β has similar effect. The high value of the labor force employed, N, and the low value of the number of labor allocated to the new capital lead to higher asymptotic output.

For various values of n one may illustrate the property of the present model diagrammatically as in Figure 11. Assume n_1 n_2 n_3. Corresponding to these three values there are three different values of the right hand side of the inequality III-14, which are designated by L_1, L_2, L_3. The three curves X_1, X_2, and X_3 corresponding to the respective values of n are all represented as starting at the same initial output X_o and rising at an increasing rate for some time. The curve for the lowest value of n is shown to approach its limit less rapidly than the others. The reason for this is that with low n the replacement of old capital takes a longer time.

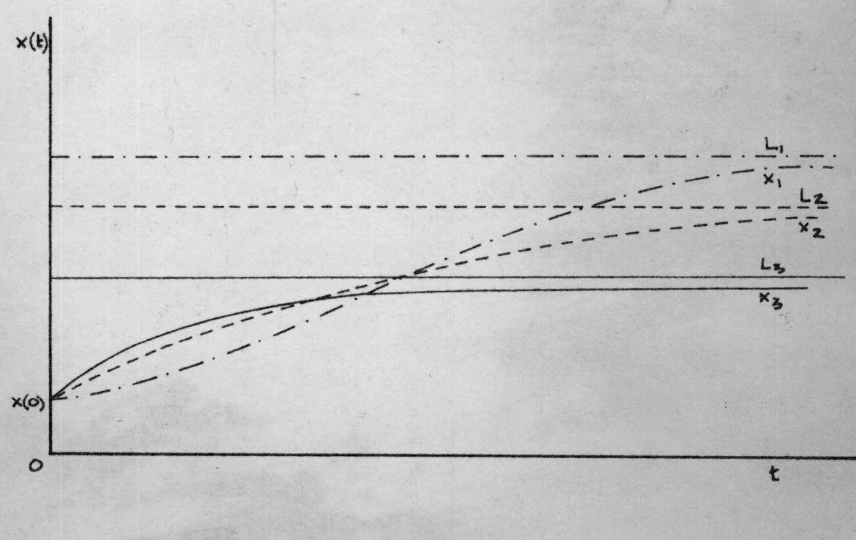

Figure 11.

From the above analysis it is known that the net saving of the economy could be absorbed in the production process only by scrapping old capital. This would lead to growth in output until a finite limit is reached when the saving available becomes just sufficient for the replacement of capital. But this argument is hard to accept, because under the assumption of a static technology there does not seem any justification for choosing new equipment to replace the old when the old and new have the same productivity. One might say that when new capital goods are being produced they are employed relatively with less amount of labor than in the previous cases. This is true for the increasing portion of the output function. In this case the familiar notion that the rate of interest has to decline is implied. This may be given as one of the explanations for the replacement of old capital for new. Even if this were true it is not plausible for saying why the life of capital goods should be finite. There are other ways in which the capital-labor ratio could be raised and the declining rate of interest would lead to less and less numbers of equipment being replaced such that output would eventually rise beyond all limit, although at an ever decreasing rate. The possibility of the convergence of output to a finite limit with constant life of capital is true only if entrepreneurs have the illusion that new capital would be more economical than the old. Or it may also happen that the prices of goods are behaving in ways which provide an impetus for production of capital in the investment sector and its demand in other sectors of the economy, the balance on the whole being perfectly maintained. If the limiting output is reached and investment activity is not maintained at the required level, profit will decline and output has to decline. To sum up, the genuine equilibrium in which the entrepreneurs know that there has been no improvement in technology and still supply and demand are equated is not possible in the case considered here.

It was mentioned above that the falling interest rate could lead to a progressive rise in the capital-labor ratio and the rate of capital accumulation. Now the case in which the factor prices move in such a way that an optimal replacement plan is adopted in the economy will be illustrated. The optimal replacement plan means that in each period capital-labor ratio is so chosen that output is increased to its maximum. The assumption of static technology is maintained and emphasis is placed on the movement along a production function.

Scrapping of Capital for Maximization of Output in Successive Periods. In the above case it was found that increase in output could be possible with rising capital-labor ratio. But the constant replacement of capital was not convincing. Now another model is formulated which will be put in period form. The general solution of the model will not be attempted here. But the arguments will be provided for showing the nature of the solution in general.

It is assumed that in the initial period there were $K(0)$ units of capital employed with N units of labor. Each labor is employed with a given amount of capital so that there is a homogeneous capital-labor ratio, written as $R(0)$, for all sectors of the economy. Output at time 1 is assumed to be due to the capital and labor employed at time 0.

Using the Cobb-Douglas production function one has:

$$\text{III-15} \qquad X(1) = N^{1-\beta}K^{\beta}(0)$$

Of the total output of period 1 the amount sX(1) becomes available in the succeeding period as capital. If the decision making body were the central planning board, it is most possible that it would find a way of utilizing this capital so that in the next period output will be maximized. It is not clear whether market forces will be favorable for the entrepreneurs to do so. However, it is assumed that wage rates were rising in such a way that entrepreneurs desire to increase capital-labor ratio in a similar way. To see what would be the replacement policy in the next period it is assumed that n(1) units of labor from the previous employment are released for working with the new capital. This involves the scrapping of $\frac{K(0)}{N}$ n(1) units of old capital. The aggregate output in period 2 is the equivalent of total output of period 1 plus the net increment of output in period 2. One has:

$$\text{III-16} \qquad X(2) = X(1) + \left\{n(1)\right\}^{1-\beta} s^{\beta} X^{\beta}(1) - n(1)\left\{\frac{K(0)}{N}\right\}^{\beta}$$

where K(0)/N is equal to R(0). Differentiating X(2) with respect to n(1) and setting it equal to zero one has the condition for maximum output at time 2, which is given by the following expression:

$$\text{III-17} \qquad \frac{sX(1)}{n(1)} = R(1) = R(0) \cdot \frac{1}{(1-\beta)}\,1/\beta$$

The value of n(1) derived from the above equation is given below:

$$\text{III-18} \qquad n(1) = \frac{sX(1)}{R(0)}\,(1-\beta)^{1/\beta}$$

The generalization of formulas III-17 and III-18 is not possible, because they do not say when the capital of the past period is going to be exhausted. However, it can be conjectured that if at any period t the amount of saving available is such that it can just replace the capital of time t-k without lowering the capital-labor ratio which has been increasing throughout the past period, the term R(1) in equation III-17 can be replaced by R(t) and R(0) by R(t-k).

From III-17 it may be argued that in the present case the capital-labor ratio will rise over time. Similarly it can be argued that the value of n may fall in the long run. This is sure to be the case, because as the capital-labor ratio rises the increase in output takes place at a diminishing rate according to the present technological assumption. Thus after a long time the difference between the amounts of saving of any two consecutive periods tends to be narrowed. Since capital-labor ratio will increase, this increase can be achieved only by allocating a smaller amount of labor. As the growth rate of output diminishes it is possible that the amount of labor allocated to new capital tends to zero. This suggests that a growth model could be obtained with the static technique assumed here by giving some appropriate form to the functions n(t) and a(t) in III-2. Here it will be shown that this is perfectly possible.

A Model with n(t) as a Diminishing Function of Time. Let it be assumed that the amount of labor allocated to employment with new capital at time t is n/t. This form of n(t) is rather hard to accept at least for the small value of t. For illustration, however, one may assume that the initial point is higher than t = 0. Assuming a fixed labor force, III-3 gives the value of a(t) as $t/e^{N/n}$. Using these functions in III-11 one has the following expression for output at time t:

III-19 $$X(t) = \int_{t/e^{N/n}}^{t} (n/u)^{1-\beta} s^{\beta} X^{\beta}(u)\, du$$

This is a somewhat difficult equation to solve. A possible solution can be found by assuming $X(t) = Qt^{z}$, where z is a constant. By trial it is easy to find that $z = \beta/(1-\beta)$. Q is a constant which can be determined from the integral III-19. Using the value X(t) in III-19 and integrating it one has:

III-20 $$Q = \left\{ \frac{1-\beta}{\beta} s^{\beta} n^{1-\beta} \left(1 - \frac{1}{e^{N/n.\ \beta/(1-\beta)}}\right) \right\}^{1/(1-\beta)}$$

In the present solution there is no finite limit to the output. There is, however, a definite trend which the rate of growth of output takes. Thus differentiation of X(t) yields:

III-21 $$\frac{dX}{dt} = \frac{\beta}{1-\beta} Qt^{2\beta/(1-\beta)}$$

Dividing III-21 by output at time t, $Qt^{\beta/(1-\beta)}$, the growth rate at time t is derived as:

III-22 $$g(t) = \frac{1-\beta}{\beta} \frac{1}{t}$$

where g(t) is the growth rate at t. Here the rate of growth tends to zero as time increases indefinitely. Growth rate is not affected by any parameters other than β and by t. However, the level of output is affected by the values of all the parameters and other constants. Since output is growing at a slower and slower pace it is natural that with a constant saving coefficient the available saving increases at a decreasing rate too. But this leads to the production of capital which is employed with less and less amount of labor. Another significant property of the model is that as time increases, the life of capital also increases. The life of capital at time t is given by $t-t/e^{N/n}$, which tends to infinity as t tends to infinity. In an economy in which labor supply is increasing the conception of infinite life of capital with net capital accumulation is easy. But in cases of fixed labor supply the infinite life is an asymptote which is theoretically possible. This is precisely the point which the above argument establishes.

As is known, the above argument could be presented in a much simpler way by assuming that each time depreciation of old capital is going on and the increment in output at time t, i.e., $\frac{dX}{dt}$, could be represented solely as a function of investment at time t, the constant labor force being assumed as in the case in which it is assumed that the supply of land is given and the output is supposed to depend simply on labor. Assuming diminishing return to new investment one might have the following equation:

$$\text{III-23} \qquad \frac{dX}{dt} = \left\{\frac{dK}{dt}\right\}^{\beta} = (sX)^{\beta}$$

whose solution is:

$$\text{III-24} \qquad X(t) = \left\{(1-\beta)\ s^{\beta}\, t\ +C\right\}^{\frac{1}{1-\beta}}$$

where C is determined by the initial conditions. The economic interpretation of this model is not essentially different from the previous ones.

So far a definite type of technology described by the Cobb-Douglas type of production function is being assumed. Even in this case one may have a situation of exponentially increasing output with fixed population if there were economic reasons which would offset the operation of the law of diminishing returns due to the fixity of the labor supply. One such situation which can be imagined for the model with static technology is that in which an increase in the scale of production diminishes the cost per unit of output.

One way to introduce the effect of increasing returns to scale is to make the aggregate output at any time not only a function of labor and capital but also of the output itself. As before, it is assumed that a fixed number n, of labor is reallocated to new capital at time t. For output at time t the following equation is used:

$$\text{III-25} \qquad X(t) = \int_{t-N/n}^{t} n^{1-\beta} s^{\beta} X^{\beta}(u)\, q\, X^{\alpha}(u) \; du \qquad \text{constant}$$

In the above equation the factor $qX(t)$ is supposed to introduce the effect of scale. It is perhaps reasonable to assume in the present case that α is a positive fraction which remains constant.

One might doubt if the introduction of a scale factor would give rise to a constant finite life of capital. Such doubt is perfectly justified, because it is necessary to assume that the entrepreneur evaluates the reduction in cost which an increment in investment would bring about and he might not be led to replace each capital at a constant time interval. However, if it is assumed that a fixed life of capital has resulted, one may get a solution for the equation III-25 with constant exponential rate of growth σ only if $\alpha+\beta = 1$. This can be seen by writing III-25 in the form of a differential equation:

$$\text{III-26} \qquad \frac{dX}{dt} = n^{1-\beta} s^{\beta} q \left\{ X^{\alpha+\beta}(t) - X^{\alpha+\beta}(t-N/n) \right\}$$

Try solution $Ce^{\sigma t}$. This solution holds if $\alpha + \beta = 1$. The value of σ is given by:

$$\text{III-27} \qquad \sigma = qn^{1-\beta} s^{\beta}(1-e^{-\sigma N/n})$$

In the present case it is apparent the scale of output has an offsetting effect against the pressure of diminishing returns which would otherwise lead to a falling rate of growth of output. This shows that capital cannot have infinite life in cases in which the effect of the scale of output toward decreasing the cost per unit of output is predominant. This might be explained for general cases with the help of a model. But elaboration of this is not intended here. Further, it should be noted that the way in which the effect of the scale of output is introduced in this model may not be the only possible one. In fact there may be a large number of ways in which this could be done. In many cases it would be possible to assume that production is homogeneous of a degree greater than one which might provide a desired model. But at the moment it is found to be more complicating.

Moreover it does not seem justifiable to assume increasing return to scale for a very wide range of output. For the present purposes it is more suitable, perhaps, to maintain the assumption of constant returns to scale and for any change in the cost by varying the scale of output, the changes in organization of industry, and so forth, could be brought as explanation. It should be noted that in the model above the existence of a changing technology was implied in some way.

The arguments on the basis of constant returns may now be resumed. It is worth observing that even in this case one is free to select a type of function and still obtain growing output. This is the case when, as will be seen presently, the function is linear.

Linear and Homogeneous Production Function. Let the production function for gross increment, $y(t)$, of output due to capital $k(t)$ and labor n, which is constant, be defined as follows:

$$\text{III-28} \qquad y(t) = ak(t) + bn \qquad a, b, \text{ constants} > 0$$

For aggregate output at t one has:

$$\text{III-29} \qquad X(t) = \int_{t-N/n}^{t} (ak(u) + bn)\, du$$

On integration III-29 gives

$$\text{III-30} \qquad X(t) = aK(t) + bN$$

It is being assumed that $k(t) = sX(t)$. Using this equation one obtains the following expression for the rate of increase of $k(t)$:

$$\text{III-31} \qquad \frac{dk}{dt} = s\frac{dX}{dt} = sa\,\frac{dK}{dt}$$

or

$$\text{III-32} \qquad \frac{dk}{dt} = sa\left\{k(t) - k(t-N/n)\right\}$$

III-32 is a mixed difference-differential equation whose detailed solution need not be discussed here. The solution that is of interest is one that would yield growth. Assume $Ce^{\sigma t}$ to be a solution. For σ one has:

$$\text{III-33} \qquad \sigma = as(1-e^{-\sigma N/n})$$

The above equation is similar to III-27. However, the nature of the present solution may be explained. III-33 may be written in the following equivalent form:

$$\text{III-34} \qquad 1 - \frac{\sigma}{as} = e^{-\sigma N/n}$$

The right hand side of the above equation is denoted by Z_1 and the left hand side by Z_2. In Figure 12 the two curves for Z_1 and Z_2 are drawn. The relevant roots are those values of σ at which the two curves intersect. One such value is obviously $\sigma = 0$. There may be one more solution of III-34 depending upon the slopes of Z_1 and Z_2 at $\sigma = 0$. If the two slopes are not equal there is another solution. For positive solution it is necessary that $asN/n > 1$, which implies that the slope of Z_1 is greater than that of Z_2 at $\sigma = 0$. In the figure the solution $\sigma = \sigma^*$ is such that $0 < \sigma^* < as$.

The solution of k(t) may be written as:

III-35 $$k(t) = C_1 + C_2 e^{\sigma^* t}$$

Integration of III-35 over the interval (t, t-N/n) yields:

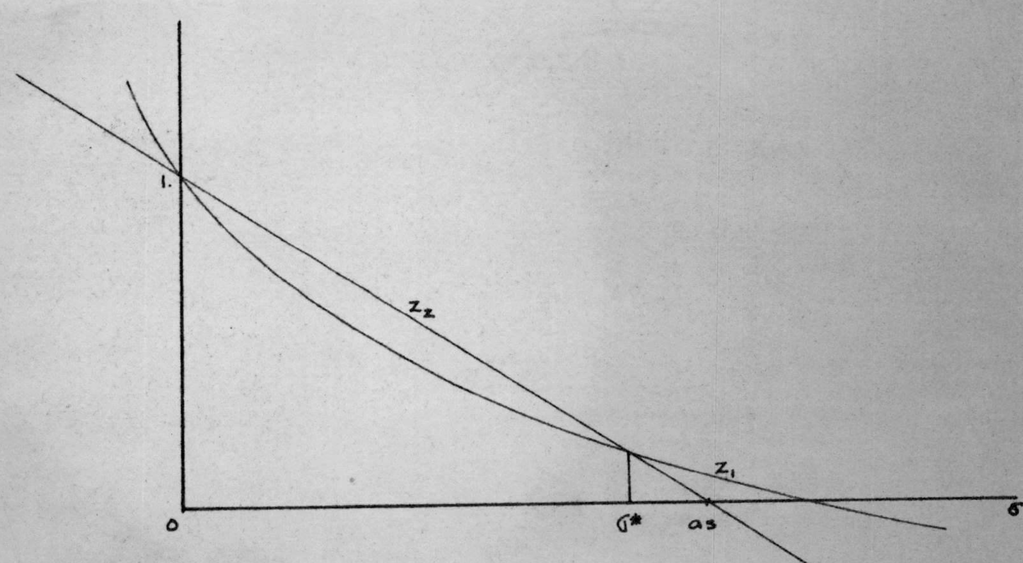

Figure 12.

III-36 $\quad K(t) = D_1 + D_2\, e^{\sigma^* t}$

where D_1 and D_2 follow from the C's of III-35 and may be determined by initial conditions. To derive the expression for the total output one may insert III-36 in III-30 which gives:

III-37 $\quad X(t) = aD_1 + aD_2 e^{\sigma^* t} + bN$

Thus in the present case there is an exponentially growing output. The reason for this is to be found in the type of production technology assumed. It may be argued that the production function is not realistic. This is true, because it implies that capital and labor are perfect substitutes. In such case it can be seen that some nonsensical situations in which production can take place with only one of the two factors of production may arise. Changes in factor ratio do not change the marginal productivity of any factor. Therefore, the above model cannot be taken too seriously.

However, this type of model is of some value if it is given a different interpretation. One may easily see that there may be growth in the type of economy assumed in the present section if there were innovations going on such that the growing capital-labor ratio left the marginal productivity of each factor involved invariant. This leads to a model in which one should actually assume that there is a set of functions F_t from which the producers are continuously making choices in order that a given profit rate is at least maintained. This problem will be discussed in the next section. In the remainder of the present section the arguments presented in the preceding discussion will be summarized and the conclusions will be given.

Summary and Conclusion. The discussion in this section was begun with Johansen's model. It was argued that if infinite life of capital is to be assumed one has to rule out the improvement of techniques of production concomitant with the installation of a new capital equipment. The reason for this is obvious. A lowering of the cost of production due to the installation of new capital would make the previous capital obsolete. This may be illustrated with the help of a simple example. Assume that technological progress of a neutral variety is going on. At time t let the production function for output produced with capital installed be:

III-38 $$y = e^{gt}k^{\beta}n^{1-\beta}$$

All terms in the above equation are as defined above. The marginal productivity of capital is then $\beta e^{gt}k^{\beta-1}n^{1-\beta}$, which means that the total return from this capital is $\beta e^{gt}k^{\beta}n^{1-\beta}$ per unit of time. If the capital yields this rate of return (which is to be proved false) for an infinite length of time the value of such capital would be:

III-39 $$V(k) = \int_0^{\infty} \beta e^{gt}k^{\beta}n^{1-\beta}e^{-ru}\,du$$
$$= (e^{gt}k^{\beta}n^{1-\beta}\beta)/r$$

where r is the discount rate. Now assume that instead of keeping this equipment forever the entrepreneur replaces it at the end of m years of its life. He replaces it by a new capital of similar amount k. The new capital produces $e^{g(t+m)}k^{\beta}n^{1-\beta}$ per unit of time. Let it be assumed that this latter equipment is kept for all time. In order to calculate the income stream one should allow depreciation on previous capital, say, k/m per unit of time. Now one has the following expression for the value of capital:

$$\text{III-40} \quad V(k) = \int_0^m (\beta e^{gt} k^{\beta} n^{1-\beta} - k/m) e^{-ru} du + \int_m^{\infty} (\beta e^{g(t+m)} k^{\beta} n^{1-\beta} e^{-ru} du$$

$$= \frac{\beta}{r} e^{gt} k^{\beta} n^{1-\beta} + \frac{\beta}{r} e^{gt} k^{\beta} n^{1-\beta} e^{-rm} (e^{gm} - 1) - k/m(1 - e^{-rm})$$

It is obvious that the last two terms in the above equation represent the net gain due to replacement. One may derive the conditions for maximizing the value of a production process which would establish the relationship between the rate of interest, m and g. For the present it is sufficient to note that there are possibilities of raising the return on capital by replacement. If the cost of operating a capital equipment some years old increases because of changing factor prices which progress in technique brings about it is more certain that capital becomes obsolete.

Even if technological progress is neglected, a static technology considered, changes in factor prices affect the durability of capital. Thus the assumption of infinite durability of capital is valid only under extremely static condition regarding technology and factor prices.

It was seen in the course of this argument that under static technology and fixed labor one may have an asymptotic trend toward infinite durability of capital if the situations become such that at a given rate of saving the construction of capital which would increase the product of the economy takes a longer and longer time. This was the case in which each period a decreasing amount of labor was allocated to new capital. In the case of increasing population this need not be necessary. If the rate of capital accumulation is equal to the rate of population growth the infinite life of capital is achieved.

It was also observed that if an increase in the scale of output reduces cost the economy might grow even with fixed labor supply if replacement of capital takes place. Perhaps this situation brings about a dynamic element as far as technology is concerned. In such a case one cannot speak of infinite life of capital goods.

If all possibilities of technological improvement, increase in labor supply, economy of scale, and change in factor prices are ruled out, the determination of equilibrium output becomes a little hard. It may be possible for the economy to be led into a situation in which replacement of capital is made at some regular interval. In that case equilibrium may be determinate. But under the present assumption on infinite life of capital it is doubtful if such equilibrium is worth the name.

So far an unrealistic case of capital goods having infinite life has been dealt with. With the help of this case at least some of the elements of the real economy were known, apart from the destructible nature of the material goods, that affect the life of capital. This has importance in the analysis, because in the actual economy when choice is to be made among various capital goods of various durability such choice has to be based on the productivity of such equipment, factor prices at the time of installation and during the life of the equipment together with the prices of goods it produces. In the next section the more realistic case in which capital goods have finite life will be discussed.

Section IV: Finite Life of Capital Goods

Introduction. Already some idea has been obtained about the nature of the problem that would arise if fixed durability of capital is assumed. In the last section it was assumed in various places that there is replacement of capital of some given age in order to allow for the fact that only by doing so or using some alternative method of replacement of old capital the saving of the community would find its way into the economy if the labor force is fixed in supply. In this section it is assumed that capital goods are perishable.

The assumption of a uniform durability of capital for the whole economy is somewhat hard to accept. In one industry of the real world there are equipment and other assets that have different durabilities. For the economy as a whole there are various commodities produced with capital of different durabilities. The assumption of one commodity is continued. But still there is the possibility that the durability of capital in any one employment is determined by the technology used. The choice of any production technique is the choice of any given durability in capital form corresponding to it. It is assumed generally that durability of capital is a separate property of capital not determined by technological progress alone. The case in which durability of capital is related to change in technique such that an increase in productivity is caused by change in technique with corresponding change in durability is rather less convincing and may in general turn out to be less valid. However it will be mentioned later that for maximizing output there are conditions which relate durability of capital to rate of increase in productivity due to neutral technological progress specifically.

The discussions in this section are divided into five subsections. In Subsection A the case of fixed labor force and no technological progress is considered. In Subsection B the case in which both the expansion of labor supply and neutral technological progress are allowed will be investigated. In Subsection C the arguments will be based on cost items constituting the total product of the economy over time. This will yield some results on the basis of which one may make observations about the technological developments over time. In Subsection D the case in which the intensity of work in the plant decreases due to obsolescence as it grows older will be discussed. In Subsection E conclusions will be given.

In all the arguments below the symbols carry the same meaning as assigned to them in the previous sections of this chapter unless otherwise stated. The durability of capital is denoted by the symbol θ as Johansen[1] has done.

A. Fixed Labor Supply and No Progress in Technique. Unless specifically mentioned, the fixed durability of capital should be taken to mean that capital installed at any point of time t will yield a uniform rate of output until $t + \theta$ which implies that the amount of labor employed with that capital remains constant at that level throughout the life of the capital. Assuming full employment one may then

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[1]See Johansen, op. cit., pp. 170-174, where the case being discussed here is dealt with.

write equation III-3 for the present model as follows:

$$\text{III-41} \qquad \int_{t-\theta}^{t} n(u)\,du = N$$

This equation may be differentiated to yield a constant solution for n(t). The constant n(t) satisfying III-41 is obviously N/θ. Under the present assumption of fixed technology one will get the equation for total output at time t which has the same form as III-11, using the Cobb-Douglas type of production function, of course.

$$\text{III-42} \qquad X(t) = \int_{t-\theta}^{t} (N/\theta)^{1-\beta} s^{\beta} X^{\beta}(u)\,du$$

This type of equation has already become familiar since the discussion of the properties of III-11. However, in the present discussion emphasis will be on the equilibrium aspect of the solution of the model. Use is made of the theory that in equilibrium the value of capital is equated to the cost of capital. The capital purchased and installed at t is k(t) which earns, according to the marginal productivity theory of distribution, the amount $\beta k^{\beta}(N/\theta)^{1-\beta}$ per unit of time for its whole life θ. The entrepreneurs discount this earning at the rate r which is assumed to remain constant. Thus the following equilibrium condition is derived using $k(t) = sX(t)$.

$$\text{III-43} \qquad sX(t) = \int_{t-\theta}^{t} \beta (N/\theta)^{1-\beta} s^{\beta} X^{\beta}(t)\, e^{-r(u-t)}du$$

The solution of the above equation is:

$$\text{III-44} \qquad sX(t) = \frac{1}{r} (N/\theta)^{1-\beta} s^{\beta} X^{\beta}(t)\,(1-e^{-r\theta})\,\beta$$

The solution for the output derived from this equation is designated $\overline{X}(t)$ and is obtained by simple manipulation of terms in III-44.

III-45 $$\bar{X}(t) = \left\{\frac{1}{r}(N/\theta)^{\frac{1-\beta}{\beta}} s^{\beta-1}(1-e^{-r\theta})\right\}^{1/(1-\beta)}$$

The above equation gives the stationary solution for equilibrium output which is constant depending only upon the supply of labor and other parameters of the model. This solution may be regarded as an asymptote towards which the economy tends in the long run. The reason for the constancy of output is to be found in the constancy of technology and the constancy of productivity of capital of all times implied in the present model.

It is worth noting that if price is introduced and if the value of capital in money terms is equated to the cost in money terms using the money rate of interest the above solution has to be modified. If q is the rate of increase of price over time, then in the solution above r is to be replaced by r-q. The reason for taking the change in price into account is to use the fact that if an entrepreneur wants to sell the equipment there is reason for him to sell this at a higher price if the price of the product is going to rise, according to his expectation, given a constant rate of interest with which he discounts the product. If this fact is used the solution of the above equation holds only for $r \neq q$. If $r = q$ the effect is similar to that of the case in which the rate of discount is zero. In such a case the equilibrium solution will be as follows:

III-46 $$\bar{X}(t) = \left\{(N/\theta)^{\frac{1-\beta}{\beta}} s^{\beta-1}\ \theta\right\}^{1/(1-\beta)}$$

The cases of r greater than or less than q may similarly be discussed. However, the introduction of price effects is not simple enough to handle adequately in the context of long run growth. Moreover it seems better that in a one commodity model the pricing problem be entirely avoided.

Now the properties of the solution III-45 may be discussed. It should be noted that the constant asymptotic output, if one may call it, which satisfies III-42 is:

III-47 $$X(t) = \left\{ (N/\theta)^{1-\beta} s^{\beta} \theta \right\}^{1/(1-\beta)}$$

III-47 is the output which is technologically feasible. If the equilibrium exists one should have equality between III-45 and III-47. For such equality to hold the following is to be satisfied:

III-48 $$\theta = \frac{1}{r} \frac{\beta}{s} (1-e^{-r\theta})$$

At $\theta = 0$ III-48 has a solution which does not interest us. The other positive solution of θ should satisfy the following condition:

III-49 $$\frac{\beta}{s} > 1$$

The above condition is usually satisfied in the real world. The reason for this which will not be elaborated upon is that β is the determinant of the share of capital in the total product and s is the determinant of the part of the product that goes into capital. Hence if the capital is to be productive the former has to exceed the latter. This may or may not be generally valid. In the static solution it is apparent that if s and β were equal it would mean that capital is earning the amount equivalent to the one which goes into producing capital. If discounted, capitalists may suffer loss.

In the argument so far θ is considered to be a parameter. This rules out the possibility of seeing how entrepreneurs should choose a durability of capital. It may, however, be asked whether the entrepreneur seeks to maximize output for a given value of θ. If he did one would have a condition for entrepreneurial optimum. Assume that it is so. Then by differentiating III-45 with respect to θ and setting it equal to zero one may obtain:

$$\text{III-50} \qquad e^{r\theta} = 1 + \frac{r\theta}{1-\beta}$$

The equation III-50 gives the equilibrium relationship between r and θ for a given β. If θ could be changed by the entrepreneurs at will the relationship between r and θ would be as follows:

$$\text{III-51} \qquad \frac{d\theta}{dr} \cdot \frac{r}{\theta} = -1$$

The above equation follows from III-50 and states that the elasticity of θ with respect to r is unity. It is quite clear that in a static economy the inverse relationship between r and θ exists. This implies that the marginal productivity of lengthening of the period of production diminishes, from the point of view of the entrepreneur of course, because of the discounting of the future income; from the point of view of the economy the diminishing marginal productivity of θ is obvious from equation III-47. It shows that increasing θ adds less and less to the product of the economy.

It should be noted that the asymptotic level of output is higher the greater the durability of capital.

From the arguments above it should not be inferred that the changes in any of the parameters would lead to growth of output or fall in the output. What is being said here is not related to the dynamics of growth. The above are simply the arguments that there are relationships between the parameters of the model which have specific economic meaning. These relationships were studied above on the basis of a particular equilibrium solution. If there is any change in the value of any parameter one is in a different model which leads to a different equilibrium solution. However, it is not impossible to say that the change in any particular parameter, if not offset by a corresponding change of others, has effects on the total output which can be easily understood from the solutions above.

Since the properties of the static model have been studied at some length and since the interest of the author lies more in the dynamic problems the next step is to take up the case of changing technology and expanding labor force together in one model.

B. Expanding Labor Force and Neutral Technological Progress. Labor force is assumed to be growing at the rate λ. It is assumed that the technological progress is taking place at the rate g per unit of time. Thus if the new capital at time t is k(t) which is employed with labor n(t), the output due to this combination of capital and labor is written as:

III-52 $$y(t) = e^{gt}\, k^{\beta}(t) \cdot n(t)^{1-\beta}$$

The introduction of technological progress leads to some difficult problems regarding the productivity of capital as a function of its age. It is clear that a capital good installed at time t incorporates the latest technology which gives rise to higher productivity of both labor and capital than in the case of a capital equipment installed some years ago. One is faced with the question whether capital does not become obsolete before it has lived θ length of time. If so, it is necessary to find the precise way in which this obsolescence is to be introduced. It is also necessary to know the pricing of factors which is not simple when this complication is introduced.

However, in the present subsection a case which appears to be inconsistent is discussed. It is assumed that a capital good that is installed at any time lasts for θ units of time and is operated with full intensity. The amount of product it yields remains uniform throughout its life. In Subsection D a more consistent approach will be attempted. In the fixed life case, without obsolescence, a simple model may then be developed as follows:

The equation III-3 for the present model is written as:

$$\text{III-53} \qquad N_o e^{\lambda t} = \int_{t-\theta}^{t} n(u)\, du$$

It should be noted that full employment is assumed. As a solution of III-53 the following form of n(t) is obtained:

$$\text{III-54} \qquad n(t) = \lambda N_o/(1-e^{-\lambda\theta}) \quad e^{\lambda t}$$

Then the aggregate output at t is given by:

$$\text{III-55} \qquad X(t) = \int_{t-\theta}^{t} e^{gu}\left\{N_o \lambda/(1-e^{-\lambda\theta})e^{\lambda u}\right\}^{1-\beta} s^{\beta} X^{\beta}(u)\, du$$

It is easy to obtain a function X(t) satisfying III-55, namely one which may be written in the form $Ce^{\sigma t}$. Such a solution may be regarded as an asymptotic solution. Here σ is the rate of growth of output which is a constant determined by the parameters g, λ, and β. C is a constant involving the constants of the equation III-55. It is easily verified that C and σ have the following values:

III-56 (a) $$C = \left[\frac{1}{\sigma}\left\{\lambda N_o / (1-e^{-\lambda\theta})\right\}^{1-\beta} s^{\beta}(1-e^{-\sigma\theta})\right]^{1/1-\beta}$$

(b) $$= g/(1-\beta) + \lambda$$

A more general solution of III-55 would take into account the initial conditions. Moreover, the equation is too complicated for a complete analysis regarding the various types of solution one may obtain. However for the present purpose III-56 may be regarded as adequate for studying the more significant aspects of the model being discussed from the point of view of long run growth.

From III-56 it is obvious that the rate of growth of output in the economy is determined by the rate of technological progress and the rate of growth of population. It should be remarked that in the models considered in the previous chapter it was indicated that the rate of growth of output is the sum of the two rates, if natural rate of growth is realized, of course. But in the present case it is not a simple sum. As III-56(b) shows the rate of technological has to be multiplied by $1/(1-\beta)$ before adding to the rate of population growth. This implies that the higher the value of β the higher the growth rate, and, from III-56(a), the higher the levels of total output. However, one may or one may not take this conclusion seriously, because although β is a technical parameter in the long run it may be subject to choice, and this choice is affected by economic situations that develop in the long run.

Now the condition of equilibrium may be explored. As before a situation is considered to be equilibrium if the amount of capital measured by its cost is equal to the discounted stream of income, which is determined by the marginal productivity of capital. The application of this rule here may be questionable because it is assumed that capital will be earning a uniform rate of return during its life time. Even if it were assumed that the capitalists were always in a situation to operate a plant at uniform intensity for its life time, it would be meaningless to assume that they would earn a uniform rate of return during the life time of the plant, despite the fact that increase in productivity is raising the rate of earnings in the new plants. However, since only the simplest versions of the problem are being studied in the present subsection the more acceptable line of approach is postponed until Subsection D. Thus it is assumed that the rate of return on $k(t) = sX(t)$ per unit of time is given by $\beta\left\{\lambda N_o/(1-e^{-\lambda\theta})\right\}^{1-\beta} e^{\lambda(1-\beta)t} s^{\beta}X^{\beta}(t)\ e^{gt}$, which is uniform for θ. If this amount is discounted continuously for θ at the rate r the following condition for equilibrium is obtained:

$$\text{III-57} \quad sX(t) = \int_{t-\theta}^{t} e^{gt}\beta\left\{N_o\lambda/(1-e^{-\lambda\theta})\right\}^{1-\beta} e^{\lambda(1-\beta)t}\ s^{\beta}X^{\beta}(t)\ e^{-r(u-t+\theta)}\ du$$

From the above the equilibrium output $\overline{X}(t)$ is:

$$\text{III-58} \qquad \overline{X}(t) = \left[1/r.\beta\ s^{\beta-1}.\left\{\lambda N_0/(1-e^{-\lambda\theta})\right\}^{1-\beta}.(1-e^{-r\theta})\right]^{1/(1-\beta)} e^{\sigma t}$$

where σ is as in III-56(b) and $r \neq 0$. It is apparent that the only difference between III-56 and III-58 is to be found in the constant parts of the solution. Then if the equilibrium output has to satisfy the equation III-55 the following equation should be satisfied:

$$\text{III-59} \qquad \frac{1}{\sigma}(1-e^{-\sigma\theta}) = \frac{1}{r}\frac{\beta}{s}(1-e^{-r\theta})$$

The equation III-59 shows that the rate of interest has to be an adjustment factor for the equalization of the two solutions for output. Since the above argument is based on the assumption of a nonzero rate of interest a solution of III-59 for this condition alone is sufficient. The case of negative rate of interest is ruled out. Hence the condition under which the solution of III-59 exists with positive rate of interest is given by:[1]

$$\text{III-60} \qquad \frac{\beta\sigma\theta}{s}(1-e^{-\sigma\theta}) > e^{r\theta}$$

One important observation which has to be made in this connection is that r could equal σ only in the case in which $\beta = s$, according to III-59. But this condition may not be acceptable. Hence it may be maintained that r need not equal the rate of growth.[2] Interest rate

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[1]The condition III-60 is derived as follows. Multiply both sides of III-59 by r, to make the matter simple. Plot both sides of the resulting equation on a graph with r measured along the horizontal axis. For the intersection of the two curves for the two functions at a positive r it is necessary that the slope of the right hand side of the equation is greater than that of the left hand side at $r = 0$. III-60 means just this.

[2]This conclusion may be contrasted with that of von Neumann. See von Neumann, loc. cit.

in the present model is to be determined with reference to the value of β, s, θ and growth rate. Moreover, since the durability of capital is also an important parameter in the model one may derive the relationship between this and the rate of interest from the conditions as well.

The way in which this is being done is by studying the condition for maximum output for any given value of θ. In the Subsection A above the condition for such maximum was easily found. In the case in which the labor force is constant but there is neutral technological progress taking place the condition III-50 would hold. It may be recalled that such a condition assures the optimal durability of capital from the point of view of entrepreneurial equilibrium. This should be understood as a determinant of the rate of interest and not of the durability, the latter being assumed in this model as given. If such an equation is taken with the equation of the type III-59 one may have two equations to determine one unknown. But if r and θ were variable it would result in a complete system. If it is objectionable to consider θ as a variable for the purpose of analysis simply because of the assumption made here that it is fixed the condition derived for entrepreneurial optimum will be redundant. However, one might conceive of two different economic systems such that their difference lies in all other parameters except for the life of capital goods. The choice of any one system depends on the effect of durability of capital goods. This then establishes the optimal condition for durability of capital goods. Only in such a case the second condition becomes necessary.

If it is admitted that variation of θ would lead to change in the level of output, other things remaining equal, one might as well obtain another condition for the maximum output derived from the equation of the type III-56, when the technological progress is taking place. Such a maximum exists at some finite value of θ in the model considered here, because a too small θ would, while allowing the entrepreneurs to take advantage of the latest improvements in technology, reduce the total product of the economy by reducing net addition to the stock of capital. If θ is too large the economy is accumulating more capital, but it is not taking advantage of the improvement in the technology that is taking place in the economy at a certain given rate.

Thus, in the case in which technological progress of the kind being considered here is taking place, one has two optima. One is for the technically possible output of the economy. Another is for the equilibrium output of the economy. The conditions are not written here. But it may be remarked that in the present model the two optima are equal if and only if the rate of interest is equal to the rate of growth of output. However, as argued above, it may not be possible that the two rates would be equal. If not, one should consider that the optimum condition derived from the equation for equilibrium output is the relevant one.

Since enough has been said about the model based on an unconvincing assumption above, it is now considered worthwhile to approach the problem posed by the assumption of the present section from a different standpoint. In the next subsection the model is studied by introducing factor prices into the picture, and with the help of the conclusions regarding the long run trend of the economy, some inferences will be derived about the possible trend in the technology of the economy.

C. Factor-Prices, Growth and Technology. Only a simple case may be taken now. In this case the entrepreneurs allow a fixed proportion of the capital invested as depreciation, and the price of capital goods of the same kind does not vary over time. The depreciation formula is arrived at by allowing the rate of interest. Thus if the amount of capital invested at time t is k(t), it is assumed that $rk(t)/(e^{r\theta}-1)$ is the depreciation allowed per unit of time throughout the life time of the capital. When investment in capital k(t) is made, the entrepreneur is committed to incur certain expenditures apart from the depreciation. The amount of labor that is employed with the capital k(t) is assumed to be n(t) which is equal to, say, $n_o e^{\lambda t}$. Assume that this level of employment for k(t) is maintained throughout the life time of the capital. It is assumed that wage rate is fixed at w and the other expenditures including the rate of profit is also assumed to be a constant, equal to v per unit of k(t). The total cost is assumed to be always covered. Then:

III-61 $$y(t) = k(t)\left(v + \frac{r}{e^{r\theta}-1}\right) + wn_o e^{\lambda t}$$

It is further assumed that net saving at t is a constant fraction of net income at t. Under the present assumption the total depreciation at t is equal to $rK(t)/(e^{r\theta}-1)$, where K(t) is the total stock of capital at t. According to the present assumption about saving the following relation holds:

III-62 $$s \left\{X(t) - RK(t)\right\} = k(t) - RK(t)$$

where $R = r/(e^{r\theta}-1)$. III-61 may be integrated over the interval $(t, t-\theta)$ to get $X(t)$ and use the integral in III-62, writing $wN_o e^{\lambda t}$ for the integral of $wn_o e^{\lambda t}$. Rearranging the terms of the equation:

III-63 $$k(t) = K(t)\ (sv+R) + wN_o e^{\lambda t}$$

Differentiation of III-64 with respect to t gives:

III-64 $$\frac{dk}{dt} = (sv+R) \left\{k(t) - k(t-\theta)\right\} + wN_o e^{\lambda t}$$

Without entering into a discussion regarding the solution of the above equation it may be stated that if there is any solution yielding positive real rate of growth of capital and of output, it should vary directly with sv+R and λ. If such growth rate is equal to λ there is no innovation of any kind in the economy. But if it is different from λ there is an innovation going on in the required direction such that the factor prices are kept constant. It should be noted that the above argument rules out the neutral rise in productivity of factors.

The object of the above argument is to show that if the factor prices are given and the rate of growth of employment of labor force is given a rate of increase of output or capital can be greater or less only if there exist technologies which assure this and if the entrepreneurs are motivated to employ such technology. In many works on growth theories such possibilities have either been assumed or neglected without much explanation. This is not without reason. One cannot single out some among a vast multitude of factors and say that such and such factors are mainly responsible for a given type of innovation. In the above analysis it might be held that if the profit rate v is the minimum below which the entrepreneurs would not be willing to invest, the maintenance of a given (minimum) profit rate might be regarded as a motivation for innovation. On the whole there may be some truth in such an argument. But for any situation the appropriate direction of innovation is a complex thing to analyze. It is not the intention here to study any further argument on this subject. There is the case of neutral technological progress which may be taken up now.

In the preceding analysis of this subsection not only the case of neutral progress in technique was avoided but also the possibility of changes in factor price was argued away. The reason is that if such changes are allowed the effect of such changes on the employment of current input with old capital goods has to be clearly incorporated in the model. In what follows the author attempts the analysis under the assumption of neutral technological progress which would raise the productivity of factors and their prices over time.

D. A More Realistic Approach to the Theory of Growth: A Suggestion. The object of this subsection is to suggest some modification of Johansen's model which has been seen to be extremely rigid in its assumptions. The rigidity of the model became apparent when it was found that rising productivity of the factors has been entirely neglected in the model as affecting the profitability of a choice of factor proportion in the plant which is already in operation.

It does not seem realistic to disregard the possibility that when a plant is in operation entrepreneurs are free to some extent to vary the employment of current inputs of labor and materials. In such a situation it is necessary to find out how far the entrepreneurs are able to respond to the changing factor prices to maintain the level of activity that would guarantee a minimum profit rate.

To be specific, it is assumed that productivity of the factors is increasing exponentially for new capital forms at the rate g per unit of time. If at any time a new capital is employed with a given amount of labor in some given proportion, such proportion may be taken to be ideal only at a given set of factor prices. But when factor prices are changing the entrepreneurs will be able to change the level of employment of variable inputs. Although one may not know precisely what rules of behavior should be consistent it may be taken for granted that when a fixed factor is given, any increase in the employment of the variable factors will lead to diminishing returns. Using the content of this rule it is assumed that when wage rate is rising, the entrepreneurs will employ less labor with the given stock of capital and that may lead to a reduction in the output of a plant less than in proportion to the reduction in the employment of labor.

It is assumed that the total labor $n(t)$ employed at time t with the capital $k(t)$ will be reduced at the rate δ per unit of time because of rising wage. Thus at the time of replacement of the capital, there will be only $e^{-\delta\theta}n(t)$ units of labor employed with $k(t)$. However, this rate of reduction of labor will not lead to a proportionate rate of reduction of output of the plant. It is assumed that the output is reduced at the rate $\alpha\delta$ where $0<\alpha<1$. The equation for deriving the form of n(t) is as follows:

$$\text{III-65} \qquad N_o e^{\lambda t} = \int_{t-\theta}^{t} n(u)\, e^{-\delta(t-u)} du$$

The solution of III-65 can easily give the value of n(t) as $n_o e^{\lambda t}$, where $n_o = \frac{N_o\,(\delta+\lambda)}{1-e^{-(\delta+\lambda)\theta}}$.

The equation for the total output may be written as:

$$\text{III-66} \qquad X(t) = \int_{t-\theta}^{t} \{n_o e^{\lambda u}\}^{1-\beta} s^{\beta} X^{\beta}(u)\, e^{gu}\, e^{-\alpha\delta(t-u)}\, du$$

where we have $k(t) = sX(t)$. By the procedure which will be illustrated in the next section one may obtain a solution for X(t). If the solution is taken to be one with exponentially growing output, it shall have an asymptotic solution of the type which had been given before to be of the form $Ce^{\sigma t}$. In the present case also the value of σ is the same as in III-56. But the value of C is given by:

$$\text{III-67} \qquad C = \left\{ \frac{1/(\alpha\delta+\sigma)\; s^{\beta} n_o^{1-\beta} (1-e^{-(\alpha\delta+\sigma)\theta})}{\alpha\delta+1} \right\}^{1/(1-\beta)}$$

From III-67 it is clear that depreciation of capital has an important role in the determination of output over time. The term depreciation is used for here. Now some important points should be noted about the model designed above.

Two types of situations that exist in the economy are imbedded in one technological constraint III-66. One is that in the real world when a new plant is built the proportion of factors is chosen with proper regard for the existing factor prices and the current technology.

This is represented by the factor representing technological progress and the Cobb-Douglas production function. Another situation included in the model is that in the short run after the plant is started, output becomes solely the function of current input. One might be inclined to object to it on the ground that this has closer resemblance to neo-classical model which Johansen's model seeks to improve. But according to the present author the above model does not resemble the neo-classical situation, because in the latter there is no explicit introduction of short period production function while dealing with long run growth problems. Furthermore this model is more realistic than that of Johansen in assuming that short run costs determine the level of utilization of a plant.

It should be remarked that it is with the help of the parameter α which is derived from the short run production function that the value of δ is determined taking account of the technological progress and therefore the tendencies of the current costs. In other words, δ should be such that given the parameter α marginal cost is always equated to price. Although the author does not intend to be too emphatic one may consider the short run production function to be of the form, $y = n^{\alpha}$, both the variables being time functions. Taking logarithm of both sides of this equation and differentiating with respect to time growth rate of y is found to be equal to α times the growth rate of n. The negative of such growth rate is the shrinkage rate used in the above model. Here it is held that α has no relation at all with β, because they are assumed to belong to two different universes.

This completes the discussion of some of the essential features of the model proposed in this section. Many problems remain to be discussed, of course. The toughness of the subject and limitations of space and time prohibit the author from embarking on further controversies at present. Before passing to another section some conclusions of this section are summarized.

E. Conclusions. The main conclusions of this section are as follows:

1. Johansen's model is very rigid in that, if competitive assumption regarding the uniformity of factor prices has to be fulfilled, the changes in factor prices, and therefore, all changes in technology resulting in the change in productivity of factors, are to be ruled out. Under the assumption of constant technique and constant factor prices the case of fixed labor supply was studied and it was found that, given an initial condition, output may grow such that in the long run it will tend to an asymptote which is finite and determined by the parameters of the system. It was also observed that output at the asymptotic level is higher, higher the θ. However, it was found that the rate of interest is related to the value of θ such that with a given value of θ there is a rate of interest which determines the asymptotic equilibrium.

2. Although the introduction of neutral technological progress in Johansen's model was found to be inconsistent, the effect of such introduction along with the assumption of expanding labor force was studied. It was found that in such case the economy would asymptotically

converge to a level of output which would grow at a rate determined by the rate of technological progress and the rate of increase of labor supply. The relation between the rate of interest and the durability of capital was considered. From the point of view of entrepreneurial equilibrium, the relation does not seem to differ from what was observed in the previous model, namely that there is an inverse relationship between the two. The problem that from the point of view of the economy the rate of technological progress imposes another condition of optimum output was also discussed. However, this condition was regarded as irrelevant if one were concerned with equilibrium.

3. The purpose in C was to discuss the model on the assumption of constant factor prices and by representing the aggregate output in terms of the earnings of factors. It was found that the rate of growth in such a case would be determined by the savings coefficient, quasi-rent and the rate of growth of labor. If the rate of growth so determined involves change in the factor-ratios, it follows that innovations are taking place in the economy such that the prices of factors are maintained at constant level. Such a situation would have been worth considering in some detail. But at the moment it was found difficult to conceive of a production model that could accomplish the objective. Moreover, to be realistic one has to pay attention to the rising productivity and increasing factor prices, which was neglected there.

4. The last section was devoted to the case in which rising productivity is allowed with an important modification of Johansen's model. The author introduced the assumption that after a capital is

installed the intensity with which it is operated varies with factor prices. It was held that capital in that case becomes a limitational factor and output becomes a function of current input. With this assumption the model was solved to yield output level which was influenced significantly by the rate of shrinkage of output, or as it may be said, by the rate of shrinkage of activity, related to a given capital good over time.

These in brief are the conclusions of the present section. A similar problem with a different assumption regarding the productive life of capital good will be considered in the next section.

Section V: Exponential Depreciation of Capital Goods[1]

In this section is studied the case of capital goods that are depreciating at an exponential rate γ . This implies that of the total amount of capital K(t) at time t only $K(t)e^{-\gamma\theta}$ will be available for use after θ units of time. Under Johansen's assumption the productivity of such capital will shrink at the same rate and the amount of labor employed with the capital at any time will also shrink at the same rate.

In the present case also there is the problem of finding out the effect of changing factor prices in the case of rising productivity. This will be tackled with the same method as the one used in D of the last section. First the implications of Johansen's model will be discussed with the assumption that there has been no change in productivity

[1]The production model in its elementary form is due to Johansen but the modifications are the author's own.

of factors of production. This will be the object of Subsection A in what follows. In Subsection B the assumptions are modified allowing for increase in the productivity of the factors of production. In C this section is concluded.

A. Growth without Technological Progress. A consequence of the present assumption regarding depreciation of capital goods is that the capital goods produced at any time live for an infinite length of time though the proportion of them living decreases exponentially over time. Thus the amount of labor employed with capital at any time also decreases in a similar way. Assuming full-employment of labor the amount of labor employed at t is distributed to capital goods produced at different points of time in such a way that the following relation holds:

$$\text{III-68} \qquad N_o e^{\lambda t} = \int_{-\infty}^{t} n(u) e^{-\gamma(t-u)} du$$

The above equation uses the assumption that the labor force is expanding at the exponential rate. The function n(t) is also used to designate the amount of labor employed with the capital produced at time t. Since the labor employed with capital of earlier periods is released for reemployment with the new capital goods the resulting expression III-68 is derived. It should be noted that labor employed with capital of time u, n(u), is reduced to $n(u)e^{-\gamma(t-u)}$ at time t, $(t>u)$. In order to solve the equation III-68 it may be differentiated at first to yield:

$$\text{III-69} \qquad \lambda N_o e^{\lambda t} = n(t) - \gamma N_o e^{\lambda t}$$

from which follows:

III-70 $$n(t) = N_o e^{\lambda t} (\lambda+\gamma)$$

The assumption that $k(t) = sX(t)$ and the use of the Cobb-Douglas production function leads to the following equation for the aggregate output at time t:

III-71 $$X(t) = \int_{-\infty}^{t} (n' e^{\lambda u})^{1-\beta} s^{\beta} X^{\beta}(u)\, e^{-\gamma(t-u)}\, du$$

where $n' = N_o\,(\lambda+\gamma)$. In order to solve the above equation it may be differentiated first to give:

III-72 $$\frac{dX}{dt} = (n' e^{\lambda t})^{1-\beta} s^{\beta} X^{\beta}(t) - \gamma X(t)$$

The solution of III-72 is the following:

III-73 $$X(t) = \left\{ \frac{n'^{(1-\beta)} s^{\beta}}{\lambda+\gamma} e^{(1-\beta)\lambda t} + Ce^{-(1-\beta)\gamma t} \right\}^{1/(1-\beta)}$$

where C is a constant determined from initial conditions. It is easily observed that the growth rate will tend to λ as time passes. For high γ the asymptotic level of output becomes low. So far the writer is with Johansen.[1] Now the other aspect of the model may be observed.

The intention here is to find out the equilibrium rate of output which satisfies the condition that the value of capital at each point of time is equated to the amount of saving that is involved in producing it. The marginal productivity of capital $k(t) = sX(t)$ at time t from the production function is to be derived and from it the total earnings of this capital per unit of time from t to ∞ is to be deduced

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[1]See Johansen, op. cit., pp. 168-70. The treatment of the problem above is essentially similar to his.

allowing the shrinkage of this earning at the rate γ. The earning is then discounted at the rate r. This gives the following equation:

$$\text{III-74} \qquad sX(t) = \int_t^{\infty} \beta (n'e^{\lambda t})^{1-\beta} s^{\beta} X^{\beta}(t)\; e^{-(r+\gamma)(u-t)}\, du$$

By integrating III-74 and rearranging the terms of the resulting equation the equation for r follows:

$$\text{III-75} \qquad r = \frac{\beta (n'e^{\lambda t})^{1-\beta} s^{\beta} X^{\beta}(t) - s\gamma X(t)}{sX(t)}$$

From III-72 it may be seen that the first term in the numerator of the right hand side of III-75 is equal to $\left\{\frac{dX}{dt} + \gamma X(t)\right\}\beta$. Using this in the equation III-75 one has:

$$\text{III-76} \qquad r = \frac{1}{s}\left\{\beta g(t) + \gamma(\beta - s)\right\}$$

where g(t) is equal to $\frac{dX}{dt}\frac{1}{X}$, or the growth rate at time t. Since the equation III-76 contains all other terms, except g(t), that are assumed to be constant g(t) also has to be constant. Otherwise it has to be assumed that at time t the entrepreneurs discount their future income at a rate of interest which would change with the passage of time. If such is the case the rate of interest will have the same tendency as the rate of growth of output to approach a constant asymptotic value as the growth rate will have. However, the author considers it better to regard the constant rate of growth λ as the valid solution. One can see that the integration of the expression in III-75 would give λ as the rate of growth of output, provided the rate of interest does not vary with time, which may be assumed to have been satisfied. In any case the equation III-76 is the condition which is to be satisfied if equilibrium output, defined according to the assumption underlying III-74, is to be realized.

From III-76 it is apparent that the rate of interest can be equal to the rate of growth if and only if $\beta = s$. This condition was true in the models considered in the last section also. An economy with $\beta - s > 0$ experiences a higher rate of interest than a similar economy with $\beta = s$ assuming γ and g to be the same for both of them.

In interpreting III-76 r should be considered as a dependent variable. Further one should not use the ceteris paribus assumption in order to explain the effect of change in any of the parameters on g or r, since all the terms are interrelated.

The case of an economy in which neutral technological progress is taking place will be investigated now.

B. Neutral Technological Progress and Growth. The author will not repeat here the arguments why in a competitive system Johansen's approach is questionable in the case of increasing productivity of factors leading to increased factor prices over time. But for the sake of emphasis it is worth repeating that while new investment in capital goods is being made, one may be able to choose an optimal factor combination considering the factor prices and prices of goods though such choice is not possible after the type of plant is chosen and set in operation. But after the capital is installed it is not necessary that it should be used to produce output using the planned combination of factors. The fact that a capital good has to live a certain length of life prohibits an entrepreneur from introducing new technology whenever he desires. But this does not necessarily prevent him from varying current inputs whenever the situation necessitates this. This necessity of varying the amount of current inputs arises whenever the prices of inputs are changing. In the present model when wage rate is changing it does not pay an entrepreneur to employ the same amount of labor as planned during the installation of the capital. He has to vary the amount of labor if he is to produce profitably at all. For an entrepreneur the fixed capital becomes a limitational factor and the rate of output becomes a function of the current output.

If this idea is used in the present model one has to consider a production function which depicts the short run technical possibilities of variation of output by varying the inputs of labor. It is misleading to suppose that variation of labor input will proportionately change the amount of output. It is equally misleading to suppose that the production function which describes the possibilities of varying output by changing inpus is the same as in the case when fixed capital outlays are being made and the appropriate choice of factor combination is made on the basis of the factor prices ruling at a given time. When new capital is being installed the different capital output ratios yielding a given amount of output are examined with the view that such output results from the normal utilization of capacity. Otherwise a production function has little meaning, when the planning of a production process with an optimal combination of factors is concerned. But when the amount of capital is given as already in the shape of a plant in operation the variation of product by varying inputs cannot be definitely described in terms of a production function of that type. In such a case the variation of inputs may reduce or increase outputs more or less sharply than in the case mentioned above. At the moment the author does not intend to elaborate upon this subject. Here the model that will be developed will utilize the same ideas as in D of the previous section.

The assumption of the previous discussions that technology of a neutral variety is advancing at a constant exponential rate g is used here. This advance in technology is leading to a continuous increase in the factor prices. The consequence of this is that in industries which utilize older capital goods less and less labor is employed. But the output from such capital is reduced less than in proportion to the reduction of labor at each moment of time. δ and $\alpha\delta$ are the respective rates at which labor and output are shrinking. α is determined by the short run production function and δ by α and g together. It is perhaps reasonable to suppose that δ is an increasing function of g.

It is now clear that there are two types of what one may call shrinkage functions. For labor they are δ and γ. For output they are $\alpha\delta$ and γ. One may find the function n(t) by solving the following integral equation:

$$\text{III-77} \qquad N_0 e^{\lambda t} = \int_{-\infty}^{t} n(u) e^{-(\delta+\gamma)(t-u)} \, du$$

Using similar arguments as in the case of III-68 the following solution is arrived at:

III-78 $$n(t) = n^* e^{\lambda t}$$

where $n^* = N_o(\gamma+\lambda+\delta)$.

For aggregate output the following equation holds:

III-79 $$X(t) = \int_{-\infty}^{t} (n^* e^{\lambda u})^{1-\beta} s^{\beta} X^{\beta}(u)\, e^{gu}\, e^{-(\alpha\delta+\gamma)(t-u)}\, du$$

The solution of III-79 may be found by using analogous argument as in the case of III-71. The solution in this case is:

III-80 $$X(t) = \left\{ \frac{(1-\beta) n^{*1-\beta} s^{\beta}}{g+(1-\beta)(\alpha\delta+\lambda+\gamma)} e^{\{g+(1-\beta)\lambda\}t} + Ce^{-(1-\beta)(\alpha\delta+\gamma)\, t} \right\}^{\frac{1}{1-\beta}}$$

The above solution does not differ in form from the previous solution in III-73. Moreover, as in the model discussed in the last section, growth rate is determined by the rate of growth of labor supply and the rate of technological progress in the long run. But there are some characteristics in the present solution which require attention.

In the previous model of this section it could be seen that the rate of growth of output is not affected in any significant way by the determinants of such rate of growth. The rate of growth of labor force could exert some influence on the asymptotic level of output as on the rate of growth. The rate of physical depreciation of capital, γ, could exert downward pressure on the level of output in the long run and also was one important determinant of the speed at which the asymptote is approached. In the present case, however, the rate of technological progress not only affects the rate of growth of output but while exerting downward pressure on the level of long run output partly by itself and partly by influencing the rate of shrinkage of output, due to the obsolescence effect of the technological progress, it also affects the speed with which the asymptote is reached. Thus g plays a somewhat complicated role in the above model.

Now the equilibrium condition may be taken as in the previous subsection. In the place of g(t) write the asymptotic growth rate of III-80 and use $\frac{g}{1+\beta}+\lambda$ in the place of λ . The following condition then holds for the present model which is analogous to III-76:

III-81 $$r = \frac{1}{s}\left\{\beta\left(\frac{g}{1-\beta}+\lambda\right) + (\alpha\delta+\gamma)\ (\beta-s)\right\}$$

The rate r is in this case doubly affected by the rate of technological progress, firstly as a component of the rate of growth of output and secondly as a determinant of the obsolescence rate. If possibly the more realistic situation, $\beta > s$, is assumed in both ways g raises r. Similarly, r is raised by α, which may be called short-run-marginal-cost-coefficient, in the event of its rise.

The validity of the above model depends upon how far the reality agrees with it. It is not the object of the author to test the conclusion in the present work. But from a purely analytical point of view the author believes that the line taken up in the last model appears to be justifiable both on grounds of consistency and greater degree of realism.

C. Conclusion. The main conclusions of this section are that the rate of physical depreciation of capital has downward pressure on the long run development of output. It also affects the speed with which the long run asymptotic level of output is reached. It was argued, as in the previous section, that the assumption of technological progress which raises factor prices cannot be valid in Johansen's model if the existence of short run production functions, which allows variation in current inputs employed per unit of existing capital goods without leading to proportionate variation in output, is denied. This argument is valid if uniformity of factor prices in all employment is assumed, which, of course, is the consequence of competitive assumption which is made throughout the present work. On the basis of this argument the last model was developed.

Once the assumption that by varying inputs, output could be varied according to the implied operation of the law of diminishing returns, is introduced an obsolescence function is derived, the shape of which is determined by the rate of technological progress (of the type discussed here), and the coefficient or coefficients of the short-run production function. The writer here considered a coefficient α which, together with g was assumed to determine the rate δ which was considered as the rate at which the employment of labor in capital installed at any time will decline even if the rate of physical depreciation were zero. The rate of shrinkage of output was similarly defined to be $\alpha\delta$.

The consequence of this assumption was that the rate of technological progress influenced directly the rate of growth and indirectly both the long run tendency of the economy and the speed of the economy toward this. Its effect on the obsolescence of capital is to be regarded as the important aspect of the last model.

Now the important features of Johansen's model of economic growth have been examined. In the next section the present chapter is concluded with some remarks.

Section VI: Concluding Remarks

In the introduction of this chapter one would find that the purpose of the chapter was to derive the implications of the growth model due to Johansen, because of the realism it wanted to bring in the technological assumption. The author was in favor of the new type of model because of its recognition of the fact that capital goods are durable, and factor substitution or innovation of any kind is not possible as long as old capital goods exists. Changes in technique can occur and, if desired, one factor can be substituted for another only at the time old capital goods are replaced by the new. Under these assumptions various models were studied with varying conditions of labor supply and technological progress. But, it is needless to repeat in detail here that the model in the form as presented by Johansen was found to be so rigid that changing productivity of factors could not be accommodated in it without sacrificing much of the consistency. This consideration led the author to formulate the last models in the last two sections.

It should, however, be admitted that the models referred to, which in the assumption that the short run and the long run production functions are different and that both of them have to be included in dealing with the theory of growth, are meant to illustrate the nature of the attempted synthesis of Johansen's model with the model of the neo-classical economists. A much more elaborate study of this aspect of the problem would have produced some more interesting results, perhaps. But the limitation of time and space relative to the amount of research required prevented the author from taking up this task.

Since the main conclusions of the different sections above are already noted at the end of those sections it is found unnecessary to repeat them here.

CHAPTER IV

SUMMARY AND CONCLUSION

Without going into methodological detail or specific analytic technique the arguments of the preceding chapters are summarized below in a generally simplified manner.

It may be recalled here that the purpose of this study has been to explain the distribution of income between labor and capital in the context of the theory of economic growth. It may also be recalled that the marginal analysis has been used as the means to accomplish the objective. It is needless to repeat here that the marginal productivity theory of income distribution is directly applicable to the neoclassical and Johansen's models of economic growth, whereas this is not the case with the fixed-coefficient model of Harrod. In the latter case income distribution could be explained with the help of a multisectoral model, using the various coefficients of production characteristic of different sectors of the economy and the demand functions for the goods.

The purpose of explaining income-distribution for all the models considered in the present work has been to see whether and under what conditions full-employment growth equilibrium exists. It is clear that growth of output is indeterminate unless the level of employment of factors and technology is known. Given the state of technology and the supply of factors, the level of employment of factors and output of goods depend upon demand for the total output. Demand and supply of output are brought into equality through adjustment of prices of goods and factors. Keeping these basic and elementary facts in mind it is worthwhile to consider how they are applied to the three models discussed in this work.

In connection with Harrod's model the discussions centered mainly around the question as to whether there is any explanation of factor pricing when there is full-employment growth equilibrium. In a one sector model it was not possible to explain this unless Kaldor's theory was accepted. Another approach that could consistently be taken was to have a multisectoral model based on the assumption of fixed coefficients of production. The existence of more than one sector with different techniques of production assures a wide range of output for which full employment of factors could be achieved. Specifically the difference between a one sector model and a model consisting of more than one sector with different techniques lies in the fact that in the former case there is one and only one resource vector for full employment to exist at any moment of time, whereas in the latter case there is a set of infinite vectors of resources for which this is true.

In the approach taken in Section VI, Chapter II, of the present work capital goods were treated for the most part as intermediate goods. The treatment of capital goods as stocks of durable goods does not present any significantly new problem concerning income-distribution. In both cases the solution of the model will lead to positive full-employment output of goods in the long run determined solely by the availability of labor and technology. The income distribution part of the model comes from the demand functions and the equations relating prices to costs in which the coefficients of production become relevant. The crucial part of the thesis is that since in the model being considered the substitution between factors is not allowed on the production side the existence of full-employment equilibrium for all economically meaningful values of production coefficients and for all initial values of the resources and their growth rates, requires that the substitution of the consumers as between all goods available or that between the present goods and future ones should be perfectly flexible. Thus the rigidity of the one side of the economy has to be completely compensated by the perfect flexibility of the other.

Kaldor has introduced some degree of flexibility in his model at least by assuming that the propensity to save varies from one group of factor owners to another. But this is done by assuming that within one group of factor owners there is some constant propensity to save. This is one step toward explaining the existence of full-employment equilibrium. However, this is not necessarily the right step. One can easily imagine cases in which Kaldor's theory is not applicable, not only in its original version but also in more elaborate versions applied to more than one sector.[1] The defects of Kaldor's assumption are

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[1]The writer came across an interesting and illuminating article by R. M. Davis, "Income Distribution in a Two Sector Model," Oxford Economic Papers (forthcoming), when the present work was already completed. In the article Kaldor's theory was applied to a model consisting of consumption goods and investment goods sectors. With given savings coefficients, there is the problem that though arbitrary initial

clearly removed by the procedure followed in the present work, in which savings coefficients are the consequences of the working of the totality of market forces. The defects referred to are the possibility of economically unacceptable prices of factors and goods at full-employment which can result from the assumption of constant saving coefficients.

The neoclassical theory of economic growth is brought into the present work as a contrast to Harrod's model. On grounds of realism the adherents of the neoclassical approach assume that the production function at any state of technique consists of alternative processes of producing a given output. The variability of coefficients of production which is thus assumed allows any arbitrary amount of factors to be fully employed with positive rate of return. The factor pricing is most easily explained if the production obeys the law of constant returns to scale. The assumption of constant returns to scale, besides leading to convenience in analysis, has an interesting property which is contained in Morishima's or Samuelson's substitution theorem. If it is assumed that factors are employed according to the rule that the marginal productivity of each factor equals its price then the fixed coefficient models appear to be assuming that the relative factor-prices remain invariant over long period. If this is assumed to be true of

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endowment of factors may lead to positive outputs which grow at the rate determined by the rate of growth of labor and the parameters of the production functions, the existence of positive profit has to satisfy certain constraints relating the saving coefficients and the total available labor. Hence the proposition that perfect flexibility of consumer behavior is required for full-employment equilibrium is to be assumed.

Harrod's model then it follows that a less than full-employment situation is due to the rigidity of factor-prices. How far this is true is not clear from Harrod's model. But in the neoclassical model the assumption of constant returns to scale results in the conclusion that stationary growth process at full-employment can be achieved in the long run.

It is true, as Solow has shown, that the assumption of constant returns to scale need not necessarily lead to a stationary growth process. It may also be true that the coefficients of production need not be determined according to the substitution theorem under conditions of instability. If one assumes, however, a nicely behaved production of the Cobb-Douglas type, and if movements of prices and output fluctuations do not disturb the working of the economic system it is possible that full-employment equilibrium will be realized. It may be expected that the fluctuation in output and prices which may occur in the short run would be unimportant compared to the effect of the persistent tendencies in factor supply and technology. Thus a neoclassical model may be regarded as fairly stable at full-employment. It may be remarked that in the case of the neoclassical model even if there are rigidities in consumer behavior the assumption of flexibility on the production side is sufficient to explain income distribution.

The assumption of variability of factor-proportions in a neoclassical model appears to be an exaggeration of reality in the sense that plant and equipment existing at any point of time are of definite nature and may not allow substitution between factors and also the introduction of new technique. Johansen's model attempts a synthesis of Harrod's model, which is regarded valid for the short run, and the neoclassical model by assuming that new techniques or changes in factor proportions can be introduced only at the time when old capital is replaced or new investment is made. With some different simple assumptions regarding the shrinkage or the durability of capital mathematical models are built which yield solution for the growth of output over time. But the income distribution aspect of the problem is entirely left out. This leaves the question as to what conditions are required for the existence of long run full-employment equilibrium unanswered.

It has been the purpose of the last chapter of the present work to formulate Johansen's model with sufficient simplifying assumptions such that the distribution aspect of the model could be clearly explained. It was assumed that in the long run the rule of substitution between factors would be the one given by the Cobb-Douglas production function and regarding the progress of technology it was considered as resulting in the neutral shift of the production function. The explanation of income distribution is easy once such assumption is made. The durability of capital, however, posed an additional problem, which will be noted at the end of this chapter. Using the same assumptions as Johansen has done regarding the shrinkage of capital both the production and distribution aspects of the model were studied.

As far as the growth of output over the long run is concerned Johansen's model supplies little additional information which may not be found in the neoclassical theory. In interpreting the distribution part of the model the marginal productivity theory was applied as in the case of the neoclassical model. The following equations of which the first refers to the case of capital goods having finite life and the second refers to the case in which the capital goods depreciate at an exponential rate, show the relationship between the rate of interest and growth rate and other parameters:

$$1. \quad \frac{1}{\sigma}(1-e^{-\sigma\theta}) = \frac{1}{r} \ \frac{\beta}{s}(1-e^{-r\theta})$$

$$2. \quad r = \frac{1}{s}\left\{\beta g(t) + \gamma(\beta - s)\right\}$$

where σ and g(t) are growth rates, r, the rate of interest, β , the exponent of the term capital in the production function which measures the proportion of income going to capital, s, the gross-saving-coefficient, γ the rate of depreciation and θ , the durability of capital.

The above equations show the relationship between the rate of interest and the other parameters of the model assumed. One may contrast the relationship explained by them with the one which von Neumann[1] has established in his model and also with the relationship which Solow[2] has explained for Harrod's model. In von Neumann's case the rate of interest is equal to the rate of growth, thereby implying that in a

[1]See J. von Neumann, "A Model of General Economic Equilibrium," Review of Economic Studies, XIII (1945-46), 8.

[2]R. M. Solow, "A Note on the Price Level and Interest Rate in a Growth Model," Review of Economic Studies, XXI (1953-54), 74-9.

stationary economy the rate of interest is zero. In the present case Equation 1 cannot be used for a stationary condition. However, from Equation 2 above it is clear that if g(t) is zero r can be zero only if $\beta =$ s for all acceptable values of γ. Perhaps the important point of difference between the present position and the position of von Neumann lies in the relation between β and s. If the von Neumann economy is stationary with depreciating capital goods, depreciation of capital per unit of time being taken as constant the return on capital is just enough to replace it. This would make $\beta = s$. But if capital gets more than the amount just sufficient to cover its cost of production (excluding interest charge) there exists a rate of interest even in the stationary state and in the case of a growing state the rate of interest exceeds the rate of growth. Similar comparison may be made between the conclusion reached here and that of Solow in whose model too the rate of interest is equal to the rate of growth of output if the price level is stationary, which is here assumed to be the case.

The equations 1 and 2 mentioned above are derived from the condition for competitive equilibrium. Thus for full-employment equilibrium to exist Johansen's model requires those conditions to be satisfied. But there is one more point elaborated in the last chapter though not to a sufficient length which deserves particular notice. The point is that in order to present the condition of competitive equilibrium more consistently the present writer had to depart from the assumption of Johansen regarding technology. The departure is necessitated by the problem which arises in the case of technological progress, especially, when the rise in productivity of factors raises their prices. In Johansen's model it is assumed that the old plants are always operated at the planned level of intensity throughout their life irrespective of the change in factor prices that might be going on. This is not necessarily true. If in the assumed economy a continuous rise in productivity is taking place and factor prices are rising it is not profitable for the old plants to produce at their original capacity. Hence plants become obsolete simply because of the changing prices of factors. Therefore some flexibility is necessary in the model if uniformity of factor prices throughout the economy and the equilibrium of the producing firms is to be realized.

Toward this end the writer has found it helpful to assume a short run production function which has different parameter from that in the long run one. The short run production function is defined for any given fixed plant in terms of current input alone as variable. The diminution in the product per unit of current input, namely labor in the model considered, when its employment is reduced and the increment of the product in the opposite case do not follow the same pattern as in the case of the variation of factor-proportion along the long run production function. With this assumption an obsolescence function was defined as depending upon the parameter of the short run production function and the rate of change in productivity occurring exogenously. The model which has resulted has resemblance with all other models discussed. But the equilibrium pattern of output over time is now affected by the additional factor, namely, obsolescence, which is different from the exogenously given physical life or depreciation of capital goods.

It is now clear that there are two aspects of the problem of factor-price-determination in the new model which might not appear clearly demarcated in the formulation given in the last chapter. One is that the long run production function (which includes in it the improvement in technology) explains the change in factor price going on in the section of the economy in which new plants are being operated. Another aspect is that the old plants are trying to adapt to such changes continuously by making the best of what they have by possible short-run adjustments open to them which are described by the short run production function. This latter part is more a problem of allocation of labor among existing fixed items of capital.

One might probably expect a much clearer view of the economic events if the effect of technological progress on the structure of the economy were explained by taking discrete intervals of time. It would, perhaps, have been much more illuminating if a major structural change were assumed to have taken place at one interval of time and if its effect on employment, output, and obsolescence of capital in subsequent intervals were studied. This would have led to the problem of structural disequilibrium of the economy.[1] However, the study of the various types of disequilibrium and the forces that restore equilibrium lies

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[1]In this connection one might consider Schumpeter's arguments about the innovations and their consequences resulting in long waves of business cycle. See his The Theory of Economic Development (Cambridge: Harvard University Press, 1934) and Business Cycles (New York: McGraw-Hill Book Co., Inc., 1939).

outside the scope of the present work. The task of the present study has been solely to study the conditions for continuous steady equilibrium growth. This is sufficient to justify the present treatment of the continuous adjustment process.

Since the points mentioned in the preceding pages summarize the important arguments of the present work, it may finally be mentioned that the three models discussed here describe three aspects of the same world. Which one of them is more relevant depends upon the issues one is confronted with.

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