REPRESENTATIONS OF PARTITION CATEGORIES

by

MAX VARGAS

A DISSERTATION

Presented to the Department of Mathematics
and the Division of Graduate Studies of the University of Oregon
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

March 2023

DISSERTATION APPROVAL PAGE

Student: Max Vargas

Title: Representations of Partition Categories

This dissertation has been accepted and approved in partial fulfillment of the requirements for the Doctor of Philosophy degree in the Department of Mathematics by:

Jonathan Brundan	Chair
Ben Elias	Core Member
Alexander Kleshchev	Core Member
Victor Ostrik	Core Member
Michael Kellman	Institutional Representative

and

Krista Chronister	Vice Provost for Graduate Studies

Original approval signatures are on file with the University of Oregon Division of Graduate Studies.

Degree awarded March 2023

DISSERTATION ABSTRACT

Max Vargas

Doctor of Philosophy

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March 2023

Title: Representations of Partition Categories

We explain a new approach to the representation theory of the partition category based on a reformulation of the definition of the Jucys-Murphy elements introduced originally by Halverson and Ram and developed further by Enyang. Our reformulation involves a new graphical monoidal category, the *affine partition category*, which is defined here as a certain monoidal subcategory of Khovanov's Heisenberg category. We use the Jucys-Murphy elements to construct some special projective functors, then apply these functors to give self-contained proofs of results of Comes and Ostrik on blocks of Deligne's category $\underline{\mathrm{Re}}\mathrm{p}(S_t)$. We then study a restriction functor $\underline{\mathrm{Re}}\mathrm{p}(S_t) \to \underline{\mathrm{Re}}\mathrm{p}(S_{t-1})$ and prove a conjecture of Comes and Ostrik involving this functor. Finally, we use the restriction functor to verify a criterion of Benson, Etingof, and Ostrik, thereby identifying the abelian envelope of $\underline{\mathrm{Re}}\mathrm{p}(S_t)$ with the Ringel dual of the category of locally finite-dimensional $\mathcal{P}ar_t$-modules.

This dissertation includes published co-authored material.

CURRICULUM VITAE

NAME OF AUTHOR: Max Vargas

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:

University of Oregon, Eugene, OR, USA
Massachusetts Institute of Technology, Cambridge, MA, USA

DEGREES AWARDED:

Doctor of Philosophy, Mathematics, 2023, University of Oregon
Bachelor of Science, Mathematics, 2018, Massachusetts Institute of Technology

AREAS OF SPECIAL INTEREST:

Representation Theory
Machine Learning

PROFESSIONAL EXPERIENCE:

Graduate Employee, University of Oregon, 2018-2023
Research Intern, Pacific Northwest National Labs, 2022-2023

GRANTS, AWARDS AND HONORS:

Promising Scholar Award, University of Oregon, 2018

PUBLICATIONS:

Brundan, J., & Vargas, M., A New Approach to the Representation Theory of the Partition Category. *Journal of Algebra*, **601** (2022), pp. 198–279.

ACKNOWLEDGEMENTS

Foremost I must thank my advisor, Jonathan Brundan. Not only for your mathematical lessons and guidance throughout the years, but also the unending patience, kindness, and positivity you have shown.

My family — mother, father, brother, and sister. You have all set examples for me to follow and have led me to boundless opportunity through your love and support. It is thanks to you that I have all that I do. And to Boomer, woof woof rRRrr bark woof rRRrRr bark.

Lastly, I want to thank Jules, Corey, Carol, Mitkins, and LobsterCucumber for the remaining things not mentioned here.

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CHAPTER I

INTRODUCTION

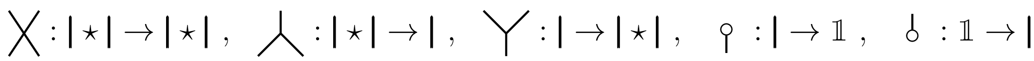
Throughout this dissertation, fix an algebraically closed field $\Bbbk$ of characteristic 0 as well as a parameter $t \in \Bbbk$. Chapters II and III both contain material that has been published and co-authored with Jonathan Brundan in [BV22].

1.1 Overview

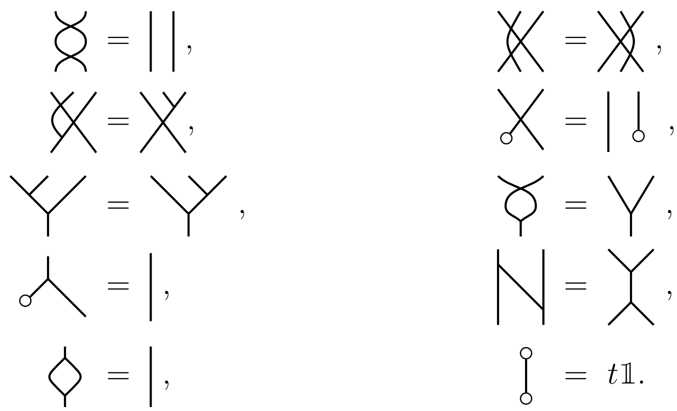
In [Del07], Deligne introduced a class of diagrammatic monoidal categories which generalize a number of classical groups. His motivation was to construct examples of tensor categories which do not exhibit the property of moderate growth. That is, he wanted his categories to contain objects X for which the number of composition factors of $X^{\otimes n}$ should be a super-exponential function of n. The first of these examples is the category $\underline{\mathrm{Rep}}(S_t)$, which may be defined as the additive Karoubi envelope of the partition category, $\mathcal{P}\!ar_t$, presented below. When t is a natural number, the usual category $\mathrm{Rep}(S_t)$ of finite-dimensional representations of the symmetric group is a certain quotient category of $\underline{\mathrm{Rep}}(S_t)$ (its semisimplification). For values of $t \notin \mathbb{N}$, $\underline{\mathrm{Rep}}(S_t)$ can be thought to interpolate between the usual categories $\mathrm{Rep}(S_t)$. From a representation theoretic perspective, however, $\underline{\mathrm{Rep}}(S_t)$ remains most interesting when $t \in \mathbb{N}$; otherwise this category is a semisimple abelian category.

This dissertation focuses primarily on the underlying category $\mathcal{P}\!ar_t$, which we now proceed to define via a monoidal presentation. This approach is based on Comes [Com20, Thm. 2.11]; see also [LSR21, Prop. 2.1]. We use the string calculus for strict monoidal categories with the convention that vertical composition $f \circ g$ is given by stacking f on top of g and horizontal composition $f \star g$ is given by placing f to the left of g.

Definition 1.1.1. The *partition category* $\mathcal{P}ar_t$ is the strict $\Bbbk$-linear monoidal category generated by one object $\mid$ and the morphisms

 (1.1.1)

subject to the following relations, as well as the ones obtained from these by horizontal and vertical flips:



(1.1.2)

(1.1.3)

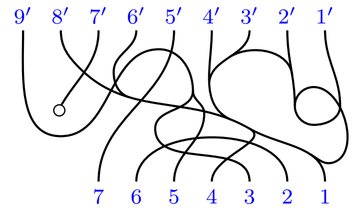
(1.1.4)

(1.1.5)

(1.1.6)

We will sometimes denote the object $\mid^{\star a}$ simply by a. Note $\mid^{\star 0} = \mathbb{1}$.

Traditionally, e.g. in [CO11, Def. 2.11] or [Del07, § 8], the partition category is thought of in terms of set partitions instead of generators and relations as above. In particular, in loc. cit., the morphism spaces $\mathrm{Hom}_{\mathcal{P}ar_t}(a,b)$ are shown to have bases given by set partitions of $\{1,\dots,a,1',\dots,b'\}$. To establish the connection between their approach and ours, we make use of partition diagrams. By a $b \times a$ *partition diagram*, we mean the diagram of a morphism $f \in \mathrm{Hom}_{\mathcal{P}ar_t}(a,b)$ built from vertical and horizontal compositions of the generators in Definition 1.1.1 which has no "floating" components (any such component can be removed at the cost of the scalar t by the "dimension relation" (1.1.6)). Labeling the boundary points (from right to left) along the bottom and top rows of f by $1,\dots,a$ and $1',\dots,b'$, such a partition diagram encodes a set partition of $\{1,\dots,a,1',\dots,b'\}$. Here is an example of a 9×7 partition diagram.

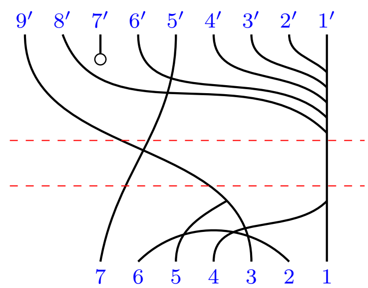


(1.1.7)

This partition diagram determines the set partition

$$\{1, 4, 1', 2', 3', 4', 6', 8'\} \sqcup \{2, 6\} \sqcup \{3, 5, 9'\} \sqcup \{7, 5'\} \sqcup \{7'\}.$$

There is an equivalence relation on the set of partition diagrams where two diagrams are equivalent if they determine the same partition of their boundaries. For example, the morphism inside $\mathcal{P}ar_t$ represented by (1.1.7) is equal to the one represented by the tidier diagram below because they determine the same partition on the labels of the endpoints.



(1.1.8)

From the defining relations, any diagrams which are equivalent in this sense are equal as morphisms in $\mathcal{P}ar_t$. In fact, the converse holds and one obtains a basis for $\mathrm{Hom}_{\mathcal{P}ar_t}(a, b)$ by fixing a set of representatives for the equivalence classes of $b \times a$ partition diagrams. The special case when $a = b = 0$ implies that

$$\mathrm{End}_{\mathcal{P}ar_t}(\mathbb{1}) = \Bbbk. \tag{1.1.9}$$

In this dissertation we take a module-theoretic approach to the representation theory of $\mathcal{P}ar_t$, working in terms of the associated path algebra:

$$Par_t := \bigoplus_{a,b \in \mathbb{N}} \mathrm{Hom}_{\mathcal{P}ar_t}(a, b)$$

Our analysis takes advantage of the fact that Par_t has a *split triangular decomposition*, following the definition in [BS, Rmk. 5.32]. This is discussed in §2.8 and the key principle is that for any partition diagram, there is an equivalent partition diagram which is a composition of various "merges" at the bottom, then crossings, then "splits" at the top as in (1.1.8). It follows that the category Par_t-Mod$_{\mathrm{lfd}}$ of locally finite-dimensional Par_t-modules is an upper finite highest weight category in the sense of [BS]. In particular, there are *standard* and *costandard* modules, indecomposable projective modules have standard flags satisfying BGG reciprocity, and so on. In fact, $\mathcal{P}ar_t$ is a *monoidal triangular category* in the sense of Sam and Snowden [SS22] who have also developed these ideas in a general setting.

With standard modules in hand, our story follows the ideas presented by Okounkov and Vershik in their approach to the representation theory of the symmetric groups [OV05]. In the main part of the thesis, we study an induction functor D induced by the monoidal operation $|\star ?$ on $\mathcal{P}ar_t$ given by "multiplication with the generating object". This plays analogy to the induction functors in the setting of symmetric groups $\mathrm{ind}_{S_n}^{S_{n+1}}(?) = \Bbbk S_{n+1} \otimes ? : \mathrm{Rep}(S_n) \to \mathrm{Rep}(S_{n+1})$. We introduce a new monoidal category, the *affine partition category*, in order to better understand the *Enyang-Jucys-Murphy* elements of [Eny13]. These let us split the functor D into indecomposable constituents. Together with some facts about a new family of central elements of $\mathcal{P}ar_t$, this affords a new and self-contained analysis of the block structure of $\mathcal{P}ar_t$. This was originally worked out by Comes and Ostrik, although their proof ultimately appealed to results about the partition algebras due to Martin [CO11, § 6.3].

The final chapters study a well-known restriction functor $F^t_{t-1} : \underline{\mathrm{Rep}}(S_t) \to \underline{\mathrm{Rep}}(S_{t-1})$. After reinterpreting this in terms of modules over path algebras to obtain a functor $R^t_{t-1} : Par_{t-1}\text{-Mod} \to Par_t\text{-Mod}$ in the other direction, we prove a conjecture of Comes and Ostrik. Namely, restriction gives an equivalence between the principal blocks of the corresponding categories [CO11, Conj. 6.8]. Finally, we study the behavior of R^t_{t-1} on tilting modules. This allows us to apply a recent theorem from [BEO23] to identify the abelian envelope of $\underline{\mathrm{Rep}}(S_t)$ with the Ringel dual of $Par_t\text{-Mod}_{\mathrm{lfd}}$ in the sense of [BS].

1.2 Organization and main results

Now we go into more detail regarding the layout of the thesis and formulate some of the main results more precisely. Chapter II is dedicated mostly to introducing the general techniques that will be used in the sequel. We set up a dictionary to pass from the framework of $\Bbbk$-linear categories and functors to that of modules over their associated path algebras. Using the theory of highest weights granted by the split triangular decomposition of Par_t, we quickly re-prove a classic result of Deligne which was also proven by Comes and Ostrik using different methods: the isomorphism classes of irreducible, indecomposable projective, indecomposable injective, standard, and costandard modules are all indexed by the set of integer partitions, $\mathcal{P}$. Those modules corresponding to a partition $\lambda \in \mathcal{P}$ are denoted by $L(\lambda), P(\lambda), I(\lambda), \Delta(\lambda)$, and $\nabla(\lambda)$, respectively.

The main content of this disseration starts with chapter III. We pass from the functor $|\star? : \mathcal{P}ar_t \to \mathcal{P}ar_t$ to the induction functor $D : Par_t\text{-Mod}_{\mathrm{lfd}} \to Par_t\text{-Mod}_{\mathrm{lfd}}$, and then show that it respects modules with a filtration by standard objects. Specifically, we establish the following combinatorial rule for determining the sections of $D\Delta(\lambda)$. Comes and Ostrik produced a similar result for generic parameter values in their proof to classify blocks [CO11, Prop. 5.15].

Theorem (See Th. 3.9.1). *For $\lambda \in \mathcal{P}$, there is a filtration $0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 = D\Delta(\lambda)$ such that*

$$V_3/V_2 \cong \bigoplus_{a \in \mathrm{add}(\lambda)} \Delta\left(\lambda + \boxed{a}\right),$$

$$V_2/V_1 \cong \Delta(\lambda) \oplus \bigoplus_{b \in \mathrm{rem}(\lambda)} \bigoplus_{a \in \mathrm{add}(\lambda - \boxed{b})} \Delta\left((\lambda - \boxed{b}) + \boxed{a}\right),$$

$$V_1/V_0 \cong \bigoplus_{b \in \mathrm{rem}(\lambda)} \Delta\left(\lambda - \boxed{b}\right),$$

where $\mathrm{add}(\lambda)$ *is the set of addable boxes to λ and* $\mathrm{rem}(\lambda)$ *is the set removable boxes from λ.*

This chapter also discusses the *affine partition category*, which we construct as a certain subcategory of Khovanov's Heisenberg category. A key feature of $\mathcal{AP}ar$ is its two new generating morphisms: the 'left dot' and the 'right dot' . It turns out $\mathcal{P}ar_t$ is a "cyclotomic" quotient of $\mathcal{AP}ar$ and the images of the left and right dots under this homomorphism $\mathcal{AP}ar \to \mathcal{P}ar_t$ produce elements in the partition category which are closely related to *Enyang-Jucys-Murphy* elements for the partition algebras. Much the same as how Jucys-Murphy elements for the symmetric group algebras provide endomorphisms (i.e., natural transformations) of the induction functors $\mathrm{ind}_{S_n}^{S_{n+1}}$, the Enyang-Jucys-Murphy elements provide endomorphisms of D. From this we find a functorial decomposition $D = \bigoplus_{a,b \in \Bbbk} D_{b|a}$ into summands. Each $D_{b|a}$ also respects standardly-filtered modules, with a more refined combinatorial rule than the one provided above.

We also construct a family of central elements for the partition algebra in order to study a Harish-Chandra homomorphism $Z(Par_t) \to Z(Sym)$ from the center of Par_t to the center of its Cartan subalgebra. This Harish-Chandra homomorphism allows us to recover Deligne's result that Par_t is semisimple if and only if $t \notin \mathbb{N}$. In the non-semisimple case, we use the functors $D_{b|a}$ to rediscover the block structure of Par_t-$\mathrm{Mod}_{\mathrm{lfd}}$. The classification of blocks is summarized below, but more detailed statements lie throughout §3.10.

Theorem (See Th. 3.10.5). *The locally unital algebra Par_t is semisimple if and only if $t \notin \mathbb{N}$. In the case $t \in \mathbb{N}$, the non-simple blocks of Par_t are in bijection with isomorphism classes of irreducibles in $\Bbbk S_t$-$\mathrm{Mod}_{\mathrm{fd}}$. All of the non-simple blocks are Morita equivalent, having infinitely many isomorphism classes of irreducible modules parametrized by $\mathbb{N}$. If $L(n), \Delta(n)$, and $P(n)$ are the nth irreducible, standard, and indecomposable projective of some non-simple block, then:*

(i) For each $n \geq 0$, $\Delta(n)$ is of length two with head $L(n)$ and socle $L(n+1)$.

(ii) $P(0)$ is isomorphic to $\Delta(0)$, while for $n \geq 1$ the module $P(n)$ has a two step Δ-flag with top section $\Delta(n)$ and bottom section $\Delta(n-1)$.

(iii) For each $n \geq 1$, $P(n)$ is self-dual with irreducible head and socle isomorphic to $L(n)$ and completely reducible heart $\operatorname{rad} P(n)/\operatorname{soc} P(n) \cong L(n-1) \oplus L(n+1)$.

The theorem immediately implies that each non-simple block is equivalent to the category of finite-dimensional modules over the algebra defined by the following quiver with relations:

$$0 \underset{y_0}{\overset{x_0}{\rightleftarrows}} 1 \underset{y_1}{\overset{x_1}{\rightleftarrows}} 2 \underset{y_2}{\overset{x_2}{\rightleftarrows}} 3 \cdots, \qquad y_0 x_0 = 0, x_{i+1}x_i = y_i y_{i+1} = x_i y_i - y_{i+1}x_{i+1} = 0.$$

This equivalence was already proven in [CO11, Th.6.4], though the approach here is independent of the results of Martin.

In chapter IV, we turn our attention to the restriction functor R^t_{t-1} : Par_{t-1}-$\mathrm{Mod}_{\mathrm{lfd}} \to Par_t$-$\mathrm{Mod}_{\mathrm{lfd}}$. Since t is now changing, we use $\Delta_t(\lambda)$ to mean the standard Par_t-module corresponding to λ, and similarly for other families mentioned above. The main result in this chapter describes the effect of R^t_{t-1} on standard modules:

Theorem (See Th. 4.3.10). *For $\lambda \in \mathcal{P}$, there is a short exact sequence*

$$0 \to \Delta_t(\lambda) \to R^t_{t-1}\Delta_{t-1}(\lambda) \to \bigoplus_{a \in \mathrm{add}(\lambda)} \Delta_t\left(\lambda + \boxed{a}\right) \to 0$$

From this point, the remainder of the dissertation focuses on the consequenes of this filtration. The first is an affirmative answer to the conjecture of Comes and Ostrik involving the principal blocks of Par_t-$\mathrm{Mod}_{\mathrm{lfd}}$ and Par_{t-1}-$\mathrm{Mod}_{\mathrm{lfd}}$ — the indecomposable subcategories containing the irreducible module $L(\varnothing)$ corresponding to the empty partition.

Theorem (See Th. 4.4.4). *The restriction functor R^t_{t-1} induces an equivalence between the principal blocks of Par_{t-1}-$\mathrm{Mod}_{\mathrm{lfd}}$ and Par_t-$\mathrm{Mod}_{\mathrm{lfd}}$.*

Finally, chapter V gives an alternate description of the abelian envelope of $\underline{\mathrm{Rep}}(S_t)$. This was originally constructed in [CO14] by considering the heart of a certain t-structure inside the homotopy category of $\underline{\mathrm{Rep}}(S_t)$, but more general constructions have appeared recently (eg. as in [BEO23, Cou21, HS22]). In the case of Par_t-$\mathrm{Mod}_{\mathrm{lfd}}$, we show that the tilting modules familiar in highest weight theory are the same as the splitting objects of [BEO23, § 2.2]. Proceeding to check the critera provided in [BEO23, Thm. 2.42] regarding a characterization of abelian envelopes, this gives our last result:

Theorem (See Th. 5.2.3). *The Ringel dual of Par_t-$\mathrm{Mod}_{\mathrm{lfd}}$ is the abelian envelope of $\underline{Rep}(S_t)$.*

CHAPTER II

FOUNDATIONS

Fix an algebraically closed field $\Bbbk$ of characteristic 0. Many of the definitions in this chapter make sense over any field, but several results require these assumptions so we fix them now for simplicity. We also establish the standing assumption that all categories will be $\Bbbk$-linear (and small) and algebras will be over $\Bbbk$, unless specified otherwise. Functors between these categories are also assumed to be $\Bbbk$-linear. This chapter summarizes the general background and theoretical techniques to be used throughout the rest of the dissertation. This includes a recollection of $\Bbbk$-linear categories and their path algebras, monoidal categories, and finally the triangular decomposition of the partition category. In the final section, we use the machinery of this triangular decomposition to provide a quick classification of irreducible modules for the partition category, first proven by Deligne in his seminal paper [Del07]. Most sections of this chapter (except §§2.1 and 2.4) contain previously published co-authored material (with J. Brundan) appearing in [BV22].

2.1 Categories and their path algebras

Here we discuss a dictionary between $\Bbbk$-linear categories and locally unital $\Bbbk$-algebras. Let A be a locally unital $\Bbbk$-algebra. That is, A is a non-unital $\Bbbk$-algebra which is equipped with distinguished system of mutually orthogonal idempotents $\{e_i \in A \mid i \in I\}$ indexed by some set I so that

$$A = \bigoplus_{i,j \in I} e_j A e_i.$$

From A, one can construct a $\Bbbk$-linear category $\mathcal{C}(A)$ whose objects are given by the indexing set I and whose morphisms are given by $\mathrm{Hom}_{\mathcal{C}(A)}(i,j) := e_j A e_i$ for $i, j \in I$. The identity morphism of the object $i \in I$ is given by e_i, and composition is induced by multiplication: for any two morphisms $a \in \mathrm{Hom}_{\mathcal{C}(A)}(i,j)$ and $b \in \mathrm{Hom}_{\mathcal{C}(A)}(j,k)$, the composition $b \circ a$ is given by the product ba. Whenever A is *locally finite-dimensional*, in the sense that $e_j A e_i$ is finite-dimensional for all $i, j \in I$, then $\mathcal{C}(A)$ is locally finite, in the sense that $\mathrm{Hom}_{\mathcal{C}(A)}(i, j)$ is finite-dimensional for all $i, j \in I$.

Conversely, if $\mathcal{C}$ is a $\Bbbk$-linear category then we can construct a locally unital $\Bbbk$-algebra $A(\mathcal{C})$ which we call the *path algebra* of $\mathcal{C}$. Letting $\mathbb{O}(\mathcal{C})$ denote the object set of $\mathcal{C}$, define $A(\mathcal{C}) := \bigoplus_{X,Y \in \mathbb{O}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(X, Y)$. The distinguished idempotents of $A(\mathcal{C})$ are the identity morphisms 1_X for all $X \in \mathbb{O}(\mathcal{C})$. Given $a \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $b \in \mathrm{Hom}_{\mathcal{C}}(W, Z)$, the product ba is defined as below.

$$ba = \begin{cases} b \circ a & \text{if } Y = W \\ 0 & \text{else.} \end{cases}$$

The full algebra structure on $A(\mathcal{C})$ is obtained by linearly extending the above rule. Notice that for any $X, Y \in \mathbb{O}(\mathcal{C})$, $1_Y A(\mathcal{C}) 1_X = \mathrm{Hom}_{\mathcal{C}}(X, Y)$. If $\mathcal{C}$ is locally finite, then $A(\mathcal{C})$ is locally finite-dimensional.

Under these constructions, notice that we have $\mathcal{C}(A(\mathcal{C})) = \mathcal{C}$ and $A(\mathcal{C}(A)) = A$ for any $\Bbbk$-linear categories $\mathcal{C}$ and locally unital $\Bbbk$-algebras A. So, the data of A is fundamentally equivalent to the data contained in the original category $\mathcal{C}$. Henceforth we will drop the notation $A(\mathcal{C})$ for the path algebra of $\mathcal{C}$, opting instead to just use A (or some other symbol whenever appropriate).

Now fix a $\Bbbk$-linear category $\mathcal{C}$ with path algebra A. Given a left A-module $M = \bigoplus_{X \in \mathbb{O}(\mathcal{C})} 1_X M$, there is an associated covariant $\Bbbk$-linear functor $F : \mathcal{C} \to \mathcal{V}ec$ from $\mathcal{C}$ to the category of $\Bbbk$-vector spaces. On objects $X \in \mathbb{O}(\mathcal{C})$, $F(X) := 1_X M$. Given a morphism $a \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, the linear map $F(a) : 1_X M \to 1_Y M$ is obtained by acting on the left with a. This construction can be performed in the opposite direction too. Starting with a ($\Bbbk$-linear) covariant functor $F : \mathcal{C} \to \mathcal{V}ec$, there is an associated A-module M built as follows. Define for each $X \in \mathbb{O}(\mathcal{C})$ the vector space

$1_X M := F(X)$. Then $M := \bigoplus_{X \in \mathbb{O}(\mathcal{C})} 1_X M$. Given $a \in 1_Y A 1_X$ and $m \in 1_Z M$, the left A-module structure on M is given by

$$a \cdot m = \begin{cases} F(a)(m) & \text{if } X = Z \\ \\ 0 & \text{else.} \end{cases}$$

The data of the module M is equivalent to the data of the original functor F. That is to say, these constructions provide isomorphisms between the category A-Mod of left A-modules and the category of $\Bbbk$-linear covariant functors $\mathcal{C} \to \mathcal{V}ec$. Because of this, we will refer to the latter category as the category of *left $\mathcal{C}$-modules*, denoted $\mathcal{C}$-Mod.

Imitating the above constructions with right A-modules and contravariant functors $\mathcal{C} \to \mathcal{V}ec$ yields an isomorphism between the category of right A-modules and the category of contravariant functors from $\mathcal{C}$ to $\mathcal{V}ec$. Following as above, contravariant functors $\mathcal{C} \to \mathcal{V}ec$ will be called *right $\mathcal{C}$-modules* and the category of such modules will be denoted Mod-$\mathcal{C}$.

Restricting our attention to the category $\mathcal{V}ec_{\text{fd}}$ of finite-dimensional $\Bbbk$-vector spaces, let $\mathcal{C}$-Mod_{lfd} (resp. Mod_{lfd}-$\mathcal{C}$) be the category of covariant (resp. contravariant) functors $\mathcal{C} \to \mathcal{V}ec_{\text{fd}}$. Also let A-Mod_{lfd} (resp. Mod_{lfd}-A) be the category of *locally finite-dimensional* left (resp. right) A-modules: those A-modules M for which each subspace $1_X M$ (resp. $M 1_X$) is finite-dimensional for all $X \in \mathbb{O}(\mathcal{C})$. Then there is an equivalence $\mathcal{C}$-$\text{Mod}_{\text{lfd}} \simeq A$-$\text{Mod}_{\text{lfd}}$ (resp. Mod_{lfd}-$\mathcal{C} \simeq \text{Mod}_{\text{lfd}}$-$A$).

Finally, there is the category A-Proj (resp. Proj-A) of finitely generated projective left (resp. right) A-modules. If A is locally finite-dimensional then this is a subcategory of A-Mod_{lfd} (resp. Mod_{lfd}-A). Under the contravariant (resp. covariant) Yoneda embedding, A-Proj (resp. Proj-A) is equivalent to the *Karoubi envelope* $\text{Kar}(\mathcal{C})$ of $\mathcal{C}$ — this is the idempotent completion of the additive envelope $\text{Add}(\mathcal{C})$.

2.2 Restriction and induction along functors

Consider a functor $F : \mathcal{A} \to \mathcal{B}$ between two locally finite categories with path algebras $A = \bigoplus_{X,Y\in\mathbb{O}(\mathcal{A})} 1_Y A 1_X$ and $B = \bigoplus_{X,Y\in\mathbb{O}(\mathcal{B})} 1_Y B 1_B$. Precomposition with F allows us to naturally restrict $\mathcal{B}$-modules $G : \mathcal{B} \to \mathcal{V}ec$ to recover $\mathcal{A}$-modules $G \circ F : \mathcal{A} \to \mathcal{V}ec$. This easy construction on the categorical level translates to the algebra level to give a restriction functor

$$\mathrm{res}_F : B\text{-Mod} \to A\text{-Mod}. \tag{2.2.1}$$

To describe this functor, consider a B-module $M = \bigoplus_{X\in B} 1_X M$. Then define

$$\mathrm{res}_F M = 1_F M := \bigoplus_{Y\in\mathbb{O}(\mathcal{A})} 1_{F(Y)} M. \tag{2.2.2}$$

The left action of A on $1_F M$ is defined so that $a \in A1_Y$ acts on the summand $1_{F(Y)}M$ by the linear map $F(a)$ and zero on all other summands. For a B-module homomorphism $\phi : M \to N$, we obtain the restricted A-module morphism $\mathrm{res}_F(\phi) : 1_F(M) \to 1_F(N)$. For the summand corresponding to $Y \in \mathbb{O}(\mathcal{A})$, $\mathrm{res}_F(\phi)$ just applies the evident linear map $1_{F(Y)}M \to 1_{F(Y)}N, m \mapsto \phi(m)$. It is easy to see that res_F is an exact functor.

An analogous construction can be applied to get an exact functor between categories of right modules:

$${}_F\,\mathrm{res} : \text{Mod-}B \to \text{Mod-}A. \tag{2.2.3}$$

In particular, ${}_F\,\mathrm{res}$ sends a right B-module M to $M1_F := \bigoplus_{Y\in\mathbb{O}(\mathcal{A})} M1_{F(Y)}$. An alternate notation for these functors (e.g. used by Sam and Snowden in [SS22]) is F^* and $(F^{\mathrm{op}})^*$, respectively.

View B itself as a (B, B)-bimodule and restrict on the right to get $B1_F = \bigoplus_{X\in\mathbb{O}(\mathcal{A})} B1_{F(X)}$. This is a (B, A)-bimodule, and moreover, the functor res_F : B-Mod $\to$ A-Mod is isomorphic to $\bigoplus_{X\in\mathbb{O}(\mathcal{A})} \mathrm{Hom}_B(B1_{FX}, -)$. The tensor-Hom adjunction in the setting of module categories for locally unital algebras (eg. as in [BS, Lem. 2.2]) gives a left adjoint to res_F, namely

$$\mathrm{ind}_F := B1_F \otimes_A - : A\text{-Mod} \to B\text{-Mod}. \tag{2.2.4}$$

Having already remarked that res_F is exact, it follows that ind_F is right exact and sends projective A-modules to projective B-modules. Choosing $X \in \mathbb{O}(\mathcal{A})$ and examining $\mathrm{ind}_F A1_X$, we have

$$\mathrm{ind}_F A1_X := B1_F \otimes_A A1_X \cong B1_{F(X)}.$$

From this observation along combined with the fact that left adjoints are cocontinuous, it follows that if M is a finitely generated A-module, then $\mathrm{ind}_F M$ is a finitely generated B-module. Alternatively, for any such M there is a surjection $\bigoplus_{i\in I} A1_{X_i} \to M$ for I a finite set. Then apply the right exactness of ind_F.

Here is another construction, starting with a left restriction to get the module $1_F B = \bigoplus_{X\in\mathbb{O}(\mathcal{A})} 1_{FX} B$. This is a (A, B)-bimodule. Since res_F is also isomorphic to $1_F B \otimes_B -$, the tensor-Hom adjuction in this setting gives a right adjoint, the *coinduction along* F:

$$\mathrm{coind}_F := \bigoplus_{Y\in\mathbb{O}(B)} \mathrm{Hom}_A(1_F B1_Y, -) : A\text{-Mod} \to B\text{-Mod}$$

The functor coind_F is right exact and sends injectives to injectives, being a right adjoint to the exact functor res_F. In [SS22, §3.6], the same functors were studied under the names $F_!$ and F_*, respectively. Similar constructions can be made with right modules instead of left modules to recover functors ${}_F\,\mathrm{ind}$ and ${}_F\,\mathrm{coind}$. These are called $(F^{\mathrm{op}})_!$ and $(F^{\mathrm{op}})_*$ in [SS22], respectively.

Lemma 2.2.1. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor as above.*

(1) If $B1_F$ is a projective right A-module then ind_F *and* ${}_F\mathrm{coind}$ *are exact functors.*

(2) If $1_F B$ is a projective left A-module then ${}_F\mathrm{ind}$ and coind_F are exact functors.

Proof. This follows from the definitions of these functors and basic properties of projective modules in abelian categories. □

Lemma 2.2.2. *Let $\mathcal{A}$ be a category (with path algebra A) having additive envelope $\widehat{\mathcal{A}} = \mathrm{Add}(\mathcal{A})$ (with path algebra $\widehat{A}$). The natural inclusion $I : \mathcal{A} \to \widehat{\mathcal{A}}$ extends to an equivalence $\mathrm{Kar}(I) : \mathrm{Kar}(\mathcal{A}) \to \mathrm{Kar}(\widehat{\mathcal{A}})$. Hence*

$$\mathrm{ind}_I : A\text{-Mod} \to \widehat{A}\text{-Mod}$$

is an equivalence.

Proof. The first statement follows since $\mathrm{Kar}(I)$ is fully faithful and dense. By Yoneda, $\mathrm{ind}_I : A\text{-Proj} \to \widehat{A}\text{-Proj}$ is an equivalence. It immediately follows that $\mathrm{ind}_I : A\text{-Mod} \to \widehat{A}\text{-Mod}$ is an equivalence (see [BD17, Cor. 2.5]) But this is true by combining the first statement with the Yoneda equivalence. □

It follows in the case of Lemma 2.2.2 that the right adjoint, res_I, is also an equivalence. So the right adjoint of res_I (being coind_I) will also be an equivalence and $\mathrm{ind}_I \cong \mathrm{coind}_I$.

Suppose that $F, G : \mathcal{A} \to \mathcal{B}$ are functors. A natural transformation $\alpha : F \Rightarrow G$ induces natural transformations $\mathrm{res}_\alpha : \mathrm{res}_F \Rightarrow \mathrm{res}_G$, $\mathrm{ind}_\alpha : \mathrm{ind}_G \Rightarrow \mathrm{ind}_F$ and $\mathrm{coind}_\alpha : \mathrm{coind}_G \Rightarrow \mathrm{coind}_F$. In particular, restricting to weight spaces, we can explicitly define these natural transformations as follows. Starting with res_α, for any B-module M and any $Y \in \mathbb{O}(\mathcal{A})$, we have:

$$\mathrm{res}_\alpha(M) : 1_{F(Y)}M \to 1_{G(Y)}M, \qquad m \mapsto \alpha_Y \cdot m \tag{2.2.5}$$

For ind_F and coind_F, take any A-module N and objects $X \in \mathbb{O}(\mathcal{A})$ and $Y \in \mathbb{O}(\mathcal{B})$. Then we have for each $1_{F(X)}$-subspace

$$\mathrm{ind}_\alpha(M) : B1_{G(X)} \otimes_A 1_X N \to B1_{F(X)} \otimes_A 1_X N \tag{2.2.6}$$
$$b \otimes m \mapsto b \cdot \alpha_X \otimes m$$
$$\mathrm{coind}_\alpha(M) : \mathrm{Hom}_A(1_{G(X)} B 1_Y, N) \to \mathrm{Hom}_A(1_{F(X)} B 1_Y, N) \tag{2.2.7}$$
$$\varphi \mapsto (b \mapsto \varphi(\alpha_X \cdot b))$$

Similarly, α induces natural transformations ${}_\alpha\mathrm{res} : {}_G\mathrm{res} \Rightarrow {}_F\mathrm{res}$, ${}_\alpha\mathrm{ind} : {}_F\mathrm{ind} \Rightarrow {}_G\mathrm{ind}$ and ${}_\alpha\mathrm{coind} : {}_F\mathrm{coind} \Rightarrow {}_G\mathrm{coind}$. Assuming for simplicity[1] that $\mathcal{A} = \mathcal{B}$, so that F and G are $\Bbbk$-linear endofunctors of $\mathcal{A}$, these constructions define $\Bbbk$-linear monoidal functors

$$\mathrm{res}_* : \mathcal{E}nd_\Bbbk(\mathcal{A}) \to \mathcal{E}nd_\Bbbk(A\text{-Mod})^{\mathrm{rev}}, \qquad \mathrm{ind}_*, \mathrm{coind}_* : \mathcal{E}nd_\Bbbk(\mathcal{A})^{\mathrm{op}} \to \mathcal{E}nd_\Bbbk(A\text{-Mod}), \tag{2.2.8}$$

$${}_*\mathrm{res} : \mathcal{E}nd_\Bbbk(\mathcal{A})^{\mathrm{op}} \to \mathcal{E}nd_\Bbbk(\text{Mod-}A)^{\mathrm{rev}}, \quad {}_*\mathrm{ind}, {}_*\mathrm{coind} : \mathcal{E}nd_\Bbbk(\mathcal{A}) \to \mathcal{E}nd_\Bbbk(\text{Mod-}A). \tag{2.2.9}$$

Here, $\mathcal{E}nd_\Bbbk(\mathcal{A})$ denotes the strict $\Bbbk$-linear monoidal category of ($\Bbbk$-linear) endofunctors and natural transformations, "op" means the opposite category with the same monoidal product, and "rev" means the same category with the reversed monoidal product.

2.3 Duality

Continue with A and B be the path algebras of locally-finite categories $\mathcal{A}$ and $\mathcal{B}$, respectively. There is a contravariant functor

$$?^{\circledast} : A\text{-Mod} \to \text{Mod-}A \tag{2.3.1}$$

[1] To formulate analogs of (2.2.8) and (2.2.9) without this assumption, one needs to work in the strict 2-category of $\Bbbk$-linear categories.

taking $V = \bigoplus_{X\in\mathbb{O}_A} 1_X V$ to $V^{\circledast} := \bigoplus_{X\in\mathbb{O}_A}(1_X V)^*$, the direct sum of the linear duals of the "weight spaces" $1_X V$. The restriction of this to locally finite-dimensional modules is an equivalence, with quasi-inverse given by the restriction of the analogously-defined duality functor

$$^{\circledast}? : \text{Mod-}A \to A\text{-Mod} \tag{2.3.2}$$

in the other direction. To obtain a duality (= contravariant auto-equivalence) on $A\text{-Mod}_{\text{lfd}}$ from (2.3.1) and (2.3.2), one also needs a $\Bbbk$-linear equivalence $\sigma : \mathcal{A} \to \mathcal{A}^{\text{op}}$. Restriction along σ gives equivalences $\text{res}_\sigma : \text{Mod}_{\text{lfd}}\text{-}A \to A\text{-Mod}_{\text{lfd}}$ and ${}_\sigma\text{res} : A\text{-Mod}_{\text{lfd}} \to \text{Mod}_{\text{lfd}}\text{-}A$, hence, we obtain the duality functor

$$?^{\circledcirc} := \text{res}_\sigma \circ ?^{\circledast} = {}^{\circledast}? \circ {}_\sigma\text{res} : A\text{-Mod}_{\text{lfd}} \to A\text{-Mod}_{\text{lfd}}. \tag{2.3.3}$$

Given a functor $F : \mathcal{A} \to \mathcal{B}$, we obviously have that

$$?^{\circledast} \circ \text{res}_F \cong {}_F\text{res} \circ ?^{\circledast} \tag{2.3.4}$$

as functors from B-Mod to Mod-A. We deduce that

$$^{\circledast}? \circ {}_F\text{ind} \cong \text{coind}_F \circ {}^{\circledast}?, \qquad {}^{\circledast}? \circ {}_F\text{coind} \cong \text{ind}_F \circ {}^{\circledast}? \tag{2.3.5}$$

as functors from Mod-A to B-Mod.

2.4 Monoidal categories

By a *monoidal category*, we mean a ($\Bbbk$-linear) category $\mathcal{C}$ equipped with a bifunctor $\star : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $\Bbbk$-linear in both variables of course, along with associativity and unit contraints satisfying the pentagon and triangle axioms (see [EGNO15]). Typically in this dissertation we require that $\text{End}_{\mathcal{C}}(\mathbb{1}) \cong \Bbbk$. Usually our monoidal categories will also be *strict* monoidal, defined diagrammatically by generators and relations. Note that we use the symbol $\star$ for the monoidal product, reserving the more traditionally-used $\otimes$ for the tensor product $\otimes_\Bbbk$ over the ground field $\Bbbk$.

Given another category $\mathcal{A}$, we say that $\mathcal{A}$ is a *(strict) $\mathcal{C}$-module category* if there is a (strict) monoidal functor $\Psi : \mathcal{C} \to \mathcal{E}nd_{\Bbbk}(\mathcal{A})$ (see [EGNO15] for an alternate definition). A *left (resp. right, two-sided) tensor ideal* $\mathcal{I}$ of $\mathcal{C}$ is the data of subspaces $\mathcal{I}(X,Y) \leq \mathrm{Hom}_{\mathcal{C}}(X,Y)$ for all $X, Y \in \mathbb{O}(\mathcal{C})$, such that these subspaces are closed in the obvious sense under the monoidal product on the left (resp. right, two-sides) as well as composition either before or after with any morphism. Then $\mathcal{C}/\mathcal{I}$ is the category with the same objects as $\mathcal{C}$ and morphisms that are the quotient spaces $\mathrm{Hom}_{\mathcal{C}}(X,Y)/\mathcal{I}(X,Y)$. If $\mathcal{I}$ is a left (resp. right) tensor ideal, then $\mathcal{C}/\mathcal{I}$ is a left (resp. right) $\mathcal{C}$-module category. If $\mathcal{I}$ is a two-sided tensor ideal, then $\mathcal{C}/\mathcal{I}$ is again a monoidal category.

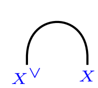
An object $X \in \mathbb{O}(C)$ has a *left dual* if there is some $X^\vee \in \mathbb{O}(C)$ with evaluation and coevaluation morphisms $\mathrm{ev}_X : X^\vee \star X \to \mathbb{1}$ and $\mathrm{coev}_X : \mathbb{1} \to X \star X^\vee$ satisfying the *zig-zag relations*. Nameley, the zig-zag relations state that the following compositions are the identity on X and $X^\vee$, respectively.

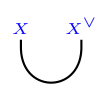
$$X \xrightarrow{\mathrm{coev}_X \star \mathrm{id}_X} X \star X^\vee \star X \xrightarrow{\mathrm{id}_X \star \mathrm{ev}_X} X \tag{2.4.1}$$

$$X^\vee \xrightarrow{\mathrm{id}_X \star \mathrm{coev}_X} X^\vee \star X \star X^\vee \xrightarrow{\mathrm{ev}_X \star \mathrm{id}_X} X^\vee \tag{2.4.2}$$

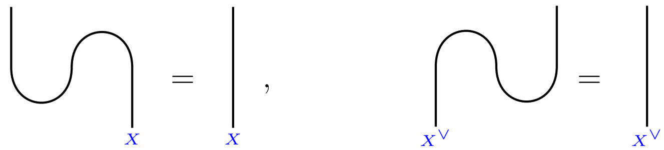
There is another notion that X has a *right dual* if there is some ${}^\vee X$ with morphisms ${}_X\mathrm{ev} : X \star {}^\vee X \to \mathbb{1}$ and ${}_X\mathrm{coev} : \mathbb{1} \to {}^\vee X \star X$ satisfying similar zig-zag identities. We say that $\mathcal{C}$ is *rigid* if every object has a left and right dual.

In adopting the diagrammatic notation for categorical calculations in this dissertation, the identity morphism of X is often represented by $|_X$. Evaluation and coevaluation are depicted as caps and cups, where the "phase change" between X and $X^\vee$ occurs at the critical point.

$\mathrm{ev}_X =$ 

, $\mathrm{coev}_X =$ 

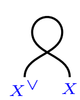
The zig-zag equations 2.4.1 turn into the following, explaining the terminology.

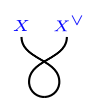


(2.4.3)

A related notion is that of a *pivotal* category; this is when $\mathcal{C}$ is a rigid monoidal category equipped with the data of isomorphisms $\alpha_X : X \xrightarrow{\sim} (X^\vee)^\vee$ natural in $X \in \mathbb{O}(\mathcal{C})$. Moreover, the morphisms α_X must respect the monoidal structure of $\mathcal{C}$, in that $\alpha_{X\star Y} = \alpha_X \star \alpha_Y$ for all $X, Y \in \mathbb{O}(\mathcal{C})$. For a pivotal category, the notions of left and right duals coincide. In such a category, assuming also that $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) \cong \Bbbk$, the *dimension* of an object X is defined by the value of the morphism $\mathrm{ev}_{X^\vee} \circ (\alpha_X \otimes 1_{X^\vee}) \circ \mathrm{coev}_X \in \Bbbk$.

By a *symmetric* monoidal category, we mean a monoidal category equipped with isomorphisms $\sigma_{X,Y} : X \star Y \to Y \star X$ satisfying the hexagon relation, and moreover, $\sigma_{Y,X} \circ \sigma_{X,Y} = \mathrm{id}_{X\star Y}$ for all $X, Y \in \mathbb{O}(C)$. See [EGNO15] for the full definition. A rigid monoidal category $\mathcal{C}$ which is also symmetric can readily be equipped with a pivotal structure. Such a structure can be described by identifying, for any $X \in \mathbb{O}(\mathcal{C})$, its left and right duals. In order to perform this identification, it is enough to prescribe the evaluation map $X \star X^\vee \to \mathbb{1}$ and coevaluation map $\mathbb{1} \to X^\vee \star X$. They are diagrammatically defined as follows:

$\mathrm{ev}_{X^\vee} =$ 

, $\qquad$ $\mathrm{coev}_{X^\vee} =$ 

In other words, we have $\mathrm{ev}_{X^\vee} = \mathrm{ev}_X \circ \sigma_{X,X^\vee}$ and $\mathrm{coev}_{X^\vee} = \sigma_{X^\vee,X} \circ \mathrm{coev}_X$. In this case, the dimension of X is the value of the morphism $\mathbb{1} \to \mathbb{1} : \mathrm{ev}_X \circ \sigma_{X,X^\vee} \circ \mathrm{coev}_X$.

2.5 Induction product

Let $\mathcal{C}$ be a strict monoidal category with path algebra C. The monoidal product $\star$ on $\mathcal{C}$ extends canonically to the Karoubian envelope, $\mathrm{Kar}(\mathcal{C})$. This section describes a monoidal product $\circledast$ which makes C-Mod into a (no longer strict) monoidal category.

It turns out that C-Proj is closed under $\circledast$ and this product agrees with the extension of $\star$ to $\mathrm{Kar}(\mathcal{C})$ through the contravariant Yoneda equivalence. The algebraist's formulation of the definition for $\circledast$ is given in the next paragraph; see also [SS15, (2.1.14)], [SS22, §3.10], where $\circledast$ is called *Day convolution*. Using $\circledast$, we can make the split Grothendieck group $K_0(C)$ of the category C-Proj into a ring with multiplication

$$[P][Q] := [P \circledast Q]. \tag{2.5.1}$$

Its identity element is the isomorphism class of the distinguished projective module $C1_{\mathbb{1}}$, where $\mathbb{1} \in \mathbb{O}(\mathcal{C})$ is the unit object.

Here is the detailed definition of $\circledast$. Let $\mathcal{C} \boxtimes \mathcal{C}$ be the $\Bbbk$-linearization of the Cartesian product $\mathcal{C} \times \mathcal{C}$. The objects in $\mathcal{C} \boxtimes \mathcal{C}$ are pairs $(X_1, X_2) \in \mathbb{O}(\mathcal{C}) \times \mathbb{O}(\mathcal{C})$, and the morphism space from (X_1, X_2) to (Y_1, Y_2) is $\mathrm{Hom}_{\mathcal{C}}(X_1, Y_1) \otimes \mathrm{Hom}_{\mathcal{C}}(X_2, Y_2)$. We denote its path algebra by

$$C \boxtimes C = \bigoplus_{X_1, X_2, Y_1, Y_2 \in \mathbb{O}(\mathcal{C})} 1_{Y_1} C 1_{X_1} \otimes 1_{Y_2} C 1_{X_2}.$$

Multiplication in $C \boxtimes C$ is the obvious "tensor-wise" product just like for a tensor product of algebras. If C is locally finite, so too is $C \boxtimes C$. Given $V_1, V_2 \in C$-Mod, let

$$V_1 \boxtimes V_2 = \bigoplus_{X_1, X_2 \in \mathbb{O}(\mathcal{C})} 1_{X_1} V_1 \otimes 1_{X_2} V_2$$

be their tensor product over $\Bbbk$ viewed as a left $C \boxtimes C$-module in the obvious way. In fact, this defines a functor $\boxtimes : C\text{-Mod} \boxtimes C\text{-Mod} \to C \boxtimes C\text{-Mod}$. The monoidal product on $\mathcal{C}$ is a functor $\star : \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$. Let

$$C1_\star = \bigoplus_{X_1, X_2 \in \mathbb{O}(\mathcal{C})} C1_{X_1 \star X_2}$$

be the $(C, C \boxtimes C)$-bimodule obtained by restricting the right C-module C along this functor. Induction along $\star$, that is, the functor $\mathrm{ind}_\star = C1_\star \otimes_{C \boxtimes C} : C \boxtimes C\text{-Mod} \to C\text{-Mod}$ from (2.2.4), is left adjoint to the restriction functor $\mathrm{res}_\star$ from (2.2.1). Then the induction product is the composition

$$\circledast := \mathrm{ind}_\star \circ \boxtimes : C\text{-Mod} \boxtimes C\text{-Mod} \to C\text{-Mod}. \tag{2.5.2}$$

Thus, for $V_1, V_2 \in C$-Mod, we have that $V_1 \circledast V_2 = C1_\star \otimes_{C \boxtimes C} (V_1 \boxtimes V_2)$. Associativity of $\circledast$ (up to natural isomorphism) follows from "transitivity of induction", i.e., the associativity of tensor products of modules over locally unital algebras. We obviously have that

$$C1_{X_1} \circledast C1_{X_2} \cong C1_{X_1 \star X_2} \tag{2.5.3}$$

for $X_1, X_2 \in \mathbb{O}(\mathcal{C})$. This justifies our earlier assertion that $\circledast$ extends the monoidal product $\star$ on $\mathrm{Kar}(\mathcal{C})$. It also follows that $V_1 \circledast V_2$ is finitely generated if both V_1 and V_2 are finitely generated. The induction product $\circledast$ is right exact in both arguments, but in general it is not left exact.

Lemma 2.5.1. *If $C1_\star$ is a projective right $C \boxtimes C$-module then the induction product $\circledast$ is biexact.*

Proof. This follows from Lemma 2.2.1. □

Finally suppose that $\mathcal{C}$ is a strict symmetric monoidal category, so that there is given a symmetric braiding σ. From this, we obtain a braiding ind_σ making C-Mod into a $\Bbbk$-linear symmetric monoidal category too.

Remark 2.5.2. There is a second convolution product $\underline{\circledast}$ which we call the *coinduction product*. This is defined by replacing $\mathrm{ind}_\star$ with $\mathrm{coind}_\star$ in (2.5.2). It is easy to understand on injective rather than projective modules. It will not often be used subsequently, but note that the induction and coinduction products are interchanged by duality.

2.6 Projective functors

Suppose that $\mathcal{C}$ is a strict monoidal category and $\mathcal{A}$ is a strict $\mathcal{C}$-module category, denoting their path algebras by C and A as usual. The data of the functor $\Psi : \mathcal{C} \to \mathcal{E}nd_{\Bbbk}(\mathcal{A})$ is equivalent to the data of a strictly associative and unital $\Bbbk$-linear monoidal functor $\star : \mathcal{C} \boxtimes \mathcal{A} \to \mathcal{A}$. For $f \in \mathrm{Hom}_{\mathcal{C}}(X, X')$, we sometimes denote the evaluation of the natural transformation $\Psi(f)$ on $Y \in \mathbb{O}(\mathcal{A})$ simply by $f_Y : X \star Y \to X' \star Y$.

The definition of the induction product $\star$ from (2.5.2) extends naturally to this setting, thereby defining a functor

$$\circledast := \mathrm{ind}_{\star} \circ \boxtimes : C\text{-Mod} \boxtimes A\text{-Mod} \to A\text{-Mod} \tag{2.6.1}$$

which makes A-Mod into a (no longer strict) C-Mod-module category. For objects $X \in \mathbb{O}(\mathcal{C})$ and $Y \in \mathbb{O}(\mathcal{A})$, we have that

$$C1_X \circledast A1_Y \cong A1_{X\star Y}, \tag{2.6.2}$$

i.e., $\circledast$ extends $\star : \mathcal{C} \boxtimes \mathcal{A} \to \mathcal{A}$. Using $\circledast$ to define the action as in (2.5.1), the split Grothendieck group $K_0(A)$ becomes a left module over the split Grothendieck ring $K_0(C)$.

Now fix $X \in \mathbb{O}(\mathcal{C})$ and consider the functor $X\star : \mathcal{A} \to \mathcal{A}$. There is an adjoint pair of endofunctors $(\mathrm{ind}_{X\star}, \mathrm{res}_{X\star})$ of A-Mod defined by induction and restriction along $X\star$:

$$\mathrm{ind}_{X\star} := A1_{X\star} \otimes_A \qquad \text{where} \qquad A1_{X\star} := \bigoplus_{Y \in \mathbb{O}_A} A1_{X\star Y}, \tag{2.6.3}$$

$$\mathrm{res}_{X\star} := 1_{X\star}A \otimes_A \qquad \text{where} \qquad 1_{X\star}A := \bigoplus_{Y \in \mathbb{O}_A} 1_{X\star Y}A. \tag{2.6.4}$$

The general properties discussed earlier give that $\mathrm{res}_{X\star}$ is exact, and $\mathrm{ind}_{X\star}$ is right exact and sends (finitely generated) projectives to (finitely generated) projectives. Thus, $\mathrm{ind}_{X\star}$ restricts to a well-defined functor $\mathrm{ind}_{X\star} : A\text{-Proj} \to A\text{-Proj}$. Note also that

$$\mathrm{ind}_{X\star}(A1_Y) \cong A1_{X\star Y} \tag{2.6.5}$$

for all $Y \in \mathbb{O}(\mathcal{A})$. One can also interpret $\mathrm{ind}_{X\star}$ as a special induction product, thanks to the following lemma.

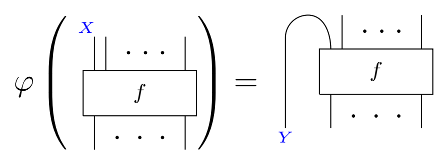
Lemma 2.6.1. *For any $X \in \mathbb{O}(\mathcal{C})$, we have that $\mathrm{ind}_{X\star} \cong C1_X \circledast$.*

Proof. This follows from the chain of isomorphisms

$$\begin{aligned} A1_{X\star} \otimes_A V &\cong (A1_\star \otimes_{C\boxtimes A} (C1_X \boxtimes A)) \otimes_A V \\ &\cong A1_\star \otimes_{C\boxtimes A} ((C1_X \boxtimes A) \otimes_A V) \\ &\cong A1_\star \otimes_{C\boxtimes A} (C1_X \boxtimes V) = C1_X \circledast V \end{aligned}$$

for $V \in A$-Mod. □

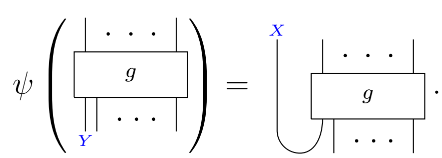
Lemma 2.6.2. *If X has a left dual Y in $\mathcal{C}$ then there is an isomorphism $\phi : 1_{X\star}A \xrightarrow{\sim} A1_{Y\star}$ of (A, A)-bimodules given explicitly by*



(2.6.6)

Hence, the functors $\mathrm{res}_{X\star}$ and $\mathrm{ind}_{Y\star}$ are isomorphic.

Proof. It is easily checked that ϕ is a bimodule homomorphism. It is an isomorphism because it has a two-sided inverse ψ defined by



□

Corollary 2.6.3. *If X has a left dual Y in $\mathcal{C}$ then $(\mathrm{ind}_{X\star}, \mathrm{ind}_{Y\star})$ and $(\mathrm{res}_{X\star}, \mathrm{res}_{Y\star})$ are adjoint pairs of functors.*

From the corollary, we deduce that if X is rigid, then both of the functors $\mathrm{ind}_{X\star}$ and $\mathrm{res}_{X\star}$ have both a right and a left adjoint. Moreover, as discussed earlier, both of these functors are exact and they preserve finitely generated projectives. We will refer to finite direct sums of direct summands of endofunctors of A-Mod of this sort as *projective functors*.

2.7 The symmetric category

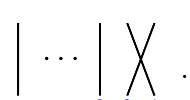
It is worth recalling the following basic example. The *symmetric category* $\mathcal{S}ym$ is the free strict symmetric monoidal category on one object. Our analysis of the partition category in chapter III is largely motivated by the Okounkov-Vershik approach to the representation theory of symmetric groups which we review here [OV05]. In string diagrams, we denote this generating object simply by $|$; then an arbitrary object is the monoidal product $|^{\star n}$ for some $n \geq 0$. Morphisms in $\mathcal{S}ym$ are generated by a single morphism depicted by the crossing

$$: | \star | \to | \star | \tag{2.7.1}$$

subject to the relations

(2.7.2)

Sometimes it is convenient to identify objects in $\mathcal{S}ym$ with natural numbers, so that the object set $\{|^{\star n} \,|\, n \in \mathbb{N}\}$ of $\mathcal{S}ym$ is identified with $\mathbb{N}$. For $m, n \geq 0$, the morphism space $\mathrm{Hom}_{\mathcal{S}ym}(n, m)$ is $\{0\}$ if $m \neq n$, while if $m = n$ it consists of $\Bbbk$-linear combinations of string diagrams representing permutations in the symmetric group S_n, i.e., we have that $\mathrm{End}_{\mathcal{S}ym}(n) = \Bbbk S_n$. Note our general convention here is to number strings by $1, \ldots, n$ from *right to left*, so that the transposition $(1\,2) \in S_n$ is represented by the string diagram



Let Sym be the path algebra of $\mathcal{S}\mathit{ym}$. Thus, we have that

$$Sym = \bigoplus_{n \geq 0} \Bbbk S_n. \tag{2.7.3}$$

Since $\Bbbk$ is of characteristic zero, we deduce from Maschke's theorem that Sym is a *semisimple* locally unital algebra. In this case, the induction product $\circledast$ making Sym-Mod into a monoidal category is nothing more than the usual induction product on representations of the symmetric groups: we have that

$$V \circledast W = \operatorname{ind}_{S_n \times S_m}^{S_{n+m}} (V \boxtimes W)$$

for $V \in \Bbbk S_n$-Mod and $W \in \Bbbk S_m$-Mod viewed as Sym-modules using (2.7.3). In fact, the induction product $\circledast$ and the coinduction product $\underline{\circledast}$ on Sym-Mod are isomorphic as $\operatorname{ind}_{S_n \times S_m}^{S_{n+m}} \cong \operatorname{coind}_{S_n \times S_m}^{S_{n+m}}$ (as always for finite groups).

Recall that the irreducible $\Bbbk S_n$-modules are the *Specht modules* $S(\lambda)$ parametrized by the set $\mathcal{P}_n$ of partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ of n. Hence, the irreducible Sym-modules are the Specht modules $S(\lambda)$ parametrized by all partitions $\lambda \in \mathcal{P} = \bigsqcup_{n \geq 0} \mathcal{P}_n$. We sometimes write $|\lambda|$ for the size $\lambda_1 + \lambda_2 + \cdots$ of a partition $\lambda \in \mathcal{P}$, and $\ell(\lambda)$ for its length, that is, the number of non-zero parts. We will often identify $\lambda \in \mathcal{P}$ with its Young diagram. For example, the partition $(5, 3^2, 2)$ is identified with

$$\begin{array}{|c|c|c|c|c|}
\hline
0 & 1 & 2 & 3 & 4 \\
\hline
-1 & 0 & 1 \\
\cline{1-3}
-2 & -1 & 0 \\
\cline{1-3}
-3 & -2 \\
\cline{1-2}
\end{array}$$

The *content* of the node in row i and column j of a Young diagram is the integer $c = j - i$ (as above). Let $\operatorname{add}(\lambda)$ be the set consisting of the contents of the *addable nodes* of λ, that is, the places in the Young diagram where a node can be added to the diagram to obtain a new Young diagram. Similarly, let $\operatorname{rem}(\lambda)$ be the set of contents of the *removable nodes* of λ, that is, the places in the Young diagram where a node can be removed from the diagram to obtain a new Young diagram. Note that all of the addable and removable nodes of a Young diagram are of different contents (another of the benefits of working in characteristic zero). For $a \in \mathrm{add}(\lambda)$, let $\lambda + \boxed{a}$ be the partition obtained by adding the unique addable node of content a to the diagram. For $b \in \mathrm{rem}(\lambda)$, let $\lambda - \boxed{b}$ be the partition obtained by removing the unique removable node of content b from the diagram.

The combinatorial notions just introduced arise naturally on considering *branching rules* for the symmetric group. In our setup, the sums over all $n \geq 0$ of the usual restriction and induction functors $\mathrm{res}_{S_n}^{S_{n+1}}$ and $\mathrm{ind}_{S_n}^{S_{n+1}} = \Bbbk S_{n+1} \otimes_{\Bbbk S_n}$ are isomorphic to the functors

$$F := \mathrm{res}_{|\,\star} : \mathcal{S}ym\text{-}\mathrm{Mod}_{\mathrm{fd}} \to \mathcal{S}ym\text{-}\mathrm{Mod}_{\mathrm{fd}}, \quad E := \mathrm{ind}_{|\,\star} : \mathcal{S}ym\text{-}\mathrm{Mod}_{\mathrm{fd}} \to \mathcal{S}ym\text{-}\mathrm{Mod}_{\mathrm{fd}}, \tag{2.7.4}$$

notation as in (2.6.3) and (2.6.4). This follows because the functor

$$|\,\star : \mathcal{S}ym \to \mathcal{S}ym, \qquad \begin{matrix} |\cdots| \\ \boxed{\;g\;} \\ |\cdots| \end{matrix} \mapsto \Bigg| \begin{matrix} |\cdots| \\ \boxed{\;g\;} \\ |\cdots| \end{matrix}. \tag{2.7.5}$$

coincides with the natural inclusion $S_n \hookrightarrow S_{n+1}$ on permutations $g \in S_n \subset \mathrm{End}_{\mathcal{S}ym}(n)$. The canonical adjunction makes (E, F) into an adjoint pair of functors. In fact, these functors are are *biadjoint*, i.e., there is also an adjunction making (F, E) into an adjoint pair. The effect of the functors F and E on the Specht module $S(\lambda)$ is well known: we have that

$$FS(\lambda) \cong \bigoplus_{b \in \mathrm{rem}(\lambda)} S(\lambda - \boxed{b}), \qquad ES(\lambda) \cong \bigoplus_{a \in \mathrm{add}(\lambda)} S(\lambda + \boxed{a}). \tag{2.7.6}$$

We finally recall a bit about the Jucys-Murphy elements in $\mathcal{S}ym$. One natural way to obtain these is to start from the *affine symmetric category* $\mathcal{AS}ym$, which is the strict $\Bbbk$-linear monoidal category obtained from $\mathcal{S}ym$ by adjoining an extra generator

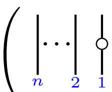
 subject to the equivalent relations

(2.7.7)

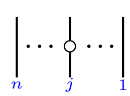
The path algebra $ASym$ is isomorphic to $\bigoplus_{n\geq 0} AH_n$ where AH_n is the nth *degenerate affine Hecke algebra*. There is an obvious faithful strict $\Bbbk$-linear monoidal functor $i : \mathcal{Sym} \to \mathcal{ASym}$. There is also a unique (non-monoidal) full $\Bbbk$-linear functor

$$p : \mathcal{ASym} \to \mathcal{Sym} \tag{2.7.8}$$

such that $p \circ i = \mathrm{Id}_{\mathcal{Sym}}$ and

$p\Bigg($  $\Bigg) = 0$ (2.7.9)

for all $n \geq 1$. For $1 \leq j \leq n$, the jth Jucys-Murphy element of the symmetric group S_n is

$x_j = p\Bigg($  $\Bigg) = \sum_{i=1}^{j-1} (i\ j) \in \Bbbk S_n,$ (2.7.10)

i.e., it is the sum of the transpositions "ending" in j. Whenever we use this notation, it should be clear from context exactly which symmetric group we have in mind. Note $x_1 = 0$ always. We may also occasionally write x_0, which should be interpreted as zero by convention.

The Jucys-Murphy elements $x_1, \ldots, x_n$ generate a commutative subalgebra of $\Bbbk S_n$ known as the *Gelfand-Tsetlin subalgebra*. As concisely explained by [OV05], for $\lambda \in \mathcal{P}_n$, each Jucys-Murphy element acts diagonalizably on the Specht module $S(\lambda)$, and the *Gelfand-Tsetlin character* of $S(\lambda)$ recording the dimensions of the simultaneous generalized eigenspaces of $x_1, \ldots, x_n$ may be obtained from the contents of standard λ-tableaux. Indeed, Young's orthonormal basis $\{v_{\mathrm{T}}\}$ for $S(\lambda)$ indexed by standard λ-tableaux T is a basis of simultaneous eigenvectors for $x_1, \ldots, x_n$, with x_j

acting on v_{T} as the content $\mathrm{cont}_j(\mathrm{T})$ of the node labelled by j in T. We will assume the reader is familiar with these ideas without giving any further explanation.

The functor p induces an isomorphism $\mathcal{ASym}/\mathcal{I} \xrightarrow{\sim} \mathcal{Sym}$ where $\mathcal{I}$ is the left tensor ideal of $\mathcal{ASym}$ generated by the morphism $\mid\!\!\!\bullet$. It follows that $\mathcal{Sym}$ is a strict left $\mathcal{ASym}$-module category. The functors E and F from (2.7.4) are also the induction and restriction functors $\mathrm{ind}_{\mid\star}$ and $\mathrm{res}_{\mid\star}$ defined using this categorical action of $\mathcal{ASym}$ on $\mathcal{Sym}$. The advantage of passing from $\mathcal{Sym}$ to $\mathcal{ASym}$ here is that the object $\mid$ of $\mathcal{ASym}$ has the endomorphism defined by the dot, giving us a natural transformation

$$\alpha := \mid\!\!\!\bullet\,\star \;:\; \mid\star \Rightarrow \mid\star.$$

Applying the general construction from (2.2.8) to this, we obtain endomorphisms

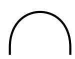
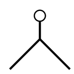
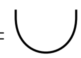
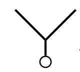
$$x := \mathrm{res}_\alpha : F \Rightarrow F, \qquad\qquad x^\vee := \mathrm{ind}_\alpha : E \Rightarrow E. \tag{2.7.11}$$

Explicitly, on a $\Bbbk S_n$-module V, x_V is the endomorphism of $FV = \mathrm{res}^{S_n}_{S_{n-1}} V$ defined by multiplying on the left by $x_n \in \Bbbk S_n$, while $x_V^\vee$ is the endomorphism of $EV = \Bbbk S_{n+1} \otimes_{\Bbbk S_n} V$ defined by multiplying the bimodule $\Bbbk S_{n+1}$ on the right by $x_{n+1} \in \Bbbk S_{n+1}$. For $c \in \Bbbk$, let F_c and E_c be the c eigenspaces of $x : F \Rightarrow F$ and $x^\vee : E \Rightarrow E$, respectively — x and $x^\vee$ are diagonalizable as $\mathrm{char}(\Bbbk) = 0$. Since $x^\vee$ is the mate of x and E and F are biadjoint, it follows that E_c and F_c are biadjoint endofunctors of $\mathcal{Sym}\text{-}\mathrm{Mod}_{\mathrm{fd}}$ for each $c \in \Bbbk$. The description of Gelfand-Tsetlin characters of Specht modules from the previous paragraph is equivalent to the assertion that the functors E_a and F_b take the Specht module $S(\lambda)$ to exactly the summands $S(\lambda + \boxed{a})$ and $S(\lambda - \boxed{b})$ in (2.7.6), or to zero if $a \notin \mathrm{add}(\lambda)$ or $b \notin \mathrm{rem}(\lambda)$, respectively. It follows that

$$F = \bigoplus_{b \in \mathbb{Z}} F_b, \qquad\qquad E = \bigoplus_{a \in \mathbb{Z}} E_a. \tag{2.7.12}$$

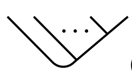
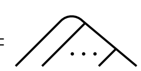
2.8 Triangular decomposition of the partition category

In this section we return to the partition category defined in the introduction, expanding on its triangular structure as illustrated by the discussion surrounding (1.1.7) and (1.1.8). As a preliminary, observe that $\mathcal{P}ar_t$ is a rigid monoidal category. The justification of this fact requires only to specify the evaluation $\mathrm{ev} : |\star| \to \mathbb{1}$ and coevaluation $\mathrm{coev} : \mathbb{1} \to |\star|$ morphisms for the single generating object of $\mathcal{P}ar_t$. They are defined below in terms of the generating morphisms provided in Definition 1.1.1.

ev =  :=  , coev =  :=  . (2.8.1)

In particular, it can easily be checked using (1.1.5) that these definitions satisfy the zig-zag identites of (2.4.3).

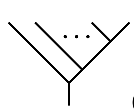
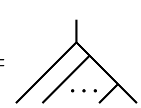
Let c be a connected component in some partition diagram representing a morphism in $\mathcal{P}ar_t$. We call c an *upward branch* if c has at least two endpoints on its top boundary and no endpoints on its bottom boundary, and a *downward branch* if it has at least two endpoints on its bottom boundary but no endpoints at the top:

$c =$  (upward) or $c =$  (downward).

We call c an *upward leaf* if it has exactly one endpoint at the top and no endpoints at the bottom, and a *downward leaf* if it has no endpoints at the top and exactly one at the bottom:

$c =$  (upward) or $c =$  (downward).

We refer to c as an *upward tree* if it has more than one endpoint at the top and exactly one endpoint at the bottom, and a *downward tree* if it has exactly one endpoint at the top and more than one endpoint at the bottom:

$c =$  (upward) or $c =$  (downward)

We say that c is a *double tree* if c has more than one endpoint at the top and more than one endpoint at the bottom. In that case, it is equivalent to the composition of an upward tree and a downward tree; for example, the rightmost connected component in (1.1.8) is a double tree. Finally we say that c is a *trunk* if c has exactly one endpoint both at the top and at the bottom:

$$c = \Big| .$$

Any connected component of a partition diagram can be represented either as an upward branch, an upward leaf, an upward tree, a downward branch, a downward leaf, a downward tree, a double tree, or a trunk.

Let f be an $m \times n$ partition diagram. We say f is

- a *permutation diagram* if all of its connected components are trunks, in which case we must have that $m = n$;
- an *upward partition diagram* if its connected components are trunks, upward branches, upward leaves and upward trees, in which case we must have that $m \geq n$;
- a *downward partition diagram* if its connected components are trunks, downward branches, downward leaves and downward trees, in which case we must have that $m \leq n$.

Let f be an upward $m \times n$ partition diagram. We say that it is *strictly upward* if $m > n$. Let $c_1, \ldots, c_k$ be the connected components of f that are either trunks or upward trees, indexing them so that their bottom endpoints are in order from right to left in f. We say that f is *normally ordered* if the rightmost of the top endpoints of each of $c_1, \ldots, c_k$ are also in order from right to left in f. In other words, f is normally ordered if it can be drawn so that the right edges of all of the upward trees and trunks in f are non-crossing. Similarly, we define *strictly downward* and *normally ordered* downward partition diagrams.

Now we can define some monoidal subcategories of $\mathcal{P}ar_t$. Let $\mathcal{S}ym$ be the symmetric category as defined in §2.7. There is a strict $\Bbbk$-linear symmetric monoidal functor

$$i_t^{\circ} : \mathcal{S}ym \to \mathcal{P}ar_t \tag{2.8.2}$$

sending the generating object and the generating morphism of $\mathcal{S}ym$ to the generating object and the generating morphism of $\mathcal{P}ar_t$ that is represented by the crossing. By the discussion in §1.1 stating that $\mathcal{P}ar_t$ has basis consisting of partition diagrams, it follows that this functor is faithful. We use it to *identify* $\mathcal{S}ym$ with a monoidal subcategory of $\mathcal{P}ar_t$. In other words, $\mathcal{S}ym$ is identified with the subcategory of $\mathcal{P}ar_t$ consisting of all objects and all the morphisms which can be written as linear combinations of permutation diagrams.

Next, let $\mathcal{P}ar^{\flat}$ be the strict $\Bbbk$-linear monoidal category generated by one object $|$ and the morphisms

$$\times : | \star | \to | \star | , \qquad \curlyvee : | \to | \star | , \qquad \mathring{|} : \mathbb{1} \to | \tag{2.8.3}$$

subject to the relations (1.1.2) to (1.1.4) and their flips in a vertical axis. We call this the *upward partition category.* The cup can also be defined in $\mathcal{P}ar^{\flat}$ as in (2.8.1). Any upward partition diagram can be interpreted as a string diagram representing a morphism in $\mathcal{P}ar^{\flat}$. Moreover, the defining relations in $\mathcal{P}ar^{\flat}$ imply that two upward $m \times n$ partition diagrams which are equivalent in the sense that they define the same partition of the set $\{1, \dots, n, 1', \dots, m'\}$ labelling the endpoints are also equal as morphisms in $\mathrm{Hom}_{\mathcal{P}ar^{\flat}}(n, m)$. There is a strict $\Bbbk$-linear monoidal functor

$$i_t^{\flat} : \mathcal{P}ar^{\flat} \to \mathcal{P}ar_t \tag{2.8.4}$$

sending the generating morphisms of $\mathcal{Par}^\flat$ to the corresponding ones in $\mathcal{Par}_t$. Note that any diagram built from compositions and monoidal products of the generating morphisms (2.8.3) can be interpreted as an upward $m \times n$ partition diagram. Hence, equivalence classes of upward $m \times n$ partition diagrams span $\mathrm{Hom}_{\mathcal{Par}^\flat}(n, m)$. Since their images in $\mathrm{Hom}_{\mathcal{Par}_t}(n, m)$ are linearly independent too, this functor is faithful. We use it to *identify* $\mathcal{Par}^\flat$ with a monoidal subcategory of $\mathcal{Par}_t$. In other words, $\mathcal{Par}^\flat$ is identified with the monoidal subcategory of $\mathcal{Par}_t$ consisting of all objects and all of the morphisms which can be written as linear combinations of upward partition diagrams. Also let $\mathcal{Par}^-$ be the monoidal subcategory of $\mathcal{Par}^\flat$ consisting of all objects and all of the morphisms which can be written as linear combinations of normally ordered upward partition diagrams.

Similarly to the previous paragraph, we define $\mathcal{Par}^\sharp$, the *downward partition category*, to be the strict $\Bbbk$-linear monoidal category generated by one object $|$ and the morphisms that are the flips of (2.8.3) in a horizontal axis, subject to the relations that are the flips of the ones for $\mathcal{Par}^\flat$. The cap can also be defined in $\mathcal{Par}^\sharp$ as in (2.8.1). Evidently, $\mathcal{Par}^\sharp \cong (\mathcal{Par}^\flat)^{\mathrm{op}}$ with isomorphism being defined by the flip σ in a horizontal axis. There is a strict $\Bbbk$-linear monoidal functor

$$i_t^\sharp : \mathcal{Par}^\sharp \to \mathcal{Par}_t \tag{2.8.5}$$

sending the generating morphisms of $\mathcal{Par}^\sharp$ to the corresponding ones in $\mathcal{Par}_t$. We have that $i_t^\sharp = \sigma \circ i_t^\flat \circ \sigma$, so we deduce from the previous paragraph that $i_t^\sharp$ is faithful too. We use it to *identify* $\mathcal{Par}^\sharp$ with a monoidal subcategory of $\mathcal{Par}_t$. In other words, $\mathcal{Par}^\sharp$ is identified with the monoidal subcategory of $\mathcal{Par}_t$ consisting of all objects and all of the morphisms which can be written as linear combinations of downward partition diagrams. Also let $\mathcal{Par}^+$ be the monoidal subcategory of $\mathcal{Par}^\sharp$ consisting of all objects and all of the morphisms which can be written as linear combinations of normally ordered downward partition diagrams.

Finally we let Par_t be the path algebra of $\mathcal{P}ar_t$. It is a locally unital algebra with distinguished idempotents $\{1_n \,|\, n \in \mathbb{N}\}$ arising from the identity endomorphisms of the objects of $\mathcal{P}ar_t$. We also have the path algebras $Par^\flat, Par^-, Sym, Par^+, Par^\sharp$ of $\mathcal{P}ar^\flat, \mathcal{P}ar^-, \mathcal{S}ym, \mathcal{P}ar^+, \mathcal{P}ar^\sharp$, which we may view as locally unital subalgebras of Par_t via the embeddings (2.8.2), (2.8.4) and (2.8.5). The following theorem is the *triangular decomposition* of Par_t.

Theorem 2.8.1. *Let $\mathbb{K} := \bigoplus_{n \geq 0} \Bbbk 1_n$ viewed as a locally unital subalgebra of Par_t. Multiplication defines a linear isomorphism*

$$Par^- \otimes_{\mathbb{K}} Sym \otimes_{\mathbb{K}} Par^+ \xrightarrow{\sim} Par_t. \tag{2.8.6}$$

Hence, we also have isomorphisms

$$Par^- \otimes_{\mathbb{K}} Sym \xrightarrow{\sim} Par^\flat, \tag{2.8.7}$$

$$Sym \otimes_{\mathbb{K}} Par^+ \xrightarrow{\sim} Par^\sharp, \tag{2.8.8}$$

$$Par^\flat \otimes_{Sym} Par^\sharp \xrightarrow{\sim} Par_t. \tag{2.8.9}$$

Proof. Any partition diagram is equivalent to a diagram that is the composition of a normally ordered upward partition diagram, a permutation diagram, and a normally ordered downward partition diagram; see (1.1.8) for an example of such a decomposition. Moreover, equivalence classes of these sorts of diagrams give bases for Par_t, Par^-, Sym and Par^+. This implies that (2.8.6) is an isomorphism. Then (2.8.7) to (2.8.9) follow as in [BS, Rem. 5.32]. □

Theorem 2.8.1 is all that is needed to see that the locally finite-dimensional locally unital algebra

$$Par_t = \bigoplus_{m,n \in \mathbb{N}} 1_m Par_t 1_n$$

has a *split triangular decomposition* in the sense of [BS, Rem. 5.32]. Spelling this out in the case of Par_t, we have:

- distinguished idempotents that are indexed by an upper finite poset. In this case, $\mathbb{N}$ equipped with the ordering reverse to the usual one.
- locally unital subalgebras Par^-, Sym, and Par^+.
- subspaces $Par^\flat = Par^- \cdot Sym$ and $Par^\sharp = Sym \cdot Par^+$ which are subalgebras, and not merely subspaces.
- the multiplication map of (2.8.6) is an isomorphism.
- for $n, m \in \mathbb{N}$, $1_m Sym 1_n$ is zero unless $m = n$, $1_m Par^- 1_n$ and $1_n Par^+ 1_m$ are zero unless $n < m$ (with the reverse ordering), and $1_n Par^- 1_n = 1_n Par^+ 1_n = \Bbbk 1_n$ for all $n \in \mathbb{N}$.

The subalgebras $Par^\flat$ and $Par^\sharp$ are referred to as the negative and positive *Borel subalgebras*, respectively. Additionally, $Sym = Par^\flat \cap Par^\sharp$ is the *Cartan subalgebra*. We stress that our imposition that the ordering on $\mathbb{N}$ be reversed gives an upper finite poset, conforming to the general conventions of [BS].

2.9 Classification of irreducible modules and highest weight structure

As Par_t has a triangular decomposition with Cartan subalgebra Sym being semisimple, we can appeal to the general results of [BS, §5.5] to obtain the classification of irreducible Par_t-modules. Alternatively, this follows from the results in [SS22, §5.5], but note that Sam and Snowden use the language of *lowest weight* rather than *highest weight* categories. Since isomorphism classes of irreducible Par_t-modules are in bijection with isomorphism classes of indecomposable projective Par_t-modules, and the latter are identified with isomorphism classes of indecomposable objects in $\mathrm{Kar}(\mathcal{P}ar_t)$, the results discussed in this section are equivalent to the classification obtained originally in [CO11, Th. 3.7].

The algebra Par_t is $\mathbb{Z}$-graded with $1_m Par_t 1_n$ being in degree $m-n$. The induced gradings on the subalgebras $Par^\flat$ and $Par^\sharp$ make these into positively and negatively graded algebras, respectively, with degree zero components in both cases being the semisimple algebra Sym. It follows that the Jacobson radicals of $Par^\flat$ and $Par^\sharp$ are the direct sums of their non-zero graded components. Moreover, the quotients by their Jacobson radicals are naturally identified with Sym, i.e., there are locally unital algebra homomorphisms

$$\pi^\flat : Par^\flat \twoheadrightarrow \mathit{Sym}, \qquad\qquad \pi^\sharp : Par^\sharp \twoheadrightarrow \mathit{Sym}. \tag{2.9.1}$$

Let $\mathrm{infl}^\sharp : \mathit{Sym}\text{-}\mathrm{Mod}_{\mathrm{fd}} \to Par^\sharp\text{-}\mathrm{Mod}_{\mathrm{fd}}$ and $\mathrm{infl}^\flat : \mathit{Sym}\text{-}\mathrm{Mod}_{\mathrm{fd}} \to Par^\flat\text{-}\mathrm{Mod}_{\mathrm{fd}}$ be the functors defined by restriction along these homomorphisms. The modules

$$\left\{ S^\flat(\lambda) := \mathrm{infl}^\flat S(\lambda) \;\middle|\; \lambda \in \mathcal{P} \right\}, \qquad \left\{ S^\sharp(\lambda) := \mathrm{infl}^\sharp S(\lambda) \;\middle|\; \lambda \in \mathcal{P} \right\} \tag{2.9.2}$$

give full sets of pairwise inequivalent irreducible modules for $Par^\flat$ and $Par^\sharp$, respectively.

As in [BS, (5.13)–(5.14)], we define the *standardization* and *costandardization functors*

$$j_! := \mathrm{ind}_{Par^\sharp}^{Par_t} \circ \mathrm{infl}^\sharp : \mathit{Sym}\text{-}\mathrm{Mod}_{\mathrm{fd}} \to Par_t\text{-}\mathrm{Mod}_{\mathrm{lfd}}, \tag{2.9.3}$$

$$j_* := \mathrm{coind}_{Par^\flat}^{Par_t} \circ \mathrm{infl}^\flat : \mathit{Sym}\text{-}\mathrm{Mod}_{\mathrm{fd}} \to Par_t\text{-}\mathrm{Mod}_{\mathrm{lfd}}, \tag{2.9.4}$$

where $\mathrm{ind}_{Par^\sharp}^{Par_t} := Par_t \otimes_{Par^\sharp}$ and $\mathrm{coind}_{Par^\flat}^{Par_t} := \bigoplus_{n\in\mathbb{N}} \mathrm{Hom}_{Par^\flat}(Par_t 1_n, ?)$. From (2.8.6) to (2.8.8) it follows that Par_t is projective both as a right $Par^\sharp$-module and as a left

$Par^\flat$-module, hence, these functors are exact. Then we define the *standard* and *costandard modules* for Par_t by

$$\Delta(\lambda) := j_! S(\lambda) = \operatorname{ind}_{Par^\sharp}^{Par_t} S^\sharp(\lambda), \qquad \nabla(\lambda) = j_* S(\lambda) = \operatorname{ind}_{Par^\flat}^{Par_t} S^\flat(\lambda), \tag{2.9.5}$$

respectively.

Theorem 2.9.1. *The Par_t-modules $\{L(\lambda) \mid \lambda \in \mathcal{P}\}$ defined from*

$$L(\lambda) := \operatorname{hd} \Delta(\lambda) \cong \operatorname{soc} \nabla(\lambda)$$

give a complete set of pairwise inequivalent irreducible left Par_t-modules. Moreover, Par_t-$\mathrm{Mod}_{\mathrm{lfd}}$ is an upper finite highest weight category in the sense of [BS, Def. 3.34] with weight poset $(\mathcal{P}, \preceq)$, where $\preceq$ is the partial order on $\mathcal{P}$ defined by $\lambda \preceq \mu$ if and only if either $\lambda = \mu$ or $|\lambda| > |\mu|$. Its standard and costandard objects are the modules $\Delta(\lambda)$ and $\nabla(\lambda)$, respectively.

Proof. This follows immediately from [BS, Cor. 5.39] using the triangular decomposition from Theorem 2.8.1 and the semisimplicity of *Sym*; see also [SS22, §5.5]. □

The fact established in Theorem 2.9.1 that Par_t-$\mathrm{Mod}_{\mathrm{lfd}}$ is an upper finite highest weight category has several significant consequences. As is the case for any category equivalent to A-$\mathrm{Mod}_{\mathrm{lfd}}$ for a locally finite locally unital algebra A ('Schurian' in the language of [BS]), $L(\lambda)$ has a projective cover we denote by $P(\lambda) \in Par_t$-$\mathrm{Mod}_{\mathrm{lfd}}$. Let Par_t-Mod_Δ be the exact subcategory of Par_t-$\mathrm{Mod}_{\mathrm{lfd}}$ consisting of all modules with a Δ-flag, that is, a finite filtration whose sections are of the form $\Delta(\lambda)$ for $\lambda \in \mathcal{P}$. For any $V \in Par_t$-Mod_Δ, the multiplicity $(V : \Delta(\mu))$ of $\Delta(\mu)$ as a section of some Δ-flag in V is well-defined independent of the flag, indeed, it can be calculated from

$$(V : \Delta(\mu)) = \dim \operatorname{Hom}_{Par_t}(V, \nabla(\mu)). \tag{2.9.6}$$

This follows from the fundamental Ext-vanishing property of highest weight categories, namely, that

$$\dim \operatorname{Ext}^i_{Par_t}(\Delta(\lambda), \nabla(\mu)) = \delta_{i,0}\delta_{\lambda,\mu} \tag{2.9.7}$$

for any $\lambda, \mu \in \mathcal{P}$ and $i \geq 0$; see [BS, Lem. 3.48]. The definition of highest weight category gives that $P(\lambda)$ has a Δ-flag, so that Par_t-Proj is a full subcategory of Par_t-Mod_Δ. Moreover, from (2.9.6), one obtains the usual *BGG reciprocity formula*

$$(P(\lambda) : \Delta(\mu)) = [\nabla(\mu) : L(\lambda)]. \tag{2.9.8}$$

The functor $\sigma : \mathcal{P}ar_t \xrightarrow{\sim} (\mathcal{P}ar_t)^{\mathrm{op}}$ defined by flipping diagrams in a horizontal axis can also be viewed as a locally unital anti-involution of the algebra Par_t. It interchanges the subalgebras $Par^\flat$ and $Par^\sharp$, and restricts to an anti-involution also denoted σ on the subalgebra Sym. Let $?^{\circledast}$ be the duality on Sym-$\mathrm{Mod}_{\mathrm{fd}}$ taking a finite-dimensional left Sym-module to its linear dual viewed again as a left module using the anti-automorphism σ. Since $\sigma(g) = g^{-1}$ for a permutation $g \in S_n \subset Sym$, this is the usual duality on each of the subcategories $\Bbbk S_n$-$\mathrm{Mod}_{\mathrm{fd}}$. It is well known that the irreducible $\Bbbk S_n$-modules are self-dual, hence,

$$S(\lambda)^{\circledast} \cong S(\lambda) \tag{2.9.9}$$

for all $\lambda \in \mathcal{P}$. There is also a duality $?^{\circledast}$ on Par_t-$\mathrm{Mod}_{\mathrm{lfd}}$ defined as in (2.3.3). Similarly, as σ interchanges $Par^\flat$ and $Par^\sharp$, we get contravariant equivalences also denoted $?^{\circledast}$ between $Par^\sharp$-$\mathrm{Mod}_{\mathrm{lfd}}$ and $Par^\flat$-$\mathrm{Mod}_{\mathrm{lfd}}$. Similarly to (2.3.4) and (2.3.5), we have that

$$?^{\circledast} \circ \mathrm{infl}^\flat \cong \mathrm{infl}^\sharp \circ ?^{\circledast}, \qquad \mathrm{ind}^{Par_t}_{Par^\sharp} \circ ?^{\circledast} \cong ?^{\circledast} \circ \mathrm{coind}^{Par_t}_{Par^\flat}. \tag{2.9.10}$$

Hence:

$$j_! \circ ?^{\circledast} \cong ?^{\circledast} \circ j_*, \qquad j_* \circ ?^{\circledast} \cong ?^{\circledast} \circ j_! \tag{2.9.11}$$

as functors from Sym-Mod$_{\mathrm{fd}}$ to Par_t-Mod$_{\mathrm{lfd}}$. Then from (2.9.9) and (2.9.11), we deduce that

$$\Delta(\lambda)^{\circledcirc} \cong \nabla(\lambda), \qquad \nabla(\lambda)^{\circledcirc} \cong \Delta(\lambda), \qquad L(\lambda)^{\circledcirc} \cong L(\lambda) \tag{2.9.12}$$

for $\lambda \in \mathcal{P}$.

The duality $?^{\circledcirc}$ will be called the *Chevalley duality* on Par_t-Mod$_{\mathrm{lfd}}$, following the language of [BS, Def. 4.49]

CHAPTER III

BLOCKS OF THE PARTITION CATEGORY

This chapter contains previously published co-authored material from [BV22]. We introduce an auxiliary monoidal category $\mathcal{APar}$, the *affine partition category*. We define this as a certain monoidal subcategory of the Heisenberg category $\mathcal{Heis}$, exploiting an observation of Likeng and Savage from [LSR21]. We then use $\mathcal{APar}$ to give a new approach to the definition of the Jucys-Murphy elements of Par_t. These were first defined in the context of the partition algebra by Halverson and Ram [HR05] and computed recursively by Enyang [Eny13]. We also construct more general central elements.

The second half of this capter studies these central elements and their images under an analog of the Harish-Chandra homomorphism. This affords us a decomposition of $\mathcal{Par}_t$-Mod as a product of subcategories, which turn out to be precisely the blocks. In fact, Par_t is semisimple if and only if $t \notin \mathbb{N}$, while if $t \in \mathbb{N}$ the non-simple blocks are in bijection with partitions of t. We also determine the structure of the non-simple blocks and explicitly show that they are all equivalent, recovering the results of Comes and Ostrik [CO11]. Our approach is similar to the Okounkov-Vershik approach to representations of $\mathcal{Sym}$ as reviewed in §2.7.

3.1 Schur-Weyl duality

Recall the generators and relations for the partition category from Definition 1.1.1. The following theorem of Deligne will play a key role in this section; see e.g. [Com20, Th. 2.3] for a proof.

Theorem 3.1.1. *Suppose that $t \in \mathbb{N}$. Let U_t be the natural permutation representation of the symmetric group S_t with standard basis $u_1, \ldots, u_t$. Viewing $\Bbbk S_t$-$\mathrm{Mod}_{\mathrm{fd}}$ as a symmetric monoidal category via the usual Kronecker tensor product*

$\otimes$, there is a full symmetric monoidal functor $\psi_t : \mathcal{P}ar_t \to \Bbbk S_t\text{-Mod}_{\mathrm{fd}}$ sending the generating object $\mid$ to U_t and defined on generating morphisms by

$$
\begin{aligned}
&\psi_t(\times) : U_t \otimes U_t \to U_t \otimes U_t\ , && u_i \otimes u_j \mapsto u_j \otimes u_i\ ,\\
&\psi_t(\curlywedge) : U_t \otimes U_t \to U_t, && u_i \otimes u_j \mapsto \delta_{i,j} u_i\ ,\\
&\psi_t(\curlyvee) : U_t \to U_t \otimes U_t, && u_i \mapsto u_i \otimes u_i\ ,\\
&\psi_t(\text{⫯}) : U_t \to \Bbbk\ , && u_i \mapsto 1\ ,\\
&\psi_t(\text{⫰}) : \Bbbk \to U_t && 1 \mapsto u_1 + \cdots + u_t\ .
\end{aligned}
$$

Furthermore, the linear map $\mathrm{Hom}_{\mathcal{P}ar_t}(n,m) \to \mathrm{Hom}_{\Bbbk S_t}(U_t^{\otimes n}, U_t^{\otimes m})$, $f \mapsto \psi_t(f)$ is an isomorphism whenever $t \geq m+n$.

For the next corollary, we assume some basic facts about semisimplification of monoidal categories; e.g., see [BEEO20, Sec. 2] which gives a concise summary of everything needed here.

Corollary 3.1.2. *When $t \in \mathbb{N}$, the functor ψ_t induces a monoidal equivalence $\overline{\psi}_t$ between the semisimplification of $\mathrm{Kar}(\mathcal{P}ar_t)$ and $\Bbbk S_t\text{-Mod}_{\mathrm{fd}}$. In particular, Par_t is* not *a semisimple locally unital algebra in these cases.*

Proof. The functor ψ_t extends canonically to a functor $\mathrm{Kar}(\mathcal{P}ar_t) \to \Bbbk S_t\text{-Mod}_{\mathrm{fd}}$. It is well known that every irreducible $\Bbbk S_t$-module appears as a constituent of some tensor power of U_t, hence, this functor is dense. Now the first statement follows from the fullness of the functor ψ_t using [BEEO20, Lem. 2.6]; see also [Del07, Th. 2.18] and [CO11, Th. 3.24]. Since $\mathrm{Kar}(\mathcal{P}ar_t)$ has infinitely many isomorphism classes of irreducible objects, it is definitely not equivalent to its semisimplification $\Bbbk S_t\text{-Mod}_{\mathrm{fd}}$. This shows that $\mathrm{Kar}(\mathcal{P}ar_t)$ is not a semisimple Abelian category as it contains non-zero negligible morphisms. Equivalently, the path algebra Par_t is not semisimple in these cases, which is the second statement. $\square$

Remark 3.1.3. Continue to assume that $t \in \mathbb{N}$. By the general theory of semisimplification, the irreducible objects in the semisimplification of $\mathrm{Kar}(\mathcal{P}ar_t)$ correspond to the indecomposable projective Par_t-modules $P(\lambda)$ of non-zero categorical dimension. In [Del07, Prop. 6.4], Deligne showed that $P(\lambda)$ has non-zero categorical dimension if and only if $t - |\lambda| > \lambda_1 - 1$, in which case the irreducible object of the semisimpliciation arising from $P(\lambda)$ corresponds under the equivalence $\overline{\psi}_t$ to the irreducible $\Bbbk S_t$-module $S(\kappa)$ where $\kappa := (t - |\lambda|, \lambda_1, \lambda_2, \dots)$.

Through the next few sections, our goal is to introduce the affine partition category. In one sense, this is an extension of the partition category with extra generators and relations. Though it would be nice to view $\mathcal{P}ar_t$ as embedding into its affine version, our realization of the affine partition category is as a subcategory of the Heisenberg category of [Kho14] and does not satisfy the final relation in (1.1.6) which is dependent on t. So we introduce the *generic partition category* $\mathcal{P}ar$ as the strict monoidal category with the same generating object and generating morphisms as $\mathcal{P}ar_t$ subject to all of the same relations except for the final relation in (1.1.6), which is omitted. The morphism

$$T := \quad \in \mathrm{End}_{\mathcal{P}ar}(\mathbb{1}) \tag{3.1.1}$$

is strictly central in $\mathcal{P}ar$, so that $\mathcal{P}ar$ can be viewed as a $\Bbbk[T]$-linear monoidal category. For $t \in \Bbbk$, let

$$\mathrm{ev}_t : \mathcal{P}ar \to \mathcal{P}ar_t \tag{3.1.2}$$

be the canonical functor taking T to $t1_{\mathbb{1}}$. Using the basis theorem for $\mathcal{P}ar_t$ for infinitely many values of t, one obtains a basis theorem for the generic partition category: each morphism space $\mathrm{Hom}_{\mathcal{P}ar}(n, m)$ is free as a $\Bbbk[T]$-module with basis given by a set of representatives for the equivalence classes of $m \times n$ partition diagrams. From this, we see that ev_t induces an isomorphism $\Bbbk \otimes_{\Bbbk[T]} \mathcal{Par} \cong \mathcal{Par}_t$, where on the left hand side we are viewing $\Bbbk$ as a $\Bbbk[T]$-module so that T acts as t. This point of view is often useful since it can be used to prove a statement involving relations in $\mathcal{Par}_t$ for all values of t just by checking it for all sufficiently large positive integers, in which case Theorem 3.1.1 can often be applied to reduce to a question about symmetric groups. To make a precise statement, let

$$\phi_t := \psi_t \circ \mathrm{ev}_t : \mathcal{Par} \to \Bbbk S_t\text{-Mod}_{\mathrm{fd}}, \tag{3.1.3}$$

assuming $t \in \mathbb{N}$.

Lemma 3.1.4. *If $f \in \mathrm{Hom}_{\mathcal{Par}}(n, m)$ satifies $\phi_t(f) = 0$ for infinitely many values of $t \in \mathbb{N}$ then $f = 0$.*

Proof. We can write $f = \sum_i p_i(T) f_i$ for polynomials $p_i(T) \in \Bbbk[T]$ and f_i running over a set of representatives for the equivalence classes of $m \times n$ partition diagrams. Since $\phi_t(f) = 0$ we have that $\sum_i p_i(t)\phi_t(f_i) = 0$ for infinitely many values of t. By the final assertion in Theorem 3.1.1, this implies that $\sum_i p_i(t)\,\mathrm{ev}_t(f_i) = 0$ for infinitely many values of $t \geq m+n$. By the basis theorem in $\mathcal{Par}_t$, this means for each i that $p_i(t) = 0$ for infinitely many values of t. Hence, $p_i(T) = 0$ for each i. $\square$

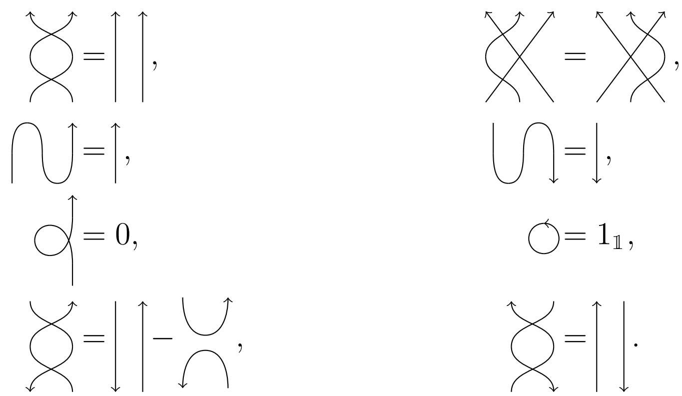
We note that the proof of Lemma 3.1.4 depends on our standing assumption that the ground field $\Bbbk$ is of characteristic zero.

3.2 Heisenberg category

Next we recall the definition of the Heisenberg category $\mathcal{Heis}$ which was introduced by Khovanov in [Kho14]. We follow the approach of [Bru18]; Khovanov's category is denoted $\mathcal{Heis}_{-1}(0)$ in the more general setup developed there.

Definition 3.2.1 ([Bru18, Rem. 1.5(2)])**.** The Heisenberg category $\mathcal{H}eis$ is the strict monoidal category with two generating objects $\uparrow$ and $\downarrow$ and five generating morphisms

subject to the following relations:



(3.2.1)

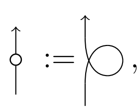
(3.2.2)

(3.2.3)

(3.2.4)

Here, we have used the the sideways crossings which are defined from

It is also convenient to introduce the shorthand

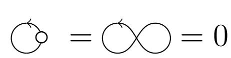


(3.2.5)

which automatically satisfies the degenerate affine Hecke algebra relation as in (2.7.7):

(3.2.6)

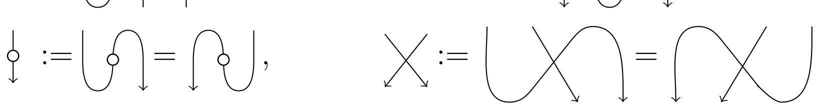
Note by (3.2.3) that



(3.2.7)

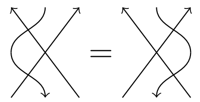
In addition, the following relations hold, so that $\mathcal{H}eis$ is strictly pivotal with duality functor defined by rotating diagrams through 180°:

(3.2.8)



(3.2.9)

Then we obtain further variations on (3.2.6) by rotating through $90°$ or $180°$ using this strictly pivotal structure. One more useful consequence of the defining relations is that



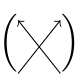
$$, \tag{3.2.10}$$

There is also a symmetry $\sigma : \mathcal{H}eis \to \mathcal{H}eis^{\mathrm{op}}$, which is the strict monoidal functor that is the identity on objects and sends a morphism to the morphism obtained by reflecting in a horizontal axis and then reversing all orientations of strings.

Khovanov constructed a categorical action of $\mathcal{H}eis$ on $\mathcal{S}ym\text{-Mod}_{\mathrm{fd}} = \bigoplus_{n\geq 0} \Bbbk S_n\text{-Mod}_{\mathrm{fd}}$, i.e., a strict monoidal functor

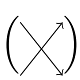
$$\Theta : \mathcal{H}eis \to \mathcal{E}nd_{\Bbbk}(\mathcal{S}ym\text{-Mod}_{\mathrm{fd}}). \tag{3.2.11}$$

Explicitly, this takes the generating objects $\uparrow$ and $\downarrow$ to the induction functor E and the restriction functor F, respectively, notation as in (2.7.4), and Θ takes generating morphisms for $\mathcal{H}eis$ to the natural transformations defined on a $\Bbbk S_n$-module V as follows (where g is an element of the appropriate symmetric group):

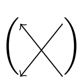


$$_V : \Bbbk S_{n+2} \otimes_{\Bbbk S_{n+1}} \Bbbk S_{n+1} \otimes_{\Bbbk S_n} V \to \Bbbk S_{n+2} \otimes_{\Bbbk S_{n+1}} \Bbbk S_{n+1} \otimes_{\Bbbk S_n} V,$$

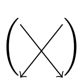
$$g \otimes 1 \otimes v \mapsto g(n{+}1\; n{+}2) \otimes 1 \otimes v,$$



$$_V : \Bbbk S_n \otimes_{\Bbbk S_{n-1}} V \to \Bbbk S_{n+1} \otimes_{\Bbbk S_n} V, \quad g \otimes v \mapsto g(n\; n{+}1) \otimes v,$$



$$_V : \Bbbk S_{n+1} \otimes_{\Bbbk S_n} V \to \Bbbk S_n \otimes_{\Bbbk S_{n-1}} V, \quad g \otimes v \mapsto \begin{cases} g_2 \otimes g_1 v & \text{if } g = g_2(n\; n{+}1)g_1 \text{ for } g_i \in S_n, \\ 0 & \text{otherwise,} \end{cases}$$



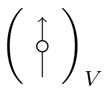
$$_V : V \to V, \qquad v \mapsto (n{-}1\; n)v,$$

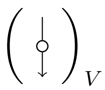
$$_V : \Bbbk S_n \otimes_{\Bbbk S_{n-1}} V \to V, \qquad g \otimes v \mapsto gv,$$

$$_V : V \to \Bbbk S_n \otimes_{\Bbbk S_{n-1}} V \qquad v \mapsto \sum_{i=1}^{n} (i\; n) \otimes (i\; n)v,$$

 $: \Bbbk S_{n+1} \otimes_{\Bbbk S_n} V \to V, \qquad g \otimes v \mapsto \begin{cases} gv & \text{if } g \in S_n, \\ 0 & \text{otherwise,} \end{cases}$

 $: V \to \Bbbk S_{n+1} \otimes_{\Bbbk S_n} V \qquad v \mapsto 1 \otimes v,$

 $: \Bbbk S_{n+1} \otimes_{\Bbbk S_n} V \to \Bbbk S_{n+1} \otimes_{\Bbbk S_n} V, \quad g \otimes v \mapsto g x_{n+1} \otimes v,$

 $: V \to V, \qquad v \mapsto x_n v.$

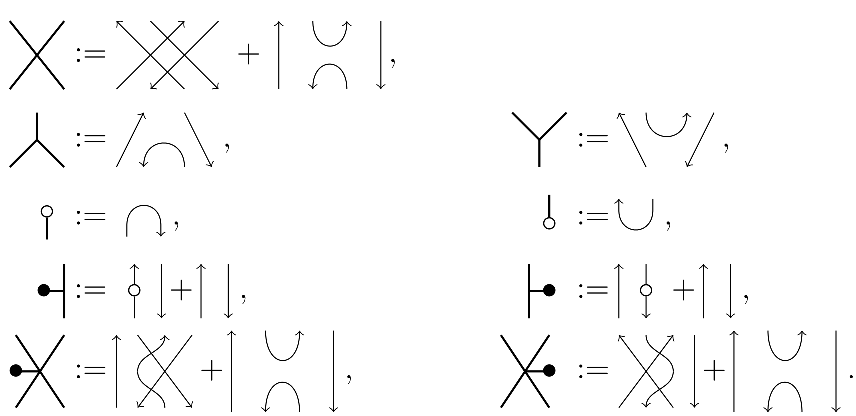
In the last two formulae, we have used the Jucys-Murphy elements $x_{n+1} \in \Bbbk S_{n+1}$ and $x_n \in \Bbbk S_n$ from (2.7.10), respectively; the natural transformations here are the endomorphisms of E and F denoted x and $x^\vee$ just before (2.7.12). All of the other formulae displayed here can also be found in [LSR21, §3]. Note in particular that the clockwise bubble  acts as multiplication by n on any $V \in \Bbbk S_n\text{-Mod}_{\text{fd}}$.

It is known moreover that the functor Θ is faithful. Indeed, in [Kho14], Khovanov uses the functor Θ to prove a basis theorem for morphism spaces in $\mathcal{H}eis$, and the argument implicitly establishes the faithfulness of Θ over fields of characteristic zero. We will not use this here in any essential way.

3.3 The affine partition category

Now the background is in place and we can make a new definition.

Definition 3.3.1. The *affine partition category* $\mathcal{AP}ar$ is the monoidal subcategory of $\mathcal{H}eis$ generated by the object $| := \uparrow \star \downarrow$ and the following morphisms



(3.3.1)

(3.3.2)

(3.3.3)

(3.3.4)

(3.3.5)

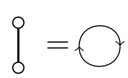
We refer to the morphisms in (3.3.4) as the *left dot* and the *right dot*, and the morphisms in (3.3.5) as the *left crossing* and the *right crossing*, respectively. The other shorthands for the generating morphisms of $\mathcal{AP}ar$ introduced in Definition 3.3.1 are the same as the symbols used for generators of the partition category. This is deliberate, indeed, the morphisms (3.3.1) to (3.3.3) generate a copy of the generic partition category $\mathcal{P}ar$ as a monoidal subcategory of $\mathcal{H}eis$. This important observation is due to Likeng and Savage; see Corollary 3.4.4 below. For now, we just need the following, which is proved in [LSR21] by a direct calculation using the defining relations in $\mathcal{H}eis$.

Lemma 3.3.2 ([LSR21, Th. 4.1])**.** *There is a strict monoidal functor*

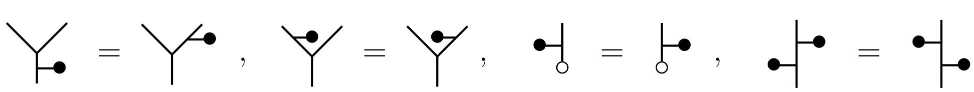
$$i : \mathcal{P}ar \to \mathcal{AP}ar \tag{3.3.6}$$

sending the generating object and generating morphisms of $\mathcal{P}ar$ to the generating object and generating morphisms in $\mathcal{AP}ar$ denoted by the same diagrams.

Because of the symmetry of the generators of $\mathcal{AP}ar$ under rotation through 180°, the strictly pivotal structure on $\mathcal{H}eis$ restricts to a strictly pivotal structure on $\mathcal{AP}ar$. The left and right dots are duals, as are the left and right crossings. Moreover, the cap and the cup making | into a self-dual object are given by the same formula (2.8.1) as we had before in $\mathcal{P}ar$, hence, i is a pivotal monoidal functor. Note also that

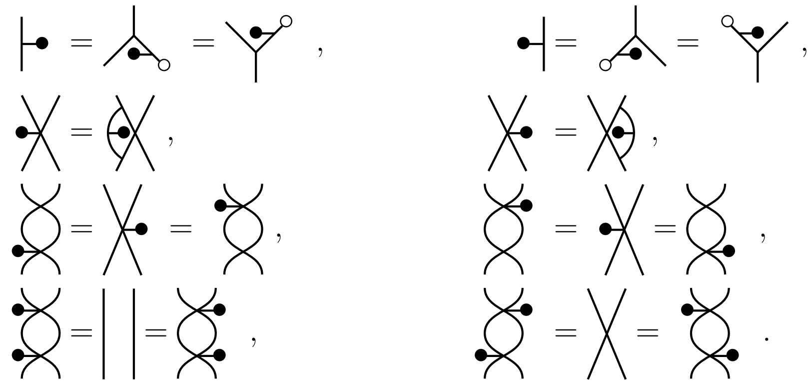
$T =$  . (3.3.7)

Also, the symmetry σ on $\mathcal{H}eis$ restricts to $\sigma : \mathcal{AP}ar \to \mathcal{AP}ar^{\mathrm{op}}$. This just reflects affine partition diagrams in a horizontal axis, just like the earlier anti-automorphism σ on $\mathcal{P}ar_t$. Here are some further relations, all of which are easily proved using the defining relations in $\mathcal{H}eis$:

 . (3.3.8)

Of course, the horizontal and vertical flips of all of these also hold. The next two lemmas establish some less obvious relations.

Lemma 3.3.3. *The following relations hold in $\mathcal{APar}$:*



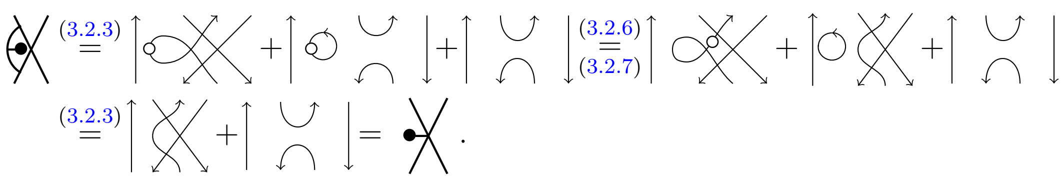
(3.3.9)

(3.3.10)

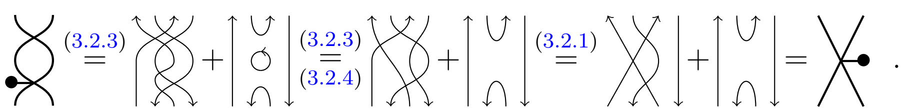
(3.3.11)

(3.3.12)

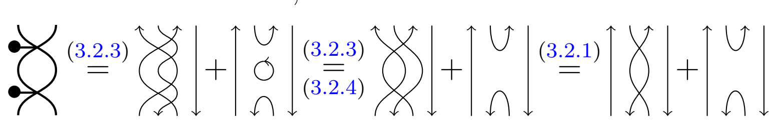
Proof. For each of (3.3.9) to (3.3.11), it suffices just to prove the first equality, and then all the others follow using σ and duality to reflect in horizontal and/or vertical axes. For (3.3.9), use (3.3.8) and (1.1.5). To prove (3.3.10), we expand as morphisms in $\mathcal{Heis}$ to see that

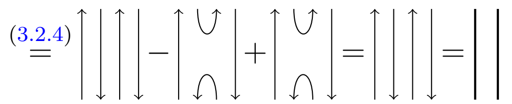


For (3.3.11), we again expand the left hand side as a morphism in $\mathcal{Heis}$:



Finally, to prove (3.3.12), the second set of relations follows from the first set of relations by composing on the bottom with a crossing and using (3.3.11). For the first set of relations, it suffices to prove the first equality, the second then follows by duality. Expanding both of the left crossings as morphisms in $\mathcal{Heis}$ produces a sum of four terms, two of which are zero, so we obtain:



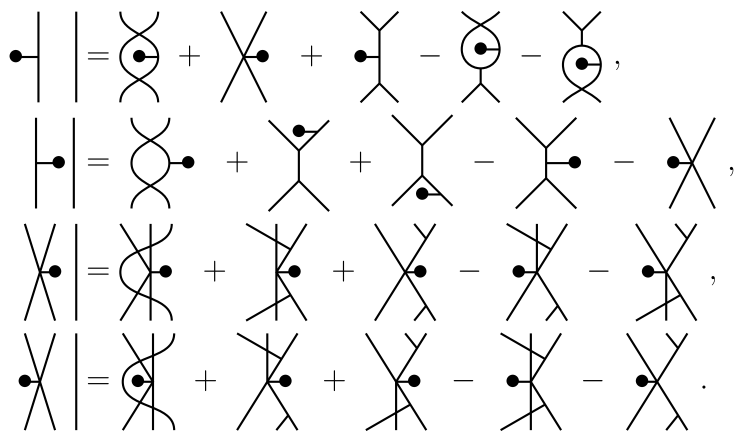


□

Corollary 3.3.4. *As a monoidal category, the subcategory* $\mathcal{APar}$ *of* $\mathcal{Heis}$ *is generated by the object* $|$, *the five undotted generators* (3.3.1) *to* (3.3.3), *and the left dot.*

Proof. The relations (3.3.9) and (3.3.10) together show that the right dot and the left and right crossings may be written in terms of the left dot and the other undotted generating morphisms. □

Lemma 3.3.5. *The following relations hold:*



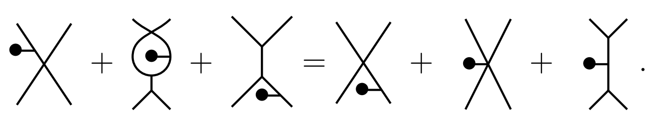
(3.3.13)

(3.3.14)

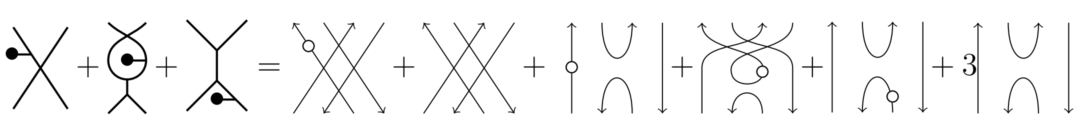
(3.3.15)

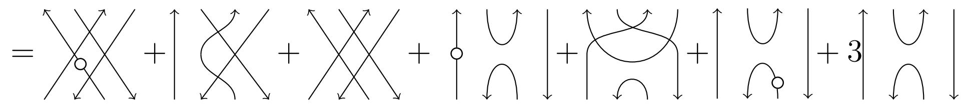
(3.3.16)

Proof. To prove (3.3.13), we observe by composing on the bottom with the crossing and using relations from $\mathcal{Par}$ plus (3.3.11) that the relation we are trying to prove is equivalent to

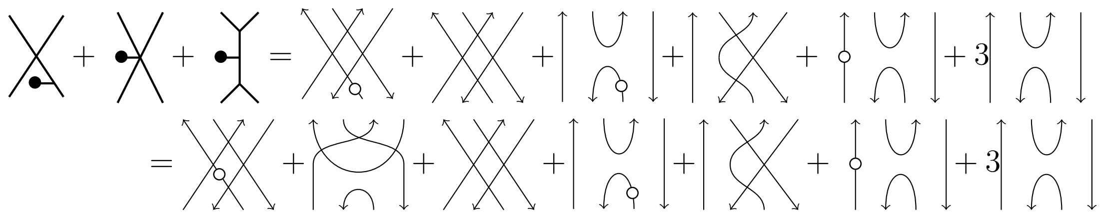


Now we expand the left hand side in terms of morphisms in $\mathcal{Heis}$ using (3.2.3) and (3.2.7), then we use (3.2.6) to commute the dot past a crossing in the first and fourth terms:





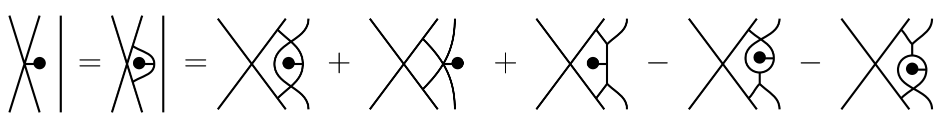
Similarly, the expansion of the right hand side is



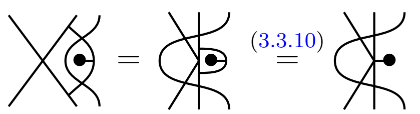
where we commuted the dot past a crossing just in the first term. These are equal.

To deduce (3.3.14), first apply duality to (3.3.13), i.e., rotate through 180°. Then compose on the top and bottom with a crossing and simplify using relations in $\mathcal{P}ar$ together with (3.3.11).

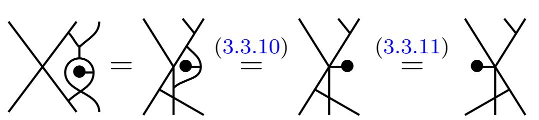
To prove (3.3.15), we rewrite its left hand side, replacing the right crossing with a left dot using (3.3.10), then we apply (3.3.13) to push this left dot past the right hand string:



Now we simplify the five terms on the right hand side of the equation just displayed to obtain the five terms on the right hand side of (3.3.15) (there is no need here to expand in terms of morphisms in $\mathcal{H}eis$). The following treats the first term:

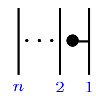


The second and third terms are easy to handle, we omit the details. For the fourth and the fifth terms, it suffices by symmetry to consider the fifth term, which we rewrite as follows:



Finally (3.3.16) follows easily from (3.3.15) on composing on the bottom with the crossing of the leftmost two strings and using (3.3.11). □

Corollary 3.3.6. *The category* $\mathcal{APar}$ *has object set* $\{\mid^{\star n} \mid n \in \mathbb{N}\}$ *(which we often identify simply with* $\mathbb{N}$*) and morphisms that are linear combinations of vertical compositions of morphisms in the image of* $i : \mathcal{Par} \to \mathcal{Heis}$ *together with the morphisms*



(3.3.17)

for all $n \geq 1$*.*

Proof. In view of Corollary 3.3.4, we just need to show that one can obtain the endomorphism of n defined by the left dot on the mth string ($m = 1, \ldots, n$) by taking a linear combinations of compositions of morphisms in the image of i and the given morphism (3.3.17) (in which the left dot is on the first string). This follows by induction on m using relations (3.3.13) and (3.3.10). $\square$

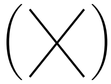
Remark 3.3.7. We have not attempted to formulate or prove a basis theorem for the morphism spaces in $\mathcal{APar}$. Creedon and De Visscher do this by combining their work with results of Khovanov [CD23]. They also show that $\mathcal{APar}$ is actually the full monoidal subcategory is $\mathcal{Heis}$ generated by the object $\mid$.

3.4 Action of $\mathcal{APar}$ on $\Bbbk S_t$-Mod$_{\mathrm{fd}}$

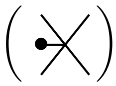
Suppose that $t \in \mathbb{N}$. The restriction of the functor Θ from (3.2.11) to the subcategory $\mathcal{APar}$ is a strict monoidal functor $\mathcal{APar} \to \mathcal{End}_{\Bbbk}(\mathcal{Sym}\text{-Mod}_{\mathrm{fd}})$ sending the generating object $\mid$ to the endofunctor $E \circ F$ (induction after restriction). Since $E \circ F$ takes $\Bbbk S_t$-modules to $\Bbbk S_t$-modules, the restriction of Θ gives strict monoidal functors

$$\Theta_t : \mathcal{APar} \to \mathcal{End}_{\Bbbk}(\Bbbk S_t\text{-Mod}_{\mathrm{fd}}). \tag{3.4.1}$$

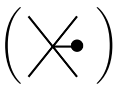
The functor Θ_t takes $\mid$ to the endofunctor $\mathrm{ind}_{S_{t-1}}^{S_t} \circ \mathrm{res}_{S_{t-1}}^{S_t} = \Bbbk S_t \otimes_{\Bbbk S_{t-1}}$ of $\Bbbk S_t$-Mod$_{\mathrm{fd}}$; this should be interpreted as the zero functor in the case $t = 0$. The natural transformations arising by applying Θ_t to the other generating morphisms of $\mathcal{APar}$ may be computed using the formulae after (3.2.11) (taking $n := t$). Explicitly, one obtains the following for $V \in \Bbbk S_t\text{-Mod}_{\mathrm{fd}}$ and $g, h \in S_t$:



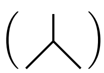
$$_V : \Bbbk S_t \otimes_{\Bbbk S_{t-1}} \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V \to \Bbbk S_t \otimes_{\Bbbk S_{t-1}} \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V, \qquad g \otimes h \otimes v \mapsto gh \otimes h^{-1} \otimes hv,$$



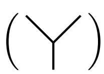
$$_V : \Bbbk S_t \otimes_{\Bbbk S_{t-1}} \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V \to \Bbbk S_t \otimes_{\Bbbk S_{t-1}} \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V, \qquad g \otimes h \otimes v \mapsto g \otimes h \otimes (h^{-1}(t)\ t)v,$$



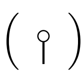
$$_V : \Bbbk S_t \otimes_{\Bbbk S_{t-1}} \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V \to \Bbbk S_t \otimes_{\Bbbk S_{t-1}} \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V, \qquad g \otimes h \otimes v \mapsto gh \otimes (h^{-1}(t)\ t) \otimes v,$$



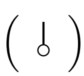
$$_V : \Bbbk S_t \otimes_{\Bbbk S_{t-1}} \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V \to \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V, \quad g \otimes h \otimes v \mapsto \delta_{h(t),t} gh \otimes v,$$



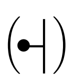
$$_V : \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V \to \Bbbk S_t \otimes_{\Bbbk S_{t-1}} \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V, \qquad g \otimes v \mapsto g \otimes 1 \otimes v,$$



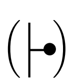
$$_V : \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V \to V, \qquad g \otimes v \mapsto gv,$$



$$_V : V \to \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V, \qquad v \mapsto \sum_{i=1}^{t} (i\ t) \otimes (i\ t)v,$$



$$_V : \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V \to \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V, \qquad g \otimes v \mapsto \sum_{j=1}^{t} g(j\ t) \otimes v,$$



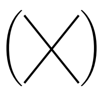
$$_V : \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V \to \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V, \qquad g \otimes v \mapsto \sum_{j=1}^{t} g \otimes (j\ t)v.$$

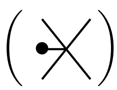
Recall the natural $\Bbbk S_d$-module U_t from Theorem 3.1.1; in particular, U_0 is the zero module. Using the Kronecker product, we can consider $U_t \otimes$ as an endofunctor of $\Bbbk S_t\text{-Mod}_{\mathrm{fd}}$. Also let triv_{S_t} be the trivial module.

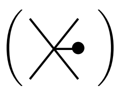
Lemma 3.4.1. *The functor Θ_t is monoidally isomorphic to the strict monoidal functor*

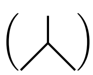
$$\Phi_t : \mathcal{APar} \to \mathcal{End}_{\Bbbk}(\Bbbk S_t\text{-Mod}_{\mathrm{fd}}) \tag{3.4.2}$$

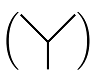
which sends the generating object $|$ *to the endofunctor $U_t \otimes$ and taking the generating morphisms for $\mathcal{APar}$ to the natural transformations defined as follows on $V \in \Bbbk S_t\text{-Mod}_{\mathrm{fd}}$ and $1 \leq i, j \leq t$:*

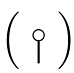
 $_V : U_t \otimes U_t \otimes V \to U_t \otimes U_t \otimes V, \qquad u_i \otimes u_j \otimes v \mapsto u_j \otimes u_i \otimes v,$

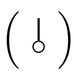
 $_V : U_t \otimes U_t \otimes V \to U_t \otimes U_t \otimes V, \qquad u_i \otimes u_j \otimes v \mapsto u_i \otimes u_j \otimes (i\ j)v,$

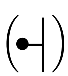
 $_V : U_t \otimes U_t \otimes V \to U_t \otimes U_t \otimes V, \qquad u_i \otimes u_j \otimes v \mapsto u_j \otimes u_i \otimes (i\ j)v,$

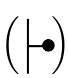
 $_V : U_t \otimes U_t \otimes V \to U_t \otimes V, \qquad u_i \otimes u_j \otimes v \mapsto \delta_{i,j} u_i \otimes v,$

 $_V : U_t \otimes V \to U_t \otimes U_t \otimes V, \qquad u_i \otimes v \mapsto u_i \otimes u_i \otimes v,$

 $_V : U_t \otimes V \to V, \qquad u_i \otimes v \mapsto v,$

 $_V : V \to U_t \otimes V, \qquad v \mapsto \sum_{i=1}^{t} u_i \otimes v,$

 $_V : U_t \otimes V \to U_t \otimes V, \qquad u_i \otimes v \mapsto \sum_{j=1}^{t} u_j \otimes (i\ j)v,$

 $_V : U_t \otimes V \to U_t \otimes V, \qquad u_i \otimes v \mapsto \sum_{j=1}^{t} u_i \otimes (i\ j)v.$

Proof. There is an isomorphism $\Bbbk S_t \otimes_{\Bbbk S_{t-1}} \mathrm{triv}_{S_{t-1}} \xrightarrow{\sim} U_t, g \otimes 1 \mapsto g u_t$. Combining this with the tensor identity, we obtain a natural $\Bbbk S_t$-module isomorphism

$$(\alpha_1^{(t)})_V : \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V \xrightarrow{\sim} U_t \otimes V, \qquad g \otimes v \mapsto g u_t \otimes g v \tag{3.4.3}$$

for $V \in \Bbbk S_t\text{-Mod}_{\mathrm{fd}}$. This defines an isomorphism $\alpha_1^{(t)} : \Bbbk S_t \otimes_{\Bbbk S_{t-1}} \overset{\sim}{\Rightarrow} U_t \otimes$. Let $\alpha_n^{(t)} := \alpha_1^{(t)} \cdots \alpha_1^{(t)}$ be the n-fold horizontal composition of $\alpha_1^{(t)}$. This is a natural isomorphism $\alpha_n^{(t)} : (\Bbbk S_t \otimes_{\Bbbk S_{t-1}})^{\circ n} \overset{\sim}{\Rightarrow} (U \otimes)^{\circ n}$ whose value on a $\Bbbk S_t$-module V is given explicitly by the map

$$\left(\alpha_n^{(t)}\right)_V : g_n \otimes \cdots \otimes g_1 \otimes v \mapsto g_n u_t \otimes g_n g_{n-1} u_t \otimes \cdots \otimes g_n g_{n-1} \cdots g_1 u_t \otimes g_n g_{n-1} \cdots g_1 v.$$

Now *define* $\Phi_t : \mathcal{APar} \to \mathcal{End}_{\Bbbk}(\Bbbk S_t\text{-Mod}_{\text{fd}})$ to be the strict monoidal functor taking the object n to $(U_t \otimes)^{\circ n}$, and defined on a morphism $f \in \operatorname{Hom}_{\mathcal{APar}}(n, m)$ by $\Phi_t(f) := \alpha_m^{(t)} \circ \Theta_t(f) \circ \left(\alpha_n^{(t)}\right)^{-1}$. It is immediate from this definition that $\alpha^{(t)} = \left(\alpha_n^{(t)}\right)_{n \geq 0} : \Theta_t \Rightarrow \Phi_t$ is an isomorphism of strict monoidal functors.

It remains to check that Φ_t as defined in the previous paragraph is equal to the functor Φ_t defined on generating morphisms in the statement of the lemma. So we need to check for each generating morphism $f \in \operatorname{Hom}_{\mathcal{APar}}(n, m)$ that the formula for $\Phi_t(f)_V$ written in the statement of the lemma is equal to $\left(\alpha_m^{(t)}\right)_V \circ \Theta_t(f)_V \circ \left(\alpha_n^{(t)}\right)_V^{-1}$ for $V \in \Bbbk S_t\text{-Mod}_{\text{fd}}$ and $t \in \mathbb{N}$. This is a routine but lengthy calculation. We just go through a couple of the cases.

If f is the crossing, we need to show that $\left(\left(\alpha_2^{(t)}\right)_V \circ \Theta_t(f)_V \circ \left(\alpha_2^{(t)}\right)_V^{-1}\right)(u_i \otimes u_j \otimes v) = u_j \otimes u_i \otimes v$. Now we consider four cases. If $t \neq i \neq j \neq t$ we have that

$$\begin{aligned}
\left(\left(\alpha_2^{(t)}\right)_V \circ \Theta_t(f)_V \circ \left(\alpha_2^{(t)}\right)_V^{-1}\right)&(u_i \otimes u_j \otimes v) \\
&= \left(\left(\alpha_2^{(t)}\right)_V \circ \Theta_t(f)_V\right)((i\,t) \otimes (j\,t) \otimes (j\,t)(i\,t)v) \\
&= \left(\alpha_2^{(t)}\right)_V ((i\,t)(j\,t) \otimes (j\,t) \otimes (i\,t)v) \\
&= u_j \otimes u_i \otimes v.
\end{aligned}$$

If $i = j$ we have that

$$\begin{aligned}
\left(\left(\alpha_2^{(t)}\right)_V \circ \Theta_t(f)_V \circ \left(\alpha_2^{(t)}\right)_V^{-1}\right)&(u_i \otimes u_j \otimes v) \\
&= \left(\left(\alpha_2^{(t)}\right)_V \circ \Theta_t(f)_V\right)((i\,t) \otimes 1 \otimes (i\,t)v) \\
&= \left(\alpha_2^{(t)}\right)_V ((i\,t) \otimes 1 \otimes (i\,t)v) \\
&= u_j \otimes u_i \otimes v.
\end{aligned}$$

If $i = t \neq j$ we have that

$$\begin{aligned}
\left(\left(\alpha_2^{(t)}\right)_V \circ \Theta_t(f)_V \circ \left(\alpha_2^{(t)}\right)_V^{-1}\right)&(u_i \otimes u_j \otimes v) \\
&= \left(\left(\alpha_2^{(t)}\right)_V \circ \Theta_t(f)_V\right)(1 \otimes (j\ t) \otimes (j\ t)v) \\
&= \left(\alpha_2^{(t)}\right)_V ((j\ t) \otimes (j\ t) \otimes v) \\
&= u_j \otimes u_i \otimes v.
\end{aligned}$$

Finally if $i \neq t = j$ we have that

$$\begin{aligned}
\left(\left(\alpha_2^{(t)}\right)_V \circ \Theta_t(f)_V \circ \left(\alpha_2^{(t)}\right)_V^{-1}\right)&(u_i \otimes u_j \otimes v) \\
&= \left(\left(\alpha_2^{(t)}\right)_V \circ \Theta_t(f)_V\right)((i\ t) \otimes (i\ t) \otimes v) \\
&= \left(\alpha_2^{(t)}\right)_V (1 \otimes (i\ t) \otimes (i\ t)v) \\
&= u_j \otimes u_i \otimes v.
\end{aligned}$$

This completes the check in this case.

If f is the left dot, we have that

$$\begin{aligned}
\left(\left(\alpha_1^{(t)}\right)_V \circ \Theta_t(f)_V \circ \left(\alpha_1^{(t)}\right)_V^{-1}\right)(u_i \otimes v) &= \left(\left(\alpha_1^{(t)}\right)_V \circ \Theta_t(f)_V\right)((i\ t) \otimes (i\ t)v) \\
= \sum_{j=1}^{t} \left(\alpha_1^{(t)}\right)_V((i\ t)(j\ t) \otimes (i\ t)v) &= \sum_{j=1}^{t} (i\ t)(j\ t)u_t \otimes (i\ t)(j\ t)(i\ t)v.
\end{aligned}$$

If $i = t$ this is $\sum_{j=1}^{t} u_j \otimes (j\ t)v$ which is right. If $i \neq t$ we pull out the $j = i$ and $j = t$ terms of the sum, simplify the three types of terms separately, then recombine to get the desired expression $\sum_{j=1}^{t} u_j \otimes (i\ j)v$. □

We now have in our hands monoidal functors ϕ_t from (3.1.3), i from (3.3.6), and Φ_t from (3.4.2). Let

$$\mathrm{Act} : \Bbbk S_t\text{-}\mathrm{Mod}_{\mathrm{fd}} \to \mathcal{E}\!nd_{\Bbbk}(\Bbbk S_t\text{-}\mathrm{Mod}_{\mathrm{fd}}) \tag{3.4.4}$$

be the monoidal functor induced by the Kronecker product, i.e., $\mathrm{Act}(V) = V \otimes$ for a $\Bbbk S_t$-module V and $\mathrm{Act}(f) = f \otimes$ for a homomorphism $f : V \to V'$.

Lemma 3.4.2. *For every $t \in \mathbb{N}$, the following diagram commutes up to the obvious canonical isomorphism of monoidal functors:*

$$\begin{tikzcd} \mathcal{APar} \arrow[r, "\Phi_t"] & \mathcal{End}_{\Bbbk}(\Bbbk S_t\text{-}\mathrm{Mod}_{\mathrm{fd}}) \\ \mathcal{Par} \arrow[u, "i"] \arrow[r, "\phi_t"] & \Bbbk S_t\text{-}\mathrm{Mod}_{\mathrm{fd}}. \arrow[u, "\mathrm{Act}"'] \end{tikzcd} \tag{3.4.5}$$

Proof. The composition $\Phi_t \circ i$ takes the nth object of $\mathcal{APar}$ to $(U_t \otimes)^{\circ n}$, while $\mathrm{Act} \circ \phi_t$ takes it to $U_t^{\otimes n} \otimes$. Let

$$\beta_n^{(t)} : (U_t \otimes)^{\circ n} \overset{\sim}{\Rightarrow} U_t^{\otimes n} \otimes$$

be the canonical isomorphism between these functors defined by associativity of tensor product. Then $\beta^{(t)} = \left(\beta_n^{(t)}\right)_{n \geq 0} : \Phi_t \circ i \Rightarrow \mathrm{Act} \circ \phi_t$ is an isomorphism of monoidal functors. To see this, we need to check naturality. This follows because the five formulae defining ϕ_t from Theorem 3.1.1 tensored on the right with a vector v are exactly the same as the formulae defining Φ_t on these five generating morphisms from Lemma 3.4.1. $\square$

Now we can prove the main theorem justifying the significance of the affine partition category. Let

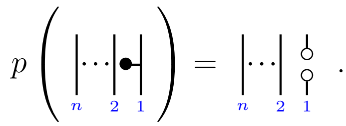
$$\mathrm{Ev} : \mathcal{End}_{\Bbbk}(\Bbbk S_t\text{-}\mathrm{Mod}_{\mathrm{fd}}) \to \Bbbk S_t\text{-}\mathrm{Mod}_{\mathrm{fd}} \tag{3.4.6}$$

be the (non-monoidal) functor defined by evaluating on triv_{S_t}. There is an obvious isomorphism of functors $\mathrm{Ev} \circ \mathrm{Act} \overset{\sim}{\Rightarrow} \mathrm{Id}_{\Bbbk S_t\text{-}\mathrm{Mod}_{\mathrm{fd}}}$ defined on V by the isomorphism $V \otimes \mathrm{triv}_{S_t} \to V, v \otimes 1 \mapsto v$.

Theorem 3.4.3. *There is a unique (non-monoidal) functor*

$$p : \mathcal{APar} \to \mathcal{Par} \tag{3.4.7}$$

such that $p \circ i = \mathrm{Id}_{\mathcal{Par}}$ *and*

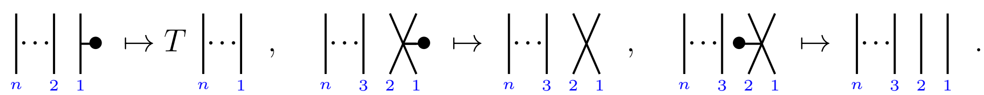


(3.4.8)

Moreover, for any $t \in \mathbb{N}$*, the following diagram of functors commutes up to natural isomorphism:*

$$\begin{tikzcd} \mathcal{APar} \arrow[r, "\Phi_t"] \arrow[d, "p"'] & \mathcal{End}_{\Bbbk}(\Bbbk S_t\text{-}\mathbf{Mod}_{\mathrm{fd}}) \arrow[d, "\mathrm{Ev}"] \\ \mathcal{Par} \arrow[r, "\phi_t"] & \Bbbk S_t\text{-}\mathbf{Mod}_{\mathrm{fd}}. \end{tikzcd} \tag{3.4.9}$$

The functor p *also maps*



(3.4.10)

Proof. For $t \in \mathbb{N}$, let $\gamma_n^{(t)} : U_t^{\otimes n} \otimes \mathrm{triv}_{S_t} \xrightarrow{\sim} U_t^{\otimes n}$ be the obvious isomorphism sending $u_{i_n} \otimes \cdots \otimes u_{i_1} \otimes 1 \mapsto u_{i_n} \otimes \cdots \otimes u_{i_1}$. We say that $f \in \mathrm{Hom}_{\mathcal{APar}}(n,m)$ is *good* if there exists a morphism $\bar{f} \in \mathrm{Hom}_{\mathcal{Par}}(n,m)$ such that

$$\phi_t(\bar{f}) = \gamma_m^{(t)} \circ \mathrm{Ev}(\Phi_t(f)) \circ \left(\gamma_n^{(t)}\right)^{-1} \tag{3.4.11}$$

for all $t \in \mathbb{N}$. If f is good, there is a *unique* $\bar{f}$ such that (3.4.11) holds for all t. To see this, suppose that $\bar{f}$ and $\bar{f}'$ both satisfy (3.4.11) for all $t \in \mathbb{N}$. Then $\phi_t(\bar{f}) = \gamma_m^{(t)} \circ \mathrm{Ev}(\Phi_t(f)) \circ \left(\gamma_n^{(t)}\right)^{-1} = \phi_t(\bar{f}')$, so that $\phi_t(\bar{f} - \bar{f}') = 0$ for all $t \in \mathbb{N}$. In view of Lemma 3.1.4 this implies that $\bar{f} = \bar{f}'$ as claimed.

Suppose that $f \in \mathrm{Hom}_{\mathcal{APar}}(n,m)$ and $g \in \mathrm{Hom}_{\mathcal{APar}}(l,m)$ are both good. Then $f \circ g$ is good with $\overline{f \circ g} := \bar{f} \circ \bar{g}$. This follows because

$$\phi_t(\bar{f}\circ\bar{g}) = \gamma_m^{(t)}\circ\mathrm{Ev}(\Phi_t(f))\circ\left(\gamma_m^{(t)}\right)^{-1}\circ\gamma_m^{(t)}\circ\mathrm{Ev}(\Phi_t(f))\circ\left(\gamma_l^{(t)}\right)^{-1} = \gamma_m^{(t)}\circ\mathrm{Ev}(\Phi_t(f\circ g))\circ\left(\gamma_l^{(t)}\right)^{-1}.$$

Similarly, sums of good morphisms are good with $\overline{f+g} := \bar{f} + \bar{g}$.

In this paragraph, we show that every morphism in $\mathcal{APar}$ is good. In view of the previous paragraph, it suffices to show that some family of generating morphisms for

$\mathcal{APar}$ are all good. Hence, in view of Corollary 3.3.6, it is enough to show that $i(f)$ is good for every morphism f in $\mathcal{Par}$ and that the morphisms (3.3.17) are good for all n. For $f \in \mathrm{Hom}_{\mathcal{Par}}(n, m)$, the morphism $i(f)$ is good with $\overline{i(f)} := f$. This follows from the following calculation using Lemma 3.4.2:

$$\gamma_m^{(t)} \circ \mathrm{Ev}(\Phi_t(i(f))) \circ \left(\gamma_m^{(t)}\right)^{-1} = \gamma_m^{(t)} \circ \mathrm{Ev}(\mathrm{Act}(\phi_t(f))) \circ \left(\gamma_m^{(t)}\right)^{-1} = \phi_t(f).$$

Also the morphism f from (3.3.17) is good for every n. To see this, let $\bar{f}$ be the morphism on the right hand side of (3.4.8). Using the definition in Theorem 3.1.1, $\phi_t(\bar{f})$ is the map $u_{i_n} \otimes \cdots \otimes u_{i_1} \mapsto \sum_{j=1}^{t} u_{i_n} \otimes \cdots \otimes u_{i_2} \otimes u_j$. Also using the definition in Lemma 3.4.1, $\mathrm{Ev}(\Phi_t(f))$ is the map $u_{i_n} \otimes \cdots \otimes u_{i_1} \otimes 1 \mapsto \sum_{j=1}^{t} u_{i_n} \otimes \cdots \otimes u_{i_2} \otimes u_j \otimes 1$. On contracting the final $\otimes 1$ using $\gamma_n^{(t)}$, these are equal, as required to prove that f is good.

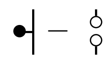
Now we can define a functor p making (3.4.9) commute (up to natural isomorphism) for all $t \in \mathbb{N}$. On objects, define p by declaring that $p(n) = n$ for each $n \geq 0$. On a morphism $f \in \mathrm{Hom}_{\mathcal{APar}}(n, m)$, we define $p(f) := \bar{f}$. The checks made so far imply that this is a well-defined functor satisfying (3.4.8). The equation (3.4.11) shows that $\gamma^{(t)} = \left(\gamma_n^{(t)}\right)_{n \geq 1} : \mathrm{Ev} \circ \Phi_t \Rightarrow \phi_t \circ p$ is a natural isomorphism. We have also already shown that $p \circ i = \mathrm{Id}_{\mathcal{Par}}$ and that (3.4.8) holds. Thus, we have established the existence of a functor $p : \mathcal{APar} \to \mathcal{Par}$ satisfying all of the properties in the statement of the theorem. The uniqueness of p follows from Corollary 3.3.6.

It remains to check the three properties (3.4.10). These can be checked using the commutativity of (3.4.9) in the same way as we just established (3.4.8). Alternatively, and possibly quicker, they can be deduced directly from (3.4.8) using the relations (3.3.9) to (3.3.11), respectively. We leave the details to the reader. $\square$

The faithfulness of i in the following corollary was already proved in two different ways in [LSR21]. Our approach is similar in spirit to the first proof given in *loc. cit.*, i.e., the argument used to prove [LSR21, Th. 5.2].

Corollary 3.4.4. *The functor $i : \mathcal{Par} \to \mathcal{APar}$ is faithful and the functor $p : \mathcal{APar} \to \mathcal{Par}$ is full.*

Proof. This follows because $p \circ i = \mathrm{Id}_{\mathcal{Par}}$. $\square$

Corollary 3.4.5. *The functor p induces an isomorphism $\mathcal{APar}/\mathcal{I} \xrightarrow{\sim} \mathcal{Par}$ where $\mathcal{I}$ is the left tensor ideal of $\mathcal{APar}$ generated by the morphism*  .

Proof. The left tensor ideal $\mathcal{I}$ is the data of subspaces $\mathcal{I}(n,m)$ of $\mathrm{Hom}_{\mathcal{APar}}(n,m)$ for each $m,n \geq 0$ which are closed under vertical composition on the top or bottom with any morphism and closed under horizontal composition on the left with any morphism. It is clear from (3.4.8) that p sends morphisms in $\mathcal{I}$ to zero, hence, p induces a functor $\bar{p} : \mathcal{APar}/\mathcal{I} \to \mathcal{Par}$. This is surjective on objects and full. To see that it is faithful, suppose that $f + \mathcal{I}(n,m) \in \mathrm{Hom}_{\mathcal{APar}/\mathcal{I}}(n,m) = \mathrm{Hom}_{\mathcal{APar}}(n,m)/\mathcal{I}(n,m)$ is a morphism sent to zero by $\bar{p}$, hence, $p(f) = 0$. In view of Corollary 3.3.6 and the definition of $\mathcal{I}$, we may assume that $f = i(\bar{f})$ for some $\bar{f} \in \mathrm{Hom}_{\mathcal{Par}}(n,m)$. Then $\bar{f} = p(i(\bar{f})) = p(f) = 0$, so that $f = i(\bar{f}) = 0$. $\square$

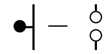
Composing the functor $p : \mathcal{APar} \to \mathcal{Par}$ with evaluation at any $t \in \Bbbk$ gives a full functor

$$p_t := \mathrm{ev}_t \circ p : \mathcal{APar} \to \mathcal{Par}_t \tag{3.4.12}$$

such that

 (3.4.13)

 (3.4.14)

Like in Corollary 3.4.5, the functor p_t induces an isomorphism $\mathcal{APar}/\mathcal{I}_t \xrightarrow{\sim} \mathcal{Par}_t$ where $\mathcal{I}_t$ is the left tensor ideal of $\mathcal{APar}$ generated by $T - t1_{\mathbb{1}}$ and  .

3.5 Jucys-Murphy elements for partition algebras

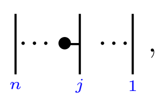
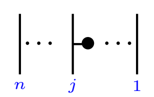
Now we can explain how affine partition category is related to the works of Enyang [Eny13] and Halverson-Ram [HR05]. These are concerned with the *partition algebra*, which is the endomorphism algebra

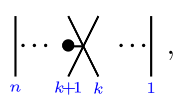
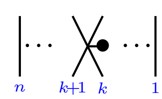
$$P_n(t) := \mathrm{End}_{\mathcal{Par}_t}(n) = 1_n Par_t 1_n. \tag{3.5.1}$$

By analogy, we define the *affine partition algebra* to be

$$AP_n := \mathrm{End}_{\mathcal{APar}}(n) = 1_n APar 1_n. \tag{3.5.2}$$

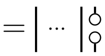
Let us denote the elements of AP_n defined by the left and right dots on the jth string by X_j^L and X_j^R, and the elements defined by the left and right crossings of the kth and $(k+1)$th strings by S_k^L and S_k^R:

$X_j^L :=$  , $\quad X_j^R :=$  , (3.5.3)

$S_k^L :=$  , $\quad S_k^R :=$  (3.5.4)

for $1 \leq j \leq n$ and $1 \leq k \leq n-1$. We note that $\{X_j^L, X_j^R \mid j = 1, \ldots, n\}$ are algebraically independent, so they generate a free polynomial algebra of rank $2n$ inside $AP_n(t)$; this follows easily from the basis theorem for morphism spaces $\mathcal{Heis}$ proved in [Kho14]. Taking the images of the elements (3.5.3) and (3.5.4) under the functor p_t from (3.4.12) gives us elements of $P_n(t)$ denoted

$$x_j^L := p_t(X_j^L), \qquad x_j^R := p_t(X_k^R), \qquad s_k^L := p_t(S_k^L), \qquad s_k^R := p_t(S_k^R). \tag{3.5.5}$$

The notation here depends implicitly on the values of n and t, which should be clear from the context. By (3.4.13) and (3.4.14), we have that $x_1^L =$  , $x_1^R = t$, $s_1^L = 1$ and $s_1^R = (1\,2) \in S_n \subset P_n(t)$.

Theorem 3.5.1. *Suppose that $t \in \mathbb{N}$ and let $\psi_t : P_n(t) \to \mathrm{End}_{\Bbbk S_t}(U_t^{\otimes n})$ be the homomorphism induced by the functor ϕ_t from Theorem 3.1.1. The elements $x_j^L, x_j^R, s_k^L, s_k^R \in P_n(t)$ satisfy*

$$\psi_t(x_j^L)(u_{i_n} \otimes \cdots \otimes u_{i_1}) = \sum_{i=1}^{t} u_{i_n} \otimes \cdots \otimes u_{i_{j+1}} \otimes (i \;\; i_j) \left[u_{i_j} \otimes \cdots \otimes u_{i_2} \otimes u_{i_1}\right], \tag{3.5.6}$$

$$\psi_t(x_j^R)(u_{i_n} \otimes \cdots \otimes u_{i_1}) = \sum_{i=1}^{t} u_{i_n} \otimes \cdots \otimes u_{i_j} \otimes (i \;\; i_j) \left[u_{i_{j-1}} \otimes \cdots \otimes u_{i_2} \otimes u_{i_1}\right], \tag{3.5.7}$$

$$\psi_t(s_k^L)(u_{i_n} \otimes \cdots \otimes u_{i_1}) = u_{i_n} \otimes \cdots \otimes u_{i_k} \otimes (i_k \;\; i_{k+1}) \left[u_{i_{k-1}} \otimes \cdots \otimes u_{i_2} \otimes u_{i_1}\right], \tag{3.5.8}$$

$$\psi_t(s_k^R)(u_{i_n} \otimes \cdots \otimes u_{i_1}) = u_{i_n} \otimes \cdots \otimes u_{i_{k+2}} \otimes (i_k \;\; i_{k+1}) \left[u_{i_{k+1}} \otimes \cdots \otimes u_{i_2} \otimes u_{i_1}\right] \tag{3.5.9}$$

for $1 \le i_1, \ldots, i_n \le t$, where we are using the diagonal action of S_t on tensor powers of U_t.

Proof. This follows from the commutativity of (3.4.9), (3.5.5) and the formulae in Lemma 3.4.1. □

Corollary 3.5.2. *Identifying $P_n(t)$ with the partition algebra in [Eny13] by reflecting diagrams through a vertical axis to account for the fact that we number vertices from right to left rather than from left to right, the elements (3.5.5) are related to the elements $L_{\frac{1}{2}}, L_1, \ldots$ and $\sigma_{\frac{3}{2}}, \sigma_2, \ldots$ of the partition algebra $P_n(t)$ defined in [Eny13]* *according to the dictionary*

$$x_j^L \leftrightarrow L_j, \qquad t - x_j^R \leftrightarrow L_{j-\frac{1}{2}}, \qquad s_k^L \leftrightarrow \sigma_{k+\frac{1}{2}}, \qquad s_k^R \leftrightarrow \sigma_{k+1}. \tag{3.5.10}$$

Hence, by [Eny13, Th. 5.5], the elements x_j^L and $t - x_j^R$ are identified with the Jucys-Murphy elements introduced originally by Halverson and Ram in [HR05].

Proof. Enyang's elements are defined by a recurrence relation which is independent of the value of the parameter t. Hence, his elements can be viewed as specializations at $T = t$ of corresponding elements of the generic partition algebra $\mathrm{End}_{\mathcal{P}ar}(n)$. To identify them with our elements, we can use Lemma 3.1.4 to see that it suffices to check that they act in the same way on $U_t^{\otimes n}$ for infinitely many values of the parameter $t \in \mathbb{N}$. This follows on comparing (3.5.6) to (3.5.9) to the formulae in [Eny13, Prop. 5.2, Prop. 5.3]. □

Remark 3.5.3. Alternatively, one can prove Corollary 3.5.2 inductively, using the recurrence relations in Lemma 3.3.5 which are equivalent to Enyang's recurrence relations [Eny13, (3.1)–(3.4)]. In fact, all of the relations derived in *loc. cit.* can now be deduced easily using the relations in $\mathcal{AP}ar$ derived in the previous section.

Remark 3.5.4. Recently, Creedon [Cre21] has introduced a renormalization of the Jucys-Murphy elements, which he denotes by $N_1, N_2, \ldots, N_{2n} \in P_n(t)$. They are defined in terms of the Enyang-Halverson-Ram elements simply by $N_{2j-1} := L_{j-\frac{1}{2}} - \frac{t}{2}$ and $N_{2j} := L_j - \frac{t}{2}$. The dictionary between Creedon's elements and ours is

$$x_j^L - \tfrac{t}{2} \leftrightarrow N_{2j}, \qquad \tfrac{t}{2} - x_j^R \leftrightarrow N_{2j-1}. \tag{3.5.11}$$

The motivation for such a renormalization will be discussed further in Remark 3.6.5 below.

3.6 Central elements

By the *center* of a ($\Bbbk$-linear) category $\mathcal{A}$, we mean the (unital) commutative algebra $Z(\mathcal{A}) := \mathrm{End}_{\Bbbk}(\mathrm{Id}_{\mathcal{A}})$ of endomorphisms of the identity endofunctor of $\mathcal{A}$. Thus, an element $z \in Z(\mathcal{A})$ is a tuple $(z_X)_{X\in\mathbb{O}(\mathcal{A})}$ such that $z_Y \circ f = f \circ z_X$ for all morphisms $f : X \to Y$ in $\mathcal{A}$. Equivalently, in terms of the path algebra A, it is the algebra

$$Z(A) := \left\{ z = (z_X)_{X\in\mathbb{O}(\mathcal{A})} \in \prod_{X\in\mathbb{O}(\mathcal{A})} 1_X A 1_X \;\middle|\; za = az \text{ for all } a \in A \right\}, \tag{3.6.1}$$

interpreting the products in the obvious way. We note that there is an algebra isomorphism

$$\mathrm{End}_{A\boxtimes A^{\mathrm{op}}}(A) \xrightarrow{\sim} Z(A), \qquad \zeta \mapsto (\zeta(1_X))_{x\in\mathbb{O}(\mathcal{A})} \in \prod_{X\in\mathbb{O}(\mathcal{A})} 1_X A 1_X, \tag{3.6.2}$$

where the algebra on the left is the endomorphism algebra of the $A \boxtimes A^{\mathrm{op}}$-module associated to the (A, A)-bimodule A. If A is locally finite-dimensional, then it is a locally finite-dimensional $A \boxtimes A^{\mathrm{op}}$-module, hence, by [BS, Lem. 2.10], the endomorphism algebra $\mathrm{End}_{A\boxtimes A^{\mathrm{op}}}(A) \cong Z(A)$ is a pseudo-compact topological algebra with respect to the pro-finite topology. That is, the topology of $Z(A)$ is such that the ideals of finite codimension form a base of neighborhoods of 0. Pseudo-compactness means that $Z(A)$ is isomorphic to $\varprojlim Z(A)/J$ where the inverse limit is over all ideals of finite codimension.

In the locally finite-dimensional case, $Z(A)$ is isomorphic to the algebra $C(A)^*$ that is the linear dual of the *cocenter* $C(A)$. The cocenter is a cocommutative coalgebra isomorphic to $\mathrm{Coend}_{A\boxtimes A^{\mathrm{op}}}(A)$ in the notation of [BS, (2.15)]. To define $C(A)$ explicitly, note that the space $D := \bigoplus_{X,Y\in\mathbb{O}_A}(1_X A 1_Y)^*$ is naturally an (A, A)-bimodule with $1_Y D 1_X = (1_X A 1_Y)^*$. Also each $1_X D 1_X$ is a coalgebra as it is the dual of the finite-dimensional algebra $1_X A 1_X$. Hence, $\bigoplus_{X\in\mathbb{O}_A} 1_X D 1_X$ is a coalgebra.

Then the cocenter is

$$C(A) := \Big(\bigoplus_{X \in \mathbb{O}_A} 1_X D 1_X \Big) \Big/ J \tag{3.6.3}$$

where J is the coideal spanned by the elements

$$\left\{ af - fa \;\middle|\; X, Y \in \mathbb{O}_A, a \in 1_X A 1_Y, f \in 1_Y D 1_X \right\}.$$

To identify $C(A)^*$ with $Z(A)$, note that the linear dual of the coalgebra $\bigoplus_{X \in \mathbb{O}_A} 1_X D 1_X$ is the algebra $\prod_{X \in \mathbb{O}_A} 1_X A 1_X$; the annihilator J° of the coideal J defines a subalgebra of $\prod_{X \in \mathbb{O}_A} 1_X A 1_X$ which is exactly the center $Z(A)$ according to the original definition (3.6.1).

In this subsection, we are going to construct a family of elements $(z^{(r)})_{r \geq 1}$ in the center $Z(APar_t)$ of the affine partition category $\mathcal{APar}_t$. We start by introducing some convenient shorthand. Given a monomial $x^r y^s \in \Bbbk[x, y]$, we use the notation

$$\big|\!\bullet_{x^r y^s} := \Big(\big|\!\!-\!\bullet \Big)^{\circ r} \circ \Big(\bullet\!\!-\!\!\big| \Big)^{\circ s} \tag{3.6.4}$$

to denote the element of $\mathrm{End}_{\mathcal{APar}}(\,|\,)$ on the right hand side, that is, it is the rth power of the right dot (represented by x) composed with the sth power of the left dot (represented by y). It then makes sense to label dots by polynomials $f(x) \in \Bbbk[x, y]$, meaning the linear combination of the morphisms $\big|\!\bullet_{x^r y^s}$ just as $f(x)$ is the linear combination of its monomials. We are also going to use generating functions in the same way as explained in the context of $\mathcal{Heis}$ in [BSW20, §3.1]. For these, u will be a formal variable which should always be interpreted by expanding as formal Laurent series in $\Bbbk(\!(u^{-1})\!)$, e.g., $(u - x)^{-1} = u^{-1} + u^{-2}x + u^{-3}x^2 + \cdots$.

Let

$$\bigcirc(u) := u 1_{\mathbb{1}} - \overset{\circ}{\underset{\circ}{\big|}}\!\bullet_{(u-x)^{-1}} = u 1_{\mathbb{1}} - \overset{\circ}{\underset{\circ}{\big|}}\!\bullet_{(u-y)^{-1}} \in u 1_{\mathbb{1}} + u^{-1} \,\mathrm{End}_{\mathcal{APar}}(\mathbb{1})[\![u^{-1}]\!]. \tag{3.6.5}$$

For $r \geq 0$, the coefficient of u^{-r-1} in this formal Laurent series is $-\,\bullet_{x^r}$; the x^r here can be replaced by y^r due to the third relation in (3.3.8). Also introduce the rational function

$$\alpha_x(u) := \frac{(u-(x+1))(u-(x-1))}{(u-x)^2} \in \Bbbk(x,u). \tag{3.6.6}$$

The expansion of this as a power series in $\Bbbk[x][\![u^{-1}]\!]$ is

$$\alpha_x(u) = 1-(u-x)^{-2} = 1-u^{-2}-2xu^{-3}-3x^2u^{-4}-4x^3u^{-5}-\cdots, \tag{3.6.7}$$

$$\alpha_x(u)^{-1} = 1+u^{-2}+2xu^{-3}+(3x^2+1)u^{-4}+(4x^3+4x)u^{-5}+\cdots. \tag{3.6.8}$$

The following elementary lemma will play a fundamental role in the rest of the article. It would be hard to formulate this without the aid of generating functions.

Lemma 3.6.1. *The following bubble slide relations hold in* $\mathcal{APar}$:

$$\bigcirc(u)\,\Big| = \tfrac{\alpha_y(u)}{\alpha_x(u)}\bullet\!\Big|\;\bigcirc(u)\,, \qquad \Big|\,\bigcirc(u) = \bigcirc(u)\;\Big|\!\bullet\tfrac{\alpha_x(u)}{\alpha_y(u)}\,. \tag{3.6.9}$$

Proof. The two equations are equivalent, so we just prove the first one. When working with $\mathcal{H}eis$, we adopt the notation of [BSW20, §3.1]: an open dot labelled by x^r means the rth power of the open dot in $\mathcal{H}eis$, and $\circlearrowright(u)$ is the formal Laurent series from [BSW20, (3.13)]. Under the embedding of $\mathcal{APar}$ into $\mathcal{H}eis$, we have that

$$\bigcirc(u+1) = (u+1)1_{\mathbb{1}} - \bullet_{(u-(y-1))^{-1}} = (u+1)1_{\mathbb{1}} - {}_{(u-x)^{-1}}\!\circ\!\circlearrowright = 1_{\mathbb{1}} + \circlearrowright(u).$$

The bubble slide relation for $\mathcal{H}eis$ from [BSW20, (3.18)] gives that

$$\circlearrowright(u)\;\uparrow\;\downarrow = {}_{\alpha_x(u)}\!\circ\!\!\uparrow\;\circlearrowright(u)\;\downarrow = {}_{\alpha_x(u)}\!\circ\!\!\uparrow\;\;\downarrow\!\!\circ_{\alpha_x(u)^{-1}}\circlearrowright(u)\;.$$

According to (3.3.4), the label x on the open dot on the $\downarrow$ string translates into the label $x-1$ on a closed dot in $\mathcal{APar}$, and the label x on the open dot on the $\uparrow$ string translates into the label $y-1$ on a closed dot in $\mathcal{APar}$. So the relation just recorded can be written equivalently as

$$\bigcirc(u+1)\,\Big| = \frac{\alpha_{y-1}(u)}{\alpha_{x-1}(u)}\,\Big|\!\!\bullet\ \bigcirc(u+1) \;=\; \frac{\alpha_y(u+1)}{\alpha_x(u+1)}\,\Big|\!\!\bullet\ \bigcirc(u+1)\quad .$$

Replacing u by $u-1$ everywhere gives the desired relation. ☐

The rational function $\alpha_y(u)/\alpha_x(u) \in \Bbbk(x,y,u)$ will also be important later on. The low degree terms of its expansion as a power series in u^{-1} can be computed using (3.6.7) and (3.6.8):

$$\frac{\alpha_y(u)}{\alpha_x(u)} = 1+2(x-y)u^{-3}+3(x^2-y^2)u^{-4}+\left[4(x^3-y^3)+2(x-y)\right]u^{-5}+\cdots. \tag{3.6.10}$$

For $n \geq 0$, let

$$C_n(u) = \sum_{r\geq 0} C_n^{(r)} u^{-r} := \bigcirc(u)\star 1_n \star \bigcirc(u)^{-1} = \underset{n}{\frac{\alpha_y(u)}{\alpha_x(u)}\Big|\!\!\bullet}\ \cdots\ \underset{2}{\Big|\!\!\bullet\frac{\alpha_y(u)}{\alpha_x(u)}}\ \underset{1}{\Big|\!\!\bullet\frac{\alpha_y(u)}{\alpha_x(u)}} \;\in 1_n APar 1_n[\![u^{-1}]\!], \tag{3.6.11}$$

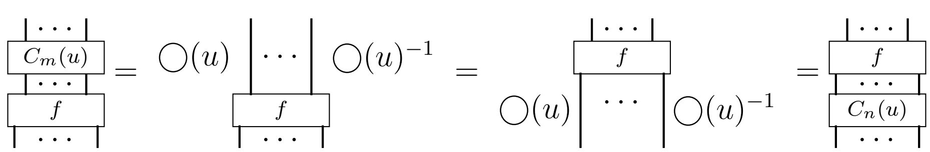
where the final equality follows by applying the bubble slide relation repeatedly. Then we define

$$C(u) = \sum_{r\geq 0} C^{(r)} u^{-r} := (C_n(u))_{n\geq 0} \in \prod_{n\geq 0} 1_n APar 1_n[\![u^{-1}]\!]. \tag{3.6.12}$$

Note by (3.6.10) that $C^{(0)} = 1$ and $C^{(1)} = C^{(2)} = 0$.

Theorem 3.6.2. $C(u) \in Z(APar)[\![u^{-1}]\!]$.

Proof. The interchange law immediately gives that



for any $f \in \mathrm{Hom}_{\mathcal{APar}}(n,m)$. ☐

The proof of the following corollary is similar to an argument used to simplify some analogous central elements in the quantum Heisenberg category in [MS22, Prop. 4.3].

Corollary 3.6.3. *For each $r \geq 1$, the element $Z^{(r)} = (Z_n^{(r)})_{n\geq 0} \in \prod_{n\geq 0} 1_n APar 1_n$ defined from*

$$Z_n^{(r)} := \sum_{i=1}^{n} \left((X_i^L)^r - (X_i^R)^r \right) = \left(X_1^L\right)^r + \cdots + \left(X_n^L\right)^r - \left(X_1^R\right)^r - \cdots - \left(X_n^R\right)^r$$

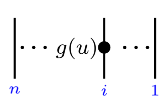
belongs to $Z(APar)$ (notation as in (3.5.3) and (3.5.4)). Moreover, the elements $Z^{(1)}, Z^{(2)}, \ldots$ generate the same subalgebra $Z_0(APar)$ of $Z(APar)$ as the elements $C^{(3)}, C^{(4)}, \ldots$.

Proof. Let $f(u) := \alpha_y(u)/\alpha_x(u)$ for short. Then define $g(u) := f'(u)/f(u) = \frac{d}{du}(\ln f(u))$ to be its logarithmic derivative. We have that

$$\begin{aligned} g(u) = &\left(-\frac{1}{u-(x-1)} + \frac{2}{u-x} - \frac{1}{u-(x+1)}\right) \\ &\qquad - \left(-\frac{1}{u-(y-1)} + \frac{2}{u-y} - \frac{1}{u-(y+1)}\right) \\ &= 2\cdot 3(y-x)u^{-4} + 2\cdot 6(y^2-x^2)u^{-5} + 2\cdot[10(y^3-x^3)+5(y-x)]u^{-6} + \cdots . \end{aligned}$$

We deduce for $r \geq 1$ that the u^{-r-3}-coefficient of $g(u)$ is equal to $2\binom{r+2}{2}(y^r - x^r)$ plus a linear combination of terms $(y^s - x^s)$ for $1 \leq s < r$ with $s \equiv r \pmod 2$.

The coefficients of the power series $C'(u)/C(u)$ are polynomials in the coefficients of the series $C(u)$. Hence, by the theorem, these coefficients are all central. To compute them, we take logarithmic derivatives of (3.6.11) to obtain the identity

$$C_n'(u)/C_n(u) = \sum_{i=1}^{n}$$ 

.

Using the previous paragraph and the definition of $Z^{(r)}$, we deduce for $r \geq 1$ that the central element defined by the u^{-r-3}-coefficient of $C'(u)/C(u)$ is equal to $2\binom{r+2}{2}Z^{(r)}$ plus a linear combination of $Z^{(s)}$ for $1 \leq s < r$ with $s \equiv r \pmod 2$. Finally, induction on r shows that each $Z^{(r)}$ is central.

The argument just given shows that each $Z^{(r)}$ lies in the subalgebra generated by $C^{(3)}, C^{(4)}, \ldots$. Conversely, by exponentiating an anti-derivative of the series

$C'(u)/C(u)$, one shows that each $C^{(r)}$ can be expressed as a polynomial in $Z^{(1)}, Z^{(2)}, \ldots$. Hence, the two families of elements generate the same subalgebra of $Z(APar)$. $\square$

Taking the images of $C(u)$ and each $Z^{(r)}$ under the functor p_t from (3.4.12) give

$$c(u) = \sum_{r\geq 0} c^{(r)} u^{-r} := (c_n(u))_{n\geq 0} \in Z(Par_t)[\![u^{-1}]\!], \tag{3.6.13}$$

$$\text{where } c_n(u) = \sum_{r\geq 0} c_n^{(r)} u^{-r} := p_t(C_n(u)),$$

$$z^{(r)} := \left(z_n^{(r)}\right)_{n\geq 0} \in Z(Par_t), \qquad \text{where } z_n^{(r)} := p_t(Z_n^{(r)}). \tag{3.6.14}$$

The elements $c_n^{(r)}$ and $z_n^{(r)}$ belong to the center $Z(P_n(t))$ of the partition algebra $P_n(t)$. In terms of the Jucys-Murphy elements (3.5.5), we have that

$$z_n^{(r)} = \sum_{i=1}^{n} \left[\left(x_i^L\right)^r - \left(x_i^R\right)^r\right] = (x_1^L)^r + \cdots + (x_n^L)^r - (x_1^R)^r - \cdots - (x_n^R)^r. \tag{3.6.15}$$

From Corollary 3.5.2, it follows that $z_n^{(1)}$ equals $z_n - nt$ where z_n is the central element from [Eny13, Th. 3.10(2)]. In fact, $z_n^{(1)}$ is closely related to the central elements of the group algebras $\Bbbk S_t$ defined by sums of transpositions:

Lemma 3.6.4 ([Eny13, Prop. 5.4]). *If $t \in \mathbb{N}$ then $\psi_t(z_n^{(1)}) : U_t^{\otimes n} \to U_t^{\otimes n}$ is equal to the endomorphism defined by the action of $\sum_{1\leq i<j\leq t} ((i\ j) - 1) \in Z(\Bbbk S_t)$.*

Remark 3.6.5. After constructing the elements $z_n^{(r)} \in Z(P_n(t))$ in the manner explained above, we came across a recent paper of Creedon which constructs similar central elements; see [Cre21, Th. 3.2.6]. To explain the connection, recall that the *rth supersymmetric power sum* in variables $x_1, \ldots, x_n, y_1, \ldots, y_m$ is $p_r(x_1, \ldots, x_n | y_1, \ldots, y_m) = x_1^r + \cdots + x_n^r - y_1^r - \cdots - y_m^r$. The expression on the right hand side of (3.6.15) is $p_r(x_1^L, \ldots, x_n^L | x_1^R, \ldots, x_n^R)$. It is easy to see that these elements belong to $Z(P_n(t))$ for all $r \geq 1$ if and only if $p_r(x_1^L - t/2, \ldots, x_n^L - t/2 | x_1^R -$

$t/2, \ldots, x_n^R - t/2) \in Z(P_n(t))$ for all $r \geq 1$. Moreover, $p_r(x_1^L - t/2, \ldots, x_n^L - t/2 | x_1^R - t/2, \ldots, x_n^R - t/2) \in Z(P_n(t))$ coincides with the rth supersymmetric power sum $p_r(N_2, N_4, \ldots, N_{2n} | -N_1, -N_3, \ldots, -N_{2n-1})$ in Creedon's renormalized Jucys-Murphy elements from (3.5.11). Creedon showed that his elements are central in $P_n(t)$ by a direct check of relations. This gives an independent way to verify that the eleemnts $z^{(r)} = (z_n^{(r)})_{n \geq 0}$ belong to $Z(\mathcal{P}ar_t)$: Creedon's calculations show that they commute with all crossings (a surprisingly hard calculation), and after that it is easy to see that they commute with all other generators of $\mathcal{P}ar_t$.

Remark 3.6.6. In [CO11, Def. 4.5], Comes and Ostrik define another family of central elements $\omega^r(t) = (\omega_n^r(t))_{n \geq 0}$ which lift the central elements of the group algebras $\Bbbk S_t$ defined by the sums of all r-cycles. We expect that our elements $z_n^{(r)}$ and their elements $\omega_n^r(t)$ are closely related, but we do not know any explicit formula. In particular, the Comes-Ostrik elements should generate the same subalgebra of $Z(Par_t)$ as our elements.

3.7 Harish-Chandra homomorphism

Although we just explain in the case of $\mathcal{P}ar_t$, the general development in this section is valid for any monoidal triangular category, replacing $\mathcal{S}ym$ with the (semisimple) Cartan subcategory and replacing the set $\mathcal{P}$ of partitions by a set parametrizing isomorphism classes of irreducible representations of the Cartan subcategory.

According to the general definition (3.6.1), the center of the partition category is a subalgebra of the unital algebra $\prod_{n \geq 0} 1_n Par_t 1_n$. Let K^+ (resp., K^-) be the left ideal (resp., right ideal) of Par_t generated by the strictly downward partition diagrams (resp., the strictly upward partition diagrams). From the triangular basis, it is easy to see that $1_n K^+ 1_n = 1_n K^- 1_n$. We denote this by K_n. It is a two-sided ideal of the finite-dimensional algebra $1_n Par_t 1_n$, and we have that

$$1_n Par_t 1_n = \Bbbk S_n \oplus K_n. \tag{3.7.1}$$

Equivalently, K_n is the two-sided ideal of $1_n Par_t 1_n$ spanned by morphisms that factor through objects $m < n$. By analogy with Lie theory, we define the *Harish-Chandra homomorphism*

$$\widehat{\mathrm{HC}} : \prod_{n \geq 0} 1_n Par_t 1_n \to \prod_{n \geq 0} \Bbbk S_n, \qquad (z_n)_{n \geq 0} \mapsto (\mathrm{HC}_n\, z_n)_{n \geq 0}, \tag{3.7.2}$$

where $\mathrm{HC}_n : 1_n Par_t 1_n \twoheadrightarrow \Bbbk S_n$ is the projection along the direct sum decomposition (3.7.1). It is obvious from (3.7.1) that the restriction of $\widehat{\mathrm{HC}}$ to $Z(Par_t)$ defines an algebra homomorphism

$$\mathrm{HC} : Z(Par_t) \to Z(Sym) = \prod_{n \geq 0} Z(\Bbbk S_n). \tag{3.7.3}$$

As each $\Bbbk S_n$ is semisimple with its isomorphism classes of irreducible representations parametrized by $\mathcal{P}_n$, we can identify the algebra appearing on the right hand side of (3.7.3) with the algebra $\Bbbk[\mathcal{P}]$ of all functions from the set $\mathcal{P}$ to the field $\Bbbk$ with pointwise operations. Under this identification, the tuple $(z_n)_{n \geq 0} \in \prod_{n \geq 0} Z(\Bbbk S_n)$ corresponds to the function $f : \mathcal{P} \to \Bbbk$ such that $f(\lambda)$ is the scalar that z_n acts by on the Specht module $S(\lambda)$ for each $\lambda \in \mathcal{P}_n$. Then the Harish-Chandra homorphism becomes a homomorphism

$$\mathrm{HC} : Z(Par_t) \to \Bbbk[\mathcal{P}]. \tag{3.7.4}$$

To describe HC more explicitly in these terms, let $\lambda \in \mathcal{P}_n$ be a partition. As we have that $\mathrm{End}_{Par_t}(\Delta(\lambda)) \cong \mathrm{End}_{Sym}(S(\lambda)) \cong \Bbbk$, an element $z = (z_n)_{n \geq 0} \in Z(Par_t)$ acts on the standard module $\Delta(\lambda)$ as multiplication by a scalar denoted $\chi_\lambda(z)$. This defines an algebra homomorphism

$$\chi_\lambda : Z(Par_t) \to \Bbbk. \tag{3.7.5}$$

To compute $\chi_\lambda(z)$, note that it is the scalar by which z_n acts on the highest weight space $1_n\Delta(\lambda)$, which is the scalar arising from the action of $\mathrm{HC}_n(z_n) \in Z(\Bbbk S_n)$ on $S(\lambda)$. It follows that

$$\chi_\lambda(z) = \mathrm{HC}_n(z_n)(\lambda) = \mathrm{HC}(z)(\lambda). \tag{3.7.6}$$

Recall that $Z(Par_t)$ is a commutative pseudo-compact topological algebra with respect to the profinite topology. Let $\mathrm{Spec}(Z(Par_t))$ be its set of open (= finite-codimensional) maximal ideals.

Lemma 3.7.1. $\mathrm{Spec}(Z(Par_t)) = \{\ker \chi_\lambda \mid \lambda \in \mathcal{P}\}$.

Proof. Points in $\mathrm{Spec}(Z(Par_t))$ parametrize isomorphism classes of finite-dimensional irreducible modules for $Z(Par_t)$ Let L_λ be the irreducible $Z(Par_t)$-module associated to $\chi_\lambda : Z(Par_t) \to \Bbbk$. Then we need to show that any finite-dimensional irreducible $Z(Par_t)$-module L is isomorphic to L_λ for some $\lambda \in \mathcal{P}$. To see this, we find it easiest to work equivalently in terms of irreducible comodules over the cocenter $C := C(Par_t)$ defined in (3.6.3). So let L be an irreducible C-comodule and L^* be the dual comodule, there being no need to distinguish between left or right since C is cocommutative. By definition, C is a quotient of the coalgebra D that is the direct sum of the coalgebras $(1_n Par_t 1_n)^*$ for all $n \geq 0$. Since L^* is isomorphic to a subcomodule of the regular C-comodule, it follows that L^* is isomorphic to a subquotient of the restriction of the regular D-comodule to C. Hence, L^* is isomorphic to a subquotient of $(1_n Par_t 1_n)^*$ for some n. So L is isomorphic to a subquotient of $1_n Par_t 1_n$. Now recall that the left Par_t-module $Par_t 1_n$ has a Δ-flag, and $z \in Z(Par_t)$ acts on $\Delta(\lambda)$ as multiplication by the scalar $\chi_\lambda(z)$. Hence, all composition factors of the finite-dimensional $Z(Par_t)$-module $1_n Par_t 1_n$ are of the form L_λ for $\lambda \in \mathcal{P}$. $\square$

Let $\approx_t$ be the equivalence relation on $\mathcal{P}$ defined by

$$\lambda \approx_t \mu \Leftrightarrow \chi_\lambda = \chi_\mu. \tag{3.7.7}$$

From Lemma 3.7.1, we see that the equivalence classes $\mathcal{P}/\approx_t$ parametrize the points in $\mathrm{Spec}(Z(Par_t))$.

Lemma 3.7.2. *The image of* $\mathrm{HC} : Z(Par_t) \to \Bbbk[\mathcal{P}]$ *consists of of all functions* $f \in \Bbbk[\mathcal{P}]$ *which are constant on* $\approx_t$*-equivalence classes. Moreover, for each subset* S *of* $\mathcal{P}$ *that is a union of* $\approx_t$*-equivalence classes, there is a unique central idempotent* $1_S \in Z(Par_t)$ *such that*

$$\mathrm{HC}(1_S)(\lambda) = \begin{cases} 1 & \text{if } \lambda \in S, \\ 0 & \text{otherwise.} \end{cases} \tag{3.7.8}$$

If S *is a single equivalence class then* 1_S *is a primitive idempotent, and* $Z(Par_t) = \prod_{S \in \mathcal{P}/\approx_t} 1_S Z(Par_t)$.

Proof. It is clear from (3.7.6) that any function in the image of HC is constant on $\approx_t$-equivalence classes. Conversely, take a function $f \in \Bbbk[\mathcal{P}]$ which is constant on equivalence classes. For an equivalence class $S \in \mathcal{P}/\approx_t$, let L_S be the irreducible $Z(Par_t)$-module associated to the central character χ_λ $(\lambda \in S)$. The previous lemma shows that these give a full set of pairwise inequivalent irreducible finite-dimensional $Z(Par_t)$-modules. It follows that the cocommutative coalgebra $C(Par_t)$ decomposes as a direct sum of indecomposable coideals

$$C(Par_t) = \bigoplus_{S \in \mathcal{P}/\approx_t} C_S,$$

where C_S is the injective hull of L_S. Then we consider the linear map $\theta : C(Par_t) \to C(Par_t)$ defined by multiplication by the scalar $f(\lambda)$ $(\lambda \in S)$ on the summand C_S. This is a comodule homomorphism. Now we use that

$$Z(Par_t) = C(\mathcal{P}ar_t)^* \cong \mathrm{End}_{C(Par_t)}(C(Par_t))^{\mathrm{op}}$$

as holds for any coalgebra, e.g., see [BS, Lem. 2.2]. It implies that θ defines an element of $Z(Par_t)$. The image of this element under HC is the function $f \in \Bbbk[\mathcal{P}]$.

To prove the existence of the idempotent 1_S for any S that is a union of $\approx_t$-equivalence classes, we apply the construction in the previous paragraph to obtain $1_S \in Z(Par_t)$ such that 1_S acts as the identity on the indecomposable summands $C_{S'}$ of $C(Par_t)$ for all $\approx_t$-equivalence classes $S' \subseteq S$ and as zero on all other summands. This is an idempotent satisfying (3.7.8), and it is a primitive idempotent if and only if S is a single equivalence class. We then have that

$$Z(Par_t) = \prod_{S \in \mathcal{P}/\approx_t} 1_S Z(Par_t)$$

as this is the algebra decomposition that is dual to the decomposition of $C(Par_t)$ as the direct sum of its indecomposable coideals. □

For $S \in \mathcal{P}/\approx_t$, the primitive central idempotent $1_S \in Z(Par_t)$ from Lemma 3.7.2 is not an element of Par_t, but we have that $1_S = (1_{S,n})_{n \geq 0}$ for idempotents $1_{S,n} = 1_S 1_n = 1_n 1_S \in 1_n Par_t 1_n$. Moreover, for a fixed n the idempotent $1_{S,n}$ is zero for all but finitely many S, so that $1_n = \sum_{S \in \mathcal{P}/\approx_t} 1_{S,n}$. The locally unital algebras $1_S Par_t = \bigoplus_{m,n \geq 0} 1_{S,m} Par_t 1_{S,n}$ are the *blocks* of the partition algebra Par_t, and we have the block decompositions

$$Par_t = \bigoplus_{S \in \mathcal{P}/\approx_t} 1_S Par_t, \qquad Par_t\text{-Mod} = \prod_{S \in \mathcal{P}/\approx_t} 1_S Par_t\text{-Mod}. \tag{3.7.9}$$

Representatives for the isomorphism classes of irreducible $1_S Par_t$-modules are given by the modules $L(\lambda)$ for all $\lambda \in S$.

Lemma 3.7.3. *The following properties are equivalent:*

(i) Par_t is semisimple.

(ii) All of the $\approx_t$-equivalence classes are singletons.

(iii) $\mathrm{HC} : Z(Par_t) \to \Bbbk[\mathcal{P}]$ *is surjective.*

(iv) $\mathrm{HC} : Z(Par_t) \to \Bbbk[\mathcal{P}]$ *is an isomorphism.*

Proof. If (i) holds, then Par_t is a direct sum of locally unital matrix algebras indexed by the set $\mathcal{P}$ that labels its irreducible representations. Hence, its center is the direct product $\prod_{\lambda \in \mathcal{P}} \Bbbk_\lambda$. It follows easily that HC is an isomorphism, i.e., (iv) holds.

Obviously, (iv) implies (iii).

The equivalence of (ii) and (iii) follows from Lemma 3.7.2.

It remains to show that (ii) implies (i). Assuming (ii), Lemma 3.7.2 shows for any $\lambda \in \mathcal{P}$ that there is a primitive central idempotent in $Z(Par_t)$ which acts as the identity on $\Delta(\lambda)$ and as zero on $L(\mu)$ for all $\mu \neq \lambda$. We deduce that all composition factors of $\Delta(\lambda)$ are isomorphic to $L(\lambda)$. Since this is a highest weight module we have that $[\Delta(\lambda) : L(\lambda)] = 1$, so actually $\Delta(\lambda)$ is irreducible. This is the case for all $\lambda \in \mathcal{P}$, so by BGG reciprocity we deduce that $P(\lambda) = \Delta(\lambda) = L(\lambda)$ for all λ, and (i) holds. $\square$

Remark 3.7.4. When Par_t is semisimple, the standardization functor $j_! : Sym\text{-Mod}_{\mathrm{fd}} \to \mathcal{P}ar_t\text{-Mod}_{\mathrm{lfd}}$ sends the irreducible Sym-modules $S(\lambda)$ to the irreducible Par_t-modules $\Delta(\lambda) = L(\lambda)$ for all $\lambda \in \mathcal{P}$. It follows easily that $j_!$ is an equivalence of categories in the semisimple case (although it is not a monoidal equivalence). Since the center is a Morita invariant, it follows that $Z(Sym) \cong Z(Par_t)$ in the semisimple case. Recalling that $Z(Sym) \cong \Bbbk[\mathcal{P}]$, this gives another way to understand the equivalence (i)⇒(iv) of Lemma 3.7.3.

Remark 3.7.5. As $Par_t\text{-Mod}_{\mathrm{lfd}}$ is an upper finite highest weight category, there is also a canonical partial order on $\mathcal{P}$, called the *minimal order* in [BS, Rem. 3.68], which we denote here by $\succeq_t$. By definition, this is the partial order generated by the relation $\lambda \succeq_t \mu$ if $[\Delta(\lambda) : L(\mu)] \neq 0$. As always for highest weight categories, the equivalence relation $\approx_t$ defining the blocks of Par_t is the transitive closure of the minimal order $\succeq_t$. We will describe $\succeq_t$ explicitly in Corollary 3.10.7 below.

3.8 "Blocks"

In the previous section, we introduced an equivalence relation $\approx_t$ on $\mathcal{P}$ whose equivalence classes parametrize the blocks of Par_t. The relation $\approx_t$ was defined in terms of the central characters $\chi_\lambda : Z(Par_t) \to \Bbbk$ arising from the irreducible Par_t-modules $L(\lambda)$; see (3.7.7). On the other hand, in (3.6.13) and (3.6.14), we constructed some explicit central elements of Par_t. Let $\sim_t$ be the equivalence relation on $\mathcal{P}$ defined from

$$\lambda \sim_t \mu \Leftrightarrow \chi_\lambda|_{Z_0(Par_t)} = \chi_\mu|_{Z_0(Par_t)} \tag{3.8.1}$$

where $Z_0(Par_t)$ is the subalgebra of $Z(Par_t)$ generated by the elements $\left\{c^{(r)} \;\middle|\; r \geq 3\right\}$ (equivalently, by the elements $\left\{z^{(r)} \;\middle|\; r \geq 1\right\}$). We refer to the $\sim_t$-equivalence classes as "blocks". We obviously have that

$$\lambda \approx_t \mu \Rightarrow \lambda \sim_t \mu, \tag{3.8.2}$$

i.e., "blocks" are unions of blocks. Defining 1_S as in Lemma 3.7.2, there are induced "block" decompositions

$$Par_t = \bigoplus_{S \in \mathcal{P}/\sim_t} 1_S Par_t, \qquad Par_t\text{-Mod} = \prod_{S \in \mathcal{P}/\sim_t} 1_S Par_t\text{-Mod}. \tag{3.8.3}$$

In this section, we are going to describe the relation $\sim_t$ in explicit combinatorial terms.

Lemma 3.8.1. *The images of the elements $x_j^L, x_j^R, s_k^L, s_k^R \in 1_n Par_t 1_n$ from (3.5.5) under the Harish-Chandra homomorphism $\widehat{\mathrm{HC}}$ from (3.7.2) are*

$$\mathrm{HC}_n(x_j^L) = x_j, \qquad \mathrm{HC}_n(x_j^R) = t - j + 1, \tag{3.8.4}$$

$$\mathrm{HC}_n(s_k^L) = 1, \qquad \mathrm{HC}_n(s_k^R) = (k\ k{+}1), \tag{3.8.5}$$

where $x_j \in \Bbbk S_n$ is the Jucys-Murphy element from (2.7.10).

Proof. Applying HC_n to the relations (3.3.15) and (3.3.16) (on the kth, $(k+1)$th and $(k+2)$th strings) we deduce that $\mathrm{HC}_n(s_{k+1}^L) = (k\ \ k{+}1\ \ k{+}2)\,\mathrm{HC}_n(s_k^L)(k{+}2\ \ k{+}1\ \ k)$ and $\mathrm{HC}_n(s_{k+1}^R) = (k\ \ k{+}1\ \ k{+}2)\,\mathrm{HC}_n(s_k^R)(k{+}2\ \ k{+}1\ \ k)$. Now (3.8.5) follows by induction on k, the base case $k = 1$ being immediate from (3.4.14). Note for this that $(k\ \ k{+}1\ \ k{+}2)(k\ \ k{+}1)(k{+}2\ \ k{+}1\ \ k) = (k{+}1\ \ k{+}2)$.

Applying HC_n to the relations (3.3.13) and (3.3.14) (on the jth and $(j+1)$th strings), using also (3.3.10), we deduce that $\mathrm{HC}_n(x_{j+1}^L) = (j\ \ j{+}1)\,\mathrm{HC}_n(x_j^L)(j\ \ j{+}1) + \mathrm{HC}_n(s_j^R)$ and $\mathrm{HC}_n(x_{j+1}^R) = (j\ \ j{+}1)\,\mathrm{HC}_n(x_j^R)(j\ \ j{+}1) - \mathrm{HC}_n(s_j^L)$. Now (3.8.4) follows using (3.8.5) and induction on j, the base case $j = 1$ being immediate from (3.4.13). Note for this that $(j\ \ j{+}1)x_j(j\ \ j{+}1) + (j\ \ j{+}1) = x_{j+1}$. $\square$

Lemma 3.8.2. *For $\lambda \in \mathcal{P}_n$, we have that*

$$\chi_\lambda(c(u)) = \prod_{i=1}^{n} \frac{\alpha_{\mathrm{cont}_i(\mathrm{T})}(u)}{\alpha_{t-i+1}(u)} \tag{3.8.6}$$

where T is some fixed standard λ-tableau and $\alpha_x(u)$ is as in (3.6.6).

Proof. Note by (3.7.6) that $\chi_\lambda(c(u)) = \mathrm{HC}_n(c_n(u))(\lambda) \in \Bbbk[\![u^{-1}]\!]$. To compute this, we use Lemma 3.8.1 and the explicit formula for $c_n(u) = p_t(C_n(u))$ arising from (3.6.11) to deduce that

$$\mathrm{HC}_n(c_n(u)) = \prod_{i=1}^{n} \frac{\alpha_{x_i}(u)}{\alpha_{t-i+1}(u)} \in Z(\Bbbk S_n)[\![u^{-1}]\!].$$

To evaluate this at λ, we act on the basis vector v_T from Young's orthonormal basis for $S(\lambda)$, remembering that $x_i v_\mathrm{T} = \mathrm{cont}_i(\mathrm{T}) v_\mathrm{T}$. $\square$

Lemma 3.8.1 suggests some combinatorics of weights. Let P be the free Abelian group on basis $\{\Lambda_c \mid c \in \Bbbk\}$. Let $\varepsilon_c := \Lambda_c - \Lambda_{c+1}$ and $\alpha_c := \varepsilon_c - \varepsilon_{c-1}$. We define the *weight* of a rational function $f(u) \in \Bbbk(u)$ to be

$$\operatorname{wt} f(u) := \sum_{c \in \Bbbk} \left[\begin{pmatrix} \text{Multiplicity of } c \\ \text{as a pole of } f(u) \end{pmatrix} - \begin{pmatrix} \text{Multiplicity of } c \\ \text{as a zero of } f(u) \end{pmatrix} \right] \Lambda_c \in P. \quad (3.8.7)$$

For example, $\operatorname{wt} \alpha_c(u) = -\Lambda_{c-1} + 2\Lambda_c - \Lambda_{c+1} = \alpha_c$. For $\lambda \in \mathcal{P}_n$, let $\operatorname{wt}_t(\lambda)$ be the weight of the rational function appearing on the right hand side of (3.8.6). As the coefficients of the power series $c(u)$ generate the subalgebra $Z_0(Par_t)$, the equivalence relation $\sim_t$ defined by (3.8.1) satisfies

$$\lambda \sim_t \mu \Leftrightarrow \operatorname{wt}_t(\lambda) = \operatorname{wt}_t(\mu). \quad (3.8.8)$$

This suggests using elements of P rather than $\sim_t$-equivalence classes to index the "blocks" from (3.8.3): for any $\gamma \in P$, let

$$S(\gamma) := \left\{ \lambda \in \mathcal{P} \;\middle|\; \operatorname{wt}_t(\lambda) = \gamma \right\}. \quad (3.8.9)$$

Then define

$$\operatorname{pr}_\gamma : Par_t\text{-Mod} \to Par_t\text{-Mod} \quad (3.8.10)$$

to be the projection functor defined by multiplication by the central idempotent $1_{S(\gamma)}$ from Lemma 3.7.2. In other words, pr_γ projects a Par_t-module V to its largest submodule all of whose irreducible subquotients are of the form $L(\lambda)$ for $\lambda \in \mathcal{P}$ with $\operatorname{wt}_t(\lambda) = \gamma$. The admissible $\gamma \in P$ which parametrize "blocks" are the ones with $S(\gamma) \neq \varnothing$; if $S(\gamma) = \varnothing$ then pr_γ is the zero functor.

Lemma 3.8.3. *For $\lambda \in \mathcal{P}_n$ and any standard λ-tableau T, we have that*

$$\operatorname{wt}_t(\lambda) = \sum_{i=1}^{n} (\alpha_{\operatorname{cont}_i(\mathrm{T})} - \alpha_{t-i+1}) = (\varepsilon_{t-|\lambda|} - \varepsilon_t) + (\varepsilon_{\lambda_1 - 1} - \varepsilon_{-1}) + \cdots + (\varepsilon_{\lambda_k - k} - \varepsilon_{-k}) \quad (3.8.11)$$

for any $k \geq \ell(\lambda)$. Moreover, given another partition $\mu \in \mathcal{P}$, we have that $\mathrm{wt}_t(\lambda) = \mathrm{wt}_t(\mu)$ *if and only if the infinite sequences* $(t-|\lambda|, \lambda_1 - 1, \lambda_2 - 2, \dots)$ *and* $(t-|\mu|, \mu_1 - 1, \mu_2 - 2, \dots)$ *are rearrangements of each other.*

Proof. The first equality in (3.8.11) follows immediately from Lemma 3.8.2. To deduce the second equality, take $k \geq \ell(\lambda)$. For $1 \leq r \leq k$ the contents of the nodes in the rth row of the Young diagram of λ are $1-r, 2-r, \dots, \lambda_r - r$, and we have that $\alpha_{1-r} + \cdots + \alpha_{\lambda_r - r} = \varepsilon_{\lambda_r - r} - \varepsilon_{-r}$. Also $\alpha_t + \alpha_{t-1} + \cdots + \alpha_{t-n+1} = \varepsilon_t - \varepsilon_{t-n}$. Now the desired formula follows easily.

Rearranging the right hand side of (3.8.11) gives that $\varepsilon_{t-|\lambda|} + \varepsilon_{\lambda_1 - 1} + \varepsilon_{\lambda_2 - 2} + \cdots + \varepsilon_{\lambda_k - k} = \mathrm{wt}_t(\lambda) + \varepsilon_{-1} + \varepsilon_{-2} + \cdots + \varepsilon_{-k} + \varepsilon_t$ for any $k \geq \ell(\lambda)$. Hence, we have that $\mathrm{wt}_t(\lambda) = \mathrm{wt}_t(\mu)$ if and only if

$$\varepsilon_{t-|\lambda|} + \varepsilon_{\lambda_1 - 1} + \varepsilon_{\lambda_2 - 2} + \cdots + \varepsilon_{\lambda_k - k} = \varepsilon_{t-|\mu|} + \varepsilon_{\mu_1 - 1} + \varepsilon_{\mu_2 - 2} + \cdots + \varepsilon_{\mu_k - k}$$

for all $k \gg 0$. This is clearly equivalent to saying that the infinite sequences $(t - |\lambda|, \lambda_1 - 1, \lambda_2 - 2, \dots)$ and $(t - |\mu|, \mu_1 - 1, \mu_2 - 2, \dots)$ may be obtained from each other by permuting the entries. $\square$

The final assertion from Lemma 3.8.3 shows that $\sim_t$ is exactly the same as the equivalence relation on partitions defined in [CO11, Def. 5.1]. The equivalence classes of this relation were investigated in detail in [CO11, §5.3]. The following summarizes the results obtained there. For the statement, we say that $\lambda \in \mathcal{P}$ is *typical* if it is the only partition in its $\sim_t$-equivalence class; otherwise we say that λ is *atypical*. Of course, these notions depend on the fixed value of the parameter t.

Theorem 3.8.4 (Comes-Ostrik). *If $t \notin \mathbb{N}$ then all partitions are typical. If $t \in \mathbb{N}$ then there is a bijection $\mathcal{P}_t \xrightarrow{\sim} \{$atypical $\sim_t$-equivalence classes$\}$ taking $\kappa \in \mathcal{P}_t$ to the*

$\sim_t$-equivalence class $\{\kappa^{(0)}, \kappa^{(1)}, \kappa^{(2)}, \dots\}$ where

$$\kappa^{(n)} := (\kappa_1 + 1, \dots, \kappa_n + 1, \kappa_{n+2}, \kappa_{n+3}, \dots) \in \mathcal{P}_{t+n-\kappa_{n+1}}, \tag{3.8.12}$$

i.e., it is the partition obtained from κ by adding a node to the first n rows of its Young diagram then removing its $(n+1)$th row. Moreover, still assuming $t \in \mathbb{N}$, a partition $\lambda \in \mathcal{P}$ is typical if and only if $t - |\lambda| = \lambda_i - i$ for some $i \geq 1$.

Example 3.8.5. For any $t \in \mathbb{N}$, the $\sim_t$-equivalence class associated to $\kappa = (t) \in \mathcal{P}_t$ is

$$S = \left\{ \varnothing, (t{+}1), (t{+}1, 1), (t{+}1, 1^2), \cdots \right\}.$$

For $t \in \mathbb{N} - \{0, 1\}$, the $\sim_t$-equivalence class associated to $\kappa = (1^t) \in \mathcal{P}_t$ is

$$S = \left\{ (1^{t-1}), (2, 1^{t-2}), (2^2, 1^{t-3}), \cdots, (2^{t-1}), (2^t), (2^t, 1), (2^t, 1^2), \cdots \right\}.$$

As noted in [CO11, Cor. 5.23] (using a different argument for the forward implication), the first assertion of Theorem 3.8.4 allows us to recover the following well known result of Deligne [Del07, Th. 2.18]: $\underline{\mathrm{Rep}}(S_t)$ is semisimple if and only if $t \notin \mathbb{N}$. In terms of the path algebra Par_t, Deligne's result can be stated as follows.

Corollary 3.8.6 (Deligne)**.** *Par_t is semisimple if and only if $t \notin \mathbb{N}$.*

Proof. We already know that Par_t is not semisimple if $t \in \mathbb{N}$ by Corollary 3.1.2. Conversely, if $t \notin \mathbb{N}$, we apply the criterion from Lemma 3.7.3, noting that all $\approx_t$-equivalence classes are singletons thanks to (3.8.2) and the first part of Theorem 3.8.4. □

Remark 3.8.7. When $t \notin \mathbb{N}$, the above arguments show for $\lambda, \mu \in \mathcal{P}$ with $\lambda \neq \mu$ that there is a central element in the subalgebra $Z_0(Par_t)$ of $Z(Par_t)$ which acts by different scalars on the irreducible modules $L(\lambda)$ and $L(\mu)$. It follows in these cases that $Z_0(Par_t)$ is a dense subalgebra of the pseudo-compact topological algebra

$Z(Par_t)$[1]. We do not expect that this is the case when $t \in \mathbb{N}$, but nevertheless $Z_0(Par_t)$ is still sufficiently large to separate blocks. This will be established in Corollary 3.10.6 below, which shows for any value of t that the relations $\sim_t$ and $\approx_t$ coincide, so that "blocks" are blocks, and (3.8.3) is always the same decomposition as (3.7.9); see also [CO11, Th. 5.3].

3.9 Special projective functors

From now on, we will primarily be interested in parameter values $t \in \mathbb{N}$, so that Par_t is not semisimple. Consider the atypical block $\{\kappa^{(0)}, \kappa^{(1)}, \kappa^{(2)}, \dots\}$ associated to $\kappa \in \mathcal{P}_t$. From (3.8.12), it follows that $\kappa^{(n)}$ is obtained from $\kappa^{(n-1)}$ by adding $\kappa_n - \kappa_{n+1} + 1$ nodes to the nth row of its Young diagram, leaving all other rows unchanged. The partition $\kappa^{(0)}$ is the smallest of all of the $\kappa^{(n)}$, hence, it is maximal in the highest weight ordering from Theorem 2.9.1. It follows that

$$P(\kappa^{(0)}) \cong \Delta(\kappa^{(0)}). \tag{3.9.1}$$

The indecomposable projectives $\Delta(\kappa^{(0)})$ are exactly the ones of non-zero categorical dimension mentioned already in Remark 3.1.3, with the irreducible $\Bbbk S_t$-module associated to the image of $\Delta(\kappa^{(0)})$ under the equivalence $\overline{\psi}_t$ between the semisimplification of $\mathrm{Kar}(\mathcal{P}ar_t)$ and $\Bbbk S_t\text{-Mod}_{\mathrm{fd}}$ being the Specht module $S(\kappa)$. It is also useful to note for $t \in \mathbb{N}$ and $\kappa \in \mathcal{P}_t$ that the associated block $\{\kappa^{(0)}, \kappa^{(1)}, \dots\}$ is the set $S(\gamma)$ from (3.8.9) for

$$\gamma := (\varepsilon_{\kappa_1} - \varepsilon_t) + (\varepsilon_{\kappa_2 - 1} - \varepsilon_{-1}) + \cdots + (\varepsilon_{\kappa_t - t + 1} - \varepsilon_{-t}) \in P. \tag{3.9.2}$$

This is follows easily using (3.8.11) and (3.8.12).

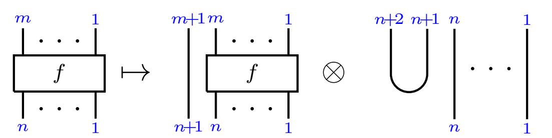
[1]These algebras are certainly not equal since $Z(Par_t) \cong \prod_{\lambda \in \mathcal{P}} \Bbbk_\lambda$ is of uncountable dimension.

In order to understand the structure of the atypical blocks more fully, we are going to use the endofunctor $\mid\star : \mathcal{P}\!ar_t \to \mathcal{P}\!ar_t$. Let

$$D := \mathrm{res}_{\mid\star} = 1_{\mid\star} Par_t \otimes_{Par_t} : Par_t\text{-Mod} \to Par_t\text{-Mod} \tag{3.9.3}$$

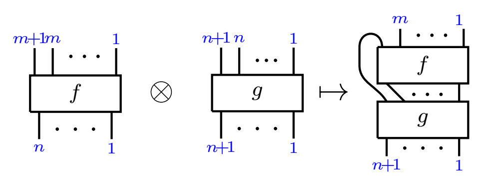
be the corresponding restriction functor from (2.6.4). This obviously preserves locally finite-dimensional modules. The object $\mid$ is self-dual so, by Lemma 2.6.2, the restriction functor D is isomorphic to the induction functor $\mathrm{ind}_{\mid\star}$. By Corollary 2.6.3, D is a self-adjoint projective functor, so it preserves finitely generated projectives (and finitely cogenerated injectives). To make the canonical adjunction as explicit as possible, we note that its unit and counit are induced by the bimodule homomorphisms

$$\eta : Par_t \to 1_{\mid\star} Par_t \otimes_{Par_t} 1_{\mid\star} Par_t,$$



(3.9.4)

$$\varepsilon : 1_{\mid\star} Par_t \otimes_{Par_t} 1_{\mid\star} Par_t \to Par_t,$$



(3.9.5)

Using (2.3.4), it follows that D commutes with the duality $?^{\circledast}$ on $Par_t\text{-Mod}_{\mathrm{lfd}}$.

Theorem 3.9.1. *For $\lambda \in \mathcal{P}$, there is a filtration $0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 = D\Delta(\lambda)$ such that*

$$V_3/V_2 \cong \bigoplus_{a \in \mathrm{add}(\lambda)} \Delta\left(\lambda + \boxed{a}\right),$$

$$V_2/V_1 \cong \Delta(\lambda) \oplus \bigoplus_{b \in \mathrm{rem}(\lambda)} \bigoplus_{a \in \mathrm{add}(\lambda - \boxed{b})} \Delta\left((\lambda - \boxed{b}) + \boxed{a}\right),$$

$$V_1/V_0 \cong \bigoplus_{b\in\mathrm{rem}(\lambda)} \Delta\left(\lambda - \boxed{b}\right).$$

Proof. This follows from Lemma 3.11.2 below, which constructs the filtration explicitly. The proof of Lemma 3.11.2 is valid over fields of positive characteristic. $\square$

Now we are going to use the affine partition category $\mathcal{APar}$ to decompose the endofunctor D as a direct sum of *special projective functors* $D_{b|a}$. The approach here is analogous to the way the affine symmetric category $\mathcal{ASym}$ was used to decompose E and F as direct sums of E_a and F_b in (2.7.12). As noted at the end of §3.3, $\mathcal{Par}_t$ is isomorphic to the quotient of $\mathcal{APar}$ by a left tensor ideal. Hence, $\mathcal{Par}_t$ is a strict $\mathcal{APar}$-module category. The self-adjoint functor D is also the restriction functor $\mathrm{res}_{\vert\star}$ arising from this categorical action of $\mathcal{APar}$ on $\mathcal{Par}_t$. Now the left and right dots give us natural transformations

$$\alpha := \vert\!\!\bullet\,\star\ : \vert\star \Rightarrow \vert\star, \qquad\qquad \beta := \bullet\!\!\vert\,\star\ : \vert\star \Rightarrow \vert\,\star.$$

Applying the general construction from (2.2.8) to these, we obtain commuting endomorphisms

$$x := \mathrm{res}_\alpha : D \Rightarrow D, \qquad\qquad y := \mathrm{res}_\beta : D \Rightarrow D. \tag{3.9.6}$$

Let $D_{b|a}$ be the summand of D that is the simultaneous generalized eigenspace of x and y of eigenvalues a and b, respectively. Explicitly, $D = \mathrm{res}_{\vert\star}$ is defined by tensoring with the bimodule $1_{\vert\star}Par_t$, and the endomorphisms x and y of D are induced by the bimodule endomorphisms ρ and λ of $1_{\vert\star}Par_t$ given by left multiplication by x^R_{m+1} and x^L_{m+1}, respectively, on the summand $1_{m+1}Par_t$ of $1_{\vert\star}Par_t$ for each $m \geq 0$. Then, $D_{b|a}$ is the functor defined by tensoring with the summand of $1_{\vert\star}Par_t$ that is the simultaneous generalized eigenspaces of ρ and λ for the eigenvalues a and b, respectively. As $1_{m+1}Par_t = \bigoplus_{n\geq 0} 1_{m+1}Par_t1_n$ with each $1_{m+1}Par_t1_n$ being finite-dimensional, these endomorphisms are locally finite, so we have that

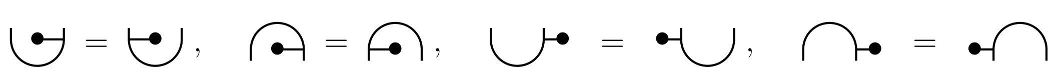
$$D = \bigoplus_{a,b \in \Bbbk} D_{b|a}. \tag{3.9.7}$$

Lemma 3.9.2. *For $a, b \in \Bbbk$, the endofunctor $D_{b|a}$ commutes with the duality $?^{\circledcirc}$, i.e., $D_{b|a} \circ ?^{\circledcirc} \cong ?^{\circledcirc} \circ D_{b|a}$.*

Proof. This follows from the fact that D commutes with the duality $?^{\circledcirc}$, and σ fixes both the left dot and the right dot. $\square$

Lemma 3.9.3. *For $a, b \in \Bbbk$, the endofunctors $D_{b|a}$ and $D_{a|b}$ are biadjoint.*

Proof. The adjunction $(D_{a|b}, D_{b|a})$ is induced by the self-adjunction of D. The unit $\bar{\eta}$ of adjunction comes from the bimodule homomorphism that is the composition of the unit η from (3.9.4) with the projection onto the generalized a and b eigenspaces of ρ and λ on the left tensor factor and the generalized b and a eigenspaces of ρ and λ on the right tensor factor. The counit $\bar{\varepsilon}$ of adjunction comes from the composition of the counit ε from (3.9.5) with the inclusion of the generalized b and a eigenspaces of ρ and λ on the left tensor factor and the generalized a and b eigenspaces of ρ and λ on the right tensor factor. To check the zig-zag identities, one just needs to use the relations



i.e., the fact that the left and right dots are duals. $\square$

When $a \neq b$, Lemma 3.9.3 can also be proved a bit more easily using the description of $D_{b|a}$ given in the following lemma, since the projection functor pr_γ commutes with $?^{\circledcirc}$ thanks to (2.9.12).

Lemma 3.9.4. *Let pr_γ be the projection functor defined by (3.8.10). If $a \neq b$ then*

$$D_{b|a} \cong \bigoplus_{\gamma \in P} \mathrm{pr}_{\gamma + \alpha_a - \alpha_b} \circ D \circ \mathrm{pr}_\gamma .$$

Also $\bigoplus_{\gamma\in P}\mathrm{pr}_\gamma \circ D\circ \mathrm{pr}_\gamma \cong \bigoplus_{a\in\Bbbk} D_{a|a}$.

Proof. Take a module V in the "block" parametrized by $\gamma\in P$, so that $\mathrm{wt}_t(\lambda)=\gamma$ for all irreducible subquotients of V. We need to show that $D_{b|a}V$ is in the "block" parametrized by $\gamma+\alpha_a-\alpha_b$. Since $D_{b|a}$ is exact, we may assume that V is irreducible, so $V=L(\lambda)$ for $\lambda\in\mathcal{P}$ with $\mathrm{wt}_t(\lambda)=\gamma$. The module $DV=1_{|\star}Par_t\otimes_{Par_t}V\cong 1_{|\star}V$ is generated by the finite-dimensional vector spaces $1_{m+1}V$ for all $m\geq 0$. Hence, $D_{b|a}V$ is generated by the simultaneous generalized eigenspaces of x^R_{m+1} and x^L_{m+1} on $1_{m+1}V$ of eigenvalues a and b, respectively. Consequently, if $L(\mu)$ is an irreducible subquotient of $D_{b|a}V$, then $c(u)$ must act on $L(\mu)$ in the same way as $|\star c_m(u)$ acts on a simultaneous eigenvector $v\in 1_{m+1}V$ for x^R_{m+1} and x^L_{m+1} of eigenvalues a and b. Also $c_{m+1}(u)$ acts on $v\in V$ as multiplication by $\chi_\lambda(c(u))$, the rational function displayed on the right hand side of (3.8.6). Using (3.6.9), we deduce that

$$\chi_\mu(c(u))=\frac{\alpha_a(u)}{\alpha_b(u)}\times\chi_\lambda(c(u)).$$

Hence, $\mathrm{wt}_t(\mu)=\mathrm{wt}_t(\lambda)+\alpha_a-\alpha_b$. □

Our main combinatorial result about the functors $D_{b|a}$ is as follows.

Theorem 3.9.5. *For $\lambda\in\mathcal{P}$ and $a,b\in\Bbbk$, there is a filtration $0=V_0\subseteq V_1\subseteq V_2\subseteq V_3=D_{b|a}\Delta(\lambda)$ such that*

$$V_3/V_2\cong\begin{cases}\Delta(\lambda+\boxed{a}) & \text{if } a\in\mathrm{add}(\lambda) \text{ and } b=t-|\lambda|\\ 0 & \text{otherwise,}\end{cases}$$

$$V_2/V_1\cong\begin{cases}\Delta(\lambda)\oplus\Delta(\lambda) & \text{if } t-|\lambda|=a=b\in\mathrm{rem}(\lambda)\\ \Delta(\lambda) & \text{if } t-|\lambda|\neq a=b\in\mathrm{rem}(\lambda) \text{ or } t-|\lambda|=a=b\notin\mathrm{rem}(\lambda)\\ \Delta\left((\lambda-\boxed{b})+\boxed{a}\right) & \text{if } a\neq b\in\mathrm{rem}(\lambda) \text{ and } a\in\mathrm{add}(\lambda-\boxed{b})\\ 0 & \text{otherwise,}\end{cases}$$

$$V_1/V_0 \cong \begin{cases} \Delta(\lambda - \boxed{b}) & \textit{if } a = t - |\lambda| + 1 \textit{ and } b \in \mathrm{rem}(\lambda) \\ 0 & \textit{otherwise.} \end{cases}$$

In particular, when $t \in \mathbb{Z}$, the functor $D_{b|a}$ is zero unless both a and b are integers.

Proof. See §3.11 below. □

The following corollary is an immediate consequence of the theorem, but actually it has a much easier proof which we include below.

Corollary 3.9.6. *For $\lambda \in \mathcal{P}$ and $a, b \in \Bbbk$ with $a \neq b$, there is a filtration $0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 = D_{b|a}\Delta(\lambda)$ such that*

$$V_3/V_2 \cong \begin{cases} \Delta(\lambda + \boxed{a}) & \textit{if } a \in \mathrm{add}(\lambda) \textit{ and } b = t - |\lambda| \\ 0 & \textit{otherwise,} \end{cases}$$

$$V_2/V_1 \cong \begin{cases} \Delta\left((\lambda - \boxed{b}) + \boxed{a}\right) & \textit{if } b \in \mathrm{rem}(\lambda) \textit{ and } a \in \mathrm{add}(\lambda - \boxed{b}) \\ 0 & \textit{otherwise,} \end{cases}$$

$$V_1/V_0 \cong \begin{cases} \Delta(\lambda - \boxed{b}) & \textit{if } a = t - |\lambda| + 1 \textit{ and } b \in \mathrm{rem}(\lambda) \\ 0 & \textit{otherwise.} \end{cases}$$

Direct proof avoiding Theorem 3.9.5. Let $\gamma := \mathrm{wt}_t(\lambda)$. By Lemma 3.9.4, we can compute $D_{b|a}\Delta(\lambda)$ by applying $\mathrm{pr}_{\gamma+\alpha_a-\alpha_b}$ to the Δ-flag for $D\Delta(\lambda)$ from Theorem 3.9.1. This produces a module with a Δ-flag consisting of all $\Delta(\mu)$ in the original Δ-flag such that $\mathrm{wt}_t(\mu) - \mathrm{wt}_t(\lambda) = \alpha_a - \alpha_b$. It just remains to compute $\mathrm{wt}_t(\mu) - \mathrm{wt}_t(\lambda)$ for the various possible μ. If $\mu = \lambda + \boxed{c}$ for $c \in \mathrm{add}(\lambda)$ then, by a computation using the first equality from (3.8.11), we have that $\mathrm{wt}_t(\mu) - \mathrm{wt}_t(\lambda) = \alpha_c - \alpha_{t-|\lambda|}$; for this to equal $\alpha_a - \alpha_b$ we must have $b = t - |\lambda|$ and $c = a$. If $\mu = \lambda - \boxed{d}$ for $d \in \mathrm{rem}(\lambda)$ then, by a similar computation, $\mathrm{wt}_t(\mu) - \mathrm{wt}_t(\lambda) = \alpha_{t-|\lambda|+1} - \alpha_d$; for this to equal $\alpha_a - \alpha_b$ we must have $d = b$ and $a = t - |\lambda| + 1$. Finally if $\mu = (\lambda - \boxed{d}) + \boxed{c}$ for $d \in \mathrm{rem}(\lambda)$

and $c \in \operatorname{add}(\lambda - \boxed{d})$ then $\mathrm{wt}_t(\mu) - \mathrm{wt}_t(\lambda) = \alpha_c - \alpha_d$; for this to equal $\alpha_a - \alpha_b$ we must have $c = a$ and $d = b$. $\square$

3.10 Blocks

We assume throughout the section that $t \in \mathbb{N}$. We are going to describe the structure of the atypical "blocks," revealing in particular that they are indecomposable, hence, they are actually blocks. Recall from Theorem 3.8.4 that the atypical "blocks" are parametrized by partitions $\kappa \in \mathcal{P}_t$, with the irreducible modules in the "block" being the ones labelled by the partitions $\{\kappa^{(0)}, \kappa^{(1)}, \dots\}$. This is the set $S(\gamma)$ from (3.8.9) where $\gamma \in P$ is obtained from κ according to (3.9.2).

The first step is to show that all of the atypical "blocks" are equivalent to each other. The proof of this uses the special projective functors $D_{b|a}$ with $a \neq b$. These are the ones which can be defined just using information about central characters rather than requiring the Jucys-Murphy elements; cf. Lemma 3.9.4 and Corollary 3.9.6. In view of Remark 3.6.6, this sort of information was already available to Comes and Ostrik in an equivalent form, and indeed they were also able to prove a similar result by an analogous argument; see [CO11, Lem. 5.18(2)] and [CO11, Prop. 6.6].

Lemma 3.10.1. *Let κ and $\tilde{\kappa}$ be partitions of t such that $\tilde{\kappa}$ is obtained from κ by moving a node from the first row of its Young diagram to its $(r+1)$th row for some $r \geq 1$. Let $a := \kappa_{r+1} - r + 1$ and $b := \kappa_1$. Then for all $n \geq 0$ we have that $D_{b|a}\Delta(\kappa^{(n)}) \cong \Delta(\tilde{\kappa}^{(n)})$ and $D_{a|b}\Delta(\tilde{\kappa}^{(n)}) \cong \Delta(\kappa^{(n)})$.*

Proof. Let $\gamma, \tilde{\gamma} \in P$ be defined from κ and $\tilde{\kappa}$ according to (3.9.2). From this formula it follows that $\tilde{\gamma} = \gamma + \alpha_a - \alpha_b$ where $a = \kappa_{r+1} - r + 1$ and $b = \kappa_1$ as in the statement of the lemma. Note that $a \neq b$. So we can apply Lemma 3.9.4 to see that $D_{b|a}\Delta(\kappa^{(n)}) = \mathrm{pr}_{\gamma+\alpha_a-\alpha_b}(D\Delta(\kappa^{(n)}))$ and $D_{a|b}\Delta(\tilde{\kappa}^{(n)}) = \mathrm{pr}_{\gamma-\alpha_a+\alpha_b}(D\Delta(\tilde{\kappa}^{(n)}))$.

Now we use this description to show that $D_{b|a}\Delta(\kappa^{(n)}) \cong \Delta(\tilde{\kappa}^{(n)})$. The proof that $D_{a|b}\Delta(\tilde{\kappa}^{(n)}) \cong \Delta(\kappa^{(n)})$ is similar and we leave this to the reader.

Fix $n \geq 0$ and let B_n be the set of $\mu \in \mathcal{P}$ which are obtained from $\kappa^{(n)}$ by removing a node, removing a node then adding a different node, or adding a node. Bearing in mind that $a \neq b$, the standard modules $\Delta(\mu)$ for $\mu \in B_n$ include all of the ones which are sections of the Δ-flag from Theorem 3.9.1 which could possibly be in the same block as $\Delta(\tilde{\kappa}^{(n)})$. Now it suffices to show for $m \geq 0$ that $\tilde{\kappa}^{(m)} \in B_n$ if and only if $m = n$. There are four cases to consider.

Case one: $n = 0$. We have that $\kappa^{(0)} = (\kappa_2, \kappa_3, \dots, \kappa_{r+1}, \dots)$ and $\tilde{\kappa}^{(0)} = (\kappa_2, \kappa_3, \dots, \kappa_{r+1}{+}1, \dots)$, which is $\kappa^{(0)}$ with one node added to the rth row of its Young diagram. We definitely have that $\tilde{\kappa}^{(0)} \in B_0$. All other $\mu \in B_0$ satisfy $|\mu| \leq |\tilde{\kappa}^{(0)}|$. Since all $\tilde{\kappa}^{(m)}$ with $m > 0$ have $|\tilde{\kappa}^{(m)}| > |\tilde{\kappa}^{(0)}|$, none of these belong to B_0.

Case two: $1 \leq n < r$. We have that $\kappa^{(n)} = (\kappa_1{+}1, \kappa_2{+}1, \dots, \kappa_n{+}1, \dots, \kappa_{r+1}, \dots)$ and $\tilde{\kappa}^{(n)} = (\kappa_1, \kappa_2{+}1, \dots, \kappa_n{+}1, \dots, \kappa_{r+1}{+}1, \dots)$, which is $\kappa^{(n)}$ with a node removed from the first row and a node added to the rth row of its Young diagram. We definitely have that $\tilde{\kappa}^{(n)} \in B_n$. For $m < n$, $\tilde{\kappa}^{(m)}$ is of smaller size than $\kappa^{(n)}$ and its rth row is of length $\kappa_{r+1} + 1$. This cannot be obtained from $\kappa^{(n)}$ by removing a node since $\kappa^{(n)}$ has rth row of length κ_{r+1}. So it does not belong to B_n. For $m > n$, $\tilde{\kappa}^{(m)}$ is of greater size than $\kappa^{(n)}$ and its first row is of length κ_1. This cannot be obtained from $\kappa^{(n)}$ by adding a node since $\kappa^{(n)}$ has first row of length $\kappa_1 + 1$. So again it does not belong to B_n.

Case three: $n = r$. We have that $\kappa^{(n)} = (\kappa_1 + 1, \kappa_2 + 1, \dots, \kappa_r + 1, \kappa_{r+2}, \dots)$ and $\tilde{\kappa}^{(n)} = (\kappa_1, \kappa_2 + 1 \dots, \kappa_r + 1, \kappa_{r+2}, \dots)$, which is $\kappa^{(n)}$ with a node removed from the first row of its Young diagram. We definitely have that $\tilde{\kappa}^{(n)} \in B_n$. The $\tilde{\kappa}^{(m)}$ with $m < n$ have $|\tilde{\kappa}^{(m)}| \leq |\tilde{\kappa}^{(n)}| - 1 = |\kappa^{(n)}| - 2$ so are not elements of B_n. The $\tilde{\kappa}^{(m)}$ with

$m > n$ have $(r+1)$th row of length $\kappa_{r+1}+2$, so these are not elements of B_n either since this is at least two more than the length of the $(r+1)$th row of $\kappa^{(n)}$.

Case four: $n > r$. We have that $\kappa^{(n)} = (\kappa_1+1, \kappa_2+1, \dots, \kappa_{r+1}+1, \dots)$ and $\tilde{\kappa}^{(n)} = (\kappa_1, \kappa_2+1, \dots, \kappa_{r+1}+2, \dots)$, which is $\kappa^{(n)}$ with a node removed from its first row and a node added to its $(r+1)$th row. We definitely have that $\tilde{\kappa}^{(n)} \in B_n$. The $\tilde{\kappa}^{(m)}$ with $m > n$ are of greater size than $\kappa^{(n)}$ and have first row of length κ_1; these cannot be obtained by adding a node to $\kappa^{(n)}$. The $\tilde{\kappa}^{(m)}$ with $r+1 \leq m < n$ are of smaller size than $\kappa^{(n)}$ and have $(r+1)$th row of length $\kappa_{r+1}+2$; these cannot be obtained by removing a node from $\kappa^{(n)}$. The $\tilde{\kappa}^{(m)}$ with $m \leq r$ have first row of length $\leq \kappa_1$ and $(r+1)$th row of length κ_{r+2}, whereas these two rows of $\kappa^{(n)}$ are of lengths κ_1+1 and $\kappa_{r+1}+1 > \kappa_{r+2}$, so these are not elements of B_n. □

Theorem 3.10.2 (Comes-Ostrik)**.** *Let κ and $\tilde{\kappa}$ be partitions of t, denoting the associated $\sim_t$-equivalence classes by $S := \{\kappa^{(0)}, \kappa^{(1)}, \dots\}$ and $\widetilde{S} := \{\tilde{\kappa}^{(0)}, \tilde{\kappa}^{(1)}, \dots\}$. There is an equivalence of categories*

$$\Sigma : 1_S Par_t\text{-Mod} \to 1_{\widetilde{S}} Par_t\text{-Mod}$$

between the corresponding "blocks" such that $\Sigma L(\kappa^{(n)}) \cong L(\tilde{\kappa}^{(n)})$ for all $n \geq 0$. The functor Σ is a composition of the special projective functors $D_{b|a}$ $(a \neq b)$, hence, it is a projective functor.

Proof. We may assume that $\tilde{\kappa}$ is obtained from κ by moving a node from the first row of its Young diagram to its $(r+1)$th row for some $r \geq 1$. Thus, we are in the situation of Lemma 3.10.1. The lemma gives us functors $D_{b|a} : 1_S Par_t\text{-Mod} \to 1_{\widetilde{S}} Par_t\text{-Mod}$ and $D_{a|b} : 1_{\widetilde{S}} Par_t\text{-Mod} \to 1_S Par_t\text{-Mod}$ such that $D_{b|a}\Delta(\kappa^{(n)}) \cong \Delta(\tilde{\kappa}^{(n)})$ and $D_{a|b}\Delta(\tilde{\kappa}^{(n)}) \cong \Delta(\kappa^{(n)})$. These functors are also biadjoint thanks to Lemma 3.9.3. It follows easily that they are quasi-inverse equivalences of categories as claimed in the theorem. In more detail, the unit and counit of one of the adjunctions gives natural transformations $D_{a|b} \circ D_{b|a} \Rightarrow \mathrm{Id}$ and $\mathrm{Id} \Rightarrow D_{b|a} \circ D_{a|b}$. We claim that these natural transformations are isomorphisms. They are non-zero, hence, they are isomorphisms on all standard modules. The functors are exact and indecomposable projectives have finite Δ-flags, so it follows that the natural transformations are isomorphisms on all indecomposable projectives. Then we get that they are isomorphisms on an arbitrary module by considering a two step projective resolution and applying the Five Lemma. $\square$

The next lemma does use the functors $D_{b|a}$ in the case $a = b$, i.e., it definitely requires the full strength of Theorem 3.9.5 rather than merely Corollary 3.9.6.

Lemma 3.10.3. *Let $\kappa \in \mathcal{P}_t$ and $S := \{\kappa^{(0)}, \kappa^{(1)}, \dots\}$ be the corresponding $\sim_t$-equivalence class. For each $n \geq 0$, there is an endofunctor $\Pi_n : Par_t\text{-Mod} \to Par_t\text{-Mod}$ such that $\Pi_n \Delta(\kappa^{(m)}) = 0$ for $m \neq n, n+1$, and moreover there exist short exact sequences $0 \to \Delta(\kappa^{(n)}) \to \Pi_n \Delta(\kappa^{(n)}) \to \Delta(\kappa^{(n+1)}) \to 0$ and $0 \to \Delta(\kappa^{(n)}) \to \Pi_n \Delta(\kappa^{(n+1)}) \to \Delta(\kappa^{(n+1)}) \to 0$. The functor Π_n is a composition of the special projective functors $D_{b|a}$ $(a, b \in \mathbb{Z})$, hence, it is a projective functor.*

Proof. In view of Theorem 3.10.2, it suffices to prove the lemma in the special case that $\kappa = (t)$, when $S = \{\varnothing, (t+1), (t+1, 1), (t+1, 1^2), \dots\}$ as in Example 3.8.5. Then we take $\Pi_0 := D_{0|t} \circ \cdots \circ D_{t-1|1} \circ D_{t|0}$ and $\Pi_n := D_{-n|-n}$ for $n > 0$. Now it is just a matter of applying Theorem 3.9.5 to see that these functors have the stated properties.

The situation for Π_0 is the most interesting. To understand this, let $u := \lceil \frac{t}{2} \rceil$ and $v := \lfloor \frac{t}{2} \rfloor$. Then one checks that $D_{v+1|u-1} \circ \cdots \circ D_{t-1|1} \circ D_{t|0}(\Delta(\varnothing)) \cong \Delta((u))$; each of these functors adds a single node to the first row of the Young diagram. After that we apply $D_{v|u}$ to get a module with a two step Δ-flag, with a copy of $\Delta((u+1))$ at the top and a copy of $\Delta((v))$ at the bottom. Note this is obtained from Theorem 3.9.5 in a slightly different way according to whether $u = v$ (i.e., t is even) or $u = v + 1$ (i.e., t is odd). Also, this is now a module in an atypical block. Finally we apply $D_{0|t} \circ D_{1|t-1} \circ \cdots D_{v-1|u+1}$ to end up with the desired two step Δ-flag with a copy of $\Delta(\kappa^{(1)}) = \Delta((t+1))$ at the top and $\Delta(\kappa^{(0)}) = \Delta(\varnothing)$ at the bottom; each of these functors adds a single node to the first row of the Young diagram labelling the module at the top and removes a node from the Young diagram labelling the module at the bottom. This is what Π_0 is meant to do to $\Delta(\varnothing)$. A similar argument shows that $\Pi_0\Delta((t+1))$ has a Δ-flag with the same two sections. It is also easy to check that $\Pi_0\Delta(\kappa^{(m)}) = 0$ for $m > 1$, indeed, $D_{t|0}$ already annihilates these standard modules.

The functors $\Pi_n = D_{-n|-n}$ for $n > 0$ are easier to analyze. Noting that $\kappa^{(n)} = \Delta((t+1, 1^{n-1}))$, the module $\Pi_n\Delta(\kappa^{(n)})$ has a two step Δ-flag with $\Delta(\kappa^{(n+1)}) = \Delta((t+1, 1^n))$ at the top and $\Delta(\kappa^{(n)})$ at the bottom; this uses the $t - |\lambda| = a = b \notin \mathrm{rem}(\lambda)$ case from Theorem 3.9.5. Similarly, $\Pi_n\Delta(\kappa^{(n+1)})$ has a Δ-flag with the same two sections. Finally, one checks that $\Pi_n\Delta(\kappa^{(m)}) = 0$ for $m \neq n, n+1$. $\square$

Remark 3.10.4. In the proof of the next theorem, we will show that the functor Π_n from Lemma 3.10.3 satisfies $\Pi_n\Delta(\kappa^{(n)}) \cong \Pi_n\Delta(\kappa^{(n+1)}) \cong \Pi_n L(\kappa^{(n+1)}) \cong P(\kappa^{(n+1)})$ for all $n \geq 0$.

Now we can prove the main result about blocks. This can also be deduced from [CO11, Th. 6.10], but the proof of that appealed to results of Martin [Mar96] in order to obtain the precise submodule structure of the indecomposable projectives, whereas we are able to establish this by exploiting the highest weight structure and the Chevalley duality $?^{\circledcirc}$.

Theorem 3.10.5. *Let $\kappa \in \mathcal{P}_t$ and $S := \{\kappa^{(0)}, \kappa^{(1)}, \dots\}$ be the corresponding $\sim_t$-equivalence class.*

(i) For each $n \geq 0$, the standard module $\Delta(\kappa^{(n)})$ is of length two with head $L(\kappa^{(n)})$ and socle $L(\kappa^{(n+1)})$.

(ii) The indecomposable projective module $P(\kappa^{(0)})$ is isomorphic to $\Delta(\kappa^{(0)})$, while for $n \geq 1$ the module $P(\kappa^{(n)})$ has a two step Δ-flag with top section $\Delta(\kappa^{(n)})$ and bottom section $\Delta(\kappa^{(n-1)})$.

(iii) For each $n \geq 1$, $P(\kappa^{(n)})$ is self-dual with irreducible head and socle isomorphic to $L(\kappa^{(n)})$ and completely reducible heart $\operatorname{rad} P(\kappa^{(n)}) / \operatorname{soc} P(\kappa^{(n)}) \cong L(\kappa^{(n-1)}) \oplus L(\kappa^{(n+1)})$.

Proof. To improve the readability, we write simply $P(n), \Delta(n)$ and $L(n)$ in place of $P(\kappa^{(n)}), \Delta(\kappa^{(n)})$ and $L(\kappa^{(n)})$. For $n \geq 0$, Lemma 3.10.3 shows that the module $P_n := \Pi_{n-1} \circ \cdots \Pi_1 \circ \Pi_0(\Delta(0))$ has a two step Δ-flag with top section $\Delta(n)$ and bottom section $\Delta(n-1)$. Since $\Delta(0)$ is projective by the minimality observed in (3.9.1) and each Π_i is a projective functor, P_n is projective. Since P_n has $L(n)$ in its head, it must contain the indecomposable projective $P(n)$ as a summand, so we either have that $P(n) \cong P_n$ if P_n is indecomposable, or $P(n) \cong \Delta(n)$ otherwise. In the former case, $(P(n) : \Delta(m)) = \delta_{m,n} + \delta_{m,n-1}$, while $(P(n) : \Delta(m)) = \delta_{m,n}$ in the latter situation. Now we apply BGG reciprocity to deduce for any $m \geq 0$ that $[\Delta(m) : L(n)] = \delta_{n,m} + \delta_{n,m+1}$ if P_n is indecomposable and $[\Delta(m) : L(n)] = \delta_{n,m}$ otherwise. Hence, for each $m \geq 0$, we either have that $\Delta(m) \cong L(m)$, or $\Delta(m)$ is of composition length two with composition factors $L(m)$ and $L(m+1)$.

We claim for any $n \geq 0$ that $\Delta(n) \cong L(n)$ if and only if $\Delta(n+1) \cong L(n+1)$. Suppose first that $\Delta(n) \cong L(n)$. Since Π_n commutes with duality by Lemma 3.9.2, this implies that $\Pi_n\Delta(n)$ is self-dual. But this module has a two step Δ-flag with top section $\Delta(n+1)$ and bottom section $\Delta(n) \cong L(n)$. The only way such a module can be self-dual is if $\Delta(n+1) \cong L(n+1)$ (and the module must be completely reducible). Conversely, suppose for a contradiction that $\Delta(n+1) \cong L(n+1)$ but $\Delta(n) \not\cong L(n)$. Then $\Delta(n)$ is of length two with composition factors $L(n)$ and $L(n+1)$, so that $P(n+1)$ has a two step Δ-flag with top section $\Delta(n+1) \cong L(n+1)$ and bottom section $\Delta(n)$. Since $\Pi_{n+1}\Delta(n) = 0$ according to Lemma 3.10.3 and Π_{n+1} is exact, we must have that $\Pi_{n+1}L(n+1) = 0$. Since $\Delta(n+1) \cong L(n+1)$, this implies that $\Pi_{n+1}\Delta(n+1) = 0$, which contradicts Lemma 3.10.3.

From the claim, we see that if $\Delta(n)$ is irreducible for any one $n \geq 0$, then it is irreducible for all $n \geq 0$. Since all atypical "blocks" are equivalent by Theorem 3.10.2, it follows in that case that the standard modules $\Delta(\lambda)$ for *all* $\lambda \in \mathcal{P}$ are irreducible. This implies that the minimal ordering $\succeq_t$ from Remark 3.7.5 is trivial, hence, the blocks are trivial and Par_t is semisimple, which contradicts Corollary 3.8.6. Thus, we have proved that $\Delta(n)$ must be of length two for every $n \geq 0$, and (i) is proved.

Property (ii) follows immediately from (i) and BGG reciprocity as noted earlier.

It remains to prove (iii). Take $n \geq 1$. By Lemma 3.10.3, we have that $\Pi_{n-1}\Delta(n+1) = 0$. Since Π_{n-1} is exact and $L(n+1)$ is a composition factor of $\Delta(n+1)$, it follows that $\Pi_{n-1}L(n+1) = 0$ too. From this, we deduce that $\Pi_{n-1}\Delta(n) \cong \Pi_{n-1}L(n)$. By Lemma 3.10.3 again, $\Pi_{n-1}\Delta(n-1)$ has the same composition length as $\Pi_{n-1}\Delta(n) \cong \Pi_{n-1}L(n)$. Also $\Delta(n-1)$ has $L(n)$ as a constituent. Using the exactness of Π_{n-1} again, we must therefore have that $\Pi_{n-1}\Delta(n-1) \cong \Pi_{n-1}L(n)$. As observed earlier in the proof, this module is isomorphic to $P(n)$, so using that $L(n)$ is self-dual and Π_{n-1} commutes with duality, we now see that $P(n)$ is self-dual. We also know that it has length four with irreducible head $L(n)$, $[P(n) : L(n)] = 2$ and

$[P(n) : L(n-1)] = [P(n) : L(n+1)] = 1$. The only possible structure is the one claimed. □

Corollary 3.10.6 (Comes-Ostrik)**.** *All "blocks" of Par_t-*Mod *are indecomposable, hence, they coincide with the blocks.*

Corollary 3.10.7. *The minimal ordering $\succeq_t$ from Remark 3.7.5 is the partial order such that $\kappa^{(m)} \succeq_t \kappa^{(n)}$ for each $\kappa \in \mathcal{P}_t$ and $m \leq n$, with all other pairs of partitions being incomparable.*

In general, in an upper finite highest weight category, the standard objects can have infinite length. Our final corollary, which is also noted in [SS22, Rem. 6.4], shows that this is not the case in Par_t-Mod$_{\text{lfd}}$. Consequently, the full subcategory consisting of all modules of finite length has enough projectives and injectives, indeed, this subcategory is an essentially finite highest weight category in the sense of [BS, Def. 3.7].

Corollary 3.10.8. *The locally unital algebra Par_t is locally Artinian, i.e., the left ideals $Par_t 1_n$ and the right ideals $1_n Par_t$ are of finite length for all $n \geq 0$.*

Proof. Theorem 3.10.5 shows that all indecomposable projective left Par_t-modules are of finite length, hence, all finitely generated projectives are of finite length too. This includes all of the left ideals $Par_t 1_n$. Since there is a duality $?^{\circledast}$, it also follows that all fintely cogenerated injective left Par_t-modules are of finite length. This includes all of the duals $(1_n Par_t)^{\circledast}$, hence, each $1_n Par_t$ is of finite length as a right module. □

3.11 Proof of Theorem 3.9.5

It just remains to prove Theorem 3.9.5. In fact, we will prove the following slightly stronger result, from which Theorem 3.9.5 follows easily on applying the functors involved to the Specht module $S(\lambda)$. To state this stronger result, let $j_! : \mathit{Sym}\text{-Mod}_{\mathrm{fd}} \to \mathit{Par}_t\text{-Mod}_{\mathrm{lfd}}$ be the standardization functor from (2.9.3), E_a and F_b be the refined induction and restriction functors from (2.7.12), $D_{b|a}$ be the special projective functor from (3.9.7), and $\mathrm{pr}_c : \mathit{Sym}\text{-Mod}_{\mathrm{fd}} \to \mathit{Sym}\text{-Mod}_{\mathrm{fd}}$ be the functor defined by multiplication by the identity element of the symmetric group S_c if $c \in \mathbb{N}$, i.e., it is the projection onto $\Bbbk S_c\text{-Mod}_{\mathrm{fd}}$ followed followed by the inclusion of $\Bbbk S_c\text{-Mod}_{\mathrm{fd}}$ into $\mathit{Sym}\text{-Mod}_{\mathrm{fd}}$, or the zero functor if $c \in \Bbbk - \mathbb{N}$.

Theorem 3.11.1. *For $a, b \in \Bbbk$, there is a filtration of the functor $D_{b|a} \circ j_! : \mathit{Sym}\text{-Mod}_{\mathrm{fd}} \to \mathit{Par}_t\text{-Mod}_{\mathrm{lfd}}$ by subfunctors $0 = S_0 \subseteq S_1 \subseteq S_2 \subseteq S_3 \subseteq S_4 = D_{b|a} \circ j_!$ such that*

$$S_4/S_3 \cong j_! \circ E_a \circ \mathrm{pr}_{t-b},$$

$$S_3/S_2 \cong j_! \circ \mathrm{pr}_{t-a} \circ \mathrm{pr}_{t-b},$$

$$S_2/S_1 \cong j_! \circ E_a \circ F_b,$$

$$S_1/S_0 \cong j_! \circ \mathrm{pr}_{t-a} \circ F_b.$$

(Recall that a subfunctor *S of a functor $T : \mathit{Sym}\text{-Mod}_{\mathrm{fd}} \to \mathit{Par}_t\text{-Mod}_{\mathrm{lfd}}$ is a functor $S : \mathit{Sym}\text{-Mod}_{\mathrm{fd}} \to \mathit{Par}_t\text{-Mod}_{\mathrm{lfd}}$ such that SV is a submodule of TV for all $V \in \mathit{Sym}\text{-Mod}_{\mathrm{fd}}$ and $Sf = Tf|_{SV}$ for all $f \in \mathrm{Hom}_{\mathit{Sym}}(V, V')$; then the quotient T/S is the obvious functor with $(T/S)(V) := TV/SV$.)*

The proof will take up the rest of the subsection. We begin by constructing a filtration of the functor $D \circ j_! : \mathit{Sym}\text{-Mod}_{\mathrm{fd}} \to \mathit{Par}_t\text{-Mod}_{\mathrm{lfd}}$. Note that $D \circ j_! \cong M \otimes_{\mathit{Sym}}$ where M is the $(\mathit{Par}_t, \mathit{Sym})$-bimodule

$$M := 1_{|\star} \mathit{Par}_t \otimes_{\mathit{Par}^\sharp} \mathrm{infl}^\sharp\, \mathit{Sym}. \tag{3.11.1}$$

We also have the (Par_t, Sym)-bimodules

$$N_4 = Par_t \otimes_{Par^\sharp} \mathrm{infl}^\sharp(Sym1_{|\star}), \tag{3.11.2}$$

$$N_3 := Par_t \otimes_{Par^\sharp} \mathrm{infl}^\sharp\, Sym, \tag{3.11.3}$$

$$N_2 := Par_t \otimes_{Par^\sharp} \mathrm{infl}^\sharp(Sym1_{|\star} \otimes_{Sym} 1_{|\star}Sym) \tag{3.11.4}$$

$$N_1 := Par_t \otimes_{Par^\sharp} \mathrm{infl}^\sharp(1_{|\star}Sym). \tag{3.11.5}$$

The functors $Sym\text{-Mod}_{\mathrm{fd}} \to Par_t\text{-Mod}_{\mathrm{lfd}}$ defined by tensoring with N_4, N_3, N_2 and N_1 are isomorphic to $j_! \circ E$, $j_!$, $j_! \circ E \circ F$ and $j_! \circ F$, respectively.

For $m \geq n \geq 0$, let $B_{m,n}$ be the basis for $1_m Par^- 1_n$ defined by representatives for the equivalence classes of normally ordered upward partition diagrams. By Theorem 2.8.1, the vector space M is isomorphic to $1_{|\star}Par^- \otimes_{\mathbb{K}} Sym$, hence, it has basis

$$\left\{ f \otimes g \,\middle|\, m \geq 0, n \geq 0, m+1 \geq n, f \in B_{m+1,n}, g \in S_n \right\}. \tag{3.11.6}$$

For any $f \in B_{m+1,n}$, let $c(f)$ be the connected component of the diagram containing the top left vertex. In the language from §2.8, this component could be a trunk, an upward tree, an upward leaf, or an upward branch. Then we introduce the following subspaces of M:

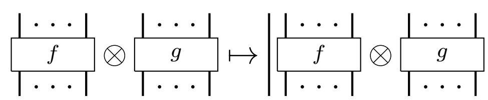
- Let M_1 be the subspace of M spanned by all $f \otimes g$ in this basis such that $c(f)$ is a trunk.
- Let M_2 be the subspace spanned by all $f \otimes g$ such that $c(f)$ is either a trunk or an upward tree.
- Let M_3 be the subspace spanned by all $f \otimes g$ such that $c(f)$ is either a trunk, an upward tree, or an upward leaf.
- Let $M_0 := 0$ and $M_4 := M$.

The following is a generalization of Theorem 3.9.1.

Lemma 3.11.2. *The subspaces* $0 = M_0 \subset M_1 \subset M_2 \subset M_3 \subset M_4 = M$ *are sub-bimodules of the* (Par_t, Sym)*-bimodule* M*. Moreover, there are bimodule isomorphisms* $\theta_i : N_i \xrightarrow{\sim} M_i/M_{i-1}$ *for each* $i = 1, \ldots, 4$*.*

Proof. The fact that each M_i is a sub-bimodule of M is easily checked by vertically composing a basis vector $f \otimes g$ with an arbitrary partition diagram on the top and with any permutation diagram on the bottom. One just needs to note that the action on top involves $\mathrm{res}_{|\star}$, so that the top left vertex is untouched. This implies that the type $c(f)$ does not change if it is a trunk or an upward leaf, while if it is an upward tree it can only be changed to another upward tree or to a trunk.

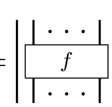
We show in this paragraph that there is a bimodule isomorphism

$\theta_1 : N_1 \to M_1,$  (3.11.7)

for any $m \geq 0, n > 0,\ f \in 1_m Par_t 1_{n-1}$ and $g \in S_n$. This is a well-defined bimodule homomorphism. By Theorem 2.8.1, N_1 is isomorphic as a vector space to $Par^- \otimes_{\mathbb{K}} 1_{|\star} Sym$, hence, it has basis

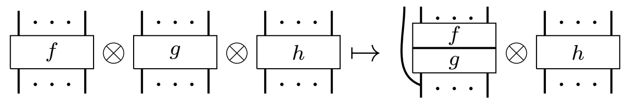
$$\left\{ f \otimes g \,\middle|\, m \geq n-1 \geq 0, f \in B_{m,n-1}, g \in S_n \right\}. \tag{3.11.8}$$

The vector space M_1 has basis given by all $f_1 \otimes g$ for $m+1 \geq n > 0, f_1 \in B_{m+1,n}$ and $g \in S_n$ such that $c(f_1)$ is a trunk. As it is normally ordered, any such f_1 is of the form

$f_1 =$ 

for a unique $f \in B_{m,n-1}$. Moreover, $f_1 \otimes g = \theta_1(f \otimes g)$ for every $g \in S_n$. It follows that θ_1 takes a basis for N_1 to a basis for M_1, so it is an isomorphism.

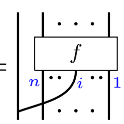
Next we show that there is a bimodule isomorphism

$\theta_2 : N_2 \to M_2/M_1,$  $+ M_1$ (3.11.9)

for $m \geq 0, n > 0, f \in 1_m Par_t 1_n$ and $g, h \in S_n$. Again, this is a well-defined bimodule homomorphism. By Theorem 2.8.1, N_2 is isomorphic to $Par^- \otimes_{\mathbb{K}} Sym 1_{|\star} \otimes_{Sym} 1_{|\star} Sym$. Also $\Bbbk S_n$ is free as a right $\Bbbk S_{n-1}$-module with basis given by $\{(i\ i{+}1\ \cdots\ n) \mid 1 \leq i \leq n\}$, which is a set of S_n/S_{n-1}-cosets. It follows that N_2 has basis

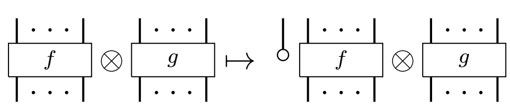
$$\left\{ f \otimes (i\ i{+}1\ \cdots\ n) \otimes g \,\middle|\, m \geq n > 0, f \in B_{m,n}, 1 \leq i \leq n, g \in S_n \right\}. \tag{3.11.10}$$

The vector space M_2/M_1 has a basis given by all $f_2 \otimes g + M_1$ for $m+1 \geq n > 0, f_2 \in B_{m+1,n}$ and $g \in S_n$ such that $c(f_2)$ is an upward tree. Any such f_2 is equal to

$f_2 =$ 

for a unique $f \in B_{m,n}$ and a unique $1 \leq i \leq n$ (the index of the string at which the component $c(f_2)$ meets the bottom of f). Moreover, $f_2 \otimes g = \theta_2(f \otimes (i\ i{+}1\ \cdots\ n) \otimes g)$ for each $g \in S_n$. It follows that θ_2 takes a basis for N_2 to a basis for M_2/M_1, so it is an isomorphism.

The isomorphism θ_3 is defined by

$\theta_3 : N_3 \to M_3/M_2,$  $+ M_2$ (3.11.11)

for $m \geq 0$, $n \geq 0$, $f \in 1_m Par_t 1_n$ and $g \in S_n$. This is obviously a well-defined bimodule homomorphism. It is an isomorphism because it takes the basis

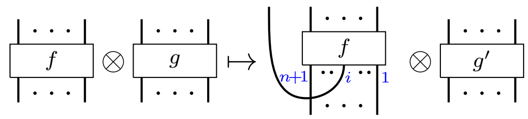
$$\{f \otimes g \mid m \geq n \geq 0, f \in B_{m,n}, g \in S_n\} \tag{3.11.12}$$

for N_3 to the basis for M_3/M_2 consisting of all $f_3 \otimes g + M_2$ for $m+1 > n \geq 0, f_3 \in B_{m+1,n}$ and $g \in S_n$ such that $c(f_3)$ is an upward leaf.

Finally, we construct the isomorphism θ_4. The vector space N_4 has basis

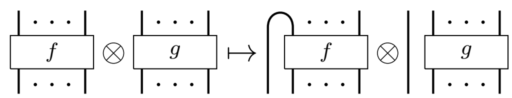
$$\{f \otimes g \,|\, m-1 \geq n \geq 0, f \in B_{m,n+1}, g \in S_{n+1}\}. \tag{3.11.13}$$

We define the linear map

$\theta_4 : N_4 \to M_4/M_3,$ 

$+ M_3,$ (3.11.14)

where $f \otimes g$ is a vector from the basis for N_4 just displayed, and $g' \in S_n$ and $1 \leq i \leq n+1$ are defined from the equation $g = (i\ i{+}1\ \cdots\ n{+}1)g'$. To see that this linear map is actually a bimodule isomorphism, we construct a bimodule homorphism in the other direction and show that it is a two-sided inverse of θ_4. Consider the map

$\phi : M \to N_4,$ 

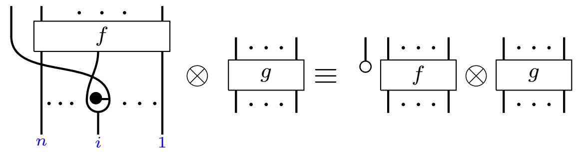
(3.11.15)

for $m \geq 0, n \geq 0, f \in 1_{m+1} Par_t 1_n$ and $g \in S_n$. It is easy to show that this is a well-defined bimodule homomorphism. Moreover, $M_3 \subseteq \ker \phi$ since applying ϕ to any basis vector $f \otimes g \in M_3$ produces a downward leaf, a cap or a downward tree which can be pushed across the tensor to act as zero on $\mathrm{infl}^\sharp\, Sym$. Hence, ϕ induces a homomorphism $\bar{\phi} : M_4/M_3 \to N_4$. It remains to check that $\bar{\phi} \circ \theta_4$ and $\theta_4 \circ \bar{\phi}$ are both identity morphisms, which is straightforward. □

In the next two lemmas, we finally need to make some explicit calculations with the relations involving the left and right dots in the affine partition category. However, we are working now with $\mathcal{P}ar_t$, not with $\mathcal{AP}ar$, so all string diagrams from now on should be interpreted as the canonical images of these morphisms in $\mathcal{AP}ar$ under the functor $p_t : \mathcal{AP}ar \to Par_t$ from (3.4.12). We will also use the notation from (2.7.10) for an open dot on the interior of a string, meaning the canonical image of this morphism in $\mathcal{AS}ym$ under the functor $p : \mathcal{AS}ym \to \mathcal{S}ym$ from (2.7.8). This is quite different from an open dot at the end of a string!

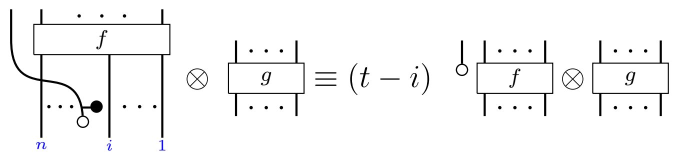
Lemma 3.11.3. *Suppose that $m \geq 0, n \geq 0, f \in 1_m Par_t 1_n$ and $g \in S_n$.*

(i) The following holds in the bimodule $M = 1_{|\star} Par_t \otimes_{Par^\sharp} \mathrm{infl}^\sharp\, Sym$ for $i = 1, \ldots, n$:



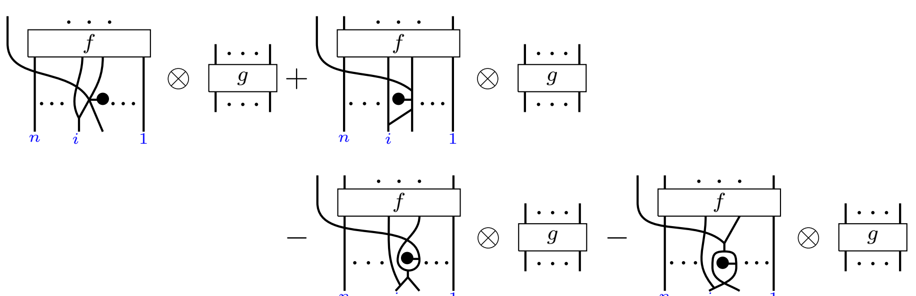
(mod M_2).

(ii) The following holds in the bimodule M for $i = 0, 1, \ldots, n$ (the case $i = 0$ is when there are no strings to the right of the dangling dots):

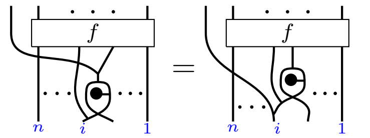


(mod M_2).

Proof. (i) We proceed by induction on $i = 1, \ldots, n$. The base case $i = 1$ follows from (3.4.13). For the induction step, we take $i > 1$ and assume the result has been proved for $i - 1$. Then we apply (3.3.13) to commute the left dot past the string to its right. This produces a sum of five terms. Ordering these terms in the same way as they appear on the right hand side of (3.3.13), the induction hypothesis can be applied to the first term, to produce the right hand side that we are after. It remains to show that the other four terms lies in M_2. These terms are as follows:

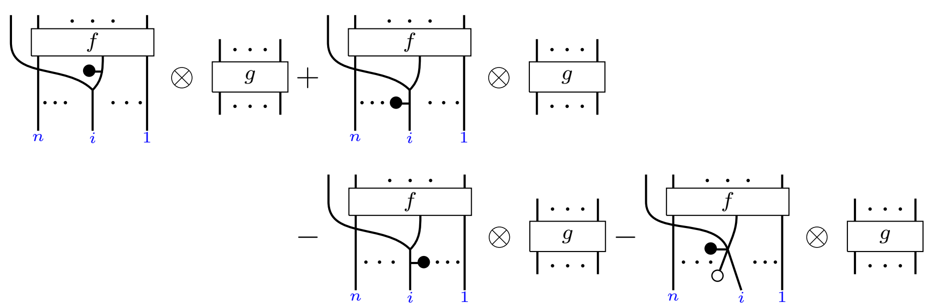


The second and third terms here are zero already in M because, in both of them, the diagram to the left of the tensor is equivalent to a diagram with a downward tree at the bottom. It remains to show that the first and the fourth terms lie in M_2. For the fourth term, we note that

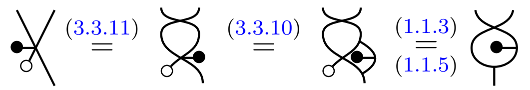


The left dot can now be absorbed into the morphism f, changing it to some other morphism f'. The result is a linear combination of morphisms in all of which the top left vertex is connected to the bottom edge, so that the connected component containing this vertex is either a tree or a trunk, and it belongs to the sub-bimodule M_2. The reason the first term lies in M_2 is very similar, one just needs to rewrite the right crossing using (3.3.10), and then it is easy to see that the top left vertex is again connected to the bottom edge.

(ii) Again we proceed by induction. The base case $i = 0$ follows from (3.4.13) using that $T = t1_{\mathbb{1}}$. For the induction step, we consider some $i > 0$. Then we apply (3.3.14) to commute the right dot past the string to its right. This produces a sum of five terms. This time, the induction hypothesis can be applied to the first term, to produce the vector that we are after but scaled by $(t - i + 1)$ rather than the desired $(t - i)$. The remaining four terms are as follows:



In the first term here, the left dot is some morphism in Par_t, which has the effect of changing f to some other morphism f'. After doing that, it is clear that the top left vertex is still connected to the bottom edge, so the first term lies in M_2. For the second and third terms, the left and right dots can be commuted across the tensor using (3.8.4), then again we see that these morphisms lie in M_2 since the top left vertex is connected to the bottom edge again. For the final term, we note that



Making this substitution in the middle of the picture reveals that the final term is exactly the expression studied in (i). On applying the conclusion of (i), we deduce that it contributes exactly the needed correction to complete the proof. $\square$

Lemma 3.11.4. *Consider the bimodule endomorphisms ρ and λ of M defined on $1_{m+1}Par_t 1_n \otimes \Bbbk S_n$ by the left action of $x^R_{m+1} \otimes 1_n$ and $x^L_{m+1} \otimes 1_n$, respectively, for each $m, n \geq 0$. These endomorphisms preserve each of the sub-bimodules M_i $(i = 1, 2, 3, 4)$, hence, ρ and λ induce endomorphisms also denoted ρ and λ of each of the subquotients M_i/M_{i-1}. Moreover, for each i, the isomorphism θ_i from Lemma 3.11.2 satisfies*

$$\theta_i \circ \rho_i = \rho \circ \theta_i, \qquad \theta_i \circ \lambda_i = \lambda \circ \theta_i, \tag{3.11.16}$$

where $\rho_i, \lambda_i : N_i \to N_i$ are defined as follows:

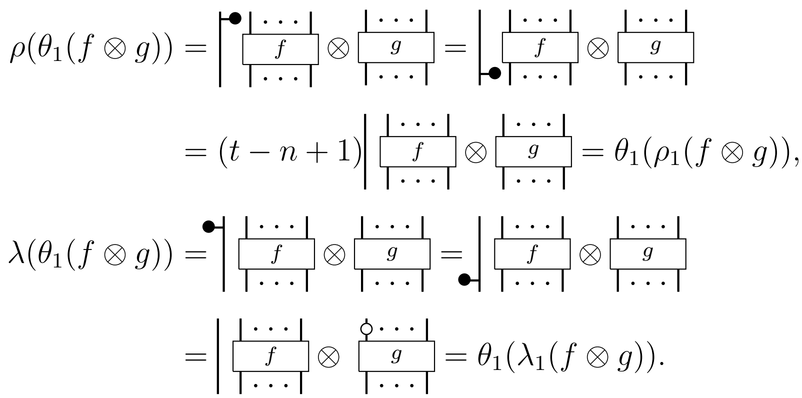
(i) ρ_1 and λ_1 are the bimodule endomorphisms of N_1 defined on the subspace $1_m Par_t 1_{n-1} \otimes \Bbbk S_n$ by the left actions of $(t-n+1)1_m \otimes 1_n$ and $1_m \otimes x_n$, respectively, for each $m \geq 0, n > 0$.

(ii) ρ_2 and λ_2 are the bimodule endomorphisms of N_2 defined on $1_m Par_t 1_n \otimes \Bbbk S_n \otimes \Bbbk S_n$ by the right action of $1_n \otimes x_n \otimes 1_n$ and the left action of $1_m \otimes 1_n \otimes x_n$, respectively.

(iii) ρ_3 and λ_3 are both equal to the bimodule endomorphism of N_3 defined on $1_m Par_t 1_n \otimes \Bbbk S_n$ by multiplication by $(t - n)$.

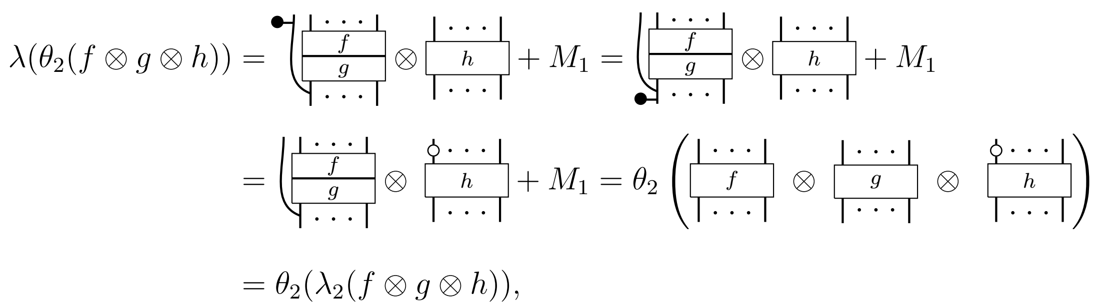
(iv) ρ_4 and λ_4 are the bimodule endomorphisms of N_4 defined on $1_m Par_t 1_{n+1} \otimes \Bbbk S_{n+1}$ by the right actions of $1_{n+1} \otimes x_{n+1}$ and $(t-n)1_{n+1} \otimes 1_{n+1}$, respectively.

Proof. (i) Recall the definition of θ_1 from (3.11.7). Take a vector $f \otimes g$ in the basis for N_1 from (3.11.8). By (3.8.4), we have that $x_n^L \equiv x_n \pmod{K_n}$ and $x_n^R \equiv (t-n+1)1_n \pmod{K_n}$ where K_n is the two sided ideal of $1_n Par_t 1_n$ from (3.7.1). Since the strictly downward partition diagrams which generate K^+ are zero on $\inf\!l^\sharp\, Sym$, it follows that

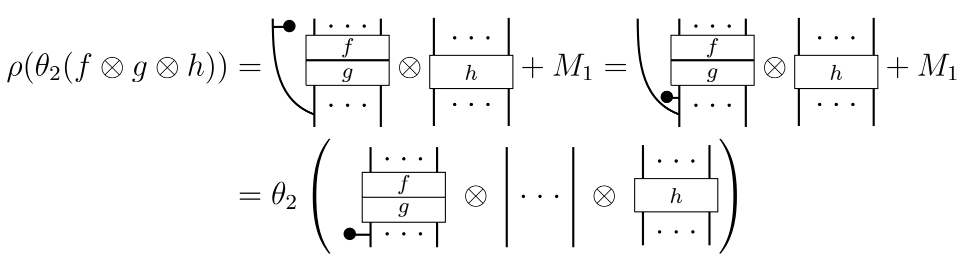


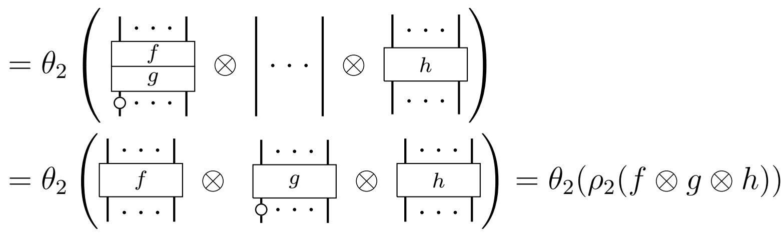
This shows as the same time that ρ and λ both leave M_1 invariant.

(ii) Recall the definition of θ_2 from (3.11.9). The argument for λ is similar to in (i). It follows from the calculation



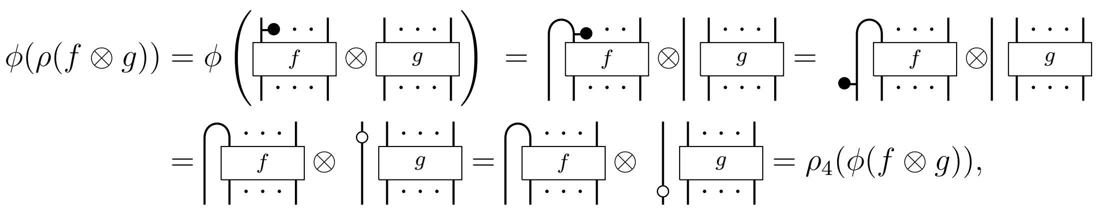
where $f \otimes g \otimes h$ is one of the basis vectors for N_2 from (3.11.10). For ρ, we instead have that

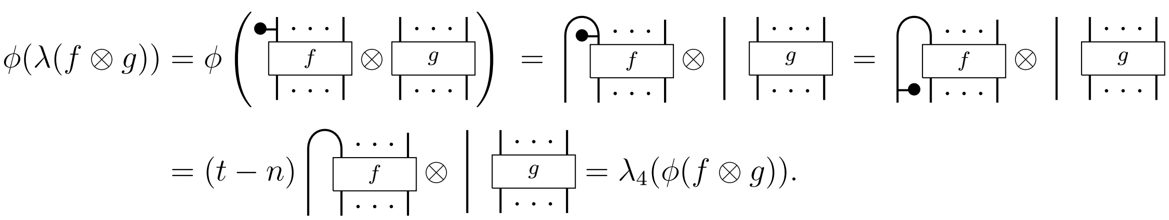




(iii) Recall the definition of θ_3 from (3.11.11). Note that $\rho = \lambda$ by the third relation from (3.3.8). For ρ, we need to show that $\rho(\theta_3(f \otimes g)) = (t-n)\theta_3(f \otimes g)$ for any basis vector $f \otimes g \in 1_m Par_t 1_n \otimes \Bbbk S_n \subset N_3$ from (3.11.12). This follows from Lemma 3.11.3(ii) taking $i = n$.

(iv) Instead of working with θ_4 from (3.11.14), it is easier to use the inverse map $\bar{\phi}$ induced by the homomorphism $\phi : M \to N_4$ from (3.11.15). We need to show that $\phi \circ \rho = \rho_4 \circ \phi$. This follows from the following calculations for $f \otimes g \in 1_{m+1} Par_t 1_n \otimes \Bbbk S_n$ and $m, n \geq 0$:





$\square$

Proof of Theorem 3.11.1. The functor $D_{b|a} \circ j_!$ is defined by tensoring with the bimodule $\overline{M}$ that is the simultaneous generalized a eigenspace of the endomorphism ρ and generalized b eigenspace of the endomorphism λ defined in Lemma 3.11.4. Lemma 3.11.2 defines a filtration of M with sections $M_i/M_{i-1} \cong N_i$ for $i = 1, \ldots, 4$. Then Lemma 3.11.4 shows that the endomorphisms ρ and λ preserve this filtration, hence, the filtration of M induces a filtration of the summand $\overline{M}$. Moreover, for each

i, $\overline{M}_i/\overline{M}_{i-1}$ is isomorphic to the summand $\overline{N}_i$ of N defined by the simultaneous generalized a-eigenspace of the endomorphism ρ_i and generalized b eigenspace of the endomorphism λ_i. By the descriptions of ρ_i and λ_i, it follows that $\overline{N}_i \otimes_{Sym}$ is isomorphic to the functor $j_! \circ E_a \circ \mathrm{pr}_{t-b}$, $j_! \circ \mathrm{pr}_{t-a} \circ \mathrm{pr}_{t-b}$, $j_! \circ E_a \circ F_b$ or $j_! \circ \mathrm{pr}_{t-a} \circ F_b$ for $i = 4, 3, 2, 1$, respectively. It remains to observe that Sym is semisimple, so every Sym-module is flat. This means that the filtration of $\overline{M}$ induces a filtration $0 = S_0 \subseteq S_1 \subseteq S_2 \subseteq S_3 \subseteq S_4 = D_{b|a} \circ j_!$ such that $S_i \cong \overline{M}_i/\overline{M}_{i-1} \otimes_{Sym} \cong \overline{N}_i \otimes_{Sym}$. $\square$

CHAPTER IV

RESTRICTION FUNCTOR

There is a well-known 'restriction' functor $F^t_{t-1} : \underline{\mathrm{Re}}\mathrm{p}(S_t) \to \underline{\mathrm{Re}}\mathrm{p}(S_{t-1})$ which interpolates between Deligne's categories for different parameters of $t \in \Bbbk$. Comes and Ostrik conjectured that this restriction functor provides an equivalence of categories between the nontrivial principle blocks of $\underline{\mathrm{Re}}\mathrm{p}(S_t)$ and $\underline{\mathrm{Re}}\mathrm{p}(S_{t-1})$ whenever $t \in \mathbb{Z}_{>0}$. Recall that Deligne's category $\underline{\mathrm{Re}}\mathrm{p}(S_t)$ is monoidally equivalent to the full subcategory Par_t-Proj of $\mathcal{P}ar_t$-$\mathrm{Mod}_{\mathrm{lfd}}$ consisting of finitely generated projectives. In this light, the first portion of this chapter provides an interpretation of this restriction functor in terms of path algebras and bimodules over them. To this end, we introduce a new intermediate category $\mathcal{P}ar^{\times}_{t-1}$ along with functors $\mathcal{P}ar_t \to \mathrm{Add}(\mathcal{P}ar^{\times}_{t-1})$ and $\mathcal{P}ar_{t-1} \to \mathrm{Add}(\mathcal{P}ar^{\times}_{t-1})$. The composition of these functors, at the level of module categories, will allow us to induce, then restrict, locally finite-dimensional $\mathcal{P}ar_{t-1}$-modules to get locally finite-dimensional $\mathcal{P}ar_t$-modules and vice-versa. This construction gives a module-theoretic version of the restriction functor $R^t_{t-1} : Par_{t-1}\text{-}\mathrm{Mod}_{\mathrm{lfd}} \to Par_t\text{-}\mathrm{Mod}_{\mathrm{lfd}}$. After studying R^t_{t-1} and its interaction with standardly-filtered modules, we show that it gives an equivalence between the principal blocks of Par_{t-1}-$\mathrm{Mod}_{\mathrm{lfd}}$ and Par_t-$\mathrm{Mod}_{\mathrm{lfd}}$, thereby proving the Comes-Ostrik conjecture.

4.1 Phantom partitions

Fix some $t \in \Bbbk$. Throughout this chapter, it is sometimes useful to distinguish between objects of $\mathcal{P}ar_t$ (or Par_t-Mod) and of $\mathcal{P}ar_{t-1}$ (or Par_{t-1}-Mod, or any other category $\mathcal{P}ar_u$ for another value of $u \in \Bbbk$). In particular, we use subscripts $X_t \in \mathbb{O}(\mathcal{P}ar_t)$ when there may be potential confusion for which category an object X is contained in. We also fix the notations $|_t$ and $\mathbb{1}_t \in \mathbb{O}(\mathcal{P}ar_t)$ for the monoidal generator and unit object of $\mathcal{P}ar_t$, actually reserving the unscripted $\mathbf{|}$ and $\mathbb{1}$ to mean the monoidal generator and unit of $\mathcal{P}ar_{t-1}$.

Definition 4.1.1. The *phantom partition category* $\mathcal{P}ar_{t-1}^{\times}$ is the strict symmetric monoidal category obtained from $\mathcal{P}ar_{t-1}$ by adjoining a new generating object $\times$ and generating morphisms:

$$\begin{matrix}\cdot\\ \\ \times\end{matrix} : \times \to \mathbb{1}, \qquad\qquad \begin{matrix}\times\\ \\ \cdot\end{matrix} : \mathbb{1} \to \times$$

subject to the relation that these morphisms are two-sided inverses of each other:

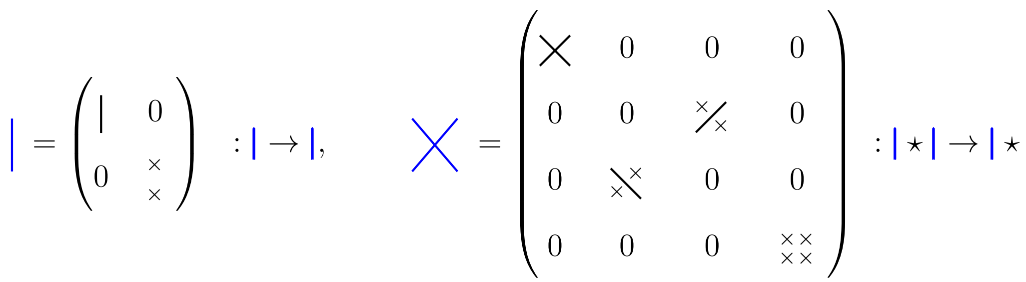
$$\begin{matrix}\times\\ \\ \times\end{matrix} = \left(\begin{matrix}\times\\ \\ \cdot\end{matrix}\right) \circ \left(\begin{matrix}\cdot\\ \\ \times\end{matrix}\right) = \begin{matrix}\times\\ |\\ \times\end{matrix} = \mathrm{id}_{\times}, \qquad \begin{matrix}\cdot\\ \times\\ \cdot\end{matrix} = \left(\begin{matrix}\cdot\\ \\ \times\end{matrix}\right) \circ \left(\begin{matrix}\times\\ \\ \cdot\end{matrix}\right) = \begin{matrix}\cdot\\ \\ \cdot\end{matrix} = \mathrm{id}_{\mathbb{1}}, \tag{4.1.1}$$

Remark 4.1.2. The dots appearing in these new morphisms are purely decorative and serve only to help distinguish the top or bottom of a diagram.

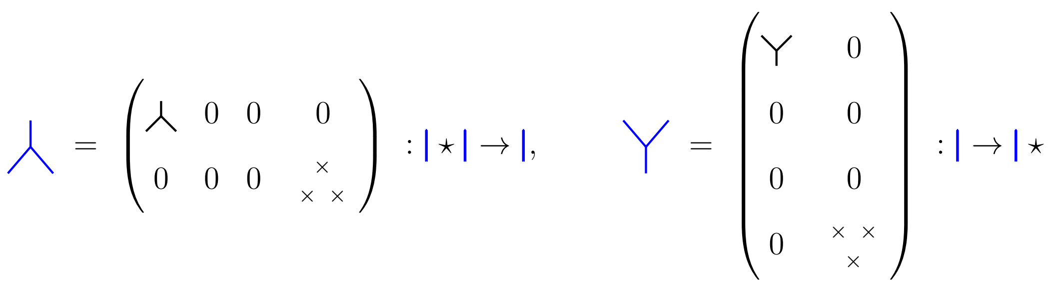
Introducing new nomenclature, a *phantom partition diagram* is a diagram obtained by taking a partition diagram, f, and inserting some (perhaps zero) symbols $\times$ along the top and bottom rows of f. Given a phantom partition diagram f, say that f is a partition diagram *with phantoms* if it contains at least one symbol $\times$. In the other case, we say f is phantomless.

Given $n \in \mathbb{N}$, let W^n be the set of words in the letters $\{\mathbf{|}, \times\}$ of length n and let $W := \bigcup_{n\in\mathbb{N}} W^n$ be the set of all words. Then $\mathbb{O}(\mathcal{P}ar_{t-1}^{\times}) = W$. Suppose w and w' are any two words with $k(w)$ and $k(w')$ of the letters being $\mathbf{|}$, respectively. Then the isomorphism between $\mathbb{1}$ and $\times$ induces an isomorphism $\mathrm{Hom}_{\mathcal{P}ar_{t-1}}(\mathbf{|}^{\otimes k(w)}, \mathbf{|}^{\otimes k(w')}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{P}ar_{t-1}^{\times}}(w, w')$. This map is described on a basis of partition diagrams by inserting phantoms $\times$ in the unique way along the bottom and top rows to spell the words w and w', respectively. Hence, the path algebra of $\mathcal{P}ar_{t-1}^{\times}$ inherits a basis consisting of phantom partition diagrams.

The restriction functor of the Comes and Ostrik conjecture is defined using Deligne's categories $\underline{\mathrm{Rep}}(S_t)$ and $\underline{\mathrm{Rep}}(S_{t-1})$. In order to understand the induced functor at the level of modules over Par_t and Par_{t-1}, we pass through additive envelopes. Introducing the object ${\color{blue}|} := | \oplus \times \in \mathrm{Add}(\mathcal{P}ar^{\times}_{t-1})$, the following special matrices are crucial for building the restriction functor in this setting. These matrices are morphisms in $\mathrm{Add}(\mathcal{P}ar^{\times}_{t-1})$, i.e., matrices of morphisms in $\mathcal{P}ar^{\times}_{t-1}$.



(4.1.2)



(4.1.3)

(4.1.4)

Remark 4.1.3. In the above definitions, we implicitly make the following identification

$$ {\color{blue}|} \star {\color{blue}|} = (| \oplus \times)^{\star 2} = (| \star |) \oplus (| \star \times) \oplus (\times \star |) \oplus (\times \star \times). $$

More generally, the summands of ${\color{blue}|}^{\star n}$ are words $w \in W^n$ for any $n \in \mathbb{N}$. The ordering of the summands comes from the strict monoidal structure on $\mathrm{Add}(\mathcal{P}ar^{\times}_{t-1})$ which is inherited by that on $\mathcal{P}ar^{\times}_{t-1}$, described as follows. Consider objects $X_1 \oplus \ldots \oplus X_n, Y_1 \oplus \ldots \oplus Y_m, X'_1 \oplus \ldots \oplus X'_{n'}$ and $Y'_1 \oplus \ldots \oplus Y'_{m'} \in \mathbb{O}(\mathrm{Add}(\mathcal{P}ar^{\times}_{t-1}))$ along with morphisms $f : X_1 \oplus \cdots X_n \to Y_1 \oplus Y_m$ and $g : X'_1 \oplus \cdots \oplus X'_{n'} \to Y'_1 \oplus \cdots \oplus Y'_{n'}$. The monoidal product is defined on objects by

$$(X_1 \oplus \cdots \oplus X_n) \star (X'_1 \oplus \cdots \oplus X'_{n'}) = \bigoplus_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n'}} X_i \star X'_j,$$

where the summands on the right hand side are ordered lexicographically with respect to the indices (i, j). To compute the product $f \star g$, recall that f is a $m \times n$ matrix whose (p, q)-entry is given by a morphism $(f)_{p,q} : X_q \to Y_p$, and similarly for g. Then

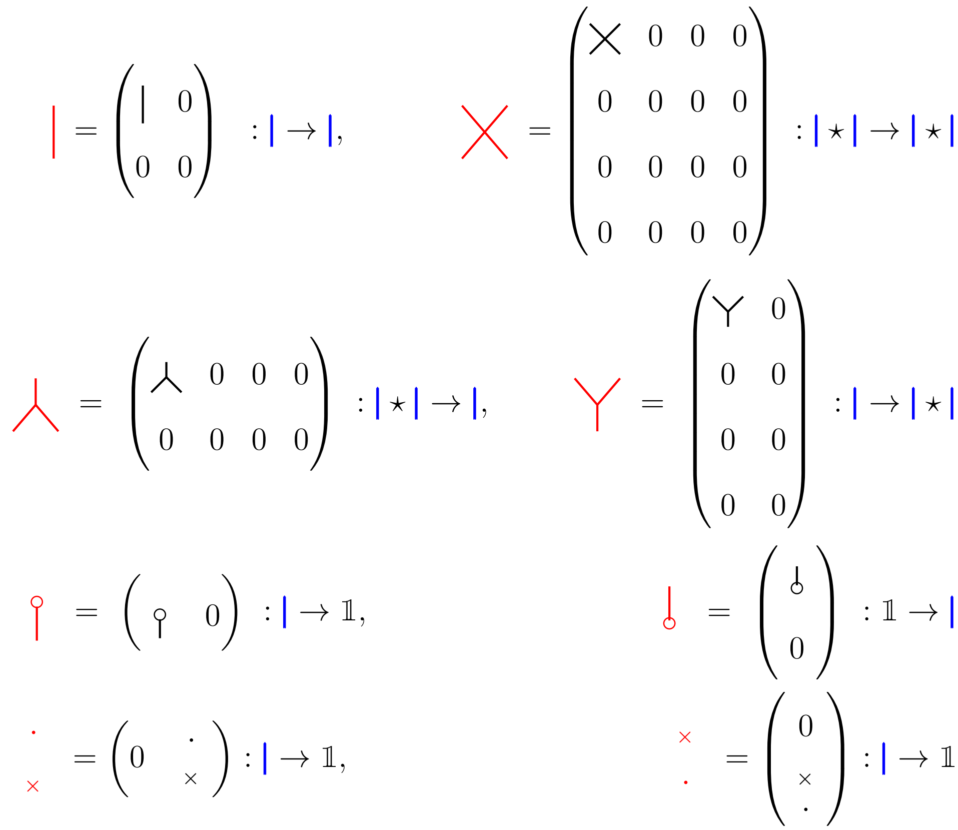
$$f \star g : \bigoplus_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n'}} X_i \star X'_j \to \bigoplus_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m'}} Y_i \star Y'_j$$

is a $(mm') \times (nn')$ matrix whose entries are given by the monoidal products $f_{p,q} \star g_{r,s}$. We refer to this monoidal structure on $\mathrm{Add}(\mathcal{P}ar^{\times}_{t-1})$ as the *Kronecker structure* as the product $f \star g$ is given by the Kronecker product of matrices. Following the usual convention with diagrams, the Kronecker product of morphisms introduced in (4.1.5)-(4.1.8) will be denoted by horizontal juxtaposition.

Remark 4.1.4. Obviously $\mathrm{Add}(\mathcal{P}ar^{\times}_{t-1})$ contains $\mathcal{P}ar^{\times}_{t-1}$ as the full subcategory of objects being all words $w \in W$ (rather than finite direct sums of words). It also contains $\mathcal{P}ar_{t-1}$ as the full subcategory whose objects are $\vert^{\star n} = \underbrace{\vert \cdots \vert}_{n}$ for all $n \in \mathbb{N}$. So there are faithful inclusion functors

$$I : \mathcal{P}ar_{t-1} \to \mathrm{Add}(\mathcal{P}ar^{\times}_{t-1}), \qquad J : \mathcal{P}ar^{\times}_{t-1} \to \mathrm{Add}(\mathcal{P}ar^{\times}_{t-1})$$

In performing calculations, it is useful to express the blue morphisms of (4.1.2)-(4.1.3) in terms of the following elementary matrices:



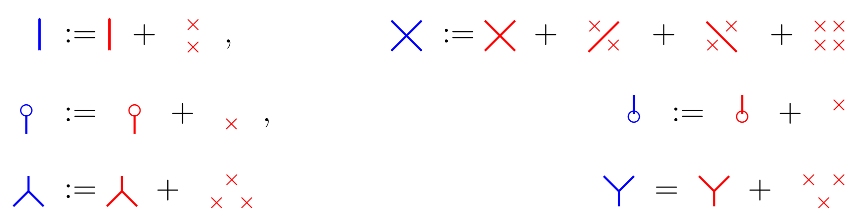
(4.1.5)

(4.1.6)

(4.1.7)

(4.1.8)

These matrices are just another set of special morphisms inside of $\mathrm{Add}(\mathcal{P}ar^{\times}_{t-1})$. They allow us to write the blue morphisms of (4.1.2)-(4.1.3) as sums of Kronecker products of these. Compositions and Kronecker products of any matrices in (4.1.5)-(4.1.8) are again elementary matrices.

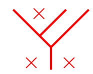


(4.1.9)

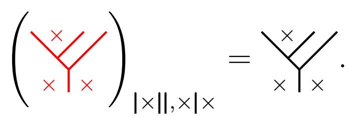
(4.1.10)

(4.1.11)

Example 4.1.5. As an example, consider the following matrix built from Kronecker products and compositions of those in (4.1.5)-(4.1.8).

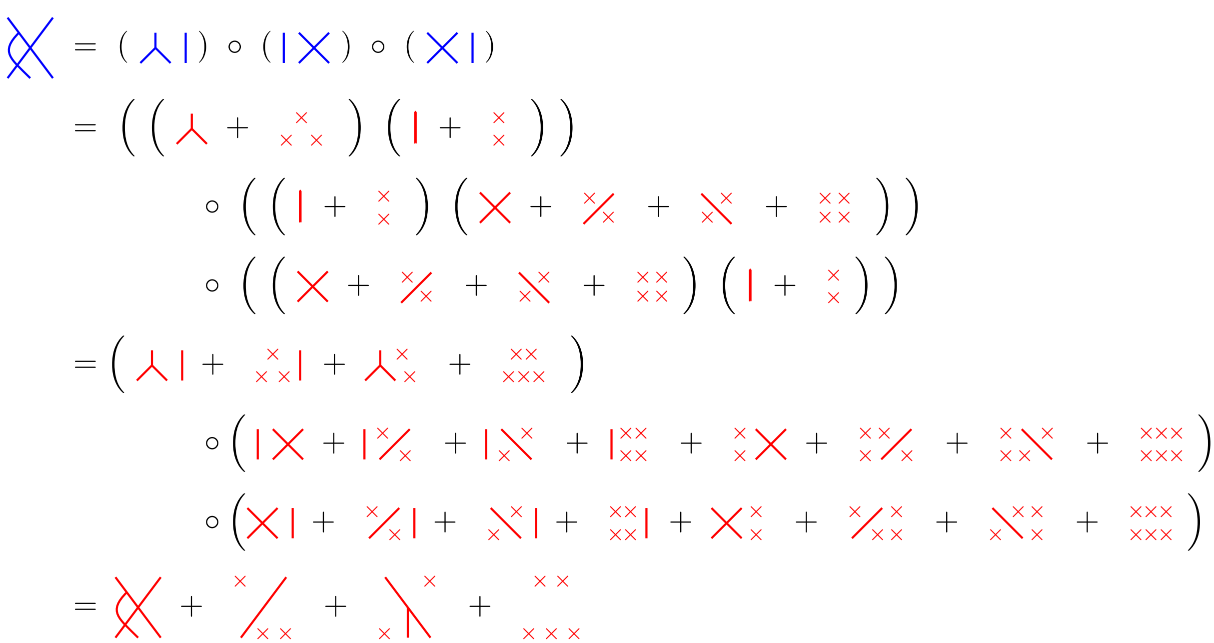


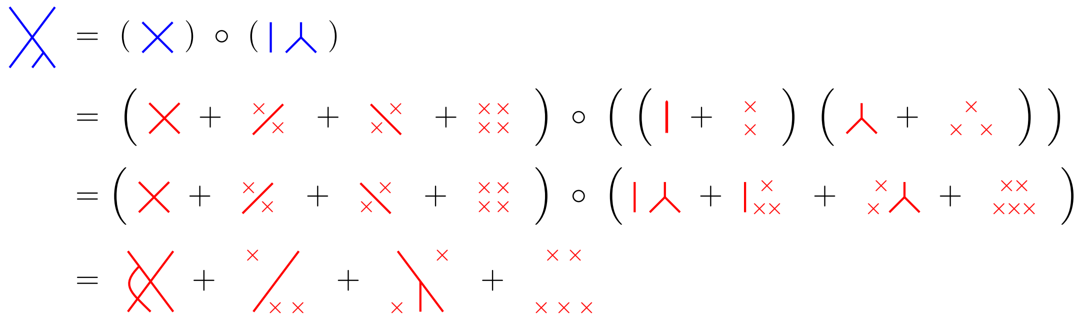
This is a matrix with rows indexed by W^4 and columns indexed by W^3 with a single non-zero entry:



Lemma 4.1.6. *The object* $\mathbf{|} \oplus \times \in \mathbb{O}(\mathrm{Add}(\mathcal{P}ar^{\times}_{t-1}))$ *is a special symmetric Frobenius object of dimension* t*.*

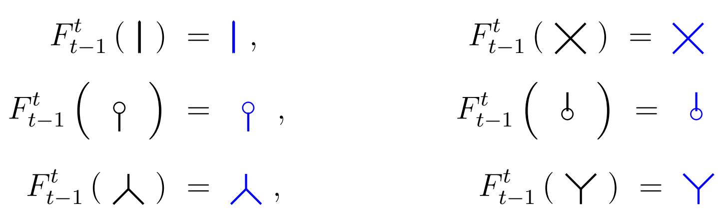
Proof. This is straightforward and is mostly relation-checking. The fact that $\mathbf{|} \oplus \times$ has the appropriate dimension is simply due to the additivity of dimension. We claim that $\mathbf{|} \oplus \times$ is a special Frobenius object whose structure maps are given by (4.1.9)-(4.1.11). To prove this, it amounts to checking that these morphisms satisfy the relations (1.1.2)-(1.1.6). This is straightforward, so we do not scribe the calculations for each relation — only for (1.1.3).





$\square$

By Lemma 4.1.6 and the universal property of $\mathcal{P}ar_t$, there is a symmetric tensor functor $F^t_{t-1} : \mathcal{P}ar_t \to \mathrm{Add}(\mathcal{P}ar^{\times}_{t-1})$ given from the assignment $|_t \mapsto |_{t-1} \oplus \times$ and

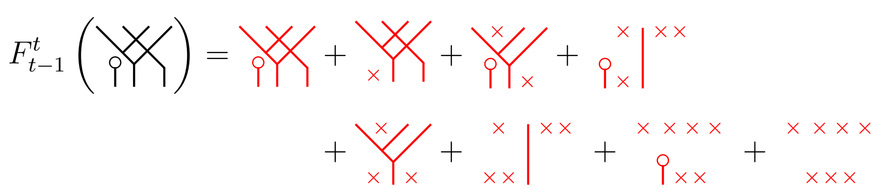


(4.1.12)

(4.1.13)

(4.1.14)

Remark 4.1.7. It will be useful to know how F^t_{t-1} is defined in terms of the Kronecker products and compositions of elementary matrices in (4.1.5)-(4.1.8), generalizing (4.1.9)-(4.1.11). Suppose for convenience that $f \in \mathcal{P}ar_t$ is a partition diagram. Then $F^t_{t-1}(f)$ is the sum of all matrices obtained by erasing connected components of f, replacing the erased boundary points on the bottom and top edges by the symbol $\times$ and coloring the resulting picture red. As an example,



In order to handle some of the calculations in §4.3 involving general partition diagrams with arbitrary amounts of connected components, it will be convenient to introduce a bit more notation to deal with the sums described above. Let $C(f)$ denote the set of connected components of f and let $S \subseteq C(f)$ be any subset. Define

$f^{[S\rightsquigarrow\times]}$ as the diagram obtained from erasing those connected components $c \in S$ and replacing the top and bottom boundary points by $\times$. Then $F^t_{t-1}(f)$ is given by the following formula.

$$f^\times := F^t_{t-1}(f) = \sum_{S\subseteq C(f)} f^{[S\rightsquigarrow\times]} \tag{4.1.15}$$

4.2 Restriction and induction functors

Having obtained functors $F^t_{t-1} : \mathcal{P}ar_t \to \mathrm{Add}(\mathcal{P}ar^\times_{t-1})$ and $I : \mathcal{P}ar_{t-1} \to \mathrm{Add}(\mathcal{P}ar^\times_{t-1})$, restriction and induction allows us to construct a new functor from the category of locally finite-dimensional left Par_t-modules to the category of locally finite-dimensional left Par_{t-1}-modules. In the formalism from §2.2, inducing along I then restricting along F^t_{t-1} defines a functor $R^t_{t-1} : Par_{t-1}\text{-Mod}_{\mathrm{lfd}} \to Par_t\text{-Mod}_{\mathrm{lfd}}$ on module categories.

$$R^t_{t-1} := \mathrm{res}_{F^t_{t-1}} \circ \mathrm{ind}_I : Par_{t-1}\text{-Mod}_{\mathrm{lfd}} \to Par_t\text{-Mod}_{\mathrm{lfd}} \tag{4.2.1}$$

Under this definition, it is obvious that R^t_{t-1} has a right adjoint given by the functor $\mathrm{res}_I \circ \mathrm{coind}_{F^t_{t-1}}$ and a left adjoint $\mathrm{res}_I \circ \mathrm{ind}_{F^t_{t-1}}$.

Since the functor $I : \mathcal{P}ar_{t-1} \to \mathrm{Add}(\mathcal{P}ar^\times_{t-1})$ is given by composing an equivalence $(\mathcal{P}ar_{t-1} \to \mathcal{P}ar^\times_{t-1})$ with the inclusion $J : \mathcal{P}ar^\times_{t-1} \hookrightarrow \mathrm{Add}(\mathcal{P}ar^\times_{t-1})$, Lemma 2.2.2 provides that ind_I itself is an equivalence. So it has quasi-inverse given by its right adjoint res_I. Consequently, coind_I is an equivalence by the same reason and then $\mathrm{ind}_I \cong \mathrm{coind}_I$.

Lemma 4.2.1. *The functor R^t_{t-1} is exact.*

Proof. This follows since R^t_{t-1} is a composition of exact functors. $\square$

Recall the Chevalley duality functor $?^{\circledcirc}$ from §2.9. Following (2.3.4) and (2.3.5), R^t_{t-1} commutes with duality:

$$\begin{aligned}\operatorname{res}_{F^t_{t-1}} \circ \operatorname{ind}_I \circ ?^{\circledcirc} &\cong \operatorname{res}_{F^t_{t-1}} \circ ?^{\circledcirc} \circ \operatorname{coind}_I \\ &\cong ?^{\circledcirc} \circ \operatorname{res}_{F^t_{t-1}} \circ \operatorname{coind}_I \\ &\cong ?^{\circledcirc} \circ \operatorname{res}_{F^t_{t-1}} \circ \operatorname{ind}_I\end{aligned}$$

So there is an isomorphism

$$R^t_{t-1} \circ ?^{\circledcirc} \cong ?^{\circledcirc} \circ R^t_{t-1} \tag{4.2.2}$$

Let A_{t-1} be the path algebra of $\operatorname{Add}(\mathcal{P}ar^{\times}_{t-1})$. The functors F^t_{t-1} and I provide A_{t-1} with a (Par_t, Par_{t-1})-bimodule structure. Then define

$$M := 1_{F^t_{t-1}} A_{t-1} 1_I. \tag{4.2.3}$$

Lemma 4.2.2. *There is an isomorphism of functors*

$$R^t_{t-1}(-) \cong M \otimes_{Par_{t-1}} -.$$

Proof. Recall that there are natural isomorphisms below, for any $N \in Par_{t-1}$-Mod and any $N' \in A_{t-1}$-Mod.

$$\operatorname{ind}_I N = A_{t-1} 1_I \otimes_{Par_{t-1}} N, \qquad \operatorname{res}_{F^t_{t-1}} N' \cong 1_{F^t_{t-1}} A_{t-1} \otimes_{A_{t-1}} N'$$

Hence for any $N \in Par_{t-1}$-Mod,

$$\begin{aligned}R^t_{t-1} N &\cong 1_{F^t_{t-1}} A_{t-1} \otimes_{A_{t-1}} A_{t-1} 1_I \otimes_{Par_{t-1}} N \\ &\cong \left(\bigoplus_{X \in \mathbb{O}(\operatorname{Add}(\mathcal{P}ar^{\times}_{t-1}))} 1_{F^t_{t-1}} A_{t-1} 1_X \right) \otimes_{A_{t-1}} \left(\bigoplus_{Y \in \mathbb{O}(\operatorname{Add}(\mathcal{P}ar^{\times}_{t-1}))} 1_Y A_{t-1} 1_I \right) \otimes_{Par_{t-1}} N \\ &\cong \bigoplus_{X \in \mathbb{O}(\operatorname{Add}(\mathcal{P}ar^{\times}_{t-1}))} \left(1_{F^t_{t-1}} A_{t-1} 1_X \right) \otimes_{1_X A_{t-1} 1_X} (1_X A_{t-1} 1_I) \otimes_{Par_{t-1}} N \\ &\cong 1_{F^t_{t-1}} A_{t-1} 1_I \otimes_{Par_{t-1}} N \\ &\cong M \otimes_{Par_{t-1}} N\end{aligned}$$

□

Going back to the definition restriction as in (2.2.2) and (2.2.3), observe that there is a vector space decomposition as below where $1_{n} := 1_{\vert^{\star n}}$. Therefore, M has a vector space decomposition

$$M \cong \bigoplus_{n,m \in \mathbb{N}} 1_{n} A_{t-1} 1_m. \tag{4.2.4}$$

Viewing M as a left Par_t-module, we can cut with any of the local units $1_n \in Par_t$, $n \in \mathbb{N}$. Obviously, $1_n M = 1_{n} M$.

Lemma 4.2.3. *Let $m, n \in \mathbb{N}$ and let $v = \underbrace{\vert \cdots \vert}_{m} \in W^m$. With the lexicographical ordering on W^n, the subspace $1_{n} A 1_m$ of M has a basis consisting of elementary column vectors $(0, \ldots, 0, f_w, 0, \ldots, 0)^T$ of length $|W^n| = 2^n$ with a single nonzero entry corresponding to some word w. Fixing $w \in W^n$ and collecting the basis elements whose nonzero entries lie in the w-th entry, those f_w form a basis of $\mathrm{Hom}_{\mathcal{P}ar^{\times}_{t-1}}(m, w)$. Such a basis for $1_{n} M 1_m$ is identified with a set of phantom partition diagrams whose top row is a word of length n and bottom row is v.*

Proof. Starting with (4.2.4), fix some $n, m \in \mathbb{N}$. By definition,

$$\begin{aligned}
1_{n} A_{t-1} 1_m &= \mathrm{Hom}_{\mathrm{Add}(\mathcal{P}ar^{\times}_{t-1})}(\vert^{\star m}, \vert^{\star n}) \\
&= \mathrm{Hom}_{\mathrm{Add}(\mathcal{P}ar^{\times}_{t-1})}(\vert^{\star m}, (\vert \oplus \times)^{\star n}) \\
&= \mathrm{Hom}_{\mathrm{Add}(\mathcal{P}ar^{\times}_{t-1})}\left(\vert^{\star m}, \bigoplus_{w \in W^n} w\right) \\
&\cong \bigoplus_{w \in W^n} \mathrm{Hom}_{\mathrm{Add}(\mathcal{P}ar^{\times}_{t-1})}(\vert^{\star m}, w) \\
&\cong \bigoplus_{w \in W^n} \mathrm{Hom}_{\mathcal{P}ar^{\times}_{t-1}}(\vert^{\star m}, w)
\end{aligned}$$

All that remains to note is that each summand $\mathrm{Hom}_{\mathcal{P}ar^{\times}_{t-1}}(\mathbf{|}^{\star m}, w)$ has basis consisting of phantom partition diagrams f with bottom row spelling $v = \mathbf{|}\cdots\mathbf{|}$ and top row spelling the word w. $\square$

Remark 4.2.4. For any $t \in \Bbbk$ and $s \in \mathbb{N}$, one can just as well define more general restriction functors F^t_{t-s} with the assistance of a new category $\mathcal{P}ar^{\times,s}_{t-s}$. This category $\mathcal{P}ar^{\times,s}_{t-s}$ can be constructed from $\mathcal{P}ar_{t-s}$ by adjoining s new generating objects $\times_{t-1}, \times_{t-2}\ldots, \times_{t-s}$ which are all isomorphic to $\mathbb{1}_{t-s}$, and then taking the additive envelope. Then the functor $F^t_{t-s} : \mathcal{P}ar_t \to \mathrm{Add}(\mathcal{P}ar^{\times,s}_{t-s})$ is defined by sending $\mathbf{|}_t$ to $\mathbf{|}_{t-s} \oplus \times_{t-s} \oplus \cdots \oplus \times_{t-1}$. Similar to (4.1.15), applying this functor to any partition diagram f results in a sum over all ways of erasing connected components of f while replacing the boundary points of an erased component by phantoms with a common index. Then one obtains functors $R^t_{t-s} : Par_{t-s}\text{-Mod}_{\mathrm{lfd}} \to Par_t\text{-Mod}_{\mathrm{lfd}}$.

For a pair $s, r \in \mathbb{N}$, F^{t-s}_{t-s-r} has an canonical extension to the additive envelope, named with the same symbol by an abuse of notation:

$$F^{t-s}_{t-s-r} : \mathrm{Add}(\mathcal{P}ar^{\times,s}_{t-s}) \to \mathrm{Add}(\mathcal{P}ar^{\times,s+r}_{t-s-r})$$

This functor sends $\mathbf{|}_{t-s} \mapsto \mathbf{|}_{t-s-r} \oplus \times_{t-s-r} \oplus \cdots \times_{t-s-1}$ and $\times_{t-i} \mapsto \times_{t-i}$ for $i = 1, \ldots, s$. It is easy to see that there is a natural isomorphism $F^t_{t-s-r} \cong F^{t-s}_{t-s-r} \circ F^t_{t-s}$. Consequently, $R^t_{t-s} \circ R^{t-s}_{t-s-r} \cong R^t_{t-s-r}$. Later on, the functors R^t_{-1} will be of particular interest for $t \in \mathbb{Z}_{\geq 0}$.

4.3 A filtration on restriction

This section studies R^t_{t-1} and its behavior on standard modules. Let $\Delta_t : Sym\text{-Mod}_{\mathrm{lfd}} \to Par_t\text{-Mod}_{\mathrm{lfd}}$ denote the standardization functor (2.9.3). Fixing $m \in \mathbb{N}$, the main goal of this section is understanding $R^t_{t-1}\Delta_{t-1}(\Bbbk S_m)$ as a (Par_t, Sym)-bimodule.

By (2.8.7), along with the fact that Par^{-} has a $\Bbbk$-basis comprised of normally ordered upwards partition diagrams, it follows that $\Delta_{t-1}(\Bbbk S_m)$ has basis indexed indexed by pairs (f, σ) where $\sigma \in S_m$ and f is a normally ordered upwards partition diagram.

Lemma 4.3.1. *Fix $m \in \mathbb{N}$. Then $R^t_{t-1}\Delta_{t-1}(\Bbbk S_m)$ has a basis over $\Bbbk$ indexed by pairs (f, σ) where $\sigma \in S_m$ and $f \in M1_m$ is a phantom partition diagram whose underlying partition diagram is normally ordered and upwards. Hence, as a free right S_m-module, $R^t_{t-1}\Delta_{t-1}(\Bbbk S_m)$ has basis consisting of phantom parition diagrams whose underlying partition diagram is normally ordered and upwards.*

Proof. Given $w \in W^n$ for some $n \in \mathbb{N}$, let 1_w be the idempotent which is the identity on the summand w of $|$. There is a decomposition $1_n = \sum_{w \in W^n} 1_w$ into mutually orthogonal idempotents. From this, we recover the following decomposition of $R^t_{t-1}\Delta_{t-1}(\Bbbk S_m)$ making use of Lemma 4.2.3.

$$\begin{aligned}
R^t_{t-1}\Delta_{t-1}(\Bbbk S_m) &\cong M \otimes_{Par_{t-1}} \Delta_{t-1}(\Bbbk S_m) \\
&= \left(\bigoplus_n 1_n M \right) \otimes_{Par_{t-1}} \Delta_{t-1}(\Bbbk S_m) \\
&= \left(\bigoplus_{n \in \mathbb{N}} \bigoplus_{w \in W^n} 1_w M \right) \otimes_{Par_{t-1}} \Delta_{t-1}(\Bbbk S_m) \\
&\cong \bigoplus_{n \in \mathbb{N}} \bigoplus_{w \in W^n} \left(1_w M \otimes_{Par_{t-1}} \Delta_{t-1}(\Bbbk S_m) \right) \\
&= \bigoplus_{n \in \mathbb{N}} \bigoplus_{w \in W^n} \left(1_w M \otimes_{Par_{t-1}} Par_{t-1} \otimes_{Par^\sharp} \operatorname{infl}^\sharp \Bbbk S_m \right) \\
&\cong \bigoplus_{n \in \mathbb{N}} \bigoplus_{w \in W^n} \left(1_w M \otimes_{Par^\sharp} \operatorname{infl}^\sharp \Bbbk S_m \right)
\end{aligned}$$

Fixing some $n \in \mathbb{N}$ and $w \in W^n$, let $k(w)$ denote the number of letters $\mid$ appearing in w. Then there is an isomorphism of right Par_{t-1}-modules $\gamma : 1_w M \xrightarrow{\sim} 1_{k(w)} Par_{t-1}$ defined by erasing all phantom $\times$'s. Making use of the triangular decomposition $Par_{t-1} \cong Par^- \otimes_{\mathbb{K}} Sym \otimes_{\mathbb{K}} Par^+$, we recover an isomorphism

$$\begin{aligned} 1_w M \otimes_{Par^\sharp} \operatorname{infl}^\sharp \Bbbk S_m &\cong 1_{k(w)} Par^- \otimes_{\mathbb{K}} Par^\sharp \otimes_{Par^\sharp} \operatorname{infl}^\sharp \Bbbk S_m \\ &\cong 1_{k(w)} Par^- 1_m \otimes \Bbbk S_m. \end{aligned}$$

Observing that $1_{k(w)} Par^- 1_m \otimes \Bbbk S_m$ is nonzero if and only if $k(w) \geq m$,

$$R^t_{t-1} \Delta_{t-1}(\Bbbk S_m) \cong \bigoplus_{n \in \mathbb{N}} \bigoplus_{\substack{w \in W^n \\ k(w) \geq m}} \left(1_{k(w)} Par^- 1_m \otimes \Bbbk S_m\right)$$

In conclusion, for every word w with $k(w) \geq m$, there is a nonzero summand $1_{k(w)} Par^- 1_m \otimes \Bbbk S_m$ which has basis indexed by pairs (f, σ) where $\sigma \in S_m$ and f is a normally ordered upwards partition diagram. Since the isomorphism γ was just given by erasing the phantom $\times$'s, the result follows. $\square$

The notation $\mathbf{B}_m$ will denote a chosen basis for $\Delta_t(\Bbbk S_m)$ corresponding to pairs (f, σ) with $\sigma \in S_m$ and $f \in Par^- 1_m$ a normally ordered upwards partition diagram. The basis element corresponding to some (f, σ) is just the composition $f\sigma \in \mathbf{B}_m$. Similarly, $\mathbf{B}_m^\times$ will denote a chosen basis for $R^t_{t-1}(\Bbbk S_m)$ corresponding to pairs (f, σ), this time with $f \in M 1_m$ a phantom partition diagram which is normally ordered and upwards. Once again, $f\sigma \in \mathbf{B}_m^\times$ is the basis element corresponding to some pair (f, σ). Diagrammatically,

$$f\sigma = \begin{array}{c} \boxed{\;f\;} \\ \boxed{\cdots} \\ \boxed{\;\sigma\;} \end{array}$$

Notice that the bases $\mathbf{B}_m$ and $\mathbf{B}_m^\times$ do not depend on t.

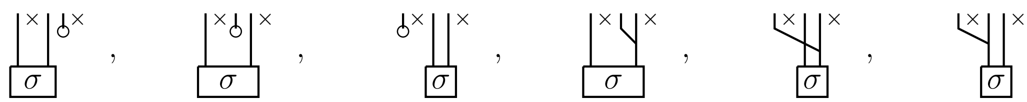
Remark 4.3.2. There is always an evident inclusion $\mathbf{B}_m \to \mathbf{B}_m^\times$ given by sending some $f\sigma$ to the same diagram in $\mathbf{B}_m^\times$. There is also another map $\zeta : \mathbf{B}_m^\times \to \mathbf{B}_m$, where

$\zeta(f\sigma)$ is the (phantomless) partition diagram obtained by erasing all phantoms; ζ just picks out the underlying partition diagram of $f\sigma$.

Example 4.3.3. The weight space $1_5 R^t_{t-1}\Delta_{t-1}(\Bbbk S_2) \cong 1_5 M 1_2 \otimes \Bbbk S_2$ has the following vector-space decomposition,

$$1_5 R^t_{t-1}\Delta_{t-1}(\Bbbk S_2) \cong \bigoplus_{\substack{w \in W^5 \\ k(w) \geq 2}} 1_{k(w)}\Delta_{t-1}(\Bbbk S_2)$$

A $\Bbbk$-basis for the summand corresponding to the word $|\times||\times$ is given by the following set of diagrams, where σ runs over S_2, for a total of 12 basis elements.



The proofs of the next few lemmas make use of several linear maps, the first of which is defined here. These will only be homomorphisms over $\Bbbk$, not as Par_t-modules. There is a surjection $\varphi : R^t_{t-1}\Delta_{t-1}(\Bbbk S_m) \to \Delta_t(\Bbbk S_m)$ defined as follows. Let $f\sigma \in \mathbf{B}^{\times}_m$ be a basis element corresponding to a pair (f, σ) as in Lemma 4.3.1. Then

$$\varphi(f\sigma) = \begin{cases} f\sigma & \text{if } f\sigma \text{ is phantomless} \\ 0 & \text{otherwise} \end{cases} \tag{4.3.1}$$

Proposition 4.3.4. *For $m \in \mathbb{N}$, there is an injection of Par_t-modules*

$$0 \to \Delta_t(\Bbbk S_m) \xrightarrow{\iota} R^t_{t-1}\Delta_{t-1}(\Bbbk S_m)$$

Proof. Consider the Sym-modules $1_m\Delta_t(\Bbbk S_m)$ and $1_m R^t_{t-1}\Delta_{t-1}(\Bbbk S_m)$. Since the only word $w \in W^m$ with $k(w) \geq m$ is $w = |\cdots|$, there is an isomorphism

$$1_m R^t_{t-1}\Delta_{t-1}(\Bbbk S_m) \cong 1_m M \otimes_{Par_{t-1}} \Delta_{t-1}(\Bbbk S_m) \cong 1_m\Delta_{t-1}(\Bbbk S_m)$$

Hence, as right Sym-modules, the spaces $1_m\Delta_t(\Bbbk S_m)$ and $1_m R^t_{t-1}\Delta_{t-1}(\Bbbk S_m)$ are isomorphic. Additionally, it is an easy observation that any vector in these subspaces is a highest weight vector. It follows that there is a homomorphism $\Delta_t(\Bbbk S_m) \to R^t_{t-1}\Delta_{t-1}(\Bbbk S_m)$ which is an isomorphism on the 1_m-weight spaces.

To see that ι is injective, consider a basis element $f\sigma$ of $\Delta_t(\Bbbk S_m)$. By the action of Par_t through (4.1.15), it follows that

$$\begin{aligned}
\iota(f\sigma) &= (f\sigma)^{\times} \cdot (\, \underbrace{| \cdots |}_{m} \,) \\
&= \sum_{S \subseteq C(f\sigma)} (f\sigma)^{[S \rightsquigarrow \times]} \cdot (\, \underbrace{| \cdots |}_{m} \,) \\
&= f\sigma + \sum_{\substack{S \subseteq C(f\sigma) \\ S \neq \varnothing}} (f\sigma)^{[S \rightsquigarrow \times]} \cdot (\, \underbrace{| \cdots |}_{m} \,)
\end{aligned}$$

Each term in the final summation apart from $f\sigma$ contains a partition diagram with phantoms. With (4.3.1), it now follows that $\varphi \circ \iota = \mathrm{id}_{\Delta_t(\Bbbk S_m)}$. So ι is injective. $\square$

Observe that, by Lemma 4.3.1, $R^t_{t-1}\Delta_{t-1}(\Bbbk S_m)$ has a vector space filtration $0 = N_{-1} \subseteq N_0 \subseteq N_1 \subseteq \cdots \subseteq R^t_{t-1}$ where

$$N_i := \Bbbk \left\langle f\sigma \in \mathbf{B}^{\times}_m \mid f \text{ has at most } i \text{ phantoms.} \right\rangle \tag{4.3.2}$$

There are projections $p_\ell : R^t_{t-1}\Delta_{t-1}(\Bbbk S_m) \to N_\ell$ defined on basis elements by $p_\ell(f\sigma) = f\sigma$ if $f\sigma \in N_\ell$ and $p_\ell(f\sigma) = 0$ otherwise. Letting $\overline{N}_\ell = N_\ell / N_{\ell-1}$ and $\overline{p}_\ell$ be the composition of p_ℓ with this quotient, the next graded decomposition is immediate.

$$R^t_{t-1}\Delta_{t-1}(\Bbbk S_m) \cong \bigoplus_{\ell \in \mathbb{N}} \overline{N}_\ell \tag{4.3.3}$$

Let $Q := R^t_{t-1}\Delta_{t-1}(\Bbbk S_m)/\iota(\Delta_t(\Bbbk S_m))$ and also let $\pi : R^t_{t-1}\Delta_{t-1}(\Bbbk S_m) \to Q$ be the quotient map.

Lemma 4.3.5. *The quotient Q has basis given by the images of $f\sigma \in \boldsymbol{B}^{\times}_m$ where f has at least one phantom. Denote this set by $\boldsymbol{B}^{Q}_m$.*

Proof. By Lemma 4.3.1, the images of all $f\sigma \in \mathbf{B}^{\times}_m$ certainly span the quotient. Additionally, since $\overline{p}_0 : R^t_{t-1}\Delta_{t-1}(\Bbbk S_m) \to \overline{N}_0$ is defined by killing all diagrams with phantoms, it follows that all those $f\sigma \in \mathbf{B}^{\times}_m$ with $\pi(f\sigma) \in \mathbf{B}^{Q}_m$ lie in the kernel of $\overline{p}_0$.

Also, the composition $\overline{p}_0 \circ \iota : \Delta_t(\Bbbk S_m) \to \overline{N}_0$ is an isomorphism as it sends a basis to a basis. Since the $\Bbbk$-span of those $f\sigma$ where f has phantoms isomorphically projects onto $\bigoplus_{\ell \geq 1} \overline{N}_\ell$, it follows that $\sum_{\ell \geq 0} \overline{p}_\ell$ is an isomorphism.

$$\iota(\Delta_t(\Bbbk S_m)) + \ker(\overline{p}_0) \xrightarrow{\overline{p}_0 + \sum_{\ell \geq 1} \overline{p}_\ell} \cong \overline{N}_0 \oplus \left(\bigoplus_{\ell \geq 1} \overline{N}_\ell \right)$$

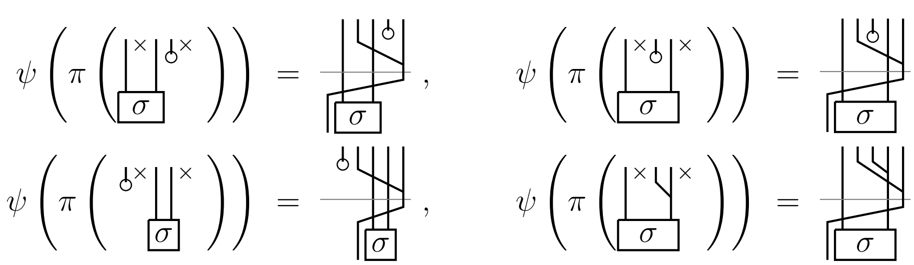
By (4.3.3), we conclude there is an internal decomposition $R^t_{t-1}\Delta_{t-1}(\Bbbk S_m) \cong \iota(\Delta_t(\Bbbk S_m)) \oplus \ker(\overline{p}_0)$. The result now follows. $\square$

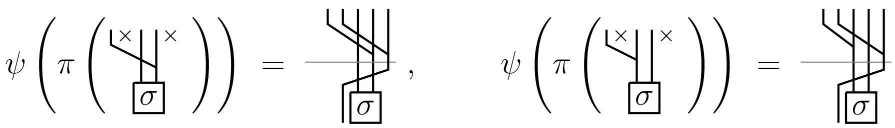
Although Lemma 4.3.5 allows us to identify $\mathbf{B}^Q_m$ with $\mathbf{B}^\times_m$, we choose to keep the two distinct. However, whenever speaking of some $\pi(f\sigma) \in \mathbf{B}^Q_m$, we allow ourselves to access the associated phantom partition diagram $f\sigma \in \mathbf{B}^\times_m$

Now enters the next linear map with the aid of Lemma 4.3.5. Given $\pi(f\sigma) \in \mathbf{B}^Q_m$, define $\tilde{f}\tilde{\sigma} \in \mathbf{B}_{m+1}$ to be the unique (phantomless) partition diagram obtained by collecting all phantoms of $f\sigma$ into a single tree, connected to the bottom left corner of $f\sigma$. The new diagram has $m+1$ components attached to the bottom row. This assignment produces a linear map

$$\psi : Q \to \Delta_t(\Bbbk S_{m+1}), \qquad \pi(f\sigma) \mapsto \tilde{f}\tilde{\sigma} \tag{4.3.4}$$

Example 4.3.6. Consider the summand of $1_5 R\Delta_{t-1}(\Bbbk S_2)$ corresponding to the word $w = |\times||\times$ from the previous example. Then ψ is defined on this subspace as follows. There are decoratory gray lines above which is the normally ordered $\tilde{f}$ and below which is $\tilde{\sigma}$.





Lemma 4.3.7. *For all $n \in \mathbb{N}$, the dimensions of 1_nQ and $1_n\Delta_t(\Bbbk S_{m+1})$ over $\Bbbk$ are equal.*

Proof. Notice first that for $\pi(f\sigma) \in \mathbf{B}_m^Q$, it is true that $\pi(f\sigma) \in 1_nQ$ if and only if the word along the top row of $f\sigma$ is of length n. So, it follows that ψ is a weight-space preserving linear map.

With the above consideration in mind, it is enough to provide a two-sided inverse to ψ. Given $f\sigma \in \mathbf{B}_{m+1}$, let $\hat{f}\hat{\sigma} \in \mathbf{B}_m^\times$ be the unique phantom partition diagram obtained by erasing the connected component of $f\sigma$ attached to the bottom left corner of $f\sigma$, replacing those erased boundary points along the top row by phantoms. Now define $\hat{\psi} : \Delta_t(\Bbbk S_{m+1}) \to Q$ on a basis by $f\sigma \mapsto \pi(\hat{f}\hat{\sigma})$ for $f\sigma \in \mathbf{B}_{m+1}$. It is easily seen that for any $\pi(f\sigma) \in \mathbf{B}_m^Q$, $(\hat{\psi} \circ \psi)(\pi(f\sigma)) = \pi(f\sigma)$. Similarly, for any $f\sigma \in \mathbf{B}_{m+1}$, $(\psi \circ \hat{\psi})(f\sigma) = f\sigma$. $\square$

Before getting into the next proposition, here is a summary of the previous handful of results. By working with explicit bases, proposition 4.3.4 and Lemma 4.3.5 allow us to view $\Delta_t(\Bbbk S_m)$ and Q as complimentary linear subspaces of $R_{t-1}^t\Delta_{t-1}(\Bbbk S_m)$. Then Lemma 4.3.7 shows that ψ is a weight-space preserving isomorphism $\Delta_t(\Bbbk S_{m+1}) \cong Q$ by constructing an inverse $\hat{\psi}$ which lifts through $R_{t-1}^t\Delta_{t-1}(\Bbbk S_m)$, making the following diagram commute.

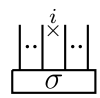
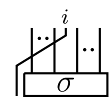
$$\begin{tikzcd}
\Delta_t(\Bbbk S_m) \arrow[r, hook] & R_{t-1}^t\Delta_{t-1}(\Bbbk S_m) \arrow[d, two heads, "\pi"] & \\
& Q \arrow[r, shift left, "\psi"] & \Delta_t(\Bbbk S_{m+1}) \arrow[l, shift left, "\hat{\psi}"] \arrow[ul, dashed]
\end{tikzcd}$$

The next proposition shows that ψ can be replaced by a proper isomorphism of Par_t modules.

Theorem 4.3.8. *For $m \in \mathbb{N}$, there is a short exact sequence of (Par_t, Sym)-bimodules*

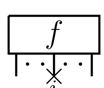
$$0 \to \Delta_t(\Bbbk S_m) \xrightarrow{\iota} R^t_{t-1}\Delta_{t-1}(\Bbbk S_m) \xrightarrow{\pi} \Delta_t E(\Bbbk S_m) \to 0$$

Proof. Throughout this lemma, we remind ourselves that $\pi(f\sigma) = f\sigma + \mathrm{im}(\iota)$ for any $\pi(f\sigma) \in \mathbf{B}^Q_m$. Thanks to Lemma 4.3.7, it is enough to exhibit a surjection $\Delta_t(\Bbbk S_m) \to Q$ as Par_t-modules. First, it is easily seen that (4.3.4) restricts to an isomorphism of left $\Bbbk S_{m+1}$-modules $1_{m+1}Q \cong 1_{m+1}\Bbbk S_{m+1}$.

$1_{m+1}\psi : 1_{m+1}Q \to \Bbbk S_{m+1},$  $+ \ \mathrm{im}(\iota) \ \mapsto$ 

Frobenius reciprocity now gives a homomorphism of Par_t-modules $\Psi : \Delta_t(\Bbbk S_{m+1}) \to Q$ which restricts to an isomorphism between the 1_{m+1} weight spaces.

Before proving surjectivity of Ψ, here is some notation. Let f be a phantom partition diagram with ℓ letters $|$ along the bottom row. If we want to erase the connected component attached to the ith letter $|$ and replace the erased boundary points by $\times$, we will denote this by placing a symbol $\times$ at the bottom of the ith letter $\times$. Illustratively, the diagram



(4.3.5)

denotes the phandom partition diagram obtained from f by erasing the component labeled 'i', and replacing all boundary points by $\times$.

The proof of surjectivity of Ψ is inductively for each weight space. Fix some $n \in \mathbb{Z}_{>m}$ and define subspaces $1_nQ[p] \subseteq 1_nQ$ for $m \leq p < n$ as below, recalling

Remark 4.3.2.

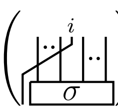
$$1_nQ[p] := \Bbbk \left\langle f\sigma \in 1_nQ \cap \mathbf{B}_m^{\times} \mid \zeta(f\sigma) \text{ has at most } p \text{ connected components} \right\rangle$$

This provides a vector space filtration

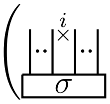
$$0 \subseteq 1_nQ[m] \subseteq 1_nQ[m+1] \subseteq \cdots \subseteq 1_nQ[n-1] = 1_nQ.$$

Now enters the inductive argument. Consider first $1_nQ[m]$. For any basis element $f\sigma \in 1_nQ[m] \cap \mathbf{B}_m^{\times}$, $\zeta(f\sigma)$ consists entirely of upwards trees and trunks. Recycling the construction in (4.3.4) and using (4.1.15) shows that

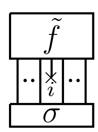
$$\Psi(\tilde{f}\tilde{\sigma}) = \tilde{f}\Psi(\tilde{\sigma})$$

$$= \sum_{S \subseteq C(\tilde{f})} \tilde{f}[S^{\times}] \cdot \Psi \left(\right.$$


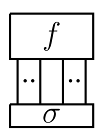
$$\left. \right)$$

$$= \sum_{S \subseteq C(\tilde{f})} \tilde{f}[S^{\times}] \cdot \left(\right.$$


$$\left. + \ \mathrm{im}(\iota) \right)$$

$$=$$


$$+ \ \mathrm{im}(\iota)$$

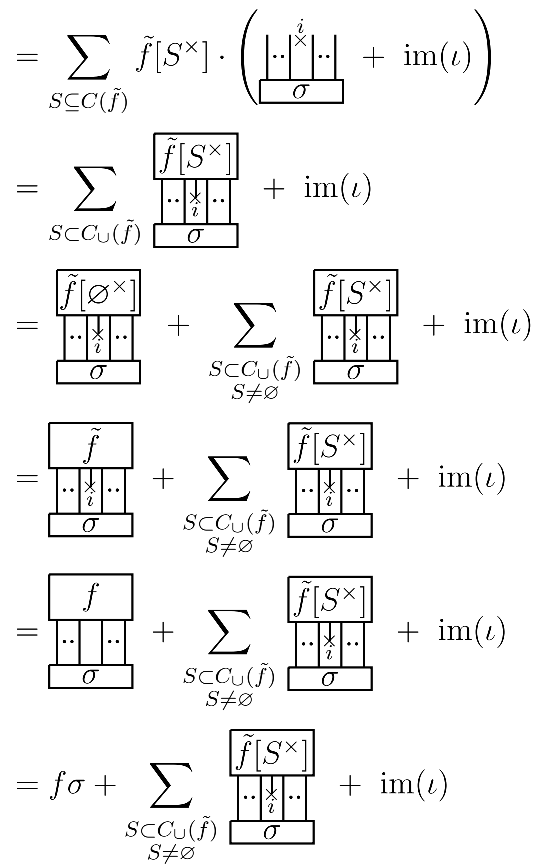
$$=$$


$$+ \ \mathrm{im}(\iota)$$

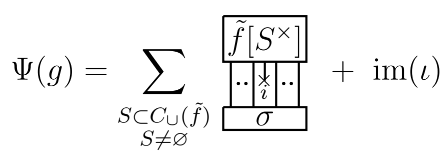
$$= f\sigma + \mathrm{im}(\iota)$$

Suppose now that for some $m < p \leq n-1$, it is known that $1_nQ[p-1] \subseteq \mathrm{im}(\Psi)$. Choose one of our basis elements $f\sigma \in 1_nQ[p] \cap \mathbf{B}_m^{\times}$. Without loss of genereality, suppose $\zeta(f\sigma)$ contains exactly p connected components. Let $C_{\cup}(\tilde{f}) \subset C(\tilde{f})$ be the set of components of $\tilde{f}$ which are branches or leaves. A similar calculation as the base case shows

$$\Psi(\tilde{f}\tilde{\sigma}) = \tilde{f}\Psi(\tilde{\sigma})$$



Every term appearing in the final summation is a diagram with less than p connected components. So the summation is in the image of Ψ by the inductive hypothesis and hence there is some $g \in \Delta_t(\Bbbk S_{m+1})$ for which



Finally, it follows that $f\sigma + \mathrm{im}(\iota) = \Psi(\tilde{f}_{\tilde{\sigma}} - g)$. The case that $p = n - 1$ completes the proof. $\square$

Corollary 4.3.9. *There is a short exact sequence of functors from Sym-*Mod_{fd} *to Par_t-*Mod_{lfd}*.*

$$0 \to \Delta_t \to R^t_{t-1} \circ \Delta_{t-1} \to \Delta_t \circ E \to 0$$

Proof. The functors here are given by tensoring with the bimodules appearing in Theorem 4.3.8. $\square$

Theorem 4.3.10. *For $\lambda \in \mathcal{P}$, there is a short exact sequence of (Par_t, Sym)-bimodules*

$$0 \to \Delta_t(\lambda) \to R^t_{t-1}\Delta_{t-1}(\lambda) \to \bigoplus_{a \in \mathrm{add}(\lambda)} \Delta_t\left(\lambda + \boxed{a}\right) \to 0$$

Proof. Immediate from Corollary 4.3.9 and the fact that the bimodules appearing in Theorem 4.3.8 are flat right *Sym*-modules, seeing as *Sym* is semisimple. $\square$

Recall that a module N has an *standard* flag if there is a filtration $0 = N_0 \subset N_1 \subset \cdots N_\ell = N$ with $N_i/N_{i-1} \cong \Delta(\lambda_i)$ for all $i = 1, \ldots, \ell$ and some partitions $\lambda_1, \ldots, \lambda_\ell$.

Corollary 4.3.11. *If N is a left Par_{t-1}-module with a standard flag, then $R^t_{t-1}N$ is a Par_t-module with a standard flag.*

Proof. This follows from Lemma 4.2.1 and Theorem 4.3.10.

$\square$

4.4 The Comes-Ostrik conjecture

We are now in a position to prove the Comes-Ostrik conjecture. Assume that $t \in \mathbb{Z}_{>0}$. Here is a general (yet specific) lemma to the case at hand. Recall that an *essentially finite* algebra is a locally unital algebra A so that each 1_iA and $A1_i$ is finite-dimensional.

Lemma 4.4.1. *Let A and B be essentially finite algebras and suppose there is an equivalence $F : A\text{-}\mathrm{Mod}_{\mathrm{fd}} \to B\text{-}\mathrm{Mod}_{\mathrm{fd}}$. If there is an exact functor $\widetilde{F} : A\text{-}\mathrm{Mod}_{\mathrm{fd}} \to B\text{-}\mathrm{Mod}_{\mathrm{fd}}$ so that $\widetilde{F}L \cong FL$ for all irreducibles $L \in A\text{-}\mathrm{Mod}_{\mathrm{fd}}$, then $\widetilde{F}$ is an equivalence too.*

Proof. We first note that $\widetilde{F}$ is fully faithful. To see faithfulness, take any nonzero A-module homomorphism $f : V \to W$ so that $\widetilde{F}(f) = 0$. Let $L \subseteq W$ be an irreducible submodule in the image of f and let $V' = f^{-1}(L)$. There is a surjection $g : V' \twoheadrightarrow L$ and an injection $h : L \hookrightarrow W$. The restriction $f|_{V'}$ has a factorization $f_{V'} = h \circ g$. Either $\widetilde{F}(h) = 0$ or $\widetilde{F}(g) = 0$. But since $\widetilde{F}$ is exact and $\widetilde{F}L \cong FL$, this would be impossible. So $\widetilde{F}$ has to be faithful. Consequently, it is also full since Hom-spaces in these two categories are of the same (finite) dimension due to F being an equivalence.

Now let P_L be the projective cover of the irreducible module $L \in A\text{-Mod}_{\text{fd}}$. Similarly, let $P_{\widetilde{F}L}$ be the projective cover of $\widetilde{F}L \in B\text{-Mod}_{\text{fd}}$. We claim that $\widetilde{F}P_L \cong P_{\widetilde{F}L}$. Since $\widetilde{F}$ is fully faithful, $\mathrm{Hom}_B(\widetilde{F}P_L, \widetilde{F}L) \cong \mathrm{Hom}_A(P_L, L) \neq 0$. So $\widetilde{F}P_L$ is indecomposable (by exactness) and has $\widetilde{F}L \cong FL$ as an irreducible quotient. Consequently, there is a surjection $P_{FL} \twoheadrightarrow \widetilde{F}P_L$. But in the Grothendieck group of B, $\widetilde{F}P_L$ and $FP_L \cong P_{FL}$ are equal. So the surjection $P_{FL} \twoheadrightarrow \widetilde{F}P_L$ must be an isomorphism as both modules have the same dimension.

It now follows that $\widetilde{F}$ is essentially surjective when restricted to the subcategories of projective A- and B-modules. Hence it is an equivalence since it is fully faithful too. Knowing that $\widetilde{F}$ restricts to an equivalence $A\text{-Proj} \to B\text{-Proj}$, we deduce the full equivalence $A\text{-Mod} \to B\text{-Mod}$ (see [BD17, Cor. 2.5]). $\square$

The rest of this section has a bit of bookkeeping, so here is some setup. Let $\mathrm{pr}_{0,t}$ denote the projection of $Par_t\text{-Mod}_{\text{lfd}}$ onto the principal block containing the irreducible module $L_t(\varnothing)$, as in (3.8.10). Also let $(Par_t\text{-Mod}_{\text{lfd}})_0$ be said principal block. Taking the partition $\kappa_t = (t)$, the indecomposable projectives in $(Par_t\text{-Mod}_{\text{lfd}})_0$ are given by $P_t(n) := P_t(\kappa_t^{(n)})$ as described in Theorem 3.10.5 and ordered by Corollary 3.10.7. So $\kappa_t^{(0)} = \varnothing, \kappa_t^{(1)} = (t+1), \kappa_t^{(2)} = (t+1, 1)$, and so on. Similarly, let $\Delta_t(n) := \Delta_t(\kappa_t^{(n)})$. Also let $L_t(n) = L_t(\kappa^{(n)})$ be the nth irreducible module in the principal block. Recall that $\Delta_t(n)$ is indecomposable with two composition factors: irreducible socle $L_t(n+1)$ and irreducible head $L_t(n)$ for $n \geq 0$.

Lemma 4.4.2. *For any $n \in \mathbb{N}$,*

$$(\mathrm{pr}_{0,t} \circ R^t_{t-1})(\Delta_{t-1}(n)) \cong \Delta_t(n).$$

Proof. This is a case analysis using the combinatorial rule provided by Theorem 4.3.10. If $n = 0$ then $\kappa^{(0)}_{t-1} = \varnothing$ and the Δ-factors of $R^t_{t-1}\Delta_{t-1}(n)$ are $\Delta_t(\varnothing)$ and $\Delta_t((1))$. Since $t > 0$, only $\Delta_t(\varnothing) = \Delta_t(0)$ is in the principal block. So $(\mathrm{pr}_{0,t} \circ R^t_{t-1})(\Delta_{t-1}(0)) \cong \Delta_t(0)$.

If $n = 1$, then $\kappa^{(n)}_{t-1} = (t)$ and the Δ-factors of $R^t_{t-1}\Delta_{t-1}(1)$ are $\Delta_t((t+1))$ and $\Delta_t((t,1))$. Only $\Delta_t((t+1)) = \Delta_t(1)$ is in the principal block.

If $n \geq 1$, then $\kappa^{(n)}_{t-1} = (t, 1^{n-1})$ and the Δ-factors of $R^t_{t-1}\Delta_{t-1}(n)$ are $\Delta_t((t+1, 1^{n-1}))$, $\Delta_t((t, 1^{n-1}, 1))$, and $\Delta_t((t, 1^n))$. Only $\Delta_t((t+1, 1^{n-1})) = \Delta_t(n)$ is in the principal block. $\square$

Lemma 4.4.3. *For any $n \in \mathbb{N}$,*

$$(\mathrm{pr}_{0,t} \circ R^t_{t-1})(L_{t-1}(n)) \cong L_t(n).$$

Proof. Suppose for the sake of contradiction that $(\mathrm{pr}_{0,t} \circ R^t_{t-1})(L_{t-1}(n)) \ncong L_t(n)$ for some $n \in \mathbb{N}$. Let $\ell \in \mathbb{N}$ be minimal so that $(\mathrm{pr}_{0,t} \circ R^t_{t-1})(L_{t-1}(\ell)) \ncong L_t(\ell)$. From Lemma 4.2.1 and Lemma 4.4.2, there is an exact sequence

$$0 \to (\mathrm{pr}_{0,t} \circ R^t_{t-1})(L_{t-1}(\ell+1)) \to \Delta_t(\ell) \to (\mathrm{pr}_{0,t} \circ R^t_{t-1})(L_{t-1}(\ell)) \to 0.$$

Since $\Delta_t(\ell)$ has length 2, either $(\mathrm{pr}_{0,t} \circ R^t_{t-1})(L_{t-1}(\ell)) = 0$ or $(\mathrm{pr}_{0,t} \circ R^t_{t-1})(L_{t-1}(\ell)) \cong \Delta_t(\ell)$. In the first case, it follows that $(\mathrm{pr}_{0,t} \circ R^t_{t-1})(L_{t-1}(\ell+1)) \cong \Delta_t(\ell)$. But there is also an exact sequence

$$0 \to (\mathrm{pr}_{0,t} \circ R^t_{t-1})(L_{t-1}(\ell+2)) \to \Delta_t(\ell+1) \to (\mathrm{pr}_{0,t} \circ R^t_{t-1})(L_{t-1}(\ell+1)) \to 0.$$

These last two observations show there is a surjection $\Delta_t(\ell+1) \to \Delta_t(\ell)$, impossible.

In the case where $(\mathrm{pr}_{0,t} \circ R^t_{t-1})(L_{t-1}(\ell)) \cong \Delta_t(\ell)$, one has $(\mathrm{pr}_{0,t} \circ R^t_{t-1})(L_{t-1}(\ell+1)) = 0$. Repeating the above argument would give a surjection $\Delta_t(\ell+2) \to \Delta_t(\ell+1)$, another contradiction. So it must be that $(\mathrm{pr}_{0,t} \circ R^t_{t-1})(L_{t-1}(n)) \cong L_t(n)$ for all $n \in \mathbb{N}$. $\square$

Theorem 4.4.4. *For $t \geq 1$, the composition*

$$\mathrm{pr}_{0,t} \circ R^t_{t-1} : (Par_{t-1}\text{-}\mathrm{Mod}_{\mathrm{lfd}})_0 \to (Par_t\text{-}\mathrm{Mod}_{\mathrm{lfd}})_0$$

is an equivalence.

Proof. By Theorem 3.10.5, there is an equivalence $(Par_{t-1}\text{-}\mathrm{Mod}_{\mathrm{lfd}})_0 \xrightarrow{\sim} (Par_t\text{-}\mathrm{Mod}_{\mathrm{lfd}})_0$ and both these categories are *essentially finite*, in that they are equivalent to categories of finite-dimensional modules over essentially-finite algebras (see [BS, Cor. 2.20]). Now apply Lemma 4.4.3 and Lemma 4.4.1. $\square$

CHAPTER V

THE ABELIAN ENVELOPE

This chapter provides an alternate perspective of the abelian envelope of $\underline{\mathrm{Rep}}(S_t)$ involving tilting modules and Ringel duality. After briefly reviewing some basics about abelian envelopes and splitting objects, we use the restriction functor R_{-1}^t from chapter IV to show that tilting modules for the partition category can be identified with splitting objects. This allows us to connect the Benson-Etingof-Ostrik construction of abelian envelopes in [BEO23] to tilting theory and Ringel duality studied in [BS].

5.1 Review of abelian envelopes

Throughout this section let $\mathcal{C}$ be a locally-finite Karoubian rigid monoidal category with $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) \cong \Bbbk$. Generally, $\mathcal{C}$ is not abelian and so is not a full-fledged *tensor category* in the sense of [EGNO15]. However, one can ask to find an *abelian envelope* of $\mathcal{C}$ if it exists. Such an abelian envelope is the data of a tensor category $\mathcal{D}$ and a monoidal functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ so that for any other tensor category $\mathcal{D}'$, composition with $\mathcal{F}$ induces an equivalence between the category of faithful monoidal functors $\mathcal{C} \to \mathcal{D}'$ and the category of exact monoidal functors $\mathcal{D} \to \mathcal{D}'$. That is, for each $\mathcal{G} : \mathcal{C} \to \mathcal{D}'$ there exists a unique functor (up to isomorphism) $\mathcal{G}' : \mathcal{D} \to \mathcal{D}'$ making the following diagram commute up to isomorphism:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\mathcal{F}"] \arrow[dr, "\mathcal{G}"'] & & \mathcal{D} \arrow[dl, dashed, "\exists \mathcal{G}'"] \\ & \mathcal{D}' & \end{tikzcd}$$

, $\mathcal{G}$ is faithful monoidal, $\mathcal{G}'$ is exact monoidal.

Any two abelian envelopes of $\mathcal{C}$ are equivalent, so we often speak of *the* abelian envelope of $\mathcal{C}$.

Remark 5.1.1. In the case of Deligne's category $\underline{\mathrm{Rep}}(S_t)$, Comes and Ostrik were the first to construct the abelian envelope by examining the heart of a certain t-structure on the homotopy category of $\underline{\mathrm{Rep}}(S_t)$ [CO14]. Later, the abelian envelope of $\underline{\mathrm{Rep}}(GL_t)$ was constructed by Entova, Hinich, and Serganova by clever use of inverse and direct limits of representations of the general linear supergroup $GL(m|n)$ with $m - n = t$ [EHS18]. More recently, Harman and Snowden build abelian envelopes for a class of Oligomorphic groups as a kind of completed group algebra [HS22].

The Benson-Etingof-Ostrik construction (and also Coulembier's approach in [Cou21]) of abelian envelopes involves the use of *splitting objects*. With $\mathcal{C}$ as above, an object $S \in \mathbb{O}(\mathcal{C})$ is splitting if for each morphism $f : X \to Y$ in $\mathcal{C}$, the morphism $f \otimes \mathrm{id}_S : X \otimes S \to Y \otimes S$ is split. That is, $f \otimes \mathrm{id}_S$ is the direct sum of a zero morphism and an isomorphism.

It is proven in [BEO23] that the splitting objects of $\mathcal{C}$ form a thick tensor ideal; the full subcategory consisting of splitting objects of $\mathcal{C}$ is Karoubian and closed under taking tensor products with arbitrary objects of $\mathcal{C}$. Let $(S_i)_{i \in I}$ be a family of irredundant representatives for the isomorphism classes of indecomposable splitting objects and suppose the following *finiteness property*: for any splitting object S, there are finitely many $i \in I$ for which $\mathrm{Hom}_{\mathcal{C}}(S_i, S)$ is nonzero. Under this assumption, Benson, Etingof, and Ostrik build the coalgebra $C := \bigoplus_{i,j \in I} \mathrm{Hom}_{\mathcal{C}}(S_i, S_j)^*$ and a functor $\mathcal{F} : \mathcal{C} \to C\text{-Comod}_{\mathrm{fd}}$ from $\mathcal{C}$ to the category of finite-dimensional C-comodules.

After proving that $C\text{-Comod}_{\mathrm{fd}}$ has a monoidal structure, Benson, Etingof, and Ostrik show that under certain conditions, $C\text{-Comod}_{\mathrm{fd}}$ is the abelian envelope of $\mathcal{C}$. The first condition is that $\mathcal{C}$ must be of *finite type*: there exists a splitting object $S \in \mathbb{O}(\mathcal{C})$ so that every indecomposable splitting object S_i $(i \in I)$ appears as a summand of $X \otimes S$ for some $X \in \mathbb{O}(\mathcal{C})$. The second condition is that $\mathcal{C}$ is *separated*, meaning that $\mathcal{F}$ is faithful. Lastly, $\mathcal{C}$ must be *complete*, meaning that $\mathcal{F}(\mathcal{C})$ is equivalent to $\mathrm{Kar}(\mathcal{F}(\mathcal{C}))$. Their main theorem on abelian envelopes is summarized below. More details can be found in [BEO23].

Theorem 5.1.2 (Benson-Etingof-Ostrik)**.** *Let $\mathcal{C}$ be a monoidal category with the finiteness property (as well as the other properties listed at the beginning of this section). Also suppose $\mathcal{C}$ is of finite type. Then $\mathcal{C}$ admits a fully faithful monoidal functor $E : \mathcal{C} \to \mathcal{D}$ into a (multi-)tensor category $\mathcal{D}$ with enough projectives if and only if $\mathcal{C}$ is separated and complete. Moreover, in this case there exists a tensor embedding $E' : C\text{-Comod}_{\mathrm{fd}} \to \mathcal{D}$ so that $E \cong E' \circ F$ and $C\text{-Comod}_{\mathrm{fd}}$ is the abelian envelope of $\mathcal{C}$.*

5.2 Ringel duality and the abelian envelope

Since many of the projective modules for Par_t are self-dual (Theorem 3.10.5), they have a standard flag and (finite) costandard flag. Hence, they are *tilting modules*[1]. A classification of indecomposable tilting modules up to isomorphism is provided by Brundan and Stroppel [BS, Thm.4.18]. In the setting for $Par_t\text{-Mod}_{\mathrm{lfd}}$, the classifcation states that there is exactly one indecomposable tilting module $T(\lambda)$ for each partition $\lambda \in \mathcal{P}$, up to isomorphism. Specifically, $T(\lambda)$ is characterized uniquely by the property that it has $\Delta(\lambda)$ as the bottom section in any standard flag. Recalling the definition (3.8.12), the tilting module for $Par_t\text{-Mod}_{\mathrm{lfd}}$ corresponding to the partition $\lambda \in \mathcal{P}$ is:

$$T_t(\lambda) = \begin{cases} P_t(\kappa^{(n+1)}) & \lambda = \kappa^{(n)} \text{ for some } \kappa \in \mathcal{P}_t \text{ and } n \in \mathbb{N} \\ P_t(\lambda) & \text{otherwise.} \end{cases} \tag{5.2.1}$$

[1] In a general upper-finite highest weight category, tilting modules potentially have infinite 'descending' costandard flags.

Note that whenever λ is not of the form $\kappa^{(n)}$ for some $\kappa \in \mathcal{P}_t$ and $n \in \mathbb{N}$, $P_t(\lambda) = L(\lambda)$ is irreducible.

Lemma 5.2.1. *If $T \in Par_{t-1}$-$\mathrm{Mod}_{\mathrm{lfd}}$ is tilting, then $R^t_{t-1}T$ is tilting too.*

Proof. This follows from (4.2.2) and Corollary 4.3.11, noting that standard flags turn into costandard flags upon applying Chevalley duality. $\square$

In the next lemma, we make use of the restriction functors $R^t_{-1} : Par_{-1}$-$\mathrm{Mod}_{\mathrm{lfd}} \to Par_t$-$\mathrm{Mod}_{\mathrm{lfd}}$ as in Remark 4.2.4. It is an easy consequence of the fact that F^t_{t-1} is monoidal that

$$(\mathbf{|} \oplus \times) \star F^t_{t-1}(-) \cong F^t_{t-1} \circ (\mathbf{|} \star -).$$

More generally,

$$(\mathbf{|} \oplus \times_{-1} \oplus \cdots \oplus \times_{t-1}) \star F^t_{-1}(-) \cong F^t_{-1} \circ (\mathbf{|} \star -).$$

Bringing back the functor $D_t : Par_t$-Mod $\to Par_t$-Mod of (3.9.3), it follows that there is a natural isomorphism

$$D_t \circ R^t_{-1} \cong R^t_{-1} \circ (D_{-1} \oplus \underbrace{\mathrm{Id} \oplus \cdots \oplus \mathrm{Id}}_{t+1}), \tag{5.2.2}$$

where Id denotes the identity functor on Par_{t-1}-Mod. Letting $D^n_t = \underbrace{D_t \circ \cdots \circ D_t}_{n}$, we have

$$D^n_t \circ R^t_{t-1} \cong R^t_{-1} \circ \left(\bigoplus_{\ell=0}^{n} (D^{\ell}_{-1})^{\oplus \binom{n}{\ell} (t+1)^{(n-\ell)}} \right) \tag{5.2.3}$$

Lemma 5.2.2. *An indecomposable $X \in Par_t$-Proj is splitting if and only if it is tilting.*

Proof. We start by showing that every tilting is splitting. Since every projective appears as a summand of a direct sum of finitely many $Q_t(n) := Par_t 1_n$, it is enough to check that tiltings split morphisms $f : Q_t(n) \to Q_t(m)$ for any $n, m \in \mathbb{N}$. By the combinatorial rule provided in Theorem 4.3.10, the module $V := R^t_{-1}\Delta_{-1}(\varnothing)$ has a section $\Delta(\varnothing)$. Since V is tilting by Lemma 5.2.1, it follows from (5.2.1) that V contains the tilting module $T_t(\varnothing)$ as a summand. Consider the following commuting diagram, using Lemma 2.6.1:

$$\begin{tikzcd}
Q_t(n)\circledast V \arrow[r, equal, "\sim"] \arrow[d, "f\circledast 1_V"] & (D^n \circ R^t_{-1})\Delta_{-1}(\varnothing) \arrow[r, "\sim"] \arrow[d] & R^t_{t-1}\left(\left(\bigoplus_{\ell=0}^{n}(D^{\ell}_{-1}\Delta_{-1}(\varnothing))^{\oplus\binom{n}{\ell}(t+1)^{(n-\ell)}}\right)\right) \arrow[d] \\
Q_t(m)\circledast V \arrow[r, equal, "\sim"] & (D^m \circ R^t_{-1})\Delta_{-1}(\varnothing) \arrow[r, "\sim"] & R^t_{t-1}\left(\left(\bigoplus_{\ell=0}^{m}(D^{\ell}_{-1}\Delta_{-1}(\varnothing))^{\oplus\binom{m}{\ell}(t+1)^{(m-\ell)}}\right)\right)
\end{tikzcd}$$

The right-most vertical arrow, given by the natural isomorphism (5.2.3), is split as Par_{-1}-Mod is semisimple. Hence, it follows that V is a splitting object and so is $T_t(\varnothing)$, being a summand of a splitting object. Moreover, since splitting objects form a thick tensor ideal, all $T_t(\lambda)$ for $\lambda \in \mathcal{P}$ are splitting since they appear as a summand of $D^m_t V = Q_t(m)\circledast V$ for a suitable $m \in \mathbb{N}$.

It remains to see that those indecomposable projectives which are not tilting are also not splitting. First consider $P_t(\varnothing) = \Delta(\varnothing)$. This is the unit with respect to $\circledast$. Since there is a non-split map $f : P_t(\varnothing) \to P_t(\varnothing^{(1)})$, it is immediate that $1_{P_t(\varnothing)}\circledast f = f$ is not split. So $P_t(\varnothing)$ is not a splitting object. Consider now $P_t(\kappa^{(0)})$ for any $\kappa \in \mathcal{P}_t$ and look at the morphism $1_{P_t(\kappa^{(0)})}\circledast f : P_t(\kappa^{(0)})\circledast P_t(\varnothing) \to P_t(\kappa^{(0)})\circledast P_t(\varnothing^{(1)})$. By Lemma 3.10.1, $D^m_t P_t(\kappa^{(0)}) = Q_t(m)\circledast P_t(\kappa^{(0)})$ will contain a summand of $P_t(\varnothing)$ for an appropriate m and we get a commuting diagram below, where the first two vertical arrows are non-split. Consequently, $P_t(\kappa^{(0)})$ cannot be a splitting object.

$$\begin{tikzcd}
P_t(\varnothing) \arrow[r, "\sim"] \arrow[d, "f"] & P_t(\varnothing)\circledast P_t(\varnothing) \arrow[r, hook, "\oplus"] \arrow[d, "1\circledast f"] & Q_t(m)\circledast P_t(\kappa^{(0)})\circledast P_t(\varnothing) \arrow[d, "1\circledast 1\circledast f"] \\
P_t(\varnothing^{(t)}) \arrow[r, "\sim"] & P_t(\varnothing)\circledast P_t(\varnothing^{(1)}) \arrow[r, hook, "\oplus"] & Q_t(m)\circledast P_t(\kappa^{(0)})\circledast P_t(\varnothing^{(1)}).
\end{tikzcd}$$

□

Lemma 5.2.3. *The category* Par_t-Proj *has the finiteness property.*

Proof. By Lemma 5.2.2, any splitting object S if a finite direct sum of indecomposable tilting modules. It follows from Theorem 3.10.5 that there are only finitely many $T_t(\lambda)$ with $\mathrm{Hom}_{Par_t}(T(\lambda), S)$ being nonzero. □

Consider now $T := \bigoplus_{\lambda\in\mathcal{P}} T_t(\lambda)$ and the algebra $B := \mathrm{End}_{Par_t}(T)^{\mathrm{op}} = \left(\bigoplus_{\lambda,\mu\in\mathcal{P}} \mathrm{Hom}_{Par_t}(T_t(\lambda), T_t(\mu))\right)^{\mathrm{op}}$. In the language of [BS], T is a tilting generator. Equipping B with the profinite topology, Brundan and Stroppel build the coalgebra

$$\mathrm{Coend}(T) := \{f \in B^* \mid f \text{ vanishes on some two-sided ideal of finite codimension}\}.$$

There is an identification $\mathrm{Comod}_{\mathrm{fd}}\text{-}\,\mathrm{Coend}(T) = B\text{-}\mathrm{Mod}_{\mathrm{fd}}$. Brundan and Stroppel also construct the Ringel duality functor $G : Par_t\text{-}\mathrm{Mod} \to \mathrm{Comod}_{\mathrm{fd}}\text{-}\,\mathrm{Coend}(T)$, defined on any $N \in Par_t$-Mod below.

$$G(N) := \left\{ f \in \mathrm{Hom}_{Par_t}(N, T)^* \;\middle|\; {\scriptstyle f \text{ vanishes on a submodule} \atop \scriptstyle \text{of } \mathrm{Hom}_{Par_t}(N,T) \text{ with finite codimension}} \right\}$$

We can now show that B-$\mathrm{Mod}_{\mathrm{fd}}$, equipped with the functor G, is the abelian envelope of Par_t, and hence, also of $\underline{\mathrm{Rep}}(S_t)$.

Theorem 5.2.4. *The Ringel dual* B-$\mathrm{Mod}_{\mathrm{fd}}$ *of* Par_t-$\mathrm{Mod}_{\mathrm{lfd}}$ *is the abelian envelope of* $\underline{Rep}(S_t)$*.*

Proof. As usual, identify $\underline{\mathrm{Rep}}(S_t)$ with Par_t-Proj by means of the Yoneda equivalence. We just need to check the conditions provided in Theorem 5.1.2. The first is that $\underline{\mathrm{Rep}}(S_t)$ is of finite type. We claim the module $T_t(\varnothing)$ is a generator for the splitting ideal. From Lemma 3.10.1 and Lemma 3.10.3, any splitting (=tilting) object in a nontrivial block of Par_t-$\mathrm{Mod}_{\mathrm{lfd}}$ is a summand of $D^n T_t(\varnothing) \cong Q_t(n) \circledast T_t(\varnothing)$ for some $n \in \mathbb{N}$. For those tiltings $T_t(\lambda)$ in a trivial block, the same is true and is easily deduced from Theorem 3.9.1.

The other conditions we have to check is that $\underline{\mathrm{Rep}}(S_t)$ is both separated and complete. Let $G : Par_t\text{-Mod}_{\mathrm{lfd}} \to B\text{-Mod}_{\mathrm{fd}}$ be the Ringel duality functor and let $\mathcal{T} = G(\underline{\mathrm{Rep}}(S_t))$ be the image of this functor. Completeness amounts to showing that $\mathcal{T}$ is equivalent to its Karoubi envelope, and separatendess requires us to show that F is faithful. This, however, is true by [BS, Thm.4.27]. So $\underline{\mathrm{Rep}}(S_t)$ is separated and complete. $\square$

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