# NON-$\mathcal{Z}$-STABLE SIMPLE AH ALGEBRAS

by

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A DISSERTATION

Presented to the Department of Mathematics
and the Division of Graduate Studies of the University of Oregon
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

September 2023

DISSERTATION APPROVAL PAGE

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Title: Non-$\mathcal{Z}$-Stable Simple AH Algebras

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Degree awarded September 2023.

DISSERTATION ABSTRACT

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Department of Mathematics

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We consider the problem of dimension growth in AH algebras $\mathcal{A}$ defined as inductive limits $\mathcal{A} = \lim_{n\to\infty}(M_{R_n}(C(X_n)), \phi_n)$ over finite connected CW-complexes $X_n$. We show that given any sequence $(X_n)$ of finite connected CW-complexes and matrix sizes $(R_n)$ with $R_n \big| R_{n+1}$ satisfying the dimension growth condition $\lim_{n\to\infty} \dfrac{\dim(X_n)}{R_n} = c$ with $c \in (0, \infty)$, there always exists an AH algebra with injective connecting homomorphisms over a subsequence which does not have Blackadar's strict comparison of positive elements, and therefore does not absorb tensorially the Jiang-Su algebra $\mathcal{Z}$. This demonstrates that no regularity condition can be placed on the spaces $X_n$ in order to stabilize AH algebras over them - there always exists a pathological construction.

Dedicated to my brothers.

TABLE OF CONTENTS

# CHAPTER 1

# PRELIMINARY NOTIONS

## 1.1 Introduction

At the end of the 1980's, George Elliott conjectured that a large class of simple, nuclear $C^*$-algebras can be classified from their $K$-theory and tracial state space, along with the pairing between them, known as the *Elliott Invariant.* For a unital $C^*$-algebra $\mathcal{A}$, this is

$$\mathrm{Ell}(\mathcal{A}) = ((K_0(\mathcal{A}), K_0(\mathcal{A})^+, [1_{\mathcal{A}}]), K_1(\mathcal{A}), T^+(\mathcal{A}), \rho),$$

where $\rho : K_0(\mathcal{A}) \to T(\mathcal{A})$ is the pairing by evaluation at a $K_0$ class. One major example of interest were the approximately homogeneous (AH) algebras: those obtained from inductive limits of matrix algebras $\mathcal{A}_n = M_{R_n}(C(X_n))$. Throughout the 1990's, much progress was made on the overall classification of nuclear $C^*$-algebras. Among these major advances included

- Kirchberg-Phillips' classification of purely infinite $C^*$-algebras satisfying the Universal Coefficient Theorem.
- Elliott-Gong-Li's classification of AH algebras of very slow dimension growth.
- Lin's classification of tracially AF algebras, and later in 2003 of real rank zero algebras.

But, in 2002, Rørdam [Rør03] gave a stark counterexample to Elliott's conjecture, exhibiting a simple, nuclear $C^*$-algebra $\mathcal{A}$ which contained both a finite and infinite projection. It was shown that by tensoring with an algebra called the Jiang-Su algebra, denoted $\mathcal{Z}$ and introduced in [JS99], that $\mathrm{Ell}(\mathcal{A} \otimes \mathcal{Z}) = \mathrm{Ell}(\mathcal{A})$, although their real rank satisfied $\mathrm{RR}(\mathcal{A}) \neq 0$, while $\mathrm{RR}(\mathcal{A} \otimes \mathcal{Z}) = 0$.

Meanwhile, the 1998 construction of J. Villadsen in [Vil98] exhibited topological obstructions in the form of perforation of the $K_0$ group of certain AH algebras. These algebras failed to have slow dimension growth: the property that

$$\lim_{n\to\infty} \frac{\dim(X_n)}{R_n} = 0,$$

eventually leading to more counterexamples. Results such as those furthered by Gong-Jiang-Su in [GJS00], showed the connection between perforation and absorbing $\mathcal{Z}$ tensorially, deemed *$\mathcal{Z}$-stability.* It became clear that the Elliott invariant, as it existed, was not sufficient for classification of many seemingly tractible $C^*$-algebras, even among AH algebras. In fact, it was later realized $\mathrm{Ell}(\mathcal{Z}) = \mathrm{Ell}(\mathbb{C})$ although $\mathcal{Z} \not\cong \mathbb{C}$.

Throughout the 2000's, Toms and Winter conjectured that for AH algebras, this so-called $\mathcal{Z}$-stability was related to the dimension growth - this quantity $\lim\limits_{n\to\infty} \dfrac{\dim(X_n)}{R_n}$. They also studied notions of divisibility, nuclear dimension, and comparison in the Cuntz semigroup, which generalizes the notion of $K$-theory of a $C^*$-algebra $\mathcal{A}$ to general positive elements in $\mathcal{A} \otimes \mathcal{K}$. An important example are the Villadsen algebras of the first type, built from coordinate projections over increasing sequences of spaces $X_n = (X_{n-1})^{k_n}$ from some seed space $X_1$ and sequence $(k_n)$ in $\mathbb{N}$, which they analyzed in detail in [TW09].

In this thesis, we investigate inductive limits over algebras $\mathcal{A}_n = M_{R_n}(C(X_n))$ for some increasing sequence $R_n$ and potentially quite general finite CW-complexes $X_n$. Given any quasitrace $\tau$ on $\mathcal{A} = \lim\limits_{n\to\infty}(\mathcal{A}_n, \phi_n)$, we associate its *dimension function* $d_\tau$ as follows: given $a \leq 1$ in $\mathcal{A}_+$, we define

$$d_\tau(a) = \lim_{n\to\infty} \tau(a^{\frac{1}{n}}).$$

Elliott-Robert-Santiago showed in [ERS11] that $d_\tau$ is a so-called functional on the Cuntz semigroup, and B. Blackadar introduced the important notion of *strict comparison:* when $d_\tau(a) < d_\tau(b)$ for every quasitrace $\tau$ implies $a \precsim b$ in the Cuntz semigroup. For the AH algebras, Toms and Winter showed that strict comparison is equivalent to $\mathcal{Z}$-stability, along with other equivalent notions such as slow dimension growth for Villadsen algebras of the first type, throughout the thesis just referred to as Villadsen algebras. In the initial stages of the project, it was hoped that a line could be drawn for different values of $c \in (0,\infty)$, possibly getting strict comparison for certain cases.

This turned out not to be possible. The main result of the thesis explains how regardless of the choice of finite CW-complexes $X_n$, and regardless of the matrix growth, i.e. the constant $c \in (0,\infty)$, there is always a subsequence which admits a construction which is not $\mathcal{Z}$-stable. Often, AH algebras are considered with diagonal connecting homomorphisms, requiring that the size of the matrices increases multiplicatively.

**Theorem 1.1.1.** Let $X_n$ be given CW-complexes with dimension $\dim(X_n) = d_n$, and let $(R_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}$. Suppose that $(d_n)$ and $(R_n)$ are monotonically increasing, with $R_n \big| R_{n+1}$ and $d_n \to \infty$, and suppose

$$\liminf_{n\to\infty} \frac{d_n}{R_n} = c \in (0, \infty).$$

Then, there exists a subsequence $(a_n)$ of $\mathbb{N}$ and connecting homomorphisms

$$\phi_{n,n+1} : M_{R_{a_n}}(C(X_{a_n})) \to M_{R_{a_{n+1}}}(C(X_{a_{n+1}}))$$

such that $\mathcal{A} := \lim\limits_{n\to\infty}(M_{R_{a_n}}(C(X_{a_n})), \phi_{n,n+1})$ is a simple, unital AH algebra which is not $\mathcal{Z}$-stable.

A small modification allows for injective connecting maps, i.e. the information of each CW-complex is preserved across connecting homomorphisms. We also show a few other facts surrounding these interesting algebras.

## 1.2 AH Algebras and Dimension Growth

**Definition 1.2.1.** Let $X_n$ be finite CW-complexes, $P_n \in C(X_n, M_{R_n})$ be projections for some sequence $R_n \in \mathbb{N}$, and $\mathcal{A}_n = P_n C(X_n, M_{R_n}) P_n$ An inductive limit

$$\mathcal{A} = \lim_{n\to\infty}(\mathcal{A}_n, \phi_n)$$

is known as an *approximately homogeneous (AH) algebra.*

Throughout this paper, without loss of generality, we will consider the case $P_n = I$, and $X_n$ connected and nontrivial of monotonically increasing dimension. We also consider those algebras $\mathcal{A}$ which have *diagonal connecting homomorphisms:* those which have connecting homomorphisms $\phi_n : \mathcal{A}_n \to \mathcal{A}_{n+1}$ of the form

$$\phi_n(a) = \text{diag}(a \circ f_1, ..., a \circ f_{N_{n,n+1}})$$

for some continuous functions $f_j : X_{n+1} \to X_n$. Note that we require $N_{n,n+1} := \frac{R_{n+1}}{R_n} \in \mathbb{N}$, i.e. $R_n \big| R_{n+1}$. We can consider the *dimension growth* of an AH algebra $\mathcal{A}$ as

$$\lim_{n\to\infty} \frac{\dim(X_n)}{R_n} = c \in [0, \infty].$$

**Definition 1.2.2.** An AH algebra $\mathcal{A}$ is said to have *slow dimension growth* if it admits an inductive decomposition $(\mathcal{A}_n, \phi_n)$ such that

$$\lim_{n\to\infty} \frac{\dim(X_n)}{R_n} = 0.$$

One may recall the next theorem from algebraic topology (c.f. Husemoller: Chapter 8 [Hus94]).

**Theorem 1.2.3.** Let $X$ be a compact metrizable Hausdorff space, and let $\omega, \xi$ be complex vector bundles over $X$. If the fiber-dimension of $\omega$ exceeds the fiber-dimension of $\xi$ by at least $\lfloor \frac{\dim(X)}{2} \rfloor$ at every point of $X$, then $\xi$ is isomorphic to a sub-bundle of $\omega$.

By the Serre-Swan theorem, complex vector bundles over such a space $X$ are identified with Murray-von Neumann equivalence classes of projections in $M_\infty(C(X))$. Note that $\|p - q\| < 1$ implies $p \sim q$, which implies $\mathrm{Rank}(p) = \mathrm{Rank}(q)$. Thus, all projections in AH algebras are equivalent to one in $\phi_{n,\infty}\mathcal{A}_n$ coming from a projection in $\mathcal{A}_n$ having finite rank. When $X_n$ are contractible, $\mathcal{A}_n = M_{R_n}(C(X_n))$ has only trivial projections, i.e. $K_0(\mathcal{A}) \simeq \mathbb{Z}$ coming from the rank of the projection in the finite stage. In such a case, $p, q \in \mathcal{A}_n$ with $\mathrm{Rank}(p) < \mathrm{Rank}(q)$ implies $[p] \leq [q]$ in $K_0(\mathcal{A})$. This motivates a notion of comparison of projections.

**Definition 1.2.4.** Let $\mathcal{A}$ be an AH algebra. We say $\mathcal{A}$ has *strict comparison of projections* if, for all projections $p, q \in \mathcal{A}_n$, we have $\mathrm{Rank}(p) < \mathrm{Rank}(q)$ implies $\phi_{n,\infty}p \leq \phi_{n,\infty}q$ in $\mathcal{A}$, i.e. $[p] \leq [q]$ in $K_0(\mathcal{A})$.

Certainly, not every AH algebra has strict comparison of projections.

**Example 1.2.5.** Let $\mathcal{A} = C(S^2)$, then, $\mathcal{A}$ does not have strict comparison of projections. For, if $\xi$ is the Hopf fibration, and

$$\theta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in M_4(C(S^2 \times S^2))$$

is a trivial rank 1 projection, then $\xi \times \xi \in M_4(C(S^2 \times S^2))$ defined by

$$\xi \times \xi(x, y) = \begin{bmatrix} \xi(x) & 0 \\ 0 & \xi(y) \end{bmatrix}$$

has $\mathrm{Rank}(\theta) = 1 < 2 = \mathrm{Rank}(\xi \times \xi)$, but we have $[\theta] \not\leq [\xi \times \xi]$ in $K_0(C(S^2 \times S^2))$. In fact, Villadsen was able to notice a key generalization of this fact in Lemma 1 of [Vil98], which is stated in a further section as Theorem 2.3.1.

Note that if $p \in C(X, M_n) \simeq M_n(C(X))$ is a projection, and $f : Y \to X$ is continuous, then $p \circ f \in M_n(C(Y))$ is a projection, and $\mathrm{Rank}(p) = \mathrm{Rank}(p \circ f)$ since rank of projections is constant on connected components. The next proposition is well-known and straightforward to prove.

**Proposition 1.2.6.** Let $\mathcal{A} = \lim\limits_{n\to\infty}(M_{R_n}(C(X_n)), \phi_n)$ be an AH algebra with slow dimension growth and diagonal connecting homomorphisms. Then, $\mathcal{A}$ has strict comparison of projections.

*Proof.* Clearly, $R_n \to \infty$ as $n \to \infty$ if $\dim(X_n) > 0$. Let $p, q \in M_{R_i}(C(X_i))$ be projections such that $\mathrm{Rank}(p) = n < \mathrm{Rank}(q) = m$. Then, for all $j \geq i$,

$$\mathrm{Rank}(\phi_{i,j}p) = \frac{R_j}{R_i}n < \frac{R_j}{R_i}m = \mathrm{Rank}(\phi_{i,j}q).$$

By slow dimension growth, there exists $N \in \mathbb{N}$ such that for all $j \geq N$ we have $\dfrac{\dim(X_j)}{2} < \dfrac{R_j}{R_i}$. Thus, for all $j \geq N$

$$\mathrm{Rank}(\phi_{i,j}p) + \frac{\dim(X_j)}{2} < \frac{R_j}{R_i}n + \frac{R_j}{R_i} \leq \frac{R_j}{R_i}n + \frac{R_j}{R_i}(m-n) = \frac{R_j}{R_i}m = \mathrm{Rank}(\phi_{i,j}q).$$

We conclude $\phi_{i,j}p \leq \phi_{i,j}q$ for all $j \geq N$, thus $\phi_{i,\infty}p \leq \phi_{i,\infty}q$, so $\mathcal{A}$ has strict comparison of projections. □

We will occasionally have need to talk about the quasitraces in such inductive limits of algebras $M_{R_n}(C(X_n))$. Haagerup showed quite generally in Theorem 5.11 of [Haa14] that quasitraces on exact unital $C^*$-algebras are traces. This includes the AH algebras analyzed in this thesis, so from now on we omit mention of quasitraces and just deal with traces.

## 1.3 Stable and Real Rank

Stable and real rank are generalizations of dimension to potentially noncommutative settings. In this section, we summarize the main properties and characterizations. Recall the following classic characterizations of covering dimension (c.f. Engelking [Eng78] and Brown-Pedersen [BP91]).

**Theorem 1.3.1.** Let $X$ be a compact Hausdorff space; then $\dim(X)$ is the least integer $n$ such that every continuous function from $X$ into $\mathbb{R}^{n+1}$ can be approximated arbitrarily closely with never-vanishing functions.

This definition gives $n$-tuples $(f_1, ..., f_n)$, which can't vanish simultaneously. So this is the same as

**Theorem 1.3.2.** Let $X$ be a compact Hausdorff space. Then $\dim(X)$ is the least integer $n$ such that every $(n+1)$-tuple of elements in $C_{\mathbb{R}}(X)$ (real-valued functions) can be approximated arbitrarily closely by $(n+1)$-tuples of elements which generate $C_{\mathbb{R}}(X)$ as an ideal.

These motivate the following definitions:

**Definition 1.3.3.** Let $\mathcal{A}$ be a Banach algebra; then $Lg_n(\mathcal{A})$ and $Rg_n(\mathcal{A})$ are the set of $n$-tuples in $\mathcal{A}$ which generate $\mathcal{A}$ as a left/right ideal. These are called *unimodular rows.*

**Definition 1.3.4.** Let $\mathcal{A}$ be a unital Banach algebra; then $\mathrm{ltsr}(\mathcal{A})$ is the least integer $n$ such that $Lg_n(\mathcal{A})$ is dense in $\mathcal{A}^n$ in the product topology, called the *left topological stable rank.* We define $\mathrm{rtsr}(\mathcal{A})$ analogously. If $\mathcal{A}$ is not unital, we unitize first.

**Remark 1.3.5.** If $\mathcal{A}$ is a Banach algebra with continuous involution (e.g. a $C^*$-algebra), then $\mathrm{ltsr}(\mathcal{A}) = \mathrm{rtsr}(\mathcal{A}) := \mathrm{tsr}(\mathcal{A})$, called the *stable rank* of $\mathcal{A}$.

The following is Proposition 1.7 of Rieffel [Rie83], giving the connection to the dimension of $X$:

**Proposition 1.3.6.** Suppose $X$ is a compact Hausdorff space; then

$$\mathrm{tsr}(C(X)) = \lfloor \frac{\dim(X)}{2} \rfloor + 1.$$

Likewise, the following result, Proposition 3.1 of Rieffel [Rie83], is of fundamental importance to the case of stable rank one:

**Theorem 1.3.7.** Let $\mathcal{A}$ be a Banach algebra; the following are equivalent:

(i) $\mathrm{ltsr}(\mathcal{A}) = 1$;
(ii) $\mathrm{rtsr}(\mathcal{A}) = 1$;
(iii) The invertible elements of $\mathcal{A}$ are dense in $\mathcal{A}$.

*Proof.* $Lg_1(\mathcal{A})$ and $Rg_1(\mathcal{A})$ consist of invertible elements, so (3) $\implies$ (1), (2). Suppose $\mathrm{ltsr}(\mathcal{A}) = 1$, and let $a$ be a left-invertible element with left-inverse $b$. We have $b \approx c$ for some left-invertible element $c$, so $ca \approx 1$, which implies $ca$ is invertible, or in particular $|ca - 1| < 1$. Hence $a$ is invertible. The other implication is analogous. □

Moreover, it is shown in Theorem 6.1 of Rieffel [Rie83]:

**Theorem 1.3.8.** Let $\mathcal{A}$ be a $C^*$-algebra. Then for every $m \in \mathbb{N}$

$$\mathrm{tsr}(M_m(\mathcal{A}) = \lfloor \frac{\mathrm{tsr}(\mathcal{A}) - 1}{m} \rfloor + 1.$$

In particular, $\mathrm{tsr}(\mathcal{A}) = 1$ if and only if $\mathrm{tsr}(M_n(\mathcal{A})) = 1$ for every $n$.

Theorem 4.1 of Elliott-Ho-Toms [EHT09] gives the following characterization of simple, unital diagonal AH algebras:

**Theorem 1.3.9.** Let $\mathcal{A} = \lim_{n\to\infty} (\mathcal{A}_n, \phi_n)$ be a simple, unital diagonal AH algebra. Then, $\mathcal{A}$ has stable rank one.

We will need a notion of cancellation for projections for analyzing comparison in $K_0$. This motivates the following definition:

**Definition 1.3.10.** A $C^*$-algebra $\mathcal{A}$ has *cancellation of projections* if for all projections $p, q, e, f \in \mathcal{A}$ with $pe = qf = 0$, $e \sim f$, and $p + e \sim q + f$, then we get $p \sim q$. $\mathcal{A}$ is said to have *cancellation* if $M_n(\mathcal{A})$ has cancellation of projections for every $n$.

**Remark 1.3.11.** Notice that $\mathcal{A}$ has cancellation of projections if and only if $p \sim q$ implies $(1 - p) \sim (1 - q)$. That is, $\mathcal{A}$ has cancellation of projections if and only if $p \sim q$ implies $p$ and $q$ are unitarily equivalent.

The following theorem and its proof can be found as Theorem 3.1.14 of H. Lin's book [Lin01]:

**Theorem 1.3.12.** Every unital $C^*$-algebra $\mathcal{A}$ with $\mathrm{tsr}(\mathcal{A}) = 1$ has cancellation.

*Proof.* Let $p, q \in M_n(\mathcal{A})$ be projections with $p \sim q$; hence there exists a partial-isometry $v \in M_n$ such that $v^*v = p$, $vv^* = q$. Since $\mathrm{tsr}(\mathcal{A}) = 1$, there exists $x \in GL_n(\mathcal{A})$ (invertible) such that

$$\|x - v\| \leq \frac{1}{8}.$$

We have $x = u(x^*x)^{\frac{1}{2}}$ as given by the polar decomposition, for $u \in U_n(\mathcal{A})$. We calculate

$$\|x^*x - p\| \leq \|x^*x - x^*v\| + \|x^*v - v^*v\| < \frac{1}{4},$$

which implies

$$upu^* \approx_{\frac{1}{4}} u(x^*x)u^* = x^*x \approx_{\frac{1}{4}} q,$$

hence $\|upu^* - q\| < 1$. Thus there exists a unitary $w$ such that $w^*upu^*w = q$. $\square$

Real rank is another notion of noncommutative dimension defined as follows:

**Definition 1.3.13.** Let $\mathcal{A}$ be a unital $C^*$-algebra; then $\mathrm{RR}(\mathcal{A})$ is the smallest integer such that for every $n$-tuple $(x_1, ..., x_n) \in \mathcal{A}_{SA}^n$ with $n \leq \mathrm{RR}(A) + 1$, and for every $\varepsilon > 0$, there exists an $n$-tuple $(y_1, ..., y_n) \in \mathcal{A}_{SA}^n$ such that $\sum_{k=1}^{n} y_k^2$ is invertible and $\|\sum_{k=1}^{n}(x_k - y_k)^2\| < \varepsilon$.

The following, Proposition 1.1 in Brown-Pedersen [BP91], gives the connection to the dimension of a space $X$:

**Proposition 1.3.14.** Let $X$ be a compact Hausdorff space; then

$$\mathrm{RR}(C(X)) = \dim(X).$$

*Proof.* The covering dimension of $X$ is the smallest integer $n$ such that every continuous function $f : X \to \mathbb{R}^{n+1}$ is approximated by $g$ such that $g(x) \neq 0$ for every $x$. Since $g = (g_1, ..., g_{n+1})$, we have $g(x) \neq 0$ for every $x$ if and only if $\sum_{k=1}^{n+1} g_k(x)^2 > 0$ for every $x$, i.e. this sum is invertible. In this context, real rank and covering dimension are the same notion. $\square$

Proposition 1.2 in Brown-Pedersen [BP91] importantly relates real rank to stable rank:

**Proposition 1.3.15.** Let $\mathcal{A}$ be a $C^*$-algebra. Then

$$\mathrm{RR}(\mathcal{A}) \leq 2\mathrm{tsr}(\mathcal{A}) - 1.$$

Brown-Pedersen [BP91] obtain in Theorem 2.6 a characterization of real rank:

**Theorem 1.3.16.** Let $\mathcal{A}$ be a $C^*$-algebra. The following are equivalent:

(i): $\mathrm{RR}(\mathcal{A}) = 0$;
(ii): The set of elements of $\mathcal{A}_{SA}$ with finite spectrum are dense in $\mathcal{A}_{SA}$ (mutually orthogonal projections);
(iii): For every hereditary subalgebra $\mathcal{B} \subset \mathcal{A}$, $b_1, ..., b_n \in \mathcal{B}$, and $\varepsilon > 0$, there exists a projection $p \in \mathcal{B}$ such that for all $j \in \{1, ..., n\}$

$$\|b_j p - b_j\| < \varepsilon.$$

## 1.4 The Cuntz Semigroup and Strict Comparison

The Cuntz semigroup provides a generalization of Murray-von Neumann comparison for a $C^*$-algebra to positive elements.

**Definition 1.4.1.** Let $\mathcal{A}$ be a $C^*$-algebra, $a, b \in \mathcal{A}_+$. We write $a \precsim b$ if there exists a sequence $(x_n)$ in $\mathcal{A}$ such that

$$x_n b x_n^* \to a.$$

We write $a \sim b$ if $a \precsim b$ and $b \precsim a$. The *Cuntz Semigroup* of $\mathcal{A}$, denoted $\mathrm{Cu}(\mathcal{A})$, is the set of Cuntz equivalence classes in $(\mathcal{A} \otimes \mathcal{K})_+$. It is an ordered semigroup under $\precsim$ and with $[a] + [b] = [a' + b']$, where $a', b'$ are orthogonal with $a' \sim a$ and $b' \sim b$.

**Remark 1.4.2.** Let $p, q \in (\mathcal{A} \otimes \mathcal{K})_+$ be projections with $p \precsim q$. Then, due to perturbation properties of projections, $p \leq q$ in Murray-von Neumann subequivalence.

In [CEI08], it is shown by Coward, Elliott, and Ivanescu that Cu is a covariant functor from the category of $C^*$-algebras to a subcategory of the category of ordered abelian groups, known as **Cu**. Particularly, if $\phi : \mathcal{A} \to \mathcal{B}$ is a homomorphism, then $[\phi] : \mathrm{Cu}(\mathcal{A}) \to \mathrm{Cu}(\mathcal{B})$ is a homomorphism with $[\phi][a] := [\phi(a)]$.

**Example 1.4.3.** Let $f, g \in C([0,1])$ be defined by

$$f(x) = \sqrt{x}, \qquad g(x) = x.$$

Suppose there exists $h \in C([0,1])$ such that $f(x) = h(x)g(x)h(x)$. Then, necessarily we have $h(x) = \pm\frac{1}{\sqrt[4]{x}}$ for all $x \neq 0$. Thus, $h \notin C([0,1])$. No element can satisfy the condition of Murray-von Neumann subequivalence for $f, g$. However, there does exist a sequence $(h_n)$ in $C[0,1]$ such that $h_n(x)g(x)h_n(x) \to f(x)$ for every $x \in [0,1]$, for example

$$h_n(x) = \begin{cases} \frac{1}{\sqrt[4]{x}} & \text{if } x \in [\frac{1}{n}, 1] \\ (n^{\frac{5}{4}})x & \text{if } x \in [0, \frac{1}{n}] \end{cases}.$$

Thus, $f(x) \precsim g(x)$. Clearly $g(x) \precsim f(x)$, since $k_n(x) = \sqrt[4]{x}$ satisfies $g(x) = k_n(x)f(x)k_n(x)$ for all $n \in \mathbb{N}$.

There is another category $\mathbf{C}$ of non-cancellative compact Hausdorff cones with jointly continuous $+$ and scaling in $[0, \infty]$, with morphisms being continuous linear maps between cones, which is intimately related to the Cuntz semigroup and the traces on $\mathcal{A}$. This is analyzed in full detail by Elliott, Robert, and Santiago in [ERS11].

In summary, let $F(Cu(\mathcal{A}))$ be the set of additive, order preserving maps on $Cu(\mathcal{A})$ sending $0 \to 0$ and which preserve suprema of increasing sequences: the so-called *linear functionals* on $Cu(\mathcal{A})$. Let $T(\mathcal{A})$ be the space of lower-semicontinuous traces on $\mathcal{A}$. Then, $F : \mathbf{Cu} \to \mathbf{C}$ and $T : \mathbf{C^*}\text{-}\mathbf{Alg} \to \mathbf{C}$ are continuous (with respect to sequential inductive limits) contravariant functors. Moreover, $F(Cu(\mathcal{A})) \simeq \mathrm{QT}_2(\mathcal{A})$ where $\mathrm{QT}_2(\mathcal{A})$ is the cone of lower-semicontinuous 2-quasitraces on $\mathcal{A}$. Lastly, there is a dual cone $L(F(Cu(\mathcal{A}))) \in \mathbf{Cu}$ to $F(Cu(\mathcal{A}))$, yielding a covariant functor $L(F(Cu(\cdot)))$ from $\mathbf{C^*}\text{-}\mathbf{Alg}$ to $\mathbf{Cu}$. Altogether, we have the functorial diagram

$$\begin{tikzcd} & \mathbf{Cu} \arrow[d, shift right, "F"'] \\ \mathbf{C^*}\text{-}\mathbf{Alg} \arrow[ur, "Cu"] \arrow[dr, "T"'] & \\ & \mathbf{C} \arrow[u, dotted, shift right, "L"'] \end{tikzcd}$$

**Remark 1.4.4.** Following [CEI08], the Cuntz semigroup has addition given by

$$[a] \oplus [b] = [a' + b'],$$

where $a', b' \in (\mathcal{A} \otimes \mathcal{K})_+$ are orthogonal to each other and Cuntz equivalent to $a$ and $b$ respectively. Let $\psi : \mathcal{A} \to \mathcal{B}$ be a homomorphism between $C^*$-algebras $\mathcal{A}$ and $\mathcal{B}$. Then, $\mathrm{Cu}(\psi) : \mathrm{Cu}(\mathcal{A}) \to \mathrm{Cu}(\mathcal{B})$ is a well-defined semigroup homomorphism given by

$$\mathrm{Cu}(\psi)[a] = [\psi(a)].$$

In particular, $\mathrm{Cu}(\psi)$ is a so-called Cu-morphism, meaning among other properties that it is *order preserving*: if $[a] \leq [b]$ in the Cuntz semigroup $\mathrm{Cu}(\mathcal{A})$, then

$$\mathrm{Cu}(\psi)[a] = [\psi(a)] \leq [\psi(b)] = \mathrm{Cu}(\psi)[b].$$

Critically is the notion of a lower-semicontinuous dimension function $d_\tau$ associated to a trace $\tau$, which is a linear functional on the Cuntz semigroup.

**Definition 1.4.5.** Let $\tau \in QT_2(\mathcal{A})$ be a quasitrace. The dimension function $d_\tau$ associated to $\tau$ is given by acting on $a \in \mathcal{A}_+$ with $a \leq 1$ by

$$d_\tau(a) = \lim_{k\to\infty} \tau(a^{\frac{1}{k}}).$$

B. Blackadar proposed the notion of strict comparison of general positive elements, much in the spirit of strict comparison of projections.

**Definition 1.4.6.** Let $\mathcal{A}$ be a $C^*$-algebra. We say $\mathcal{A}$ has *Blackadar's strict comparison of positive elements*, or simply say $\mathcal{A}$ has *strict comparison* or *strict comparison of positive elements*, if, for every $a, b \in \mathcal{A}_+$, we have $d_\tau(a) < d_\tau(b)$ for every $\tau \in QT_2(\mathcal{A})$ implies $a \precsim b$.

Note that, trivially, strict comparison of positive elements implies strict comparison of projections.

**Example 1.4.7.** $M_n(\mathbb{C})$ has strict comparison of positive elements. In fact, every element $A \in M_n(\mathbb{C})_+$ is Cuntz equivalent to a trivial projection $P$ with $\mathrm{Rank}(P) = \mathrm{Rank}(A)$.

*Proof.* There is a unique trace $\tau$ on $M_n(\mathbb{C})$, which is the standard one $\tau = \mathrm{Tr}$, and $d_\tau(A) = \mathrm{Tr}(P) = \mathrm{Rank}(P)$ (or $\dfrac{\mathrm{Rank}(P)}{n}$ for the normalized trace) when $P \in M_n(\mathbb{C})$ is a projection. Note that $A \precsim B$ in $M_n(\mathbb{C})$ implies $\mathrm{Rank}(A) \leq \mathrm{Rank}(B)$ from standard linear algebra, for there are $C_n \in M_n(\mathbb{C})$ such that $C_n B C_n^* \to A$.

Let $A \in M_n(\mathbb{C})_+$; then the functional calculus gives $A = \mathrm{id}_{\sigma(A)}$ in $C(\sigma(a))$, i.e. $A(x) = x$. Let $B \in C(\sigma(a))$ be defined by

$$B(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } x \in \sigma(A) \setminus \{0\} \\ 0 & \text{if } x = 0 \end{cases}.$$

Then, $P = BAB^* = 1$ on $\sigma(A) \setminus \{0\}$, and $BAB^*(0) = 0$; in particular, $P$ is a trivial projection in $C^*(A)$ with $A \precsim P$. Let $C = \sqrt{A}$; then, $CPC(x) = x$ on $\sigma(A)$, thus $P \precsim A$. Therefore, $\mathrm{Rank}(P) = \mathrm{Rank}(A)$ and $[A] = [P]$ in the Cuntz semigroup. Since $d_\tau$ agrees on Cuntz equivalence classes, we have

$$d_\tau(P) = \mathrm{Rank}(P) = \mathrm{Rank}(A) = d_\tau(A).$$

Let $A, B \in M_n(\mathbb{C})$ satisfy $d_\tau(A) < d_\tau(B)$. Then, $[A] = [P]$ and $[B] = [Q]$ in the Cuntz semigroup for some trivial projections $P, Q \in M_n(\mathbb{C})$, with $\mathrm{Rank}(P) < \mathrm{Rank}(Q)$. Since they are trivial projections, $P \leq Q$ in $K_0(M_n(\mathbb{C})) \simeq \mathbb{Z}$, thus $A \sim P \precsim Q \sim B$ in the Cuntz semigroup. □

A similar example to the following appears in Toms [Tom08a]:

**Example 1.4.8.** $M_6(C([0,1]^6))$ fails to have strict comparison of positive elements.

*Proof.* The traces in $M_n(C(X))$ have extreme points given by evaluation maps at each $x \in X$. In other words,

$$d_\tau(A) < d_\tau(B)$$

for $A, B \in M_n(C(X))_+$ if and only if

$$\mathrm{Rank}(A(x)) < \mathrm{Rank}(B(x))$$

for every $x \in M_n(C(X))$. Let $\xi' \in C(S^2, M_2(\mathbb{C})) \simeq M_2(C(S^2))$ be a nontrivial Rank 1 projection, e.g. the Hopf fibration. Let $c = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \in [0,1]^3$ be the center of the cube,

$$\mathcal{S} = \{x \in [0,1]^3 \mid \|x - c\| = \frac{1}{4}\}$$

be the sphere in $[0,1]^3$ of radius $\frac{1}{4}$ in with center $c$, and $\zeta : \mathcal{S} \to S^2$ a homeomorphism. Thus, $\xi = \xi' \circ \zeta \in C(\mathcal{S}, M_2(\mathbb{C}))$ is a nontrivial projection. Let

$$U_{\mathcal{S}} = \{x \in [0,1]^3 \mid \|x - c\| \in (\frac{1}{8}, \frac{3}{8})$$

be an annular neighborhood of $\mathcal{S}$ in $[0,1]^3$. We have, $\xi \times \xi \in C(\mathcal{S} \times \mathcal{S}, M_6(\mathbb{C}))$ is a projection, as given by

$$(\xi \times \xi)(x,y) = \begin{bmatrix} \xi(x) & 0_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & \xi(y) & 0_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2} & 0_{2\times 2} \end{bmatrix}.$$

Let $\rho : [0,1]^3 \setminus \{c\} \to \mathcal{S}$ be projection along radial lines emanating from $c = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $f$ a positive function in $C_0(U_{\mathcal{S}})$ with $f(x) = 1$ for $x \in \mathcal{S}$, and $g$ a positive function in $C_0([0,1]^6 \setminus (\mathcal{S} \times \mathcal{S}))$. Define

$$G(x) = \begin{bmatrix} g(x) & 0 \\ 0 & g(x) \end{bmatrix} \in M_2([0,1]^6).$$

Thus, for $x, y \in [0,1]^3$,

$$\Phi(x,y) = \begin{cases} \begin{bmatrix} 0_{4\times 4} & 0_{4\times 2} \\ 0_{2\times 4} & G(x) \end{bmatrix} & \text{if } (x,y) = (c,c) \\ \begin{bmatrix} f(x)\cdot\xi(\rho(x)) & 0 & 0 \\ 0 & f(y)\cdot\xi(\rho(y)) & 0 \\ 0 & 0 & G(x) \end{bmatrix} & \text{if } (x,y) \neq (c,c) \end{cases}$$

defines a function $\Phi \in M_6(C([0,1]^6))$ such that $\Phi\big|_{\mathcal{S}\times\mathcal{S}} = \xi \times \xi$ and $\mathrm{Rank}(\Phi(x)) \in \{2,4\}$ for all $x \in [0,1]^6$. Put

$$\theta_1 = \begin{bmatrix} 1 & 0_{1x5} \\ 0_{5x1} & 0_{5\times 5} \end{bmatrix},$$

which is a trivial Rank 1 projection in $M_6([0,1]^6)$. Then, by Villadsen [Vil98] Lemma 1, stated in a further section as Theorem 2.3.1, we have $\theta_1\big|_{\mathcal{S}\times\mathcal{S}} \not\lesssim \xi \times \xi$ in $K_0(\mathcal{S} \times \mathcal{S}) \simeq K_0(S^2 \times S^2)$.

For the above elements, we have

$$\mathrm{Rank}(\theta_1(x)) = 1 < 2 \leq \mathrm{Rank}(\Phi(x))$$

for every $x \in [0,1]^6$, i.e. $d_\tau(\theta_1) < d_\tau(\Phi)$ for every trace $\tau$. Suppose $\theta_1 \precsim \Phi$ in $\mathrm{Cu}(M_6(C([0,1]^6)))$; this would imply $\theta_1\big|_{\mathcal{S}\times\mathcal{S}} \precsim \Phi\big|_{\mathcal{S}\times\mathcal{S}} = \xi \times \xi$, which would imply $\theta_1 \leq \xi \times \xi$ in $K_0(\mathcal{S} \times \mathcal{S})$. That is a contradiction, therefore $M_6(C([0,1]^6))$ does not have strict comparison of positive elements. $\square$

## 1.5 Perforation and the Jiang-Su Algebra $\mathcal{Z}$

In [JS99], Jiang and Su constructed a simple, separable, nuclear, infinite-dimensional algebra $\mathcal{Z}$ with the same Elliott invariant as $\mathbb{C}$. It is an inductive limit of sums of so-called dimension drop algebras. It was later shown that it is intimately related to the notion of slow dimension growth and Blackadar's strict comparison of positive elements.

**Definition 1.5.1.** Suppose $\mathcal{A}$ is a $C^*$-algebra. Then $K_0(\mathcal{A})$ is called *weakly unperforated* if $nx \in K_0(\mathcal{A})_+ \setminus \{0\}$ implies $x \in K_0(\mathcal{A})_+$.

**Example 1.5.2.** Let $\mathcal{A} = M_n(C(X))$, with $X$ contractible. Then $K_0(\mathcal{A})$ is weakly unperforated.

**Example 1.5.3.** Let $\mathcal{A}$ be a $C^*$-algebra with strict comparison of projections and cancellation of projections. Then, $K_0(\mathcal{A})$ is weakly unperforated.

**Remark 1.5.4.** It is shown by Elliott-Ho-Toms [EHT09] that all simple AH algebras have stable rank one, therefore have cancellation of projections by Theorem 1.3.12. Since strict comparison in the Cuntz semigroup implies strict comparison of projections, all simple AH algebras with strict comparison of positive elements necessarily have $K_0(\mathcal{A})$ is weakly unperforated. The converse is not true, as we will investigate more later.

The following important notion is shown by Gong-Jiang-Su in [GJS00]:

**Theorem 1.5.5.** Let $\mathcal{A}$ be a simple, unital $C^*$-algebra with $K_0(\mathcal{A})$ weakly unperforated. Then $\mathrm{Ell}(\mathcal{A}) \simeq \mathrm{Ell}(\mathcal{A} \otimes \mathcal{Z})$.

Moreover, Rørdam [Rør04] was able to show that $\mathcal{Z}$-stability, that $\mathcal{A} \otimes \mathcal{Z} \simeq \mathcal{A}$, generally implied strict comparison for simple, unital, exact, finite $C^*$-algebras.

Toms and Winter conjectured, and later proved for the AH algebras, a strong equivalence known as the Toms-Winter Conjecture. A generalization of the next theorem can be found in [Tom11].

**Theorem 1.5.6.** Let $\mathcal{A}$ be an AH algebra. Then $\mathcal{A} \otimes \mathcal{Z} \simeq \mathcal{A}$ if and only if $\mathcal{A}$ has Blackadar's strict comparison of positive elements.

Strict comparison has an important connection to almost unperforation.

**Definition 1.5.7.** Let $\mathcal{A}$ be a $C^*$-algebra. We say $\mathrm{Cu}(\mathcal{A})$ is *almost unperforated* if $(n+1)[x] \leq n[y]$ in $\mathrm{Cu}(\mathcal{A})$ for some $n$ implies $[x] \leq [y]$.

In Proposition 3.2.4 of [Bla+12], B. Blackadar, L. Robert, A. Tikuisis, A. Toms, and W. Winter show an important connection to strict comparison:

**Theorem 1.5.8.** Let $\mathcal{A}$ be a simple, stably finite, unital $C^*$-algebra. Then, $\mathcal{A}$ has strict comparison if and only if $\mathrm{Cu}(\mathcal{A})$ is almost unperforated.

**Example 1.5.9.** $\mathrm{Cu}(M_6(C([0,1]^6)))$ is not almost unperforated.

*Proof.* We know that this is true from Example 1.4.8. But, on the other hand, let $\theta_1, \Phi$ be the elements constructed in this example. Then, $101[\theta_1] \leq 100[\Phi]$ in $\mathrm{Cu}(M_6(C([0,1]^6)))$ from Toms' result in [Tom08b], because

$$\mathrm{Rank}(101[\theta_1](x)) + \frac{1}{2}\dim([0,1]^6) = 101 + 3 < 200 \leq \mathrm{Rank}(100[\Phi](x))$$

for every $x \in [0,1]^6$. But, as we know based on the example, $[\theta_1] \not\lesssim [\Phi]$. Therefore, $\mathrm{Cu}(M_6(C([0,1]^6)))$ is not almost unperforated. $\square$

# CHAPTER 2

# CONSTRUCTIONS OF GOODEARL, VILLADSEN AND TOMS

## 2.1 Goodearl Algebras

In [Goo92], K. R. Goodearl constructed an AH algebra with a curious property, later expanded on by J. Villadsen.

**Definition 2.1.1.** A *Goodearl algebra* over seed space $X$ is an inductive limit $\mathcal{A} = \lim_{n\to\infty}(M_{R_n}(C(X)), \phi_n)$ where

$$\phi_n(a) = \mathrm{diag}(a, a, ..., a, \delta_n(a), ...., \delta_n(a))$$

for $\delta_n(a) := x_n$ for some $x_n \in X$ are (constant) evaluation maps, and at least one identity and evaluation map occur.

**Remark 2.1.2.** We have $\mathcal{A}$ is simple if and only if the evaluation points are dense in $X$. Thus, when $\mathcal{A}$ is simple and $\dim(X) < \infty$, $R_n \to \infty$, therefore $\mathcal{A}$ has slow dimension growth.

A. Toms (c.f [Tom08b]) has shown the analogous result to Theorem 1.2.3 for Cuntz subequivalence for the commutative $C^*$-algebra $C(X)$, and the following generalization for $M_n(C(X))$:

**Theorem 2.1.3.** Let $X$ be a compact metric space with $\dim(X)$ being the covering dimension. Let $a, b \in M_n(C(X))$ be positive, and suppose for all $x \in X$

$$\mathrm{Rank}(a(x)) + \frac{1}{2}\dim(X) < \mathrm{Rank}(b(x)).$$

Then we have $a \precsim b$.

The next proposition and its proof illustrates the Toms-Winter conjecture for the simple Goodearl algebras over a finite-dimensional space $X$.

**Proposition 2.1.4.** Let $X$ satisfy $\dim(X) = d$, and let $\mathcal{A}$ be a simple Goodearl algebra constructed from $X$. Then $\mathcal{A}$ has strict comparison of positive elements.

*Proof.* Suppose $a, b \in \mathcal{A}_+$, and $d_\tau(b) - d_\tau(a) = \delta > 0$ for every 2-quasitrace $\tau$. Since $d_\tau$ is lower-semicontinuous, we have

$$U = \{c \in \mathcal{A}_+ \mid d_\tau(c) - d_\tau(a) > \frac{\delta}{2}\}$$

is an open set containing $b$; thus, there exists $r > 0$ such that

$$\|b - c\| < r \quad \text{implies} \quad d_\tau(c) > d_\tau(a) + \frac{\delta}{2}.$$

Let $\varepsilon \in (0, r)$ be arbitrary. A result of Kirchberg and Rørdam states that $\|a - b\| < \eta$ implies $(a - \eta)_+ \precsim b$. Moreover, Cuntz inequality is preserved under $*$-homomorphisms, and there are increasing sequences $(a'_n), (b'_n)$ such that $\phi_{n,\infty} a'_n \to a$ and $\phi_{n,\infty} b'_n \to b$, with $a'_n, b'_n \in \mathcal{A}_n$ for each $n \in \mathbb{N}$. That is, there exists $n \in \mathbb{N}$ and $a', b' \in \mathcal{A}_n$ such that

(i) $\|\phi_{n,\infty} a' - a\| < \varepsilon$;
(ii) $\|b - \phi_{n,\infty} b'\| < \varepsilon < r$;

which satisfy

(iii) $(a - \varepsilon)_+ \precsim \phi_{n,\infty} a' \precsim a$;
(iv) $\phi_{n,\infty} b' \precsim b$.

So, we have

$$d_\tau(b) \geq d_\tau(\phi_{n,\infty} b') > d_\tau(a) + \frac{\delta}{2} \geq d_\tau(a') + \frac{\delta}{2} > d_\tau(a - \varepsilon)_+.$$

Let $\tilde{\tau}$ be any 2-quasitrace, with $\tau_n = \tilde{\tau} \circ \phi_{n,\infty}$ its pullback to $\mathcal{A}_n$, and let $\|a\| = 1$ with $a \in \mathcal{A}_+$. Write $\tau := \tau_{n-1}$ for brevity. For a typical element $a \in \mathcal{A}_{n-1}$, let us compute a trace of

$$\phi_{n-1}(a) = \operatorname{diag}(\alpha_n.a, \beta_n.(\delta_n(a))) \in \mathcal{A}_n,$$

where $\alpha_n.a$ denotes $a$ repeated on the diagonal $\alpha_n$ times, and analogously for $\beta_n.(\delta_n(a))$. We find

$$\begin{aligned}
\tau_n(\phi_{n-1,n} a) &= \frac{1}{\nu(n)} \left[\alpha_n \tau(a) + \beta_n \tau(\delta_n(a))\right], \\
\tau_{n+1}(\phi_{n-1,n+1} a) &= \frac{1}{\nu(n+1)} \cdot [\alpha_n \alpha_{n+1} \tau(a)] \\
&\qquad + \frac{1}{\nu(n)} \cdot \beta_n \cdot \tau(\delta_n(a)) + \frac{1}{\nu(n+1)} \cdot \beta_{n+1} \alpha_n \cdot \tau(\delta_{n+1}(a)), \\
&\cdots \\
\tau_s(\phi_{n-1,s}(a)) &= \frac{1}{\nu(n)} \left[\omega_{s+1,n} \cdot \tau(a)\right] + \sum_{k=n}^{s} \frac{\beta_k}{\alpha_k} \cdot \frac{1}{\nu(k)} \left(\prod_{j=n}^{k} \alpha_j\right) \cdot \tau(\delta_k(a)).
\end{aligned}$$

In particular, we have that

$$\tau_s(\phi_{n-1,s}(a)) = c_0\tau(a) + \sum_{k=n}^{s} c_k \tau(\delta_k(a))$$

for coefficients $c_k \in (0,\infty)$ where $c_k \to 0$ as $k \to \infty$ and $c_0$ is decreasing; indeed for all $s$ we have $c_0 + \sum_{k=n}^{s} c_k = 1$.

The function $d_n := d_{\tau_n} = d_{\tilde{\tau}} \circ \phi_{n,\infty}$ is a lower-semicontinuous dimension function on $\mathcal{A}_n$ with $d_n(b') - d_n(a') > \frac{\delta}{4} > 0$. Therefore, there exists an open set $V \subset X$ such that for every $x \in V$ we have $\mathrm{Rank}(b'(x)) > \mathrm{Rank}(a'(x))$. Because the sequence $(x_k)_{k\geq M}$ is dense in $X$ for every $M \in \mathbb{N}$, and $\sum_k c_k < \infty$, with $c_k > 0$ for all $k$, there exists a subsequence $(x_j)_{j\in I}$ for some finite index set $I \subset \mathbb{N}$ such that

(i) $x_j \in V$ for every $j \in I$;
(ii) $\sum_{j\in I} c_j < \frac{\delta}{8}$;
(iii) $|I| > \frac{1}{2}d$.

Let $a_\gamma = \bigoplus_{k\in I} \delta_k(a')$ and likewise $b_\gamma = \bigoplus_{k\in I} \delta_k(b')$ as block diagonals (along with zero blocks) such that, for sufficiently large $s$,

$$a_\gamma \leq \phi_{n,s}a' \quad \text{and} \quad b_\gamma \leq \phi_{n,s}b'.$$

We have

$$\begin{aligned}
\tau_s(\phi_{n,s}b' - b_\gamma) &= c_0\tau(b') + \sum_{k=n,\ k\notin I}^{s} c_k\tau(\delta_k(b')) \\
&> c_0\tau(a') + \sum_{k=n,\ k\notin I}^{s} c_k\tau(\delta_k(a')) + \frac{\delta}{8} \\
&= \tau_s(\phi_{n,s}a' - a_\gamma) + \frac{\delta}{8};
\end{aligned}$$

therefore, for all $x \in X$, $\mathrm{Rank}((\phi_{n,s}b' - b_\gamma)(x)) \geq \mathrm{Rank}((\phi_{n,s}a' - a_\gamma)(x))$.

On the other hand, we have selected more than $\frac{1}{2}d$ elements $x_j$ which are all constant matrices satisfying $\mathrm{Rank}(\delta_j(b')) \geq \mathrm{Rank}(\delta_j(a')) + 1$. Whence,

$$\mathrm{Rank}(b_\gamma(x)) > \mathrm{Rank}(a_\gamma(x)) + \frac{1}{2}d;$$

we conclude, since $a_\gamma \perp \phi_{n,s}a'$ and $b_\gamma \perp \phi_{n,s}b'$, that for all $x \in X$

$$\mathrm{Rank}(\phi_{n,s}b'(x)) > \mathrm{Rank}(\phi_{n,s}a'(x)) + \frac{1}{2}d.$$

By Toms' result, since $\varepsilon > 0$ is arbitrary, we have $\phi_{n,\infty}a' \precsim \phi_{n,\infty}b'$. Putting it all together, we have found

$$(a-\varepsilon)_+ \precsim \phi_{n,\infty}a' \precsim \phi_{n,\infty}b' \precsim b.$$

Since $\varepsilon > 0$ is arbitrary, we conclude $a \precsim b$.

$\square$

It is not an obvious fact that even Goodearl algebras over an infinite-dimensional seed space $X$ have strict comparison. It follows from sufficient divisibility properties of these algebras, as investigated by Fu, Li, and Lin in [FLL22].

**Definition 2.1.5.** Let $\mathcal{A}$ be a simple $C^*$-algebra. $\mathcal{A}$ is said to have property (TAD) if given any $\varepsilon > 0$, $s \in \mathcal{A}_+$ with $s \neq 0$, $n \in \mathbb{N}$, and any finite subset $\mathcal{F} \subset \mathcal{A}$, there exists $\theta \in \mathcal{A}_+$ and a $C^*$ subalgebra $\mathcal{D} \otimes M_n \subset \mathcal{A}$ such that

(i) $\theta x \approx_\varepsilon x\theta$ for all $x \in \mathcal{F}$;
(ii) $(1-\theta)x \in_\varepsilon \mathcal{D} \otimes 1_n$ for all $x \in \mathcal{F}$;
(iii) $\theta \precsim s$.

**Definition 2.1.6.** Let $\mathcal{A}$ be a simple $C^*$-algebra. $\mathcal{A}$ is said to be *tracially approximately divisible* if for any $\varepsilon > 0$, finite $\mathcal{F} \subset \mathcal{A}$, element $e_F \in \mathcal{A}_+^1$ with $e_F x \approx_{\varepsilon/4} x \approx_{\varepsilon/4} xe_F$ for all $x \in \mathcal{F}$, every nonzero $s \in \mathcal{A}_+$ and $n \in \mathbb{N}$, there exists $\theta \in \mathcal{A}_+^1$ and a $C^*$ algebra $\mathcal{D} \otimes M_n \subset \mathcal{A}$, and a c.p.c. map $\beta : \mathcal{A} \to \mathcal{A}$ such that

(i) $x \approx_\varepsilon x_1 + \beta(x)$ for all $x \in \mathcal{F}$, with $\|x_1\| \leq \|x\|$, $x_1 \in \mathrm{Her}(\theta)$;
(ii) $\beta(x) \in_\varepsilon \mathcal{D} \otimes 1_n$, and $e_F\beta(x) \approx_\varepsilon \beta(x) \approx_\varepsilon \beta(x)e_F$ for all $x \in \mathcal{F}$;
(iii) $\theta \precsim s$.

If $\mathcal{A}$ has property (TAD) it is shown to be tracially approximately divisible in [FLL22]. In this work, simple $C^*$ algebras with property (TAD) are shown to have strict comparison.

**Proposition 2.1.7.** Let $\mathcal{A}$ be a real rank one Goodearl algebra over an infinite-dimensional space $X$. Then $\mathcal{A}$ has property (TAD). Therefore, it has strict comparison of positive elements.

*Proof.* Let $\varepsilon > 0$, $\mathcal{F} = \{x^1, ..., x^\ell\} \subset \mathcal{A}$, $s \in \mathcal{A}_+$ nonzero, and $n \in \mathbb{N}$. There exists some large $N \in \mathbb{N}$ and $s_N, x_N^1, ..., x_N^\ell \in \mathcal{A}_N$ such that

(i) $b_N x_N \approx_{\frac{\varepsilon}{3}} x_N b_N$ for every $b_N \in \mathcal{A}_N$. Such an $N$ exists because $\mathcal{A}$ is real rank one and so the connecting homomorphisms are approximately $\mathrm{id} \otimes 1_N$ as $N \to \infty$;
(ii) $\phi_{N,\infty} x_N^j \approx_{\frac{\varepsilon}{3}} x$ for all $j = 1, ..., \ell$;
(iii) $\phi_{N,\infty} s_N \approx_{\frac{\varepsilon}{3}} s$ with $s_N \precsim s$.

Put $\theta_N = \frac{1}{\|s_N\|} s_N$, where and $D = \mathcal{A} \simeq \mathcal{A} \otimes M_n$. Then, with $\theta = \phi_{N,\infty}\theta_N$, we have

$$\begin{aligned}
\|\theta x - x\theta\| \leq \|\theta x - \theta\phi_{n,\infty}x_N\| + \|\phi_{n,\infty}x_N\theta - x\theta\| + \|\phi_{n,\infty}x_N\theta - \theta\phi_{n,\infty}x_N\| \\
\leq 2\|\theta\|\|x - \phi_{N,\infty}x_N\| + \|\phi_{n,\infty}(x_N\theta_N - \theta_N x_N)\| \\
\leq 2 \cdot 1 \cdot \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

□

**Corollary 2.1.8.** Let $\mathcal{A}$ be a simple Goodearl algebra; then $K_0(\mathcal{A})$ is weakly unperforated.

*Proof.* $\mathcal{A}$ has cancellation of projections, since it has stable rank one, and so we have $\tau(p) < \tau(q)$ implies $p \precsim q$ for any projections $p, q$. □

## 2.2 Villadsen Algebras

The Villadsen algebras employ an illustrative construction first investigated by J. Villadsen in [Vil98].

**Definition 2.2.1.** A map $\phi_j : C(X_j) \to M_{R_j}(C(X_{j+1}))$ is called *diagonal* if it has the form

$$\phi_j(f) = \mathrm{diag}(f \circ \lambda_1, \ldots f \circ \lambda_{R_j}) := \begin{bmatrix} f \circ \lambda_1 & 0 & & \cdots & 0 \\ 0 & f \circ \lambda_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & f \circ \lambda_{R_j - 1} & 0 \\ 0 & \cdots & & 0 & f \circ \lambda_{R_j}) \end{bmatrix},$$

where each $\lambda_j : X_{j+1} \to X_j$ is a continuous map, called the *eigenvalue maps* of $\phi_j$.

**Remark 2.2.2.** Let $X$ be a locally compact Hausdorff space with $\dim(X) \geq 1$; put

$$X_1 = X,$$
$$X_{j+1} = X_j^{n_j} \quad (n > 1)$$

for some sequence $(n_j)$ in $\mathbb{N}$. For each $i, j \in \mathbb{N}$ with $i \leq j$, put $D_{i,j} = \prod_{k=i}^{j-1} n_k$. Notice

$$\dim(X_n) = \dim(X_1) \cdot D_{1,n} \quad \text{and} \quad D_{i,j} \cdot D_{j,\ell} = D_{i,\ell}.$$

Thus, $D_{i,j}$ is the number of *distinct* coordinate projections from $X_j \to X_i$, for we have

$$D_{i,j} = \frac{\dim(X_j)}{\dim(X_i)} \quad \text{and} \quad X_j = (X_i)^{D_{i,j}}.$$

**Definition 2.2.3.** Let $(k_j)$ be an increasing sequence in $\mathbb{N}$ with $k_j \big| k_{j+1}$ for all $j$; a unital diagonal homomorphism

$$\phi_j : M_{R_j} \otimes C(X^{k_j}) \to M_{R_{j+1}}(C(X^{k_{j+1}}))$$

is called a *Villadsen map of the first type* if each eigenvalue map is either a point evaluation or a coordinate projection. An inductive limit $\mathcal{A} = \lim\limits_{n\to\infty}(\mathcal{A}_n, \phi_n)$ over $\mathcal{A}_j = M_{R_j}(C(X^{k_j}))$, where $\phi_n$ are Villadsen maps of the first type, is called a *Villadsen algebra (of the first type)*.

Throughout this paper, we will refer to a Villadsen algebra to mean a Villadsen algebra of the first type where $\dim(X_1) \geq 1$ and $X_1$ is connected. As with the Goodearl algebras, a Villadsen algebra $\mathcal{A}$ is simple when the evaluation maps are sufficiently dense when included in $X^{\mathbb{N}}$.

Put $M_{i,j} = \dfrac{R_j}{R_i}$ and denote $D_{i,j}$ as the number of distinct coordinate projections appearing in the connecting homomorphisms from stage $i$ to $j$. Thus $M_{i,j} \cdot M_{j,\ell} = M_{i,\ell}$, so

$$\frac{D_{i,j}}{M_{i,j}} \cdot \frac{D_{j,j+1}}{M_{j,j+1}} = \frac{D_{i,j+1}}{M_{i,j+1}}$$

and generally

$$\frac{D_{i,j}}{M_{i,j}} = \frac{D_{i,i+1}}{M_{i,i+1}} \cdot \ldots \cdot \frac{D_{j-1,j}}{M_{j-1,j}}$$

is a product of terms which have infinitely many terms strictly less than 1 when $\mathcal{A}$ is simple, as there are infinitely many point-evaluation maps. Therefore $\left(\dfrac{D_{i,j}}{M_{i,j}}\right)$ is a decreasing positive sequence in $j$, so this sequence converges. We will mostly be interested, without loss of generality, in the case $D_{i,j} = N_{i,j}$, i.e. all the distinct coordinate projections are included. In Toms-Winter [TW09], it is shown that when $\lim\limits_{j\to\infty} \dfrac{D_{i,j}}{M_{i,j}} = \varepsilon > 0$, then $\mathcal{A}$ doesn't have strict comparison.

Similarly, let $\alpha_n$ denote the number of total projections in the definition of $\phi_n$, including multiplicity. Put

$$\omega_{i,j} = \alpha_i \cdot \ldots \cdot \alpha_{j-1} \frac{R_i}{R_j}.$$

Analogously, we have

$$\omega_{i,j} \cdot \omega_{j,\ell} = \omega_{i,\ell},$$

so $\omega_{i,j}$ is a decreasing positive sequence in $j$ which also converges. In Goodearl's paper [Goo92], it is shown that in the case $X_n = X$ for all $n$, i.e. the Goodearl algebras, when $\lim\limits_{j\to\infty} \omega_{1,j} = \varepsilon > 0$, then $\mathcal{A}$ has real rank one. It turns out that this phenomenon with real rank is typical of the Villadsen algebras as well. Note that many of the following results are shown in one form or another throughout the literature. The next lemma's proof is inspired by a similar proof in Goodearl's paper.

**Lemma 2.2.4.** Suppose that $\mathcal{A} = \lim\limits_{n\to\infty} (M_{R_n}(C(X_n)), \phi_n)$ is a simple Villadsen algebra of the first type. Let $\alpha_n$ be the number of projections appearing in the connecting homomorphism $\phi_n$ and put

$$\omega_{i,j} = \alpha_i \cdot \ldots \cdot \alpha_{j-1} \frac{R_i}{R_j}.$$

If

$$\lim_{j\to\infty} \omega_{1,j} = \varepsilon > 0,$$

then $\mathcal{A}$ has real rank one.

*Proof.* Since $\mathcal{A}$ has stable rank one, it has real rank zero or one by Proposition 1.3.15. There exists $x, y$ in $X$ and $f : X \to \mathbb{C}$ continuous such that $f(x) = 1$ and $f(y) = 0$; put $a = \mathrm{diag}(f, ..., f) \in \mathcal{A}_1$. Appealing to Theorem 1.3.16, suppose that there exists $b \in \mathcal{A}$ such that $b$ is a linear combination of projections and $\|b - \phi_{1,\infty}(a)\| < \dfrac{\varepsilon}{4}$. Then there exists $s \in \mathbb{N}$ and $c \in \mathcal{A}_s$ such that

(i) $c$ is a linear combination of projections;

(ii) $\|c - \phi_{1,s}(a)\| < \frac{\varepsilon}{2}$.

There is a particular $\tilde{x} \in X^{n_s}$ such that $\pi_1^{r_1}\pi_2^{r_2} \circ \ldots \circ \pi_s^{r_s} : X^{n_s} \to X$ has

$$\pi_1^{r_1}\pi_2^{r_2} \circ \ldots \circ \pi_s^{r_s}(\tilde{x}) = x$$

for all choices of $r_j$. In particular, that element is $\tilde{x} = (x, x, \ldots, x)$. Likewise, $\tilde{y} = (y, y, \ldots, y)$ has the analogous statement. Thus, $\phi_{1,s}(\tilde{x}) - \phi_{1,s}(\tilde{y})$ is a diagonal matrix which has at least $\varepsilon \cdot \nu(s)$ entries of 1 and the rest are zero; i.e.

$$|\operatorname{Tr}(\phi_{1,s}(\tilde{x}) - \phi_{1,s}(\tilde{y})| > \varepsilon\nu(s).$$

On the other hand, since Tr is a continuous map on linear combinations of projections into $\mathbb{Z}$, one has $\operatorname{Tr}(c(\tilde{x})) = \operatorname{Tr}(c(\tilde{y}))$. Therefore

$$|\operatorname{Tr}(c(z)) - \operatorname{Tr}(\phi_{1,s}(a)(z))| \le \|c(z) - \phi_{1,s}(a)(z)\| \cdot \nu(s) < \frac{\varepsilon}{2}\nu(s)$$

for all $z \in X^{n_s}$. But since $\operatorname{Tr}(c(\tilde{x})) = \operatorname{Tr}(c(\tilde{y}))$ we have

$$|\operatorname{Tr}(\phi_{1,s}a(\tilde{x})) - (\phi_{1,s}a(\tilde{y}))| < \varepsilon \cdot \nu(s).$$

This is a contradiction; therefore such an element $a$ cannot have been approximated by a linear combination of projections. Therefore, $\mathcal{A}$ cannot be real rank zero by Proposition 1.3.16. $\square$

Note that $D_{i,j} \le \alpha_{i,j}$ for all $i, j$, since we are just discounting multiplicity. The next statement is proved in Toms, Winter [TW09]; their proof includes elements of both the above and below proofs.

**Corollary 2.2.5.** Suppose $\mathcal{A} = \lim_{n\to\infty}(\mathcal{A}_n, \phi_n)$ is a simple Villadsen algebra with real rank zero. Then, $\mathcal{A}$ has strict comparison of positive elements.

*Proof.* Suppose $\mathcal{A}$ does not have strict comparison. Let $D_{1,j}$ denote the number of distinct projections from stage 1 to $j$, and $M_{1,j} = \frac{R_j}{R_1}$ denote the relative matrix size. By Toms [TW09] Lemma 5.1,

$$\lim_{j\to\infty} \frac{D_{1,j}}{M_{1,j}} \to \varepsilon > 0.$$

But, $D_{1,j} \le \alpha_{i,j}$ since we are just ignoring the distinction; therefore

$$\lim_{j\to\infty} \omega_{1,j} \ge \varepsilon > 0.$$

By Lemma 2.2.4, $\mathcal{A}$ has real rank one. $\square$

The converse is essentially also proved in Christensen [Chr18], though not stated explicitly. He cites A. Toms as well:

**Lemma 2.2.6.** Suppose that $\mathcal{A} = \lim_{n\to\infty}(M_{R_n}(C(X_n)), \phi_n)$ is a simple Villadsen algebra of the first type. Let $\alpha_n$ be the number of projections appearing in the connecting homomorphism $\phi_n$, and put

$$\omega_{i,j} = \alpha_i \cdot \ldots \cdot \alpha_{j-1}\frac{R_i}{R_j}.$$

If

$$\lim_{j\to\infty} \omega_{1,j} = 0$$

then $\mathcal{A}$ has real rank zero.

*Proof.* We have $D_{1,j} \leq \alpha_{1,j}$. Therefore if $\omega_{1,j} \to 0$, $\mathcal{A}$ has slow dimension growth. Since $\mathcal{A}$ is simple and has slow dimension growth, it has real rank zero if and only if the projections separate the traces (see Definition 2.4.1 later on) of $\mathcal{A}$ (c.f. Blackadar, Dardalat, Rørdam [BDR91]). But, when the product is zero, then $\mathcal{A}$ has a unique trace, as illustrated in Example 2.1.4 (c.f. also Christensen [Chr18] - Theorem 3.6). Therefore, $\mathcal{A}$ has real rank zero. One can also surely use practically the same direct proof as appears in Goodearl's paper [Goo92]. □

These lemmas witness the following generalization of the phenomenon of real rank one appearing in the Goodearl algebras to the Villadsen algebras:

**Theorem 2.2.7.** Suppose that $\mathcal{A} = \lim_{n\to\infty}(M_{R_n}(C(X_n)), \phi_n)$ is a simple Villadsen algebra of the first type. Let $\alpha_n$ be the number of projections appearing in the connecting homomorphism $\phi_n$, and put

$$\omega_{i,j} = \alpha_i \cdot \ldots \cdot \alpha_{j-1}\frac{R_i}{R_j}.$$

We have $\lim_{j\to\infty} \omega_{1,j} = 0$ if and only if $\mathcal{A}$ has real rank zero.

## 2.3 Villadsen's Chern Class Obstruction

Toms exhibited in [Tom08a] a simple Villadsen algebra with weakly unperforated $K_0$ group, in particular over a contractible seed space $X_1$, but which failed to have strict comparison of positive elements. It showed the necessity of the Cuntz semigroup for classification, but also suggested the difficulty in explicitly finding the

Cuntz semigroup of even modest algebras. This was largely based on the work of Villadsen, who showed the following in Lemma 1 of [Vil98].

**Theorem 2.3.1.** Let $\zeta$ be a complex line bundle over a finite CW-complex $B$, and let $n \in \mathbb{N}$. Let $\theta_1$ denote the trivial line bundle. If $[\zeta^{\times n}] - [\theta_1] \in K_0(B^n)_+$, then the n-th tensor power of the Euler class of $\zeta$ is zero.

The lack of strict comparison illustrated in Example 1.4.8 can be preserved across the Villadsen algebras if the number of projections is sufficiently large. This phenomenon is known as *Villadsen's Chern class obstruction.* Let $\mathcal{A} = \lim\limits_{n\to\infty}(C_{R_n}(X_n))$ be a simple Villadsen algebra, and let $N_{i,j}$ and $M_{i,j}$ denote the number of distinct projections and the relative matrix size $M_{i,j} = \dfrac{R_j}{R_i}$, respectively, in the connecting homomorphisms $\phi_{i,j}$ from stages $i$ to $j$. In [TW09], Toms and Winter were able to prove the dichotomy that $\mathcal{A}$ has strict comparison if and only if

$$\lim_{j\to\infty} \frac{N_{i,j}}{M_{i,j}} = 0$$

for all $i$, and $\mathcal{A}$ doesn't have strict comparison if and only if

$$\lim_{i\to\infty} \lim_{j\to\infty} \frac{N_{i,j}}{M_{i,j}} = 1.$$

In particular, in this case

$$\lim_{j\to\infty} \frac{N_{i,j}}{M_{i,j}} = \varepsilon_i > 0$$

for all $i$; this sequence $\left(\dfrac{N_{i,j}}{M_{i,j}}\right)_{j=1}^{\infty}$ is a decreasing sequence in $j$, and $(\varepsilon_i)_{i=1}^{\infty}$ is an increasing sequence in $i$ with $\lim\limits_{i\to\infty} \varepsilon_i = 1$ from the multiplicative properties. This number $\varepsilon_i$ is the limit of the proportion of projection maps from $\mathcal{A}_i$ compared to the size of the matrices. When it is positive, as we will see later, the projection block manages to preserve enough information to prevent strict comparison.

## 2.4 Tracial States in Villadsen Algebras

The extremal tracial states in $M_n(C(X))$ come from point evaluations at $x$: given $A \in M_n(C(X))$, $x \in X$, the function

$$\tau_x(A) = \mathrm{Tr}(A(x))$$

defines an extreme tracial state on $M_n(C(X))$. Moreover, for the Villadsen algebras, we find for $a \in A_n = M_{R_n}(C(X_n))_+$, and $x \in X_n$,

$$\tau_{x'}(\phi_{n,\infty}a) := \lim_{k\to\infty} \mathrm{Tr}(\phi_{n,k}a(x,x,\ldots,x)) = \lim_{k\to\infty} \tau_{(x,x,\ldots,x)}(\phi_{n,k}a)$$

where $(x,x,\ldots,x)$ has $D_{n,k} := \dfrac{\dim(X_k)}{\dim(X_n)}$ copies, defines a tracial state on $\bigcup_{i=n}^{\infty} \mathcal{A}_i$ which extends to $\mathcal{A}$. Thus the extreme traces from point evaluations on $\mathcal{A}_n$ extend naturally to traces on $\mathcal{A}$. In this section, we make a summary of some results informing the Villadsen algebras.

**Definition 2.4.1.** Let $\mathcal{A}$ be a $C^*$ algebra. We say the *projections in $\mathcal{A}$ separate the traces* if $\tau_1(p) = \tau_2(p)$ for every projection $p \in \mathcal{A}$ implies $\tau_1 = \tau_2$.

The following statement is concluded by Theorem 1.3 in Blackadar, Bratteli, Elliott, and Kumjian's paper [Bla+92]:

**Theorem 2.4.2.** Let $\mathcal{A}$ be a simple Villadsen algebra. Consider the following statements:

(i) The projections in $\mathcal{A}$ separate the traces;
(ii) For every $a \in \mathcal{A}_{i,SA}$, $\varepsilon > 0$, there is $j \geq i$ such that $TV(\phi_{i,j}(a)) < \varepsilon$, where $TV$ is the *variation of the normalized trace*, denoted for $b \in \mathcal{A}_j$

$$TV(b) = \sup\{|\,\mathrm{Tr}(b(x)) - \mathrm{Tr}(b(y))| \mid x, y \in X_j\},$$

(iii) $\mathcal{A}$ has real rank zero.

We have (iii) $\implies$ (ii) $\iff$ (i).

Along similar lines, the following was concluded in T. Ho's Ph.D Thesis as applied to the Villadsen algebras:

**Theorem 2.4.3.** Let $\mathcal{A}$ be a simple Villadsen algebra. The following are equivalent:

(i) $\mathcal{A}$ has real rank zero;
(ii) $\omega_{1,\infty} := \lim_{j\to\infty} \omega_{1,j} = 0$;
(iii) $TR(\mathcal{A}) = 0$, where $TR$ is the *tracial topological rank*;
(iv) $\mathcal{A}$ has slow dimension growth and projections in $\mathcal{A}$ separate the traces.

We can strengthen these results for the Villadsen algebras.

**Lemma 2.4.4.** Let $\mathcal{A}$ be a simple Villadsen algebra with real rank one. Then the projections in $\mathcal{A}$ do not separate the traces.

*Proof.* Let

$$\omega_{i,\infty} := \lim_{j\to\infty} \omega_{i,j} = \lim_{j\to\infty} \alpha_i \cdot ... \cdot \alpha_{j-1}\frac{R_i}{R_j};$$

We have $\omega_{i,\infty} > 0$ for all $i$ since $\mathcal{A}$ is real rank one, by Theorem 2.2.7. For $x \in X_n$ for some $n \in \mathbb{N}$, let $\tau_{x'}$ be the tracial states on $\mathcal{A}$ coming from evaluation at $(x, x, \ldots)$.

Let $p \in \mathcal{A}$ be a projection. Then, there exists a projection $p_n \in \mathcal{A}_n$ for some large $n$ such that $p \sim p_n$. For projections, Tr is a continuous map from $\mathcal{A} \to \mathbb{Z}$. In fact,

$$\mathrm{Tr}(p_n(x)) = \mathrm{Tr}(\phi_{n,k}p_n(x, x, \ldots, x)) = \mathrm{Tr}(\phi_{n,k}p_n(y, y, \ldots, y)) = \mathrm{Tr}(p_n(y))$$

for every $k \geq n$ and $x, y \in X_n$. Thus,

$$\tau_{x'}(p) = \mathrm{Tr}(p_n(x)) = \mathrm{Tr}(p_n(y)) = \tau_{y'}(p)$$

for every projection $p \in \mathcal{A}$ and $x, y \in X_n$. But, generally $\tau_{x'} \neq \tau_{y'}$ for given $x, y$. For example, if $x, y \in X_1$ and $f \in C(X_1)$ is a function which has $f(x) = 0$ and $f(y) = 1$, putting $a = \mathrm{diag}(f, f, ..., f) \in \mathcal{A}_1$ we can see

$$\begin{aligned}|\tau_{x'}(\phi_{1,\infty}a) - \tau_{y'}(\phi_{1,\infty}a) &= \lim_{k\to\infty} |\,\mathrm{Tr}(\phi_{1,k}a(x, x, \ldots, x)) - \mathrm{Tr}(\phi_{1,k}a(y, y, \ldots, y)| \\ &\geq \omega_{1,\infty} > 0.\end{aligned}$$

Thus, the projections in $\mathcal{A}$ do not separate the traces.

□

**Corollary 2.4.5.** For simple Villadsen algebras, in Theorem 2.4.2, (i), (ii), and (iii) are equivalent.

*Proof.* If $\mathcal{A}$ does not have real rank zero, it has real rank one, since it has stable rank one, by Theorem 1.3.15. Thus, the negation of (iii), i.e. real rank one, implies the negation of (ii) and (i). □

Below we summarize a general statement about Villadsen algebras, which generalizes in some cases to suitable diagonal AH algebras (c.f Theorem 3.4 of [TW09]). Equivalence of (vi)-(xv) is well established by Toms, Winter, Niu, Lin, and others.

**Corollary 2.4.6.** Let $\mathcal{A}$ be a simple Villadsen algebra. The following are equivalent:

(i) $\mathcal{A}$ has real rank zero;
(ii) $\omega_{1,\infty} = 0$;
(iii) $TR(\mathcal{A}) = 0$
(iv) For every $a \in \mathcal{A}_{i,SA}$, $\varepsilon > 0$, there is $j \geq i$ such that $TV(\phi_{i,j}(a)) < \varepsilon$;
(v) The projections in $\mathcal{A}$ separate the traces.

Any of the above implies the following, which are equivalent:

(vi) $\mathcal{A}$ has Blackadar's strict comparison of positive elements;
(vii) $\mathcal{A}$ has slow dimension growth;
(viii) $\mathcal{A}$ tensorially absorbs the Jiang-Su algebra $\mathcal{Z}$;
(ix) $\mathcal{A}$ has finite nuclear dimension;
(x) $\mathcal{A}$ has finite decomposition rank;
(xi) $\mathcal{A}$ is tracially approximately divisible;
(xii) $\mathcal{A}$ is tracially $\mathcal{Z}$-absorbing;
(xiii) $\mathcal{A}$ has Niu's mean dimension zero;
(xiv) $\mathcal{A}$ has Lin's tracial approximate oscillation zero;
(xv) $\mathcal{A}$ has almost unperforated Cuntz semigroup.

These each imply

(xvi) $K_0(\mathcal{A})$ is weakly unperforated.

Lastly, there are examples witnessing:

(vi)-(xv) $\not\Longrightarrow$ (i)-(v): Goodearl [Goo92];
(xvi) $\not\Longrightarrow$ (vi)-(xv): Toms [Tom08a].

CHAPTER 3

# GENERALIZATION TO AH ALGEBRAS OVER CW-COMPLEXES

## 3.1 Embedded Spheres in the Cube

In this section, we establish some basics about how some sets homeomorphic to spheres embed naturally into sets homeomorphic to cubes.

**Lemma 3.1.1.** For each $i \in \mathbb{N}$, let $X_i$ be a CW-complex and $Q_i \subset X_i$ be subsets such that $Q_i \simeq [0,1]^{N_i}$ for some $N_i$ with $N_i \big| N_{i+1}$. Let $\eta_i : Q_i \to [0,1]^{N_i}$ be these homeomorphisms, for each $j > i$ put $N_{i,j} = \dfrac{N_j}{N_i}$, and let

$$\pi_k^{i,j} : [0,1]^{N_j} \simeq \left([0,1]^{N_i}\right)^{N_{i,j}} \to [0,1]^{N_i}$$

for $k \in \{1, ..., N_{i,j}\}$ be the coordinate projections. Let $\gamma_j : X_j \to Q_j$ be a retract, and for each $k \in \{1, ..., N_{i,j}\}$, put

$$\lambda_k^{i,j} := \eta_i^{-1} \circ \pi_k^{i,j} \circ \eta_j \circ \gamma_j : X_j \to Q_i \subset X_i.$$

Then,

(i) There exist injections $\iota_k^{i,j} : Q_i \to Q_j \subset X_j$ such that for all $i, j$ and $x \in Q_i$,

$$(\lambda_{k_1}^{i,j} \circ \iota_{k_2}^{i,j})(x) = \begin{cases} x & \text{if } \ k_1 = k_2 \\ \eta_i^{-1}(0) & \text{if } \ k_1 \neq k_2 \end{cases}.$$

(ii) Given $i \in \mathbb{N}$ and $D \in \mathbb{N}$ such that $D \leq N_i - 1$, there exists a subset $\tilde{S}^D \subset Q_i$ such that $S^D \simeq \tilde{S}^D$, a relatively open neighborhood $U_i \subset Q_i$ of $\tilde{S}^D$, and a retract $\tau : \overline{U_i} \to \tilde{S}^D$.

(iii) Given $i \in \mathbb{N}$, $j > i$, $k \in \{1, ..., N_{i,j}\}$, and $D \in \mathbb{N}$ with $D \leq N_i - 1$, let $\tilde{S}^D \subset Q_i$ be the subset in (ii), and $\iota_k^{i,j}$ be the injections in (i). There exists a subset $\mathcal{S}_j \subset Q_j$ and a homeomorphism $\zeta_{i,j} : \prod_{k=1}^{N_{i,j}} \iota_k^{i,j}(Q_i) \to Q_j$ such that $\mathcal{S} \simeq (S^D)^{\times N_{i,j}}$, and for all $\ell \in \{1, ..., N_{i,j}\}$ and $(x_1, ..., x_{N_{i,j}}) \in (\tilde{S}^D)^{N_{i,j}}$,

$$\lambda_k^{i,j}(\mathcal{S}_j) = \tilde{S}^D \qquad \text{and} \qquad \lambda_k^{i,j} \circ \zeta_{i,j}(\iota_1^{i,j}(x_1), ..., \iota_{N_{i,j}}^{i,j}(x_{N_{i,j}}) = x_k.$$

We will call $\zeta_{i,j}$ the *rectifying homeomorphism* of $\mathcal{S}_j$.

*Proof of (i).* For each $k \in \{1, ..., N_{i,j}\}$, let $(\iota_k^{i,j})' : [0,1]^{N_i} \to [0,1]^{N_j} \simeq \big([0,1]^{N_i}\big)^{N_{i,j}}$ be inclusion in the $k$ slot with 0 in the other coordinates. Let $\iota_1^{i,j}, ..., \iota_{N_{i,j}}^{i,j} : Q_i \to Q_j$ be defined by

$$\iota_k^{i,j} = \eta_j^{-1} \circ (\iota_k^{i,j})' \circ \eta_i.$$

Note that, since $\gamma_j : X_j \to Q_j$ is a retract, we have $\gamma_j\big|_{Q_j} = \mathrm{id}_{Q_j}$, so $\gamma_j \circ \iota_k^{i,j} = \iota_k^{i,j}$ for each $k$. Thus, for each $x \in Q_i$, we have

$$\begin{aligned}(\lambda_{k_1}^{i,j} \circ \iota_{k_2}^{i,j})(x) &= (\eta_i^{-1} \circ \pi_{k_1}^{i,j} \circ \eta_j \circ \eta_j^{-1} \circ (\iota_{k_2}^{i,j})' \circ \eta_i)(x) \\ &= \eta_i^{-1} \circ (\pi_{k_1}^{i,j} \circ (\iota_{k_2}^{i,j})')(\eta_i(x)) \\ &= \begin{cases} x & \text{if } \ k_1 = k_2 \\ \eta_i^{-1}(0) & \text{if } \ k_1 \neq k_2 \end{cases}.\end{aligned}$$

□

*Proof of (ii).* Let

$$U_i' = \{(x_1, ..., x_{N_i}) \in [0,1]^{N_i} \mid \left(\sum_{k=1}^{D+1} (x_k - \frac{1}{2})^2\right)^{\frac{1}{2}} \in (\frac{1}{8}, \frac{3}{8})\},$$

$$S' = \{(x_1, ..., x_{D+1}, \frac{1}{2}, ..., \frac{1}{2}) \in [0,1]^{N_i} \mid \left(\sum_{k=1}^{D+1} (x_k - \frac{1}{2})^2\right)^{\frac{1}{2}} = \frac{1}{4}\}$$

and let $\tau' : \overline{U_i}' \to S'$ be retraction along radial lines emanating from $(\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2})$ in the first $D+1$ coordinates, and projection to $\frac{1}{2}$ in the rest, i.e. if $d = \left(\sum_{k=1}^{D+1} (x_k - \frac{1}{2})^2\right)^{\frac{1}{2}}$, then

$$\tau'(x_1, ..., x_{N_i}) = (\frac{1}{4d} x_1, ..., \frac{1}{4d} x_{D+1}, \frac{1}{2}, ..., \frac{1}{2}).$$

Put

$$\tilde{S}^D = \eta_i^{-1} S', \qquad , U_i = \eta_i^{-1}(U_i') \qquad \tau = \eta_i^{-1} \circ \tau' \circ \eta_i.$$

Then, since $\eta$ is a homeomorphism and $(\tau')^2 = \tau$ as a retract, we have $S^D \simeq S' \simeq \eta_i^{-1} S'$, that $U_i$ is an open neighborhood, and $\tau^2 = \tau$ is a retract. □

*Proof of (iii).* Let $\zeta_{i,j} : \prod_{k=1}^{N_{i,j}} \iota_k^{i,j}(Q_i) \to Q_j$ be the map

$$\zeta_{i,j}(\iota_1^{i,j}(x_1), ..., \iota_{N_{i,j}}^{i,j}(x_{N_{i,j}}) = \zeta_{i,j}\left(\eta_j^{-1} \circ (\iota_1^{i,j})') \circ \eta_i(x_1), ..., \eta_j^{-1} \circ (\iota_{N_{i,j}}^{i,j})') \circ \eta_i(x_{N_{i,j}})\right)$$
$$:= \eta_j^{-1}\left(\eta_i(x_1), ..., \eta_i(x_{N_{i,j}})\right),$$

which is well-defined since $\eta_i, \eta_j$ and each $\iota_k^{i,j}$ are injections for each $i, j, k$. Thus, for all $k \in \{1, ..., N_{i,j}\}$,

$$\begin{aligned}
\lambda_k^{i,j} \circ \zeta(\iota_1^{i,j}(x_1), ..., \iota_{N_{i,j}}^{i,j}(x_{N_{i,j}}) &= \lambda_k^{i,j}\left(\eta_j^{-1}(\eta_i(x_1), ..., \eta_i(x_{N_{i,j}}))\right) \\
&= \eta_i^{-1} \circ \pi_k \circ \eta_j \circ \left(\eta_j^{-1}(\eta_i(x_1), ..., \eta_i(x_{N_{i,j}}))\right) \\
&= \eta_i^{-1}(\eta_i(x_\ell)) \\
&= x_\ell.
\end{aligned}$$

The continuous map $\zeta_{i,j}$ has inverse

$$\zeta_{i,j}^{-1}(x) = \left(\iota_1^{i,j}(\lambda_1^{i,j}(x)), ..., \iota_{N_{i,j}}^{i,j}(\lambda_{N_{i,j}}^{i,j}(x)\right),$$

for we have

$$\begin{aligned}
&\zeta_{i,j}^{-1} \circ \zeta_{i,j}(\iota_1^{i,j}(x_1), ..., \iota_{N_{i,j}}^{i,j}(x_{N_{i,j}}) = \zeta_{i,j}^{-1}(\eta_j^{-1}(\eta_i(x_1), ..., \eta_i(x_{N_{i,j}}))) \\
&\quad = (\iota_1^{i,j}(\lambda_1^{i,j}(\eta_j^{-1}(\eta_i(x_1), ..., \eta_i(x_{N_{i,j}})), ..., \iota_{N_{i,j}}^{i,j}(\lambda_{N_{i,j}}^{i,j}(\eta_j^{-1}(\eta_i(x_1), ..., \eta_i(x_{N_{i,j}})) \\
&\quad = (\iota_1^{i,j}(x_1), ..., \iota_{N_{i,j}}^{i,j}(x_{N_{i,j}})).
\end{aligned}$$

Since $Q_i, Q_j$ are compact Hausdorff spaces, and $\zeta$ is a continuous bijection, then it is indeed a homeomorphism. Letting $\mathcal{S} = \zeta_{i,j}(\prod_{k=1}^{N_{i,j}} \iota_k^{i,j}(\tilde{S}^D)) \subset Q_j$, we have

$$\mathcal{S} \simeq \prod_{k=1}^{N_{i,j}} \iota_k^{i,j}(\tilde{S}^D) \simeq (S^D)^{\times N_{i,j}}$$

and $\lambda_k^{i,j}(\mathcal{S}_j) = \tilde{S}^D$, specifically the sphere $\tilde{S}^D$ coming from the $k$-th coordinate. □

**Porism 3.1.2.** For each $i \in \mathbb{N}$, let $X_i$ be a CW-complex and $Q_i \subset X_i$ be subsets such that $Q_i \simeq [0,1]^{N_i}$ for some $N_i$ with $N_i \big| N_{i+1}$. Let $\lambda_k^{i,j} : X_j \to Q_i$ and $\iota_k^{i,j} : Q_i \to X_j$ be as described in Lemma 3.1.1 for each $k \in \{1, ..., N_{i,j}\}$. Then, for each $k$, $\iota_k \circ \lambda_k$ is a local retract from $X_j \to \iota_k(Q_i) \subset Q_j$.

*Proof.* We have $\lambda_k \circ \iota_k = \mathrm{id}$, so $(\iota_k \circ \lambda_k) \circ (\iota_k \circ \lambda_k) = \iota_k \circ \lambda_k$. $\square$

In the next theorem, we establish how projections over a sphere carry across connecting homomorphisms in Villadsen algebras. This allows us to see the failure of strict comparison, similar to Example 1.4.8.

**Theorem 3.1.3.** Suppose the following setup:

(i) For $k \in \mathbb{N}$, put $X_k = [0,1]^{D_k}$, where $D_k \big| D_{k+1}$, with $D_1 \geq 3$, and put $\mathcal{A}_k = M_{R_k}(C(X_k))$. Suppose $\mathcal{A} = \lim\limits_{n\to\infty}(\mathcal{A}_n, \phi_{n,n+1})$ is a simple Villadsen algebra with seed space $X_1$.
(ii) Let $j \in \mathbb{N}$, and suppose $N_{1,j}$ is the number of projection maps from $\mathcal{A}_1$ to $\mathcal{A}_j$ in the connecting homomorphisms $\phi_{1,j}$, and $M_{1,j} = \dfrac{R_j}{R_1}$ is the total number of eigenvalue maps. Suppose for simplicity $N_{1,j} = \dfrac{D_j}{D_1}$, i.e. all projection maps appear from $X_1 \to X_j$.
(iii) Let $U \subset X_1$ be an open neighborhood of some set $\mathcal{S} \subset X_1$, such that $\mathcal{S} \simeq S^2$ and there exists a retract $\tau : \overline{U} \to \mathcal{S} \simeq S^2$. Without loss of generality, we just refer to $\mathcal{S}$ as $S^2$.
(iv) Let $f \in C_0(X_1)$ have support $\overline{U}$ and equal 1 on $S^2$.
(v) Let $\eta \in M_{R_1}(C(S^2))$ be a projection.
(vi) Let $\rho_j : M_{R_j}(C(X_j)) = M_{R_j}(C(X_1^{N_{1,j}})) \to M_{R_j}(C((S^2)^{N_{1,j}}))$ be restriction, and let $\tau_f : M_{R_1}(C(S^2)) \to M_{R_1}(C(X_1))$ be the function

$$\tau_f(\xi)(x) = \begin{cases} f(x)\xi(\tau(x)) & \text{if } x \in \overline{U} \\ 0 & \text{if } x \notin \overline{U} \end{cases}.$$

Then, up to rearrangement of the blocks,

$$\rho_j \circ \phi_{1,j} \circ \tau_f(\eta) = \mathrm{diag}((\eta)^{\times N_{1,j}}, K_j(\eta)),$$

where $K_j(\eta)$ is a constant block of rank at most $\mathrm{Rank}(\eta) \cdot (M_{1,j} - N_{1,j})$, and $(\eta)^{\times N_{1,j}}$ is the projection in $M_{R_1 \cdot N_{1,j}}(C((S^2)^{N_{1,j}}))$ given by

$$(\eta)^{\times N_{1,j}}(y_1, ..., y_{N_{1,j}}) := \mathrm{diag}(\eta(y_1), \ldots, \eta(y_{N_{1,j}})).$$

*Proof.* Put $\psi_{1,j} = \rho_j \circ \phi_{1,j} \circ \tau_f$. We follow the diagram

$$\begin{tikzcd} M_{R_1}(C(X_1)) \arrow[r, "\phi_{1,j}"] & M_{R_j}(C(X_j)) \arrow[d, "\rho_j"] \\ M_{R_1}(C(S^2)) \arrow[u, "\tau_f^*"] \arrow[r, dotted, "\psi_{1,j}"] & M_{R_j}(C((S^2)^{\times N_{1,j}})) \end{tikzcd}$$

When $(x_1, ..., x_{N_{1,j}}) \in X_j = X_1^{N_{1,j}}$, up to rearrangement of the blocks,

$$\begin{aligned}(\phi_{1,j} \circ \iota \circ \tau_f^*(\eta))(x_1, ..., x_{N_{1,j}}) &= \operatorname{diag}(f(\pi_1(x_1, ..., x_{N_{1,j}})) \cdot \eta(\tau(\pi_1(x_1, ..., x_{N_{1,j}}))), \dots, \\ &\quad \dots, f(\pi_{N_{1,j}}(x_1, ..., x_{N_{1,j}})) \cdot \eta(\tau(\pi_{N_{1,j}}(x_1, ..., x_{N_{1,j}}))), K_j(\eta)) \\ &= \operatorname{diag}(f(x_1)\eta(\tau(x_1)), ..., f(x_{N_{1,j}})\eta(\tau(x_{N_{1,j}})), K_j(\eta))\end{aligned}$$

where

$$K_j(\eta) = \operatorname{diag}(\tau_f^* \circ \delta_1, ..., \tau_f^* \circ \delta_{\beta_j})$$

is a constant block of $\beta_j := M_{1,j} - N_{1,j}$ point evaluations of $\tau_f(\eta)$ in the connecting homomorphisms from $\mathcal{A}_1$ to $\mathcal{A}_j$. In particular, $\operatorname{Rank}(\tau_f(\eta)(x))$ is either $\operatorname{Rank}(\eta)$ when $x \in U$ or 0 when $x \notin U$, so the rank of this constant point evaluation block $K_j(\eta)$ is at most

$$\operatorname{Rank}(K_j(\eta)) \leq \operatorname{Rank}(\eta) \cdot (M_{1,j} - N_{1,j}) \leq R_j - \operatorname{Rank}(\eta) \cdot N_{1,j}.$$

Given $(x_1, ..., x_{N_{1,j}}) \in X_j$ for $x_1, ..., x_{N_{1,j}} \in X_1$, then $(x_1, ..., x_{N_{1,j}}) \in (S^2)^{N_{1,j}}$ if and only if $f(x_k) = 1$ for all $k$. Likewise, $\tau\big|_{S^2} = \mathrm{id}_{S^2}$ as a retract, thus $(x_1, ..., x_{N_{1,j}}) \in (S^2)^{N_{1,j}}$ implies

$$\begin{aligned}\operatorname{diag}(f(x_1)\eta(\tau(x_1)), ..., f(x_{N_{1,j}})\eta(\tau(x_{N_{1,j}})), K_j(\eta)) &= \operatorname{diag}(\eta(x_1), \dots, \eta(x_{N_{1,j}}), K_j(\eta)) \\ &= \operatorname{diag}((\eta)^{\times N_{1,j}}, K_j(\eta))(x_1, ..., x_{N_{1,}}).\end{aligned}$$

□

**Remark 3.1.4.** Note that, when $x \in S^2$, we have $\tau, f$ are trivial, so if all the point evaluations in the connecting homomorphisms $\delta_1, ..., \delta_{\beta_j}$ from $\mathcal{A}_1$ to $\mathcal{A}_j$ are in $S^2$ or outside $\overline{U}$, then the block

$$K_j(\eta) = \operatorname{diag}(\tau_f^* \circ \delta_1, ..., \tau_f^* \circ \delta_{\beta_j})$$

is actually a constant (trivial) projection, and so for $x \in (S^2)^{N_{1,j}}$,

$$
\begin{aligned}
\psi_{1,j}(\zeta \cdot \xi)(x) &= \phi_{1,j}(\zeta \cdot \xi)(x) \\
&= \phi_{1,j}(\zeta)(x) \cdot \phi_{1,j}(\xi)(x) \\
&= \psi_{1,j}\zeta(x) \cdot \psi_{1,j}\xi(x).
\end{aligned}
$$

So, $\psi_{1,j}$ is a homomorphism in this case. Since there are finitely many point evaluations, such a particular neighborhood $U$ could always be chosen for any given $j$. Generally, we note that the constant block $K_j(\eta)$ is Cuntz equivalent to a constant projection by Example 1.4.7.

**Remark 3.1.5.** Suppose (i) and (ii) in the theorem above. Let

$$\iota_1, ..., \iota_{N_{i,j}} : [0,1]^{D_i} \to [0,1]^{D_j}$$

be inclusion satisfying

$$\pi_k^{i,j} \circ \iota_k = \mathrm{id}\,.$$

Define

$$U \subset X_i, \quad f \in C_0(U), \quad \xi \in M_{R_i}(C((S^D))), \quad \text{and} \quad \tau_f : M_{R_i}(C(S^D)) \to M_{R_i}(C(X_i))$$

in (iii) - (vi) analogously for $S^D \subset X_i = [0,1]^{D_i}$, for some $D \in \mathbb{N}$. Since $f = 1$ on $S^D$ with support $\overline{U}$, for $(x_1, ..., x_{N_{i,j}}) \in U^{N_{i,j}}$ the restriction to $(S^D)^{N_{i,j}} \subset [0,1]^{D_j}$ of $\phi_{i,j} \circ \tau_f(\xi)$ is

$$
\begin{aligned}
\phi_{i,j}\circ\tau_f\xi\big|_{U^{N_{i,j}}}(x_1, \ldots, x_{N_{i,j}}) &= \mathrm{diag}\left(\tau_f\xi \circ \pi_1^{i,j}, \ldots, \tau_f\xi \circ \pi_{N_{i,j}}^{i,j}, K_j(\xi)\right)(x_1, \ldots, x_{N_{i,j}}) \\
&= \mathrm{diag}(\tau_f\xi \circ \pi_1^{i,j}(x_1, \ldots, x_{N_{i,j}}), \ldots, \tau_f\xi \circ \pi_{N_{i,j}}^{i,j}(x_1, \ldots, x_{N_{i,j}}), K_j(\xi)) \\
&= \mathrm{diag}(\tau_f\xi \circ \pi_1^{i,j}(x_1, , 0, 0, \ldots), \ldots, \tau_f\xi \circ \pi_{N_{i,j}}^{i,j}(0, \ldots, 0, x_{N_{i,j}}), K_j(\xi)) \\
&= \mathrm{diag}(\tau_f\xi \circ \pi_1^{i,j} \circ \iota_1(x_1), \ldots, \tau_f\xi \circ \pi_{N_{i,j}}^{i,j} \circ \iota_{N_{i,j}}(x_{N_{i,j}}), K_j(\xi)).
\end{aligned}
$$

This formulation is useful for generalizing this calculation to sets $Q_i$ homeomorphic to cubes.

Thanks to H. Lin for his particularly pointed advice on honing these next formulations.

**Remark 3.1.6.** For each $i \in \mathbb{N}$, let $X_i$ be a finite connected CW-complex, and $(R_n)$ be a sequence in $\mathbb{N}$ with $R_n \big| R_{n+1}$ for all $n$; put $\mathcal{A}_n = M_{R_n}(C(X_n))$. Let $C_n : [0,1] \to X_n$ and $F_n : X_{n+1} \to [0,1]$ be continuous maps, and $\delta'_n := C_n \circ F_n : X_{n+1} \to X_n$, i.e. $\delta'_n$ is a map which factors through an interval. Assuming $\dim(X_j) \geq 1$, one can choose $F_n$ to be surjective (as a retraction) and $C_n : [0,1] \to X_n$ to be an embedding. In this case, we obtain a unital homomorphism $\psi'_{I,n} : A_n \to M_{R_n}(C([0,1]))$ which is surjective and a homomorphism $\psi'_n : M_{R_n}(C([0,1])) \to A_{n+1}$ which is injective, so $\delta'_n$ induces $\psi'_n \circ \psi'_{I,n} : A_n \to A_{n+1}$.

Let $\delta_n : X_n \to X_{n-1}$ be a constant map, i.e. $\delta_n(y) = x_n$ for some $x_n \in X_{n-1}$ and for all $y \in X_n$. Suppose $\phi_{n,n+1} : \mathcal{A}_n \to \mathcal{A}_{n+1}$ is of the form

$$\phi_{n,n+1}(a) = \operatorname{diag}(a \circ \lambda_1^{n,n+1}, \ldots, a \circ \lambda_{\alpha_{n,n+1}}^{n,n+1}, a \circ \Delta_1^{n,n+1}, ..., a \circ \Delta_{\beta_n}^{n,n+1})$$

for each $n$, where $\beta_n = \dfrac{R_{n+1}}{R_n} - \alpha_{n,n+1}$ and $\Delta_k^{n,n+1} = \delta_n$ or $\Delta_k^{n,n+1} = \delta'_n$ for each $k \in \{1, ..., \beta_n\}$ and each $\lambda_i^{n,n+1} : X_{n+1} \to X_n$ is a continuous map. Then with $\mathcal{A}_n = M_{R_n}(C(X_n))$, we may form the AH algebra $\mathcal{A} = \lim\limits_{n\to\infty}(\mathcal{A}_n, \phi_{n,n+1})$.

**Definition 3.1.7.** Let $X_n$ be finite connected CW-complexes for each $n \in \mathbb{N}$ with $\dim(X_n) \geq 1$ and suppose $\mathcal{A} = \lim\limits_{n\to\infty}(\mathcal{A}_n, \phi_{n,n+1})$, where $\phi_{n,n+1}$ is of the form

$$\phi_{n,n+1}(a) = \operatorname{diag}(a \circ \lambda_1^{n,n+1}, \ldots, a \circ \lambda_{\alpha_{n,n+1}}^{n,n+1}, a \circ \Delta_1^{n,n+1}, ..., a \circ \Delta_{\beta_n}^{n,n+1}).$$

We will say $\mathcal{A}$ is a *generalized interval Villadsen algebra* if each $\lambda_i^{n,n+1} : X_{n+1} \to X_n$ is a continuous map and every $\Delta_j^{n,n+1}$ is either a point-evaluation, or induces $\psi'_n \circ \psi'_{I,n}$, where $\psi'_{I,n} : A_n \to M_{R_n}(C([0,1]))$ is a surjective unital homomorphism and a homomorphism $\psi'_n : M_{R_n}(C([0,1])) \to A_{n+1}$ which is injective.

Note that one may write

$$\phi_{n,n+1}(f) = \operatorname{diag}(\Psi_n(f), \psi_n \circ \psi_{I,n}(f)) \ \text{ for all} f \in \mathcal{A}_n,$$

where $\Psi_n : A_n \to M_{\alpha_{n,n+1}R_n}(C(X_{n+1}))$ is a unital homomorphism and $\psi_{I,n} : A_n \to C_n$ is a unital surjective homomorphism and $\psi_n : C_n \to A_{n+1}$ is an injective homomorphism, and where $C_n$ is a finite direct sum of full matrix algebras and $C^*$-algebras of the form $M_r(C([0,1]))$.

**Lemma 3.1.8.** Let $X, Y$ be CW-complexes, $\mathcal{A}_1 = M_R(C(X))$, $\mathcal{A}_2 = M_R(C(Y))$, $\omega : Y \to [0,1]$ and $P : [0,1] \to X$ be continuous maps. Then, for every $a \in \mathcal{A}_1$,

$$a \circ P \circ \omega \precsim I_{R\times R}$$

where $I_{R\times R} \in \mathcal{A}_2$ is the $R \times R$ identity matrix.

*Proof.* By Theorem 1 of L. Robert [Rob13], $\mathrm{Cu}(M_n(C([0,1]))) \simeq LSC(X, \overline{\mathbb{N}})$, thus $M_n(C([0,1]))$ has strict comparison. Given $a \in M_n(C[0,1])$, the value of $[a] \in LSC(X, \overline{\mathbb{N}})$ is $[a](x) = \mathrm{Rank}(a(x)) \leq R$, in particular. Let $a \in M_R(C(X))$; then,

$$a \circ P \in M_R(C([0,1])) \precsim I_{R\times R} \in M_R(C([0,1])).$$

Let $\Omega : M_R(C([0,1]))$ be the homomorphism $\Omega(A)(x) = A(\omega(x))$. Thus, as a Cuntz-morphism, $\mathrm{Cu}(\Omega)$ preserves Cuntz inequality, and certainly $\Omega(I_{R\times R}) = I_{R\times R}$ as a constant matrix. Hence

$$\Omega(a \circ P) = a \circ P \circ \omega \precsim I_{R\times R}.$$

□

Recall that, by Theorem 2.2.7, when $\mathcal{A}$ is a simple Villadsen algebra of the first type with $\dfrac{\alpha_{i,j}}{M_{i,j}} \to 0$, where $\alpha_{i,j}$ is the total number of projection maps (including multiplicity) from $\mathcal{A}_i$ to $\mathcal{A}_j$, then $\mathcal{A}$ has real rank zero. Therefore, it is Tracially AF by Theorem 2.1 of H. Lin [Lin03]. An analogous result can be shown for the generalized interval Villadsen algebras. Special thanks to H. Lin who assisted greatly in the following proof.

Denote by $\mathcal{I}$ the class of $C^*$-algebras which are finite direct sums of $C^*$-algebras of the form $M_r$ or $M_k(C(X))$ for some compact subset $X \subset [0,1]$.

**Lemma 3.1.9.** Let $\mathcal{A}$ be a unital simple generalized interval Villadsen algebra as in Definition 3.1.7, and $a \in \mathcal{A}_+ \setminus \{0\}$. Suppose that for all $n$, we have $\alpha_{n,n+1} < \dfrac{R_{n+1}}{R_n}$, i.e. at least one interval or evaluation map occurs. Then, there is a nonzero projection $e \in \mathcal{A}_n$ for some $n$ such that $\phi_{n,\infty}(e) \precsim a$ and $e(x) = e(x')$ for all $x, x' \in X_n$.

*Proof.* First, we show that $\mathcal{A}$ has the following property: given any $\varepsilon > 0$, and finite subset $\mathcal{F} \subset A$ which contains nonzero $b$, there exists a nonzero projection $q \in \mathcal{A}$ such that for all $f \in \mathcal{F}$,

(i) $\|qf - fq\| < \varepsilon$;
(ii) $qfq \in_\varepsilon \mathcal{C}$ for some $\mathcal{C} \in \mathcal{I}$;
(iii) $\|qbq\| \geq (1-\varepsilon)\|b\|$.

To prove this, we may assume without loss of generality $b \in \mathcal{A}_+$ with $\|b\| = 1$. Let $\varepsilon > 0$, and define $f \in C([0,1])_+$ such that

$$0 \leq f \leq 1, \quad f(t) = 1 \text{ for } t \in [1 - \tfrac{\varepsilon}{4}, 1], \quad \text{and} \quad f(t) = 0 \text{ for } t \in [0, 1 - \tfrac{\varepsilon}{2}].$$

Put $b_\varepsilon = f(b)$; since $\mathcal{A}$ is simple, there exist $x_1, \ldots x_\ell \in \mathcal{A}$ such that

$$\sum_{i=1}^{\ell} x_i^* b_\varepsilon x_i = 1.$$

We may assume $\mathcal{F} = \phi_{m,\infty}(\mathcal{G})$ for some finite set $\mathcal{G} \subset \mathcal{A}_m$ and for some $m \in \mathbb{N}$. In particular, there exists $b' \in \mathcal{G} \cap (\mathcal{A}_m)_+$ such that $b_\varepsilon = \phi_{m,\infty}(b')$. We may further assume that there exists $x_i' \in \mathcal{A}_m$ for $i \in \{1, \ldots, \ell\}$ such that

$$\|\sum_{i=1}^{\ell} (x_i')^* b' x_i' - 1_{\mathcal{A}_m}\| < \frac{1}{4}.$$

We may then find $y \in \mathcal{A}_m$ such that

$$\sum_{i=1}^{\ell} y^* (x_i')^* b' x_i' y = 1_{\mathcal{A}_m} \qquad (*).$$

Choose $\mathcal{G}_1 = \mathcal{G} \cup \{y, y^*, x_i, x_i', y^*(x_i')^*, x_i y\}$. Since $\beta_{m,m+1} = \dfrac{R_{m+1}}{R_m} - \alpha_{m,m+1} \geq 1$, we may write

$$\phi_m(f) = \operatorname{diag}(\Psi_m(f), \psi_m \circ \psi_{I,m}(f))$$

for all $f \in \mathcal{A}_m$, where $\Psi_m : \mathcal{A}_m \to M_{R_m \cdot \alpha_{m,m+1}}(C(X_{m+1}))$ is a unital homomorphism and $\psi_{I,m} : \mathcal{A}_m \to \mathcal{C}'$ and $\psi_m : \mathcal{C}' \to M_{\beta_m}(C(X_{m+1}))$ unital homomorphisms which are surjective and injective respectively, for some $\mathcal{C}' \in \mathcal{I}$.

Put $\mathcal{C} = \phi_{m+1,\infty}(\psi_m(\mathcal{C}'))$ and $q_0 = \psi_m \circ \psi_{I,m}(1_{\mathcal{A}_m})$. Thus, $\mathcal{C} \in \mathcal{I}$ and $q_0 \in \mathcal{A}_{m+1}$ is a projection. Putting $q = \phi_{m+1,\infty}(q_0)$ we have $qg = gq$ for all $g \in \mathcal{G}_1$. In particular, $qb = bq$, so we have shown (i) and (ii) hold. From $(*)$ we have

$$\sum_{i=1}^{\ell} \phi_{m,\infty}(y^*(x_i')^*)q\phi_{m,\infty}(b')q\phi_{m,\infty}(x_i'y) = q,$$

which implies $f(qbq) = qb_\varepsilon q \neq 0$. Hence,

$$\|qbq\| \geq (1-\varepsilon) = (1-\varepsilon)\|b\|,$$

which shows (iii). Since $\mathcal{A}$ is a simple $C^*$-algebra, by (i)-(iii) and by the proof of Theorem 3.2 of [Lin07], $\mathcal{A}$ has property (SP): that every non-zero hereditary $C^*$-subalgebra of $\mathcal{A}$ contains a nonzero projection. We may apply property (SP) to get the desired conclusion.

Let $a \in \mathcal{A}_+ \setminus \{0\}$, and assume without loss of generality $\|a\| = 1$. Since $\mathcal{A}$ has property (SP), there is a nonzero projection $q \in \overline{a\mathcal{A}a}$, so $q \precsim a$. By Lemma 2.7.2 in [Lin01], there is $i \in \mathbb{N}$ and a projection $q' \in \mathcal{A}_k$ such that $\phi_{k,\infty}(q') \sim q$. As before, we may write

$$\phi_m(f) = \mathrm{diag}(\Psi_m(f), \psi_m \circ \psi_{I,m}(f))$$

for all $f \in \mathcal{A}_m$, where $\Psi_m : \mathcal{A}_m \to M_{R_m \cdot \alpha_{m,m+1}}(C(X_{m+1}))$ is a unital homomorphism and $\psi_{I,m} : \mathcal{A}_m \to \mathcal{C}''$ and $\psi_m : \mathcal{C}'' \to M_{\beta_m}(C(X_{m+1}))$ unital homomorphisms which are surjective and injective respectively, for some $\mathcal{C}'' \in \mathcal{I}$.

Put $p_1 = \psi_k \circ \psi_{I,k}(q') \in \psi_k(\mathcal{C}'')$. Since $\psi_k \circ \psi_{I,k}$ is injective, $p_1 \neq 0$. Assume $\mathrm{Rank}(q') = r \in (0, \beta_k R_k)$ and choose a constant projection $q_0 \in \mathcal{C}''$ such that $\mathrm{Rank}(\psi_k(q_0)) = r$. Put $\mathcal{C}_1 = \psi_k(\mathcal{C}'')$; then, in $\mathcal{C}_1$, we have $q_1 := \psi_k(q_0) \sim p_1$. Thus, $e$ must also be constant in $M_{\beta_k}(C(X_{m+1}))$. With $n = k+1$, $p = \phi_{k+1,\infty}(q_0)$, we then have

$$p \sim \phi_{k+1,\infty}(p_1) \precsim \phi_{k+1,\infty}(q') \sim q \precsim a.$$

□

**Theorem 3.1.10.** Let $\mathcal{A}$ be a unital simple generalized interval Villadsen algebra as in Definition 3.1.7. Suppose that for all $j$ we have

$$\lim_{n\to\infty} \alpha_{j,n} \cdot \frac{R_j}{R_n} = 0.$$

Then, $\mathcal{A}$ has tracial rank at most one.

*Proof.* Following Definition 5.3 of [Lin07], we show the following: given any $\varepsilon > 0$, element $a \in \mathcal{A}_+ \setminus \{0\}$, and finite subset $\mathcal{F} \subset \mathcal{A}$, there exists a projection $p \in \mathcal{A}$ and a $C^*$-algebra $\mathcal{C} \in \mathcal{I}$ such that

(i) $\|pf - fp\| < \varepsilon$ for all $f \in \mathcal{F}$,
(ii) $pfp \in \mathcal{C}$ for all $f \in \mathcal{F}$, and
(iii) $1 - p \precsim a$.

To show this, we may assume without loss of generality that $\mathcal{F} = \phi_{m,\infty}(\mathcal{F}_1)$ for some $m \in \mathbb{N}$ and some finite subset $\mathcal{F}_1 \subset \mathcal{A}_m$. By Lemma 3.1.9, choose a nonzero projection $e \in \mathcal{A}_n$ such that $e$ is constant in $M_{R_m}(C(X_m))$ and $q := \phi_{m,\infty}(e) \precsim a$. Since $\mathcal{A}$ is simple, we have

$$\sigma = \inf\{\tau(q) \mid \tau \in T(\mathcal{A})\} > 0.$$

We claim there exists $n_0 \in \mathbb{N}$ such that

$$\tau(\phi_{m,n}(e)) > \frac{\sigma}{2}$$

for all $\tau \in T(\mathcal{A}_n)$ and $n \geq n_0$. Otherwise, there would be a subsequence $\{n_k\}$ such that $n_k \to \infty$ and $\tau_{n_k} \in T(\mathcal{A}_{n_k})$ with

$$\tau_{n_k}(\phi_{m,n_k}(e)) < \frac{\sigma}{2}$$

for all $k$.

Since each $\mathcal{A}_n$ is nuclear, by Theorem 2.3.13 of [Lin01], there is a sequence of completely-positive contractive (cpc) maps $L_k : \mathcal{A} \to \phi_{k,\infty}(\mathcal{A}_k) \simeq \mathcal{A}_k$ such that for any $m \in \mathbb{N}$ we have

$$\lim_{k\to\infty} \|L_k(\phi_{m,\infty}(a) - a\| = 0 \qquad (*)$$

for all $a \in \mathcal{A}_m$. Define $t_k : \mathcal{A} \to \mathbb{C}$ by

$$t_k(b) = \tau_{n_k} \circ L_k(b)$$

for each $b \in \mathcal{A}, k \in \mathbb{N}$. Let $t$ be a weak-$*$ limit of $\{t_k\}$. Note that each $t_k$ is a state, and $\mathcal{A}$ is unital, and so is $t$, so by $(*)$, we have $t$ is a tracial state on $\mathcal{A}$. However, we have

$$t(\phi_{m,\infty}(e)) \leq \frac{\sigma}{2}$$

from the above estimates, contradicting that $\sigma = \inf\{\tau(q) \mid \tau \in T(\mathcal{A})\} > 0$. Whence, the claim is proved.

Now, choose $n \geq m$ such that

$$\alpha_{m,n} \cdot \frac{R_m}{R_n} < \frac{\sigma}{2}$$

and

$$t(\phi_{m,n+1}(e)) \geq \frac{\sigma}{2}$$

for all $t \in T(\mathcal{A}_{n+1})$. We can write

$$\phi_{m,n+1}(f) = \mathrm{diag}(\Psi(f), \psi \circ \psi_I(f))$$

for all $f \in \mathcal{A}_m$, where $\Psi : \mathcal{A}_m \to \mathcal{A}_{n+1}$ is a homomorphism, $\Psi(1_{\mathcal{A}_m})$ is a constant projection of rank $R_m \cdot \alpha_{m,n+1}$, $\psi_I : \mathcal{A}_m \to \mathcal{C}_0$ and $\psi : \mathcal{C}_0 \to \mathcal{A}_{n+1}$ are homomorphisms which are surjective and injective, respectively, for some $\mathcal{C}_0 \in \mathcal{I}$. Put
$\mathcal{C} = \phi_{n+1,\infty} \circ \psi(\mathcal{C}_0)$; then, $\mathcal{C} \in \mathcal{I}$. Put $q_0 = \Psi(1_{\mathcal{A}_m})$ and

$$p = \phi_{n+1,\infty}(\psi \circ \psi_I(1_{\mathcal{A}_m})) = 1 - \phi_{n+1,\infty}(q_0).$$

Thus, $1 - p = \phi_{n+1,\infty}(q_0)$, and for any $f \in \mathcal{F}_1 \subset \mathcal{A}_m$ we have

(a) $(1 - q_0)\phi_{m,n+1}(f) = \phi_{m,n+1}(f)(1 - q_0)$ and
(b) $(1 - q_0)\phi_{m,n+1}(f)(1 - q_0) \in \psi(\mathcal{C}_0)$.

Hence, we have (i) and (ii) above. Since $\Psi(1_{\mathcal{A}_m})$ has rank $\alpha_{m,n+1} \cdot R_m$, we have $t(q_0) = \alpha_{m,n+1} \cdot R_m$. Thus, since $\alpha_{m,n} \cdot \dfrac{R_m}{R_n} < \dfrac{\sigma}{2}$ and $t(\phi_{m,n+1}(e)) \geq \dfrac{\sigma}{2}$, we find

$$t(q_0) < t(\phi_{m,n+1}(e))$$

for all $t \in T(\mathcal{A}_{n+1})$. Since both $q_0$ and $e$ are constant projections in $M_{R_{n+1}}(C(X_{n+1}))$, we conclude

$$q_0 \precsim \phi_{m,n+1}(e).$$

Thus, (iii) follows from

$$1 - p = \phi_{n+1,\infty}(q_0) \precsim \phi_{m,\infty}(e) \precsim a.$$

Therefore, $\mathcal{A}$ has tracial rank at most one. $\square$

## 3.2 Prescribed Dimension Growth over CW-Complexes

In this section we pose and answer two questions:

**Question 3.2.1.** Given any number $c \in (0,\infty)$, does there always exist an AH algebra $\mathcal{A} = \lim_{n\to\infty}(M_{R_n}(C(X_n)))$ such that $\lim_{n\to\infty}\dfrac{\dim(X_n)}{R_n} = c$ and $\mathcal{A}$ does not have strict comparison?

**Question 3.2.2.** Given a sequence $(X_n)$ of CW-complexes, and a sequence $R_n$ such that $\dfrac{\dim(X_n)}{R_n} \to c \in (0,\infty)$ and $\dim(X_n) \to \infty$, does there always exist an AH construction $\mathcal{A} = \lim_{i\to\infty}(M_{R_{n_i}}(C(X_{n_i})), \phi_{n_i})$, at least over some subsequence of $(X_n)$ and $(R_n)$, which fails to have strict comparison?

We will answer both questions in the affirmative by generalizing the Villadsen construction to the setting of CW-complexes by factoring $X_j$ through a set $Q_j \subset X_j$, a set which is homeomorphic to a cube. First, we exhibit how to get to a particular constant $c \in (0,\infty)$ with a simple Villadsen algebra.

**Theorem 3.2.3.** Let $c \in (0,\infty)$. There exists a simple Villadsen algebra $\mathcal{A} = \lim_{n\to\infty}(M_{R_n}(C(X_n)), \phi_n)$ such that

$$\lim_{n\to\infty}\frac{\dim(X_n)}{R_n} = c$$

and which does not have strict comparison of positive elements.

*Proof.* Suppose $c \in (0,1)$. Let us define a sequence $q_n$ in $\mathbb{Q}$. There exists a rational number $q_1 \in (c,1)$. Let $q_2$ satisfy $\dfrac{c}{q_1} < q_2 < 1$ with $q_1q_2 - c < \dfrac{1}{2}$. Likewise, there exists $q_3$ such that $\dfrac{c}{q_1q_2} < q_3 < 1$ and $q_1q_2q_3 - c < \dfrac{1}{3}$. Continuing in this fashion recursively, we have

$$q_n:\ q_1...q_2q_n - c < \frac{1}{n} \quad \text{with} \quad \frac{c}{\prod_{j=1}^{n-1} q_j} < q_n < 1.$$

Then, $c < \displaystyle\prod_{j=1}^{n} q_j < 1$ is a decreasing sequence with limit $c$. We necessarily have $q_k \to 1$ as $k \to \infty$.

Put

$$
\begin{aligned}
q_k &= \frac{n_k}{m_k}, \\
R_k &= \prod_{j=1}^{k} m_j, \text{ and} \\
N_j &= n_j, \\
X_1 &= I := [0,1], \\
X_k &= (X_{k-1})^{\times N_k}.
\end{aligned}
$$

Thus,

$$\dim(X_k) = \dim(X_1) \cdot \prod_{j=1}^{k} N_j = \dim(X_1) \cdot N_{1,k}.$$

Let $(x_j)$ be a dense sequence in $X_1^\infty$ with $x_k \in I^{\prod_{j=1}^{k} N_j} := X_k$ without loss of generality. Put

$$\mathcal{A}_k = M_{R_k}(C(X_k)) = M_{\prod_{j=1}^{k} m_j} C(I^{\prod_{j=1}^{k} n_j})$$

and define $\phi_{k-1,k} : \mathcal{A}_{k-1} \to \mathcal{A}_k$ by

$$\phi_{k-1,k}(a) = \operatorname{diag}(a \circ \pi_1^k, ..., a \circ \pi_{N_k}^k, \delta_k(a), ..., \delta_k(a))$$

where $\pi_j^k$ are the projection maps from $X_k \to X_{k-1}$ for each $j \in \{1, ..., N_k\}$ and $\delta_k(a) = a(x_k)$ with multiplicity $m_k - n_k \geq 1$. Then, $\mathcal{A} \simeq (\mathcal{A}_n, \phi_n)$ is a simple Villadsen algebra with

$$\lim_{k\to\infty} \frac{N_{1,k}}{M_{1,k}} = \lim_{k\to\infty} \prod_{j=1}^{k} \frac{n_j}{m_j} = \frac{\dim(X_k)}{R_k} = \dim(I) \cdot \prod_{k=1}^{\infty} q_k = c.$$

Since $c > 0$, we have $\mathcal{A}$ does not have strict comparison by Toms-Winter [TW09] Lemma 5.1.

Suppose $c \geq 1$; there exists an integer $M > c$; thus $\dfrac{c}{M} \in (0,1)$. Let $X_1$ be any compact Hausdorff space of dimension $M$; obtain $(x_j), n_j, m_j$ as before. Then the analogue of the above gives

$$\lim_{k\to\infty} \frac{N_{1,k}}{M_{1,k}} = \lim_{k\to\infty} \prod_{j=1}^{k} \frac{n_j}{m_j} = \dim(X_1) \cdot \frac{\dim(X_k)}{R_k} = M \cdot \prod_{k=1}^{\infty} q_k = M \cdot \frac{c}{M} = c$$

which, likewise, fails to have strict comparison by Toms-Winter [TW09] Lemma 5.1. $\square$

Next, we present a particular construction by factoring through cubes embedded in given CW-complexes.

**Construction 3.2.4.** For each $i \in \mathbb{N}$, let $X_i$ be a finite connected CW-complex with $1 \leq \dim(X_i) \leq \dim(X_{i+1})$, and $(R_n)$ be a sequence in $\mathbb{N}$ with $R_n \big| R_{n+1}$ for all $n$; put $\mathcal{A}_n = M_{R_n}(C(X_n))$. As in Lemma 3.1.1, let $Q_i \subset X_i$ be subsets such that $Q_i \simeq [0,1]^{N_i}$ for some $N_i$ with $N_i \big| N_{i+1}$. Further, let $\eta_i : Q_i \to [0,1]^{N_i}$ be these homeomorphisms, for each $j > i$ put $N_{i,j} = \dfrac{N_j}{N_i}$ and let

$$\pi_k^{i,j} : [0,1]^{N_j} \simeq \left([0,1]^{N_i}\right)^{N_{i,j}} \to [0,1]^{N_i}$$

for $k \in \{1, ..., N_{i,j}\}$ be the coordinate projections. Let $\gamma_j : X_j \to Q_j$ be a retract, and for each $k \in \{1, ..., N_{i,j}\}$, put

$$\lambda_k^{i,j} := \eta_i^{-1} \circ \pi_k^{i,j} \circ \eta_j \circ \gamma_j : X_j \to Q_i \subset X_i$$

and

$$\phi_{n,n+1}(a) = \operatorname{diag}(a \circ \lambda_1^{n,n+1}, \ldots, a \circ \lambda_{N_{n,n+1}}^{n,n+1}, a \circ \Delta_1^{n,n+1}, ..., a \circ \Delta_{\beta_n}^{n,n+1}),$$

where every $\Delta_j^{n,n+1}$ is either a point-evaluation, or induces the composition $\psi'_n \circ \psi'_{I,n}$, where $\psi'_{I,n} : A_n \to M_{R_n}(C([0,1]))$ is a surjective unital homomorphism and $\psi'_n : M_{R_n}(C([0,1])) \to A_{n+1}$ is an injective homomorphism. Therefore, we have $\mathcal{A} = \lim\limits_{n\to\infty}(\mathcal{A}_n, \phi_{n,n+1})$ is a generalized interval Villadsen algebra as in Definition 3.1.7.

**Remark 3.2.5.** Every Villadsen algebra over $X_i = [0,1]^{N_i}$ for some $D \in \mathbb{N}$ is trivially a generalized interval Villadsen algebra produced by Construction 3.2.4, with $Q_i = [0,1]^{N_i}$ and $\eta_i = \mathrm{id}$. Of course the maps $\Delta_k^{n,n+1}$ are all just constant maps in this case. We allow for factoring through an interval because it both trivialize vector bundles and allows us to make an argument later to produce injectivity in connecting homomorphisms between potentially quite different CW-complexes.

**Theorem 3.2.6.** Let $X_n$ be given CW-complexes with dimension $\dim(X_n) = d_n$, and let $(R_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}$. Suppose that $(d_n)$ and $(R_n)$ are monotonically increasing, with $R_n \big| R_{n+1}$ and $d_n \to \infty$, and suppose

$$\liminf_{n\to\infty} \frac{d_n}{R_n} = c \in (0, \infty).$$

Then, there exists a subsequence $(a_n)$ of $\mathbb{N}$ and connecting homomorphisms

$$\phi_{n,n+1} : M_{R_{a_n}}(C(X_{a_n})) \to M_{R_{a_{n+1}}}(C(X_{a_{n+1}}))$$

such that $\mathcal{A} := \lim\limits_{n\to\infty}(M_{R_{a_n}}(C(X_{a_n})), \phi_{n,n+1})$ is a simple, unital AH algebra which is not $\mathcal{Z}$-stable.

*Proof.* Using the fact that $R_n, d_n \to \infty$, let $(q_n)$ be a sequence in $\mathbb{Q} \cap \left(\frac{3}{4}, 1\right)$, writing $q_n = \dfrac{N_n}{M_n}$ for $N_n, M_n \in \mathbb{Z}$, satisfying the following properties:

(i) $\displaystyle\prod_{j=1}^{\infty} q_j > \frac{2}{3}$;

(ii) $M_k = \dfrac{R_{n_{k+1}}}{R_{n_k}} > 2^k$ for some increasing subsequence $(n_k)$ of $\mathbb{N}$;

(iii) $q_k \cdot c \cdot R_{n_k} < d_{n_k} - 2 < d_{n_k} < \dfrac{1}{q_k} \cdot c \cdot R_{n_k}$.

Restricting $(R_k)$ and $(d_k)$ to the subsequence $(n_k)$, we regard $R_{n_j}$ as $R_j$ and $d_{n_j}$ as $d_j$ from here on. Write $\mathcal{A}_n := M_{R_n}(C(X_n))$. Without loss of generality, we may assume assume $\frac{1}{2}d_1 > 6$ and $M_1 \geq 4$.

Put $D_1 = 6 \leq \dfrac{1}{2}d_1$ and

$$D_k = D_1 \cdot N_1 \cdot ... \cdot N_{k-1};$$

note that

$$R_k = M_1 \cdot M_2 \cdot ... \cdot M_{k-1} \cdot R_1.$$

Therefore,

$$\begin{aligned}
D_2 &= D_1 \cdot N_1 \\
&< \frac{1}{2} d_1 \cdot q_1 \cdot M_1 \\
&< \frac{1}{2} \cdot \frac{1}{q_1} \cdot c \cdot R_1 \cdot q_1 \cdot M_1 \\
&= \frac{1}{2} \cdot \frac{1}{q_1} \cdot c \cdot R_2 \cdot q_1 \\
&< \frac{1}{2} \cdot \frac{1}{q_1} \cdot d_2.
\end{aligned}$$

In general,

$$D_k < \frac{1}{2} \cdot \prod_{j=1}^{k-1} \frac{1}{q_k} d_k < \frac{3}{4} d_k$$

with $3 \big| D_k$. Note that there exists $L > 1$ such that $D_1 > \frac{1}{L} d_1$, so we have for all $k$

$$\frac{D_k}{R_k} = \frac{D_1}{R_1} \cdot q_1 \cdot ... \cdot q_{k-1} > \frac{2}{3} \cdot \frac{D_1}{R_1} > \frac{2}{3} \cdot \frac{1}{L} \cdot \frac{d_1}{R_1} > \frac{2}{3} \cdot \frac{1}{L} \cdot q_1 \cdot c > \frac{c}{2L} > 0 \qquad (*).$$

Since $X_n$ is a CW-complex with $D_n \leq \dim(X_n) = d_n$, there exists a compact subset $Q_n \subset X_n$ for each $n$ such that

$$Q_n \simeq I^{D_n} = (I^{D_{n-1}})^{\times N_{n-1}}.$$

Let $\eta_n : Q_n \to I^{D_n}$ be such a homeomorphism. For $k \in \{1, ..., N_{n-1}\}$ let

$$\pi^n_k : I^{D_n} \to I^{D_{n-1}}$$

be projection to the $k$-$I^{D_n}$ factor for each $k \in \{1, \ldots, N_{n-1}\}$. Since $X_n$ is a CW-complex, there exists a retract

$$\gamma_n : X_n \to Q_n$$

for each $n$. For $j \geq i$, put $D_{i,j} = \dfrac{D_j}{D_i} = \displaystyle\prod_{k=i}^{j-1} N_k$. When $j > i$ we have $I^{D_j} = (I^{D_i})^{\times D_{i,j}}$; let $\pi^{i,j}_k : I^{D_j} \to I^{D_i}$ be projection onto the $k$-$I^{D_i}$ factor, for each $k \in \{1, ...D_{i,j}\}$.

For each $i \in \mathbb{N}$, $j > i$, $k \in \{1, \ldots, D_{i,j}\}$, put

$$\lambda^{i,j}_k := \eta_i^{-1} \circ \pi^{i,j}_k \circ \eta_j \circ \gamma_j : X_j \to Q_i \subset X_i.$$

Let $(x_n)$ be a dense sequence in $I^\infty$ such that $x_n \in I^{D_n} \subset I^\infty$ for each $n$. Define $\delta_n : X_{n+1} \to X_n$ to be the constant map

$$\delta_n(x) = x_n;$$

thus, $a \circ \delta_n(x) = a(x_n)$ is a constant matrix. Put $\mathcal{A}_n = M_{R_n}(C(X_n))$ and define $\phi_{n,n+1} : \mathcal{A}_n \to \mathcal{A}_{n+1}$ by

$$\phi_{n,n+1}(a) = \operatorname{diag}(a \circ \lambda_1^{n,n+1}, ..., a \circ \lambda_{N_n}^{n,n+1}, a \circ \delta_n, \ldots, a \circ \delta_n)$$

with $\mathcal{A} = \lim\limits_{n\to\infty}(\mathcal{A}_n, \phi_{n,n+1})$. Thus, $\mathcal{A}$ has been produced via Construction 3.2.4. But, in fact, we can show that $\mathcal{A}$ itself is isomorphic to a simple Villadsen algebra over the cube, noting that the above maps $\phi_{n,n+1}$ are *not injective*.

Consider the maps

$$\iota_n : M_{R_n}(C(I^{D_n})) \to M_{R_n}(C(X_n)) \quad \text{and} \quad \rho_n : M_{R_n}(C(X_n)) \to M_{R_n}(C(I^{D_n}))$$

defined by

$$\iota_n(a)(x) = a(\eta_n \circ \gamma_n(x)), \qquad \rho_n(a)(x) = a\big|_{Q_n}(\eta_n^{-1}(x)),$$

and $\tilde{\phi}_{n,n+1} : M_{R_n}(C(I^{D_n})) \to M_{R_{n+1}}(C(I^{D_{n+1}}))$ defined by

$$\tilde{\phi}_{n,n+1}(a) = \operatorname{diag}(a \circ \pi_1^{n,n+1}, ..., a \circ \pi_{D_{n,n+1}}^{n,n+1}, a \circ \delta'_n, ..., a \circ \delta'_n),$$

where $a \circ \delta'_n(x) = a(x_n)$ is again point evaluation. Put $\mathcal{B}_n = M_{R_n}(C(I^{D_n}))$; then by density of the sequence $(x_n)$, we have $\mathcal{B} := \lim\limits_{n\to\infty}(\mathcal{B}_n, \tilde{\phi}_{n,n+1})$ is a simple Villadsen algebra of the first type. By construction, in particular $(*)$ above, $D_{i,j}$ is the number of projections appearing in eigenvalue maps from $\mathcal{B}_j$ to $\mathcal{B}_i$ and $M_{i,j} := \dfrac{R_j}{R_i}$ is the total number of eigenvalue maps, then for every $i \in \mathbb{N}$, with

$$\lim_{j\to\infty} \frac{N_{i,j}}{M_{i,j}} > \frac{c}{2L} > 0.$$

Therefore, by Toms-Winter [TW09] Lemma 5.1, $\mathcal{B}$ does not have strict comparison.

Consider the following diagram:

$$\begin{tikzcd}
M_{R_1}(C(X_1)) \arrow[r, "\phi_{1,2}"] \arrow[d, shift left, "\rho_1"] & M_{R_2}(C(X_2)) \arrow[r, "\phi_{2,3}"] \arrow[d, shift left, "\rho_2"] & M_{R_3}(C(X_3)) \arrow[r, "\phi_{3,4}"] \arrow[d, shift left, "\rho_3"] & \dots \\
M_{R_1}(C(I^{D_1})) \arrow[r, "\tilde{\phi}_{1,2}"] \arrow[u, shift left, "\iota_1"] & M_{R_2}(C(I^{D_2})) \arrow[r, "\tilde{\phi}_{2,3}"] \arrow[u, shift left, "\iota_2"] & M_{R_3}(C(I^{D_3})) \arrow[r, "\tilde{\phi}_{3,4}"] \arrow[u, shift left, "\iota_3"] & \dots
\end{tikzcd}$$

For $a \in M_{R_n}(C(X_n)), x \in X_{n+1}$, the map $\phi_n := \phi_{n,n+1}$ is given by the diagram:

$$\begin{aligned}
(\iota_{n,n+1} \circ \tilde{\phi}_n \circ \rho_n)(a)(x) &= (\tilde{\phi}_n \circ \rho_n)(a)(\eta_{n+1} \circ \gamma_{n+1}(x)) \\
&= \tilde{\phi}_n(a\big|_{Q_n} \circ \eta_n^{-1})(\eta_{n+1} \circ \gamma_{n+1}(x)) \\
&= \mathrm{diag}((a\big|_{Q_n} \circ \eta_n^{-1}) \circ \pi_1^{n,n+1}, \dots, a\big|_{Q_n} \circ \eta_n^{-1}) \circ \pi_{D_{n,n+1}}^{n,n+1}, \delta_n'((a\big|_{Q_n} \circ \eta_n^{-1})), \dots, \\
&\qquad , \dots, \delta_n'((a\big|_{Q_n} \circ \eta_n^{-1})))(\eta_{n+1} \circ \gamma_{n+1}(x)) \\
&= \mathrm{diag}(a \circ \eta_n^{-1} \circ \pi_1^{n,n+1} \circ \eta_{n+1} \circ \gamma_{n+1}(x), \dots, \\
&\qquad , \dots, a \circ \eta_n^{-1} \circ \pi_1^{n,n+1} \circ \eta_{n+1} \circ \gamma_{n+1}(x), \delta_n(a), \dots, \delta_n(a)) \\
&= \phi_n(a)(x).
\end{aligned}$$

Note that we have $\eta_n \circ \gamma_n \circ \eta_n^{-1} = \mathrm{id}_{I^{D_n}}$ for each $n$, since $\gamma_n = \mathrm{id}_{Q_n}$ as a retract. Thus, for $a \in M_{R_n}(C(I^{D_n}))$ and $x \in I^{D_{n+1}}$,

$$\begin{aligned}
&(\rho_{n+1} \circ \phi_n \circ \iota_n)(a)(x) = (\phi_n \circ \iota_n)(a)\big|_{Q_{n+1}}(\eta_{n+1}^{-1}(x)) \\
&= \mathrm{diag}(\iota_n(a) \circ \eta_n^{-1} \circ \pi_1^{n,n+1} \circ \eta_{n+1} \circ \gamma_{n+1}, \dots, \\
&\qquad , \dots, \iota_n(a) \circ \eta_n^{-1} \circ \pi_{D_n}^{n,n+1} \circ \eta_{n+1} \circ \gamma_{n+1}, \delta_n(\iota_n(a)), \dots, \delta_n(\iota_n(a)))\big|_{Q_{n+1}}(\eta_{n+1}^{-1}(x)) \\
&= \mathrm{diag}(a \circ (\eta_n \circ \gamma_n \circ \eta_n^{-1}) \circ \pi_1 \circ (\eta_{n+1} \circ \gamma_{n+1} \circ \eta_{n+1}^{-1})(x), \dots, \\
&\qquad , \dots, a \circ (\eta_n \circ \gamma_n \circ \eta_n^{-1}) \circ \pi_1 \circ (\eta_{n+1} \circ \gamma_{n+1} \circ \eta_{n+1}^{-1})(x), \\
&\qquad\qquad a(\eta_n \circ \gamma_n \circ \eta_n^{-1}(x_n)), \dots, a(\eta_n \circ \gamma_n \circ \eta_n^{-1}(x_n))) \\
&= \mathrm{diag}(a \circ \pi_1^{n,n+1}(x), \dots, a \circ \pi_{D_n}^{n,n+1}(x), a(x_n), \dots, a(x_n)) \\
&= \mathrm{diag}(a \circ \pi_1^{n,n+1}, \dots, a \circ \pi_{D_n}^{n,n+1}, \delta_n'(a), ..., \delta_n'(a)) \\
&= \tilde{\phi}_n(a)(x).
\end{aligned}$$

Indeed, the diagram is fully intertwining; whence $\mathcal{A} \simeq \mathcal{B}$ and $\mathcal{A}$ does not have strict comparison.

$\square$

## 3.3 Injectivity

In the finale, we answer a followup questions about modifying such a construction to preserve information between the connecting homomorphisms, answering again in the affirmative:

**Question 3.3.1.** Can the connecting homomorphisms $\phi_n$ be made injective without losing significatnt information about a lack of $\mathcal{Z}$-stability?

**Lemma 3.3.2.** For each $i \in \mathbb{N}$, let $X_i$ be a CW-complex, and $(R_n)$ be a sequence in $\mathbb{N}$ with $R_n \big| R_{n+1}$ for all $n$; put $\mathcal{A}_n = M_{R_n}(C(X_n))$. Suppose $\mathcal{A} = \lim_{n\to\infty}(\mathcal{A}_n, \phi_{n,n+1})$ is a generalized interval Villadsen algebra as in Definition 3.1.7 which is produced by Construction 3.2.4, assuming this construction's notation along with that of Lemma 3.1.1. Thus, the connecting homomorphisms $\phi_{n,n+1}$ are obtained from factoring through an interval, or projections over a cube. Let

(i) $\eta \in M_{R_i}(C(\tilde{S}^D))$ be a projection, for some $i, D \in \mathbb{N}$ where $\tilde{S}^D \subset Q_i \subset X_i$ has $\tilde{S}^D \simeq S^D$ as given by Lemma 3.1.1 (ii),
(ii) $U_i \subset X_i$ be an open neighborhood of $\tilde{S}^D$ with retract $\tau : \overline{U_i} \to \tilde{S}^D$, as given by Lemma 3.1.1 (ii),
(iii) $j > i$, and $\mathcal{S}_j \subset Q_j$ be the subset given in Lemma 3.1.1 (iii) with rectifying homeomorphism $\zeta_{i,j}$,
(iv) $\zeta' : (\tilde{S}^D)^{N_{i,j}} \to \mathcal{S}_j$ be the homeomorphism

$$\zeta'(x_1, ..., x_{N_{i,j}}) = \zeta_{i,j}(\iota_1^{i,j}(x_1), \dots \iota_{N_{i,j}}^{i,j}(x_{N_{i,j}}),$$

where $\iota_1^{i,j}, ..., \iota_{N_{i,j}}^{i,j}$ are the injections given in Lemma 3.1.1 (i) with respect to which the rectifying homeomorphism $\zeta_{i,j}$ satisfies, for all $k$,

$$\lambda_k^{i,j} \circ \zeta'(x_1, \dots, x_k) = \lambda_k^{i,j} \circ \zeta_{i,j}(\iota_1^{i,j}(x_1), ..., \iota_{N_{i,j}}^{i,j}(x_{N_{i,j}}) = x_k.$$

(v) $f \in C_0(U_i)$ be equal to 1 on $\tilde{S}^D$ and 0 off $U_i$.
(vi) $\tau_f^*\eta \in M_{R_i}(C_0(U_i)) \subset M_{R_i}(C(X_i))$ be the function

$$\tau_f^*\eta(x) = \begin{cases} 0 & \text{if } x \notin U_i \\ f(x)\eta(\tau(x)) & \text{if } x \in U_i \end{cases}.$$

Then, $\left(\phi_{i,j}\tau_f^*\eta\right)\big|_{\mathcal{S}_j} \in M_{R_j}(C(\mathcal{S}_j)) \simeq C(\mathcal{S}_j, M_{R_j})$ satisfies

$$\left(\phi_{i,j}\tau_f^*\eta\right)\big|_{\mathcal{S}_j} \circ \zeta'(x_1, ..., x_{N_{i,j}}) = \operatorname{diag}(\eta(x_1), \dots \eta(x_{N_{i,j}}), K_{i,j}(\eta)(x_1, ..., x_{N_{i,j}})),$$

where $K_{i,j}(\eta)$ is a block of rank at most $\operatorname{Rank}(\eta)\left(\dfrac{R_j}{R_i} - \cdot N_{i,j}\right)$ comprised of either maps which are factored through an interval or which are constant.

*Proof.* Put $\beta_{i,j} = \dfrac{R_j}{R_i} - N_{i,j}$, which is the total number of maps which are constant or factored through an interval from stage $i$ to stage $j$. Therefore, the diagonal block $K_{i,j}(\eta)' \in M_{\beta_{i,j}}(C(X_j))$ appearing in $\phi_{i,j}\tau_f^*\eta$ of such maps has rank at most $\operatorname{Rank}(\eta) \cdot \beta_{i,j}$. Specifically, $K_{i,j}(\eta)'$ is

$$K_{i,j}(\eta)' = \operatorname{diag}(\tau_f^*\eta \circ \Delta_1^{i,j}, \dots \tau_f^*\eta \circ \Delta_{\beta_{i,j}}^{i,j}),$$

where $\Delta_1^{i,j}, \dots \Delta_{\beta_{i,j}}^{i,j} : X_j \to X_i$ are the continuous maps which are constant or factor through an interval from Construction 3.2.4. Thus, putting $K_{i,j}(\eta) := K_{i,j}(\eta)'\big|_{\mathcal{S}_j} \circ \zeta' \in M_{\beta_{i,j}}(C((\tilde{S}^D)^{N_{i,j}})$, we have $K_{i,j}(\eta)$ has at most the same rank. Since $i, j$ are fixed, for the main calculation we may suppress the symbol "$i, j$" and put

$$\lambda_k := \lambda_k^{i,j}, \;\; \iota_k := \iota_k^{i,j}, \;\; \zeta := \zeta_{i,j}, \;\; K(\eta) := K_{i,j}(\eta), \; K(\eta)' := K_{i,j}(\eta)' \text{ and } N := N_{i,j}.$$

Let $(x_1, ..., x_N) \in (\tilde{S}^D)^N$ be arbitrary; put

$$y = \zeta'(x_1, ..., x_N) = \zeta(\iota_1(x_1), ..., \iota_N(x_N)),$$

which is unique in $\mathcal{S}_j$ since $\zeta'$ is a homeomorphism. Analogously to Remark 3.1.5:

$$\begin{aligned}
\left(\phi_{i,j}\tau_f^*\eta\right)\big|_{S_j} \circ \zeta'(x_1, ..., x_N) &= \phi_{i,j}\tau_f^*\eta\big|_{\mathcal{S}_j}(y) \\
&= \operatorname{diag}\left(\tau^*\eta \circ \lambda_1, \dots, \tau^*\eta \circ \lambda_N, K(\eta)'\big|_{\mathcal{S}_j})\right)(y) \\
&= \operatorname{diag}\left(\tau^*\eta \circ \lambda_1(y), \dots, \tau^*\eta \circ \lambda_N(y), K(\eta)'\big|_{\mathcal{S}_j}(y))\right) \\
&= \operatorname{diag}(\tau^*\eta \circ \lambda_1 \circ \zeta(\iota_1(x_1), \dots, \iota_N(x_N)), \dots \\
&\qquad \dots, \tau^*\eta \circ \lambda_N \circ \zeta(\iota_1(x_1), \dots, \iota_N(x_N)), K(\eta)'\big|_{\mathcal{S}_j} \circ \zeta'(x_1, ..., x_N)). \\
&= \operatorname{diag}(\tau^*\eta(y_1), \dots, \tau^*\eta(y_N), K(\eta)(x_1, ..., x_N)). \\
&= \operatorname{diag}(\eta(x_1), \dots, \eta(x_N), K(\eta)(x_1, ..., x_N))
\end{aligned}$$

□

**Lemma 3.3.3.** Suppose $\mathcal{A} = \lim_{n\to\infty}(\mathcal{A}_n, \phi_{n,n+1})$ is a generalized interval Villadsen algebra as in Definition 3.1.7 which is produced by Construction 3.2.4, assuming this construction's notation along with that of Lemma 3.1.1. Thus, the connecting homomorphisms $\phi_{n,n+1}$ are obtained from factoring through an interval, or projections over a cube. Let $\eta \in M_{R_i}(C(\tilde{S}^D))$ be the projection in (i), for some $i, D$. Suppose $B \in M_{R_i}(C(X_i))$ has $B\big|_{\tilde{S}^D} = 0$, $B$ is orthogonal to $\tau_f^*\eta$, and $A \in M_{R_i}(C(X_i))$ is given by

$$A = \tau_f^*\eta + B.$$

Let $\psi : (S^D)^{N_{i,j}} \to (\tilde{S}^D)^{N_{i,j}}$ be the natural coordinate-wise homeomorphism.

With this setup, if for some $Q \in \mathbb{N}$, $k \geq i$, we have $\Theta_Q \in M_{R_k}(C(X_k))$ satisfying $[\Theta_Q\big|_{\mathcal{S}_k}] = [\theta_Q]$, a trivial rank $Q$ projection, and

$$(\Theta_Q) \precsim (\phi_{i,k}A),$$

then, for $j \geq k$, in $K_0(M_{R_j}(C((S^D)^{N_{i,j}})))$

$$[\theta_Q]^{\times N_{k,j}} = [\theta_{Q\times N_{k,j}}] \leq [\eta \circ \psi]^{\times N_{i,j}} \oplus \theta_{\ell_{i,j}},$$

where $[\sigma]^{N_{i,j}}$ denotes the equivalence class of $\sigma^{\times N_{i,j}} \in M_{R_i \cdot N_{i,j}}C((S^D)^{N_{i,j}}))$ given by

$$\sigma^{\times N_{i,j}}(x_1, ..., x_{N_{i,j}}) := \mathrm{diag}(\sigma(x_1), \ldots, \sigma(x_{N_{i,j}})),$$

and $\theta_{\ell_{i,j}}$ is a trivial projection of rank $\ell_{i,j}$ with

$$\ell_{i,j} := 2(R_j - R_i \cdot N_{i,j})$$

*Proof.* Let $\Psi : M_{R_j}(C(\tilde{S}^D)^{N_{i,j}}) \to M_{R_j}(C((S^D)^{N_{i,j}})) \simeq C((S^D)^{N_{i,j}}, M_{R_j})$ be the isomorphism

$$\Psi A = A \circ \psi.$$

Note that since $B\big|_{\tilde{S}^2} = 0$, we have $\phi_{i,j}B\big|_{\tilde{S}^2}$ is a matrix factored through $[0,1]$ of rank no more than $\frac{1}{2}\ell_{i,j} = R_i(R_j - N_{i,j})$, for all $j \geq i$. Likewise, so is $K_{i,j}(\eta)$. By Lemma 3.1.8, these elements are comparable to trivial ones after having factored through $\mathrm{Cu}(M_n(C[0,1]))$: we have

$$\phi_{i,j}B\big|_{\tilde{S}^D} \precsim \theta_{\frac{1}{2}\ell_{i,j}} \qquad \text{and} \qquad K_{i,j}(\eta) \precsim \theta_{\frac{1}{2}\ell_{i,j}}.$$

Likewise, we have $[\phi_{k,j}\Theta_Q\big|_{S_j}] \geq [\theta_Q]^{\times N_{k,j}} = [\theta_{Q\cdot N_{k,j}}]$ by cutting down by the trivial block.

By functoriality of $\mathrm{Cu}(\cdot)$ (c.f. Remark 1.4.4), we have $(\Theta_Q) \precsim (\phi_{i,k}A)$ implies in the Cuntz semigroup $\mathrm{Cu}(M_{R_j}(C((S^D)^{N_{i,j}})))$

$$\begin{aligned}
\mathrm{Cu}(\Psi)\left[(\phi_{k,j}\Theta_Q)\big|_{\mathcal{S}_j}\right] &\leq \mathrm{Cu}(\Psi)\left[(\phi_{i,j}A)\big|_{\mathcal{S}_j}\right] \\
&= \mathrm{Cu}(\Psi)\left[\left(\phi_{i,j}\tau_f^*\eta + \phi_{i,j}B)\right)\big|_{\mathcal{S}_j}\right] \\
&= \mathrm{Cu}(\Psi)\left([(\phi_{i,j}\eta)] \oplus \left[(\phi_{i,j}B))\big|_{\mathcal{S}_j}\right]\right) \\
&= \mathrm{Cu}(\Psi)\left([\eta]^{\times N_{i,j}}\right) \oplus \left[K_{i,j}(\eta)] \oplus \left[(\phi_{i,j}B))\big|_{\mathcal{S}_j}\right]\right) \\
&\leq [\eta\circ\psi]^{\times N_{i,j}} \oplus [\theta_{\frac{1}{2}\ell_{i,j}}] \oplus [\theta_{\frac{1}{2}\ell_{i,j}}] \\
&= [\eta\circ\psi]^{\times N_{i,j}} \oplus [\theta_{\ell_{i,j}}].
\end{aligned}$$

As noted, we have

$$\mathrm{Cu}(\Psi)\left[(\phi_{k,j}\Theta_Q)\big|_{\mathcal{S}_j}\right] \geq [\theta_{Q\cdot N_{k,j}}].$$

So, putting these inequalities together, we have that if $(\Theta_Q) \precsim (\phi_{i,k}A)$, then

$$[\theta_Q]^{\times N_{k,j}} = [\theta_{Q\times N_{k,j}}] \leq [\eta\circ\psi]^{\times N_{i,j}} \oplus \theta_{\ell_{i,j}},$$

in $\mathrm{Cu}(M_{R_j}(C((S^D)^{N_{i,j}})))$. But, for projections Cuntz inequality is the same as Murray-von Neumann inequality, therefore we conclude in $K_0((C((S^D)^{QN_{i,j}})))$

$$[\theta_Q]^{\times N_{k,j}} = [\theta_{Q\cdot N_{k,j}}] \leq [\eta\circ\psi]^{\times N_{i,j}} \oplus \theta_{\ell_{i,j}}.$$

$\square$

In the final theorem, we establish that the construction in Theorem 3.2.6 can be modified to allow for injective connecting homomorphisms, thereby preserving the information of each CW-complex.

**Theorem 3.3.4.** Let $X_n$ be given finite connected CW-complexes and let $(R_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}$. Suppose that $(d_n)$ and $(R_n)$ are monotonically increasing, with $R_n \big| R_{n+1}$ and $d_n \to \infty$, and suppose

$$\liminf_{n\to\infty} \frac{\dim(X_n)}{R_n} = c \in (0, \infty).$$

Then, there exists a subsequence $(a_n)$ of $\mathbb{N}$ and *injective* connecting homomorphisms

$$\phi_{n,n+1} : M_{R_{a_n}}(C(X_{a_n})) \to M_{R_{a_{n+1}}}(C(X_{a_{n+1}}))$$

such that $\mathcal{A} := \lim\limits_{n\to\infty}(M_{R_{a_n}}(C(X_{a_n})), \phi_{n,n+1})$ is a simple, unital AH algebra which is not $\mathcal{Z}$-stable.

*Proof.* Construct the subsequence and algebras $\mathcal{A}_k = R_{a_k}(C(X_{a_k}))$ with continuous maps $\lambda_k^{n,n+1} : X_{a_{k+1}} \to X_{a_k}$ factoring through a cube and $\delta_n$ evaluation on the dense sequence $(x_n)$ in $I^\infty$ as in the proof of Theorem 3.2.6. Relabeling as before, we may replace each $\mathcal{A}_{a_n}$ with just $\mathcal{A}_n$. Without loss of generality, we may choose the sequence $q_n$ such that $q_n = \dfrac{N_n}{M_n}$ has $M_n - N_n \geq 2$.

Since $X_n$ is a finite connected CW-complex, it is a *Peano space*; thus, there exists a continuous surjective map $\omega_n : [0,1] \to X_n$, a *Peano curve,* by the Hahn-Mazurkiewicz Theorem. Let $P_n : X_{n+1} \to [0,1]$ be a surjective map. Replace one of the constant maps $\delta_n$ in each connecting homomorphism $\phi_{n,n+1}$ with the map $\delta_n' := \omega_n \circ P_n : X_n \to X_{n-1}$.

Thus, $\mathcal{A} = \lim\limits_{n\to\infty}(\mathcal{A}_n, \phi_{n,n+1})$ is a simple, unital generalized interval Villadsen algebra as in Definition 3.1.7 which is produced by Construction 3.2.4. The connecting homomorphisms $\phi_{n,n+1}$ are obtained from factoring through an interval, or projections over a cube, and are injective thanks to surjectivity of the maps $\delta_n'$.

Let $\eta' \in M_2(C(\tilde{S^2}))$ be the Hopf fibration, a nontrivial line bundle for which no tensor power of the Euler class is zero. Thus, by Villadsen's Lemma, Theorem 2.3.1, we have

$$[\theta_1] \not\leq [\eta' \circ \psi]^{\times n} \qquad (*)$$

in $K_0(C((S^2)^n))$, for any $n \in \mathbb{N}$.

Without loss of generality, we may assume $i$ is large enough that $R_i \geq 3$, $\dim(Q_i) \geq 3$, and $D_{i,i+1} := \dfrac{\dim(Q_{i+1})}{\dim(Q_i)} \geq 2$.

By construction of $\mathcal{A}$, the number $N_{i,j}$ of projections, and $\ell_{i,j}$, which is twice the number of constant maps along with those factored through an interval, satisfies $N_{i,j} >> \ell_{i,j}$, since $\dfrac{R_i}{R_j} N_{i,j} > \dfrac{c}{2L} > 0$ for some $L > 0$, and for all $i$ and $j \geq i$. Without loss of generality we may assume $i$ is also large enough that $2 \leq \ell_{i,j} < \dfrac{1}{2} N_{i,j}$ for every $j \geq i$.

Assume all the notation of Lemma 3.3.2, with $S^2 \simeq \tilde{S}^2 \subset Q_i \subset X_i$. Put $\eta = \operatorname{diag}(\eta', 0, \ldots, 0) \in M_{R_i}(C(\tilde{S}^2))$. So, by Lemma 3.3.2, in $K_0(C((S^2)^{N_{i,i+1}}))$,

$$[\phi_{i,i+1}\tau_f^*\eta\big|_{\mathcal{S}_{i+1}}] \leq [\eta]^{\times N_{i,i+1}} \oplus [\theta_{\ell_{i,i+1}}],$$

where $\ell_{i,i+1}$ is twice the number of trivial maps in $\phi_{i,i+1} : \mathcal{A}_i \to \mathcal{A}_{i+1}$.

Appealing to Lemma 3.3.3, let $\theta_{Q'} \in M_{R_{i+1}}(C(\mathcal{S}_{i+1}))$ be a trivial bundle of rank $Q' = N_{i,i+1} > \ell_{i,i+1}$. For each $j \geq i$, let $\psi : (S^2)^{N_{i,j}} \to (\tilde{S}^2)^{N_{i,j}}$ be the natural coordinate-wise homeomorphism and put $A, B \in M_{R_i}(C(X_i))$ as

$$B = \operatorname{diag}(0_{2\times 2}, h, 0, \ldots), \qquad A = \tau_f^*\eta + B,$$

where $h \in C_0(X_i \setminus \tilde{S}^2)$ is a strictly positive element. Thus, $B\big|_{\tilde{S}^2} = 0$, $B$ is orthogonal to $\tau_f^*\eta$, and $A\big|_{\tilde{S}^2} = \eta$. Let $g \in C_0(U_{i+1})$ be a strictly positive function, where $U_{i+1}$ is a neighborhood of $\mathcal{S}_{i+1} \simeq (S^2)^{N_{i,i+1}}$ obtained from the rectifying homeomorphism $\zeta_{i,i+1}(\prod \iota_{i,i+1}(U_i))$, i.e. one such that the support agrees with the nontrivial part of $\phi_{i,i+1}\tau_f^*\eta$, and with $g\big|_{\mathcal{S}_{i+1}} = 1$. Thus, we have in $M_{R_{i+1}}(C(X_{i+1}))$

$$\operatorname{Rank}(g\theta_{Q'}(x)) = \begin{cases} 0 & \text{if } \ x \notin U_{i+1} \\ N_{i,i+1} & \text{if } \ x \in U_{i+1} \end{cases}$$

and

$$\operatorname{Rank}(\phi_{i,i+1}A(x)) = \begin{cases} 1 & \text{if } \ x \notin U_{i+1} \\ N_{i,i+1} + \ell_{i,i+1} & \text{if } \ x \in U_{i+1} \end{cases},$$

so we find

$$\mathrm{Rank}(\phi_{i+1,j} g\theta_{Q'}(x)) < \mathrm{Rank}(\phi_{i,j} A(x))$$

for all $x \in X_j$, $j \geq i$, i.e. $d_{t_x}(\phi_{i,j} g\theta_{Q'}) < d_{t_x}(\phi_{i,j} A)$ for all $j \geq i$ and all extreme traces $t_x \in T(\mathcal{A}_j)$. Hence,

$$d_t(\phi_{i+1,\infty} g\theta_{Q'}) < d_t(\phi_{i,\infty} A)$$

holds for all $t \in T(\mathcal{A})$.

Suppose that $\phi_{i+1,j} g\theta_{Q'} \precsim \phi_{i,j} A$ for sufficiently large $j$. Put $\Theta_Q = \phi_{i+1,j} g\theta_{Q'}$ in Lemma 3.3.3, where $Q = Q' \cdot N_{i+1,j} = N_{i,j}$. Then, $(\phi_{i+1,j} g\theta_Q)\big|_{\mathcal{S}_j} \precsim (\phi_{i,j} A)\big|_{\mathcal{S}_j})$ for sufficiently large $j$, where $\mathcal{S}_j \simeq (S^2)^{N_{i,j}}$. By Lemma 3.3.3, we have

$$[\theta_{N_{i,i+1}}]^{\times N_{i+1,j}} = [\theta_{N_{i,j}}] \leq [\eta \circ \psi]^{\times N_{i,j}} \oplus \theta_{\ell_{i,j}}$$

in $K_0((C(S^2)^{N_{i,j}}))$, where

$$\mathrm{Rank}(\theta_{\ell_{i,j}}) = \ell_{i,j} = 2R_i(R_j - N_{i,j}) < N_{i,j}.$$

By cancellation of projections, since $\ell_{i,j} < N_{i,j}$, we reduce to

$$[\theta_V] \leq [\eta \circ \psi]^{\times N_{i,j}}$$

in $K_0((S^2)^{N_{i,j}}$, where $V \geq 1$. But, this contradicts $(*)$ above - the phenomenon known as Villadsen's Chern class obstruction. Therefore, $\mathcal{A}$ does not have strict comparison of positive elements. $\square$

**Remark 3.3.5.** One could dispense with the need for connectedness by adjusting the construction to include maps $\zeta_n$ to each component.

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