AN APPROACH TO THE IRREDUCIBLE REPRESENTATIONS OF FINITE GROUPS OF LIE TYPE THROUGH BLOCK THEORY AND SPECIAL CONJUGACY CLASSES

by

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This dissertation is concerned with the study of certain irreducible representations, over the field of complex numbers, of finite groups of Lie type, and especially with the characters afforded by these representations. The methods are based on the theory of blocks with cyclic defect groups for certain primes different from the characteristic, called special primes, for which the groups have cyclic Sylow subgroups.

To be more specific, let T be a maximal torus of a finite group G of Lie type, whose order is divisible by at least one special prime. Then a family of irreducible characters of G is constructed from the local character theory of G relative to T. In other words, the characters in the family are parameterized by certain characters of

$N_G(T)$.

Chapter I is a collection of the results on finite groups of Lie type, special conjugacy classes, block theory, and the Deligne-Lusztig theory which are used later. In chapter II, special primes are shown to exist for a wide class of tori.

In chapter III, using Brauer's theory of blocks as developed further by Dade, the corresponding families of characters are shown to exist. Suzuki's theory of special conjugacy classes (as described by G. Higman) is then employed to discuss the compatibility of the block theories relative to distinct special primes, and to investigate the properties of the irreducible characters in the above families. Indeed, if T is a maximal torus of G whose order is divisible by at least one special prime, and if θ is an irreducible representation of T, set $N = N_G(T)$ and $e = [N_\theta : T]$, where N_θ is the stabilizer of θ in N, and let X be the set of elements of T whose orders are divisible by some special prime. Then the above methods are used to show that there exist irreducible characters $\chi_1, \ldots, \chi_e$ of G and signs $\varepsilon_1, \ldots, \varepsilon_e$ such that for each i, $\chi_i|_X = (\varepsilon_i/e)\theta^N|_X$. Character values for the characters χ_i are also given, up to a particular congruence, on the other elements of T. This includes, of course, a result on character degrees.

One of the primary objectives in the study of the representations of finite groups of Lie type is the decomposition of the Deligne-Lusztig virtual characters in cases where this has not yet been accomplished. This is the subject of chapter IV. If $\underline{G}$ is a connected reductive affine algebraic group giving rise to a finite group G of Lie type, and if $\underline{T}$ is a maximal torus of $\underline{G}$ giving rise to a maximal torus T of G such that $|T|$ is divisible by some special prime, then using the previous notation, the result obtained is that $R_{\underline{T}}^{\underline{G}}(\theta) = \sum_{i=1}^{e} \varepsilon_i \chi_i$. The Deligne-Lusztig theory is then used to sharpen some of the results of chapter III.

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NOTATION

We establish here some notational conventions which we shall use throughout.

If X and Y are sets, we shall denote the containment of X as a subset of Y by $X \subseteq Y$.

Let G be a group. G^* will denote the set of non-trivial elements of G. If $x, g \in G$ and $X \subseteq G$, then $g^{-1}xg$ (respectively $\{g^{-1}yg: y \in X\}$) will be denoted exponentially by x^g (respectively X^g). If r is a prime number, then we shall call $x \in G$ an r'-element of G if the order of x in G is relatively prime to r. If G acts on a set C and $c \in C$, then by $\mathrm{Stab}_G(c)$ we shall mean the isotropy group $\{g \in G: gc = c\}$.

Assume now that G is a finite group. Then by Irr(G) we shall denote the set of irreducible characters of G over $\mathbb{C}$. If G happens to be abelian, then Irr(G) has a natural group structure (under which it is isomorphic to G). We shall emphasize this by using the symbol $G^\wedge$ to denote this group. Let H be a subgroup of G, let $g \in G$, and let χ be a class function of H over $\mathbb{C}$. Then by χ^g we shall mean the class function of H^g defined by

$$\chi^g(h^g) = \chi(h)$$

for all $h \in H$. Of course if $\chi \in \mathrm{Irr}(H)$, then $\chi^g \in \mathrm{Irr}(H^g)$.

Let K be a field. We shall denote by K* the group of units in K. Let $\underline{G}$ be an affine algebraic group over K. Then $\underline{G}^{\circ}$ will denote the connected component of the identity element of $\underline{G}$. If $F: \underline{G} \rightarrow \underline{G}$ is a morphism of affine algebraic groups and $\underline{H}$ is an F-stable closed subgroup of $\underline{G}$, then by $\underline{H}_F$ we shall mean the set $\{h \in \underline{H}: F(h) = h\}$.

CHAPTER I. PRELIMINARIES FROM GROUP THEORY AND REPRESENTATION THEORY

§1. Finite Groups of Lie Type

Given a prime number p and a positive integer α, set $q = p^{\alpha}$ and denote by $\mathbb{F}_q$ the finite field of order q, viewed as a subfield of its algebraic closure K.

Let $\underline{G}$ be a connected reductive affine algebraic group over K with affine coordinate ring A, and let $A_{\mathbb{F}_q} \leq A$ be an $\mathbb{F}_q$-rational structure such that the induced Frobenius map $F: \underline{G} \to \underline{G}$ is a homomorphism of abstract groups. It follows that F is a bijective morphism of algebraic groups such that its fixed point set $\underline{G}_F$ is finite. In particular, $\underline{G}_F$ is a finite group, which we denote by G.

(1.1) DEFINITION. Any group G arising in the above manner is called a finite group of Lie type.

We retain the preceding notation throughout. It is shown by Springer and Steinberg in [13] that F-stable maximal tori of $\underline{G}$ exist, and that if $x \in \underline{G}$ is semisimple and $F(x) = x$, then there exists an F-stable maximal torus of $\underline{G}$ containing x.

(1.2) DEFINITION. Let $\underline{T}$ be an F-stable maximal torus of $\underline{G}$. Then the abelian subgroup $\underline{T}_F$ of $G = \underline{G}_F$ is called a

<u>maximal torus</u> of G.

The maximal tori of G can be classified in the following manner. Let $\underline{T}'$ be an F-stable maximal torus of $\underline{G}$, set $\underline{N}' = N_{\underline{G}}(\underline{T}')$, and let $W(\underline{T}') = \underline{N}'/\underline{T}'$ be the corresponding Weyl group. Clearly G permutes its maximal tori by conjugacy. Denote by D the set of orbits under this action. Now since $\underline{T}'$ is F-stable, F induces in the obvious manner an endomorphism of $W(\underline{T}')$, which we shall continue to denote by F. We say that two elements $w_1, w_2 \in W(\underline{T}')$ are F-conjugate if there exists an element $w_3 \in W(\underline{T}')$ such that $w_2 = w_3 w_1 F(w_3)^{-1}$. F-conjugacy is clearly an equivalence relation, and we denote the set of equivalence classes by C. Given $w = n_w\underline{T}' \in W(\underline{T}')$, where $n_w \in \underline{N}'$, Lang's Theorem (see [13]) guarantees the existence of an element $g \in \underline{G}$ satisfying $n_w = gF(g)^{-1}$. It follows easily that $\underline{T} = g^{-1}\underline{T}'g$ is an F-stable maximal torus of $\underline{G}$, hence we obtain a maximal torus $T = \underline{T}_F$ of G. It can be shown that the G-conjugacy class of T depends only upon the F-conjugacy class of w. Our discussion therefore produces a well-defined function $f: C \rightarrow D$, which can in fact be shown to be bijective. In the following definition we abuse the language slightly since T is unique only up to G-conjugacy.

(1.3) DEFINITION. T is called the maximal torus of G obtained from $\underline{T}'$ by <u>twisting</u> by $w \in W(\underline{T}')$.

If F induces the trivial endomorphism on $W(\underline{T}')$, then the F-conjugacy classes of $W(\underline{T}')$ are just the usual conjugacy classes, so in this case the maximal tori of G are (up to G-conjugacy) in bijective correspondence with the conjugacy classes of $W(\underline{T}')$. Suppose we have singled out a fundamental system in the set of roots relative to $\underline{T}'$. If $J \subseteq W(\underline{T}')$ is the corresponding set of fundamental reflections, then $(W(\underline{T}'), J)$ is a Coxeter system, so we may speak of the conjugacy class of Coxeter elements in $W(\underline{T}')$ (see Bourbaki [1]).

(1.4) DEFINITION. Suppose F acts trivially on $W(\underline{T}')$, and let $w \in W(\underline{T}')$ be a Coxeter element. If T is a maximal torus of G obtained from $\underline{T}'$ by twisting by w, then we call T the _Coxeter torus_ of G relative to $W(\underline{T}')$.

In chapter II we shall need the following result.

(1.5) PROPOSITION. _Let_ $T = \underline{T}_F$ _be a maximal torus of_ $G = \underline{G}_F$ _obtained from_ $\underline{T}'$ _by twisting by_ $w = n\underline{T}' \in W = W(\underline{T}')$, _where_ $n \in N_{\underline{G}}(\underline{T}')$. _Assume that_

$$N_{\underline{G}}(\underline{T})_F = N_G(T), \tag{1.6}$$

and let

$$T_1 = \{t_1 \in \underline{T}' : n^{-1}t_1 n = F(t_1)\},$$
$$N_1 = \{n_1 \in N_{\underline{G}}(\underline{T}') : n^{-1}n_1 n = F(n_1)\},$$

and

$$C'(w) = \{w_1 \in W: w_1 w F(w_1)^{-1} = w\}.$$

Then T_1 *is a normal subgroup of* N_1, *and the following conditions hold:*

(a) *There exists an element* $a \in \underline{G}$ *such that* $aTa^{-1} = T_1$ *and* $aNa^{-1} = N_1$.

(b) $N_G(T)/T \cong N_1/T_1 \cong C'(w)$.

(c) *If* F *acts trivially on* W *and* $C_W(w) = \langle w \rangle$, *then there exists an element* $n_1 \in N_1$ *such that* $N_1/T_1 = \langle n_1 T_1 \rangle$ *and* $n_1^{-1} t_1 n_1 = F(t_1)$ *for all* $t_1 \in T_1$.

PROOF. Clearly T_1 is a normal subgroup of N_1. In view of (1.6), (a) and (b) follow from the discussion in chapter II of Srinivasan [14].

As for (c), suppose that F acts trivially on W and that $C_W(w) = \langle w \rangle$. Then $C'(w) = \langle w \rangle$, so by (b), $N_1/T_1 \cong \langle w \rangle$. Choose $n_1 \in N_1$ such that $n_1 T_1$ generates N_1/T_1 and set $w_1 = n_1 \underline{T}'$ in W. Then $n^{-1} n_1 n = F(n_1)$ implies that $w^{-1} w_1 w = F(w_1) = w_1$, so that w_1 lies in $C_W(w) = \langle w \rangle$. Moreover, the order of w_1 in W is equal to the order of $n_1 T_1$ in N_1/T_1, so $\langle w_1 \rangle = \langle w \rangle$. It follows that we may choose $n_1 \in N_1$ in such a way that $N_1/T_1 = \langle n_1 T_1 \rangle$ and $n_1 \underline{T}' = w = n\underline{T}'$. Thus, for all $t_1 \in T_1$, $n_1^{-1} t_1 n_1 = n^{-1} t_1 n = F(t_1)$, as desired.

The condition (1.6) is not very restrictive. Indeed, it is shown in [13] that (1.6) holds if no root relative to $\underline{T}$ contains $\underline{T}_F$ in its kernel. In case $\underline{G}$ is semisimple, Veldkamp ([17]) lists the exceptions to the latter condition.

Curtis [3], Springer-Steinberg [13], and Srinivasan [14] serve as good general references for this section.

§2. The Theory of Special Conjugacy Classes

In the remaining sections of this chapter, we discuss some results from representation theory of finite groups which will prove useful in the sequel. From now on, unless we explicitly indicate an exception, "character" will mean "complex character."

(2.1) DEFINITION. (a) Let G be a finite group and $H \leq G$ a subgroup containing elements $h_1, \ldots, h_n$ whose respective conjugacy classes in H are $\mathfrak{C}_1, \ldots, \mathfrak{C}_n$, and assume that the following conditions hold:

(i) For all i, $C_G(h_i) \leq H$.

(ii) If $i \neq j$, then h_i and h_j are not conjugate in G.

(iii) If for some i, $h \in H$ satisfies $\langle h \rangle = \langle h_i \rangle$, then $h \in \mathfrak{C}_j$ for some j.

Then $\mathfrak{C}_1, \ldots, \mathfrak{C}_n$ are called _special conjugacy classes_ of H in G.

(b) A _trivial intersection set_ (T.I. set) of G is a non-empty subset X of G such that if $g \notin N_G(X)$, then $X^g \cap X \leq \{1\}$.

(2.2) PROPOSITION (Suzuki, Higman). Let G, H, and $h_i \in C_i$ $(1 \leq i \leq n)$ be as in (2.1a). Then

(a) $X = \bigcup\{C_i: 1 \leq i \leq n\}$ is a T.I. set of G such that $N_G(X) = H$.

(b) There is a basis $\theta_1, \ldots, \theta_n$ of virtual characters of H for the $\mathbb{C}$-space of class functions of H vanishing off X.

(c) Let $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_s\}$, let $\mathrm{Irr}(H) = \{\varphi_1, \ldots, \varphi_m\}$, and set $\theta_i = \sum_{j=1}^{m} a_{ij}\varphi_j$ and $\theta_i^G = \sum_{j=1}^{s} b_{ij}\chi_j$ $(1 \leq i \leq n)$. Then

$$\sum_{k=1}^{s} b_{ik}b_{jk} = \sum_{k=1}^{m} a_{ik}a_{jk}$$

for all $i, j \in \{1, \ldots, n\}$. In particular,

$$\sum_{k=1}^{s} b_{ik}^2 = \sum_{k=1}^{m} a_{ik}^2$$

for all $i \in \{1, \ldots, n\}$.

(d) There exist unique complex numbers c_{jk} $(1 \leq j,k \leq n)$ satisfying $\varphi_i(h_j) = \sum_{k=1}^{n} c_{jk}a_{ki}$ $(1 \leq i \leq m,$ $1 \leq j \leq n)$. Moreover, the c_{jk} also satisfy $\chi_i(h_j) = \sum_{k=1}^{n} c_{jk}b_{ki}$ $(1 \leq i \leq s,\ 1 \leq j \leq n)$.

PROOF. See Dornhoff [8].

(2.3) PROPOSITION. Let X be a T.I. set in G and let $N = N_G(X)$. Let θ and γ be class functions of N vanishing off X, and assume that $\theta(1) = 0$. Then

(a) $\theta^G|_X = \theta|_X$.

(b) $(\theta^G, \gamma^G)_G = (\theta, \gamma)_N$.

PROOF. See [8].

We do not include here the standard results on exceptional character theory, because there is no need to invoke them explicitly in the sequel. We will use Dade's results on blocks with a cyclic defect group however ([6]), and his proofs do employ exceptional character theory.

§3. Block Theory

We discuss now the results we shall need later concerning Brauer's organization of Irr(G) into blocks. For reference, see Curtis and Reiner [4,5], Dornhoff [8], or Isaacs [10].

Let G be a finite group and r a prime number. Then there exists a complete Noetherian local integral domain S with Jacobson radical J(S) and quotient field L such that

(a) $r \in J(S)$.

(b) Both L and $\bar{S} = S/J(S)$ are splitting fields for every subgroup of G.

(c) The integral closure L_0 of $\mathbb{Z}$ in L may be assumed to be contained in $\mathbb{C}$.

For example, extend the r-adic valuation from $\mathbb{Q}$ to $\mathbb{Q}(\sqrt[g]{1})$, where $g = |G|$ and $\sqrt[g]{1}$ denotes the set of g^{th} roots of 1 in $\mathbb{C}$. If L is the completion of $\mathbb{Q}(\sqrt[g]{1})$ and S the corresponding valuation ring in L, then S and L satisfy the above conditions.

Let IBr(G) denote the set of irreducible Brauer characters of G. Then for each $\chi \in \mathrm{Irr}(G)$ and each $\varphi \in \mathrm{IBr}(G)$, there is a non-negative integer $d_{\chi\varphi}$ such that

$$\chi(x) = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\varphi}\varphi(x)$$

for all r'-elements x of G.

(3.1) DEFINITION. The integers $d_{\chi\varphi}$ are called the <u>decomposition</u> <u>numbers</u> of G.

Let $\mathcal{C}_1,\ldots,\mathcal{C}_n$ be the conjugacy classes of G and for each i, let $\hat{\mathcal{C}}_i$ denote the sum of the elements of $\mathcal{C}_i$ in the group algebra LG. Then $\{\hat{\mathcal{C}}_i : 1 \leq i \leq n\}$ forms an L-basis for Z(LG). For each i, fix $g_i \in \mathcal{C}_i$, and for each $\chi \in \mathrm{Irr}(G)$, define a linear map $\omega_\chi : Z(LG) \to L$ by

$$\omega_\chi(\hat{\mathcal{C}}_i) = |\mathcal{C}_i|\,\chi(g_i)/\chi(1) \qquad (1 \leq i \leq n).$$

Then ω_χ is an L-algebra homomorphism and $\omega_\chi(\hat{\mathcal{C}}_i) \in S$ for all i. Let the image of $a \in SG$ under the projection $SG \to \bar{S}G$ be denoted by $\bar{a}$ (or by a^- when it is convenient). Now for each $\chi \in \mathrm{Irr}(G)$, we may view ω_χ as an S-algebra homomorphism from Z(SG) to S. As such, it induces a well-defined $\bar{S}$-algebra homomorphism $\overline{\omega_\chi} : Z(\bar{S}G) \to \bar{S}$ via $\overline{\omega_\chi}(\bar{z}) = \overline{\omega_\chi(z)}$ where $\bar{z}$ is the element of $Z(\bar{S}G)$ represented by $z \in Z(SG)$. We may define an equivalence relation $\sim$ on Irr(G) by declaring $\chi_1 \sim \chi_2$ if and only if $\overline{\omega_{\chi_1}} = \overline{\omega_{\chi_2}}$. If, for any field k, a central character of kG is defined to be a k-algebra homomorphism from Z(kG) onto k, then $\{\overline{\omega_\chi}\colon \chi \in \mathrm{Irr}(G)\}$ is the set of all central characters of $\bar{S}G$.

(3.2) DEFINITION. An <u>r-block</u> of G is any subset $B = B(r)$ of $\mathrm{Irr}(G) \dot{\cup} \mathrm{IBr}(G)$ such that

(a) $B \cap \mathrm{Irr}(G)$ is an equivalence class under $\sim$.

(b) $B \cap \mathrm{IBr}(G) = \{\varphi \in \mathrm{IBr}(G)\colon d_{\chi\varphi} \neq 0$ for some $\chi \in B \cap \mathrm{Irr}(G)\}$.

We denote by $\mathrm{Bl}(G) = \mathrm{Bl}_r(G)$ the set of all r-blocks of G, and if $B \in \mathrm{Bl}(G)$, we denote by B' (respectively B") the set $B \cap \mathrm{Irr}(G)$ (respectively $B \cap \mathrm{IBr}(G)$).

Clearly we have disjoint unions

$$(3.3)\qquad \mathrm{Irr}(G) = \bigcup\{B'\colon B \in \mathrm{Bl}(G)\},$$
$$\mathrm{IBr}(G) = \bigcup\{B''\colon B \in \mathrm{Bl}(G)\}.$$

Moreover, there is a bijection between Bl(G) and the set of central characters of $\bar{S}G$, given by

$$(3.4)\qquad B \longleftrightarrow \overline{\omega_\chi}$$

where χ is any element of B'.

Let $B \in \mathrm{Bl}(G)$. It can be shown that there exists a class $\mathcal{C}_1$ of G such that (a) if $\chi \in B'$, then $\overline{\omega_\chi}((\hat{\mathcal{C}}_1)^-) \neq 0$ and (b) if $x \in \mathcal{C}_1$, then $(1/|G|)\sum_{\chi\in B'} \chi(1)\chi(x^{-1})^- \neq 0$. Select $x \in \mathcal{C}_1$ and let $D \in \mathrm{Syl}_r(C_G(x))$. Then the set of all D arising from B in this manner forms a conjugacy class of subgroups of G.

(3.5) DEFINITION. D is called a <u>defect group</u> of B, and $|D|$ the <u>defect</u> of B.

Now let H be a subgroup of G and let $b \in Bl(H)$. Then an irreducible character $\psi \in b'$ gives rise to an algebra homomorphism $\overline{\omega_\psi}: Z(\bar{S}H) \to \bar{S}$. Since $\{(\hat{\mathcal{C}}_i)^-: 1 \leq i \leq n\}$ forms an $\bar{S}$-basis of $Z(\bar{S}G)$, we may determine a unique $\bar{S}$-linear map $\overline{\omega_\psi}^G: Z(\bar{S}G) \to \bar{S}$ by setting

$$\overline{\omega_\psi}^G((\hat{\mathcal{C}}_i)^-) = \overline{\omega_\psi}(\textstyle\sum_{h \in \mathcal{C}_i \cap H} h)$$

for all i, where the argument of $\overline{\omega_\psi}$ is taken to be an element of $\bar{S}H$ (which plainly lies in $Z(\bar{S}H)$) in case $\mathcal{C}_i \cap H \neq \emptyset$, and is interpreted as 0 if $\mathcal{C}_i \cap H = \emptyset$. Now if $\overline{\omega_\psi}^G$ is in fact a central character of $\bar{S}G$, then (3.4) implies that $\overline{\omega_\psi}^G = \overline{\omega_\chi}$ for some $\chi \in Irr(G)$, thus determining a unique block $B \in Bl(G)$ such that $\chi \in B'$.

(3.6) DEFINITION. Whenever the map $\overline{\omega_\psi}^G$ is a central character of $\bar{S}G$, we say that b^G is <u>defined</u>, and we set it equal to the block B, which is said to be <u>induced</u> from b.

Induction of blocks is transitive: If $H_1 \leq H_2 \leq G$ and $b \in Bl(H_1)$ is such that b^{H_2} and $(b^{H_2})^G$ are both defined, then b^G is also defined, and it coincides with $(b^{H_2})^G$. Henceforth, statements such as $b^G = B$, made without further qualification, will be taken to mean both that b^G is defined, and that it is equal to B.

(3.7) PROPOSITION. (a) Let $H \leq G$ be a subgroup, let $b \in \mathrm{Bl}(H)$ be such that b^G is defined, let D_1 be a defect group of b, and let D be a defect group of b^G. Then there exists an element $g \in G$ such that $D_1{}^g \leq D$.

(b) (Brauer's Second Main Theorem) Let x be an r-element of G. Then for each $\chi \in \mathrm{Irr}(G)$ and each $\varphi \in \mathrm{IBr}(C_G(x))$, there is a uniquely determined algebraic integer $d^x_{\chi,\varphi}$ (called a higher decomposition number) such that for all $\chi \in \mathrm{Irr}(G)$ and all r'-elements $y \in C_G(x)$,

$$\chi(xy) = \sum_{\varphi \in \mathrm{IBr}(C_G(x))} d^x_{\chi,\varphi}\varphi(y).$$

Moreover, if $\varphi \in b''$ for some $b \in \mathrm{Bl}(C_G(x))$, $\chi \in B'$ for some $B \in \mathrm{Bl}(G)$, and $d^x_{\chi,\varphi} \neq 0$, then $b^G = B$.

PROOF. See [4], [8], or [10].

Let R be an r-subgroup of G, let $b \in \mathrm{Bl}(C_G(R))$, and let $y \in N_G(R)$. Then since $C_G(R) \lhd N_G(R)$, y induces a permutation of $\mathrm{Bl}(C_G(R))$ as follows: for all $\chi \in \mathrm{Irr}(C_G(R))$ and all $\varphi \in \mathrm{IBr}(C_G(R))$, it is not difficult to show that $\chi^y \in \mathrm{Irr}(C_G(R))$ and $\varphi^y \in \mathrm{IBr}(C_G(R))$. It can be argued then that there is a unique block $b_1 \in \mathrm{Bl}(C_G(R))$ such that $b_1{}' = (b')^y$ and $b_1{}'' = (b'')^y$ so we set $b^y = b_1$. In this manner, $N_G(R)$ acts on $\mathrm{Bl}(C_G(R))$. Note that if $b \in \mathrm{Bl}(C_G(R))$, then $C_G(R) \leq \mathrm{Stab}_{N_G(R)}(b)$.

(3.8) PROPOSITION (Dade). Let $B \in Bl(G)$ have non-trivial cyclic defect group D and let $|D| = r^a > 1$. If $0 \leq k \leq a$, define subgroups D_k, C_k, and N_k of G by $[D:D_k] = r^k$, $C_k = C_G(D_k)$, and $N_k = N_G(D_k)$. Then for all $k \leq a - 1$, we have $D_{k+1} \leq D_k$, $C_k \leq C_{k+1}$, and $N_k \leq N_{k+1}$, and for all k, we have $D_k \leq C_k \leq N_k$. There is at least one block b_0 of C_0 satisfying $b_0{}^G = B$. Set $E = \mathrm{Stab}_{N_0}(b_0)$, and $e = [E:C_0]$. For $1 \leq k \leq a - 1$, $b_0{}^{C_k}$ is defined, and we denote it by b_k. The following assertions hold:

(a) Let $b \in Bl(C_0)$. Then b^G is defined, and $b^G = B \Leftrightarrow b^h = b_0$ for some $h \in N_0$.

(b) If $k < a$, then $b_k{}''$ consists of a unique irreducible Brauer character φ_k, and for all $h \in N_k$, $(b_k{}^h)''$ consists of the unique irreducible Brauer character $(\varphi_k)^h$.

(c) B' contains certain distinct irreducible characters $\chi_1,\ldots,\chi_e$ of G for which the non-zero higher decomposition numbers satisfy the following: there exist signs $\varepsilon_1,\ldots,\varepsilon_e,\gamma_0,\gamma_1,\ldots,\gamma_{a-1}$ such that if $\langle x\rangle = D_k$ and $h \in N_k$, then $d^{x}_{\chi_i,(\varphi_k)h} = \varepsilon_i\gamma_k$.

(d) The ε_i and γ_k may be chosen so that $\gamma_0 = 1$. Having done this, replace G by C_{a-1}. The block b_{a-1} of C_{a-1} has defect group D, hence the foregoing results hold for C_{a-1} and b_{a-1}, giving us new signs $(\gamma_0)',\ldots,(\gamma_{a-1})'$, where we may assume that $(\gamma_0)' = 1$. Then $\gamma_0 = (\gamma_0)',\ldots,\gamma_{a-1} = (\gamma_{a-1})'$.

PROOF. See Dade [6].

(3.9) LEMMA. <u>Retaining the notation of (3.8), assume that</u> $G = C_0$. <u>Then</u> $e = 1$ <u>and</u> (3.8) <u>holds with</u> $\varepsilon_1 = \gamma_0 = \gamma_1 = \cdots = \gamma_{a-1} = 1$.

PROOF. This is Proposition 2.1 of [6].

§4. The Deligne-Lusztig Theory

We return now to the notation of §1. In [7], Deligne and Lusztig establish the existence of a set of virtual representations of $G = \underline{G}_F$ (i.e., elements of the Grothendieck group of finite dimensional representations of G over $\overline{Q_l}$, the algebraic closure of the field Q_l of l-adic numbers, where l is any prime distinct from p) parameterized by pairs $(\underline{T}, \theta)$ where $\underline{T}$ is an F-stable maximal torus of G and θ an irreducible representation of $\underline{T}_F$. For the precise definition of these representations as alternating sums, in the Grothendieck group, of certain $\overline{Q_l}$-subspaces of the l-adic cohomology groups of a particular variety, and for their properties, we refer the reader to [7] or to [14].

We shall confine our attention here to the corresponding virtual characters of G, the one associated with the pair $(\underline{T}, \theta)$ being denoted by $R_{\underline{T}}^{\underline{G}}(\theta)$ (or by $R_{\underline{T}}(\theta)$, or even by $R(\theta)$ if the references to $\underline{G}$ and to $\underline{T}$ are understood). With their work in [7], Deligne and Lusztig solved the well known Macdonald conjectures (see [11]) and provide a framework crucial to the problem of finding all irreducible representations of all finite groups of Lie type. Since each irreducible character of G is a constituent of some $R_{\underline{T}}^{\underline{G}}(\theta)$, the problem of finding all irreducible characters of G may be solved by decomposing each $R_{\underline{T}}^{\underline{G}}(\theta)$ as a $\mathbb{Z}$-linear combination of its irreducible constituents. In chapter IV this is done for particular maximal tori $\underline{T}$. The remainder of this section is devoted to the results from the Deligne-Lusztig theory which are necessary for this task.

We begin with notation and several definitions. If $\underline{T}$ and $\underline{T}'$ are F-stable maximal tori of $\underline{G}$, define $N(\underline{T}, \underline{T}')$ to be the set $\{n \in \underline{G}: \underline{T}^n = \underline{T}'\}$. Then it is easily shown that $N(\underline{T}, \underline{T}')$ is F-stable and that $T = \underline{T}_F$ acts by left translation on the set $N(\underline{T}, \underline{T}')_F$ of fixed points of $N(\underline{T}, \underline{T}')$. If $\theta \in T^\wedge$ and $n \in N(\underline{T}, \underline{T}')_F$, then $\theta^n \in (T')^\wedge$, where $T' = \underline{T}'_F$. Since θ^n depends only upon the orbit w of n in the orbit space $T\backslash N(\underline{T}, \underline{T}')_F$, we may write θ^w. Now in case $\underline{T}' = \underline{T}$, $T\backslash N(\underline{T}, \underline{T}')_F$ becomes the quotient group $N_{\underline{G}}(\underline{T})_F/T$.

(4.1) DEFINITION. If $\underline{T}$ is an F-stable maximal torus of $\underline{G}$ and $T = \underline{T}_F$, then $\theta \in T^\wedge$ is said to be in <u>general position</u> if $\{w \in N_{\underline{G}}(\underline{T})_F/T: \theta^w = \theta\} = \{1\}$.

Now let $\underline{T}$ be an F-stable maximal torus of $\underline{G}$ with $T = \underline{T}_F$, and denote by u a unipotent element of G (i.e., u is unipotent as an element of the affine algebraic group $\underline{G}$).

Then Deligne and Lusztig ([7]) have shown that $R_{\underline{T}}^{\underline{G}}(\theta)(u)$ is independent of $\theta \in T^\wedge$, i.e., that $R_{\underline{T}}^{\underline{G}}(\theta)(u) = R_{\underline{T}}^{\underline{G}}(1_T)(u)$ for all $\theta \in T^\wedge$.

(4.2) DEFINITION. Let $\underline{T}$ be as above, and let U be the set of all unipotent elements of G. Then the function $Q_{\underline{T}}^{\underline{G}}: U \to \mathbb{C}$ defined by $Q_{\underline{T}}^{\underline{G}}(u) = R_{\underline{T}}^{\underline{G}}(1_T)(u)$ for all $u \in U$ is called <u>Green's function</u> (relative to $\underline{T}$).

For any closed, connected, reductive, F-stable subgroup $\underline{H}$ of $\underline{G}$, denote by $\sigma(\underline{H})$ the common dimension of all maximal $\mathbb{F}_q$-split tori of $\underline{H}$.

(4.3) PROPOSITION (<u>Deligne and Lusztig</u>). <u>Let</u> $\underline{T}$ <u>and</u> $\underline{T}'$ <u>be</u> F-<u>stable maximal tori of</u> $\underline{G}$, <u>and let</u> $\theta \in (\underline{T}_F)^\wedge$ <u>and</u> $\theta' \in (\underline{T}'_F)^\wedge$. <u>Then the following assertions hold:</u>

(a) $(R_{\underline{T}}(\theta), R_{\underline{T}'}(\theta'))_G = \left|\{w \in \underline{T}_F \backslash N(\underline{T}, \underline{T}')_F : \theta^w = \theta'\}\right|$.

(b) $R_{\underline{T}}(\theta)$ <u>is (up to sign) an irreducible character if and only if</u> θ <u>is in general position. If this is the case, then</u> $\varepsilon R_{\underline{T}}(\theta) \in \mathrm{Irr}(G)$, <u>where</u> $\varepsilon = (-1)^{\sigma(\underline{T})}(-1)^{\sigma(\underline{G})}$.

(c) <u>If</u> $x = su$ (s <u>semisimple</u>, u <u>unipotent</u>) <u>is the Jordan decomposition in</u> $\underline{G}$ <u>of</u> $x \in G$, <u>then</u> $C_{\underline{G}}(gsg^{-1})^\circ$ <u>is reductive and</u>

$$R_{\underline{T}}^{\underline{G}}(\theta)(x) = \left|C_G(s)^\circ{}_F\right|^{-1} \sum_{g \in G} Q_{\underline{T}}^{C_{\underline{G}}(gsg^{-1})^\circ}(gug^{-1})\dot{\theta}(gsg^{-1})$$

<u>where</u> $\dot{\theta}$ <u>coincides with</u> θ <u>on</u> T <u>and is</u> 0 <u>on</u> $G \setminus T$.

(d) $R_{\underline{T}}(\theta)(x) = \theta^G(x)$ <u>for all</u> $x \in \underline{T}_F$ <u>satisfying</u> $C_{\underline{G}}(x)^\circ = \underline{T}$.

PROOF. See [7].

CHAPTER II. REGULAR SEMISIMPLE ELEMENTS AND SPECIAL PRIMES

Throughout this chapter (and succeeding ones), let $G = \underline{G}_F$ be a finite group of Lie type with notation as in §1.

§5. Definitions and Properties

If $x \in G$ is semisimple, then x is contained in some F-stable maximal torus $\underline{T}$ of $\underline{G}$, and if in addition $C_{\underline{G}}(x)^\circ = \underline{T}$, then $\underline{T}$ is clearly the unique maximal torus of $\underline{G}$ which contains x.

(5.1) DEFINITION. Let $x \in G$ be semisimple and let $\underline{T}$ be an F-stable maximal torus of $\underline{G}$ such that $x \in T = \underline{T}_F$. Set $N = N_G(T)$. Then

(a) x is called regular if $C_{\underline{G}}(x)^\circ = \underline{T}$.

(b) x is called locally regular (relative to T) if $C_N(x) = T$.

We observe that by our preceding remark, the notion of regularity is well-defined.

(5.2) LEMMA. *Let $T = \underline{T}_F$ be a maximal torus of G, and let $x \in T$. Then the following assertions hold:*

(a) *If x is locally regular (relative to T), then x is regular.*

(b) x *is locally regular* (*relative to* T) *if and only if* $C_G(x) = T$.

PROOF. (a) is proved by Springer in [12]. Plainly $C_G(x) = T$ implies that $C_N(x) = T$, so only the converse of this remains to be shown. If $C_N(x) = T$, then by (a), $C_{\underline{G}}(x)^\circ = \underline{T}$. The connected component of an affine algebraic group is a normal subgroup, so $\underline{T} \lhd C_{\underline{G}}(x)$, and it follows that $C_{\underline{G}}(x) \leq N_{\underline{G}}(\underline{T})$. Now $N_{\underline{G}}(\underline{T})_F \leq N_G(T)$, therefore since $C_G(x) = C_{\underline{G}}(x)_F$, it follows that $C_G(x) \leq N_G(T)$. Hence $C_G(x) = C_N(x) = T$, and the proof is complete.

It is implied by (5.2a) that if a semisimple element $x \in G$ is locally regular (relative to T), then $\underline{T}$ is the unique maximal torus of $\underline{G}$ which contains x. Hence the phrase "relative to T" is superfluous, and we shall omit it.

(5.3) DEFINITION. Let T be a maximal torus of G. A prime number r is called a *special prime* of G relative to T (or simply a special prime if the references to G and T are understood) if the following conditions hold:

(a) $r \mid |T|$.

(b) For all $x \in T$, $r \mid |x|$ implies that x is locally regular.

We denote by S(G,T) the set of all special primes of G relative to T.

The next result, for the proof of which the author is indebted to Gary Seitz, implies that blocks of G relative to a special prime r have cyclic defect groups, thereby enabling us to invoke (3.8).

(5.4) PROPOSITION. *Let* $G = \underline{G}_F$ *be a finite group of Lie type with maximal torus* $T = \underline{T}_F$. *Assume that* $\underline{T} \subsetneq \underline{G}$, *and let* $r \in S(G,T)$. *Then each* $R \in Syl_r(G)$ *is cyclic, and there exists a unique such* R *contained in* T.

PROOF. We may choose $x \in T$ such that $|x| = r$. Then $x \in R$ for some $R \in Syl_r(G)$. We show first that $R \leq T$. Let $1 \neq z \in Z(R)$. $C_G(x) = T$ by (5.2b) since x is locally regular. Therefore z, which centralizes x, must lie in T. Now since z has order divisible by r, z is locally regular, so $R \leq C_G(z) = T$ by (5.2b). Since T is abelian, it follows by Sylow theory that R is the unique element of $Syl_r(G)$ contained in T.

Now we show that R is cyclic, from which it follows that the same holds for all elements of $Syl_r(G)$, thus concluding the proof. Suppose that R is not cyclic, and recalling that $\underline{T} \subsetneq \underline{G}$, let $\beta: \underline{T} \to K^*$ be a root. Let $\underline{U}_\beta$ be the corresponding root group of $\underline{G}$ and $x_\beta: K \to \underline{U}_\beta$ an isomorphism of affine algebraic groups (where K is viewed additively as an affine algebraic group). Then for all $t \in \underline{T}$ and all $a \in K$, $tx_\beta(a)t^{-1} = x_\beta(\beta(t)a)$. Since $\beta(R)$ is a finite subgroup of K*, it is a cyclic subgroup, therefore since R is not cyclic there exists a non-trivial element $y \in$

$R \cap \mathrm{Ker}(\beta)$. It follows that $yx_\beta(a)y^{-1} = x_\beta(a)$ for all $a \in K$, so $\underline{U}_\beta \leq C_{\underline{G}}(y)$. Now $\underline{U}_\beta \leq C_{\underline{G}}(y)^\circ$ since $\underline{U}_\beta$ is connected. But y is locally regular, so (5.2a) implies that $\underline{U}_\beta \leq \underline{T}$, a contradiction. Hence R is cyclic.

It need not be true that all locally regular elements of T have order divisible by some $r \in S(G,T)$. For example, let $\underline{G} = SL(2,K)$ where K is the algebraic closure of $\mathbb{F}_q$ with $q = 27$, and let F be the q^{th} power map $g \mapsto g^{(q)}$ for all $g \in \underline{G}$ (where $g^{(q)}$ is the matrix obtained from g by raising each entry to the power q). Then $G = \underline{G}_F$ is SL(2,27). Let T be the Coxeter torus of G. Then T is cyclic of order $q + 1 = 2^2 \cdot 7$ (see 1.10 in chapter II of [13]) and the only elements of T which are not locally regular are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

(see Theorem 38.1 and Step 1 of its proof in [8]). Therefore, $S(G,T) = \{7\}$, and an element of T having order 4 completes our example.

Now let G = SL(2,q) with $q = 5^3$. Then as above, the Coxeter torus T is cyclic of order $q + 1 = 2 \cdot 3^2 \cdot 7$ and $S(G,T) = \{3,7\}$. Thus in general, we may have $|S(G,T)| > 1$. It is also possible however, that $S(G,T) = \emptyset$. For example, arguing as before, we see that the Coxeter torus T of G = SL(2,3) has order 2^2, but the element

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

of T, which has order divisible by 2, is not locally regular. Therefore $S(G,T) = \emptyset$. We shall see in the next section however, that this last example is exceptional.

§6. Existence of Special Primes

We give in this section conditions which lead to the existence of special primes in a variety of examples.

(6.1) DEFINITION. Given a prime number s and a positive integer a, the pair (s,a) is said to be compatible if none of the following conditions hold:

(a) $s = 2$ and $a = 1$.

(b) $s = 2$ and $a = 6$.

(c) $a = 2$ and $s = 2^m - 1$ for some positive integer $m \geq 2$.

Otherwise (s,a) is said to be incompatible.

(6.2) PROPOSITION (Zsigmondy). *If (s,a) is a compatible pair, then there exists a prime number r such that* $r \mid s^a - 1$, *but for all positive integers* $b < a$, $r \nmid s^b - 1$.

PROOF. See [18].

As in §1, let $G = \underline{G}_F$ be a finite group of Lie type where F is the Frobenius map associated with an $\mathbb{F}_q$-rational structure of $\underline{G}$, $q = p^{\alpha}$, T is a maximal torus of G, and $N = N_G(T)$. For all positive integers m, denote by $f_m(X)$ the m^{th} cyclotomic polynomial.

(6.3) PROPOSITION. <u>In order that there exist a special prime</u> r <u>of</u> G <u>relative to</u> T, <u>it is sufficient that</u> N/T <u>be cyclic with a generator</u> nT ($n \in N$) <u>of order</u> m <u>satisfying the following conditions</u>:

(a) $n^{-1}tn = t^q$ <u>for all</u> $t \in T$.

(b) $f_m(q) \mid |T|$.

(c) (p,m) <u>is a compatible pair</u>.

PROOF. Suppose $n \in N$, $N = \langle nT \rangle$, $|nT| = m$, and (a), (b), and (c) hold. Since (p,m) is compatible, (6.2) implies that there exists a prime number r such that $r \mid q^m - 1$, but for all $b < m$, $r \nmid q^b - 1$. Since $q^m - 1 = \prod f_b(q)$ where the product is taken over all positive integers b dividing m, r must divide $f_b(q)$ for some b. If $b < m$, then $r \mid q^b - 1$ contradicts the property defining r. Hence $r \mid f_m(q)$, and by (b) it follows that $r \mid |T|$.

To complete the proof, we must show that if $x \in T$ satisfies $r \mid |x|$, then $C_N(x) = T$. But if such is not the case, then $T \subsetneq C_N(x)$ since T is abelian. Thus $b = [N:C_N(x)]$ is less than m. Now by (a), the conjugacy class of N containing x is

$$\{x, x^q, \ldots, x^{q^{b-1}}\},$$

and $x^{q^b} = x$. Hence $x^{q^b-1} = 1$, and it follows that $r \mid q^b - 1$, contrary to the property defining r. Therefore x is locally regular, and the proposition is proved.

We indicate now how (6.3) can be used to establish the existence of special primes in certain cases.

First, let G be a finite (untwisted) Chevalley group over $\mathbb{F}_q$ as defined in Steinberg [15]. Then $G = \underline{G}_F$ where $\underline{G}$ is the corresponding Chevalley group over the algebraic closure K of $\mathbb{F}_q$ and F is the usual Frobenius map induced on $\underline{G}$ by the field automorphism $a \mapsto a^q$ of K. We assume for convenience that the root system associated with this group is indecomposable. The diagonal subgroup $\underline{H}$ of $\underline{G}$ is a maximal torus and F acts trivially on the Weyl group $W = W(\underline{H})$. Let $T = \underline{T}_F$ be the Coxeter torus of G relative to $W(\underline{H})$, so that T is obtained from $\underline{H}$ by twisting by a Coxeter element $w \in W$. Assume that $N_{\underline{G}}(\underline{T})_F = N_G(T)$ (see §1), and denote this group by N. We assert that if $q = p^\alpha$ is chosen so that $(p, |w|)$ is a compatible pair, then $S(G,T) \neq \emptyset$.

To prove this, we apply (1.5) (and borrow its notation) to see that

$$N/T \cong N_1/T_1 \cong \langle w \rangle,$$

and that there exists an element $n_1 \in N_1$ such that $N_1/T_1 = \langle n_1 T_1 \rangle$ and $n_1^{-1} t_1 n_1 = F(t_1)$ for all $t_1 \in T_1$. Now $F(x) = x^q$ for all $x \in \underline{H}$, so $n_1^{-1} t_1 n_1 = t_1^{\,q}$ for all $t_1 \in T_1$. Again by

(1.5), it follows that there is an element $n \in N$ such that $N/T = \langle nT \rangle$ and

$$n^{-1}tn = t^q$$

for all $t \in T$. Let $m = |w|$, and let $X(\underline{H})$ denote the character group of $\underline{H}$. By Theorem 2.10 of [14], $|T| = |P_w(q)|$, where $P_w(X)$ is the characteristic polynomial of w acting as a linear transformation on the $\mathbb{R}$-space $X(\underline{H}) \otimes_{\mathbb{Z}} \mathbb{R}$. It can be checked by consulting Carter's list (table 3 of [2]) of polynomials $P_w(X)$ for all indecomposable root systems, that $f_m(q) \,\big|\, |T|$. Now (6.3) finishes the proof of our assertion.

The algebraic groups considered above are all semisimple. However, by essentially the same discussion, $S(G,T)$ can be shown to be non-empty for certain finite groups $G = \underline{G}_F$ of Lie type where $\underline{G}$ is not semisimple. For example, let $G = GL(n,q)$, where $q = p^{\alpha}$. If K is the closure of $\mathbb{F}_q$, then $G = GL(n,K)_F$, where F is the usual Frobenius map. Let $\underline{H}$, $w \in W(\underline{H})$, and $T = \underline{T}_F$ be as in the previous discussion. Then provided that $N_{GL(n,K)}(\underline{T})_F = N_G(T)$ and that $(p, |w|)$ is compatible, $S(G,T) \neq \emptyset$. It is not difficult to show that $|T| = q^n - 1$, and that $|w| = n$ (w may be viewed as an n-cycle in S_n).

We return now to the case where $G = \underline{G}_F$ is a finite (untwisted) Chevalley group over $\mathbb{F}_q$ ($q = p^{\alpha}$), $\underline{G}$ the corresponding Chevalley group over K, and F the usual

Frobenius map. Again we assume for convenience that the associated root system is indecomposable, and again we base our classification of the maximal tori of G on the diagonal subgroup $\underline{H}$ of $\underline{G}$. Our discussion for the Coxeter torus provides us with a method for using (1.5) and (6.3) to produce other maximal tori $T = \underline{T}_F$ of G, corresponding to elements $w \in W = W(\underline{H})$, for which $S(G,T) \neq \emptyset$. Let $P_w(X)$ be the characteristic polynomial of w on $X(\underline{H}) \otimes_{\mathbb{Z}} \mathbb{R}$. Since F acts trivially on W and $F(x) = x^q$ for all $x \in \underline{H}$, verification that the following conditions hold will suffice:

(a) $N_{\underline{G}}(\underline{T})_F = N_G(T)$.

(b) $C_W(w) = \langle w \rangle$.

(c) $f_{|w|}(q) \,\big|\, P_w(q)$.

(d) $(p, |w|)$ is a compatible pair.

Conditions (b) and (c) are the important ones since the failure of (a) or (d) is incidental and rare (see the remarks at the end of §1, and see (6.1)). The Weyl group elements w for which (b) and (c) hold can be determined by using the tables in [2]. Below in tabular form, we give the verification that (b) and (c) hold for four examples where w is not a Coxeter element of W. Using Carter's notation, the second column gives the admissible diagram Γ associated with the conjugacy class of w in W.

(6.4) TABLE

root system	Γ	$\|w\|$	$\|C_W(w)\|$	$f_{\|w\|}(q)$	$P_w(q)$
E_6	$E_6(a_1)$	9	9	q^6+q^3+1	q^6+q^3+1
E_7	$E_7(a_1)$	14	14	$q^6-q^5+q^4-q^3+q^2-q+1$	$(q^6-q^5+q^4-q^3+q^2-q+1)(q+1)$
E_8	$E_8(a_1)$	24	24	q^8-q^4+1	q^8-q^4+1
E_8	$E_8(a_2)$	20	20	$q^8-q^6+q^4-q^2+1$	$q^8-q^6+q^4-q^2+1$

Using a slightly more general version of (6.3), our discussion applies to the twisted analogues of the finite Chevalley groups. Lemma 1 of Surowski in [16] suggests the manner in which (6.3) should be altered.

Our general method (6.3) for showing that $S(G,T) \neq \emptyset$ requires that $N_G(T)/T$ be cyclic. This is not an accident.

(6.5) PROPOSITION. *Let $G = G_F$ be a finite group of Lie type and let $T = T_F$ be a maximal torus of G such that $S(G,T) \neq \emptyset$. Then $N_G(T)/T$ is cyclic.*

PROOF. Let $r \in S(G,T)$, and let $x \in T$ have order r. Set $N = N_G(T)$. Then since x is locally regular (relative to T), we have

$$C_N(\langle x \rangle) = C_N(x) = T.$$

Therefore N/T is embedded in $\mathrm{Aut}(\langle x \rangle)$. But x is cyclic of prime order, so $\mathrm{Aut}(\langle x \rangle)$ is cyclic, and the result follows.

We conclude this section with the remark that $S(G,T) \neq \emptyset$ implies by (5.2a) that T contains regular elements.

CHAPTER III. MAIN RESULTS

In this chapter and in the next, let $G = \underline{G}_F$ be a finite group of Lie type. Fix a maximal torus $T = \underline{T}_F$ of G, and set $N = N_G(T)$. Assume that $\underline{T} \subsetneq \underline{G}$, and that the set $S(G,T) = \{r_1,\ldots,r_n\}$ of special primes of G relative to T is not empty. The assumption that $\underline{T} \subsetneq \underline{G}$ will be used, without further reference, as justification for invoking (5.4).

§7. Special Conjugacy Classes and Special Primes

N acts on $T^\wedge$ by conjugation since $T \lhd N$. Denote by N_θ the stabilizer in N of $\theta \in T^\wedge$.

(7.1) DEFINITION. $\theta \in T^\wedge$ is said to be <u>regular</u> if $N_\theta = T$.

Note that $N_{\underline{G}}(\underline{T})_F \leq N$, so that if $\theta \in T^\wedge$ is regular, then it is in general position. The two notions are equivalent if $N_{\underline{G}}(\underline{T})_F = N$.

For each j $(1 \leq j \leq n)$, let R_j be the unique r_j-Sylow subgroup of G contained in T whose existence is guaranteed by (5.4). Let $R = R_1 \times \cdots \times R_n$, and let Q be the unique subgroup of T satisfying $T = Q \times R$. Set $Q^\sim = \{\psi \in T^\wedge;\ R \leq \ker(\psi)\}$ and $R^\sim = \{\lambda \in T^\wedge;\ Q \leq \ker(\lambda)\}$. Let Y denote the set of regular elements in T, and $Y^\wedge$ the set of regular characters in $T^\wedge$. As remarked earlier, $Y \neq \emptyset$. Set

$$X = \{x \in T: \; r_j \,\big|\, |x| \text{ for some } j\},$$

and

$$X^\wedge = \{\theta \in T^\wedge: \; r_j \,\big|\, |\theta| \text{ for some } j\}.$$

By the definition of special prime and (5.2a), $X \trianglelefteq Y$. Each element $x \in T$ can be expressed uniquely in the form

$$x = ab \qquad\qquad (a \in Q,\ b \in R).$$

It follows from the definitions that $x \in X$ if and only if $b \neq 1$.

Analogous conditions hold in $T^\wedge$, which is isomorphic to T. Clearly $T^\wedge = Q^\sim \times R^\sim$, so that each character $\theta \in T^\wedge$ can be written uniquely in the form

$$\theta = \psi\lambda \qquad\qquad (\psi \in Q^\sim,\ \lambda \in R^\sim).$$

Moreover, $\theta \in X^\wedge$ if and only if $\lambda \neq 1_T$.

(7.2) LEMMA. $X^\wedge \trianglelefteq Y^\wedge$.

PROOF. Let $\theta = \psi\lambda \in X^\wedge$ $(\psi \in Q^\sim, \lambda \in R^\sim)$. Then $\lambda \neq 1_T$. Let $h \in N_\theta$. By (7.1), we must show that $h \in T$. Now $\psi\lambda = (\psi\lambda)^h = \psi^h\lambda^h$. $\psi^h \in Q^\sim$ and $\lambda^h \in R^\sim$ since $Q \lhd N$ and $R \lhd N$, so by uniqueness of expression, $h \in N_\lambda$, and it follows that $h^{-1} \in N_\lambda$ as well.

$R = R_1 \times \cdots \times R_n$ is cyclic because each R_j is cyclic and the orders $|R_j|$ are pairwise relatively prime. It follows that $R^{\sim}$, which is isomorphic to R, is also cyclic. Let $R = \langle x \rangle$ and $R^{\sim} = \langle \lambda_0 \rangle$. Choose i such that $0 < i < |R|$ and $\lambda = \lambda_0{}^i$. There is an integer k such that $h^{-1}xh = x^k$. It follows that $\lambda_0{}^{h^{-1}} = \lambda_0{}^k$. Therefore $\lambda_0{}^i = (\lambda_0{}^i)^{h^{-1}} = \lambda_0{}^{ik}$, so $\lambda_0{}^{ik-i} = 1_T$, whence $|R| \bigm| ik - i$. Now $x^{ik-i} = 1$, and $x^i = x^{ik} = h^{-1}x^i h$ follows. Thus $h \in C_N(x^i)$. But $C_N(x^i) = T$ since $x^i \in R^*$ by our choice of i. Therefore $h \in T$, and the proof is complete.

(7.3) LEMMA. X <u>is the union of a set of special conjugacy classes of</u> N <u>in</u> G.

PROOF. This result is a consequence of the fact that the set Y of regular elements in T is the union of a set of special classes of N in G. We prove the latter first, beginning with the observation that Y is in fact a union of conjugacy classes of N. Indeed, if $y \in Y$ and $h \in N$, then since conjugation by h induces an automorphism of the affine algebraic group $\underline{G}$, we have $C_{\underline{G}}(h^{-1}yh)^{\circ} = h^{-1}C_{\underline{G}}(y)^{\circ}h = h^{-1}\underline{T}h$. Now $h^{-1}\underline{T}h$ is an F-stable maximal torus of $\underline{G}$ containing $h^{-1}yh$, so by definition (5.1), $h^{-1}yh$ is regular, and our observation follows. Let $\{y_1, \ldots, y_m\}$ be a complete set of representatives of the classes of N contained in Y.

For each i, $\underline{T} = C_{\underline{G}}(y_i)^{\circ} \lhd C_{\underline{G}}(y_i)$, thus forcing $C_{\underline{G}}(y_i) \leq N_{\underline{G}}(\underline{T})$. It follows that $C_G(y_i) = C_{\underline{G}}(y_i)_F \leq N_{\underline{G}}(\underline{T})_F \leq N$.

Now if $g^{-1}y_i g = y_k$ for some i, some k, and some $g \in G = \underline{G}_F$, then $g^{-1}\underline{T}g = g^{-1}C_{\underline{G}}(y_i)^\circ g = C_{\underline{G}}(g^{-1}y_i g)^\circ = C_{\underline{G}}(y_k)^\circ = \underline{T}$, so that $g \in N_{\underline{G}}(\underline{T})_F \leq N$.

Finally, suppose that $\langle y \rangle = \langle y_i \rangle$ for some $y \in N$ and some i. Then $C_{\underline{G}}(y)^\circ = C_{\underline{G}}(\langle y \rangle)^\circ = C_{\underline{G}}(\langle y_i \rangle)^\circ = C_{\underline{G}}(y_i)^\circ = \underline{T}$, and it follows that y is conjugate in N to some y_k. We have shown that Y is the union of a set of special conjugacy classes of N in G.

X is clearly a union of conjugacy classes of N, and we represent these classes by elements $x_1, \ldots, x_s$ of X. In order to show that these classes form a set of special classes of N in G, we must verify that the x_i satisfy (i), (ii), and (iii) of (2.1a). But (i) and (ii) follow by inheritance from Y since $X \subseteq Y$. As for (iii), if $x \in N$ satisfies $\langle x \rangle = \langle x_i \rangle$ for some i, then $|x| = |x_i|$ implies that $r_j \,\Big|\, |x|$ for some j. Hence $x \in X$, so x is conjugate in N to some x_k. This concludes the proof of the lemma.

Now (2.2b) guarantees the existence of a basis of virtual characters of N for the $\mathbb{C}$-space of class functions of N which vanish off X. Our goal is the construction of such a basis.

Since $Q \lhd N$, N acts on $Q^\sim$ by conjugation. Fix a complete set Ω of orbit representatives for this action. $R \lhd N$, so for each $\psi \in \Omega$, N_ψ acts on $R^\sim \setminus \{1_T\}$ by conjugation. Fix a complete set Λ_ψ of orbit representatives for this action.

(7.4) DEFINITION. For each $\psi \in \Omega$, and each $\lambda \in \Lambda_\psi$, set $\theta_{\psi,\lambda} = \psi^N - (\psi\lambda)^N$.

It is plain that N acts by conjugation on $X^\wedge$.

(7.5) LEMMA. $\{\psi\lambda : \psi \in \Omega,\ \lambda \in \Lambda_\psi\}$ _is a complete set of representatives for the orbits of_ $X^\wedge$ _under the action of_ N.

PROOF. If $\psi \in \Omega$ and $\lambda \in \Lambda_\psi$, then $\lambda \neq 1_T$, so that by an earlier comment, $\psi\lambda \in X^\wedge$. Now let $\psi_1, \psi_2 \in \Omega$, $\lambda_1 \in \Lambda_{\psi_1}$, and $\lambda_2 \in \Lambda_{\psi_2}$, and suppose that $(\psi_1\lambda_1)^h = \psi_2\lambda_2$ for some $h \in N$. Then $\psi_1{}^h\lambda_1{}^h = \psi_2\lambda_2$, hence by uniqueness of expression, $\psi_1{}^h = \psi_2$ and $\lambda_1{}^h = \lambda_2$. Therefore $\psi_1 = \psi_2$, forcing $h \in N_{\psi_1}$. Now by our definition of Λ_{ψ_1}, it follows that $\lambda_1 = \lambda_2$. We have shown that $\psi_1\lambda_1$ and $\psi_2\lambda_2$ represent distinct orbits unless $\psi_1 = \psi_2$ and $\lambda_1 = \lambda_2$.

Now let $\psi \in Q^\sim$ and $\lambda \in R^\sim \setminus \{1_T\}$, so that $\psi\lambda$ is an arbitrary member of $X^\wedge$. We show that there exists an element $\psi_1 \in \Omega$ and an element $\lambda_1 \in \Lambda_{\psi_1}$ such that $\psi\lambda$ and $\psi_1\lambda_1$ are conjugate under the action of N. Choose $\psi_1 \in \Omega$ and $h \in N$ such that $\psi_1 = \psi^h$. Then $(\psi\lambda)^h = \psi_1\lambda^h$. Now choose $\lambda_1 \in \Lambda_{\psi_1}$ and $h' \in N_{\psi_1}$ such that $(\lambda^h)^{h'} = \lambda_1$. Then $(\psi\lambda)^{hh'} = (\psi_1\lambda^h)^{h'} = \psi_1\lambda^{hh'} = \psi_1\lambda_1$, as desired.

(7.6) COROLLARY. _As_ ψ _ranges over_ Ω, _and for each such_ ψ, _as_ λ _ranges over_ Λ_ψ, _the characters in_ $T^\wedge$ _of the form_

ψ and $\psi\lambda$ are pairwise non-conjugate under N and comprise a complete set of representatives for the orbits of $T^\wedge$ under the action of N.

PROOF. $T^\wedge = Q^\sim \cup X^\wedge$ (disjoint), so the corollary follows from (7.5) and the definition of Ω.

We are now prepared to discuss the irreducible characters of N. Henceforth, for each $\psi \in \Omega$, we denote by C_ψ the set of irreducible constituents of ψ^{N_ψ}, and by n_ψ the index $[N_\psi : T]$.

(7.7) PROPOSITION. (a) *For all $\psi \in \Omega$, ${\delta_1}^N \neq {\delta_2}^N$ if $\delta_1, \delta_2 \in C_\psi$ are distinct, and $(\psi\lambda_1)^N \neq (\psi\lambda_2)^N$ if $\lambda_1, \lambda_2 \in \Lambda_\psi$ are distinct.*

(b) $\mathrm{Irr}(N) = \bigcup_{\psi\in\Omega}(\{\delta^N : \delta \in C_\psi\} \cup \{(\psi\lambda)^N : \lambda \in \Lambda_\psi\})$, *and all of these unions are disjoint.*

(c) *For each $\psi \in \Omega$, $\psi^N = \sum_{\delta\in C_\psi} \delta(1)\delta^N$ and $n_\psi = \sum_{\delta\in C_\psi} \delta(1)^2$.*

(d) *Let $\psi \in \Omega$. Then*

$$\psi^N|_X = -n_\psi \sum_{\lambda\in\Lambda_\psi} (\psi\lambda)^N|_X,$$

and for each $\delta \in C_\psi$,

$$\delta^N|_X = -\delta(1) \sum_{\lambda\in\Lambda_\psi} (\psi\lambda)^N|_X.$$

PROOF. Denote by Δ the set of representatives of the orbits of $T^\wedge$ under the action of N which are given in (7.6).

Since $T \lhd N$, the Clifford theory (see [10]) implies that all irreducible characters are obtained, each once, in the form $\delta_\theta{}^N$, where θ ranges over Δ, and for each $\theta \in \Delta$, δ_θ ranges over the irreducible constituents of θ^{N_θ}. By (7.2), each $\theta \in \Delta$ which takes the form $\psi\lambda$ for some $\psi \in \Omega$ and some $\lambda \in \Lambda_\psi$, has stabilizer T, so θ^N is irreducible. Hence (a) and (b) are proved.

Let $\psi \in \Omega$. Then again by the Clifford theory, $\psi^N = \sum_{\delta \in C_\psi} a_\delta \delta^N$ where for each $\delta \in C_\psi$, a_δ is an integer satisfying $\delta|_T = a_\delta \psi$. Therefore, $\delta(1) = a_\delta \psi(1) = a_\delta$. This proves that $\psi^N = \sum_{\delta \in C_\psi} \delta(1)\delta^N$. It follows that $[N:T] = [N:T]\psi(1) = \psi^N(1) = \sum_{\delta \in C_\psi} \delta(1)\delta^N(1) = \sum_{\delta \in C_\psi} \delta(1)^2[N:N_\psi] = [N:N_\psi]\sum_{\delta \in C_\psi} \delta(1)^2$. Hence $[N_\psi:T] = \sum_{\delta \in C_\psi} \delta(1)^2$, and this completes the proof of (c).

Now fix $\psi \in \Omega$, let B be a right transversal of T in N, C a right transversal of T in N_ψ, and D a right transversal of N_ψ in N. Then $\{yh: y \in C, h \in D\}$ is also a right transversal of T in N, therefore $\sum_{\lambda \in \Lambda_\psi} (\psi\lambda)^N|_X =$

$$\sum_{\lambda \in \Lambda_\psi} (1/|T|) \sum_{h \in N} (\psi\lambda)^h|_X = \sum_{\lambda \in \Lambda_\psi} \sum_{h \in B} (\psi\lambda)^h|_X =$$

$$\sum_{\lambda \in \Lambda_\psi} \sum_{h \in D} \sum_{y \in C} (\psi\lambda)^{yh}|_X = \sum_{h \in D} \sum_{\lambda \in \Lambda_\psi} \sum_{y \in C} (\psi\lambda)^{yh}|_X =$$

$$\sum_{h \in D} \sum_{\lambda \in R^\sim \setminus \{1_T\}} (\psi\lambda)^h|_X = \sum_{h \in D} \sum_{\lambda \in R^\sim \setminus \{1_T\}} (\psi^h\lambda^h)|_X =$$

$$\sum_{h \in D} \psi^h|_X \sum_{\lambda \in R^\sim \setminus \{1_T\}} \lambda^h|_X = \sum_{h \in D} \psi^h|_X \sum_{\lambda \in R^\sim \setminus \{1_T\}} \lambda|_X.$$

Let ρ denote the regular character of R. A typical element of X has the form xy where $1 \neq x \in R$ and $y \in Q$. Now

$\sum_{\lambda \in R^{\sim} \setminus \{1_T\}} \lambda(xy) = \sum_{\lambda \in R^{\sim} \setminus \{1_T\}} \lambda(x) = \rho(x) - 1 = -1$

since $1 \neq x \in R$. Hence $\sum_{\lambda \in \Lambda_\psi} (\psi\lambda)^N|_X = -\sum_{h \in D} \psi^h|_X =$

$-(1/|N_\psi|) \sum_{h \in N} \psi^h|_X = -(1/n_\psi)\psi^N|_X$, which proves the first

equality of (d).

If $\psi \in \Omega$ and $\delta \in C_\psi$, then as shown in the proof of (c), $\delta|_T = \delta(1)\psi$, whence for each $x \in X$, $\delta^N(x) =$

$(1/n_\psi)\sum_{h \in N} \delta(x^h) = (1/n_\psi)(1/|T|)\sum_{h \in N} \delta(1)\psi(x^h) =$

$(\delta(1)/n_\psi)\psi^N(x)$. The second equality of (d) now follows from the first.

In view of the fact that by (6.5), N/T is cyclic, it can be shown that for $\psi \in \Omega$ and $\delta \in C_\psi$, $\delta(1) = 1$. This fact is not necessary however, in the sequel, so we do not wish to emphasize it.

(7.8) PROPOSITION. *The set $\{\theta_{\psi,\lambda} : \psi \in \Omega, \lambda \in \Lambda_\psi\}$ of virtual characters of N forms a basis for the $\mathbb{C}$-space of class functions of N vanishing off X.*

PROOF. If $\psi \in \Omega$ and $\lambda \in \Lambda_\psi$, then $\theta_{\psi,\lambda} = \psi^N - (\psi\lambda)^N = (\psi - \psi\lambda)^N$ is clearly a virtual character of N and vanishes on $N \setminus X$ since $\psi - \psi\lambda$ vanishes on $T \setminus X = Q$.

We use the fact that Irr(N) forms a basis for the

$\mathbb{C}$-space of all class functions of N to show that the functions $\theta_{\psi,\lambda}$ are linearly independent. Suppose that

$$\Sigma_{\psi\in\Omega}\Sigma_{\lambda\in\Lambda_\psi} a_{\psi,\lambda}\theta_{\psi,\lambda} = 0$$

where $a_{\psi,\lambda} \in \mathbb{C}$ for each $\psi \in \Omega$ and each $\lambda \in \Lambda_\psi$. Then

$$\Sigma_{\psi\in\Omega}\Sigma_{\lambda\in\Lambda_\psi} a_{\psi,\lambda}[(\Sigma_{\delta\in C_\psi}\delta(1)\delta^N) - (\psi\lambda)^N] = 0,$$

or equivalently,

$$\Sigma_{\psi\in\Omega}\Sigma_{\lambda\in\Lambda_\psi}\Sigma_{\delta\in C_\psi} a_{\psi,\lambda}\delta(1)\delta^N - \Sigma_{\psi\in\Omega}\Sigma_{\lambda\in\Lambda_\psi} a_{\psi,\lambda}(\psi\lambda)^N = 0.$$

It is clear now that $a_{\psi,\lambda} = 0$ for all $\psi \in \Omega$ and all $\lambda \in \Lambda_\psi$. Hence the functions $\theta_{\psi,\lambda}$ are linearly independent.

Since the set of functions, each of which arises as the characteristic function on some class of N contained in X, forms a basis for the $\mathbb{C}$-space V of class functions of N vanishing off X,

$$\dim(V) = |\{\mathfrak{C} : \mathfrak{C} \text{ is a class of } N, \mathfrak{C} \subseteq X\}|.$$

Each $x \in X$ is locally regular, hence the size of the class of N containing x is $[N:T]$. It follows that the number of N-classes contained in X is $|X|/[N:T]$. A similar argument shows that the number of N-orbits of $T^\wedge$ contained in $X^\wedge$ is $|X^\wedge|/[N:T]$. Now $T \cong T^\wedge$, hence $|X^\wedge| = |X|$, and it follows that

$$\dim(V) = |\{\mathcal{O} : \mathcal{O} \text{ is an N-orbit}, \mathcal{O} \subseteq X^\wedge\}|.$$

which, by (7.5), is equal to $|\{\psi\lambda : \psi \in \Omega, \lambda \in \Lambda_\psi\}| = |\{\theta_{\psi,\lambda} : \psi \in \Omega, \lambda \in \Lambda_\psi\}|$. Hence the functions $\theta_{\psi,\lambda}$ form a basis for V, and the proof is complete.

§8. Block Theory for Special Primes

In this section we employ the notation of §3, with the agreement that r is replaced by some r_j.

(8.1) LEMMA. *Fix $r_j \in S(G,T)$ and let $B(r_j) \in Bl_{r_j}(G)$ have non-trivial defect. Then R_j is a defect group of $B(r_j)$.*

PROOF. Since the set of defect groups of $B(r_j)$ forms a conjugacy class of r_j-subgroups of G, there is a defect group D of $B(r_j)$ such that $\{1\} \neq D \leq R_j$. D is cyclic because R_j is, and $C_G(D) = T$, for if $\langle x \rangle = D$, then x is locally regular, so by (5.2b) $C_G(D) = C_G(x) = T$.

By (3.8) there exists an r_j-block $b(r_j)$ of T satisfying $b(r_j)^G = B(r_j)$. Let D' be a defect group of $b(r_j)$. Since T is abelian, (3.7a) implies that $D' \leq D$. But by the definition (3.5) of defect group, $D' \in Syl_{r_j}(C_T(t))$ for some $t \in T$, hence $D' = R_j$ since $C_T(t) = T$. It follows that $D = R_j$, and the lemma is proved.

Our objective now is to apply Dade's results (3.8) to G. (8.2c) and (8.4) constitute what is essentially a translation of these results to our present setting.

For each j, let Q_j be the unique subgroup of T satisfying $T = Q_j \times R_j$. Set $Q_j^{\sim} = \{\theta \in T^{\wedge}: R_j \leq \ker(\theta)\}$ and $R_j^{\sim} = \{\theta \in T^{\wedge}: Q_j \leq \ker(\theta)\}$. Since $Q_j \lhd N$, N acts on $Q_j^{\sim}$ by conjugation. Let Ω_j be a complete set of orbit representatives for this action. We may, and henceforth we shall, assume that $\Omega \subseteq \Omega_j$. Set $X_j = \{x \in T: r_j \mid |x|\}$. Then clearly $X_j \subseteq X$.

(8.2) LEMMA. (a) For each j, the set

$$\{b(r_j)': b(r_j) \in Bl_{r_j}(T)\}$$

coincides with the set of cosets of $R_j^{\sim}$ in $T^{\wedge}$.

(b) For each j, if $b(r_j) \in Bl_{r_j}(T)$, then $b(r_j)''$ contains precisely one element φ which is uniquely determined by the property that for all $\theta \in b(r_j)'$, $\varphi = \theta|_{Q_j}$.

(c) For each j, there is a bijection

$$\Omega_j \longleftrightarrow \{B(r_j) \in Bl_{r_j}(G): B(r_j) \text{ has non-trivial defect}\}$$

given by

$$\psi \longleftrightarrow b(r_j)^G,$$

where $b(r_j)$ is the unique r_j-block of T satisfying $\psi \in b(r_j)'$. Moreover, if $b_1, b_2 \in Bl_{r_j}(T)$, then $b_1^G = b_2^G$ if and only if $b_2 = b_1^h$ for some $h \in N$.

PROOF. Note that the arguments given here for (a) and (b) are valid for any finite abelian group T and any prime

r_j. Recall from §3 the properties of the ring S.

(a) Given $b(r_j) \in Bl_{r_j}(T)$ and $\theta_1 \in b(r_j)'$, a character $\theta_2 \in T^{\wedge}$ satisfies $\theta_2 \in b(r_j)'$ if and only if $\overline{\omega_{\theta_1}} = \overline{\omega_{\theta_2}}$ (see (3.2)). View T as embedded in $Z(\bar{S}T)$. Then since T is abelian, $\overline{\omega_{\theta_i}}|_T = \bar{\theta}_i$ $(i = 1,2)$, where for $\theta \in T^{\wedge}$, $\bar{\theta}$ denotes the linear character of T over $\bar{S}$ defined by $\bar{\theta}(t) = \overline{\theta(t)}$ for all $t \in T$. Let $\bar{T}$ denote the group of linear characters of T over $\bar{S}$. Since $\overline{\omega_{\theta_1}} = \overline{\omega_{\theta_2}}$ if and only if $\bar{\theta}_1 = \bar{\theta}_2$, it follows that the set $\{b(r_j)' : b(r_j) \in Bl_{r_j}(T)\}$ coincides with the set of all cosets in $T^{\wedge}$ of the kernel of the group homomorphism $f: T^{\wedge} \rightarrow \bar{T}$ defined by $f(\theta) = \bar{\theta}$ for all $\theta \in T^{\wedge}$. Hence to prove (a), it suffices to show that $\ker(f) = R_j^{\sim}$. But since $\mathrm{char}(\bar{S}) = r_j$,

$$\bar{\theta} = \bar{1}_T \Leftrightarrow Q_j \leq \ker(\theta) \Leftrightarrow \theta \in R_j^{\sim}.$$

(b) The irreducible $\bar{S}$-characters of T are precisely those of the form $\bar{\theta}$ $(\theta \in T^{\wedge})$. If φ is the irreducible Brauer character afforded by $\bar{\theta}$, then it follows easily from the definition of Brauer character (see [5]) that $\varphi = \theta|_{Q_j}$. Let $b(r_j)$ be the unique element of $Bl_{r_j}(T)$ such that $\theta \in b(r_j)'$. Then by (3.2), $\varphi \in b(r_j)''$, and by (a), $b(r_j)' = \{\theta_1 \in T^{\wedge}: \theta_1|_{Q_j} = \varphi\}$. It is clear now that φ is the unique element of $b(r_j)''$, and (b) is proved.

(c) The last assertion of (c) is a consequence of (3.8a) (Here D is replaced by R_j and C_0 by T. Recall that we may write $R_j = \langle x \rangle$ where x is locally regular, so that by

(5.2b), $C_G(R_j) = C_G(x) = T$).

Now let $\psi \in \Omega_j$, and let $b(r_j)$ be the unique element of $Bl_{r_j}(T)$ satisfying $\psi \in b(r_j)'$. By (3.8a), $b(r_j)^G$ is defined. Since T is abelian and $R_j \in Syl_{r_j}(T)$, R_j is a defect group of $b(r_j)$. Therefore, since $R_j \in Syl_{r_j}(G)$, (3.7a) implies that R_j is also a defect group of $b(r_j)^G$. In particular, $b(r_j)^G$ has non-trivial defect, hence we may define a function

$$\beta: \Omega_j \longrightarrow \{B(r_j) \in Bl_{r_j}(G): B(r_j) \text{ has non-trivial defect}\}$$

by $\beta(\psi) = b(r_j)^G$, where $b(r_j) \in Bl_{r_j}(T)$, $\psi \in b(r_j)'$. By (3.8a) and the definition of Ω_j, β is one-to-one. One of the opening assertions of (3.8), together with (3.8a), implies that β is onto. This completes the proof of (8.2).

(8.3) DEFINITION. For each j, and each $\psi \in \Omega_j$, denote by $b_\psi(r_j)$ the unique r_j-block of T satisfying $b_\psi(r_j)' = \psi R_j^\sim$, and denote by $B_\psi(r_j)$ the r_j-block of G given by $B_\psi(r_j) = b_\psi(r_j)^G$.

(8.4) PROPOSITION. _Fix_ $\psi \in \Omega_j$. _Then_ $B_\psi(r_j)'$ _contains certain distinct irreducible characters_

$$\chi_{\psi,1}, \ldots, \chi_{\psi,n_\psi}$$

_of_ G _whose higher decomposition numbers are given as follows:_

<u>there exist signs</u> $\varepsilon_{\psi,1},\ldots,\varepsilon_{\psi,n_\psi}$ <u>such that for each</u> $x \in R_j^*$, <u>each</u> i, <u>and each</u> $\varphi \in IBr(T)$,

$$d^x_{\chi_{\psi,i},\varphi} = \begin{cases} \varepsilon_{\psi,i} \text{ \underline{if} } \varphi \in (b_\psi(r_j)^h)\text{" \underline{for some} } h \in N \\ 0 \text{ \underline{otherwise}.} \end{cases}$$

PROOF. By (8.2c), (8.1), and (5.4), $B = B_\psi(r_j)$ has non-trivial cyclic defect group R_j, so we may apply (3.8) with $r = r_j$ and $D = R_j$. We begin by translating some of the notation of (3.8).

For $0 \leq k < a$, $C_k = T$ and $N_k = N_G(T)$ (= N). We prove this assertion in steps (i)-(iii) below, making implicit use of (5.2b).

(i) $N_G(T) \leq N_k$: If $h \in N_G(T)$, then $D^h \in Syl_{r_j}(T)$, so that $D^h = D$ because T is abelian. Thus $D_k \leq D$ implies that $D_k{}^h \leq D$. But D_k is the unique subgroup of D of order $|D_k|$, hence $D_k{}^h = D_k$. Therefore $h \in N_G(D_k) = N_k$.

(ii) $C_k = T$: $\{1\} \neq D_k = \langle x \rangle$ for some $x \in D$, hence $C_k = C_G(D_k) = C_G(x)$. But x is locally regular since $r_j \mid |x|$, so $C_G(x) = T$.

(iii) $N_k \leq N_G(T)$: $C_k = C_G(D_k) \triangleleft N_G(D_k) = N_k$. But by (ii), $C_k = T$, so $T \triangleleft N_k$ implies that $N_k \leq N_G(T)$.

This proves our assertion, so we replace each C_k by T and each N_k by $N = N_G(T)$ in (3.8). Now $b_\psi(r_j)^G = B$, so we may take $b_0 = b_\psi(r_j)$ in (3.8). From (a) and (b) of (8.2), and from the fact that $T^\wedge = Q_j^\sim \times R_j^\sim$, it follows that

$$E = \mathrm{Stab}_{N_0}(b_\psi(r_j)) = \mathrm{Stab}_N(b_\psi(r_j)) = N_\psi$$

in (3.8). Hence $e = [N_\psi : T] = n_\psi$ in (3.8).

Now by (b) and (c) of (3.8), there exist characters $\chi_{\psi,1},\ldots,\chi_{\psi,n_\psi} \in B'$ and signs $\varepsilon_{\psi,1},\ldots,\varepsilon_{\psi,n_\psi},\gamma_0,\gamma_1,\ldots,\gamma_{a-1}$ such that for all $k < a$, if $\langle x\rangle = D_k$ and if $\varphi \in \mathrm{IBr}(T) = \mathrm{IBr}(C_G(x))$, then

$$d^x_{\chi_{\psi,i},\varphi} = \begin{cases} \varepsilon_{\psi,i}\gamma_k \text{ if } \varphi \in (b_\psi(r_j)^h)'' \text{ for some } h \in N \\ 0 \text{ otherwise.}\end{cases}$$

Therefore it suffices to show that we may choose the signs $\varepsilon_{\psi,i}$ and γ_k in such a way that $\gamma_0 = \gamma_1 = \cdots = \gamma_{a-1} = 1$.

Invoking (3.8d), we choose the $\varepsilon_{\psi,i}$ and the γ_k so that $\gamma_0 = 1$, and we apply (3.8) to C_{a-1} and b_{a-1}. But since $C_{a-1} = T = C_0$ and $b_{a-1} = b_\psi(r_j)$, we are in effect applying (3.8) to C_0 and $b_\psi(r_j)$. Therefore by (3.9) we obtain new signs $(\gamma_0)' = (\gamma_1)' = \cdots = (\gamma_{a-1})' = 1$, which, by (3.8d), forces $\gamma_0 = \gamma_1 = \cdots = \gamma_{a-1} = 1$, as desired. This concludes the proof of the proposition.

The characters $\chi_{\psi,i}$ of (8.4) are called by Dade the "non-exceptional characters" in $B_\psi(r_j)$ (see [6]).

(8.5) COROLLARY. <u>With notation as in</u> (8.4),

$$\chi_{\psi,i}|_{X_j} = (\varepsilon_{\psi,i}/n_\psi)\,\psi^N|_{X_j}$$

<u>for all</u> $\psi \in \Omega_j$ <u>and all</u> $i \in \{1,\ldots,n_\psi\}$.

PROOF. A typical element of X_j has the form xy, where $x \in R_j{}^*$ and $y \in Q_j$. Let φ be the unique element of $b_\psi(r_j)''$ (see (8.2b)). Then $\varphi = \psi|_{Q_j}$ and for all $h \in N$, the unique element of $(b_\psi(r_j)^h)''$ is φ^h. Since $\mathrm{Stab}_N(b_\psi(r_j)) = N_\psi$, the conjugates of $b_\psi(r_j)$ under the action of N are obtained, each once, in the form $b_\psi(r_j)^h$, as h ranges over a left transversal $\mathcal{L}$ of N_ψ in N. Therefore we may calculate as follows:

$$
\begin{aligned}
\chi_{\psi,i}(xy) &= \textstyle\sum_{h\in\mathcal{L}} d^x_{\chi_{\psi,i},\varphi^h}\,\varphi^h(y) && \text{(by (3.7b), (8.4))}\\
&= \textstyle\sum_{h\in\mathcal{L}} \varepsilon_{\psi,i}\,\varphi(y^{h^{-1}}) && \text{(by (8.4))}\\
&= (\varepsilon_{\psi,i}/|N_\psi|)\textstyle\sum_{h\in N}\psi(y^h) && \text{(by (8.2b))}\\
&= (\varepsilon_{\psi,i}/|N_\psi|)\textstyle\sum_{h\in N}\psi((xy)^h) && \text{(since } \psi\in Q_j{}^\sim)\\
&= (\varepsilon_{\psi,i}/n_\psi)(1/|T|)\textstyle\sum_{h\in N}\psi((xy)^h)\\
&= (\varepsilon_{\psi,i}/n_\psi)\psi^N(xy),
\end{aligned}
$$

as desired.

Recall that for all $j \in \{1,\ldots,n\}$, $\Omega \subseteq \Omega_j$ and $X_j \subseteq X$, so that a class function of T which vanishes on $T \setminus X$ must also vanish on $T \setminus X_j$.

(8.6) COROLLARY. <u>With notation as in</u> (8.4), <u>if</u> $\psi \in \Omega$ <u>and</u> $\lambda \in \Lambda_\psi$, <u>then</u> $(\chi_{\psi,i}, \psi^G - (\psi\lambda)^G)_G = \varepsilon_{\psi,i}$ <u>for all</u> $i \in \{1,\ldots,n_\psi\}$.

PROOF. For the following calculation, we recall that C_ψ denotes the set of irreducible constituents of ψ^{N_ψ}, and we observe that $\psi - \psi\lambda$ vanishes on $Q_j = T \setminus X_j$:

$$
\begin{aligned}
(\chi_{\psi,i}, \psi^G - (\psi\lambda)^G)_G &= (\chi_{\psi,i}|_T, \psi - \psi\lambda)_T && \text{(Frobenius reciprocity)} \\
&= (\varepsilon_{\psi,i}/n_\psi)(\psi^N|_T, \psi - \psi\lambda)_T && \text{(by (8.5))} \\
&= (\varepsilon_{\psi,i}/n_\psi)(\psi^N, \psi^N - (\psi\lambda)^N)_N && \text{(Frobenius reciprocity)} \\
&= (\varepsilon_{\psi,i}/n_\psi)(\psi^N, \psi^N)_N && \text{(by (7.7))} \\
&= (\varepsilon_{\psi,i}/n_\psi) \textstyle\sum_{\delta \in C_\psi} \delta(1)^2 && \text{(by (7.7))} \\
&= \varepsilon_{\psi,i} && \text{(by (7.7)).}
\end{aligned}
$$

In §8, we have dealt with an individual prime $r_j \in S(G,T)$. It is desirable however, to treat all primes in $S(G,T)$ simultaneously. For example, it is natural to ask whether the formula of (8.5) holds on all of X, and whether the set $\{\chi_{\psi,1}, \ldots, \chi_{\psi,n}\}$ of non-exceptional characters in $B_\psi(r_j)'$ is independent of $r_j \in S(G,T)$. In the next section we shall answer these questions affirmatively and introduce some additional characters in Irr(G) for which the formula holds.

§9. Character Values

In this section we make liberal use of the notation introduced in §7 and in §8. Denote by E the set

$$\{g \in G: r_j \mid |g| \text{ for some } j \in \{1,\dots,n\}\}.$$

Then $E = \bigcup_{y \in G} X^y$. Indeed, it is obvious that $X \subseteq E$. On the other hand, if $r_j \mid |g|$, then we may write $g = ab$ where $|a| = r_j^{\,s}$ for some positive integer s, $(|b|, r_j) = 1$, and $ab = ba$. Since the r_j-Sylow subgroup R_j of G is contained in T, Sylow's theorem implies that there exists an element $y \in G$ such that $a^y \in T$. Now $b \in C_G(a)$ implies that $b^y \in C_G(a^y)$. But a^y is locally regular in T since $r_j \mid |a^y|$, so by (5.2b), $b^y \in T$. Hence $g^y = a^y b^y \in T$, and it follows that $g^y \in X$, whence $g \in X^{y^{-1}}$. This proves the assertion, which implies that if the values of a class function of G are known on X, then they are known on E as well.

(9.1) THEOREM. <u>Let $\psi \in \Omega$. Then there exists a sign ε_ψ, for each $i \in \{1,\dots,n_\psi\}$ there exists a sign $\varepsilon_{\psi,i}$ and an irreducible character $\chi_{\psi,i}$ of G, and for each $\lambda \in \Lambda_\psi$ there exists an irreducible character $\chi_{\psi,\lambda}$ of G, such that the following assertions hold:</u>

(a) <u>For each $\psi \in \Omega$ and each $\lambda \in \Lambda_\psi$,</u>

$$\psi^G - (\psi\lambda)^G = \Big(\sum_{i=1}^{n_\psi} \varepsilon_{\psi,i}\chi_{\psi,i}\Big) - \varepsilon_\psi \chi_{\psi,\lambda}.$$

Moreover, the map

$$f:\{(\Psi,\lambda):\ \Psi\in\Omega,\ \lambda\in\Lambda_\psi\}\ \dot\cup\ \{(\Psi,i):\ \psi\in\Omega,\ i\in\{1,\dots,n_\psi\}\}\to \mathrm{Irr}(G),$$

given by

$$f((\Psi,\beta)) = \chi_{\psi,\beta},$$

is injective.

(b) *For each* $r_j \in S(G,T)$, *the set* $\{\chi_{\psi,i}:\ 1\le i\le n_\psi\}$ *coincides with the set of non-exceptional characters in* $B_\psi(r_j)$ *given by* (8.4).

(c) *For each* $\psi\in\Omega$ *and each* $i\in\{1,\dots,n_\psi\}$,

$$\varepsilon_{\psi,i}\chi_{\psi,i}|_X = (1/n_\psi)\Psi^N|_X = -\varepsilon_\psi\sum_{\lambda\in\Lambda_\psi}\chi_{\psi,\lambda}|_X.$$

In particular, $\varepsilon_{\psi,i_1}\chi_{\psi,i_1}|_E = \varepsilon_{\psi,i_2}\chi_{\psi,i_2}|_E$ *for all* $\psi\in\Omega$ *and all* $i_1,\ i_2\in\{1,\dots,n_\psi\}$.

(d) *For each* $\psi\in\Omega$ *and each* $\lambda\in\Lambda_\psi$,

$$\chi_{\psi,\lambda}|_X = \varepsilon_\psi(\Psi\lambda)^N|_X,$$

and

$$\varepsilon_\psi\chi_{\psi,\lambda}|_{G\smallsetminus E} = \sum_{i=1}^{n_\psi}\varepsilon_{\psi,i}\chi_{\psi,i}|_{G\smallsetminus E}.$$

In particular, $\chi_{\psi,\lambda_1}|_{G\smallsetminus E} = \chi_{\psi,\lambda_2}|_{G\smallsetminus E}$ *for all* $\psi\in\Omega$ *and all* $\lambda_1,\ \lambda_2\in\Lambda_\psi$.

(e) *Let* $\chi\in\mathrm{Irr}(G)$ *be distinct from all* $\chi_{\psi,i}$ *and all* $\chi_{\psi,\lambda}$. *Then* χ *vanishes on* E.

PROOF. (a) Let $\psi \in \Omega$ and $\lambda \in \Lambda_\psi$. Since $\Omega \subseteq \Omega_1$, (8.4), (8.5), and (8.6) imply that there exist signs $\varepsilon_{\psi,1}, \dots, \varepsilon_{\psi,n_\psi}$ and distinct irreducible characters $\chi_{\psi,1}, \dots, \chi_{\psi,n_\psi}$ in $B_\psi(r_1)'$ such that for all i,

$$(9.2) \qquad \chi_{\psi,i}|_{X_1} = (\varepsilon_{\psi,i}/n_\psi)\psi^N|_{X_1} \text{ and } (\chi_{\psi,i}, \theta_{\psi,\lambda}{}^G)_G = \varepsilon_{\psi,i},$$

where $\theta_{\psi,\lambda} = \psi^N - (\psi\lambda)^N$. For each $\zeta \in \mathrm{Irr}(N)$ and each $\chi \in \mathrm{Irr}(G)$, let $a_\zeta = (\zeta, \theta_{\psi,\lambda})_N$ and $b_\chi = (\chi, \theta_{\psi,\lambda}{}^G)_G$, so that $\theta_{\psi,\lambda} = \sum_{\zeta\in\mathrm{Irr}(N)} a_\zeta \zeta$ and $\theta_{\psi,\lambda}{}^G = \sum_{\chi\in\mathrm{Irr}(G)} b_\chi \chi$. Then, since by (7.6) X is a union of special classes of N in G, and since by (7.8) all such $\theta_{\psi,\lambda}$ form a basis of virtual characters of N for the $\mathbb{C}$-space of class functions of N vanishing off X, we may apply (2.2c) to obtain

$$\sum_{\chi\in\mathrm{Irr}(G)} b_\chi{}^2 = \sum_{\zeta\in\mathrm{Irr}(N)} a_\zeta{}^2.$$

But by (7.7), $\theta_{\psi,\lambda} = (\sum_{\delta\in C_\psi} \delta(1)\delta^N) - (\psi\lambda)^N$ is a decomposition of $\theta_{\psi,\lambda}$ as a linear combination of its distinct irreducible constituents, hence by (7.7c), $n_\psi + 1 = \sum_\zeta a_\zeta{}^2 = \sum_\chi b_\chi{}^2$. Now by (9.2), for $i \in \{1,\dots,n_\psi\}$, $b_{\chi_{\psi,i}} = \varepsilon_{\psi,i} = \pm 1$, so that $\sum_{i=1}^{n_\psi} b_{\chi_{\psi,i}}{}^2 = n_\psi$. It follows that there exists a sign $\varepsilon_{\psi,\lambda}$ and an irreducible character $\chi_{\psi,\lambda}$ of G, distinct from each of the $\chi_{\psi,i}$, such that $\theta_{\psi,\lambda}{}^G = (\sum_{i=1}^{n_\psi} \varepsilon_{\psi,i}\chi_{\psi,i}) - \varepsilon_{\psi,\lambda}\chi_{\psi,\lambda}$.

Now we apply (2.2c) again to show that for each $\psi \in \Omega$, the signs $\varepsilon_{\psi,\lambda}$ are independent of $\lambda \in \Lambda_\psi$. Indeed, since for each $\psi \in \Omega$ and $\lambda_k \in \Lambda_\psi$ $(k = 1,2)$, $\theta_{\psi,\lambda_k} = (\sum_{\delta \in C_\psi} \delta(1)\delta^N) - (\psi\lambda_k)^N$ and ${\theta_{\psi,\lambda_k}}^G = (\sum_{i=1}^{n_\psi} \varepsilon_{\psi,i}\chi_{\psi,i}) - \varepsilon_{\psi,\lambda_k}\chi_{\psi,\lambda_k}$, (2.2c) asserts that $(\sum_{i=1}^{n_\psi} {\varepsilon_{\psi,i}}^2) + \varepsilon_{\psi,\lambda_1}\varepsilon_{\psi,\lambda_2} = (\sum_{\delta \in C_\psi} \delta(1)^2) + 1$. By (7.7c), this becomes $\varepsilon_{\psi,\lambda_1}\varepsilon_{\psi,\lambda_2} = 1$, which forces $\varepsilon_{\psi,\lambda_1} = \varepsilon_{\psi,\lambda_2}$. Thus we are justified in replacing $\varepsilon_{\psi,\lambda}$ by ε_ψ, and the proof of the first part of (a) is complete.

Although its proof is relatively long, the second part of (a) will play a crucial role in our arguments for (c) and (d). We have remarked already that for each $\psi \in \Omega$ and each $\lambda \in \Lambda_\psi$, the characters $\chi_{\psi,1}, \ldots, \chi_{\psi,n_\psi}, \chi_{\psi,\lambda}$ are distinct. Moreover, if $\psi \in \Omega$ and λ_1, $\lambda_2 \in \Lambda_\psi$, then $\chi_{\psi,\lambda_1} = \chi_{\psi,\lambda_2}$ implies by the first part of (a) that ${\theta_{\psi,\lambda_1}}^G = {\theta_{\psi,\lambda_2}}^G$. Now by (2.3b), induction is an isometry from the $\mathbb{C}$-space of class functions of N vanishing off X into the $\mathbb{C}$-space of class functions of G. Therefore it is a monomorphism, and we conclude that $\theta_{\psi,\lambda_1} = \theta_{\psi,\lambda_2}$. It follows that $(\psi\lambda_1)^N = (\psi\lambda_2)^N$, and by (7.7a), this forces $\lambda_1 = \lambda_2$.

Thus to finish the proof of the second part of (a), we must show that if ψ_1, $\psi_2 \in \Omega$ are distinct with $\lambda_1 \in \Lambda_{\psi_1}$ and $\lambda_2 \in \Lambda_{\psi_2}$, then any irreducible constituent of $\theta_{\psi_1,\lambda_1}$ cannot be an irreducible constituent of $\theta_{\psi_2,\lambda_2}$. But due to the way in which the characters $\chi_{\psi_1,i}$ and $\chi_{\psi_2,k}$ arose, $\{\chi_{\psi_1,i}: 1 \leq i \leq n_{\psi_1}\} \subseteq B_{\psi_1}(r_1)'$, and $\{\chi_{\psi_2,k}: 1 \leq k \leq n_{\psi_2}\} \subseteq B_{\psi_2}(r_1)'$. Since the unions (3.3) are disjoint, $B_{\psi_1}(r_1)' \cap B_{\psi_2}(r_1)' = \emptyset$ because by (8.2c), $B_{\psi_1}(r_1) \neq B_{\psi_2}(r_1)$. Therefore the problem reduces to showing that we cannot have $\chi_{\psi_1,i} = \chi_{\psi_2,\lambda}$ for some $i \in \{1,\ldots,n_{\psi_1}\}$ and some $\lambda \in \Lambda_{\psi_2}$.

In order to do this, we shall invoke (2.2d). The set $\{\theta_{\psi,\lambda}: \psi \in \Omega, \lambda \in \Lambda_\psi\}$ may be indexed by the set $\{(\psi,\lambda): \psi \in \Omega, \lambda \in \Lambda_\psi\}$. Let C be a set of representatives of the classes of N contained in X. By (2.2d), there exist uniquely determined complex numbers $c_{x,(\psi,\lambda)}$ $(x \in C, \psi \in \Omega, \lambda \in \Lambda_\psi)$ satisfying

$$(9.3) \qquad \zeta(x) = \sum_{\psi\in\Omega} \sum_{\lambda\in\Lambda_\psi} c_{x,(\psi,\lambda)} a_{(\psi,\lambda),\zeta}$$

for all $\zeta \in \mathrm{Irr}(N)$ and all $x \in C$, where $a_{(\psi,\lambda),\zeta} = (\zeta, \theta_{\psi,\lambda})_N$. Moreover, the $c_{x,(\psi,\lambda)}$ also satisfy

$$(9.4) \qquad \chi(x) = \sum_{\psi\in\Omega} \sum_{\lambda\in\Lambda_\psi} c_{x,(\psi,\lambda)} b_{(\psi,\lambda),\chi}$$

for all $\chi \in \mathrm{Irr}(G)$ and all $x \in C$, where $b_{(\psi,\lambda),\chi} = (\chi, \theta_{\psi,\lambda}{}^G)_G$. Now by (7.7), for $\psi \in \Omega$ and $\lambda \in \Lambda_\psi$,

$$(9.5) \qquad a_{(\psi,\lambda),\zeta} = \begin{cases} \delta(1) & \text{if } \zeta = \delta^N \ (\delta \in C_\psi) \\ -1 & \text{if } \zeta = (\psi\lambda)^N \\ 0 & \text{if } \zeta \in \mathrm{Irr}(N) \text{ is otherwise.} \end{cases}$$

And by the first part of (a), for all $\psi \in \Omega$ and all $\lambda \in \Lambda_\psi$,

$$(9.6) \qquad b_{(\psi,\lambda),\chi} = \begin{cases} \varepsilon_{\psi,i} & \text{if } \chi = \chi_{\psi,i} \ (i \in \{1,\ldots,n_\psi\}) \\ -\varepsilon_\psi & \text{if } \chi = \chi_{\psi,\lambda} \\ 0 & \text{if } \chi \in \mathrm{Irr}(G) \text{ is otherwise.} \end{cases}$$

We are now prepared to finish the proof of (a). Fix

$\psi_0 \in \Omega$ and $i \in \{1,\ldots,n_{\psi_0}\}$. Set

$$B = \{\psi \in \Omega: \chi_{\psi_0,i} = \chi_{\psi,\lambda} \text{ for some } \lambda \in \Lambda_\psi\}.$$

We must show that $B = \emptyset$. By our previous discussion, we know that $\psi_0 \notin B$, and that if $\psi \in B$, then there is a unique $\lambda_\psi \in \Lambda_\psi$ such that $\chi_{\psi_0,i} = \chi_{\psi,\lambda_\psi}$. Let $1 \neq x \in X_1 \cap R$. Then by (9.2), (9.4), (9.6), (9.5), (9.3), and (7.7d) respectively, $(\varepsilon_{\psi_0,i}/n_{\psi_0})\psi_0{}^N(x) = \chi_{\psi_0,i}(x)$

$$= \sum_{\psi\in\Omega}\sum_{\lambda\in\Lambda_\psi} c_{x,(\psi,\lambda)}{}^{b}(\psi,\lambda),\chi_{\psi_0,i}$$

$$= \sum_{\lambda\in\Lambda_{\psi_0}} c_{x,(\psi_0,\lambda)}{}^{\varepsilon_{\psi_0,i}} + \sum_{\psi\in B} c_{x,(\psi,\lambda_\psi)}{}^{(-\varepsilon_\psi)}$$

$$= -\varepsilon_{\psi_0,i}\sum_{\lambda\in\Lambda_{\psi_0}} c_{x,(\psi_0,\lambda)}{}^{a}(\psi_0,\lambda),(\psi_0\lambda)^N +$$

$$\sum_{\psi\in B}\varepsilon_\psi\sum_{\lambda\in\Lambda_\psi} c_{x,(\psi,\lambda)}{}^{a}(\psi,\lambda),(\psi\lambda_\psi)^N$$

$$= -\varepsilon_{\psi_0,i}\sum_{\lambda\in\Lambda_{\psi_0}}(\psi_0\lambda)^N(x) + \sum_{\psi\in B}\varepsilon_\psi(\psi\lambda_\psi)^N(x)$$

$$= (\varepsilon_{\psi_0,i}/n_{\psi_0})\psi_0{}^N(x) + \sum_{\psi\in B}\varepsilon_\psi(\psi\lambda_\psi)^N(x).$$

It follows that $\sum_{\psi\in B}\varepsilon_\psi(\psi\lambda_\psi)^N(x) = 0$.

Next we observe that for each $\psi \in B$, $\varepsilon_\psi(\psi\lambda_\psi)^N(x) = \chi_{\psi,\lambda_\psi}(x)$. Indeed, by the first part of (a), (9.2), and (2.3a), we have $\varepsilon_\psi\chi_{\psi,\lambda_\psi}(x) = (\sum_{k=1}^{n_\psi}\varepsilon_{\psi,k}\chi_{\psi,k}(x)) - (\psi - \psi\lambda_\psi)^G(x) = \sum_{k=1}^{n_\psi}(1/n_\psi)\psi^N(x) - (\psi^N(x) - (\psi\lambda_\psi)^N(x)) =$

$(\psi\lambda_\psi)^N(x)$, and the observation follows. Therefore, $0 = \sum_{\psi \in B} \chi_{\psi,\lambda_\psi}(x) = \sum_{\psi \in B} \chi_{\psi_0,i}(x) = |B|\chi_{\psi_0,i}(x) = |B|(\varepsilon_{\psi_0,i}/n_{\psi_0})\psi_0^N(x)$ by (9.2). Finally, since $\psi_0 \in Q^\sim$ and $x \in R$, $\psi_0^N(x) = (1/|T|)\sum_{h \in N} \psi_0(x^h) = |N|/|T|$, so we have $0 = |B|\,\varepsilon_{\psi_0,i}[N:N_{\psi_0}]$, which forces $|B| = 0$. Hence $B = \emptyset$, and the proof of (a) is complete.

(b) In the proof of (a), the characters $\chi_{\psi,i}$ $(1 \leq i \leq n_\psi)$ arose as the characters in $B_\psi(r_1)$ given by (8.4). So if $S(G,T) = \{r_1\}$, there is nothing left to prove. Assume otherwise, and let $r_j \in S(G,T)$ be distinct from r_1. Let χ_i $(1 \leq i \leq n_\psi)$ be the characters in $B_\psi(r_j)'$ given by (8.4). Then arguing as in the proof of (a), for all i and all $\lambda \in \Lambda_\psi$, χ_i is an irreducible constituent of $\psi^G - (\psi\lambda)^G$. Suppose that $\chi_i \notin \{\chi_{\psi,k}: 1 \leq k \leq n_\psi\}$ for some i. It follows from the first part of (a) that $\chi_i = \chi_{\psi,\lambda}$ for all $\lambda \in \Lambda_\psi$. By the second part of (a), this forces $|\Lambda_\psi| = 1$.

Now $R \triangleleft N$, and since R is cyclic, each subgroup of R is also normal in N. Thus in view of the definition of Λ_ψ, $|\Lambda_\psi| = 1$ implies that $R^\sim = R_1^\sim$. Since $R \cong R^\sim$, it follows that $S(G,T) = \{r_1\}$, contradicting the existence of r_j. This concludes the proof of (b).

(c),(d) Let $\psi_0 \in \Omega$, $i \in \{1,\ldots,n_{\psi_0}\}$, and $x \in C$. Then by (9.4), the first part of (a) together with (9.6), (9.5), (9.3), and (7.7d) respectively, $\chi_{\psi_0,i}(x) =$

$$\sum_{\psi \in \Omega} \sum_{\lambda \in \Lambda_\psi} c_{x,(\psi,\lambda)} b_{(\psi,\lambda),\chi_{\psi_0,i}} = \sum_{\lambda \in \Lambda_{\psi_0}} c_{x,(\psi_0,\lambda)} \varepsilon_{\psi_0,i}$$

$$= -\varepsilon_{\Psi_0,1} \sum_{\lambda\in\Lambda_{\Psi_0}} c_{x,(\Psi_0,\lambda)^a(\Psi_0,\lambda),(\Psi_0\lambda)^N}$$

$$= -\varepsilon_{\Psi_0,1} \sum_{\lambda\in\Lambda_{\Psi_0}} (\Psi_0\lambda)^N(x) = (\varepsilon_{\Psi_0,1}/n_{\Psi_0})\Psi_0{}^N(x),$$

and this establishes the first equality of (c).

Now let $\Psi_0 \in \Omega$, $\lambda_0 \in \Lambda_{\Psi_0}$, and $x \in C$. We proceed similarly to obtain $\chi_{\Psi_0,\lambda_0}(x) =$

$$\sum_{\Psi\in\Omega} \sum_{\lambda\in\Lambda_\Psi} c_{x,(\Psi,\lambda)^b(\Psi,\lambda),\chi_{\Psi_0,\lambda_0}} = c_{x,(\Psi_0,\lambda_0)^{(-\varepsilon_{\Psi_0})}}$$

$$= \varepsilon_{\Psi_0} c_{x,(\Psi_0,\lambda_0)^a(\Psi_0,\lambda_0),(\Psi_0\lambda_0)^N} = \varepsilon_{\Psi_0}(\Psi_0\lambda_0)^N(x),$$

thus establishing the first equality of (d).

We use this equality, together with (7.7d), to establish the second equality of (c). If $\Psi \in \Omega$, then

$$-\varepsilon_\Psi \sum_{\lambda\in\Lambda_\Psi} \chi_{\Psi,\lambda}\Big|X = -\varepsilon_\Psi \sum_{\lambda\in\Lambda_\Psi} \varepsilon_\Psi (\Psi\lambda)^N\Big|X =$$

$$-\sum_{\lambda\in\Lambda_\Psi} (\Psi\lambda)^N\Big|X = (1/n_\Psi)\Psi^N\Big|X.$$

The proof of (c) is now complete.

Since by (7.8), $\Psi^N - (\Psi\lambda)^N$ vanishes on $N \setminus X$ for each $\Psi \in \Omega$ and each $\lambda \in \Lambda_\Psi$, $\Psi^G - (\Psi\lambda)^G = (\Psi^N - (\Psi\lambda)^N)^G$ vanishes on $G \setminus \bigcup_{g\in G} X^g = G \setminus E$. By (a), this establishes the second equality of (d), and the proof of (d) is complete.

(e) If $\chi \in \mathrm{Irr}(G)$ is distinct from all $\chi_{\Psi,1}$ and all $\chi_{\Psi,\lambda}$, then (a) implies that for all $\Psi \in \Omega$ and all $\lambda \in \Lambda_\Psi$,

$b_{(\psi,\lambda),\chi} = 0$. Thus (9.4) implies that χ vanishes on X, and it follows that χ vanishes on E. This concludes the proof of the theorem.

(9.7) COROLLARY. <u>The virtual characters of</u> G <u>in</u>

$$\{\varepsilon_{\psi,i}\chi_{\psi,i} + \varepsilon_\psi \sum_{\lambda\in\Lambda_\psi} \chi_{\psi,\lambda} \colon \psi \in \Omega,\ i \in \{1,\ldots,n_\psi\}\},$$

<u>together with all</u> $\chi \in \mathrm{Irr}(G)$ <u>which are distinct from all</u> $\chi_{\psi,i}$ <u>and all</u> $\chi_{\psi,\lambda}$, <u>form a basis for the</u> $\mathbb{C}$-<u>space of class functions of</u> G <u>which vanish on</u> E.

PROOF. Irr(G) forms a basis for the space of all class functions of G, hence by (9.1a), the set $\mathcal{B}$ of class functions given in (9.7) is linearly independent. The elements of $\mathcal{B}$ vanish on E by (9.1c) and (9.1e).

Now $E = \bigcup_{g\in G} X^g$ is a union of conjugacy classes of G, hence the same is true of $G \setminus E$, and since by (7.3) X is the union of a set of special classes of N in G, the number of classes of G contained in E is equal to the number of classes of N contained in X. Since the set of functions, each of which arises as the characteristic function on some class of G contained in $G \setminus E$, forms a basis for the $\mathbb{C}$-space V of class functions of G vanishing on E, we have dim(V) =

$$|\{\mathfrak{C} \colon \mathfrak{C} \text{ is a class of } G,\ \mathfrak{C} \subseteq G \setminus E\}| =$$

$$|\{\mathfrak{C} \colon \mathfrak{C} \text{ is a class of } G\}| - |\{\mathfrak{C} \colon \mathfrak{C} \text{ is a class of } N,\ \mathfrak{C} \subseteq X\}|.$$

Now since $|\{\mathcal{C} : \mathcal{C}$ is a class of $G\}| = |\mathrm{Irr}(G)|$, since by (7.9) $|\{\mathcal{C} : \mathcal{C}$ is a class of N, $\mathcal{C} \subseteq X\}| = |\{(\psi,\lambda) : \psi \in \Omega,\ \lambda \in \Lambda_\psi\}|$, and by (9.1a), we obtain $\dim(V) =$

$$|\mathrm{Irr}(G)| - |\{(\psi,\lambda) : \psi \in \Omega,\ \lambda \in \Lambda_\psi\}| =$$

$$|\mathrm{Irr}(G)| - |\{\chi_{\psi,\lambda} : \psi \in \Omega,\ \lambda \in \Lambda_\psi\}| .$$

But again by (9.1a), this number is equal to $|\mathcal{B}|$. It follows that $\mathcal{B}$ spans V, and this concludes the proof.

In chapter IV, we shall see that the sign ε_ψ appearing in (9.1) is independent of $\psi \in \Omega$, and that it is in fact equal to $\varepsilon = (-1)^{\sigma(\underline{T})}(-1)^{\sigma(\underline{G})}$ (see (4.3b)). We shall see also that we may extend to all of Y the formulas

(9.8)
$$\chi_{\psi,i}|X = (\varepsilon_{\psi,i}/n_\psi)\psi^N|X,$$
$$\chi_{\psi,\lambda}|X = \varepsilon_\psi(\psi\lambda)^N|X,$$
$$\chi|X = 0$$

(for all $\psi \in \Omega$, all $i \in \{1,\dots,n_\psi\}$, all $\lambda \in \Lambda_\psi$, and all $\chi \in \mathrm{Irr}(G)$ which are distinct from all $\chi_{\psi,i}$ and all $\chi_{\psi,\lambda}$) which are given in (9.1).

Assuming that the structure of T, the structure of N/T, and the manner in which N/T acts on T are known, (9.8) provides (up to sign) certain values for all irreducible characters of G. This assumption is often quite reasonable. That the structure of T is known in many cases can be seen by reference to Springer and Steinberg [13] and to Gager [9]. By (6.5), N/T is cyclic, so we need only its order to determine its structure entirely. Finally, (1.5c) gives the action of N/T on T under commonly occurring conditions.

By way of contrast with (9.8), which treats character values on E, we now discuss character values on the elements of $G \setminus E$ which are conjugate to some element of T. We retain the notation of (9.1). Let $\mathcal{A}$ be the subring of $\mathbb{C}$ obtained by adjoining to $\mathbb{Z}$ all $|Q|^{\text{th}}$ roots of 1. Then $\mathcal{A}\cdot|R|$ is an ideal in $\mathcal{A}$.

(9.9) THEOREM. (a) Let $\psi \in \Omega$ and $i \in \{1,\ldots,n_\psi\}$. Then in $\mathcal{A}$,

$$\chi_{\psi,i}(t) \equiv (\varepsilon_{\psi,i}/n_\psi)\psi^N(t) \pmod{\mathcal{A}\cdot|R|}$$

for all $t \in Q^*$, and in $\mathbb{Z}$,

$$\chi_{\psi,i}(1) \equiv \varepsilon_{\psi,i}[N:N_\psi] \pmod{|R|}.$$

(b) Let $\psi \in \Omega$ and $\lambda \in \Lambda_\psi$. Then in $\mathcal{A}$,

$$\chi_{\psi,\lambda}(t) \equiv \varepsilon_\psi(\psi\lambda)^N(t) \pmod{\mathcal{A}\cdot|R|}$$

for all $t \in Q^*$, and in $\mathbb{Z}$,

$$\chi_{\psi,\lambda}(1) \equiv \varepsilon_\psi[N:T] \pmod{|R|}.$$

(c) Let $\chi \in \mathrm{Irr}(G)$ be distinct from all $\chi_{\psi,i}$ and all $\chi_{\psi,\lambda}$. Then in $\mathcal{A}$,

$$\chi(t) \equiv 0 \pmod{\mathfrak{a}\cdot|R|}$$

<u>for</u> <u>all</u> $t \in Q^*$, <u>and</u> <u>in</u> $\mathbb{Z}$,

$$\chi(1) \equiv 0 \pmod{|R|}.$$

PROOF. (a) Fix $\psi_0 \in \Omega$ and $i \in \{1,\ldots,n_{\psi_0}\}$. Set $\Omega' = \Omega \setminus \{\psi_0\}$. By Frobenius reciprocity and (9.1a), for all $\psi \in \Omega$ and all $\lambda \in \Lambda_\psi$,

$$(\psi - \psi\lambda,\ \chi_{\psi_0,i})_T = (\psi^G - (\psi\lambda)^G,\ \chi_{\psi_0,i})_G$$

$$= \begin{cases} \varepsilon_{\psi_0,i} & \text{if } \psi = \psi_0 \\ 0 & \text{if } \psi \neq \psi_0. \end{cases}$$

Therefore, setting $a_\psi = (\psi,\ \chi_{\psi_0,i})_T$ for each $\psi \in \Omega$, we have $(\psi\lambda,\ \chi_{\psi_0,i})_T = a_\psi$ for all $\lambda \in \Lambda_\psi$ if $\psi \in \Omega'$, and $(\psi_0\lambda,\ \chi_{\psi_0,i})_T = a_{\psi_0} - \varepsilon_{\psi_0,i}$ for all $\lambda \in \Lambda_{\psi_0}$.

For each $\psi \in \Omega$, let D_ψ be a right transversal of N_ψ in N. Recall that each element of $T^\wedge$ can be expressed uniquely in the form $\psi\lambda$ ($\psi \in Q^\sim$, $\lambda \in R^\sim$), and that if $t \in Q$, then $\psi\lambda(t) = \psi(t)$. If $\theta \in T^\wedge$ and $h \in N$, then by Frobenius reciprocity, $(\theta^h,\ \chi_{\psi_0,i})_T = ((\theta^h)^N,\ \chi_{\psi_0,i})_N = (\theta^N,\ \chi_{\psi_0,i})_N = (\theta,\ \chi_{\psi_0,i})_T$. By virtue of these considerations we compute that for each $t \in Q$, $\chi_{\psi_0,i}(t) = \sum_{\theta\in T^\wedge} (\theta,\ \chi_{\psi_0,i})_T\,\theta(t)$

$$= \sum_{\psi\in\Omega'}\ \sum_{h\in D_\psi}\ \sum_{\lambda\in R^\sim} a_\psi \psi^h\lambda(t) \ +$$

$$\sum_{h\in D_{\psi_0}} \Big[a_{\psi_0}\psi_0{}^h(t) + \sum_{\lambda\in(R^\sim)^*} (a_{\psi_0} - \varepsilon_{\psi_0,i})\psi_0{}^h\lambda(t)\Big]$$

$$= \sum_{\psi\in\Omega'} \sum_{h\in D_\psi} |R|\, a_\psi \psi^h(t) \;+$$

$$\sum_{h\in D_{\psi_0}} \left[a_{\psi_0}\psi_0^{\,h}(t) + (|R| - 1)(a_{\psi_0} - \varepsilon_{\psi_0,1})\psi_0^{\,h}(t)\right]$$

$$= |R|\Big[\sum_{\psi\in\Omega} \sum_{h\in D} a_\psi \psi^h(t) - \varepsilon_{\psi_0,1} \sum_{h\in D_{\psi_0}} \psi_0^{\,h}(t)\Big] \;+$$

$$\varepsilon_{\psi_0,1} \sum_{h\in D_{\psi_0}} \psi_0^{\,h}(t).$$

All summands of the last expression lie in $\mathfrak{a}$, so $\chi_{\psi_0,1}(t) \in \mathfrak{a}$. Since $\sum_{h\in D_{\psi_0}} \psi_0^{\,h}(t) = (1/n_{\psi_0})\psi_0^{\,N}(t)$, the first part of (a) follows. Setting $t = 1$ gives the second part.

(b) Fix $\psi_0 \in \Omega$ and $\lambda_0 \in \Lambda_{\psi_0}$. Then by Frobenius reciprocity and (9.1a), for all $\psi \in \Omega$ and all $\lambda \in \Lambda_\psi$,

$$(\psi - \psi\lambda,\ \chi_{\psi_0,\lambda_0})_T = \begin{cases} -\varepsilon_\psi & \text{if } \psi = \psi_0 \text{ and } \lambda = \lambda_0 \\ 0 & \text{otherwise.} \end{cases}$$

Now an argument very much in the spirit of the proof of (a) proves (b).

(c) Let $\chi \in \mathrm{Irr}(G)$ be distinct from all $\chi_{\psi,1}$ and all $\chi_{\psi,\lambda}$. Then for all $\psi \in \Omega$ and all $\lambda \in \Lambda_\psi$, (9.1e) and the fact that $\psi - \psi\lambda$ vanishes on $T \setminus X$ imply that $(\psi - \psi\lambda, \chi)_T = 0$. Again we conclude the proof by arguing as for (a).

(9.10) COROLLARY. <u>Let</u> $\chi \in \mathrm{Irr}(G)$. <u>Then</u> $|R| \mid \chi(1)$ <u>if</u> χ <u>is distinct from all</u> $\chi_{\psi,1}$ <u>and all</u> $\chi_{\psi,\lambda}$, <u>and</u> $\chi(1)$ <u>is relatively</u>

<u>prime to</u> $|R|$ <u>otherwise</u>.

PROOF. Since $R \lhd N$ and each element of R^* is locally regular, R^* is a union of conjugacy classes of N, each of which has order $[N:T]$. Therefore $[N:T] \mid (|R| - 1)$, and it follows that $([N:T], |R|) = 1$. In view of this remark, the corollary is an immediate consequence of the theorem.

CHAPTER IV. THE CONNECTION WITH THE DELIGNE-LUSZTIG THEORY

The notation and assumptions with which we opened chapter III remain in force throughout this chapter. In addition, we shall use the notation introduced in sections 4, 7, and 9.

§10. Decomposing $R_{\underline{T}}^{G}(\theta)$

In §9 we utilized the fact that if γ is a class function of N vanishing off X, then $\gamma^G|_X = \gamma|_X$ (see (2.3a)). In this section however, we shall need the following stronger result.

(10.1) LEMMA. <u>If γ is any class function of</u> N, <u>then</u> $\gamma^G|_X = \gamma|_X$.

PROOF. Let $x \in X$. Then $\gamma^G(x) = (1/|N|)\sum_{g\in G} \dot{\gamma}(g^{-1}xg)$ where $\dot{\gamma}$ is the function on G which agrees with γ on N and takes the value 0 elsewhere. Hence it suffices to show that if $g^{-1}xg \in N$, then $g \in N$.

Let $y = g^{-1}xg \in N$. $x \in X$ implies that $r_j \,\big|\, |y|$ for some j. Therefore there exist elements a, b $\in$ N such that $y = ab = ba$ and $|a| = r_j^{\,s}$ for some positive integer s. Now $R_j \in \mathrm{Syl}_{r_j}(G)$, so $R_j \in \mathrm{Syl}_{r_j}(N)$ as well. Moreover, $R_j \lhd T \lhd N$, and R_j is characteristic in T, hence $R_j \lhd N$. It follows that $a \in R_j^* \leq X$, therefore a is locally regular in T. Now by (5.2b), $b \in C_G(a) = T$, so $g^{-1}xg = ab \in X$. Since by (7.3) and (2.2a) X is a T.I. set in G with normalizer N, it follows that $g \in N$, as desired.

Let $\varepsilon = (-1)^{\sigma(\underline{T})}(-1)^{\sigma(\underline{G})}$, as in §4.

(10.2) THEOREM. <u>For each</u> $\psi \in \Omega$, <u>each</u> $i \in \{1,\ldots,n_\psi\}$, <u>and each</u> $\lambda \in \Lambda_\psi$, <u>let</u> $\chi_{\psi,i}$, $\varepsilon_{\psi,i}$, $\chi_{\psi,\lambda}$ <u>and</u> ε_ψ <u>be as in</u> (9.1). <u>Then</u> $\varepsilon_\psi = \varepsilon$. <u>Moreover</u>, (<u>after a possible relabeling of the characters</u> $\chi_{\psi,1},\ldots,\chi_{\psi,n_\psi},\chi_{\psi,\lambda}$ <u>and the signs</u> $\varepsilon_{\psi,1},\ldots,\varepsilon_{\psi,n_\psi}$, ε <u>in case</u> $|\Lambda_\psi| = 1$) $R^{\underline{G}}_{\underline{T}}(\psi\lambda) = \varepsilon\chi_{\psi,\lambda}$ <u>and</u> $R^{\underline{G}}_{\underline{T}}(\psi) = \sum_{i=1}^{n_\psi} \varepsilon_{\psi,i}\chi_{\psi,i}$.

PROOF. For any $\theta \in T^\wedge$, we write $R^{\underline{G}}_{\underline{T}}(\theta)$ in the abbreviated form $R(\theta)$. Fix $\psi \in \Omega$ and $\lambda_0 \in \Lambda_\psi$. Since by (5.2a) each element of X is regular, (4.3d) implies that for all $\lambda \in \Lambda_\psi$, $R(\psi) - R(\psi\lambda)$ and $\psi^G - (\psi\lambda)^G$ agree on $E = \bigcup_{g\in G} X^g$. From the fact that for all $\lambda \in \Lambda_\psi$, $\psi - \psi\lambda$ vanishes on $T \setminus X$, it follows immediately that $\psi^G - (\psi\lambda)^G$ vanishes on $G \setminus E$, and it follows from the character formula (4.3c) that $R(\psi) - R(\psi\lambda)$ vanishes on $G \setminus E$ as well. Thus by (9.1a),

$$(10.3) \qquad R(\psi) - R(\psi\lambda) = \left(\sum_{i=1}^{n_\psi} \varepsilon_{\psi,i}\chi_{\psi,i}\right) - \varepsilon_\psi\chi_{\psi,\lambda}$$

for all $\lambda \in \Lambda_\psi$.

Now it suffices to show that $R(\psi) = \sum_{i=1}^{n_\psi} \varepsilon_{\psi,i}\chi_{\psi,i}$. Indeed, since $\psi\lambda$ is regular for all $\lambda \in \Lambda_\psi$, it is in general position, hence by (4.3b) $\varepsilon R(\Psi\lambda) \in \mathrm{Irr}(G)$. Since by (4.3a) $(R(\Psi), R(\Psi\lambda_0)) = 0$, it will follow at once that $R(\Psi\lambda_0) = \varepsilon\chi_{\Psi,\lambda_0}$. Moreover, using (10.1) and (9.1d), $\varepsilon_\Psi = \varepsilon$ will then be a consequence of the fact that

$$\varepsilon\chi_{\Psi,\lambda_0}|_X = R(\Psi\lambda_0)|_X = (\Psi\lambda_0)^G|_X = (\Psi\lambda_0)^N|_X = \varepsilon_\Psi\chi_{\Psi,\lambda_0}|_X.$$

So suppose that $R(\Psi) \neq \sum_{i=1}^{n_\Psi} \varepsilon_{\Psi,i}\chi_{\Psi,i}$. For all $\lambda \in \Lambda_\Psi$, $\varepsilon R(\Psi\lambda) \in \mathrm{Irr}(G)$ and $(R(\Psi), R(\Psi\lambda)) = 0$. Thus (10.3) implies that $\varepsilon R(\Psi\lambda_0) = \chi_{\Psi,k}$ for some k. It follows that $\chi_{\Psi,k}$ is not an irreducible constituent of $R(\Psi)$. But then by (10.3) again, $\chi_{\Psi,k} = \varepsilon R(\Psi\lambda)$ for all $\lambda \in \Lambda_\Psi$. By (9.1a), this forces $|\Lambda_\Psi| = 1$.

Now R is a cyclic normal subgroup of N, so the same holds for all subgroups of R. Therefore from $|\Lambda_\Psi| = 1$, which implies that the number of conjugacy classes of N contained in R^* is also 1, we may conclude that R is simple, i.e., that $R = \langle x\rangle$, where $x \in R$ has order r and r is the unique member of S(G,T).

It follows that the irreducible characters in $B_\Psi(r)'$ consist precisely of $\chi_{\Psi,1},\ldots,\chi_{\Psi,n_\Psi}$ (the non-exceptional characters in $B_\Psi(r)'$), and χ_{Ψ,λ_0} (the unique exceptional character in $B_\Psi(r)'$). Since $S(G,T) = \{r\}$, all of our previous results follow as a direct consequence of Dade's work in [6] without the intercession of a result such as (9.1) to reconcile the information provided by the block theories relative to several distinct special primes. But

$\Lambda_\psi = \{\lambda_0\}$, hence Dade's results are independent of which character among $\chi_{\psi,1},\ldots,\chi_{\psi,n_\psi},\chi_{\psi,\lambda_0}$ is called the exceptional character in $B_\psi(r)$' (see p. 38 of [6]). Therefore all of our previous results hold if we relabel these characters (and the corresponding signs) in such a way that $\chi_{\psi,\lambda_0} = \varepsilon R(\psi\lambda_0)$. After such a relabeling, it follows that

$$R(\psi) = \sum_{i=1}^{n_\psi} \varepsilon_{\psi,i}\chi_{\psi,i},$$

as desired. This concludes the proof of the theorem.

It should be remarked that the information contained in (10.2) is complete in the sense that

$$\{R_{\underline{T}}^{G}(\theta):\ \theta \in T^\wedge\} = \{R_{\underline{T}}^{G}(\psi\lambda):\ \psi\in\Omega,\ \lambda\in\Lambda_\psi\}.$$

This follows by (7.6) and the character formula (4.3c).

Thanks to (10.2), the multiplicity of each $\chi \in \mathrm{Irr}(G)$ in each $R_{\underline{T}}^{G}(\theta)$ is known (up to sign). Therefore for $\chi \in \mathrm{Irr}(G)$, we may apply the formula 7.6.2 of Deligne-Lusztig [7], which states that for all regular elements y in T,

$$\chi(y) = \sum_{\theta\in T^\wedge} \theta(y)(\chi, R_{\underline{T}}^{G}(\theta)).$$

A computation which is by now familiar then shows that the formulas (9.8) are valid with X replaced by Y.

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