# POLYNOMIAL ROOT DISTRIBUTION AND ITS IMPACT ON SOLUTIONS TO THUE EQUATIONS

by

GREG KNAPP

A DISSERTATION

Presented to the Department of Mathematics
and the Division of Graduate Studies of the University of Oregon
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

June 2023

DISSERTATION APPROVAL PAGE

Student: Greg Knapp

Title: Polynomial Root Distribution and Its Impact on Solutions to Thue Equations

This dissertation has been accepted and approved in partial fulfillment of the requirements for the Doctor of Philosophy degree in the Department of Mathematics by:

| | |
|---|---|
| Shabnam Akhtari | Chair |
| Ellen Eischen | Core Member |
| Weiyong He | Core Member |
| Ben Young | Core Member |
| Brittany Erickson | Institutional Representative |

and

| | |
|---|---|
| Krista Chronister | Vice Provost for Graduate Studies |

Original approval signatures are on file with the University of Oregon Division of Graduate Studies.

Degree awarded June 2023.

# DISSERTATION ABSTRACT

Greg Knapp

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In this study, we focus on two topics in classical number theory. First, we examine Thue equations—equations of the form $F(x, y) = h$ where $F(x, y)$ is an irreducible, integral binary form and $h$ is an integer—and we give improvements to both asymptotic and explicit bounds on the number of integer pair solutions to Thue equations. These improved bounds largely stem from improvements to a counting technique associated with "The Gap Principle," which describes the gap between denominators of good rational approximations to an algebraic number. Next, we will take inspiration from the impact of polynomial root distribution on solutions to Thue equations and we examine polynomial root distribution as its own topic. Here, we will look at the relation between the separation of a polynomial—the minimal distance between distinct roots—and the Mahler measure of a polynomial—a height function which connects the roots of a polynomial with its coefficients. We make a conjecture about how separation can be bounded above by the Mahler measure and we give data supporting that conjecture along with proofs of the conjecture in some low-degree cases.

ACKNOWLEDGMENTS

I would like to thank my advisor, Shabnam Akhtari, for her help in selecting interesting and accessible problems and for all of her support throughout the challenges of graduate school. She showed me, above all, how to be an incredible mentor.

I would like to thank the others who contributed to the mathematics contained in this dissertation. Ben Young gave helpful feedback on the code that went into optimizing the parameters $T$ and $Z$. Joe Webster gave me the regular reminder to look at as many examples as possible and he inspired much of the progress that I made on relating Mahler measure and separation. Chris Sinclair gave me specific suggestions to enable my work on Mahler measure and separation. This research was supported by NSF grant number 2001281.

I would like to thank my more general mathematical role models. Cathy Hsu was incredibly supportive throughout my time in graduate school, from helping me find a social group in my first year, to helping me get myself organized in my fourth year, to helping me with my job application materials in my sixth year (among many other things). Ellen Eischen set a great example of what a successful researcher and effective teacher looks like. Joe Webster always modeled the value of being a clear expositor and teacher. Ken Barrett gave me the drive to open up every black box and ask why the metaphorical machinery is constructed the way it is.

I would like to thank my therapist for regularly reminding me about the importance of balance. He helped me make the last six years bearable (and more than that, fun) and showed me that the best dissertation is a done dissertation.

I would like to thank my family and friends for continuously being there for me. Jake Anderson, Kyle Bahnsen, Marissa Masden, Johanna Meven, Kelly Pohland, Austin Ray, Melissa Ruszczyk, Joseph Tate, Steven Tokar, Michael Volkovitsch, and Eli Wolff have consistently been able to pull me away from my work to play games and I'm so happy that they try so hard to do that. Samantha Platt asked hard questions and helped me reflect on so many issues I didn't know I was taking for granted. Alex Roesch was the one person outside of math who was genuinely excited about my research in pure mathematics. Gabrielle Cohen helped me stay in shape and reminded me that there's a lot that happens outside of the math department. Jamie Piatka was my confidante long before graduate school and I am grateful for her steadfast presence in my life. Kyla Pohl has been an incredible partner and she does a great job cheering me up after tough days. My parents have continued to be role models and have done so much to help me grow into a more mature person.

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# CHAPTER 1

# INTRODUCTION

The main thesis of our study is that properties of polynomial roots can be understood through knowledge about the coefficients of the polynomial. It is well known that the the roots of a polynomial of degree at least five cannot necessarily be expressed in an elementary way in terms of the coefficients of that polynomial, but the inability to express the roots of the polynomial exactly does not prohibit us from understanding certain features of the roots.

One of the interesting properties we can ask about is which rational numbers are close to the roots of a polynomial. The question of finding good rational approximations to real numbers has been thoroughly explored through the field of Diophantine approximation, which we describe in the introduction. Chapter 2 will discuss an application of Diophantine approximation to the study of Thue equations and we will prove some new results bounding the number of solutions to Thue equations.

Another question we can ask about the roots of a polynomial is how they are distributed in the complex plane. In Chapter 3, we discuss previous work finding lower bounds on the distance between roots of polynomials in terms of the coefficients and we turn that question around to ask about upper bounds on the distances between roots. We show how those upper bounds provide us with some ability to quantify the statement "the roots of polynomials are not randomly distributed," we conjecture what the sharpest upper bounds might look like, and we prove sharp upper bounds in some low-degree settings.

## 1.1 Using Approximation to Categorize Real Numbers

We begin our exploration of Diophantine approximation by showing how approximation results can be used to classify real numbers. One of the foundational theorems in this area is Dirichlet's Approximation Theorem.

**Theorem 1.1** (Dirichlet)**.** *Let $\alpha \in \mathbb{R}$ and $Q \in \mathbb{Z}_{>0}$. Then there exist integers $p, q \in \mathbb{Z}$ with $1 \leqslant q \leqslant Q$ so that*

$$|q\alpha - p| < \frac{1}{Q}.$$

Dirichlet's proof of this theorem is clever [Dir42].

*Proof.* For any real number $x$, let $\{x\}$ denote the *fractional part* of $x$, namely $\{x\} := x - \lfloor x \rfloor$. Subdivide the half-open unit interval $[0, 1)$ into $Q$ intervals of the form

$J_n = \left[\frac{n-1}{Q}, \frac{n}{Q}\right)$. Now consider the sequence of $Q+1$ numbers $0, \{\alpha\}, \{2\alpha\}, \ldots, \{Q\alpha\}$. These $Q+1$ numbers all lie in $[0,1)$ and by the pigeonhole principle (also known as Dirichlet's box principle), there exist $n, s_1, s_2 \in \mathbb{N}$ with $0 \leqslant s_1 < s_2 \leqslant Q$ so that $\{s_1\alpha\}, \{s_2\alpha\} \in J_n$. Hence,

$$|\{s_1\alpha\} - \{s_2\alpha\}| < \frac{1}{Q}.$$

But writing $\{s_1\alpha\} = s_1\alpha - r_1$ and $\{s_2\alpha\} = s_2\alpha - r_2$ for integers $r_1$ and $r_2$ yields

$$|(s_2 - s_1)\alpha - (r_2 - r_1)| < \frac{1}{Q}$$

and so we take $q = s_2 - s_1$ and $p = r_2 - r_1$. We note that the fact that $1 \leqslant q \leqslant Q$ follows from the fact that $0 \leqslant s_1 < s_2 \leqslant Q$. □

On the face of it, it is not immediately obvious that Dirichlet's Theorem has anything to do with approximation. However, it has the following immediate corollary. Before stating the corollary, we give a quick definition.

**Definition 1.2.** A pair $(p,q) \in \mathbb{Z}^2$ is *primitive* if $\gcd(p,q) = 1$.

**Corollary 1.3.** *Let $\alpha \in \mathbb{R}$. Then $\alpha$ is irrational if and only if there are infinitely many primitive pairs $(p,q) \in \mathbb{Z}^2$ with $q > 0$ so that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}. \tag{1.1}$$

*Proof.* Suppose first that $\alpha$ is rational, so write $\alpha = \frac{r}{s}$ for integers $r$ and $s$. Note that for any integers $p$ and $q$ with $\frac{p}{q} \neq \frac{r}{s} = \alpha$, we have

$$\left|\alpha - \frac{p}{q}\right| = \left|\frac{rq - sp}{qs}\right| \geqslant \frac{1}{|qs|}$$

since $|rq - sp|$ is a nonzero, positive integer. Hence, if the primitive pair $(p,q)$ satisfies inequality (1.1), then it must satisfy

$$\frac{1}{|qs|} < \frac{1}{q^2},$$

implying that $|q| < |s|$. Then there are only finitely many possible values for $q$. It is now easy to check that for each $q$ there are only finitely many possible values of $p$ that satisfy (1.1) and hence, there are only finitely many primitive pairs $(p,q)$ that satisfy (1.1).

Next, suppose that $\alpha$ is irrational. Suppose by contradiction that there are only finitely many primitive pairs $(p_1, q_1), \ldots, (p_n, q_n)$ satisfying (1.1). Observe first that because $\alpha$ is irrational,

$$m := \min_{1 \leqslant i \leqslant n} |q_i\alpha - p_i| > 0.$$

Then set

$$Q := 1 + \lceil \frac{1}{m} \rceil$$

so that

$$\frac{1}{Q} < |q_i\alpha - p_i|$$

for all $1 \leqslant i \leqslant n$. Apply Dirichlet's Theorem to this value of $Q$ to find $p, q \in \mathbb{Z}$ so that $1 \leqslant q \leqslant Q$ and for all $1 \leqslant i \leqslant n$,

$$|q\alpha - p| < \frac{1}{Q} < |q_i\alpha - p_i|.$$

Writing $d = \gcd(p, q)$ and setting $p' = p/d$ and $q' = q/d$, we see that the pair $(p', q')$ is primitive and we have $q' = \frac{q}{d} \leqslant Q$. Further,

$$|q'\alpha - p'| \leqslant d|q'\alpha - p'| = |q\alpha - p| < \frac{1}{Q}$$

implying that

$$\left|\alpha - \frac{p'}{q'}\right| < \frac{1}{Qq'} \leqslant \frac{1}{(q')^2}.$$

Hence, the pair $(p', q')$ satisfies (1.1). Moreover, $(p', q')$ is distinct from all of the pairs $(p_i, q_i)$ because $|q'\alpha - p'| < |q_i\alpha - p_i|$. This contradicts the hypothesis that we had listed all pairs satisfying (1.1); hence, there must be infinitely many primitive pairs satisfying (1.1). □

Note that the "if" direction of the corollary can actually be improved somewhat: $\alpha$ is irrational if for any $\varepsilon > 0$, there are infinitely many primitive pairs $(p, q) \in \mathbb{Z}^2$ with $q \neq 0$ so that $|\alpha - \frac{p}{q}| < \frac{1}{q^{1+\varepsilon}}$.

Corollary 1.3 gives our first indication that the classification of $\alpha$ as rational or irrational can be encoded in the language of rational approximation. The first question one might ask about this corollary is if it can be improved. The answer is yes, but only by a constant factor. This combines Theorem 5B with Lemma 2E in Chapter I.2 of [Sch80].

**Proposition 1.4.** *Let $\alpha \in \mathbb{R}$. Then $\alpha$ is irrational if and only if there are infinitely many primitive pairs $(p,q) \in \mathbb{Z}^2$ with $q > 0$ so that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}.$$

*Moreover, the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$ is irrational, yet for any constant $C < \frac{1}{\sqrt{5}}$, there are only finitely many primitive primitive pairs $(p,q) \in \mathbb{Z}^2$ with $q > 0$ so that*

$$\left|\varphi - \frac{p}{q}\right| < \frac{C}{q^2}.$$

There is another sense in which Corollary 1.3 can be somewhat improved. Observe that since irrational numbers comprise almost all of the real line according to Lebesgue measure, Corollary 1.3 implies that for almost all real numbers $\alpha$, there are infinitely many primitive pairs $(p,q) \in \mathbb{Z}^2$ with $q > 0$ so that (1.1) holds. However, Khinchin in [Khi64] shows that

**Theorem 1.5** (Khinchin)**.** *For almost all real numbers $\alpha$, there are infinitely many primitive pairs $(p,q) \in \mathbb{Z}^2$ with $q > 0$ so that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2 \log q}.$$

From these results, it is natural to ask what sort of information we can learn about rational approximations if we start with finer hypotheses about the algebraicity of $\alpha$. Liouville's Theorem on approximation is one of the first along these lines. This is Theorem 1A of Chapter V.1 in [Sch80].

**Theorem 1.6** (Liouville)**.** *Suppose that $\alpha \in \mathbb{R}$ is algebraic* of degree $n$. Then there exists a constant $C(\alpha) > 0$ so that for any primitive pair $(p,q) \in \mathbb{Z}^2$ with $q > 0$,*

$$\left|\alpha - \frac{p}{q}\right| > \frac{C(\alpha)}{q^n}.$$

Liouville's Theorem indicates that a real algebraic $\alpha$ of degree $n$ cannot be approximated by infinitely many rationals according to the law $|\alpha - \frac{p}{q}| < \frac{1}{q^{n+\varepsilon}}$ when $\varepsilon$ is any positive real number. However, Liouville's Theorem can be radically improved, as Roth showed in [Rot55].

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*Of course, Liouville's Theorem also holds for algebraic $\alpha \notin \mathbb{R}$, but we will generally not focus on this case in this chapter.

**Theorem 1.7** (Roth)**.** *Suppose that $\alpha \in \mathbb{R}$ is algebraic and $\varepsilon > 0$. Then there are only finitely many primitive pairs $(p, q) \in \mathbb{Z}^2$ with $q > 0$ so that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\varepsilon}}. \tag{1.2}$$

In summary, we note that if $\alpha$ is rational, then for any $C > 1$, there are infinitely many rationals $\frac{p}{q}$ which satisfy $|\alpha - \frac{p}{q}| < \frac{C}{q}$. However, there are only finitely many rationals which satisfy $|\alpha - \frac{p}{q}| < \frac{C}{q^{1+\varepsilon}}$. If $\alpha$ is an irrational algebraic number, then for any $C > 1$, there are infinitely many rationals $\frac{p}{q}$ which satisfy $|\alpha - \frac{p}{q}| < \frac{C}{q^2}$. However, there are only finitely many rationals $\frac{p}{q}$ which satisfy $|\alpha - \frac{p}{q}| < \frac{C}{q^{2+\varepsilon}}$.

Based on this pattern, one might expect that if $\alpha$ is transcendental, then there would be infinitely many solutions to some inequality like $|\alpha - \frac{p}{q}| < \frac{C}{q^3}$. However, a law like this is dramatically false. Theorem 32 in Khinchin's [Khi64] implies that

**Theorem 1.8** (Khinchin)**.** *For almost all real numbers $\alpha$, there are only finitely many primitive pairs $(p, q) \in \mathbb{Z}^2$ with $q > 0$ so that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2(\log q)^{1+\varepsilon}}.$$

So while the theory of rational approximation is currently able to completely distinguish between rational and irrational numbers, we do not currently have a way of completely distinguishing between algebraic and transcendental numbers.

## 1.2 (In)effectiveness of Approximation Results

Note that the proof of Corollary 1.3 is *effective* for rational $\alpha$ in the sense that it gives a method for finding the finitely many primitive pairs $(p, q)$ which can satisfy (1.1): the size of $q$ can be bounded in terms of the denominator of $\alpha$ and for each $q$, one can easily bound the size of $p$ in terms of $q$ and $\alpha$. Hence, there are not only finitely many primitive pairs $(p, q)$ satisfying (1.1), they all live in a finite search space which can be easily expressed in terms of $\alpha$. Liouville's Theorem is also effective in the sense that $C(\alpha)$ can be expressed in terms of $\alpha$ (see Theorem 6.1 in [Eve21]). However, Roth's Theorem is highly ineffective: the proof gives no method of finding the finitely many solutions to (1.2).

One of the main goals of modern Diophantine approximation is then to find effective improvements of Liouville's Theorem. Fel'dman manages to improve Liouville's general theorem in an effective way.

**Theorem 1.9** (Fel'dman). *Let $\alpha$ be algebraic of degree $n \geqslant 3$. Then there exist effectively computable constants $C(\alpha) > 0$ and $a(\alpha) > 0$ so that for any primitive pair $(p, q) \in \mathbb{Z}^2$ with $q > 0$,*

$$\left|\alpha - \frac{p}{q}\right| > \frac{C(\alpha)}{q^{n-a(\alpha)}}.$$

In [Fel71], Fel'dman notes that he plans to estimate the sizes of $C(\alpha)$ and $a(\alpha)$ in a future paper, but no such paper later appears, nor do other authors appear to take up this task. As it stands then, we shall have to be satisfied with knowing that $C(\alpha)$ and $a(\alpha)$ can be computed with enough patience.

That said, even if it is quite difficult to improve upon Fel'dman's result about all rational approximations to $\alpha$ in an effective way, there are effective methods which can improve upon Fel'dman's result in certain settings. In particular, solutions to Thue equations provide a natural setting where the quality of the corresponding rational approximations can be effectively measured.

## 1.3 Thue Equations

### *1.3.1 Connecting Thue Equations to Diophantine Approximation*

Of course, before we can see how solutions to Thue equations give rise to good rational approximations of algebraic numbers which can be effectively described, we must first discuss what a Thue equation is. Our first definition will help us concisely state the hypotheses we regularly assume:

**Definition 1.10.** An *integral binary form* is a homogeneous polynomial in two variables with integer coefficients.

Next, we define what a Thue equation is.

**Definition 1.11.** Let $F(x, y) \in \mathbb{Z}[x, y]$ be an irreducible integral binary form of degree $n \geqslant 3$ and let $h \in \mathbb{Z}$. Then the equation

$$F(x, y) = h \tag{1.3}$$

is known as a *Thue equation*.

A major number-theoretic goal is to find all of the solutions to (1.3) in integers. In this document, whenever we refer to a solution to a Thue equation, we specifically mean a pair of *integers* $(p,q)$ for which $F(p,q)=h$. The solutions to Thue equations tend to produce good rational approximations to the roots of the one-variable polynomials $F(x,1)$ and $F(1,y)$. For instance, by dehomogenizing, consider that a solution $(p,q)\in\mathbb{Z}^2$ to $F(x,y)=h$ with $q\neq 0$ yields

$$F\left(\frac{p}{q},1\right)=\frac{h}{q^n}.$$

Writing $f(x)=F(x,1)$ and factoring $f(x)$ over $\mathbb{C}[x]$ as

$$f(x)=b\prod_{i=1}^{n}(x-\alpha_i),$$

we note that this implies that

$$\frac{|h|}{|q|^n}=|b|\prod_{i=1}^{n}\left|\frac{p}{q}-\alpha_i\right|$$

and hence, for sufficiently large $q$, $\frac{p}{q}$ gives a good rational approximation of some root $\alpha_i$ of $f(x)$. Likewise, by reversing the roles of $x$ and $y$, one can see that for sufficiently large $p$, $\frac{q}{p}$ gives a good rational approximation of some root $\alpha_i^*$ of $F(1,y)$.

The rational approximations to a root $\alpha$ of $F(x,1)$ which arise from solutions to Thue equations tend to satisfy much stronger approximation laws than a generic rational approximation to $\alpha$. Moreover, those laws tend to include effective constants, like the *height* of the polynomial $f(x)$. The following definition is the only definition of height that we will use throughout this paper. However, it is worth observing that this height is the naïve height of a polynomial and this is related to, but not identical to, other heights such as the Weil height.

**Definition 1.12.** Given a polynomial $g(x_1,\ldots,x_n)\in\mathbb{C}[x_1,\ldots,x_n]$ of the form

$$g(x_1,\ldots,x_n)=\sum_{i_1,\ldots,i_n}a_{i_1,\ldots,i_n}x_1^{i_1}\cdots x_n^{i_n},$$

the *height* of $g(x_1,\ldots,x_n)$ is

$$H(g)=\max_{i_1,\ldots,i_n}|a_{i_1,\ldots,i_n}|.$$

The height of a polynomial with integer coefficients gives a sense of its complexity: the larger the height, the more bits of information required to represent the polynomial. The height of a single-variable polynomial $f(x)$ is a useful, effectively computable constant often found in results about approximations of the roots of $f(x)$. Consider this result from Bombieri and Schmidt in [BS87] on the rational approximations one can obtain from solutions to Thue equations:

**Proposition 1.13** (Bombieri and Schmidt)**.** *Suppose that* $(p, q) \in \mathbb{Z}^2$ *is a solution to the Thue equation* $F(x, y) = h$*, where* $n \geqslant 3$ *denotes the degree of* $F$ *and* $p, q \neq 0$*. Then there exists a root* $\alpha$ *of* $F(x, 1)$ *with*

$$\min\left(1, \left|\alpha - \frac{p}{q}\right|\right) \leqslant \frac{((2n+2)H(F))^n\, h}{|q|^n}. \tag{1.4}$$

*By symmetry, there exists a root* $\beta$ *of* $F(1, y)$ *with*

$$\min\left(1, \left|\beta - \frac{q}{p}\right|\right) \leqslant \frac{((2n+2)H(F))^n\, h}{|p|^n}. \tag{1.5}$$

Note that the exponents on $|q|$ and $|p|$ in this theorem are far better than we would expect from something like Dirichlet's Theorem. In fact, Roth's Theorem guarantees that there can be only finitely many rational numbers satisfying inequalities (1.4) and (1.5)[†] and hence, finitely many integer-pair solutions to the Thue equation $F(x, y) = h$.[‡]

It was Mahler's realization in [Mah33] that inequalities like (1.4) can actually provide bounds on the *number* of solutions to the Thue equation $F(x, y) = h$. Mahler did not give any kind of method for finding the solutions, but instead found bounds for the number of good rational approximations to the roots of $F(x, 1)$ and $F(1, y)$ and translated those into bounds on the number of solutions to the Thue inequality $F(x, y) = 1$. From there Mahler was able to estimate the number of solutions to $F(x, y) = h$. This exploration instigated one of the major projects in the study of Thue equations: finding good bounds on the number of solutions to $F(x, y) = h$, which is the topic we analyze in chapter 2.

Given the large size of the constant factor $((2n+2)H(F))^n h$, Proposition 1.13 only gives tight bounds on the quality of the approximation (and hence, the number of solutions to $F(x, y) = h$) when $|q|$ is large. For smaller values of $q$, Mueller and Schmidt in [MS88] are able to improve the size of the constant at the cost of reducing the exponent on $|q|$:

**Proposition 1.14** (Mueller and Schmidt)**.** *Suppose that* $(p, q) \in \mathbb{Z}^2$ *is a solution to the Thue equation* $F(x, y) = h$ *where* $n \geqslant 3$ *denotes the degree of* $F(x, y)$*,* $s + 1$ *denotes the*

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[†]Roth's Theorem only guarantees that there are finitely many rationals with $|\alpha - p/q| < ((2n+2)H(F))^n h/|q|^n$ and in fact, it is not hard to see that there are infinitely many rationals satisfying (1.4). However, those infinitely many rationals have bounded $|q|$ and hence unbounded $|p|$. Only finitely many of those rationals can then satisfy (1.5).

[‡]Both Roth's Theorem and Bombieri and Schmidt's lemma came rather later than Thue's realization that the equations bearing his name have only finitely many solutions, but they provide a quick proof here so we will not worry overmuch about this.

*number of nonzero coefficients of $F(x, y)$, and $q \neq 0$ with $|q|$ larger than an explicit constant depending only on $F$ and $h$.*[§] *Then there is a set $S$ of roots of $F(x, 1)$ and a set $S^*$ of roots of $F(1, y)$, both with cardinalities less than or equal to $6s + 4$ so that either*

$$\left|\alpha - \frac{p}{q}\right| < \frac{K(F, h)}{|q|^{n/s}}$$

*for some $\alpha \in S$ or*

$$\left|\alpha^* - \frac{q}{p}\right| < \frac{K(F, h)}{|p|^{n/s}}$$

*for some $\alpha^* \in S^*$.*

In Mueller and Schmidt's proposition, the value of $K(F, h)$ is explicit and it is meaningfully smaller than the constant $((2n + 2)H(F))^n h$ that appears in Bombieri and Schmidt's proposition. Moreover, $K(F, h)$ is a multiple of a negative power of $H(F)$, indicating that for polynomials with large height, solutions to the corresponding Thue equation must produce particularly good rational approximations of the roots of $F(x, 1)$ and $F(1, y)$.

The role of the parameter $s$ is not apparent, but it will be explained in chapter 2. However, it is worth noting that $s$ can be small while $n$ is large, so the Mueller and Schmidt's proposition still provides a good exponent on $|q|$ (or $|p|$, depending on which is the denominator) and it has a smaller constant than Bombieri and Schmidt's proposition.

### *1.3.2 New Results on Thue Equations*

In chapter 2, we will explore how this parameter $s$ impacts the number of solutions to Thue equations. In particular, we will improve bounds on the number of solutions to general Thue equations given by Mueller and Schmidt in [MS88], and Saradha and Sharma in [SS17]. The following theorem is our main asymptotic[¶] result. Before we state the result, however, we introduce some notation that will be useful throughout the remainder of this paper.

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[§]This constant is smaller than the constant needed to make Bombieri and Schmidt's result useful.

[¶]Our use of the word "asymptotic" does not always indicate that some parameter is going to infinity. Rather, we use the word "asymptotic" to refer to a bound where we only give the main term and we disregard error terms and constants.

*Notation* 1.15. For any set $X$ and functions $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$, the notation $f(x) \ll g(x)$ means that there exists a constant $C > 0$ so that $f(x) \leqslant Cg(x)$ for all $x \in X$. Sometimes, we will also write $f(x) = O(g(x))$ to mean $f(x) \ll g(x)$ or we may simply write $O(g(x))$ to refer to some function $f(x)$ which satisfies $f(x) \ll g(x)$. If $f(x)$ and $g(x)$ both depend on some parameter $n$, then the notation $f(x) \ll_n g(x)$ means that there exists a constant $C > 0$ which may depend on $n$ so that $f(x) \leqslant Cg(x)$ for all $x \in X$. In these cases, when using big-oh notation, we will write $O_n(g(x))$ to indicate that the implicit constant depends on $n$.

The following theorem makes use of the parameter $\Phi$, which will be defined at the beginning of Section 2.3. For now, it is only important to know that it satisfies $\log^3(s) \ll e^{\Phi} \ll s$.

**Theorem 1.16.** *Let $F(x, y)$ be an irreducible integral binary form of degree $n \geqslant 3$ with $s + 1$ nonzero coefficients and let $h$ be a positive integer. Then if $n > 4se^{2\Phi}$, the total number of primitive solutions to $|F(x, y)| \leqslant h$ satisfies*

$$N(F, h) \ll se^{\Phi} h^{2/n}.$$

We will also examine the case where $s = 2$ (in this case, $F(x, y)$ is called a trinomial because it is the sum of three nonzero terms) and improve explicit bounds on the number of solutions to the particular Thue equations $F(x, y) = \pm 1$. The following theorem is our main explicit result:

**Theorem 1.17.** *Let $F(x, y) = h_n x^n + h_k x^k y^{n-k} + h_0 y^n$ where $h_n, h_k, h_0, n, k \in \mathbb{Z}$ with $0 < k < n$. Suppose that $F(x, y)$ is irreducible over $\mathbb{Z}[x, y]$ and $n \geqslant 6$. Then there are at most $2v(n)z(n) + 8$ distinct integer pair solutions to the equation $|F(x, y)| = 1$ where*

$$v(n) = \begin{cases} 3 & \text{if } n \text{ is odd} \\ 4 & \text{if } n \text{ is even} \end{cases}$$

*and $z(n)$ is defined by the following table.*

| $n$ | 6 | 7 | 8 | 9 | 10–11 | 12–16 | 17–37 | 38–216 | $\geqslant 217$ |
|---|---|---|---|---|---|---|---|---|---|
| $z(n)$ | 15 | 12 | 11 | 9 | 8 | 7 | 6 | 5 | 4 |

The main method by which we are able to achieve these results is an improvement in a counting technique associated with The Gap Principle. The Gap Principle is not a single explicit result, but rather a series of related results that states that when two rational numbers approximate the same algebraic number, their denominators must be exponentially far apart. The counting technique improvement is Lemma 2.19. We apply this technique to the versions of The Gap Principle which we give towards the beginning of the proof of Lemma 2.23 and which we give as Theorem 2.42.

### 1.4 Root Distribution

Another feature to observe about Proposition 1.14 is that solutions to the Thue equation $F(x, y) = h$ provide good rational approximations to one of relatively few roots of $F(x, 1)$ or $F(1, y)$. There are $2n$ such roots, but solutions to $F(x, y) = h$ provide good rational approximations (in the sense of Proposition 1.14) to elements of some subset of those roots with size at most $12s + 8$. This gives us some information about how the roots of $F(x, 1)$ and $F(1, y)$ are distributed in the complex plane when $s$ is small relative to $n$: either

1. there are few roots which are able to be well-approximated by rational numbers or
2. the roots come in clusters so that a good rational approximation of one such root is a good approximation of the roots in the nearby cluster.

While we do not discuss the full generality needed to understand root clustering and equidistribution, we will explore the notion of root distribution by studying how near two roots of the same polynomial can get.

The study of polynomial root distribution originates in the broader undertaking to find solutions to equations. Of course, in antiquity, "equations" referred to "polynomial equations" and as we now know, "finding solutions" is often too much to ask. However, we can glean information about the roots of a polynomial from information about the coefficients, and that is the essence of the study of polynomial root distribution. Major questions of this field include:

**Question 1.18.** For a polynomial $f(x) \in \mathbb{R}[x]$, how many of its roots are real?

**Question 1.19.** For a polynomial $f(x) \in \mathbb{C}[x]$, is there an "easily computed" compact region of the plane in which its roots must live?

**Question 1.20.** For a polynomial $f(x) \in \mathbb{R}[x]$, how close together can its roots be?

Our goal for this discussion will be to understand polynomials with integer coefficients, but as is often the case in number theory, it is helpful to consider the fields $\mathbb{C}$ and $\mathbb{R}$.

### *1.4.1 Number of Real Roots*

We can address Question 1.18 without introducing any new terminology. Let $f(x) \in \mathbb{R}[x]$ be a polynomial of degree $n$. We begin by examining upper bounds on the number of real roots of $f(x)$.

Naïvely, $f(x)$ can have no more than $n$ real roots because $f(x)$ has no more than $n$ roots. However, this is not necessarily a good bound and in many cases, we can do better. The following lemma is a corollary of Descartes' rule of signs and is also given in [Sch87].

**Lemma 1.21.** *Suppose that $f(x) \in \mathbb{R}[x]$ is a polynomial with $s+1$ nonzero summands. Then $f(x)$ has no more than $2s+2$ real roots.*

This lemma gives some indication of why Proposition 1.14 is true. After all, solutions to the Thue equation $F(x, y) = h$ produce good rational approximations of the roots of $F(x, 1)$ and $F(1, y)$. However, rational numbers can only produce arbitrarily good approximations of real numbers, so we expect that solutions to $F(x, y) = h$ produce good rational approximations to only the roots of $F(x, 1)$ and $F(1, y)$ that lie near or on the real axis. Lemma 1.21 indicates that the number of such roots is controlled by $s$, so we would expect that solutions to $F(x, y) = h$ produce approximations to some number of roots of $F(x, 1)$ or $F(1, y)$ where that number is controlled by $s$.

On the other hand, finding lower bounds on the number of real roots of $f(x)$ is a much more difficult subject. It is difficult in general to detect if $f(x)$ has any real roots at all. Of course, if the degree of $f(x)$ is odd, then $f(x)$ has real roots, but if the degree of $f(x)$ is even, this becomes much more difficult. However, it is possible to do this for $f(x)$ with rational coefficients in an effective manner.

The polynomial $f(x)$ has a real root if and only if the number field $K := \mathbb{Q}[x] / (f(x))$ has an embedding into $\mathbb{R}$. Letting $Q(x_1, x_2, x_3, x_4, x_5) = \sum_{i=1}^{5} x_i^2$, Meyer's Theorem combined with the Hasse-Minkowski Theorem (see [Ser73]) combine to indicate that there exists an embedding $K \to \mathbb{R}$ if and only if $Q$ does not nontrivially represent 0 over $K$. However, Raghavan in [Rag75] gives an effective algorithm for detecting whether or not $Q$ represents 0 nontrivially over $K$ and so we have an effective method for checking whether or not $f(x)$ has a real root.

### *1.4.2 Roots in a Compact Region*

Question 1.19 is also straightforward to address.

**Proposition 1.22.** *Let $f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{C}[x]$ where $a_n, a_0 \neq 0$. If $\alpha \neq 0$ is a root of $f(x)$, then*

$$\frac{1}{1 + H(f)/|a_0|} < |\alpha| < 1 + \frac{H(f)}{|a_n|}.$$

*Proof.* Note that since $\alpha$ is a root of $f(x)$, we can write

$$-a_n\alpha^n = \sum_{j=0}^{n-1} a_j\alpha^j.$$

Taking absolute values on both sides and applying the triangle inequality yields

$$|a_n||\alpha|^n \leqslant \sum_{j=0}^{n-1} |a_j||\alpha|^j \leqslant H(f)\sum_{j=0}^{n-1} |\alpha|^j = \frac{H(f)(|\alpha|^n - 1)}{|\alpha| - 1} < \frac{H(f)|\alpha|^n}{|\alpha| - 1}.$$

Dividing

$$|a_n||\alpha|^n < \frac{H(f)|\alpha|^n}{|\alpha| - 1}$$

by $|\alpha|^n$ and rearranging yields the desired inequality

$$|\alpha| < 1 + \frac{H(f)}{|a_n|}.$$

To get the lower bound on $\alpha$, observe that the reciprocal polynomial $f^R(x) = x^n f(1/x)$ has height $H(f^R) = H(f)$ and has $1/\alpha$ as a root. By the first part of the proof then,

$$\frac{1}{|\alpha|} < 1 + \frac{H(f^R)}{|a_0|}$$

and taking reciprocals yields the desired lower bound on $|\alpha|$. □

Moreover, it is not difficult to see that these are good bounds on the roots of $f(x)$. For example, for $t \geqslant 1$, the family of polynomials $f_t(x) = (x^{n-1} - 1)(x - t)$ has height $t$, leading coefficient 1, and a root located at $x = t$. Proposition 1.22 indicates that the roots of $f_t(x)$ must live in the circle with radius $1 + t$ centered at 0. As $t$ tends to infinity then, the ratio of the size of the largest root of $f_t(x)$ to the upper bound on root size given by Proposition 1.19 tends to 1, so the bound is sharp. A similar construction can be given to show that the lower bound is sharp.

### 1.4.3 Separation of Polynomial Roots

Question 1.20 is trickier to address and we give an overview to this question here before covering it in more depth in chapter 3. We want to understand the question more precisely, however, so we will introduce the following quantities.

**Definition 1.23.** Given a polynomial $f(x_1, \ldots, x_m) \in \mathbb{C}[x_1, \ldots, x_m]$, the *Mahler measure* of $f(x_1, \ldots, x_m)$ is the quantity

$$M(f) = \exp\left(\int_0^1 \int_0^1 \cdots \int_0^1 \log|f(e^{2\pi i t_1}, \ldots, e^{2\pi i t_m})|\, dt_1\, dt_2 \cdots dt_m\right).$$

**Definition 1.24.** Given a polynomial $f(x) \in \mathbb{C}[x]$ of degree $n$, with roots $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, and with leading coefficient $b$, the *discriminant* of $f(x)$ is the quantity

$$\Delta_f = b^{2n-2} \prod_{1 \leqslant i < j \leqslant n} (\alpha_i - \alpha_j)^2.$$

**Definition 1.25.** Given a polynomial $f(x) \in \mathbb{C}[x]$ with roots $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, the *separation* of $f(x)$ is the quantity

$$\operatorname{sep}(f) = \min_{\alpha_i \neq \alpha_j} |\alpha_i - \alpha_j|.$$

Before giving any answers to Question 1.20, we will address some of the connections between the roots, the height, the Mahler measure, and the discriminant of a polynomial $f(x) \in \mathbb{C}[x]$.

**Lemma 1.26.** *For a single-variable polynomial $f(x) \in \mathbb{C}[x]$ with roots $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and leading coefficient $b$, we have*

$$M(f) = |b| \prod_{j=1}^{n} \max(1, |\alpha_j|).$$

This is Proposition 1.6.5 in [BG06]. This result is a corollary of the more general Jensen's formula (see [Rud74]) and it gives an important connection between the Mahler measure of a polynomial and the polynomial's roots. Moreover, the Mahler measure satisfies the following dual inequalities, stated as Lemma 1.6.7 in [BG06]:

**Lemma 1.27.** *Suppose that $f(x) \in \mathbb{C}[x]$ has degree $n$ and $s + 1$ nonzero summands. Then*

$$\binom{n}{\lfloor n/2 \rfloor}^{-1} H(f) \leqslant M(f) \leqslant H(f)\sqrt{s+1} \leqslant H(f)\sqrt{n+1}. \tag{1.6}$$

This lemma shows that, up to a constant factor depending on the degree of $f(x)$, the Mahler measure and height of $f(x)$ are the same. This is what makes the Mahler measure a key quantity in number theory: it connects the known coefficients to the unknown roots.

The discriminant of $f(x)$ is important for the same reason. Its definition is given in terms of the roots of $f(x)$, but it has a key connection to the coefficients of $f(x)$ as well. Note that the definition of the discriminant given in 1.24 has $\Delta_f/b^{2n-2}$ given as a polynomial in $\alpha_1, \ldots, \alpha_n$. Moreover, $\Delta_f/b^{2n-2}$ is a symmetric polynomial in $\alpha_1, \ldots, \alpha_n$ and hence, by Milne's proof of the Fundamental Theorem of Symmetric Polynomials (see Theorem 2.2 in [Mil20]), $\Delta_f/b^{2n-2}$ can be expressed as a polynomial of degree equal to $2n-2$ in the elementary symmetric polynomials, $e_1(\alpha_1, \ldots, \alpha_n), \ldots, e_n(\alpha_1, \ldots, \alpha_n)$. However, if the leading coefficient of $f(x)$ is $b$, then the coefficients of $f(x)$ are equal to (up to sign) $be_1(\alpha_1, \ldots, \alpha_n), \ldots, be_n(\alpha_1, \ldots, \alpha_n)$. Letting $b_i$ denote $be_i(\alpha_1, \ldots, \alpha_n)$, we find that $\Delta_f/b^{2n-2}$ can be expressed as a polynomial of degree $2n-2$ in the variables $b_i/b$. Hence, $\Delta_f$ can be expressed as a polynomial of degree $2n-2$ in the variables $b_i$, i.e. the coefficients of $f(x)$.

While it is helpful to know that the discriminant can be defined in terms of the roots or in terms of the coefficients of $f(x)$, the previous paragraph gives no indication of how the discriminant can be expressed in terms of the coefficients. A tool known as the *resultant* helps with that particular concern:

**Definition 1.28.** Let $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j$ be polynomials with complex coefficients with $a_n, b_m \neq 0$. Then the *resultant* of $f(x)$ and $g(x)$ is the determinant of the $(n+m) \times (n+m)$ Sylvester matrix

$$\operatorname{Res}(f, g) := \det \begin{pmatrix} a_n & 0 & \cdots & 0 & b_m & 0 & \cdots & 0 \\ a_{n-1} & a_n & \ddots & \vdots & b_{m-1} & b_m & \ddots & \vdots \\ a_{n-2} & a_{n-1} & \ddots & 0 & b_{m-2} & b_{m-1} & \ddots & 0 \\ \vdots & \vdots & \ddots & a_n & \vdots & \vdots & \ddots & b_m \\ a_0 & a_1 & \cdots & \vdots & b_0 & b_1 & \cdots & \vdots \\ 0 & a_0 & \ddots & \vdots & 0 & b_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_1 & \vdots & \vdots & \ddots & b_1 \\ 0 & 0 & \cdots & a_0 & 0 & 0 & \cdots & b_0 \end{pmatrix}.$$

It turns out that the discriminant of a polynomial $f(x)$ is closely related to the resultant of $f(x)$ and its derivative $f'(x)$. The following lemma can be found after the proof of Proposition 2.34 in [Mil20].

**Lemma 1.29.** *If $f(x) \in \mathbb{C}[x]$ is a polynomial with degree $n$ and leading coefficient $b \neq 0$, then*

$$\Delta_f = (-1)^{n(n-1)/2} \cdot \frac{\operatorname{Res}(f, f')}{b}.$$

This gives a straightforward way to compute the discriminant as a polynomial in the coefficients of $f(x)$. Before we begin to address Question 1.20, we will look at an important class of examples.

**Example 1.30.** Consider the family of polynomials $Q_{n,r}(x) = x^n - r$ for real $r \geqslant 1$ and integer $n \geqslant 2$. Let $\zeta_n$ denote a primitive $n$th root of unity. Then we have:

$$\begin{aligned}
H(Q_{n,r}) &= \max(1, r) = r \\
M(Q_{n,r}) &= \prod_{j=1}^{n} \max(1, |\zeta_n^j r^{1/n}|) = r \\
\Delta_{Q_{n,r}} &= (-1)^{n-1} n^n r^{n-1} \\
\operatorname{sep}(Q_{n,r}) &= r^{1/n} |e^{2\pi i/n} - 1| \\
&= r^{1/n} \sqrt{(1 - \cos(2\pi/n))^2 + \sin^2(2\pi/n)} \\
&= r^{1/n} \sqrt{2 - 2\cos(2\pi/n)} \\
&= r^{1/n} \sqrt{2 - 2(\cos^2(\pi/n) - \sin^2(\pi/n))} \\
&= 2r^{1/n} \sin(\pi/n)
\end{aligned}$$

Importantly, note that for this class of examples,

$$H(Q_{n,r} = M(Q_{n,r})$$

and

$$\operatorname{sep}(Q_{n,r}) = 2\sin\left(\frac{\pi}{n}\right) \cdot M(Q_{n,r})^{1/n}.$$

A key relation which helps to address Question 1.20 was shown by Mahler in [Mah64]:

**Theorem 1.31** (Mahler)**.** *Let $f(x) \in \mathbb{C}[x]$ have degree $n \geqslant 2$. Then*

$$\operatorname{sep}(f) > \frac{\sqrt{3|\Delta_f|}}{n^{(n+2)/2} M(f)^{n-1}}.$$

In particular, if we suppose that $f(x)$ is separable, then $|\Delta_f| > 0$ by Definition 1.24. Furthermore, $|\Delta_f|$ is a polynomial in the coefficients of $f(x)$. So if $f(x) \in \mathbb{Z}[x]$—the primary case we are interested in—then $|\Delta_f|$ must be a positive integer. Hence, we must have $|\Delta_f| \geqslant 1$. As a result, we have the following corollary.

**Corollary 1.32** (Mahler)**.** *Suppose that $f(x) \in \mathbb{Z}[x]$ is separable of degree $n \geqslant 2$. Then*

$$\operatorname{sep}(f) > \frac{\sqrt{3}}{n^{(n+2)/2} M(f)^{n-1}}.$$

This corollary is the standard to which other theorems on polynomial root separation are often compared. Philosophically, it indicates that the roots of polynomials with integer coefficients repel each other to some extent.

### *1.4.4 New Results on Polynomial Root Separation*

In Chapter 3, we reverse the question that Mahler answers with Theorem 1.31 and we ask about upper bounds for $\operatorname{sep}(f)$ in terms of $M(f)$.

Naïvely, Lemma 1.22 gives an upper bound on separation in terms of the height of the polynomial, which can then be translated into a bound on separation in terms of the Mahler measure. Lemma 1.22 indicates that any two roots of $f(x) \in \mathbb{C}[x]$ of degree $n$ must satisfy $|\alpha - \beta| < 2 + 2H(f)$ and by Lemma 1.27, we find that

$$\operatorname{sep}(f) < 2 + 2\binom{n}{\lfloor n/2 \rfloor} M(f). \tag{1.7}$$

However, this is a crude estimate and can be dramatically improved, as we will show in Proposition 3.4. We conjecture that this estimate can be improved in the following way:

**Conjecture 1.33.** *Suppose $f(x) \in \mathbb{R}[x]$ is monic and separable of degree $n \geqslant 2$. If $f(x)$ has any real roots, then*

$$\operatorname{sep}(f) \ll_n M(f)^{1/(n-1)}.$$

*If $f(x)$ has only nonreal roots, then*

$$\operatorname{sep}(f) \ll_n M(f)^{1/n}.$$

We support this conjecture with data, families of examples, and the following theorem.

**Theorem 1.34.** *Let $f(x) \in \mathbb{R}[x]$ be monic and separable with* $\deg(f) = n \geqslant 2$ *and suppose that any of the following conditions is met.*

1. $\deg(f) = 2$.

2. $\deg(f) = 3$.

3. $\deg(f) = 4$ *and $f(x)$ has no real roots.*

4. *Every root of $f(x)$ is real.*

*Then if $f(x)$ has any real roots,*

$$\operatorname{sep}(f) \ll_n M(f)^{1/(n-1)}.$$

*If $f(x)$ has only nonreal roots, then*

$$\operatorname{sep}(f) \ll_n M(f)^{1/n}.$$

We prove this theorem in pieces and we consider each condition separately. We mainly prove this by analyzing the geometry of possible root locations in the complex plane.

# CHAPTER 2

# THUE EQUATIONS

## 2.1 Introduction

Axel Thue, in [Thu09], showed that when $h$ is an integer and when $F(x, y)$ is an irreducible integral binary form (recall Definition 1.10) with degree $n \geqslant 3$, the equation

$$F(x, y) = h$$

has only finitely many integer-pair solutions. We have previously stated this equation as equation (1.3) and we will continue to use this number to refer to it during this chapter. Recall that we will only use the word "solution" to refer to integer-pair solutions.

It is worth noting first that the each of the hypotheses is necessary. The famous Pell equation $x^2 - dy^2 = 1$ has infinitely many integer solutions and it meets every hypothesis except for $\deg(F) \geqslant 3$. If $F(x, y)$ is not required to be irreducible, then it may have a linear factor, say $mx - ny$. But then any multiple of the pair $(n, m)$ will satisfy $F(x, y) = 0$ and this will give infinitely many solutions to $F(x, y) = 0$. If $F(x, y)$ is not required to be homogeneous, then again we can acquire infinitely many solutions to $F(x, y) = 0$ as, for example, the equation $x^6 + y^3 = 0$ has infinitely many solutions of the form $(n, -n^2)$.

As a consequence of Thue's result, the inequality

$$|F(x, y)| \leqslant h \tag{2.1}$$

also has finitely many integer-pair solutions. This inequality—called *Thue's Inequality*—can often be easier to work with because it treats the solutions to the equations

$$|F(x, y)| = 1, \ldots, |F(x, y)| = h$$

in aggregate so we deal primarily with Thue's inequality in this paper.

Several natural questions arise from Thue's results such as

1. How many solutions are there to (2.1)?

2. How large are solutions to (2.1)?

3. On which features of $F$ and $h$ do the solutions to (2.1) depend?

This chapter largely handles the first question, though of course the second and third questions are related. In particular, the number of nonzero summands of $F(x, y)$ significantly impacts the number of solutions to (2.1).

The rough reasoning for this is as follows: a solution $(p, q)$ to (2.1) gives a good rational approximation $p/q$ to a root of $f(X) := F(X, 1)$ as we saw previously in Proposition 1.13. The only roots of $f(X)$ which ought to allow good rational approximations are the real roots of $f(X)$. It is the number of nonzero coefficients of $f(X)$ that controls the number of real roots of $f(X)$, as we previously saw in Lemma 1.21, so we expect the number of nonzero summands of $f(X)$ to play a role in bounding the number of solutions to (2.1).

Moreover, this connection between solutions to (2.1) and rational approximations to roots of $f(X)$ gives us reason to initially count only the solutions $(p, q)$ with $\gcd(p, q) = 1$: pairs $(p, q)$ with $\gcd(p, q) = 1$ and $q > 0$ are in bijection with rational numbers $p/q$. This motivates the following definition.

**Definition 2.1.** A pair $(p, q) \in \mathbb{Z}^2$ is said to be *primitive* if $\gcd(p, q) = 1$.

Once we have a bound on the number of primitive solutions to (2.1), we can often use these to bound the total number of solutions to (2.1) with partial summation techniques (see the discussion after the statement of Proposition 3 in [MS88], for instance).

Our first theorem regards asymptotic bounds on the number of primitive solutions to (2.1), so take a moment to review the definition of the $\ll$ symbol from Notation 1.15. Additionally, we introduce the following piece of notation:

*Notation* 2.2. Let $N(F, h)$ denote the total number of primitive integer-pair solutions to (2.1).

Mueller and Schmidt in [MS88] prove that

**Theorem 2.3** (Mueller and Schmidt)**.** *Let $F(x, y)$ be an irreducible integral binary form of degree $n \geqslant 3$ with $s + 1$ nonzero coefficients and let $h$ be a positive integer. The number of primitive integer solutions of the inequality $|F(x, y)| \leqslant h$ satisfies*

$$N(F, h) \ll s^2 h^{2/n}(1 + \log h^{1/n}). \tag{2.2}$$

*Moreover, if $n \geqslant s \log^3 s$, then*

$$N(F, h) \ll s^2 h^{2/n}.$$

In the same paper, Mueller and Schmidt conjecture that the exponent on $s$ can be improved. We later explain the heuristics behind this conjecture and state their conjecture as Conjecture 2.5. Our theorem (which is stated first as Theorem 1.16, but reprinted here for convenience) improves the exponent on $s$ at the cost of a stronger assumption on $n$.

The parameter $\Phi$ will be defined at the beginning of Section 2.3 and it satisfies $\log^3(s) \ll e^{\Phi} \ll s$.

**Theorem.** *Let $F(x, y)$ be an irreducible integral binary form of degree $n \geqslant 3$ with $s + 1$ nonzero coefficients and let $h$ be a positive integer. Then if $n > 4se^{2\Phi}$, the total number of primitive solutions to $|F(x, y)| \leqslant h$ satisfies*

$$N(F, h) \ll se^{\Phi} h^{2/n}. \tag{2.3}$$

Since the number of nonzero summands plays such an important role, we have names for polynomials which meet specific values of $s$.

**Definition 2.4.** If a polynomial has exactly two nonzero summands, it is called a *binomial*; if it has exactly three nonzero summands, it is called a *trinomial*; and if it has exactly four nonzero summands, it is called a *tetranomial*.

Authors such as Bennett [Ben01], Evertse [Eve82], Grundman and Wisniewski [GW13], Hyyrö [Hyy64], Mueller [Mue87], Mueller and Schmidt [MS87], and Thomas [Tho00] have examined binomial, trinomial, and tetranomial Thue equations in hopes to get a better handle on how the number of nonzero summands of $F(x, y)$ affect the number of solutions to Thue equations.

In particular, Bennett's result that there is at most one solution in positive integers to $F(x, y) = 1$ for binomial $F(x, y)$ is worth mentioning because that is the best possible result: the infinite family of binomial Thue equations

$$(a + 1)x^n - ay^n = 1$$

always has a solution at $x = 1$, $y = 1$.

In this chapter, we will also improve explicit bounds for the number of solutions to the Thue equation

$$|F(x, y)| = 1 \tag{2.4}$$

in the particular case that $F(x, y)$ is a trinomial. The case where $h = \pm 1$ is foundational to the study of Thue equations as many bounds for general $h$ can be derived from knowing the number of solutions to $|F(x, y)| = 1$. We will discuss this in more detail at the beginning of Section 2.2.

In the setting where $F(x, y)$ is a trinomial, Thomas showed in [Tho00] that there are no more than $2v(n)w(n) + 8$ distinct integer pair solutions to $|F(x, y)| = 1$ when $F(x, y) \in \mathbb{Z}[x, y]$ is a trinomial irreducible binary form of degree $n \geqslant 3$ where

$$v(n) = \begin{cases} 3 & \text{if } n \text{ is odd} \\ 4 & \text{if } n \text{ is even} \end{cases}$$

and $w(n)$ is defined by the following table.

| $n$ | 5[1] | 6 | 7 | 8 | 9 | 10–11 | 12–16 | 17–37 | $\geqslant 38$ |
|---|---|---|---|---|---|---|---|---|---|
| $w(n)$ | 27[1] | 16 | 13 | 11 | 9 | 8 | 7 | 6 | 5 |

[1] There is an error in the proof of Lemma 4.1 in [Tho00]: it is claimed that $\frac{b^t-1}{b-1} < b^t$ which is not the case for the choice of $b = 1.5$ when $n = 5$. Tracing this error through to its conclusion, the author believes that this is not correctable and that Thomas' work does not yield a result when $n = 5$.

We are able to improve the bounds that Thomas provides and we have the following theorem (stated first as Theorem 1.17, but reprinted here for convenience).

**Theorem.** *Let* $F(x, y) = h_n x^n + h_k x^k y^{n-k} + h_0 y^n$ *where* $h_n, h_k, h_0, n, k \in \mathbb{Z}$ *with* $0 < k < n$. *Suppose that* $F(x, y)$ *is irreducible over* $\mathbb{Z}[x, y]$ *and* $n \geqslant 6$. *Then there are at most* $2v(n)z(n) + 8$ *distinct integer pair solutions to the equation* $|F(x, y)| = 1$ *where*

$$v(n) = \begin{cases} 3 & \textit{if } n \textit{ is odd} \\ 4 & \textit{if } n \textit{ is even} \end{cases}$$

*and* $z(n)$ *is defined by the following table.*

| $n$ | 6 | 7 | 8 | 9 | 10–11 | 12–16 | 17–37 | 38–216 | $\geqslant 217$ |
|---|---|---|---|---|---|---|---|---|---|
| $z(n)$ | 15 | 12 | 11 | 9 | 8 | 7 | 6 | 5 | 4 |

Both of our main results are primarily derived from an improvement in efficiency to a counting technique associated with the gap principle (see Lemma 2.19).

## 2.2 Conjectures and Heuristics

Baker in [Bak67] showed that integer solutions to (2.1) all satisfy $\max(|x|, |y|) < Ce^{(\log h)^\kappa}$ where $\kappa$ is any real number larger than the degree of $F$ and $C$ is an effectively computable constant depending only on $F$ and $\kappa$. This result provides some insight to the number of solutions to (2.1) in that it gives an upper bound for the number of solutions. Explicitly, the total number of integer pairs $(x, y)$ satisfying

$$\max(|x|, |y|) < Ce^{(\log h)^\kappa}$$

is

$$(2\lfloor Ce^{(\log h)^\kappa} \rfloor + 1)^2.$$

Then the number of integer pair solutions to (2.1) must also be no larger than

$$(2\lfloor Ce^{(\log h)^\kappa} \rfloor + 1)^2,$$

though this upper bound is not sharp.

In particular, we can see that the dependence of the upper bound on $h$ can be much improved by considering the geometry of the situation. The set

$$S(F, h) := \{(x, y) \in \mathbb{R}^2 : |F(x, y)| \leqslant h\}$$

is a star body of finite volume (the latter claim is true, though non-obvious) and intuition from the geometry of numbers indicates that the number of integer pair solutions to (2.1)—which correspond exactly to the integer lattice points inside $S(F, h)$—should be approximately equal to $\text{vol}(S(F, h))$. But if $F$ has degree $n$, then

$$\begin{aligned}
\text{vol}(S(F, h)) &= \text{vol}\{(x, y) \in \mathbb{R}^2 : |F(x, y)| \leqslant h\} \\
&= \text{vol}\left\{(x, y) \in \mathbb{R}^2 : \left|F\left(\frac{x}{h^{1/n}}, \frac{y}{h^{1/n}}\right)\right| \leqslant 1\right\} \\
&= \text{vol}\{(h^{1/n}x, h^{1/n}y) \in \mathbb{R}^2 : |F(x, y)| \leqslant 1\} \\
&= h^{2/n}\,\text{vol}(S(F, 1)).
\end{aligned}$$

Based on this fact, we might guess that the number of solutions to to (2.1) is bounded above by a constant depending on $F$ times $h^{2/n}$. This turns out to be the case as Mahler in [Mah33] showed that the number of solutions to (2.1) is equal to

$$\text{vol}(S(F, 1))h^{2/n} + O_F(h^{1/(n-1)}).$$

Since this essentially gives the dependence on $h$ of the number of solutions to (2.1) when $h$ is large, it more or less remains to examine how many solutions there are to (2.4) and multiply that result by $h^{2/n}$.

From here, note that we have already observed in our discussion after Lemma 1.21 that solutions to (2.4) yield good rational approximations of relatively few roots of $F(x, 1)$

and $F(1, y)$. Specifically, solutions yield good rational approximations to one of no more than $8s + 12$ algebraic numbers in the sense of Proposition 1.14. If there are boundedly many good rational approximations per root (as seems plausible, though we have not argued for this) and the number of roots which solutions can approximate is no more than a constant times $s$, then we would expect that the number of solutions to (2.4) would be no more than a constant times $s$. Adding in dependence on $h$, we arrive at the following conjecture which Mueller and Schmidt made in [MS88].

**Conjecture 2.5** (Mueller and Schmidt)**.** *Let $F(x, y)$ be an irreducible integral binary form of degree $n \geqslant 3$ with $s + 1$ nonzero coefficients and let $h$ be a positive integer. The number of primitive solutions of $|F(x, y)| \leqslant h$ satisfies*

$$N(F, h) \ll sh^{2/n}.$$

This remains a conjecture at the moment, though there are quite a few results moving in the direction of this bound. Initial work was done by Schmidt in [Sch87] to show that the total number of solutions to (2.1) satisfies

$$N(F, h) \ll (ns)^{1/2} h^{2/n} (1 + \log h^{2/n}). \tag{2.5}$$

Mueller and Schmidt later eliminated the dependence of the bound on the degree $n$ and replaced it by a more suitable dependence on $s$. This is where Theorem 2.3, which is from [MS88], enters the picture.

Thunder improves the logarithmic factor of (2.2) in [Thu95] when $h$ is large relative to the discriminant of $F$ and Akhtari and Bengoechea in [AB20] have improved both the exponent on $s$ and the logarithmic factor when $h$ is small relative to the discriminant of $F$. Saradha and Sharma in [SS17] improve the exponent on $s$ without assuming any restrictions on $h$.

When $h$ is small compared to the coefficients of $F$, one might expect that neither $h$ nor the specific coefficients of $F$ play a serious role in the number of solutions to (2.1). To that end, letting $H(F)$ denote the height of $F(x, y)$ (recall Definition 1.12), Mueller and Schmidt in [MS88] gave the following conjecture:

**Conjecture 2.6** (Mueller and Schmidt)**.** *Let $F(x, y)$ be an irreducible integral binary form of degree $n \geqslant 3$ with $s + 1$ nonzero coefficients and let $h$ be a positive integer. If $\rho > 0$ and $h \leqslant H(F)^{1-\frac{s}{n}-\rho}$, then the number of primitive solutions to $|F(x, y)| \leqslant h$ is less than or equal to a function of only $s$ and $\rho$.*

Mueller in [Mue87] and Mueller and Schmidt in [MS87] have confirmed Conjecture 2.6 in the cases of $s = 1, 2$ while Thomas in [Tho00], and Grundman and Wisniewski in [GW13] provide evidence for this conjecture in the cases of $s = 2, 3$, but the conjecture remains open otherwise. Techniques in these cases are varied and often rely on the improved approximation results one can acquire by fixing a small value of $s$.

Alternatively, one could compare the size of $h$ to the discriminant of $F$ and in that case, Akhtari and Bengoechea in [AB20] show that

**Theorem 2.7** (Akhtari and Bengoechea)**.** *Let $F(x, y)$ be an irreducible integral binary form of degree $n \geqslant 3$ with $s + 1$ nonzero coefficients and let $h$ be a positive integer. If $F$ has discriminant $\Delta_F$ and*

$$0 < h < \frac{|\Delta_F|^{\frac{1}{8(n-1)}}}{(3n^{800\log^2 n})^{n/2}(ns)^{2s+n}},$$

*then the number of primitive solutions to the inequality $|F(x, y)| \leqslant h$ satisfies*

$$N(F, h) \ll s \log s \min\left(1, \frac{1}{\log n - \log s}\right).$$

## 2.3 Notation and Definitions

Throughout the chapter, we use the following notation.

Suppose that $F(x, y) = \sum_{i=0}^{s} a_i x^{n_i} y^{n-n_i} \in \mathbb{Z}[x, y]$ is an irreducible binary form of degree $n \geqslant 3$ with each $a_i \neq 0$ so that $F(x, y)$ has exactly $s + 1$ nonzero coefficients. Set $H = H(F)$ to be the height of $F$ and let $h$ be a positive integer. Following Akhtari and Bengoechea in [AB20], define

$$R = n^{800\log^2 n} = e^{800\log^3 n},$$
$$C = Rh(2H\sqrt{n(n+1)})^n.$$

Following Saradha and Sharma in [SS17], define

$$\Psi = \max_{0\leqslant i\leqslant s} \max\left(\sum_{w=0}^{i-1} \frac{1}{n_i - n_w}, \sum_{w=i+1}^{s} \frac{1}{n_w - n_i}\right),$$
$$\Phi = \max(\Psi, 3\log\log s).$$

Saradha and Sharma in [SS17] note that $e^{\Psi} \ll s$, so that $\log^3 s \ll e^{\Phi} \ll s$.

Again following Akhtari and Bengoechea in [AB20], select constants $a$ and $b$ so that $0 < a < b < 1$ and $\frac{\sqrt{2}\sqrt{3+a^2}}{1-b} < 3$. Then set

$$\begin{aligned}
\lambda &= \frac{\sqrt{2(n+a^2)}}{1-b},\\
Y_S &= \left((12e^{\Psi})^n R^{2s} h\right)^{\frac{1}{n-2s}},\\
Y_L &= (2C)^{\frac{1}{n-\lambda}}(4H\sqrt{n+1}e^{n/2})^{\lambda/((n-\lambda)a^2)},\\
K &= 2R(ns)^2(12e^{\Psi})^{n/s}h^{1/s}H^{\frac{1}{n}-\frac{1}{s}}.
\end{aligned}$$

For a pair $\mathbf{x} = (x, y) \in \mathbb{Z}^2$, define

$$\begin{aligned}
\langle \mathbf{x} \rangle &= \min(|x|, |y|),\\
|\mathbf{x}| &= \max(|x|, |y|).
\end{aligned}$$

Then we make the following definitions.

**Definition 2.8.** A pair $\mathbf{x} \in \mathbb{Z}^2$ is *small* if $\langle \mathbf{x} \rangle \leqslant Y_S$, *medium* if $Y_S \leqslant \langle \mathbf{x} \rangle$ and $|\mathbf{x}| < Y_L$, and *large* if $|\mathbf{x}| \geqslant Y_L$.

To count the solutions of each type, we use the following notation.

*Notation* 2.9. Let $N_L(F, h)$ denote the number of primitive large solutions to (2.1). Let $N_M(F, h)$ denote the number of primitive medium solutions to (2.1). Let $N_S(F, h)$ denote the number of primitive small solutions to (2.1).

Observe that the terms “small,” “medium,” and “large” are all dependent on $F(x, y)$ and $h$ and moreover, they are not necessarily disjoint categories.

The essential strategy for counting solutions to (2.1) is to find bounds separately for the numbers of primitive small, medium, and large solutions. Even though there is some overlap in the classification of large, medium, and small solutions (and hence, some overcounting of the number of solutions), the existing techniques for counting the different types of solutions are so disparate that it is difficult to count the overlap in any meaningful way.

## 2.4 Previous Results

Describing the best existing results is difficult because of the issue of “moving goalposts.” Authors often try to prove something along the lines of “the number of solutions to (2.1) is bounded by $f(n, s, h)$" for an appropriate function $f(n, s, h)$. To do this, they come up with an appropriate function $g(s)$ and note something like "if $n \leqslant g(s)$, then (2.5) immediately implies that the number of solutions is bounded by $f(n, s, h)$." Then the author will proceed to count small, medium, and large solutions under the assumption that $n > g(s)$. Hence, the "best counts" for small, medium, and large solutions often depend on the author's intended upper bound, $f(n, s, h)$ and the best counts often have the form "if $n > g(s)$, then the number of small/medium/large solutions is bounded $f(n, s, h)$." In particular, if we wish to prove something like Conjecture 2.5, then we first must eliminate the logarithmic factor from (2.5) and even then, we can only use (2.5) to show Conjecture 2.5 under the assumption that $n \ll s$.

What follows is a necessarily incomplete list of some of the best existing bounds for counts of (primitive) large, medium, and small solutions, where we make sure to be clear about what assumptions the authors use when it comes to the size of $n$ relative to $s$. Recall that $N_L(F, h)$, $N_M(F, h)$, and $N_S(F, h)$ denote the number of primitive large, medium, and small solutions to (2.1) respectively. For each type of solution, we make sure to include a result where the author merely assumes $n \gg s$.

### *2.4.1 Large Solutions*

Mueller and Schmidt in [MS88] show that

**Lemma 2.10** (Mueller and Schmidt)**.** *For all n,*

$$N_L(F, h) \ll s.$$

Of the three types of solutions, this is the smallest upper bound and also the one most closely aligned with Conjectures 2.5 and 2.6, so it has received little more attention than what is stated here.

### *2.4.2 Medium Solutions*

Turning our attention to medium solutions, Saradha and Sharma in [SS17] show that

**Lemma 2.11** (Saradha and Sharma)**.** *When $n > 4se^{2\Phi}$,*

$$N_M(F, h) \ll \frac{s}{\Phi}\left(\log s + \log\left(1 + h^{1/n}\right)\right).$$

Alternatively, Bengoechea in [Ben22] shows that

**Lemma 2.12** (Bengoechea)**.** *When $n \geqslant 3s$,*

$$N_M(F,h) \ll \begin{cases} s\left(1+\frac{\log h^{1/n}}{\log H}\right) & \text{if } n \geqslant s^4 \\ (s\log s)\left(1+\frac{\log h^{1/n}}{\log H}\right) & \text{if } 9s^2 \leqslant n < s^4 \\ (s\log s)\left(1+\frac{s+\log h^{1/n}}{\log H}\right) & \text{if } n < 9s^2 \end{cases}.$$

These two counts of medium solutions are hard to compare since

$$\log\log s \ll \Phi \ll \log s,$$

so Saradha and Sharma's result is better when $4se^{2\Phi} < n < s^4$ and Bengoechea's is better otherwise. Additionally, Bengoechea's result also incorporates the height $H$, which is advantageous toward proving Conjecture 2.6.

Akhtari and Bengoechea in [AB20] consider the special case when $h$ is small relative to the discriminant of $F$. They show the following lemma.

**Lemma 2.13** (Akhtari and Bengoechea)**.** *If $F$ has discriminant $\Delta_F$ and*

$$0 < h < \frac{|\Delta_F|^{\frac{1}{8(n-1)}}}{(3n^{800\log^2 n})^{n/2}(ns)^{2s+n}},$$

*then*

$$N_M(F,h) \ll s\log s \min\left(1, \frac{1}{\log n - \log s}\right).$$

### *2.4.3 Small Solutions*

Next, we examine results for small solutions. Saradha and Sharma in [SS17] show

**Lemma 2.14** (Saradha and Sharma)**.** *There exist constants $c_7$ and $c_8$ so that if*

$$n > 4se^{2\Phi},$$

*then*

$$N_S(F,h) \ll e^{c_7(\log s)e^{-2\Phi}} h^{\frac{2}{n}} + se^{\Phi + c_8\left(\log^3 s\right)e^{-\Phi}} h^{\frac{1}{n-2s}}.$$

After simplifying, we get

**Corollary 2.15.** *When $n > 4se^{2\Phi}$,*

$$N_S(F,h) \ll se^{\Phi} h^{2/n}.$$

With fewer restrictions on the values of $n$, Mueller and Schmidt in [MS88] show that

**Lemma 2.16** (Mueller and Schmidt)**.** *Let* $c_2 = (ns^2)^{2s/n}$ *and* $c_3 = s^2 e^{(3300s \log^3 n)/n}$. *When* $n \geqslant 4s$,

$$N_S(F, h) \ll c_2 h^{2/n} + c_3 h^{1/(n-2s)}.$$

*If in addition* $n \geqslant s \log^3 s$, *then*

$$N_S(F, h) \ll h^{2/n} + s^2 h^{1/(n-2s)}.$$

In the most general $n \geqslant 4s$ case, we may use the fact that $c_2$ and $c_3$ are decreasing functions of $n$ to acquire the corollary

**Corollary 2.17** (Mueller and Schmidt)**.** *When* $n \geqslant 4s$,

$$N_S(F, h) \ll s^2 e^{825 \log^3 4s} h^{2/n}.$$

Again, when $h$ is small relative to the discriminant of $F$, Akhtari and Bengoechea show in [AB20] that

**Lemma 2.18** (Akhtari and Bengoechea)**.** *If* $F$ *has discriminant* $\Delta_F$ *and*

$$0 < h < \frac{|\Delta_F|^{\frac{1}{8(n-1)}}}{(3n^{800 \log^2 n})^{n/2} (ns)^{2s+n}},$$

*then*

$$N_S(F, h) \leqslant 12s + 16.$$

## 2.5 Technical Results

The main technical accomplishment of this chapter is the following version of a counting technique often used in conjunction with "The Gap Principle."

**Lemma 2.19.** *Suppose that* $L, M, T, p, y_0, \ldots, y_\ell$ *are positive real numbers satisfying the following conditions:*

1. $L \leqslant y_0 \leqslant \ldots \leqslant y_\ell \leqslant M$
2. $p > 2$
3. $L^{p-2} > T$

4. $y_{i+1} \geqslant T^{-1} y_i^{p-1}$ *for each* $0 \leqslant i < \ell$

*Then*

$$\ell \leqslant \frac{\log\left[\frac{\log(MT^{-1/(p-2)})}{\log\left(LT^{-1/(p-2)}\right)}\right]}{\log(p-1)}.$$

The purpose of this lemma is to bound the number of real numbers $y_0, \ldots, y_\ell$ that could live between two fixed bounds $L$ and $M$ under certain assumptions on how far apart $y_0, \ldots, y_\ell$ must be. If, for instance, we knew that $L \leqslant y_0 < y_1 < \cdots < y_\ell \leqslant M$ and we knew that there were a $\delta > 0$ so that $y_i \geqslant y_{i-1} + \delta$ (the condition giving the "gap"), then we would know that $\ell \leqslant \frac{M-L+1}{\delta}$. Lemma 2.19 instead counts the number of $y_i$ which could live between $L$ and $M$ under the gap condition that $y_{i+1} \geqslant T^{-1} y_i^{p-1}$.

Lemma 2.19 is comparable to Lemma 1 in [SS17]. However, by fixing any $L > 0$, $p > 2$, $\ell \in \mathbb{Z}_{>0}$, and $0 < T < L^{p-2}$, then setting $y_0 = L$, $y_i = T^{-1} y_{i-1}^{p-1}$, and $M = y_\ell$, one can see that this upper bound is sharp where the upper bound in Lemma 1 of [SS17] is not.

*Proof.* We first argue by induction that for each $1 \leqslant i \leqslant \ell$, we have

$$y_\ell \geqslant \frac{y_{\ell-i}^{(p-1)^i}}{T^{\sum_{j=0}^{i-1}(p-1)^j}}. \tag{2.6}$$

This is clearly true for $i = 0$ so now suppose that it is true for generic $0 \leqslant i < \ell$. Then we have

$$\begin{aligned} y_\ell &\geqslant \frac{y_{\ell-i}^{(p-1)^i}}{T^{\sum_{j=0}^{i-1}(p-1)^j}} \\ &\geqslant \frac{\left(\frac{y_{\ell-i-1}^{p-1}}{T}\right)^{(p-1)^i}}{T^{\sum_{j=0}^{i-1}(p-1)^j}} \\ &= \frac{y_{\ell-i-1}^{(p-1)^{i+1}}}{T^{\sum_{j=0}^{i}(p-1)^j}} \end{aligned}$$

which completes the induction.

Hence, we can take $i = \ell$ in inequality (2.6) to get

$$\begin{aligned} M &\geqslant y_\ell \\ &\geqslant \frac{y_0^{(p-1)^\ell}}{T^{\sum_{j=0}^{\ell-1}(p-1)^j}} \end{aligned}$$

$$= \frac{y_0^{(p-1)^\ell}}{T^{\frac{(p-1)^\ell - 1}{p-2}}}$$

$$\geqslant \frac{L^{(p-1)^\ell}}{T^{\frac{(p-1)^\ell - 1}{p-2}}}.$$

We then multiply both sides of

$$M \geqslant \frac{L^{(p-1)^\ell}}{T^{\frac{(p-1)^\ell - 1}{p-2}}}$$

by $T^{-1/(p-2)}$ to get

$$MT^{-1/(p-2)} \geqslant \left(LT^{-1/(p-2)}\right)^{(p-1)^\ell}.$$

Taking a log on both sides (and using the fact that $LT^{-1/(p-2)} > 1$) yields

$$\frac{\log\left(MT^{-1/(p-2)}\right)}{\log\left(LT^{-1/(p-2)}\right)} \geqslant (p-1)^\ell$$

and taking logs again and using the fact that $p > 2$ yields the desired inequality. □

There is another technical lemma that we will regularly use in our later estimations. In the following lemma, the condition that $g(x) > 1 + \delta$ is necessary and it adds unfortunate complication to the statements of some later results.

**Lemma 2.20.** *Suppose that $f(x)$ and $g(x)$ are functions $\mathbb{R} \to \mathbb{R}_{>0}$ and that there exist absolute constants $\delta, k > 0$ so that $g(x) > 1 + \delta$ for all $x$ and $f(x) \ll g(x)^k$. Then $\log f(x) \ll \log g(x)$.*

*Proof.* The statement $f(x) \ll g(x)^k$ means that there exists $c$ so that $f(x) \leqslant cg(x)^k$. Taking logs of both sides gives $\log f(x) \leqslant k \log g(x) + \log c$ and dividing by $\log g(x)$ gives

$$\frac{\log f(x)}{\log g(x)} \leqslant \frac{k \log g(x) + \log c}{\log g(x)} = k + \frac{\log c}{\log g(x)} \leqslant k + \frac{\log c}{\log(1+\delta)}$$

which implies that $\log f(x) \ll \log g(x)$. □

We next prove a technical inequality that we will use later.

**Lemma 2.21.** *Suppose that $a, b > 0$. Then $\log(a+b) \leqslant \log^+ a + \log^+ b + \log 2$.*

*Proof.* Suppose that $a < 1$. If $b < 1$, then $\log(a + b) < \log(2)$ and we are done. Hence, we may assume that $b \geqslant 1$. Then we have

$$\log(a + b) \leqslant \log(1 + b) \leqslant \log(2b) = \log b + \log 2 \leqslant \log^+ a + \log^+ b + \log 2$$

and we are done. A similar argument works if $b < 1$, so assume that $a, b \geqslant 1$ for the rest of the proof.

Note that

$$a + b \leqslant 2 \max(a, b) \leqslant 2ab$$

and taking logs on both sides yields the desired result. □

Finally, we separate out another technical lemma that will be useful exactly once later.

**Lemma 2.22.** *Suppose that $A, C > 0$, $B \geqslant 1$ and that $AC > 1$. Let $\varepsilon_1, \varepsilon_2 > 0$ and suppose further that $A^{\varepsilon_2/\varepsilon_1} \leqslant C$. Then*

$$\frac{\log(AB^{\varepsilon_1})}{\log(AB^{\varepsilon_1+\varepsilon_2}C)} \leqslant \frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}.$$

*Proof.* Observe that since $A^{\varepsilon_2/\varepsilon_1} \leqslant C$, we have

$$A^{\frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1}} \leqslant AC$$

and hence,

$$A^{\frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1}} B^{\varepsilon_1+\varepsilon_2} \leqslant AB^{\varepsilon_1+\varepsilon_2}C.$$

Taking logs on both sides yields

$$\log(A^{\frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1}} B^{\varepsilon_1+\varepsilon_2}) \leqslant \log(AB^{\varepsilon_1+\varepsilon_2}C)$$

and using the facts that $AC > 1$ and $B \geqslant 1$ to conclude that the log on the right-hand side is positive, we then conclude that

$$\frac{\left(\frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1}\right)\log(AB^{\varepsilon_1})}{\log(AB^{\varepsilon_1+\varepsilon_2}C)} = \frac{\log\left(A^{\frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1}} B^{\varepsilon_1+\varepsilon_2}\right)}{\log\left(AB^{\varepsilon_1+\varepsilon_2}C\right)} \leqslant 1.$$

Therefore,

$$\frac{\log(AB^{\varepsilon_1})}{\log(AB^{\varepsilon_1+\varepsilon_2}C)} \leqslant \frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}$$

as desired. □

### 2.6.1 Improved Bounds on the Number of Medium Solutions to Thue's Inequality

In this section, we focus on improving the bounds on the number of primitive medium solutions to (2.1). This will eventually allow us to prove Theorem 1.16 by using our new bounds for the number of primitive medium solutions along with others' bounds for the number of primitive small and large solutions.

The least restrictive claim we are able to make about the number of primitive medium solutions to (2.1) is the following:

**Lemma 2.23.** *Let $N_M(F, h)$ be the number of primitive medium solutions of $|F(x, y)| \leqslant h$. Suppose further that $n \geqslant 3s$. Then*

$$N_M(F, h) \ll s \cdot \frac{\log\left(n^{3/2} + \frac{\log h}{\max(1, \log H)}\right)}{\log(\frac{n}{s} - 1)}.$$

After showing this lemma, we will show:

**Lemma 2.24.** *In the region of the ns-plane cut out by $n \geqslant 3s$, $s \geqslant 1$, the function*

$$g(n, s) = \frac{\log\left(n^{3/2} + \frac{\log h}{\max(1, \log H)}\right)}{\log\left(\frac{n}{s} - 1\right)}$$

*has $\frac{\partial g}{\partial n} < 0$. Hence, for any fixed $s$ and subset $I \subseteq [3s, \infty)$ with $I$ having a minimal element $n_0$,*

$$\max_{n \in I} g(n, s) = g(n_0, s).$$

From this, we will acquire a number of corollaries, including the following theorem, which we will immediately prove.

**Theorem 2.25.** *Let $F(x, y)$ be an irreducible integral binary form of degree $n \geqslant 3$ which has $s + 1$ nonzero coefficients. Let $H$ be a positive integer. Then if $n \geqslant 3s$,*

$$N_M(F, h) \ll s\left(1 + \log\left(s + \frac{\log h}{\max(1, \log H(F))}\right)\right). \tag{2.7}$$

*Proof.* By Lemma 2.24 with $I = [3s, \infty)$, we can substitute $n = 3s$ into the result from Lemma 2.23 and follow that with an application of Lemma 2.20 using

$$f(s) = 3^{3/2}s^{3/2} + \frac{\log h}{\max(1, \log H)},\ g(s) = e\left(s + \frac{\log h}{\max(1, \log H)}\right),\ k = \frac{3}{2},\ \text{and } \delta = 1.$$

This yields

$$\begin{aligned} N_M(F,h) &\ll s \cdot \frac{\log(3^{3/2}s^{3/2} + \frac{\log h}{\max(1,\log H)})}{\log 2} \\ &\ll s\left(1 + \log\left(s + \frac{\log h}{\max(1,\log H)}\right)\right). \end{aligned}$$

□

In the context of trying to prove Conjecture 2.6 and in analogue to Lemma 2.13, we can say

**Corollary 2.26.** *In addition to the hypotheses of Theorem 2.25, suppose that $h \leqslant H$. Then*

$$N_M(F,h) \ll s(1 + \log s).$$

This means that in pursuit of Conjecture 2.6, one only need prove the conjecture for small solutions.

**Corollary 2.27.** *In addition to the hypotheses of Theorem 2.25, suppose that $n \geqslant s^{1+\varepsilon}$ where $\varepsilon > 0$. Then*

$$N_M(F,h) \ll_\varepsilon s\left(1 + \log\left(\frac{\log h}{\max(1,\log H)}\right)\right).$$

*Proof.* By Lemma 2.24, we can substitute $n = s^{1+\varepsilon}$ into the result from Lemma 2.23. □

**Corollary 2.28.** *In addition to the hypotheses of Theorem 2.25, suppose that $n \geqslant s^{1+\varepsilon}$ and $h \leqslant H$. Then*

$$N_M(F,h) \ll_\varepsilon s.$$

Note that these corollaries together give a strict improvement on Lemma 2.12 when $n < s^4$ and they give a comparable result when $n \geqslant s^4$. Also observe that this corollary yields the expected heuristic for medium solutions from our discussion before Conjecture 2.5 and from the statement of Conjecture 2.6.

In other contexts (with other assumptions), we can acquire more specific results. In direct comparison to Lemma 2.11, we have

**Corollary 2.29.** *Under the assumption that $n > 4se^{2\Phi}$,*

$$N_M(F,h) \ll \frac{s}{\Phi}\left(1 + \log\left(s + \frac{\log h}{\max(1,\log H)}\right)\right).$$

*Proof.* If $n \leqslant s^2$, then from Lemma 2.23, we have

$$N_M(F,h) \ll s \cdot \frac{\log\left(s^3 + \frac{\log h}{\max(1,\log H)}\right)}{\log(\frac{n}{s} - 1)}.$$

Now we can apply the fact that $n > 4se^{\Phi}$ to acquire

$$N_M(F,h) \ll s \cdot \frac{1 + \log\left(s + \frac{\log h}{\max(1,\log H)}\right)}{\Phi}.$$

If $n > s^2$, then $\frac{n}{s} > \sqrt{n}$ and we have

$$N_M(F,h) \ll s \cdot \frac{\log\left(n^{3/2} + \frac{\log h}{\max(1,\log H)}\right)}{\log(\sqrt{n} - 1)}.$$

Now substituting $n = \max(4se^{2\Phi}, s^2)$, using the fact that $\Phi \ll \max(1, \log s)$, and observing that

$$\left(s + \frac{\log h}{\max(1,\log H)}\right)^3 \geqslant s^3 + \frac{\log h}{\max(1,\log H)},$$

we find

$$N_M(F,h) \ll s \cdot \frac{1 + \log\left(s + \frac{\log h}{\max(1,\log H)}\right)}{\Phi}.$$

□

### *2.6.2 Proofs of Lemmas 2.23 and 2.24*

Before we prove Lemmas 2.23 and 2.24, it is helpful to set up some initial estimates.

**Lemma 2.30.** $\log Y_L \ll \log H + \log h^{1/n} + \sqrt{n}$.

*Proof.* We have $\lambda \asymp \sqrt{n}$ and $n - \lambda \asymp n - \sqrt{n}$, so we can conclude that

$$\begin{aligned}
\log Y_L &= \frac{1}{n-\lambda}\log(2C) + \frac{\lambda}{(n-\lambda)a^2}\log(4H(n+1)^{1/2}e^{n/2}) \\
&= \frac{1}{n-\lambda}\log 2 + \frac{1}{n-\lambda}\log(Rh) + \frac{n}{n-\lambda}\log(2H\sqrt{n(n+1)}) + \\
&\qquad + \frac{\lambda}{(n-\lambda)a^2}\log(4H(n+1)^{1/2}e^{n/2}) \\
&\ll 1 + \frac{\log R}{n-\sqrt{n}} + \frac{\log h}{n-\sqrt{n}} + \log H + \log(\sqrt{n(n+1)}) +
\end{aligned}$$

$$+\frac{\sqrt{n}}{n-\sqrt{n}}\log H+\frac{\sqrt{n}}{n-\sqrt{n}}\log(\sqrt{n+1})+\frac{n^{3/2}}{2(n-\sqrt{n})}.$$

We next apply the facts that $\frac{\sqrt{n}}{n-\sqrt{n}}\log H \ll \log H$ and $\frac{\sqrt{n}}{n-\sqrt{n}}\log(\sqrt{n+1}) \ll \log(\sqrt{n(n+1)})$. Then we have

$$\log Y_L \ll 1+\frac{\log^3 n}{n-\sqrt{n}}+\frac{\log h}{n-\sqrt{n}}+\log H+\log(\sqrt{n(n+1)})+\frac{n^{3/2}}{2(n-\sqrt{n})}.$$

Using the fact that

$$1,\ \frac{\log^3 n}{n-\sqrt{n}},\ \log(\sqrt{n(n+1)}),\ \frac{n^{3/2}}{2(n-\sqrt{n})} \ll \sqrt{n},$$

we get

$$\log Y_L \ll \log H+\log h^{1/n}+\sqrt{n}.$$

□

Using this result, we conclude that there exists an absolute constant $c$ so that

$$Y_L \leqslant (Hh^{1/n}e^{\sqrt{n}})^c. \tag{2.8}$$

Now we prove Lemma 2.23

*Proof of Lemma 2.23.* We want to apply Lemma 2.19 to count primitive medium solutions to $|F(x,y)| \leqslant h$. Saradha and Sharma's Lemma 4.4 in [SS17] gives that there exists a set $S$ of roots of $F(x,1)$ and a set $S^*$ of roots of $F(1,y)$ with cardinalities $|S|,|S^*| \leqslant 6s+4$ so that any solution $(x,y)$ of (2.1) with $\min(|x|,|y|) \geqslant 12e^{\Psi}(ns)^{2s/n}h^{1/n}$ either has

$$\left|\alpha-\frac{x}{y}\right| \leqslant \frac{K}{2|y|^{n/s}}$$

or

$$\left|\alpha^*-\frac{y}{x}\right| \leqslant \frac{K}{2|x|^{r/s}}$$

for some $\alpha \in S$ or $\alpha^* \in S^*$.

Since $Y_S \geqslant 12e^{\Psi}(ns)^{2s/n}h^{1/n}$, Saradha and Sharma's Lemma 4.4 applies to any medium solution to (2.1). Fix an $\alpha \in S$ and enumerate all of the primitive medium solutions $(x_i,y_i)$ which satisfy

$$\left|\alpha-\frac{x}{y}\right| \leqslant \frac{K}{2|y|^{n/s}} \tag{2.9}$$

so that $Y_S \leqslant |y_0| \leqslant |y_1| \leqslant \cdots \leqslant |y_t| \leqslant Y_L$. For any $0 \leqslant i < t$, we can conclude that

$$\begin{aligned}\frac{K}{|y_i|^{n/s}} &\geqslant \left|\frac{x_i}{y_i} - \frac{x_{i+1}}{y_{i+1}}\right| \\ &= \left|\frac{x_i y_{i+1} - y_i x_{i+1}}{y_i y_{i+1}}\right| \\ &\geqslant \frac{1}{|y_i y_{i+1}|}.\end{aligned}$$

Rearranging yields "The Gap Principle,"

$$|y_{i+1}| \geqslant \frac{|y_i|^{n/s-1}}{K}$$

and we are now in a position to apply Lemma 2.19 with $L = Y_S$, $M = Y_L$, $T = K$, and $p = \frac{n}{s}$.

We first observe that $p > 2$ because $n \geqslant 3s$. Next, we have that $L^{p-2} > T$ because

$$\begin{aligned}L^{p-2}T^{-1} &= Y_S^{\frac{n-2s}{s}} K^{-1} &(2.10)\\ &= \frac{(12e^{\Psi})^{n/s} R^2 h^{1/s}}{2R(ns)^2 (12e^{\Psi})^{n/s} h^{1/s} H^{\frac{1}{n}-\frac{1}{s}}} \\ &= \frac{R}{2(ns)^2} H^{\frac{1}{s}-\frac{1}{n}} \\ &\geqslant \frac{n^{800\log^2 n}}{2n^4} H^{\frac{1}{s}-\frac{1}{n}} &(2.11)\\ &> 1.\end{aligned}$$

Hence, we can apply Lemma 2.19 and we find that if there are $t+1$ primitive medium solutions which give good rational approximations of $\alpha$, then

$$t \ll \frac{\log\left[\frac{\log(Y_L^{\frac{n}{s}-2}K^{-1})}{\log(Y_S^{\frac{n}{s}-2}K^{-1})}\right]}{\log(\frac{n}{s}-1)}. \tag{2.12}$$

To get an upper bound on the right-hand side of this inequality, it is easiest to manipulate the individual pieces one at a time.

Handling the case where $H > 1$ first, we can apply equation (2.8) to get

$$\begin{aligned}\log\left(Y_L^{\frac{n}{s}-2}K^{-1}\right) &\leqslant \log\left(\left(Hh^{1/n}e^{\sqrt{n}}\right)^{\left(\frac{n}{s}-2\right)c} H^{\frac{1}{s}-\frac{1}{n}}\right) \\ &= \left(\left(\frac{n}{s}-2\right)c + \frac{1}{s} - \frac{1}{n}\right)\log H +\end{aligned}$$

$$+\frac{\left(\frac{n}{s}-2\right)c}{n}\log h + c\left(\frac{n}{s}-2\right)\sqrt{n}. \tag{2.13}$$

To get a lower bound on $\log(Y_S^{\frac{n}{s}-2}K^{-1})$, we note that equations (2.10) and (2.11) imply that

$$\log(Y_S^{\frac{n}{s}-2}K^{-1}) \geqslant \left(\frac{1}{s}-\frac{1}{n}\right)\log H = \left(\frac{n-s}{ns}\right)\log H. \tag{2.14}$$

Now we combine equations (2.13) and (2.14) to find

$$\begin{aligned}
\frac{\log\left(Y_L^{\frac{n}{s}-2}K^{-1}\right)}{\log\left(Y_S^{\frac{n}{s}-2}K^{-1}\right)} &\leqslant \frac{\left(\left(\frac{n}{s}-2\right)c+\frac{1}{s}-\frac{1}{n}\right)\log H + \frac{\left(\frac{n}{s}-2\right)c}{n}\log h + c\left(\frac{n}{s}-2\right)\sqrt{n}}{\left(\frac{n-s}{ns}\right)\log H} \\
&\ll \frac{n(n-2s)c+n-s}{n-s} + \frac{(n-2s)c}{n-s}\frac{\log h}{\log H} + \frac{cn(n-2s)\sqrt{n}}{n-s} \\
&\ll n^{3/2} + \frac{\log h}{\log H}
\end{aligned}$$

and inserting this result along with a use of Lemma 2.20 into equation (2.12) yields

$$\begin{aligned}
t &\ll \frac{\log\left[\frac{\log\left(Y_L^{\frac{n}{s}-2}K^{-1}\right)}{\log\left(Y_S^{\frac{n}{s}-2}K^{-1}\right)}\right]}{\log\left(\frac{n}{s}-1\right)} \\
&\ll \frac{\log\left(n^{3/2}+\frac{\log h}{\log H}\right)}{\log\left(\frac{n}{s}-1\right)}.
\end{aligned}$$

Next, we handle the $H = 1$ case. Here, we return to equations (2.10), (2.11), (2.12), and (2.13) with $H = 1$ and again use Lemma 2.20 to find

$$\begin{aligned}
t &\ll \frac{\log\left[\frac{\log\left(Y_L^{\frac{n}{s}-2}K^{-1}\right)}{\log\left(Y_S^{\frac{n}{s}-2}K^{-1}\right)}\right]}{\log\left(\frac{n}{s}-1\right)} \\
&\ll \frac{\log\left[\frac{\log\left(\left(h^{1/n}e^{\sqrt{n}}\right)^{c\left(\frac{n}{s}-2\right)}\right)}{800\log^3 n-\log(2n^4)}\right]}{\log\left(\frac{n}{s}-1\right)} \\
&\ll \frac{\log\left[c\left(\frac{1}{s}-\frac{2}{n}\right)\log(h)+c\sqrt{n}\left(\frac{n}{s}-2\right)\right]}{\log\left(\frac{n}{s}-1\right)}
\end{aligned}$$

$$\ll \frac{\log\left(\log(h) + n^{3/2}\right)}{\log\left(\frac{n}{s} - 1\right)}.$$

For every $H$ then, we conclude that

$$t \ll \frac{\log\left(n^{3/2} + \frac{\log h}{\max(1, \log H)}\right)}{\log\left(\frac{n}{s} - 1\right)}.$$

By Lemma 2.24 (whose proof is independent of this proof), the quantity

$$g(n, s) = \frac{\log\left(n^{3/2} + \frac{\log h}{\max(1, \log H)}\right)}{\log\left(\frac{n}{s} - 1\right)}$$

decreases to 3/2 as $n \to \infty$ for any fixed $s$. In particular, there exists a $\delta > 0$ so that $g(n, s) > \delta$ for all $s \geqslant 1$ and $n \geqslant 3s$, implying that we can safely conclude $t + 1 \ll g(n, s)$.

The upper bound $t + 1 \ll g(n, s)$ on the number of medium solutions which yield good rational approximations to $\alpha$ in the sense of (2.9) is independent of $\alpha$. The same bound on the number of medium solutions which yield good rational approximations of any $\alpha^* \in S^*$ holds by symmetry. Since $|S| + |S^*| \ll s$ and since we have an upper bound on the number of good rational approximations to $\alpha \in S$ and $\alpha^* \in S^*$, we conclude that

$$N_M(F, h) \ll s \cdot \frac{\log\left(n^{3/2} + \frac{\log h}{\max(1, \log H)}\right)}{\log(\frac{n}{s} - 1)}.$$

□

Next, we proceed to prove Lemma 2.24.

*Proof of Lemma 2.24.* For ease of notation, set $L = \frac{\log h}{\max(1, \log H)}$ so we have

$$g(n, s) = \frac{\log(n^{3/2} + L)}{\log(\frac{n}{s} - 1)}.$$

Note that we have

$$\begin{aligned}
\frac{\partial g}{\partial n} &= \frac{\log(\frac{n}{s} - 1)\frac{3n^{1/2}}{2(n^{3/2}+L)} - \log(n^{3/2} + L)\frac{1}{\frac{n}{s}-1}\frac{1}{s}}{\log^2(\frac{n}{s} - 1)} \\
&= \frac{3\log(\frac{n}{s} - 1)(n - s) - 2(n + Ln^{-1/2})\log(n^{3/2} + L)}{2(n + Ln^{-1/2})(n - s)\log^2(\frac{n}{s} - 1)} \\
&= \frac{\log((\frac{n}{s} - 1)^3)(n - s) - (n + Ln^{-1/2})\log((n^{3/2} + L)^2)}{2(n + Ln^{-1/2})(n - s)\log^2(\frac{n}{s} - 1)}.
\end{aligned}$$

Since the denominator in the above expression is positive, $\frac{\partial g}{\partial n}$ has the same sign as its numerator. Upon checking that $(\frac{n}{s} - 1)^3 < (n^{3/2} + L)^2$ and observing that $n - s < n + Ln^{-1/2}$, we find that $\frac{\partial g}{\partial n} < 0$. □

### 2.6.3 Improving Akhtari and Bengoechea's Medium Solution Bound

In the context of trying to prove something like Conjecture 2.6, Akhtari and Bengoechea have changed the conditions slightly so that $h$ is assumed to be small relative to $|\Delta_F|$ rather than being small relative to $H(F)$. These new assumptions on $h$ yield additional information about the size of $Y_L$, which they leverage to improve bounds on the number of primitive solutions. Our approach to bounding the number of medium solutions can be applied to their context and we have the following slight improvement of Proposition 3.2 in [AB20].* The proof of this theorem will also use a different technique than Akhtari and Bengoechea used in their paper.

**Theorem 2.31.** *Suppose that h satisfies*

$$0 < h < \frac{|D|^{\frac{1}{8(n-1)}}}{(2n^{800\log^2 n})^{n/2}(ns)^{2s+n}}$$

*and that $n > 3s$. Then*

$$N_M(F,h) \ll s\log(s).$$

*In addition, if $n \geqslant s^{1+\varepsilon}$ for some $\varepsilon > 0$, then*

$$N_M(F,h) \ll_\varepsilon s.$$

*Proof.* The first several lines of this proof are identical to the first several lines of the proof of Lemma 2.23. We modify the proof beginning at the observation that if there are $t+1$ primitive medium solutions corresponding to a root $\alpha$ of $F(x,1)$ or $F(1,y)$, then

$$t \ll \frac{\log\left[\frac{\log(Y_L^{\frac{n}{s}-2}K^{-1})}{\log(Y_S^{\frac{n}{s}-2}K^{-1})}\right]}{\log(\frac{n}{s}-1)}.$$

*The improvement lies in reducing the assumption that $n > 10s$ to $n > 3s$ and also in the fact that the definition of a medium solution in this paper is slightly broader than the definition of a medium solution in [AB20].

We now apply equation (17) from [AB20], namely that $Y_L \ll H^2$, so that there exists a constant $c$ so that $Y_L \leqslant cH^2$:

$$t \ll \frac{\log\left[\frac{\log(Y_L^{\frac{n}{s}-2}K^{-1})}{\log(Y_S^{\frac{n}{s}-2}K^{-1})}\right]}{\log(\frac{n}{s}-1)} \leqslant \frac{\log\left[\frac{\log(cH^2)^{\frac{n}{s}-2}K^{-1}}{\log Y_S^{\frac{n}{s}-2}K^{-1}}\right]}{\log\left(\frac{n}{s}-1\right)}. \tag{2.15}$$

In the case that $H = 1$, we can use equation (2.11) to obtain that $Y_S^{\frac{n}{s}-2}K^{-1} \geqslant \frac{n^{796}}{2}$ so that

$$t \ll \frac{\log\left[\frac{\log c^{\frac{n}{s}-2}K^{-1}}{\log Y_S^{\frac{n}{s}-2}K^{-1}}\right]}{\log(\frac{n}{s}-1)} \leqslant \frac{\log\left[\frac{(\frac{n}{s}-2)\log(c)}{\log(\frac{n^{796}}{2})}\right]}{\log(\frac{n}{s}-1)} \leqslant \frac{\log((\frac{n}{s}-2)\log(c))}{\log(\frac{n}{s}-1)} \ll 1.$$

In the case that $H > 1$, we return to equation (2.15) to get

$$\begin{aligned} t &\ll \frac{\log\left[\frac{\log\left((cH^2)^{\frac{n}{s}-2}H^{\frac{1}{s}-\frac{1}{n}}\right)}{\log H^{\frac{1}{s}-\frac{1}{n}}}\right]}{\log\left(\frac{n}{s}-1\right)} \\ &= \frac{\log\left[\frac{(\frac{n}{s}-2)\log(c)}{(\frac{1}{s}-\frac{1}{n})\log(H)} + \frac{\frac{2n}{s}-4+\frac{1}{s}-\frac{1}{n}}{\frac{1}{s}-\frac{1}{n}}\right]}{\log(\frac{n}{s}-1)} \\ &\leqslant \frac{\log\left[\frac{(\frac{2n}{s}-4)\log(c)}{\frac{1}{s}-\frac{1}{n}} + \frac{\frac{2n}{s}-4}{\frac{1}{s}-\frac{1}{n}} + 1\right]}{\log(\frac{n}{s}-1)} \\ &\leqslant \frac{\log\left[(\log(c)+1)\frac{2n(2n-2s)}{n-s} + 1\right]}{\log(\frac{n}{s}-1)} \\ &\ll \frac{\log(n)}{\log(\frac{n}{s}-1)}. \end{aligned}$$

Defining $g(n,s) := \frac{\log(n)}{\log(\frac{n}{s}-1)}$, we can show as in the proof of Lemma 2.23 that $t+1 \ll g(n,s)$ also holds. Additionally, for fixed $s$, the function $g(n,s)$ is minimized when $n = 3s$. Since there are $|S| + |S^*| \ll s$ roots $\alpha$ of $F(x,1)$ or $F(1,y)$ to which primitive medium solutions produce good rational approximations, we may conclude that

$$N_M(F,h) \ll s \cdot g(n,s) \leqslant s \cdot g(3s,s) = s \cdot \frac{\log(3s)}{\log(2)} \ll s\log(s).$$

Likewise, an identical argument will show that $g(n, s)$ is minimized at $n = s^{1+\varepsilon}$ when $n \geqslant s^{1+\varepsilon}$. In the case when $n \geqslant s^{1+\varepsilon}$ then, we acquire

$$N_M(F, h) \ll s \cdot g(s^{1+\varepsilon}, s) = s \cdot \frac{\log(s^{1+\varepsilon})}{\log(s^{\varepsilon} - 1)} \ll_{\varepsilon} s.$$

□

### 2.6.4 Proof of Theorem 1.16

Finally, we want to prove our main asymptotic theorem, Theorem 1.16.

*Proof.* Suppose that $n > 4se^{2\Phi}$. From Lemma 2.10, we have that

$$N_L(F, h) \ll s.$$

From Lemma 2.23 and Lemma 2.24, we have that

$$N_M(F, h) \ll s \frac{\log\left(8s^{3/2}e^{3\Phi} + \frac{\log h}{\max(1, \log H)}\right)}{\log\left(4e^{2\Phi} - 1\right)}.$$

Finally, from Lemma 2.15, we have

$$N_S(F, h) \ll se^{\Phi}h^{2/n}.$$

Combining these three inequalities yields that

$$\begin{aligned} N(F, h) &\ll se^{\Phi}h^{2/n}\left(1 + e^{-\Phi}h^{-2/n}\frac{\log\left(8s^{3/2}e^{3\Phi} + \frac{\log h}{\max(1, \log H)}\right)}{\log(4e^{2\Phi} - 1)}\right) \\ &\ll se^{\Phi}h^{2/n}\left(1 + e^{-\Phi}h^{-2/n}\log\left(8s^{9/2} + \frac{\log h}{\max(1, \log H)}\right)\right) \\ &\ll se^{\Phi}h^{2/n}\left(1 + \log^{-3}(s)h^{-2/n}\left(\log^{+}(8s^{9/2}) + \log^{+}\left(\frac{\log h}{\max(1, \log H)}\right) + \log 2\right)\right) \end{aligned}$$

where the last step uses Lemma 2.21. From this last step, we now see

$$N(F, h) \ll se^{\Phi}h^{2/n}$$

and this concludes the proof. □

## 2.7 Explicit Results for Trinomials

Now consider the particular case where $s = 2$, i.e. $F(x, y)$ is a trinomial. In addition, we only examine the Thue equation (2.4).

In this section, we follow Thomas in [Tho00] for much of our reasoning. However, we use different notation: the parameters which Thomas calls $u$ and $v$, we call $a$ and $b$[†] and the parameters which Thomas calls $b$ and $b_0$, we will call $d$ and $d_0$.

Throughout the remainder of this section, suppose that $F(x, y) = h_n x^n + h_k x^k y^{n-k} + h_0 y^n$ where $h_n, h_k, h_0, n, k \in \mathbb{Z}$, $0 < k < n$, and $n \geqslant 6$. Suppose further that $F(x, y)$ is irreducible over $\mathbb{Z}[x, y]$. Let $H = \max(|h_n|, |h_k|, |h_0|)$ be the height of $F(x, y)$. Any time we refer to a "solution," we specifically mean a solution to equation (2.4) in $\mathbb{Z}^2$.

We will not give a sophisticated bound on the number of solutions $(p, q)$ with $|pq| \leqslant 1$ and we will consider $(p, q)$ and $(-p, -q)$ to be equivalent solutions, spurring the following definition.

**Definition 2.32.** A pair $(p, q) \in \mathbb{Z}^2$ is called *regular* if $p \neq 0$, $q > 0$, and $|p| \neq q$.

If there are $r$ regular solutions to (2.4), then there will be at most $2r + 8$ distinct solutions since for every solution $(p, q)$ with $|pq| > 1$, either $(p, q)$ or $(-p, -q)$ is regular and there are at most 8 solutions with $|pq| \leqslant 1$. From this fact and Theorem 2.33 below, Theorem 1.17 will follow.

**Theorem 2.33.** *Equation* (2.4) *has at most* $v(n)z(n)$ *regular solutions where* $v(n)$ *and* $z(n)$ *are defined in Theorem 1.17.*

More specifically, let $f(x) := F(x, 1)$ and set $R_F$ to be the number of real roots of $f$. We also wish to include certain critical points, so we make the following definition:

**Definition 2.34.** A critical point $\tau \in \mathbb{R}$ of $g(x) \in \mathbb{R}[x]$ is *proper* if there exists a neighborhood $U$ of $\tau$ for which $g''(x)g(x) > 0$ for all $x \in U \setminus \{\tau\}$.

Now let $C_F$ to be the number of proper critical points of $f(x)$. Setting $N_F$ to be the number of regular solutions to (2.4), we will show the following theorem.

---

[†]This aligns with similar notation used in [AB20] for instance, and also makes clear the difference between these parameters—whose choice will depend on $n$—and the values $u_n$ to be defined later and $v(n)$ defined in Theorem 1.17

**Theorem 2.35.** *Let $F(x, y)$ be a trinomial of degree $n \geqslant 6$. Then*

$$N_F \leqslant z(n)R_F + \ell(n)C_F$$

*where $\ell(n)$ is defined by the following table.*

| $n$ | 6–7 | 8 | $\geqslant 9$ |
|---|---|---|---|
| $\ell(n)$ | 4 | 3 | 2 |

We first show that Theorem 2.35 implies Theorem 2.33. Since $\ell(n)$ is less than $z(n)$, we have that $z(n)R_F + \ell(n)C_F \leqslant z(n)(R_F + C_F)$. Moreover, one can check with calculus that polynomials with at most four real roots have $R_F + C_F \leqslant v(n)$. Since irreducible trinomials have at most four real roots, we get Theorem 2.33 from Theorem 2.35.

To prove Theorem 2.35, we need some additional setup.

**Definition 2.36.** For a polynomial $g(x) \in \mathbb{R}[x]$, an *exceptional point* of $f$ is either a real root or a proper critical point of $g(x)$

Let $\mathcal{E}(f)$ be the set of exceptional points, $\tau_1 < \tau_2 < \cdots < \tau_c$, of $f$. Note that there exist improper critical points $\eta_1 < \eta_2 < \cdots < \eta_{c-1}$ so that
$\tau_1 < \eta_1 < \tau_2 < \eta_2 < \cdots < \eta_{c-1} < \tau_c$. Setting $\eta_0 = -\infty$ and $\eta_c = +\infty$, we can define
$J_1 = (-\infty, \eta_1)$ and $J_i = [\eta_i, \eta_{i+1})$ for $1 \leqslant i \leqslant c$.

**Definition 2.37.** A real number $\rho$ *belongs to* $\tau_i$ (and $\tau_i$ *belongs to* $\rho$) if $\rho \in J_i$.

Thomas, in [Tho00], shows that the number of regular solutions $(p, q)$ of (2.4) for which there exists a critical point of $f(x) = F(x, 1)$, $\tau$, so that $\frac{p}{q}$ belongs to $\tau$ is no larger than $\ell(n)$ (see the completion of the proof of Thomas' Theorem 2.2, given after the statement of Theorem 7.1). So it only remains to show

**Lemma 2.38.** *The number of regular solutions, $(p, q)$, of* (2.4) *for which $\frac{p}{q}$ belongs to a real root of $f$ is no larger than $z(n)$.*

By Theorem 2.2 in [Tho00], it suffices to show Lemma 2.38 for real roots of $f$ which are greater than 1. Then by Lemma 2.4 of [Tho00], we conclude that any regular $(p, q)$ for which $\frac{p}{q}$ belongs to an exceptional point greater than 1 has $p > q \geqslant 1$ and so we may assume that $p > q \geqslant 1$. Defining

$$p_0(n) := \begin{cases} 3 & \text{if } 6 \leqslant n \leqslant 8 \\ 2 & \text{if } n \geqslant 9 \end{cases},$$

we note that any regular solution $(p, q)$ with $p > q \geqslant 1$ must satisfy

$$p \geqslant p_0(n) \tag{2.16}$$

except for possibly $(2, 1)$ when $n \leqslant 8$.

**Definition 2.39.** A solution, $(p, q)$ to equation (2.4) with $p > q \geqslant 1$ and $p \geqslant p_0(n)$ is called *special*.

Since at most one solution is not special in the case that $6 \leqslant n \leqslant 8$, it suffices to show the following lemma, which will be our final reduction.

**Lemma 2.40.** *Let $\alpha > 1$ be a real root of $f(x)$. Then the number of special solutions $(p, q)$ of* (2.4) *for which $\frac{p}{q}$ belongs to $\alpha$ is no greater than $z(n) - 1$ if $6 \leqslant n \leqslant 8$ and no greater than $z(n)$ if $n \geqslant 9$.*

To prove Lemma 2.40, we split solutions into two cases: small and large. For $F(x, y)$ of degree $n$ and height $H = H(F)$, we choose a constant $Y_F = H^{\chi_n} \cdot e^{\pi_n}$ (for some values $\chi_n$ and $\pi_n$ to be specified later, but which depend only on $n$) and make the following definition.

**Definition 2.41.** A special solution $(p, q)$ to (2.4) is *small* if $q \leqslant Y_F$ and is *large* otherwise.

### *2.7.1 Small Special Solutions*

One of Thomas' main achievements in [Tho00] is the following theorem (numbered 4.1 in [Tho00]), which is a version of the "Gap Principle:"

**Theorem 2.42** (Thomas)**.** *Suppose that $F(x, y) \in \mathbb{Z}[x, y]$ is an irreducible (over $\mathbb{Z}$) trinomial binary form of degree $n \geqslant 5$ and height $H = H(F)$. Let $(p, q)$ and $(p', q')$ be special solutions to* (2.4) *which belong to a real root and suppose $q' > q$. Then*

$$q' > \frac{H^{d/n} p^{n^* - d} q^d}{K_d(n)} \tag{2.17}$$

*where*

$$n^* := \frac{n-2}{2}, \tag{2.18}$$

*$d$ is chosen to be any real number satisfying $0 \leqslant d \leqslant n^*$, and*

$$K_d(n) := m_n(r_n(1 + u_n))^d \tag{2.19}$$

*where*

$$m_n = 2\sqrt{\frac{2n}{(n-1)(n-2)}}, \qquad r_n = (2.032)^{1/n}, \qquad u_n = \sqrt{\frac{2}{(n-2)p_0^n}}. \tag{2.20}$$

This approximation result will be helpful in proving the following proposition:

**Proposition 2.43.** *Let $\alpha > 1$ be a real root of $f(x)$. There are no more than*

$$T := \left\lfloor \max\left( \frac{\log\left(\frac{\chi_n n(d-1)+d}{d_0(d-1)+d}\right)}{\log d}, \frac{\log\left(\frac{\pi_n}{\log K_d(n)^{-\frac{1}{d-1}} Q_1} + \frac{d}{d_0(d-1)+d}\right)}{\log d} \right) \right\rfloor + 2 \tag{2.21}$$

*small special solutions $(p, q)$ where $p/q$ belongs to $\alpha$.*

*Proof.* If there less than 2 special solutions $(p, q)$ where $p/q$ belongs to $\alpha$, then we are done. Otherwise, suppose that there are exactly $t + 2$ small special solutions $(p, q)$ where $p/q$ belongs to $\alpha$ and $t \geqslant 0$. Label those $t + 2$ solutions as $(p_0, q_0), \ldots, (p_{t+1}, q_{t+1})$ ordered so that

$$1 \leqslant q_0 < q_1 < \ldots < q_{t+1} \leqslant Y_F$$

(the strict inequality follows from the fact that the $\frac{p_i}{q_i}$ are principal convergents to $\alpha$ by Corollary 3.2 in [Tho00]).

Choose numbers $d_0, d \in \mathbb{R}_{>0}$ and recall the definition of $n^*$ from equation (2.18) before making the following definitions:

$$c_0 := n^* - d_0, \qquad K_0 := K_{d_0}(n), \qquad Q_1 := \frac{p_0(n)^{c_0}}{K_0}.$$

In particular, choose $d$ and $d_0$ so that

$$1 < d \leqslant n^*, \tag{2.22}$$

$$0 \leqslant d_0 \leqslant \min(n^* - 1.4, d), \tag{2.23}$$

$$Q_1^{d-1} > \max(1, K_d(n)). \tag{2.24}$$

In the proof of Proposition 2.47 and in the computations in Section 2.7.4, we show by example that choosing such $d$ and $d_0$ are possible.

First, observe that by Theorem 2.42 applied to $q_1 > q_0 \geqslant 1$ (and using the observation that $p_0 \geqslant p_0(n)$), we get $q_1 > H^{d_0/n} Q_1$.

Here is where we depart from Thomas' method. We now aim to apply Lemma 2.19 to the inequalities

$$H^{d_0/n}Q_1 < q_1 < q_2 < \cdots < q_{t+1} \leqslant Y_F$$

and

$$q_{i+1} > \frac{H^{d/n}q_i^d}{K_d(n)}.$$

In the notation of Lemma 2.19, we have $L = H^{d_0/n}Q_1$, $M = Y_F$, $p = d+1$, and $T = \frac{K_d(n)}{H^{d/n}}$. To apply the conclusion of Lemma 2.19, we need to check that $p > 2$ (trivial based on the fact that $d$ is chosen to be greater than 1 from (2.22)) and we need to check that $L^{p-2} > T$. But this occurs if and only if

$$\left(H^{d_0/n}Q_1\right)^{d-1} > \frac{K_d(n)}{H^{d/n}},$$

i.e.

$$H^{(d_0(d-1)+d)/n}Q_1^{d-1} > K_d(n),$$

which is guaranteed by (2.24).

Now applying Lemma 2.19 and using the fact that $t$ is an integer yields

$$\begin{aligned} t &\leqslant \left\lfloor \frac{\log\left[\frac{\log\left(Y_F\left(\frac{K_d(n)}{H^{d/n}}\right)^{-\frac{1}{d-1}}\right)}{\log\left(H^{d_0/n}Q_1\left(\frac{K_d(n)}{H^{d/n}}\right)^{-\frac{1}{d-1}}\right)}\right]}{\log d}\right\rfloor \\ &= \left\lfloor \frac{\log\left[\frac{\log\left(Y_F K_d(n)^{-\frac{1}{d-1}} H^{\frac{d}{n(d-1)}}\right)}{\log\left(K_d(n)^{-\frac{1}{d-1}} H^{\frac{d_0}{n}+\frac{d}{n(d-1)}} Q_1\right)}\right]}{\log d}\right\rfloor \\ &= \left\lfloor \frac{\log\left[\frac{\log(Y_F)}{\log\left(K_d(n)^{-\frac{1}{d-1}} H^{\frac{d_0}{n}+\frac{d}{n(d-1)}} Q_1\right)} + \frac{\log\left(K_d(n)^{-\frac{1}{d-1}} H^{\frac{d}{n(d-1)}}\right)}{\log\left(K_d(n)^{-\frac{1}{d-1}} H^{\frac{d_0}{n}+\frac{d}{n(d-1)}} Q_1\right)}\right]}{\log d}\right\rfloor. \end{aligned} \tag{2.25}$$

We now claim that we can apply Lemma 2.22 to the logarithmic quantity

$$\frac{\log\left(K_d(n)^{-\frac{1}{d-1}} H^{\frac{d}{n(d-1)}}\right)}{\log\left(K_d(n)^{-\frac{1}{d-1}} H^{\frac{d_0}{n}+\frac{d}{n(d-1)}} Q_1\right)}$$

with $A = K_d(n)^{-\frac{1}{d-1}}$, $B = H$, $C = Q_1$, $\varepsilon_1 = \frac{d}{n(d-1)}$, and $\varepsilon_2 = \frac{d_0}{n}$. All of the hypotheses are clear except possibly that $AC > 1$ which follows from (2.24) and $A^{\varepsilon_2/\varepsilon_1} < C$.

To verify this second hypothesis, we want to first note that

$$\frac{\varepsilon_2}{\varepsilon_1} = \frac{d_0(d-1)}{d}$$

and so, recalling the definition of $K_d(n)$ from (2.19),

$$\begin{aligned} A^{\varepsilon_2/\varepsilon_1} &= K_d(n)^{-\frac{d_0}{d}}. \\ &= (m_n(r_n(1+u_n))^d)^{-\frac{d_0}{d}} \\ &= m_n^{-\frac{d_0}{d}}(r_n(1+u_n))^{-d_0} \end{aligned}$$

whereas

$$\begin{aligned} C &= Q_1 \\ &= \frac{p_0(n)^{c_0}}{K_{d_0}(n)} \\ &= p_0(n)^{n^*-d_0} m_n^{-1}(r_n(1+u_n))^{-d_0}. \end{aligned}$$

Hence, $A^{\varepsilon_2/\varepsilon_1} < C$ if and only if

$$m_n^{1-\frac{d_0}{d}} < p_0(n)^{n^*-d_0}.$$

But this follows from the selection of $d_0 \leqslant d$ (by (2.23)) along with the fact that $m_n$ is a decreasing function of $n$ for $n \geqslant 6$ (which we can see from (2.20)), so

$$m_n^{1-\frac{d_0}{d}} < \max(1, m_n) \leqslant 2\sqrt{3/5} < 2^{1.4} < p_0(n)^{n^*-d_0}.$$

Now that we have verified that we may apply Lemma 2.22, we continue the estimation we left off in inequality (2.25):

$$\begin{aligned} t &\leqslant \left\lfloor \frac{\log\left[\frac{\log(Y_F)}{\log\left(K_d(n)^{-\frac{1}{d-1}} H^{\frac{d_0}{n}+\frac{d}{n(d-1)}} Q_1\right)} + \frac{\log\left(K_d(n)^{-\frac{1}{d-1}} H^{\frac{d}{n(d-1)}}\right)}{\log\left(K_d(n)^{-\frac{1}{d-1}} H^{\frac{d_0}{n}+\frac{d}{n(d-1)}} Q_1\right)}\right]}{\log d} \right\rfloor \\ &\leqslant \left\lfloor \frac{\log\left[\frac{\log(Y_F)}{\log\left(K_d(n)^{-\frac{1}{d-1}} H^{\frac{d_0}{n}+\frac{d}{n(d-1)}} Q_1\right)} + \frac{\frac{d}{n(d-1)}}{\frac{d_0}{n}+\frac{d}{n(d-1)}}\right]}{\log d} \right\rfloor \end{aligned}$$

$$= \left\lfloor \frac{\log\left[\frac{\log(Y_F)}{\log\left(K_d(n)^{-\frac{1}{d-1}} H^{\frac{d_0}{n}+\frac{d}{n(d-1)}} Q_1\right)} + \frac{d}{d_0(d-1)+d}\right]}{\log d} \right\rfloor.$$

Now using the definition $Y_F = H^{\chi_n} \cdot e^{\pi_n}$, we have

$$\begin{aligned}
t &\leqslant \left\lfloor \frac{\log\left[\frac{\chi_n \log H + \pi_n}{\frac{d_0(d-1)+d}{n(d-1)} \log H + \log\left(K_d(n)^{-\frac{1}{d-1}} Q_1\right)} + \frac{d}{d_0(d-1)+d}\right]}{\log d} \right\rfloor \\
&\leqslant \left\lfloor \max\left( \frac{\log\left(\frac{\chi_n}{\frac{d_0(d-1)+d}{n(d-1)}} + \frac{d}{d_0(d-1)+d}\right)}{\log d}, \frac{\log\left(\frac{\pi_n}{\log K_d(n)^{-\frac{1}{d-1}} Q_1} + \frac{d}{d_0(d-1)+d}\right)}{\log d} \right) \right\rfloor \\
&= \left\lfloor \max\left( \frac{\log\left(\frac{\chi_n n(d-1)+d}{d_0(d-1)+d}\right)}{\log d}, \frac{\log\left(\frac{\pi_n}{\log K_d(n)^{-\frac{1}{d-1}} Q_1} + \frac{d}{d_0(d-1)+d}\right)}{\log d} \right) \right\rfloor \\
&= T - 2.
\end{aligned}$$

Therefore, the number of small special solutions $(p, q)$ for which $p/q$ belongs to $\alpha$ is $t + 2 \leqslant T$. □

### *2.7.2 Large Special Solutions*

Here we follow Thomas in [Tho00] as he follows Bombieri-Schmidt in [BS87]. If we choose numbers $a$ and $b$ satisfying

$$0 < a < b < 1 - \sqrt{2 \cdot \frac{n + a^2}{n^2}} \tag{2.26}$$

then we can define

$$L = \frac{\sqrt{2(n + a^2)}}{1 - b} \qquad D = \frac{L}{n - L} \qquad A = \frac{1}{a^2} \qquad E = \frac{1}{2(b^2 - a^2)}.$$

Now we choose

$$\chi_n = D(A + 1) + 1, \tag{2.27}$$

$$\pi_n = (D(4+A)+2)\log(2) + \frac{(D+1)\log(n)}{2} + \frac{nAD}{2}. \tag{2.28}$$

With these choices of $\pi_n$ and $\chi_n$, we aim to apply Lemma 2 of [BS87] and conclude the following:

**Proposition 2.44.** *Suppose $\alpha > 1$ is a real root of $f(x)$. If $\chi_n \geqslant 2$ and $\pi_n \geqslant 5\log(2) + 2\log(n)$, then there are at most*

$$Z := \left\lfloor \frac{\log E + 2\log(n) - \log(L-2)}{\log(n-1)} \right\rfloor + 2 \tag{2.29}$$

*large special solutions belonging to $\alpha$.*

The proof of this proposition relies on the two following lemmas:

**Lemma 2.45.** *$Y_F$ as defined here is greater than or equal to $Y_0$ as defined in [BS87].*

Lemma 2.45 ensures that any large solution in the sense of this paper is a large solution in the sense of Bombieri and Schmidt.

**Lemma 2.46.** *Suppose that $\chi_n \geqslant 2$ and $\pi_n \geqslant 5\log(2) + 2\log(n)$. If $\alpha > 1$ is a real root of $f(x)$ and $(p, q)$ is a large special solution of* (2.4) *so that $p/q$ belongs to $\alpha$, then $\alpha$ is the closest (complex) root of $f(x)$ to $p/q$.*

Given an algebraic $\beta$, Lemma 2 of [BS87] only counts the number of rational numbers which are nearest to $\beta$ out of all of the conjugates of $\beta$ and which form good approximations of $\beta$. If there were a real root $\alpha > 1$ of $f(x)$ and a large special solution $(p, q)$ of (2.4) for which $p/q$ belonged to $\alpha$ yet there was a root $\beta$ of $f(x)$ with $\beta$ closer to $p/q$ than $\alpha$, Lemma 2 of [BS87] would not count $p/q$. However, Lemma 2.46 confirms that this is not the case if we choose $a$ and $b$ carefully enough to make $\chi_n \geqslant 2$ and $\pi_n \geqslant 5\log(2) + 2\log(n)$.

We first prove these two lemmas:

*Proof of Lemma 2.45.* $Y_0$ depends on the Mahler measure $M(F)$ rather than the height $H(F)$. These are related (for trinomials $F(x, y)$) by $M(F) \leqslant 3^{1/2}H(F)$, which follows from the fact that $M(F) \leqslant \ell_2(F)$ (see Lemma 1.6.7 in [BG06]). Now, using Thomas' notation in Bombieri and Schmidt's notation, we have that

$$Y_0 := (2C)^{\frac{1}{n-\lambda}} \left(4e^{A_1}\right)^{\frac{\lambda}{n-\lambda}}$$

where

$$C = (2n^{1/2}M(F))^n,$$
$$t = \sqrt{\frac{2}{n+a^2}},$$
$$A_1 = \frac{t^2}{2-nt^2}\left(\log M(F) + \frac{n}{2}\right),$$
$$\lambda = \frac{2}{t(1-b)}.$$

Some of our other constants regularly appear in the estimation which shows $Y_0 < Y_F$ and we list them here for simplicity:

$$A = \frac{1}{a^2} = \frac{2}{2(n+a^2)-2n} = \frac{2}{n+a^2}\cdot\frac{1}{2-\frac{2n}{n+a^2}} = \frac{t^2}{2-nt^2},$$
$$L = \frac{\sqrt{2(n+a^2)}}{1-b} = \frac{2}{1-b}\sqrt{\frac{n+a^2}{2}} = \frac{2}{t(1-b)} = \lambda,$$
$$D = \frac{L}{n-L} = \frac{\lambda}{n-\lambda}.$$

Note also that this implies that $D+1 = \frac{n}{n-\lambda}$.

Before making the final estimate, we take a moment to observe that

$$\left(\sqrt{3}\right)^{\frac{n}{n-\lambda}+AD} < 2^{\frac{n-1}{n-\lambda}}.$$

This estimate is tedious, but not difficult. One can show that

$$\left(\sqrt{3}\right)^{\frac{n}{n-\lambda}+AD} < 2^{\frac{n-1}{n-\lambda}}$$

occurs if and only if

$$2 < \left(\frac{2}{\sqrt{3}}\right)^{n+A\lambda}.$$

Estimating $A$ from below by

$$A > \frac{1}{(1-\sqrt{2(n+1)/n^2})^2}$$

and estimating $\lambda$ from below by $\lambda > \sqrt{2n}$ gives that

$$2 < \left(\frac{2}{\sqrt{3}}\right)^{n+A\lambda}$$

is implied by

$$2 < \left(\frac{2}{\sqrt{3}}\right)^{n+\frac{n^2\sqrt{2n}}{n^2-2n\sqrt{2n+2}+2n+2}}.$$

Upon observing that

$$n + \frac{n^2\sqrt{2n}}{n^2 - 2n\sqrt{2n+2} + 2n + 2} \geqslant 20$$

when $n \geqslant 6$ for instance, one can now see that

$$2 < \left(\frac{2}{\sqrt{3}}\right)^{n+\frac{n^2\sqrt{2n}}{n^2-2n\sqrt{2n+2}+2n+2}}$$

and as a result, we must have

$$\left(\sqrt{3}\right)^{\frac{n}{n-\lambda}+AD} < 2^{\frac{n-1}{n-\lambda}}.$$

We can now conclude

$$\begin{aligned}
Y_0 &= (2C)^{\frac{1}{n-\lambda}}(4e^{A_1})^{\frac{\lambda}{n-\lambda}} \\
&= (2(2n^{1/2}M(F))^n)^{\frac{1}{n-\lambda}}(4e^{A(\log M(F)+\frac{n}{2})})^{\frac{\lambda}{n-\lambda}} \\
&= 2^{\frac{1+n+2\lambda}{n-\lambda}} \cdot n^{\frac{n}{2(n-\lambda)}} \cdot M(F)^{\frac{n}{n-\lambda}} e^{AD(\log M(F)+\frac{n}{2})} \\
&= 2^{\frac{1+n+2\lambda}{n-\lambda}} \cdot n^{\frac{n}{2(n-\lambda)}} \cdot M(F)^{\frac{n}{n-\lambda}+AD} \cdot e^{\frac{ADn}{2}} \\
&\leqslant 2^{\frac{1+n+2\lambda}{n-\lambda}} \cdot n^{\frac{n}{2(n-\lambda)}} \cdot (\sqrt{3}H(F))^{\frac{n}{n-\lambda}+AD} \cdot e^{\frac{ADn}{2}}.
\end{aligned}$$

Next we use the fact that $\left(\sqrt{3}\right)^{\frac{n}{n-\lambda}+AD} < 2^{\frac{n-1}{n-\lambda}}$ to find that

$$\begin{aligned}
Y_0 &< 2^{\frac{1+n+2\lambda}{n-\lambda}+\frac{n-1}{n-\lambda}+AD} \cdot n^{\frac{n}{2(n-\lambda)}} \cdot H(F)^{\frac{n}{n-\lambda}+AD} \cdot e^{\frac{ADn}{2}} \\
&= 2^{\frac{2n+2\lambda}{n-\lambda}+AD} \cdot n^{\frac{D+1}{2}} \cdot H(F)^{1+D+AD} \cdot e^{\frac{ADn}{2}} \\
&= H(F)^{\chi_n} \cdot \exp\left(\left(\frac{2n+2\lambda}{n-\lambda} + AD\right)\log(2) + \frac{D+1}{2}\log n + \frac{ADn}{2}\right) \\
&= H(F)^{\chi_n} \cdot \exp\left(\left(\frac{4\lambda+2(n-\lambda)}{n-\lambda} + AD\right)\log(2) + \frac{D+1}{2}\log n + \frac{ADn}{2}\right) \\
&= H(F)^{\chi_n} \cdot \exp\left((4D+2+AD)\log(2) + \frac{D+1}{2}\log n + \frac{ADn}{2}\right) \\
&= H(F)^{\chi_n} \cdot e^{\pi_n} \\
&= Y_F.
\end{aligned}$$

□

*Proof of Lemma 2.46.* Since $\frac{p}{q}$ is a large special solution, we have

$$p > q \geqslant Y_F = H^{\chi_n} e^{\pi_n}.$$

Since $\frac{p}{q}$ belongs to $\alpha$, Thomas' Corollary 3.1 in [Tho00] indicates that

$$\left|\frac{p}{q} - \alpha\right| < \frac{1}{p^{n^*} q} < \frac{1}{Y_F^{n/2}}.$$

if we recall that $n^* = \frac{n-2}{2}$ by its definition in (2.18).

Suppose, by contradiction, that there exists $\beta \in \mathbb{C}$ with $f(\beta) = 0$ and $\left|\frac{p}{q} - \beta\right| < \left|\frac{p}{q} - \alpha\right|$. Then by the triangle equality, we find that

$$\begin{aligned} |\alpha - \beta| &\leqslant \left|\frac{p}{q} - \beta\right| + \left|\frac{p}{q} - \alpha\right| \\ &< \frac{2}{Y_F^{n/2}}. \end{aligned} \tag{2.30}$$

Since $\alpha$ and $\beta$ are distinct roots of $f$, Theorem 4 in [Rum79] indicates that

$$|\alpha - \beta| > \frac{1}{2n^{n/2+2}(4H)^n} = \frac{1}{2^{2n+1} n^{n/2+2} H^n}. \tag{2.31}$$

Combining (2.30) and (2.31), we find that

$$\frac{1}{2^{2n+1} n^{n/2+2} H^n} < \frac{2}{Y_F^{n/2}}$$

and rearranging yields

$$\frac{Y_F^{n/2}}{2^{2n+2} n^{n/2+2} H^n} < 1.$$

From here, we can use the fact that $Y_F = H^{\chi_n} e^{\pi_n}$ to find

$$\begin{aligned} 1 &> \frac{H^{n(\chi_n/2-1)} e^{n\pi_n/2}}{2^{2n+2} n^{n/2+2}} \\ &\geqslant \frac{e^{n\pi_n/2}}{2^{2n+2} n^{n/2+2}} \end{aligned}$$

where the last inequality follows because $\chi_n \geqslant 2$. After rearranging, this implies that

$$\begin{aligned} \pi_n &< \frac{(4n+4)\log(2) + (n+4)\log(n)}{n} \\ &\leqslant 5\log(2) + 2\log(n) \end{aligned}$$

where the last inequality follows from the fact that $n \geqslant 6$. However, the last inequality contradicts our hypothesis that $\pi_n \geqslant 5\log(2) + 2\log(n)$, so no such $\beta$ can exist and the closest root of $f(x)$ to $p/q$ is $\alpha$. □

Finally, we prove Proposition 2.44.

*Proof of Proposition 2.44.* Let $\alpha > 1$ be a real root of $f(x)$. By Lemma 2.45, every large special solution $(p, q)$ so that $p/q$ belongs to $\alpha$ is large in the sense of [BS87]. Moreover, by Lemma 2.46, any large special solution $(p, q)$ so that $p/q$ belongs to $\alpha$ has

$$\left|\frac{p}{q} - \alpha\right| = \min_{f(\beta)=0} \left|\frac{p}{q} - \beta\right|.$$

Hence, every large special solution $(p, q)$ so that $p/q$ belongs to $\alpha$ is large (in the sense of [BS87]) and is nearest to $\alpha$ among all the roots of $f(X)$. Lemma 2 of [BS87] indicates that there are no more than

$$Z = \left\lfloor \frac{\log E + 2\log(n) - \log(L-2)}{\log(n-1)} \right\rfloor + 2$$

large solutions $(p, q)$ so that $p/q$ is nearest to $\alpha$ among all the roots of $f(X)$ and so we conclude that there are no more than $Z$ large special solutions $(p, q)$ with $p/q$ belonging to $\alpha$. □

### *2.7.3 Choosing Parameters for Large Degrees*

Begin by assuming $n \geqslant 507$. We handle all smaller instances of $n$ computationally. Recall that $n^* = \frac{n-2}{2}$ from its definition in (2.18).

**Proposition 2.47.** *For $n \geqslant 507$, we can take $d_0 = \frac{n^*}{2}$, $d = n^*$, $a = \frac{1}{4}$, $C = 7/6$, $c = \frac{8}{9C^2-1}$, $b = 1 - \frac{\sqrt{2n+\frac{1}{8}}}{\frac{cn^2}{n-1}+2}$ and obtain $T = 2$ and $Z = 2$.*

Observe first that these are the smallest possible values of $T$ and $Z$.

*Proof.* To show this, we first must show that these choices of $d_0, d, a$, and $b$ meet the requirements listed in (2.22), (2.23), (2.24), and (2.26).

Certainly $0 \leqslant d_0 \leqslant \min(n^* - 1.4, d)$ and $1 < d \leqslant n^*$. All that remains to show for $d_0$ and $d$ is (2.24). We have

$$\begin{aligned} Q_1^{d-1} &= \left(\frac{p_0^{c_0}}{K_0}\right)^{d-1} \\ &\geqslant \left(\frac{2^{n^*/2}}{K_{n^*/2}(n)}\right)^{\frac{n^*}{2}-1}. \end{aligned}$$

But observe that

$$\begin{aligned}
K_{n^*/2}(n) &= 2\sqrt{\frac{2n}{(n-1)(n-2)}}\left(2.032^{1/n}\left(1+\sqrt{\frac{2}{(n-2)p_0^n}}\right)\right)^{n^*/2} \\
&\leqslant 2\cdot 2.032\left(1+\frac{1}{\sqrt{(n-2)2^{n-1}}}\right)^{\frac{n-2}{4}} \\
&\leqslant 5\left(1+\frac{1}{n-2}\right)^{\frac{n-2}{4}} \\
&\leqslant 5e^{1/4}
\end{aligned}$$

so $Q_1^{d-1}$ is certainly greater than 1. Similar reasoning shows that $K_d(n) \leqslant 5e^{1/2}$, so it is certainly also the case that

$$K_d(n) \leqslant 5e^{1/2} < \left(\frac{2^{n^*/2}}{5e^{1/4}}\right)^{\frac{n^*}{2}-1} \leqslant \left(\frac{2^{n^*/2}}{K_{n^*/2}(n)}\right)^{\frac{n^*}{2}-1} = Q_1^{d-1}.$$

Hence, our choices of $d$ and $d_0$ are valid.

Next, we wish to check that our choices for $a$ and $b$ are valid. To check $0 < a < b$, note that

$$\begin{aligned}
b = 1 - \frac{\sqrt{2n+\frac{1}{8}}}{\frac{cn^2}{n-1}+2} &\geqslant 1 - \frac{2\sqrt{n}}{\frac{cn^2}{n-1}} = 1 - \frac{2(n-1)\sqrt{n}}{cn^2} \geqslant 1 - \frac{2}{c\sqrt{n}} \\
&\geqslant 1 - \frac{2}{c\sqrt{507}} > \frac{1}{4} = a. \qquad (2.32)
\end{aligned}$$

To check that $b < 1 - \frac{\sqrt{2n+2a^2}}{n}$, it suffices to show that $n > \frac{cn^2}{n-1} + 2$. But this occurs if and only if $(1-c)n^2 - 3n + 2 > 0$, i.e. $n > \frac{3+\sqrt{9-8(1-c)}}{2(1-c)} \approx 9.66$, which we certainly have.

To show that $T = 2$, we claim that we have the following two inequalities (and from equation (2.21) together with the fact that $\frac{d}{d_0(d-1)+d} \leqslant 1$, it will follow that $T = 2$):

$$\frac{\chi_n n(d-1)}{d_0(d-1)+d} + 1 < d, \qquad (2.33)$$

$$\frac{\pi_n}{\log K_d(n)^{-\frac{1}{d-1}} Q_1} + 1 < d. \qquad (2.34)$$

We first show (2.33). Substituting $d_0 = \frac{n^*}{2}$ and $d = n^*$, observe that (2.33) is equivalent to

$$\chi_n < \frac{\left(\frac{n-2}{4}\right)\left(\frac{n-4}{2}-1\right)+\frac{n-2}{2}}{n} = \frac{n-2}{8} = \frac{n^*}{4}.$$

Keeping an eye on the definition of $\chi_n$ given in equation (2.27), we have that

$$\begin{aligned}
A &= 16,\\
C &= \frac{7}{6},\\
c &= \frac{8}{9C^2 - 1} = \frac{32}{45},\\
b &= 1 - \frac{\sqrt{2n + \frac{1}{8}}}{\left(\frac{cn^2}{n-1} + 2\right)}.
\end{aligned}$$

All of these together yield

$$L = \frac{cn^2}{n-1} + 2 = \frac{32}{45}\left(n + 1 + \frac{1}{n-1}\right) + 2 \tag{2.35}$$

and it is now easy to check that

$$\frac{32}{45}n \leqslant L \leqslant \frac{32}{45}n + 3.$$

From here we have

$$\begin{aligned}
D &= \frac{L}{n-L} \leqslant \frac{\frac{32}{45}n + 3}{n - \left(\frac{32}{45}n + 3\right)} = \frac{\frac{32}{45}n + 3}{\frac{13}{45}n - 3} = \frac{32}{13} + \frac{6075}{13(13n - 135)} \leqslant 2.54\\
D &\geqslant \frac{\frac{32}{45}n}{n - \frac{32}{45}n} = \frac{32}{13} \approx 2.46
\end{aligned}$$

when we use the fact that $n \geqslant 507$. To convert these into estimates on $\chi_n$, we have

$$\chi_n = 17D + 1 \leqslant \frac{94853}{2152} \leqslant 44.08, \tag{2.36}$$

$$\chi_n = 17D + 1 \geqslant \frac{557}{13} \geqslant 42.8. \tag{2.37}$$

Since $n \geqslant 507$, we now have $\chi_n \leqslant 44.08 < \frac{n-2}{8}$ which confirms equation (2.33).

Equation (2.34) is more complicated to handle. Observe that by equation (2.28), we have

$$\begin{aligned}
\pi_n &= (D(4 + A) + 2)\log 2 + \frac{(D+1)\log n}{2} + \frac{ADn}{2}\\
&\leqslant 36.6 + 1.77\log n + 20.28n\\
&\leqslant 37 + 21n.
\end{aligned} \tag{2.38}$$

For reference later, we will also note

$$\begin{aligned}\pi_n &\geqslant 35.5 + 1.7\log n + 19.6n \\ &\geqslant 46 + 19n. \end{aligned} \tag{2.39}$$

It will additionally be helpful for us to have an estimate on $K_d(n)$. We have

$$\begin{aligned} K_d(n) &= 2\sqrt{\frac{2n}{(n-1)(n-2)}}\left(2.032^{1/n}\left(1+\sqrt{\frac{2}{(n-2)p_0^n}}\right)\right)^d \\ &\leqslant 4\sqrt{\frac{3n-3}{(n-1)(n-2)}}\left(1+\sqrt{\frac{2n-4}{(n-2)p_0^n}}\right)^d \\ &\leqslant 4\sqrt{\frac{3}{n-2}}\left(1+\sqrt{\frac{2}{p_0^n}}\right)^{n^*}. \end{aligned}$$

Now, since $p_0 \geqslant 2$, we have

$$\begin{aligned} K_d(n) &\leqslant 4\sqrt{\frac{3}{n-2}}\left(1+\sqrt{\frac{2}{2^n}}\right)^{\frac{n-2}{2}} \leqslant 4\sqrt{\frac{3}{n-2}}\left(1+\frac{1}{2^{\frac{n-2}{2}}}\right)^{\frac{n-2}{2}} \\ &\leqslant 4\sqrt{\frac{3}{n-2}}\left(1+\frac{1}{\left(\frac{n-2}{2}\right)}\right)^{\frac{n-2}{2}} \leqslant 4e\sqrt{\frac{3}{n-2}} \end{aligned} \tag{2.40}$$

and similar reasoning yields

$$K_{d_0}(n) \leqslant 4\sqrt{\frac{3}{n-2}}\left(1+\frac{1}{2^{n^*}}\right)^{n^*/2}. \tag{2.41}$$

Combining the upper bounds in equation (2.38) with the fact that for $n \geqslant 270$,

$$37 + 21n < \frac{\log 1.9}{8}(n-2)(n-4)$$

yields

$$\pi_n < \frac{\log(1.9)}{8}(n-2)(n-4).$$

Now inequalities (2.40) and (2.41) give

$$\begin{aligned} (d-1)\log\left[K_d(n)^{-\frac{1}{d-1}}Q_1\right] &= \log\left[K_d(n)^{-1}Q_1^{d-1}\right] \\ &\geqslant \log\left[\frac{1}{4}\cdot e^{-1}\cdot\left(\frac{n-2}{3}\right)^{1/2}\cdot\left(\frac{p_0^{c_0}}{K_0}\right)^{n^*-1}\right] \end{aligned}$$

$$
\begin{aligned}
&\geqslant \log\left[\frac{1}{4}\cdot e^{-1}\cdot\left(\frac{n-2}{3}\right)^{1/2}\left(\frac{2^{n^*/2}}{K_{d_0}(n)}\right)^{n^*-1}\right]\\
&\geqslant \log\left[\frac{1}{4}\cdot e^{-1}\cdot\left(\frac{n-2}{3}\right)^{1/2}\left(\frac{2^{n^*/2}}{(1+2^{-n^*})^{n^*/2}}\right)^{\frac{n-4}{2}}\left(\frac{1}{4}\sqrt{\frac{n-2}{3}}\right)^{\frac{n-4}{2}}\right]\\
&\geqslant \log\left[\left(\frac{1}{4}\right)^{\frac{n-2}{2}}\left(\frac{n-2}{3}\right)^{\frac{n-2}{4}} e^{-1}\left(\frac{2}{1+2^{-n^*}}\right)^{\left(\frac{n-2}{4}\right)\left(\frac{n-4}{2}\right)}\right]\\
&\geqslant \log\left[\left(\frac{n-2}{48}\right)^{\frac{n-2}{4}} e^{-1}\cdot 1.9^{\frac{(n-2)(n-4)}{8}}\right]\\
&\geqslant \frac{\log 1.9}{8}(n-2)(n-4)+\frac{n-2}{4}\log\left(\frac{n-2}{48}\right)-1\\
&\geqslant \frac{\log(1.9)}{8}(n-2)(n-4)\\
&> \pi_n
\end{aligned}
$$

which now implies that equation (2.34) is satisfied. Hence, we conclude that $T = 2$.

Finally, we check that $Z = 2$. In order to use Proposition 2.44, we must verify that $\chi_n \geqslant 2$ and $\pi_n \geqslant 5\log(2) + 2\log(n)$. However, these quickly follow from (2.37) and (2.39).

As before in equation (2.32), we have $b \geqslant 1 - \frac{2}{c\sqrt{507}} > 0.87509$, so

$$
E = \frac{1}{2(b^2 - a^2)} < 0.711 < c
$$

and so (also using (2.35))

$$
\frac{\log E + 2\log(n) - \log(L-2)}{\log(n-1)} < \frac{\log\left(\frac{cn^2}{L-2}\right)}{\log(n-1)} = 1.
$$

Therefore, by (2.29), we note that $Z = 2$. □

Note that Proposition 2.47 proves Lemma 2.40 for $n \geqslant 507$.

### *2.7.4 Choosing Parameters for Small Degrees*

For $n \leqslant 506$, we make parameter choices listed in table A.1. One can check that the parameter choices satisfy (2.24), (2.26), (2.22), and (2.23) along with the necessary bounds on $\pi_n$ and $\chi_n$ in order to use Proposition 2.44, and yield the $T$ and $Z$ values giving

$z(n) = T + Z + 1$ when $6 \leqslant n \leqslant 8$ and $z(n) = T + Z$ for $n \geqslant 9$. A Jupyter notebook, whose code is contained in appendix B.1 produced these parameters and verified that these parameters are valid and yield the conclusion of Lemma 2.40 for $n \leqslant 506$, which concludes our investigation.

In brief, the code picks a value of $n$, sets $d = n^*$, brute force loops over a number of valid values for the parameters $d_0, a, b$, computes the corresponding $T$ and $Z$ values defined in equations (2.21) and (2.29), and records the values of $d_0, a$, and $b$ which minimize $T + Z$. The following table contains some of the more interesting data points in table A.1.

| $n$ | $d_0$ | $d$ | $a$ | $b$ | $T$ | $Z$ |
|---|---|---|---|---|---|---|
| 6 | 0 | 2 | 0.18 | 0.29 | 10 | 4 |
| 7 | 0.539 | 2.5 | 0.2 | 0.28 | 7 | 4 |
| 8 | 0.992 | 3 | 0.16 | 0.41 | 7 | 3 |
| 9 | 0.882 | 3.5 | 0.17 | 0.4 | 6 | 3 |
| 10 | 1.17 | 4 | 0.23 | 0.41 | 5 | 3 |
| 11 | 1.674 | 4.5 | 0.14 | 0.37 | 5 | 3 |
| 12 | 2.088 | 5 | 0.27 | 0.41 | 4 | 3 |
| 13 | 2.255 | 5.5 | 0.2 | 0.37 | 4 | 3 |
| 14 | 2.484 | 6 | 0.16 | 0.35 | 4 | 3 |
| 15 | 2.958 | 6.5 | 0.13 | 0.34 | 4 | 3 |
| 16 | 3.136 | 7 | 0.11 | 0.32 | 4 | 3 |
| 17 | 3.904 | 7.5 | 0.32 | 0.42 | 3 | 3 |
| 18 | 4.158 | 8 | 0.27 | 0.39 | 3 | 3 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 36 | 8.268 | 17 | 0.08 | 0.26 | 3 | 3 |
| 37 | 7.728 | 17.5 | 0.08 | 0.25 | 3 | 3 |
| 38 | 11.454 | 18 | 0.44 | 0.48 | 2 | 3 |
| 39 | 11.799 | 18.5 | 0.4 | 0.45 | 2 | 3 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 215 | 59.907 | 106.5 | 0.03 | 0.87 | 3 | 2 |
| 216 | 59.136 | 107 | 0.03 | 0.87 | 3 | 2 |
| 217 | 67.3735 | 107.5 | 0.389816 | 0.881816 | 2 | 2 |
| 218 | 67.9042 | 108 | 0.399038 | 0.883038 | 2 | 2 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |

| $n$ | $d_0$ | $d$ | $a$ | $b$ | $T$ | $Z$ |
|---|---|---|---|---|---|---|
| 506 | 218.022 | 252 | 0.517076 | 0.927076 | 2 | 2 |
| 507 | 218.457 | 252.5 | 0.517138 | 0.927138 | 2 | 2 |

*Table 2.1.* Summary of parameter choices which minimize $T + Z$

### *2.7.5 Attaining Bounds With Examples*

Theorem 1.17 indicates that for $n \geqslant 217$, there are no more than 40 distinct solutions to equation (2.4) when $F(x, y)$ is a trinomial. For smaller $n$, this upper bound is even larger. Of interest is whether or not it is possible to find a particular trinomial for which (2.4) has 40 distinct solutions. The computer algebra system GP has a method called `thue` which, on input a Thue equation, will output the solutions to that Thue equation‡. The author has used this method to create a function in Sage which, on input a degree $n$ and height $H$, will compute the solutions to every trinomial Thue equation of degree $n$ and height $H$. The method can be found in appendix B.1 and the raw data can be found on the author's website at:

https://pages.uoregon.edu/gknapp4/files/trinomial_solution_data.zip

The maximal number of solutions to equation (2.4) for a trinomial $F(x, y)$ of degree $n$ and height $H$ are found in the two tables below. Notably, no trinomial has been found with more than 12 solutions to (2.4), which is far from the upper bound of 40. Moreover, while much of the data in the tables give the notion that the maximal number of solutions only depends on $H$, the column $H = 16$ confirms that the data supporting such a hypothesis is coincidental. A hyphen in the table means that the case in question has not yet been computed.

‡While the accuracy of GP's `thue` method relies on the Generalized Riemann Hypothesis to solve the Thue equation $F(x, y) = h$, our use of it does not because `thue` does not assume GRH when solving the specific equation $F(x, y) = \pm 1$.

|  | $H = 1$ | $H = 2$ | $H = 3$ | $H = 4$ | $H = 5$ | $H = 6$ | $H = 7$ | $H = 8$ | $H = 9$ |
|---|---|---|---|---|---|---|---|---|---|
| $n = 6$ | 8 | 6 | 8 | 8 | 6 | 6 | 6 | 6 | 8 |
| $n = 7$ | 8 | 6 | 8 | 8 | 6 | 6 | 6 | 6 | 8 |
| $n = 8$ | 8 | 6 | 8 | 8 | 6 | 6 | 6 | 6 | 8 |
| $n = 9$ | 8 | 6 | 8 | 8 | 6 | 6 | 6 | 6 | 8 |
| $n = 10$ | 8 | 6 | 8 | 8 | 6 | 6 | 6 | 6 | 8 |
| $n = 11$ | 8 | 6 | 8 | 8 | 6 | 6 | 6 | 6 | 8 |
| $n = 12$ | 8 | 6 | 8 | 8 | 6 | 6 | 6 | - | - |
| $n = 13$ | 8 | 6 | 8 | 8 | 6 | 6 | - | - | - |
| $n = 14$ | 8 | 6 | 8 | 8 | 6 | 6 | - | - | - |
| $n = 15$ | 8 | 6 | 8 | 8 | 6 | - | - | - | - |
| $n = 16$ | 8 | 6 | 8 | 8 | 6 | - | - | - | - |
| $n = 17$ | 8 | 6 | 8 | 8 | - | - | - | - | - |
| $n = 19$ | 8 | 6 | 8 | - | - | - | - | - | - |
| $n = 20$ | 8 | 6 | 8 | - | - | - | - | - | - |

*Table 2.2.* The maximal number of solutions to equation (2.4) for a trinomial $F(x, y)$ of degree $n$ and height $H \leqslant 9$.

|  | $H = 10$ | $H = 11$ | $H = 12$ | $H = 13$ | $H = 14$ | $H = 15$ | $H = 16$ | $H = 17$ |
|---|---|---|---|---|---|---|---|---|
| $n = 6$ | 6 | 6 | 6 | 6 | - | 6 | 12 | 6 |
| $n = 7$ | 6 | 6 | 6 | 6 | 6 | 6 | 8 | 6 |
| $n = 8$ | 6 | 6 | 6 | 6 | 6 | - | 12 | - |
| $n = 9$ | 6 | 6 | 6 | - | - | - | 8 | - |
| $n = 10$ | - | 6 | - | - | - | - | - | - |

*Table 2.3.* The maximal number of solutions to equation (2.4) for a trinomial $F(x, y)$ of degree $n$ and height $H \geqslant 10$.

# CHAPTER 3

# ROOT SEPARATION

## 3.1 Introduction

Recall that we used Rump's result (2.31) in our proof of Lemma 2.46. Rump's result bounding the distances between distinct roots of a given polynomial is not just useful as a technical tool, but also is part of a well-studied tradition regarding the geometry of roots of polynomials. This tradition includes Descartes' Rule of Signs, the Gauss-Lucas Theorem, and the Schinzel-Zassenhaus Conjecture, to name a few key results. Bounds on root separation in particular have computational applications to root-finding algorithms as Koiran outlines in [Koi19]. In this chapter, we focus primarily on bounding the separation between roots of a fixed polynomial. Recall the definition of a polynomial's separation, given in Definition 1.25 and restated here for convenience.

**Definition.** Given a polynomial $f(x) \in \mathbb{C}[x]$ with roots $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, the *separation* of $f(x)$ is the quantity

$$\operatorname{sep}(f) = \min_{\alpha_i \neq \alpha_j} |\alpha_i - \alpha_j|.$$

In particular, we look at upper bounds on the separation of a polynomial. Our main conjecture is Conjecture 1.33, which is reprinted here for convenience:

**Conjecture.** *Suppose $f(x) \in \mathbb{R}[x]$ is monic and separable of degree $n \geqslant 2$. If $f(x)$ has any real roots, then*

$$\operatorname{sep}(f) \ll_n M(f)^{1/(n-1)}.$$

*If $f(x)$ has only nonreal roots, then*

$$\operatorname{sep}(f) \ll_n M(f)^{1/n}.$$

In the course of this chapter, we prove Theorem 1.34, which we restate here for convenience.

**Theorem.** *Let $f(x) \in \mathbb{R}[x]$ be monic and separable with $\deg(f) = n \geqslant 2$ and suppose that any of the following conditions is met.*

1. $\deg(f) = 2$.

2. $\deg(f) = 3$.

3. $\deg(f) = 4$ *and* $f(x)$ *has no real roots.*

4. *Every root of* $f(x)$ *is real.*

*Then if* $f(x)$ *has any real roots,*

$$\operatorname{sep}(f) \ll_n M(f)^{1/(n-1)}.$$

*If* $f(x)$ *has only nonreal roots, then*

$$\operatorname{sep}(f) \ll_n M(f)^{1/n}.$$

In fact, we prove something more precise: Propositions 3.7 through 3.10 give (sometimes sharp) explicit bounds on $\operatorname{sep}(f)$ for the cases given in Theorem 1.34.

## 3.2 Context

### *3.2.1 Lower Bounds on Separation*

The starting point for much research around bounds on separation is Corollary 1.32, stated again here for convenience.

**Theorem** (Mahler)**.** *Suppose that* $f(x) \in \mathbb{Z}[x]$ *is separable of degree* $n \geqslant 2$*. Then*

$$\operatorname{sep}(f) > \frac{\sqrt{3}}{n^{(n+2)/2} M(f)^{n-1}}.$$

Mahler's theorem gives an important lower bound for separation in terms of Mahler measure. Many others have developed this theory further. For example, Rump removed the separability hypothesis when he proved in [Rum79] that

**Theorem 3.1** (Rump)**.** *Suppose that* $f(x) \in \mathbb{Z}[x]$ *has degree* $n$ *and let* $S$ *denote the sum of the absolute values of the coefficients of* $f(x)$*. Then*

$$\operatorname{sep}(f) > \frac{1}{2n^{n/2+2}(S+1)^n}.$$

Since $S \leqslant 2^n M(f)$, this yields the relation

$$\operatorname{sep}(f) > \frac{1}{2n^{n/2+2}(2^n M(f)+1)^n}$$

which is numerically worse than Mahler's theorem, but does encompass more cases.

Others—especially Bugeaud, Dujella, Pejković—have focused on improving the numerics of Mahler's theorem. To examine their results, we introduce some new notation.

*Notation* 3.2. Let $f(x) \in \mathbb{C}[x]$ be a polynomial. Then let

$$e(f) := -\frac{\log \operatorname{sep}(f)}{\log M(f)} \tag{3.1}$$

so that $e(f)$ always satisfies the relation

$$\operatorname{sep}(f) = \frac{1}{M(f)^{e(f)}}.$$

Then, for an integer $n \geqslant 2$, let

$$\begin{aligned} e(n) &:= \sup\{e(f) : f(x) \in \mathbb{Z}[x],\ \deg(f) = n,\ f \text{ is separable}\}, \\ e_{\mathrm{irr}}(n) &:= \sup\{e(f) : f(x) \in \mathbb{Z}[x],\ \deg(f) = n,\ f \text{ is irreducible}\}, \\ e^*_{\mathrm{red}}(n) &:= \sup\{e(f) : f(x) \in \mathbb{Z}[x],\ \deg(f) = n,\ f \text{ is reducible and monic}\} \end{aligned}$$

Now, $e(n)$ represents the minimal value of $e$ which makes the following statement true: there exists a constant $C(n)$ so that for every separable $f(x) \in \mathbb{Z}[x]$ of degree $n$,

$$\operatorname{sep}(f) > \frac{C(n)}{M(f)^e}.$$

The quantity $e_{\mathrm{irr}}(n)$ plays the same role, but for the more restrictive class of irreducible polynomials, so $e_{\mathrm{irr}}(n) \leqslant e(n)$ for all $n$. Mahler's theorem then implies that $e(n) \leqslant n - 1$, but it remains possible that $e(n)$ could be smaller. Likewise, Theorem 3.1 implies that $e^*_{\mathrm{red}}(n) \leqslant n$.

By constructing families of examples, Bugeaud and Dujella in [BD11] showed that

$$e_{\mathrm{irr}}(n) \geqslant \frac{n}{2} + \frac{n-2}{4(n-1)},$$

demonstrating that $n$ is at least the right order of magnitude for $e(n)$ and $e_{\mathrm{irr}}(n)$. On the other hand, Dujella and Pejković in [DP17] showed that

$$e^*_{\mathrm{red}}(n) \leqslant n - 2,$$

which indicates that Rump's Theorem 3.1 can very likely be improved.

Of course, the degree is not the only quantity which impacts the separation of a polynomial. The number of nonzero summands of $f(x)$ also impacts the separation, as we note that Example 1.30 shows that for monic binomials $f(x)$,

$$\operatorname{sep}(f) > 2\left(\frac{\pi}{n} + O\left(\frac{1}{n^3}\right)\right) M(f)^{1/n}.$$

Koiran, in [Koi19], uses Baker's bounds on linear forms in logarithms to show:

**Theorem 3.3** (Koiran). *Suppose $f(x) \in \mathbb{Z}[x]$ is a trinomial and let*

$$Y = \log \max(H(f), \deg(f)).$$

*Then for some absolute constant C,*

$$\operatorname{sep}(f) > \exp(-CY^3).$$

Each of these results has thus far explored lower bounds on the separation in terms of the Mahler measure. In the next section, we turn the question around and ask about upper bounds on separation in terms of the Mahler measure.

### 3.2.2 Upper Bounds on Separation

In the introduction to this dissertation, we noted that there is a trivial upper bound on separation given by the Mahler measure via inequality (1.7). This can be improved upon without too much work from a surprising source. Theorem 1.31 gives a lower bound on separation prima facie, but can be manipulated to produce an upper bound on separation.

**Proposition 3.4.** *Suppose that $f(x) \in \mathbb{C}[x]$ is separable of degree $n \geqslant 2$ and leading coefficient $b \neq 0$. Then*

$$\operatorname{sep}(f) < \frac{n^{\frac{n+2}{n^2-n-2}}}{3^{1/(n^2-n-2)}} \cdot \left(\frac{M(f)}{|b|}\right)^{\frac{2(n-1)}{n^2-n-2}}.$$

*If $n \geqslant 4$, then*

$$\operatorname{sep}(f) < n^{\frac{1}{n-3}} \left(\frac{M(f)}{|b|}\right)^{2/\left(n-\frac{1}{2}\right)}.$$

*Proof.* Observe first that since $f(x)$ is separable, we have

$$\begin{aligned} |\Delta_f| &= |b|^{2n-2} \prod_{1 \leqslant i < j \leqslant n} |\alpha_i - \alpha_j|^2 \\ &\geqslant |b|^{2n-2} \operatorname{sep}(f)^{n^2-n}. \end{aligned}$$

We can apply this fact along with Theorem 1.31 to find that

$$\begin{aligned} \operatorname{sep}(f) &> \frac{\sqrt{3|\Delta_f|}}{n^{(n+2)/2} M(f)^{n-1}} \\ &\geqslant \frac{\sqrt{3}|b|^{n-1} \operatorname{sep}(f)^{(n^2-n)/2}}{n^{(n+2)/2} M(f)^{n-1}} \end{aligned}$$

and rearranging yields

$$\operatorname{sep}(f) < \frac{n^{\frac{n+2}{n^2-n-2}}}{3^{1/(n^2-n-2)}} \cdot \left(\frac{M(f)}{|b|}\right)^{\frac{2(n-1)}{n^2-n-2}}.$$

If $n \geqslant 4$, it is easy to verify that $\frac{n+2}{n^2-n-2} \leqslant \frac{1}{n-3}$ and $\frac{2(n-1)}{n^2-n-2} \leqslant \frac{2}{n-\frac{1}{2}}$, which concludes the proof. ◻

Observe that Proposition 3.4 gives a much better bound that we found in (1.7).

Moreover, Proposition 3.4 lends more credence to the philosophy that "polynomial roots are not randomly distributed." Vaguely, what we mean by this is that while the roots of $f(x)$ must lie in the complex disk of radius $1 + \frac{H(f)}{|b|}$, they are not uniformly distributed within that disk. If they were, separation would satisfy a much different bound. Here, we examine the expected value of the minimum separation between two points when $n$ random points are selected inside a disk of specified radius. We follow a similar line of reasoning that can be found in Hernan Gonzalaz' StackExchange answer given at the URL in the footnotes.*

**Proposition 3.5.** *Suppose that n points $S = \{\alpha_1, \ldots, \alpha_n\}$ are scattered independently with uniform probability in the complex disk of radius $R > 0$, centered at $0$. Then the expected value of* $\operatorname{sep}(S) := \min_{i \neq j} |\alpha_i - \alpha_j|$ *is at least*

$$\frac{2\sqrt{2}}{3} \cdot \frac{R}{n}.$$

*Proof.* Let $p_n(x)$ denote the probability that $\operatorname{sep}(S) \geqslant x$. Then the expected separation is

$$E_{n,R} := \int_0^\infty p_n(x)\, dx.$$

We will place $n$ balls with diameter $x$ randomly uniformly inside $B_R(0)$. Say that those balls are $B_1, \ldots, B_n$ which have respective centers $\alpha_1, \ldots, \alpha_n$. For every $j \in \{\{i,k\} : 1 \leqslant i < k \leqslant n\}$, let $S_j = S_{i,k}$ denote the event that $B_i \cap B_k \neq \emptyset$. Now, $P(S_j)$ is equal to the probability that $\alpha_k$ is placed within distance $x$ of $\alpha_i$, which is at most $\frac{\pi x^2}{\pi R^2} = \frac{x^2}{R^2}$. Hence, $P(S_j) \leqslant \left(\frac{x}{R}\right)^2$.

Then we note that

$$p_n(x) = P(\cap_j S_j^c)$$

*https://math.stackexchange.com/questions/2005775/average-minimum-distance-between-n-points-generate-i-i-d-uniformly-in-the-bal

$$
\begin{aligned}
&= 1 - P(\cup_j S_j) \\
&\geqslant 1 - \sum_j P(S_j) \\
&\geqslant \max\left(1 - \binom{n}{2}\left(\frac{x}{R}\right)^2, 0\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
E_{n,R} &= \int_0^\infty p_n(x)\,dx \\
&\geqslant \int_0^{R/\sqrt{\binom{n}{2}}} 1 - \binom{n}{2}\left(\frac{x}{R}\right)^2\,dx \\
&= \frac{2\sqrt{2}R}{3\sqrt{n(n-1)}} \\
&\geqslant \frac{2\sqrt{2}}{3}\cdot\frac{R}{n}.
\end{aligned}
$$

□

Hence, if the roots of monic $f(x) \in \mathbb{C}[x]$ of degree $n$ were randomly uniformly distributed in the complex ball of radius $1 + H(f)$, we would expect their separation to be at least $\frac{2\sqrt{2}}{3} \cdot \frac{1+H(f)}{n}$. However, Proposition 3.4 shows this not to be the case: the separation is always much smaller for polynomials of large height.

The main question that we explore is whether or not Proposition 3.4 can be improved. Like those who study lower bounds on separation, we largely focus on the exponent of $M(f)$ in the relation $\operatorname{sep}(f) < C(n)M(f)^e$. This will give us a quantification for how nonuniform the distribution of polynomial roots is.

### 3.3 Data and Conjectures

For the remainder of this chapter, we will examine polynomials with real coefficients, keeping in mind the fact that our ultimate goal is to understand polynomials with integer coefficients. We are searching for upper bounds on separation in terms of the Mahler measure and we aim to produce those bounds by selecting a large number of "random" polynomials with a certain set of characteristics, plotting the separations against the Mahler measures of those polynomials, then examining the data to see what region of the plane these points can lie in.

One of the major challenges with this approach is determining what we mean by a "random" polynomial. For polynomials in $\mathbb{R}[x]$, one could choose a polynomial of degree

$n$ by choosing a random vector from a box in $\mathbb{R}^{n+1}$ to use as the coefficients. Alternatively, one could select a polynomial of degree $n$ by choosing a random element of the star-body of polynomials of degree $n$ with Mahler measure at most 1 (or some other bound), as Sinclair and Yattselev do in [SY15]. Here, we choose to select a random polynomial by choosing its roots uniformly from a specified compact subset of the complex plane. The major advantage of this approach is the speed with which we can compute Mahler measure and separation. Were we to select the coefficients of the random polynomial, we would first have to find the roots of the polynomial before we could compute the separation and Mahler measure, a notoriously difficult problem.

To produce this data, we wrote Sage code which is fully specified and documented in Appendix B.3. The most important method in that Appendix is `PlotMahlerVSep` which takes as input a number of trials $N$, a number of roots $n$, a "radius" $R$, a number of real roots $r$ (which defaults to $r = 0$), a discriminant lower bound $d$ (which defaults to $d = 0$), and a string indicating the region of the complex plane from which the roots will be chosen. The method then creates an empty plot $P$ and conducts the following experiment $N$ times:

1. Select $r$ random elements from the specified region of $\mathbb{R}$ uniformly and select $\frac{n-r}{2}$ random elements and their complex conjugates from the specified region of $\mathbb{C}$ uniformly.

2. Set $f(x)$ to be the monic polynomial with those $n$ elements as roots. If $|\Delta_f| \geqslant d$, proceed to the next step. Otherwise, return to the previous step.

3. Compute the separation and Mahler measure of $f(x)$ and add the point $(M(f), \text{sep}(f))$ to the plot $P$.

If the region is selected to be a box, the real roots are chosen from the interval $[-R, R]$ while the complex roots are chosen from the box $\max(|\mathfrak{R}[z]|, |\mathfrak{I}[z]|) \leqslant R$. If the region is selected to be a ball, the real roots are chosen from the interval $[-R, R]$, while the complex roots are chosen from the ball $|z| \leqslant R$. If the region is selected to be an annulus, the real roots are chosen from the set $[-R, -1/R] \cup [1/R, R]$ while the complex roots are chosen from the region $1/R \leqslant |z| \leqslant R$.

The user is able to specify an annular region because Sinclair and Yattselev show in [SY15] that a "typical" polynomial has roots clustering around the unit circle in $\mathbb{C}$. This allows us to potentially spot differences between data sets for typical and atypical polynomials. However, since we want to prove results for all polynomials, we typically select our region to be a box or a ball.

Finally, we note that since we are interested in identifying the optimal exponent $e$ in a relation like $\operatorname{sep}(f) < C(n) \cdot M(f)^e$, we often find it helpful to plot separation against Mahler measure on log-log axes.

To identify the most general patterns available, we begin with the least restrictive computations in the least expensive cases. Figure 3.1 comes from the command

```
PlotMahlerVSep(50000,2,2,numRealRoots=2),
```

which selects 50,000 quadratic polynomials whose roots lie in the real interval $[-2, 2]$ and plots their separations against their Mahler measures:

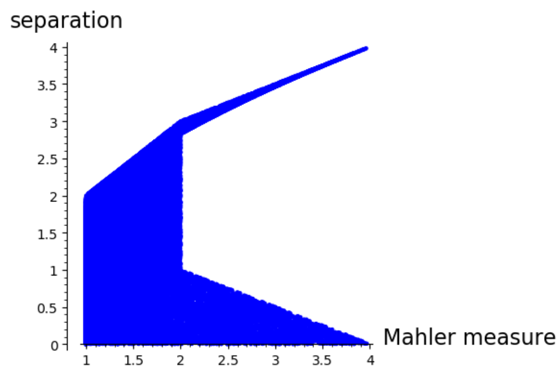


*Figure 3.1.* Separation against Mahler measure for monic quadratic polynomials with two real roots

In contrast, Figure 3.2 comes from the command

```
PlotMahlerVSep(50000,2,2,numRealRoots=0),
```

which selects 50,000 quadratic polynomials whose roots are distinct complex conjugates which lie in the complex ball of radius 2, and plots their separations against their Mahler measures.

Immediately, we can see the role that the presence of real roots plays in determining the relation between separation and Mahler measure. Some of these differences are caused by the anomalies present in the degree two case: for instance, the “empty” space in Figure 3.1 comes from the fact that attaining a Mahler measure near four for a degree two polynomial requires both roots to have absolute value approximately two, so the two roots must be either very close together (both near 2 or both near $-2$) or very far apart (one root

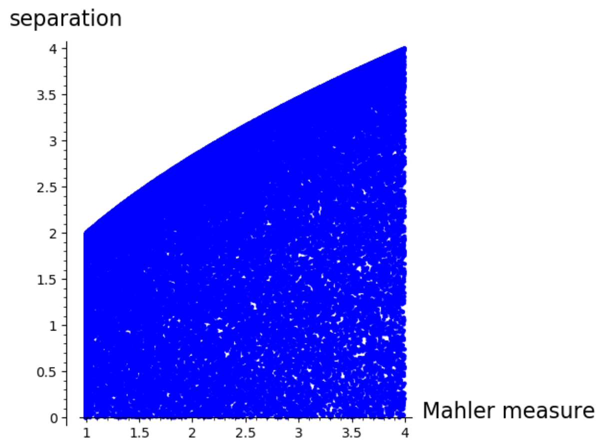


*Figure 3.2.* Separation against Mahler measure for monic quadratic polynomials with no real roots

near 2 and the other near $-2$). If we allowed larger roots, no such "empty space" would appear: the polynomial $g(x) = x^2 - 2\sqrt{5}x + 4$ has $M(g) = 4$ and $\mathrm{sep}(g) = 2$, for instance.

However, the sense that the real roots impact the relationship between separation and Mahler measure is borne out in further examples. Consider the degree three case, where a polynomial in $\mathbb{R}[x]$ can either have one real root or three real roots. Figure 3.3 displays the results of the command

```
PlotMahlerVSep(50000,3,2,numRealRoots=3),
```

which selected 50,000 cubic polynomials with three real roots coming from the interval $[-2, 2]$. In some contrast, Figure 3.4 displays the results of the command

```
PlotMahlerVSep(50000,3,2,numRealRoots=1),
```

which selected 50,000 cubic polynomials with one real root from the interval $[-2, 2]$ and a single pair of complex conjugate roots from the complex ball $|z| \leqslant 1$.

Some of the differing appearance on these two plots comes from the different scaling factors on the axes and some is due to a weaker version of the "empty space effect" that we saw in the degree two case, but there are meaningful differences between the graphs that are easy to spot: one of those is that the curve which bounds the blue region from above appears to pass through the point $(1, 1)$ in Figure 3.3 in contrast to the point $(1.7, 1)$ in Figure 3.4. In fact, this is because a polynomial $f(x)$ with $M(f) = 1$ must have all of its roots satisfying $|\alpha| \leqslant 1$. The maximum separation of the roots of a cubic polynomial if all three roots are real is then 1 (if the roots are located at $-1, 0, 1$). On the other hand, if one

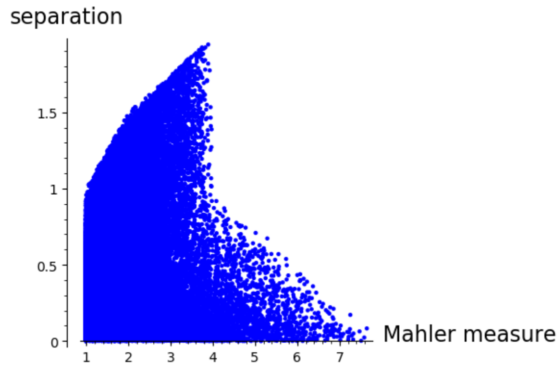


*Figure 3.3.* Separation against Mahler measure for monic cubic polynomials with three real roots

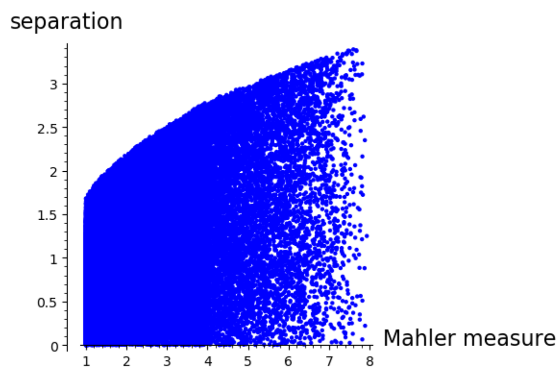


*Figure 3.4.* Separation against Mahler measure for monic cubic polynomials with one real root

root is real and two are complex, the maximum separation is then $2\sin(\pi/3) = \sqrt{3}$ when the roots are the three cube roots of unity.

Hence, if we are trying to understand the relation $\text{sep}(f) < C(n)M(f)^{e(n)}$, we should aim to incorporate differences for polynomials with different signatures.

**Definition 3.6.** For a polynomial $f(x) \in \mathbb{R}[x]$, the *signature* of $f(x)$ is the pair $(r, s)$ if $f(x)$ has exactly $r$ roots in $\mathbb{R}$ and $s$ pairs of distinct complex conjugate roots.

Moreover, we should begin using log-log plots to illustrate our data if we specifically want to examine the relation $\text{sep}(f) < C(n)M(f)^{e(n)}$ since on a log-log plot, this bound is linear with slope $e(n)$:

$$\log \text{sep}(f) < \log C(n) + e(n) \log M(f).$$

In fact, this is the exact behavior we see when plotting data on log-log axes. Figure 3.5 is the result of the same operation as Figure 3.2, but plotted on log-log axes and Figure 3.6 is the result of the same operation as Figure 3.1, but plotted on log-log axes.

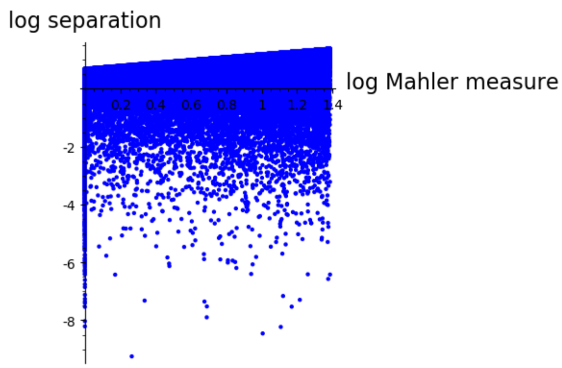


*Figure 3.5.* Logarithmic separation against logarithmic Mahler measure for monic quadratic polynomials with no real roots

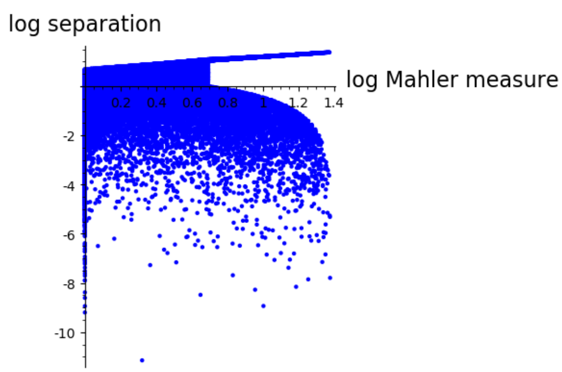


*Figure 3.6.* Logarithmic separation against logarithmic Mahler measure for monic quadratic polynomials with two real roots

As expected, the upper bounds on these two regions appears to be linear. More than that, the upper bound on the blue region in Figure 3.5 appears to approximately be the line $y = x + 0.7$ and the upper bound on the blue region in Figure 3.6 appears to approximately be the line $y = \frac{x}{2} + 0.6$. Because this is the relatively simple quadratic case, we are able to precisely determine these bounds later and we do so later in Proposition 3.7.

Continuing to the cubic case, we again see that when plotted on log-log axes, the upper bound on the blue region appears linear. Figure 3.7 recreates the data of Figure 3.3 on log-log axes and Figure 3.8 recreates the data of Figure 3.4 on log-log axes.

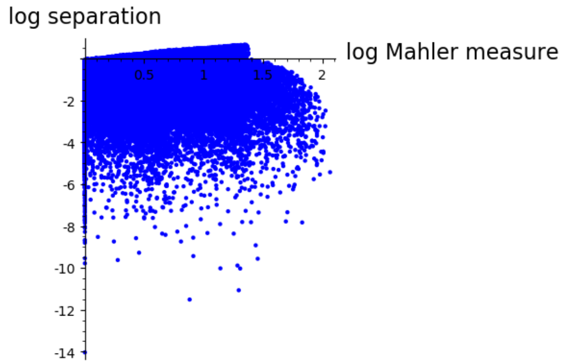


*Figure 3.7.* Logarithmic separation against logarithmic Mahler measure for monic cubic polynomials with three real roots

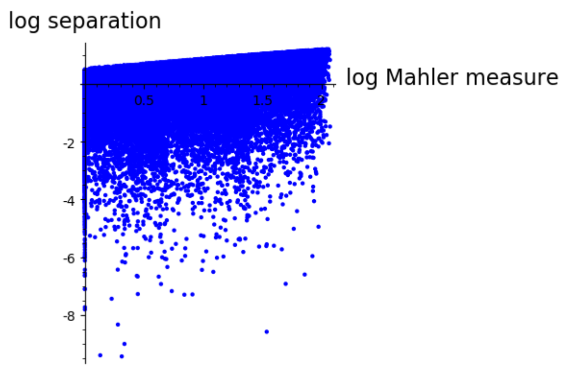


*Figure 3.8.* Logarithmic separation against logarithmic Mahler measure for monic cubic polynomials with one real root

Very loose estimates of the upper bounds for the blue regions in these figures indicate that the blue region in Figure 3.7 is bounded above by the line $y = x/2$ and the blue region in Figure 3.8 is bounded above by the line $y = x/2 + 0.5$.

We continue to the quartic case to finish identifying the pattern. Figure 3.9 shows the results of selecting 100,000 polynomials of degree 4 with four real roots in the interval $[-10, 10]$ and plotting their logarithmic separations against their logarithmic Mahler measures. Figure 3.10 does the same thing, but for polynomials with two real roots in the interval $[-10, 10]$ and a single pair of complex conjugate roots in the ball $|z| \leqslant 10$. Figure 3.11 does the same, but for polynomials with two pairs of complex conjugate roots in the ball $|z| \leqslant 10$.

Again eyeballing the upper bounds of the blue region, it appears that the blue region in Figure 3.9 is bounded above by the line $y = x/3 - 1/6$, the blue region in Figure 3.10 is bounded above by the line $y = x/3 + 1/6$, yet the blue region in Figure 3.11 is bounded

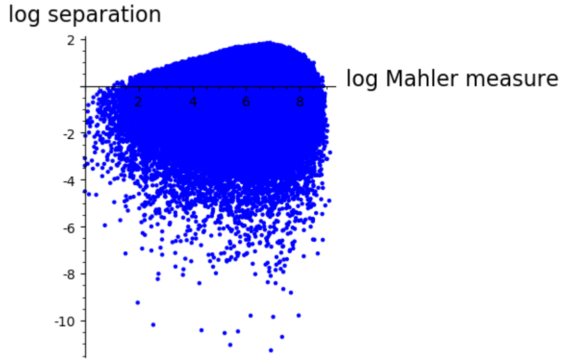


*Figure 3.9.* Logarithmic separation against logarithmic Mahler measure for monic quartic polynomials with four real roots

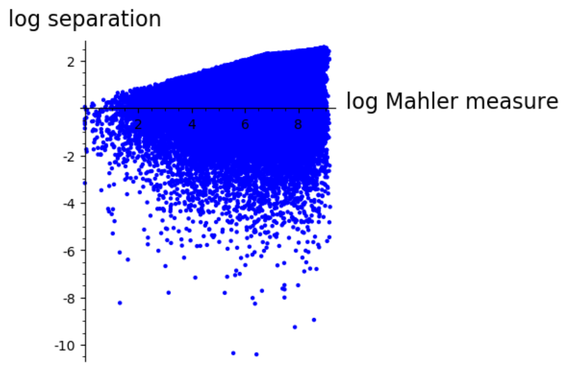


*Figure 3.10.* Logarithmic separation against logarithmic Mahler measure for monic quartic polynomials with two real roots

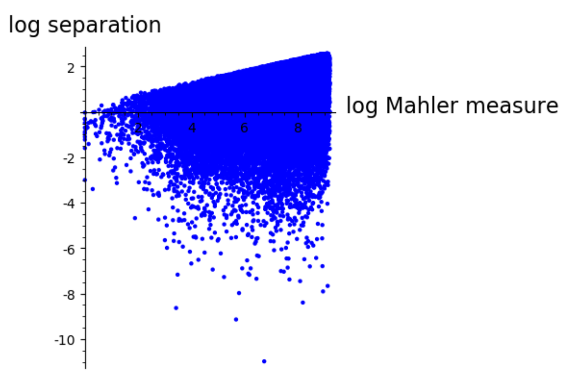


*Figure 3.11.* Logarithmic separation against logarithmic Mahler measure for monic quartic polynomials with no real roots

above by the line $y = x/4 + 1/3$.

Since the slope of the linear upper bound corresponds to the exponent $e(n)$ in the relation $\text{sep}(f) < C(n)M(f)^{e(n)}$, we are now able to see the data that led to (and supports)

Conjecture 1.33. This conjecture is true in certain cases and we verify this in our proof of Theorem 1.34.

### 3.4 Proof of Theorem 1.34

Our main goal of this section is to prove Theorem 1.34, which we will do in pieces. We will moreover give specific constants for various degrees and signatures of $f(x)$ in the inequality $\operatorname{sep}(f) \ll_n M(f)^{1/(n-1)}$. We prove the following propositions:

**Proposition 3.7.** *Let $f(x) \in \mathbb{R}[x]$ have degree 2 and leading coefficient $b$. If $f(x)$ has no real roots, then*

$$\operatorname{sep}(f) \leqslant 2\left(\frac{M(f)}{|b|}\right)^{1/2}.$$

*If $f(x)$ has two real roots, then*

$$\operatorname{sep}(f) \leqslant \left(\frac{M(f)}{|b|}\right) + 1.$$

*Moreover, these bounds are sharp.*

**Proposition 3.8.** *Let $f(x) \in \mathbb{R}[x]$ be separable of degree $n \geqslant 3$ with leading coefficient $b$. Suppose further that all $n$ of the roots of $f$ are real. Then*

$$\operatorname{sep}(f) \leqslant \frac{8.2}{n} \cdot \left(\frac{M(f)}{|b|}\right)^{1/(n-1)}.$$

**Proposition 3.9.** *Let $f(x) \in \mathbb{R}[x]$ have degree 3 and leading coefficient $b$. If $f(x)$ has exactly one real root, then*

$$\operatorname{sep}(f) < \sqrt{3}\left(\frac{M(f)}{|b|}\right)^{1/2}.$$

**Proposition 3.10.** *Suppose that $f(x) \in \mathbb{R}[x]$ has degree 4, leading coefficient $b$ and no real roots. Then*

$$\operatorname{sep}(f) \leqslant \sqrt{2}\left(\frac{M(f)}{|b|}\right)^{1/4}.$$

*Moreover, this bound is sharp.*

These propositions cover each of the cases stated in Theorem 1.34, so once we have given proofs of these four propositions, we will have proved Theorem 1.34.

*Proof of Proposition 3.7.* Suppose that the two roots of $f(x)$ are $\alpha_1$ and $\alpha_2$.
Case 1: First, suppose that $f(x)$ has no real roots. In this case, $|\alpha_1| = |\alpha_2|$ and hence,

$$\operatorname{sep}(f) = |\alpha_1 - \alpha_2| \leqslant 2|\alpha_1| = 2|\alpha_1|^{1/2}|\alpha_2|^{1/2} \leqslant 2\left(\frac{M(f)}{|b|}\right)^{1/2}.$$

This bound is achieved by the family of polynomials $P_r(x) = (x - ir)(x + ir)$ for $r \geqslant 1$.
Case 2: Second, suppose that $f(x)$ has two real roots. Without loss of generality, we can assume that $\alpha_2 \geqslant \max(1, |\alpha_1|)$. If $\alpha_1 \geqslant -1$, then we have

$$\operatorname{sep}(f) \leqslant \alpha_2 + 1 \leqslant \frac{M(f)}{|b|} + 1$$

and we are done. Else, $\alpha_1 < -1$ and we now have

$$\operatorname{sep}(f) = \alpha_2 - \alpha_1$$

and

$$\frac{M(f)}{|b|} + 1 = -\alpha_1\alpha_2 + 1.$$

We now observe that the inequality $\operatorname{sep}(f) \leqslant \frac{M(f)}{|b|} + 1$ is equivalent to the inequality $\alpha_2 - 1 \leqslant -\alpha_1(\alpha_2 - 1)$, which is true by virtue of the fact that $\alpha_2 \geqslant 1$ and $\alpha_1 \leqslant -1$. This bound is achieved by the family of polynomials $Q_r(x) = (x + 1)(x - r)$ for $r \geqslant 1$. □

With the proof of Proposition 3.7, we can now re-consider Figures 3.1 and 3.2 with the added upper bounds.

In Figure 3.12, we can see that the upper bound $y = x + 1$ is attained regularly; it only doesn't appear sharp for $M(f) > 2$ for the following reason. If you choose a polynomial $f(x)$ with $M(f) > 2$ where both roots are chosen from the interval $[-2, 2]$, this forces both roots to have opposite signs and live outside the interval $[-1, 1]$, which artificially inflates the Mahler measure relative to the separation.

In Figure 3.13, we can see that the upper bound $y = 2\sqrt{x}$ is attained regularly.

We next consider the totally real case for polynomials of degree at least 3.

*Proof of Proposition 3.8.* We may assume that $|b| = 1$; if not, replace $M(f)$ everywhere in this proof by $\frac{M(f)}{|b|}$ and all statements will still be true.

First, denote the closest root of $f(x)$ to 0 by $\alpha$ and let $r = \operatorname{sep}(f)$. Suppose that there are $s$ roots of $f(x)$ which are less than $\alpha$ and $t$ roots of $f(x)$ which are greater than $\alpha$. Define

$$g(x) = \prod_{i=1}^{t}(x - \alpha - ri) \cdot (x - \alpha) \cdot \prod_{j=1}^{s}(x - \alpha + rj)$$

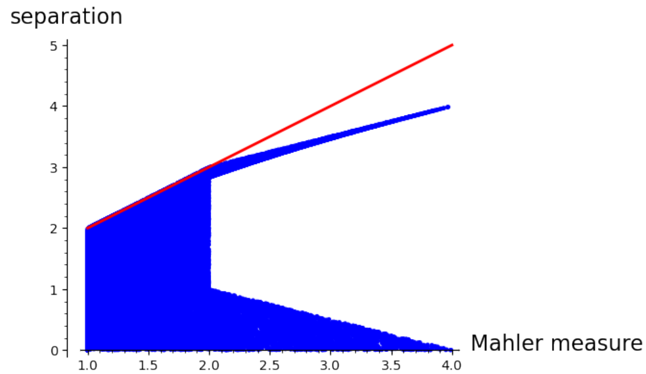


*Figure 3.12.* Mahler measure against separation for monic quadratic polynomials with two real roots and the sharp upper bound of $y = x + 1$.

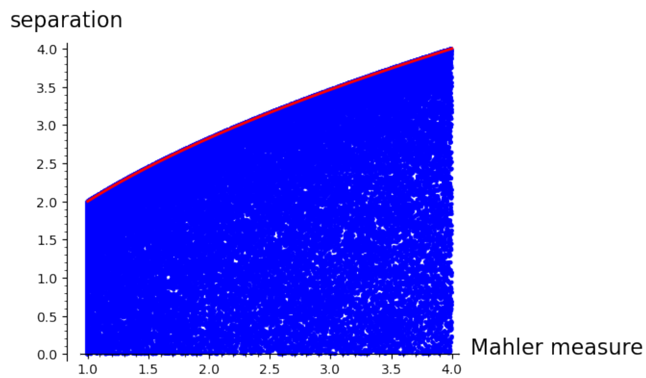


*Figure 3.13.* Mahler measure against separation for monic quadratic polynomials with no real roots and the sharp upper bound of $y = 2x^{1/2}$.

and observe that we have $M(f) \geqslant M(g)$ because the roots of $g$ are no further from the origin than the corresponding roots of $f$. Furthermore, we have $\operatorname{sep}(f) = r = \operatorname{sep}(g)$, so it suffices to prove the Proposition for $g(x)$.

Let $\beta$ be the closest root of $g(x)$ to the origin and write

$$g(x) = \prod_{i=1}^{T}(x - \beta - ri) \cdot (x - \beta) \cdot \prod_{j=1}^{S}(x - \beta + rj).$$

Since $\beta$ is the closest root of $g(x)$ to 0, we note that $|\beta| \leqslant r/2$. We may also assume without loss of generality that $\beta \leqslant 0$ (else, we may apply the same proof to $g(-x)$). We now have

$$M(g) \geqslant \prod_{i=1}^{T} |\beta + ri| \cdot \prod_{j=1}^{S} |\beta - rj|$$

$$\geqslant \prod_{i=1}^{T}\left(ri - \frac{r}{2}\right) \cdot \prod_{j=1}^{S}\left(rj + \frac{r}{2}\right)$$

$$= r^{n-1} \cdot \prod_{i=1}^{T}\left(i - \frac{1}{2}\right) \cdot \prod_{j=1}^{S}\left(j + \frac{1}{2}\right). \tag{3.2}$$

Getting a handle on the lower bound given in equation (3.2) will require some use of the gamma function. We use the fact that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ together with the usual fact that $\Gamma(x) = (x-1)\Gamma(x-1)$ to note that

$$\prod_{i=1}^{T}\left(i - \frac{1}{2}\right) = \frac{\Gamma\left(T + \frac{1}{2}\right)}{\sqrt{\pi}}$$

and

$$\prod_{j=1}^{S}\left(j + \frac{1}{2}\right) = \frac{2\Gamma\left(S + \frac{3}{2}\right)}{\sqrt{\pi}}.$$

Now applying Gautschi's inequality yields

$$\prod_{i=1}^{T}\left(i - \frac{1}{2}\right) = \frac{\Gamma\left(T + \frac{1}{2}\right)}{\sqrt{\pi}} > \frac{\Gamma(T+1)}{\sqrt{\pi(T+1)}} = \frac{T!}{\sqrt{\pi(T+1)}},$$

$$\prod_{j=1}^{S}\left(j + \frac{1}{2}\right) = \frac{2\Gamma\left(S + \frac{3}{2}\right)}{\sqrt{\pi}} > \frac{2\Gamma(S+2)}{\sqrt{\pi(S+2)}} = \frac{2(S+1)!}{\sqrt{\pi(S+2)}}.$$

Now we have

$$M(g) \geqslant r^{n-1} \cdot \frac{T!}{\sqrt{\pi(T+1)}} \cdot \frac{2(S+1)!}{\sqrt{\pi(S+2)}} \tag{3.3}$$

from inequality (3.2) and we aim to estimate the right-hand side of this inequality from below in terms of $n$. We have the restrictions $0 \leqslant S, T \leqslant n$ and $S + T + 1 = n$, so we can replace $S + 1$ in equation (3.3) by $n - T$ to find

$$\begin{aligned} \frac{T!}{\sqrt{\pi(T+1)}} \cdot \frac{2(S+1)!}{\sqrt{\pi(S+2)}} &= \frac{T!}{\sqrt{\pi(T+1)}} \cdot \frac{2(n-T)!}{\sqrt{\pi(n-T+1)}} \\ &= \frac{2n!}{\pi\binom{n}{T}\sqrt{(T+1)(n-T+1)}} \\ &\geqslant \frac{2n!}{\pi\binom{n}{\lfloor n/2 \rfloor}\sqrt{(n/2+1)(n-n/2+1)}} \end{aligned}$$

Now, using the fact that $\binom{n}{\lfloor n/2 \rfloor} \leqslant \frac{2^{n+1}}{\sqrt{\pi(2n+1)}}$ gives

$$\begin{aligned}
\frac{T!}{\sqrt{\pi(T+1)}} \cdot \frac{2(S+1)!}{\sqrt{\pi(S+2)}} &\geqslant \frac{2n!}{\pi\binom{n}{\lfloor n/2 \rfloor}\sqrt{(n/2+1)(n-n/2+1)}} \\
&\geqslant \frac{n! \cdot \sqrt{2n+1}}{\sqrt{\pi}2^{n-1}(n+2)} \\
&> \frac{\sqrt{2n}(n/e)^n e^{\frac{1}{12n+1}}\sqrt{2n+1}}{2^{n-1}(n+2)} \\
&\geqslant \frac{n(n/e)^n e^{\frac{1}{12n+1}}}{2^{n-2}(n+2)} \\
&\geqslant 4\left(\frac{n}{2e}\right)^n e^{\frac{1}{12n+1}}\left(1-\frac{2}{n+2}\right) \\
&\geqslant 2.4\left(\frac{n}{2e}\right)^n
\end{aligned}$$

by way of Stirling's approximation and the fact that $n \geqslant 3$.

Finally, we are able to return to (3.3) to find that that

$$\begin{aligned}
M(g) &\geqslant r^{n-1} \cdot \frac{n! \cdot \sqrt{2n+1}}{\sqrt{\pi}2^{n-1}(n+2)} \\
&\geqslant \operatorname{sep}(g)^{n-1} 2.4\left(\frac{n}{2e}\right)^n
\end{aligned} \tag{3.4}$$

which concludes the proof. In particular, it yields

$$\operatorname{sep}(g) \leqslant \left(\frac{M(g)}{2.4}\right)^{\frac{1}{n-1}}\left(\frac{2e}{n}\right)^{\frac{n}{n-1}} \leqslant \frac{(2e)^{3/2}}{\sqrt{2.4}n^{\frac{n}{n-1}}} M(g)^{\frac{1}{n-1}} \leqslant \frac{8.2}{n} M(g)^{\frac{1}{n-1}}$$

under the assumption that $n \geqslant 3$. □

With this proof complete, this gives us a chance to revisit Figures 3.3 and 3.9. We can replicate the results of the degree 3 case in Figure 3.3 with the upper bound given by the more precise bound

$$\operatorname{sep}(f) \leqslant \left(\frac{\sqrt{\pi}2^{n-1}(n+2)}{n! \cdot \sqrt{2n+1}}\right)^{1/(n-1)} M(f)^{1/(n-1)} \tag{3.5}$$

given as inequality (3.4) in the proof of Proposition 3.8. However, Figure 3.14 indicates that the constant is not optimal.

Similarly, we replicate the results of the degree 4 case in Figure 3.9 with the upper bound coming from (3.5) and again, Figure 3.15 demonstrates that the constant is not optimal. However, we suspect that for large $n$, the inequality becomes sharper.

Next, we consider cubic polynomials which have only 1 real root.

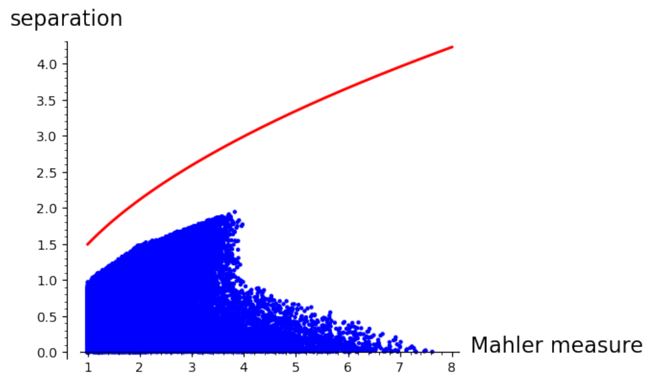


*Figure 3.14.* Mahler measure against separation for monic cubic polynomials with 3 real roots and the upper bound $y = \sqrt{10\sqrt{\pi}/(3\sqrt{7})} * \sqrt{x}$.

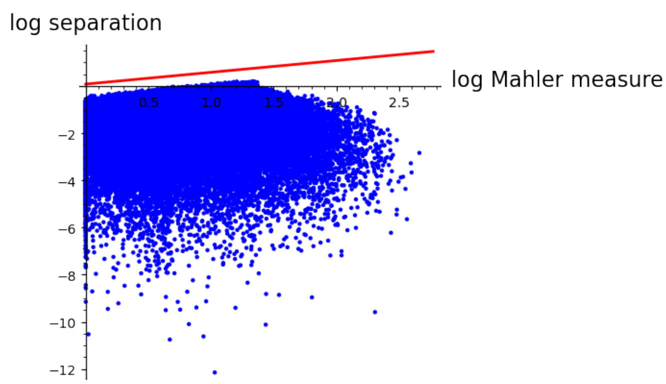


*Figure 3.15.* Mahler measure against separation for monic quartic polynomials with 4 real roots and the upper bound $y = \log(\sqrt{2\sqrt{\pi}/3}) + x/2$.

*Proof of Proposition 3.9.* We may assume that $|b| = 1$; if not, replace $M(f)$ everywhere in this proof by $\frac{M(f)}{|b|}$ and all statements will still be true. Suppose that the real root of $f(x)$ is $\alpha$ and without loss of generality, we may assume that $\alpha \geqslant 0$. Let $\beta$ denote the complex root of $f(x)$ with positive imaginary part.

We claim that we may assume that $\Re[\beta] \leqslant 0$. If not, then set $\beta' = -\Re[\beta] + i\Im[\beta]$ and $g(x) = (x-\alpha)(x-\beta')(x-\overline{\beta'})$. Since $\mathrm{sep}(g) \geqslant \mathrm{sep}(f)$ and $M(g) = M(f)$, proving the proposition for $g(x)$ will prove it for $f(x)$. Hence, we only need prove the proposition under the assumption that $\Re[\beta] \leqslant 0$.

We next make a few reductions.

Let $R = |\beta|$. Note that if $\Im[\beta] \leqslant \frac{\sqrt{3}R}{2}$, then

$$\mathrm{sep}(f) \leqslant |\beta - \overline{\beta}| = 2\Im[\beta] \leqslant \sqrt{3}R \leqslant \sqrt{3}M(f)^{1/2}$$

and we are done.

Hence, for the rest of the proof, assume that $\Im[\beta] > \frac{\sqrt{3}R}{2}$. Note that this automatically implies that $|\Re[\beta]| < \frac{R}{2}$. This means that

$$\beta \in S := \left\{ z \in \mathbb{C} \mid \Re[z] \leqslant 0, \Im[z] \geqslant \frac{\sqrt{3}|z|}{2} \right\}$$

which, in the real plane, is the slice of the second quadrant bounded by the lines $x = 0$ and $y = -\sqrt{3}x$.

Our next goal is to reduce to the case where $\alpha, \beta$, and $\overline{\beta}$ form an equilateral triangle in the complex plane. Before we can do this, we claim that if $t$ is the unique point in $\mathbb{R}$ with $t > \Re[\beta]$ which forms an equilateral triangle with $\beta$ and $\overline{\beta}$, then $t \geqslant 0$. It can be easily checked that $t = \sqrt{3}\Im[\beta] + \Re[\beta]$ and using the fact that $\Im[\beta] \geqslant \frac{\sqrt{3}R}{2}$ and $\Re[\beta] \geqslant -\frac{R}{2}$ shows that in fact, $t \geqslant \frac{R}{2} > 0$.

Suppose then, that $\alpha \geqslant t$. Then set $h(x) = (x-t)(x-\beta)(x-\overline{\beta})$. Then $\operatorname{sep}(h) = 2\Im[\beta] = \operatorname{sep}(h)$ and $M(f) \geqslant M(h)$, so it suffices to prove the proposition for $h(x)$. Hence, we may assume that $\alpha \leqslant t$.

Suppose that $0 \leqslant \alpha \leqslant t$ so that $|\alpha - \beta| = \operatorname{sep}(f)$. Let $\beta'$ be the unique complex number with $\Re[\beta'] = \Re[\beta]$, which has $\Im[\beta] > 0$, and which forms an equilateral triangle with $\alpha$ and $\overline{\beta'}$. Note that this point is the intersection of the lines $\alpha = \sqrt{3}\Im[z] + \Re[z]$ and $\Re[z] = \Re[\beta]$. Note also that $\Im[\beta'] \leqslant \Im[\beta]$ because

$$\sqrt{3}\Im[\beta'] + \Re[\beta'] = \alpha \leqslant t = \sqrt{3}\Im[\beta] + \Re[\beta]$$

and $\Re[\beta'] = \Re[\beta]$. Set $j(x) = (x-\alpha)(x-\beta')(x-\overline{\beta'})$ and note that $\operatorname{sep}(j) = |\alpha - \beta| = \operatorname{sep}(f)$ and $M(j) \leqslant M(f)$ since $\Re[\beta'] = \Re[\beta]$ and $\Im[\beta'] \leqslant \Im[\beta]$, so $|\beta'| \leqslant |\beta|$. Hence, it suffices to prove the proposition for $j(x)$ and we may assume that $\alpha \geqslant t$.

For the remainder of the proof then, we have $\alpha = t = \sqrt{3}\Im[\beta] + \Re[\beta]$. So $\beta$ lies on the line in the complex plane defined by $\alpha = \sqrt{3}\Im[z] + \Re[z]$ and above the line $\Im[z] = -\sqrt{3}\Re[z]$. Letting $x$ denote the real part of $\beta$ and letting $y$ denote the imaginary part of $\beta$, we then have $\alpha = \sqrt{3}y + x$ and $y \geqslant -\sqrt{3}x$. The intersection of these two lines is the point $\frac{-\alpha}{2} + i\frac{\sqrt{3}\alpha}{2}$. Since $x \leqslant 0$, it must be the case that $\beta$ lies on the line segment between the points $i\frac{\alpha}{\sqrt{3}}$ and $-\frac{\alpha}{2} + i\frac{\sqrt{3}\alpha}{2}$. This implies that $|\beta| \leqslant \alpha$ and moreover, that

$$y \leqslant \frac{\sqrt{3}\alpha}{2}.$$

Note also that if $y < \frac{\sqrt{3}}{2}$, then

$$\operatorname{sep}(f) \leqslant 2y < \sqrt{3} \leqslant \sqrt{3}M(f)^{1/2}$$

and we are done. So we may assume that $y \geqslant \frac{\sqrt{3}}{2}$.

Hence, we are left to prove that $\operatorname{sep}(f) \leqslant \sqrt{3}M(f)^{1/2}$ under the assumption that

$$\frac{\sqrt{3}}{2} \leqslant y \leqslant \frac{\sqrt{3}\alpha}{2}.$$

We consider the ratio

$$\frac{3M(f)}{\operatorname{sep}(f)^2} \geqslant \frac{3\alpha|\beta|^2}{4y^2} := Y$$

and we aim to show that $Y \geqslant 1$. We first wish to express $Y$ in terms of $\alpha$ and $y$:

$$\begin{aligned} Y &= \frac{3\alpha|\beta|^2}{4y^2} \\ &= \frac{3\alpha(x^2 + y^2)}{4y^2} \\ &= \frac{3\alpha\left((\alpha - \sqrt{3}y)^2 + y^2\right)}{4y^2} \\ &= \frac{3(\alpha^3 - 2\sqrt{3}\alpha^2 y + 4y^2\alpha)}{4y^2}. \end{aligned}$$

We claim that under our assumptions, $Y$ is a nondecreasing function of $\alpha$. To see this, note that

$$\begin{aligned} \frac{\partial Y}{\partial \alpha} &= \frac{9\alpha^2 - 12\sqrt{3}\alpha y + 12y^2}{4y^2} \\ &= \frac{\left(3\alpha - 2\sqrt{3}y\right)^2}{4y^2} \end{aligned}$$

is never negative. Since $\alpha \geqslant \frac{2}{\sqrt{3}}y$, we must have

$$\begin{aligned} Y &= \frac{3\alpha^2 - 2\sqrt{3}\alpha y + 4y^2}{4y^2}\alpha \\ &\geqslant \frac{4y^2 - 4y^2 + 4y^2}{4y^2} \cdot \frac{2}{\sqrt{3}}y \\ &\geqslant 1 \end{aligned}$$

since $y \geqslant \frac{\sqrt{3}}{2}$, which concludes the proof that $\operatorname{sep}(f) \leqslant \sqrt{3}M(f)^{1/2}$. □

With this proof complete, we have the chance to revisit Figure 3.4 along with its upper bound. We recreate this in Figure 3.16.

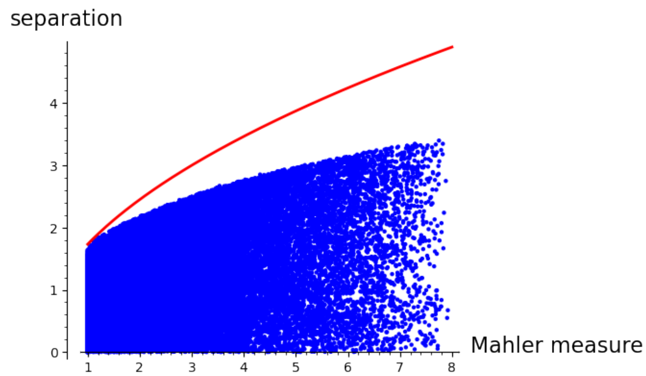


*Figure 3.16.* Mahler measure against separation for monic cubic polynomials with only one real root and the upper bound $y = \sqrt{3x}$.

Here, we see different behavior than we saw in Figures 3.14 and 3.15. In Figure 3.16, we see that the constant is optimal, though it appears that the exponent is not optimal. However, the family of polynomials

$$f_t(x) = (x-1)\left(x - (1 - t\sqrt{3} + it)\right)\left(x - (1 - t\sqrt{3} - it)\right)$$

has $\operatorname{sep}(f_t) \sim M(f_t)^{1/2}$, so the exponent is also optimal. As a result, we see that a relation of the form $\operatorname{sep}(f) \leqslant a \cdot M(f)^b$ is not subtle enough to capture the relation between separation and Mahler measure for cubic polynomials with signature $(1,1)$. This also explains why the proof of Proposition 3.9 is as complicated as it is.

However, the proof of Proposition 3.10 is more natural and this will show up in the sharpness of the bound we prove.

*Proof of Proposition 3.10.* For simplicity, assume $|b| = 1$. If $|b| \neq 1$, replace every instance of $M(f)$ by $\frac{M(f)}{|b|}$ and the proof is identical. Suppose that the roots of $f(x)$ are $\alpha, \bar{\alpha}, \beta, \bar{\beta}$ where $\Im[\alpha], \Im[\beta] > 0$ and $|\alpha| = r \leqslant |\beta| = R$. We first note that

$$\operatorname{sep}(f) = \min(2\Im[\alpha], 2\Im[\beta], |\alpha - \beta|).$$

We have the following two cases:

Case 1: $2r \leqslant R$

In this case, we note that

$$\operatorname{sep}(f) \leqslant 2\Im[\alpha] \leqslant 2r \leqslant 2r^{1/2}\left(\frac{R}{2}\right)^{1/2} = \sqrt{2}r^{1/2}R^{1/2} \leqslant \sqrt{2}M(f)^{1/4}.$$

Case 2: $r \leqslant R < 2r$

In this case, we first observe that if $\Im[\alpha] < \frac{\sqrt{2}}{2} r^{1/2} R^{1/2}$ or if $\Im[\beta] < \frac{\sqrt{2}}{2} r^{1/2} R^{1/2}$, then we are done because

$$\operatorname{sep}(f) \leqslant \min(2\Im[\alpha], 2\Im[\beta]) \leqslant \sqrt{2} r^{1/2} R^{1/2} \leqslant \sqrt{2} M(f)^{1/4}.$$

Hence,

$$\alpha \in \left\{ z \in \mathbb{C} : |z| = r \text{ and } \Im[z] \geqslant \frac{\sqrt{2}}{2} r^{1/2} R^{1/2} \right\} =: S$$

and

$$\beta \in \left\{ z \in \mathbb{C} : |z| = R \text{ and } \Im[z] \geqslant \frac{\sqrt{2}}{2} r^{1/2} R^{1/2} \right\} =: T.$$

As a result, we have

$$\begin{aligned} \operatorname{sep}(f) &\leqslant |\alpha - \beta| \\ &\leqslant \sup_{\substack{z_1 \in S \\ z_2 \in T}} |z_1 - z_2| \\ &= \left| \left( \sqrt{r^2 - \frac{rR}{2}} + i \cdot \frac{\sqrt{2}}{2} r^{1/2} R^{1/2} \right) - \left( -\sqrt{R^2 - \frac{rR}{2}} + i \cdot \frac{\sqrt{2}}{2} r^{1/2} R^{1/2} \right) \right| \\ &= \sqrt{r^2 - \frac{rR}{2}} + \sqrt{R^2 - \frac{rR}{2}}. \end{aligned}$$

We claim that

$$\sqrt{r^2 - \frac{rR}{2}} + \sqrt{R^2 - \frac{rR}{2}} \leqslant \sqrt{2} r^{1/2} R^{1/2}.$$

To see this, divide both sides of the inequality by $r^{1/2} R^{1/2}$ to obtain the equivalent inequality

$$\sqrt{\frac{r}{R} - \frac{1}{2}} + \sqrt{\frac{R}{r} - \frac{1}{2}} \leqslant \sqrt{2}.$$

Observe that this new inequality only depends on the ratio $x = \frac{R}{r}$, which we have bounded by $1 \leqslant x < 2$. It is now a simple calculus problem to show that

$$\sqrt{\frac{1}{x} - \frac{1}{2}} + \sqrt{x - \frac{1}{2}} \leqslant \sqrt{2}$$

for all $1 \leqslant x < 2$ and the proof that $\operatorname{sep}(f) \leqslant \sqrt{2} M(f)^{1/4}$ is complete.

To see that the bound is sharp, consider the family of polynomials given in Example 1.30. □

Finally, we revisit Figure 3.11 and add on the upper bound from Proposition 3.10. As we predicted before beginning the proof, Figure 3.17 shows that the bound $\text{sep}(f) \leqslant \sqrt{2}M(f)^{1/4}$ is attained regularly.

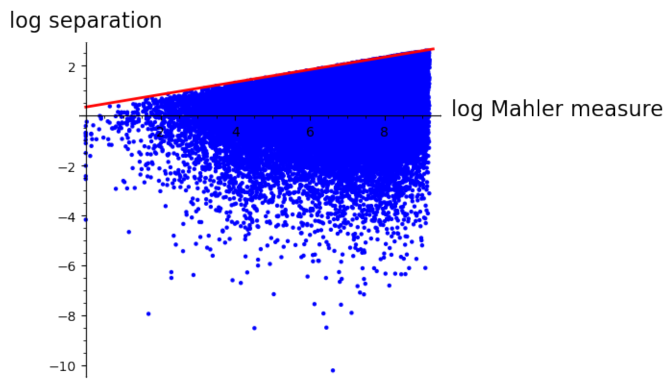


*Figure 3.17.* Logarithmic Mahler measure against logarithmic separation for monic quartic polynomials with no real roots and the upper bound $y = \log(\sqrt{2}) + x/4$

We conclude by noting that these proofs are more or less the limits of the elementary approach. An elementary proof of Conjecture 1.33 is likely possible for quartic polynomials with signature $(2, 1)$, but will be tedious and long. A full proof of Conjecture 1.33 is likely to require a more clever approach: perhaps an induction which relates $\text{sep}(f)$ to $\text{sep}(f')$ and $M(f)$ to $M(f')$ or an appeal to a sphere-packing problem that has already been solved. However, it is encouraging for the truth of the conjecture that it holds in nontrivial, low-degree cases.

# APPENDIX A

## PARAMETER CHOICES FOR SECTION 2.7.4

| $n$ | $d_0$ | $d$ | $a$ | $b$ | $T$ | $Z$ |
|---|---|---|---|---|---|---|
| 6 | 0 | 2 | 0.18 | 0.29 | 10 | 4 |
| 7 | 0.539 | 2.5 | 0.2 | 0.28 | 7 | 4 |
| 8 | 0.992 | 3 | 0.16 | 0.41 | 7 | 3 |
| 9 | 0.882 | 3.5 | 0.17 | 0.4 | 6 | 3 |
| 10 | 1.17 | 4 | 0.23 | 0.41 | 5 | 3 |
| 11 | 1.674 | 4.5 | 0.14 | 0.37 | 5 | 3 |
| 12 | 2.088 | 5 | 0.27 | 0.41 | 4 | 3 |
| 13 | 2.255 | 5.5 | 0.2 | 0.37 | 4 | 3 |
| 14 | 2.484 | 6 | 0.16 | 0.35 | 4 | 3 |
| 15 | 2.958 | 6.5 | 0.13 | 0.34 | 4 | 3 |
| 16 | 3.136 | 7 | 0.11 | 0.32 | 4 | 3 |
| 17 | 3.904 | 7.5 | 0.32 | 0.42 | 3 | 3 |
| 18 | 4.158 | 8 | 0.27 | 0.39 | 3 | 3 |
| 19 | 4.544 | 8.5 | 0.23 | 0.36 | 3 | 3 |
| 20 | 4.712 | 9 | 0.21 | 0.35 | 3 | 3 |
| 21 | 4.86 | 9.5 | 0.19 | 0.33 | 3 | 3 |
| 22 | 5.418 | 10 | 0.17 | 0.32 | 3 | 3 |
| 23 | 5.369 | 10.5 | 0.16 | 0.31 | 3 | 3 |
| 24 | 5.664 | 11 | 0.15 | 0.31 | 3 | 3 |
| 25 | 5.858 | 11.5 | 0.14 | 0.3 | 3 | 3 |
| 26 | 6.148 | 12 | 0.13 | 0.29 | 3 | 3 |
| 27 | 6.66 | 12.5 | 0.12 | 0.29 | 3 | 3 |
| 28 | 7.308 | 13 | 0.11 | 0.28 | 3 | 3 |
| 29 | 6.776 | 13.5 | 0.11 | 0.28 | 3 | 3 |
| 30 | 7.56 | 14 | 0.1 | 0.27 | 3 | 3 |
| 31 | 7.074 | 14.5 | 0.1 | 0.27 | 3 | 3 |
| 32 | 8.296 | 15 | 0.09 | 0.27 | 3 | 3 |
| 33 | 7.614 | 15.5 | 0.09 | 0.26 | 3 | 3 |
| 34 | 7.154 | 16 | 0.09 | 0.26 | 3 | 3 |
| 35 | 8.758 | 16.5 | 0.08 | 0.26 | 3 | 3 |

| $n$ | $d_0$ | $d$ | $a$ | $b$ | $T$ | $Z$ |
|---|---|---|---|---|---|---|
| 36 | 8.268 | 17 | 0.08 | 0.26 | 3 | 3 |
| 37 | 7.728 | 17.5 | 0.08 | 0.25 | 3 | 3 |
| 38 | 11.454 | 18 | 0.44 | 0.48 | 2 | 3 |
| 39 | 11.799 | 18.5 | 0.4 | 0.45 | 2 | 3 |
| 40 | 11.968 | 19 | 0.38 | 0.43 | 2 | 3 |
| 41 | 11.946 | 19.5 | 0.37 | 0.42 | 2 | 3 |
| 42 | 12.462 | 20 | 0.35 | 0.41 | 2 | 3 |
| 43 | 12.797 | 20.5 | 0.33 | 0.39 | 2 | 3 |
| 44 | 12.936 | 21 | 0.32 | 0.38 | 2 | 3 |
| 45 | 13.266 | 21.5 | 0.31 | 0.37 | 2 | 3 |
| 46 | 13.596 | 22 | 0.3 | 0.37 | 2 | 3 |
| 47 | 13.926 | 22.5 | 0.29 | 0.36 | 2 | 3 |
| 48 | 14.256 | 23 | 0.28 | 0.35 | 2 | 3 |
| 49 | 14.144 | 23.5 | 0.28 | 0.35 | 2 | 3 |
| 50 | 14.464 | 24 | 0.27 | 0.34 | 2 | 3 |
| 51 | 15.015 | 24.5 | 0.26 | 0.33 | 2 | 3 |
| 52 | 14.868 | 25 | 0.26 | 0.33 | 2 | 3 |
| 53 | 15.183 | 25.5 | 0.25 | 0.32 | 2 | 3 |
| 54 | 15.006 | 26 | 0.25 | 0.32 | 2 | 3 |
| 55 | 16.064 | 26.5 | 0.24 | 0.32 | 2 | 3 |
| 56 | 16.64 | 27 | 0.23 | 0.31 | 2 | 3 |
| 57 | 16.443 | 27.5 | 0.23 | 0.31 | 2 | 3 |
| 58 | 17.29 | 28 | 0.22 | 0.3 | 2 | 3 |
| 59 | 17.073 | 28.5 | 0.22 | 0.3 | 2 | 3 |
| 60 | 16.836 | 29 | 0.21 | 0.8 | 3 | 2 |
| 61 | 16.579 | 29.5 | 0.2 | 0.8 | 3 | 2 |
| 62 | 16.588 | 30 | 0.19 | 0.8 | 3 | 2 |
| 63 | 16.878 | 30.5 | 0.18 | 0.8 | 3 | 2 |
| 64 | 17.168 | 31 | 0.17 | 0.8 | 3 | 2 |
| 65 | 18.06 | 31.5 | 0.16 | 0.8 | 3 | 2 |
| 66 | 18.972 | 32 | 0.15 | 0.8 | 3 | 2 |
| 67 | 17.416 | 32.5 | 0.15 | 0.8 | 3 | 2 |
| 68 | 18.644 | 33 | 0.14 | 0.8 | 3 | 2 |

| $n$ | $d_0$ | $d$ | $a$ | $b$ | $T$ | $Z$ |
|---|---|---|---|---|---|---|
| 69 | 17.655 | 33.5 | 0.14 | 0.8 | 3 | 2 |
| 70 | 19.234 | 34 | 0.13 | 0.8 | 3 | 2 |
| 71 | 19.86 | 34.5 | 0.15 | 0.81 | 3 | 2 |
| 72 | 18.144 | 35 | 0.15 | 0.81 | 3 | 2 |
| 73 | 19.437 | 35.5 | 0.14 | 0.81 | 3 | 2 |
| 74 | 18.338 | 36 | 0.14 | 0.81 | 3 | 2 |
| 75 | 20.007 | 36.5 | 0.13 | 0.81 | 3 | 2 |
| 76 | 18.868 | 37 | 0.13 | 0.81 | 3 | 2 |
| 77 | 20.938 | 37.5 | 0.12 | 0.81 | 3 | 2 |
| 78 | 19.764 | 38 | 0.12 | 0.81 | 3 | 2 |
| 79 | 22.26 | 38.5 | 0.11 | 0.81 | 3 | 2 |
| 80 | 21.056 | 39 | 0.11 | 0.81 | 3 | 2 |
| 81 | 19.812 | 39.5 | 0.11 | 0.81 | 3 | 2 |
| 82 | 23.16 | 40 | 0.1 | 0.81 | 3 | 2 |
| 83 | 21.896 | 40.5 | 0.1 | 0.81 | 3 | 2 |
| 84 | 20.988 | 41 | 0.1 | 0.81 | 3 | 2 |
| 85 | 20.05 | 41.5 | 0.1 | 0.81 | 3 | 2 |
| 86 | 21.924 | 42 | 0.11 | 0.82 | 3 | 2 |
| 87 | 20.961 | 42.5 | 0.11 | 0.82 | 3 | 2 |
| 88 | 24.128 | 43 | 0.1 | 0.82 | 3 | 2 |
| 89 | 22.734 | 43.5 | 0.1 | 0.82 | 3 | 2 |
| 90 | 21.726 | 44 | 0.1 | 0.82 | 3 | 2 |
| 91 | 25.86 | 44.5 | 0.09 | 0.82 | 3 | 2 |
| 92 | 24.852 | 45 | 0.09 | 0.82 | 3 | 2 |
| 93 | 23.814 | 45.5 | 0.09 | 0.82 | 3 | 2 |
| 94 | 22.746 | 46 | 0.09 | 0.82 | 3 | 2 |
| 95 | 22.099 | 46.5 | 0.09 | 0.82 | 3 | 2 |
| 96 | 26.904 | 47 | 0.08 | 0.82 | 3 | 2 |
| 97 | 25.816 | 47.5 | 0.08 | 0.82 | 3 | 2 |
| 98 | 25.164 | 48 | 0.08 | 0.82 | 3 | 2 |
| 99 | 24.021 | 48.5 | 0.08 | 0.82 | 3 | 2 |
| 100 | 23.324 | 49 | 0.08 | 0.82 | 3 | 2 |
| 101 | 22.607 | 49.5 | 0.08 | 0.82 | 3 | 2 |

| $n$ | $d_0$ | $d$ | $a$ | $b$ | $T$ | $Z$ |
|---|---|---|---|---|---|---|
| 102 | 28.674 | 50 | 0.07 | 0.82 | 3 | 2 |
| 103 | 27.987 | 50.5 | 0.08 | 0.83 | 3 | 2 |
| 104 | 27.28 | 51 | 0.08 | 0.83 | 3 | 2 |
| 105 | 26.052 | 51.5 | 0.08 | 0.83 | 3 | 2 |
| 106 | 25.3 | 52 | 0.08 | 0.83 | 3 | 2 |
| 107 | 24.528 | 52.5 | 0.08 | 0.83 | 3 | 2 |
| 108 | 30.96 | 53 | 0.07 | 0.83 | 3 | 2 |
| 109 | 29.697 | 53.5 | 0.07 | 0.83 | 3 | 2 |
| 110 | 28.93 | 54 | 0.07 | 0.83 | 3 | 2 |
| 111 | 28.143 | 54.5 | 0.07 | 0.83 | 3 | 2 |
| 112 | 27.336 | 55 | 0.07 | 0.83 | 3 | 2 |
| 113 | 26.509 | 55.5 | 0.07 | 0.83 | 3 | 2 |
| 114 | 25.662 | 56 | 0.07 | 0.83 | 3 | 2 |
| 115 | 24.795 | 56.5 | 0.07 | 0.83 | 3 | 2 |
| 116 | 32.804 | 57 | 0.06 | 0.83 | 3 | 2 |
| 117 | 31.977 | 57.5 | 0.06 | 0.83 | 3 | 2 |
| 118 | 31.13 | 58 | 0.06 | 0.83 | 3 | 2 |
| 119 | 30.263 | 58.5 | 0.06 | 0.83 | 3 | 2 |
| 120 | 29.376 | 59 | 0.06 | 0.83 | 3 | 2 |
| 121 | 29.05 | 59.5 | 0.06 | 0.83 | 3 | 2 |
| 122 | 28.128 | 60 | 0.06 | 0.83 | 3 | 2 |
| 123 | 26.595 | 60.5 | 0.07 | 0.84 | 3 | 2 |
| 124 | 35.164 | 61 | 0.06 | 0.84 | 3 | 2 |
| 125 | 34.257 | 61.5 | 0.06 | 0.84 | 3 | 2 |
| 126 | 33.33 | 62 | 0.06 | 0.84 | 3 | 2 |
| 127 | 32.383 | 62.5 | 0.06 | 0.84 | 3 | 2 |
| 128 | 31.416 | 63 | 0.06 | 0.84 | 3 | 2 |
| 129 | 30.429 | 63.5 | 0.06 | 0.84 | 3 | 2 |
| 130 | 30.048 | 64 | 0.06 | 0.84 | 3 | 2 |
| 131 | 29.026 | 64.5 | 0.06 | 0.84 | 3 | 2 |
| 132 | 28.62 | 65 | 0.06 | 0.84 | 3 | 2 |
| 133 | 27.563 | 65.5 | 0.06 | 0.84 | 3 | 2 |
| 134 | 27.132 | 66 | 0.06 | 0.84 | 3 | 2 |

| $n$ | $d_0$ | $d$ | $a$ | $b$ | $T$ | $Z$ |
|---|---|---|---|---|---|---|
| 135 | 38.409 | 66.5 | 0.05 | 0.84 | 3 | 2 |
| 136 | 37.392 | 67 | 0.05 | 0.84 | 3 | 2 |
| 137 | 36.355 | 67.5 | 0.05 | 0.84 | 3 | 2 |
| 138 | 35.964 | 68 | 0.05 | 0.84 | 3 | 2 |
| 139 | 34.892 | 68.5 | 0.05 | 0.84 | 3 | 2 |
| 140 | 34.476 | 69 | 0.05 | 0.84 | 3 | 2 |
| 141 | 33.369 | 69.5 | 0.05 | 0.84 | 3 | 2 |
| 142 | 32.928 | 70 | 0.05 | 0.84 | 3 | 2 |
| 143 | 31.786 | 70.5 | 0.05 | 0.84 | 3 | 2 |
| 144 | 31.32 | 71 | 0.05 | 0.84 | 3 | 2 |
| 145 | 30.844 | 71.5 | 0.05 | 0.84 | 3 | 2 |
| 146 | 30.358 | 72 | 0.05 | 0.84 | 3 | 2 |
| 147 | 38.394 | 72.5 | 0.05 | 0.85 | 3 | 2 |
| 148 | 37.948 | 73 | 0.05 | 0.85 | 3 | 2 |
| 149 | 36.771 | 73.5 | 0.05 | 0.85 | 3 | 2 |
| 150 | 36.3 | 74 | 0.05 | 0.85 | 3 | 2 |
| 151 | 35.088 | 74.5 | 0.05 | 0.85 | 3 | 2 |
| 152 | 34.592 | 75 | 0.05 | 0.85 | 3 | 2 |
| 153 | 34.086 | 75.5 | 0.05 | 0.85 | 3 | 2 |
| 154 | 32.824 | 76 | 0.05 | 0.85 | 3 | 2 |
| 155 | 32.293 | 76.5 | 0.05 | 0.85 | 3 | 2 |
| 156 | 31.752 | 77 | 0.05 | 0.85 | 3 | 2 |
| 157 | 31.201 | 77.5 | 0.05 | 0.85 | 3 | 2 |
| 158 | 30.64 | 78 | 0.05 | 0.85 | 3 | 2 |
| 159 | 30.069 | 78.5 | 0.05 | 0.85 | 3 | 2 |
| 160 | 46.56 | 79 | 0.04 | 0.85 | 3 | 2 |
| 161 | 45.298 | 79.5 | 0.04 | 0.85 | 3 | 2 |
| 162 | 44.802 | 80 | 0.04 | 0.85 | 3 | 2 |
| 163 | 43.505 | 80.5 | 0.04 | 0.85 | 3 | 2 |
| 164 | 42.984 | 81 | 0.04 | 0.85 | 3 | 2 |
| 165 | 42.453 | 81.5 | 0.04 | 0.85 | 3 | 2 |
| 166 | 41.912 | 82 | 0.04 | 0.85 | 3 | 2 |
| 167 | 40.55 | 82.5 | 0.04 | 0.85 | 3 | 2 |

| $n$ | $d_0$ | $d$ | $a$ | $b$ | $T$ | $Z$ |
|---|---|---|---|---|---|---|
| 168 | 39.984 | 83 | 0.04 | 0.85 | 3 | 2 |
| 169 | 39.408 | 83.5 | 0.04 | 0.85 | 3 | 2 |
| 170 | 38.822 | 84 | 0.04 | 0.85 | 3 | 2 |
| 171 | 38.226 | 84.5 | 0.04 | 0.85 | 3 | 2 |
| 172 | 37.62 | 85 | 0.04 | 0.85 | 3 | 2 |
| 173 | 37.004 | 85.5 | 0.04 | 0.85 | 3 | 2 |
| 174 | 36.378 | 86 | 0.04 | 0.85 | 3 | 2 |
| 175 | 35.742 | 86.5 | 0.04 | 0.85 | 3 | 2 |
| 176 | 35.096 | 87 | 0.04 | 0.85 | 3 | 2 |
| 177 | 44.772 | 87.5 | 0.04 | 0.86 | 3 | 2 |
| 178 | 44.166 | 88 | 0.04 | 0.86 | 3 | 2 |
| 179 | 43.55 | 88.5 | 0.04 | 0.86 | 3 | 2 |
| 180 | 42.924 | 89 | 0.04 | 0.86 | 3 | 2 |
| 181 | 42.288 | 89.5 | 0.04 | 0.86 | 3 | 2 |
| 182 | 41.642 | 90 | 0.04 | 0.86 | 3 | 2 |
| 183 | 40.986 | 90.5 | 0.04 | 0.86 | 3 | 2 |
| 184 | 40.32 | 91 | 0.04 | 0.86 | 3 | 2 |
| 185 | 39.644 | 91.5 | 0.04 | 0.86 | 3 | 2 |
| 186 | 38.958 | 92 | 0.04 | 0.86 | 3 | 2 |
| 187 | 38.262 | 92.5 | 0.04 | 0.86 | 3 | 2 |
| 188 | 37.556 | 93 | 0.04 | 0.86 | 3 | 2 |
| 189 | 36.84 | 93.5 | 0.04 | 0.86 | 3 | 2 |
| 190 | 36.114 | 94 | 0.04 | 0.86 | 3 | 2 |
| 191 | 36.309 | 94.5 | 0.04 | 0.86 | 3 | 2 |
| 192 | 35.568 | 95 | 0.04 | 0.86 | 3 | 2 |
| 193 | 34.817 | 95.5 | 0.04 | 0.86 | 3 | 2 |
| 194 | 34.056 | 96 | 0.04 | 0.86 | 3 | 2 |
| 195 | 34.236 | 96.5 | 0.04 | 0.86 | 3 | 2 |
| 196 | 33.46 | 97 | 0.04 | 0.86 | 3 | 2 |
| 197 | 32.674 | 97.5 | 0.04 | 0.86 | 3 | 2 |
| 198 | 31.878 | 98 | 0.04 | 0.86 | 3 | 2 |
| 199 | 57.289 | 98.5 | 0.03 | 0.86 | 3 | 2 |
| 200 | 56.608 | 99 | 0.03 | 0.86 | 3 | 2 |

| $n$ | $d_0$ | $d$ | $a$ | $b$ | $T$ | $Z$ |
|---|---|---|---|---|---|---|
| 201 | 55.917 | 99.5 | 0.03 | 0.86 | 3 | 2 |
| 202 | 54.23 | 100 | 0.03 | 0.86 | 3 | 2 |
| 203 | 53.514 | 100.5 | 0.03 | 0.86 | 3 | 2 |
| 204 | 52.788 | 101 | 0.03 | 0.86 | 3 | 2 |
| 205 | 52.052 | 101.5 | 0.03 | 0.86 | 3 | 2 |
| 206 | 52.312 | 102 | 0.03 | 0.86 | 3 | 2 |
| 207 | 51.561 | 102.5 | 0.03 | 0.86 | 3 | 2 |
| 208 | 50.8 | 103 | 0.03 | 0.86 | 3 | 2 |
| 209 | 50.029 | 103.5 | 0.03 | 0.86 | 3 | 2 |
| 210 | 49.248 | 104 | 0.03 | 0.86 | 3 | 2 |
| 211 | 48.457 | 104.5 | 0.03 | 0.86 | 3 | 2 |
| 212 | 47.656 | 105 | 0.03 | 0.86 | 3 | 2 |
| 213 | 47.886 | 105.5 | 0.03 | 0.86 | 3 | 2 |
| 214 | 47.07 | 106 | 0.03 | 0.86 | 3 | 2 |
| 215 | 59.907 | 106.5 | 0.03 | 0.87 | 3 | 2 |
| 216 | 59.136 | 107 | 0.03 | 0.87 | 3 | 2 |
| 217 | 67.3735 | 107.5 | 0.389816 | 0.881816 | 2 | 2 |
| 218 | 67.9042 | 108 | 0.399038 | 0.883038 | 2 | 2 |
| 219 | 68.4369 | 108.5 | 0.408258 | 0.884258 | 2 | 2 |
| 220 | 69.0792 | 109 | 0.408477 | 0.884477 | 2 | 2 |
| 221 | 69.4002 | 109.5 | 0.417694 | 0.885694 | 2 | 2 |
| 222 | 70.047 | 110 | 0.417909 | 0.885909 | 2 | 2 |
| 222 | 69.504 | 110 | 0.404909 | 0.884909 | 2 | 2 |
| 223 | 69.824 | 110.5 | 0.405123 | 0.885123 | 2 | 2 |
| 224 | 70.144 | 111 | 0.405336 | 0.885336 | 2 | 2 |
| 225 | 70.464 | 111.5 | 0.405547 | 0.885547 | 2 | 2 |
| 226 | 71.89 | 112 | 0.405757 | 0.885757 | 2 | 2 |
| 227 | 72.215 | 112.5 | 0.405966 | 0.885966 | 2 | 2 |
| 228 | 72.54 | 113 | 0.406173 | 0.886173 | 2 | 2 |
| 229 | 73.986 | 113.5 | 0.406379 | 0.886379 | 2 | 2 |
| 230 | 74.316 | 114 | 0.406583 | 0.886583 | 2 | 2 |
| 231 | 74.646 | 114.5 | 0.406786 | 0.886786 | 2 | 2 |
| 232 | 74.976 | 115 | 0.406988 | 0.886988 | 2 | 2 |

| $n$ | $d_0$ | $d$ | $a$ | $b$ | $T$ | $Z$ |
|---|---|---|---|---|---|---|
| 233 | 76.447 | 115.5 | 0.407188 | 0.887188 | 2 | 2 |
| 234 | 76.782 | 116 | 0.407387 | 0.887387 | 2 | 2 |
| 235 | 77.117 | 116.5 | 0.407585 | 0.887585 | 2 | 2 |
| 236 | 77.452 | 117 | 0.407782 | 0.887782 | 2 | 2 |
| 237 | 78.948 | 117.5 | 0.407977 | 0.887977 | 2 | 2 |
| 238 | 79.288 | 118 | 0.408171 | 0.888171 | 2 | 2 |
| 239 | 79.628 | 118.5 | 0.408364 | 0.888364 | 2 | 2 |
| 240 | 79.968 | 119 | 0.408556 | 0.888556 | 2 | 2 |
| 241 | 81.489 | 119.5 | 0.408747 | 0.888747 | 2 | 2 |
| 242 | 81.834 | 120 | 0.408936 | 0.888936 | 2 | 2 |
| 243 | 82.179 | 120.5 | 0.409124 | 0.889124 | 2 | 2 |
| 244 | 82.524 | 121 | 0.409311 | 0.889311 | 2 | 2 |
| 245 | 84.07 | 121.5 | 0.409497 | 0.889497 | 2 | 2 |
| 246 | 84.42 | 122 | 0.409682 | 0.889682 | 2 | 2 |
| 247 | 84.77 | 122.5 | 0.409865 | 0.889865 | 2 | 2 |
| 248 | 85.12 | 123 | 0.410048 | 0.890048 | 2 | 2 |
| 249 | 85.47 | 123.5 | 0.410229 | 0.890229 | 2 | 2 |
| 250 | 87.046 | 124 | 0.410409 | 0.890409 | 2 | 2 |
| 251 | 87.401 | 124.5 | 0.410588 | 0.890588 | 2 | 2 |
| 252 | 87.756 | 125 | 0.410766 | 0.890766 | 2 | 2 |
| 253 | 88.111 | 125.5 | 0.410943 | 0.890943 | 2 | 2 |
| 254 | 88.466 | 126 | 0.411119 | 0.891119 | 2 | 2 |
| 255 | 90.072 | 126.5 | 0.411294 | 0.891294 | 2 | 2 |
| 256 | 90.432 | 127 | 0.411468 | 0.891468 | 2 | 2 |
| 257 | 90.792 | 127.5 | 0.411641 | 0.891641 | 2 | 2 |
| 258 | 91.152 | 128 | 0.411813 | 0.891813 | 2 | 2 |
| 259 | 91.512 | 128.5 | 0.411984 | 0.891984 | 2 | 2 |
| 260 | 93.148 | 129 | 0.412154 | 0.892154 | 2 | 2 |
| 261 | 93.513 | 129.5 | 0.412323 | 0.892323 | 2 | 2 |
| 262 | 93.878 | 130 | 0.412491 | 0.892491 | 2 | 2 |
| 263 | 94.243 | 130.5 | 0.412658 | 0.892658 | 2 | 2 |
| 264 | 94.608 | 131 | 0.412824 | 0.892824 | 2 | 2 |
| 265 | 96.274 | 131.5 | 0.412989 | 0.892989 | 2 | 2 |

| $n$ | $d_0$ | $d$ | $a$ | $b$ | $T$ | $Z$ |
|---|---|---|---|---|---|---|
| 266 | 96.644 | 132 | 0.413153 | 0.893153 | 2 | 2 |
| 267 | 97.014 | 132.5 | 0.413316 | 0.893316 | 2 | 2 |
| 268 | 97.384 | 133 | 0.413479 | 0.893479 | 2 | 2 |
| 269 | 97.754 | 133.5 | 0.41364 | 0.89364 | 2 | 2 |
| 270 | 98.124 | 134 | 0.413801 | 0.893801 | 2 | 2 |
| 271 | 99.825 | 134.5 | 0.41396 | 0.89396 | 2 | 2 |
| 272 | 100.2 | 135 | 0.414119 | 0.894119 | 2 | 2 |
| 273 | 100.575 | 135.5 | 0.414277 | 0.894277 | 2 | 2 |
| 274 | 100.95 | 136 | 0.414434 | 0.894434 | 2 | 2 |
| 275 | 101.325 | 136.5 | 0.41459 | 0.89459 | 2 | 2 |
| 276 | 88.14 | 137 | 0.494745 | 0.904745 | 2 | 2 |
| 277 | 88.465 | 137.5 | 0.4949 | 0.9049 | 2 | 2 |
| 278 | 88.79 | 138 | 0.495053 | 0.905053 | 2 | 2 |
| 279 | 89.115 | 138.5 | 0.495206 | 0.905206 | 2 | 2 |
| 280 | 90.816 | 139 | 0.495358 | 0.905358 | 2 | 2 |
| 281 | 91.146 | 139.5 | 0.495509 | 0.905509 | 2 | 2 |
| 282 | 91.476 | 140 | 0.49566 | 0.90566 | 2 | 2 |
| 283 | 91.806 | 140.5 | 0.495809 | 0.905809 | 2 | 2 |
| 284 | 92.136 | 141 | 0.495958 | 0.905958 | 2 | 2 |
| 285 | 93.867 | 141.5 | 0.496106 | 0.906106 | 2 | 2 |
| 286 | 94.202 | 142 | 0.496253 | 0.906253 | 2 | 2 |
| 287 | 94.537 | 142.5 | 0.4964 | 0.9064 | 2 | 2 |
| 288 | 94.872 | 143 | 0.496545 | 0.906545 | 2 | 2 |
| 289 | 95.207 | 143.5 | 0.49669 | 0.90669 | 2 | 2 |
| 290 | 96.968 | 144 | 0.496834 | 0.906834 | 2 | 2 |
| 291 | 97.308 | 144.5 | 0.496978 | 0.906978 | 2 | 2 |
| 292 | 97.648 | 145 | 0.49712 | 0.90712 | 2 | 2 |
| 293 | 97.988 | 145.5 | 0.497262 | 0.907262 | 2 | 2 |
| 294 | 98.328 | 146 | 0.497403 | 0.907403 | 2 | 2 |
| 295 | 98.668 | 146.5 | 0.497544 | 0.907544 | 2 | 2 |
| 296 | 100.464 | 147 | 0.497684 | 0.907684 | 2 | 2 |
| 297 | 100.809 | 147.5 | 0.497823 | 0.907823 | 2 | 2 |
| 298 | 101.154 | 148 | 0.497961 | 0.907961 | 2 | 2 |

| $n$ | $d_0$ | $d$ | $a$ | $b$ | $T$ | $Z$ |
|---|---|---|---|---|---|---|
| 299 | 101.499 | 148.5 | 0.498099 | 0.908099 | 2 | 2 |
| 300 | 101.844 | 149 | 0.498236 | 0.908236 | 2 | 2 |
| 301 | 103.67 | 149.5 | 0.498372 | 0.908372 | 2 | 2 |
| 302 | 104.02 | 150 | 0.498508 | 0.908508 | 2 | 2 |
| 303 | 104.37 | 150.5 | 0.498642 | 0.908642 | 2 | 2 |
| 304 | 104.72 | 151 | 0.498777 | 0.908777 | 2 | 2 |
| 305 | 105.07 | 151.5 | 0.49891 | 0.90891 | 2 | 2 |
| 306 | 105.42 | 152 | 0.499043 | 0.909043 | 2 | 2 |
| 307 | 107.281 | 152.5 | 0.499176 | 0.909176 | 2 | 2 |
| 308 | 107.636 | 153 | 0.499307 | 0.909307 | 2 | 2 |
| 309 | 107.991 | 153.5 | 0.499438 | 0.909438 | 2 | 2 |
| 310 | 108.346 | 154 | 0.499569 | 0.909569 | 2 | 2 |
| 311 | 108.701 | 154.5 | 0.499698 | 0.909698 | 2 | 2 |
| 312 | 109.056 | 155 | 0.499827 | 0.909827 | 2 | 2 |
| 313 | 110.952 | 155.5 | 0.499956 | 0.909956 | 2 | 2 |
| 314 | 111.312 | 156 | 0.500084 | 0.910084 | 2 | 2 |
| 315 | 111.672 | 156.5 | 0.500211 | 0.910211 | 2 | 2 |
| 316 | 112.032 | 157 | 0.500338 | 0.910338 | 2 | 2 |
| 317 | 112.392 | 157.5 | 0.500464 | 0.910464 | 2 | 2 |
| 318 | 112.752 | 158 | 0.500589 | 0.910589 | 2 | 2 |
| 319 | 113.112 | 158.5 | 0.500714 | 0.910714 | 2 | 2 |
| 320 | 115.048 | 159 | 0.500838 | 0.910838 | 2 | 2 |
| 321 | 115.413 | 159.5 | 0.500962 | 0.910962 | 2 | 2 |
| 322 | 115.778 | 160 | 0.501085 | 0.911085 | 2 | 2 |
| 323 | 116.143 | 160.5 | 0.501208 | 0.911208 | 2 | 2 |
| 324 | 116.508 | 161 | 0.50133 | 0.91133 | 2 | 2 |
| 325 | 116.873 | 161.5 | 0.501451 | 0.911451 | 2 | 2 |
| 326 | 117.238 | 162 | 0.501572 | 0.911572 | 2 | 2 |
| 327 | 119.214 | 162.5 | 0.501692 | 0.911692 | 2 | 2 |
| 328 | 119.584 | 163 | 0.501812 | 0.911812 | 2 | 2 |
| 329 | 119.954 | 163.5 | 0.501931 | 0.911931 | 2 | 2 |
| 330 | 120.324 | 164 | 0.50205 | 0.91205 | 2 | 2 |
| 331 | 120.694 | 164.5 | 0.502168 | 0.912168 | 2 | 2 |

| $n$ | $d_0$ | $d$ | $a$ | $b$ | $T$ | $Z$ |
|---|---|---|---|---|---|---|
| 332 | 121.064 | 165 | 0.502286 | 0.912286 | 2 | 2 |
| 333 | 121.434 | 165.5 | 0.502403 | 0.912403 | 2 | 2 |
| 334 | 123.45 | 166 | 0.502519 | 0.912519 | 2 | 2 |
| 335 | 123.825 | 166.5 | 0.502635 | 0.912635 | 2 | 2 |
| 336 | 124.2 | 167 | 0.502751 | 0.912751 | 2 | 2 |
| 337 | 124.575 | 167.5 | 0.502866 | 0.912866 | 2 | 2 |
| 338 | 124.95 | 168 | 0.50298 | 0.91298 | 2 | 2 |
| 339 | 125.325 | 168.5 | 0.503094 | 0.913094 | 2 | 2 |
| 340 | 125.7 | 169 | 0.503207 | 0.913207 | 2 | 2 |
| 341 | 126.075 | 169.5 | 0.50332 | 0.91332 | 2 | 2 |
| 342 | 128.136 | 170 | 0.503433 | 0.913433 | 2 | 2 |
| 343 | 128.516 | 170.5 | 0.503545 | 0.913545 | 2 | 2 |
| 344 | 128.896 | 171 | 0.503656 | 0.913656 | 2 | 2 |
| 345 | 129.276 | 171.5 | 0.503767 | 0.913767 | 2 | 2 |
| 346 | 129.656 | 172 | 0.503878 | 0.913878 | 2 | 2 |
| 347 | 130.036 | 172.5 | 0.503988 | 0.913988 | 2 | 2 |
| 348 | 130.416 | 173 | 0.504097 | 0.914097 | 2 | 2 |
| 349 | 130.796 | 173.5 | 0.504206 | 0.914206 | 2 | 2 |
| 350 | 132.902 | 174 | 0.504315 | 0.914315 | 2 | 2 |
| 351 | 133.287 | 174.5 | 0.504423 | 0.914423 | 2 | 2 |
| 352 | 133.672 | 175 | 0.504531 | 0.914531 | 2 | 2 |
| 353 | 134.057 | 175.5 | 0.504638 | 0.914638 | 2 | 2 |
| 354 | 134.442 | 176 | 0.504745 | 0.914745 | 2 | 2 |
| 355 | 134.827 | 176.5 | 0.504851 | 0.914851 | 2 | 2 |
| 356 | 135.212 | 177 | 0.504957 | 0.914957 | 2 | 2 |
| 357 | 135.597 | 177.5 | 0.505062 | 0.915062 | 2 | 2 |
| 358 | 135.982 | 178 | 0.505167 | 0.915167 | 2 | 2 |
| 359 | 138.138 | 178.5 | 0.505272 | 0.915272 | 2 | 2 |
| 360 | 138.528 | 179 | 0.505376 | 0.915376 | 2 | 2 |
| 361 | 138.918 | 179.5 | 0.505479 | 0.915479 | 2 | 2 |
| 362 | 139.308 | 180 | 0.505583 | 0.915583 | 2 | 2 |
| 363 | 139.698 | 180.5 | 0.505685 | 0.915685 | 2 | 2 |
| 364 | 140.088 | 181 | 0.505788 | 0.915788 | 2 | 2 |

| $n$ | $d_0$ | $d$ | $a$ | $b$ | $T$ | $Z$ |
|---|---|---|---|---|---|---|
| 365 | 140.478 | 181.5 | 0.50589 | 0.91589 | 2 | 2 |
| 366 | 140.868 | 182 | 0.505991 | 0.915991 | 2 | 2 |
| 367 | 141.258 | 182.5 | 0.506092 | 0.916092 | 2 | 2 |
| 368 | 141.648 | 183 | 0.506193 | 0.916193 | 2 | 2 |
| 369 | 143.859 | 183.5 | 0.506293 | 0.916293 | 2 | 2 |
| 370 | 144.254 | 184 | 0.506393 | 0.916393 | 2 | 2 |
| 371 | 144.649 | 184.5 | 0.506493 | 0.916493 | 2 | 2 |
| 372 | 145.044 | 185 | 0.506592 | 0.916592 | 2 | 2 |
| 373 | 145.439 | 185.5 | 0.506691 | 0.916691 | 2 | 2 |
| 374 | 145.834 | 186 | 0.506789 | 0.916789 | 2 | 2 |
| 375 | 146.229 | 186.5 | 0.506887 | 0.916887 | 2 | 2 |
| 376 | 146.624 | 187 | 0.506984 | 0.916984 | 2 | 2 |
| 377 | 147.019 | 187.5 | 0.507081 | 0.917081 | 2 | 2 |
| 378 | 147.414 | 188 | 0.507178 | 0.917178 | 2 | 2 |
| 379 | 147.809 | 188.5 | 0.507274 | 0.917274 | 2 | 2 |
| 380 | 150.08 | 189 | 0.50737 | 0.91737 | 2 | 2 |
| 381 | 150.48 | 189.5 | 0.507466 | 0.917466 | 2 | 2 |
| 382 | 150.88 | 190 | 0.507561 | 0.917561 | 2 | 2 |
| 383 | 151.28 | 190.5 | 0.507656 | 0.917656 | 2 | 2 |
| 384 | 151.68 | 191 | 0.50775 | 0.91775 | 2 | 2 |
| 385 | 152.08 | 191.5 | 0.507844 | 0.917844 | 2 | 2 |
| 386 | 152.48 | 192 | 0.507938 | 0.917938 | 2 | 2 |
| 387 | 152.88 | 192.5 | 0.508032 | 0.918032 | 2 | 2 |
| 388 | 153.28 | 193 | 0.508125 | 0.918125 | 2 | 2 |
| 389 | 153.68 | 193.5 | 0.508217 | 0.918217 | 2 | 2 |
| 390 | 154.08 | 194 | 0.508309 | 0.918309 | 2 | 2 |
| 391 | 156.411 | 194.5 | 0.508401 | 0.918401 | 2 | 2 |
| 392 | 156.816 | 195 | 0.508493 | 0.918493 | 2 | 2 |
| 393 | 157.221 | 195.5 | 0.508584 | 0.918584 | 2 | 2 |
| 394 | 157.626 | 196 | 0.508675 | 0.918675 | 2 | 2 |
| 395 | 158.031 | 196.5 | 0.508766 | 0.918766 | 2 | 2 |
| 396 | 158.436 | 197 | 0.508856 | 0.918856 | 2 | 2 |
| 397 | 158.841 | 197.5 | 0.508946 | 0.918946 | 2 | 2 |

| $n$ | $d_0$ | $d$ | $a$ | $b$ | $T$ | $Z$ |
|---|---|---|---|---|---|---|
| 398 | 159.246 | 198 | 0.509035 | 0.919035 | 2 | 2 |
| 399 | 159.651 | 198.5 | 0.509124 | 0.919124 | 2 | 2 |
| 400 | 160.056 | 199 | 0.509213 | 0.919213 | 2 | 2 |
| 401 | 160.461 | 199.5 | 0.509302 | 0.919302 | 2 | 2 |
| 402 | 160.866 | 200 | 0.50939 | 0.91939 | 2 | 2 |
| 403 | 163.262 | 200.5 | 0.509478 | 0.919478 | 2 | 2 |
| 404 | 163.672 | 201 | 0.509565 | 0.919565 | 2 | 2 |
| 405 | 164.082 | 201.5 | 0.509652 | 0.919652 | 2 | 2 |
| 406 | 164.492 | 202 | 0.509739 | 0.919739 | 2 | 2 |
| 407 | 164.902 | 202.5 | 0.509826 | 0.919826 | 2 | 2 |
| 408 | 165.312 | 203 | 0.509912 | 0.919912 | 2 | 2 |
| 409 | 165.722 | 203.5 | 0.509998 | 0.919998 | 2 | 2 |
| 410 | 166.132 | 204 | 0.510083 | 0.920083 | 2 | 2 |
| 411 | 166.542 | 204.5 | 0.510169 | 0.920169 | 2 | 2 |
| 412 | 166.952 | 205 | 0.510254 | 0.920254 | 2 | 2 |
| 413 | 167.362 | 205.5 | 0.510338 | 0.920338 | 2 | 2 |
| 414 | 167.772 | 206 | 0.510423 | 0.920423 | 2 | 2 |
| 415 | 168.182 | 206.5 | 0.510507 | 0.920507 | 2 | 2 |
| 416 | 168.592 | 207 | 0.51059 | 0.92059 | 2 | 2 |
| 417 | 171.063 | 207.5 | 0.510674 | 0.920674 | 2 | 2 |
| 418 | 171.478 | 208 | 0.510757 | 0.920757 | 2 | 2 |
| 419 | 171.893 | 208.5 | 0.51084 | 0.92084 | 2 | 2 |
| 420 | 172.308 | 209 | 0.510922 | 0.920922 | 2 | 2 |
| 421 | 172.723 | 209.5 | 0.511005 | 0.921005 | 2 | 2 |
| 422 | 173.138 | 210 | 0.511086 | 0.921086 | 2 | 2 |
| 423 | 173.553 | 210.5 | 0.511168 | 0.921168 | 2 | 2 |
| 424 | 173.968 | 211 | 0.51125 | 0.92125 | 2 | 2 |
| 425 | 174.383 | 211.5 | 0.511331 | 0.921331 | 2 | 2 |
| 426 | 174.798 | 212 | 0.511411 | 0.921411 | 2 | 2 |
| 427 | 175.213 | 212.5 | 0.511492 | 0.921492 | 2 | 2 |
| 428 | 175.628 | 213 | 0.511572 | 0.921572 | 2 | 2 |
| 429 | 176.043 | 213.5 | 0.511652 | 0.921652 | 2 | 2 |
| 430 | 176.458 | 214 | 0.511732 | 0.921732 | 2 | 2 |

| $n$ | $d_0$ | $d$ | $a$ | $b$ | $T$ | $Z$ |
|---|---|---|---|---|---|---|
| 431 | 179.004 | 214.5 | 0.511811 | 0.921811 | 2 | 2 |
| 432 | 179.424 | 215 | 0.51189 | 0.92189 | 2 | 2 |
| 433 | 179.844 | 215.5 | 0.511969 | 0.921969 | 2 | 2 |
| 434 | 180.264 | 216 | 0.512048 | 0.922048 | 2 | 2 |
| 435 | 180.684 | 216.5 | 0.512126 | 0.922126 | 2 | 2 |
| 436 | 181.104 | 217 | 0.512204 | 0.922204 | 2 | 2 |
| 437 | 181.524 | 217.5 | 0.512282 | 0.922282 | 2 | 2 |
| 438 | 181.944 | 218 | 0.512359 | 0.922359 | 2 | 2 |
| 439 | 182.364 | 218.5 | 0.512436 | 0.922436 | 2 | 2 |
| 440 | 182.784 | 219 | 0.512513 | 0.922513 | 2 | 2 |
| 441 | 183.204 | 219.5 | 0.51259 | 0.92259 | 2 | 2 |
| 442 | 183.624 | 220 | 0.512667 | 0.922667 | 2 | 2 |
| 443 | 184.044 | 220.5 | 0.512743 | 0.922743 | 2 | 2 |
| 444 | 184.464 | 221 | 0.512819 | 0.922819 | 2 | 2 |
| 445 | 184.884 | 221.5 | 0.512894 | 0.922894 | 2 | 2 |
| 446 | 185.304 | 222 | 0.51297 | 0.92297 | 2 | 2 |
| 447 | 185.724 | 222.5 | 0.513045 | 0.923045 | 2 | 2 |
| 448 | 188.36 | 223 | 0.51312 | 0.92312 | 2 | 2 |
| 449 | 188.785 | 223.5 | 0.513194 | 0.923194 | 2 | 2 |
| 450 | 189.21 | 224 | 0.513269 | 0.923269 | 2 | 2 |
| 451 | 189.635 | 224.5 | 0.513343 | 0.923343 | 2 | 2 |
| 452 | 190.06 | 225 | 0.513417 | 0.923417 | 2 | 2 |
| 453 | 190.485 | 225.5 | 0.513491 | 0.923491 | 2 | 2 |
| 454 | 190.91 | 226 | 0.513564 | 0.923564 | 2 | 2 |
| 455 | 191.335 | 226.5 | 0.513637 | 0.923637 | 2 | 2 |
| 456 | 191.76 | 227 | 0.51371 | 0.92371 | 2 | 2 |
| 457 | 192.185 | 227.5 | 0.513783 | 0.923783 | 2 | 2 |
| 458 | 192.61 | 228 | 0.513855 | 0.923855 | 2 | 2 |
| 459 | 193.035 | 228.5 | 0.513927 | 0.923927 | 2 | 2 |
| 460 | 193.46 | 229 | 0.513999 | 0.923999 | 2 | 2 |
| 461 | 193.885 | 229.5 | 0.514071 | 0.924071 | 2 | 2 |
| 462 | 194.31 | 230 | 0.514143 | 0.924143 | 2 | 2 |
| 463 | 194.735 | 230.5 | 0.514214 | 0.924214 | 2 | 2 |

| $n$ | $d_0$ | $d$ | $a$ | $b$ | $T$ | $Z$ |
|---|---|---|---|---|---|---|
| 464 | 195.16 | 231 | 0.514285 | 0.924285 | 2 | 2 |
| 465 | 195.585 | 231.5 | 0.514356 | 0.924356 | 2 | 2 |
| 466 | 198.316 | 232 | 0.514426 | 0.924426 | 2 | 2 |
| 467 | 198.746 | 232.5 | 0.514497 | 0.924497 | 2 | 2 |
| 468 | 199.176 | 233 | 0.514567 | 0.924567 | 2 | 2 |
| 469 | 199.606 | 233.5 | 0.514637 | 0.924637 | 2 | 2 |
| 470 | 200.036 | 234 | 0.514707 | 0.924707 | 2 | 2 |
| 471 | 200.466 | 234.5 | 0.514776 | 0.924776 | 2 | 2 |
| 472 | 200.896 | 235 | 0.514845 | 0.924845 | 2 | 2 |
| 473 | 201.326 | 235.5 | 0.514914 | 0.924914 | 2 | 2 |
| 474 | 201.756 | 236 | 0.514983 | 0.924983 | 2 | 2 |
| 475 | 202.186 | 236.5 | 0.515052 | 0.925052 | 2 | 2 |
| 476 | 202.616 | 237 | 0.51512 | 0.92512 | 2 | 2 |
| 477 | 203.046 | 237.5 | 0.515188 | 0.925188 | 2 | 2 |
| 478 | 203.476 | 238 | 0.515256 | 0.925256 | 2 | 2 |
| 479 | 203.906 | 238.5 | 0.515324 | 0.925324 | 2 | 2 |
| 480 | 204.336 | 239 | 0.515391 | 0.925391 | 2 | 2 |
| 481 | 204.766 | 239.5 | 0.515459 | 0.925459 | 2 | 2 |
| 482 | 205.196 | 240 | 0.515526 | 0.925526 | 2 | 2 |
| 483 | 205.626 | 240.5 | 0.515593 | 0.925593 | 2 | 2 |
| 484 | 206.056 | 241 | 0.515659 | 0.925659 | 2 | 2 |
| 485 | 206.486 | 241.5 | 0.515726 | 0.925726 | 2 | 2 |
| 486 | 209.322 | 242 | 0.515792 | 0.925792 | 2 | 2 |
| 487 | 209.757 | 242.5 | 0.515858 | 0.925858 | 2 | 2 |
| 488 | 210.192 | 243 | 0.515924 | 0.925924 | 2 | 2 |
| 489 | 210.627 | 243.5 | 0.51599 | 0.92599 | 2 | 2 |
| 490 | 211.062 | 244 | 0.516055 | 0.926055 | 2 | 2 |
| 491 | 211.497 | 244.5 | 0.516121 | 0.926121 | 2 | 2 |
| 492 | 211.932 | 245 | 0.516186 | 0.926186 | 2 | 2 |
| 493 | 212.367 | 245.5 | 0.51625 | 0.92625 | 2 | 2 |
| 494 | 212.802 | 246 | 0.516315 | 0.926315 | 2 | 2 |
| 495 | 213.237 | 246.5 | 0.51638 | 0.92638 | 2 | 2 |
| 496 | 213.672 | 247 | 0.516444 | 0.926444 | 2 | 2 |

| $n$ | $d_0$ | $d$ | $a$ | $b$ | $T$ | $Z$ |
|---|---|---|---|---|---|---|
| 497 | 214.107 | 247.5 | 0.516508 | 0.926508 | 2 | 2 |
| 498 | 214.542 | 248 | 0.516572 | 0.926572 | 2 | 2 |
| 499 | 214.977 | 248.5 | 0.516635 | 0.926635 | 2 | 2 |
| 500 | 215.412 | 249 | 0.516699 | 0.926699 | 2 | 2 |
| 501 | 215.847 | 249.5 | 0.516762 | 0.926762 | 2 | 2 |
| 502 | 216.282 | 250 | 0.516825 | 0.926825 | 2 | 2 |
| 503 | 216.717 | 250.5 | 0.516888 | 0.926888 | 2 | 2 |
| 504 | 217.152 | 251 | 0.516951 | 0.926951 | 2 | 2 |
| 505 | 217.587 | 251.5 | 0.517014 | 0.927014 | 2 | 2 |
| 506 | 218.022 | 252 | 0.517076 | 0.927076 | 2 | 2 |
| 507 | 218.457 | 252.5 | 0.517138 | 0.927138 | 2 | 2 |

*Table A.1.* Parameter choices which minimize $T + Z$.

# APPENDIX B

# PYTHON AND SAGE CODE

## B.1 Python Code for Section 2.7.4

Here are the sequence of methods which minimize the value of $T + Z$ for a given parameter $n$. For brevity, the version here does not include the clarifying comments which are present in the actual Jupyter notebook.

```
from math import sqrt,floor,log
from collections import deque
import decimal
decimal.getcontext().prec = 10**4

def pZero(n):
    if (n <= 8):
        return 3.0
    else:
        return 2.0

def K(d,n):
    return 2*sqrt(float((2*n)/((n-1)*(n-2))))*(2.032**(1/n)*(1+sqrt(
    float(2/((n-2)* pZero(n)**n)))))**d

def star(n):
    return (n-2) * 0.5

def qOne(dZero,d,n):
    return pZero(n)**(star(n)-dZero)/K(dZero,n)

def validSmall(dZero,d,n):
    return (0<=dZero) and (dZero <= star(n)-1.4) and (dZero <= d) and (1
     < d) and (d <= star(n)) and (qOne(dZero,d,n)>max(1,K(d,n)**(1/(d-1)
    )))

def validLarge(a,b,n):
    return (0<a) and (a < b) and (b < 1 - sqrt(float(2*(n + a**2)/n**2))
    )

def L(a,b,n):
    return sqrt(float(2*(n+a**2)))/(1-b)
```

```
def D(a,b,n):
    return L(a,b,n)/(n - L(a,b,n))

def A(a,b,n):
    return 1/a**2

def chiN(a,b,n):
    return D(a,b,n)*(A(a,b,n) + 1)+1

def piN(a,b,n):
    return (D(a,b,n)*(4 + A(a,b,n)) + 2)*float(log(2))+(D(a,b,n) + 1)*
   float(log(n))/2 + n*A(a,b,n)*D(a,b,n)/2

def E(a,b,n):
    return 1/(2*(b**2-a**2))

def Z(dZero,d,a,b,n):
    return floor((float(log(E(a,b,n))) + 2*float(log(n)) - float(log(L(a
   ,b,n) - 2)))/float(log(n-1)))+2

def T(dZero,d,a,b,n):
    firstQuantity = float(log((chiN(a,b,n) * n * (d - 1) + 1) / (dZero *
    (d-1) + d)))/float(log(d))
    secondQuantity = float(log(piN(a,b,n)/float(log(K(d,n)**(-1/(d-1))*
   qOne(dZero,d,n))) + d / (dZero * (d - 1) + d)))/float(log(d))
    return floor(max(firstQuantity,secondQuantity))+2

def minTPlusZ(nMin,nMax,prec):
    toReturn = []
    for n in range(nMin,nMax):
        nStar = star(n)
        aUpper = (2*n**2 - sqrt(4*n**4 - 4*(n**2 - 2*n)*(n**2 - 2)))
   /(2*(n**2 - 2))
        a = prec
        tempMinA=prec
        dZero = 0
        tempMinDZero = 0
        b = a + prec
        tempMinB = a + prec
        tempMinT = T(dZero,nStar,a,b,n)
        tempMinZ = Z(dZero,nStar,a,b,n)
        tempMinSum = tempMinT + tempMinZ
```

```
        while ((a <= aUpper) and (tempMinSum > 4)):
            while ((dZero <= nStar - 1.4) and (tempMinSum > 4)):
                while ((b < 1-sqrt(2*(n + a**2)/n**2)) and (tempMinSum >
 4)):
                    tempT = T(dZero,nStar,a,b,n)
                    tempZ = Z(dZero,nStar,a,b,n)
                    if (tempT + tempZ < tempMinSum):
                        tempMinA = a
                        tempMinB = b
                        tempMinDZero = dZero
                        tempMinT = tempT
                        tempMinZ = tempZ
                        tempMinSum = tempT + tempZ
                    b += prec
                dZero += prec * (nStar - 1.4)
                b = a + prec
            a += prec
            dZero = 0
            b = a + prec
        assert validSmall(tempMinDZero,nStar,n), "d0,d,n are invalid"
        assert validLarge(tempMinA,tempMinB,n), "a,b,n are invalid"
        assert chiN(tempMinA,tempMinB,n) >= 2, "chiN is too small"
        assert piN(tempMinA,tempMinB,n) >= 5*log(2) + 2*log(n), "piN is
too small"
        toReturn.append([n,tempMinDZero,nStar,tempMinA,tempMinB,tempMinT
,tempMinZ])
    return toReturn

def minNWithMinTPlusZ(nMax,prec):
    n = nMax
    nStar = star(n)
    aUpper = (2*n**2 - sqrt(4*n**4 - 4*(n**2 - 2*n)*(n**2 - 2)))/(2*(n
**2 - 2))
    a = aUpper - prec
    b = aUpper
    dZero = nStar - 1.4
    tempTPlusZ = 4
    listOfParams = deque([])
    while ((n >= 6) and (tempTPlusZ == 4)):
        tempT = T(dZero,nStar,a,b,n)
        tempZ = Z(dZero,nStar,a,b,n)
        tempTPlusZ = tempT + tempZ
```

```
        while ((a > 0) and (tempTPlusZ > 4)):
            while ((b > a) and (tempTPlusZ > 4)):
                while ((dZero >= 0) and (tempTPlusZ > 4)):
                    tempT = T(dZero,nStar,a,b,n)
                    tempZ = Z(dZero,nStar,a,b,n)
                    tempTPlusZ = tempT + tempZ
                    if (tempTPlusZ == 4):
                        assert validSmall(dZero,nStar,n), "d0,d,n are
invalid"
                        assert validLarge(a,b,n), "a,b,n are invalid"
                        assert chiN(a,b,n) >= 2, "chiN is too small"
                        assert piN(a,b,n) >= 5*log(2) + 2*log(n), "piN
is too small"
                        listOfParams.appendleft([n,dZero,nStar,a,b,tempT
,tempZ])
                    dZero -= prec*(nStar - 1.4)
                b -= prec
                dZero = nStar - 1.4
            a -= prec
            b = aUpper
        n -= 1
        aUpper = (2*n**2 - sqrt(4*n**4 - 4*(n**2 - 2*n)*(n**2 - 2)))
/(2*(n**2 - 2))
        nStar = star(n)
        a = aUpper - prec
        b = aUpper
        dZero = nStar - 1.4
    return listOfParams
```

## B.2 Sage Method For Section 2.7.5

Here is the specific command which takes as input a degree $n$ and height $H$. It finds every irreducible trinomial $F(x, y)$ with degree $n$ and height $H$, solves the Thue equation $|F(x, y)| = 1$, then stores the trinomials and their solution lists in a .csv file called degree_n_height_H_thue_equations.csv.

```
import itertools
import csv

R.<x> = ZZ[]
```

```
def TrinomialThueWriter(degree,height):
    filename = "thue_equation_solution_data/degree_{}_height_{}
   _thue_equations.csv".format(degree,height)
    columnHeads = ["Number of Solutions to |F(x,y)| = 1", "Leading
   Coefficient", "Middle Coefficient", "Constant Coefficient", "Middle
   Degree", "List of Solutions to |F(x,y)| = 1"]
    rows = []
    # Note that we only need to check positive leading coefficients
   since F(x,y) will have the same solutions as -F(x,y).
    # Note also that if the leading coefficient is larger than the
   absolute value of the constant coefficient, then we will have
   already computed the reciprocal polynomial F(y,x).  Hence, we can
   skip polynomials where the constant coefficient has absolute value
   less than the leading coefficient.
    for leadCoeff in range(1,height + 1):
        for midCoeff in itertools.chain(range(-height,0),range(1,height
   +1)):
            for constantCoeff in itertools.chain(range(-height,-
   leadCoeff+1), range(leadCoeff,height+1)):
                if (abs(leadCoeff) == height or abs(midCoeff)== height
   or abs(constantCoeff)== height) and (gcd(gcd(leadCoeff,midCoeff),
   constantCoeff)==1):
                    for midDegree in range(1,degree):
                        P = leadCoeff * x^degree + midCoeff * x^
   midDegree + constantCoeff
                        if P.is_irreducible():
                            thueInfo = gp.thueinit(P)
                            negSolns = gp.thue(thueInfo,-1)
                            posSolns = gp.thue(thueInfo,1)
                            totalSolns = len(negSolns)+len(posSolns)
                            rows.append([totalSolns, leadCoeff, midCoeff
   , constantCoeff, midDegree, gp.concat(posSolns,negSolns)])
    with open(filename,'w') as csvfile:
        csvwriter = csv.writer(csvfile)
        csvwriter.writerow(columnHeads)
        csvwriter.writerows(rows)
```

## B.3 Sage Methods for Section 3.3

The following methods, written for Sage, produce a specified number of polynomials of fixed degree whose roots come from a specified region of the complex plane and may satisfy have other specified properties (e.g. exceed a specified bound on the discriminant or have a particular signature). These methods allow one to then plot those polynomials' Mahler measures against their separations.

```
import itertools

# The below function computes the absolute value of the discriminant
   based on the entries of roots (accounting for multiple roots).
def AbsoluteDiscriminantFromRoots(roots):
    n = len(roots)
    return abs(prod([(roots[i] - roots[j])^2 for i,j in itertools.
   product(range(n),range(n)) if i < j]))

# On input $n$ (an even integer) and $R$, the function below generates a
    set of $n/2$ points uniformly distributed in the box $|\Im[z]| < R,
    |\Re[z]| < R$ and returns the list of those points and their
   complex conjugate.  If a number of real roots is specified, it
   chooses that number of real values in the interval $[-R,R]$ and then
    chooses the remaining roots from the same complex box and includes
   their complex conjugates.  Finally, if a lower bound on the (
   absolute) discriminant is specified, the method will ensure that the
    absolute discriminant of the set of roots is large enough before
   returning the set of roots.
def GenerateComplexRootsInBox(n, R, numRealRoots = 0,
   discriminantLowerBound = 0):
    assert (n - numRealRoots) % 2 ==0, "invalid signature chosen"
    listOfRoots = []
    while (len(listOfRoots) < numRealRoots):
        listOfRoots.append(RR.random_element(-R,R))
    while(len(listOfRoots) < n):
        listOfRoots.append(CDF.random_element(-R,R,-R,R))
    if AbsoluteDiscriminantFromRoots(listOfRoots) >=
   discriminantLowerBound:
        return listOfRoots
    else:
        return GenerateComplexRootsInBox(n,R,numRealRoots,
   discriminantLowerBound)

# Same as the previous method, but now chooses points in the annulus $1/
   R < |z| < R$
def GenerateComplexRootsInAnnulus(n, R, numRealRoots = 0,
   discriminantLowerBound = 0):
```

```
    assert (n - numRealRoots) % 2 ==0, "invalid signature chosen"
    assert R > 1, "invalid annulus chosen"
    listOfRoots = []
    while (len(listOfRoots) < numRealRoots):
        testPoint = RR.random_element(-R,R)
        if abs(testPoint) >= 1/R:
            listOfRoots.append(testPoint)
    while(len(listOfRoots) < n):
        testPoint = CDF.random_element(-R,R,-R,R)
        tpnorm = testPoint.norm()
        if tpnorm <= R^2 and tpnorm >= 1/R^2:
            listOfRoots.append(testPoint)
            listOfRoots.append(testPoint.conj())
    if AbsoluteDiscriminantFromRoots(listOfRoots) >=
   discriminantLowerBound:
        return listOfRoots
    else:
        return GenerateComplexRootsInAnnulus(n,R,numRealRoots,
   discriminantLowerBound)

# Same as the previous method, but now chooses points in the ball $|z| <
    R$
def GenerateComplexRootsInBall(n, R, numRealRoots = 0,
   discriminantLowerBound = 0):
    assert (n - numRealRoots) % 2 ==0, "invalid signature chosen"
    listOfRoots = []
    while (len(listOfRoots) < numRealRoots):
        listOfRoots.append(RR.random_element(-R,R))
    while(len(listOfRoots) < n):
        testPoint = CDF.random_element(-R,R,-R,R)
        tpnorm = testPoint.norm()
        if tpnorm <= R^2:
            listOfRoots.append(testPoint)
            listOfRoots.append(testPoint.conj())
    if AbsoluteDiscriminantFromRoots(listOfRoots) >=
   discriminantLowerBound:
        return listOfRoots
    else:
        return GenerateComplexRootsInBall(n,R,numRealRoots,
   discriminantLowerBound)
```

```
# The below method takes a list of complex numbers as input and outputs
    the minimal distance between distinct elements.
def SeparationFromRoots(listOfRoots):
     return min([abs(i - j) for i,j in itertools.product(listOfRoots,
    listOfRoots) if i < j])

# The below method takes a list of complex numbers as input and outputs
    the Mahler measure of the monic polynomial with those roots.
def MahlerMeasureFromRoots(listOfRoots):
     return prod([abs(listOfRoots[i]) for i in range(len(listOfRoots)) if
      abs(listOfRoots[i]) > 1])

# The following method takes as input a number of trials to run the
    following experiment.  Randomly choose numRoots points from the
    specified region (default: ball) satisfying the specified
    constraints, then plot the Mahler measure of the polynomial with
    those points as roots versus the separation of that same polynomial
def PlotMahlerVSep(numTrials,numRoots,radius,numRealRoots = 0,
    discriminantLowerBound = 0,region = "ball"):
     heightVSsepList = []
     if region == "ball":
          for j in range(numTrials):
               roots = GenerateComplexRootsInBall(numRoots,radius,
    numRealRoots,discriminantLowerBound)
               mahlerMeasure = MahlerMeasureFromRoots(roots)
               sep = SeparationFromRoots(roots)
               heightVSsepList.append((mahlerMeasure,sep))
          return point(heightVSsepList,axes_labels = ["Mahler measure","
    separation"])
     if region == "annulus":
          for j in range(numTrials):
               roots = GenerateComplexRootsInAnnulus(numRoots,radius,
    numRealRoots,discriminantLowerBound)
               mahlerMeasure = MahlerMeasureFromRoots(roots)
               sep = SeparationFromRoots(roots)
               heightVSsepList.append((mahlerMeasure,sep))
          return point(heightVSsepList,axes_labels = ["Mahler measure","
    separation"])
     if region == "box":
          for j in range(numTrials):
               roots = GenerateComplexRootsInBox(numRoots,radius,
    numRealRoots,discriminantLowerBound)
```

```
            mahlerMeasure = MahlerMeasureFromRoots(roots)
            sep = SeparationFromRoots(roots)
            heightVSsepList.append((mahlerMeasure,sep))
        return point(heightVSsepList,axes_labels = ["Mahler measure","
    separation"])

# Same as the previous method, but plotted on log-log axes
def PlotLogMahlerVLogSep(numTrials,numRoots,radius,numRealRoots = 0,
    discriminantLowerBound = 0,region = "ball"):
    heightVSsepList = []
    if region == "ball":
        for j in range(numTrials):
            roots = GenerateComplexRootsInBall(numRoots,radius,
    numRealRoots,discriminantLowerBound)
            mahlerMeasure = MahlerMeasureFromRoots(roots)
            sep = SeparationFromRoots(roots)
            heightVSsepList.append((log(mahlerMeasure),log(sep)))
        return point(heightVSsepList,axes_labels = ["log Mahler measure"
    ,"log separation"])
    if region == "annulus":
        for j in range(numTrials):
            roots = GenerateComplexRootsInAnnulus(numRoots,radius,
    numRealRoots,discriminantLowerBound)
            mahlerMeasure = MahlerMeasureFromRoots(roots)
            sep = SeparationFromRoots(roots)
            heightVSsepList.append((log(mahlerMeasure),log(sep)))
        return point(heightVSsepList,axes_labels = ["log Mahler measure"
    ,"log separation"])
    if region == "box":
        for j in range(numTrials):
            roots = GenerateComplexRootsInBox(numRoots,radius,
    numRealRoots,discriminantLowerBound)
            mahlerMeasure = MahlerMeasureFromRoots(roots)
            sep = SeparationFromRoots(roots)
            heightVSsepList.append((log(mahlerMeasure),log(sep)))
        return point(heightVSsepList,axes_labels = ["log Mahler measure"
    ,"log separation"])
```

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