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VALUATIONS ON A COMMUTATIVE RING

by

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TABLE OF CONTENTS

Introduction 1

1. Valuations and Valuation Pairs. 4

2. Independence and Extensions 21

3. The Inverse Property, Approximation Theorems 36

4. Galois Extensions 57

Bibliography. 75

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INTRODUCTION

The purpose of this paper is to extend some results from the theory of valuations on a field to an arbitrary commutative ring with identity. The results obtained are well known when interpreted in the context of a field and comprise only a bare introduction to the theory for fields, however, the modified proofs give some added insight even in this case.

In Section I of Chapter 1, the concept of valuation on a field is extended to an arbitrary commutative ring and a natural correspondence is obtained between valuations and what we call valuation pairs. Section II shows that these valuation pairs are the same as those in [1], where they are the subject of an exercise.

Sections III and IV of Chapter 1 relate valuation pairs to the generalized primes of D. K. Harrison. The results presented here predate the rest of this paper, being developed to investigate primes [3]. Proposition 1.14 evolved during the course of a seminar given by Professor Harrison during the fall of 1965, while 1.12 and 1.13 appeared in the unrevised form of [3].

The outline followed for Chapters 2, 3 and 4 is essentially that used in [6] in developing the theory for fields. Many of the arguments used are almost verbatum those used in this source. Standard results for finite rank valuations are not given since their proofs (and statements) are not significantly altered in the more general context.

Section I of Chapter 2 deals with the concept of independence of valuations and Section II with the concept of extension. Section III combines these to obtain some results essential to Chapter 3.

Sections I and II of Chapter 3 are used to develop the machinery and setting for the "approximation theorems" of Section III. The approximation theorem is applied in Section IV to obtain the classical inequality "$\Sigma e_i f_i \leq n$".

The paper ends with the proof of the classical equation $efg\pi^d = |G|$ in the context of a commutative ring R which is Galois over a ring K with group G. The generalized Galois theory necessary for this result is outlined in Section I of Chapter 4. The rest of the chapter is devoted to the definitions and relations (which are interesting in their own right) necessary for its statement and proof.

General ring theory comparable to that found in [5] is assumed, but beyond that the treatment is largely self contained. A notable exception is Section I of Chapter 4 where several results are quoted from [2] without proof.

In order to cut down on verbiage, much notation is assumed as standard once it is introduced. Thus R is always a commutative ring with identity, K is always a subring of R, V is always a valuation on R, etc.

"Ring" will always mean "commutative ring with identity". "K a subring of R" means the same thing as "R is an extension of K"; in both cases meaning that K is a ring, $K \subseteq R$ and the identity of K is the identity of R. Ring Homomorphisms will always take identity to identity. Prime ideals are always proper.

The word "iff" is a contraction of "if and only if", and is sometines denoted by $\Leftrightarrow$. "$A \Rightarrow B$" means "A implies B", "$\exists$" means "there exists" and "$\forall$" means "for all".

If A and B are sets, $A \setminus B = \{x \mid x \in A, x \notin B\}$ and should not be confused with A/B, which denotes a quotient of rings, groups, etc.

1. VALUATIONS AND VALUATION PAIRS

Section I

By an ordered group, we mean an abelian group Γ^* (written multiplicatively) which is linearly ordered by a relation " $<$ " satisfying $\alpha < \beta \Rightarrow \alpha\gamma < \beta\gamma$ for all $\alpha, \beta, \gamma \in \Gamma^*$. We will always denote the identity of an ordered group by e, and we admit the group $\{e\}$ as an ordered group.

<u>DEFINITION 1.1</u>. A valuation semigroup Γ is the disjoint union of an ordered group Γ^* and an element 0, where the order and multiplication of Γ^* is extended to Γ by:

i.) $0 \leq \alpha$ for all $\alpha \in \Gamma$

ii.) $0 \cdot \alpha = \alpha \cdot 0 = 0$ for all $\alpha \in \Gamma$

<u>DEFINITION 1.2</u>. A valuation V on a commutative ring R is a map V from R to a valuation semigroup Γ satisfying

i.) $V(xy) = V(x)V(y)$ for all $x,y \in R$

ii.) $V(x+y) \leq \max\{V(x),V(y)\}$ for all x,y in R

iii.) V is onto Γ.

We shall sometimes like to think of Γ as embedded in a larger valuation semi-group, in which case we relax iii.) to "$V(R)\setminus V(0)$ is a group".

One can check that $V(1) = e$ and $V(0) = 0$ for all valuations. If R is a field and i.) holds, then $V(R\setminus\{0\})$ is always a group so iii.) can be replaced by $V(1) \neq V(0)$ or one can work with ordered groups rather than semi-groups. Condition ii.) is the non-Archimedian condition in a field.

PROPOSITION 1.3. Let V be a valuation on a ring R, set

$$A_V = \{x \in R \mid V(x) \leq e\}$$
$$P_V = \{x \in R \mid V(x) < e\}$$
$$\sigma_V = \{x \in R \mid V(x) = 0\}$$

Then A_V is a subring of R, P_V is a prime ideal of A_V and σ_V is a prime ideal of R. Further, if σ is an ideal of R, $\sigma \subset A_V$, $A_V \neq R$, then $\sigma \subset \sigma_V$.

PROOF. Note that $V(-x) = V(-1)V(x)$ and $V(-1) = V(-1)^{-1}$, thus that $V(-1) = e$ and $V(x) = V(-x)$ for all $x \in R$. Thus we have $A_V = -A_V$, $P_V = -P_V$ and $\sigma_V = -\sigma_V$. By condition ii.) of Definition 1.2 we have $A_V + A_V \subset A_V$, $P_V + P_V \subset P_V$ and $\sigma_V + \sigma_V \subset \sigma_V$. By i.) we have $A_VP_V \subset P_V$ and $R\sigma_V \subset \sigma_V$, thus A_V is a subring of R, P_V is an ideal of A_V and σ_V is an ideal of R. If $a \cdot b \in P_V$, then $e > V(ab) = V(a)V(b)$ so either $e > V(a)$ or $e > V(b)$. Thus P_V is a prime ideal of A_V ($V(1) = e$ so $1 \notin P_V$). If $ab \in \sigma_V$, then $0 = V(ab) = V(a)V(b)$, so $V(a) = 0$ or $V(b) = 0$, so σ_V is a prime ideal of R.

Finally, suppose $A_V \neq R$ and σ is an ideal of R. If $\sigma \not\subset \sigma_V$, then $V(a) \neq 0$ for some $a \in \sigma$. But then $V(b) = V(a)^{-1}$ for some $b \in R$ and $V(c) > e$ for some $c \in R$ (since $A_V \neq R$ by hypothesis). But then $abc \in \sigma$ while $V(abc) = V(a)V(b)V(c) = eV(c) = V(c) > e$ so $\sigma \not\subset A_V$.

<u>PROPOSITION 1.4</u>. If V is a valuation on a ring R, $x \in R \setminus A_V$, then there is a $y \in P_V$ with $xy \in A_V \setminus P_V$.

<u>PROOF</u>. If $x \in R \setminus A_V$, then $V(x) > e$ and for some $y \in R$, $V(y) = V(x)^{-1}$. $e = V(x)^{-1}V(x) > V(x)^{-1}e = V(x)^{-1}$ so $y \in P_V$. Now $V(xy) = V(x)V(y) = V(x)V(x)^{-1} = e$ so $xy \in A_V \setminus P_V$.

<u>DEFINITION 1.5</u>. By a valuation pair of a ring R, we mean a pair (A,P), where A is a subring of R and P is a prime ideal of A, such that $x \in R \setminus A \Rightarrow xy \in A \setminus P$ for some $y \in P$.

Note that (A_V, P_V) is a valuation pair of R for any valuation V or R. We have the converse;

<u>PROPOSITION 1.6</u>. If (A,P) is a valuation pair of R, then there is a valuation V of R with $A = A_V$ and $P = P_V$. Furthermore if V_1 is another valuation of R with either $P = P_{V_1}$, <u>or</u> with $A = A_{V_1}$ and $A \neq R$, then there is an order preserving isomorphism $\phi : \Gamma_{V_1} \to \Gamma_V$ with $\phi \circ V_1 = V$.

PROOF. Let (A,P) be a valuation pair of R. For $x,y \in R$ define $x \sim y$ if $\{z \in R \mid xz \in P\} = \{z \in R \mid yz \in P\}$. "$\sim$" is clearly an equivalence relation on R. Let $V(x) = \{y \mid y \sim x\}$ and $\Gamma_v = \{V(x) \mid x \in R\}$.

CLAIM 1. $V(xy) = V(x'y')$ for all $x' \in V(x)$, $y' \in V(y)$. Thus defining $V(x)V(y) = V(xy)$ makes Γ_v into a semi group. Furthermore, $\Gamma_v \setminus \{V(0)\}$ is a group with $e = V(1) = A \setminus P$.

SUBPROOF. Suppose $x' \in V(x)$, $y' \in V(y)$. Then $(xy)z \in P$ iff $x(yz) \in P$ iff $x'(yz) \in P$ iff $y(x'z) \in P$ iff $y'(x'z) \in P$ iff $(x'y')z \in P$, so $V(xy) = V(x'y')$. The operation $V(x)V(y)$ is thus well defined; it is associative and commutative since multiplication in R is. $V(1)$ is clearly an identity and $V(1) \neq V(0)$ since $1 \cdot 0 \in P$ but $1 \cdot 1 \notin P$.

If $x \notin A$, then $xy \in A \setminus P$ for some $y \in P$. Since $1 \cdot y \in P$, $x \not\sim 1$. Thus $V(x) \neq V(1)$ and $V(1) \subset A$. If $z \in P$, then $z \cdot 1 \in P$, $1 \cdot 1 \notin P$ so $V(1) \neq V(z)$ and $V(1) \subset A \setminus P$. (Note that we have also shown that $V(x) \cap A = \phi$ if $x \notin A$.)

Suppose $x \in A \setminus P$ and $xy \in P$. If $y \notin P$ (i.e., if $x \notin V(1)$), then $y \notin A$, since P is a prime ideal of A. But then $yz \in A \setminus P$ for some $z \in P$, while $x(yz) = (xy)z \in P$, contradicting P a prime ideal of A. Thus $A \setminus P \subset V(1)$, which gives $V(1) = A \setminus P$.

Finally, if $x \notin V(0)$, then we have $xy \notin P$ for some $y \in R$. If $xy \in A \setminus P$, we have $V(xy) = V(x)V(y) = V(1)$; otherwise $xy \notin A$ and $xyz \in A \setminus P$ for some $z \in P$ and $V(xyz) = V(x)V(yz) = V(1)$. Thus $\Gamma_V \setminus \{V(0)\}$ is a group.

<u>CLAIM 2</u>. Define $V(x) < V(y)$ if $\exists\, z \in R$ with $xz \in P$, $yz \in A \setminus P$. Then "<" is a linear order on Γ_V, $\Gamma_V \setminus \{V(0)\}$ is an ordered group and Γ_V is a valuation semi-group.

<u>SUBPROOF</u>. Note that $V(x) < V(y)$ iff $V(xz) \subset P$ while $V(yz) = V(1)$ for some $z \in P$ iff $V(y) \neq 0$ and $V(x)V(y)^{-1} \subset P$. Thus "<" is well defined.

If $V(x) \neq V(y)$, then for some $z \in R$, either $xz \in P$ and $yz \notin P$, or $xz \notin P$ and $yz \in P$. Suppose $xz \in P$ and $yz \notin P$. If $yz \in A \setminus P$, we have $V(x) < V(y)$. Otherwise $yz \notin A$ so $yzw \in A \setminus P$ for some $w \in P$; then $(xz)w \in P$ and again $V(x) < V(y)$. Thus "<" is a linear order on Γ_V.

Let $V(x) < V(y)$ and $V(z) \neq 0$. Let $V(w) = V(z)^{-1}$. Now $xt \in P$ and $yt \in A \setminus P$ for some $t \in R$, so we have $(xt)(zw) = xz(tw) \in P$ and $(yt)(zw) = (yz)(tw) \in A \setminus P$ so that $V(x)V(z) < V(y)V(z)$. Thus $\Gamma_V \setminus \{V(0)\}$ is an ordered group.

Clearly $V(0) \leq V(x)$ and $V(0)V(x) = V(0)$ for all $x \in R$, so Γ_V is a valuation semi-group.

Thus V is a valuation on R. By construction, $A = A_V$ and $P = P_V$.

Now suppose V_1 is a valuation on R with $A = A_{V_1} \neq R$

or $P = P_{v_1}$. If $P = P_{v_1}$, then $A_{v_1} = \{x \in R \mid xP_1 \subset P_1\}$ $= \{x \in R \mid xP \subset P\} = A$. If $A = A_{v_1} \neq R$, then

$P = \{x \in A \mid zy \in A \text{ for some } y \notin A\}$
$= \{x \in A_{v_1} \mid xy \in A_{v_1} \text{ for some } y \notin A_{v_1}\} = P_{v_1}$. Thus if $A = A_{v_1} \neq R$ or $P = P_{v_1}$, then $(A,P) = (A_{v_1}, P_{v_1})$.

Claim for for $x \in R$ that $V_1^{-1}(\{V_1(x)\}) = V(x)$.

<u>SUBPROOF</u>. This is clear if $V(x) = V(0)$.

Let $0 \neq V(x)$, $V(z) = V(x)^{-1}$. Then $y \in V(x)$ iff $yz \in A \setminus P$ iff $V_1(zy) = e$ iff $V_1(z)V_1(y) = e$ iff $V_1(y) = V_1(z)^{-1}$ iff $V_1(y) = V_1(x)$. Thus $V(x) = V_1^{-1}(\{V_1(x)\})$.

Now $V_1^{-1}(\{V_1(xy)\}) = V_1^{-1}(\{V_1(x)V_1(y)\}) = V(xy)$ $= V(x)V(y)$, so

$$\Gamma_{v_1} \xrightarrow{V_1^{-1}(\{\ \})} \Gamma_v$$

is an isomorphism. Also $V_1(x) < V_1(y)$ iff $V_1(x)V_1(y)^{-1} < e$ iff $V_1^{-1}(\{V_1(x)V_1(y)^{-1}\}) = V(x)V(y)^{-1} \subset P$ iff $V(x)V(y)^{-1} < e$ iff $V(x) < V(y)$, so order is preserved.

Thus $V_1^{-1}(\{\ \})$ is the order preserving isomorphism claimed in the proposition.

Henceforth, we will speak of <u>the</u> valuation determined by (A,P) and will refer to the coset representation of Γ_v derived above as the normal representation and wherever desired assume this is the representation under consideration.

COROLLARY 1.7. If (A,P) is a valuation pair of R, then

i.) $R \setminus A$ is closed under multiplication

ii.) $R \setminus P$ is closed under multiplication

iii.) $xy \in A \Rightarrow x \in A$ or $y \in P$

iv.) $x^n \in A \Rightarrow x \in A$

v.) $x^n \in A \setminus P \Rightarrow x \in A \setminus P$

vi.) $A = \{x \in R \mid xP \subset P\}$

vii.) $A = R$ or $P = \{x \in A \mid xy \in A$ for some $y \notin A\}$.

PROOF. Let V be the valuation associated with (A,P) in 1.8. Translating, we have

i.) $V(x)V(y) > e$ if $V(x) > e$ and $V(y) > e$;

ii.) $V(x)V(y) \geq e$ if $V(x) \geq e$ and $V(y) \geq e$;

iii.) $V(x)V(y) \leq e \Rightarrow V(x) \leq e$ or $V(y) < e$;

iv.) $V(x)^n \leq e \Rightarrow V(x) \leq e$;

v.) $V(x)^n = e \Rightarrow V(x) = e$;

vi.) $V(x) \leq e \Leftrightarrow V(x)V(y) < e$ for all $V(y) < e$;

vii.) If $V(z) > e$ for some z then

$V(x) < e \Leftrightarrow V(x)V(t) \leq e$ for some $V(t) > e$.

Section II

DEFINITION 1.8. For R a commutative ring, let $T = T(R) = \{(A,\delta) \mid A$ is a subring of R, δ is a prime ideal of $A\}$. For (A,δ), $(B,\sigma) \in T$ write $(A,\delta) \leq (B,\sigma)$ if $A \subset B$

and $\delta = A \cap \sigma$. "$\leq$" is clearly an inductive partial order on T, so by Zorns lemma, T has maximal elements. We (temporarily) call maximal elements of T maximal pairs. Note that if $(A,\delta) \in T$, then there is a maximal pair (B,σ) with $(B,\sigma) \geq (A,\delta)$.

PROPOSITION 1.9. If (A,δ) is a maximal pair of R, then A is integrally closed in R.

PROOF. Let $\overline{A}$ be the integral closure of A in R. Then there is a prime ideal σ of $\overline{A}$ with $\sigma \cap A = \delta$ (see [5], p. 257). That is $(\overline{A},\sigma) \geq (A,\delta)$ so $A = \overline{A}$.

PROPOSITION 1.10. (A,δ) is a maximal pair of R, iff (A,δ) is a valuation pair of R.

PROOF. It is clear that valuation pairs are maximal pairs, so it is the converse that is of interest.

Let (A,δ) be a maximal pair of R, $x \notin A$, $B = A[x]$ and $\sigma = B\delta$. σ is an ideal of B with $\delta \subset \sigma \cap A$. If $\delta = A \cap \sigma$, then $A \setminus \delta$ is a multiplicative subset of B with $(A \setminus \delta) \cap \sigma = \phi$. Then by Krulls lemma, there is a prime ideal σ' of B with $\sigma \subset \sigma'$ and $(A \setminus \delta) \cap \sigma' = \phi$. That is $\delta = \sigma' \cap A$ and $(B,\sigma') \geq (A,\delta)$. But since $A \neq B$, this is a contradiction, hence $\sigma \cap A \neq \delta$.

Thus there are $p_i \in \delta$, $a \in A \setminus \delta$ with (*) $\sum_{i=0}^{n} x^i p_i = a$.

We can assume n is minimal for an expression of this form.

We have $ap_n^{n-1} = (xp_n)^n + \sum_{i=0}^{n-1} (xp_n)^i p_n^{n-1-i} p_i$, an integral expression for xp_n, thus $xp_n \in A$ by 1.9.

If $xp_n \in \delta$, then $(xp_n + p_{n-1})x^{n-1} + \sum_{i=0}^{n-2} x^i p_i = a$ is an expression of form (*) with lower degree, contradicting the choice of n. Thus $xp_n \in A \setminus \delta$ and (A, δ) is a valuation pair.

We now drop the terminology "maximal pair" in favor of "valuation pair".

Section III

<u>DEFINITION 1.11</u>. We call a valuation pair (A,P) of R an H (Harrison) pair (P is what is called a finite prime in [3]) if A/P is a locally finite field. That is if every finite subset of A/P is contained in a finite subfield of A/P.

<u>PROPOSITION 1.12</u>. (See [3]): $(A,P) \in T$ is an H pair of R iff

i.) Q is closed under $\cdot$ and $-$, and $P \subset Q \Rightarrow P = Q$ or $1 \in Q$,

ii.) $A = \{x \mid xP \subset P\}$.

PROOF. Let (A,P) be an H pair. Then (A,P) is a valuation pair so ii.) is clear. Suppose Q is closed under multiplication and $P \not\subseteq Q$. Let $x \in Q \setminus P$. If $x \notin A$, then for some $y \in P$, $xy \in A \setminus P$, since (A,P) is a valuation pair and $(xy)^n = 1 + z$ for some integer $n > 0$, $z \in P$, since A/P is a locally finite field. But then $xy \in Q$, $(xy)^n \in Q$, $z \in Q$, so $(xy)^n - z = 1 \in Q$. If $x \in A$ then $x^n = 1 + z$ for some $n > 0$, $z \in P$ and $x^n - z = 1 \in Q$.

Conversely, suppose (A,P) satisfies i.) and ii.). If $(B,\sigma) \in T$ and $(B,\sigma) \geq (A,P)$, then $\sigma = P$ by i.) and then $B = A$ by ii.), so (A,P) is a valuation pair.

Assume (A,P) satisfies i.) and ii.) and let $\rho : A \to A/P$ be the natural map. Then if σ is a non-zero subset of A/P (e.g., an ideal of A/P) closed under $\cdot$ and $-$, then $1 \in \rho^{-1}(\sigma)$ and $1 \in \sigma$. Hence A/P is a field. Also $1 \in Z \cdot \rho(1) \cdot p$ for all prime integers p with $\rho(1) \cdot p \neq 0$, so $Z \cdot \rho(1) = Z_p = Z/(p)$ for some prime integer p.

Also, if $x \in A/P$, $x \neq 0$, then $1 \in xZ_p[x]$ so x is algebraic over Z_p, hence is in the finite field $Z_p[x]$ of A/P. This gives case $n = 1$ of the inductive hypothesis: "If E is a finite subset of A/P and $|E| = n$, then E is contained in a finite subfield of A/P."

Assume the hypothesis true for n and let $|E| = n + 1$, $a \in E$. Then $|E \setminus \{a\}| = n$ so there is a finite subfield F of A/P with $E \setminus \{a\} \subset F$. If $a = 0$ we are done, otherwise $1 \in aF[a]$ so a is algebraic over F, hence $F[a]$ is a finite subfield of A/P containing E.

Thus A/P is locally finite.

<u>COROLLARY 1.13</u>. If S is a subset of R closed under $-$ and $\cdot$, and $1 \notin S$, then there is a H pair (A,P) of R with $S \subset P$. If $B = \{x \in R \mid xS \subset S\}$ and B/S is a locally finite field, then $(A,P) \geq (B,S)$.

<u>PROOF</u>. $\{\sigma \subset R \mid S \subset \sigma, \sigma - \sigma \subset \sigma, \sigma \cdot \sigma \subset \sigma, 1 \notin \sigma\}$ is inductively partially ordered by $\subset$, thus by Zorns lemma contains a maximal element P. Then if $A = \{x \mid xP \subset P\}$, (A,P) satisfies i.) and ii.) of 1.12. By the maximality of P, P is a maximal ideal of A, hence a prime ideal so $(A,P) \in T$. Thus (A,P) is an H pair of R.

If B/S is a locally finite field, $x \in B \setminus S$, then $x^n = 1 + S$ for some integer $n > 0$, some $s \in S$, thus $x^n \in A \setminus P$. But then by 1.7, $x \in A \setminus P$. Thus $B \setminus S \subset A \setminus P$ so $B \subset A$ and $S = B \cap P$.

<u>PROPOSITION 1.14</u>. Let E be a finite subset of R, σ a subset of R with $\sigma - \sigma \subset \sigma$, $\sigma\sigma \subset \sigma$. If σ and the multiplicative subset generated by E have void intersection, and $E\sigma \subset \sigma$, then there is an H pair (A,P) of R with $\sigma \subset P$ and $E \subset A \setminus P$.

<u>PROOF</u>. Consider the finitely generated subring $S = Z \cdot 1[E]$ of R. $\mu = \sigma \cap S$ is an ideal of S which has void intersection with the multiplicative subset generated

by the finite subset E of S. Thus by the integer version of the Nullstellensatz (see [1], pp. 67,68) there is a maximal ideal δ of S with $E \cap \delta = \phi$ and $\mu \subset \delta$. Also by the Nullstellensatz, S/δ is a locally finite field.

Now $\delta + \sigma$ is closed under $\cdot$ and $-$ (since $S\sigma \subset \sigma$), and $1 \notin \delta + \sigma$ (for $1 = p + a$ gives $a = 1 - p \in \mu = \sigma \cap S$ and $1 = p + a \in \delta$). Thus by 1.3, there is an H pair (A,P) of R with $\delta + \sigma \subset P$. Then $\delta \subset P$ so again by 1.3, $S \setminus \delta \subset A \setminus P$ since S/δ is a locally finite field. But then $E \subset S \setminus \delta \subset A \setminus P$.

<u>COROLLARY 1.15</u>. If $N_1 = \bigcap\{P \mid (A,P)$ is a valuation pair of $R\}$, $N_2 = \bigcap\{P \mid (A,P)$ is an H pair of $R\}$, $N = \{x \in R \mid x^n = 0$ for some integer $n\}$, then $N = N_1 = N_2$.

<u>PROOF</u>. $N \subset N_1$ by 1.7 and $N_1 \subset N_2$ since the set being intersected to obtain N_1 contains that being intersected to obtain N_2.

If $x \notin N$, then with $E = \{x\}$ and $\sigma = (0)$ in 1.14, we have $x \in A \setminus P$ for some H pair (A,P). That is, $x \notin N_2$. Thus since $N \subset N_2$, we have $N = N_2$.

<u>COROLLARY 1.16</u>. If (B,Q) is a valuation pair of R, then $Q = \bigcap\{P \mid (A,P)$ is an H pair of $R, Q \subset P\}$. Further, if E is a finite subset of R with $E \cap Q = \phi$, then there is an H pair (A,P) of R with $E \cap B \subset A \setminus P$ and $(E \setminus B) \cap A = \phi$.

PROOF. It suffices to prove the second statement. Let E be a finite subset of R with $E \cap Q = \phi$. Let $E_1 = E \cap B$, $E_2 = E \setminus B$. For $x \in E_2$, chose $q_x \in Q$ with $xq_x \in B \setminus Q$, and let $E'_2 = \{xq_x \mid x \in E_2\}$.

Applying 1.14 to $E_1 \cup E'_2$ and Q, there is an H pair (A,P) of R with $Q \subset P$ and $E_1 \cup E'_2 \subset A \setminus P$. But then if $x \in E_2$, $xq_x \in A \setminus P$, so $x \notin A$. That is $E_1 \subset A \setminus P$ and $E_2 \cap A = \phi$.

COROLLARY 1.17. If $A = R$ for all H pairs (A,P) of R, then $A = R$ for all valuation pairs (A,P) of R.

PROOF. If (A,P) is a valuation pair of R, then by 1.16 and hypothesis, P is the intersection of (maximal) ideals of R, hence is an ideal of R. That is, $A = \{x \mid xP \subset P\} = R$.

Section IV

DEFINITION 1.18. Let A be a subring of a ring R. If $\rho : A \to S$ is a homomorphism we call ρ a partial homomorphism on R. If, whenever B is a subring of R, $A \subset B$, $\tau : B \to T$ a homomorphism, $\mu :(\text{image } \rho) \to T$ a homomorphism and $\tau|_A = \mu \circ \rho$ one also has $B = A$, then we call ρ maximal.

One can show using 1.10 that if A is a subring of R, P an ideal of A, then (A,P) is a valuation pair of R if and only if the natural map $A \to A/P$ is a maximal partial homomorphism of R into a domain.

A place on a field is a maximal partial homomorphism into a field. However, if (A,P) is a valuation pair of a field F, $x \in A \setminus P$, then $xx^{-1} \in A \setminus P$ gives $x^{-1} \in A \setminus P$ by 1.17, so A/P is a field. That is, a pair (A,P) of a field F is a valuation pair if and only if the natural map $A \to A/P$ is a place.

Thus at first glance one might expect "maximal partial homomorphism into a domain" to generalize "place". This generalization is unsatisfying since such maps do not compose (see 1.20) as do places on a field. The generalized places of [3] do compose and satisfy the hypothesis of 1.20.

DEFINITION 1.19. A valuation pair (A,P) of R is called a prime pair if A/P is a field.

PROPOSITION 1.20. Let ρ be a partial homomorphism from R to S with dom $\rho \neq R$. If the composite partial homomorphism $R \xrightarrow{\rho} S \to A/P$ is maximal for all H pairs (A,P) of S, then (dom ρ, $\sqrt{\ker \rho}$) is a prime pair of R. Conversely, if (dom ρ, $\sqrt{\ker \rho}$) is a prime pair of R the composite is maximal for all valuation pairs (A,P) of S.

PROOF. Suppose $R \xrightarrow{\rho} S \to A/P$ is maximal whenever (A,P) is an H pair of S. Let B = dom ρ, $\sigma = \ker \rho$.

CLAIM 1. Every H pair (A',P') of B with $\sqrt{\sigma} \subset P'$

is a valuation pair of R.

<u>SUBPROOF.</u> Let (A',P') be an H pair of B, $\sqrt{\sigma} \subset P'$. By 1.13 there is an H pair (A,P) of S with $(A,P) \geq (\rho(A'), \rho(P'))$. Then since $R \xrightarrow{\rho} S \to A/P$ is maximal, (A',P') is a maximal pair of R.

<u>CLAIM 2.</u> If $x \notin B$, then $xy \in B$ for some $y \in B$. Also of $xy \in B$, then $y \in \sqrt{\sigma}$.

<u>SUBPROOF.</u> Let (A',P') by any H pair of B with $\sqrt{\sigma} \subset P'$. Then since (A',P') is a valuation pair of R, there is a $y \in P'$ with $xy \in A' \setminus P' \subset B$. If $y \notin \sqrt{\sigma}$, then $\{xy,y\} \cap \sqrt{\sigma} = \phi$, so by 1.14, there is an H pair (A'',P'') of B with $\{xy,y\} \subset A'' \setminus P''$ and $\sqrt{\sigma} \subset P''$. But since $x \notin A''$, this cannot happen by 1.7.

<u>CLAIM 3.</u> $\sqrt{\sigma}$ is a maximal ideal of B.

<u>SUBPROOF.</u> Suppose δ is a maximal ideal of B with $\sqrt{\sigma} \subset \delta$ and $\delta \setminus \sqrt{\sigma} \neq \phi$, say $y \in \delta \setminus \sqrt{\sigma}$. Let $x \in R \setminus B$. Then $xy \notin B$ by Claim 2. Let (A',P') be an H pair of B with $\delta \subset P'$. Then $xy \notin P'$ so $z(xy) \in A' \setminus P'$ for some $z \in P'$. But $(zx)y \in A' \subseteq B$ implies $zx \in B$ by Claim 2, and then $(zx)y \in \delta \subset P'$, a contradiction.

Thus $(B,\sqrt{\sigma})$ is a valuation pair of R and $B/\sqrt{\sigma}$ is a field.

Now suppose $(B,\sqrt{\sigma})$ is a valuation pair of R and

$B/\sqrt{\sigma}$ is a field. Let (A,P) be a valuation pair of S. Since $\rho(\sqrt{\sigma})$ is nil, $\rho(\sqrt{\sigma}) \subset P$. Let $x \in \rho(B)\setminus A$. Since $B/\sqrt{\sigma}$ is a field, there is an $x' \in \rho(B)$, $y \in \rho(\sqrt{\sigma})$ with $xx' = 1 + y \in A\setminus P$. Thus $x' \in P \cap \rho(B)$; i.e., $(\rho(B) \cap A, \rho(B) \cap P)$ is a valuation pair of $\rho(B)$.

Thus $(\rho^{-1}(A), \rho^{-1}(P))$ is a valuation pair of B, for if $(A',P') \geq (\rho^{-1}(A), \rho^{-1}(B))$, then $(\rho(A'), \rho(P')) \geq (A \cap \rho(B), P \cap \rho(B))$. Also $\sqrt{\sigma} \subset \rho^{-1}(P)$.

CLAIM. Every valuation pair (A',P') of B with $\sqrt{\sigma} \subset P$ is a valuation pair of R.

SUBPROOF. Suppose $x \notin A'$. If $x \in B$, then $\exists\, x' \in B\setminus\sqrt{\sigma}$ with $xx' \in 1 + \sqrt{\sigma}$; since $x \notin A'$, $x' \in P'$. If $x \notin B$, then $\exists\, y \in \sqrt{\sigma}$ with $xy \in B\setminus\sqrt{\sigma}$. $xy(xy)' = x(y(xy)') \in 1 + \sqrt{\sigma}$, and since $x \notin B$, $y(xy)' \in \sqrt{\sigma} \subset P'$.

Thus $(\rho^{-1}(A), \rho^{-1}(P))$ is a valuation pair of R so the composite $R \xrightarrow{\rho} S \rightarrow A/P$ is maximal.

Proposition 1.20 gives some insight into generalized places as defined in [3] and these provide motivation for occasionally including special results for valuations corresponding to prime pairs.

EXAMPLE 1.21. Not all valuation pairs (A,P) are prime pairs, even when one requires $A \neq R$.

PROOF. If A = R is allowed, one needs only produce a ring R that has a non maximal prime ideal.

For the second case, let $R = Q[x]$ where Q is the rational numbers and x is an indeterminate. Let p be a prime integer, $A_p = \{\frac{m}{n} \mid (m,n) = 1 = (n,p), \text{or } m = 0\}$, $A = A_p[x]$, $P = A \cdot p$, $\sigma = P + Ax$. One can easily check that (A,P) is a valuation pair of R and that σ is a proper ideal of A with $P \subseteq \sigma$. Thus (A,P) is not a prime pair of R.

2. INDEPENDENCE AND EXTENSIONS

Section I

Throughout this section, V is a fixed valuation on a fixed ring R.

Let ϕ be an order homomorphism of Γ_V into a valuation semi-group Γ with $\phi(e) = e$, $\phi(0) = 0$. Since ϕ is a homomorphism, $\phi(\Gamma_V \setminus \{0\})$ is an ordered group (inherited order), and since $e \neq 0$, $\phi \circ V$ is a valuation on R. With this notation we have:

<u>PROPOSITION 2.1.</u> $\phi^{-1}(e)$ is an isolated subgroup of Γ_V and $P_{\phi \circ V}$ is a prime V-closed ideal of A_V, where

<u>DEFINITION 2.2.</u> A subgroup H of a valuation semi-group Γ is said to be isolated if $0 \notin H$ and whenever $\alpha, \beta, \gamma \in \Gamma$ with $\alpha \leq \beta \leq \gamma$ and $\alpha, \gamma \in H$ then $\beta \in H$.

<u>DEFINITION 2.3.</u> An ideal σ of A_V is said to be V-closed if $x \in \sigma$, $y \in R$ and $V(y) \leq V(x)$ implies $y \in \sigma$.

<u>PROOF.</u> If $\alpha, \beta, \gamma \in \Gamma_V$, $\alpha \leq \beta \leq \gamma$ and $\phi(\alpha) = \phi(\gamma) = e$, then $e = \phi(\alpha) \leq \phi(\beta) \leq \phi(\gamma) = e$ since ϕ is order preserving. Also $e = \phi(e)\phi(\gamma) = \phi(\alpha\alpha^{-1})\phi(\gamma) = \phi(\alpha)\phi(\alpha^{-1}\gamma) = \phi(\alpha^{-1}\gamma)$ so $\alpha^{-1}\gamma \in \phi^{-1}(e)$ and $\phi^{-1}(e)$ is a group, hence an isolated subgroup of Γ_V.

Let $\beta \in \Gamma_V$. If $\phi(\beta) < e$, then $\beta < e$ so $P_{\phi \circ V} \subseteq P_V$. If $\beta \leq e$, then $\phi(\beta) \leq \phi(e) = e$ so $A_V \subseteq A_{\phi \circ V}$. Thus $P_{\phi \circ V} \subseteq A_V \subseteq A_{\phi \circ V}$ and since $P_{\phi \circ V}$ is a prime ideal of $A_{\phi \circ V}$, $P_{\phi \circ V}$ is a prime ideal of A_V. If $x \in P_{\phi \circ V}$, $y \in R$ and $V(y) \leq V(x)$ then $\phi \circ V(y) \leq \phi \circ V(x) < e$ so $y \in P_{\phi \circ V}$. That is $P_{\phi \circ V}$ is a V-closed ideal of A_V.

The first step towards a converse is:

PROPOSITION 2.4. If H is an isolated subgroup of Γ_V, then there is an order homomorphism ϕ of Γ_V onto a valuation semigroup $\Gamma_{\phi \circ V}$ with $\phi^{-1}(e) = H$.

PROOF. Set $\phi(\alpha) = \alpha H$ for all $\alpha \in \Gamma_V$. Then $\phi(\Gamma_V \setminus \{0\}) = \Gamma_V \setminus \{0\} / H$ is a group and $H \neq 0 = 0 \cdot H$.

Suppose $\alpha < \beta$ and $\alpha H \neq \beta H$. Then if $h_1, h_2 \in H$, $h_1\alpha < h_2\beta$, for otherwise $h_1\alpha \geq h_2\beta$ gives $e \geq \beta^{-1}\alpha \geq h_1^{-1}h_2$ and $\beta^{-1}\alpha \in H$, since $e, h_1^{-1}h_2 \in H$ and H is isolated. Thus the order "$\alpha H \leq \beta H \Leftrightarrow \alpha \leq \beta$" is well defined on $\phi(\Gamma_V)$. One can easily check now that $\phi(\Gamma_V)$ is a valuation semigroup with the usual coset multiplication and that ϕ is an order homomorphism onto.

Set $V_H = \phi \circ V$ and note that $P_{V_H} = \{x \in R \mid V_H(x) < e\}$ $= \{x \in R \mid V(x)H < H\} = \{x \in R \mid V(x) < \alpha, \forall \alpha \in H\}$. If $\beta \in \Gamma_V$ and $\beta \notin H$, then $\beta H < H$ or $\beta^{-1}H < H$, so $\beta \in V(P_{V_H})$ or $\beta^{-1} \in V(P_{V_H})$. That is $H = \{\alpha \in \Gamma_V \mid V(x) <$

min $\{\alpha,\alpha^{-1}\}, \forall x \in P_{v_H}\}$.

PROPOSITION 2.5. $H \to P_{v_H}$ and

$\sigma \to \{\alpha \in \Gamma_v \mid V(x) < \min\{\alpha,\alpha^{-1}\}, \forall x \in \sigma\}$ is a one-one correspondence between isolated subgroups H of Γ_v and V-closed prime ideals σ of A_v. The correspondence is order inverting, where order is $\subseteq$ in both cases.

PROOF. With the preceeding remarks, all that remains to be shown is that $T = \{\alpha \in \Gamma_v \mid V(x) < \min\{\alpha,\alpha^{-1}\}, \forall x \in \sigma\}$ is an isolated subgroup of Γ_v and that the correspondence is order inverting.

$0 \notin T$ since $0 = V(0)$ and $0 \in \sigma$. $e \in T$ since $e \leq V(x)$, $x \in \sigma$, gives $1 \in \sigma$ since σ is V-closed; a contradiction, since σ is a prime ideal (hence proper). By definition of T, $\alpha \in T \Rightarrow \alpha^{-1} \in T$.

Let $\alpha,\beta \in T$, $\alpha = V(x)$, $\beta = V(y)$. If $\alpha\beta \notin T$, then $\alpha\beta = V(xy) \leq V(z)$ for some $z \in \sigma$ and $xy \in \sigma$ since σ is v-closed. But $x \notin \sigma$ and $y \notin \sigma$ so $x \notin A_v$ or $y \notin A_v$ since σ is a prime ideal of A_v. Suppose $x \notin A_v$. Then $\exists\, x' \in A_v$ with $V(x'xy) = V(x')V(x)V(y) = \alpha\alpha^{-1}\beta = \beta$, a contradiction since $x'xy \in \sigma$. Thus $\alpha\beta \in T$ so T is a group.

If $\alpha,\beta \in T$, $\gamma \in \Gamma_v$ with $\alpha \leq \gamma \leq \beta$, then $\beta^{-1} \leq \gamma^{-1} \leq \alpha^{-1}$, and if $x \in \sigma$, then $V(x) < \min\{\alpha,\beta^{-1}\} \leq \min\{\gamma,\gamma^{-1}\}$. This gives $\gamma \in T$ and T is an isolated subgroup of Γ_v.

If $\sigma_1 \subseteq \sigma_2$, then it is clear that $\{\alpha \in \Gamma_V \mid V(x) < \min\{\alpha,\alpha^{-1}\}, \forall x \in \sigma_2\} \subseteq \{\alpha \in \Gamma_V \mid V(x) < \min\{\alpha,\alpha^{-1}\}, \forall x \in \sigma_1\}$, so the correspondence is order inverting.

The correspondence above is clarified further by

PROPOSITION 2.6. A prime ideal σ of A_V is V-closed iff $\sigma_V \subseteq \sigma \subseteq P_V$. Further, the V-closed ideals of A_V are linearly ordered by inclusion.

PROOF. If σ is a V-closed ideal of A_V, then $0 \in \sigma$ gives $\sigma_V \subseteq \sigma$ and $1 \notin \sigma$ gives $\sigma \subseteq P_V$ (since $V(A_V \setminus P_V) = e$).

Now suppose $\sigma_V \subseteq \sigma \subseteq P_V$ and σ is a prime ideal of A_V. Let $x \in \sigma$, $y \in R$ with $V(y) \leq V(x)$. If $V(y) = 0$, then $y \in \sigma$, so assume $V(y) \neq 0$. Then $V(x) > 0$ so $\exists x' \in R$ with $V(x') = V(x)^{-1}$. Now $V(y) \leq V(x) < e$ gives $y \in P_V$ and $V(yx') \leq V(xx') = e$ gives $yx' \in A_V$. Thus $xyx' \in \sigma$, but $xx' \notin \sigma$, so $y \in \sigma$ since σ is a prime ideal of A_V. Thus σ is V-closed.

Now suppose σ and δ are V-closed ideals of A_V. Suppose $x \in \sigma \setminus \delta$ and $y \in \delta \setminus \sigma$. Then $V(x) \leq V(y)$ gives $x \in \delta$ while $V(y) \leq V(x)$ gives $y \in \sigma$. Thus $\delta \subseteq \sigma$ or $\sigma \subseteq \delta$.

In particular the V-closed prime ideals are linearly ordered by $\subseteq$.

PROPOSITION 2.7. The set of V-closed ideals of A_V and the set of V-closed prime ideals of A_V are order complete with respect to the order $\subseteq$.

<u>DEFINITION 2.9.</u> If V' is a valuation with $A_V \subseteq A_{V'}$ and $\sigma_V \subseteq P_{V'} \subseteq P_V$, we say V' dominates V and write $V' \geq V$. We say V and V'' are dependent if $V' \geq V$ and $V' \geq V''$ for some V' with $V'(R) \neq \{e, 0\}$, and independent otherwise.

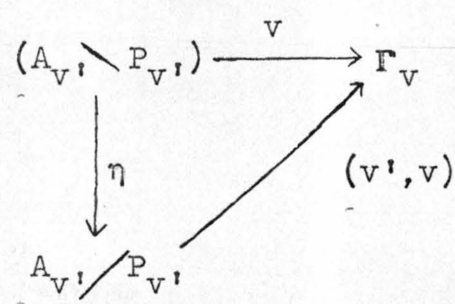
<u>PROPOSITION 2.10.</u> If $V' \geq V$ then $P_{V'}$ is a V-closed prime ideal of A_V. If V' and V are dependent, then $\sigma_V = \sigma_{V'}$.

<u>PROOF.</u> Let $V' \geq V$. Then $P_{V'} \subseteq A_V \subseteq A_{V'}$ shows that $P_{V'}$ is a prime ideal of A_V. Then since $\sigma_V \subseteq P_{V'} \subseteq P_V$, $P_{V'}$ is V-closed by 2.6.

Since σ_V is an ideal of R, $\sigma_V \subseteq P_{V'}$, $\sigma_V \subseteq \sigma_{V'}$ by 1.3. $\sigma_{V''} \subseteq P_V$ gives $\sigma_{V'} \subseteq \sigma_V$ also by 1.3, so $\sigma_V = \sigma_{V'}$.

Now if V' and V are dependent, say $V'' \geq V$, $V'' \geq V'$, then $\sigma_V = \sigma_{V''} = \sigma_{V'}$.

<u>PROPOSITION 2.11.</u> If $V' \geq V$, then there is an order homomorphism $\phi : \Gamma_V \to \Gamma_{V'}$ with $V' = \phi \circ V$. Also, there is a valuation (V', V) on $A_{V'}/P_{V'}$ such that if $\eta : A_{V'} \to A_{V'}/P_{V'}$ is the natural homomorphism, then the following diagram commutes.



Further $(A_{(V',V)}, P_{(V',V)}) = (A_V/P_{V'}, P_V/P_{V'})$;

$\sigma_{(V',V)} = \eta(P_{V'})$; and $(V',V)(A_{V'}/P_{V'}) = \phi^{-1}(e) \cup \{0\}$.

(V',V) is called the induced valuation.

PROOF. Using 2.10 and preceeding results, $V' = V_{P_{V'}}$ and $\phi^{-1}(e) = H_{P_{V'}}$. $V''^{-1}(e) = A_{V'} \setminus P_{V'} = V^{-1}(\phi^{-1}(e))$, so $V(A_V \setminus P_V) = \phi^{-1}(e)$.

The remainder of the proposition is clear once it is shown that $x \in A_{V'} \setminus P_{V'}$ implies $V(x) = V(y)$ for all $y \in x + P_{V'}$. But if $x \in A_{V'} \setminus P_{V'}$, $p \in P_{V'}$, then $V(x) > V(p)$, so $V(x) = V(x + p - p) \leq \max\{V(x + p), V(-p)\}$ $\leq \max\{V(x), V(p)\} = V(x)$.

PROPOSITION 2.12. Let V_1 and V_2 be distinct dependent valuations on R. Then there is a valuation V on R with $V \geq V_1$ and $V \geq V_2$ such that (V,V_1) and (V,V_2) are independent valuations on A_V/P_V.

PROOF. Since V_1 and V_2 are dependent, $\mathbb{A} = \{P_{V'} \mid V' \text{ a valuation on } R, V' \geq V_1, V' \geq V_2\}$ is non-empty. Thus $P_V = \inf \mathbb{A}$ is a V_i closed ideal of A_{V_i} and $V \geq V_i$, $i = 1, 2$.

Now suppose $\overline{V}$ is a valuation on A_V/P_V with $\overline{V} \geq (V,V_1)$ and $\overline{V} \geq (V,V_2)$. Let $P = \{x \in A_V \mid x + P_V \in P_{\overline{V}}\}$. Since $P_{\overline{V}}$ is a prime ideal of A_{V_i}/P_V, $i = 1, 2$, and

$P_{\overline{V}} \subset (P_{V_1}/P_V) \cap (P_{V_2}/P_V)$, it follows that

$\sigma_{V_1} = \sigma_{V_2} \subseteq P_V \subseteq P \subseteq P_{V_1} \cap P_{V_2}$ and P is a prime ideal of

A_{V_i}, and thus V_i closed by 2.6, i = 1, 2. Thus $P = P_{V'}$

for some valuation V' on R with $V' \geq V_i$, i = 1, 2, by 2.6 and 2.5. But then $V' \in \mathbb{A}$ so $V' \geq V$ and $P_V \subseteq P$.

Thus $P = P_V$ and $P_{\overline{V}} = P/P_V = P_V/P_V$ is an ideal (zero) of A_V/P_V so $\overline{V}(A_V/P_V) = \{e, 0\}$. That is (V, V_1) and (V, V_2) are independent.

Section II

Throughout this section, let V_0 be a fixed valuation on a ring K and let R be an extension of K. We will consider the problem of "extending" V_0 to R.

<u>PROPOSITION 2.13</u>. There is a valuation pair (A,P) of R with $A_{V_0} = A \cap K$ and $P_{V_0} = P \cap K$. Further, if (A_{V_0}, P_{V_0}) is a prime (H) pair of K, then (A,P) can be chosen as a prime (H) pair of R.

<u>PROOF</u>. $(A_{V_0}, P_{V_0}) \in T(R)$ so there is a valuation pair $(A,P) \in T(R)$ with $(A,P) \geq (A_{V_0}, P_{V_0})$. Then since $(A \cap K, P \cap K) \geq (A_{V_0}, P_{V_0})$, the first statement follows.

If (A_{v_0}, P_{v_0}) is an H pair of K, the existence of an H pair (A,P) with $(A,P) \geq (A_{v_0}, P_{v_0})$ is given by 1.13.

Now suppose (A_{v_0}, P_{v_0}) is a prime pair of K. Let $S = \{(A,P) \mid (A,P) \geq (A_{v_0}, P_{v_0})$ and (A,P) a valuation pair of R$\}$. For (A_1,P_1), $(A_2,P_2) \in S$, define a partial order $\gtrsim$ on S by $(A_1,P_1) \gtrsim (A_2,P_2)$ if $A_2 \subseteq A_1$ and $P_2 \subseteq P_1$. If $\mathbb{A}$ is a chain in S, then $A_{\mathbb{A}} = \bigcup \{A \mid (A,P) \in \mathbb{A}$ for some P$\}$ is a ring, $P_{\mathbb{A}} = \bigcup \{P \mid (A,P) \in \mathbb{A}$ for some A$\}$ is a prime ideal of $A_{\mathbb{A}}$, and $(A_{\mathbb{A}}, P_{\mathbb{A}}) \geq (A_{v_0}, P_{v_0})$. Now there is a valuation pair (A,P) of R with $(A,P) \geq (A_{\mathbb{A}}, P_{\mathbb{A}})$. Then (A,P) is in S and (A,P) is an upper bound for $\mathbb{A}$. That is, $\gtrsim$ is an inductive partial order on S, hence S has maximal elements by Zorns lemma.

Let (A,P) be maximal in S. Let δ be a maximal ideal of A with $P \subseteq \delta$. Then $P_{v_0} \subseteq \delta \cap A_{v_0}$, and since (A_{v_0}, P_{v_0}) is a prime pair, $P_{v_0} = \delta \cap A_{v_0}$. Thus if (A',P') is a valuation pair of R with $(A',P') \geq (A,\delta)$, then $(A',P') \in S$ and $(A',P') \gtrsim (A,P)$. Since (A,P) is maximal in S, A = A', $P' = P \subseteq \delta \subseteq P'$. That is, P is a maximal ideal of A.

Since $A_{v_0}/P_{v_0} = A \cap K / P \cap K \simeq (A_{v_0} + P)/P$, 2.13 shows that a maximal partial homomorphism of K into a domain (field, locally finite field) can always be extended to a maximal partial homomorphism of R into a domain (field, locally finite field). This is a classical result on extension of places. Due to the trivial ideal structure of a field, this also is an extension theorem for valuations on fields, as will be seen from the following interesting, but misleading result.

PROPOSITION 2.14. If V_1 is a valuation on R with $(A_{V_1}, P_{V_1}) \geq (A_{V_0}, P_{V_0})$, then there is an order isomorphism ϕ of $(\Gamma_{V_0} \setminus \{0\})$ into Γ_{V_1} such that $\phi \circ V_0(x) = V_1(x)$ for all $x \in K$ with $V_0(x) \neq 0$.

PROOF. Let $z \in K$, $V_0(x) \neq 0$. Using the standard representation of Γ_{V_0} and Γ_{V_1}, it will suffice to show that $V_0(x) = V_1(x) \cap K$, for then $\phi(V_0(x)) = V_1(x)$ is as advertised.

Let $x' \in V_0(x)^{-1}$, $y \in V_0(x)$. Then $x'y \in A_{V_0} \setminus P_{V_0}$ $= V_0(1) \subseteq A_{V_1} \setminus P_{V_1} = V_1(1)$, so $V_1(x')^{-1} = V_1(y)$. That is $V_1(y) = V_1(V_0(x)) = V_1(x) \supseteq V_0(x)$. If $z \in V_1(x) \cap K$, then $zx' \in V_1(1) \cap K = (A_{V_1} \setminus P_{V_1}) \cap K = A_{V_0} \setminus P_{V_0}$ $= V_0(1)$, so $V_0(z) = V_0(x')^{-1} = V_0(x)$. Thus $V_1(x) \cap K = V_0(x)$.

The above result is misleading since in general there are many $x \in \sigma_{V_0}$ with $V_1(x) \neq 0$.

DEFINITION 2.15. Let R be an extension of K, V_0 a valuation of K. A valuation V_1 on R is called an extension of V_0 to R if there is an order isomorphism ϕ of Γ_{V_0} into Γ_{V_1} such that $\phi \circ V_0(x) = V_1(x)$ for all $x \in K$.

By the proof of 2.14, an immediate result is iii.) $\Rightarrow$ i.) of the following.

PROPOSITION 2.16. Let R be an extension of K, V_0 a valuation on K, V_1 a valuation on R. Then the following are equivalent.

i.) V_1 is an extension of V_0 to R.

ii.) $(A_{V_1}, P_{V_1}) \geq (A_{V_0}, P_{V_0})$ and $V_1|_K$ is a valuation on K.

iii.) $(A_{V_1}, P_{V_1}) \geq (A_{V_0}, P_{V_0})$ and $\sigma_{V_0} \subseteq \sigma_{V_1}$.

PROOF. If V_1 is an extension of V_0 to R, then $V_1(K) = \phi \circ V_0(K)$ is a valuation semi-group contained in Γ_{V_1}, so $V_1|_K$ is a valuation on K. If $x \in K$, then $V_1(x) \leq e$ iff $\phi \circ V_0(x) \leq e$ iff $V_0(x) \leq e$ so $A_{V_1} \cap K = A_{V_0}$. Also $V_1(x) < e$ iff $\phi \circ V_0(x) < e$ iff $V_0(x) < e$ so $P_{V_1} \cap K = P_{V_0}$. That is $(A_{V_1}, P_{V_1}) \geq (A_{V_0}, P_{V_0})$, which gives i.) $\Rightarrow$ ii.).

If $(A_{v_1}, P_{v_1}) \geq (A_{v_0}, P_{v_0})$ and $V_1|_K$ is a valuation on K, then $(A_{v_1}|_K, P_{v_1}|_K) = (A_{v_0}, P_{v_0})$ and $\sigma_{v_1}|_K = \sigma_{v_0}$ by 1.6. But $\sigma_{v_1}|_K = \sigma_{v_1} \cap K \subset \sigma_{v_1}$, so ii.) $\Rightarrow$ iii.).

<u>THEOREM 2.17</u>. (Extension Theorem) V_0 has extensions to R iff $K \cap R\sigma_{v_0} = \sigma_{v_0}$. Further, if V_0 has extensions to R and (A_{v_0}, P_{v_0}) is a prime (H) pair of K, then V_0 has an extension V_1 such that (A_{v_1}, P_{v_1}) is a prime (H) pair of R.

<u>PROOF</u>. If V_0 has an extension V_1 to R, then $\sigma_{v_0} \subseteq \sigma_{v_1}$ by 2.16 so $K \cap R\sigma_{v_0} \subseteq K \cap R\sigma_{v_1} = K \cap \sigma_{v_1} = \sigma_{v_0}$.

Conversely, suppose $K \cap R\sigma_{v_0} = \sigma_{v_0}$. Then $\delta = P_{v_0} + R\sigma_{v_0}$ is an ideal of $B = A_{v_0} + R\sigma_{v_0}$ with $A_{v_0} = B \cap K$ and $P_{v_0} = \delta \cap K$. One can check that $A_{v_0} / P_{v_0} \simeq B / \delta$, so δ is a prime ideal of B and $(B, \delta) \geq (A_{v_0}, P_{v_0})$. Now if (A_{v_1}, P_{v_1}) is any valuation pair of R with $(A_{v_1}, P_{v_1}) \geq (B, \delta)$, then $(A_{v_1}, P_{v_1}) \geq (A_{v_0}, P_{v_0})$ and $\sigma_{v_0} \subseteq R\sigma_{v_0} \subseteq A_{v_1}$, so $R\sigma_{v_0} \subseteq \sigma_{v_1}$ by 1.3. That is $\sigma_{v_0} \subseteq \sigma_{v_1}$, so V_1 is an extension of V_0 by 2.16.

Finally, if (A_{V_0}, P_{V_0}) is a prime (H) pair of K, then (B, δ) is a prime (H) pair of B, and since R is an extension of B, (A_{V_1}, P_{V_1}) (above) may be chosen as a prime (H) pair of R by 2.11.

PROPOSITION 2.18. Let R be an extension of K, and suppose R is integral over K. Then every valuation on K has extensions to R. In particular, if V_0 is a valuation on K, V_1 a valuation on R with $(A_{V_1}, P_{V_1}) \geq (A_{V_0}, P_{V_0})$, then V_1 extends V_0.

PROOF. It suffices to prove the last statement, and for this proof we are indebted to D. K. Harrison.

By 2.16 and 2.17 we need only show $(A_{V_1}, P_{V_1}) \geq (A_{V_0}, P_{V_0})$ implies $R\sigma_{V_0} \subseteq A_{V_1}$. Let $\alpha \in \sigma_{V_0}$, $x \in R$. Since R is integral over K, there are $a_i \in K$, $n > 0$ with $x^n + \sum_{i=0}^{n-1} a_i x^i = 0$. Then $\alpha^n \cdot 0 =$

$(\alpha x)^n + \sum_{i=0}^{n-1} a_i \alpha^{n-1} (\alpha x)^i = 0$. But $a_i \alpha^{n-i} \in \sigma_{V_0} \subset A_{V_1}$ for $i = 0, 1, \cdots, n-1$, that is, αx is integral over A_{V_1}. Since A_{V_1} is integrally closed, $\alpha x \in A_{V_1}$.

Section III

In this section, we assume R is an extension of K. The results obtained will be needed in Chapter 3.

PROPOSITION 2.19. Let V_0', V_0 be valuations on K with $V_0' \geq V_0$. Then

i.) V_0 has extensions to R iff V_0' has extensions to R,

ii.) If V_1 is an extension of V_0 to R, then the set of extensions V' of V_0' to R with $V' \geq V_1$ is non-empty, linearly ordered and has a smallest element. If $(\Gamma_{V_1}/\Gamma_{V_0}) \setminus \{0\}$ is torsion, then there is a unique extension V' of V_0' to R with $V' \geq V_1$.

PROOF. $\sigma_{V_0'} = \sigma_{V_0}$ by 2.9, so $R\sigma_{V_0'} \cap K = \sigma_{V_0'}$ iff $R\sigma_{V_0} \cap K = \sigma_{V_0}$. Thus i.) follows by 2.15.

Let $V_{0'} = \phi \circ V_0$, ϕ an order homomorphism of Γ_{V_0} onto $\Gamma_{V_{0'}}$. Then $\phi^{-1}(e)$ is an isolated subgroup of Γ_{V_0} and $H = \{\gamma \in \Gamma_{V_1} \mid \exists\, \alpha, \beta \in \phi^{-1}(e) \text{ with } \alpha \geq \gamma \geq \beta\}$ is an isolated subgroup of Γ_{V_1}. If θ is the natural map $\Gamma_{V_1} \to \Gamma_{V_1}/H$, then $V_1' = \theta \circ V_1$ is a valuation on R with $V_1' \geq V_1$. Since $H \cap \Gamma_{V_0} = \phi^{-1}(e)$, V_1' extends V_0'.

Let $V' = \theta' \circ V_1$ be an extension of V_0'. Then $\theta'^{-1}(e) \cap \Gamma_{v_0} = \phi^{-1}(e)$ so $H \subseteq \theta'^{-1}(e)$. That is $V' \geq V_1'$. The linear order property now follows from 2.6 and 2.9.

Now suppose $(\Gamma_{v_1}/\Gamma_{v_0}) \setminus \{0\}$ is torsion and $\theta'(\alpha) = e$. Then there is an integer $n > 0$ with $\alpha^n \in \Gamma_{v_0}$, so $\alpha^n \in \phi^{-1}(e)$. If $\alpha \geq e$, then $\alpha^n \geq \alpha \geq e$ so $\alpha \in H$, while if $e \geq \alpha$, then $(\alpha^n)^{-1} \geq \alpha^{-1} \geq e$ so $\alpha^{-1} \in H$. Thus $\theta'^{-1}(e) = H$ and $V' = V_1'$.

<u>PROPOSITION 2.20</u>. Let V_1, V_1' be valuations on R with $V_1' \geq V_1$. If $V_1 \mid_K$ is a valuation on K, then so is $V_1' \mid_K$ and $V_1' \mid_K \geq V_1 \mid_K$.

<u>PROOF</u>. Let $V_1' = \phi \circ V_1$, where ϕ is an order homomorphism of Γ_{v_1} onto $\Gamma_{v_1'}$. Then $V_1(K)$ a valuation semi-group gives $V_1'(K) = \phi \circ V_1(K)$ a valuation semi-group. Since $V_1' \mid_K = \phi \circ V_1 \mid_K$, $V_1' \mid_K \geq V_1 \mid_K$.

<u>PROPOSITION 2.21</u>. Let V_0, V_0' be valuations on K with $V_0' \geq V_0$ and V_1, V_1' be corresponding extensions to R with $V_1' \geq V_1$. Then the induced valuation (V_1', V_1) is an extension of the induced valuation (V_0', V_0).

<u>PROOF</u>. $A_{v_0'}/P_{v_0'} = A_{v_1'} \cap K / P_{v_1'} \cap K \cong$

$(A_{V_0'} + P_{V_1'})/P_{V_1'} \subset A_{V_1'}/P_{V_1'}$, so the proposition is meaningful. Using the fact that V_1' extends V_0' and 2.11 one sees that the solid part of the diagram

$$\begin{array}{ccccc}
(A_{V_0'} \setminus P_{V_0'}) & & & & \\
 & \searrow & & & \\
\Big\downarrow & & \{(A_{V_0'}/P_{V_0'}) \setminus \{P_{V_0'}\}\} & \rightarrow & \{(\Gamma_{V_0}/H_{P_{V_0}}) \setminus \{H_{P_{V_0}}\}\} \\
 & & \Big\downarrow & & \vdots\downarrow \\
 & & \{(A_{V_1'}/P_{V_1'}) \setminus \{P_{V_1'}\})\} & \rightarrow & \{(\Gamma_{V_1}/H_{P_{V_1'}}) \setminus \{H_{P_{V_1'}}\}\} \\
 & \nearrow & & & \\
(A_{V_1'} \setminus P_{V_1'}) & & & &
\end{array}$$

commutes. This induces the dotted part. The proposition is that the zeros of the inside parts can be included and the diagram will remain commutative. This is clear.

PROPOSITION 2.22. Let V_0 be a valuation on K, V_1, V_2 dependent extensions of V_0 to R. Then there is a valuation V' of R with $V' \geq V_1$ and $V' \geq V_2$ such that the induced valuations (V,V_1) and (V',V_2) are independent extensions of $(V'\mid_K, V_0)$.

PROOF. There is a $V' \geq V_i$, $i = 1, 2$ with (V',V_1) and (V',V_2) independent by 2.12. $V'\mid_K$ is a valuation on K and $V'\mid_K \geq V_0$ by 2.20 and (V',V_i) extends $(V'\mid_K, V_0)$, $i = 1, 2$ by 2.21.

3. THE INVERSE PROPERTY, APPROXIMATION THEOREMS

Section I

A key fact about fields that is indispensible in proving theorems about valuations is that the set of all valuations on a field satisfy:

DEFINITION 3.1. We say that a set $\mathbb{A}$ of valuations on a ring R has the inverse property if for every x in R there is an x' in R such that $V(xx') = e$ whenever V is in $\mathbb{A}$ and $V(x) \neq 0$. $\mathbb{A}$ is said to have the strong inverse property if for every x in R there is an x' in R with $V(xx' - 1) < e$ whenever V is in $\mathbb{A}$ and $V(x) \neq 0$.

Note that {V} has the strong inverse property iff (A_V, P_V) is a prime pair of R.

PROPOSITION 3.2. Let $\mathbb{A}$ be a set of valuations on R which has the inverse property, $\mathbb{A}'$ a set of valuations on R such that for every V' in $\mathbb{A}'$ there is a V in $\mathbb{A}$ with $V' \geq V$. Then $\mathbb{A} \cup \mathbb{A}'$ has the inverse property. In particular, $\mathbb{A}'$ has the inverse property.

PROOF. Let $x, x' \in R$ with $V(xx') = e$ whenever $V \in \mathbb{A}$ with $V(x) \neq 0$. Let $V' \in \mathbb{A}'$ and suppose $V' \geq V$, $V \in \mathbb{A}$ and $V'(x) \neq 0$. Then $V(x) \neq 0$ by 2.10 so $V(xx') = e$. Then $xx' \in A_V \setminus P_V \subset A_{V'} \setminus P_{V'}$ so $V'(xx') = e$.

PROPOSITION 3.3. Let $\mathbb{A}$ be a set of valuations on R with the inverse property and V' a valuation on R such that $V' \geq V$ for all V in $\mathbb{A}$. Then $\mathbb{A}^* = \{(V',V) \mid V \in \mathbb{A}\}$ has the inverse property.

PROOF. Let ρ be the natural map $A_{V'} \to A_{V'}/P_{V'}$. For $x \in A_{V'}$, let $V(xx') = e$ whenever $V \in \mathbb{A}$, $V(x) \neq 0$. Since $(V',V)(\rho(x)) = V(x)$ if $x \in A_{V'} \setminus P_{V'}$; $(V',V)(\rho(x)) = 0$ if $x \in P_{V'}$; we have $(V',V)(\rho(xx')) = e$ if $(V',V)(\rho(x)) \neq 0$. Thus it remains only to show that $x' \in A_{V'}$ if $x \in A_{V'} \setminus P_{V'}$. Since $xx' \in A_V \setminus P_V \subset A_{V'} \setminus P_{V'}$, this follows by 1.7.

In general the set of all valuations on a ring does not satisfy the inverse condition. In order to discover some sets which do, some preliminary results are needed.

PROPOSITION 3.4. Let V be a valuation on a ring R, $a,b \in R$ with $V(a) \neq V(b)$. Then $V(a + b) = \max\{V(a),V(b)\}$.

PROOF. Without loss of generality, we may assume $V(a) > V(b)$. Then $V(a) = V(a' + b - b) \leq \max\{V(a+b),V(b)\}$ $\leq \max\{V(a),V(b)\} = V(a)$, so $\max\{V(a+b),V(b)\} = V(a+b) = V(a)$.

COROLLARY 3.5. Let V be a valuation on a ring R, $a_i \in R$, $i = 1, 2, \cdots, m$. If $V(\sum_{i=1}^{n} a_i) < \max V(a_i)$, then $V(a_j) = \max V(a_i) = V(a_k)$ for some $j \neq k$.

PROOF. Let $V(a_j) = \max V(a_i)$. Then since $V(\sum_{i=1}^{n} a_i)$

$= V(\sum_{\substack{i=1 \\ i \neq j}}^{n} a_i + a_j) \leq \max\{V(\sum_{\substack{i=1 \\ i \neq j}}^{n} a_i), V(a_j)\}$,

$V(\sum_{\substack{i=1 \\ i \neq j}}^{n} a_i) = V(a_j)$ by 3.4. But $V(\sum_{\substack{i=1 \\ i \neq j}}^{n} a_i) \leq \max_{i \neq j} V(a_i)$, so

$\max_{i \neq j} V(a_i) \geq V(a_j)$, that is $V(a_k) = \max_{i \neq j} V(a_i) = V(a_j)$ for some $k \neq j$.

COROLLARY 3.6. Let V be a valuation on a ring R, $a_i \in R$, $i = 1, 2, \cdots, n, n+1, \cdots, k$, with $V(a_i) = 0$ for $n < i \leq k$. Then $V(\sum_{i=1}^{k} a_i) = V(\sum_{i=1}^{n} a_i)$.

PROOF. $V(\sum_{i=1}^{k} a_i) = V(\sum_{i=1}^{n} a_i + \sum_{i=n+1}^{k} a_i) \leq$

$\max\{V(\sum_{i=1}^{n} a_i), V(\sum_{i=n+1}^{k} a_i)\} = V(\sum_{i=1}^{n} a_i)$. The last equality

holds since $V(\sum_{i=n+1}^{k} a_i) = 0$. Equality now follows from 3.2.

Section II

For the remainder of this chapter, R is assumed to be an extension of a ring K and V_0 a valuation on K which has extensions to R. If V_α is any extension of V_0 to R, we will consider Γ_{V_0} as a sub-semi-group of Γ_{V_α}.

PROPOSITION 3.7. Let $\mathbb{A}$ be a set of valuations on R extending V_0, δ an ideal of R contained in $\bigcap\{\sigma_V \mid V \in \mathbb{A}\}$, such that $\delta \bigcap K = \sigma_{V_0}$. If $x \in R$ has $x + \delta$ algebraic over K/σ_{V_0}, then there is an $x' \in R$ with $V(xx') = e$ for all $V \in \mathbb{A}$ with $V(x) \neq 0$. If (A_{V_0}, P_{V_0}) is a prime pair of K, then x' may be chosen so that $V(xx' - 1) < e$.

PROOF. Note that $V(t) = 0$ for all $t \in \delta$, $V \in \mathbb{A}$. If $x + \delta$ is algebraic over K/σ_{V_0}, then there are $a_i \in K$, $t \in \delta$ with $a_r \notin \delta$ and $\sum_{i=0}^{n} a_i x^i = t$, $(V(a_n) \neq 0)$. Let $s = \min\{i \mid V(a_i) \neq 0\}$.

Then for $V \in \mathbb{A}$, $0 = V(t) = V(\sum_{i=0}^{n} a_i x^i) = V(\sum_{i=s}^{n} a_i x^i)$ $= V(x^s)V(\sum_{i=s}^{n} a_i x^{i-s})$. Thus if $V(x) \neq 0$, then $V(\sum_{i=s}^{n} a_i x^{i-s}) = 0$ $= V(\sum_{i=s+1}^{n} a_i x^{i-s} + a_s) < \max\{V(\sum_{i=s+1}^{n} a_i x^{i-s}), V(a_s)\}$, so by 3.4, $V(\sum_{i=s+1}^{n} a_i x^{i-s}) = V(a_s) = V(x)V(\sum_{i=s+1}^{n} a_i x^{i-s-1})$.

Choose $a' \in K$ with $V_0(a'a_s) = e$, $(V_0(a'a_s + 1) < e$ if (A_{V_0}, P_{V_0}) is a prime pair of K). Then with $x' = a' \cdot \sum_{i=s+1}^{n} a_i x^{i-s-1}$, $V(xx') = e$ whenever $V \in \mathbb{A}$ with

$V(x) \nmid 0$.

If (A_{v_0}, P_{v_0}) is a prime pair, $V(x) \nmid 0$, $V \in \mathbb{A}$, then $V(xx' + a'a_s) = V(a')V(\sum_{i=s}^{n} a_i x^i) = 0$ so by 3.6, $V(xx' - 1)$ $= V(xx' - 1 - (xx' + a'a_s)) = V(a'a_s + 1) < e$.

COROLLARY 3.8. Let $\mathbb{A}$ be a set of valuations on R extending V_0; $\delta = \bigcap\{\sigma_V \mid V \in \Lambda\}$, and suppose R/δ is algebraic over $K/K\cap\delta$. Then $\mathbb{A}$ has the inverse property; $\mathbb{A}$ has the strong inverse property if (A_{v_0}, P_{v_0}) is a prime pair of K.

PROOF. This is clear by 3.7. Note that $K \cap \delta = \sigma_{v_0}$.

If V extends V_0, then there is a natural homomorphism $\rho : \Gamma_V \to \{(\Gamma_V \setminus \{0\})/(\Gamma_{v_0} \setminus \{0\})\} \cup \{0\}$, namely $\rho(\alpha) = \alpha(\Gamma_V \setminus \{0\})$. Rather than carry the zeros, we denote $\rho(\Gamma_V)$ by Γ_V/Γ_{v_0} and $\rho(x)$ by $x\Gamma_{v_0}$. We say $\alpha\Gamma_{v_0}$ is torsion if $(\alpha\Gamma_{v_0})^n$ is $e\Gamma_{v_0}$ or $0\Gamma_{v_0}$ for some $n > 0$, and that Γ_V/Γ_{v_0} is torsion if every element is torsion. Note that $\alpha\Gamma_V$ is torsion iff $\alpha^n \in \Gamma_{v_0}$ for some integer $n > 0$.

With this notation, we have a companion proposition to 3.8.

PROPOSITION 3.9. Let $\mathbb{A}$ be a set of valuations on R extending V_0, $\delta = \bigcap\{\sigma_V \mid V \in \mathbb{A}\}$, and suppose R/δ is algebraic over $K/K \cap \delta$. Then Γ_V/Γ_{V_0} is torsion for all $V \in \mathbb{A}$.

PROOF. This is immediate from 3.10.

PROPOSITION 3.10. Let V extend V_0, δ be an ideal of R with $\sigma_{V_0} \subset \delta \subset \sigma_V$. Let $x \in R$ with $x + \delta$ algebraic over $K/K \cap \delta$. Then $x\Gamma_{V_0}$ is torsion.

PROOF. If $V(x) = 0$, there is nothing to show, so suppose $V(x) \neq 0$. Then $\exists\, a_i \in K$, $t \in \delta$, $a_r \notin \delta$ with $\sum_{i=0}^{r} a_i x^i = t$. Since $V(a_r x^r) \neq 0$, we have $0 = V(t)$

$= V(\sum_{i=0}^{r} a_i x^i) < \max\{V(a_i x^i)\}$, so by 3.5, $V(a_i x^i)$

$= \max\{V(a_i x^i)\} = V(a_j x^j) \neq 0$ for some $i \neq j$.

Assume $i > j$ and let $V(x') = V(x)^{-1}$, $V(a') = V(a_i)^{-1}$. Then $V(x^{i-j}) = V(a_i x^i)V(x')^j V(a') = V(a_j x^j)V(x')^j V(a')$ $= V(a_j)V(a') \in \Gamma_{V_0}$.

PROPOSITION 3.11. Let V be an extension of V_0, $V' \geq V$ and $V_0' = V'|_K$. If Γ_V/Γ_{V_0} is torsion, then so is $\Gamma_{V'}/\Gamma_{V_0'}$ and $\Gamma_{(V',V)}/\Gamma_{(V_0',V_0)}$.

PROOF. Let $\phi: \Gamma_V \rightarrow \Gamma_{V'}$ be the homomorphism such that $V' = \phi \circ V$. Then $V_{0'} = \phi \circ V_0$, $\Gamma_{(V',V)} = (\ker \phi) \cup \{0\}$

and $\Gamma_{(v_0',v_0)} = \{(\ker\phi)\cap\Gamma_{v_0}\}\cup\{0\} = \Gamma_{(v',v)}\cap\Gamma_{v_0}$.

If $\phi(\alpha)\in\Gamma_{v}'$, then $\alpha^n\in\Gamma_{v_0}$ for some $n > 0$ so $\phi(\alpha^n) = \phi(\alpha)^n\in\Gamma_{v_0'}$ so $\Gamma_{v_0'}/\Gamma_{v_0'}$ is torsion. If $\alpha\in\Gamma_{(v',v)}$ then $\alpha^n\in\Gamma_{v_0}\cap\Gamma_{(v',v)} = \Gamma_{(v_0',v_0)}$ for some $n > 0$ so $\Gamma_{(v',v)}/\Gamma_{(v_0',v_0)}$ is torsion.

A trivial but useful remark is

REMARK 3.12. If V is an extension of V_0 and Γ_v/Γ_{v_0} is torsion, then $V(R) = \{e,0\}$ iff $V_0(K) = \{e,0\}$.

REMARK 3.13. If R is integral over K, δ any ideal of R, then R/δ is integral (hence algebraic) over $K/K\cap\delta$.

Section III

In this section we assume R is an extension of K, V_0 is a valuation on K and $\mathbb{A}$ is a set of extensions of V_0 to K with the inverse property and such that Γ_v/Γ_{v_0} is torsion for each $V\in\mathbb{A}$. In some of the results we also require $P_v\not\subseteq P_{v'}$ if $V, V'\in\mathbb{A}$ and $V\neq V'$. The following proposition indicates the effect of this last restriction.

PROPOSITION 3.14. Let V_1 and V_2 be distinct elements of $\mathbb{A}$ with $P_{v_1}\subseteq P_{v_2}$. Then P_{v_0} is an ideal of K and R is not integral over K.

PROOF. If P_{v_0} is an ideal of K then P_{v_1} and P_{v_2} are ideals of R by 3.12. Then $A_{v_1} = A_{v_2} = R$, and if R were integral over K we would also have $P_{v_1} = P_{v_2}$ (see [5], page 259), contradicting P_{v_1} and P_{v_2} distinct.

It remains only to show that if P_{v_0} is not an ideal of K then $P_{v_1} \nsubseteq P_{v_2}$.

If P_{v_0} is not an ideal of K, then P_{v_1} and P_{v_2} are not ideals of R, so by 1.6, $A_{v_1} \neq A_{v_2}$.

CASE 1. $A_{v_1} \setminus A_{v_2} \neq \phi$. Let $y \in A_{v_1} \setminus A_{v_2}$. Then $V_1(y) \leq e < V_2(y)$. Since $\Gamma_{v_j}/\Gamma_{v_0}$ is torsion, there is an integer $n > 0$, and $a \in K$ with $V_2(y^n) = V_0(a)$. Then $V_2(y) = V_2(y^{n+1}a') > e$ while $V_1(y^{n+1}a') = V_1(y^{n+1})V_1(a') < e$, since $V_0(a') < e$. Thus $y^{n+1}a' \in P_{v_1} \setminus P_{v_2}$.

CASE 2. $A_{v_2} \setminus A_{v_1} \neq \phi$. By Case 1, there is $y \in R$ with $V_1(y) > e > V_2(y)$. Then $V_1(1 + y) = V_1(y) > e$ while $V_2(1 + y) = V_2(1) = e$ so $V_1((1 + y)') < e$ while $V_2((1 + y)') = e$. Thus $(1 + y)' \in P_{v_1} \setminus P_{v_2}$.

PROPOSITION 3.15. Let $V_1, V_2, \cdots, V_n$ be distinct elements of $\mathbb{A}$ with $P_{V_i} \not\subseteq P_{V_1}$ if $i \neq 1$. Then there is an $x \in R$ with $V_1(x) \geq e$ and $V_i(x) < e$ for $i \neq 1$. Further, if P_{V_0} is not an ideal of K one can require $V_1(x) > e$.

PROOF. Case 1: P_{V_0} an ideal of K. Then P_{V_i} is a prime ideal of R, $i = 1, 2, \cdots, n$. Choose $x_i \in P_{V_i} \setminus P_{V_1}$, $i = 2, 3, \cdots, n$ and let $x = \prod_{i=2}^{n} x_i$.

Case 2: P_{V_0} not an ideal of K. Proof by induction on n.

$n = 2$, Choose $y \in P_{V_2} \setminus P_{V_1}$. Then $V_1(y) \geq e > V_2(y)$. Since $\Gamma_{V_2}/\Gamma_{V_0}$ is torsion and $\Gamma_{V_0} \neq \{0,e\}$, there is an $n > 0$ and $a \in K \setminus \mathfrak{q}_{V_0}$ with $e > V_2(a) > V_2(y^n)$. Then with $x = a'y^n$ we have $V_1(x) \geq V_1(a') > e$ while $V_2(aa') = e > V_2(x)$.

Now assume 3.15 holds for $r = n - 1$, $n > 2$. For $i = 2, 3$, choose $y_i \in R$ with $V_1(y_i) > e$ and $V_j(y_i) < e$ if $j \neq 1$ and $j \neq i$. If $V_i(y_i) \leq e$, let $x_i = y_i$, otherwise let $x_i = (1 + y_i)'y_i$.

CLAIM. $V_1(x_i) \geq e$, $V_i(x_i) \leq e$, $V_j(x_i) < e$, $j \neq 1, i$.

SUBPROOF. This is automatic if $x_i = y_i$. Otherwise

$V_1(1 + y_i) = V_1(y_i) > e$ and $V_1((1 + y_i)'y_i) = e$;

$V_i(1 + y_i) = V_i(y_i) > e$ and $V_i((1 + y_i)'y_i) = e$ if $j \neq 1,i$

$V_j(1 + y_i) = V_j(1) = e$ and $V_j((1 + y_i)'y_i) = V_j(y_i) < e$.

Thus we have $V_1(x_2x_3) \geq e$ and $V_i(x_2x_3) < e$ if $i \neq 1$. Let $z = x_2x_3$. Again since $\Gamma_{V_i}/\Gamma_{V_0}$ is torsion and $\Gamma_{V_0} \neq \{0,e\}$, there is an $n > 0$ and an $a \in K\setminus\sigma_{V_0}$ with $e > V_i(a) > V_i(z^n)$ for all $i \neq 1$ and $x = a'z^n$ has $V_1(x) > e$, $V_i(x) < e$ for all $i \neq 1$.

<u>PROPOSITION 3.16</u>. Assume P_{V_0} is not an ideal of K and $V_1, V_2, \cdots, V_n \in \mathbb{A}$ are pairwise independent. Then if $\alpha_i \in \Gamma_{V_i}\setminus\{0\}$, $i = 2, 3, \cdots, n$, there is an $x \in R$ with $V_1(x) \geq e$ and $V_i(x) < \alpha_i$, $i = 2, 3, \cdots, n$.

<u>PROOF</u>. Since $\Gamma_{V_i}/\Gamma_{V_0}$ is torsion for $i = 2, \cdots, n$, there are $n_i > 0$ with $\alpha_i^{n_i} \in \Gamma_{V_0}\setminus\{0\}$. Let $\alpha = \min\{\{e\} \cup \{\alpha_i^{n_i} \mid i = 2, \cdots, n\}\}$. It suffices to show there is an $x \in R$ with $V_1(x) \geq e$ and $V_i(x) < \alpha$, $i = 2, 3, \cdots, n$.

Let $H = \{\alpha \in \Gamma_{V_0} \mid \exists x \in R$ with $V_1(x) \geq e$, $V_i(x) < \min\{\alpha,\alpha^{-1}\}$ $i \neq 1\}$. Then $e \in H$ by 3.15, and it is easily checked that H is an isolated subgroup of Γ_{V_0}. The proposition will be established if $H = \Gamma_{V_0}\setminus\{0\}$, or equivalently, that if V_0' is the valuation determined by H, then $V_0'(K) = \{e,0\} = \Gamma_{V_0}/H$.

Since $V_0' \geq V_0$ and $\Gamma_{v_i}/\Gamma_{v_0}$ is torsion for each i, by 2.19 there is a unique $V_i' \geq V_i$ which extends V_0'', $i = 1, 2, \cdots, n$. Since the V_i are independent, either $V_i'(R) = \{e,0\}$ for some i, in which case $V_0'(K) = \{e,0\}$ by 3.12 and the proposition is established; or the V_i' are distinct.

Assume the V_i' are distinct. By 3.2 and 3.11, 3.15 applies to $V_1', V_2', \cdots, V_n'$. Thus there is an $x \in R$ with $V_1'(x) > e$ and $V_i'(x) < e$, $i = 2, 3, \cdots, n$.

There is an integer $r > 0$ and b in K with $V_i'(x^r) > V_i'(b) = V_0'(b) < e$ for $i \neq 2, 3, \cdots, n$. That is $V_i(x^r) < V_0(b) < a < V_0(b)^{-1}$ for all $\alpha \in H$, $i = 2, 3, \cdots, n$, while $V_1(x^r) > e$. This is a contradiction since then $V_0(b) \in H$, $V_0'(b) = e$. Thus $V_0'(K) = \{e,0\}$.

<u>COROLLARY 3.17</u>. (Approximation Theorem) Suppose P_{v_0} is not an ideal of K and $V_1, V_2, \cdots, V_n \in \mathbb{A}$ are pairwise independent. Then if $\alpha_i \in \Gamma_{v_i} \setminus \{0\}$, $i = 1, 2, \cdots, n$, then there is an $x \in R$ with $V_i(x) = \alpha_i$, $i = 1, 2, \cdots, n$.

<u>PROOF</u>. For each i, choose $z_i \in R$ with $V_i(z_i) = \alpha_i$. Choose $x_i \in R$ with $V_i(x_i) > e$, and for $j \neq i$, with

$V_j(x_i) < \min\{\alpha_j V_j(z_i)^{-1}, e\}$, if $V_j(z_i) \neq 0$, with $V_j(x_i) < e$ if $V_j(z_i) = 0$. Let $t_i = x_i(1 + x_i)'$. Then $V_i(t_i) = e$ and $V_j(t_i) = V_j(x_i)$ if $i \neq j$.

Now $V_i(t_i z_i) = V_i(z_i) = \alpha_i$, and if $i \neq j$, $V_j(t_i z_i)$

$$= V_j(t_i)V_j(z_i) = \begin{cases} 0 \text{ if } V_j(z_i) = 0, \\ V_j(x_i)V_j(z_i) < \alpha_j, \text{ if } V_j(z_i) \neq 0. \end{cases}$$

That is $V_j(t_i z_i) = \max_k V_j(t_k z_k)$ only if $i = j$, so

$$V_j\left(\sum_{i=1}^{n} t_i z_i\right) = V_j(t_j z_j) = \alpha_j, \; j = 1, 2, \cdots, n \text{ by } 3.2.$$

COROLLARY 3.18. (Strong Approximation Theorem) Suppose $\mathbb{A}$ has the strong inverse property and $V_1, V_2, \cdots, V_n \in \mathbb{A}$ are pairwise independent. If $a_i \in R$ have $V_i(a_i) \neq 0$, $i = 1, 2, \cdots, n$, then there is an $x \in R$ with $V_i(x) = V_i(a_i) > V_i(x - a_i)$, $i = 1, 2, \cdots, n$.

PROOF. Case 1: P_{V_0} an ideal of K. Then the P_{V_i} are maximal ideals of R so $P_{V_i} \not\subseteq P_{V_j}$ if $i \neq j$, and 3.15 applies. For each i, choose $x_i \in R$ with $V_i(x_i) = e$, $V_j(x_i) = 0$, $i \neq j$. Choose $x'_i \in A_{V_i} \setminus P_{V_i}$ with $x_i x'_i = 1 + t_i$, $t_i \in P_{V_i}$. Then $V_j(x_i x'_i a_i) = 0$ if $i \neq j$, while $V_i(x_i x'_i a_i - a_i) = V_i(a_i t_i) = 0 < V_i(a_i) = V_i(x_i x'_i a_i) = e$.

Let $x = \sum_{i=1}^{n} x_i x'_i a_i$, then $V_i(x - a_i)$

$= V_i(x_i x'_i a_i - a_i + \sum_{j \neq i} x_j x'_j a_j) = 0.$

Case 2: P_{V_0} not an ideal of K. Choose a'_i so that $V_j(a_i a'_i) = e$ whenever $V_j(a_i) \neq 0$. For each i, choose $x_i \in R$ with $V_i(x_i) > e$; $V_j(x_i) < \min\{V_j(a_j)V_j(a'_i), e\}$ if $V_j(a_i) \neq 0$, $V_j(x_i) < e$ if $V_j(a_i) = 0$. Choose $y_i \in R$ with $V_j(y_i) = V_j(1 + x_i)^{-1}$ if $V_j(1 + x_i) \neq 0$ and so that $V_i(y_i(1 + x_i) - 1) < e$.

Then $y_i(1 + x_i) = 1 + t_i$ where $V_i(t_i) < e$;

$(x_i y_i - 1)(1 + x_i) = x_i y_i (1 + x_i) - 1 - x_i = x_i t_i - 1;$

$V_i(x_i y_i - 1) V_i(1 + x_i) \leq \max V_i(x_i y_i), V_i(1) < V_i(x_i)$
$= V_i(1 + x_i)$; so $V_i(x_i y_i - 1) < e$ and $V_i(x_i y_i a_i - a_i)$
$< V_i(a_i)$.

Also if $i \neq j$, $V_j(y_i) = V_j(1 + x_i)^{-1} = V_j(1)^{-1} = e$, so $V_j(x_i y_i a_i) = V_j(x_i)V_j(a_i) < V_j(a_j)$.

Now if $x = \sum_{j=1}^{n} x_j y_j a_j$ we have $V_i(x - a_i)$

$= V_i((x_i y_i a_i - a_i) + \sum_{j \neq i} x_j y_j a_j) \leq \max\{\{V_i(x_i y_i a_i - a_i)\}$
$\cup \{V_i(x_j y_j a_j) \mid i \neq j\}\} < V_i(a_i).$

Section IV

DEFINITION 3.19. Let D be a domain, D^* its field of quotients and S an extension of D. Then the ring of quotients $S_{D \setminus \{0\}}$ (see [5]) is a vector space over D^*. Set $[S;D] = \dim_{D^*} S_{D \setminus \{0\}}$. $[S;D]$ is called the rank of S over D.

One can show using "common denominator" arguments, that if $r \leq [S;D]$, there are $a_1, a_2, \cdots, a_r \in S$ such that $\sum_{i=1}^{n} d_i a_i = 0$, $d_i \in D$ implies $d_i = 0$, $i = 1, 2, \cdots, r$. If $s > [S;D]$ and $a_1, a_2, \cdots, a_s \in S$, there are $d_i \in D$, not all zero, with $\sum_{i=1}^{s} d_i a_i = 0$. In the first case we call the a_i "independent", and in the second, "dependent".

DEFINITION 3.20. Let R be an extension of K, V_0 a valuation on K with extensions to R. Let $\mathbb{A}_0 = \{V \mid V \text{ extends } V_0 \text{ to } R\}$, and for $\mathbb{A} \subseteq \mathbb{A}_0$ let $\sigma_{\mathbb{A}} = \bigcap \{\sigma_V \mid V \in \mathbb{A}\}$. Set $n_{\mathbb{A}} = [R/\sigma_{\mathbb{A}} ; K/\sigma_{V_0}]$, and note that $\mathbb{A} \subseteq \mathbb{A}'$ gives $n_{\mathbb{A}} \leq n_{\mathbb{A}'}$.

For $V \in \mathbb{A}_0$, set $f_V = [A_b/P_V; A_{V_0}/P_{V_0}]$. $f_{V_{n_{\mathbb{A}}}}$ is called the relative degree of V (with respect to V_0). Set $e_V = (\Gamma_V : \Gamma_{V_0})$ (the index of the group $\Gamma_{V_0} \setminus \{0\}$ in $\Gamma_V \setminus \{0\}$). e_V is called reduced ramification index of V (with respect to V_0).

Note that if $n_{\mathbb{A}} < \infty$ then for each $x \in R/\sigma_{\mathbb{A}}$ the set $x, x^2, \cdots, x^{n_{\mathbb{A}}}$ is dependent over K/σ_{V_0}. Thus $R/\sigma_{\mathbb{A}}$ is algebraic over K/σ_{V_0}, so that $\mathbb{A}$ has the inverse property by 3.8 and Γ_V/Γ_{V_0} is torsion for each $V \in \mathbb{A}$ by 3.9.

<u>PROPOSITION 3.21</u>. Let R be an extension of K with $n_{\mathbb{A}} < \infty$, where $\mathbb{A} = \{V_1, V_2, \cdots, V_n\} \subset \mathbb{A}_0$. Suppose the V_i are pairwise independent and if P_{V_0} is an ideal of K, also assume $P_{V_i} \not\subseteq P_{V_j}$ for $i \neq j$. Then $\sum_{i=1}^{n} e_{V_i} f_{V_i} \leq n_{\mathbb{A}}$.

<u>PROOF</u>. First suppose P_{V_0} is not an ideal of K. For each i, $i = 1, 2, \cdots, n$, choose $y_{1i}, y_{2i}, \cdots, y_{n_i i}$ in R such that the cosets $V_i(y_{ki})\Gamma_{V_0}$ are non zero and distinct. Note that $n_i \leq e_{V_i}$. Since $\Gamma_{V_i}/\Gamma_{V_0}$ is torsion for each i, there is $\beta \in \Gamma_{V_0}$ with $0 < \beta < V_t(y_{kt})$ for all t, k, and an $\alpha_i \in \Gamma_{V_0}$ with $\alpha_i V_j(y_{ri}) < \beta$ for all $j \neq i$ and all r. By 3.16 there is an $a_i \in R$ with $V_i(a_i) = e$, $V_j(a_i) < \alpha_i$ if $i \neq j$.

Set $b_{ki} = a_i y_{ki}$. $V_i(b_{ki}) = V_i(y_{ki})$, so the cosets

$V_i(b_{ki})\Gamma_{v_0}$ are non zero and distinct, $k = 1, 2, \cdots, n_i$.

Also if $k \neq i$, $V_j(b_{ki}) = V_j(a_i)V_j(y_{ki}) < \alpha_i V_j(y_{ki}) < \beta$ $< V_t(y_{st})$ for all t, s. That is, since $V_t(y_{st}) = V_t(b_{st})$ we have:

(a) $V_j(b_{ki}) < V_t(b_{st})$ for all s,t if $i \neq j$.

Let $x_{1i}, x_{2i}, \cdots, x_{m_i i}$ be in A_{v_i} with the $x_{ki} + P_{v_i}$ linearly independent over A_{v_0}/P_{v_0}. Note that $m_i \leq f_{v_i}$.

As in the above argument, there is an $\alpha_i \in \Gamma_{v_0}$ with $\alpha_i \neq 0$ and $\alpha_i V_j(x_{ri}) < e$ if $i \neq j$. Choose $b_i \in R$ with $V_i(b_i) = e$ and $V_j(b_i) < \alpha_i$ if $i \neq j$.

Set $a_{ki} = b_i x_{ki}$. Then $V_i(a_{ki}) = e$ and if $t_k \in A_{v_0}$, then $V(\sum_{k=1}^{m_i} t_k a_{ki}) = V(b_i)V(\sum_{k=1}^{m_i} t_k x_{ki}) < e$ only if $V(\sum_{k=1}^{m_i} t_k x_{ki}) < e$, so the $a_{ki} + P_{v_0}$ are linearly independent over A_{v_0}/P_{v_0}. Also $V_j(a_{ki}) = V_j(b_i)V_j(x_{ki})$ so

(b) $V_j(a_{ki}) < e$ iff $i \neq j$, for all k.

If P_{v_0} is an ideal of K, using 3.15 (note $n_i = 1$ for all i), one can chose a_{ki}, b_{ki} with the properties described above, including (a) and (b). The arguments are similar but simpler.

The proof of the proposition will be complete if we can show that the $\sum_{i=1}^{n} n_i m_i$ elements $a_{ki} b_{ji} + \sigma_{\mathbb{A}}$ are linearly independent over K/σ_{v_0}. To show this, it suffices to show that if $\alpha_{kij} \in K$ has $V_t(\sum_{k,j,i} \alpha_{kji} a_{ki} b_{ji}) = 0$, $t = 1, 2, \cdots, n$, then $V_0(\alpha_{kji}) = 0$ for all k, i, j.

Without loss of generality, we can assume $V_0(\alpha_{111}) = \max_{ijk} V_0(\alpha_{ijk})$. We have $V_1\left(\sum_j (\sum_k \sigma_{kj1} a_{k1}) b_{j1} + \sum_{\substack{k,j \\ i<1}} \alpha_{kji} a_{ki} b_{ji}\right) = 0$, so that

(c) $V_1(\sum_j (\sum_k \alpha_{kj1} a_{k1}) b_{j1}) = V_1(\sum_{\substack{k,j \\ i>1}} \alpha_{kji} a_{ki} b_{ji})$ by 3.4.

Consider the second term of (c). For $i \neq 1$, $V_1(\alpha_{kji}) \leq V_1(\alpha_{111})$ by assumption; $V_1(a_{ki}) < e$ by (b); and $V_1(b_{ji}) < V_1(b_{j1})$ for all j, by (a). In particular then, unless $V_1(\alpha_{111}) = V_0(\alpha_{111}) = 0$, since $V_1(b_{11}) \neq 0$, one has $V_1(\alpha_{kji} a_{ki} b_{ji}) < V_1(\alpha_{111}) V_1(b_{11})$ for all k, j, when $i \neq 1$, so that $V_1(\sum_{\substack{k,j \\ i>1}} \alpha_{kji} a_{ki} b_{ji}) < V_1(\alpha_{111}) V_1(b_{11})$.

Thus if we show $V_1(\sum_j (\sum_k \alpha_{kj1} a_{k1}) b_{j1}) \geq V_1(\alpha_{111}) V_1(b_{11})$, it will follow that $V_1(\alpha_{111}) = 0$ and the proposition will be established.

Note $V_1((\sum_k \alpha_{kj1} a_{k1}) b_{j1}) = V_1(\sum_k \alpha_{kj1} a_{k1}) V_1(b_{j1})$.

<u>CLAIM D</u>. $V_1(\sum_k \alpha_{kj1} a_{k1}) = \max_k V_1(\alpha_{kj1})$.

If Claim D is true, since $\max_k V_1(\alpha_{kj1}) \in \Gamma_{V_0}$ and the $V_1(b_{j1})$ determine distinct cosets $V_1(b_{j1})\Gamma_{V_0}$, we have $V_1(\sum_k \alpha_{kj1}a_{k1})b_{j1}) \neq V_1(\sum_k \alpha_{ks1}a_{k1})V_1(b_{s1})$ if $s \neq j$. Then

$$V_1(\sum_j(\sum_k \alpha_{kj1}a_{k1})b_{j1}) = \max_j V_1((\sum_k \alpha_{kj1}a_{k1})b_{j1}) \geq V_1(\alpha_{111})V_1(b_{11})$$

by 3.5 and Claim D.

Thus all that remains is to establish D. Let $V_1(\alpha_{1j1}) = \max_k V_1(\alpha_{kj1})$. D is certainly true if $V_1(\alpha_{1j1}) = 0$, so assume $V_1(\alpha_{1j1}) \neq 0$. Let $t \in K$ with $V_1(t) = V_1(\alpha_{1j1})^{-1}$. Then $V_1(t\alpha_{1j1}) = e$ and $V_1(t\alpha_{kj1}) \leq e$ if $k \neq 1$, so $V_1(\sum_k t\alpha_{kj1}a_{k1}) < e$, since $V_1(a_{k1}) = e$ for all k.

Let ρ be the natural map $A_{V_1} \to A_{V_1}/P_{V_1}$. If $V_1(\sum_k t\alpha_{kj1}a_{k1}) < e$ then $\sum_k \rho(t\alpha_{kj1})\rho(a_{k1}) = 0$, but $t\alpha_{kj1} \in A_{V_0}$ for all k, j, $\rho(t\alpha_{1j1}) \neq 0$ and the $\rho(a_{k1})$ were chosen linearly independent over $\rho(A_{V_1})$ which gives a contradiction.

Thus $V_1(\sum_k t\alpha_{kj1}a_{k1}) = e = V_1(t)(\sum_k \alpha_{kj1}a_{k1})$, so

$$V_1(\sum_k \alpha_{kj1}a_{k1}) = V_1(t)^{-1} = V_1(\alpha_{1j1}) = \max_k V_1(\alpha_{kj1}).$$

Let V be an extension of V_0 to R and $V' \geq V$. By 2.20 $V'|_K$ is a valuation on K with $V'|_K \geq V_0$. Let $e_{V'}$ and $f_{V'}$ be the reduced ramification index and the relative degree for (V',V).

PROPOSITION 3.22. With the above notation, we have $e_{V'}e_{(V',V)} = e_V$ and $f_{(V',V)} = f_V$.

PROOF. $\Gamma_V / \Gamma_{(V',V)} \simeq \Gamma_{V'}$, $\Gamma_{V_0} / \Gamma_{(V'|_K, V_0)} \simeq \Gamma_{V'|_K}$ by 2.11, so $(\Gamma_{V'} : \Gamma_{V'|_K}) = e_{V'} = (\Gamma_V / \Gamma_{(V',V)} : \Gamma_{V_0} / \Gamma_{(V'|_K, V_0)})$

$= e_V / e_{(V',V)}$.

$f_{(V',V)} = [T;S]$ where $S = (A_{V_0} / P_{V'|_K}) / (P_{V_0} / P_{V'|_K})$

$\simeq A_{V_0} / P_{V_0}$ and $T = (A_V / P_{V'}) / (P_V / P_{V'}) \simeq A_V / P_V$,

so $f_{(V',V)} = f_V$.

PROPOSITION 3.23. Suppose R is an extension of K, $\mathbb{A} \subseteq \mathbb{A}_0$, $n_{\mathbb{A}} < \infty$ and $P_V \not\subseteq P_{V'}$ whenever V and V' are independent elements of $\mathbb{A}$. Then if $V_1, V_2, \cdots, V_n$ are distinct elements of $\mathbb{A}$, one has $\sum_{i=1}^{n} e_{V_i} f_{V_i} \leq n_{\mathbb{A}}$. In particular $\mathbb{A}$ is a finite set.

PROOF. (Note that by 3.14 the restriction $P_V \not\subseteq P_{V'}$ applies only when $\sigma_{V_0} = P_{V_0}$.) By induction on n. Proposition 3.21 gives n = 1, so assume the proposition holds for $n \geq 1$. We distinguish three cases, the first which is also covered by 3.21.

CASE 1. $V_1, V_2, \cdots, V_{n+1}$ are independent.

CASE 2. V_1 and V_2 are independent.

CASE 3. V_i and V_j are dependent for all i, j.

In case 2, assume V_1 and V_{n+1} are dependent. By 2.12, there is a valuation V_1' on R with $V_{1'} \geq V_1$ and $V_1' \geq V_{n+1}$ and (V_1', V_1), (V_1', V_{n+1}) independent. By 2.19, for i = 2, 3, $\cdots$, r, there are unique $V_i' \geq V_i$ which extend $V_1'|_K$. Let $V_{i1}, V_{i2}, \cdots, V_{is}$, be the distinct valuations thus obtained, $V_{1'} = V_{i1}$.

Now (is) $\leq$ n, and by 3.2 and 3.11 the inductive hypothesis applies to $\{V_{i1}, V_{i2}, \cdots, V_{is}\} = \mathbb{A}'$, so

$$\sum_{j=1}^{s} e_{v_{ij}} f_{v_{ij}} \leq n_{\mathbb{A}'}.$$

Let $S_{ij} = \{k \mid V_{ij} \geq V_k\}$ and let $\mathbb{A}_{ij} = \{(V_{ij}, V_k) \mid k \in S_{ij}\}$. Since V_1 and V_2 are independent each $\mathbb{A}_{ij}$ has n or fewer elements and by 3.3 and 3.11, the inductive hypothesis applies to give $\sum_{k \in S_{ij}} e_{(v_{ij}, v_k)} f_{(v_{ij}, v_k)} \leq n_{\mathbb{A}_{ij}}$ (*).

Now $\sigma_{\mathbb{A}_{ij}} = (0)$ so $n_{\mathbb{A}_{ij}} = [A_{v_{ij}} / P_{v_{ij}} ; A_{v_{ij}}|_K / P_{v_{ij}}|_K]$ $= f_{v_{ij}}$; $\sigma_{\mathbb{A}'} = \sigma_{\{V_1, V_2, \cdots, V_n\}} \supseteq \sigma_{\mathbb{A}}$ so $n_{\mathbb{A}'} \leq n_{\mathbb{A}}$. Now using 3.12 and the above, $n_{\mathbb{A}} \geq n_{\mathbb{A}'} \geq \sum_{j=1}^{s} e_{v_{ij}}$

$$\geq \sum_{j=1}^{s} e_{v_{ij}} \sum_{k \in S_{ij}} e_{(v_{ij}, v_k)} f_{(v_{ij}, v_k)}$$

$$= \sum_{j=1}^{s} \sum_{k \in S_{ij}} e_{v_{ij}} e_{(v_{ij}, v_k)} f_{(v_{ij}, v_k)}$$

$$= \sum_{i=1}^{n+1} e_{v_i} f_{v_i} .$$

This completes case 2.

For case 3, chose $V' \geq V_1$ and $V' \geq V_2$ such that (V', V_1) and (V', V_2) are independent (2.12). Continue as in case 2, noting that A_{i1} may have n + 1 elements, but that two distinct ones are independent, so case 2 allows us to get the equation (*) and complete the argument.

4. GALOIS EXTENSIONS

Section I

DEFINITION 4.1. Let R be a ring, G a finite group of automorphisms on R and $R^G = \{x \in R \mid \sigma(x) = x \text{ for all } \sigma \in G\} = K$. We say R is Galois over K with group G if either of the following conditions hold:

(1) There are $x_i, y_i \in R$ such that $\sum_{i=1}^{n} x_i \sigma(y_i) = \delta_{\sigma 1}$, where $\delta_{\sigma 1} = 1$ if $\sigma = 1$ (the identity of G) and $\delta_{\sigma 1} = 0$ if $\sigma \in G$, $\sigma \neq 1$.

(2) For every ideal δ of R and $\sigma \in G$, with $\delta \neq R$, $\sigma \neq 1$, there is an $x \in R$ with $x - \sigma(x) \notin \delta$.

For the equivalence of the above two conditions, and for the equivalence of either to the "usual" definition of "R Galois over K with group G", the interested reader is referred to [2], page 18.

For the main results of this chapter, we will need an assortment of specialized results. [2] will be quoted freely as a source of proofs.

LEMMA 4.2. If R is Galois over K with group G, then there is an $a \in R$ with $\sum_{\sigma \in G} \sigma(a) = 1$.

PROOF. See [2], page 21.

PROPOSITION 4.3. If R is Galois over K with group G, and δ is a prime ideal of K, then $R/R\delta$ is Galois over K/δ with group $\hat{G} \simeq G$.

PROOF. R is integral over K (see [2] or 4.12) so $R\delta$ is an ideal of R with $R\delta \cap K = \delta$ ([5], page 257), thus we can identify K/δ with a subring of $R/R\delta$.

For $\sigma \in G$, $\sigma(R\delta) = \sigma(R)\sigma(\delta) = R\delta$, so setting $\overline{\sigma}(x + R\delta) = \sigma(x) + R\delta$, for all $x \in R$, gives an automorphism of $R/R\delta$. The map $G \to \{\overline{\sigma} \mid \sigma \in G\} = \hat{G}$ is clearly a group homomorphism, and by (2) of 4.1, if $\sigma \in G$, $\sigma \neq 1$, there is an $x \in R$ with $\sigma(x) - x \notin R\delta$, so $\overline{\sigma} \neq \overline{1}$ and the map is one-one.

Let $\rho : R \to R/R\delta$ be the natural map. If $x_i, y_i \in R$ satisfy (1) of 4.1, then $\sum_{i=1}^{n} \rho(x_i)\overline{\sigma}(\rho(y_i)) = \rho(\sum_{i=1}^{n} x_i\sigma(y_i)) = \delta_{\overline{\sigma}\overline{1}}$, so $R/R\delta$ is Galois over $(R/R\delta)^{\hat{G}}$ with group $\hat{G}$.

Now suppose $x \in R$ and $\overline{\sigma}(\rho(x)) = \rho(x)$ for all $\overline{\sigma} \in \hat{G}$. Then for each $\sigma \in G$ there are $t_\sigma \in R\delta$ with $x = \sigma(x) + t_\sigma$. Let $a \in R$ have $1 = \sum_{\sigma \in G} \sigma(a)$ as in 4.2. Then $\sigma(a)x = \sigma(ax) + \sigma(a)t_\sigma$; $x = \sum_{\sigma \in G} \sigma(a)x = \sum_{\sigma \in G} \sigma(ax) + \sum_{\sigma \in G} \sigma(a)t_\sigma$; $\rho(x) = \rho(\sum_{\sigma \in G} \sigma(ax))$. Since $\tau(\sum_{\sigma \in G} \sigma(ax)) = \sum_{\sigma \in G} \sigma(ax)$ for all $\tau \in G$, $\sum_{\sigma \in G} \sigma(ax) \in K$ and $\rho(x) \in \rho(K) = K/\delta$. That is $(R/R\delta)^{\hat{G}} = K/\delta$.

PROPOSITION 4.4. Let R be a ring, K a subring which is a domain, R_K be the ring of quotients of R with respect to the multiplicative set $K \setminus \{0\}$. Then if σ is an automorphism of R with $\sigma(x) = x$ for all $x \in K$, there is a unique extension of σ to an automorphism $\overline{\sigma}$ on R_K. Further $\overline{\sigma}(x) = x$ for all $x \in K_K$.

PROOF. Clear. See [5] for definition and existence of R_K.

PROPOSITION 4.5. If R is a Galois over K with group G and K is a domain, then R_K is Galois over K_K with group $\widehat{G} = \{\overline{\sigma} \mid \sigma \in G\} \simeq G$.

PROOF. Clear using (1) of 4.1, and 4.4.

LEMMA 4.6. If R is Galois over K with group G and K is a field, then $\dim_K R = |G|$. ($|S|$ = number of elements in S.)

PROOF. See [2], page 27.

COROLLARY 4.7. If R is Galois over K with group G and δ is a prime ideal of K, then $[R/R\delta \,;\, K/\delta] = |G|$.

PROOF. Clear by 4.3, 4.5 and 4.6.

LEMMA 4.8. If R is Galois over K with group G and R is a domain, then G is the set of all automorphisms of R such that $\sigma(x) = x$ for all $x \in K$.

PROOF. See [2].

PROPOSITION 4.9. If R is a domain, G a finite group of automorphisms on R with $K = R^G$, then

(1) $R_K = R_R$

(2) R_K is Galois over K_K with group $\hat{G} \simeq G$.

(3) $[R;K] = |G|$

(4) Every automorphism σ of R with $\sigma(x) = x$ for all $x \in K$, is an element of G.

PROOF. Let $\hat{G} = \{\overline{\sigma} \mid \sigma \in G\}$, where $\overline{\sigma}$ is as in 4.4. Then ${R_K}^{\hat{G}} = K_K$, so R_K is an integral extension of a field, and is a domain, thus R_K is a field and $R_K = R_R$. (2) of 4.1. is then satisfied, so R_K is Galois over K_K with group $\hat{G}$. Since $|\hat{G}| = |G|$, (3) follows from 4.6 and the definition of $[R;K]$.

If σ is an automorphism of R satisfying (4), then the extension $\overline{\sigma}$ (as in 4.4) has $\overline{\sigma}(x) = x$ for all $x \in K_K$, so $\overline{\sigma} \in G$ by 4.8. But then $\sigma = \overline{\sigma}|_K \in G$.

PROPOSITION 4.10. If R is Galois over K with group G and H is a subgroup of G, then

(1) R is Galois over R^H with group H.

(2) If H is normal in G, then R^H is Galois over K with group G/H, where $(\sigma H)(x) = \sigma(x)$ for all $\sigma \in G$, $x \in R^H$.

PROOF. See [2], page 22.

PROPOSITION 4.11. Suppose R is Galois over K with group $G \neq 1$, δ is a prime ideal of R, $b \in R$ and $(bx - \sum_{\sigma \in G} \sigma(x)) \in \delta$ for all $x \in R$. Then $b \in \delta$.

PROOF. There is an $x \in R$, $\tau \in G$ with $x - \tau(x) \notin \delta$ by (1) of 4.1. But $bx - \sum_{\sigma \in G} \sigma(x) \in \sigma$;

$b\tau(x) - \sum_{\sigma \in G} \sigma(\tau(x)) = b\tau(x) - \sum_{\sigma \in G} \sigma(x) \in \delta$ gives

$b(x - \tau(x)) \in \delta$, so $b \in \delta$.

This completes the preliminaries.

Section II

For the remainder of this chapter we will assume that G is a finite group of automorphisms on a ring R, with $R^G = \{x \in R \mid \sigma(x) = x \text{ for all } \sigma \in G\} = K$. We let $|G| = n$. Let V_0 be a fixed valuation on K.

PROPOSITION 4.12. R is integral over K.

PROOF. For $a \in R$ let $f_a(x) = \prod_{\sigma \in G} (x - \sigma(a))$

$= x^n + \sum_{i=0}^{n-1} a(i)x^i$. One computes that $a(i) = \sum_{S \in \mathbb{A}_i} \prod_{\sigma \in S} \sigma(a)$, where $\mathbb{A}_i$ is the set of all subsets of G containing $n - i$ elements, and that $a(i) \in K$ for $i = 0, 1, 2, \cdots, n-1$. Since $f_a(a) = 0$, a is integral over K.

Thus V_0 has extensions to R by 2.18.

PROPOSITION 4.13. Let V be a fixed extension of V_0 to R, and for $\sigma \in G$, $x \in R$ define $V_\sigma(x) = V(\sigma(x))$. Then V_σ is a valuation on R extending V_0 and $\{V_\sigma \mid \sigma \in G\} = \{V' \mid V' \text{ is a valuation on } R, (A_{V'}, P_{V'}) \geq (A_{V_0}, P_{V_0})\}$.

Furthermore $A = \{x \in R \mid V(\sigma(x)) \leq e, \forall \sigma \in G\} = \bigcap_{\sigma\in G} A_{V_\sigma}$, is the integral closure of A_{V_0} in R;

$\{x \in R \mid V(\sigma(x)) < e, \forall \sigma \in G\} = \bigcap_{\sigma\in G} P_{V_\sigma} = \sqrt{AP_{V_0}}$;

and $\{x \in R \mid V(\sigma(x)) = 0, \forall \sigma \in G\} = \bigcap_{\sigma\in G} \sigma_{V_\sigma} = \sqrt{R\sigma_{V_0}}$.

PROOF. $V_\sigma = V \circ \sigma$ is a multiplicative homomorphism of R onto Γ_V so it is a valuation. $V_\sigma(x) = V(\sigma(x)) = V(x) = V_0(x)$ for $x \in K$, so V_σ extends V_0.

Since A_{V_σ} is integrally closed, the integral closure of A_{V_0} in R is contained in A. However from the form of f_a in 4.12, if $a \in A$, $a(i) \in A_{V_0}$, $i = 0, 1, \cdots, n-1$, so that a is integral over A_{V_0}.

It is clear that $P_{V_0} \subset \bigcap_{\sigma\in G} P_{V_\sigma}$ so that $\sqrt{AP_{V_0}} \subset \bigcap_{\sigma\in G} P_{V_\sigma}$. Conversely, by the form of f_a in 4.12, if $a \in \bigcap_{\sigma\in G} P_{V_\sigma}$, then $a(i) \in P_{V_\sigma}$, $i = 0, 1, \cdots, n-1$, so that

$a^n = -\sum_{i=0}^{n-1} a(i)a^i \in AP_{V_0}$, and $a \in \sqrt{AP_{V_0}}$. The argument that

$\bigcap_{\sigma \in G} \sigma_{V_\sigma} = \sqrt{R\sigma_{V_0}}$ is similar.

Now let $(A_{V'}, P_{V'})$ be a valuation pair of R with $(A_{V'}, P_{V'}) \geq (A_{V_0}, P_{V_0})$. By 2.18, V' extends V_0.

Now by 3.15, if $P_{V'} \nsubseteq P_{V_0} \nsubseteq P_{V'}$ for all $\sigma \in G$, there is an $x \in R$ with $V'(x) = e$, $V_\sigma(x) < e$ for all $\sigma \in G$, contradicting $\sqrt{P_{V_0} A} = \bigcap_{\sigma \in G} P_{V_\sigma} \subset P_{V'}$. Thus $P_{V'} \subseteq P_{V_\sigma}$ (or $P_{V_\sigma} \subseteq P_{V'}$) for some $\sigma \in G$. If P_{V_0} is not an ideal, this gives $V' = V_\sigma$ by 3.14. If P_{V_0} is an ideal, then so are $P_{V'}$ and P_{V_σ}, and since R is integral over K, $P_{V'} = P_{V_\sigma}$ (see [5], page 259), so $V' = V_\sigma$.

<u>COROLLARY 4.14</u>. V_0 has a finite number g of extensions and for any two extensions V and V' of V_0, $e_V = e_{V'}$ and $f_V = f_{V'}$.

<u>PROOF</u>. Since G is finite, the number of extensions is also finite by 4.13. If V and V' are two extensions of V_0, $V' = V_\sigma$ for some σ in G. The map $\hat{\sigma} : \Gamma_V \to \Gamma_{V'}$ given by $\hat{\sigma}(\alpha) = V_\sigma(V^{-1}(\alpha))$ is an isomorphism with $\hat{\sigma}(\alpha) = \alpha$ for all $\alpha \in \Gamma_{V_0}$, so $e_V = (\Gamma_V : \Gamma_{V_0}) = (\hat{\sigma}(\Gamma_V) : \hat{\sigma}(\Gamma_{V_0}))$ $= (\Gamma_{V'} : \Gamma_{V_0}) = e_{V'}$.

The map $\bar{\sigma} : A_V/P_V \to A_{V'}/P_{V'}$ given by $\bar{\sigma}(x + P_V)$

$= \sigma^{-1}(x) + \sigma^{-1}(P_V) = \sigma^{-1}(x) + P_{V'}$, is an isomorphism with $\overline{\sigma}(x + P_V) = x + P_{V'}$ for all $x \in A_{V_0}$, so $\overline{\sigma}(A_{V_0}/P_{V_0})$ $= A_{V_0}/P_{V_0}$. Thus $f_V = [A_V/P_V ; A_{V_0}/P_{V_0}]$ $= [\overline{\sigma}(A_V/P_V) ; \overline{\sigma}(A_{V_0}/P_{V_0})] = [A_{V'}/P_{V'} ; A_{V_0}/P_{V_0}] = f_{V'}$.

Let $e = (\Gamma_V : \Gamma_{V_0})$ and $f = [A_V/P_V ; A_{V_0}/P_{V_0}]$, where V is any extension of V_0. The letter e is traditional when used in this way and we rely on the context to distinguish it from $V(1)$.

We can "count" the number g of extensions of V_0. Let V be a fixed extension of V_0 and set

$$G_Z \overset{\text{def}}{=} \{\sigma \in G \mid V = V_\sigma\} = \{\sigma \in G \mid \sigma(P_V) = P_V\}.$$

For the second equality, note that $V_\sigma \neq V$ iff $V(x) < e$ and $V_\sigma(x) \geq e$ for some $x \in R$. For $\sigma, \tau \in G$, $V_\sigma = V_\tau$ iff $P_{V_\sigma} = P_{V_\tau}$ iff $\sigma^{-1}(P_V) = \tau^{-1}(P_V)$ iff $\sigma^{-1}\tau(P_V) = P_V$ iff $\sigma^{-1}\tau \in G_Z$ iff $\sigma G_Z = \tau G_Z$.

That is $g = (G : G_Z)$.

PROPOSITION 4.15. Let S be any subring of R with $K \subset S \subset R$, V an extension of V_0 to R. Then $V|_S$ is a valuation on S extending V_0 and $\{V' \mid V' \text{ a valuation on S extending } V_0\} = \{V_\sigma|_S \mid \sigma \in G\}$.

PROOF. To show $V|_S$ is a valuation on S we need to show that if $x \in S$, $V(x) \neq 0$ then there is a $y \in S$ with $V(y) = V(x)^{-1}$. Since Γ_V/Γ_{V_0} is torsion, $x \in S$, $V(x) \neq 0$, there is an $r > 0$ with $V(x^r) \in \Gamma_{V_0}$. If $a \in K$ with $V(a) = V(x^r)^{-1}$, then $V(ax^{r-1}) = V(x)^{-1}$ and $ax^{r-1} \in S$.

Let V' be an extension of V_0 to S. Then R is integral over S, so V' has an extension $\overline{V}$ to R. $\overline{V}$ is an extension of V_0 to R, so $\overline{V} = V_\sigma$ for some $\sigma \in G$. But then $V' = \overline{V}|_S = V_\sigma|_S$.

NOTATION. For the remainder of this chapter V will be a fixed extension of V_0 to R and $G_Z = \{\sigma \in G \mid V = V_\sigma\}$ as above. We will denote subgroups of G by subscripts such as G_B. We let $K_B = \{x \in R \mid \sigma(x) = x \text{ for all } \sigma \in G_B\} = R^{G_B}$, $V_B = V|_{K_B}$, $k_B = A_{V_B}/P_{V_B}$, $k_0 = A_{V_0}/P_{V_0}$, $k = A_V/P_V$, $e_B = (\Gamma_V : \Gamma_{V_B})$ and $f_B = [k ; k_B]$.

PROPOSITION 4.16. V is the unique extension of V_Z to R, and if G_A is a subgroup of G such that V is the unique extension of V_A to R, then $G_A \leq G_Z$ so that $K_Z \subseteq K_A$.

PROOF. By 4.13 and the definition of G_Z, $\{V' \mid V'$ extends V_Z to R$\} = \{V_\sigma \mid \sigma \in G_Z\} = \{V\}$. In the same way, if $G_A \leq G$, $\{V' \mid V'$ extends V_A to R$\} = \{V_\sigma \mid \sigma \in G_A\}$, and the latter set is $\{V\}$ iff $V_\sigma = V$ for all $\sigma \in G_A$ iff $G_A \leq G_Z$.

Section III

For the remainder of the chapter, the additional assumption that R is Galois over K with group G will be made.

PROPOSITION 4.17. $\Gamma_{V_Z} = \Gamma_{V_0}$ and $k_Z = k_0$. That is, $e_{V_Z} = f_{V_Z} = 1$.

PROOF. Let $V_1, V_2, \cdots, V_g$ be the extensions of V_0 to K_Z. By 4.15, each V_i is of the form $(V_Z)_\sigma$ for some $\sigma \in G$, so $e_{V_i} = e_{V_Z}$, $f_{V_i} = f_{V_Z}$ for each i. Thus

$$\sum_{i=1}^{g} e_{V_i} f_{V_i} = e_{V_Z} f_{V_Z} g \leq [K_Z / \bigcap_{i=1}^{g} \sigma_{V_i} ; K/\sigma_{V_0}]$$

$\leq [K_Z/K_Z\sigma_{V_0} ; K/\sigma_{V_0}]$. The first inequality follows from 3.23 and the second holds since $K_Z\sigma_{V_0} \subseteq \bigcap_{i=1}^{g}\sigma_{V_i}$.

Let $r = |G_Z|$. Since R is Galois over K_Z with group G_Z, $[R/R\sigma_{V_Z} ; K_Z/\sigma_{V_Z}] = r$ by 4.7. Thus there are $x_1, x_2, \cdots, x_r \in R$ with the $x_i + R\sigma_{V_0}$ linearly independent over K_Z/σ_{V_Z}. Let $e_{V_Z} f_{V_Z} g = h$. The inequality in the first paragraph gives the existence of $y_1, y_2, \cdots, y_h \in K_Z$, such that the $y_i + K_Z\sigma_{V_0}$ are linearly independent over K/σ_{V_0}. Thus the hr elements $x_i y_j + R\sigma_{V_0}$ are linearly independent over K/σ_{V_0}, and $hr \leq [R/R\sigma_{V_0} ; K/\sigma_{V_0}] = n$ by 4.7, since R is Galois over K.

That is $e_{v_Z} f_{v_Z} gr = e_{v_Z} f_{v_Z} n \leq n$, thus $e_{v_Z} = f_{v_Z} = 1$.

Since $\sigma(A_v) = A_v$ and $\sigma(P_v) = P_v$ for every $\sigma \in G_Z$, we have a natural map $\sigma \to \overline{\sigma}$ of G_Z into the group of automorphisms of $k = A_v/P_v$. We have $\overline{\sigma} = \overline{1}$ iff $\sigma(x) - x \in P_v$ for all $x \in A_v$, so $G_T \overset{\text{def}}{=} \{\sigma \in G_Z \mid \overline{\sigma} = \overline{1}\} =$ $\{\sigma \in G_Z \mid \sigma(x) - x \in P_v \text{ for all } x \in A_v\}$, is a normal subgroup of G_Z. Note that $V(x - \sigma(x)) < e$ whenever $V(x) \leq e$, gives $x - \sigma(x) \in P_v$ whenever $x \in A_v$, and this gives $\sigma(x) \in P_v$ whenever $x \in P_v$, so $G_T = \{\sigma \in G \mid V(x - \sigma(x)) < e$ for all $x \in A_v\}$.

Let $\pi = 1$ if the characteristic of k_0 is zero, and let π be the characteristic of k_0 otherwise.

For D a domain, let D^* be its field of quotients.

PROPOSITION 4.18. With the notation above, set $\hat{G} = \{\overline{\sigma} \mid \sigma \in G_Z\} \simeq G_Z/G_T$. Then

(1) k^* is purely inseparable over k_T^*,

(2) k_T^* is Galois over k_0^*,

(3) $\hat{G} \overset{\text{nat}}{\simeq} \mathrm{Aut}_{k_0} k_T$,

(4) $\hat{G} = \mathrm{Aut}_{k_0} k$,

(5) $e_{v_T} = 1$,

(6) $e = (\Gamma_v; \Gamma_{v_T})$,

(7) $f = |G_Z/G_T| \pi^r$, for some integer r.

PROOF. Let $\rho : A_V \to A_V / P_V = k$ be the natural map. Note that for $a \in A_V$, $\sigma \in G_T$, that $a - \sigma(a) \in P_V$, so that $\rho(a) = \rho(\sigma(a))$. Let $t = |G_T|$. Recall now (replacing G with G_T) the polynomial $f_a(x)$ of 4.12, and note that for $a \in A_V$ that $\rho(a(i)) = \binom{t}{i}\rho(a)^{t-i} \in k_T$, so that $\rho(f_a(x)) = (x - \rho(a))^t$. That is every element $\rho(a)$ of k is either in k_T or has a purely inseparable minimal polynomial over k_T^*. But $k^* = k_T^* k$ by 4.9, so k^* is purely inseparable over k_T^*.

Since k^* is purely inseparable over k_T^*, the restriction map $\mathrm{Aut}_{k_0^*} k^* \to \mathrm{Aut}_{k_0^*} k_T^*$ is an isomorphism, thus the restriction map $\mathrm{Aut}_{k_0} k \to \mathrm{Aut}_{k_0} k_T$ is also an isomorphism. Let $\hat{\hat{G}} = \{\bar{\sigma}|_{k_T} \mid \bar{\sigma} \in \hat{G}\}$. Let $S = k_T^{\hat{\hat{G}}}$. Then by 4.9, k_T^* is Galois over S^* with group $\hat{\hat{G}}$, so by 4.7 $[k_T;S] = [k_T^*;S] = |\hat{G}|$. Now $k_Z \subseteq S$, so $k_Z^* \subseteq S^*$, so $[k_T^*;k_Z^*] = [k_T;S^*][S^*:k_Z^*]$ $= [k_T;S][S;k_Z] = |\hat{G}|[S;k_Z] = |G_Z/G_T|[S;k_Z] = [k_T;k_Z]$.

Now R is Galois over K_Z with group G_Z and $G_T \lhd G_Z$, so K_T is Galois over K_Z with group G_Z/G_T by 4.10; so by 4.7 $|G_Z/G_T| = [K_T/K_T\sigma_{v_Z}; K_Z/\sigma_{v_Z}]$. But $[K_T/K_T\sigma_{v_Z}; K_Z/\sigma_{v_Z}] \geq (\Gamma_{v_T};\Gamma_{v_Z})[k_T;k_Z]$ by 3.23, so $|G_Z/G_T| \geq (\Gamma_{v_T};\Gamma_{v_Z})|G_Z/G_T|[S;k_Z]$, so $(\Gamma_{v_T};\Gamma_{v_Z}) = [S;k_Z] = 1$.

But $\Gamma_{v_Z} = \Gamma_{v_0}$ by 4.17, so $\Gamma_{v_T} = \Gamma_{v_0}$ (gives 5 and 6), and $[S;k_Z] = 1$, gives $S^* = k_Z^*$; but $k_0 = k_Z$ by 4.17, so

$S^* = k_0^*$ (giving 2). If σ is an automorphism of k_T with $\sigma(x) = x$ for all $x \in k_0 = k_Z$, σ can be extended to an automorphism $\overline{\sigma}$ of k_σ^* with $\overline{\sigma}(x)$ for all $x \in k_T^*$, so $\sigma \in G$ by 4.8 (which gives 3).

Since k^* has no automorphisms fixing every element of k_T^* other than the identity, we have (4). Also k^* purely inseparable over k_T^* gives $[k\ ;k_T] = \pi^r$ for some integer r. But then $f = [k;k_0] = [k;k_Z] = [k^*;k_Z^*] = [k^*;k_T^*][k_T^*;k_Z^*] = \pi^r|G_Z/G_T|$.

Section IV

Let $\rho: A_V \to A_V/P_V$ be the natural map and let A^* be the group of units of the field of quotients for A_V/P_V. Let $\Gamma^* = \Gamma_V \setminus \{0\}$. For $\alpha \in \Gamma^*$, $\sigma \in G_T$, let $(\alpha,\sigma) = \rho(\sigma(a)a')\rho(aa')^{-1}$, where $V(a) = \alpha$, $V(a') = a^{-1}$.

<u>PROPOSITION 4.19</u>. (α,σ) is independent of choice of a, a', and for all $\alpha,\beta \in \Gamma^*$, $\sigma,\tau \in G_T$ we have

(1) $(\alpha\beta,\sigma) = (\alpha,\sigma)(\beta,\sigma)$

(2) $(\alpha,\sigma\tau) = (\alpha,\sigma)(\alpha,\tau)$.

Thus we have homomorphisms

$\Gamma^* \xrightarrow{\psi} \mathrm{Hom}(G_T,A^*)$, where $\psi(\alpha)(\sigma) = (\alpha,\sigma)$,

$G_T \xrightarrow{\phi} \mathrm{Hom}(\Gamma^*,A^*)$, where $\phi(\sigma)(\alpha) = (\alpha,\sigma)$.

<u>PROOF</u>. Recall that $V(x) = V(\sigma(x))\ \forall\, \sigma \in G_T$. Thus $\phi(a)a' \in A_V$ and $\rho(\sigma(a)a') \neq 0$, so $(\alpha,\sigma) \in A^*$. Also

$\rho(x) = \rho(\sigma(x))\ \forall x \in A_V$.

Now suppose $V(a) = V(b) = \alpha$, $V(a') = V(b') = \alpha^{-1}$.

Then: $\rho(\sigma(a)a')\rho(aa')^{-1} = \rho(\sigma(a)a')\rho(ba')\rho(ba')^{-1}\rho(aa')^{-1}$

$$= \rho(\sigma(a)a')\rho(\sigma(ba'))\rho(aa')^{-1}\rho(ba')^{-1}$$

$$= \rho(\sigma(a)a'\sigma(b)\sigma(a'))\rho(aa')^{-1}\rho(ba')^{-1}$$

$$= \rho(\sigma(b)a')\rho(\sigma(aa'))\rho(aa')^{-1}\rho(ba')^{-1}$$

$$= \rho(\sigma(b)a')\rho(ba')^{-1}$$

$$= \rho(\sigma(b)a')\rho(bb')\rho(bb')^{-1}\rho(ba')^{-1}$$

$$= \rho(\sigma(b)a'bb')\rho(ba')^{-1}\rho(bb')^{-1}$$

$$= \rho(\sigma(b)b')\rho(bb')^{-1}$$

so (α,σ) is well defined.

Now let $V(a) = \alpha$, $V(b) = \beta$. Then

$$(\alpha\beta,\sigma) = \rho(\sigma(ab)a'b')\rho(aba'b')^{-1}$$

$$= \rho(\sigma(a)a')\rho(\sigma(b)b')\{\rho(aa')\rho(bb')\}^{-1}$$

$$= \rho(\sigma(a)a')\rho(aa')^{-1}\rho(\sigma(b)b')\rho(bb')^{-1}$$

$$= (\alpha,\sigma)(\beta,\sigma)$$

which gives (1).

$$(\alpha,\sigma\tau) = \rho(\sigma\tau(a)a')\rho(aa')^{-1}$$

$$= \rho(\sigma(\tau(a))a')\rho(\tau(a))a')^{-1}\rho(\tau(a)a')\rho(aa')^{-1}$$

$$= \rho(\sigma(b)b')\rho(bb')^{-1}\rho(\tau(a)a')\rho(aa')^{-1}$$

$$= (\alpha,\sigma)(\alpha,\tau),$$

where $b = \tau(a)$, $b' = a'$ in the third step. This gives (2).

It is clear by (1) and (2) that ϕ and ψ are homomorphisms.

Now $\sigma \in \ker \phi$ iff $\rho(\sigma(a)a') = \rho(aa')$ $\forall a \in R \setminus \sigma_v$

iff $V(\sigma(a)a' - aa') < e$ $\forall a \in R \setminus \sigma_v$

iff $V(\sigma(a) - a)V(a') < e$ $\forall a \in R \setminus \sigma_v$

iff $V(\sigma(a) - a) < V(a)$ $\forall a \in R \setminus \sigma_v$

So $G_V \overset{def}{=} \operatorname{Ker} \phi = \{\sigma \in G_T \mid V(\sigma(a) - a) < V(a) \ \forall a \in R \setminus \sigma_v\}$

PROPOSITION 4.20. If $\alpha^{\pi^r} \in \Gamma^*_{v_0}$ for some integer $r \geq 0$, then $\alpha \in \operatorname{Ker} \psi$.

PROOF. First suppose $\alpha \in \Gamma_{v_0}$. Then in defining (α,σ), we can chose $a,a' \in K$. Then $(\alpha,\sigma) = \rho(\sigma(a)a')\rho(aa')^{-1} = 1$, since $\sigma(a) = a$. Thus $\Gamma^*_{v_0} \subseteq \operatorname{Ker} \psi$. If $\alpha^{\pi^r} \in \operatorname{Ker} \psi$ for some $r \geq 0$, then $\psi(\alpha)$ has order π^t, for some t. But the only element of $\operatorname{Hom}(G_T, A^*)$ of order a multiple of π is 1, so $\alpha \in \operatorname{Ker} \psi$.

The finite abelian group $\Gamma^*_v / \Gamma^*_{v_0}$ may be expressed as the sum of the π group Γ_π and a group $\Gamma_{\pi'}$ with order e_0 prime to π. The above proposition shows that $\alpha \in \operatorname{Ker} \phi$ if $\alpha\Gamma^*_{v_0} \in \Gamma_\pi$, so there are induced homomorphisms

$$G_T / G_V \xrightarrow{\bar{\phi}} \operatorname{Hom}(\Gamma_{\pi'}, A^*)$$

$$\Gamma_{\pi'} \xrightarrow{\bar{\psi}} \operatorname{Hom}(G_T / G_V, A^*)$$

since $(\alpha,\sigma) = (\beta,\tau)$, whenever $\alpha \operatorname{Ker} \psi = \beta \operatorname{Ker} \psi$, or $\sigma G_V = \tau G_V$.

PROPOSITION 4.21. G_T / G_V is abelian with order prime to π and G_V is a π group.

<u>PROOF</u>. $\overline{\Phi}$ above is one-one, and since Hom$(\Gamma_{\pi'}, A^*)$ is abelian and has order prime to π, the same holds for G_T/G_V.

Now let $\sigma \in G_V$ and suppose σ has prime order q. Let $H = \{\sigma^i \mid i = 1, 2, \cdots, q\}$. Since R is Galois over R^H with group H, by Proposition 4.11, either $V(q) = 0$ (and $q = \pi$) or there is an $x \in R$ with

$V(qx - \sum_{i=1}^{q} \sigma(x)) \neq 0$. Let $y = qx - \sum_{i=1}^{q} \sigma^i(x)$ and note that

$\sum_{i=1}^{q} \sigma^i(y) = 0.$

But if $V(yy') = e$, in the second case we have

$\sum_{i=1}^{q} y'\sigma^i(y) = 0, \ 0 = \rho(\sum_{i=1}^{q} y'\sigma^i(y)) = \sum_{i=1}^{q} \rho(y'\sigma^i(y))$

$= q\rho(yy')$. Since $\rho(y'y) \neq 0$ and A_V/P_V is a domain, $\rho(q) = 0$ and $q = \pi$.

PROPOSITION 4.22. $\overline{\psi}$ is one-one.

<u>PROOF</u>. Suppose $\alpha \in \Gamma^*_V$ and $(\alpha, \sigma) = 1$ for all $\sigma \in G_T$, i.e., that $V(\sigma(a) - a) < V(a)$ for all $\sigma \in G_T$, whenever $V(a) = \alpha$. Let $V(a) = \alpha$, $y = \prod_{\sigma \in G_V} \sigma(a)$. Then $y \in R^{G_V} = K_V$ and $V(y) = \prod_{\sigma \in G_V} V(\sigma(a)) = \alpha^{\pi^u}$, where π^u is the order of G_V. Since $\alpha \in \operatorname{Ker} \psi$, so is α^{π^u}. We wish to show that $\alpha^{\pi^u} \in \Gamma_{V_0}$.

Since $G_V \triangleleft G_T$, K_V is Galois over K_T with group G_T/G_V, and $y \in K_V$ gives $\Sigma\sigma G_V(y) = \sum_{i=1}^{e'_0} \sigma_i(y) \in K_T$, where e'_0 is the order of G_T/G_V and $\sigma_i G_V$, $i = 1, 2, \cdots, e'_0$, are the distinct elements of G_T/G_V. Now e'_0 is prime to π, so $V(e'_0) = 1$. Since $\alpha \in \text{Ker } \psi$, so is α^{π^u}, hence $V(\sigma_i(y) - y) < V(y)$, $i = 1, 2, \cdots, e'_0$.

That is $\sigma_i(y) = y + t_i$ with $V(t_i) < V(y)$, $i = 1, 2, \cdots, e'_0$, so $\sum_{i=1}^{e'_0} \sigma_i(y) = e'_0 y + \sum_{i=1}^{e'_0} t_i = r \in K_T$. This gives $e'_0 y - r = \sum_{i=1}^{e'_0} t_i$; $V(e'_0 y - r) \leq \max\{V(t_i)\} < V(e'_0 y) = V(y)$. Thus $V(e'_0 y - r) < \max\{V(e'_0 y), V(r)\}$, so $V(y) = V(e'_0 y) = V(r)$. But $r \in K_T$ and $V(K_T) = \Gamma_{v_0}$, so $V(y) = \alpha^{\pi^u} \in \Gamma_{v_0}$.

Thus the map $h: G_T/G_Z \to \Gamma_{\pi'} \longrightarrow A^*$ given by the pairing (α, σ) is faithful in the sense that $h(\sigma G_Z, \overline{\alpha}) = 1$ for all $\overline{\alpha} \in \Gamma_{\pi'}$ iff $\sigma G_Z = G_Z$. Also, $h(\sigma G_Z, \overline{\alpha}) = 1$ for all $\sigma G_Z \in G_T/G_V$ iff $\overline{\alpha} = \overline{1}$. Also h takes its values in the cyclic group of e'_0th roots of unity in A^*, which is cyclic of order prime to π.

Regarding G_T/G_V as a group of characters on $\Gamma_{\pi'}$ and conversely, the theory of characters for finite abelian groups ([4], page 189) shows that G_T/G_V is the entire character group of $\Gamma_{\pi'}$ and conversely. That is, $\overline{\psi}$ and $\overline{\Phi}$ are isomorphisms, and $\Gamma_{\pi'}$ and G_T/G_V are isomorphic.

In particular, $e_0 = |\Gamma_{\pi'}| = |G_T/G_V| = e'_0$. Let $|\Gamma_\pi| = \pi^s$ (and note that $s \leq u$ by proof of Proposition 4.22 above).

PROPOSITION 4.23. Let R be a Galois extension of K with group G. Then efg divides the order of G, in fact $efg\pi^{d} = |G|$ for some integer $d \geq 0$.

PROOF.
$$\begin{aligned}|G| &= (G:G_Z)(G_Z:G_T)(G_T:G_V)(G_V:1)\\ &= g \cdot f\pi^{-r} \cdot e\pi^{-s} \cdot \pi^{u}\\ &= efg\pi^{u-r-s},\end{aligned}$$
and since $efg \leq |G|$, we must have $d = u - r - s \geq 0$.

COROLLARY 4.24. If A_{v_0}/P_{v_0} is of characteristic zero, then $efg = |G|$, $G_V = 1$ and $\Gamma^*_v/\Gamma^*_{v_0} \simeq G_T$.

PROOF. $efg = |G|$ by 4.23, since $\pi = 1$. G_V is a π group, so $G_V = 1$. Γ_π is a π group so $\Gamma_\pi = 1$, and $G_T = G_T/G_V \simeq \Gamma_{\pi'} = \Gamma^*_V/\Gamma^*_{V_0}$.

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TYPED

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