LINES AND CONIC SECTIONS IN A FAMILY OF QUARTIC SURFACES

by

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A THESIS

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This thesis is primarily concerned with the fact that the general quartic surface does not contain a line or a conic section. More formally, the set of quartic surfaces containing lines or conic sections has measure zero.

This result follows from an important result in algebraic geometry, the Noether-Lefschetz theorem; however, this paper will use elemenary techniques in algebra and analysis to come to the same conclusion. In addition, this paper will introduce the idea of curves, surfaces, and their parameter spaces.

Acknowledgements

Thank you to my committee members, Dr. Addington, Corey Brooke, and Dr. Shoop. I also have to thank all the friends and family on which I tested my thesis. Finally, thank you to the Mathematics department and the Clark Honors College for facilitating this process.

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1 Introduction

Geometric shapes can be described by equations. This one simple concept is the foundation of a variety of complex mathematical ideas, discovered over centuries. The idea that the geometric properties of a shape are related to the algebraic properties of a respective equation came to the fore in the work of Descartes and Fermat in the 17th century, but these connections were being made for thousands of years prior [1]. In this paper, we will explore some of these connections, culminating in a discussion of a specific type of geometric shapes and their properties. The first half of this paper is intended to be accessible to an audience without much background in mathematics and to give an intuitive explanation of the results of the paper. The second half will require knowledge of calculus, analysis, and algebra, in order to prove these results.

2 Curves

2.1 Lines in a plane

Pairs of numbers can be represented as points on a plane. Specifically, if we identify a point in a plane as the origin or center of the plane, we can link the pair of numbers, $x = -1, y = 2$, with the point that is 1 unit left of the origin and two units up from the origin (see figure 1).

Consider the equation $4x + y - 6 = 0$. Given pairs of number, we can ask whether they satisfy this equation or not.

For example, $x = 1, y = 2$ gives us $4 + 2 - 6 = 0$, which is true.

On the other hand, $x = -1, y = 3$ gives us $-4 + 3 - 6 = -7$, so this point does not satisfy the equation.

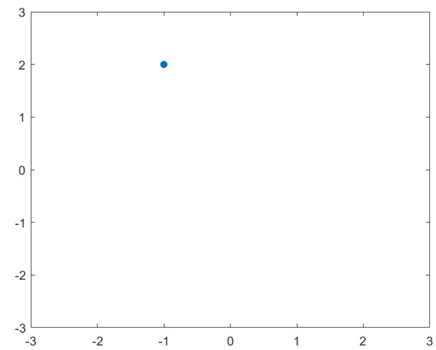

Figure 1: The point given by $(-1,2)$

Since we know which pairs of numbers satisfy $4x+y-6=0$, we can ask which points on the plane correspond to these solutions. As it turns out, these points will exactly make up a line on the plane. This equation might be more familiar in the form $y=-4x+6$ (see figure 2).

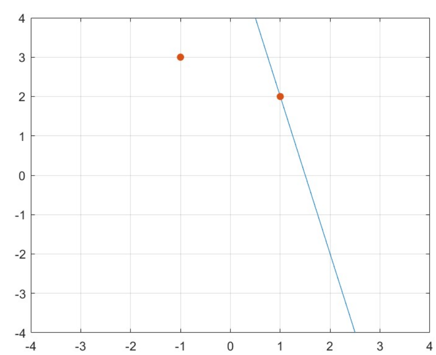

Figure 2: $4x+y-6=0$ or $y=-4x+6$ with 1 point on and 1 point off

In fact, any equation of the form $ax+by+c=0$ will give a line on the xy-plane.

2.2 Conic sections

Next, we can consider an equation such as $y-x^2=0$. Just as before, we can consider the points that satisfy this equation. $x=0, y=0$ give us $0-0=0$, so it satisfies the equation. Similarly $x=1, y=1$ satisfies the equation. However, $x=2, y=2$ does not satisfy the equation, so the set of solutions is not a line. It will actually be a curve known as a parabola (see figure 3).

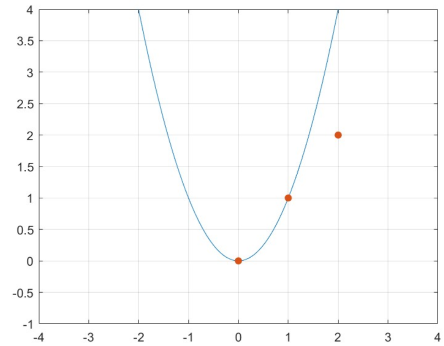

Figure 3: $y - x^2 = 0$ or $y = x^2$

An interesting property of a parabola allows it to be used in devices like satellite dishes, whose cross section are often parabolas. This has to do with the idea of reflections. All possible vertical rays that reflect off the inside of a parabola will intersect at a single focus point (see figure 4). In the case of satellite dishes, this allows signals to reflect off of the surface of dish and converge at whatever device is actually receiving them.

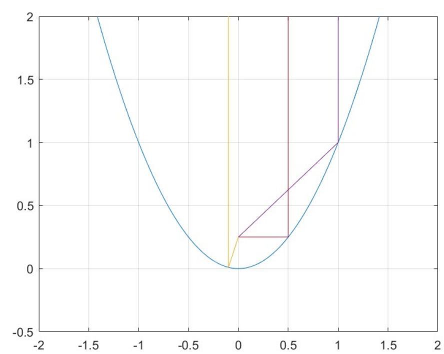

Figure 4: $y - x^2 = 0$ or $y = x^2$

Another familiar curve is given by the equation $x^2 + y^2 - 1 = 0$. This is simply a circle of radius 1 (see figure 5).

A circle is an example of a more general type of conic section called an ellipse, given by $ax^2 +$

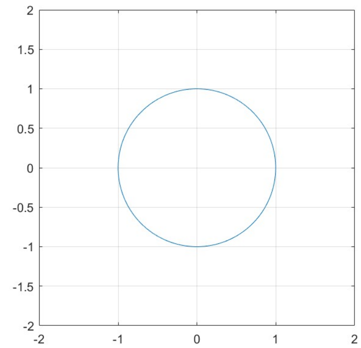

Figure 5: $x^2 + y^2 - 1 = 0$ or $x^2 + y^2 = 1$

$by^2 - 1 = 0$ (see figure 6).

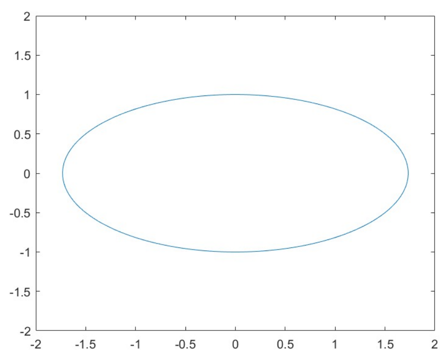

Figure 6: $x^2/3 + y^2 - 1 = 0$ or $x^2 + 3y^2 = 3$

All ellipses have a property similar to that of parabolas. There are two special points in each ellipse with the property that every ray emanating from one point will be reflected towards the other. This has been famously used to construct whispering galleries.

Similarly, the curve given by $xy - 1 = 0$ is an example of a hyperbola (see figure 7).

In general, we call $ax^2 + bxy + cy^2 + dx + ey + f$, a polynomial of degree 2 or a quadratic polynomial. (We will explain this idea of degree later.) Any curve described by an equation of the form $ax^2 + bxy + cy^2 + dx + ey + f = 0$ is called a conic section. This is because each curve can be drawn as the intersection of a cone and a plane. Conic sections were extensively studied by Ancient Greek

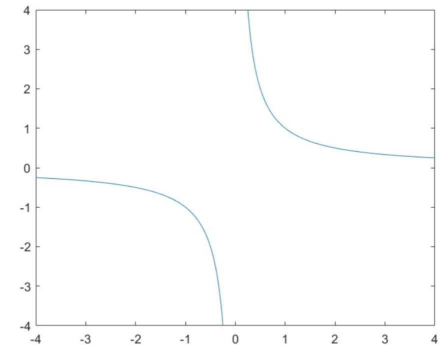

Figure 7: $xy - 1 = 0$ or $y = \frac{1}{x}$

mathematicians [4]. In fact, Apollonius of Perga first classified them into hyperbolas, parabolas, and ellipses. It is possible that Greek interest in conic sections partly originated from sundials. The reason is that the rays originating from the sun that are blocked by the point of a sundial form a cone as the sun moves through the sky. If the shadow is cast on a plane, it should trace a conic through the day.

2.3 Cubic curves

We could also consider the points satisfied by the equation $y = x^3$ (see figure 8).

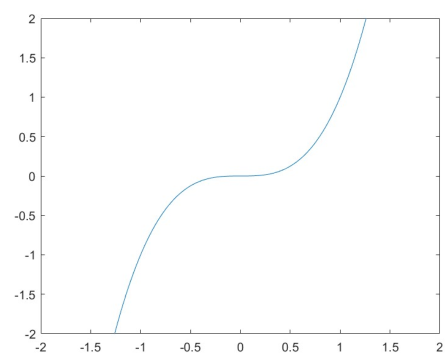

Figure 8: $y - x^3 = 0$ or $y = x^3$

Such a curve, satisfied in general by $ax^3 + bx^2y + cxy^2 + dy^3 + ex^2 + fxy + gy^2 + hx + iy + j = 0$ is called a cubic curve. A specific type of cubic curve, called the elliptic curve, has important applications in cryptography. Specifically, mathematicians are interested in the group of solutions to elliptic curves and an interesting geometric operation that allows us to "add" two points on the curve to get another one. Figure 9 is an example of an elliptic curve.

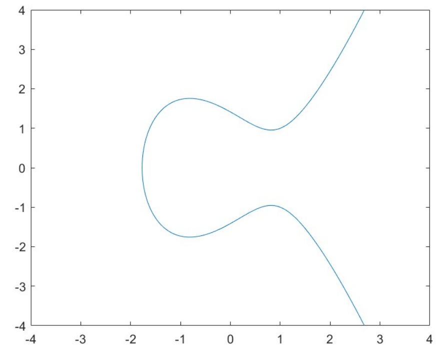

Figure 9: $-y^2+x^3-2x+2=0$

2.4 Degree of curves

So far we have categorized our curves as lines given by $ax+by+cz-d=0$, conic sections given $ax^2+bxy+cy^2+dx+ey+f=0$, and cubic curves given by $ax^3+bx^2y+cxy^2+dy^3+ex^2+fxy+gy^2+hx+iy+j=0$. Observe that the variables in the first case have exponents that are at most 1, while conic sections can also have exponents of 2, and cubics can have exponents that are up to 3. This is the notion of the degree of a polynomial. It is important to note that we refer here not the the individual exponents of both variables in each term, but rather their sum. So an xy term is allowed in a degree-2 polynomial, but an x^2y term is not allowed, because $2+1$ is bigger than 2. In this way we can continue to categorize curves given by polynomials of degree 4, 5, 6, and beyond. However, we will not explore these higher degree curves in this paper. The reason is that, while higher-degree curves are more complex, lower degree curves have much more structure. For example conic sections are already quite complex, but they can still be categorized into 3 main types. It becomes more and more difficult to make similar generalizations if we increase the degree of our polynomial.

3 Surfaces in space

Just as we identify a pair of numbers with a point on the plane, we can identify a triple of numbers with a point in space. Likewise, we can also consider polynomial equations in 3 variables, often x, y, z. Then, we can consider which points in space correspond to solutions to a given polynomial equation.

We will find that the solutions are surfaces, instead of curves. We will categorize these surfaces by the degree of their polynomial equations as well, beginning with first degree polynomials.

3.1 Planes

The solutions to the equation, $ax+by+cz+d=0$, is just a plane. The polynomial, $ax+by+cz+d$, is a first degree polynomial, also called a linear polynomial (see figure 10). This is the simplest type of surface we can construct.

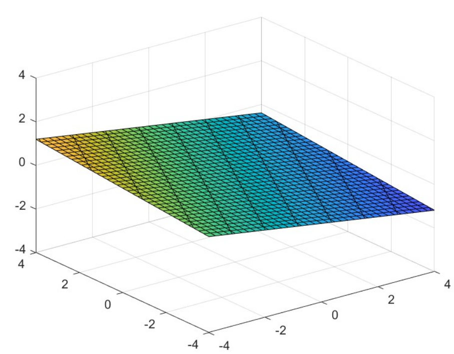

Figure 10: $2x-y+10z+0$

3.2 Quadric surfaces

The solution to a degree-2 polynomial equation in 3 variable, $ax^2+by^2+cz^2+dxy+exz+fyz+gx+hy+iz+j=0$, is a quadric surface.

The study of various quadric surfaces is very rich. Interestingly, when quadric surfaces contain a line, they contain infinitely many other lines. In fact, some quadric surfaces, such as hyperbolic paraboloids, are ruled surfaces, meaning that every point of the surface is contained in a line. (A plane is also a ruled surface, which can be easily determined visually.) These surfaces are commonly used in architecture [7]. The reason is relatively practical. You can make a complex and visually appealing surface using just straight beams. Some of the most significant examples of hyperboloid architecture come from the architect, Vladimir Shukhov, who pioneered the use of metal lattices. In 1896, he made his first of many hyperboloid towers at the All-Russian Industrial Art Exhibition in Nizhniy Novgorod (see figure 11) [3]. While Shukhov created many of the earliest hyperboloid structures, these types of quadric surfaces continue to inspire modern-day architects.

Figure 11: Shukhov's 1896 tower [6]

Below is a video of the lines that make up a cone given by $x^2+y^2+z^2=0$, which is a simple example of a quadric ruled surface: https://youtu.be/1qIeMC57-v4 Its lines are all those that pass through the origin with a 45° angle with the xy-plane.

Also linked below is a video of the lines that make up the quadric surface given by $x^2+y^2-z^2-1$: https://youtu.be/mfOafDzA08k. This example is perhaps more surprising. As an interesting side note, the lines on the surface can be parameterized as $(\cos(\theta)+\sin(\theta)t, \sin(\theta)-\cos(\theta)t, t)$, where θ ranges from 0 to 2π, and t ranges from $-\infty$ to ∞.

A familiar example of a quadric surface (one without any lines) is a sphere, as it is the solution of the polynomial equation $x^2+y^2+z^2-1=0$ (see figure 12). This is an example of a quadric surface that is certainly not a ruled surface, as it contains no lines.

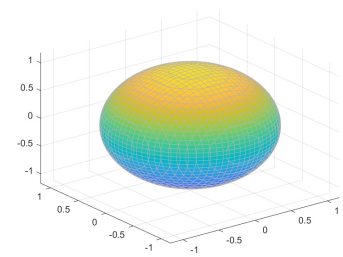

Figure 12: $x^2+y^2+z^2-1$

As an aside, while it is evident visually, the fact that a circle contains no lines can be found algebraically from only the equation.

Quadric surfaces always contain conic sections. In fact, every vertical cross section through the origin of the below quadric surface is a parabola (see figure 13). This is an example of the type of quadric surface that might approximate the shape of a satellite dish.

3.3 Cubic surfaces

Next, we can consider the types of surfaces that are solutions to cubic polynomials, which are simply called cubic surfaces. Figure 14 is an example of a cubic surface.

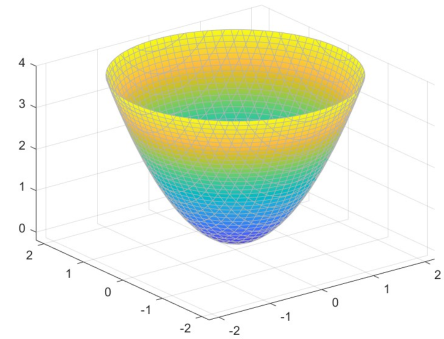

Figure 13: $x^2+y^2-z=0$

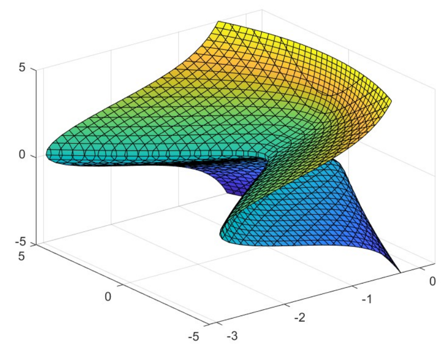

Figure 14: $x^3+y^2+3xz^2+z+1$

Unlike quadrics, these contain only finitely many lines. Specifically, a cubic will contain exact 27 complex lines [5]. This means that every cubic surfaces will contain at least 1 real line, since properly complex lines come in complex conjugate pairs. In fact, any cubic curve contains exactly 3, 7, 15, or 27 real lines. (The set of the complex conjugates of a line forms its own line.) The surface below is called the Clebsch surface, which contains 27 real lines (see figure 15). The equation of this special surface is a bit long:

$$81(x^3+y^3+z^3)-189(x^2y+x^2z+y^2x+y^2z+z^2x+z^2y)+54xyz+126(xy+xz+yz)-9(x^2+y^2+z^2)-9(x+y+z)+1=0$$

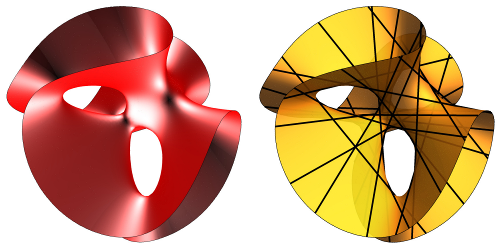

Figure 15: Clebsch surface [8]

3.4 Quartic surfaces

All the types of curves and surfaces we have discussed so far are fascinating in their own right. However, in this paper, we will be focusing specifically on quartic surfaces. These are solutions to fourth degree polynomials (see figure 16).

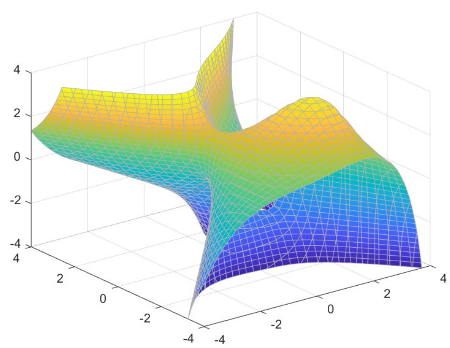

Figure 16: $x^4 - y^3z + 3xyz - 5y^2 + z^2 + xy + yz + 3xz - 1 = 0$

While some special quartic surfaces contain lines or conic sections, most do not, and the majority of the rest of the paper will be devoted to explaining this fact.

As we close our general description of the various types of curves and surfaces, it is worth noting that we can certainly look at the solutions to polynomials in more than 3 variables; however, these hypersurfaces cannot be visualized if the number of variables is greater than 3.

4 Spaces of curves and surfaces

At the beginning of this paper, we characterized all lines as solutions to equations of the form $ax+by+c=0$. Every point on this line exists on a plane, which we call a 2-dimensional space, corresponding to the two variables x and y. However, each polynomial $ax+by+c$ can be viewed as a point in 3-dimensional space with variables, a, b, and c. We can call this 3-dimensional space a parameter space.

We can consider the spaces of other curves as well. Knowing that every conic section can be expressed as the solution to $ax^2+bxy+cy^2+dx+ey+f=0$ for some a, b, c, d, e, and f. So we can think of every conic section (or every second degree polynomial in x and y) as a point in 6-dimensional space. We cannot visualize a 6-dimensional space, like we can for 2 or 3 dimensions.

It is actually incredibly useful to consider the spaces of various types of curves and surfaces and the dimensions of these spaces. For example, if we write a conic section as $ax^2+bxy+cy^2+dx+ey+f=0$, it turns out that the expression b^2-4ac can determine what kind of curve we have. Plugging in our coefficients, if it's negative, then our conic is an ellipse or circle; if positive, then it is a hyperbola; and if it's zero, then the conic is a parabola.

In the example of a circle, we observe that $b^2-4ac=0^2-4\times1\times1=-4$ (see figure 17).

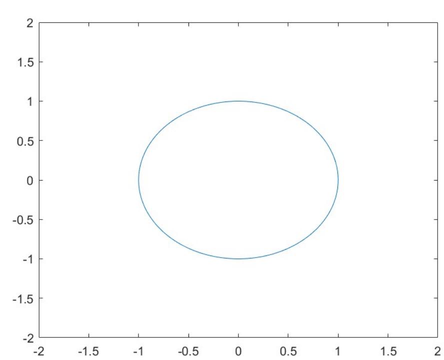

Figure 17: $x^2+y^2-1=0$: $a=1$, $b=0$, and $c=1$

In the example of the parabola, given by $y-x^2=0$, $b^2-4ac=0^2-4\times1\times0=0$ (see figure 18).

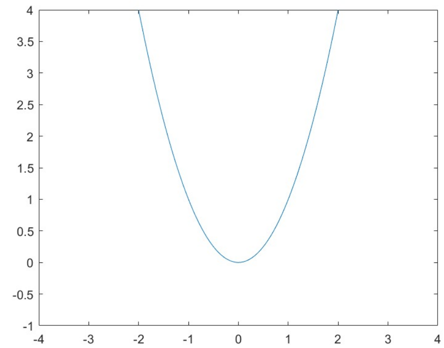

Figure 18: $-x^2+y=0$: $a=-1$, $b=0$, and $c=0$

In the case of the hyperbola, given by $xy-1$, $b^2-4ac=1-4\times 0\times 0=1$ (see figure 19).

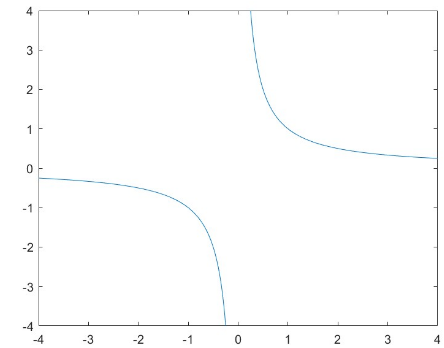

Figure 19: $xy-1=0$: $a=0$, $b=1$, and $c=0$

There is an interesting distinction to be made between parabolas, which occur for only one value of b^2-4ac, and the other two types, which occur for the rest. We can uniquely identify each parabola with a choice of a,b,d,e,f, since given a,b and the fact that $b^2-4ac=0$, we can solve for the only value of c that gives us a parabola. So the space of quadratic polynomials in 2 variables that correspond to parabolas is only 5-dimensional.

If we graph the surface defined by $b^2-4ac=0$ in space with points (a,b,c), we get the surface in figure 20.

Every point in the space above represents a choice of a,b,c, which in turn gives us lots of conic sections (since we have not chosen d,e,f). However, only the points that happen to fall exactly on the surface give us

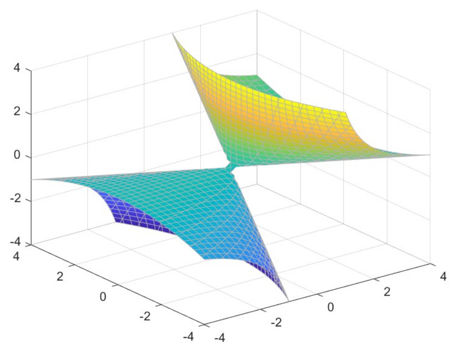

Figure 20: Graph of $b^2 - 4ac$

parabolas.

The video below show a family of conic sections and the point in (a,b,c) corresponding to each one. As the point passes through the surface $b^2 - 4ac$, our conic becomes a parabola: https://youtu.be/yrZjofM4nPg.

Informally, if we were to choose a random conic section, you would not expect that it happens to be a parabola. Just as there is a 0% chance that a random point on a plane (a 3-dimensional space) happens to land on a line (a 2-dimensional subspace), it will not land on a point in 6-dimensional space happens to land on a 5-dimensional subspace.

One way we can express this notion is by saying that a general conic section is not a parabola. This again illustrates the idea that parabolas are incredibly rare in the space of all conic sections.

We can see the rarity of a parabola can be seen more visually by considering conic sections as the intersection of planes and cones. To generate a parabola, we have to choose a plane that is parallel to the side of the cone (see figure 21). Any other orientation of the plane will give us a hyperbola or ellipse (see figure 22 and 23).

Later in this paper, we will argue that quartic surfaces that contain lines as well as quartic surfaces that contain conic sections are rare compared to all quartic surfaces in the same sense as the parabola being rare compared to all conic sections. Specifically, we argue that the space of all quartic polynomials in 3 variables is 35-dimensional, whereas the space of quartic polynomials corresponding to surfaces containing lines is 34-

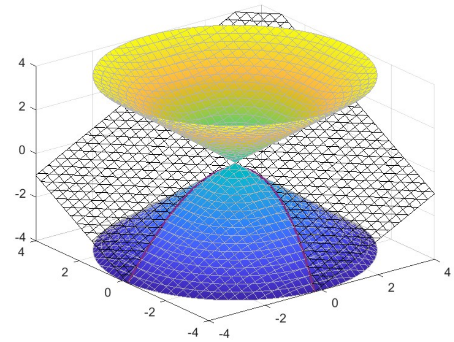

Figure 21: A parabola given as an intersection of a plane and a cone

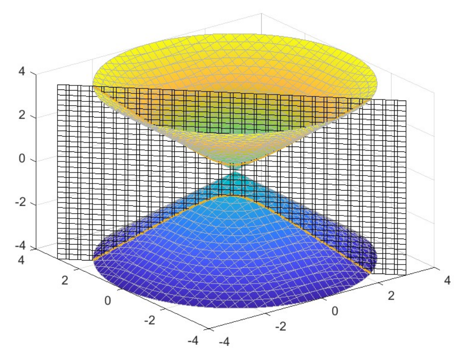

Figure 22: A hyperbola given as an intersection of a plane and a cone

dimensional. In fact, the space of quartic surfaces containing conic sections is also 34-dimensional.

5 Curves in surfaces

In this section, we will explain more thoroughly what we mean by saying that a quartic contains a line or a quartic contains a conic. Or more generally, what does it mean for a surface to contain a curve?

We talked about curves existing in a plane as solutions to polynomials in two variables; however, it also makes sense to consider curves running through space or along a surface that is not perfectly flat like a plane. For example, if a point travels around the equator of a sphere, a type of surface, it will trace out a circle, a type of curve (see figure 24).

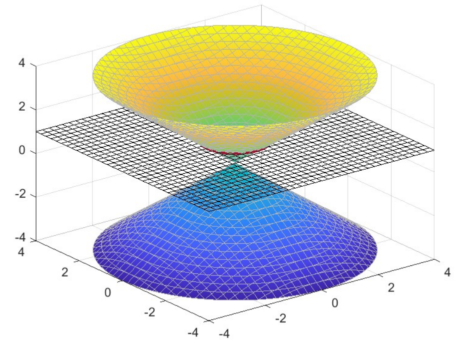

Figure 23: An ellipse given as an intersection of a plane and a cone

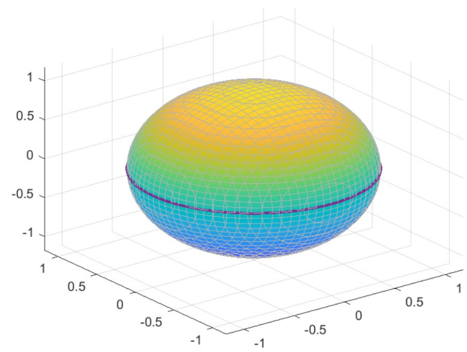

Figure 24: Sphere with circle running along the equator

Can a quartic surface contain a line? To answer this question, simply consider, as an example, the quartic polynomial given by $x^2y^2 + xyz^2 - x - y = 0$ (see figure 25). The line parameterized by t as $x = 0, y = 0, z = t$ is contained in the quartic surface, since $x = 0$, $y = 0$, and $z = t$ satisfy the equation, $x^2y^2 + xyz^2 - x - y = 0$, for each value of t. This is the line running along the z axis.

Later in this paper, we will answer the more difficult question of whether an arbitrary quartic surface contains a line. This is the question that the idea of dimension will help us answer.

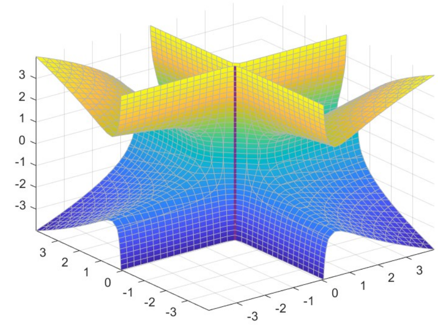

Figure 25: $x^2y^2 + xyz^2 - x - y$

6 Description of research

Over the summer of 2022, I had the chance to conduct research within the University of Oregon mathematics department's program for undergraduate research. I primarily worked with Dr. Nicolas Addington, who specializes in algebraic geometry, as well as Corey Brooke. Specifically, I worked on creating an animation of a one-parameter family of quartic surfaces using the Matlab software. Our family of quartic polynomials, h, is defined by the following equations:

$$f = 2x^3y + 2x^2y^2 + 2xy^3 + y^4 + y^3z - x^2z^2 + 2xyz^2 + y^2z^2 + yz^3 - 2z^4 + x^3 + 2xy^2 + 2y^3 + 2xyz + 2xz^2 + yz^2 + xy + z^2 - x$$

$$g = (y^2 + z^2 - 1)(-3 + 1x + 1y + 2z + 2xy - 2yz + 2xz + 1x^2 + 0z^2) +$$

$$+x(1 - 2x + 1y + 2z - 2xy + 1yz - 2x^2 + 2y^2 - 2z^2 + 0x^2y + 0x^2z - 2xyz + 2x^3 - 1xy^2 - 2xz^2 - 1y^2z + 0yz^2 + 1y^3 + 0z^3)$$

$$h = f\cos(t) + g\sin(t);$$

Two versions of the animation are linked here: the first with pauses to highlight the lines and conic sections: https://www.youtube.com/watch?v=6-zxnahvpis, and the second without: https://www.youtube.com/watch?v=

4OCYuKdmjMI.

These equations were chosen so that they would contain conic sections. Specifically, f contains two conic sections in the plane $y = 0$. The first one can be parameterized as $x = s^2$, $y = 0$, and $z = s$. It's a parabola. The second one is the ellipse, given by $x = cos(s)$, $y = 0$, $z = \frac{1}{\sqrt{2}} sin(s)$.

g has two conic sections in the plane $x = 0$. The first is just the unit circle given by $x = 0, y = cos(s), z = sin(s)$. The second is the hyperbola given by $x = 0, y = s, z = \frac{s-3}{2s-2}$.

We used a random number generator to choose integers between -2 and 2. These numbers were used as coefficients of parts of the polynomials that we had not yet determined. For example, the $(-3 + 1x + 1y + 2z + 2xy - 2yz + 2xz + 1x^2 + 0z^2)$ was mostly chosen using this method. The exception is the -3 on the constant term. We tested whether the polynomials we generated were smooth. If they weren't smooth, we made some change and tested it again. The -3 term is an artifact of this process. We wanted them to be smooth for aesthetic purposes. There will unavoidably be quartic surfaces with singularities elsewhere within our family.

The reason we chose both f and g to have conic sections in them is purely practical. In a future section, we will construct a method to find quartic surfaces with conics in a family of quartic surfaces. Unfortunately, this process is very computationally intense. We simply weren't able to find a solution that would allow a computer to solve that equation for us in a reasonable amount of time. On the other hand, the process for finding quartic surfaces with lines is much simpler. We used a software called mSolve to find the lines that were elsewhere in the family. In the end, our video contains the two points with conic sections we manually included and 5 lines, chosen mostly for aesthetic purposes from those found by mSolve.

7 Choosing a quartic with lines

In this section, we will explain why exactly the space of quartic surfaces containing lines is 34 dimensional, and why the space of all quartic surfaces is 35-dimensional. The rest of the paper will be more technical than the preceding sections.

To choose all quartic polynomials we have to choose all possible coefficients. In other words, we have a choice to make for each monic monomial in x, y, z with degree less than or equal to 4 such as y^4 or xz^2 [2]. We can count the number of these terms using combinatorics. The easiest way is to homogenize these monomials with a dummy variable. We will use w. So, for example, xz^2 become xz^2w, and 1 becomes w^4. In other words, our $0, 1, 2, 3, 4$-degree monomials in 3 variable are in one-to-one correspondence with 4-degree monomials in 4-variables. We can then count the number of quartic monomials in 4-variables with a common combinatorial trick. Counting this number is the same as counting the number of ways to distribute 4 identical things into 4 boxes. (The 4 exponents, if you will, have to be distributed to x, y, z, w.) This is the same as the number of ways to choose 3 boundaries out of 7 total slots. So, $*|**||*$ gives us xy^2z^0w, for example. We can count this using the choose function as $\binom{7}{3} = 35$.

So, we have found how many values we need to choose to get all quartic polynomials. In other words, the family of all quartic polynomials is a 35-parameter family. How many parameters do we need to cover all quartic polynomials with lines in them? The first step is to choose what line is contained in the quartic. To simplify this, let's assume that this line is not parallel to the xy-plane. This is true for almost all arbitrary lines. Then the line passes through the plane $z = 0$ and the plane $z = 1$ in one spot each. Let's call these points $(x_0, y_0, 0)$ and $(x_1, y_1, 1)$. Then we can parameterize the line as $x = x_0(1-s) + x_1 s$, $y = y_0(1-s) + y_1 s$ and $z = s$, as s goes from $-\infty$ to ∞. So we need 4 numbers to uniquely determine a line in space. Consider using polynomial long division to divide some quartic polynomial f by $(x_0(1-s) + x_1 s))$, which is a plane containing the line. If we want this quartic to contain the line, the remainder will not necessarily be 0; however, the remainder should be divisible by $(y - (y_0(1-z) + y_1 z))$. We should be able to write our quartic as $g(x - x_0(1-z) + x_1 z)) + h(y - (y_0(1-z) + y_1 z))$, where g and h have degree 3 or less, since the degree of the product of two polynomials is the sum of the degrees.

Any line is just the intersection of these two planes. Now we simply have to choose g and h. Using the combinatorial method described earlier, there are $\binom{6}{3} = 20$ parameters-worth of degree 3 or less polynomials in 3 variables. So, we may be tempted to say that was must choose 44 numbers to get all quartic surfaces with lines; however, something has clearly gone wrong, since quartic surfaces with lines are just a special type of quartic surfaces. In other words, if we need 44 parameters to choose each quartic with a line, we clearly need at least 44 parameters to choose every quartic with or without a line. The reason for this discrepancy is that we double-counted lots of these quartic surfaces, because there may be more than one way to write the same quartic as $g(x-x_0(1-z)+x_1z))+h(y-(y_0(1-z)+y_1z))$, where g and h are cubics.

Let's enumerate how many times we overcounted. Imagine that we had two pairs of cubics, g,h and g',h', giving us the same quartic. For brevity, let's rewrite $l_1 = x-x_0(1-z)+x_1z)$ and $l_2 = y-(y_0(1-z)+y_1z)$.

$$gl_1+hl_2 = g'l_1+h'l_2$$

$$\iff l_1(g-g') = l_2(h'-h)$$

l_1 and l_2 are different monic linear polynomials, which are irreducible . Since $\mathbb{R}[x,y,z]$ is a unique factorization domain, they are also prime [2]. We know that l_2 must divide $l_1(g-g')$, but $l_2 \neq l_1$, so l_2 does not divide l_1, since l_1 has an x in it and l_2 does not. Then, since l_1 is prime, l_1 must divide $(g-g')$.

By the same argument, l_1 must divide $h-h'$. So both sides of $l_1(g-g') = l_2(h'-h)$ will have a factor of l_1 and l_2, but each side should be a quartic. So, if we divide out by l_1l_2 on each side, we should get the same 2-degree polynomial.

In other words, we can write all duplicates of g and h as

$$g' = g+l_2i$$

and

$$h' = h-l_1i$$

where i is a quadratic (or less).

So for each quartic written as $gl_1 + hl_2$, every quadratic or less gives us a duplicate quartic that also contains the line. We can use our combinatorial trick again to count the number of degree $0,1,2$ polynomials, as $\binom{5}{3} = 10$. To get the actual dimension of our subspace of quartic surfaces containing lines, we simply take $44 - 10 = 34$. Notice that this is one less than the the number of values we need to chose to get all quartic surfaces, so an arbitrary quartic will not contain a line.

There are a few more details to consider. Firstly, to parameterize the line using only 4 variable, we had to assume that it wasn't parallel to the xy-plane. If it is parallel to the xy-plane, we could parameterize the line with 3-variables instead. Specifically, we could choose the z-value, z_0, the whole line is contained in. Then, we would have to assume that it's not also parallel to the xz plane, allowing us to uniquely determine the line by choosing points $(x_0, 0, z_0)$ and $(x_1, 1, z_0)$. Since the rest of the computation works out the same, this is a 33-dimensional vector space of quartic surfaces with lines with this specific property. Since it's one less than the 34-dimensional subspace, it doesn't increase the dimension of quartic surfaces with lines at all.

However, we are still missing some quartic surfaces with lines, since we just assumed that our quartic surfaces with lines parallel to the xy-plane weren't also parallel to the xz-plane. If they are, then we can parameterize these lines using just two variables: the y-coordinate of the line and the z-coordinate of the line. This specific case gives us a 32-dimensional subspace, which is again not going to add to our overall parameter count.

It is also worth noting that we implicitly assumed in this parameter count that a random quartic with a line only contains one line. We can certainly choose quartic surfaces that contain more than one line, as seen in figure 26.

So, it is possible that we might be counting a quartic more than once, since we pick a line first and then choose a quartic containing that line. However, we can already see that we need to choose only 34 values to get every quartic with a line. If we are still over-counting quartic surfaces, it just means that they are even more

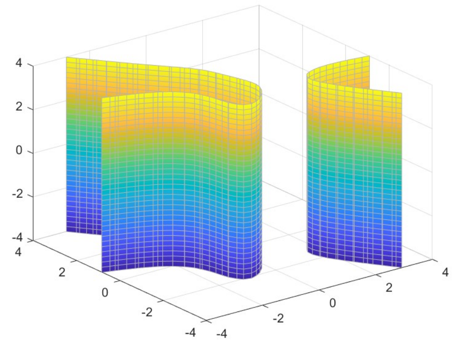

Figure 26: Quartic given by $x^4 + y^4 + 4x^3y - x^2y^2 - xy^3 + 10x + 3y + 1$

rare. As a result, it does not change our conclusion that a general quartic does not contain a line.

8 Choosing a quartic with conic sections

Let's also ask whether an arbitrary quartic contains conic sections. As with lines, we can construct quartic surfaces containing conic sections. Figure 27 is an example of such a quartic.

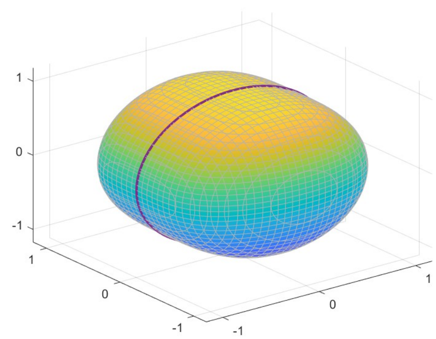

Figure 27: $x^2 + y^4 + z^2 - 1$

We will now proceed with a very similar parameter count for the subspace of quartic surfaces containing conic sections. We can choose the plane in which it's contained. A general plane can be written as $ax + by + cz + d$; however, we will assume that $a \neq 0$. This just means that the plane is not parallel to the x-axis.

Then, we can divide through by a, and write the plane as $l = x + b'y + c'z + d'$. Since it's not parallel to the x-axis, we can use yz-coordinates for the plane. So we can write the conic as some quadratic, q, in yz coordinates. If we view this quadratic in yz as a surface in space (which doesn't depend on x), this quadric surface intersects the plane at exactly the conic. Let's assume that the y^2 term is nonzero, and divide through. This allows us to uniquely determine our conic using 8 parameters, $\binom{4}{2} - 1 = 5$ for the quadratic and 3 for the plane.

If we divide our quartic by q, the remainder must be divisible by our plane l. So, similar to lines, we can write our quartic as

$$lg + qh$$

where g is cubic and h is quadratic.

Just like lines, we might prematurely assume that the parameter count is $8 + 10 + 20 = 38$. However, there may be more than one pair of g and h where $lg + qh$ is our quartic polynomial. Let g' be a cubic and h' be a quadratic such that

$$lg + qh = lg' + qh'$$

$$l(g - g') = q(h' - h)$$

Since l must have an x term in it and q does not have an x in any of its terms, l cannot divide q. Since l is irreducible, l just divides $h - h'$. Likewise, q divides $g - g'$, since q divides $l(g - g')$, but q and 1 don't share any factors. Since $(g - g')$ is cubic and k is quadratic, we can write $(g - g')$ as some qk, where k is linear. Then, we see that $q(h' - h) = l(g - g') = qlk \implies (h' - h) = lk$. Rearranging these expression, we can conclude that any duplicate is $lg' + qh'$ where $g' = g - qk$ and $h' = h + lk$. So our duplicates are exactly determined by our linear k, which has dimension $\binom{4}{3} = 4$. So the actual number of values we need to choose to get every quartic with conic sections is $8 + 10 + 20 - 4 = 34$. This is one dimension less than the 35-dimensional space of quartic surfaces, therefore we conclude that the general quartic does not contain a conic.

Just like with lines, we need to address some of the assumptions we made. If the plane is parallel to the x-axis, we can choose either y or z, and the argument would proceed exactly the same differing only in the names of the variables. Crucially, at least one of the coefficients for x, y, z must be nonzero, otherwise it wouldn't actually be a plane. For the conic, we will still be writing it as a quadratic in two of our 3 variable, so without loss of generality, we can still call them y and z. We assumed the y^2 term is nonzero. If it is 0, then let's assume that z^2 is nonzero. Then, we can parameterize our quadratic using only 4 variables. Likewise, if both the y^2 and z^2 terms are 0, we then assume the yz term is nonzero. Now our quadratic can be parameterized with 3 variables. We can continue this for all terms $y^2, z^2, yz, y, z, 1$. For all of them, the dimension is strictly less than the dimension of the 5-dimensional space of quadratics with a nonzero y^2 term, so we can disregard them.

It is also important to note that these quartic surfaces will contain a second conic in the same plane. It's just the intersection of the h we constructed earlier with the plane.

9 Solving for lines and conic sections

The video shown earlier specifically shows a 1-parameter family of quartic surfaces. This just means that we can index all of the quartic polynomials in our family with a single scalar [5]. We can construct such a family by allowing the coefficients of the quartic polynomial to each be a function of one variable $t \in \mathbb{R}$. For now, we will assume the coefficients of our family be linear functions of t.

We know that an arbitrary quartic does not contain a line or a conic. But given a specific family of quartic surfaces, we can solve for points that contain lines or conic sections. We will find that there are only finitely many of these points in a general family of quartic surfaces. To be clear, we can certainly construct a family of quartic surfaces that all contain lines. For example, we could take inspiration from our parameter count for lines and define a family as follows

$$f_t(x, y, z) = l_1 g_t + l_2 g_t,$$

where l_1, l_2 are planes that are not parallel and g_t, h_t are cubics with coefficients depending on t. For any t, $f_t(x,y,z)$ will contain the line that is the intersection of l_1 and l_2. We could make a similar construction for a family of quartic surfaces that always contain a certain conic. Instead, we will consider families of quartic surfaces, where we claim somewhat vaguely that their coefficients are arbitrary linear functions of t. Let's begin by showing this with lines.

Let's parameterize our family of quartic surfaces with t and call it $f_t(x,y,z)$. We will again assume that our line isn't parallel to the xy-plane. This allows us to parameterize it as some $x = as+b$, $y = cs+d$, and $z = s$. For this example, our coefficients depend on t so let's call them $l(t)$. Then evaluate the polynomial at $x = as+b$, $y = cs+d$, and $z = s$. Specifically, we want that $f_t(as+b, cs+d, s) = 0$. If we view this as a polynomial of s, we will see that its degree is still 4. If we have some term of $f_t(x,y,z)$ of degree 4, we can write it as $l(t)x^{e_x}y^{e_y}+z^{e_z}$, where $e_x, e_y, e_z \in \mathbb{Z}_{\geq 0}$ such that $e_x+e_y+e_z = 4$. Then, if we evaluate this term at $x = as+b$, $y = cs+d$, and $z = s$, we get $l(t)(as+b)^{e_x}(cs+d)^{e_y}s^{e_z}$. If we expand this, the term with the greatest degree will be $a^{e_x}b^{e_y}s^{e_x+e_y+e_z}$. Since $e_x+e_y+e_z = 4$, this term, viewed as a polynomial of s, is still degree 4.

Furthermore, this polynomial over s must just be the 0 polynomial, since varying s gives us different points on the line. This means that each of our coefficients must be 0. These coefficients are expressions in a,b,c,d,t which we can then set equal to 0. A general quartic polynomial in s will have 5 coefficient, one each for the degrees $0,1,2,3$, and 4 terms. So we have 5 variables in 5 linear equations. This means that, if these equations are actually "arbitrary," we should be able to solve these equations and get a finite number of solutions. Specifically, the values for a,b,c,d will determine what line, and the value of t will give us the quartic that contains this line.

We can set up a similar system of equations for conic sections. Let's choose our plane $x+ay+bz+c$, again assuming that the plane is not parallel to the x-axis. Then, we divide $f_t(x,y,z)$ by $x+ay+bz+c$ using polynomial long division. The remainder of this will be a quartic in only y and z, with coefficients in t,a,b,c. Next, we can divide this remainder by our equation for the conic in y and z, just as in the parameter count. We will again assume that the y^2 term is nonzero, so we can write this conic with just 5 coefficients: $y^2 + d_1x^2 + d_2xy + d_3x + d_4y + d_5$.

Then we can use polynomial long division once again. This time, the remainder is a quartic where none of the terms can have y raised to a power of more than 1, since that would have been divided out. So we are left with only the terms $1, x, x^2, x^3, x^4, y, xy, x^2y, x^3y$. Their coefficients are some combination of $d_1, d_2, d_3, d_4, d_5, a, b, c, t$. For this quartic to contain the conic, this remainder must just be the 0 function, so we get 9 equations for each coefficient in 9 variables. We can theoretically solve these equations and find a finite number of conic sections in our family.

These systems of equations also mean that a general quartic family will also not contain the same line or conic more than once. If we find a line or conic determined by the a, b, c, d or the $d_1, d_2, d_3, d_4, d_5, a, b, c$ respectively, there should only be 1 value of t that solves the system of equations.

When we were solving these system of equations, we did not parameterize our family as $h = f\cos(t) + g\sin(t)$. Instead, we parameterized it as

$$h = (1-t)f + tg$$

where t runs from $-\infty \to \infty$. This parameterization made more sense from a computational perspective; however, we found that the trigonometric parameterization was more aesthetically pleasing.

This may seem like it generates a different family, but rescaling any point in our family does not change the surface, except when we are rescaling by 0 or ∞. So, unless $t = 1$, we can rescale the equation by $\frac{1}{1-t}$, to get a new parameterization of our family

$$h = f + \frac{t}{1-t}g$$

We see that $\frac{t}{1-t}$ ranges all the way from $-\infty$ to ∞, with the exception of the point at -1. Specifically, for any

$k \in \mathbb{R}$ where $k \neq -1$, we can solve the equation $\frac{t}{1-t} = k$:

$$\frac{t}{1-t} = k \implies t = k - kt \implies t = \frac{k}{k+1}$$

We approach $k = -1$ as t goes to $\pm\infty$. The other problem is that we have lost the point at $t = 1$. In the family, $h = (1-t)f + tg$, the point $t = 1$ did exist. It's just the point $h = g$. However, we can approach this point by letting $t \to 1$, in our new family.

We can change our family once more. Consider

$$h = f + \tan(t)g$$

as t ranges over one period of length 2π. This is just a rescaling of $h = \cos(t)f + \sin(t)g$, by a factor of $\frac{1}{\cos t}$. For any $k \in \mathbb{R}$, we can find t such that $k = tan(t)$, by simply using the arctan. The only point we are missing from $h = \cos(t)f + \sin(t)g$ is when $t = \frac{\pi}{2}$, which is exactly the point $h = g$.

So, if we choose some quartic polynomial in the family $h = (1-t)f + tg$, a scalar multiple of it exists in $h = \cos(t)f + \sin(t)g$.

As a result, for our project, we used our systems of equations to find t that gives a quartic with a line in $h = (1-t)f + tg$. Then, we can simply use some inverse trigonometric functions to find t' such that $h' = \cos(t')f + \sin(t')g$ is a scalar multiple of our desired quartic polynomial. Thus the quartic surfaces given are the same.

10 Justification of dimension count

In the above section, we used the idea of dimension to argue that a general quartic will not contain a line or a conic. We mean, more precisely, that the set of quartic surfaces containing lines or conic sections will have measure 0 in the 35-dimensional space of all quartic surfaces. To prove this, we will use the idea of Lipschitz continuity.

10.1 Analytic argument

We will prove in the following section that the quartic surfaces containing lines are the image of a polynomials map from $\mathbb{R}^{35} \to \mathbb{R}^{34}$. Let's show that the image of such a map will have measure 0. In fact, we will generalize this argument to any map f from $\mathbb{R}^m \to \mathbb{n}$, with polynomial components and $m < n$.

Firstly, we claim that such a map $f : \mathbb{R}^m$ to $\mathbb{R}^n$ is Lipschitz continuous on every compact convex subspace, $K \subset \mathbb{R}^{34}$. Lipschitz continuity means that there exists some constant $d \in \mathbb{R}$ such for all $x, y \in K$ such that $|x - y| < \delta$, then $|f(x) - f(y)| < d\delta$.

We know it's Lipschitz continuous since each component of the map is a polynomial, and so each partial derivative is continuous. Thus the matrix $f'(x)$ exists, and we can apply the mean value theorem to produce that for every $x, y \in K$

$$|f(x) - f(y)| \leq \sup ||f'(x)|| |x - y|$$

where $||f'(x)||$ is the matrix norm of $f'(x)$. To satisfy the definition of Lipschitz continuity, we can just take d to be the supremum of $||f'(x)||$ on K. The supremum exists, since each partial derivative is continuous, so the matrix norm is continuous. Thus, over a compact set K, we can find the supremum of the matrix norm.

Now we use the fact that f is Lipschitz continuous to prove that the image of any interval $[a, b]^m$ has measure 0.

If we prove this for any such interval, the entire image also trivially has measure 0, since the measure of a countable union of sets is less than or equal to the sum of the measures.

Let $\varepsilon > 0$. Choose $k \in \mathbb{N}$ such that $k \geq \frac{b-a}{\varepsilon}$. Then we can partition $[b - a]^m$ into k^m hypercubes, whose side lengths are less than ε. Since the side length is less than ε, if x and y are in the same cube, $|x - y| < \sqrt{m}\varepsilon$

which is the length of the diagonal of the cube. Since f is Lipschitz continuous, for x, y in one of the cubes, $|f(x) - f(y)| < d\sqrt{m}\varepsilon$. So the image of each of these cubes can be contained in a cube of side length $d\sqrt{m}\varepsilon$ (we can imagine inscribing the circle with diameter $d\sqrt{m}\varepsilon$ into this cube).

So the measure of the image of $[a,b]^m$ is less than

$$\textit{number of cubes} \times \textit{volume of each} = k^m (d\sqrt{m}\varepsilon)^n \geq \left(\frac{b-a}{\varepsilon}\right)^m \left(d\sqrt{m}\varepsilon\right)^n = (b-a)^m d\sqrt{m}\varepsilon^{n-m}$$

Since $n - m > 0$, choosing small values of ε will give us small values of $(b-a)^m d\sqrt{m}\varepsilon^{n-m}$. So the image of f must have measure 0.

10.2 Lines in conic sections

While we expect the curve described by a second degree polynomial in two variables to look like a parabola, hyperbola, or ellipse, there is a degenerate case when the second degree polynomials looks like a line or pair of lines. This is specifically when the polynomial is reducible. For example, a point satisfies the following equation $4x^2 + 3xy - y^2 - 4x + y = (x + y - 1)(4x - y) = 0$, if and only if it falls on the line $y = 1 - x$ or the line $y = 4x$ (see figure 28).

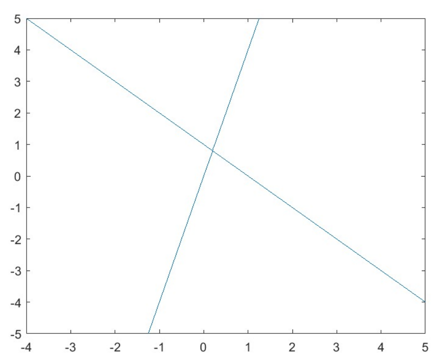

Figure 28: $4x^2 + 3xy - y^2 - 4x + y = (x + y - 1)(4x - y) = 0$

Before constructing a map from $\mathbb{R}^{34}$ to $\mathbb{R}^{35}$ whose image is all quartic with lines, let's construct a simpler map from $\mathbb{R}^5$ to $\mathbb{R}^6$ whose image is all of these degenerate conic sections containing lines. Consider the conic $ax^2+bxy+cy^2+dx+ey+f$ and the line parameterized by $x=At+B$ and $y=t$. To see if the conic contains the line, we can just evaluate $ax^2+bxy+cy^2+dx+ey+f$ at $(At+B,t)$. The conic contains the line if the resulting expression is 0 for all values of t.

$$a(At+B)^2+b(At+B)t+ct^2+d(At+B)+et+f=(aA^2+bA+c)t^2+(a2AB+dA+bB+e)t+(aB^2+dB+f)$$

If we view the expression as a polynomial of t, we see that it must be the 0 polynomial, which means that $aA^2+bA+c=0$, $a2AB+dA+bB+e=0$, and $aB^2+dB+f=0$.

So, we can pick the a,b,d coefficients of the polynomials, as well as the line defined by $x=at+b$ and $y=t$. Then, we are forced to set $c=-aA^2-bA$, $e=-2aAB-dA-bB$, and $f=-aB^2-dB$. In other words, we have defined a map from $\mathbb{R}^5$ of (a,b,d,A,B) to (a,b,c,d,e,f) whose image is all conic sections containing a line of the form $x=at+b$ and $y=t$.

Of course, we are missing the lines that are parallel to the x-axis. To account for these we can symmetrically construct a similar map using the evaluation at $(t,At+B)$ instead. In this case, our map is from (b,c,e,A,B) to (a,b,c,d,e,f) where $a=-cA^2-bA$, $d=-2cAB-eA-bB$, and $f=-cB^2-eB$.

So, if we take the union of these two images, we see that the measure of the union must be less than or equal to the sum of the measures. However, by our argument using Lipschitz continuity, both of these images have measure 0.

So, in the 6-dimensional space of 2nd degree polynomials, the set of reducible ones has measure 0. It is worth noting that this statement is subtly different from the claim that the set of degenerate conic sections in the 5-dimensional space of conic sections has measure 0. However, we see that rescaling a reducible polynomial leaves it reducible, and rescaling an irreducible polynomial leaves it irreducible, so these two statements are equivalent.

It is also important to note that in this case a way of writing the 3 coefficients was relatively obvious; however, this is just because we chose a simple example. In general, we can construct a matrix with columns representing a, b, c, d, e, f respectively:

$$\begin{bmatrix} A^2 & A & 1 & 0 & 0 & 0 \\ 2AB & B & 0 & A & 1 & 0 \\ B^2 & 0 & 0 & B & 0 & 1 \end{bmatrix}$$

In general, we could hope to row reduce this matrix for an explicit way of writing 3 variables as a function of the other 3 plus A, B.

10.3 Lines in quartic surfaces

Given 30 numbers $a_1, a_2, \ldots, a_{30}$, and 4 numbers b, c, d, e, we will define a quartic polynomial whose first 30 coefficients are $a_1, a_2, \ldots, a_{30}$ that contains the line $(bt + c, dt + e)$. In other words, we will construct a map from $\mathbb{R}^{34}$ to $\mathbb{R}^{35}$ whose image is all quartic surfaces containing lines.

Similar to the case with lines in conic sections, we want to evaluate the polynomials

$$a_1x^4 + a_2x^3y + a_3x^3z + \ldots + a_{31}z^4 + a_{32}z^3 + a_{33}z^2 + a_{34}z + a_{35}$$

at the lines parameterized by

$$(At + B, Ct + D, t)$$

Plugging this equation in will get us a massive 4-dimensional polynomial in t. If the polynomial with coefficients, $a_1, \ldots, a_{35}$, contains the line $(At + b, Ct + D, t)$, then this 4-degree polynomial in t, whose coefficients are expressions of $a_1,\ldots,a_{35},A,B,C,D$, must be the zero polynomial. In other words, we get a system of 5 equations in 39 variables. The equations are linear with respect to $a_1,\ldots,a_{35}$, so we can write a 5 by 34 matrix. Crucially, we see that the last 5 columns of this matrix make up a permutation matrix. In other words, our five equations can be rewritten as equations for a_{31}, a_{32}, a_{33}, a_{34}, and a_{35}. These equations are unfortunately less concise than those for finding conic sections containing lines, but I will still include them below:

$$a_{35} = -(B^4a_1 + B^3Da_2 + B^3a_4 + B^2D^2a_5 + B^2Da_7 + B2a_{10} + BD^3a_{11} + BD^2a_{13} + BDa_{16} + Ba_{20} + D^4a_{21} + D^3a_{23} + D^2a_{26} + Da_{30})$$

$$a_{34} = -(4AB^3a_1 + (B^3C + 3AB^2D)a_2 + B^3a_3 + 3AB^2a_4 + (2B2CD + 2ABD2)a_5 + B^2Da_6 + (B^2C + 2ABD)a_7 + B^2a_8 + 2ABa_9 + 3BCD^2a_{10} + AD^3a_{11} + BD^2a_{12} +$$

$$+(2BCD + AD^2)a_{13} + BDa_{15} + (BC + AD)a_{16} + Ba_{19} + Aa_{20} + 4CD^3a_{21} + D^3a_{22} + 3CD^2a_{23} + D2a_{25} + 2CDa_{26} + Da_{29} + Ca_{30})$$

$$a_{33} = 6A^2B^2a_1 + (3AB^2Ca + 3A^2BD)a_2 + 3AB^2a_3 + 3A^2Ba_4 + (B^2C^2 + 4ABCD + A^2D^2)a_5 + (B^2C + 2ABD)a_6 + (2ABC + A^2D)a_7 + B^2a_8 + 2ABa_9 + A^2a_{10} +$$

$$+(3BC^2Da + 3ACD^2)a_{11} + (2BCD + AD^2)a_{12} + (BC^2 + 2ACD)a_{13} + BDa_{14} + (BC + AD)a_{15} + ACa_{16} + Ba_{18} + Aa_{19} + 6C^2D^2a_{21} + 3CD^2a_{22} + 3C^2Da_{23} + D^2a_{24} + 2CDa_{25} + C^2a_{26} + Da_{28} + Ca_{29}$$

$$a_{32} = 4A^3Ba_1 + (3A^2BC + A^3D)a_2 + 3A^2Ba_3 + A^3a_4 + (2ABC^2 + 2A^2CD)a_5 + (2ABC + A^2D)a_6 + A^2Ca_7 + 2ABa_8 + A^2a_9 + (BC^3 + 3AC^2D)a_{11} + (BC^2 + 2ACD)a_{12} +$$

$$+AC^2a_{13} + (BC + AD)a_{14} + ACa_{15} + Ba_{17} + Aa_{18} + 4C^3Da_{19} + 3C^2Da_{20} + C^3a_{21} + 2CDa_{22} + C2a_{23} + Da_{25} + Ca_{26}$$

$$a_{31} = A4a_1 + A^3Ca_2 + A^3a_3 + A^2C^2a_5 + A^2Ca_6 + A^2a_8 + AC^3a_{11} + AC^2a_{12} + ACa_{14} + Aa_{17} + C^4a_{21} + C^3a_{22} + C^2a_{24} + Ca_{27}$$

So, we can define a map from $\mathbb{R}^{34} \to \mathbb{R}^{35}$ sending $(a_1,a_2,\ldots,a_{30},A,B,C,D)$ to $(a_1,\ldots,a_{35})$, with $a_{31},\ldots,a_{35}$ defined as above, such that the quartic given by $a_1x^4 + a_2x^3y + a_3x^3z + \ldots + a_{31}z^4 + a_{32}z^3 + a_{33}z^2 + a_{34}z + a_{35}$ contains the line $(at+b, ct+d, t)$. As a result, the image contains all quartic surfaces containing lines that are not parallel to the xy-plane. However, by symmetry, we can construct similar maps for the lines not parallel to the yz-planes and the xz-plane.

Since each of these maps have polynomial components, they are Lipschitz continuous, and in $\mathbb{R}^{35}$ these images have measure 0. So their union, which is all quartic polynomials with lines, has measure 0 in the set of all quartic polynomials.

10.4 Conic sections in quartic surfaces

If we want to test whether a quartic surface contains a conic section, we first want to restrict it to the plane containing the conic.

We can parameterize any line that is not perpendicular to the xy-plane as $(u, v, Au + Bv + C)$, where A, B, C are constants.

So we simply evaluate the quartic, $a_1x^4 + a_2x^3y + a_3x^3z + \ldots + a_{31}z^4 + a_{32}z^3 + a_{33}z^2 + a_{34}z + a_{35}$, at $x = u$, $b = v$, $z = Au + Bv + C$, giving us a quartic planar polynomial in u and v.

Then, we can simply use polynomial long division on the resulting quartic in u and v by our conic in u and v. Specifically, let's consider the conic $u^2 + b_1uv + b_2v^2 + b_3x + b_4y + b_5$. This unfortunately excludes those conic sections whose u^2 coefficient is 0; however, it uniquely represents the rest of the conic sections in this plane. So together $(A, B, C, b_1, b_, b_3, b_4, b_5)$ uniquely determines our conic sections.

The remainder of our division cannot have any terms including u^2, since "dividing" by $u^2 + b_1uv + b_2v^2 + b_3x + b_4y + b_5$ is really equivalent to replacing every u^2 with $-(b_1uv + b_2v^2 + b_3x + b_4y + b_5)$.

So the remainder is a polynomial in u, v with $1, u, v, uv, v^2, uv^2, v^3, uv^3, v^4$. Our quartic containing the conic is equivalent to this being the 0 polynomial. But this is the same as all 9 of these coefficients, which are expressions in $a_1, a_2, \ldots, a_{35}, A, B, C, b_1, b_2, b_3, b_4, b_5$. These equations are all linear in $a_1, \ldots, a_{35}$, so we can rewrite this system of equations as a 9 by 35 matrix with entries that are expressions in $A, B, C, b_1, \ldots, b_5$. The ij entry of this matrix represents the coefficient of a_j in the ith equation.

Specifically, the first row represents the coefficient for 1, the second represents the coefficient for u, the third v, the fourth uv, the fifth v^2, the sixth, uv^2, the seventh v^3, the eighth uv^3, and the ninth v^4.

Fortunately, if we calculate this matrix, we see that we can again easily write $a_{11}, a_{13}, a_{16}, a_{20}, a_{21}, a_{23}, a_{26}, a_{30}, a_{35}$ as functions of the rest.

We will not write these functions out here since they are excessively long. For example, the $5,8$ entry of this matrix is

$$-A^2b_2b_3^2 - 2A^2b_1b_3b_4 - A^2b_1^2b_5 + 2ACb_2b_3 + 2ACb_1b_4 + 2ABb_3b_4 + A^2b_4^2 + 2ABb_1b_5 + 2A^2b_2b_5 - C^2b_2 - 2BCb_4 - B2b_5.$$

However, we know that these explicit formulas do exist, and so we can define a map from $\mathbb{R}^{34}$ sending $(a_1, \ldots, a_{10}, a_{12}, a_{14}, a_{15}, a_{17}, a_{18}, a_{19}, a_{22}, a_{24}, a_{25}, a_{27}, a_{28}, a_{29}, a_{31}, a_{32}, a_{33}, a_{34}, A, B, C, b_1, b_2, b_3, b_4, b_5)$ to $(a_1, \ldots, a_{35})$. This map has polynomial components, so we can again apply Lipschitz continuity to conclude that the image of this map has measure 0. The image of the map is all quartic polynomials containing conic sections, whose u^2 term is nonzero. So, just as with lines, we have formalized our proof that an arbitrary quartic will not contain any conic sections. For the purposes of this paper, we will leave out the arguments for the different types of conic sections, as they are essentially the same.

Conclusion

In this paper, we have approached the question of whether an arbitrary quartic contains lines or conic sections from 3 different angles. First, we showed that to get all quartic surfaces with lines or all quartic surfaces with conic sections we need only choose 34 numbers, whereas we have to choose 35 to get all quartic surfaces. Then, we demonstrated how to solve a system of equations to find the finite number of points in a family of quartic surfaces that contain lines or conic sections. Finally, we constructed two maps, one whose image is all quartic surfaces with lines and the other whose image is all quartic surfaces with conic sections, and proved that the measure of these images is 0, using the idea of Lipschitz continuity.

This is one of many fascinating ways we can use the algebra behind an equation to describe the geometry of the surface it represents. The fact that most quartic surfaces do not contain lines or conics can also be seen as a result the Noether-Lefschetz theorem, which is an important result on the homology of surfaces of degree 4 or greater.

Glossary

Combinatorics The study of counting. 19

Conic sections A curve that can be defined as a degree 2 polynomial in a plane. 22

Degree of polynomial The degree of a monomial $ax^{r_1}y^{r_2}z^{r_3}$, where a is a constant and r_1, r_2, r_3 are nonnegative integers, is $r_1 + r_2 + r_3$. Then, a polynomial's degree is the maximal degree of all its terms. 4

Irreducible In a ring, an irreducible element is any element that cannot be written as ab, where neither a nor b is invertible. 20

Linear Polynomial A polynomial of degree 1. 20

Monomial A polynomial with just one term. 19

Polynomial A polynomial in k variables is the sum of products of the k variables times a scalar multiples.. 4

Prime In a ring, a prime element p is an element such that, if p divide ab, then p divides a or p divides b . 20

Quadratic Polynomial A polynomial of degree 2. 4

Quartic Polynomial A polynomial of degree 4. 11

Vector space A set equipped with addition, additive inverses, and a 0 element, as well as "scalar" multiplication by elements from a field. 21

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