RIBBONS, SATELLITES, AND EXOTIC PHENOMENA IN HEEGAARD FLOER HOMOLOGY

by

GARY GUTH

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Student: Gary Guth

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This dissertation has been accepted and approved in partial fulfillment of the requirements for the Doctor of Philosophy degree in the Department of Mathematics by:

Robert Lipshitz	Chair
Boris Botvinnik	Core Member
Ben Elias	Core Member
Laura Fredrickson	Core Member
Daniela Vallega-Neu	Institutional Representative

and

Krista Chronister	Vice Provost for Graduate Studies

Original approval signatures are on file with the University of Oregon Division of Graduate Studies.

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DISSERTATION ABSTRACT

Gary Guth

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We study properties of surfaces embedded in 4-manifolds by way of Heegaard Floer homology. We begin by showing link Floer homology obstructs concordance through ribbon homology cobordisms; this extends the work of Zemke and Daemi-Lidman-Vela–Vick-Wong. In another direction, we consider the effect of satellite operations on concordances. We show that the map induced by a satellite concordance is determined by the pattern and the map induced by the original concordance map. As an application, we produce the first examples of stably exotic behavior in the four-ball, i.e. we produce exotic disks whose exotic behavior persists under many 1-handle stabilizations. As a second application, in joint work with Hayden-Kang-Park, we show that the positive Whitehead doubling pattern is injective on the class of $\widehat{HFK}$-distinguishable disks in B^4: we show that for any disks D, D' in B^4 which are distinguished by their induced maps on $\widehat{HFK}$, their positive Whitehead doubles are also distinguished. In particular, $\mathrm{Wh}^+(D)$ and $\mathrm{Wh}^+(D')$ are exotic.

CURRICULUM VITAE

NAME OF AUTHOR: Gary Guth

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:

University of Oregon, Eugene, OR
University of California, Berkeley, CA

DEGREES AWARDED:

Doctor of Philosophy, Mathematics, 2023, University of Oregon
Bachelors of Arts, Mathematics, 2017, University of California, Berkeley

AREAS OF SPECIAL INTEREST:

Low Dimensional Topology

PROFESSIONAL EXPERIENCE:

Graduate Student Fellow, University of Oregon, 2018-2023

PUBLICATIONS:

Guth, G. (2021). Ribbon Homology Cobordisms and Link Floer Homology. *arXiv:2111.09925*.

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CHAPTER I

INTRODUCTION

Four-manifolds and their Surfaces

In this section, we provide a brief overview of four-manifold topology, aiming to motivate the questions explored in this dissertation. Roughly, a manifold is a space which, locally, is indistinguishable from our own Euclidean space – objects like 3-space, the surface of the earth, planes, and donuts. Topology concerns itself with those features of spaces which are preserved by gentle deformations. To a topologist, a circle is equivalent to an ellipse, as the circle can simply be stretched to match the dimensions of the ellipse, or the the ellipse compressed until it forms a perfect circle. However, the surface of a ball is somehow fundamentally different from the surface of a donut; no amount of stretching or twisting will create a hole in the sphere nor close off the hole in the donut.

The notion of gentle deformation is formalized as follows: let X and Y be two n-dimensional manifolds, and let f be a function from X to Y which has an inverse. We say that f is a *homeomorphism* if both f and its inverse are continuous. If such a map between X and Y exists, we say that X and Y are *homeomorphic*. If f and its inverse are smooth as well, we say that f is a *diffeomorphism*, and that X and Y are diffeomorphic. A priori, homeomorphism is a weaker relation than diffeomorphism, yet, in dimensions 1, 2, and 3 these two relations are equivalent; any two manifolds which are homeomorphic are also diffeomorphic. In higher dimensions, however, homeomorphic manifolds need not be diffeomorphic, and the interplay between these two notions of equivalence continues to be a central theme in the study of manifolds.

In particular, much of the richness of four-dimensional topology arises from the stark difference between these two notions of equivalence; four-manifolds X and Y which are homeomorphic but not diffeomorphic are called *exotic*. Dimension four is the smallest dimension in which exotic phenomena can appear. While exotic manifolds exist in infinitely many dimensions, exotic behavior in dimension four is especially peculiar. For instance, if $n \neq 4$, Euclidean space $\mathbb{R}^n$ admits a unique smooth structure, i.e. any manifold homeomorphic to $\mathbb{R}^n$ is, in fact, diffeomorphic to $\mathbb{R}^n$; on the hand, there are uncountably many smooth four-manifolds homeomorphic to $\mathbb{R}^4$ but not diffeomorphic to it. In any dimension other than four, the n-dimensional hyper-sphere S^n admits a finite number of smooth structures, and the exact number of such structures is computable. However, the number of smooth structures on S^4 is completely mysterious.

What is it about dimension four which makes the smooth topology so complicated? In short, the answer is the way in which surfaces (two dimensional manifolds) can be embedded in four-manifolds. The question of classifying smooth, simply connected n-manifolds was revolutionized by Smale, with his proof of the h-cobordism theorem for $n > 4$. Two n-manifolds X_0, X_1 are *cobordant* if there exists an $(n+1)$-manifold W such that $\partial W = X_0 \amalg X_1$. If the inclusion of $X_i \hookrightarrow W$ is a homotopy equivalence, we say that W is an *h-cobordism*. Smale proved that if two simply connected n-manifolds X and Y, $n > 4$, can be connected by a simply connected h-cobordism W, they are, in fact, diffeomorphic.

Any smooth, compact manifold can be constructed from simple pieces called *handles*. An n-dimensional k-handle is simply an n-dimensional disk $D^k \times D^{n-k}$; n-

dimensional k-handles are attached to boundaries of n-manifolds along $S^{k-1} \times D^{n-k}$. An $(n+1)$-dimensional cobordism W between X_0 and X_1 can be decomposed into handles, so that

$$W \cong X_0 \times [0,1] \cup H_1 \cup H_2 \cup \ldots \cup H_{n-1},$$

where H_k is a union of $(n+1)$-dimensional k-handles. At times, handles can cancel in pairs: roughly a k-handle creates a k-dimensional cavity which can potentially be filled in by a $(k+1)$-handle. See Figure 1. Smale argues that if W is an h-cobordism, any handle decomposition for W can be simplified until all handles cancel in pairs, leaving a trivial decomposition of W as:

$$W \cong X_0 \times [0,1].$$

It then follows that $X_0 \cong X_1$, since $\partial W \cong \partial(X \times [0,1])$.

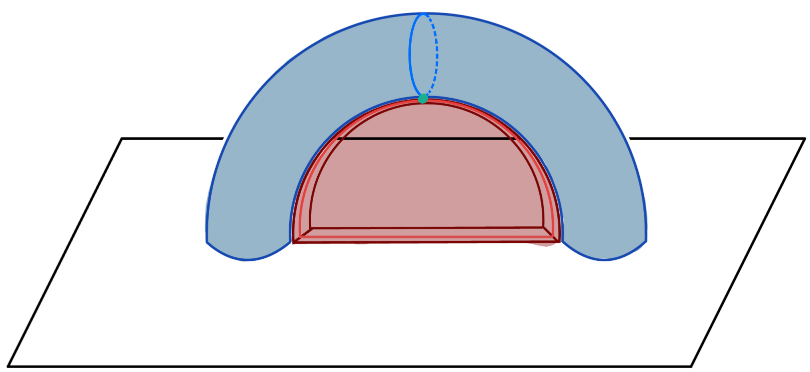


FIGURE 1 A three-dimensional 1-handle and a canceling 2-handle. The cocore $\{\text{pt}\} \times \partial D^2 \subset D^1 \times D^2$ of the 1-handle intersects the core $\partial D^2 \times \{\text{pt}\} \subset D^2 \times D^1$ of the 2-handle in a single point, so the two may be canceled.

A pair of handles $h_1 = D^k \times D^{n+1-k}$ and $h_2 = D^{k+1} \times D^{n-k}$ cancel exactly when the cocore of h_1, $S_1 = \{\text{pt}\} \times \partial D^{n+1-k} \subset h_1$ and the core of h_2, $S_2 = \partial D^{k+1} \times \{\text{pt}\} \subset h_2$ intersect in a single point. The assumption that W is an h-cobordism guarantees that, after perhaps sliding handles over one another, S_1 and S_2 will intersect once *algebraically*. Smale carefully argues that when the dimension of X_i is at least 5, h_1 and h_2 can be smoothly deformed until S_1 and S_2 intersect in a single point *geometrically*.

Smale's proof relies on the "Whitney trick". Since the algebraic intersection of S_1 and S_2 is one, all additional intersection points appear in pairs. A pair of oppositely oriented intersection points p and q determines a loop γ in W, by concatenating a path λ_1 in S_1 from p to q and a path λ_2 in S_2 from q to p. Since W is assumed to be simply connected, this loop can be contracted to a point, and this contraction determines a disk D, with boundary γ. *Since X_0 and X_1 are assumed to have dimension greater than 4*, we can deform this disk until it contains no self-intersections and is disjoint from S_1 and S_2. The intersection points p and q can then be eliminated by pushing S_1 along this disk D. For a schematic, see Figure 2.

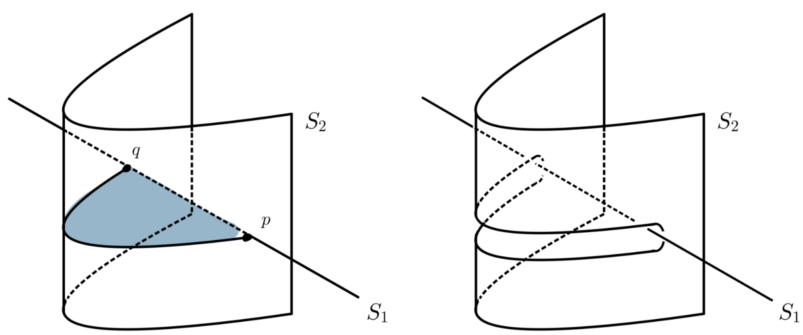


FIGURE 2 Two surfaces S_1 and S_2 intersect in a pair of points. This intersection is eliminated by finding a Whitney disk which can be used push S_2 away from S_1.

This argument uses the assumption on the dimension in an essential way: in dimension four, the Whitney disk may intersect itself or the 2- and 3-handles which we hope to cancel as there are insufficiently many dimensions to eliminate these intersections.

However, the groundbreaking work of Freedman showed that Smale's program can be carried out in dimension four by dropping the requirement that Whitney disks be *smoothly* embedded, and settle for topologically embedded disks [Fre82]. As a consequence, Freedman proved that h-cobordant four-manifolds are homeomorphic. Moreover, the homeomorphism type of a four-manifold X is determined by its *intersection form*, which is a bilinear form that records the ways in which embedded surfaces intersect in X. Just a year later, with the advent of gauge theory, Donaldson proved that the question of classifying smooth, simply connected four-manifolds was much more complicated [Don83]; many topological four-manifolds fail to admit smooth structures and those that do admit smooth structures very often admit infinitely many.

In an essential way, the landscape of four dimensional topology is shaped by the ways in which surfaces can be embedded in four-manifolds. This thesis aims to continue to tease out the relations between four-manifolds and their surfaces.

Topology in Dimension 3.5

There is much to be gained by considering the interplay of three and four-dimensional topology. Every three manifold can be realized as the boundary of some four-manifold, and much can be deduced about the four-manifold by studying its boundary. For instance, a theorem of Freedman and Quinn [FQ90] states that any three manifold Y which has the same integral homology as S^3 bounds a unique contractible four-manifold up to homeomorphism (such a three-manifold is called an *integer homology sphere*. The story is quite different smoothly, however, and many integer homology spheres do not bound smooth contractible manifolds, or even manifolds with the homology of B^4, integer homology balls. Two integer homology spheres Y_1 and Y_2 are *homology cobordant* if there is a smooth four-manifold W with $\partial W = -Y_1 \amalg Y_2$ and $H_*(W, Y_i; \mathbb{Z}) = 0$ for $i \in \{1, 2\}$. Integer homology 3-spheres which bound an integer homology ball are exactly those three-manifolds which are cobordant to S^3. The set of integer homology spheres modulo homology cobordism forms a group under connected sum, called the three-dimensional integer homology cobordism group, usually denoted $\Theta^3_{\mathbb{Z}}$.

The 3-dimensional homology cobordism group highlights yet again the peculiar nature of low dimensional topology. The analogous n-dimensional homology cobordism groups are trivial for $n \neq 3$ by work of Kervaire [Ker69]. However, $\Theta^3_{\mathbb{Z}}$ was proven to be non-trivial by Rokhlin, who exhibited a surjective homomorphism $\mu : \Theta^3_{\mathbb{Z}} \to \mathbb{Z}/2$ [Roh52]. Work of Fintushel-Stern and Furuta showed that $\Theta^3_{\mathbb{Z}}$ is infinitely generated [FS85, FS90, Fur90] and recent work of Dai-Hom-Stoffregen-Truong showed it contains a $\mathbb{Z}^\infty$ summand [DHST18].

In a similar vein, natural questions about four-space arise when studying knot theory. We define a knot K to be an embedding of the 1-sphere into a 3-manifold Y,

$$K : S^1 \hookrightarrow Y^3.$$

We say that K is the *unknot* if K bounds a disk in Y; moreover, the unknot is uniquely characterized by this property. However, if we think of Y as the boundary of a 4-manifold W, we can consider surfaces with boundary K which are allowed to occupy space in the four-manifold W. By allowing our surfaces to live in four-space, it is no longer true that the unknot is characterized by bounding a disk. In fact, many knots in S^3 bound disks which are embedded in the four-ball, B^4. Such knots are called *slice*.

Some slice disks can be visualized in three-dimensions. For example, consider the immersed disk in Figure 3. This disk has two *ribbon singularities*, where the left band of the surface passes through two separate sheets of this surface. By pushing small neighborhoods of these arcs where the surface intersects itself into the interior of the four-ball we can eliminate these singularities, leaving us with an embedded disk in the four-ball with boundary K.

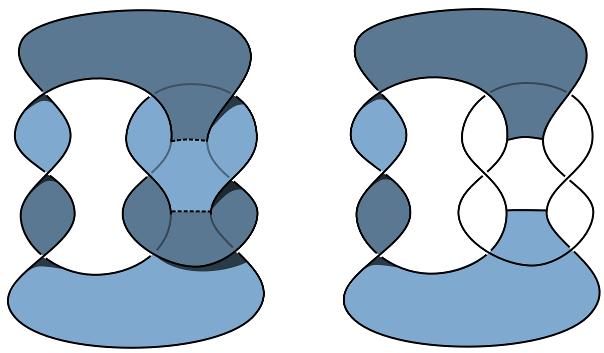


FIGURE 3 An immersed disk with boundary K. By pushing part of this disk into the interior of the four-ball, we can remove the ribbon singularities to obtain an embedded disk.

A generalization of this phenomena is called *concordance*. Knots K_1 and K_2 are called *concordant* if there is an embedded annulus $C \subset [1,2] \times S^3$ with the property that $C \cap \{i\} \times S^3 = K_i$, for $i \in \{1,2\}$. Notice that a knot is slice if and only if it is concordant to the unknot. The set of knots in S^3 modulo smooth concordance forms a group, $\mathcal{C}$, with respect to connected sums. Much work has been invested in understanding the structure of this group.

A concordance C is called *ribbon* if there are no local maxima with respect to the projection $[1,2] \times S^3 \to [1,2]$. A long standing conjecture of Gordon, which was recently proved by Agol, posits that the relation of ribbon concordance defines a partial order on the set of knots in S^3 [Ago22, Gor81].

A *homology concordance* between knots K_1 and K_2 in three-manifolds Y_1 and Y_2 respectively is a pair (W, C) such that

1. W is a homology cobordism from Y_1 to Y_2;

2. C is an annulus embedded in W so that $C \cap Y_i = k_i$ for $i \in \{1, 2\}$.

The set of knots in S^3 modulo homology concordance forms a group, $\mathcal{C}_{\mathbb{Z}}$, under connected sums, and the set of knots in arbitrary homology spheres modulo homology concordance also forms a group, denoted $\widehat{\mathcal{C}}_{\mathbb{Z}}$. There are natural maps

$$\mathcal{C} \to \mathcal{C}_{\mathbb{Z}} \to \widehat{\mathcal{C}}_{\mathbb{Z}}.$$

The first map is surjective, while the second is injective. The kernel of the first map is quite mysterious; an element of the kernel would be a knot in S^3 which is not slice in B^4, but is slice in some other homology 4-ball. No examples of such knots are known. The cokernel of the second map is also interesting to consider; an element of the cokernel of this map is a knot in a homology 3-sphere which is not homology concordant to any knot in S^3. The cokernel is highly nontrivial by work of Levine and Hom-Levine-Lidman and even contains a $\mathbb{Z}^\infty$ summand by work of Zhou [Lev16, HLL22, Zho20].

A cobordism W from Y_1 to Y_2 is called *ribbon* if it admits a handle decomposition consisting of 1- and 2-handles (i.e. contains no three handles). Equivalently, W can be equipped with a Morse function with no critical points of index three. A homology concordance (W, C) is called *simultaneously ribbon* if there is a Morse function $f : W \to \mathbb{R}$ satisfying:

1. W is ribbon with respect to f (f has no critical points of index three.)

2. The restriction of f to C is Morse

3. $f|_C$ has no critical points of index two.

Recent work of Huber and Friedl-Misev-Zenter shows that ribbon homology cobordism defines a partial order on the set of irreducible homology spheres [Hub22, FMZ22]. This and Agol's work lead to the natural question:

Question 1.2.1. Does simultaneous ribbon homology cobordism define a partial order on the set of pairs (Y, K) of knots K in homology 3-spheres?

We provide evidence for this in Chapter 3.

Satellite Operations and Exotic Behavior

In Section 1.2, we introduced the smooth concordance group, which was the set of knots in S^3 studied up to smooth concordance equipped with the operation of connected sum. This equivalence relation can be weakened by considering *locally flat* concordances. An embedded surface $\Sigma \subset X^4$ is *locally flat* if every point x in Σ contains a neighborhood U such that the pair $(U, U \cap \Sigma)$ is homeomorphic to $(\mathbb{R}^4, \mathbb{R}^2 \times \{0\})$. We will write $\mathcal{C}_{TOP}$ for the group of knots in S^3 modulo locally flat concordance under connected sum. The natural map

$$\mathcal{C} \to \mathcal{C}_{TOP},$$

has a nontrivial kernel, i.e. there are knots in S^3 which are not smoothly slice, but are *topologically slice*. In fact, by work of Freedman-Quinn [FQ90], any knot K with trivial Alexander polynomial is topologically slice. Conversely, there are many examples of knots with trivial Alexander polynomial which are not smoothly slice, as can be shown using gauge or Floer theory.

One particularly interesting class of examples arises by studying *satellite knots*. Let K be a knot in S^3 and let P be a knot in the solid torus. Define the *satellite of K with pattern P* in S^3 to be the result of removing a neighborhood of the knot K in S^3 and replacing it with the solid torus containing the knot P, gluing according to the Seifert framing of K. More succinctly,

$$(S^3, P(K)) = ((S^3 - \nu(K)) \cup (S^1 \times D^2), P).$$

The knot K is called the *companion knot* and P is called the *pattern knot*. The Alexander polynomial of a satellite knot is determined by the Alexander polynomials of the companion and the pattern:

$$\Delta_{P(K)} = \Delta_P(t) \cdot \Delta_K(t^w).$$

Here, w is the *winding number* of P, which is defined as follows: P represents a class in the first homology of $S^1 \times D^2$, and therefore $[P]$ is some multiple of a generator for $H^1(S^1 \times D^2) \cong \mathbb{Z}$; define w to be this integer.

Consider the pattern shown in Figure 4. This pattern is called the positive Whitehead double pattern, $P = \mathrm{Wh}^+$. Though Wh^+ is nontrivial in $S^1 \times D^2$, it becomes unknotted after embedding $(S^1 \times D^2, \mathrm{Wh}^+) \hookrightarrow S^3$. Therefore, since the Whitehead doubling pattern has winding number zero, we have that for any knot K,

$$\Delta_{\mathrm{Wh}^+(K)}(t) = \Delta_U(t) \cdot \Delta_K(t^0) = \Delta_K(1).$$

The Alexander polynomial for any knot K evaluated at $t = 1$ is equal to ± 1. Therefore, by Freedman-Quinn, $\mathrm{Wh}^+(K)$ is topologically slice. In many cases,

$\mathrm{Wh}^+(K)$ can be shown not to be smoothly slice. Conjecturally, a knot K is smoothly slice if and only if $\mathrm{Wh}^+(K)$ is smoothly slice.

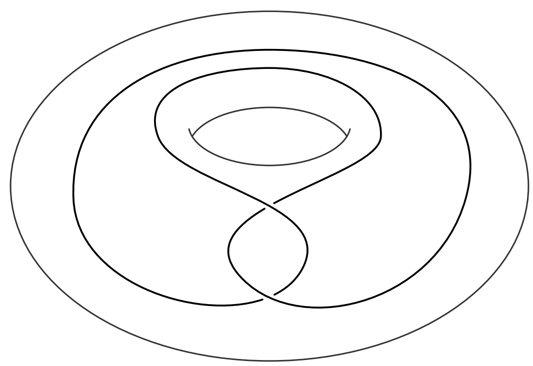


FIGURE 4 The positive Whitehead double pattern in the solid torus.

This example has a nice reformulation. Given a concordance $C : K_1 \to K_2$, and a satellite pattern $P \subset S^1 \times D^2$, define the satellite concordance $P(C)$: $P(K_1) \to P(K_2)$ as the result of deleting a neighborhood of the concordance from $S^3 \times [1, 2]$ and replacing it with $(S^1 \times D^2 \times [1, 2]), P \times [1, 2])$:

$$(S^3 \times [1, 2], P(C)) = ((S^3 \times I - \nu(C)) \cup (S^1 \times D^2 \times [1, 2]), P \times [1, 2]).$$

Therefore, a satellite pattern P determines a well-defined operator on $\mathcal{C}$ (or $\mathcal{C}_{TOP}$), namely $[K] \mapsto [P(K)]$. As an operator on $\mathcal{C}_{TOP}$, Wh^+ is trivial, since after applying the pattern, all knots become topologically slice. However, its behavior on $\mathcal{C}$ is much more mysterious; the conjecture above can be reformulated by asking whether the operator Wh_+ is injective.

There is a natural four-dimensional analogue of this phenomenon. Suppose D is a slice disk for a knot K. By puncturing D, we obtain concordance $C : U \to K$. Given a satellite pattern P, the procedure outlined above produces a concordance

$$P(C) : P(U) \to P(K).$$

If the pattern P satisfies the property that $P \hookrightarrow S^1 \times D^2 \hookrightarrow S^3$ is the unknot, we say that P is an *unknotted pattern*. In other words, P is unknotted when $P(U) = U$. Therefore, the concordance $C : U \to K$ gives rise to a concordance

$$P(C) : U \to P(K).$$

By capping off (S^3, U) with (B^4, D^2), we obtain a slice disk for $P(K)$, which we will call the *satellite of D with pattern P*, denoted $P(D)$. For an example, see Figure 6.

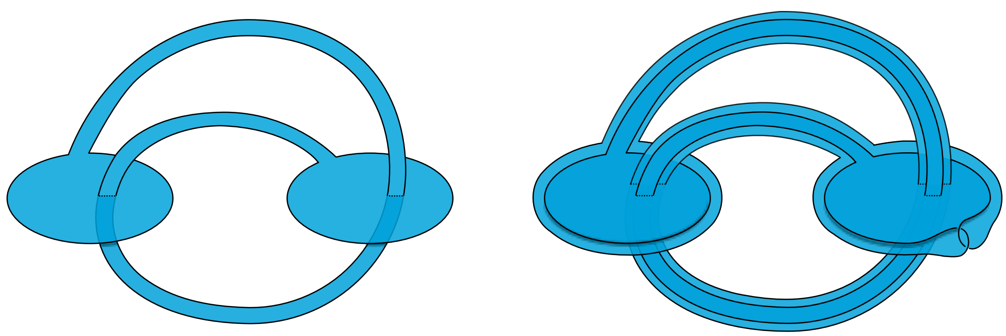


FIGURE 5 The negative Whitehead double of a slice disk for the Stevedore knot.

Extending the work of Freedman-Quinn, Conway-Powell show that if D is a slice disk for K with the additional property that

$$\pi_1(B^4 - \nu(D)) \cong \mathbb{Z}$$

then D is unique up to topological isotopy rel boundary [CP21]. Two smooth embeddings $f, g : \Sigma \to X$ are *isotopic* if there is a one parameter family of diffeomorphisms $H_t : X \to X$ such that H_0 is the identity and $H_1 \circ f = g$. A topological isotopy is analogous, replacing the family of diffeomorphisms with a family of homeomorphisms. We often will identify these embeddings with their images and say that $\Sigma_0 = f(\Sigma)$ and $\Sigma_1 = g(\Sigma)$ are isotopic (in the appropriate category). Surfaces Σ_0 and Σ_1 in X are called *exotic* if they are topologically isotopic but not smoothly. In the case in which our surfaces have boundary, we will consider isotopies which restrict to the identity on the boundaries.

Just as closed 4-manifolds may exhibit exotic behavior, surfaces embedded in a given four-manifold may be equivalent but not smoothly. In particular, given a pair of slice disks D_1 and D_2 for K, the positive Whitehead doubles of D_1 and D_2 are necessarily topologically isotopic by Conway-Powell, but can often be shown to be nonisotopic smoothly. In analogy with the case of knots in S^3, we conjecture:

Conjecture 1.3.1. Let D_1 and D_2 be slice disks for K. D_1 and D_2 are smoothly isotopic red boundary if and only if $\mathrm{Wh}^+(D_1)$ and $\mathrm{Wh}^+(D_2)$ are smoothly isotopic red boundary.

As before, this question can be reformulated in terms of satellite operations. Following [JZ21], let $\mathrm{Surf}_0(K)$ be the set of isotopy classes of connected, properly embedded genus-0 surfaces in the four-ball with boundary. An unknotted pattern P induces a map

$$P : \mathrm{Surf}_0(K) \to \mathrm{Surf}_0(P(K)),$$

in the obvious way, by taking a slice disk for K to the satellite disk for $P(K)$ determined by the pattern. In this language, we can reframe Conjecture **??** as asking whether the operator Wh^+ is injective. In Chapter 5, in joint work with Hayden-Kang-Park, we provide evidence for this conjecture (as well as evidence that much larger families of unknotted satellite operators are injective) by studying the behavior of invariants coming knot Floer homology.

Stability of Exotic Behavior

Though exotic behavior abounds in dimension four, it tends to be unstable, which is to say that exotic behavior tends to vanish after enlarging the objects in some way. The most famous example is due to Wall.

Theorem 1. *[Wal64] Let X_0 and X_1 be simply connected, oriented exotic 4-manifolds. Then for sufficiently large k, $X_0 \#^k (S^2 \times S^2)$ and $X_1 \#^k (S^2 \times S^2)$ are diffeomorphic.*

In order to motivate an analogous operation for surfaces, we will sketch a proof of this argument.

Proof. If X_0 and X_1 are exotic, they are cobordant through some 5-manifold W; we can assume that W has no 1- or 4-handles (else we can do surgery to eliminate them), meaning W is built entirely from five-dimensional 2- and 3-handles. The 2-handles are attached along nullhomotopic circles, which has the effect of splitting off $S^2 \times S^2$ summands on the boundary. Hence, $\partial_+(X_0 \times I \cup 2\text{-handles}) = X_0 \#^k S^2 \times S^2$; a dual argument shows that $\partial_-(X_1 \times I \cup 3\text{-handles}) = X_0 \#^k S^2 \times S^2$. The result then follows by observing that $\partial_+(X_0 \times I \cup 2\text{-handles}) \cong \partial_-(X_1 \times I \cup 3\text{-handles})$. $\square$

Work of Gompf [Gom84] extends Wall's result to oriented 4-manifolds with arbitrary fundamental groups. A famous open conjecture states that a *single stabilization* is always enough to eliminate such exotic behavior. The operation of taking a connected sum with $S^2 \times S^2$ is called a *stabilization*.

The work of Hosokawa-Kawauchi and Baykur-Sunukjian [HK79, BS16] shows that exotic surfaces Σ_0 and Σ_1 contained in a four-manifold X also become smoothly isotopic after increasing the genus of the two surfaces by attaching "tubes" (or, more precisely, attaching the boundaries of three-dimensional 1-handles). This operation is also called *stabilization* (or sometimes, *internal stabilization*).

Theorem 2. *Let Σ_0 and Σ_1 be smoothly embedded surfaces in a four-manifold X. Let $\widetilde{\Sigma}_0$ and $\widetilde{\Sigma}_1$ be obtained from Σ_0 and Σ_1 by k internal stabilizations. Then, for sufficiently large k, $\widetilde{\Sigma}_0$ and $\widetilde{\Sigma}_1$ are smoothly isotopic.*

The proof is nearly identical to the four-manifold case. Since Σ_0 and Σ_1 are homologous, they are cobordant through a three-manifold Y in X. Y can be built from three dimensional 1- and 2-handles, and $\widetilde{\Sigma}_0$ and $\widetilde{\Sigma}_1$ can be identified both as the union of Σ_0 together with the boundary of the 1-handles and as the union of Σ_1 with the boundary of the 2-handles. For this reason, increasing the genus of the two surfaces is a natural analogue of stabilization in the surface case.

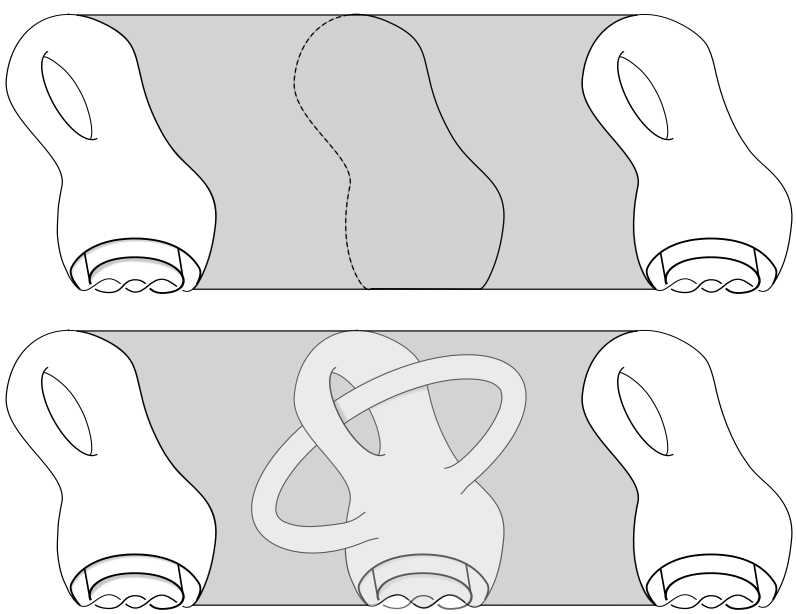


FIGURE 6 A cobordism between Σ_0 and Σ_1, where the intermediate surface $\widetilde{\Sigma}$ is visible as Σ_0 with a collection of 1-handles attached.

These examples demonstrate the general principle that exotic behavior is unstable. One naturally asks *how unstable*: how many times is it necessary to stabilize exotic pairs before they become equivalent? A long standing conjecture states that, in both of these cases, a single stabilization should always be enough to eliminate the exotic behavior. Much work has been done in providing evidence for this conjecture; for most known examples, it has been confirmed that a single stabilization is enough (for example, see [Auc03, AKMR15, BS16, Bay18]) and, under certain homological conditions, there are cases where the exotic behavior is known to dissolve under a single stabilization [AKM$^+$17]. However, in Chapter 6, we provide counterexamples in the case of surfaces with boundary.

Heegaard Floer Homology

The results of this dissertation rely heavily on a collection of invariants coming from *Heegaard Floer homology.* Heegaard Floer homology, introduced by Ozsváth and Szabó [OS04b, OS06, OS04a], is a collection of invariants of three- and four-manifolds as well as knots and surfaces embedded within them. We will provide a much more thorough review of the salient aspects of the theory in Chapter 2, though, for the casual reader, we have included a structural summary of these invariants here.

Any three-manifold Y can be represented by a combinatorial object called a *Heegaard diagram*, $\mathcal{H}$, which consists of a closed surfaces Σ of genus g decorated with two collections of embedded curves $\{\alpha_1, \dots, \alpha_g\}$ and $\{\beta_1, \dots, \beta_g\}$ with the property that when Σ is thickened and three-dimensional 2-handles are attached along the α and β curves the resulting object is homeomorphic to $Y - (B^3 \amalg B^3)$.

To this Heegaard diagram $\mathcal{H}$ (together with a choice of basepoint on Σ, Ozsváth-Szabó define a chain complex, $CF^{-}(\mathcal{H})$, which is an $\mathbb{F}_2[U]$-module. There are several variants:

$$CF^{\infty}(\mathcal{H}) = CF^{-}(\mathcal{H}) \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}],$$

$$CF^{+}(\mathcal{H}) = CF^{\infty}(\mathcal{H})/CF^{-}(\mathcal{H}),$$

$$\widehat{CF}(\mathcal{H}) = CF^{-}(\mathcal{H})/U \cdot CF^{-}(\mathcal{H}).$$

The differential counts certain holomorphic disks in the g-fold symmetric product of Σ. The homology of this complex is denoted $HF^{\circ}(Y)$ ($\circ \in \{-, \infty, +, \widehat{\ }\}$), and, up to canonical isomorphism does not depend on the choices made in the construction of the complex.

Moreover, these invariants fit into the structure of a topological quantum field theory (TQFT). A TQFT associates to an n-manifold an algebraic object, and to a cobordism between n-manifolds, a morphism between objects. (Formally, a TQFT is a symmetric monoidal functor from a cobordism category to an abelian category.) In the context of Heegaard Floer homology, a cobordism W from Y_1 to Y_2 (together with a graph embedded in W connecting the basepoints), there is an induced map

$$F_W : HF^{\circ}(Y_1) \to HF(Y_2).$$

These maps are well behaved: the product cobordism $Y \times [0, 1] : Y \to Y$ induces the identity map on $HF^{\circ}(Y)$, and given a decomposition of W as $W_1 \cup W_2$ where $W_1 : Y_0 \to Y_1$ and $W_2 : Y_1 \to Y_2$, the cobordism maps satisfy a composition law [OS06, Zem15].

Given a knot K in a three manifold Y, Ozsváth-Szabó and, independently, Rasmussen define a complex $\mathcal{CFL}^-(Y,K)$ [OS04a, Ras03]. We will follow the conventions of [Zem19d] and view $\mathcal{CFL}^-(Y)$ as an $\mathbb{F}[U,V]$-module. Just as in the closed 3-manifold case, the chain homotopy type of this complex is an invariant of the pair (Y,K). Moreover, if W is a cobordism from Y_1 to Y_2 and Σ is a surface embedded in W such that $\Sigma \cap W = K_1 \cup K_2$, where $K_i \subset Y_i$ for $i \in \{1,2\}$ there is an associated map

$$F_{W,\Sigma} : \mathcal{HFL}^-(Y_1,K_1) \to \mathcal{HFL}(Y_2,K_2).$$

This map depends on some additional data; see chapter 2 for more details. The map $F_{W,\Sigma}$ is an invariant of the pair (W,Σ) [Zem19d].

Summary of Results

In this section, we provide a brief overview of the layout of this dissertation. In Chapter 2, we review some background on Heegaard Floer homology. In Chapter 3, we analyze the cobordism maps associated to ribbon homology concordances. The main theorem is an obstruction to the existence of such cobordisms, building off of work of [Zem19b, DLVVW19].

Theorem 3. *Let $(W,\mathcal{F}) : (Y_0,K_0) \to (Y_1,K_1)$ be a ribbon $\mathbb{Z}$-homology concordance where $\mathcal{F} = (C,\mathcal{A})$ is a surface decorated by $\mathcal{A}$, a pair of parallel arcs. Then the induced map*

$$F_{W,\mathcal{F},\mathfrak{s}} : \mathcal{HFL}^-(Y_0,K_0,\mathfrak{s}|_{Y_0}) \to \mathcal{HFL}^-(Y_1,K_1,\mathfrak{s}|_{Y_1})$$

is a split injection.

We will also consider an algebraic reduction of $\mathcal{HFL}^-(Y,K,\mathfrak{s})$, denoted $HFL^-(Y,K,\mathfrak{s})$. $HFL^-(Y,K,\mathfrak{s})$ is a finitely generated $\mathbb{F}[V]$-module, and therefore can be decomposed into a free summand and a torsion summand, which is denoted $HFL^-_{red}(Y, K, \mathfrak{s})$.

Definition 1.6.1. Let K be a nullhomologous knot in a 3-manifold Y. Define the *torsion order of K in Y* to be the quantity

$$\mathrm{Ord}_V(Y, K, \mathfrak{s}) = \min\{d \in \mathbb{N} : V^d \cdot HFL^-_{red}(Y, K, \mathfrak{s}) = 0\}.$$

Juhász-Miller-Zemke [JMZ20] use the the torsion order of knots in S^3 to give bounds on many topological invariants of knots, including the fusion number, the bridge index, and the cobordism distance. We prove an analogue of [JMZ20, Theorem 1.2] in the ribbon homology cobordism setting.

Theorem 4. *Suppose $(W, \Sigma) : (Y_0, K_0) \to (Y_1, K_1)$ is a $\mathbb{Z}$-homology link cobordism such that W is ribbon with respect to a Morse function $h : W \to \mathbb{R}$ compatible with Σ. Suppose Σ has m critical points of index 0 and M critical points of index 2 with respect to $h|_\Sigma$. Then*

$$\mathrm{Ord}_V(Y_0, K_0, \mathfrak{s}|_{Y_0}) \leq \max\{M, \mathrm{Ord}_V(Y_1, K_1)\} + 2g(\Sigma).$$

When W is a product, we also have

$$\mathrm{Ord}_V(Y_1, K_1, \mathfrak{s}|_{Y_1}) \leq \max\{m, \mathrm{Ord}_V(Y_0, K_0)\} + 2g(\Sigma).$$

We use Theorem 4 to prove some results about ribbon cobordisms between knots in homology cobordant 3-manifolds, and consider some generalizations of the fusion number in the context of ribbon homology cobordisms.

In Chapter 4, we make use of bordered Floer homology to show that maps induced by satellite concordances are determined by the "companion" concordance and the type-A structure associated to the pattern knot in the solid torus.

Theorem 5. *Let $C : K \to K'$ be a smooth concordance. Then, there exists a map $F : \widehat{CFD}(S^3 - K) \to \widehat{CFD}(S^3 - K')$ induced by C, such that for any pattern knot P in the solid torus, the following diagram commutes up to homotopy:*

$$\begin{tikzcd} CFA^-(\mathcal{H}_P) \boxtimes \widehat{CFD}(S^3 - K) \arrow[r, "\simeq"] \arrow[d, "\mathrm{id} \boxtimes F"] & CFK^-(K_P) \arrow[d, "F_{C_P}"] \\ CFA^-(\mathcal{H}_P) \boxtimes \widehat{CFD}(S^3 - K') \arrow[r, "\simeq"] & CFK^-(K'_P), \end{tikzcd}$$

where $\mathcal{H}_P$ doubly pointed, bordered Heegaard diagram for $P \subset S^1 \times D^2$, K_P and K'_P are satellites of K and K', and C_P is the concordance induced by P. The horizontal arrows are given by the pairing theorem [LOT18].

After proving Theorem 13, we describe a sufficient condition on the type-A structure of the satellite pattern to determine whether the pattern is "*HFK*-injective"; in short, we give a criterion for determining whether a pair of concordances which are distinguishable by knot Floer homology will remain distinguishable after applying the satellite.

In Chapter 5, we turn to applications. In particular, we show:

Theorem 6. *Let K be a knot in S^3 with slice disks D_1, D_2. If D_1 and D_2 are $\widehat{HFK}$-distinguishable, then so are $Wh^+(D_1)$ and $Wh^+(D_2)$. Moreover, $Wh^+(D_1)$ and $Wh^+(D_2)$ are an exotic pair.*

In fact, the first statement of Theorem 6 holds for many satellite patterns (positive cables, the Mazur patterns, and generalized doubling patterns). Finally, we prove that there exist exotic surfaces in the four-ball with arbitrarily large stabilization distance.

Theorem 7. *For any p, there exists a knot J_p which bounds a pair of exotic disks D_p and D'_p which remains exotic after $p-1$ internal stabilizations.*

Theorem 7 also (somewhat trivially) gives examples of higher genus surfaces with arbitrarily large stabilization distance.

CHAPTER II

BACKGROUND MATERIAL

Heegaard Floer Homology

A *Heegaard splitting* for a three-manifold Y is a decomposition of Y as

$$Y = H_1 \cup_\phi H_2,$$

where H_1 and H_2 are handlebodies of genus g (i.e. regular neighborhoods of the g-fold wedge of circles) and ϕ is an orientation reversing homeomorphism from ∂H_1 to ∂H_2. Every three-manifold possesses such a decomposition by an application of Morse theory: simply choose a self-indexing Morse function $f : Y \to [0, 3]$ and define $H_1 = f^{-1}([0, \frac{3}{2}])$, $H_2 = f^{-1}([\frac{3}{2}, 3])$.

The data of a Heegaard splitting can be recorded combinatorially by a *Heegaard diagram.* A Heegaard diagram consists of a tuple $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, which consists of

1. A closed surface Σ of genus g;
2. Two collections of pairwise disjoint, closed curves $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_g\}$ and $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_g\}$ in Σ such that both $\{\alpha_i\}$ and $\{\beta_i\}$ represent linearly independent classes in $H_1(\Sigma, \mathbb{Z})$.

The three-manifold Y can be reconstructed from $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ as follows: thicken Σ to $\Sigma \times [0, 1]$; attach thickened disks along the α-curves in $\Sigma \times \{0\}$ and thickened disks along the β-curves in $\Sigma \times \{1\}$; the homological assumptions on the α- and β-curves

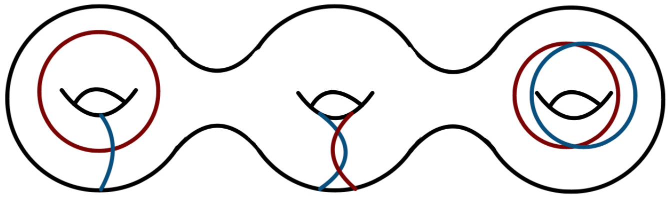


FIGURE 7 A Heegaard diagram for $S^1 \times S^2 \# S^1 \times S^2$. The α curves are drawn in red and the β curves are drawn in blue.

guarantee that the resulting three-manifold has boundary $S^2 \amalg S^2$, which can be filled with three-balls.

Heegaard Floer homology is an invariant of closed 3-manifolds defined by Ozsváth and Szabó [OS04b]. Given a pointed Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ for Y, one considers the symmetric product

$$\mathrm{Sym}^g(\Sigma) = \Sigma^{\times g}/S_g,$$

the quotient of the g-fold product of Σ by the symmetric group. $\mathrm{Sym}^g(\Sigma)$ is a smooth manifold, and in fact, inherits a complex structure from Σ. The attaching curves $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ determine half-dimensional submanifolds

$$\mathbb{T}_\alpha = \alpha_1 \times \ldots \times \alpha_g \qquad\qquad \mathbb{T}_\beta = \beta_1 \times \ldots \times \beta_g.$$

$CF^-(Y, z)$ is freely generated as an $\mathbb{F}[U]$-module by intersection points in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, and the differential counts holomorphic disks connecting intersection points, weighted by U-powers determined the algebraic intersection of the disk with the subvariety

$$V_z = \{z\} \times \mathrm{Sym}^{g-1}(\Sigma).$$

Let D^2 be the unit disk in the complex plane. Let e_1 be the arc in ∂D^2 with positive real part and let e_2 be the arc in ∂D^2 with negative real part. Given intersection points $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, a *Whitney disk* from $\mathbf{x}$ to $\mathbf{y}$ is a map

$$\phi : D^2 \to \mathrm{Sym}^g(\Sigma)$$

such that

1. $\phi(-i) = \mathbf{x}$ and $\phi(i) = \mathbf{y}$,
2. $\phi(e_1) \subset \mathbb{T}_\alpha$ and $\phi(e_2) \subset \mathbb{T}_\beta$.

The set of homotopy classes of Whitney disks from $\mathbf{x}$ to $\mathbf{y}$ is denoted $\pi_2(\mathbf{x}, \mathbf{y})$. Given $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ and a path of complex structures J_s on $\mathrm{Sym}^g(\Sigma)$, we define $\mathcal{M}_{J_s}(\phi)$ to be the moduli space of J_s-holomorphic Whitney disks representing ϕ. The complex disk D^2 has an $\mathbb{R}$-action given by translation, and we define

$$\widehat{\mathcal{M}}_{J_s}(\phi) = \mathcal{M}_{J_s}(\phi)/\mathbb{R}.$$

The *Maslov index* of ϕ, $\mu(\phi)$ is the expected dimension of this moduli space. Ozsváth and Szabó prove that for a generic path of complex structures, if $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a Heegaard diagram with attaching curves in general position, then for each $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ with $\mu(\phi) = 1$, $\widehat{\mathcal{M}}_{J_s}(\phi)$ is a compact, zero dimensional manifold.

Finally, we can define a differential on $CF^-(Y, z)$ as

$$\partial(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}), \\ \mu(\phi) = 1}} \# \widehat{\mathcal{M}}(\phi) U^{n_{\mathbf{z}}(\phi)} \mathbf{y}.$$

One proves that $\partial^2 = 0$ by studying the of the moduli spaces $\widehat{\mathcal{M}}(\phi)$ with $\phi \in \pi_2(\mathbf{x}, \mathbf{w})$ and $\mu(\phi) = 2$.

Ozsváth and Szabó prove that the chain homotopy type of $CF^-(Y, z)$ does not depend on the choices made in the construction (the complex structure and choice of Heegaard splitting) and is therefore a well-defined three-manifold invariant.

The Heegaard Floer homology groups split over $\mathrm{Spin}^{\mathrm{C}}$-structures of the three-manifold. In [OS04b], Ozsváth and Szabó define a map

$$\mathfrak{s}_z : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathrm{Spin}^{\mathrm{C}}(Y)$$

by interpreting $\mathrm{Spin}^{\mathrm{C}}$-structures on Y as homology classes of non-vanishing vector fields on Y (in the sense of [Tur97]); once we choose a Morse function inducing our Heegaard splitting, an intersection point $\mathbf{x}$ determines flowlines from the index one to index two critical points and the basepoint determines a flowline connecting the index zero and three critical points. Outside a neighborhood of these flowlines, the gradient vector field is non-vanishing, and this homology class is defined to be the $\mathrm{Spin}^{\mathrm{C}}$-structure associated to the intersection point $\mathbf{x}$. Intersection points which are connected by a Whitney disk are necessarily in the same $\mathrm{Spin}^{\mathrm{C}}$-structure, so we have that

$$CF^\circ(Y) = \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^{\mathrm{C}}(Y)} CF^\circ(Y, \mathfrak{s}).$$

Link and Knot Floer Homology

There is a refinement of this invariant for knots and links in three-manifolds. Several versions of link Floer homology exist in the literature. Knot Floer homology is an invariant of knots in 3-manifolds defined by Ozsváth and Szabó [OS04a] and independently by Rasmussen [Ras03]. The extension to links is due to Ozsváth and Szabó in [OS08a]. We review the definitions in order to establish the conventions we will be following.

We can encode the data of a knot in a three-manifold by placing additional basepoints on our Heegaard diagram. Once again, choose a self-indexing Morse function $f : Y \to [0, 3]$ which induces a Heegaard splitting of Y with one index zero critical point and index three critical point. A pair of basepoints w and z on the Heegaard surfaces determines a knot in Y in the following way: there is a unique flow-line connecting the index zero and three critical points which passes through w (and similarly for z). The union of these two paths is a knot in Y. Similarly, we can encode a link in Y by choosing more w and z basepoints on our Heegaard surface.

Definition 2.2.1. A *multi-based link* $\mathbb{L} = (L, \mathbf{w}, \mathbf{z})$ in a 3-manifold Y is an oriented link L with two collections of basepoints $\mathbf{w}$ and $\mathbf{z}$ such that each component of L has at least one $\mathbf{w}$- and one $\mathbf{z}$- basepoint and the basepoints alternate between $\mathbf{w}$ and $\mathbf{z}$ as one travels along the link.

The link Floer complexes are constructed by choosing a multi-pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ for $(Y, \mathbb{L})$, where $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_{g+n-1})$ and $\boldsymbol{\beta} = (\beta_1, ..., \beta_{g+n-1})$ are the attaching curves, g is the genus of Σ, and $n = |\mathbf{w}| = |\mathbf{z}|$. Denote by $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ the half dimensional tori $\alpha_1 \times ... \times \alpha_{g+n-1}$ and $\beta_1 \times ... \times \beta_{g+n-1}$ in $\mathrm{Sym}^{g+n-1}(\Sigma)$. The link Floer complex splits over $\mathrm{Spin}^{\mathrm{C}}$-structures for Y and is generated by intersection points in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Following [Zem18], we define $\mathcal{CFL}^-(Y, \mathfrak{s})$ to be the free $\mathbb{F}_2[U, V]$-module generated by intersection points $\mathbf{x}$ in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $\mathfrak{s}_{\mathbf{w}}(\mathbf{x}) = \mathfrak{s}$. As in the three-manifold case, the differential is defined by counting holomorphic disks of Maslov index 1, with the U and V variables recording the algebraic intersections of the disks with the basepoints: let

$$\partial(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x},\mathbf{y}), \\ \mu(\phi)=1}} \#\widehat{\mathcal{M}}(\phi) U^{n_{\mathbf{w}}(\phi)} V^{n_{\mathbf{z}}(\phi)} \mathbf{y},$$

and extend $\mathbb{F}_2[U,V]$-linearly. Note, $\mathcal{CFL}^-(Y,\mathfrak{s})$ could also have been defined as generated by intersection points $\mathbf{x}$ with $\mathfrak{s}_{\mathbf{z}}(\mathbf{x}) = \mathfrak{s}$. By [Zem18, Lemma 3.3], $\mathfrak{s}_{\mathbf{w}}(\mathbf{x}) - \mathfrak{s}_{\mathbf{z}}(\mathbf{x}) = PD[L]$, where $[L]$ is the fundamental class of the link. Hence, when the homology class of the link is trivial in $H_1(Y;\mathbb{Z})$, the maps $\mathfrak{s}_w$ and $\mathfrak{s}_z$ agree, so either choice yields the same complex. However, for links which are *not* nullhomologous, the two complexes may differ. For a more general set up, see [Zem18, Section 3].

Cobordisms and Functoriality

Heegaard Floer homology (as well as knot and link Floer homology) have the structure of a topological quantum field theory. In short, four-dimensional cobordisms between three-manifolds induced functorial maps between chain complexes. Similarly, decorated link cobordisms induce maps between link Floer complexes. Since we will primarily be interested in surfaces in this thesis, we will emphasize those maps induced by link cobordisms.

Definition 2.3.1. A *decorated link cobordism* from $(Y_0,\mathbb{L}_0) = (Y_0,(L_0,\mathbf{w}_0,\mathbf{z}_0))$ to $(Y_1,\mathbb{L}_1) = (Y_1,(L_1,\mathbf{w}_1,\mathbf{z}_1))$ is a pair $(W,\mathcal{F}) = (W,(\Sigma,\mathcal{A}))$ with the following properties:

1. W is an oriented cobordism from Y_0 to Y_1

2. Σ is an oriented surface in W with $\partial\Sigma = -L_0 \cup L_1$

3. $\mathcal{A}$ is a properly embedded 1-manifold in Σ, dividing it into subsurfaces $\Sigma_{\mathbf{w}}$ and $\Sigma_{\mathbf{z}}$ such that $\mathbf{w}_0, \mathbf{w}_1 \subset \Sigma_{\mathbf{w}}$ and $\mathbf{z}_0, \mathbf{z}_1 \subset \Sigma_{\mathbf{z}}$.

In [Zem18], it is shown that a decorated link cobordism $(W, \mathcal{F})$ from $(Y_0, \mathbb{L}_0)$ to $(Y_1, \mathbb{L}_1)$ and a Spin^{C}-structure $\mathfrak{s}$ on W, give rise to a map

$$F_{W,\mathcal{F},\mathfrak{s}} : \mathcal{CFL}^-(Y_0, \mathbb{L}_0, \mathfrak{s}|_{Y_0}) \to \mathcal{CFL}^-(Y_1, \mathbb{L}_1, \mathfrak{s}|_{Y_1}),$$

and these maps are functorial [Zem18, Theorem B] in the following sense:

1. Let $(W, \mathcal{F})$ be the trivial link cobordism, i.e. $W = Y \times [0,1]$, $\Sigma = L \times [0,1]$ and $\mathcal{A}$ is a collection of arcs $\mathbf{p} \times [0,1]$ where $\mathbf{p} \subset L - (\mathbf{w} \cup \mathbf{z})$ and consists of exactly one point in each component of $L - (\mathbf{w} \cup \mathbf{z})$. Then $F_{W,\mathcal{F},\mathfrak{s}} = \text{id}_{\mathcal{CFL}^-(Y,\mathbb{L},\mathfrak{s}|_Y)}$.

2. If $(W, \mathcal{F})$ can be decomposed into the union of two decorated link cobordisms $(W_1, \mathcal{F}_1) \cup (W_2, \mathcal{F}_2)$ and $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are Spin^{C}-structures on W_1 and W_2 respectively which agree on their common boundary, then

$$F_{W_2,\mathcal{F}_2,\mathfrak{s}_2} \circ F_{W_1,\mathcal{F}_1,\mathfrak{s}_1} \simeq \sum_{\substack{\mathfrak{s} \in \text{Spin}^{\text{C}}(W), \\ \mathfrak{s}|_{W_i} = \mathfrak{s}_i}} F_{W,\mathcal{F},\mathfrak{s}}.$$

The decorated link cobordism maps are defined as compositions of maps associated to handle attachments to the embedded surfaces and to the ambient 4-manifold.

In general, it is quite difficult to compute the decorated link cobordism maps. In some simple cases, however, the link cobordism maps can be computed in terms of the graph cobordism maps defined in [Zem15].

Definition 2.3.2. If $(Y_0, \mathbf{w}_0)$ and $(Y_1, \mathbf{w}_1)$ are 3-manifolds with a collection of basepoints $\mathbf{w}_0$ and $\mathbf{w}_1$, a ribbon graph cobordism between them is a pair (W, Γ) such that W is a cobordism from Y_0 to Y_1 and Γ is a graph embedded in W with the properties that $\Gamma \cap Y_i = \mathbf{w}_i$, each basepoint $\mathbf{w}_i$ has valence 1 in Γ, and at each vertex, the edges of Γ are given a cyclic ordering.

A ribbon graph cobordism $(W, \Gamma) : (Y_0, \mathbf{w}_0) \to (Y_1, \mathbf{w}_1)$ gives rise to two maps:

$$F^A_{W,\Gamma,\mathfrak{s}}, F^B_{W,\Gamma,\mathfrak{s}} : CF^-(Y_0, \mathbf{w}_0, \mathfrak{s}|_{Y_0}) \to CF^-(Y_1, \mathbf{w}_1, \mathfrak{s}|_{Y_1}).$$

These two maps satisfy

$$F^A_{W,\Gamma,\mathfrak{s}} \simeq F^B_{W,\overline{\Gamma},\mathfrak{s}},$$

where $\overline{\Gamma}$ is the graph obtained by reversing the cyclic ordering at each of the vertices. The map $F^A_{W,\Gamma,\mathfrak{s}}$ depends on the interaction between the graph and the $\boldsymbol{\alpha}$-curves while $F^B_{W,\overline{\Gamma},\mathfrak{s}}$ depends on the interaction of the graph and the $\boldsymbol{\beta}$-curves. When Γ is simply a path, these maps agree with the original cobordism maps defined by Ozsváth and Szabó [Zem15, Theorem B].

The graph cobordism maps encode the action of $\Lambda^* H_1(Y)/\mathrm{Tors}$ on the Heegaard Floer complexes. Recall that, given a closed loop $\gamma \subset Y$, the action of $[\gamma] \in H_1(Y)/\mathrm{Tors}$ on $HF^-(Y, \mathbf{w})$ is induced by a map

$$A_\gamma : CF^-(Y, \mathbf{w}) \to CF^-(Y, \mathbf{w})$$

defined bv

$$A_\gamma(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}), \\ \mu(\phi)=1}} a(\gamma, \phi) \# \widehat{\mathcal{M}}(\phi) U^{n_{\mathbf{w}}(\phi)} \mathbf{y}.$$

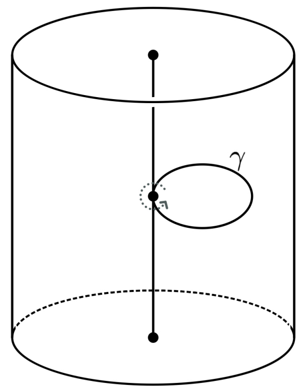


FIGURE 8 The graph Γ realizing the action of a closed curve γ in Y. The cyclic ordering is indicated by the dashed arrow.

Roughly speaking, the quantity $a(\gamma, \phi)$ is the intersection number of γ and the portion of the boundary of a domain for ϕ which lies on an $\boldsymbol{\alpha}$-curve. This map satisfies $A_\gamma^2 \simeq 0$ and can be realized by the graph cobordism $(Y \times [0,1], \Gamma)$, where the graph Γ is shown in Figure 8.

If $(W, \mathcal{F}) : (Y_0, \mathbb{L}_0) \to (Y_1, \mathbb{L}_1)$ is a decorated link cobordism and $\Gamma \subset \Sigma$ is a ribbon graph, we say that Γ is the *ribbon 1-skeleton* of $\Sigma_{\mathbf{w}}$ if $\Gamma \subset W$, $\Gamma \cap Y_i = \mathbf{w}_i$, $\Sigma_{\mathbf{w}}$ is a regular neighborhood of Γ in Σ, and the cyclic orders of Γ agree with the orientation of Σ. A *ribbon 1-skeleton* of $\Sigma_{\mathbf{z}}$ is defined in exactly the same way. There are natural chain isomorphisms

$$\mathcal{CFL}^-(Y, \mathbb{L}, \mathfrak{s}) \otimes_{\mathbb{F}_2[U,V]} \mathbb{F}_2[U,V]/(V-1) \cong CF^-(Y, \mathbf{w}, \mathfrak{t})$$

and

$$\mathcal{CFL}^-(Y, \mathbb{L}, \mathfrak{s}) \otimes_{\mathbb{F}_2[U,V]} \mathbb{F}_2[U,V]/(U-1) \cong CF^-(Y, \mathbf{z}, \mathfrak{t} - PD[L]).$$

Under these identifications, a link cobordism map $F_{W,\mathcal{F},\mathfrak{s}}$ induces two maps on $CF^-(Y)$, denoted

$$F_{W,\mathcal{F},\mathfrak{s}}|_{U=1} \qquad \text{and} \qquad F_{W,\mathcal{F},\mathfrak{s}}|_{V=1}.$$

These maps agree with the maps induced by the graph cobordism maps associated to the ribbon 1-skeletons of Σ.

Theorem 8. *[Zem18, Theorem C] If $(W,\mathcal{F})$ is a decorated link cobordism, and $\Gamma_{\boldsymbol{w}} \subset \Sigma_{\boldsymbol{w}}$ and $\Gamma_{\boldsymbol{z}} \subset \Sigma_{\boldsymbol{z}}$ are ribbon 1-skeleta, then*

$$F_{W,\mathcal{F},\mathfrak{s}}|_{U=1} \simeq F^A_{W,\Gamma_{\boldsymbol{z}},\mathfrak{s}-PD[\Sigma]} :$$

and

$$F_{W,\mathcal{F},\mathfrak{s}}|_{V=1} \simeq F^B_{W,\Gamma_{\boldsymbol{w}},\mathfrak{s}},$$

under the identifications above.

For a full discussion on Zemke's graph TQFT framework, see [Zem15] or, for an overview, see [Zem19c, Section 9.2].

In the following situation, the decorated link cobordism maps are determined by the corresponding graph cobordism map. Let $\mathcal{F}$ be a closed surface in $W : Y_0 \to Y_1$, which is decorated by $\mathcal{A}$, as in Definition 2.3.1. Choose disjoint disks D_0 and D_1 in $\mathcal{F}$ which each intersect $\mathcal{A}$ in a single arc, and perturb $\mathcal{F}$ so that it intersects Y_i in D_i. Remove each D_i, leaving a decorated cobordism $\mathcal{F}_0$ between doubly pointed unknots $\mathcal{U}_1$ and $\mathcal{U}_2$. Let p_i denote the center of the disk D_i. Identify $\mathcal{CFL}^-(Y_i,\mathcal{U}_i,\mathfrak{s})$ with $CF^-(Y_i,p_i,\mathfrak{s}) \otimes_{\mathbb{F}[W]} \mathbb{F}[U,V]$, where W acts on $\mathbb{F}[U,V]$ as UV. Under this identification, a graph cobordism map $F_{W,\Gamma,\mathfrak{s}}$ induces a map $\mathcal{CFL}^-(Y_0,\mathcal{U}_0,\mathfrak{s}|_{Y_0}) \to \mathcal{CFL}^-(Y_1,\mathcal{U}_1,\mathfrak{s}|_{Y_1})$, which we write as $F_{W,\Gamma,\mathfrak{s}}|^{\mathbb{F}_2[U,V]}$. In this case, the link cobordism map induced by $(W, \mathcal{F}_0) : (Y_0, \mathcal{U}_0) \to (Y_1, \mathcal{U}_1)$ is relatively simple.

Proposition 2.3.3. *[Zem19c, Proposition 9.7] Let $\mathcal{F} = (\Sigma, \mathcal{A})$ be a closed decorated link cobordism, and let $(W, \mathcal{F}_0)$ be the link cobordism obtained from $\mathcal{F}$ by the procedure outlined above. Define ΔA to be*

$$\frac{\langle c_1(\mathfrak{s}), \Sigma\rangle - [\Sigma] \cdot [\Sigma]}{2} + \frac{\chi(\Sigma_w) - \chi(\Sigma_z)}{2}.$$

Then,

$$F_{W,\mathcal{F}_0,\mathfrak{s}} \simeq \begin{cases} V^{\Delta A} \cdot F^B_{W,\Gamma_w,\mathfrak{s}}|^{\mathbb{F}[U,V]} & \Delta A \geq 0 \\ U^{-\Delta A} \cdot F^A_{W,\Gamma_z,\mathfrak{s}-PD[\Sigma]}|^{\mathbb{F}[U,V]} & \Delta A \leq 0, \end{cases}$$

where Γ_w and Γ_z are ribbon 1-skeleta.

It will also be useful to understand how the link cobordism maps change under surgery operations. If $(W, \mathcal{F})$ is a link cobordism and γ is a closed curve in $\mathcal{A}$, we can simultaneously do surgery on γ in W and Σ to obtain a new link cobordism $(W(\gamma), \mathcal{F}(\gamma))$, i.e. remove a regular neighborhood of $\gamma \subset (W, \Sigma)$, which can can identified with $(S^1 \times D^3, S^1 \times D^1)$ and replace it with $(D^2 \times S^2, D^2 \times S^0)$. The surface obtained by surgery on γ naturally inherits a decoration $\mathcal{A}(\gamma)$, so denote the new decorated surface $\mathcal{F}(\gamma) = (\Sigma(\gamma), \mathcal{A}(\gamma))$ (see Figure 9). If the curve γ represents a non-divisible element of $H_1(W; \mathbb{Z})$ then

$$F_{W,\mathcal{F},\mathfrak{s}} \simeq F_{W(\gamma),\mathcal{F}(\gamma),\mathfrak{s}(\gamma)}$$

by [Zem19a, Proposition 5.4]. The assumption that $[\gamma]$ is non-divisible guarantees that there is a is the unique $\text{Spin}^{\mathbb{C}}$-structure $\mathfrak{s}(\gamma)$ on $W(\gamma)$ which extends a given

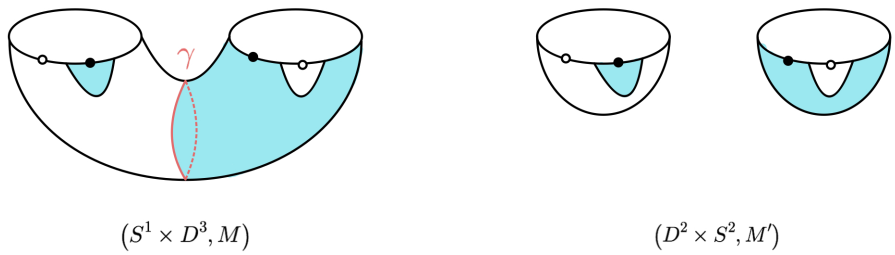


FIGURE 9 The decorated link cobordisms $(S^1 \times D^3, M)$ (left) and $(D^2 \times S^2, M')$ (right).

Spin$^{\mathrm{C}}$-structure on $W - N(\gamma)$. An analogous result holds for surgeries on collections of curves $\gamma_1, ..., \gamma_n$ which will be of use in the proof of Theorem **??**.

Proposition 2.3.4. *Let $(W, \mathcal{F})$ be a link cobordism. Let $\gamma_1, ..., \gamma_n$ be closed curves in $\mathcal{A}$ and let $(W(\gamma_1, ..., \gamma_n), \mathcal{F}(\gamma_1, ..., \gamma_n))$ be the surgered link cobordism. If the restriction map $H^1(W - \amalg N(\gamma_i)) \to H^1(\amalg \partial N(\gamma_i))$ is surjective, then there is a unique Spin^C-structure $\mathfrak{s}(\gamma_1, ..., \gamma_n)$ extending $\mathfrak{s}|_{W - \amalg N(\gamma_i)}$ for each $\mathfrak{s} \in$ Spin$^C(W)$ and*

$$F_{W,\mathcal{F},\mathfrak{s}} \simeq F_{W(\gamma_1,...,\gamma_n),\mathcal{F}(\gamma_1,...,\gamma_n),\mathfrak{s}(\gamma_1,...,\gamma_n)}.$$

Finally, recall in [OS06, Proof of Theorem 3.1], Ozsváth and Szabó define an extended cobordism map:

$$F_{W,\mathfrak{s}} : \Lambda^* H_1(W; \mathbb{Z})/\mathrm{Tors} \otimes CF^-(Y_0, \mathfrak{s}|_{Y_0}) \to CF^-(Y_1, \mathfrak{s}|_{Y_1}).$$

A Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ gives rise to a cobordism $X_{\alpha,\beta,\gamma}$. Since the natural map $H_1(\partial X_{\alpha,\beta,\gamma}) \to H_1(X_{\alpha,\beta,\gamma})$ is surjective, a given element $h \in H_1(X_{\alpha,\beta,\gamma})$ is in the image of some $(h_1, h_2, h_3) \in H_1(\partial X_{\alpha,\beta,\gamma}) \cong H_1(Y_{\alpha,\beta}) \oplus H_1(Y_{\beta,\gamma}) \oplus H_1(Y_{\alpha,\gamma})$.

Then, by utilizing the H_1/Tors-action on $\partial X_{\alpha,\beta,\gamma}$, define a map

$$\Lambda^* H_1(X_{\alpha,\beta,\gamma};\mathbb{Z})/\mathrm{Tors} \otimes CF^-(Y_{\alpha,\beta},\mathfrak{s}_{\alpha,\beta}) \otimes CF^-(Y_{\beta,\gamma},\mathfrak{s}_{\beta,\gamma}) \to CF^-(Y_{\alpha,\gamma},\mathfrak{s}_{\alpha,\gamma}),$$

by

$$F_{\alpha,\beta,\gamma}(h \otimes \mathbf{x} \otimes \mathbf{y}) = F_{\alpha,\beta,\gamma}((h_1 \cdot \mathbf{x}) \otimes \mathbf{y}) + F_{\alpha,\beta,\gamma}(\mathbf{x} \otimes (h_2 \cdot \mathbf{y})) - h_3 \cdot F_{\alpha,\beta,\gamma}(\mathbf{x} \otimes \mathbf{y}).$$

This action induces a map on homology. By decomposing $W = W_1 \cup W_2 \cup W_3$ into the 1-, 2-, and 3-handle attachment cobordisms, the extended cobordism map is defined to be

$$F_{W,\mathfrak{s}}(h \otimes \mathbf{x}) = F_{W_3,\mathfrak{s}} \circ F_{W_2,\mathfrak{s}}(h \otimes F_{W_1,\mathfrak{s}}(\mathbf{x})),$$

This map satisfies a version of the usual $\mathrm{Spin}^{\mathrm{C}}$-composition law:

Proposition 2.3.5. *[OS06, Proposition 4.20] If $W = W_1 \cup W_2$, and $\xi_1 \in \Lambda^* H_1(W_1;\mathbb{Z})/Tors$ and $\xi_2 \in \Lambda^* H_1(W_2;\mathbb{Z})/Tors$, then*

$$F_{W_2,\mathfrak{s}_2}(\xi_2 \otimes F_{W_1,\mathfrak{s}_1}(\xi_1 \otimes \cdot)) = \sum_{\substack{\mathfrak{s} \in Spin^C(W), \\ \mathfrak{s}|_{W_i} = \mathfrak{s}_i}} F_{W,\mathfrak{s}}((\xi_3 \otimes \cdot),$$

where $\xi_3 \in \Lambda^* H_1(W;\mathbb{Z})/Tors$ is the image of $\xi_1 \otimes \xi_2$ under the natural map.

There is an $H_1(Y;\mathbb{Z})$/Tors-action on multi-pointed Heegaard diagrams as well [Zem15, Equation 5.2]

$$A_\gamma : \mathcal{CFL}^-(Y,\mathbb{L},\mathfrak{s}) \to \mathcal{CFL}^-(Y,\mathbb{L},\mathfrak{s}),$$

defined

$$A_\gamma(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\substack{\phi\in\pi_2(\mathbf{x},\mathbf{y}),\\ \mu(\phi)=1}} a(\gamma,\phi)\#\widehat{\mathcal{M}}(\phi)U^{n_{\mathbf{w}}(\phi)}V^{n_{\mathbf{z}}(\phi)}\mathbf{y}.$$

Using this action, we can define extended link cobordism maps

$$F_{W,\mathcal{F},\mathfrak{s}} : (\Lambda^* H_1(W)/\mathrm{Tors}\otimes\mathbb{F}_2)\otimes\mathcal{CFL}^-(Y_0,\mathbb{L}_0,\mathfrak{s}|_{Y_0}) \to \mathcal{CFL}^-(Y_1,\mathbb{L}_1,\mathfrak{s}|_{Y_1}).$$

in exactly the same way. Note, that since the link Floer TQFT is defined with coefficients in $\mathbb{F}_2$, we need to tensor the exterior algebra generated by $H_1(Y)/\mathrm{Tors}$ with $\mathbb{F}_2$.

We will make use of an algebraic variant of $\mathcal{CFK}^-$, which is usually denoted CFK^-. Define $CFK^-(Y,K,\mathfrak{s})$ be the $\mathbb{F}_2[U]$-module obtained from $\mathcal{CFK}^-(Y,K,\mathfrak{s})$ by setting $V=0$ with differential

$$\partial(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\substack{\phi\in\pi_2(\mathbf{x},\mathbf{y}),\\ \mu(\phi)=1,\\ n_{\mathbf{z}}(\phi)=0}} \#\widehat{\mathcal{M}}(\phi)U^{n_{\mathbf{w}}(\phi)}\mathbf{y}.$$

Let $HFK^-(Y,K,\mathfrak{s})$ be the homology of this complex.

Secondary invariants of a surface

We now recall the definition of the secondary invariant τ of Juhász and Zemke [JZ21]. Let Σ be a surface in B^4 with boundary $\mathbb{K} = (K,w,z)$, a doubly based knot. Decorate Σ by a single arc such that the $\mathbf{z}$-subregion $\Sigma_{\mathbf{z}}\subset S$ is a bigon. Let $\mathcal{F}=(\Sigma,\mathcal{A})$ be the resulting decorated surface. Then, there is an induced map

$$F_{B^4,\mathcal{F}} : \mathbb{F}[U,V] \to \mathcal{CFK}^-(K,w,z).$$

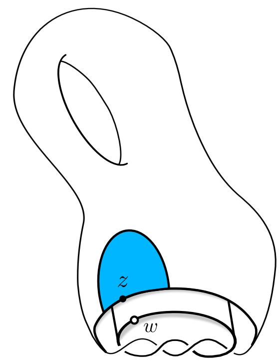


FIGURE 10 A surface Σ with $\Sigma_{\mathbf{z}}$-sub region a bigon.

Definition 2.3.6. [JZ21, Definition 4.4] Let (K, w, z) be a doubly based knot in S^3 and let Σ and Σ' be two disks in B^4 with boundary K, decorated as above. Then, define

$$\tau(\Sigma, \Sigma') = \min\{n : U^n \cdot [F_{B^4,\mathcal{F}}(1)] = U^n \cdot [F_{B^4,\mathcal{F}'}(1)] \in HFK^-(K)\}.$$

Remark 2.3.7. This invariant, τ (and its relatives ν, V_k, Υ), are called "secondary invariants" because they are defined as surface analogues of existing knot invariants.

A key result of [JZ21] states this invariant provides a lower bound for the stabilization distance.

Theorem 9. *[JZ21, Theorem 1.1] Let K be in a knot in S^3, and let Σ, Σ' be disks in B^4 with boundary K. Then,*

$$\tau(\Sigma, \Sigma') \leq \mu(\Sigma, \Sigma'),$$

where $\mu(\Sigma, \Sigma')$ is the stabilization distance between S and S'.

Bordered Floer Homology

Before defining the invariants of bordered three-manifolds, we provide a brief review of the relevant algebraic structures.

Fix a ground ring $\mathbf{k}$ of characteristic two. An $\mathcal{A}_\infty$ algebra $\mathcal{A}$ over $\mathbf{k}$ is a graded $\mathbf{k}$ -module A equipped with $\mathbf{k}$ -linear maps

$$\mu_i : A^{\otimes i} \to A[2-i],$$

for $i \geq 1$ satisfying

$$\sum_{i+j=n+1} \sum_{\ell=1}^{n-j+1} \mu_i(a_1 \otimes \ldots \otimes a_{\ell-1} \otimes \mu_j(a_\ell \otimes \ldots \otimes a_{\ell+j-1}) \otimes a_{\ell+j} \otimes \ldots \otimes a_n) = 0,$$

for $n \geq 1$. An $\mathcal{A}_\infty$ algebra is *strictly unital* if there is an element $1 \in \mathcal{A}$ such that $\mu_2(a,1) = \mu_2(1,a) = a$ and $\mu_i(a_1, \ldots, a_i) = 0$ if $i \neq 2$ and $a_j = 1$ for some j.

A *right* $\mathcal{A}_\infty$ *module* $\mathcal{M}$ over $\mathcal{A}$ is a graded $\mathbf{k}$ -module M equipped with operations

$$m_i : M \otimes A^{\otimes(i-1)} \to M[2-i],$$

satisfying

$$\sum_{i+j=n+1} m_i(m_j(x \otimes a_1 \otimes \ldots \otimes a_{j-1}) \otimes \ldots \otimes a_{n-1})$$

$$+ \sum_{i+j=n+1} \sum_{\ell=1}^{n-j} \mu_i(a, a_1 \otimes \ldots \otimes a_{\ell-1} \otimes \mu_j(a_\ell \otimes \ldots \otimes a_{\ell+j-1}) \otimes \ldots \otimes a_{n-1}) = 0.$$

We say $\mathcal{M}$ is strictly unital if for any $x \in M$, $m_2(x,1) = x$ and $m_i(x \otimes a_1 \otimes \ldots \otimes a_{i-1}) = 0$ if $i > 2$ and some $a_j = 1$.

Let $\mathcal{A}$ be a dg algebra and let N be a graded $\mathbf{k}$ -module with a map

$$\delta^1 : N \to (A \otimes N)[1],$$

such that

$$(\mu_2 \otimes \mathbb{I}_N) \circ (\mathbb{I}_A \otimes \delta^1) \circ \delta^1 + (\mu_1 \otimes \mathbb{I}_N) \circ \delta^1 = 0.$$

The pair (N, δ^1) is called a *tyep D structure* over A. A *type D structure homomorphism* is a $\mathbf{k}$ -module map $f^1 : N_1 \to A \otimes N_2$ satisfying

$$(\mu_2 \otimes \mathbb{I}_{N_2}) \circ (\mathbb{I}_A \otimes f^1) \circ \delta^2_{N_1} + (\mu_2 \otimes \mathbb{I}_{N_2}) \circ (\mathbb{I}_A \otimes \delta^1_{N_2}) \circ f^1 + (\mu_1 \otimes \mathbb{I}_{N_2}) \circ f^1 = 0.$$

The map δ^1 can be iterated to define maps

$$\delta^k : N \to (A^{\otimes k} \otimes N)[k],$$

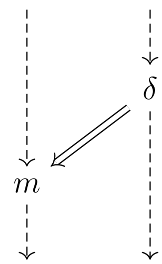
where $\delta^0 = \mathbb{I}_N$ and $\delta^i = (\mathbb{I}_{A^{\otimes (i-1)}} \otimes \delta^1) \circ \delta^{i-1}$. Similarly, a type D homomorphism f^1 can be used to define maps

$$f^k : N_1 \to (A^{\otimes k} \otimes N_2)[k-1], \quad f^k(x) = \sum_{i+j=k-1} (\mathbb{I}_{A^{\otimes (i-1)}} \otimes \delta^j_{N_2}) \circ (\mathbb{I}_{A^{\otimes i}} \otimes f^1) \circ \delta^i_{N_2}.$$

Given an $\mathcal{A}_\infty$ module $\mathcal{M}$ over $\mathcal{A}$ and a type D structure (N, δ^1), we define the $\mathbf{k}$ -module $\mathcal{M} \boxtimes N = M \otimes_{\mathbf{k}} N$, equipped with differential

$$\partial^{\boxtimes}(x \otimes y) = \sum_{k=0}^{\infty} (m_{k+1} \otimes \mathbb{I}_N)(x \otimes \delta^k(y)).$$

We can represent the differential on the box tensor product graphically as

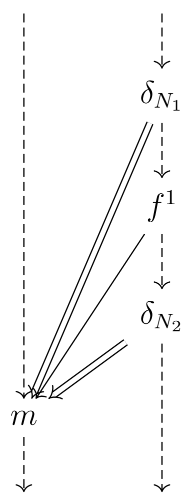
$\partial^{\boxtimes} =$ 

.

Given a map of type D structure $f^1 : N_1 \to N_2$, we define a map $\mathbb{I}_{\mathcal{M}} \boxtimes f^1 : \mathcal{M} \boxtimes N_1 \to \mathcal{M} \boxtimes N_2$ by

$$(\mathbb{I}_{\mathcal{M}} \boxtimes f^1)(x \otimes y) = \sum_{k=1}^{\infty} (m_{k+1} \otimes \mathbb{I}_{N_2})(x \otimes f^k(y)).$$

Graphically, this map is represented

$\mathbb{I}_{\mathcal{M}} \boxtimes f^1 =$ 

.

Bordered Floer homology is a package of invariants of 3-manifolds with parametrized boundary. Bordered Floer homology associates to a surface F a differential graded algebra $\mathcal{A}(F)$ and to a 3-manifold Y with boundary together with an identification $\varphi : \partial Y \to F$ a left differential graded module over

$\mathcal{A}(-F)$, the type D module of Y, denoted $\widehat{CFD}(Y)$ and a right $\mathcal{A}_\infty$-module over $\mathcal{A}(F)$, the type A module of Y, denoted $\widehat{CFA}(Y)$. Much like classical Heegaard Floer homology, the bordered Floer invariants are defined by representing a 3-manifold with parametrized boundary by a kind of Heegaard diagram and counting holomorphic disks which, in the bordered case, may asymptotically approach the boundary.

Bordered Floer homology has a pairing theorem [LOT18, Theorem 1.3], which recovers the hat-version of the Heegaard Floer homology of the manifold obtained by gluing bordered manifolds along their common boundary. Given 3-manifolds Y_1 and Y_2 with $\partial Y_1 \cong F \cong \partial Y_2$, there is a homotopy equivalence

$$\widehat{CF}(Y_1 \cup Y_2) \simeq \widehat{CFA}(Y_1) \boxtimes_{\mathcal{A}(F)} \widehat{CFD}(Y_2).$$

Bordered Floer theory also recovers knot Floer homology [LOT18, Theorem 11.21]. Given a doubly pointed bordered Heegaard diagram $(\mathcal{H}_1, w, z)$ for $(Y_1, \partial F, K)$ and a bordered Heegaard diagram $(\mathcal{H}_2, z)$ with $\partial Y_1 \cong F \cong -\partial Y_2$, then

$$HFK^-(Y_1 \cup Y_2, K) \cong H_*(CFA^-(\mathcal{H}_1, w, z) \boxtimes_{\mathcal{A}(F)} \widehat{CFD}(\mathcal{H}_2, z)).$$

Bordered Floer theory, therefore, gives an effective way to study satellites. Let K_P be the satellite of K with pattern P. $\widehat{CFD}(S^3 - K)$ is determined by $CFK^-(K)$ [LOT18, Chapter 11], $HFK^-(K_P)$ can be computed by finding a doubly pointed bordered Heegaard diagram $\mathcal{H}_P$ for the pattern P in the solid torus and computing the box tensor product $CFA^-(\mathcal{H}_P) \boxtimes \widehat{CFD}(S^3 - K)$.

The last important result from bordered Floer theory which we will utilize, is the morphism spaces pairing theorem, which gives a means of recovering classical

Heegaard Floer homology in terms of the Hom functor rather than the tensor product functor [LOT11]. If Y_1 and Y_2 are 3-manifolds with $\partial Y_1 \cong F \cong \partial Y_2$, then

$$\widehat{CF}(-Y_1 \cup Y_2) \simeq \mathrm{Mor}_{\mathcal{A}(-F)}(\widehat{CFD}(Y_1), \widehat{CFD}(Y_2)),$$

where the latter object is the chain complex of $\mathcal{A}(-F)$-linear maps from $\widehat{CFD}(Y_1)$ to $\widehat{CFD}(Y_2)$ equipped with differential

$$d(\varphi) = \partial_{\widehat{CFD}(Y_2)} \circ \varphi + \varphi \circ \partial_{\widehat{CFD}(Y_1)}.$$

Having concluded the requisite background material, we proceed towards the proofs of Theorems 3 and 4.

CHAPTER III

RIBBON HOMOLOGY CONCORDANCES

First Homology Action as Link Cobordism Maps

It is helpful in geometric arguments that the H_1/Tors-action can be realized as a graph cobordism map on CF^-. In the same way, it is beneficial to realize the $H_1(Y)/\mathrm{Tors}$-action on $\mathcal{CFL}^-$ as a link cobordism map. By Theorem 8, $\Sigma_{\mathbf{w}}$ and $\Sigma_{\mathbf{z}}$ should be ribbon 1-skeleta of the graph in Figure 8. Our strategy will be to try to compute the decorated link cobordism map obtained by tubing on a torus whose longitude is a curve which represents the class in $H_1(Y)/\mathrm{Tors}$.

Construct a closed decorated link cobordism $\mathcal{F}$ inside of $S^2 \times S^1 \times [-1,1]$ as follows. Let Σ be the boundary of a tubular neighborhood of $\{\mathrm{pt}\} \times S^1 \times \{0\} \subset S^2 \times S^1 \times \{0\}$. Let the dividing circles $\mathcal{A}$ be two parallel closed curves on the boundary of Σ obtained by isotoping $\{\mathrm{pt}\} \times S^1 \times \{0\}$ radially, so that $\mathcal{A}$ divides Σ into $\Sigma_{\mathbf{w}}$ and $\Sigma_{\mathbf{z}}$ which are both annuli. As in the text preceding Proposition 2.3.3, choose disks D_1 and D_2 which intersect the same dividing arc $\{\mathrm{pt}\} \times S^1$, and take $(W, \mathcal{F}_0)$ to be the link cobordism obtained by isotoping $\mathcal{F}$ and removing the disks D_1 and D_2. See Figure 11. Note that the graph shown in Figure 8 is a ribbon 1-skeleton for both $\Sigma_{\mathbf{w}}$ and $\Sigma_{\mathbf{z}}$.

Lemma 3.1.1. *For $(S^2 \times S^1 \times [-1,1], \mathcal{F}_0)$ the link cobordism described above*

$$F_{S^2\times S^1\times[-1,1],\mathcal{F}_0,\mathfrak{s}_0}(\theta^+) = \theta^- \text{ and } F_{S^2\times S^1\times[-1,1],\mathcal{F}_0,\mathfrak{s}_0}(\theta^-) = 0,$$

where θ^+ (θ^-) is the generator of $\mathcal{CFL}^-(S^2 \times S^1, \mathcal{U}, \mathfrak{t}_0)$ of higher (lower) grading, $\mathfrak{s}_0$ is the torsion Spin^C-structure on $S^2 \times S^1 \times [-1,1]$ and $\mathfrak{t}_0 = \mathfrak{s}_0|_{S^2\times S^1}$.

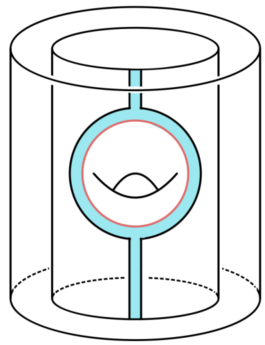


FIGURE 11 A schematic of the decorated surface in $S^2 \times S^1 \times \{0\}$ described in the text preceding Lemma 3.1.1.

Proof. By Proposition 2.3.3, $F_{S^2\times S^1\times[-1,1],\mathcal{F}_0,\mathfrak{s}_0}$ is determined by its reduction to a graph cobordism map: concretely, if $F^B_{S^2\times S^1\times[-1,1],\Gamma_w,\mathfrak{s}}$ is the corresponding graph map, then

$$F_{S^2\times S^1\times[-1,1],\mathcal{F}_0,\mathfrak{s}_0} \simeq V^{\Delta A}\cdot F^B_{S^2\times S^1\times[-1,1],\Gamma_w,\mathfrak{s}_0}|^{\mathbb{F}_2[U,V]}$$

under the identification of $\mathcal{CFL}^-(Y,\mathbb{U},\mathfrak{s}_0)$ with $CF^-(Y,w,\mathfrak{s}_0)\otimes_{\mathbb{F}_2[W]}\mathbb{F}_2[U,V]$ as before. The quantity ΔA, given by

$$\frac{\langle c_1(\mathfrak{s}_0),[\Sigma]\rangle - [\Sigma]\cdot[\Sigma]}{2} + \frac{\chi(\Sigma_w)-\chi(\Sigma_z)}{2},$$

vanishes, since $[\Sigma]$ is nullhomologous in $S^1\times S^2$, and Σ_w and Σ_z are both cylinders. By construction, $F^B_{S^2\times S^1\times[-1,1],\Gamma_w,\mathfrak{s}_0}$ is the graph cobordism shown in Figure 8. Hence, $F^B_{S^2\times S^1\times[-1,1],\Gamma_w,\mathfrak{s}_0}$ is just the map A_γ. It is straightforward to verify that the action of $[\{\mathrm{pt}\}\times S^1]\in H_1(S^2\times S^1)$ takes θ^+ to θ^- and θ^- to zero. $\square$

We now turn to the case of a nullhomologous knot K embedded in an arbitrary 3-manifold Y. Let $(Y\times[-1,1],\mathcal{F}_Y)$ be Morse-trivial, in the sense defined in Section **??**. The idea is to modify $\mathcal{F}_Y$ by by tubing on a torus with a dividing

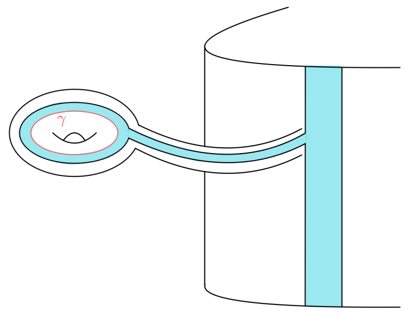


FIGURE 12 A schematic of the decorated surface $\mathcal{F}_\gamma$. In general, $K \times \{0\}$ and γ might be linked.

arc which represents a class in $H_1(Y;\mathbb{Z})/\mathrm{Tors}$. Let $\gamma \subset Y \times \{0\} - (K \times \{0\})$ be such a curve. Choose a path λ in $Y \times \{0\}$ connecting γ to a point p in $K \times \{0\}$ which lies on one of the dividing arcs.

Let T be the boundary of a tubular neighborhood of γ in $Y \times \{0\}$. Decorate T with two parallel circles isotopic to γ. Tube T and Σ together along λ. Denote the resulting surface Σ_γ. Decorate the tube with two parallel arcs, connected to one of the circles parallel to γ on one end, and to one of the dividing arcs on $\mathcal{F}_Y$ on the other end. A schematic of this decoration, which is denoted $\mathcal{A}_\gamma$, is shown in Figure 12.

Lemma 3.1.2. *For the decorated surfaces* $(Y \times [-1,1], \mathcal{F}_Y)$ *and* $(Y \times [-1,1], \mathcal{F}_\gamma)$ *described above, we have that*

$$F_{Y\times[-1,1],\mathcal{F}_\gamma,\mathfrak{s}}(\cdot) \simeq F_{Y\times[-1,1],\mathcal{F}_Y,\mathfrak{s}}([\gamma]\otimes\cdot).$$

Proof. Decompose $(Y \times [-1,1], \mathcal{F}_\gamma)$ as $(X, \mathcal{F}_X) \circ (N_\lambda(\gamma), \mathcal{F})$ where $X = ((Y \times [-1,1]) - N_\lambda(\gamma))$, $\mathcal{F} = \mathcal{F}_\gamma \cap N_\lambda(\gamma)$, and $\mathcal{F}_X = \mathcal{F}_\gamma \cap X$. $\mathcal{F}$ is a punctured torus whose decoration is shown in Figure 13. Given a $\mathrm{Spin}^{\mathrm{C}}$-structure $\mathfrak{s}$ on $Y \times [-1,1]$, its restriction to X is torsion on $\partial N_\lambda(\gamma)$, which can be extended by $\mathfrak{s}_0$, the unique

Spin$^{\mathrm{C}}$-structure on $N_\lambda(\gamma)$. By considering another Mayer-Vietoris sequence, it is not hard to see that this extension is unique. The composition law then implies that

$$F_{Y\times[-1,1],\mathcal{F}_\gamma,\mathfrak{s}} \simeq F_{X,\mathcal{F}_X,\mathfrak{s}|_X} \circ F_{N_\lambda(\gamma),\mathcal{F},\mathfrak{s}_0}.$$

The map $F_{N_\lambda(\gamma),\mathcal{F},\mathfrak{s}_0}$ associated to the cobordism $(Y,K) \to (Y \amalg S^2 \times S^1, K \amalg U)$ can be computed as the composition of a 0-handle/birth map, followed by a 1-handle map, followed by the map $F_{S^2\times S^1\times[-1,1],\mathcal{F}_0,\mathfrak{s}_0}$ computed above: given an element $\mathbf{x} \in \mathcal{CFL}^-(Y,\mathbb{L},\mathfrak{s})$, the 0-handle/birth map simply introduces a pair of intersection points c^+, c^- on a genus 0 Heegaard diagram for S^3, and takes $\mathbf{x} \mapsto \mathbf{x} \otimes c^+$. Attaching a 1-handle, with both feet attached to the new 0-handle corresponds to the map $\mathbf{x} \otimes c^+ \mapsto \mathbf{x} \otimes \theta^+$ [Zem18, Section 5]. By Lemma 3.1.1, $F_{S^2\times S^1\times[-1,1],\mathcal{F}_0,\mathfrak{s}_0}(\mathbf{x} \otimes \theta^+) = \mathbf{x} \otimes \theta^-$. All together then,

$$F_{N_\lambda(\gamma),\mathcal{F},\mathfrak{s}_0}(\mathbf{x}) = \mathbf{x} \otimes \theta^-.$$

On the other hand, consider the cobordism $F_{N_\lambda(\gamma),\mathcal{D},\mathfrak{s}_0}$ where $\mathcal{D}$ is a disk decorated with a single arc, followed by the action of $[\gamma]$: this can be computed as the composition of the 0-handle/birth and 1-handle maps followed by the action of $[\gamma]$. Just as before, the 0-handle/birth and 1-handle maps take an intersection point $\mathbf{x}$ to $\mathbf{x} \otimes \theta^+$ and the action of $[\gamma]$ takes this element to $\mathbf{x} \times \theta^-$. Hence,

$$F_{N_\lambda(\gamma),\mathcal{F},\mathfrak{s}_0}(\mathbf{x}) = [\gamma] \cdot F_{N_\lambda(\gamma),\mathcal{D},\mathfrak{s}_0}(\mathbf{x}).$$

See Figure 13 for a comparison of these two cobordism maps.

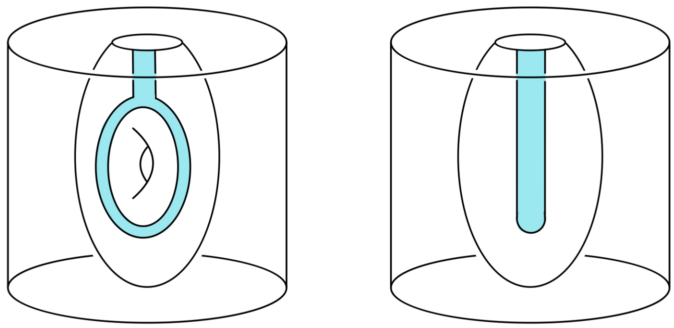


FIGURE 13 When the link cobordism map on the right is followed by the action of γ, it becomes equivalent to the map on the left.

By definition of the extended link cobordism maps, $[\gamma] \cdot F_{N_\lambda(\gamma),\mathcal{D},\mathfrak{s}_0}(\mathbf{x}) = F_{N_\lambda(\gamma),\mathcal{D},\mathfrak{s}_0}([\gamma] \otimes \mathbf{x})$. Combining these observations with the composition law for the extended cobordism maps shows

$$\begin{aligned} F_{Y\times[-1,1],\mathcal{F}_\gamma,\mathfrak{s}}(\mathbf{x}) &\simeq F_{X,\mathcal{F}_X,\mathfrak{s}|_X} \circ F_{N_\lambda(\gamma),\mathcal{F},\mathfrak{s}_0}(\mathbf{x}) \\ &\simeq F_{X,\mathcal{F}_X,\mathfrak{s}|_X} \circ F_{N_\lambda(\gamma),\mathcal{D},\mathfrak{s}_0}([\gamma] \otimes \mathbf{x}) \\ &\simeq F_{X,\mathcal{F}_X,\mathfrak{s}_X}(1 \otimes F_{N_\lambda(\gamma),\mathcal{D},\mathfrak{s}_0}([\gamma] \otimes \mathbf{x})) \\ &\simeq F_{Y\times[-1,1],\mathcal{F}_Y,\mathfrak{s}}([\gamma] \otimes \mathbf{x}) \end{aligned}$$

as desired. $\square$

Note that the choice of path λ did not matter, since the diffeomorphism type of the neighborhood $N_\lambda(\gamma)$ did not depend on λ.

Let $(W, \mathcal{F}) : (Y_0, K_0) \to (Y_1, K_1)$ be a link cobordism which is concordance Morse-trivial. Decompose W as a composition of handle attachments $W_3 \circ W_2 \circ W_1$. Let γ be a curve in W which represents an element of $H_1(W)/\text{Tors}$. Homotope the curve γ so it is contained in the boundary of W_1, which is denoted $\widetilde{Y} = \partial_+ W_1$. In this dimension and codimension, such a homotopy can be taken to be an isotopy. Let $\widetilde{\mathcal{F}}_\gamma$ be the decorated surface in $\widetilde{Y} \times [-1,1]$ described above which realizes the action of $[\gamma]$. Let $(W, \mathcal{F}_\gamma)$ be the link cobordism $(W_3, \mathcal{F}_3) \circ (W_2, \mathcal{F}_2) \circ (\widetilde{Y} \times [-1,1], \widetilde{\mathcal{F}}_\gamma) \circ (W_1, \mathcal{F}_1)$, where $\mathcal{F}_i$ is $\mathcal{F} \cap W_i$.

Lemma 3.1.3. *Let $(W, \mathcal{F}_\gamma)$ be the decorated surface described above. Then,*

$$F_{W,\mathcal{F},\mathfrak{s}}([\gamma] \otimes \cdot) \simeq F_{W,\mathcal{F}_\gamma,\mathfrak{s}}(\cdot).$$

Proof. We will make use of the decomposition of $(W, \mathcal{F}_\gamma)$ as

$$(W_3, \mathcal{F}_3) \circ (W_2, \mathcal{F}_2) \circ (\widetilde{Y} \times [-1,1], \widetilde{\mathcal{F}}_\gamma) \circ (W_1, \mathcal{F}_1).$$

$F_{W,\mathcal{F}_\gamma}$ can be computed as the composition:

$$F_{W,\mathcal{F}_\gamma,\mathfrak{s}}(x) \simeq F_{W_3,\mathcal{F}_3,\mathfrak{s}|_{W_3}} \circ F_{W_2,\mathcal{F}_2,\mathfrak{s}|_{W_2}} \circ F_{\tilde{Y}\times[-1,1],\mathcal{F}_\gamma,\mathfrak{s}|_{\tilde{Y}\times[-1,1]}} \circ F_{W_1,\mathcal{F}_1,\mathfrak{s}|_{W_1}}(x).$$

By Lemma 3.1.2, this map is chain homotopic to

$$F_{W_3,\mathcal{F}_3,\mathfrak{s}|_{W_3}} \circ F_{W_2,\mathcal{F}_2,\mathfrak{s}|_{W_2}} \circ F_{\tilde{Y}\times[-1,1],\mathcal{F}_{\tilde{Y}},\mathfrak{s}|_{\tilde{Y}\times[-1,1]}}([\gamma] \otimes F_{W_1,\mathcal{F}_1,\mathfrak{s}|_{W_1}}(x)).$$

Since $(\tilde{Y} \times [-1,1], \mathcal{F}_{\tilde{Y}})$ is the identity,

$$F_{\tilde{Y}\times[-1,1],\mathcal{F}_{\tilde{Y}},\mathfrak{s}|_{\tilde{Y}\times[-1,1]}}([\gamma] \otimes F_{W_1,\mathcal{F}_1,\mathfrak{s}|_{W_1}}(x)) = [\gamma] \cdot F_{W_1,\mathcal{F}_1,\mathfrak{s}|_{W_1}}(x) + 0,$$

and hence this map can be rewritten as:

$$F_{W_3,\mathcal{F}_3,\mathfrak{s}|_{W_3}} \circ F_{W_2,\mathcal{F}_2,\mathfrak{s}|_{W_2}}([\gamma] \cdot F_{W_1,\mathcal{F}_1,\mathfrak{s}|_{W_1}}(x)).$$

But, since γ was chosen as to lie in $\widetilde{Y}$, this is by definition equal to the map $F_{W,\mathcal{F},\mathfrak{s}}([\gamma] \otimes x)$. $\square$

Maps induced by ribbon homology concordances

As is typical in proving results of this kind, our strategy will be to compare the double of a ribbon $\mathbb{Z}$-homology concordance to a Morse-trivial link cobordism. Recall that the double of a link cobordism $(W, \Sigma) : (Y_0, K_0) \to (Y_1, K_1)$ is the link cobordism $(D(W), D(\Sigma)) = (W, \Sigma) \cup_{(Y_1,K_1)} (\overline{W}, \overline{\Sigma})$, where $(\overline{W}, \overline{\Sigma})$ is the link cobordism obtained by turning (W, Σ) around and reversing the orientation.

It will be helpful to have a version of the "sphere tubing" property of the link cobordism maps [Zem19b, Lemma 3.1], [MZ19, Lemma 4.2]; if $(W, \mathcal{F})$ is a link cobordism in a homology cobordism and S is a null-homologous 2-sphere embedded in the complement of the link cobordism, S can be tubed to the embedded surface without changing the induced map.

Proposition 3.2.1. *Let $\mathcal{F} = (\Sigma, \mathcal{A})$ be a decorated link cobordism in a homology cobordism W, and let $S \subset W$ be a smoothly embedded, nullhomologous sphere disjoint from $\mathcal{F}$. Let $\mathcal{F}'$ be a decorated cobordism obtained by connecting Σ and S by a tube whose feet are disjoint from $\mathcal{A}$. Then,*

$$F_{W,\mathcal{F},\mathfrak{s}} \simeq F_{W,\mathcal{F}',\mathfrak{s}}.$$

Proof. Factor $F_{W,\mathcal{F},\mathfrak{s}}$ and $F_{W,\mathcal{F}',\mathfrak{s}}$ through a regular neighborhood $N(S)$ of S. Since S is nullhomologous, $N(S)$ can be identified with $D^2 \times S^2$. $\mathcal{F}'$ intersects $N(S)$ in a disk D' and $\partial N(S)$ in an unknot. We can perturb $\mathcal{F}$ so it meets $N(S)$ in a disk D and $\partial N(S)$ in an unknot as well. Let $\mathcal{D}$ and $\mathcal{D}'$ be the disks D and D' decorated with a single dividing arc. Since S is nullhomologous, the restriction of a given $\mathfrak{s} \in \mathrm{Spin}^{\mathrm{C}}(W)$ to $N(S)$ will be torsion. By [Zem19b, Lemma 3.1], $F_{N(S),\mathcal{D},\mathfrak{s}|_{N(S)}}$ does not depend on the choice of embedded disk, and so $F_{N(S),\mathcal{D},\mathfrak{s}|_{N(S))}} \simeq F_{N(S),\mathcal{D}',\mathfrak{s}|_{N(S)}}$.

Moreover, since W is a homology cobordism, the map $H^2(W) \to H^2(\partial_- W)$ is an isomorphism, and, in particular, an injection. It follows from the following diagram that the map $H^2(W) \to H^2(W - N(S))$ is injective as well.

$$\begin{tikzcd}
H^2(W) \arrow[r, "\cong"] \arrow[d] & H^2(\partial_- W) \\
H^2(W - N(S)) \arrow[ur] &
\end{tikzcd}$$

Therefore, a given $\mathrm{Spin}^{\mathrm{C}}$-structure on $W - N(S)$ will extend over $N(S)$, and moreover will extend uniquely. By the composition law for the link cobordism maps,

$$\begin{aligned}
F_{W,\mathcal{F},\mathfrak{s}} &\simeq F_{W-N(S),\mathcal{F}-N(S),\mathfrak{s}|_{W-N(S)}} \circ F_{N(S),\mathcal{D},\mathfrak{s}|_{N(S))}} \\
&\simeq F_{W-N(S),\mathcal{F}-N(S),\mathfrak{s}|_{W-N(S)}} \circ F_{N(S),\mathcal{D}',\mathfrak{s}|_{N(S))}} \\
&\simeq F_{W,\mathcal{F}',\mathfrak{s}},
\end{aligned}$$

as desired. □

We will prove Theorem 3 in two steps: we will prove the theorem for link cobordisms (W, Σ) which are concordance Morse-trivial and then argue that we can always reduce to this case.

The first step will follow from Proposition 2.3.4. Let $(W, \mathcal{F}) : (Y_0, K_0) \to (Y_1, K_1)$ be a ribbon $\mathbb{Z}$-homology concordance which is concordance Morse-trivial and decorated by a pair of parallel arcs. Decompose $W = W_1 \cup W_2$ into the 1-

and 2-handle cobordisms. A key observation of [DLVVW19] is that the product cobordism $Y_0 \times [-1,1]$ and the double of W can be obtained by two different surgeries on the same intermediate manifold $X = D(W_1)$. X can be described explicitly as $(Y_0 \times [-1,1])\#^n(S^1 \times S^3)$. It is not hard to see that surgery on $S^1 \times S^3$ along $S^1 \times \{\text{pt}\}$ yields S^4, and so that $Y_0 \times [-1,1]$ can be obtained from surgery on X is straightforward. That $D(W)$ can also be obtained by surgery on X follows from the following lemma.

Lemma 3.2.2. *[DLVVW19, Proposition 5.1] Let $W : Y_0 \to Y_1$ be a cobordism corresponding to attaching 2-handles along curves $\gamma_1, \ldots, \gamma_n \subset Y_0$. Then, the double of W can be obtained from $Y_0 \times [-1,1]$ by doing surgery on $\gamma_1, \ldots, \gamma_n \subset Y_0 \times \{0\}$.*

To apply Proposition 2.3.4, there is a homological condition on the collection of surgery curves $\alpha_1, \ldots, \alpha_n$, which must be satisfied, namely the restriction map $H^1(X - \amalg N(\alpha_i)) \to H^1(\amalg \partial N(\alpha_i))$ must be surjective.

Lemma 3.2.3. *If $\alpha_1, \ldots, \alpha_n \subset X$ are either the attaching curves of the 2-handles of W or the core curves $S^1 \times \{pt\}$ of the $S^1 \times S^3$ summands, which we denote $\gamma_1, ..., \gamma_n$ and $\eta_1, ..., \eta_n$ respectively, the restriction map $H^1(X - \amalg N(\alpha_i)) \to H^1(\amalg \partial N(\alpha_i))$ is surjective.*

Proof. Let $\alpha_1, \ldots, \alpha_n$ be either set of curves. Inclusions induce the following commutative diagram:

$$\begin{tikzcd} H^1(X) \arrow[r] \arrow[d] & H^1(\amalg \partial N(\alpha_i)) \\ H^1(W - \amalg N(\alpha_i)) \arrow[ur] & \end{tikzcd}$$

The map $H^1(X - \amalg N(\alpha_i)) \to H^1(\amalg \partial N(\alpha_i))$ is surjective if the map $H^1(X) \to H^1(\amalg \partial N(\alpha_i))$ is. The curve η_i runs over the ith 1-handle geometrically once, and since W is a $\mathbb{Z}$-homology cobordism, after some handle slides, we can arrange that the curve γ_i runs over the ith 1-handle algebraically once. In either case, this implies that the composition

$$H^1(\#^n(S^1 \times S^3)) \to H^1(X) \to H^1(\amalg \partial N(\alpha_i)),$$

is an isomorphism. Therefore, the map $H^1(X) \to H^1(\amalg \partial N(\alpha_i))$ is surjective as desired.

□

We can now establish the theorem for the case where (W, Σ) is concordance Morse-trivial.

Proposition 3.2.4. *Suppose $(W, \mathcal{F}) : (Y_0, K_0) \to (Y_1, K_1)$ is a ribbon $\mathbb{Z}$-homology concordance which is concordance Morse-trivial where $\mathcal{F} = (C, \mathcal{A})$ is the concordance decorated by a pair of parallel arcs. Then, every $\mathfrak{s} \in Spin^C(W)$ has a unique extension $D(\mathfrak{s}) \in Spin^C(D(W))$ and the map induced by the double of $(W, \mathcal{F})$*

$$F_{D(W),D(\mathcal{F}),D(\mathfrak{s})} : \mathcal{HFL}^-(Y_0, K_0, \mathfrak{s}|_{Y_0}) \to \mathcal{HFL}^-(Y_0, K_0, \mathfrak{s}|_{Y_0})$$

is the identity.

Proof. Decompose W as $W_1 \cup W_2$ where W_i is the cobordism corresponding to the attachment of the i-handles, $i = 1, 2$. Let $\widetilde{Y} = \partial_+ W_1$ which can be identified with $Y_0 \#^n(S^1 \times S^2)$, where n is the number of 1-handles (and since W is a $\mathbb{Z}$-homology cobordism, n is also the number of 2-handles). Let $X = \overline{W_1} \cup W_1$ be the double of W_1, which is diffeomorphic to $(Y_0 \times [-1, 1])\#^n(S^1 \times S^3)$. Define a decorated surface $\mathcal{F}_\alpha = (\Sigma_\alpha, \mathcal{A}_\alpha)$ in X as follows: Let $\alpha_1, \ldots, \alpha_n$ be a collection of curves in

X. Isotope each α_i so that it is embedded in $\widetilde{Y}$. In Section 3.1, we constructed a decorated surface $\mathcal{F}_{\alpha_i} \subset \widetilde{Y} \times [-1,1]$ which realized the action of α_i. Define $\mathcal{F}_\alpha$ to be the decorated surface obtained stacking these surfaces on top of one another, i.e. $\mathcal{F}_\alpha = \mathcal{F}_{\alpha_n} \cup \cdots \cup \mathcal{F}_{\alpha_1}$. Let $(X, \mathcal{F}_X) = (\overline{W_1}, \overline{\mathcal{F} \cap W_1}) \cup (\widetilde{Y} \times [-1,1], \mathcal{F}_\alpha) \cup (W_1, \mathcal{F} \cap W_1)$.

Let η_i be the curve $S^1 \times \{\text{pt}\}$ in the ith $S^1 \times S^3$ summand of X. By taking α_i to be η_i, we obtain a decorated surface $\mathcal{F}_\eta$ in X, with the curves $\eta_1, \ldots, \eta_n \subset \mathcal{A}_\eta$. Apply Proposition 2.3.4 to see that

$$F_{X,\mathcal{F}_\eta,\mathfrak{s}} \simeq F_{X(\eta_1,\ldots,\eta_n),\mathcal{F}_\eta(\eta_1,\ldots,\eta_n),\mathfrak{s}(\eta_1,\ldots,\eta_n)}.$$

Doing surgery on X along the curves $\eta_1, \ldots, \eta_n$ yields $Y_0 \times [-1,1] \#^n S^4$ which is, of course, diffeomorphic to the product $Y_0 \times [-1,1]$. Recall that $\mathcal{F}_\eta$ was defined by tubing on tori decorated by parallel copies of η_i. Doing surgery on the curves η_i in these tori yields spheres embedded in the S^4 summands (and are therefore nullhomologous.) Therefore, $(Y_0 \times [-1,1], \mathcal{F}_\eta(\eta_1, \ldots, \eta_n))$ can be built from the Morse-trivial link cobordism $(Y_0 \times [-1,1], \mathcal{F}_{Y_0 \times [-1,1]})$ by tubing on a collection of spheres. By Proposition 3.2.1, the link cobordism map does not detect tubing on nullhomologous spheres, and hence

$$F_{X(\eta_1,\ldots,\eta_n),\mathcal{F}_\eta(\eta_1,\ldots,\eta_n),\mathfrak{s}(\eta_1,\ldots,\eta_n)} \simeq F_{Y_0 \times [-1,1],\mathcal{F}_{Y_0 \times [-1,1]},\mathfrak{s}(\eta_1,\ldots,\eta_n)}.$$

By Lemma 3.2.2, $D(W)$ can be obtained from X by doing surgery on the attaching curves $\gamma_1, \ldots, \gamma_n$ of the 2-handles of W. Now take the α_i to be the attaching curves $\gamma_1, ..., \gamma_n$ for the 2-handles for W, and consider the decorated link cobordism $\mathcal{F}_\gamma$ in X which realizes the action of $\gamma_1, \ldots, \gamma_n$. Just as above, Proposition 2.3.4 shows:

$$F_{X,\mathcal{F}_\gamma,\mathfrak{s}} \simeq F_{X(\gamma_1,\dots,\gamma_n),\mathcal{F}_\gamma(\gamma_1,\dots,\gamma_n),\mathfrak{s}(\gamma_1,\dots,\gamma_n)} \simeq F_{D(W),\mathcal{F}_\gamma(\gamma_1,\dots,\gamma_n),\mathfrak{s}(\gamma_1,\dots,\gamma_n)}.$$

Again, the surface $\mathcal{F}_\gamma(\gamma_1,\dots,\gamma_n)$ is obtained by tubing on the spheres that arise by doing surgery on the tori in $\mathcal{F}_\gamma$. Here is a geometric argument that these spheres are nullhomologous in $D(W)$. The torus T_i corresponding to γ_i was defined by isotoping γ_i into $\partial_+ W_1$ and taking the boundary of a regular neighborhood of γ_i in $\partial_+ W_1$. Hence, $[T_i] = 0 \in H_2(X)$. By attaching a thickened disk along a meridian of γ_i, we obtain cobordism from T_i to the sphere S_i obtained by surgery on γ. Therefore, $[S_i] = [T_i] = 0$ in $H_2(X)$. Since S_i is disjoint from γ_i it also represents a class in $H_2(X - N(\gamma_i))$. X is obtained from $X - N(\gamma_i)$ by attaching a 3- and 4-handle, so the relative homology group $H_1(X, X - N(\gamma_i)) = 0$, implying the map induced by inclusion $H_2(X - N(\gamma_i)) \to H_2(X)$ is injective. In particular, this means $[S_i]$ is trivial in $H_2(X - N(\gamma_i))$, and therefore trivial in $H_2(X(\gamma_i))$ as well. Applying this argument to each γ_i shows that all spheres S_i are nullhomologous in $D(W)$.

Therefore, we can again apply Proposition 3.2.1 to see that

$$F_{D(W),\mathcal{F}_\gamma(\gamma_1,\dots,\gamma_n),\mathfrak{s}(\gamma_1,\dots,\gamma_n)} \simeq F_{D(W),D(\mathcal{F}),\mathfrak{s}(\gamma_1,\dots,\gamma_n)},$$

where $D(\mathcal{F})$ is the decorated cobordism obtained by doubling $\mathcal{F}$. Since $(W, \mathcal{F})$ was concordance Morse-trivial, $(D(W), D(\Sigma))$ is as well.

Altogether, we have shown that the maps induced by $(Y_0 \times [-1,1], \mathcal{F}_{Y_0\times[-1,1]})$ and $(D(W), D(\mathcal{F}))$ are chain homotopic to the maps induced by the link cobordisms $(X, \mathcal{F}_\eta)$ and $(X, \mathcal{F}_\gamma)$ respectively. Hence, we simply need to show that

$$F_{X,\mathcal{F}_\eta,\mathfrak{s}} \simeq F_{X,\mathcal{F}_\gamma,\mathfrak{s}}.$$

By Lemma 3.1.3, we have that

$$F_{X,\mathcal{F}_\eta,\mathfrak{s}}(\cdot) \simeq F_{X,\mathcal{F}_X,\mathfrak{s}}([\eta_1] \wedge \cdots \wedge [\eta_n] \otimes \cdot)$$

and

$$F_{X,\mathcal{F}_\gamma,\mathfrak{s}}(\cdot) \simeq F_{X,\mathcal{F}_X,\mathfrak{s}}([\gamma_1] \wedge \cdots \wedge [\gamma_n] \otimes \cdot),$$

where $(X, \mathcal{F}_X)$ is concordance Morse-trivial. We will show that

$$F_{X,\mathcal{F}_X,\mathfrak{s}}([\eta_1] \wedge \cdots \wedge [\eta_n] \otimes \cdot) \simeq F_{X,\mathcal{F}_X,\mathfrak{s}}([\gamma_1] \wedge \cdots \wedge [\gamma_n] \otimes \cdot),$$

This is effectively proven in [DLVVW19, Theorem 4.10], but we will recall their argument for the convenience of the reader. By [DLVVW19, Proposition 5.1],

$$[\eta_1] \wedge \cdots \wedge [\eta_n] = [\gamma_1] \wedge \cdots \wedge [\gamma_n] \in (\Lambda^* H_1(X)/\mathrm{Tors}/\langle H_1(Y_0)/\mathrm{Tors}\rangle) \otimes \mathbb{F}_2,$$

where $\langle H_1(Y_0)/\mathrm{Tors}\rangle$ is the ideal generated by elements of $H_1(Y_0)/\mathrm{Tors}$. Therefore, $[\eta_1] \wedge \cdots \wedge [\eta_n]$ and $[\gamma_1] \wedge \cdots \wedge [\gamma_n]$ differ by an element of $\Lambda^n(H_1(X)/\mathrm{Tors}) \cap \langle H_1(Y_0)/\mathrm{Tors}\rangle \otimes \mathbb{F}_2$. By the linearity of the link cobordism maps, it suffices to show that $F_{X,\mathcal{F}_X,\mathfrak{s}}(\mathbf{x} \otimes \xi) = 0$ for any $\xi \in \Lambda^n(H_1(X)/\mathrm{Tors}) \cap \langle H_1(Y_0)/\mathrm{Tors}\rangle \otimes \mathbb{F}_2$.

$\Lambda^n(H_1(X)/\mathrm{Tors}) \cap \langle H_1(Y_0)/\mathrm{Tors}\rangle \otimes \mathbb{F}_2$ is generated by elements of the form $\omega \wedge \left(\bigwedge_{i\in S}[\eta_i]\right)$ where ω is a wedge of elements in $H_1(Y_0)/\mathrm{Tors}$ and S is a proper subset of $\{1, ..., n\}$. So, we can take ξ to be of this form. Since S is a proper subset of $\{1, ..., n\}$ there is some η_j which does not appear as a factor of ξ. We can realize the map $F_{X,\mathcal{F}_X,\mathfrak{s}}(\xi \otimes \cdot)$ as a link cobordism map $F_{X,\mathcal{F}_\xi,\mathfrak{s}}$. By our construction of $\mathcal{F}_\xi$, we can assume that $\mathcal{F}_\xi$ is disjoint from the jth $S^1 \times S^3$ summand, of which η_j is the core. Therefore, $(X, \mathcal{F}_\xi)$ can be decomposed as $(X - T_j, \mathcal{F}_\xi - \mathcal{D}) \cup (T_j, \mathcal{D})$, where $T_j = (S^1 \times S^3) - D^4$ is the jth $S^1 \times S^3$ summand, and $\mathcal{D}$ is a disk decorated with a single dividing arc. Hence, the map $F_{X,\mathcal{F}_\xi,\mathfrak{s}}$ factors through $F_{T_j,\mathcal{D},\mathfrak{s}|_{T_j}}$.

But, the map $F_{T_j,\mathcal{D},\mathfrak{s}|_{T_j}}$ is trivial. This map can be computed as the following composition: first, a 0-handle with a birth disk (which is identified with $\mathcal{D}$) is born, then a 1-handle is attached with both feet on the 0-handle, with a trivially embedded annulus followed by 3-handle, again with a trivial annulus. The 1- and 3-handle maps take $\mathbf{x}$ to $\mathbf{x} \otimes \theta^+$ and $\mathbf{x} \otimes \theta^+$ to zero respectively, where θ^+ is the top graded generator of $\mathcal{CFL}^-(S^1 \times S^2, \mathcal{U}, \mathfrak{t}_0)$.

Therefore, it has been established that

$$F_{Y_0\times[-1,1],\mathcal{F}_{Y_0\times[-1,1]},\mathfrak{s}(\eta_1,\ldots,\eta_n)} \simeq F_{D(W),D(\mathcal{F}),\mathfrak{s}(\gamma_1,\ldots,\gamma_n)}.$$

The map on the left is induced by a Morse-trivial link cobordism, and therefore induces the identity map on $\mathcal{HFL}^-(Y_0, K_0, \mathfrak{t}_0)$, where $\mathfrak{t}_0$ is the restriction of $\mathfrak{s}(\eta_1, \ldots, \eta_n)$ to Y_0. $\mathrm{Spin}^{\mathrm{C}}$-structures on Y_0 extend uniquely over W and $D(W)$ since they are $\mathbb{Z}$-homology cobordisms. Therefore $\mathfrak{s}(\gamma_1, ..., \gamma_n)$ *is* the unique $\mathrm{Spin}^{\mathrm{C}}$ structure extending $\mathfrak{s}$ over $D(W)$. Therefore, for any $\mathfrak{s} \in \mathrm{Spin}^{\mathrm{C}}(W)$, the map $F_{D(W),D(\mathcal{F}),D(\mathfrak{s})}$ induces the identity. □

Having completed the proof in the case that (W, Σ) is concordance Morse-trivial we show that we can always reduce to this case. Given any concordance $C \subset W$, there is a Morse function f on W with the property that $f|_C$ has no critical points (for instance, simply identify $C \cong S^1 \times I$ and extend the projection $S^1 \times I \to I$ to a Morse function on W.) However, it is not immediately clear that the extension can be chosen in such a way as to give W the structure of a ribbon cobordism. The following proposition shows that birth-saddle pairs of a concordance can be traded for four-dimensional 1- and 2-handle pairs.

Proposition 3.2.5. *Let K_0 and K_1 be ribbon concordant in a homology cobordism W. Then, there is handle decomposition of the pair (W, Σ) with the property that W is ribbon and the handle decomposition for Σ is trivial.*

Proof. Choose a Morse function on (W, Σ) giving rise to a banded link diagram. This decomposition of Σ consists of a collection of birth circles $U_1, ..., U_n$ and bands $B_1, ..., B_n$, and after some band slides, we can assume the band B_i has one foot on U_i and the other on K_0.

Consider a birth circle U_i. If a band B_j runs through U_i (for $j \neq i$), it can be swum through the band B_i, so we can assume that the only band which runs through U_i is B_i. Introduce a canceling 1- and 2-handle pair near U_i. Slide all strands of B_i which run through U_i over the newly introduced 2-handle. Each strand of B_i which ran through U_i now also runs through the 1-handle we have added. Now, contract the band B_i until it no longer runs through U_i. In doing so, we drag along an arc of K_0 through U_i, possibly many times. Slide U_i under the 1-handle, unlinking it from K_0. The band B_i can be swum past the 2-handle, at which point, U_i and B_i can be cancelled. See Figure 14. Repeat for all U_i and B_i.

This process has yielded some new ribbon $\mathbb{Z}$-homology concordance $(W, \Sigma) : (Y_0, K_0') \to (Y_1, K_1')$ which is concordance Morse-trivial. It remains to show that this process has not altered the original knots. By erasing the new 1- and 2-handle pairs, we obtain the knot $K_0' \subset Y_0$, which is clearly isotopic to K_0. On the other hand, if we do surgery on the whole diagram, we obtain a knot K_1' in a surgery diagram for Y_1. To see K_1' is isotopic to K_1, we can slide an arc of K_1' near where we canceled the birth circle and band over the black 0-framed surgery curve. Any arcs which pass through the black 0-framed surgery curve can now be slid over the red 0-framed surgery curve. Iterating this process, we obtain a knot which is isotopic to the result of band surgery on the bands $B_1, ..., B_n$ in the original diagram, which is exactly K_1. $\square$

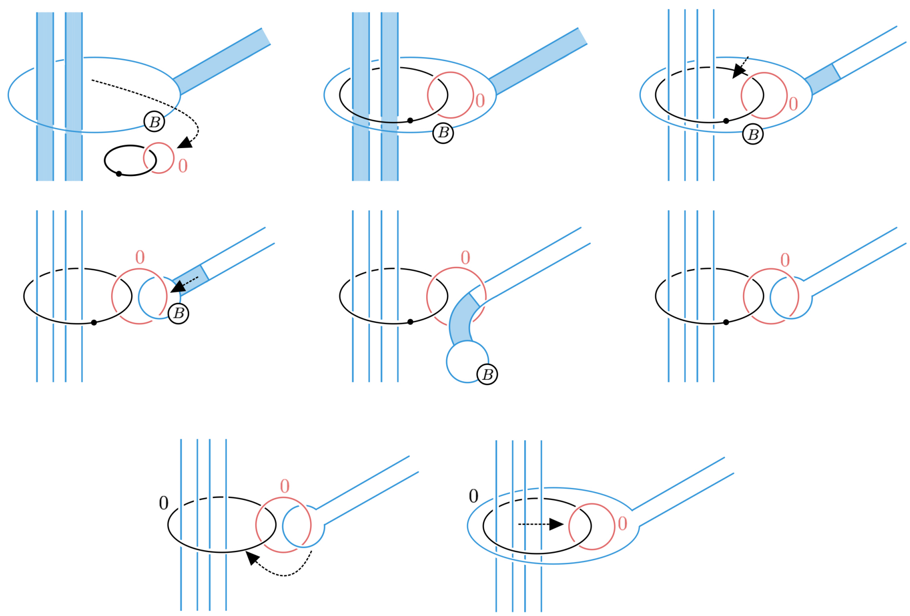


FIGURE 14 A procedure for trading handles of a concordance for handles of the ambient manifold.

Proof of Theorem 1. This now follows immediately from the previous propositions. For any ribbon $\mathbb{Z}$-homology concordance $(W, \mathcal{F})$, the doubled link cobordism induces the same map as a concordance Morse-trivial link cobordism $(D(W), \mathcal{F}')$, and this map induces the identity map on $\mathcal{HFL}^-(Y_0, K_0, \mathfrak{s}|_{Y_0})$. As $\mathfrak{s}$ extends uniquely over $D(W)$, the composition law implies that $F_{W,\mathcal{F},\mathfrak{s}}$ has a left inverse, namely $F_{\overline{W},\overline{\mathcal{F}},\mathfrak{s}}$. Hence, $F_{W,\mathcal{F},\mathfrak{s}}$ is a split injection. $\square$

Torsion and Link Floer Homology

Let $CFL^-(Y,K,\mathfrak{s})$ be the $\mathbb{F}_2[V]$-module obtained from $\mathcal{CFL}^-(Y,K,\mathfrak{s})$ by setting $U=0$ with differential

$$\partial(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\substack{\phi\in\pi_2(\mathbf{x},\mathbf{y}),\\ \mu(\phi)=1,\\ n_{\mathbf{w}}(\phi)=0}} \#\widehat{\mathcal{M}}(\phi)V^{n_{\mathbf{z}}(\phi)}\mathbf{y}.$$

Let $HFL^-(Y,K,\mathfrak{s})$ be the homology of this complex.

A key property of the link Floer TQFT which is utilized in [JMZ20] is the following.

Lemma 3.3.1. *[JMZ20, Lemma 3.1] Let $(W,\mathcal{F})$ be a decorated link cobordism. Let $\mathcal{F}_V$ be a link cobordism obtained by adding a tube to the $\Sigma_{\boldsymbol{z}}$ region. Then,*

$$F_{W,\mathcal{F}_V,\mathfrak{s}} \simeq V\cdot F_{W,\mathcal{F},\mathfrak{s}}.$$

Proof. Choose a neighborhood the tube diffeomorphic to the 4-ball. By the composition law, it suffices to show that

$$F_{B^4,\mathcal{F}_V\cap B^4} \simeq V\cdot F_{B^4,\mathcal{F}},$$

where $\mathcal{F}$ is a pair of disks which bound a 2-component unlink in ∂B^4 decorated by a dividing arc. $\mathcal{F}_V\cap B^4$ is obtained from $F_{B^4,\mathcal{F}}$ by adding a tube connecting the two disk with feet in the $\mathbf{z}$-region. This map can be computed as the composition of two $\mathbf{z}$-band maps. This computation is carried out in [Zem18, Section 8.2], and the resulting map is multiplication by V. $\square$

With this tool at our disposal, we can prove an analogue of [JMZ20, Proposition 4.1].

Proposition 3.3.2. *Let $(W,\Sigma) : (Y_0, K_0) \to (Y_1, K_1)$ be a $\mathbb{Z}$-homology link cobordism. Let $h : W \to \mathbb{R}$ be a Morse function compatible with Σ with respect to which W is ribbon. Suppose that $h|_\Sigma$ has m critical points of index 0, b critical points of index 1, and M critical points of index 2. Let $\mathcal{F}$ be a decoration of Σ such that $\Sigma_{\boldsymbol{w}}$ is a regular neighborhood of an arc from K_0 to K_1. Then,*

$$V^M \cdot F_{D(W),D(\mathcal{F}),D(\mathfrak{s})} \simeq V^{b-m} \cdot \mathrm{id}_{\mathcal{HFL}^-(Y_0,K_0,\mathfrak{s}|_{Y_0})}.$$

Proof. The Morse function h induces a movie presentation for (W,Σ).

1. m birth disks appear disjoint from K_0, with boundaries $U_1, \dots, U_m$.

2. n 4-dimensional 1-handles are attached whose feet are disjoint from Σ.

3. m fusion bands B_1, ..., B_n are attached which connect K_0 and $U_1, \dots, U_m$. After some band slides, the band B_i has one foot in U_i and the other in K_0.

4. $b - m$ additional bands $B_{m+1}, ..., B_b$ are attached.

5. n four-dimensional 2-handles are attached along curves $\gamma_1, \dots, \gamma_n$.

6. M death disks appear capping off unknotted components $U_1, \dots, U_M$ in the link obtained by doing band surgery.

By playing this movie forward, and then again in reverse, we obtain a movie for $(D(W), D(\mathcal{F}))$.

Consider the link cobordism which is obtained by deleting steps (5)-(8), i.e. remove the 2-handles, the deaths, the dual births, and the dual 2-handles. The resulting four-manifold X is the double of the cobordism $W_1 = (Y_1 \times [0,1]) \cup$ 1-handles. The resulting surface is $D(\mathcal{F} \cap W_1)$. Let $\mathcal{G}_\gamma$ be the surface obtained from $D(\mathcal{F} \cap W_1)$ by tubing on tori $T_1, ..., T_n$ which are the boundaries of regular neighborhoods of the γ_i curves in $\partial_+ W_1$ as in the proof of Proposition 3.2.4. Surgery on the curves $\gamma_1, \dots, \gamma_n$ yields a link cobordism $(D(W), \mathcal{G})$. The decorated cobordism $\mathcal{G}$ can be obtained from $D(\mathcal{F})$ by attaching M tubes from the deaths disks to the dual birth disks and tubing on n nullhomologous spheres which are the result of surgery on the tori T_i. We can arrange for the feet of the tubes to be sit in the subsurface $\Sigma_{\mathbf{z}}$. Attaching the nullhomologous spheres has no effect on the link cobordism map, and attaching the tubes has the result of multiplying by V^M by Lemma 3.3.1. Therefore, by Proposition 2.3.4,

$$F_{X,\mathcal{G}_\gamma,\mathfrak{t}} \simeq V^M \cdot F_{D(W),D(\mathcal{F}),D(\mathfrak{s})},$$

where $\mathfrak{t}$ is determined by the fact that $\mathfrak{t}(\gamma_1, \dots, \gamma_n) = D(\mathfrak{s})$.

Now consider the cobordism obtained by deleting steps (4)-(9) and again tubing on the tori corresponding to the γ_i curves. The ambient four-manifold is still X, but removing steps (4) and (9) has the effect of removing the bands $B_{m+1}, ..., B_b$ and their duals from $\mathcal{G}_\gamma$. Call this surface $\mathcal{H}_\gamma$. Since the bands $B_{m+1}, ..., B_b$ and their duals form a collection of $b - m$ tubes, another application of Lemma 3.3.1 shows

$$F_{X,\mathcal{G}_\gamma,\mathfrak{t}} \simeq V^{b-m} \cdot F_{X,\mathcal{H}_\gamma,\mathfrak{t}}.$$

But, surgery on $(X, \mathcal{H}_\gamma)$ along $\gamma_1, \ldots, \gamma_n$ yields the link cobordism $(D(W), \mathcal{H})$ which is the double of a ribbon homology concordance. So by Proposition 3.2.4

$$F_{X,\mathcal{H}_\gamma,\mathfrak{t}} \simeq \mathrm{id}_{\mathcal{CFL}^-(Y_0,K_0,\mathfrak{s}|_{Y_0})} .$$

Altogether then, we have that

$$V^M \cdot F_{D(W),D(\mathcal{F}),D(\mathfrak{s})} \simeq V^{b-m} \cdot \mathrm{id}_{\mathcal{HFL}^-(Y_0,K_0,\mathfrak{s}|_{Y_0})},$$

as desired. □

Proof of Theorem 4. This now follows by an argument identical to that of [JMZ20, Theorem 1.2]. □

Unlike the inequality of [JMZ20, Theorem 1.2], this does *not* give the symmetric result

$$\mathrm{Ord}_V(Y_1, K_1, \mathfrak{s}|_{Y_1}) \leq \max\{m, \mathrm{Ord}_V(Y_0, K_0, \mathfrak{s}|_{Y_0})\} + 2g(\Sigma),$$

since $\overline{W}$ is not ribbon as we have defined it, unless $W = Y_0 \times [0, 1]$.

Applications

We do have some immediate applications. Theorem 4 gives a clear relationship between the torsion orders of ribbon homology cobordant knots.

Corollary 3.4.1. *Let* $(W, \Sigma, \mathfrak{s}) : (Y_0, K_0) \to (Y_1, K_1)$ *be a ribbon* $\mathbb{Z}$*-homology cobordism, then*

$$\mathrm{Ord}_V(Y_0, K_0, \mathfrak{s}|_{Y_0}) - \mathrm{Ord}_V(Y_1, K_1, \mathfrak{s}|_{Y_1}) \leq 2g(\Sigma).$$

Recall that the *fusion number* $\mathcal{F}us(K)$ of a ribbon knot K in S^3 is the minimal number of bands in a handle decomposition of ribbon concordance C from the unknot U to K in $S^3 \times [0,1]$. By [JMZ20], the torsion order of K in S^3 provides a lower bound for the fusion number of K. There are a few possible generalizations we will consider.

Let K be a knot in a 3-manifold Y. Suppose that that K is ribbon in Y, in the sense that there is a concordance $(Y \times [0,1], C) : (Y \times \{0\}, U) \to (Y \times \{1\}, K)$ where U is the boundary of a disk in Y and C is an annulus which is ribbon with respect to the projection $Y \times [0,1] \to [0,1]$.

Definition 3.4.2. We define the *fusion number of K in Y*, which we denote $\mathcal{F}us_Y(K)$, to be the minimal number of bands in a ribbon concordance from U to K in $Y \times [0,1]$.

Corollary 3.4.3. *If K is ribbon in Y, then*

$$\mathrm{Ord}_V(Y, K, \mathfrak{s}) \leq \mathcal{F}us_Y(K).$$

Proof. Let $(Y \times [0,1], C) : (Y, U) \to (Y, K)$ be a ribbon concordance with $b = \mathcal{F}us_Y(K)$ bands (and therefore b local minima as well). Theorem 2 then implies

$$\mathrm{Ord}_V(Y, K, \mathfrak{s}) \leq \max\{b, \mathrm{Ord}_V(Y, U, \mathfrak{s})\} = b,$$

since $HFL^-(Y, U, \mathfrak{s})$ is torsion free. □

In another direction, one could also consider ribbon concordances in $\mathbb{Z}$-homology cobordisms. Given a $\mathbb{Z}$-homology concordance $(W, \Sigma) : (Y_0, U) \to (Y_1, K)$, one can always find a Morse function h on W compatible with Σ so that $h|_\Sigma$ is ribbon, so we will continue to require that the ambient manifold is ribbon as well. However, by imposing the condition that the ambient manifold is ribbon, we have introduced an asymmetry which makes generalizing the fusion number to ribbon $\mathbb{Z}$-homology concordances somewhat subtle; in $S^3 \times [0,1]$, concordances *from* the unknot with no local maxima can be turned around and viewed as concordances *to* the unknot with no local minima. However, this is clearly not the case for a ribbon homology concordance $(W, \Sigma) : (Y_0, U) \to (Y_1, K)$, as $\overline{W}$ is not ribbon.

Therefore, since ribbon homology link cobordisms to and from the unknot differ, we can consider both cases: on the one hand, we have ribbon $\mathbb{Z}$-homology concordances $(W, \Sigma) : (Y', U) \to (Y, K)$ (where W is a ribbon $\mathbb{Z}$-homology cobordism and Σ is an annulus with *no local maxima*), and on the other, link cobordisms $(W, \Sigma) : (Y, K) \to (Y', U)$ where W is a ribbon $\mathbb{Z}$-homology cobordism and Σ is an annulus with *no local minima*.

For the latter notion, Theorem **??** immediately implies the following.

Corollary 3.4.4. *Let K be a knot in a 3-manifold Y. If $(W, \Sigma) : (Y, K) \to (Y', U)$ is a link cobordism such that W is a ribbon $\mathbb{Z}$-homology cobordism and Σ is an annulus with no local minima and b index 1 critical points, then*

$$\mathrm{Ord}_V(Y, K, \mathfrak{s}) \leq b,$$

for any $\mathfrak{s} \in Spin^C(Y)$.

Let $\mathcal{F}us^\wedge(Y, K)$ be minimal number of bands over all link cobordisms of the form $(W, \Sigma) : (Y, K) \to (Y', U)$ where W is a ribbon $\mathbb{Z}$-homology cobordism and Σ is an annulus with no local minima. The previous result, of course, implies that $\mathrm{Ord}_V(Y, K, \mathfrak{s}) \leq \mathcal{F}us^\wedge(Y, K)$.

Let us now turn to ribbon $\mathbb{Z}$-homology concordances $(W, \Sigma) : (Y, U) \to (Y', K)$. A little care is needed in defining a fusion number in this context, as we must also take into account the handle decomposition of the ambient manifold, as the following example illustrates.

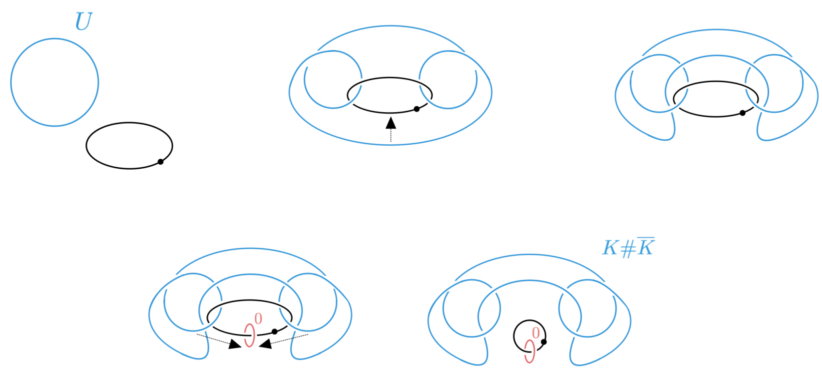


FIGURE 15 A concordance from the unknot to $K\#\overline{K}$ with no bands.

Example 3.4.5. Let K be a trefoil, and consider $K\#\overline{K}$. There is a concordance from the unknot to $K\#\overline{K}$ in $S^3 \times I$ with a single birth and band. This concordance, of course, has the minimal number of bands, else $K\#\overline{K}$ would be isotopic to the unknot. However, this birth-band pair can be eliminated in the homology cobordism obtained from $S^3 \times [0, 1]$ obtained by attaching a canceling 1- and 2-handle pair. Consider the following movie which is shown in Figure 15:

1. $t = 0$: An unknot U sits in S^3.
2. $t = 1$: A 1-handle is attached away from U. This has the result of doing 0-surgery on an unknot unlinked with U.
3. $t = 3/2$: U is isotoped in $\partial(S^3 \times [0, 1] \cup 1\text{-handle})$.
4. $t = 2$: A 2-handle is attached to $\partial(S^3 \times [0, 1] \cup 1\text{-handle})$.

5. $t = 5/2$: An isotopy in $\partial(S^3 \times [0,1] \cup 1\text{-handle} \cup 2\text{-handle})$ pulls the knot $K\#\overline{K}$ away from the 1- and 2-handle. This can be done by sliding an arc in K and in $\overline{K}$ over the 2-handle.

No bands are needed in this example, but $\mathrm{Ord}_V(S^3, K\#\overline{K}) = \mathrm{Ord}_V(S^3, K) > 0$. Hence, $\mathrm{Ord}_V(S^3, K)$ cannot possibly be a lower bound on the number of bands required in such link cobordisms.

This example illustrates that handles of the surface can be traded for handles in the ambient manifold. In light of these observations, we define $\mathcal{F}us^{\vee}(Y, K)$ to be minimal number of bands plus 2-handles over all ribbon homology concordances $(W, \Sigma) : (Y', U) \to (Y, K)$. However, $\mathrm{Ord}_V(Y, K, \mathfrak{s})$ cannot be a lower bound for $\mathcal{F}us^{\vee}(Y, K)$ by work of Hom-Kang-Park.

Example 3.4.6. Let K be a ribbon knot in S^3 with fusion number 1. By [HKP20, Theorem 1], the torsion order of the $(p, 1)$-cable of K, which we denote $K_{p,1}$, is p. However, there is a Kirby diagram for the complement of a ribbon disk of $K_{p,1}$ with a single 2-handle and two 1-handles. By replacing one of the dotted unknots with a unknot U we obtain ribbon $\mathbb{Z}$-homology cobordism from the unknot to $K_{p,1}$ with no bands and one 2-handle. Therefore, $\mathcal{F}us^{\vee}(S^3, K_{p,1}) = 1$, but $\mathrm{Ord}_V(S^3, K_{p,1}) = p$.

CHAPTER IV

SATELLITE CONCORDANCES AND BORDERED FLOER HOMOLOGY

The material in this section draws from forthcoming work joint with Hayden-Kang-Park.

In the section we prove Theorem 13. Let $C : K \to K'$ be a concordance. Given a pattern knot $P \subset S^1 \times D^2$, we obtain a concordance between the satellites of K and K' as follows. Remove a neighborhood of C in $S^3 \times I$. The Seifert framing of K determines an identification $\varphi : \partial\nu(K) \to S^1 \times \partial D^2$. Define the satellite concordance C_P to be $(S^3 \times I - C, \varnothing) \cup_{\varphi\times\mathrm{id}} (S^1 \times D^2 \times I, P \times I)$.

Since it requires no additional effort, we prove Theorem 13 for homology concordances, i.e. link cobordisms $(W, C) : (Y, K) \to (Y', K')$ where Y and Y' are integer homology spheres, C is an annulus, and W is a homology cobordism.

Let $(W, C) : (Y, K) \to (Y', K')$ be a homology concordance. Let $\mathcal{F} = (C, \mathcal{A})$ be the annulus C decorated with a pair of parallel arcs running from K to K'. This decorated concordance induces a map $CFK^-(Y, K) \to CFK^-(Y', K')$, denoted $F_{W,C}$, which is defined as a composition of elementary cobordism maps [Zem18]. Choose a Morse function $f : W \to \mathbb{R}$ on W so that $f|_C$ has no critical points and choose a gradient-like vector field which is tangent to C. This induces a handle decomposition for (W, C) which only involves four-dimensional handles.

Since the restriction of this Morse function to the surface has no critical points, this also produces a handle decomposition for the complement of C; the attaching curves for the handles are already embedded in the complement of K, so the cobordism $W - C$ is built simply by attaching the same handles to $Y - K$.

Proposition 4.0.1. *Let $(W,C) : (Y,K) \to (Y',K')$ be a homology concordance. Then, given a handle decomposition for (W,C) as above, there exists a map F_{W-C} with the property that for any pattern knot P in the solid torus, the following diagram commutes up to homotopy:*

$$\begin{tikzcd}
CFA^-(\mathcal{H}_P) \boxtimes \widehat{CFD}(Y-K) \arrow[r, "\simeq"] \arrow[d, "\mathbb{I}_{\mathcal{H}_P} \boxtimes F_{W-C}"] & CFK^-(Y, K_P) \arrow[d, "F_{W,C_P}"] \\
CFA^-(\mathcal{H}_P) \boxtimes \widehat{CFD}(Y'-K') \arrow[r, "\simeq"] & CFK^-(Y', K'_P).
\end{tikzcd}$$

Here, $\mathcal{H}_P$ is a doubly pointed Heegaard diagram for the knot P in the solid torus, K_P and K'_P are the satellites of K and K' with pattern P, (W, C_P) is the concordance induced by the pattern, and the horizontal homotopy equivalences are given by the pairing theorem for knot Floer homology [LOT18].

The map $F_{W-C} : \widehat{CFD}(Y-K) \to \widehat{CFD}(Y'-K')$ will be defined in the standard way, as a composition of maps corresponding to handle attachments [OS06].

1- and 3-handle maps:

The maps associated to 1- and 3-handle attachments are simplest to define. For simplicity, let Y be an integer homology sphere. We will write $Y(S^0)$ for the result of S^0-surgery on Y (which is, of course, diffeomorphic to $Y\#S^1 \times S^2$).

Lemma 4.1.1. *Let $F : CFK^-(Y,K_P) \to CFK^-(Y(S^0), K'_P, \mathfrak{t}_0)$ be the 1-handle cobordism map. There exists a map $\widetilde{F} : \widehat{CFD}(Y-K) \to \widehat{CFD}((Y-K)(S^0), \mathfrak{t}_0|_{(Y-K)(S^0)})$ making the following diagram commute up to homotopy:*

$$\begin{tikzcd}
CFA^-(\mathcal{H}_P) \boxtimes \widehat{CFD}(Y-K) \arrow[r, "\simeq"] \arrow[d, "\mathbb{I}_{\mathcal{H}_P} \boxtimes \widetilde{F}"] & CFK^-(Y, K_P) \arrow[d, "F"] \\
CFA^-(\mathcal{H}_P) \boxtimes \widehat{CFD}((Y-K)(S^0), \mathfrak{t}_0|_{(Y-K)(S^0)}) \arrow[r, "\simeq"] & CFK^-(Y(S^0), K_P, \mathfrak{t}_0),
\end{tikzcd}$$

where $\mathfrak{t}_0$ *is the torsion Spin*^{C}*-structure on* $Y(S^0)$*, i.e.* $c_1(\mathfrak{t}_0) = 0$.

Proof. Fix a nice Heegaard diagram $\mathcal{H} = (\Sigma_g, \alpha_1^c, \ldots, \alpha_{g-1}^c, \beta_1, \ldots, \beta_g)$ for $Y - K$. Choose curves λ and μ in Σ isotopic to a longitude and meridian of K respectively, which intersect in a single point and avoid the α circles. A bordered Heegaard diagram $\mathcal{H}_B$ for $Y - K$ is obtained from $\mathcal{H}$ by deleting a neighborhood of p, and defining $\alpha_1^a = \lambda - p$ and $\alpha_2^a = \mu - p$ to be the α-arcs parametrizing the boundary, i.e.

$$\mathcal{H}_B = (\Sigma - p, \alpha_1^a, \alpha_2^a, \alpha_1^c, \ldots, \alpha_{g-1}^c, \beta_1, \ldots, \beta_g).$$

Let $\mathcal{H}_P$ be a nice doubly pointed bordered Heegaard diagram for the pattern knot embedded in the solid torus. A doubly pointed Heegaard diagram $\mathcal{H}_{K_P}$ for (Y, K_P) is obtained by gluing $\mathcal{H}_B$ and $\mathcal{H}_P$ along their common boundary.

Recall the definition of the 1-handle map $F : CFK^-(Y, K_P) \to CFK^-(Y(S^0), K_P', \mathfrak{t}_0)$. Choose a pair of points p_1 and p_2 in $\mathcal{H}_{K_p}$ away from $\mathcal{H}_P$. Moreover, assume p_1 and p_2 lie in the same connected component of $\Sigma - (\cup_i \alpha_i) - (\cup_i \beta_i)$ as the basepoint z. Remove neighborhoods of p_1 and p_2, and attach an annulus. Add two new curves α_0 and β_0 which are homologically essential in the annulus and intersect transversely in a pair of points, which we denote θ^+ and θ^-. There are two bigons from θ^+ to θ^-. The 1-handle map is simply

$$\mathbf{x} \mapsto \mathbf{x} \otimes \theta^+.$$

In exactly the same way, S^0-surgery on the bordered Heegaard diagram for $Y - K$ gives rise to a map $\widetilde{F} : \widehat{CFD}(Y - K) \to \widehat{CFD}((Y - K)(S^0)), \mathfrak{t}_0|_{(Y-K)(S^0)})$ defined $\mathbf{x}' \mapsto \mathbf{x}' \otimes \theta^+$. Since we chose nice diagrams, the identification of $CFA^-(\mathcal{H}_P) \boxtimes \widehat{CFD}(Y - K) \xrightarrow{\simeq} CFK^-(Y, K)$ is simply the map which takes a pair of intersection points in $\mathcal{H}_P$ and $\mathcal{H}_K$ and views them as a single intersection point in $\mathcal{H}_P \cup \mathcal{H}_K$. Tautologically, then, the desired diagram commutes. $\square$

The case of the 3-handles is dual to this case, and follows similarly.

Lemma 4.1.2. *Let $H : CFK^-(Y(S^0), K_P, \mathfrak{t}_0) \to CFK^-(Y, K'_P)$ be the 3-handle cobordism map. There exists a map $\widetilde{H} : \widehat{CFD}((Y-K)(S^0), \mathfrak{t}_0|_{(Y-K)(S^0)}) \to \widehat{CFD}(Y-K')$ making the following diagram commute up to homotopy:*

$$\begin{tikzcd}
CFA^-(\mathcal{H}_P) \boxtimes \widehat{CFD}((Y-K)(S^0), \mathfrak{t}_0|_{(Y-K)(S^0)}) \arrow[r, "\simeq"] \arrow[d, "\mathbb{I}_{\mathcal{H}_P} \boxtimes \widetilde{H}"] & CFK^-(Y(S^0), K_P, \mathfrak{t}_0) \arrow[d, "H"] \\
CFA^-(\mathcal{H}_P) \boxtimes \widehat{CFD}(Y-K') \arrow[r, "\simeq"] & CFK^-(Y, K'_P).
\end{tikzcd}$$

2-handle map:

The 2-handle cobordism is the only interesting case, and follows from the pairing theorem for triangles [LOT16, Proposition 5.35]. Before we define the 2-handle cobordism maps on $\widehat{CFD}$, we recall some facts about bordered Heegaard triple diagrams.

Given a doubly pointed bordered Heegaard diagram $\mathcal{H}_{\alpha,\beta^0}$, we obtain a doubly pointed bordered Heegaard triple diagram $\mathcal{H}_{\alpha,\beta^0,\beta^1}$ by performing a Hamiltonian translation on each of the β^0-curves. By removing the β^0-curves, we obtain an ordinary doubly pointed bordered Heegaard diagram $\mathcal{H}_{\alpha,\beta^1}$. Counting holomorphic triangles defines a map

$$m_2 : CFA^-(\mathcal{H}_{\alpha,\beta^0}) \otimes CFA^-(\mathcal{H}_{\beta^0,\beta^1}) \to CFA^-(\mathcal{H}_{\alpha,\beta^1}).$$

Taking Θ_{β^0,β^1} to be the top graded generator of the homology of $\widehat{CF}(\mathcal{H}_{\beta^0,\beta^1})$, we can consider the map

$$m_2(-,\Theta_{\beta^0,\beta^1}) : \mathit{CFA}^-(\mathcal{H}_{\alpha,\beta^0}) \to \mathit{CFA}^-(\mathcal{H}_{\alpha,\beta^1}).$$

We will make use of the fact that this map is homotopic to the map $\Psi_{\mathcal{H}_{\alpha,\beta^1}\leftarrow\mathcal{H}_{\alpha,\beta^0}}$ induced by the isotopy of β^0 to β^1, and is just the "nearest point map", taking an intersection point in $\alpha\cap\beta^0$ to the closest intersection point in $\alpha\cap\beta^1$. For a proof in the classical case, see [Lip06, Proposition 11.4] or [JTZ21, Lemma 9.7].

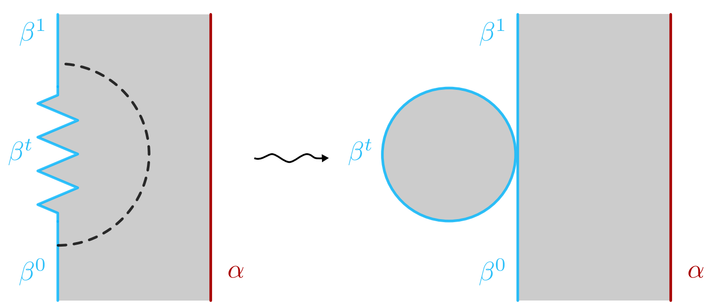


FIGURE 16 By pinching along the dotted line, we see a dynamic bigon map is homotopic to the composition of a monogon map with a triangle map.

Lemma 4.2.1. *Let $\mathcal{H}_{\alpha,\beta^0,\beta^1}$ be a bordered Heegaard triple diagram where the β^1-curves are small Hamiltonian translates of the β^0-curves. Let Θ_{β^0,β^1} be the top graded generator of $H_*(\widehat{CF}(\beta^0,\beta^1))$. Then,*

$$m_2(-,\Theta) \sim \Psi_{\mathcal{H}_{\alpha,\beta^1}\leftarrow\mathcal{H}_{\alpha,\beta^0}}.$$

Moreover, $\Psi_{\mathcal{H}_{\alpha,\beta^1}\leftarrow\mathcal{H}_{\alpha,\beta^0}}$ is the nearest point map.

Proof. The holomorphic polygon maps are defined by counting certain holomorphic maps

$$u : (S,\partial S) \to (\Sigma\times\Delta, (\alpha\times e_0)\cup(\beta^1\times e_1)\cup\ldots\cup(\beta^n\times e_n),$$

where Δ is a disk with n boundary punctures, and edges labeled $e_0, \ldots, e_n$. If $B \in \pi_2(x_0, \ldots, x_n; \rho_1, \ldots, \rho_m)$ then denote by $\mathcal{M}^B(x_0, \ldots, x_n; \rho_1, \ldots, \rho_m)$ the moduli space of embedded holomorphic maps in the homotopy class B with asymptotics $x_1, \ldots, x_n, \rho_1, \ldots, \rho_m$. The map $\Psi_{\mathcal{H}_{\alpha,\beta^1} \leftarrow \mathcal{H}_{\alpha,\beta^0}}$ is defined by counting bigons with dynamic boundary conditions, i.e. maps

$$(S, \partial S) \to (\Sigma \times [0,1] \times \mathbb{R}, (\alpha \times 1 \times \mathbb{R}) \cup \bigcup_t (\beta^t \times 0 \times \{t\}))$$

where $\beta^t = \beta^0$ for $t \leq 0$ and $\beta^t = \beta^1$ for $t \geq 1$. By a neck stretching argument, the moduli space of such maps splits into a product of $\mathcal{M}^B(x, y, z; \rho_1, \ldots, \rho_m)$ and $\mathcal{M}^A(x; \rho_1, \ldots, \rho_m)$. In other words, $\Psi_{\mathcal{H}_{\alpha,\beta^1} \leftarrow \mathcal{H}_{\alpha,\beta^0}}$ is the composition of the monogon map θ with the triangle map $m_2(-,-)$. See Figure 16.

Claim: $\theta(1) = \Theta_{\beta^0,\beta^1}$. In particular, $m_2(-,\theta(1)) = m_2(-,\Theta_{\beta^0,\beta^1})$, as desired.

If $\theta(1)$ was not equal to Θ_{β^0,β^1}, then take the standard genus g Heegaard diagram for $S^1 \times S^2$ and consider the two maps

$$m_2(-,\theta(1)), \Psi_{\mathcal{H}_{\alpha,\beta^1} \leftarrow \mathcal{H}_{\alpha,\beta^0}} : \mathit{CFA}^-(\mathcal{H}_{\alpha,\beta^0}) \to \mathit{CFA}^-(\mathcal{H}_{\alpha,\beta^0})$$

from above. The map $\Psi_{\mathcal{H}_{\alpha,\beta^1} \leftarrow \mathcal{H}_{\alpha,\beta^0}}$ is a homotopy equivalence, but is homotopic to $m_2(-,\theta(1))$. If $\theta(1)$ were not the highest graded generator, the composition would be zero, a contradiction.

Finally, as $t \to 0$, and β^t approaches β^0, holomorphic disks with Maslov index 0 limit to bigons for the pair (α, β^0). But, any bigon for (α, β^0) with Maslov index 0 must be constant, since the $\mathbb{R}$-action on moduli space of homotopy classes of non-constant bigons is free, implying the dimension of the moduli space is nonzero. $\square$

We are now ready to define the 2-handle cobordism maps. Let $\mathcal{L}$ be a framed link in Y. Let $W(\mathcal{L})$ be the cobordism corresponding to attaching 2-handles along $\mathcal{L}$. Let $Y(\mathcal{L})$ be the 3-manifold obtained by surgery on $\mathcal{L}$. As usual, the 2-handle map is defined by counting holomorphic triangles.

Lemma 4.2.2. *Fix a $Spin^C$-structure $\mathfrak{s}$ on $W(\mathcal{L})$. Let $G_{\mathfrak{s}} : CFK^-(Y, K_P, \mathfrak{s}|_Y) \to CFK^-(Y', K'_P, \mathfrak{s}|_{Y'})$ be the 2-handle cobordism map. There exists a map $\widetilde{G}_{\mathfrak{s}} : \widehat{CFD}(Y - K, \mathfrak{s}|_{Y-K}) \to \widehat{CFD}(Y' - K', \mathfrak{s}|_{Y'-K'})$ making the following diagram commute up to homotopy:*

$$\begin{tikzcd}
CFA^-(\mathcal{H}_p) \boxtimes \widehat{CFD}(Y - K, \mathfrak{s}|_{Y-K}) \arrow[r, "\simeq"] \arrow[d, "\mathbb{I}_{CFA^-(\mathcal{H}_P)} \boxtimes \widetilde{G}_{\mathfrak{s}}"] & CFK^-(Y, K_P, \mathfrak{s}|_Y) \arrow[d, "G_{\mathfrak{s}}"] \\
CFA^-(\mathcal{H}_P) \boxtimes \widehat{CFD}(Y' - K', \mathfrak{s}|_{Y'-K'}) \arrow[r, "\simeq"] & CFK^-(Y', K'_P, \mathfrak{s}|_{Y'}).
\end{tikzcd}$$

Proof. To define the map for 2-handles, let $\mathcal{L}$ be a framed, k-component link in $Y - K$, and let $\mathcal{B}$ be a bouquet for $\mathcal{L}$. Choose a Heegaard triple diagram $\mathcal{H}^{\alpha,\beta,\beta'}$ which is subordinate to this bouquet in the sense of [OS06] and also admits a decomposition into two bordered Heegaard diagrams $\mathcal{H}_P^{\alpha,\beta,\beta'} \cup \mathcal{H}_B^{\alpha,\beta,\beta'}$, where $\mathcal{H}_P^{\alpha,\beta,\beta'}$ is the bordered Heegaard triple diagram obtained from $\mathcal{H}_P$ by adding Hamiltonian translates of the β-curves. Such a diagram can be produced as follows.

1. Start with a Heegaard diagram $\mathcal{H} = (\Sigma_g, \alpha_1, \ldots, \alpha_{g-1}, \beta_{k+1}, \ldots, \beta_g)$ for $Y - K - \mathcal{B}$. For each β-curve in $\mathcal{H}$, add a β'-curve which is a Hamiltonian translate so $|\beta_i \cap \beta'_j| = 2\delta_{ij}$.

2. Add a collection $\{\beta_1, ..., \beta_k\}$ which are meridians of the components of $\mathcal{L}$. Attaching 1- and 2-handles along the α and β curves yields $Y - K$.

3. For each component of $\mathcal{L}$, add a curve β'_i according to its framing. Attaching 1- and 2-handles along the α and β' curves produces $(Y - K)(\mathcal{L}) = Y' - K'$.

4. Choose a longitude and meridian of K in Σ disjoint from the α-curves which intersect in a single point, p. Define arcs $\alpha_1^a = \lambda - p$ and $\alpha_2^a = \mu - p$.

Altogether, this defines a bordered Heegaard triple diagram

$$\mathcal{H}_B^{\alpha,\beta,\beta'} = (\Sigma_g - p, \alpha_1^a, \alpha_2^a, \alpha_1^c, \ldots, \alpha_{g-1}^c, \beta_1, \ldots, \beta_g, \beta_1', \ldots, \beta_g').$$

By construction, $\mathcal{H}_P^{\alpha,\beta,\beta'} \cup \mathcal{H}_B^{\alpha,\beta,\beta'}$ is a Heegaard triple subordinate to our chosen bouquet $\mathcal{B}$.

For $\delta, \varepsilon \in \{\alpha, \beta, \beta'\}$, let $\mathcal{H}_P^{\delta,\varepsilon}$ and $\mathcal{H}_B^{\delta,\varepsilon}$ be the various standard bordered Heegaard diagrams associated to these triples. Let $\Theta_{\beta,\beta'}$ and Θ be the top dimensional generators of $H_*(\widehat{CF}(\mathcal{H}_B^{\beta,\beta'}))$ and $H_*(\widehat{CF}(\mathcal{H}_P^{\beta,\beta'}))$ respectively. The pairing theorem for triangles [LOT16, Proposition 5.35] gives the following homotopy-commutative square:

$$\begin{tikzcd}
\mathit{CFA}^-(\mathcal{H}_P^{\alpha,\beta}) \boxtimes \widehat{CFD}(\mathcal{H}_B^{\alpha,\beta}) \arrow[r, "\simeq"] \arrow[d, "m_2(-,\Theta)\boxtimes m_2(-,\Theta_{\beta,\beta'})"'] & \mathit{CFK}^-(\mathcal{H}_P^{\alpha,\beta} \cup \mathcal{H}_B^{\alpha,\beta}) \arrow[d, "m_2(-,\Theta\otimes\Theta_{\beta,\beta'})"] \\
\mathit{CFA}^-(\mathcal{H}_P^{\alpha,\beta'}) \boxtimes \widehat{CFD}(\mathcal{H}_B^{\alpha,\beta'}) \arrow[r, "\simeq"] & \mathit{CFK}^-(\mathcal{H}_P^{\alpha,\beta'} \cup \mathcal{H}_B^{\alpha,\beta'}).
\end{tikzcd}$$

The right vertical arrow is by definition the 2-handle cobordism map G on CFK^-. After identifying $\mathit{CFA}^-(\mathcal{H}_\infty^{\alpha,\beta})$ with $\mathit{CFA}^-(\mathcal{H}_\infty^{\alpha,\beta'})$, the map $m_2(-,\Theta)$ is the identity. Define $m_2(-,\Theta_{\beta,\beta'})$ to be the 2-handle cobordism map $\widetilde{G_{\mathfrak{s}}}$ on $\widehat{CFD}$. $\square$

Since each handle attaching cobordism map was defined with respect to a particular Heegaard diagram, the last step is to ensure maps induced by the Heegaard moves relating two diagrams induce homotopy equivalences which are compatible with the bordered Floer homology pairing theorem. This is shown by Hendricks-Lipshitz in [HL19].

Lemma 4.2.3. *[HL19, Lemma 5.6] Suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ are a pair of bordered Heegaard diagrams related by a bordered Heegaard move and $\mathcal{H}_0$ is another bordered Heegaard diagram with $\partial\mathcal{H}_0 = -\partial\mathcal{H}_i$, $i \in \{1,2\}$. Then, the diagram*

$$\begin{tikzcd} CFA^-(\mathcal{H}_1) \boxtimes \widehat{CFD}(\mathcal{H}_0) \arrow[r, "\simeq"] \arrow[d] & CFK^-(\mathcal{H}_1 \cup \mathcal{H}_0) \arrow[d] \\ CFA^-(\mathcal{H}_2) \boxtimes \widehat{CFD}(\mathcal{H}_0) \arrow[r, "\simeq"] & CFK^-(\mathcal{H}_2 \cup \mathcal{H}_0) \end{tikzcd}$$

commutes up to homotopy. The vertical maps come from the proof of invariance of bordered and classical Floer homology.

Proof of Proposition 4.0.1: We have a decomposition of $W - C$ into handle-attachment cobordisms, $W_1 \cup W_2 \cup W_3$. Define F_{WC} to be the composition

$$F_{W-C} = \widetilde{H} \circ \Psi_{\mathcal{H}_3 \leftarrow \mathcal{H}_2} \circ \widetilde{G}_{\mathfrak{s}} \circ \Psi_{\mathcal{H}_2 \leftarrow \mathcal{H}_1} \circ \widetilde{F},$$

where $\Psi_{\mathcal{H}_{i+1} \leftarrow \mathcal{H}_i}$ $i \in \{1,2\}$ are the change of diagram homotopy equivalences and $\mathfrak{s}$ is the restriction of the unique $\mathrm{Spin}^{\mathrm{C}}$-structure on W to W_2. By stacking the diagrams from Lemma 4.1.1, 4.1.2, 4.2.2, and 4.2.3 the result follows. $\square$

No Cancelation Lemma

In Chapter 5, we will be interested in distinguishing maps induced by satellite concordances. Let C and C' be two concordances from K_0 to K_1. Assuming $F_C \neq F_{C'}$, we would like a sufficient condition to guarantee that $F_{P(C)} \neq F_{P(C')}$.

Given a concordance $C : U \to K$, Theorem 13 guarantees the existence of a map

$$F : \widehat{CFD}(S^3 - U) \to \widehat{CFD}(S^3 - K),$$

with the property that for any satellite pattern P, the map induced by the concordance $P(C)$ can be computed as

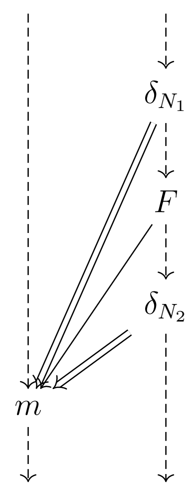
$$\mathbb{I}_P \boxtimes F : \widehat{CFA}(S^1 \times D^2, P) \boxtimes \widehat{CFD}(S^3 - U) \ \to \widehat{CFA}(S^1 \times D^2, P) \boxtimes \widehat{CFD}(S^3 - K),$$

where $\mathbb{I}_P$ is the identity map on $\widehat{CFA}(S^1 \times D^2, P)$. By definition, the map $\mathbb{I}_P \boxtimes F$ can be written

$$\mathbb{I}_P \boxtimes F(x \otimes v) = \sum_{k=1}^{\infty} (m_{k+1} \otimes \mathbb{I}_P)(x \otimes F^k(v)),$$

or, diagrammatically as

$\mathbb{I}_P \boxtimes F =$



.

As described in Chapter 11 of [LOT18], a model for $CFK^-(K)$ gives rise to a model for $\widehat{CFD}(S^3 - K)$: roughly a basis for $\widehat{HFK}(K)$ forms a basis for $\iota_0 \cdot \widehat{CFD}(S^3 - K)$ and the structure of the differential of $CFK^-(K)$ determines a basis for $\iota_1 \cdot \widehat{CFD}(S^3 - K)$ as well as the differentials.

Since we will be primarily interested in slice disks, we will restrict ourselves to unknotted patterns. If P is an unknotted pattern, then

$$\widehat{CFA}(S^1 \times D^2, P) \boxtimes \widehat{CFD}(S^3 - U) \simeq \widehat{CFK}(S^3, U).$$

Therefore, there is some element a in $\widehat{CFA}(S^1 \times D^2, P)$ such that $a \otimes v$ generates homology of $\widehat{CFK}(S^3, U)$, where v is the unique element in $\widehat{CFD}(S^3 - U)$. On homology, $\mathbb{I}_P \boxtimes F$ is therefore determined by the image of $a \otimes v$. We show that given some seemingly restrictive assumptions on the structure of $\widehat{CFA}(S^1 \times D^2, P)$, $\mathbb{I}_P \boxtimes F$ is guaranteed to be nontrivial on homology.

Lemma 4.3.1. *Let*

$$F : \widehat{CFD}(S^3 - U) \to \widehat{CFD}(S^3 - K),$$

be a morphism of type-D structures which tensors with the identity map of $\widehat{CFA}(S^1 \times D^2, \lambda)$ to give a nontrivial map, where λ is the knot $S^1 \times \{pt\} \subset S^1 \times D^2$. Let a be an element of $\widehat{CFA}(S^1 \times D^2, P)$ such that $a \otimes v$ generates the homology of $\widehat{CFA}(S^1 \times D^2, P) \boxtimes \widehat{CFD}(S^3 - U)$ and extend $\{a\}$ to a basis for $\widehat{CFA}(S^1 \times D^2, P)$. If the coefficient of a is zero in every $\mathcal{A}_\infty$ operation $m_k(b, \rho_{i_1}, \ldots, \rho_{i_{k-1}})$ of $\widehat{CFA}(S^1 \times D^2, P)$ which preserves the filtration, then the map

$$\mathbb{I}_P \boxtimes F : H_*(\widehat{CFA}(S^1 \times D^2, P) \boxtimes \widehat{CFD}(S^3 - U)) \to H_*(\widehat{CFA}(S^1 \times D^2, P) \boxtimes \widehat{CFD}(S^3 - K))$$

is nontrivial.

Proof. Let $x = F(v)$, where F is the map of type-D structures induced by F_C. We can write

$$x = 1 \cdot \theta + \sum_{I \in \{1,2,3,12,23,123\}} \rho_I \theta_I,$$

for some $\theta_I \in \widehat{CFD}(S^3 - K)$. The term θ must be nontrivial, since we have assumed that $\mathbb{I}_{\widehat{CFA}(S^1 \times D^2, \lambda)} \boxtimes F$ has nontrivial image (this follows from the fact that $\widehat{CFA}(S^1 \times D^2, \lambda)$ has no nontrivial $\mathcal{A}_\infty$-operations and so all other terms in $F(x)$ are annihilated after taking the box tensor product.) $(\mathbb{I}_P \boxtimes F)(a \otimes v)$ is defined to be

$$\begin{aligned}(\mathbb{I}_P \boxtimes F)(a \otimes v) &= \sum_{k=1}^{\infty} (m_{k+1} \otimes \mathbb{I}_P) \circ (a \otimes F^k(v)) \\ &= a \otimes \theta + \text{ other terms.}\end{aligned}$$

The term $a \otimes \theta$ could be canceled if, for some k, $F^k(v) = \rho_{i_1} \otimes \ldots \otimes \rho_{i_k} \otimes \theta$ and there is an operation of the form $m_{k+1}(a, \rho_{i_1}, \ldots, \rho_{i_k})$ in which a appears with nonzero coefficient. However, we have assumed that no such operations exist.

Moreover, when we pass to homology, there are no relation between $a \otimes \theta$ and any other elements of $H_*(\widehat{CFA}(S^1 \times D^2, P) \boxtimes \widehat{CFD}(S^3 - K))$, since $a \otimes \theta$ could only appear as a term in the boundary of another element if there were a filtration preserving operation of the form $m_{k+1}(b, \rho_{i_1}, \ldots, \rho_{i_k})$ in which a appeared with nonzero coefficient. Again, no such operations exist.

Therefore, $a \otimes \theta$ appears as a non-canceling term in the expansion of $(\mathbb{I}_P \boxtimes F)(a \otimes v) \in H_*(\widehat{CFA}(S^1 \times D^2, P) \boxtimes \widehat{CFD}(S^3 - K))$, from which it follows that $\mathbb{I}_P \boxtimes F$ has nontrivial image. $\square$

CHAPTER V

INJECTIVE SATELLITE OPERATORS

The material in this section draws from forthcoming work joint with Hayden-Kang-Park.

We turn now to applications of Theorem 13. We begin by showing that if knot Floer homology distinguishes a pair of slice disks, it will also distinguish their positive Whitehead doubles. Given a slice disk D of K, we denote the induced element $F_D(1)$ in $\widehat{HFK}(S^3, K)$ by t_D.

Theorem 10. *Let K be a knot in S^3 with slice disks D_1, D_2. If $t_{D_1} \neq t_{D_2}$, then $t_{Wh^+(D_1)} \neq t_{Wh^+(D_2)}$ as well.*

Proof. Let F_1, F_2 be the type-D morphisms determined by D_1 and D_2 respectively. Since $t_{D_1} \neq t_{D_2}$, we have that $\mathbb{I}_{\widehat{CFA}(S^1 \times D^2, \lambda)} \boxtimes (F_1 + F_2)$ is nontrivial. The type-A structure for the positive Whitehead double is computed by Levine [Lev12]:

$$\begin{tikzcd}
c & c' \arrow[l] & \\
 & b \arrow[ul, "\rho_3\rho_2\rho_1"'] \arrow[d, "\rho_1"] & b' \arrow[ul, "\rho_3\rho_2\rho_1"'] \arrow[l] \arrow[dl, "\rho_{123}"] \arrow[d, "\rho_1"] \arrow[dd, bend left, "\rho_{12}"] \\
 & a & a' \arrow[l, "1+\rho_{23}"] \arrow[d, "\rho_2"] \\
 & & d. \arrow[ul, "\rho_3"]
\end{tikzcd}$$

In the diagram above, an arrow of the form $x \xrightarrow{\rho_{i_1}\cdots\rho_{i_k}} y$ indicates that $m_{k+1}(x, \rho_{i_1}, \ldots, \rho_{i_k}) = y$. Arrows pointing left lower the filtration. A short computation illustrates that $H_*(\widehat{CFA}(S^1 \times D^2, \mathrm{Wh}^+) \boxtimes \widehat{CFD}(S^3 - U)) = \mathbb{F}\langle b \otimes v\rangle$.

There is a single arrow into b, but it lowers the filtration level. Therefore, by Lemma 4.3.1, $\mathbb{I}_{\mathrm{Wh}^+} \boxtimes (F_1 + F_2)$ is nontrivial. Therefore, by Theorem 13, $t_{Wh^+(D_1)} \neq t_{Wh^+(D_2)}$. □

We now prove that for any slice disks D_1 and D_2 for K, their positive Whitehead doubles are topologically isotopic.

A slice disk D is called a *$\mathbb{Z}$-disk* if the fundamental group of its complement is isomorphic to $\mathbb{Z}$. By the work of Conway and Powell, any two $\mathbb{Z}$-disks with common boundary are topologically isotopic rel. boundary [CP21, Theorem 1.2]. We can arrange to work in this situation by choosing appropriate satellite patterns. Recall that the *winding number* of a pattern P is the algebraic intersection of P with a generic meridional disk of the solid torus containing P.

Proposition 5.0.1. *If P is a winding number zero pattern with $P(U) = U$ and D is a slice disk, then the satellite disk $P(D)$ is a $\mathbb{Z}$-disk.*

Proof. Choose a tubular neighborhood $\nu(D)$ of D. The satellite disk $P(D)$ is contained in $\nu(D)$, so we have a splitting

$$B^4 \setminus P(D) = (B^4 \setminus \nu(D)) \cup (\nu(D) \setminus P(D)).$$

Since the homeomorphism class of $\nu(D) \setminus P(D)$ does not depend on the choice of D,

$$\nu(D) \setminus P(D) \cong \nu(D_0) \setminus P(D_0),$$

where D_0 denotes the trivial slice disk of an unknot U. Since P is unknotted $P(D_0)$ is also a trivial slice disk of $P(U) = U$ so we may take $\nu(D_0)$ to be the whole of B^4.

Therefore,

$$\nu(D_0) \smallsetminus P(D_0) \cong B^4 \smallsetminus D_0 \cong D^2 \times (D^2 \smallsetminus \{pt\}),$$

implying $\pi_1(\nu(D) \smallsetminus P(D)) \cong \mathbb{Z}$.

The fundamental group of the intersection

$$(B^4 \smallsetminus \nu D) \cap (\nu D \smallsetminus P(D)) \cong S^1 \times D^2$$

is also $\mathbb{Z}$, and the natural maps

$$\pi_1\left((B^4 \smallsetminus \nu D) \cap (\nu D \smallsetminus P(D)\right) \to \pi_1\left(B^4 \smallsetminus \nu D\right),$$
$$\pi_1\left((B^4 \smallsetminus \nu D) \cap (\nu D \smallsetminus P(D)\right) \to \pi_1\left(\nu D \smallsetminus P(D)\right),$$

are given by the inclusion of a meridional class of $\pi_1(B^4 \smallsetminus \nu D)$ and the multiplication map $\mathbb{Z} \xrightarrow{\times w(P)} \mathbb{Z}$, respectively, where $w(P)$ denotes the winding number of P. Since P has winding number 0 and $\pi_1(B^4 \smallsetminus \nu D)$ is normally generated by a meridian of D, we see

$$\pi_1(B^4 \smallsetminus P(D)) \cong \pi_1(\nu D \smallsetminus P(D)) \cong \mathbb{Z}.$$

Therefore $P(D)$ is a $\mathbb{Z}$-disk. $\square$

As the positive Whitehead double pattern satisfies all the hypotheses of Proposition 5.0.1, we have the following corollary.

Corollary 5.0.2. *Let D_1 and D_2 be any slice disks for K which are distinguished by their induced maps on $\widehat{HFK}$. Then, $Wh^+(D_1)$ and $Wh^+(D_2)$ are exotic disks.*

By considering deform spun disks, in forthcoming work, we prove the following.

Theorem 11. *For any nontrivial knot* K*, the knot* $Wh^{+}(K\#K\#-K\#-K)$ *bounds a pair of exotic disks.*

CHAPTER VI

STABLY EXOTIC SURFACES

In this section, we produce a pair of exotic disks which remain exotic after many stabilizations.

Theorem 12. *For any p, there exists a knot J_p which bounds a pair of exotic disks D_p and D'_p which remains exotic after $p-1$ internal stabilizations.*

We distinguish our surfaces by comparing their induced maps on knot Floer homology. Stabilization has a simple effect on the induced map; attaching a tube simply corresponds to multiplication by U (or V) [JMZ20, JZ21]. Juhász-Zemke make use of this fact to define their suite of secondary Heegaard Floer invariants which provide lower bounds for the stabilization distance of two surfaces. See Chapter 2 for details.

Recall that the *torsion order* $\mathrm{Ord}_U(K)$ of a knot K is the smallest power of U which annihilates the torsion submodule of $HFK^-(K)$. Since a stabilization corresponds to multiplication by U, any two maps induced by disks with boundary K become indistinguishable after multiplication by $U^{\mathrm{Ord}_U(K)}$. Therefore, these lower bounds cannot be used to show disks bounding knots with torsion order 1 have large stabilization distance. $\mathrm{Ord}_U(K)$ is bound above by the *fusion number* of K, which is the minimal number of bands occurring in ribbon disk for K [JMZ20]. Therefore, to have any hope in finding disks with large stabilization distance, it is necessary to work with knots with large fusion number.

Cabled concordances

Recent work of Hom-Kang-Park [HKP20] and Hom-Lidman-Park [HLP22] studies how cabling is related to the torsion order and fusion number of a knot. If K is ribbon with fusion number 1, then the knot Floer homology of the $(p, 1)$-cable of K has torsion order p [HKP20, Lemma 3.3]. Cabling has a natural four-dimensional extension: given a concordance $C : K \to K'$, there is a cabled concordance between the cables of K and K'. In particular, given a ribbon knot K with fusion number 1, $K_{(p,1)}$ bounds a "cabled" ribbon disk, and has fusion number p.

The knot J shown in Figure **??**, bounds an exotic pair of disks D and D' by the work of Hayden [Hay21]. We will refer to these disks as the "positron disks" since their double branched cover is the positron quark of [AM98].These disks are distinguished by their induced maps on knot Floer homology [DMS22]. But, J has fusion number 1, so the two maps become equal after a single stabilization. In fact, we will show directly that these two disks are smoothly isotopic after a single stabilization. However, the cabled disks D_p and D'_p have fusion number p, and as we show, have stabilization distance p as well.

We begin by reviewing the two definitions of stabilization distance and illustrate that the two need not agree. We then review the construction of cabled concordances and show how this operation can be used to produce disks which are topologically isotopic. We conclude this section by giving an upper bound on the stabilization distance of D_p and D'_p by explicitly showing they become isotopic after p stabilizations.

Two notions of stabilization distance

The most general notion of internal stabilization is due to [JZ21].

Definition 6.2.1. Let Σ be an oriented surface with boundary, smoothly embedded in a 4-manifold W. Let B be a 4-ball in the interior of W whose boundary intersects Σ in an n-component unlink L. Moreover, suppose $\Sigma \cap B$ is a collection of disks $D_1, ..., D_n$ which can be isotoped into ∂B relative to their boundaries. Let S_0 be a connected genus g surface in B with boundary L. The surface $\Sigma' = (\Sigma - B) \cup_L S_0$ is called the (g, n)-*stabilization of* Σ *along* (B, S_0). We call Σ the (g, n)-*destabilization of* Σ' *along* (B, S_0). See Figure 17.

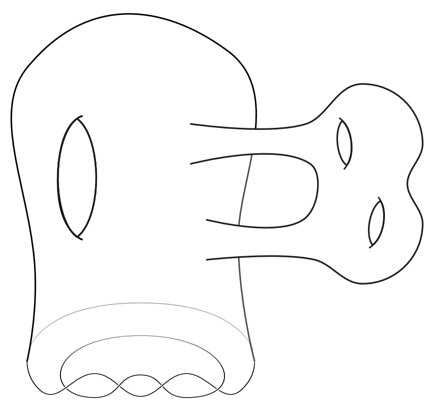


FIGURE 17 A (g, n)-stabilization along (B^4, S_0). The case $(g, n) = (2, 2)$ is shown.

Definition 6.2.2. Let Σ and W be as above, and let Σ' be a (g, n)-stabilization of Σ. When $(g, n) = (0, 2)$ and $S_0 \cup D_1 \cup D_2$ bounds a 3-dimensional 1-handle embedded in W, we say Σ' is a *1-handle stabilization* of Σ.

We will simply write "stabilization" instead of (g, n)-stabilization, and state explicitly when we mean 1-handle stabilization. We now formally define the two notions of stabilization distance. For simplicity, we will only define the stabilization distance for disks.

Definition 6.2.3. Let Σ and Σ' be disks in W such that $\partial\Sigma = \partial\Sigma'$ and $[\Sigma] = [\Sigma'] \in H_2(W, \partial W; \mathbb{Z})$. The *1-handle stabilization distance* $d(\Sigma, \Sigma')$ between Σ and Σ' is the minimal number k such that Σ and Σ' become isotopic rel boundary after each is stabilized k times.

Definition 6.2.4. Let Σ and Σ' be disks in W such that $\partial\Sigma = \partial\Sigma'$ and $[\Sigma] = [\Sigma'] \in H_2(W, \partial W; \mathbb{Z})$. The *stabilization distance* $\mu(\Sigma, \Sigma')$ between Σ and Σ' is defined to be the minimum of

$$\max\{g(\Sigma_1), \ldots, g(\Sigma_k)\}$$

over all sequences of connected surfaces from $\Sigma = \Sigma_1$ to $\Sigma' = \Sigma_k$ in W such that $\partial\Sigma_i = K$ for all i and Σ_i and Σ_{i+1} are related by a stabilization or a destabilization.

Note that necessarily $d(\Sigma, \Sigma') \geq \mu(\Sigma, \Sigma')$. However, as the next example illustrates, the two notions are distinct.

Example 6.2.5. The knot $K = 9_{46}$ is shown in Figure 19. K bounds an obvious torus in S^3. Moreover, by compressing the two circles which generate the first homology of this torus, we obtain two slice disks D and D' with boundary K. Both disks can be described as banded unlinks with a single band [Swe01, HKM20] (Figure 19). It is shown in [MP19] that $d(D, D') = 1$. This can be seen as follows.

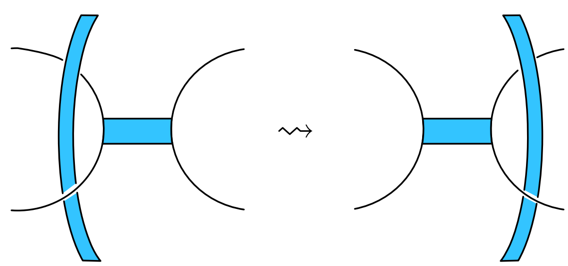


FIGURE 18 Swimming one band through another.

To show that the two disks become isotopic after a single 1-handle stabilization, it suffices to show that by attaching a tube, the relative positions

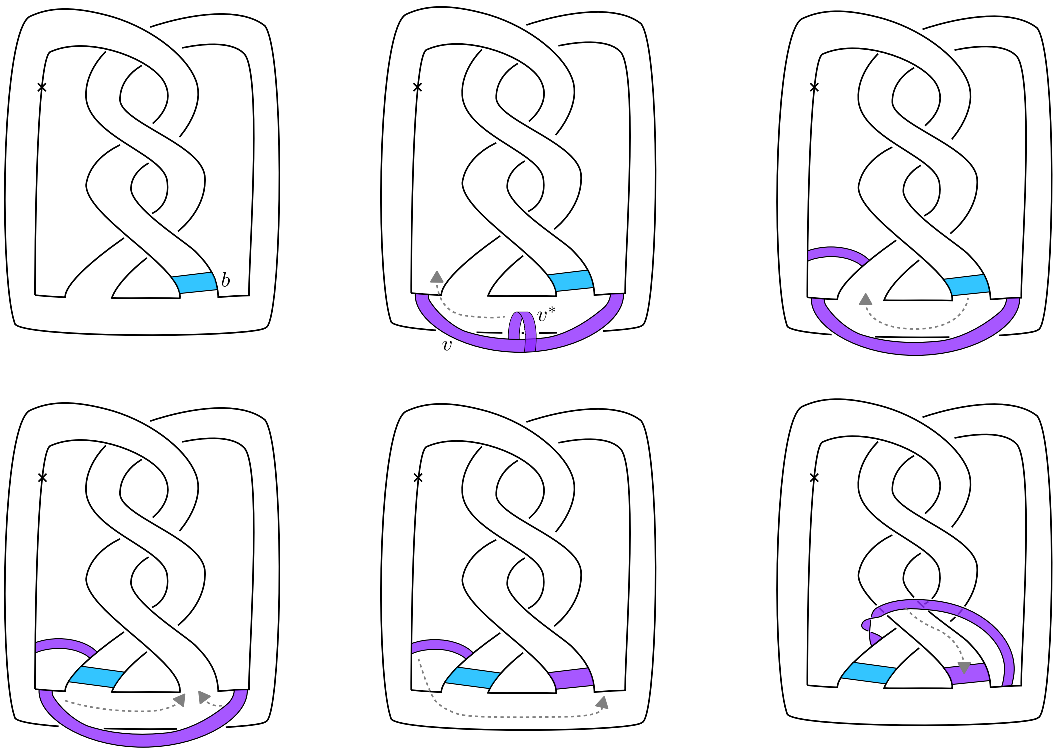


FIGURE 19 An isotopy taking a 1-handle stabilization of D to a 1-handle stabilization of D'.
A swim move occurs in frame 6. (Continued in Figure 20)
.

of the two bands can be swapped, and then that the tube can be isotoped until it is once again clearly the result of a 1-handle stabilization.

By strategically attaching a tube, we can slide the right-hand band, b, to the left (Figure 19.) However, this band slide separates the tube into two bands, v and v^*. Next, slide v into the position originally occupied by b. Now, we can drag v^* around K until it once again forms a tube with v by performing swim moves as necessary to change crossings of v^* with the diagram for K. See Figure 18 for an example of a swim move, and see 20 for the remainder of the isotopy taking the stabilization of D to the stabilization of D'.

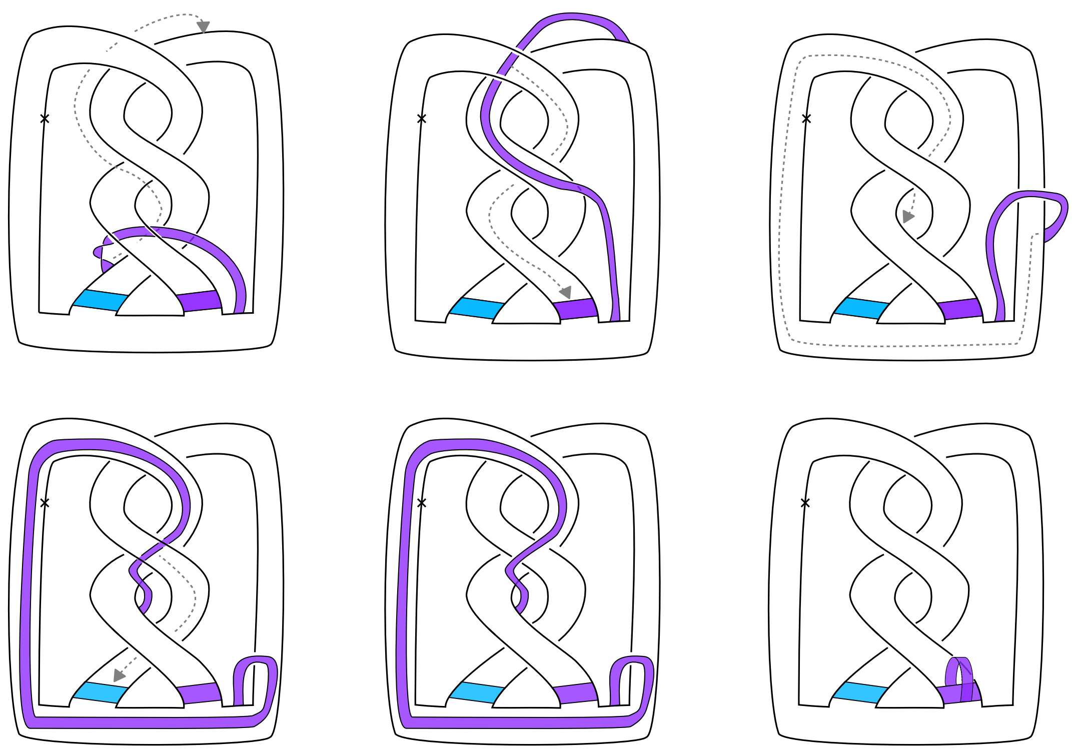


FIGURE 20 The remainder of the isotopy between the stabilizations of D and D'. Swim moves occur in frames 2 and 4.

Miller-Powell use Alexander modules to show that by taking boundary connected sums of these disks, they can produce disks with arbitrarily large 1-handle stabilization distance: $d(\natural^m D, \natural^m D') = m$. However, it is clear from Figures 19 and 20, that $\mu(\natural^m D, \natural^m D') = 1$; no band ever slid over the marked point on the diagram, so we can take the connected sums at the marked points. Since the more general stabilization distance allows us to stabilize *and* destabilize, we can attaching a single tube in order to isotope the first D summand to D', then destabilize, and then repeat the strategy on the next copy of D, until we are left with $\natural^m D'$.

It is worth noting that even though the obvious ribbon disk for $\natural^m D$ has m bands, we can in some sense "reuse" (stabilize then destabilize) the same tube m times to make $\natural D$ and $\natural D'$ isotopic. K has fusion number 1, so necessarily

$\mathrm{Ord}_U(K) = 1$. By the knot Floer homology Künneth formula,

$$\mathrm{Ord}_U(\#^m K) = \mathrm{Ord}_U(K) = 1.$$

This implies that $\tau(\natural^m D, \natural^m D') \leq 1$ (see Definition 2.3.6), and therefore τ cannot detect the large 1-handle stabilization distance of these disks.

Concordances induced by cables

Let $C : K \to K'$ be a concordance. Given a pattern knot $P \subset S^1 \times D^2$, we obtain a concordance between the satellites of K and K' as follows. Remove a neighborhood of C in $S^3 \times I$. The Seifert framing of K determines an identification $\varphi : \partial\nu(K) \to S^1 \times \partial D^2$. Define the satellite concordance C_P to be $(S^3 \times I - C, \varnothing) \cup_{\varphi\times\mathrm{id}} (S^1 \times D^2 \times I, P \times I)$.

The (p, q)-cable of a knot is a satellite with pattern $P = T_{p,q}$, the (p, q)-torus knot. Since the $(p, 1)$-torus knot is the unknot, it is clear that if a knot K is slice, so is its $(p, 1)$-cable. Moreover, by capping off the $(p, 1)$-cable of the unknot, we obtain a cabled disk for K.

By the work of Freedman and Quinn, locally flat proper submanifolds have topological normal bundles which are unique up to ambient isotopy [FQ90, Section 9.3]. Therefore, by the nature of the construction, topological isotopy is preserved by the cabling operation. Therefore, since the disks D and D' which bound the knot J are topologically isotopic by the work of Conway and Powell [CP21, Theorem 1.2], this topological isotopy also produces a topological isotopy between the $(p, 1)$-cables of D and D'. This gives us:

Lemma 6.3.1. *The cabled disks D_p and D'_p which bound K_p are topologically isotopic.*

An upper bound for the stabilization distance

We now turn to the proof that the stabilization distance between D_p and D'_p is at most p.

As a warm up case, consider the disks D and D' with boundary J. Let b be the left-hand band which defines D and let b' be the right-hand band which defines D'. As in Example 6.2.5, attach a tube $v \cup v^*$ to J, so that the band b can be slid until it becomes isotopic to the band b'. Next, slide the band v into the position originally occupied by b. We have exchanged the roles of b and b' at the cost of tangling the bands v and v^* which made up the stabilization; it is not clear whether the resulting surface is isotopic to a stabilization of D'. If can isotope v^* away from $J \cup b$ and back onto v, we will be done.

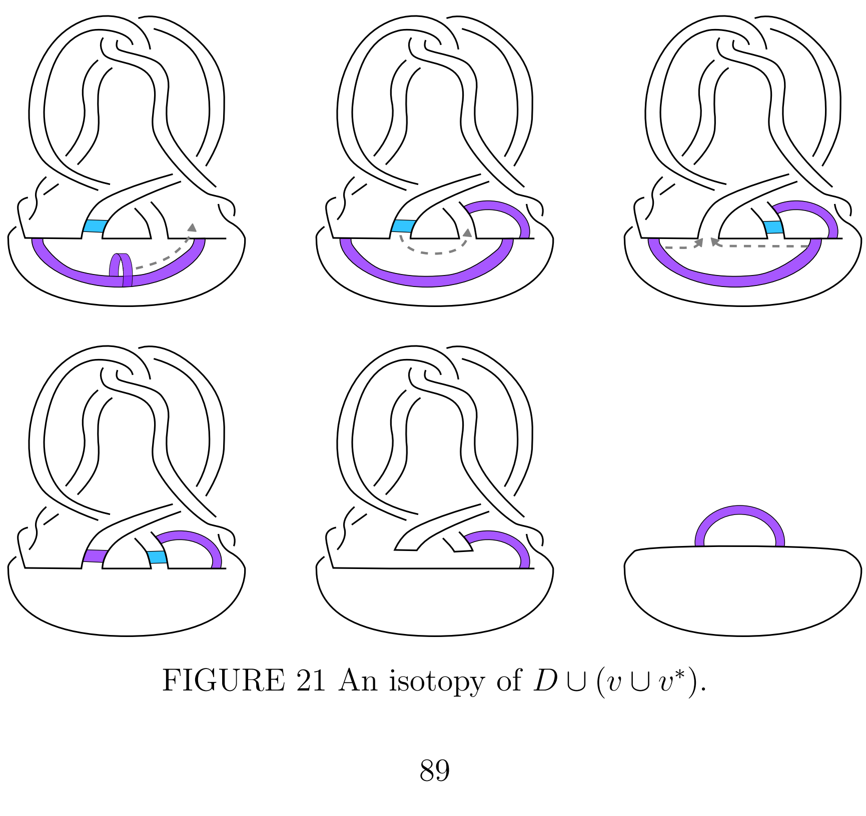


FIGURE 21 An isotopy of $D \cup (v \cup v^*)$.

As in Example 6.2.5, we can use the bands b and v to pull v^* into the correct position by a sequence of swim moves. Recall that a swim move of v^* through b corresponds to pushing v^* below the critical point for b, and performing an isotopy of v^* in $S^3 - J(b)$, i.e. in the complement of link obtained by band surgery on b. Performing the entire isotopy at this level will turn out to be easier to visualize, especially once we progress to the cabled case.

Push the band v^* into the interior of B^4, (say to radius $r = 2/3$) below the critical points for the bands b and v. Here, the level sets of the surface are isotopic to $J(b)(v) = J(b)(b')$ (the result of band surgery on both b and b') which is the unknot. Moreover, from Figure 21, we see that the band v^* is attached trivially.

At this point, the diagram is symmetric. Hence, this argument can be repeated with the disk D' stabilized with tube $u \cup u^*$: we isotope u^* into B^4 as well, until we see u^* attached to the unknot. $U \cup v^*$ and $U \cup u^*$ are clearly isotopic, so by composing the first isotopy with the inverse of the second we obtain the desired isotopy from $D \cup (v \cup v^*)$ to $D' \cup (u \cup u^*)$.

The cabled case is similar.

Proposition 6.4.1. *The disks D_p and D'_p become isotopic after p 1-handle stabilizations.*

Proof. Figure 22 gives a band presentation for D_p. Let $b_1, \ldots, b_p$ and $b'_1, \ldots, b'_p$ be the bands of D_p and D'_p. As in the case $p = 1$, attach tubes $v_1 \cup v_1^*, \ldots, v_p \cup v_p^*$ in order to move the band b_i into the position of b'_i, and then slide v_i into the original position of b_i (Figure 22). The result of band surgery on all the b_i and v_i bands is again the unknot, and as shown in Figure 23 this unknot is naturally identified with the $(p, 1)$-cable of the unknot. Moreover, at this level set, the v_i^* bands are attached trivially. So, just as before, perform the symmetric isotopy of D'_p, and

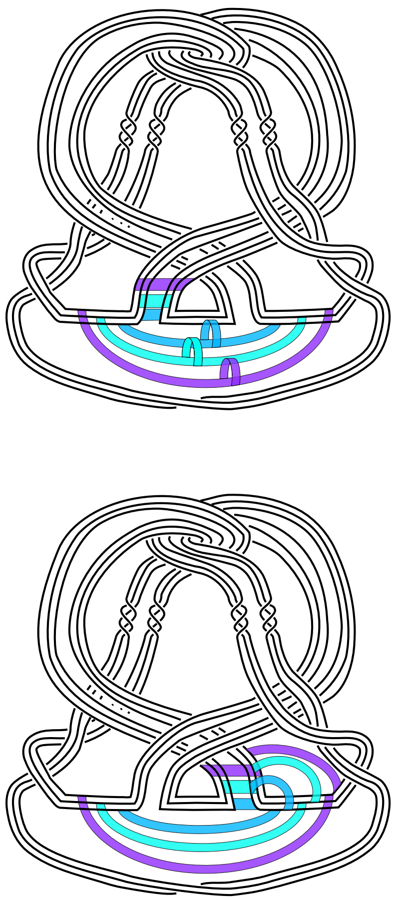


FIGURE 22 Part 1 of an isotopy between p-fold stabilizations of D_p and D'_p

concatenate to obtain an isotopy between the p-fold stabilizations of D_p and D'_p.

□

A lower bound for the stabilization distance

We now make use of the secondary invariants of Juhász-Zemke to provide a lower bound on the stabilization distance of the disks D_p and D'_p.

The positron disks D and D' are distinguished by their induced maps on knot Floer homology. In [DMS22], Dai-Mallick-Stoffregen compute $HFK^-(J)$. The summand which contains the image of this disk has the form $\mathbb{F}[U]\langle x\rangle \oplus \mathbb{F}\langle e_1, e_2\rangle$.

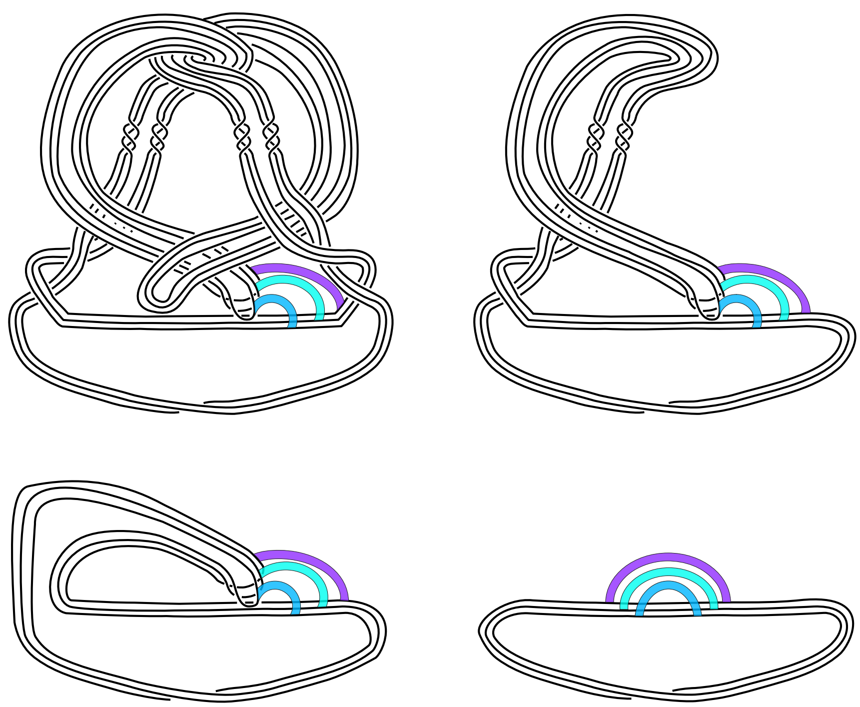


FIGURE 23 Part 2 of an isotopy between p-fold stabilizations of D_p and D'_p

The formal variable U acts trivially on the $\mathbb{F}$-summands. Using their equivariant knot Floer homology program, they prove the following.

Theorem 13. *Let D and D' be the exotic positron disks. Then, the maps F_D and $F_{D'}$ satisfy:*

$$(F_D + F_{D'})(1) = e_1 + e_2.$$

As a sanity check, notice that

$$\begin{aligned}(F_{D\#T^2} + F_{D'\#T^2})(1) &= U(F_D + F_{D'})(1) \\ &= U(e_1 + e_2) \\ &= 0,\end{aligned}$$

since D and D' become isotopic after a single stabilization.

Theorem 14. *Let D and D' be a pair of disks distinguished by their induced maps on HFK^-. Let D_p and D'_p denote their $(p,1)$-cables. Then, the the stabilization distance between D and D' is at least p.*

Proof. Let $A_p = CFA^-(S^1 \times D^2, C_{p,1})$ be the type-A structure associated to the $(p,1)$-cabling pattern. A_p is generated by $\alpha, \beta_1, \ldots, \beta_{2p-2}$. Since we are only interested in computing maps $CFA^-(S^1 \times D^2, C_{p,1}) \boxtimes \widehat{CFD}(S^3 - U) \to CFA^-(S^1 \times D^2, C_{p,1}) \boxtimes \widehat{CFD}(S^3 - J)$ and the homology of $CFA^-(\mathcal{H}_p) \boxtimes \widehat{CFD}(S^3 - U) \simeq CFK^-(U)$ is generated by $\alpha \otimes v$, it is enough to consider the $\mathcal{A}_\infty$-operations involving α.

$$m_{2+i}(\alpha, \overbrace{\rho_{12}, \ldots, \rho_{12}}^{i}, \rho_1) = \beta_{2p-i-2} \qquad 0 \le i \le p-2$$

$$m_{4+i+j}(\alpha, \rho_3, \overbrace{\rho_{23}, \ldots, \rho_{23}}^{j}, \rho_2, \overbrace{\rho_{12}, \ldots, \rho_{12}}^{i}, \rho_1) = U^{pj+i+1}\beta_{i+1} \qquad 0 \le i \le p-2, 0 \le j$$

$$m_{3+i}(\alpha, \rho_3, \overbrace{\rho_{23}, ..., \rho_{23}}^{j}, \rho_2) = U^{p(j+1)}\alpha, \qquad 0 \le j.$$

For the full collection of $\mathcal{A}_\infty$-operations, see [Pet13, Section 4]. Let F_1 and F_2 be the type-D morphisms determined by F_D and $F_{D'}$. An argument identical to that of Lemma 4.3.1 shows that, since the only arrows into α are of the form $m_{3+i}(\alpha, \rho_3, \overbrace{\rho_{23}, ..., \rho_{23}}^{j}, \rho_2) = U^{p(j+1)}\alpha$ for $0 \le j$ we must have an element $\theta \in \widehat{CFD}(S^3 - J)$ such that $\alpha \otimes \theta$ appears in $\mathbb{I}_{C_{p,1}} \boxtimes (F_1 + F_2)(\alpha \otimes v)$ which cannot be cancelled. Moreover, $U^k\alpha \otimes v$ is nontrivial in homology for $k < p$. Therefore, we have that

$$U^k(\mathbb{I}_{C_{p,1}} \boxtimes (F_1 + F_2)(\alpha \otimes v)) \neq 0,$$

for $k < p$. Therefore,

$$p \leq \mu(D_p, D'_p),$$

as claimed. $\square$

In particular, the $(p, 1)$-cables of the positron disks have stabilization distance exactly p.

APPENDIX

DIRECT COMPUTATIONS

In this section, we illustrate the how Theorem 13 can be used to carry out explicit computations. Again, let F_{D_p} and $F_{D'_p}$ be the $(p,1)$-cables of the exotic positron disks D and D'. It is more convenient to view these disks as concordances C_p and C'_p from the unknot. F_{C_p} and $F_{C'_p}$ will be computed in terms of $F_{S^3 \times I - C}$ and $F_{S^3 \times I - C'}$, which as we will see, are determined by F_C and $F_{C'}$ (up to some indeterminacy).

The complex $\mathcal{CFK}^-(J)$ consists of a singleton generator x as well as four boxes.

$$\begin{tikzcd}
x & b_i \arrow[d, "V"] & a_i \arrow[l, "U"] \arrow[d, "V"] & g_i \arrow[d, "V"] & f_i \arrow[l, "U"] \arrow[d, "V"] \\
 & e_i & c_i \arrow[l, "U"] & j_i & h_i \arrow[l, "U"]
\end{tikzcd}$$

FIGURE 24 The complex $\mathcal{CFK}^-(J)$, $i \in \{1,2\}$.

Generator	gr_U	gr_V
x	0	0
a_i	0	0
b_i	1	-1
c_i	-1	1
e_i	0	0
f_i	-1	-1
g_i	0	-2
h_i	-2	0
j_i	-1	-1

TABLE 1 Bi-gradings of generators of $\mathcal{CFK}^-(J)$.

The summands generated by the two boxes generated by $\{f_i, g_i, h_i, j_i\}$ contain no elements of bigrading $(0,0)$, and therefore do not intersect the images of the maps F_C and $F_{C'}$. For this reason, we will work primarily with the subcomplex of $\mathcal{CFK}^-(J)$ generated by $\{x, a_i, b_i, c_i, e_i\}$ to simplify the notation.

[LOT18] gives an algorithm for determining $\widehat{CFD}(S^3 - K)$ in terms of $\mathcal{CFK}^-(K)$. Figure 25 shows how a box and a singleton generator give rise to summands of $\widehat{CFD}(S^3 - J)$.

$$
\begin{tikzcd}
b_i \arrow[d, "V"] & a_i \arrow[l, "U"'] \arrow[d, "V"] \\
e_i & c_i \arrow[l, "U"']
\end{tikzcd}
\quad x \quad \rightsquigarrow \quad
\begin{tikzcd}
b_i \arrow[d, "\rho_1"] & y_i^1 \arrow[l, "\rho_2"'] & a_i \arrow[l, "\rho_3"'] \arrow[d, "\rho_1"] \\
y_i^2 & & y_i^4 \\
e_i \arrow[u, "\rho_{123}"] & y_i^3 \arrow[l, "\rho_2"'] & c_i \arrow[l, "\rho_3"'] \arrow[u, "\rho_{123}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[loop right, "\rho_{12}"]
\end{tikzcd}
$$

FIGURE 25 On the left is the summand of $\mathcal{CFK}^-(J)$ containing $F_D(1)$ and $F_{D'}(1)$. On the right is a model for the corresponding summand of $\widehat{CFD}(S^3 - J)$.

Computing the morphism complex

By Theorem 13, there exists a map $F : \widehat{CFD}(S^3 - J) \to \widehat{CFD}(S^3 - J)$ with the property that for any pattern P in the solid torus, $\mathbb{I}_{CFA^-(P)} \boxtimes F$ computes the concordance map induced by the pattern. In particular, if we take P to be the longitudinal unknot in the solid torus, which we denote (T_∞, λ) this map also computes F_C (respectively $F_{C'}$). Since we did not show this map F is unique, we will try to pin it down by computing the morphism space $\mathrm{Mor}_{\mathcal{A}(T^2)}(\widehat{CFD}(S^3 - U), \widehat{CFD}(S^3 - J))$ and considering which maps $f \in \mathrm{Mor}_{\mathcal{A}(T^2)}(\widehat{CFD}(S^3 -$

$U), \widehat{CFD}(S^3 - J))$ have the property that $\mathbb{I}_{CFA^-(T_\infty,\lambda)} \boxtimes f \simeq F_C$ (respectively $F_{C'}$).

We begin by computing the dimension of the homology of the morphism space $\mathrm{Mor}_{\mathcal{A}(T^2)}(\widehat{CFD}(S^3-U), \widehat{CFD}(S^3-J))$.

Lemma A.1.1. *The space of homotopy classes of maps from $\widehat{CFD}(S^3-U)$ to $\widehat{CFD}(S^3-J)$ is 10 dimensional.*

Proof. By [LOT11], there is a homotopy equivalence:

$$\begin{aligned}\mathrm{Mor}_{\mathcal{A}(T^2)}(\widehat{CFD}(S^3-U), \widehat{CFD}(S^3-J)) &\simeq \widehat{CF}(-(S^3-U)\cup(S^3-J)) \\ &= \widehat{CF}(S^3_0(J)).\end{aligned}$$

The mapping cone formula [OS08b] shows that $HF^+(S^3_0(J),[0])$ is the homology of the mapping cone $H_*(\mathrm{Cone}(\mathbb{A}_0^+ \xrightarrow{v_0+h_0} \mathbb{B}_0^+))$. We illustrate part of the complex.

$$\begin{tikzcd}[column sep=small]
\mathbb{A}_0^+ & & & & \mathbb{B}_0^+ & & \\
& & (UV)^{-1}x_i & & & & (UV)^{-1}\tilde{x}_i \\
& b_i \arrow[d] & (UV)^{-1}a_i \arrow[l] \arrow[d] & & & \tilde{b}_i \arrow[d] & (UV)^{-1}\tilde{a}_i \arrow[l] \arrow[d] \\
& e_i & c_i \arrow[l] & & & \tilde{e}_i & \tilde{c}_i \arrow[l] \\
& x_i & & & & \tilde{x}_i & \\
Ub_i & a_i \arrow[l] \arrow[d] & & \arrow[r, "v_0+h_0"] & \; & \tilde{a}_i \arrow[d] & \\
& Vc_i & & & & V\tilde{c}_i &
\end{tikzcd}$$

The homology of the portion of the complex shown is $\mathcal{T}^+\langle x_i\rangle \oplus \mathcal{T}^+\langle \tilde{x}_i\rangle \oplus \mathbb{F}\langle Ub_i = Vc_i\rangle$, where $\mathcal{T}^+ = \mathbb{F}[U,U^{-1}]/(U\cdot\mathbb{F}[U])$. The homology of the summand which is not shown (the remaining two boxes) is $\mathbb{F}\langle Ug_i = Vf_i\rangle$. From $HF^+(S^3_0(J))$, $\widehat{HF}(S^3_0(J))$ is obtained via the exact triangle

$$\begin{tikzcd}
HF^+(S^3_0(J)) \arrow[rr, "U"] & & HF^+(S^3_0(J)) \arrow[dl] \\
& \widehat{HF}(S^3_0(J)). \arrow[ul] &
\end{tikzcd}$$

A straightforward computation shows that $\widehat{HF}(S^3_0(J)) = \ker(U) \oplus \operatorname{coker}(U) \cong \mathbb{F}^{\oplus 10}$. □

Having computed the dimension of $H_*(\mathrm{Mor}_{\mathcal{A}(T^2)}(\widehat{CFD}(S^3 - U), \widehat{CFD}(S^3 - J)))$, our next task is to find a basis. To simplify the exposition, we introduce some notation. The summand of $\mathcal{CFK}^-(J)$ generated by x gives rise to a summand of $\widehat{CFD}(S^3 - J)$ generated by an element we will also denote x, with differential $\delta^1(x) = \rho_{12}x$. Call this type-D structure B. The unit boxes in $\mathcal{CFK}^-(J)$ correspond to boxes in $\widehat{CFD}(S^3 - J)$ as in Figure 26. Let C be such a type-D structure.

$$\begin{tikzcd}
b \arrow[d, "\rho_1"] & y^1 \arrow[l, "\rho_2"'] & a \arrow[l, "\rho_3"'] \arrow[d, "\rho_1"] \\
y^2 & & y^4 \\
e \arrow[u, "\rho_{123}"] & y^3 \arrow[l, "\rho_2"'] & c. \arrow[l, "\rho_3"'] \arrow[u, "\rho_{123}"']
\end{tikzcd}$$

FIGURE 26 The type-D structure C associated to a unit box.

If we ignore the gradings, $\widehat{CFD}(S^3 - U)$ is isomorphic to B, and $\widehat{CFD}(S^3 - J)$ is isomorphic to $B \oplus C^{\oplus 4}$. Therefore, it suffices to find bases for the homologies of $\mathrm{Mor}_{\mathcal{A}(T^2)}(B, B)$ and $\mathrm{Mor}_{\mathcal{A}(T^2)}(B, C)$.

The space of homotopy classes of maps $B \to B$ has a basis given by:

$$\phi = (x \mapsto x)$$

$$\psi = (x \mapsto \rho_{12} x).$$

Since $H_*(\mathrm{Mor}_{\mathcal{A}(T^2)}(B, B \oplus C^{\oplus 4})$ is ten dimensional, $H_*(\mathrm{Mor}_{\mathcal{A}(T^2)}(B, C))$ must be two dimensional. By inspection, a basis is given by:

$$g = (x \mapsto e + \rho_3 y^2 + \rho_1 y^3)$$

$$h = (x \mapsto \rho_1 y^4).$$

This is confirmed by Zhan's bordered Floer homology calculator [Zha].

From concordance maps to complement maps

The maps $F_{S^3 \times I - C}$ and $F_{S^3 \times I - C'}$ satisfy the property that

$$\mathbb{I}_{CFA^-(T_\infty, \lambda)} \boxtimes F_{S^3 \times I - C} \simeq F_C$$

and

$$\mathbb{I}_{CFA^-(T_\infty, \lambda)} \boxtimes F_{S^3 \times I - C'} \simeq F_{C'}.$$

Therefore, to determine these maps, we will compute $\mathbb{I}_{CFA^-(T_\infty, \lambda)} \boxtimes f$ for all basis elements, f, of the morphism space.

$CFA^-(T_\infty, \lambda)$ is a right $\mathcal{A}_\infty$-module over $\mathcal{A}(T^2)$, whose $\mathcal{A}_\infty$-operations can be computed by counting holomorphic disks in the doubly pointed bordered Heegaard diagram shown in Figure 27.

Let α be the single intersection point in $\alpha \cap \beta$. The only non-trivial $\mathcal{A}_\infty$-operations are given by

$$m_{3+j}(\alpha, \rho_3, \overbrace{\rho_{23}, \ldots, \rho_{23}}^{j}, \rho_2) = U^{j+1}\alpha.$$

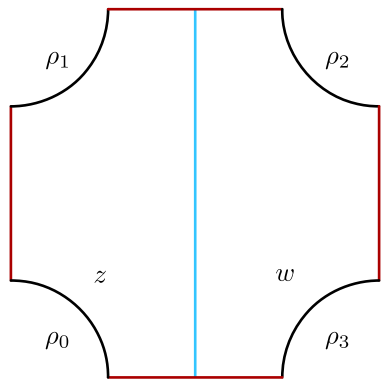


FIGURE 27 A doubly pointed bordered Heegaard diagram for the longitudinal unknot in the solid torus, (T_∞, λ).

Call this module A. Since $A \boxtimes B$ has a single generator, $\alpha \otimes x$, the maps $\mathbb{I}_A \boxtimes f$ are determined by the image of this element.

Lemma A.2.1. *Let ϕ, ψ, g, h be the basis for $H_*(\mathrm{Mor}_{\mathcal{A}(T^2)}(B, B \oplus C))$ described above. Then,*

$$\mathbb{I}_A \boxtimes \phi = (\alpha \otimes x \mapsto \alpha \otimes x)$$

$$\mathbb{I}_A \boxtimes \psi = 0$$

$$\mathbb{I}_A \boxtimes g = (\alpha \otimes x \mapsto \alpha \otimes e)$$

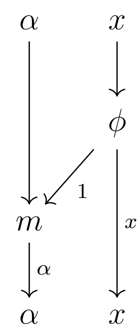
$$\mathbb{I}_A \boxtimes h = 0.$$

Proof. The differentials of C are shown again, as they are needed below.

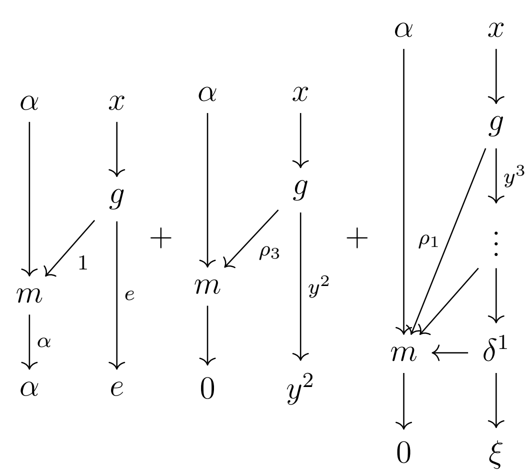
$$\begin{tikzcd}
b \arrow[d, "\rho_1"] & y^1 \arrow[l, "\rho_2"'] & a \arrow[l, "\rho_3"'] \arrow[d, "\rho_1"] \\
y^2 & & y^4 \\
e \arrow[u, "\rho_{123}"] & y^3 \arrow[l, "\rho_2"] & c. \arrow[l, "\rho_3"] \arrow[u, "\rho_{123}"']
\end{tikzcd}$$

Since many of the differentials on the tensor product are trivial, many terms will be forced to be zero. In particular, $\delta^1(x) = \rho_{12}x$, and since there are no $\mathcal{A}_\infty$-operations involving ρ_{12}, any terms involving $\delta^1(x)$ will be zero. $\mathbb{I}_A \boxtimes \psi$ is zero for this reason. We make use of the graphical notion of [LOT18, Chapter 2].

By strict unitality, $\mathbb{I}_A \boxtimes \varphi(\alpha \otimes x)$ has a single term, $\alpha \otimes x$.

$\mathbb{I}_A \boxtimes \phi =$  $= (\alpha \otimes x \mapsto \alpha \otimes x),$

To compute $\mathbb{I}_A \boxtimes g = \mathbb{I}_A \boxtimes (x \mapsto e + \rho_3 y^2 + \rho_1 y^3)$, we first note that, again, by strict unitality, the only term e could contribute is $\alpha \otimes e$. Secondly, any term contributed by $\rho_1 y^3$ will be of the form $m_\ell(\alpha, \overbrace{\rho_{12}, \dots, \rho_{12}}^{i}, \rho_1, \dots) \times \xi$. But, since none of the $\mathcal{A}_\infty$-operations involve ρ_1, any term of this form must be zero. Therefore, all that remains is to check whether $\rho_3 y^2$ contributes any nonzero terms to $\mathbb{I}_A \boxtimes g$. A nonzero term could appear, since all the $\mathcal{A}_\infty$-operations involve ρ_3, but $\delta^1(y^2) = 0$, so no ρ_2 coefficient will appear. Therefore,

$\mathbb{I}_A \boxtimes g =$  $= (\alpha \otimes x \mapsto \alpha \otimes e)$

Consider $\mathbb{I}_A \boxtimes h = \mathbb{I}_A \boxtimes (x \mapsto \rho_1 y^4)$. Much like the previous case, $\rho_1 y^4$ can contribute no nonzero terms, since ρ_1 does not appear in any of the $\mathcal{A}_\infty$-operations.

$$\mathbb{I}_A \boxtimes h = \begin{tikzcd} \alpha \arrow[dd] & v \arrow[d] \\ & h \arrow[dl, "\rho_1"'] \arrow[dd, "y^4"] \\ m \arrow[d, "0"] & \\ \alpha & y^4 \end{tikzcd} = (\alpha \otimes v \mapsto 0).$$

This concludes this computation. $\square$

We can now turn to the maps $F_{S^3 \times I - C}$ and $F_{S^3 \times I - C'}$. Recall from that Dai-Mallick-Stoffregen show that:

$$F_C(v) + F_{C'}(v) = e_1 + e_2,$$

where v is the generator for $HFK^-(U)$. A basis for $H_*(\mathrm{Mor}_{\mathcal{A}(T^2)}(\widehat{CFD}(S^3 - U), \widehat{CFD}(S^3 - J)))$ is given by $\{\phi, \psi, g_i, h_i\}$, where g_i and h_i are maps from $\widehat{CFD}(S^3 - U)$ to the ith box in $\widehat{CFD}(S^3 - J))$ which agree with the maps g and h above. The complex $\widehat{CFD}(S^3 - J)$ is shown in full in Figure 28.

$$\begin{tikzcd} b_i \arrow[d, "\rho_1"] & y_i^1 \arrow[l, "\rho_2"'] & a \arrow[l, "\rho_3"'] \arrow[d, "\rho_1"] & & g_i \arrow[d, "\rho_1"] & z_i^1 \arrow[l, "\rho_2"'] & f_i \arrow[l, "\rho_3"'] \arrow[d, "\rho_1"] & \\ y_i^2 & & y_i^4 & & z_i^2 & & z_i^4 & x \arrow[loop right, "\rho_{12}"] \\ e_i \arrow[u, "\rho_{123}"] & y_i^3 \arrow[l, "\rho_2"'] & c_i \arrow[l, "\rho_3"'] \arrow[u, "\rho_{123}"'] & & j_i \arrow[u, "\rho_{123}"] & z_i^3 \arrow[l, "\rho_2"'] & h_i \arrow[l, "\rho_3"'] \arrow[u, "\rho_{123}"'] & \end{tikzcd}$$

FIGURE 28 The full complex $\widehat{CFD}(S^3 - J)$.

Let v be the single generator for $\widehat{CFD}(S^3 - U)$. By Lemma A.2.1, we can identify which maps $f : \widehat{CFD}(S^3 - U) \to \widehat{CFD}(S^3 - J)$ have the property that $\mathbb{I}_A \boxtimes f$ is homotopic to either F_C or $F_{C'}$.

Again, $F_C(v) + F_{C'}(v) = e_1 + e_2$. Since the maps $\psi, h_1, \ldots, h_4$ satisfy $\mathbb{I}_A \boxtimes \psi = \mathbb{I}_A \boxtimes h_i = 0$, we cannot immediately rule out the possibility that they appear as terms in $F_{S^3 \times I - C}(v)$ and $F_{S^3 \times I - C'}(v)$. Therefore, we can deduce that

$$F_{S^3 \times I - C} + F_{S^3 \times I - C'} = g_1 + g_2 + \varepsilon_1 \psi + \varepsilon_2 \cdot \sum_{i=1}^{4} h_i$$

where $\varepsilon_i \in \{0, 1\}$. Surprisingly, despite the indeterminacy of these maps, this will be sufficient information to compute the sum of the maps associated to the cabled disks.

In summary, we have the following:

Proposition A.2.2. *Let C and C' be the exotic concordances from the unknot to J. Then, the maps $F_{S^3 \times I - C}$ and $F_{S^3 \times I - C'}$ satisfy:*

$$F_{S^3 \times I - C} + F_{S^3 \times I - C'} = g_1 + g_2 + \varepsilon_1 \psi + \varepsilon_2 \cdot \sum_{i=1}^{4} h_i$$

where $\varepsilon_i \in \{0, 1\}$.

Remark A.2.3. We will see that ϵ_2 can be taken to be zero. Once we compute $CFA^-(\mathcal{H}_p) \boxtimes \widehat{CFD}(S^3 - K)$, we will see that the terms contributed by h_i are in the wrong grading. However, this fact has no effect on the final result, so we do not emphasize this.

From complement maps to cabled concordance maps

Having found our candidates for the maps $F_{S^3 \times I - C}$ and $F_{S^3 \times I - C'}$, all that remains is to compute the tensor products of the candidates with the identity map for the $\mathcal{A}_\infty$-module associated to the $(p, 1)$-cable pattern in the solid torus.

A doubly pointed Heegaard diagram $\mathcal{H}_P$ for the $(p, 1)$-cable in the solid torus is shown in Figure 29. Let $A_p = CFA^-(\mathcal{H}_p)$. A_p is generated by $\alpha, \beta_1, \ldots, \beta_{2p-2}$.

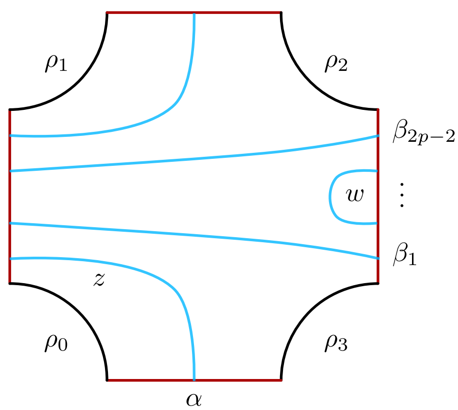


FIGURE 29 A doubly pointed bordered Heegaard diagram for the $(p, 1)$-cable in the solid torus.

Since we are only interested in computing maps $CFA^-(\mathcal{H}_p) \boxtimes \widehat{CFD}(S^3 - U) \to CFA^-(\mathcal{H}_p) \boxtimes \widehat{CFD}(S^3 - J)$ and the homology of $CFA^-(\mathcal{H}_p) \boxtimes \widehat{CFD}(S^3 - U) \simeq CFK^-(U)$ is generated by $\alpha \otimes v$, it is enough to consider the $\mathcal{A}_\infty$-operations involving α.

$$
\begin{aligned}
m_{2+i}(\alpha, \overbrace{\rho_{12}, \ldots, \rho_{12}}^{i}, \rho_1) &= \beta_{2p-i-2} && 0 \leq i \leq p-2 \\
m_{4+i+j}(\alpha, \rho_3, \overbrace{\rho_{23}, \ldots, \rho_{23}}^{j}, \rho_2, \overbrace{\rho_{12}, \ldots, \rho_{12}}^{i}, \rho_1) &= U^{pj+i+1}\beta_{i+1} && 0 \leq i \leq p-2, 0 \leq j \\
m_{3+i}(\alpha, \rho_3, \overbrace{\rho_{23}, \ldots, \rho_{23}}^{j}, \rho_2) &= U^{p(j+1)}\alpha, && 0 \leq j.
\end{aligned}
$$

For the full collection of $\mathcal{A}_\infty$-operations, see [Pet13, Section 4]. As before, we will start by computing $\mathbb{I}_{A_p} \boxtimes f$ for the basis elements of $H_*(\mathrm{Mor}_{\mathcal{A}(T^2)}(B, B \oplus C))$, and then use the fact that $\widehat{CFD}(S^3 - J)$ is isomorphic to $B \oplus C^{\oplus 4}$ to compute $F_{S^3 \times I - C_p}$ and $F_{S^3 \times I - C'_p}$.

Lemma A.3.1. *Let ϕ, ψ, h, and h be the basis of $H_*(\mathrm{Mor}_{\mathcal{A}(T^2)}(B, B \oplus C))$ as computed in Section A.2. Then,*

$$\begin{aligned}
\mathbb{I}_{A_p} \boxtimes \phi &= (\alpha \otimes x \mapsto \alpha \otimes x) \\
\mathbb{I}_{A_p} \boxtimes \psi &= (\alpha \otimes x \mapsto 0) \\
\mathbb{I}_{A_p} \boxtimes g &= \left(\alpha \otimes x \mapsto \alpha \otimes e + \sum_{i=0}^{p-2} \beta_{2p-i-2} \otimes y^3 \right) \\
\mathbb{I}_{A_p} \boxtimes h &= \left(\alpha \otimes x \mapsto \sum_{i=0}^{p-2} \beta_{2p-i-2} \otimes y^4 \right)
\end{aligned}$$

Proof. We proceed as in Lemma A.2.1. This computation is slightly more involved, as there are more $\mathcal{A}_\infty$-operations.

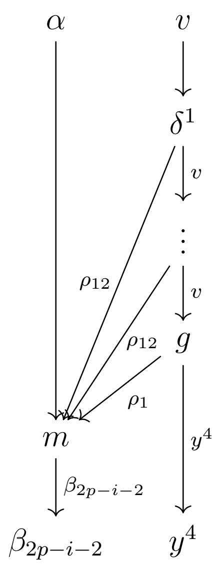
First, for the map $\psi = (x \mapsto \phi_{12} x)$, it must be that $\mathbb{I}_{A_p} \boxtimes \psi = 0$. Since $\delta^1(x) = \rho_{12} x$, any term coming from $\rho_{12} x$ will be of the form $m_k(\rho_{12}, \ldots, \rho_{12})$, which must be zero, as there are no $\mathcal{A}_\infty$-operations only involving ρ_{12}.

For the map $\phi = (x \mapsto x)$, the map $\mathbb{I}_{A_p} \boxtimes \phi$ has a single term by strict unitality:

$$\mathbb{I}_{A_p} \boxtimes \phi = \begin{tikzcd}
\alpha \arrow[dd] & v \arrow[d] \\
 & f \arrow[dl, "1"'] \arrow[dd, "v"] \\
m \arrow[d, "\alpha"] & \\
\alpha & v
\end{tikzcd} = (\alpha \otimes v \mapsto \alpha \otimes v).$$

The map $h = (x \mapsto \rho_1 y^4)$ has many nonzero terms, since there are nontrivial $\mathcal{A}_\infty$-operations involving ρ_{12} and ρ_1, namely $m_{2+i}(\alpha, \overbrace{\rho_{12}, \ldots, \rho_{12}}^{i}, \rho_1) = \beta_{2p-i-2}$ for $0 \leq i \leq p-2$. But these are the only terms that can appear, since $\delta^1(y^4) = 0$. Therefore,

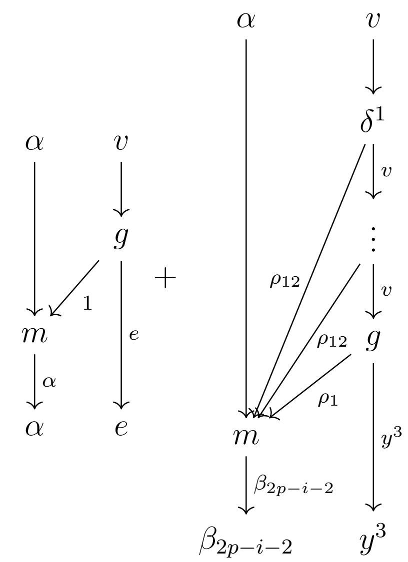
$\mathbb{I}_{A_p} \boxtimes h =$



$$= \left(\alpha \otimes v \mapsto \sum_{i=0}^{p-2} \beta_{2p-i-2} \otimes y^4 \right).$$

The map $g = (x \mapsto e + \rho_3 y^2 + \rho_1 y^3)$ is the most complicated. Once again, by strict unitality, it must be that $\alpha \otimes e$ appears as a term in $\mathbb{I}_B \boxtimes g(\alpha \otimes x)$, and no other terms will come from e. The differential of y^2 is zero, so $\rho_2 y^2$ can only contribute terms of the form $m_{2+i}(\alpha, \overbrace{\rho_{12}, \ldots, \rho_{12}}^{i}, \rho_2) \otimes y^2$, but these are all zero. Finally, the differential of y^3 is $\rho_2 e$, and the differential of e is $\rho_{123} y^2$, so there could potentially be terms of the form $m_{2+i}(\alpha, \overbrace{\rho_{12}, \ldots, \rho_{12}}^{i}, \rho_1) \otimes y^3$, $m_{3+i}(\alpha, \overbrace{\rho_{12}, \ldots, \rho_{12}}^{i}, \rho_1, \rho_2) \otimes e$ or $m_{4+i}(\alpha, \overbrace{\rho_{12}, \ldots, \rho_{12}}^{i}, \rho_1, \rho_2, \rho_{123}) \otimes y^2$, but only the first case introduces non-zero terms. Therefore,

$$\mathbb{I}_{A_p} \boxtimes g =$$



$$= \left(\alpha \otimes v \mapsto \alpha \otimes e + \sum_{i=0}^{p-2} \beta_{2p-i-2} \otimes y^3\right).$$

This concludes the computation. □

In Proposition A.2.2, we found the sum of the maps $F_{S^3\times I-C}$ and $F_{S^3\times I-C'}$. By Theorem 13, the maps F_{C_p} and $F_{C'_p}$ induced by the $(p,1)$-cables of C and C' can be computed as $\mathbb{I}_{A_p} \boxtimes F_{S^3\times I-C}$ and $\mathbb{I}_{A_p} \boxtimes F_{S^3\times I-C'}$ respectively. Lemma A.3.1 can now be applied to give us the candidates for the maps F_{C_p} and $F_{C'_p}$. As before, let $\{\phi, \psi, g_i, h_i\}$ be the basis for $H_*(\mathrm{Mor}_{\mathcal{A}(T^2)}(\widehat{CFD}(S^3 - U), \widehat{CFD}(S^3 - J)))$ from Section A.2.

Since

$$F_{S^3\times I-C} + F_{S^3\times I-C'} = g_1 + g_2 + \varepsilon_1\psi + \varepsilon_2 \cdot \sum_{i=1}^{4} h_i$$

for $\varepsilon_i \in \{0,1\}$, from Lemma A.3.1 it follows that

$$(\mathbb{I}_{A_p} \boxtimes F_{S^3\times I-C} + \mathbb{I}_{A_p} \boxtimes F_{S^3\times I-C'})(\alpha \otimes v) =$$

$$\alpha \otimes (e_1 + e_2) + \left(\sum_{k=1}^{2} \sum_{i=0}^{p-2} \beta_{2p-i-2} \otimes y_k^3 + \varepsilon_2 \cdot \beta_{2p-i-2} \otimes (y_k^4 + z_k^4) \right)$$

Summarizing our results, we have the following.

Proposition A.3.2. *Let C_p and C'_p be the $(p, 1)$-cables of the exotic concordances C and C'. Let $\mathcal{H}_p$ be the doubly pointed bordered Heegaard diagram for the $(p, 1)$-cable pattern shown in Figure 29. Then, the maps F_{C_p} and $F_{C'_p}$ satisfy:*

$$(\mathbb{I}_{A_p} \boxtimes F_{S^3 \times I - C} + \mathbb{I}_{A_p} \boxtimes F_{S^3 \times I - C'})(\alpha \otimes v) =$$

$$\alpha \otimes (e_1 + e_2) + \left(\sum_{k=1}^{2} \sum_{i=0}^{p-2} \beta_{2p-i-2} \otimes y_k^3 + \varepsilon_2 \cdot \beta_{2p-i-2} \otimes (y_k^4 + z_k^4) \right)$$

where ε is either 0 or 1.

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