

SCALAR CURVATURE AND TRANSFER MAPS IN Spin AND Spin^c BORDISM

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DISSERTATION ABSTRACT

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In 1992, Stolz proved that, among simply connected Spin-manifolds of dimension 5 or greater, the vanishing of a particular invariant α is necessary and sufficient for the existence of a metric of positive scalar curvature. More precisely, there is a map $\alpha: \Omega_*^{\text{Spin}} \rightarrow \text{ko}$ (which may be realized as the index of a Dirac operator) which Hitchin established vanishes on bordism classes containing a manifold with a metric of positive scalar curvature. Stolz showed $\ker \alpha$ is the image of a transfer map $\Omega_{*-8}^{\text{Spin}} \text{BPSp}(3) \rightarrow \Omega_*^{\text{Spin}}$. In this paper we prove an analogous result for Spin^c-manifolds and a related invariant $\alpha^c: \Omega_*^{\text{Spin}^c} \rightarrow \text{ku}$. We show that $\ker \alpha^c$ is the sum of the image of Stolz's transfer $\Omega_{*-8}^{\text{Spin}} \text{BPSp}(3) \rightarrow \Omega_*^{\text{Spin}^c}$ and an analogous map $\Omega_{*-4}^{\text{Spin}^c} \text{BSU}(3) \rightarrow \Omega_*^{\text{Spin}^c}$. Finally, we expand on some details in Stolz's original paper and provide alternate proofs for some parts.

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For Grandma Marge

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CHAPTER 1

INTRODUCTION

A major topic of differential geometry today is the study of Riemannian curvatures on a smooth manifold, and how this relates to topological invariants. Perhaps the simplest notion of curvature on a Riemannian manifold (M, g) is the scalar curvature s_g . One way to define curvature is by comparing the volume of a geodesic ball B_ε around a point $x \in M$ with a ball $B_\varepsilon^{(0)}$ of radius ε in Euclidean space. More precisely, $s_g(x)$ is such that

$$\frac{\text{Vol}(B_\varepsilon)}{\text{Vol}(B_\varepsilon^{(0)})} = \left(1 - \frac{s_g(x)}{6(n+2)}\varepsilon^2 + O(\varepsilon^4) \right).$$

In this document we address a fundamental question about Riemannian metrics of positive scalar curvature (psc metrics).

Under which conditions does a manifold admit a psc metric? (1.1)

The case for 2-dimensional manifolds is unique: here the scalar curvature is simply twice the Gauss curvature K_g . The Gauss-Bonnet theorem states

$$\int_M s_g(x) d\sigma_g = 4\pi\chi(M),$$

where σ_g is the volume element corresponding to g and $\chi(M)$ is the Euler characteristic of M . Hence, in 2 dimensions, the existence of metrics of positive scalar curvature is a matter of which surfaces have positive Euler characteristic – namely, S^2 and $\mathbb{R}P^2$.

The existence problem 1.1 has also been resolved for all manifolds of dimension five or greater which are simply connected. Under these conditions, Gromov and Lawson proved a crucial result of surgery theory. We recall some core definitions and facts of this field. By default, all manifolds we mention in this paper are compact.

A surgery on a manifold is defined as follows: let $S^p \times D^{q+1} \rightarrow M$ be an embedding of a sphere S^p with a tubular neighborhood (a “handle”) around it, where $p + q + 1$ is

the dimension of M . The surgery on M given by this embedding produces a manifold M' obtained by removing the interior of $S^p \times D^{p+1} \subset M$ and gluing in the handle $D^{p+1} \times S^q$:

$$M' = (M \setminus \text{int}(S^p \times D^{p+1})) \cup_{S^p \times S^q} (D^{p+q} \times S^q).$$

In short, the effect of this surgery is to “suture up” the p -dimensional hole given by a topologically framed embedding of S^p . We say that the codimension of this surgery is $q + 1$, since this is the codimension of S^p in M . Gromov and Lawson established an important result about Spin manifolds and metrics of positive scalar curvature:

Theorem 1.1 ([GL]). *Let (M, g) be a compact Riemannian manifold with positive scalar curvature (a psc metric). If M' is obtained from M by a surgery of codimension ≥ 3 , then M' carries a psc metric.*

Theorem 1.1 has the following important corollary and related result which, for some classes of manifolds, allows us to only consider bordism classes when discussing the existence of psc metrics. For a review of X -bordism, see [Switz, §12.12].

Corollary 1.2 ([GL]). *Let M be a simply connected Spin manifold of dimension ≥ 5 . If M is Spin-bordant to a manifold M' with a psc metric, then M also has a psc metric.*

For manifolds which are not Spin, Gromov and Lawson proved the following.

Theorem 1.3 ([GL]). *Let M be a simply connected manifold of dimension ≥ 5 which is not Spin. Then M admits a psc metric.*

Remark. There is inconsistency in the literature on the usage of “bordism” versus “cobordism.” Following Rudyak ([Rudyak, 7.19]), we tend to use “bordism” ring and say that two manifolds are “bordant.” This terminology may be more common today

since it allows authors to reserve “cobordism” for the dual cohomology theory (as in [Rudyak, 7.30] and [Switz, §12]). Historically the term “cobordism” was more common for the homology theory (as in [Stong]). This is presumably since “cobordant” has a literal interpretation of two manifolds which together bound another manifold (the French verb “*bordant*” means “to bound”).

We can summarize corollary 1.2 and theorem 1.3 and rephrase them as follows: for M a simply connected manifold of dimension $n \geq 5$,

- if M is Spin, then the existence of a psc metric on M depends only on the Spin bordism class $[M] \in \Omega_n^{\text{Spin}}$;
- if M is *not* Spin, then the existence of a psc metric on M depends only on the oriented bordism class $[M] \in \Omega_n^{\text{SO}}$.

Remark. In particular, for simply connected manifolds of dimension ≥ 5 , the existence of a psc metric is determined by purely topological data. Similarly, a manifold M is Spin if and only if $w_2(M) = 0$ and simply connected if and only if $w_1(M) = 0$ (assuming M is connected). The equivalence classes of Spin-structures on a manifold are in bijection with $H^1(M)$ (with coefficients in \mathbb{Z}_2), hence, if a simply connected manifold admits a Spin-structure, the Spin-structure is unique.

Since at least the early 1960s, it was known that the presence of a Spin-structure on a manifold was had an intimate relationship with that of a psc metric. A Riemannian manifold (M, g) which is Spin has a Dirac operator D_g acting on the bundle of spinors over M , and in 1963 Lichnerowicz ([Lich]) proved the formula

$$D_g^2 = \nabla^* \nabla + \frac{1}{4} s_g, \tag{1.2}$$

where ∇ is a covariant derivative and ∇^* is the formal adjoint. Equation (1.2) implies that if s_g is positive, D_g is invertible (i.e., there are no harmonic spinors on M). If

M is a Spin manifold of dimension n with a psc metric, the Lichnerowicz formula implies that $[M] \in \Omega_n^{\text{Spin}}$ lies in the kernel of the index

$$\alpha: \Omega_n^{\text{Spin}} \rightarrow \text{ko}_n. \quad (1.3)$$

Following Stolz's work, we had a complete answer to the existence problem 1.1 for simply connected manifolds of dimension ≥ 5 . Under these conditions,

- (i) if M is not Spin, then M admits a psc metric ([GL]);
- (ii) if M is Spin, then M has a psc metric if and only if $\alpha([M]) = 0$ ([Stolz]).

Remark. In [GL], Gromov and Lawson proved the rational version of Stolz's result, which can be stated as follows: for a simply connected Spin-manifold of dimension $n \geq 5$, if $\alpha([M]) = 0$, then some multiple $M \# M \# \cdots \# M$ admits a psc metric.

The techniques used by Stolz in his proof are central to the results in this paper. We now describe the geometric idea behind Stolz work. Let $\mathbb{H}P^2$ be the quaternionic projective space with the standard metric g_0 . It is well known that the group of isometries of $\mathbb{H}P^2$ (with respect to g_0) is the projective group $\text{PSp}(3)$ (which is the quotient of $\text{Sp}(3)$ by $\{\pm 1\}$). We now make an elementary observation.

Lemma 1.4. *Let E be the total space of a smooth bundle over a manifold M with fiber $\mathbb{H}P^2$ and structure group $\text{PSp}(3)$. Then E carries a psc metric.*

Proof. Cover M by open sets U_1, \dots, U_n over which E is locally (diffeomorphic to) $D_i \times \mathbb{H}P^2$. Let g_M be the the metric on M and g_0 be the standard metric on $\mathbb{H}P^2$. Since M is compact, s_{g_M} bounded, whence there exists sufficiently small $\lambda > 0$ such that $g := g_M \times \lambda g_0$ has positive scalar curvature on each neighborhood: $s_g = s_{g_M} + \lambda^{-1} s_{g_0}$. Again for λ sufficiently small, this holds over each of the U_i . Finally, since $\text{PSp}(3)$ acts isometrically on $\mathbb{H}P^2$, these local metrics glue to form a global psc metric. \square

Remark. It is an elementary fact that, for any $\lambda > 0$, the scaled metric λg is a Riemannian metric with scalar curvature $s_{\lambda g} = \lambda^{-1} s_g$. In fact, since scalar curvature can be defined as a sum of sectional curvature, one's geometric intuition for surfaces (roughly) shows this for higher dimensions.

Of course $\mathrm{PSp}(3)$ is a compact Lie group, so we have a classifying space $\mathrm{BPSp}(3)$ with a universal $\mathrm{PSp}(3)$ -bundle and the associated $\mathbb{H}\mathbb{P}^2$ -bundle. We now define a transfer map

$$\Psi: \Omega_n^{\mathrm{Spin}} \mathrm{BPSp}(3) \rightarrow \Omega_{n+8}^{\mathrm{Spin}} \quad (1.4)$$

as follows: a class in $\Omega_n^{\mathrm{Spin}} \mathrm{BPSp}(3)$ is represented by a pair (M, f) , where M is an n -dimensional Spin manifold and f is a map $M \rightarrow \mathrm{BPSp}(3)$. Let \widehat{M} be the total space of the pullback via f of the $\mathbb{H}\mathbb{P}^2$ -bundle. Note that $\widehat{M} \rightarrow M$ is an $\mathbb{H}\mathbb{P}^2$ -bundle which also has the structure group $\mathrm{PSp}(3)$ acting isometrically on the fibers. Now we have the following main result from [Stolz].

Theorem 1.5 (Stolz). *For the transfer map $\Psi: \Omega_n^{\mathrm{Spin}} \mathrm{BPSp}(3) \rightarrow \Omega_{n+8}^{\mathrm{Spin}}$ and index map $\alpha: \Omega_n^{\mathrm{Spin}} \rightarrow \mathrm{ko}_n$, we have $\mathrm{im} \Psi = \ker \alpha$.*

Since Lichnerowicz's formula implies $\mathrm{im} \Psi \subseteq \ker \alpha$, Stolz had to show $\ker \alpha \subseteq \mathrm{im} \Psi$. He achieved this by reducing to a homotopy-theoretic problem: to begin with, the transfer Ψ may be considered as the map induced on homotopy of a map of spectra

$$T: \mathrm{MSpin} \wedge \Sigma^8 \mathrm{BPSp}(3)_+ \rightarrow \mathrm{MSpin}. \quad (1.5)$$

The index map α is similarly induced by a map

$$D: \mathrm{MSpin} \rightarrow \mathrm{ko}. \quad (1.6)$$

The algebraic fact that $\alpha([M]) = 0$ if M has a psc metric, for example, translates to statement that DT is nullhomotopic. This means that T factors through the

homotopy fiber $\widehat{\text{MSpin}}$ of D . We write \widehat{T} for this lift and $i: \widehat{\text{MSpin}} \rightarrow \text{MSpin}$ for the inclusion of $\widehat{\text{MSpin}}$. Theorem 1.5 then follows from showing \widehat{T} induces a surjection of homotopy groups.

It is worth emphasizing here that, prior to Stolz's result in [Stolz], the same result had been proven rationally in [GL]. As was well known by that time, the map $\text{MSpin} \rightarrow \text{MSO}$ is a homotopy equivalence for any coefficient ring containing $1/2$. Hence the most difficult component is the 2-primary data. Hence, in this paper, we work entirely with coefficients in $\mathbb{Z}_2 = \mathbb{Z}/(2)$ for homology and cohomology. We may also abuse notation and write $\pi_* X$ to mean $\pi_*(X) \otimes \mathbb{Z}_{(2)}$.

1.1 TWISTED SCALAR CURVATURE FOR Spin^c MANIFOLDS

As we have seen, existence of a Spin structure (and consequently of the Dirac operator) plays a fundamental role in the existence of a psc metric. More generally, a Spin^c -structure implies the existence of a Spin^c Dirac operator.

Let M be a simply-connected manifold with $w_2(M) \neq 0$ (i.e. M is not a spin manifold). Then M has a Spin^c -structure if there is a class $c \in H^2(M; \mathbb{Z})$ which maps to $w_2(M)$ under the mod-2 reduction map $H^2(M; \mathbb{Z}) \rightarrow H^2(M; \mathbb{Z}_2)$. The class c gives a map $c: M \rightarrow \mathbb{C}P^\infty$ and, consequently, a complex line bundle $L \rightarrow M$. We use the notation (M, L) for a manifold with a choice of Spin^c -structure. We notice that a Spin -manifold M has a canonical Spin^c corresponding to the trivial complex line bundle over M .

Assume (M, L) is a non- Spin Spin^c manifold. We choose a Riemannian metric g on M , a Hermitian metric h on L , and a unitary connection A_L on the line bundle

L . These data give the Spin^c Dirac operator $D_{(M,L)}$. Then we have the Lichnerowicz formula [Lich]

$$D_{(M,L)}^2 = \nabla\nabla^* + \frac{1}{4}s_g + \mathcal{R}_L \tag{1.7}$$

here

$$\mathcal{R}_L = \frac{1}{2} \sum_{j < k} F_L(e_j, e_k) \cdot e_j \cdot e_k,$$

where one sums over an orthonormal frame, and where F_L is the curvature of the connection A_L on the line bundle L . We denote

$$s_g^L := s_g + 4\mathcal{R}_L,$$

and say that s_g^L is the L -twisted scalar curvature. Notice that s_g^L is, in fact, a zeroth order operator on spinors, and it depends on a choice of the hermitian metric h on L and on the connection A_L .

1.2 EXISTENCE OF POSITIVE L -TWISTED SCALAR CURVATURE

One of our primary goals is to resolve the following existence question: *under which conditions on a non-Spin Spin^c manifold (M, L) does there exist a Riemannian metric g , a Hermitian metric h on L , and a connection A_L such that the L -twisted scalar curvature s_g^L is positive?*

It turns out that the positivity of the L -twisted scalar curvature s_g^L is also invariant under surgeries of codimension at least three. In particular, this implies that for a simply-connected manifold M the existence of the data (g, h, A_L) as above such that s_g^L is a positive operator, depends only on the Spin^c -cobordism class $[(M, L)] \in \Omega^{\text{Spin}^c}$.

Next, we recall that the index of the Spin^c Dirac operator $D_{(M,L)}$ takes values in ku_* , where ku is the connective cover of the complex K -theory spectrum KU . In

particular, we have a Spin^c -version of the α invariant

$$\alpha^c : \Omega_n^{\text{Spin}^c} \rightarrow \text{ku}_n. \quad (1.8)$$

The formula (1.7) shows that, if the L -twisted scalar curvature s_g^L is positive, then the Spin^c -bordism class $[(M, L)] \in \Omega_n^{\text{Spin}^c}$ is in the kernel of the index map α^c . This leads to the following existence result due to Botvinnik and Rosenberg.

Theorem 1.6 ([BR22; BR18]). *Let (M, L) be a simply connected non-Spin Spin^c manifold of dimension $n \geq 5$. Then M admits a Riemannian psc metric g , a hermitian metric h and a connection A_L such that $s_g^L > 0$ if and only if the Spin^c index $\alpha^c[(M, L)]$ vanishes in ku_n .*

A proof of Theorem 1.6 is given in [BR22] Theorem 3.8 and [BR18] Corollary 32 which relies on deep results in cobordism and homotopy theory. A main goal of this thesis is to give a direct proof of Theorem 1.6 using the same technology developed by Stolz in [Stolz].

As with the Spin case, we will ignore odd-primary data. The proof for the case of odd primes also appears in [BR22]; in this paper we give a direct, constructive proof for the 2 primary case.

1.3 NEW TRANSFER MAP AND MAIN RESULT

We notice that the complex projective spaces $\mathbb{C}P^{2k}$ are non-spin manifolds. In particular, $\mathbb{C}P^2$ has a standard Spin^c -structure given by a complex line bundle $L_0 \rightarrow \mathbb{C}P^2$ with $c_1(L_0) = x$, where x is a generator of $H^2(\mathbb{C}P^2; \mathbb{Z})$. A calculation of $\alpha^c[(\mathbb{C}P^{2k}, L_0)]$ was worked out by Hattori [Hat], in particular, he shows that $\alpha^c[(\mathbb{C}P^2, L_0)] = 0$.

The projective plane $\mathbb{C}P^2$ has a remarkable property – namely, the group $G = \text{SU}(3)$ acts transitively on $\mathbb{C}P^2$, and $\mathbb{C}P^2 = G/H$, where the subgroup $H := \text{SU}(2, 1)$

is the subgroup of elements in $U(2) \times U(1) \subset U(3)$ with determinant 1. We obtain a fiber bundle $p : BH \rightarrow BG$ with fiber $\mathbb{C}P^2$ and structure group $SU(3)$. Thus given a Spin^c -manifold (M, L) and a map $f : M \rightarrow BG$, we can form the associated $\mathbb{C}P^2$ -bundle $\hat{p} : E \rightarrow M$ as a pull-back

$$\begin{array}{ccc} E & \longrightarrow & BH \\ \downarrow \hat{p} & & \downarrow p \\ M & \xrightarrow{f} & BG \end{array}$$

where $E = M \times_f \mathbb{C}P^2$ has dimension $n + 4$ and has a Spin^c structure inherited from the Spin^c structure on M defined by L and the Spin^c structure on $\mathbb{C}P^2$ defined by the bundle L . This construction defines a transfer map

$$\mathcal{T}^c : \Omega_{n-4}^{\text{Spin}^c}(\text{BSU}(3)) \rightarrow \Omega_n^{\text{Spin}^c}. \quad (1.9)$$

We also have a Spin^c -version of the Stolz transfer map:

$$\mathcal{T} : \Omega_{n-8}^{\text{Spin}^c}(\text{BPSp}(3)) \rightarrow \Omega_n^{\text{Spin}^c}. \quad (1.10)$$

Here is the main result of this thesis:

Theorem 1.7. *The transfer maps \mathcal{T} and \mathcal{T}^c are such that*

$$\text{im}(\mathcal{T}) + \text{im}(\mathcal{T}^c) = \ker \alpha^c$$

as abelian groups.

1.4 PROOF OF MAIN RESULT (THEOREM 1.7)

We summarize and expand on Stolz's original proof and give an analogous result for Spin^c bordism involving the transfer map $\mathcal{T}^c : \Omega_{*-4}^{\text{Spin}^c} \text{BSU}(3) \rightarrow \Omega_*^{\text{Spin}^c}$ of (1.9). The Steenrod algebra \mathcal{A} and basic theory of modules and comodules over \mathcal{A} (and

its dual) are needed to describe the transfer maps, so we begin with some algebraic preliminaries in chapter 2. In chapter 3 we give more details about the transfer maps \mathcal{T} and \mathcal{T}^c and compute their induced maps on cohomology. Important lemmas and other results used in the proof of theorem 1.7 appear in chapter 3, with some necessary computations included in the appendices.

The compact Lie group $\mathrm{PSp}(3)$ acts transitively on $S^{11} \subset \mathbb{H}^3$ and this descends to a transitive action on \mathbb{HP}^2 . The stabilizer of $[0 : 0 : 1]$ is $\mathrm{PSp}(2, 1) := \mathrm{P}(\mathrm{Sp}(2) \times \mathrm{Sp}(1))$, giving a fiber bundle $\mathrm{PSp}(2, 1) \rightarrow \mathrm{PSp}(3) \rightarrow \mathbb{HP}^2$. In turn this yields a bundle $\pi: \mathrm{BPSp}(2, 1) \rightarrow \mathrm{BPSp}(3)$ with fiber \mathbb{HP}^2 and structure group $\mathrm{PSp}(3)$. Note that the standard metric on \mathbb{HP}^2 has positive scalar curvature and the structure group $\mathrm{PSp}(3)$ acts via isometries with respect to the standard metric.

Using observations in the last paragraph we obtain the transfer map of Stolz $\mathcal{T}: \Omega_{n-8}^{\mathrm{Spin}} \mathrm{BPSp}(3) \rightarrow \Omega_n^{\mathrm{Spin}}$ as follows: a class in $\Omega_n^{\mathrm{Spin}} \mathrm{BPSp}(3)$ can be written $[M, f]$, where M is an n -dimensional Spin manifold and f is a map $M \rightarrow \mathrm{BPSp}(3)$. Let \widehat{M} be the total space of the pullback via f of our \mathbb{HP}^2 -bundle. Now $[\widehat{M}]$ is a class in $\Omega_{n+8}^{\mathrm{Spin}}$, and the transfer map \mathcal{T} takes $[M, f]$ to $[\widehat{M}]$. Now the natural inclusion $\mathrm{MSpin} \rightarrow \mathrm{MSpin}^c$ lets us easily translate this to Spin^c -bordism, although we abuse notation slightly and reuse the map names, e.g. $\mathcal{T}: \Omega_{n-8}^{\mathrm{Spin}^c} \mathrm{BPSp}(3) \rightarrow \Omega_n^{\mathrm{Spin}^c}$. This notation is different from that of Stolz, who writes Ψ instead of \mathcal{T} .

Analogously to the quaternionic case, the special unitary group $\mathrm{SU}(3)$ acts transitively on $S^5 \subset \mathbb{C}^3$ yielding a transitive action on \mathbb{CP}^2 . As before, the stabilizer of $[0 : 0 : 1]$ is $\mathrm{SU}(2, 1) := \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1))$ and we hence obtain a bundle $\mathrm{BSU}(2, 1) \rightarrow \mathrm{BSU}(3)$ with fiber \mathbb{CP}^2 . This bundle admits a Spin^c structure, giving a map $\mathrm{BSU}(3) \rightarrow \mathrm{BSpin}^c$. In this case we get a transfer map $\mathcal{T}: \Omega_{n-4}^{\mathrm{Spin}^c} \mathrm{BSU}(3) \rightarrow \Omega_n^{\mathrm{Spin}^c}$ with the property $\mathrm{im} \mathcal{T}^c \subseteq \ker \alpha^c$. Unfortunately, the analogy with the Spin case ends here, as the reverse containment fails. Moreover, we will see that neither \mathcal{T}

nor T^c are surjective onto $\ker \alpha^c$; instead we combine the two transfers and show $\text{im } \mathcal{T} + \text{im } T^c = \ker \alpha^c$.

Via the classical Pontrjagin-Thom construction $\Omega_n^{\text{Spin}^c}(X)$ can be identified with $\pi_n(\text{MSpin}^c \wedge X_+)$. The transfer map \mathcal{T} is the map on homotopy groups induced by a map of spectra $T: \text{MSpin}^c \wedge \Sigma^8 \text{BPSp}(3)_+ \rightarrow \text{MSpin}^c$; similarly \mathcal{T}^c is induced by a map $T^c: \text{MSpin}^c \wedge \Sigma^4 \text{BSU}(3) \rightarrow \text{MSpin}^c$. The α^c invariant corresponds to a spectrum map $D^c: \text{MSpin}^c \rightarrow \text{ku}$, and we have the following diagram of maps, where $i: \widehat{\text{MSpin}}^c \rightarrow \text{MSpin}^c$ is inclusion of the homotopy fiber of D^c and $\mu: \text{MSpin}^c \wedge \text{MSpin}^c \rightarrow \text{MSpin}^c$ is the ring spectrum product map.

$$\begin{array}{ccc}
& & \widehat{\text{MSpin}}^c \\
& \nearrow^{\widehat{T} \vee \widehat{T}^c} & \downarrow i \\
\text{MSpin}^c \wedge (\Sigma^8 \text{BPSp}(3)_+ \vee \Sigma^4 \text{BSU}(3)_+) & \longrightarrow & \text{MSpin}^c \wedge \text{MSpin}^c \xrightarrow{\mu} \text{MSpin}^c \\
& \searrow_{T \vee T^c} & \downarrow D^c \\
& & \text{ku}
\end{array}$$

The unlabeled map is $1 \wedge (t \vee t^c)$, where t and t^c are the respective bundle transfer maps composed with the Thom map. That is,

- $t: \Sigma^8 \text{BPSp}(3)_+ \rightarrow \text{MSpin}^c$ is the composition $t = \mathcal{M} \circ \mathcal{T}$;
- $\mathcal{T}: \Sigma^8 \text{BPSp}(3)_+ \rightarrow M(-\tau)$ is the bundle transfer map of [Board, §VI.6],
 - where $-\tau$ is the stable complement of the bundle of tangent vectors along the fibers of $\text{BPSp}(2, 1) \rightarrow \text{BPSp}(3)$;
- $\mathcal{M}: M(-\tau) \rightarrow \text{MSpin}^c$ is the Thom map induced by the classifying map of $-\tau$.

Our main result can now be restated as follows:

Theorem 1.8. *On homotopy groups, $T \vee T^c$ induces a surjection*

$$\Omega_*^{\text{Spin}^c} (\Sigma^8 \text{BPSp}(3)_+ \vee \Sigma^4 \text{BSU}(3)_+) \rightarrow \pi_* \widehat{\text{MSpin}^c}.$$

Remark 1.9. After localizing at 2, the spectrum MSpin^c splits into a wedge of suspensions of ku with other spectra. The map D^c is simply projection onto the lowest degree copy of ku . Details can be found in [HH, §5]. The analogous statements hold for MSpin and ko . These facts give a particularly nice description of $\widehat{\text{MSpin}^c}$ and $\widehat{\text{MSpin}}$ which make these homotopy fibers reasonable objects to work with.

To prove theorem 1.8 (at the prime 2), we use the Adams spectral sequence along with homological data. More precisely, we will show that $\widehat{T} \vee \widehat{T}^c$ induces a split surjection of \mathcal{A}_* -comodules, hence a surjection of the E_2 -terms

$$\text{Ext}_{\mathcal{A}_*}^{s,t} (\mathbb{Z}_2, H_* \text{MSpin}^c \otimes (H_{*-8} \text{BPSp}(3) \oplus H_{*-4} \text{BSU}(3))) \rightarrow \text{Ext}_{\mathcal{A}_*}^{s,t} (\mathbb{Z}_2, H_* \widehat{\text{MSpin}^c}) \quad (1.11)$$

which converge to

$$\pi_* (\text{MSpin}^c \wedge (\Sigma^8 \text{BPSp}(3)_+ \vee \Sigma^4 \text{BSU}(3)_+)) \otimes \mathbb{Z}_{(2)} \rightarrow \pi_* (\widehat{\text{MSpin}^c}) \otimes \mathbb{Z}_{(2)} \quad (1.12)$$

It turns out that the first of these spectral sequences has no nontrivial differentials, hence the surjection (1.11) at the level of E_2 pages implies the surjection (1.12) on the E_∞ pages.

To prove theorem 1.7, then, it remains to show $\widehat{T}_* \vee \widehat{T}_*^c$ is a split surjection of \mathcal{A}_* -comodules. Several algebraic maneuvers make our task easier: a useful change-of-rings construction reduces our task to working with comodules over a sub-coalgebra $E(1)_* \subset \mathcal{A}_*$. We also prefer to dualize to cohomology in order to work with \mathcal{A} -modules and $E(1)$ -modules rather than comodules, although most results are stated in terms of comodules where possible. Finally, we give an algebraic result in Theorem 2.14

which provides an easy way to show that an injection of $E(1)$ -modules (or a surjection of $E(1)_*$ -comodules) is split. Following [Pete], we also know

$$H_*\mathrm{BSpin}^c = \mathbb{Z}_2[x_i^2, x_j : \alpha(i) < 3, \alpha(j) \geq 3],$$

where $\alpha(n)$ is the number of nonzero terms in the base-2 expansion of n .

Theorem 1.10. *We have a commuting diagram as shown, where $\rho: \mathrm{BSpin}^c \rightarrow \mathrm{BO}$ is the projection and D, U and ι are orientation classes. We make some observations about this diagram. We have the standard identification $H_*H\mathbb{Z}_2 = \mathbb{Z}_2[\zeta_1, \zeta_2, \dots]$, and,*

$$\begin{array}{ccc} \mathrm{MSpin}^c & \xrightarrow{D} & \mathrm{ku} \\ \downarrow M\rho & & \downarrow \iota \\ \mathrm{MO} & \xrightarrow{U} & H\mathbb{Z}_2 \end{array}$$

via the Thom isomorphism, we can write $H_*\mathrm{MO} = \mathbb{Z}_2[x_1, x_2, \dots]$, where $\deg(x_i) = i$. The spectrum MO is $H\mathbb{Z}_2$ -oriented via a Thom class (i.e., an orientation) $\mathcal{U}: \mathrm{MO} \rightarrow H\mathbb{Z}_2$. Similarly $D^c: \mathrm{MSpin}^c \rightarrow \mathrm{ku}$ can be considered as the ku -orientation of MSpin^c . We use the Thom isomorphism to identify $H_*\mathrm{MO}$ with $H_*\mathrm{BO} = \mathbb{Z}_2[x_1, x_2, x_3, \dots]$, where $\deg(x_i) = i$. The dual Steenrod algebra \mathcal{A}_* is the polynomial ring $\mathbb{Z}_2[\xi_1, \xi_2, \dots]$, although we prefer to use the Hopf conjugate generators and write $\mathcal{A}_* = \mathbb{Z}_2[\zeta_1, \zeta_2, \dots]$ (here $\deg(\xi_i) = \deg(\zeta_i) = 2^i - 1$). Of course $\mathcal{A}_* = H_*H\mathbb{Z}_2$, and the inclusion $\mathrm{ko} \rightarrow H\mathbb{Z}_2$ induces on homology the injection

$$\mathbb{Z}_2[\zeta_1^2, \zeta_2^2, \zeta_3, \zeta_4, \dots] \rightarrow \mathbb{Z}_2[\zeta_1, \zeta_2, \zeta_3, \zeta_4, \dots].$$

1. The maps $M\rho$ and ι induce monomorphisms on homology.
2. We can identify $H_*\mathrm{MSpin}^c = \mathrm{im}(M\rho_*) = \mathbb{Z}_2[x_i^2, x_j : \alpha(i) < 3, \alpha(j) \geq 3]$.
3. Define $P \subset H_*\mathrm{MO}$ as $P = \mathbb{Z}_2[x_{2^{i-1}} : i \geq 1]$. Then U_* maps P isomorphically onto $H_*H\mathbb{Z}_2$.

4. Identifying $P \subset H_*\mathrm{MSpin}^c$, let

$$R = P \cap H_*\mathrm{MSpin}^c = \mathbb{Z}_2[x_1^2, x_3^2, x_7, x_{15}, \dots] \subset H_*\mathrm{MSpin}^c.$$

Then D_* maps R isomorphically onto $H_*\mathrm{ku}$.

Proof. The first claim is immediate from the fact that the corresponding maps on cohomology are quotients (see [Stong], for example). The computation of $H_*\mathrm{MSpin}^c$ and $H^*\mathrm{MSpin}^c$ can be found in [Switz]. The third claim is [Stolz, Corollary 4.7], and the last follows from the nontrivial fact that D_* admits a splitting $H_*\mathrm{ku} \rightarrow H_*\mathrm{MSpin}^c$ ([Hat], [Stolz2]). \square

There is a useful functorial construction for MSpin^c -module spectra which identifies H_*X with $\mathcal{A}_* \square_{E(1)_*} \overline{H_*X}$, where $\overline{H_*X} := \mathbb{Z}_2 \otimes_R H_*X$ is a subalgebra of $H_*\mathrm{MSpin}^c$ which maps isomorphically to $H_*\mathrm{ku}$ via D_*^c . The algebra $E(1)$ is generated by $Q_0 = \mathrm{Sq}^1$ and $Q_1 = \mathrm{Sq}^1 \mathrm{Sq}^2 + \mathrm{Sq}^2 \mathrm{Sq}^1$; the cotensor product \square is also standard notation and also defined in chapter 2. (Note that $R = \mathcal{A}_* \square_{E(1)_*} \mathbb{Z}_2$.) This construction, which is detailed in theorem 2.15, is functorial in the sense that maps $f: H_*X \rightarrow H_*Y$ induce maps $\bar{f}: \overline{H_*X} \rightarrow \overline{H_*Y}$. This method allows us to work with modules over $E(1)$ (or comodules over $E(1)_*$) rather than using the whole Steenrod algebra.

Applying this construction to the diagram which appeared just before theorem 1.8 yields the following diagram.

$$\begin{array}{c}
\overline{H_*\mathrm{MSpin}^c} \otimes (H_*\Sigma^8\mathrm{BPSp}(3)_+ \oplus H_*\Sigma^4\mathrm{BSU}(3)_+) \\
\downarrow 1 \otimes (t_* \oplus t_*^c) \\
\overline{H_*\mathrm{MSpin}^c} \otimes H_*\mathrm{MSpin}^c \\
\downarrow \mu \\
\overline{H_*\mathrm{MSpin}^c} \\
\downarrow \overline{D}_* \\
\overline{H_*\mathrm{ku}}
\end{array}$$

An important observation is that $\overline{H_*\mathrm{ku}} = \mathbb{Z}_2$, and we can identify \overline{D}_* with the augmentation map of $\overline{H_*\mathrm{MSpin}^c}$. We can summarize these maps in more compact notation by writing $H_1 = H_*\Sigma^8\mathrm{BPSp}(3)_+$ and $H_2 = H_*\Sigma^8\mathrm{BSU}(3)_+$.

$$\begin{array}{ccccccc}
& & \overline{T}_* & & & & \\
& & \curvearrowright & & & & \\
\overline{H_*\mathrm{MSpin}^c} \otimes (H_1 \oplus H_2) & \longrightarrow & \overline{H_*\mathrm{MSpin}^c} \otimes H_*\mathrm{MSpin}^c & \xrightarrow{\mu} & \overline{H_*\mathrm{MSpin}^c} & \xrightarrow{\overline{D}_*} & \overline{H_*\mathrm{ku}} \\
& & \downarrow 1 \otimes p & & \parallel & & \\
& & \overline{H_*\mathrm{MSpin}^c} \otimes \overline{H_*\mathrm{MSpin}^c} & \xrightarrow{m} & \overline{H_*\mathrm{MSpin}^c} & &
\end{array}$$

Recall that $\overline{M} = \mathbb{Z}_2 \otimes_R M$, and here $p: M \rightarrow \overline{M}$ is $x \mapsto 1 \otimes x$. In these terms μ is given by $(1 \otimes x) \otimes y \mapsto 1 \otimes xy$, while $m \circ (1 \otimes p)$ is $(1 \otimes x) \otimes y \mapsto (1 \otimes x) \otimes (1 \otimes y) \mapsto 1 \otimes xy$. It follows that the diagram commutes and \overline{T}_* is equal to the composition

$$\overline{H_*\mathrm{MSpin}^c} \otimes (H_1 \oplus H_2) \rightarrow \overline{H_*\mathrm{MSpin}^c} \otimes \overline{H_*\mathrm{MSpin}^c} \xrightarrow{m} \overline{H_*\mathrm{MSpin}^c}.$$

Lemma 1.11. *To show that T_* is surjective onto $\ker D_*$, it suffices to show*

$$H_n(\Sigma^8\mathrm{BPSp}(3)_+ \vee \Sigma^4\mathrm{BSU}(3)_+) \rightarrow QH_n\mathrm{MSpin}^c$$

is surjective for $n \geq 4$ with $n \neq 2^k \pm 1$, where $QH_n\mathrm{MSpin}^c$ is generated by indecomposable elements in $H_n\mathrm{MSpin}^c$

Proof. First we claim that it suffices to show

$$H_* (\Sigma^8 \text{BPSp}(3)_+ \vee \Sigma^4 \text{BSU}(3)_+) \rightarrow Q\overline{H_* \text{MSpin}^c}$$

is surjective. This amounts to the obvious fact that $\ker \overline{D}_*$ is generated (over \mathbb{Z}_2) by elements in $\overline{H_* \text{MSpin}^c}$ with at least one indecomposable factor. These generators are of the form $\mu(x \otimes y)$ for some $x \in \overline{H_* \text{MSpin}^c}$ and $y \in Q\overline{H_* \text{MSpin}^c}$ (note that x may be 1 here). Provided y is in the image of t_* , $\mu(x \otimes y)$ is in the image of \overline{T}_* . The result now follows from Lemma 1.12. \square

Lemma 1.12. *The projection $p: H_* \text{MSpin}^c \rightarrow \overline{H_* \text{MSpin}^c}$ induces a homomorphism $Qp: QH_* \text{MSpin}^c \rightarrow Q\overline{H_* \text{MSpin}^c}$ such that $Qp: QH_n \text{MSpin}^c \rightarrow Q\overline{H_n \text{MSpin}^c}$ is an isomorphism of abelian groups for $n \geq 4$ with $n \neq 2^k \pm 1$. Further, $Q\overline{H_n \text{MSpin}^c} = 0$ for $n < 4$ or $n = 2^k \pm 1$.*

Proof. Recall that $H_* \text{BSpin}^c = \mathbb{Z}_2[x_i^2, x_j : \alpha(i) < 3, \alpha(j) \geq 3]$ and we defined the subring $R = \mathbb{Z}_2[p_1^2, p_2^2, p_3, p_4, \dots]$. Here p_i is the image of x_{2^i-1} under the Thom isomorphism, so identifying $H_* \text{MSpin}^c$ with $H_* \text{BSpin}^c$ (as algebras), we can write $R = \mathbb{Z}_2[x_1^2, x_3^2, x_7, x_{15}, \dots]$. It is now easy to see that $\overline{H_* \text{BSpin}^c}$ has generators x_i^2 and x_j of $H_* \text{BSpin}^c$ except for x_1^2 , x_3^2 , and x_{2^n-1} for $n \geq 3$ (since $\alpha(2^n - 1) = n$). This shows $Q\overline{H_n \text{BSpin}^c} = 0$ for $n < 4$ and $n = 2^k - 1$; to finish the lemma, use the ring identification $H^* \text{BSpin}^c = \mathbb{Z}_2[w_n : n \geq 2, n \neq 2^k + 1]$ to show $Q\overline{H_n \text{MSpin}^c}$ is trivial for all $n = 2^k + 1$. \square

Remark. We pause here to make an important note: as an \mathcal{A} -module, $H^* \text{BSpin}^c$ is the quotient of $H^* \text{BSO}$ by the ideal generated by Sq^3 ([Stong, §XI]). As an algebra, one can show with the Adèm relations that $H^* \text{BSpin}^c$ is the polynomial ring with generators w_i for all $i \geq 2$ with $i \neq 2^k + 1$. This identification does not respect the full \mathcal{A} -module structure, however it respects the actions by Q_0 and Q_1 . The analogous

fact is true for the Spin case and the actions by Sq^1 and Sq^2 ([Stolz]). An explicit example of how this can cause issues is given in remark B.5.

After dualizing to cohomology, theorem 1.7 now follows from the following lemma.

Lemma 1.13. *The map*

$$PH^n BSpin^c \rightarrow H^n(\Sigma^8 BPSp(3)_+ \vee \Sigma^4 BSU(3)_+)$$

is injective for $n \geq 4$, $n \neq 2^k \pm 1$.

Proof. The proof of [Stolz, proposition 7.5] given by Stolz easily shows that the primitive generator y_n maps injectively for $n \geq 8$, $\alpha(n) \geq 2$, since in these degrees z_n is the image of y_n under $H^* BSpin^c \rightarrow H^* BSpin$ (the primitive generators z_n and y_n are detailed in appendix B). The remaining primitives are y_6 and y_{2^k} for $k \geq 2$. Since $\alpha(6) = 2$, $y_6 = s_{3,3} = w_2^3 + w_2 w_4 + w_6$. Of course $\alpha(2^k) = 1$, so $y_{2^k} = w_2^{2^{k-1}}$. In either case, y_n maps nontrivially, as witnessed by w_2 following theorem C.1. \square

Lemmas 1.13 and 1.11 together prove theorem 1.8 and thus complete the proof of theorem 1.7.

CHAPTER 2

ALGEBRAIC BACKGROUND

In this section we provide an algebraic overview of the Steenrod algebra and its dual. We then discuss certain subalgebras and quotient algebras relevant to our purposes. We will assume all (co)homology has coefficients in \mathbb{Z}_2 unless otherwise stated.

2.1 THE STEENROD ALGEBRA AND ITS DUAL

We denote the Steenrod algebra as $\mathcal{A} := H^*H\mathbb{Z}_2$. As an algebra, \mathcal{A} is generated by the Steenrod squares Sq^n subject to the Adem relations: for $m < 2n$,

$$\text{Sq}^m \text{Sq}^n = \sum_{0 \leq i \leq \lfloor m/2 \rfloor} \binom{n-i-1}{m-2i} \text{Sq}^{m+n-i} \text{Sq}^i.$$

Now \mathcal{A} is a Hopf algebra, and the coproduct $\Delta: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ is characterized as the algebra homomorphism satisfying the Cartan formula $\Delta(\text{Sq}^k) = \sum_{i+j=k} \text{Sq}^i \otimes \text{Sq}^j$. We write \mathcal{A}_* for the dual algebra. Milnor established that \mathcal{A}_* is the polynomial ring (over \mathbb{Z}_2) with generators ξ_i of degree $2^i - 1$ for all $i > 0$ (we also use the convention $\xi_0 = 1$). We will abuse notation and reuse Δ to denote the coproduct in \mathcal{A}_* :

$$\Delta(\xi_k) = \sum_{i+j=k} \xi_i^{2^j} \otimes \xi_j.$$

Similarly, we use μ for the product and χ for the conjugate map in either case. The latter is characterized by commutativity of the following diagrams.

The Steenrod algebra acts naturally on the \mathbb{Z}_2 -cohomology of any space and satisfies the following properties: for a space X with $x, y \in H^*X$

1. $\text{Sq}^n(x) = x^2$ if $\deg(x) = n$
2. $\text{Sq}^n(x) = 0$ if $\deg(x) < n$

$$\begin{array}{ccc}
& \mathcal{A} \otimes \mathcal{A} & \xrightarrow{1 \otimes \chi} & \mathcal{A} \otimes \mathcal{A} \\
& \nearrow & & \searrow \\
\mathcal{A} & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \mathcal{A} \\
& \searrow & & \nearrow & \\
& \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\chi \otimes 1} & \mathcal{A} \otimes \mathcal{A}
\end{array}
\qquad
\begin{array}{ccc}
& \mathcal{A}_* \otimes \mathcal{A}_* & \xrightarrow{1 \otimes \chi} & \mathcal{A}_* \otimes \mathcal{A}_* \\
& \nearrow & & \searrow \\
\mathcal{A}_* & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \mathcal{A}_* \\
& \searrow & & \nearrow & \\
& \mathcal{A}_* \otimes \mathcal{A}_* & \xrightarrow{\chi \otimes 1} & \mathcal{A}_* \otimes \mathcal{A}_*
\end{array}$$

3. $\text{Sq}(xy) = \text{Sq}(x) \text{Sq}(y)$, where $\text{Sq} = 1 + \text{Sq}^1 + \text{Sq}^2 + \dots$.

As we mentioned, third property is also called the Cartan formula. Note that $\text{Sq}(x)$ is always a finite sum due to the second property. An instructive application of the properties of the Sq^n is when x is the generator of $H^*\mathbb{RP}^\infty$. In this case $\text{Sq}(x) = 1 + x + x^2$, and this completely determines the action of \mathcal{A} on $H^*\mathbb{RP}^\infty$. Via the splitting principle, one can use this to determine the action on BO.

As an algebra, \mathcal{A} is generated by the classes Sq^n for $n \geq 0$, but this is not a minimal generating set: for example, $\text{Sq}^3 = \text{Sq}^1 \text{Sq}^2$. We demonstrate now that \mathcal{A} is generated (as an algebra) by only the classes Sq^{2^n} for all n . First we need a useful fact of arithmetic modulo 2.

Lemma 2.1. *Given positive integers a, b with base-2 expansions $a = a_0 + 2a_1 + \dots + 2^n a_n$ and $b = b_0 + 2b_1 + \dots + 2^n b_n$, the modulo-2 binomial coefficient $\binom{a}{b}$ factors as*

$$\binom{a_0 + 2a_1 + 2^2 a_2 + \dots + 2^n a_n}{b_0 + 2b_1 + 2^2 b_2 + \dots + 2^n b_n} = \binom{a_0}{b_0} \binom{a_1}{b_1} \binom{a_2}{b_2} \dots \binom{a_n}{b_n}.$$

Proof. By definition or otherwise, $\binom{a}{b}$ is the coefficient of x^b in $(1+x)^a$. Now the Frobenius map gives

$$\begin{aligned}
(1+x)^a &= (1+x)^{a_0} (1+x)^{2a_1} (1+x)^{2^2 a_2} \dots (1+x)^{2^n a_n} \\
&= (1+x)^{a_0} (1+x^2)^{a_1} (1+x^{2^2})^{a_2} \dots (1+x^{2^n})^{a_n}.
\end{aligned}$$

It follows from the Euclidean division algorithm that the only way to obtain a x^b term is if each factor $(1 + x^{2^k})^{a_k}$ contributes $x^{2^k b_k}$. Hence $\binom{a}{b}$ is nonzero precisely if every factor $\binom{a_i}{b_i}$ is nonzero. \square

Notice that a factor $\binom{a_i}{b_i}$ is 0 if and only if $a_i = 0$ and $b_i = 1$. Hence a useful interpretation of lemma 2.1 is that $\binom{a}{b} = 1$ if and only if every digit in the base-2 expansion of a is greater than or equal to the corresponding digit in the base-2 expansion of b .

Theorem 2.2. *As an algebra, \mathcal{A} is generated by the classes Sq^{2^n} for all n .*

Proof. If n is not a power of 2, pick the largest integer k such that $2^k < n$. Let $b = 2^k$ and $a = n - b$. Then

$$\text{Sq}^a \text{Sq}^b = \sum_c \binom{b-c-1}{a-2c} \text{Sq}^{a+b-c} \text{Sq}^c.$$

When $c = 0$, we have the term $\binom{b-1}{a} \text{Sq}^{a+b} \text{Sq}^0 = \binom{b-1}{a} \text{Sq}^n$. Now $b - 1 = 2^k - 1$ has the base 2 expansion $1 + 2 + 2^2 + \dots + 2^{k-1}$, and, by construction, $a = n - b$ can only contain summands 2^i with $i \leq k - 1$. By corollary (2.1), $\binom{b-1}{a} = 1$, and thus

$$\text{Sq}^a \text{Sq}^b = \text{Sq}^n + \sum_{c>0} \binom{b-c-1}{a-2c} \text{Sq}^{a+b-c} \text{Sq}^c.$$

That is,

$$\text{Sq}^n = \text{Sq}^{n-2^k} \text{Sq}^{2^k} + \sum_{c>0} \binom{2^k-c-1}{n-2^k-2c} \text{Sq}^{n-c} \text{Sq}^c.$$

Thus Sq^n can be written in terms of Sq^{2^k} and Sq^i with $i < n$. To avoid infinite descent, we see that Sq^n can be written using only Sq^{2^j} for various j . \square

Given a sequence $I = (i_1, i_2, \dots, i_r)$, we use the shorthand $\text{Sq}^I = \text{Sq}^{i_1} \text{Sq}^{i_2} \dots \text{Sq}^{i_r}$. The sequence I is called **admissible** if $i_j \geq 2i_{j+1}$ for all j ; hence the monomials Sq^I

are precisely those with no nontrivial Adém relations. It follows from the Adém relations that \mathcal{A} is generated as a vector space by the monomials Sq^I for all admissible sequences I . Serre established the linear independence of these classes, hence they form a basis known as the Serre-Cartan basis for \mathcal{A} .

The coproduct also has a particularly elegant description using the Sq^I notation: given any sequence $I = (i_1, \dots, i_n)$, we have

$$\Delta(\text{Sq}^I) = \sum_{I_1 + I_2 = I} \text{Sq}^{I_1} \otimes \text{Sq}^{I_2}, \quad (2.1)$$

where $I_1 + I_2$ denotes component-wise addition. This formula becomes clear upon examination: applying Δ to each factor, the terms in $\Delta(\text{Sq}^I)$ correspond to a choice of one term from each $\Delta(\text{Sq}^{i_j})$. These correspond to a choice of number a_j between 0 and i_j which indicates the term $\text{Sq}^{a_j} \otimes \text{Sq}^{b_j}$ in $\Delta(\text{Sq}^{i_j})$ (where $a_j + b_j = i_j$). Hence the terms of Sq^I correspond to any and all choices of sequences which are component-wise between $(0, 0, \dots, 0)$ and (i_1, i_2, \dots, i_n) . Each such sequence of course has a complimentary sequence such that their sum is I .

Remark 2.3. In the coproduct formula (2.1), it is worth emphasizing that none of I , I_1 , or I_2 are required to be admissible. Also, we are *adding* sequences together rather than concatenating: for example, $\Delta(\text{Sq}^{3,5,2})$ will a priori have 72 terms including $\text{Sq}^{3,5,2} \otimes 1$, $\text{Sq}^{1,3,2} \otimes \text{Sq}^{2,2,0}$, and $\text{Sq}^{1,1,1} \otimes \text{Sq}^{2,4,1}$. Of course, many of these terms vanish due to nontrivial relations. The second and third terms mentioned vanish since $\text{Sq}^{1,3,2} = 0$ and $\text{Sq}^{1,1,1} = 0$.

2.2 A CLOSER LOOK AT \mathcal{A}_*

One must be careful when taking the dual of an infinite dimensional space. Here, \mathcal{A}_* is the subspace of the $\text{Hom}_{\mathbb{Z}_2}(\mathcal{A}, \mathbb{Z}_2)$ generated by functionals with finite-

dimensional support. Since \mathcal{A} is infinite dimensional, the true linear dual has strictly larger dimension, but we nonetheless refer to \mathcal{A}_* as simply “dual” to \mathcal{A} and vice versa. This is possible since, by only allowing functionals with finite dimensional support, we ensure that \mathcal{A}_* is isomorphic to \mathcal{A} as a graded vector space.

Define $I_n = (2^{n-1}, 2^{n-2}, \dots, 4, 2, 1)$ for $n > 0$ and $I_0 = (0)$ (that is, $\text{Sq}^{I_0} = \text{Sq}^0 = 1$). For all $n \geq 0$, one can define ξ_n as the linear dual to Sq^{I_n} with respect to the Serre-Cartan basis for \mathcal{A} . That is, $\langle \xi_n, \text{Sq}^I \rangle$ is 1 if $I = I_n$ and is 0 if I is any other admissible sequence. Of course, this also defines $\langle \xi_n, \text{Sq}^I \rangle$ for any I after applying the Adem relations. We will write $\xi_0 = 1$. We noted earlier that cocommutativity of \mathcal{A} ensures that \mathcal{A}_* is commutative. Some relatively simple combinatorics will show that, for every n , the number of admissible sequences of degree n corresponds bijectively to the number of partitions of n using the numbers $2^k - 1 = \deg(I_k)$. This turns out to not give false hope: Serre showed that, as an algebra, \mathcal{A}_* is the polynomial ring $\mathbb{Z}_2[\xi_1, \xi_2, \dots]$. Note that $\deg(\xi_n) = 2^n - 1$, and as with the notation Sq^I , we write $\xi^J = \xi_1^{j_1} \xi_2^{j_2} \dots \xi_r^{j_r}$ for any a sequence $J = (j_1, \dots, j_r)$ of nonnegative integers. Note that using admissible sequences correspond to Serre-Cartan basis elements for \mathcal{A} , but for \mathcal{A}_* it is more convenient to use arbitrary sequences (with finite length and nonnegative integer entries).

The comultiplication on \mathcal{A}_* is given by

$$\mu^*(\xi_n) = \sum_{i=0}^n \xi_{n-i}^{2^i} \otimes \xi_i = \xi_n \otimes 1 + \xi_{n-1}^2 \otimes \xi_1 + \xi_{n-2}^4 \otimes \xi_2 + \dots + \xi_1^{2^{n-1}} \otimes \xi_{n-1} + 1 \otimes \xi_n.$$

Sometimes it is convenient to use the Hopf conjugates $\zeta_n := \chi(\xi_n)$. Then the coprod-

uct is

$$\mu^*(\zeta_n) = \sum_{i=0}^n \zeta_i \otimes \zeta_{n-i}^{2^i} = 1 \otimes \zeta_n + \zeta_1 \otimes \zeta_{n-1}^2 + \zeta_2 \otimes \zeta_{n-2}^4 + \cdots + \zeta_{n-1} \otimes \zeta_1^{2^{n-1}} + \zeta_n \otimes 1.$$

In theory, one can compute the coproduct directly by dualizing from the product on \mathcal{A} . Consider $\mu^*(\xi_2)$, for example. By definition, ξ_2 is dual to $\text{Sq}^{2,1}$. There are only two vector space generators in degree 3, and we have

$$\langle \xi_2, \text{Sq}^{2,1} \rangle = 1 \quad \text{and} \quad \langle \xi_2, \text{Sq}^3 \rangle = 0.$$

The basis elements which may be summands of $\mu^*(\xi_2)$ are

$$\xi_2 \otimes 1, \quad \xi_1^3 \otimes 1, \quad \xi_1^2 \otimes \xi_1, \quad \xi_1 \otimes \xi_1^2, \quad 1 \otimes \xi_1^3, \quad \text{and} \quad 1 \otimes \xi_2,$$

and we can simply check which of these are summands in by evaluating against basis elements in the Serre-Cartan basis elements $\text{Sq}^3 \otimes 1$, $\text{Sq}^{2,1} \otimes \text{Sq}^1$, etc. For example,

$$\langle \mu^*(\xi_2), \text{Sq}^3 \otimes 1 \rangle = \langle \xi_2, \mu(\text{Sq}^3 \otimes 1) \rangle = \langle \xi_2, \text{Sq}^3 \rangle = 0.$$

In this fashion we see that $\mu^*(\xi_2)$ evaluates to 1 on precisely the generators $\text{Sq}^{2,1} \otimes 1$, $\text{Sq}^2 \otimes \text{Sq}^1$ and $1 \otimes \text{Sq}^{2,1}$. By examining the lower dimensions one finds $\mu^*(\xi_2) = \xi_2 \otimes 1 + \xi_1^2 \otimes \xi_1 + 1 \otimes \xi_2$.

We defined ξ_n to be dual to Sq^n in the basis consisting of monomials Sq^I with I admissible; now we set $\text{Sq}(I) = \text{Sq}(i_1, i_2, \dots, i_n)$ to be dual to $\xi^I = \xi_1^{i_1} \xi_2^{i_2} \cdots \xi_n^{i_n}$ with respect to the basis consisting of all monomials ξ^J for any sequence J . Note that the degree of ξ^I is $\sum_j i_j(2^j - 1)$, and we can obtain an element of this degree in the original basis of \mathcal{A} as follows: create an admissible sequence I' such that the excess $i'_j - 2i'_{j+1}$ in position j is equal to i_j . One might hope that $\text{Sq}^{I'}$ is equal to $\text{Sq}(I)$, but this is not true in general. However, we can order these bases to obtain a bilinear

pairing which recovers $\text{Sq}^{I'}$ from $\text{Sq}(I)$ and vice versa. This ordering on both the sets of admissible sequences and the sets of sequences is lexicographic from the right. For example,

$$(100, 21, 3) < (42, 22, 3) < (16, 8, 4) < (8, 4, 2, 1).$$

2.3 PAIRING BETWEEN \mathcal{A} AND \mathcal{A}_*

Let \mathcal{I} be the set of finite sequences of nonnegative integers and let $\mathcal{J} \subset \mathcal{I}$ be the set of such sequences which are admissible. We give \mathcal{I} and \mathcal{J} total orders using the lexicographic ordering from the right. That is, longer sequences have higher order, and, if two sequences have the same length, we compare the rightmost entries where they differ (larger means higher order). For example,

$$(5) < (3, 1) < (4, 1) < (0, 2) < (0, 0, 1).$$

Note that this order applies to both \mathcal{I} and \mathcal{J} . We also have an order-preserving bijection $\sigma: J \rightarrow I$ given by $\sigma(j_1, \dots, j_n) = (j_1 - 2j_2, j_2 - 2j_3, \dots, j_{n-1} - 2j_n, j_n)$.

Lemma 2.4. *Let I and J be admissible sequences. If $I \geq J$, then*

$$\langle \xi^{\sigma(I)}, \text{Sq}^J \rangle = \begin{cases} 1, & \text{if } I = J \\ 0, & \text{if } I > J. \end{cases}$$

Proof. Assuming I is nontrivial, we can write $I = (i_1, \dots, i_n)$ with $i_n > 0$. Then we can subtract 1 from the last entry of $\sigma(I)$ to get $\tilde{I} := (i_1 - 2i_2, i_2 - 2i_3, \dots, i_{n-1} - 2i_n, i_n - 1)$, and we have

$$\begin{aligned} \langle \xi^{\sigma(I)}, \text{Sq}^J \rangle &= \left\langle \Delta^* \left(\xi^{\tilde{I}} \otimes \xi_n \right), \text{Sq}^J \right\rangle \\ &= \left\langle \xi^{\tilde{I}}, \Delta(\text{Sq}^J) \right\rangle \end{aligned}$$

$$= \sum_{J_1+J_2=J} \langle \xi^{\tilde{I}} \otimes \xi_n, \text{Sq}^{J_1} \otimes \text{Sq}^{J_2} \rangle.$$

Since ξ_n is dual to $I_n := (2^{n-1}, 2^{n-2}, \dots, 4, 2, 1)$, the only possible nonzero term here is when $J_2 = I_n$. Thus

$$\begin{aligned} \sum_{J_1+J_2=J} \langle \xi^{\tilde{I}} \otimes \xi_n, \text{Sq}^{J_1} \otimes \text{Sq}^{J_2} \rangle &= \langle \xi^{\tilde{I}}, \text{Sq}^{J-I_n} \rangle \langle \xi_n, \text{Sq}^{I_n} \rangle \\ &= \langle \xi^{\tilde{I}}, \text{Sq}^{J-I_n} \rangle. \end{aligned}$$

At this point, notice that J must have at least length n for $J - I_n$ to be nonnegative. In addition, the assumption $I \geq J$ means the length of J is at most n , so we can write $J = (j_1, \dots, j_n)$ (note that $I \geq J$ also implies $i_n \geq j_n$). If $I = J$, then we can repeat this process until reaching $\langle \xi_k, \text{Sq}^{I_k} \rangle = 1$ for some minimal k . We now assume $I > J$. It's easy to see that $J - I_n = (j_1 - 2^{n-1}, j_2 - 2^{n-2}, \dots, j_{n-1} - 2, j_n - 1)$ is admissible, and since $i_n \geq j_n$, we can repeat this process $j_n - 1$ more times. If $i_n = j_n$, we start from the beginning, replacing \tilde{I} with $(i_1 - 2i_2, \dots, i_{n-1} - 2i_n)$ and J with $(j_1 - 2^{n-1}j_n, j_2 - 2^{n-2}j_n, \dots, j_{n-2} - 4j_n, j_{n-1} - 2j_n)$. In this case, $i_{n-1} - 2i_n \geq j_{n-1} - 2j_n$, and we continue as long as possible. Eventually, we reach some value $\langle \xi^{I'}, \text{Sq}^{J'} \rangle$ where the length of I' is strictly larger than the length of J' . As we saw, in this case $\langle \xi^{I'}, \text{Sq}^{J'} \rangle = 0$, which completes the proof. \square

2.4 THE HOPF ALGEBRA ANTIPODE MAP AND HOPF SUBALGEBRAS

Since \mathcal{A}_* is also a Hopf algebra, it has an antipode map $\chi: \mathcal{A}_* \rightarrow \mathcal{A}_*$ characterized by the commutativity of the following diagram.

The diagonal maps are the obvious product and coproduct maps, and the unlabeled horizontal maps are the augmentation (counit) and unit maps. For $n \geq 0$, the middle row takes ξ_n to 0. Comparison with the top row shows $\sum_{i+j=n} \xi_i^{2^j} \chi(\xi_j) = 0$. In

$$\begin{array}{ccccc}
& & \mathcal{A}_* \otimes \mathcal{A}_* & \xrightarrow{1 \otimes \chi} & \mathcal{A}_* \otimes \mathcal{A}_* & & \\
& & \nearrow & & \searrow & & \\
\mathcal{A}_* & \xrightarrow{\quad} & \mathbb{Z}_2 & \xrightarrow{\quad} & \mathcal{A}_* & & \\
& & \searrow & & \nearrow & & \\
& & \mathcal{A}_* \otimes \mathcal{A}_* & \xrightarrow{\chi \otimes 1} & \mathcal{A}_* \otimes \mathcal{A}_* & &
\end{array}$$

particular $\chi(\xi_n)$ is determined inductively via

$$\chi(\xi_n) = \xi_1^{2^{n-1}} \chi(\xi_{n-1}) + \xi_2^{2^{n-2}} \chi(\xi_{n-2}) + \cdots + \xi_{n-2}^4 \chi(\xi_2) + \xi_{n-1}^2 \chi(\xi_1) + \xi_n.$$

Adopting the semi-standard convention $\zeta_n := \chi(\xi_n)$, we equivalently have

$$\zeta_n = \xi_n + \sum_{i=1}^{n-1} \xi_i^{2^{n-i}} \zeta_{n-i} = \xi_n + \xi_{n-1}^2 \zeta_1 + \xi_{n-2}^4 \zeta_2 + \cdots + \xi_1^{2^{n-1}} \zeta_{n-1}.$$

For $x \in \mathcal{A}$, we write $\Delta(x) = \sum_i x'_i \otimes x''_i$. The fact that \mathcal{A} is connected means that, if $x \in \mathcal{A}^+$, then the conjugate satisfies $\sum_i \chi(x'_i) x''_i = 0 = \sum_i x'_i \chi(x''_i)$, where \mathcal{A}^+ denotes the positively graded part of \mathcal{A} . In this case, $\Delta(x)$ is the sum of $1 \otimes x + x \otimes 1$ plus terms in $\mathcal{A}^+ \otimes \mathcal{A}^+$, hence one can inductively determine $\chi(x)$ by this property. For example, with the Steenrod algebra we have $\Delta(\text{Sq}^1) = \text{Sq}^1 \otimes 1 + 1 \otimes \text{Sq}^1$, hence $\mu(\chi(\text{Sq}^1) \otimes 1) + \mu(\chi(1) \otimes \text{Sq}^1) = 0$ and $\chi(\text{Sq}^1) = \text{Sq}^1$. Next $\Delta(\text{Sq}^2) = \text{Sq}^2 \otimes 1 + \text{Sq}^1 \otimes \text{Sq}^1 + 1 \otimes \text{Sq}^2$ and so $\chi(\text{Sq}^2) = \chi(\text{Sq}^1) \text{Sq}^1 + \text{Sq}^2 = \text{Sq}^2$. For Sq^3 the antipode is nontrivial: $\chi(\text{Sq}^3) = \chi(\text{Sq}^2) \text{Sq}^1 + \chi(\text{Sq}^1) \text{Sq}^2 + \text{Sq}^3 = \text{Sq}^2 \text{Sq}^1$.

We say $E \subseteq \mathcal{A}$ is a sub-Hopf algebra of \mathcal{A} if E is a Hopf algebra whose structure maps are restrictions of those for \mathcal{A} . In particular, E is a subalgebra of \mathcal{A} , the dual E_* is a sub-coalgebra of \mathcal{A}_* , and the conjugation maps preserve E and E_* as subspaces.

The correct notion of the quotient of \mathcal{A} by a Hopf subalgebra E is a bit subtle: since E is unital, the quotient by the left or right ideal generated by E gives the zero

ring. Instead, we use the semi-standard notation $\mathcal{A} // E$ to denote the Hopf algebra quotient $\mathbb{Z}_2 \otimes_E \mathcal{A}$. In light of this, the dual notion involves the cotensor product. That is, we say $\mathbb{Z}_2 \otimes_E \mathcal{A}$ is dual to $\mathbb{Z}_2 \square_E \mathcal{A}_*$.

2.5 A BRIEF REVIEW OF COMODULE THEORY

We now include a superficial review of comodules and cotensor products. Given an algebra A over a base ring k , a right A -module M has the structure of a map $\mu_M: M \otimes_k A \rightarrow M$; a left A -module has $\mu_N: A \otimes_k N \rightarrow N$. Completely dual to this, a right A -comodule C has coaction map $\Delta_C: C \rightarrow C \otimes_k A$, and a left A -comodule D similarly has $\Delta_D: D \rightarrow A \otimes_k D$. Now the tensor product $M \otimes_A N$ may be defined as the cokernel of

$$M \otimes A \otimes N \xrightarrow{\mu_M \otimes 1 - 1 \otimes \mu_N} M \otimes N,$$

where the unadorned tensor \otimes denotes \otimes_k . Dually, the cotensor product $C \square_R D$ is the kernel of

$$C \otimes D \xrightarrow{\Delta_C \otimes 1 - 1 \otimes \Delta_D} C \otimes A \otimes D.$$

As is consistent with the literature, we will refer to elements $Q_i \in \mathcal{A}$ defined as the linear dual to ξ_i with respect to the monomial basis. The following facts can be found in [Rog] 15.5.

Lemma 2.5. *For all n , $\mathcal{A}(n) := \langle \text{Sq}^1, \text{Sq}^2, \text{Sq}^4, \dots, \text{Sq}^{2^n} \rangle$ and $E(n) := \langle Q_0, Q_1, \dots, Q_n \rangle$ are sub-Hopf algebras of \mathcal{A} . Moreover, $E(n)$ is the exterior algebra on Q_0, \dots, Q_n . The quotient $\mathcal{A} // \mathcal{A}(n) = \mathbb{Z}_2 \otimes_{\mathcal{A}(n)} \mathcal{A}$ is dual to*

$$\mathbb{Z}_2 \square_{\mathcal{A}(n)_*} \mathcal{A}_* = \mathbb{Z}_2 \left[\xi_1^{2^{n+1}}, \xi_2^{2^n}, \dots, \xi_n, \xi_{n+1}^2, \xi_{n+2}, \xi_{n+3}, \xi_{n+4}, \dots \right].$$

The quotient $\mathcal{A} // E(n) = \mathbb{Z}_2 \otimes_{E(n)} \mathcal{A}$ is dual to

$$\mathbb{Z}_2 \square_{E(n)_*} \mathcal{A}_* = \mathbb{Z}_2 [\xi_1^2, \xi_2^2, \dots, \xi_n^2, \xi_{n+1}^2, \xi_{n+2}, \xi_{n+3}, \xi_{n+4}, \dots].$$

This lemma is overpowered for our purposes, but it shows $\mathcal{A} // \mathcal{A}(1)$ is dual to

$$\mathbb{Z}_2[\xi_1^4, \xi_2^2, \xi_3, \xi_4, \dots] = \mathbb{Z}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \zeta_4, \dots],$$

and $\mathcal{A} // E(1)$ is dual to

$$\mathbb{Z}_2[\xi_1^2, \xi_2^2, \xi_3, \xi_4, \dots] = \mathbb{Z}_2[\zeta_1^2, \zeta_2^2, \zeta_3, \zeta_4, \dots].$$

The topological significance is easy to state: for ko the connective cover of the (real) KO-theory spectrum (and ku the connective cover for complex K -theory),

$$H^* ko = \mathcal{A} // \mathcal{A}(1) \quad \text{and hence} \quad H_* ku = \mathbb{Z}_2[\xi_1^4, \xi_2^2, \xi_3, \xi_4, \dots].$$

On the other hand,

$$H^* ku = \mathcal{A} // E(1) \quad \text{and} \quad H_* ku = \mathbb{Z}_2[\xi_1^2, \xi_2^2, \xi_3, \xi_4, \dots].$$

Note that $E(0) = \mathcal{A}(0)$ is generated by 1 and Sq^1 , and we have also described $H^* H\mathbb{Z} = \mathcal{A} // \mathcal{A}(0)$ and $H_* H\mathbb{Z} = \mathbb{Z}_2[\xi_1^2, \xi_2, \xi_3, \dots]$.

For further reference, see [Adams] (proposition 16.6 of section III), [Stong] (page 330), and [ABP] (page 287).

2.6 $\mathcal{A}(1)$ MODULES AND $H^*BPSp(3)$

Recall that $\mathcal{A}(1)$ is the subalgebra of \mathcal{A} generated by Sq^1 and Sq^2 . It is easy to verify that $\mathcal{A}(1)$ has basis

$$\{1, Sq^1, Sq^2, Sq^3, Sq^{2,1}, Sq^{3,1}, Sq^{4,1} + Sq^5, Sq^{5,1}\}.$$

Here we have written elements in terms of Sq^I with admissible I , but note that we have several nontrivial relations: for example,

$$Sq^{1,2} = Sq^3$$

$$\begin{aligned} \mathrm{Sq}^{2,2} &= \mathrm{Sq}^{3,1} \\ \mathrm{Sq}^{2,1,2} &= \mathrm{Sq}^{2,3} = \mathrm{Sq}^{4,1} + \mathrm{Sq}^5 \\ \mathrm{Sq}^{2,2,2} &= \mathrm{Sq}^{2,1,2,1} = \mathrm{Sq}^{1,2,1,2} = \mathrm{Sq}^{5,1} . \end{aligned}$$

We can depict $\mathcal{A}(1)$ -modules as graphs with a node for each basis element, a short edge indicating left multiplication by Sq^1 , and a long edge indicating left multiplication by Sq^2 . Diagrammatically, we indicate degree with vertical position. In addition to the regular $\mathcal{A}(1)$ -module, we consider \mathbb{Z}_2 as a module concentrated in degree zero. Write I for the augmentation ideal of $\mathcal{A}(1)$. The inclusion $I \hookrightarrow \mathcal{A}(1)$ and augmentation $\mathcal{A}(1) \rightarrow \mathbb{Z}_2$ maps are of course maps of $\mathcal{A}(1)$ -modules. The other useful $\mathcal{A}(1)$ -modules are the “joker” $J := \mathcal{A}(1)/\mathcal{A}(1)(\mathrm{Sq}^3)$ and the module $K := \mathcal{A}(1)/\mathcal{A}(1)(\mathrm{Sq}^1, \mathrm{Sq}^2 \mathrm{Sq}^3)$. These submodules are depicted in figure 2.1.

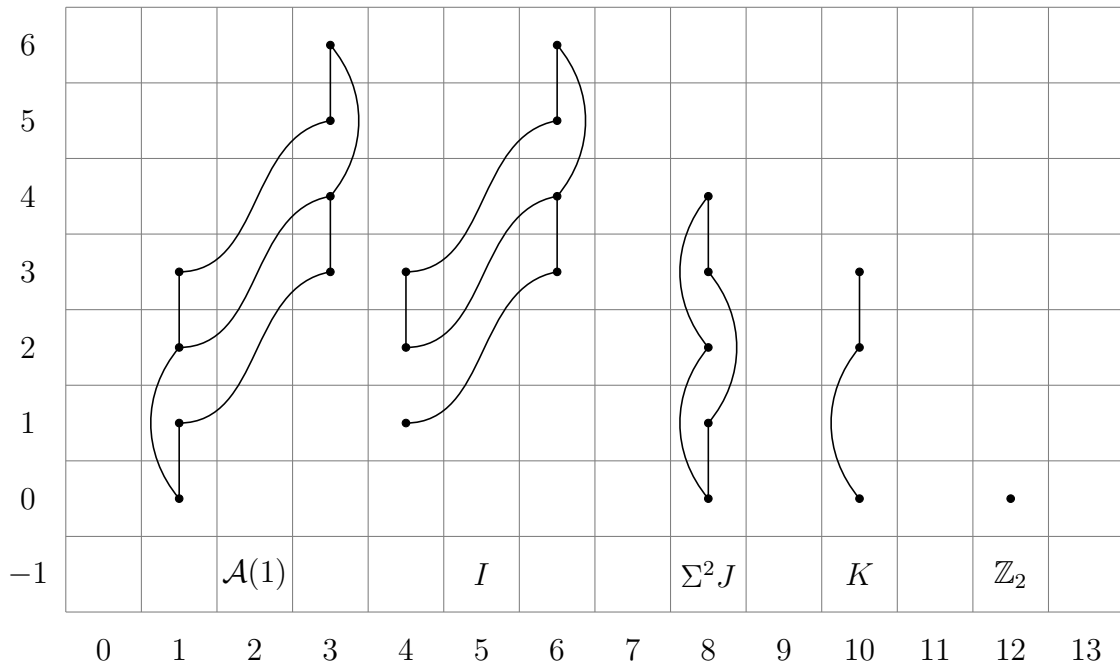


Figure 2.1. Relevant submodules of $\mathcal{A}(1)$.

A key computation in [Stolz] was the the $\mathcal{A}(1)$ -module structure of $H^* \mathrm{BPSp}(3)$.

We revisit Stolz's computation beginning with the $\mathcal{A}(1)$ action on $H^*\text{BPSp}(3)$ ([Stolz]).

We will rely on Kono's determination in [Kono] that

$$H^*\text{BPSp}(3) = \mathbb{Z}_2[t_2, t_3, t_8, t_{12}],$$

where $\deg t_i = i$. To compute the the actions of Sq^1 and Sq^2 on generators, Stolz used representations of the subgroup $\text{P}(\text{Sp}(1)^3) \subset \text{PSp}(3)$ using the geometric fact that $\text{Sp}(1)$ naturally acts on \mathbb{H} as the unit quaternions. He found that Sq^1 and Sq^2 act as follows:

$$\begin{aligned} \text{Sq}^1(t_2) &= t_3 & \text{Sq}^1(t_3) &= 0 & \text{Sq}^1(t_8) &= 0 & \text{Sq}^1(t_{12}) &= 0 \\ \text{Sq}^2(t_2) &= t_2^2 & \text{Sq}^2(t_3) &= t_2 t_3 & \text{Sq}^2(t_8) &= 0 & \text{Sq}^2(t_{12}) &= t_2 t_{12}. \end{aligned}$$

Lemma 2.6. *As $\mathcal{A}(1)$ -modules, $H^*\text{BPSp}(3) \cong \mathbb{Z}_2[t_8, t_{12}^2] \otimes (\mathbb{Z}_2[t_2, t_3] \oplus \mathbb{Z}_2[t_2 t_{12}, t_3 t_{12}])$.*

Proof. The Cartan formula shows

$$\text{Sq}^1(xy) = \text{Sq}^1(x)y + x \text{Sq}^1(y) \quad \text{and} \quad \text{Sq}^2(xy) = \text{Sq}^2(x)y + \text{Sq}^1(x) \text{Sq}^1(y) + x \text{Sq}^2(y),$$

hence to verify that $\mathbb{Z}_2[t_8, t_{12}^2]$ is an $\mathcal{A}(1)$ -submodule, it suffices to check that generators Sq^1 and Sq^2 take generators t_8 and t_{12}^2 to $\mathbb{Z}_2[t_8, t_{12}^2]$. This is immediate by Stolz's computation. The same is true for $\mathbb{Z}_2[t_2, t_3]$. Finally for $\mathbb{Z}_2[t_2 t_{12}, t_3 t_{12}]$ we check

$$\begin{aligned} \text{Sq}^1(t_2 t_{12}) &= \text{Sq}^1(t_2)t_{12} + t_2 \text{Sq}^1(t_{12}) = t_2 t_{12} \\ \text{Sq}^2(t_2 t_{12}) &= t_2^2 t_{12} + t_2^2 t_{12} = 0 \\ \text{Sq}^1(t_3 t_{12}) &= 0 \\ \text{Sq}^2(t_3 t_{12}) &= t_2 t_3 t_{12} + t_2 t_3 t_{12} = 0. \end{aligned}$$

Clearly we have an injection $\mathbb{Z}_2[t_8, t_{12}^2] \otimes (\mathbb{Z}_2[t_2, t_3] \oplus \mathbb{Z}_2[t_2, t_3]t_{12}) \rightarrow \mathbb{Z}_2[t_2, t_3, t_8, t_{12}]$.

Since these modules are finite in each degree we can verify that all elements have been

accounted for with generating functions. A generating function for $\mathbb{Z}_2[t_2, t_3, t_8, t_{12}]$ is

$$\frac{1}{(1-x^2)(1-x^3)(1-x^8)(1-x^{12})}$$

. On the other hand, for $\mathbb{Z}_2[t_8, t_{12}^2] \otimes (\mathbb{Z}_2[t_2, t_3] \oplus \mathbb{Z}_2[t_2, t_3]t_{12})$ we have the generating function

$$\begin{aligned} & \frac{1}{(1-x^8)(1-x^{24})} \left(\frac{1}{(1-x^2)(1-x^3)} + \frac{x^{12}}{(1-x^2)(1-x^3)} \right) \\ &= \frac{1}{(1-t^8)(1-t^{12})(1+t^{12})} \left(\frac{1+x^{12}}{(1-x^2)(1-x^3)} \right) \\ &= \frac{1}{(1-x^2)(1-x^3)(1-x^8)(1-x^{12})} \end{aligned}$$

□

Next we aim to write $\mathbb{Z}_2[t_8, t_{12}^2]$, $\mathbb{Z}_2[t_2, t_3]$ and $\mathbb{Z}_2[t_2t_{12}, t_3t_{12}]$ in of $\mathcal{A}(1)$, \mathbb{Z}_2 , I , J , and K . The utility of 2.6 is that $\mathbb{Z}_2[t_8, t_{12}]$ has trivial action by Sq^1 and Sq^2 , hence is the direct sum of \mathbb{Z}_2 with a trivial module. This reduces our task to finding the $\mathcal{A}(1)$ -module structure of $\mathbb{Z}_2[t_2, t_3]$ and $\mathbb{Z}_2[t_2t_{12}, t_3t_{12}]$. We refer the reader to [Stolz, proposition 6.5] for the remainder of the proof.

Theorem 2.7 (Stolz lemma 7.6). *Suppose $f: C \rightarrow D$ is a map of left $\mathcal{A}(1)$ -modules, where C is a direct sum of suspensions of \mathbb{Z}_2 , J , and $\mathcal{A}(1)$, and D is a direct sum of suspensions of \mathbb{Z}_2 , I , J , K , and $\mathcal{A}(1)$. If f is injective and f induces an injection $H(C; Q_0) \rightarrow H(D; Q_0)$, then f is a split injection.*

2.7 $E(1)$ MODULES AND $H^*\text{BSU}(3)$

Let $E(1)$ be the subalgebra of \mathcal{A} generated by $Q_0 = \text{Sq}^1$ and $Q_1 = \text{Sq}^2 \text{Sq}^1 + \text{Sq}^3$. Note that $E(1)$ is a subalgebra of $\mathcal{A}(1)$, and the inclusion $E(1) \rightarrow \mathcal{A}(1)$ yields a

restriction of scalars functor $\mathcal{A}(1)\text{-Mod} \Rightarrow E(1)\text{-Mod}$ right adjoint to induction $\mathcal{A}(1) \otimes_{E(1)} -: E(1)\text{-Mod} \Rightarrow \mathcal{A}(1)\text{-Mod}$. As before we consider \mathbb{Z}_2 as a submodules of $E(1)$ concentrated in degree 0. Let L be the augmentation ideal of $E(1)$ and let $C = E(1)/E(1)\text{Sq}^1$. These $E(1)$ -modules are shown in figure 2.2

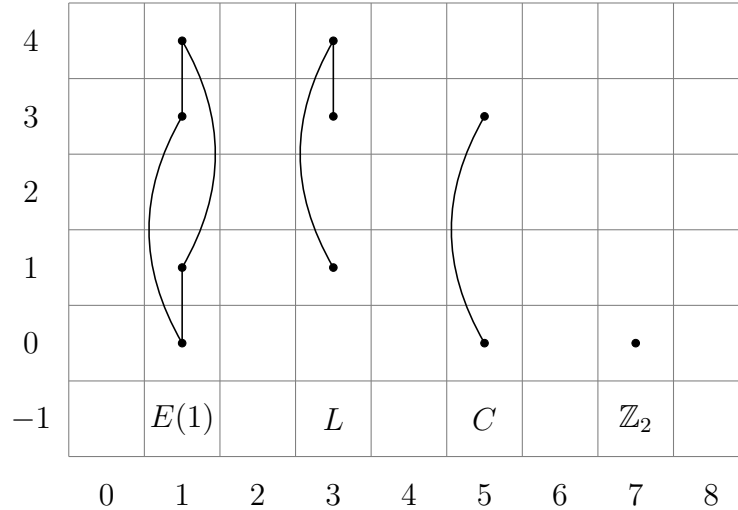


Figure 2.2. Relevant submodules of $E(1)$.

Lemma 2.8. *We have isomorphisms of $E(1)$ -modules*

$$\mathcal{A}(1) \cong E(1) \oplus \Sigma^2 E(1)$$

$$I \cong L \oplus \Sigma^2 E(1)$$

$$J \cong \mathbb{Z}_2 \oplus \Sigma^{-2} E(1)$$

$$K \cong \Sigma^{-1} L.$$

Lemma 2.9 (Stolz proposition 6.5). *As an $\mathcal{A}(1)$ -module, $H^*\text{BPSp}(3)$ is the direct sum of a free module with a direct sum of $8i$ -fold suspensions of \mathbb{Z}_2 , $\Sigma^{-1}I$, $\Sigma^4 J$, and $\Sigma^4 K$.*

Corollary 2.10. *As an $E(1)$ -module, $H^*\text{BPSp}(3)$ is a direct sum of $4i$ -fold suspensions of \mathbb{Z}_2 and $\Sigma^{-1}L$.*

Proof. Thus lemma 2.9 shows that $H^*BPSp(3)$ is, as an $E(1)$ -module, the direct sum of a free module with a direct sum of $8i$ -fold suspensions of \mathbb{Z}_2 , $\Sigma^{-1}L$, $\Sigma^4\mathbb{Z}_2$, and Σ^3L . \square

As an algebra, $H^*BSU(3)$ is the polynomial ring $\mathbb{Z}_2[y_4, y_6]$, where $\deg(y_i) = i$. In theorem A.2 we compute the actions by Sq^1 and Sq^2 , and we summarize the result here: $Sq^1(y_6) = Sq^2(y_6) = Sq^1(y_4) = 0$ and $Sq^2(y_4) = y_6$. Hence $\mathbb{Z}_2[y_4^2, y_6]$ is an $E(1)$ -submodule of $H^*BSU(3)$ with trivial action. We can now make a better description of $H^*BSU(3)$.

Theorem 2.11. *As an $E(1)$ -module, $H^*BSU(3) \cong \mathbb{Z}_2[y_4^2] \oplus C \otimes_{\mathbb{Z}_2} \mathbb{Z}_2[y_4^2, y_6]y_4$. Here $\mathbb{Z}_2[y_4^2]$ and $\mathbb{Z}_2[y_4^2, y_6]y_4$ are trivial $E(1)$ -modules which may be considered as submodules of $\mathbb{Z}_2[y_4, y_6] \cong H^*BSU(3)$.*

Proof. Knowing the actions of Sq^1 and Sq^2 on generators y_4 and y_6 , it is easy to verify $\mathbb{Z}_2[y_4^2]$ is a trivial submodule of $\mathbb{Z}_2[y_4, y_6]$. As vector spaces,

$$\mathbb{Z}_2[y_4, y_6] \cong \mathbb{Z}_2[y_4^2] \oplus \mathbb{Z}_2[y_4^2, y_6]y_4 \otimes C.$$

To see this, observe that a generating function in t for the right hand side is

$$\begin{aligned} \frac{1}{1-t^8} + \frac{t^4(1+t^2)}{(1-t^8)(1-t^6)} &= \frac{1-t^6+t^4(1+t^2)}{(1-t^4)(1+t^4)(1-t^6)} \\ &= \frac{1}{(1-t^4)(1-t^6)}. \end{aligned}$$

To make this explicit, momentarily write $\{1, x\}$ for a \mathbb{Z}_2 -basis of C and define a map

$$\varphi: \mathbb{Z}_2[y_4^2] \oplus \mathbb{Z}_2[y_4^2, y_6]y_4 \otimes C \rightarrow \mathbb{Z}_2[y_4, y_6]$$

by extending linearly from $\varphi(\theta_1, \theta_2 \otimes 1) = \theta_1 + \theta_2$ and $\varphi(\theta_1, \theta_2 \otimes x) = \theta_1 + \theta_2 y_4^{-1} y_6$. Since the domain and codomain of φ are finite dimensional in each grading, φ is a linear isomorphism provided φ is surjective. Elements of the form $y_4^{2m} y_6^n$ are obtained

as $\varphi(0, y_4^{2m} y_6^{n-1} y_4 \otimes x)$ if $n \geq 1$ and $\varphi(y_4^{2m}, 0)$ if $n = 0$. For odd powers of y_4 , observe $y_4^{2m+1} y_6^n = \varphi(0, y_4^{2m+1} y_6^n \otimes 1)$. Hence φ is an isomorphism of graded vector spaces \square

2.8 ko AND ku

Let ko be the connective cover of KO , meaning there is a map $ko \rightarrow KO$ inducing isomorphisms $\pi_n ko \rightarrow \pi_n KO$ for $n \geq 0$ and where $\pi_n ko = 0$ for $n < 0$. Similarly one may define ku as the connective cover of the complex K-theory spectrum K . Adams notes that one may take the zeroth space ku to be $\mathbb{Z} \times BU$, hence $ku^0(X) = K^0(X)$ for any CW complex space X ; in general the higher cohomology groups may differ. The similar statements apply for ku . The α and α^c genus correspond to spectrum maps $MSpin \rightarrow ko$ and $MSpin^c \rightarrow ku$, and for this reason we care about the \mathcal{A} -module structures of ko and ku .

As an \mathcal{A} -module, we can identify $H^* ko = \mathcal{A} // \mathcal{A}(1)$ and $H^* ku = \mathcal{A} // E(1)$. Thus $H_* ko$ and $H_* ku$ are as given in lemma 2.5. We prove the result for ku here, following a proof by Bruner ([Bru]) which in turn was adapted from Adams: for further reference, see proposition 16.6 of section III of [Adams].

Lemma 2.12. *As an \mathcal{A} -module, $H^* ku \cong \mathcal{A} // E(1)$.*

Proof. We begin with the fibration induced by $ku \rightarrow H\mathbb{Z}$. The total space is $\Sigma^2 ku$, giving a cofibration which extends to the right as in $\Sigma^3 \xrightarrow{i} ku \xrightarrow{j} H\mathbb{Z} \xrightarrow{k} \Sigma^3 ku$. We compare the two fiber sequences below.

$$\begin{array}{ccccc}
 & & \Sigma^2 ku & \longrightarrow & * \\
 & \nearrow \text{dashed} & & & \downarrow \\
 \Sigma^2 ku & \xrightarrow{i} & ku & \dashrightarrow & \Sigma^3 ku \\
 & & \downarrow j & & \\
 & & H\mathbb{Z} & \dashrightarrow k &
 \end{array}$$

Write $\omega_n \in H^n \Sigma^n \text{ku}$ for the fundamental cohomology class. The transgression in the upper right fibration is $\tau(\omega_2) = \omega_3$. It follows that $k^* \omega_3 = \tau \omega_2$. We compute $\tau(\omega_2)$ by looking at the third space in each spectrum. Recall that

$$\text{ku} = (\text{BU} \times \mathbb{Z}, \text{U}, \text{BU}, \text{SU}, \text{BSU}, \text{SU}\langle 5 \rangle, \text{SU}\langle 6 \rangle, \dots)$$

and

$$H\mathbb{Z} = (\mathbb{Z}, S^1, \mathbb{C}P^\infty, K(\mathbb{Z}, 3), K(\mathbb{Z}, 4), \dots).$$

Restricting to the third spaces gives $\text{SU}\langle 5 \rangle \rightarrow \text{SU} \rightarrow K(\mathbb{Z}, 3)$. Write $\iota_3 \in H^3 K(\mathbb{Z}, 3)$ for the fundamental class. We also know that $H^* \text{SU} = \bigwedge(e_3, e_5, e_7, \dots)$. Since $\text{SU}\langle 5 \rangle$ is 4-connected, the first nonzero cohomology group $H^5 \text{SU}\langle 5 \rangle = H^5 \text{SU}$ is generated by e_5 . We also know $H^* K(\mathbb{Z}, 3)$ (for example using $\mathbb{C}P^\infty \rightarrow * \rightarrow K(\mathbb{Z}, 3)$): the first several generators are $1, \iota_3, \text{Sq}^2 \iota_3$, and $\text{Sq}^3 \iota_3 = \iota_3^2$. Since $H^5 \text{SU}$ is one-dimensional (and generated by e_5), we necessarily have the transgression $\tau(e_5) = \text{Sq}^3 \iota_3$. Now we turn to the cofibration $\text{ku} \rightarrow H\mathbb{Z} \rightarrow \Sigma^3 \text{ku}$. Write $u \in H^0 H\mathbb{Z}$ for the fundamental class. We have an isomorphism $\alpha: \mathcal{A}/\mathcal{A}\text{Sq}^1 \rightarrow H^* H\mathbb{Z}$ given by $\alpha(\theta) = \theta u$ and a diagram In addition, we have the exact sequence

$$\begin{array}{ccccccc} & & & \mathcal{A}/\mathcal{A}\text{Sq}^1 & & & \\ & & & \downarrow \alpha & & & \\ 0 & \longrightarrow & H^* \Sigma^3 \text{ku} & \longrightarrow & H^*(H\mathbb{Z}) & \xrightarrow{j^*} & H^* \text{ku} \xrightarrow{i^*} 0 \end{array}$$

$$0 \rightarrow \mathcal{A}/(\text{Sq}^1, \text{Sq}^3) \xrightarrow{f} \mathcal{A}/(\text{Sq}^1) \xrightarrow{g} \mathcal{A}/(\text{Sq}^1, \text{Sq}^3) \rightarrow 0$$

where $f(x) = x \text{Sq}^3$ and g is the quotient: note that f is well defined since $\text{Sq}^3 \text{Sq}^3 = 0$ and $\text{Sq}^1 \text{Sq}^3 = 0$, and obviously $\text{im } f \subseteq \ker g$. On the other hand $\ker g$ is generated by elements of the form $x \text{Sq}^1 + y \text{Sq}^3$, where $x, y \in \mathcal{A}$. Since $f(y) = y \text{Sq}^3 = x \text{Sq}^1 + y \text{Sq}^3 \in \mathcal{A}/(\text{Sq}^1)$, the sequence is exact. Our first computation shows that k^* hits Sq^3 , so $j^* \alpha$ induces the map β . Together we get the following diagram.

$$\begin{array}{ccccccccc}
(\mathcal{A}/(\mathrm{Sq}^1, \mathrm{Sq}^3))_{n-3} & \xrightarrow{f} & (\mathcal{A}/(\mathrm{Sq}^1))_n & \xrightarrow{g} & (\mathcal{A}/(\mathrm{Sq}^1, \mathrm{Sq}^3))_n & \xrightarrow{\delta} & (\mathcal{A}/(\mathrm{Sq}^1, \mathrm{Sq}^3))_{n-2} & \xrightarrow{f} & (\mathcal{A}/(\mathrm{Sq}^1))_{n+1} \\
\downarrow \beta & & \downarrow \alpha & & \downarrow \beta & & \downarrow \beta & & \downarrow \alpha \\
H^{n-3} \mathrm{ku} & \xrightarrow{k^*} & H^n(H\mathbb{Z}) & \xrightarrow{j^*} & H^n \mathrm{ku} & \xrightarrow{\delta} & H^{n-2} \mathrm{ku} & \xrightarrow{k^*} & H^{n+1}(H\mathbb{Z})
\end{array}$$

We know that α is an isomorphism, and the Hurewicz theorem shows that the map $\beta: \mathcal{A}/(\mathrm{Sq}^1, \mathrm{Sq}^3)_n \rightarrow H^n \mathrm{ku}$ is an isomorphism for $n \leq 2$. If β is an isomorphism for $n \leq r$ with $r > 2$, it follows from the five-lemma that the middle vertical map in the diagram is an isomorphism. By induction the result follows. \square

2.9 EXTENDED MODULES AND INDECOMPOSABLE QUOTIENTS

As a general fact, if A is an algebra over a ring B and N is a left B -module, $A \otimes_B N$ is a left A -module in the obvious way. Further, if N is free as a B -module, then $A \otimes_B N$ is free as an A -module (the same is true for projective modules). In this context, we call $A \otimes_B N$ an extended B -module. In practice, B may be easier to work with than A , so it is a desirable property for an A -module to be an extended B -module.

In addition to free modules, extended modules arise in the more interesting case that A is a Hopf algebra and B is a sub-Hopf algebra of A . Two other notable extended modules are $H^* \mathrm{ko} = \mathcal{A} \otimes_{\mathcal{A}(1)} \mathbb{Z}_2$ and $H^* \mathrm{ku} = \mathcal{A} \otimes_{E(1)} \mathbb{Z}_2$.

Let A and B be sub-Hopf algebras of \mathcal{A} with $B \subseteq A$, and let N be a left B -module. The tensor product $A \otimes_B N$ is naturally a left A -module in the obvious way. If $B = \mathbb{Z}_2$, then $A \otimes_B N$ is a free A -module: generators of N over \mathbb{Z}_2 each generate a copy of A in $A \otimes_B N$. This is true more generally: if N is free as a left B -module, then $A \otimes_B N$ is a free left A -module. Lemma 2.13 is essentially an observation which appears in [Stolz].

Lemma 2.13. *The map $D: \text{MSpin} \rightarrow \text{ko}$ induces a split surjection on homology which allows us to consider $H_* \text{ko}$ as a subalgebra of $H_* \text{MSpin}$.*

Lemma 2.14. *([Stolz, corollary 5.5]) Let Y be an MSpin -module spectrum whose homology is bounded below and locally finite. Identify $H_* \text{ko}$ as a subalgebra of $H_* \text{MSpin}$ as in lemma 2.13. Then there is a functorial isomorphism*

$$H_* Y \rightarrow \mathcal{A}_* \square_{\mathcal{A}(1)_*} \overline{H_* Y},$$

where $\overline{H_* Y} := \mathbb{Z}_2 \otimes_{H_* \text{ko}} H_* Y$. Given a map $f: X \rightarrow Y$ of MSpin -module spectra, we can identify $f_*: H_* X \rightarrow H_* Y$ with $1 \otimes \bar{f}_*$, where \bar{f}_* is a map $\overline{H_* X} \rightarrow \overline{H_* Y}$. In addition, if \bar{f}_* is a split surjection of $\mathcal{A}(1)_*$ -comodules, then f_* is a split surjection of \mathcal{A}_* -comodules.

Lemma 2.15. *Let Y be an MSpin^c -module spectrum whose homology is bounded below and locally finite, and let $R \subset H_* \text{MSpin}^c$ be the subalgebra which D_* maps isomorphically to $H_* \text{ku}$. Let $\overline{H_* Y} := \mathbb{Z}_2 \otimes_R H_Y$ be the R -indecomposable quotient of $H_* Y$. Then there is a functorial isomorphism of \mathcal{A}_* -comodules*

$$H_* Y \rightarrow \mathcal{A}_* \square_{E(1)_*} \overline{H_* Y}.$$

Given a map $f: X \rightarrow Y$ of MSpin^c -module spectra, we can identify $f_*: H_* X \rightarrow H_* Y$ with $1 \otimes \bar{f}_*$, where \bar{f}_* is a map $\overline{H_* X} \rightarrow \overline{H_* Y}$. In addition, if \bar{f}_* is a split surjection of $E(1)_*$ -comodules, then f_* is a split surjection of \mathcal{A}_* -comodules.

Lemmas 2.14 and 2.15 are both particular cases of [Stolz, proposition 5.4], so we defer to his proof.

CHAPTER 3

TRANSFER MAPS

Unless specified otherwise, we will assume that all spaces have a cell structure with finitely many cells in each dimension. For two spaces X, Y , we write $[X, Y]$ for homotopy classes of maps $X \rightarrow Y$. Given based spaces (X, x_0) and (Y, y_0) , we similarly write $[(X, x_0), (Y, y_0)]$ for (basepoint preserving) homotopy classes of maps $(X, x_0) \rightarrow (Y, y_0)$. The notation X_+ means the disjoint union of X with a point.

3.1 VIRTUAL VECTOR BUNDLES

Define a virtual vector bundle over X to be a pair (X, f) , where $f: X \rightarrow \text{BO} \times \mathbb{Z}$ is an unbased map. For convenience, we sometimes omit the word “vector” and say (X, f) is a “virtual bundle.” Homotopy classes of maps $X \rightarrow \text{BO} \times \mathbb{Z}$ form the (unreduced) real K-theory of X ; i.e., $\text{KO}(X) := [X, \text{BO} \times \mathbb{Z}]$. (Note that $\text{KO}(X)$ can equivalently be defined as *based* homotopy classes of maps $X_+ \rightarrow \text{BO} \times \mathbb{Z}$.) Given a virtual bundle $\xi = (X, f)$, we call f the classifying map of ξ . Projection $\text{BO} \times \mathbb{Z} \rightarrow \mathbb{Z}$ defines the rank of ξ . When X is connected, the rank may be considered an integer $\text{rk}(\xi) \in \mathbb{Z}$. Two virtual bundles (X, f) and (X, g) are equivalent if they correspond to the same class of $\text{KO}(X)$ (i.e., if f and g are homotopic).

By contrast, a genuine vector bundle η over a connected space X is classified by a map $X \rightarrow \text{BO}(n)$, where n is the rank of the bundle. Now η yields a virtual bundle of rank n via the composition

$$X \rightarrow \text{BO}(n) \rightarrow \text{BO} \times \{n\} \hookrightarrow \text{BO} \times \mathbb{Z}.$$

Generally we abuse notation and write η for both the vector bundle and virtual bundle. However, it is important to note that a virtual bundle does not always arise from a genuine vector bundle in this way. For example, virtual bundles can have

negative rank, and, if X is not compact, a map $X \rightarrow \mathrm{BO} \times \mathbb{Z}$ may not factor through $\mathrm{BO}(n) \times \mathbb{Z}$ for any n .

While there are virtual bundles which do not arise from vector bundles, there is also some loss of information in considering vector bundles as virtual bundles. For example, a stably trivial vector bundle yields a trivial virtual bundle of the same rank: suppose $\xi \oplus \mathbf{k} \cong \mathbf{n} \oplus \mathbf{k}$, where n is the rank of ξ and \mathbf{k} is the trivial vector bundle of rank k . Then $X \xrightarrow{f} \mathrm{BO}(n) \hookrightarrow \mathrm{BO}(n+k)$ is nullhomotopic, so the resulting composition $X \rightarrow \mathrm{BO}$ is also nullhomotopic. This means that ξ , considered as a virtual bundle, is equivalent to the trivial virtual bundle of rank n .

Remark 3.1. It is immediate from the definitions that trivial virtual bundles are equivalent if and only if they have the same rank. It follows that vector bundles ξ and η over a connected base space X yield equivalent virtual bundles if and only if $\mathrm{rk}(\xi) = \mathrm{rk}(\eta)$ and $\xi \oplus \mathbf{k} \cong \eta \oplus \mathbf{k}$ for some k .

Suppose we have a genuine vector bundle ξ of rank n over a CW complex X . In particular, X is paracompact, so we can embed ξ into a trivial vector bundle \mathbf{k} for some k . The complementary bundle ξ^\perp (which can be defined topologically as the quotient bundle of \bar{k} by ξ) can be used to form an additive inverse of ξ as classes in $\mathrm{KO}(X)$. That is, $\mathrm{KO}(X)$ is an additive group, and, since $\xi + \xi^\perp = k$, we have $-\xi = \xi^\perp - k$. Hence $-\xi$ can be considered as the formal difference $\xi^\perp - k$ (which is a virtual bundle of rank $-n$). The (virtual) bundle $-\xi$ is called the **stable complement** of ξ . In particular, when X is a manifold, we define the **stable normal bundle** of X as the stable complement to the tangent bundle (considered as a virtual bundle).

Now suppose ξ is stably trivial. Without loss of generality, $\xi \oplus \mathbf{1} \cong \mathbf{n} + \mathbf{1}$. Then $X \xrightarrow{f} \mathrm{BO}(n) \hookrightarrow \mathrm{BO}(n+1)$ is nullhomotopic, so $\tilde{f}: X_+ \rightarrow \mathrm{BO} \times \mathbb{Z}$ is homotopic to

the constant map with image $\{b_0\} \times \{n\}$ and $\tilde{\xi} \equiv \tilde{\mathbf{n}}$. This shows that *stably* trivial vector bundles become trivial virtual bundles of the same rank. Similar logic shows that *stably equivalent* bundles of equal rank represent equivalent virtual bundles (of the same rank). It is important to not overlook the requirement that the genuine bundles had the same rank to begin with. For example, trivial virtual bundles of distinct ranks are *not* equivalent.

3.2 THOM SPECTRA AND DOLD'S THEOREM

Given a CW spectrum X , we can suspend or desuspend X to obtain $\Sigma^r X$ for any $r \in \mathbb{Z}$, where $(\Sigma^r X)_i = (X)_{r+i}$. A map of spectra $X \rightarrow Y$ of degree d is defined as a map $\Sigma^d X \rightarrow Y$ of degree 0 (a map $S^0 \rightarrow S^0$ of degree d , for example, is a stable map $S^d \rightarrow S^0$). To avoid mentions of degree, we will instead use suspensions and desuspensions as appropriate to make all maps have degree 0. For this reason, assume maps of spectra have degree 0 by default. We will now describe some relevant spectra.

Example 3.2 (Spectrum of a CW space). Any CW complex X can be considered a (CW) spectrum whose i -th space $(X)_i$ is the basepoint $\{x_0\}$ for $i < 0$ and $(X)_i = \Sigma^i X$ for $i \geq 0$. We use X to denote the spectrum as well as the space.

Example 3.3 (MO and similar Thom spectra). Write γ_n for the universal n -plane bundle over $\text{BO}(n)$. The inclusion $\text{BO}(n) \rightarrow \text{BO}(n+1)$ is induced by the composition $\text{O}(n) \hookrightarrow \text{O}(n) \times \text{O}(1) \rightarrow \text{O}(n+1)$, so $\gamma_{n+1}|_{\text{BO}(n)} \cong \gamma_n \oplus 1$. The classical construction of Thom spaces yields spaces $\text{MO}(n)$ and maps $\Sigma \text{MO}(n) \rightarrow \text{MO}(n+1)$ which give a spectrum MO . Similar constructions give MSO , MU , MSpin , MSpin^c , etc.

Example 3.4 (Thom spectrum of a genuine vector bundle). Suppose we have a (genuine) vector bundle ξ over a CW complex X of rank k . The classifying map $c_\xi: X \rightarrow \mathrm{BO}(k)$ induces $X^\xi \rightarrow \mathrm{MO}(k)$ (here X^ξ denotes the classical Thom space rather than a spectrum; existence of c_ξ is detailed in [Switz] 11.33). The pullback of γ_{k+1} by the composition $X \rightarrow \mathrm{BO}(k) \rightarrow \mathrm{BO}(k+1)$ yields a map $\Sigma X^\xi \cong X^{\xi+1} \rightarrow \mathrm{MO}(k+1)$, and inductively we get a spectrum map $\Sigma^{-k} X^\xi \rightarrow \mathrm{MO}$. (Equivalently, we have a map $X^\xi \rightarrow \mathrm{MO}$ of degree $-k$.)

Example 3.5 (Thom spectrum of a virtual bundle represented by a genuine bundle). Suppose now that ξ is a virtual bundle of (virtual) rank k over a CW complex X such that the classifying map $X \rightarrow \mathrm{BO}$ factors through $\mathrm{BO}(n)$ for some n . This is true, for example, when X is finite dimensional ([Switz] 6.35). Let α be the induced vector bundle of rank n . Our first construction yields a classifying map $\Sigma^{-n} X^\alpha \rightarrow \mathrm{MO}$ of degree 0. Define $X^\xi := \Sigma^{k-n} X^\alpha$ so as to have the classifying map $\Sigma^{-k} X^\xi \rightarrow \mathrm{MO}$ (just as we did when ξ was a genuine bundle of rank k).

We should ensure that, in an appropriate sense, the spectrum X^ξ and classifying map $\Sigma^{-k} X^\xi \rightarrow \mathrm{MO}$ are well-defined regardless of the choice of representative α . Suppose a (genuine) m -bundle β also represents η . The maps $c_\alpha, c_\beta: X \rightarrow \mathrm{BO}$ are then homotopic via a cellular homotopy H : to see this, apply relative cellular approximation to $(X \times I, X \times \{0, 1\})$. This means H factors through $\mathrm{BO}(\ell)$ for some ℓ ; in particular, $\alpha \oplus \mathbf{s}$ is isomorphic (as a vector bundle) to $\beta \oplus \mathbf{t}$ for some $s, t \in \mathbb{N}$. Hence

$$\Sigma^{k-n} X^\alpha \simeq \Sigma^{k-n-s} X^{\alpha \oplus \mathbf{s}} \simeq \Sigma^{k-n-s} X^{\beta \oplus \mathbf{t}} \simeq \Sigma^{k-n-s+t} X^\beta.$$

Since $\alpha \oplus \mathbf{s} \cong \beta \oplus \mathbf{t}$, we have $n+s = m+t$, so $\Sigma^{k-n-s+t} X^\beta = \Sigma^{k-m} X^\beta$. All equivalences shown are natural, so we have a natural equivalence $\Sigma^{k-n} X^\alpha \simeq \Sigma^{k-m} X^\beta$ which, after desuspension, also carries the classifying map $\Sigma^{-n} X^\alpha \rightarrow \mathrm{MO}$ to the classifying map

$\Sigma^{-m} X^\beta \rightarrow \text{MO}$.

Starting with a genuine bundle ξ , we can consider ξ to be virtual, in which case Example 3.4 and Example 3.5 give potentially distinct definitions of X^ξ . However, it is trivial to check that both constructions agree.

Example 3.6 (Thom spectrum of a negative bundle). Let ξ be a genuine vector bundle of rank k over X . As a virtual bundle, ξ has additive inverse $-\xi$ of rank $-k$. Explicitly, since X is paracompact, we can include ξ into a trivial vector bundle of rank n . The orthogonal complement ξ^\perp is then a genuine vector bundle of rank $n - k$. We simply let $-\xi$ be the virtual bundle represented by ξ^\perp and with rank $-k$ (recall that the rank is just given by $\text{BO} \times \mathbb{Z} \rightarrow \mathbb{Z}$). Alternatively, and perhaps more naturally, notice that $\xi \oplus \xi^\perp = \mathbf{n}$, so $-\xi = \xi^\perp - \mathbf{n}$. The formal difference $\xi^\perp - \mathbf{n}$ then has rank $n - k - n = -k$. Note that both definitions of $-\xi$ are equivalent: the presence of a trivial vector bundle does not affect the resulting map $X \rightarrow \text{BO}$, only the component $X \rightarrow \mathbb{Z}$.

For all our purposes, we have seen how a virtual bundle η of rank r defines an associated Thom spectrum X^η , even when $r < 0$. In this context, we have the following generalization of the classical Thom isomorphism established by Dold (details can be found in [Board, §4.4]), as we explain subsequently.

3.3 THE GENERAL BUNDLE TRANSFER MAP

Virtual bundles and generalized Thom isomorphisms

As usual, let BO denote the CW complex filtered by the classifying spaces $\text{BO}(n)$. For a connected CW complex X , a **virtual bundle** ξ over X of rank $r \in \mathbb{Z}$ is an

element of $\mathrm{KO}(X) := [X_+, \mathrm{BO} \times \mathbb{Z}]$ whose projection onto \mathbb{Z} is identically r . We can now define orientations of a virtual bundle with respect to any oriented spectrum A .

For any virtual bundle ξ over X , we have an associated **Thom spectrum** X^ξ . We will write n for the trivial (virtual) bundle of rank n over X . For $n \geq 0$, $X^n = S^n \wedge X_+$. In particular $X^0 = X_+$ is the disjoint union of X with a point.

We note a potential confusion in terminology here: a genuine vector bundle ξ of rank k over X can be considered a virtual bundle of the same rank, but the classifying map $X^\xi \rightarrow \mathrm{MO}$ has degree $-k$ (or codegree k). In this case, ξ has a genuine classifying map $X \rightarrow \mathrm{BO}(k)$, and we use the same term for the induced map of spectra $X^\xi \rightarrow \mathrm{MO}$. Classes of maps to MO are more general, since a virtual bundle cannot necessarily be classified by a map $X \rightarrow \mathrm{BO}(n)$ for some n .

Let A be a spectrum with unit $i: S^0 \rightarrow A$, and let ξ be a virtual bundle of rank n over a connected CW complex X . A spectrum map $u: X^\xi \rightarrow A$ of degree $-n$ (or codegree n) is a **fundamental class** of ξ (with respect to A) if u restricts to i on each fiber of ξ . More precisely, for each $x \in X$ the fiber of ξ in X^ξ can be identified with S^n . This gives a degree n map of spectra $S^0 \rightarrow X^\xi$ (whose 0-th component is a map $S^n \rightarrow X^\xi$), and composing with $u: X^\xi \rightarrow A$ gives a map $S^0 \rightarrow A$ of degree 0. If ξ has a fundamental class with respect to A , we also say ξ is **A -oriented**. Now we can state the following generalization (by Dold) of the classical Thom isomorphism (details can be found in [Board, §4.4]).

Theorem 3.7 (Dold). *Suppose a virtual bundle ξ of rank k over a CW complex X is oriented with respect to a spectrum A with unit $i: S^0 \rightarrow A$. In addition, let C be a spectrum with a left A -action $A \wedge C \rightarrow C$. Then for any virtual bundle η , there are “Thom isomorphisms”*

$$\Phi^\xi: \tilde{C}^*(X^\eta) \rightarrow \tilde{C}^{*+k}(X^{\eta+\xi}) \quad \text{and} \quad \Phi_\xi: \tilde{C}_*(X^{\eta+\xi}) \rightarrow \tilde{C}_{*-k}(X^\eta).$$

In particular, when $\eta = 0$, these isomorphisms become

$$\Phi^\xi: C^*(X) \rightarrow \widetilde{C}^{*+k}(X^\xi) \quad \text{and} \quad \Phi_\xi: \widetilde{C}_*(X^\xi) \rightarrow C_{*-k}(X).$$

When X is a compact smooth manifold with tangent bundle τ , we say that X is **A -oriented** if $-\tau$ is A -oriented. This definition is (naturally) equivalent to the more classical version using local homology groups ([Board] 4.7). This also allows for a notion of A -orientation for certain bundles whose fibers are smooth manifolds. Suppose $\pi: E \rightarrow B$ is a bundle whose fiber F is a closed k -manifold and whose structure group G is a compact Lie group acting smoothly on F . We say that E has an A -structure if the virtual bundle $-\tau$ is A -oriented, where τ is the bundle of tangent vectors along the fibers of π and $-\tau$ is the stable complement. Thus, an A -structure on E amounts to appropriately compatible A -orientations of each fiber $\pi^{-1}(b)$ (for all $b \in B$).

The Thom map associated to a bundle

Suppose for the remainder of this section that we have a fiber bundle $\pi: E \rightarrow B$ whose fiber F is a closed k -manifold and whose structure group G is a compact Lie group acting smoothly on F . Again, let τ be the bundle of tangent vectors along the fibers of π and let $-\tau$ denote the stable complement. I.e., τ is a vector bundle over E whose fiber over $x \in E$ is the tangent space of $\pi^{-1}(\pi(x)) \cong F$ at x .

In this setting, we have a transfer $T(\pi): \Sigma^k B_+ \rightarrow E^{-\tau}$. We summarize the construction here and refer to [Board] 6.20 for more details. First consider the case when B is a compact CW complex (then E is compact: it has an induced CW structure from F and B with finitely many cells). For some d , there is a representation space \mathbb{R}^d along with a G -equivariant smooth embedding $F \hookrightarrow \mathbb{R}^d$ ([Board] 6.19). Write E_G for the principal G -bundle underlying $\pi: E \rightarrow B$, and consider the associated

\mathbb{R}^d -bundle $\eta = E_G \times_G \mathbb{R}^d$. By compactness, η embeds (via a bundle map) into a trivial vector bundle $B \times \mathbb{R}^{n+k}$ for some n . Let U be a tubular neighborhood of F in \mathbb{R}^d and consider the bundle $E_G \times_G U$. We now have the bundle maps shown.

$$\begin{array}{ccccccc}
F & \hookrightarrow & U & \hookrightarrow & \mathbb{R}^d & \hookrightarrow & \mathbb{R}^{n+k} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E & \hookrightarrow & E_G \times_G U & \hookrightarrow & E_G \times_G \mathbb{R}^d & \hookrightarrow & B \times \mathbb{R}^{n+k}
\end{array}$$

We may consider $E_G \times_G U$ to be a fiberwise tubular neighborhood of E in $B \times \mathbb{R}^{n+k}$. In addition, we can identify $E_G \times_G U$ with the unit disk bundle of the normal bundle ν of $E \hookrightarrow B \times \mathbb{R}^{n+k}$. Collapse the complement of $E_G \times_G U$ to a point and thus obtain a map $\Sigma^{n+k} B_+ \rightarrow E^\nu$ on Thom spaces. Note that ν here is a genuine bundle of rank n over E .

For our purposes, we must modify this construction to apply when B is only filtered by compact CW complexes $B_0 \subset B_1 \subset B_2 \subset \dots$. Then each B_i has an associated bundle ν_i over $E_i := \pi^{-1}(B_i)$, say with rank n_i , and we can ensure these be compatible in the sense that the ν_i form a virtual bundle ν (of rank 0) over E . Then the maps $\Sigma^{n_i+k} B_+ \rightarrow E^{\nu_i}$ form a spectrum-level map $\Sigma^k B_+ \rightarrow E^{-\tau}$, where $-\tau$ is considered a virtual bundle over E (of rank 0). For more details see [Führ].

Constructing the umkehr map from the Thom map

Definition 3.8 ([Board] V.6.2). Let A be a ring spectrum and let C be a spectrum with a left A -action $A \wedge C \rightarrow C$. The **transfer maps** associated to a map $f: X \rightarrow Y$ of spaces refer to functorial homomorphisms

$$f_! : C^* X \rightarrow C^{*-r} Y \quad \text{and} \quad f^! : C_* Y \rightarrow C_{*+r} X$$

which are multiplicative in the sense that $f_!$ and $f^!$ are maps of C_* -modules and, for $\alpha \in C^* X$, $\beta \in C^* Y$, and $y \in C_* Y$, we have

- (a) $f_!(\alpha \smile f^*\beta) = f_!(\alpha) \smile \beta$;
- (b) $f_!(f^*(\alpha) \smile \beta) = (-1)^{r|\alpha|} \alpha \smile f_!\beta$;
- (c) $f^!(y \frown \alpha) = f^!(y) \frown f^*(\alpha)$;
- (d) $f_*(f^!(y) \frown \alpha) = (-1)^{r|y|} y \frown f_!(\alpha)$;
- (e) $\langle f^!x, \alpha \rangle = (-1)^{r|x|} \langle x, f_!\alpha \rangle$.

Theorem 3.9 ([Board] V.6.21 and V.6.2). *Let $\pi: E \rightarrow B$ be a bundle with fiber F and structure group G satisfying the following:*

- E and B are CW complexes;
- F is a compact smooth manifold of dimension k ;
- G is a compact Lie group which acts smoothly on F ;
- there is a ring spectrum A and a spectrum C with a left A -action such that $-\tau$ is A -oriented, where τ is the bundle of tangents along the fibers of π .

Then we have transfer maps

$$\pi_!: C^*(E) \rightarrow C^{*-k}(B) \quad \text{and} \quad \pi^!: C_*(B) \rightarrow C_{*+k}(E)$$

which are multiplicative in the sense of definition 3.8 and are the respective compositions

$$C^*(E) \xrightarrow{\Phi^{-\tau}} \tilde{C}^*(E^{-\tau}) \xrightarrow{T(\pi)^*} \tilde{C}^*(\Sigma^k B_+)$$

and

$$\tilde{C}_{*+k}(\Sigma^k B_+) \xrightarrow{T(\pi)_*} \tilde{C}_{*+k}(E^{-\tau}) \xrightarrow{\Phi^{-\tau}} C_{*+k}(E).$$

The $\mathbb{H}P^2$ -bundle transfer

Let us recall an earlier observation: $\mathrm{PSp}(3)$ acts transitively on $S^{11} \subset \mathbb{H}^3$ and this descends to a transitive action on $\mathbb{H}P^2$. The fiber over a point is $\mathrm{PSp}(2, 1) := \mathrm{P}(\mathrm{Sp}(2) \times \mathrm{Sp}(1))$, so we have a bundle $\mathrm{PSp}(2, 1) \rightarrow \mathrm{PSp}(3) \rightarrow \mathbb{H}P^2$. In turn this yields a bundle $\pi: \mathrm{BPSp}(2, 1) \rightarrow \mathrm{BPSp}(3)$ with fiber $\mathbb{H}P^2$ and structure group $\mathrm{PSp}(3)$ acting by isometries on $\mathbb{H}P^2$. Let τ be the bundle of tangent vectors along the fibers of π and write $-\tau$ for the stable complement. We now compute some basic (co)homological properties of this bundle.

Theorem 3.10. *For the bundle $\mathbb{H}P^2 \rightarrow \mathrm{BPSp}(2, 1) \xrightarrow{\pi} \mathrm{BPSp}(3)$ with τ the associated bundle of tangent vectors along the fiber,*

1. $H^*\mathrm{BPSp}(2, 1) \cong \mathbb{Z}[u_2, u_3, u_4, u_8]$, where $\deg(u_i) = i$;
2. $H^*\mathrm{BPSp}(3) \cong \mathbb{Z}_2[t_2, t_3, t_8, t_{12}]$, where $\deg(t_j) = j$;
3. $\pi^*(t_2) = u_2$ and $\pi^*(t_3) = u_3$, while $\pi^*(t_8) = u_4^2 + u_8$ and $\pi^*(t_{12}) = u_4u_8$;
4. $\mathrm{Sq}^1(t_2) = t_3$ and $\mathrm{Sq}^1(t_3) = \mathrm{Sq}^1(t_4) = \mathrm{Sq}^1(t_{12}) = 0$;
5. $\mathrm{Sq}^2(t_2) = t_2^2$, $\mathrm{Sq}^2(t_3) = t_2t_3$, $\mathrm{Sq}^2(t_8) = 0$, and $\mathrm{Sq}^2(t_{12}) = t_2t_{12}$;
6. $w(\tau) = 1 + (u_2^2 + u_4) + (u_2u_4 + u_3^2) + u_3u_4 + u_8$;
7. $\pi_!: H^*\mathrm{BPSp}(2, 1) \rightarrow H^{*-8}\mathrm{BPSp}(3)$ is given modulo t_{12} by

$$\pi_!(u_3^a u_2^b u_4^c u_8^d) = \begin{cases} t_3^a t_2^b t_8^{c/2-1}, & \text{if } c > 0 \text{ is even and } d = 0, \\ t_3^a t_2^b t_8^{d-1}, & \text{if } c = 0 \text{ and } d > 0, \\ 0, & \text{if otherwise.} \end{cases}$$

We delay the proof of these facts and show how they apply. Our $\mathbb{H}P^2$ -bundle has a fiberwise Spin-structure, i.e., $-\tau$ is oriented with respect to $M\text{Spin}$. Applying Boardman's construction ([Board]) to the fiber sequence $\mathbb{H}P^2 \rightarrow B\text{PSp}(2, 1) \xrightarrow{\pi} B\text{PSp}(3)$ yields a bundle transfer map $\mathcal{T}: \Sigma^8 B\text{PSp}(3)_+ \rightarrow M(-\tau)$. Meanwhile, theorem 3.7 gives identifications

$$\tilde{H}^* M(-\tau) \cong H^* B\text{PSp}(2, 1) \quad \text{and} \quad \tilde{H}_* M(-\tau) \cong H_* B\text{PSp}(2, 1).$$

In terms of spectra, the transfer of theorem 3.9 with $C = M\text{Spin}$ is then

$$M\text{Spin} \wedge \Sigma^8 B\text{PSp}(3)_+ \xrightarrow{1 \wedge \mathcal{T}} M\text{Spin} \wedge M(-\tau) \rightarrow M\text{Spin} \wedge B\text{PSp}(2, 1)_+,$$

where the righthand map induces the Thom isomorphism. The Thom isomorphism can be described more explicitly as the map induced on homotopy groups of the composition indicated below. We use $G = \text{PSp}(3)$ and $H = \text{PSp}(2, 1)$; dashed arrows indicate a map induced on homotopy groups.

$$\begin{array}{ccccccc}
& & & \xrightarrow[\Psi]{T} & & & \\
& & & \text{---} & & & \\
& & & \text{---} & & & \\
& & & \text{---} & & & \\
& & & \text{---} & & & \\
M\text{Spin} \wedge \Sigma^8 B G_+ & \xrightarrow{1 \wedge \mathcal{T}} & M\text{Spin} \wedge M(-\tau) & \xrightarrow{1 \wedge \mathcal{M}} & M\text{Spin} \wedge M\text{Spin} & \xrightarrow{\mu} & M\text{Spin} \\
& & \downarrow \pi^! & & \downarrow \Phi & & \uparrow \text{proj} \\
& & M\text{Spin} \wedge M(-\tau) \wedge B H_+ & \xrightarrow{1 \wedge \mathcal{M} \wedge 1} & M\text{Spin} \wedge M\text{Spin} \wedge B H_+ & \xrightarrow{\mu \wedge 1} & M\text{Spin} \wedge B H_+
\end{array}$$

Here \mathcal{M} is the map of Thom spectra induced by the classifying map $BH \rightarrow B\text{Spin}$ of $-\tau$, which motivates theorem 3.10 (in particular, it demonstrates why computing $w(-\tau)$ is relevant).

As mentioned earlier, we will use the same names for several of these maps with $M\text{Spin}$ replaced with $M\text{Spin}^c$. For example, $T: \Sigma^8 B\text{PSp}(3) \rightarrow M\text{Spin}^c$ and $T^c: \Sigma^4 B\text{SU}(3) \rightarrow M\text{Spin}^c$ together are used to prove theorem 1.7.

3.4 THE $\mathbb{C}P^2$ -BUNDLE TRANSFER

Analogously to the quaternionic case, the unitary group $U(3)$ acts transitively on $S^5 \subset \mathbb{C}^3$ yielding a transitive action on $\mathbb{C}P^2$. As before, the stabilizer of $[0 : 0 : 1]$ is $SU(2, 1) := S(U(2) \times U(1))$ and we hence obtain a bundle $BSU(2, 1) \rightarrow BSU(3)$ with fiber $\mathbb{C}P^2$. This bundle admits a Spin^c structure, giving a map $BSU(3) \rightarrow B\text{Spin}^c$. Now a class in $\Omega_n^{\text{Spin}^c} BSU(3)$ consists of an n -manifold P with a stable normal Spin^c structure along with a map $f: P \rightarrow BSU(3)$. Let \widehat{P} be the pullback of the bundle $BSU(2, 1) \rightarrow BSU(3)$; hence \widehat{P} is an $(n + 4)$ -dimensional Spin^c manifold fibered over P . On bordism classes, this correspondence $[P] \rightarrow [\widehat{P}]$ geometrically describes the transfer map $\Omega_n^{\text{Spin}^c} BSU(3) \rightarrow \Omega_{n+4}^{\text{Spin}^c}$.

We have the following maps of spectra.

$$\text{MSpin}^c \wedge \Sigma^4 BG_+ \xrightarrow{1 \wedge t} \text{MSpin}^c \wedge \text{MSpin}^c \xrightarrow{\mu} \text{MSpin}^c \xrightarrow{D} \text{ku}$$

$$\underbrace{\hspace{10em}}_T$$

These induce maps on homology.

$$H_* \text{MSpin}^c \otimes H_* \Sigma^4 BG_+ \xrightarrow{1 \otimes t_*} H_* \text{MSpin}^c \otimes H_* \text{MSpin}^c \xrightarrow{\mu_*} H_* \text{MSpin}^c \xrightarrow{D_*} \text{ku} \quad (3.1)$$

Recall $\mathcal{A}_* = \mathbb{Z}_2[\xi_1, \xi_2, \xi_3, \dots]$, where $\deg(\xi_i) = 2^i - 1$ with coproduct

$$\psi(\xi_n) = \sum_{i=0}^n \xi_{n-i}^{2^i} \otimes \xi_i = \xi_n \otimes 1 + \xi_{n-1}^2 \otimes \xi_1 + \cdots + \xi_1^{2^{n-1}} \otimes \xi_{n-1} + 1 \otimes \xi_n.$$

Alternatively, one can use the Hopf algebra conjugates ζ_i of ξ_i , characterized by

$$\psi(\zeta_n) = \sum_{i=0}^n \zeta_i \otimes \zeta_{n-i}^{2^i} = 1 \otimes \zeta_n + \zeta_1 \otimes \zeta_{n-1}^2 + \cdots + \zeta_{n-1} \otimes \zeta_1^{2^{n-1}} + \zeta_n \otimes 1.$$

The dual of $\text{Sq}^n = \text{Sq}(n)$ is ξ_1^n , while $Q_n = \text{Sq}(0, \dots, 0, 1)$ (with n zeros followed by a 1) is dual to ξ_{n+1} .

Note that $H_*H\mathbb{Z}_2$ is also the dual \mathcal{A}_* of the Steenrod algebra \mathcal{A} . For the subalgebra $E(1)$ generated by Q_0 and Q_1 , we have $H^*ku = \mathcal{A} // E(1) = \mathbb{Z}_2 \otimes_{E(1)} \mathcal{A}$. Dually, $H_*ku = \mathbb{Z}_2 \square_{E(1)_*} \mathcal{A}_*$.

Theorem 2.15 stated that, if Y is an $M\text{Spin}^c$ -module spectrum for which H_*Y is bounded below and of finite type, there is a functorial isomorphism of \mathcal{A}_* -comodules

$$H_*Y \rightarrow \mathcal{A}_* \square_{E(1)_*} \overline{H_*Y}.$$

Applying theorem 2.15 to (A.1) gives

$$\overline{H_*M\text{Spin}^c} \otimes H_*\Sigma^4 BG_+ \xrightarrow{1 \otimes \bar{t}_*} \overline{H_*M\text{Spin}^c} \otimes H_*M\text{Spin}^c \xrightarrow{\mu} \overline{H_*M\text{Spin}^c} \xrightarrow{\bar{D}_*} \overline{H_*ku}.$$

However, $\overline{H_*ku} = \mathbb{Z}_2$, and one can identify \bar{D}_* with the augmentation homomorphism of $\overline{H_*M\text{Spin}^c}$. It follows that $\ker \bar{D}_*$ is generated over \mathbb{Z}_2 by elements which have at least one nontrivial indecomposable factor. These elements can be written $\mu(x \otimes y)$ for some $x \in \overline{H_*M\text{Spin}^c}$ and $y \in Q\overline{H_*M\text{Spin}^c}$. (Here, x may be 1, and, for an algebra A with augmentation ideal $I(A)$, QA denotes the indecomposable quotient $I(A)/\mu(I(A) \otimes I(A))$.) Thus to show that \bar{T}_* surjects onto $\ker D_*$, it suffices to show that the composition $H_*\Sigma^4 BG_+ \rightarrow H_*M\text{Spin}^c \rightarrow \overline{H_*M\text{Spin}^c} \rightarrow Q\overline{H_*M\text{Spin}^c}$ is surjective.

We have $QH_nM\text{Spin}^c \cong \mathbb{Z}_2$ for $n \geq 2$, $n \neq 2^k + 1$ and $QH_nM\text{Spin}^c = 0$ otherwise. Further, the projection $QH_nM\text{Spin}^c \rightarrow Q\overline{H_nM\text{Spin}^c}$ is an isomorphism for $n \geq 4$, $n \neq 2^k \pm 1$, and $Q\overline{H_nM\text{Spin}^c} = 0$ for $n < 4$ or $n = 2^k \pm 1$. We have

$$\overline{H_*M\text{Spin}^c} = \mathbb{Z}_2 [x_n^{\beta(n)} : n \geq 4, n \neq 2^k \pm 1],$$

where $\beta(n) = 2$ if $\alpha(n) < 3$ and $\beta(n) = 1$ for $\alpha(n) \geq 3$ (here $\alpha(n)$ is the number of nonzero terms in the base-2 expansion of n). Ordering the generators by their lower indices,

$$H_*M\text{Spin}^c = \mathbb{Z}_2 [x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2, x_7, x_8^2, x_9^2, x_{10}^2, x_{11}, \dots].$$

Since $R = \mathbb{Z}_2[x_1^2, x_3^2, x_7, x_{15}, \dots]$, we have

$$\overline{H_*\mathrm{MSpin}^c} = \mathbb{Z}_2 \otimes_R H_*\mathrm{MSpin}^c = \mathbb{Z}_2 [x_2^2, x_4^2, x_5^2, x_6^2, x_8^2, x_9^2, x_{10}^2, x_{11}, \dots].$$

We must show that $H_n \Sigma^4 BG_+ \rightarrow QH_n \mathrm{MSpin}^c$ is surjective for $n \geq 4$, $n \neq 2^k \pm 1$. Dually, we must show $PH^n \mathrm{MSpin}^c \rightarrow H^{n-4} BG_+$ is injective for $n \geq 4$, $n \neq 2^k \pm 1$. Recall that the map $BG_+ \rightarrow \mathrm{MSpin}^c$ can be decomposed into $\Sigma^4 BG_+ \xrightarrow{T(\pi)} M(-\tau) \xrightarrow{M(c)} \mathrm{MSpin}^c$, where $\pi: BH \rightarrow BG$ is the bundle map, $T(\pi)$ is the Thom collapse map, and $c: BH \rightarrow \mathrm{BSpin}^c$ classifies the complement $-\tau$ of τ , the bundle along the fibers of π . The h -space inverse $\mathrm{BSpin}^c \rightarrow \mathrm{BSpin}^c$ provides a homotopy equivalence $\mathrm{MSpin}^c \rightarrow \mathrm{MSpin}^c$ allowing us to replace $M(\tau)$ with $M(\tau)$. On cohomology, we have the maps

$$H^n \mathrm{MSpin}^c \xrightarrow{c(\tau)^*} H^n BH \xrightarrow{T(\pi)^*} H^{n-4} BG.$$

- $H^* BH = \mathbb{Z}_2[x_2, x_4]$,
- $H^* BG = \mathbb{Z}_2[y_4, y_6]$,
- $w(\tau) = 1 + x_2 + x_4$,
- $\pi^* y_4 = x_2^2 + x_4$ and $\pi^* y_6 = x_2 x_4$,
- we can write $H^* BH$ as a free $H^* BG$ -module with basis $1, x_2, x_2^2$,
- and the transfer map $H^n BH \rightarrow H^{n-4} BG$ is the $H^* BG$ -module map taking $x = r_0(x) + r_1(x)x_2 + r_2(x)x_2^2$ to $r_2(x)$, where $r_i(x) \in H^* BG$.

APPENDIX A

THE $\mathcal{A}(1)$ -ACTION ON $H^*B\mathrm{PSp}(3)$

In this section, let $G = \mathrm{PSp}(3)$ be the quotient of $\mathrm{Sp}(3)$ by its center $\pm I$ (here I is the identity matrix), and let $H = \mathrm{PSp}(2, 1)$ be the quotient of $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ by $\pm I$. Notice that H is a subgroup of G and the inclusions

$$\mathrm{Sp}(1, 1, 1) \rightarrow \mathrm{Sp}(2, 1) \rightarrow \mathrm{Sp}(3)$$

induce inclusions

$$i_1: \mathrm{PSp}(1, 1, 1) \rightarrow H \quad \text{and} \quad i_2: H \rightarrow G$$

(here $\mathrm{PSp}(1, 1, 1) := \mathrm{P}(\mathrm{Sp}(1)^3)$). Let $i: \mathbb{Z}_2^4 \rightarrow \mathrm{PSp}(3)$ be the composition

$$\mathbb{Z}_2^2 \times \mathbb{Z}_2^2 \xrightarrow{j \times 1} \mathrm{PSp}(1) \times \mathbb{Z}_2^2 \xrightarrow{\Delta \times j_1 \times j_2} \mathrm{PSp}(1, 1, 1),$$

where j maps $(1, 0)$ to i and $(0, 1)$ to j , Δ is the diagonal, and j_n sends each generator to -1 in the n -th factor. Thus,

$$(1, 0, 0, 0) \mapsto [i, i, i]$$

$$(0, 1, 0, 0) \mapsto [j, j, j]$$

$$(0, 0, 1, 0) \mapsto [-1, 1, 1]$$

$$(0, 0, 0, 1) \mapsto [1, -1, 1],$$

etc. For example, $(1, 1, 0, 1) \mapsto [k, -k, k]$. (Note that the images of $[i, i, i]$ and $[j, j, j]$ indeed commute in $\mathrm{PSp}(1)^3$.) Writing \mathbb{Z}_2 as the multiplicative group ± 1 , we have

$$(x_1, x_2, y_1, y_2) \mapsto [(-1)^{y_1} i^{x_1} j^{x_2}, (-1)^{y_2} i^{x_1} j^{x_2}, i^{x_1} j^{x_2}].$$

Given a compact Lie group K , any representation $\rho: K \rightarrow \mathrm{GL}_n(\mathbb{R})$ gives rise to a real vector bundle over BK which we denote $E\rho$. Define a four-dimensional real

representation R_{ij} of $\mathrm{PSp}(1, 1, 1)$ acting on \mathbb{H} via $[h_1, h_2, h_3] \cdot x = h_i x \overline{h_j}$. If $i = j$, this action fixes the \mathbb{R} -span of 1, so we can decompose R_{ii} as the sum of a trivial representation and a 3-dimensional representation R_i . The inclusion $\mathbb{Z}_2^4 \times \mathbb{Z}_2^2 \xrightarrow{i} \mathrm{PSp}(3)$ then defines a bundle over $B((\mathbb{Z}/2)^2 \times (\mathbb{Z}/2)^2) \cong (\mathbb{R}P^\infty)^4$, and we can identify $w(ER_{ij})$ with its image in $H^*((\mathbb{R}P^\infty)^4; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, x_2, y_1, y_2]$.

We will now compute how ρ_{ij} transforms the elements x_i, y_j . In general, we have

$$\rho_{ij}(x_1, x_2, y_1, y_2)(h) = [(-1)^{y_1} i^{x_1} j^{x_2}, (-1)^{y_2} i^{x_1} j^{x_2}, i^{x_1} j^{x_2}] \cdot h.$$

Case 1: $i = j$. Then

$$\begin{aligned} \rho_{ii}(x_1, x_2, y_1, y_2)(h) &= [(-1)^{y_1} i^{x_1} j^{x_2}, (-1)^{y_2} i^{x_1} j^{x_2}, i^{x_1} j^{x_2}] \cdot h \\ &= i^{x_1} j^{x_2} \overline{h j^{x_1} i^{x_2}} \\ &= i^{x_1} j^{x_2} \overline{h j^{x_2} i^{x_1}} \\ &= (-1)^{x_1+x_2} i^{x_1} j^{x_2} h j^{x_2} i^{x_1}. \end{aligned}$$

In particular,

$$\rho_{ii}(x_1, x_2, y_1, y_2)(1) = (-1)^{x_1+x_2} (i^{x_1} (j^{x_2} j^{x_2}) i^{x_1}) = 1.$$

When $h = i$, note that $jij = jk = i$, so

$$\begin{aligned} \rho_{ii}(x_1, x_2, y_1, y_2)(i) &= (-1)^{x_1+x_2} i^{x_1} (j^{x_2} i j^{x_2}) i^{x_1} \\ &= (-1)^{x_1+x_2} (i^{x_1}) i (i^{x_1}) \\ &= (-1)^{x_1+x_2} i^{2x_1+1} \\ &= (-1)^{x_2} i. \end{aligned}$$

Next

$$\rho_{ii}(x_1, x_2, y_1, y_2)(j) = (-1)^{x_1+x_2} i^{x_1} (j^{x_2} j j^{x_2}) i^{x_1}$$

$$\begin{aligned}
&= (-1)^{x_1+x_2} (i^{x_1}) j^{2x_2+1} (i^{x_1}) \\
&= (-1)^{x_1} i^{x_1} j i^{x_1} \\
&= (-1)^{x_1} j
\end{aligned}$$

and

$$\begin{aligned}
\rho_{ii}(x_1, x_2, y_1, y_2)(k) &= (-1)^{x_1+x_2} i^{x_1} (j^{x_2} k j^{x_2}) i^{x_1} \\
&= (-1)^{x_1+x_2} (i^{x_1}) k (i^{x_1}) \\
&= (-1)^{x_1+x_2} k.
\end{aligned}$$

Thus the total Stiefel-Whitney class of $i^*(ER_{ii})$ is thus

$$Bi^*(w(ER_{ii})) = (1 + x_1)(1 + x_2)(1 + x_1 + x_2).$$

Case 2: $i \neq j$. We have

$$\begin{aligned}
\rho_{23}(x_1, x_2, y_1, y_2)(h) &= (-1)^{y_2} \rho_{11}(x_1, x_2, y_1, y_2)(h) \\
\rho_{13}(x_1, x_2, y_1, y_2)(h) &= (-1)^{y_1} \rho_{11}(x_1, x_2, y_1, y_2)(h) \\
\rho_{12}(x_1, x_2, y_1, y_2)(h) &= (-1)^{y_1+y_2} \rho_{11}(x_1, x_2, y_1, y_2)(h).
\end{aligned}$$

Thus, if we let $y_3 = y_1 + y_2$,

$$\begin{aligned}
Bi^*(w(ER_{23})) &= (1 + y_2)(1 + x_1 + y_2)(1 + x_2 + y_2)(1 + x_1 + x_2 + y_2) \\
Bi^*(w(ER_{13})) &= (1 + y_1)(1 + x_1 + y_1)(1 + x_2 + y_1)(1 + x_1 + x_2 + y_1) \\
Bi^*(w(ER_{12})) &= (1 + y_3)(1 + x_1 + y_3)(1 + x_2 + y_3)(1 + x_1 + x_2 + y_3).
\end{aligned}$$

Next we compute $w(\text{BG})$ and the action by Sq^1, Sq^2 . First,

$$Bi^*(w(ER_{ii})) = 1 + (x_1^2 + x_1x_2 + x_2^2) + x_1x_2(x_1 + x_2) \pmod{2}.$$

Let $t_2 = x_1^2 + x_1x_2 + x_2^2$ and $t_3 = x_1x_2(x_1 + x_2)$ so that $w(i^*ER_{ii}) = 1 + t_2 + t_3$. When $i \neq j$, we define as y_ℓ ,

$$\begin{aligned} Bi^*(w(ER_{ij})) &= 1 + x_1^2 + x_1x_2 + x_2^2 + x_1x_2(x_1 + x_2) + x_1x_2(x_1 + x_2)y_\ell + (x_1^2 + x_1x_2 + x_2^2)y_\ell^2 + y_\ell^4 \\ &= 1 + t_2 + t_3 + t_3y_\ell + t_2y_\ell^2 + y_\ell^4. \end{aligned}$$

We can write this fourth order term as $s_k = t_3y_\ell + t_2y_\ell^2 + y_\ell^4$, where $\{i, j, k\} = \{1, 2, 3\}$.

To summarize,

$$Bi^*(w(ER_{ij})) = \begin{cases} 1 + t_2 + t_3, & \text{if } i = j \\ 1 + t_2 + t_3 + s_k, & \text{if } \{i, j, k\} = \{1, 2, 3\} \end{cases} \quad (\text{A.1})$$

where

$$\begin{aligned} s_1 &= t_3y_2 + t_2y_2^2 + y_2^4 \\ s_2 &= t_3y_1 + t_2y_1^2 + y_1^4 \\ s_3 &= t_3y_3 + t_2y_3^2 + y_3^4. \end{aligned}$$

Remark A.1. There seems to be a minor indexing error in Stolz's work. He claims $\rho_{23} = (-1)^{y_1}\rho_{11}$ and $\rho_{13} = (-1)^{y_2}\rho_{11}$. Then $s_k = t_3y_k + t_2y_k^2 + y_k^4$, which cleans up some notation. This essentially amounts to switching the role of y_1 and y_2 in the map i . In his version, $(0, 0, 1, 0)$ would map to $[1, -1, 1]$ and $(0, 0, 0, 1)$ maps to $[-1, 1, 1]$.

Next we consider the adjoint representation \mathfrak{g} of G , which is equivalent to the conjugation action of G skew-Hermitian 3×3 quaternionic matrices.

We claim that, restricted to $P(\text{Sp}(1)^3)$, the representation \mathfrak{g} decomposes as

$$R_1 \oplus R_2 \oplus R_3 \oplus R_{23} \oplus R_{13} \oplus R_{12}.$$

The restriction here is from the $i_2 \circ i_1$, where

$$(\mathbb{Z}/2)^2 \times (\mathbb{Z}/2)^2 \xrightarrow{i} P(\mathrm{Sp}(1)^3) \xrightarrow{i_1} P(\mathrm{Sp}(2) \times \mathrm{Sp}(1)) \xrightarrow{i_2} \mathrm{PSP}(3).$$

It follows that

$$B(i_2 i_1 i)^*(w(E\mathfrak{g})) = t^3(t + s_1)(t + s_2)(t + s_3),$$

where $t = 1 + t_2 + t_3$. To simplify this, it helps to notice that

$$\begin{aligned} s_1 + s_2 &= t_3(y_1 + y_2) + t_2(y_1^2 + y_2^2) + y_1^4 + y_2^4 \\ &= t_3(y_1 + y_2) + t_2(y_1 + y_2)^2 + (y_1 + y_2)^4 \\ &= s_3. \end{aligned}$$

We have

$$B(i_2 i_1 i)^*(w(E\mathfrak{g})) = (s_1^2 s_2 + s_1 s_2^2) t^3 + (s_1^2 + s_1 s_2 + s_2^2) t^4 + t^6$$

and we let $t_8 = s_1^2 + s_1 s_2 + s_2^2$ and $t_{12} = s_1^2 s_2 + s_1 s_2^2$ so that

$$B(i_2 i_1 i)^*(w(E\mathfrak{g})) = t_{12} t^3 + t_8 t^4 + t^6. \quad (\text{A.2})$$

Now we can put this together. From (A.1), we know that t_2, t_3 are in the image of Bi^* . We can consider Bi_1^* and Bi_2^* as bundles with respective fibers $\mathbb{H}P^1$ and $\mathbb{H}P^2$, and Hurewicz's theorem shows that Bi_1^* and Bi_2^* are isomorphisms on cohomology groups of degree at most 3. In particular, t_2 and t_3 are in the image of $Bi_2 i_1 i^*$. Using Kono's computation of $H^* \mathrm{BPSp}(3)$, we can thus identify the generators t_2 and t_3 with those in $H^* \mathrm{BPSp}(3)$.

On the other hand, (A.2) shows t_8 and t_{12} are also in the image of $Bi_2 i_1 i^*$. These elements are polynomials in s_k , so they are not in the polynomial ring $\mathbb{Z}/2[t_2, t_3]$. We can thus identify t_8 and t_{12} with the generators in $H^* \mathrm{BPSp}(3)$.

A.1 ACTION BY $\mathcal{A}(1)$ ON $B\text{PSp}(3)$

In the last section, we identified $H^*B\text{PSp}(3) = \mathbb{Z}/(2)[t_2, t_3, t_8, t_{12}]$ with the subring of $\mathbb{Z}_2[x_1, x_2, y_1, y_2]$ via

$$t_2 = x_1^2 + x_1x_2 + x_2^2$$

$$t_3 = x_1x_2(x_1 + x_2)$$

$$t_8 = s_1^2 + s_1s_2 + s_2^2$$

$$t_{12} = s_1s_2(s_1 + s_2)$$

where

$$s_1 = t_3y_2 + t_2y_2^2 + y_2^4 \quad \text{and} \quad s_2 = t_3y_1 + t_2y_1^2 + y_1^4.$$

For a generator $x \in H^*B\mathbb{Z}_2$, we have $\text{Sq}^1 x = x^2$ and $\text{Sq}^2 x = 0$. We use this to compute the actions of Sq^1 and Sq^2 on cohomology classes.

Theorem A.2. *Identifying $H^*B\text{PSp}(3) = \mathbb{Z}_2[t_2, t_3, t_8, t_{12}]$, we have*

$$\begin{array}{llll} \text{Sq}^1 t_2 = t_3 & \text{Sq}^1 t_3 = 0 & \text{Sq}^1 t_8 = 0 & \text{Sq}^1 t_{12} = 0 \\ \text{Sq}^2 t_2 = t_2^2 & \text{Sq}^2 t_3 = t_2t_3 & \text{Sq}^2 t_8 = 0 & \text{Sq}^2 t_{12} = t_2t_{12}. \end{array}$$

Proof. Using naturality, we compute

$$\begin{aligned} \text{Sq}^1(t_2) &= \text{Sq}^1(x_1^2 + x_1x_2 + x_2^2) \\ &= \text{Sq}(x_1)x_2 + x_1\text{Sq}(x_2) \\ &= x_1^2x_2 + x_1x_2^2 \\ &= t_3 \end{aligned}$$

and

$$\text{Sq}^2(t_2) = \text{Sq}^2(x_1^2 + x_1x_2 + x_2^2)$$

$$\begin{aligned}
&= x_1^4 + \text{Sq}^2(x_1)x_2 + \text{Sq}^1(x_1)\text{Sq}^1(x_2) + x_1\text{Sq}^2(x_2) + x_2^4 \\
&= x_1^4 + x_1^2x_2^2 + x_2^4 \\
&= (x_1^2 + x_1x_2 + x_2^2)^2 \\
&= t_2^2
\end{aligned}$$

(which is expected since t_2 has degree 2). Next

$$\begin{aligned}
\text{Sq}^1(t_3) &= \text{Sq}^1(x_1x_2(x_1 + x_2)) \\
&= \text{Sq}^1(x_1x_2)(x_1 + x_2) + x_1x_2(\text{Sq}^1(x_1) + \text{Sq}^1(x_2)) \\
&= (x_1^2x_2 + x_1x_2^2)(x_1 + x_2) + x_1x_2(x_1^2 + x_2^2) \\
&= x_1x_2(x_1 + x_2)^2 + x_1x_2(x_1^2 + x_2^2) \\
&= x_1x_2(x_1^2 + x_2^2) + x_1x_2(x_1^2 + x_2^2) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\text{Sq}^2(t_3) &= \text{Sq}^2(x_1x_2(x_1 + x_2)) \\
&= \text{Sq}^2(x_1x_2)(x_1 + x_2) + \text{Sq}^1(x_1x_2)\text{Sq}^1(x_1 + x_2) + x_1x_2\text{Sq}^2(x_1 + x_2) \\
&= (x_1x_2)^2(x_1 + x_2) + (x_1^2x_2 + x_1x_2^2)(x_1^2 + x_2^2) \\
&= (x_1x_2)^2(x_1 + x_2) + (x_1x_2)(x_1 + x_2)^3 \\
&= t_3(x_1x_2 + (x_1 + x_2)^2) \\
&= t_2t_3.
\end{aligned}$$

Continuing,

$$\begin{aligned}
\text{Sq}^1(s_1) &= \text{Sq}^1(t_3y_2 + t_2y_2^2 + y_2^4) \\
&= \text{Sq}^1(t_3)y_2 + t_3y_2^2 + \text{Sq}^1(t_2)y_2^2
\end{aligned}$$

$$\begin{aligned}
&= t_3 y_2^2 + t_3 y_2^2 \\
&= 0
\end{aligned}$$

and similarly $\text{Sq}^1(s_2) = 0$. Further,

$$\begin{aligned}
\text{Sq}^2(s_1) &= \text{Sq}^2(t_3 y_2 + t_2 y_2^2 + y_2^4) \\
&= \text{Sq}^2(t_3) y_2 + t_3 \text{Sq}^2(y_2) + t_2^2 y_2^2 + t_2 \text{Sq}^2(y_2^2) + \text{Sq}^2(y_2^4) \\
&= t_2 t_3 y_2 + t_2^2 y_2^2 + t_2 y_2^4 + \text{Sq}^2(y_2^2) y_2^2 + \text{Sq}^1(y_2^2) \text{Sq}^1(y_2^2) + y_2^2 \text{Sq}^2(y_2^2) \\
&= t_2 t_3 y_2 + t_2^2 y_2^2 + t_2 y_2^4 + y_2^6 + y_2^6 \\
&= t_2(t_3 y_2 + t_2 y_2^2 + y_2^4) \\
&= t_2 s_1
\end{aligned}$$

and similarly $\text{Sq}^2(s_2) = t_2 s_2$. Finally, we can compute

$$\begin{aligned}
\text{Sq}^1(t_8) &= \text{Sq}^1(s_1^2 + s_1 s_2 + s_2^2) \\
&= \text{Sq}^1(s_1) s_2 + s_1 \text{Sq}^1(s_2) \\
&= 0
\end{aligned}$$

as well as

$$\begin{aligned}
\text{Sq}^2(t_8) &= \text{Sq}^2(s_1^2 + s_1 s_2 + s_2^2) \\
&= \text{Sq}^2(s_1) s_1 + \text{Sq}^1(s_1) \text{Sq}^1(s_1) + s_1 \text{Sq}^2(s_1) \\
&\quad + \text{Sq}^2(s_1) s_2 + \text{Sq}^1(s_1) \text{Sq}^1(s_2) + s_1 \text{Sq}^2(s_2) \\
&\quad + \text{Sq}^2(s_2) s_2 + \text{Sq}^1(s_2) \text{Sq}^1(s_2) + s_2 \text{Sq}^2(s_2)
\end{aligned}$$

$$\begin{aligned}
&= \text{Sq}^2(s_1)s_2 + s_1 \text{Sq}^2(s_2) \\
&= t_2 s_1 s_2 + t_2 s_1 s_2 \\
&= 0.
\end{aligned}$$

Then

$$\begin{aligned}
\text{Sq}^1(t_{12}) &= \text{Sq}^1(s_1 s_2 (s_1 + s_2)) \\
&= \text{Sq}^1(s_1 s_2)(s_1 + s_2) + s_1 s_2 \text{Sq}^1(s_1 + s_2) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\text{Sq}^2(t_{12}) &= \text{Sq}^2(s_1 s_2 (s_1 + s_2)) \\
&= \text{Sq}^2(s_1 s_2)(s_1 + s_2) + \text{Sq}^1(s_1 s_2) \text{Sq}^1(s_1 + s_2) + s_1 s_2 \text{Sq}^2(s_1 + s_2) \\
&= (\text{Sq}^2(s_1)s_2 + \text{Sq}^1(s_1) \text{Sq}^1(s_2) + s_1 \text{Sq}^2(s_2))(s_1 + s_2) + t_2 s_1 s_2 (s_1 + s_2) \\
&= t_2 t_{12}.
\end{aligned}$$

□

A.2 COHOMOLOGY OF $\text{BPSp}(2, 1)$

We write $H = \text{PSp}(2, 1) := \text{P}(\text{Sp}(2) \times \text{Sp}(1))$ and $G = \text{PSp}(3)$. We have the bundle

$$\mathbb{H}P^2 = G/H \rightarrow BH \xrightarrow{\pi} BG.$$

Let τ denote the corresponding vertical bundle along the fibers of π .

Also recall the maps

$$(\mathbb{Z}/2)^2 \times (\mathbb{Z}/2)^2 \xrightarrow{i} P(\mathrm{Sp}(1)^3) \xrightarrow{i_1} H \xrightarrow{i_2} G.$$

Note that $Bi_2 = \pi$.

Claim 2: restricted to $H = P(\mathrm{Sp}(2) \times \mathrm{Sp}(1))$, the representation \mathfrak{g} splits as $\mathfrak{h} \oplus \mathfrak{h}^\perp$, with \mathfrak{h} the adjoint representation of H and $\mathfrak{h}^\perp \cong \tau$. When restricted to $P(\mathrm{Sp}(1)^3)$, \mathfrak{h}^\perp splits as $R_{13} \oplus R_{23}$.

Assuming claim 2 holds,

$$\begin{aligned} B(i_1 i)^* w(\tau) &= (t + s_1)(t + s_2) \\ &= t^2 + (s_1 + s_2)t + s_1 s_2 \\ &= 1 + s_1 + s_2 + t_2^2 + t_3^2 + (s_1 + s_2)t_2 + (s_1 + s_2)t_3 + s_1 s_2. \end{aligned}$$

Define

$$u_2 = t_2 \quad u_3 = t_3 \quad u_4 = s_1 + s_2 \quad u_8 = s_1 s_2.$$

Then

$$B(i_1 i)^* w(\tau) = 1 + (u_4 + u_2^2) + (u_3^2 + u_2 u_4) + u_3 u_4 + u_8.$$

This means that the class $w_4(\tau)$ restricts to a nontrivial element in $H^4 \mathbb{H}P^2$, and thus the map $H^*BH \rightarrow H^*G/H$ is surjective. The Serre spectral sequence of π collapses, and $\pi^*: H^*BG \rightarrow H^*BH$ is thus injective. Moreover, by Leray-Hirsch, H^*BH is a free H^*BG module with basis $\{1, w_4(\tau), w_4(\tau)^2\}$. We can therefore identify H^*BH with the subring of $\mathbb{Z}/(2)[u_2, u_3, u_4, u_8]$ where $\pi^*(t_2) = u_2$, $\pi^*(t_3) = u_3$, $\pi^*(t_8) = u_4^2 + u_8$, and $\pi^*(t_{12}) = u_4 u_8$. Under this identification, we have

$$w(\tau) = 1 + (u_4 + u_2^2) + (u_3^2 + u_2 u_4) + u_3 u_4 + u_8.$$

APPENDIX B

PRIMITIVE GENERATORS

Since $\mathbb{R}\mathbb{P}^\infty$ is an h -space, the cohomology forms a Hopf algebra. As an algebra, $H^*\mathbb{R}\mathbb{P}^\infty$ is the polynomial ring $\mathbb{Z}_2[u]$, where $\deg(u) = 1$. The coalgebra structure is determined by the fact that the coproduct Δ is an algebra homomorphism and $\Delta(u) = 1 \otimes u + u \otimes 1$. For the algebra structure in homology, $H_*\mathbb{R}\mathbb{P}^\infty = \bigwedge (x_1, x_2, x_4, x_8, \dots)$ is the exterior algebra on generators x_n , where x_n is the linear dual of u^{2^n} .

We know that $H^*BO = \mathbb{Z}_2[w_1, w_2, w_3, \dots]$. Dually, $H_*BO = \mathbb{Z}_2[u_1, u_2, u_3, \dots]$, where u_i is the linear dual of w_1^i . The canonical inclusion $j: BO(1) \rightarrow BO$ then satisfies $j_*x_n = u_{2^n}$. Clearly there is one indecomposable element in each degree, hence one primitive in each degree. In this case, the primitive elements are just s_n (the symmetric polynomial generated by $x_1^n + \dots + x_k^n$ written in terms of the elementary symmetric polynomials w_i , where $k \geq n$). s

As an algebra, $H^*BSO = \mathbb{Z}_2[w_n : n \geq 2]$. There is one indecomposable element in degree n for every $n \geq 2$, so, dually, $P^n H^*BSO$ is \mathbb{Z}_2 for $n \geq 2$ and 0 otherwise. The primitives of H^*BSO are not just those for H^*BO in degrees $n \geq 2$. The reason is because s_n is equal to $(-1)^{n+1}nw_n$ plus decomposables, so in even degrees we need to know that s_n has a nontrivial decomposable term which is not a multiple of w_1 . For example,

$$s_4 = w_1^4 - 4w_1^2w_2 + 4w_1w_3 + 2w_2^2 - 4w_4 = w_1^4 \in H^*BO,$$

so s_4 vanishes when restricted to H^*BSO . However, we can use the following lemma.

Lemma B.1. *If x is primitive, then x^{2^n} is primitive for all n .*

Proof. Using the Frobenius map,

$$\Delta(x^{2^n}) = \Delta(x)^{2^n} = (1 \otimes x + x \otimes 1)^{2^n} = 1 \otimes x^{2^n} + x^{2^n} \otimes 1.$$

□

For n even, we write $n = 2^k m$ for some $k \geq 1$ and odd m . Then as long as $m \geq 1$, we know s_m is a nontrivial primitive in $H^* \text{BSO}$. The primitive in degree n is therefore $s_m^{2^k}$. We still have not accounted for the primitives whose degrees are powers of 2. For degree reasons, w_2 is primitive (note that $w_2 \neq s_2$ since $s_2 = s_1^2 \pmod{2}$). This means $w_2^{2^n}$ is the primitive element of degree 2^{n+1} . We summarize this below.

Theorem B.2. *The coalgebra $H^* \text{BSO}$ has one primitive element v_n of degree n for all $n \geq 2$, where*

$$v_n = \begin{cases} w_2, & \text{if } n = 2 \\ s_n, & \text{if } n \text{ is odd} \\ s_{n/2}^2, & \text{if } n > 2 \text{ is even.} \end{cases}$$

Proof. We already computed the following:

$$v_n = \begin{cases} s_n, & \text{if } n \text{ is odd} \\ s_m^{2^k}, & \text{if } n = 2^k m, \text{ where } m > 1 \text{ is odd and } k \geq 1 \\ w_2^{2^{k-1}}, & \text{if } n = 2^k. \end{cases}$$

If n is even and not a power of 2, we can write $n = 2^k m$ for $k \geq 1$ and $m > 1$ odd. We then have $v_n = s_m^{2^k}$. If $k > 1$, then $n/2$ is also even and not a power of 2, so $v_{n/2} = s_m^{2^{k-1}}$. In this case $v_n = v_{n/2}^2$. Now if $k = 1$ (so $n = 2m$), then $v_n = s_m^2$ and $v_{n/2} = v_m = s_m$. Again we have $v_n = v_{n/2}^2$. Next suppose $n = 2^k$. For all $k \geq 1$, we saw $v_n = w_2^{2^{k-1}}$. When $k > 1$, $n/2 = 2^{k-1}$ is also an even power of 2, so $v_{n/2} = w_2^{2^{k-2}}$. Thus $v_n = v_{n/2}^2$ as claimed. □

Theorem B.3. *The primitive elements of $H^*\text{BSpin}$ comprise one generator z_n for each $n \geq 4$ not of the form $2^s + 1$, where*

$$z_n = \begin{cases} s_n, & \text{if } n \neq 2^s + 1 \text{ is odd} \\ w_4^{n/4}, & \text{if } \alpha(n) = 1 \text{ and } n \text{ is a power of } 2 \\ s_{m,m}^{n/(2m)}, & \text{if } \alpha(n) = 2 \text{ and } n = 2^k m \text{ for } m = 2^s + 1 \\ z_{n/2}^2, & \text{if } \alpha(n) \geq 3 \text{ and } n \text{ is even.} \end{cases}$$

Here $\alpha(n)$ is the number of 2-bits in the binary expansion of n . The first definition only applies for $n \neq 2^s + 1$, but this covers all odd-degree primitive generators; the third through fourth definitions together cover the all even degrees starting with 4.

Proof. In $H^*\text{BO}$, s_n is nw_n plus decomposable elements, so s_n restricts to a nontrivial primitive in $H^*\text{BSpin}$ provided that n is odd, $n \geq 4$, and $n \neq 2^s + 1$ (these are the degrees in which w_n is nontrivial). When n is even, start by writing $n = 2^k m$ for $k \geq 1$ and for odd $m > 1$. If $m \neq 2^s + 1$ (note that this precludes $m = 3$), then s_m is a nontrivial primitive in $H^*\text{BSpin}$ and we can set $z_n = s_m^{2^k}$. Equivalently, $z_n = z_{n/2}^2$. The condition $m \neq 2^s + 1$ means $\alpha(n) \neq 2$, and $m > 1$ means $\alpha(n) \neq 1$. Finally we consider when $m = 2^s + 1$. Then $n = 2^k(2^s + 1) = 2^{k+s} + 2^k$, so $\alpha(n) = 2$. The class s_m is not primitive, and in fact is zero by lemma B.4. It follows that $s_{m,m}$ is primitive since

$$\Delta(s_{m,m}) = s_{m,m} \otimes 1 + s_m \otimes s_m + 1 \otimes s_{m,m} = s_{m,m} \otimes 1 + 1 \otimes s_{m,m}.$$

We can now define $z_n = s_{m,m}^{2^{k-1}}$ for these values of n . □

Lemma B.4. *For all k , s_{2^k+1} vanishes when restricted to $H^*\text{BSpin}$.*

Proof. Stong shows that the natural map $H^*\text{BSO} \rightarrow \mathbb{H}^*\text{BSpin}$ is epic with kernel $\mathcal{A}w_2$. In particular w_2 vanishes in $H^*\text{BSpin}$, where also $w_2 = s_2$. By definition s_{2^k+1}

is the sum $\sum_i x_i^{2^k+1}$ expressed in terms of the elementary symmetric polynomials w_j in the x_i . We can compute the total square easily:

$$\begin{aligned}
\text{Sq}(s_{2^k+1}) &= \text{Sq}\left(\sum_i x_i^{2^k+1}\right) \\
&= \sum (x_i + x_i^2)^{2^k+1} \\
&= \sum \left((x_i + x_i^2) (x_i + x_i^2)^{2^k} \right) \\
&= \sum \left((x_i + x_i^2) (x_i^{2^k} + x_i^{2^k+1}) \right) \\
&= \sum \left(x_i^{2^k+1} + x_i^{2^k+2} + x_i^{2^k+1+1} + x_i^{2^k+1+2} \right) \\
&= s_{2^k+1} + s_{2^k+2} + s_{2^k+1+1} + s_{2^k+1+2}.
\end{aligned}$$

In particular $\text{Sq}^{2^k}(s_{2^k+1}) = s_{2^k+1+1}$, so by induction s_{2^k+1} vanishes for all $k \geq 0$ in $H^*\text{BSpin}$. \square

Remark B.5. A potential point of confusion arises when we identify $H^*\text{BSpin}$ with $\mathbb{Z}_2[w_n : n > 2, n \neq 2^k + 1]$. Namely, this is an isomorphism of rings, but not of \mathcal{A} -modules, assuming the \mathcal{A} -action on Stiefel-Whitney classes is inherited from the action on $H^*\text{BO}$. It is better to therefore identify $H^*\text{BSpin}$ with the quotient of $H^*\text{BSO} = \mathbb{Z}_2[w_n : n \geq 2]$ by $\mathcal{A}w_2$. To highlight the potential issue, consider the class s_{17} . Lemma B.4 shows s_{17} is trivial in $H^*\text{BSpin}$, but direct computation (using Newton's identities, for example) shows that s_{17} does not vanish when simply setting $w_n = 0$ for all $n \geq 2, n \neq 2^k + 1$. In particular $s_{17} = w_7w_{10} + w_6w_{11} + w_4w_{13}$. This is a low-degree case which demonstrates $H^*\text{BSpin}$ is not isomorphic to $\mathbb{Z}_2[w_n : n > 2, n \neq 2^k + 1]$ as an \mathcal{A} -module.

Theorem B.6. *We have $PH^n\text{BSpin}^c \cong \mathbb{Z}_2$ whenever $n \geq 2$ and $n \neq 2^k + 1$, and $PH^n\text{BSpin}^c = 0$ for other n . Let z_n be the primitive generator of $PH^n\text{BSpin}^c$ in*

degree n for some $n \geq 2$ with $n \geq 2^k + 1$. Then

$$z_n = \begin{cases} s_n, & \text{if } \alpha(n) \geq 3, \\ s_{n/2, n/2}, & \text{if } \alpha(n) = 2, \\ w_2^{n/2}, & \text{if } \alpha(n) = 1. \end{cases}$$

Proof. The Girard formula shows that s_n is nw_n plus decomposable elements. Thus $z_n = s_n$ for all odd n . If n is a power of 2, then $z_n = w_2^{n/2}$. The remaining case is when $n = 2^i m$ for odd $m > 1$ and $i > 0$. When $m \neq 2^k + 1$, s_m is primitive, so $z_n = s_m^{2^i} = s_{2^i m} = s_n$. If $m = 2^k + 1$, one can show that $s_{m, m}$ is primitive, and thus $z_n = s_{m, m}^{2^{i-1}} = s_{n/2, n/2}$. This proves the claim. \square

Note that over \mathbb{Z} , $s_n^2 = 2s_{n, n} + s_{2n}$, so $s_{n, n} = \frac{1}{2}(s_n^2 - s_{2n})$. We can use this to compute $s_{n, n}$ modulo 2 if we know s_n over \mathbb{Z} .

APPENDIX C

COMPUTATION OF THE Spin^c TRANSFER MAP

All coefficients are taken in \mathbb{Z}_2 unless stated otherwise. In this proof we refer to integration over the fiber, which we define first here: let X be a Poincaré duality space of formal dimension n with a (known) orientation, and let $X \rightarrow E \xrightarrow{\pi} B$ be a fibration for which $\pi_1 B$ acts on X by orientation preserving homotopy equivalences. Then integration along the fiber $\pi_! : H^{k+n} E \rightarrow H^k B$ is defined as the composite

$$H^{k+n} E \twoheadrightarrow E_\infty^{k,n} \hookrightarrow E_2^{k,n} = H^k(B; H^n X) \rightarrow H^k B.$$

The first two maps come from the fact that $E_2^{k,\ell} = 0$ for $\ell > n$ and the last map comes from the orientation.

Theorem C.1. *The group $G := \text{SU}(3)$ acts transitively on the space $\mathbb{C}\text{P}^2$ with fiber $H := S(\text{U}(1) \times \text{U}(2))$. This gives a bundle*

$$\mathbb{C}\text{P}^2 \rightarrow B(S(\text{U}(2) \times \text{U}(1))) \xrightarrow{\pi} \text{BSU}(3).$$

with associated vertical bundle τ . We exhibit classes cohomology x_i, y_i and prove the following:

1. $H^* B H \cong \mathbb{Z}_2[x_2, x_4]$;
2. $H^* B G = \mathbb{Z}_2[y_4, y_6]$;
3. $B\pi^* y_4 = x_2^2 + x_4$ and $B\pi^* y_6 = x_2 x_4$;
4. *we have*
 - a) $\text{Sq}^1(y_4) = 0$ and $\text{Sq}^2(y_4) = y_6$;
 - b) $\text{Sq}^1(y_6) = 0$ and $\text{Sq}^2(y_6) = 0$;

5. $w(\tau) = 1 + x_2 + x_4$

6. $\pi_! : H^n BH \rightarrow H^{n-4} BG$ is given modulo y_6 by

$$\pi_!(x_2^a x_4^b) = \begin{cases} y_4^{a/2-1}, & \text{if } a > 0 \text{ is even and } b = 0, \\ y_4^{b-1}, & \text{if } a = 0 \text{ and } b > 0, \\ 0, & \text{if otherwise.} \end{cases}$$

Proof of (1) – (4): We have a commuting diagram as shown, where the horizontal rows are fiber bundles.

$$\begin{array}{ccccc} U(1) & \longrightarrow & BS(U(1) \times U(2)) & \xrightarrow{f} & B(U(1) \times U(2)) \\ \parallel & & \downarrow \pi & & \downarrow j \\ U(1) & \longrightarrow & BSU(3) & \xrightarrow{g} & BU(3) \\ \parallel & & \downarrow & & \downarrow \\ U(1) & \longrightarrow & EU(1) & \longrightarrow & BU(1) \end{array}$$

The map $BU(3) \rightarrow BU(1)$ is induced by the determinant. Comparing spectral sequences shows that the generator of $H^1 U(1)$ transgresses to c_2 , where $H^* BU(3) = \mathbb{Z}_2[c_2, c_4, c_6]$. Thus $H^* BSU(3) = \mathbb{Z}_2[y_4, y_6]$ where $g^* c_4 = y_4$ and $g^* c_6 = y_6$. Next, write $H^* BU(1) = \mathbb{Z}_2[a_2]$ and $H^* BU(2) = \mathbb{Z}_2[b_2, b_4]$. Using this to identify $H^* B(U(1) \times U(2)) = \mathbb{Z}_2[a_2, b_2, b_4]$, we can compute j^* via the product formula for Chern classes as shown.

$$\begin{array}{ll} g^* c_2 = 0 & j^* c_2 = a_2 + b_2 \\ g^* c_4 = y_4 & j^* c_4 = a_2 b_2 + b_4 \\ g^* c_6 = y_6 & j^* c_6 = a_2 b_4. \end{array}$$

By composing f with the projection to $BU(2)$, we obtain generators $x_2 = f^* b_2$ and $x_4 = f^* b_4$ in $H^* BS(U(1) \times U(2)) = \mathbb{Z}_2[x_2, x_4]$. Comparing spectral sequences again,

the generator of $H^1U(1)$ in the top row transgresses to $j^*c_2 = a_2 + b_2$, and this is the only nontrivial differential. Thus we also have $f^*a_2 = f^*(a_2 + b_2 + b_2) = f^*b_2 = x_2$. Finally we compute

$$\pi^*y_4 = \pi^*g^*c_4 = f^*j^*c_4 = f^*(a_2b_2 + b_4) = x_2^2 + x_4$$

and

$$\pi^*y_6 = \pi^*g^*c_6 = f^*j^*c_6 = f^*(a_2b_4) = x_2x_4,$$

thus establishing parts (1) – (3) of the theorem. For part (4), note that $\text{Sq}^1 y_4 = \text{Sq}^1 y_6 = 0$ for dimension reasons. Temporarily write x, y, z for the (mod 2) Chern roots of $\text{BU}(3)$. Then $\text{Sq}^2 c_4 = c_2c_4 + c_6$ and $\text{Sq}^2 c_6 = c_2c_6$ as shown.

$$\begin{aligned} \text{Sq}^2 c_4 &= \text{Sq}^2(xy + xz + yz) & \text{Sq}^2 c_6 &= \text{Sq}^2(xyz) \\ &= (x + y)xy + (x + z)xz + (y + z)yz & &= (x + y + z)(xyz) \\ &= (x + y + z)(xy + xz + yz) + 3xyz & &= c_2c_6 \\ &= c_2c_4 + c_6 & & \end{aligned}$$

Thus $\text{Sq}^2 y_4 = \text{Sq}^2 g^*(c_4) = g^* \text{Sq}^2(c_4) = g^*(c_2c_4 + c_6) = y_6$, and similarly $\text{Sq}^2 y_6 = 0$.

Proof of (5): Let $\tilde{\tau}$ be the vertical bundle of $B(U(2) \times U(1)) \rightarrow \text{BU}(3)$. The total space of the universal $U(3)$ -bundle is $EU(3) \times_{U(3)} \mathfrak{g}$, where \mathfrak{g} is the Lie algebra of $U(3)$. Considering $B(U(2) \times U(1))$ as $EU(3)/(U(2) \times U(1))$, we see that

$$\tilde{\tau} = EU(3) \times \mathfrak{h}^\perp$$

where $\mathfrak{h}^\perp \cong \mathfrak{g}/\mathfrak{h}$ is a chosen orthogonal complement and \mathfrak{h} is the Lie algebra of $U(2) \times U(1)$.

We now compute the action of $U(2) \times U(1)$ on \mathfrak{h}^\perp . An arbitrary element in $U(2)$ and \mathfrak{h}^\perp can be written respectively as

$$\begin{pmatrix} a & b & 0 \\ -u\bar{b} & u\bar{a} & 0 \\ 0 & 0 & z \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & v_1 \\ 0 & 0 & v_2 \\ -\bar{v}_1 & -\bar{v}_2 & 0 \end{pmatrix},$$

where $a\bar{a} + b\bar{b} = 1$ and $u\bar{u} = z\bar{z} = 1$. We write the former as (P, z) and latter as v . The adjoint action is then $(P, z)v(P, z)^{-1}$. As matrices this takes v to

$$\begin{pmatrix} 0 & 0 & \bar{z}(av_1 + bv_2) \\ 0 & 0 & u\bar{z}(\bar{a}v_2 - \bar{b}v_1) \\ -z(\bar{a}\bar{v}_1 + \bar{b}\bar{v}_2) & \bar{u}z(b\bar{v}_1 - a\bar{v}_2) & 0 \end{pmatrix}.$$

At the same time

$$\begin{pmatrix} a & b \\ -u\bar{b} & u\bar{a} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} av_1 + bv_2 \\ u(\bar{a}v_2 - \bar{b}v_1) \end{pmatrix},$$

and thus the adjoint action of (P, z) on v is $v \mapsto (P, z) \cdot v = Pv\bar{z}$. Include $\mathbb{Z}_2^3 \rightarrow U(2) \times U(1)$ with the first factor mapping to z and the second two mapping to $U(2)$ (to be consistent with earlier notation). We thus compute

$$\begin{pmatrix} (-1)^j & 0 & 0 \\ 0 & (-1)^k & 0 \\ 0 & 0 & (-1)^i \end{pmatrix} \cdot v = \begin{pmatrix} (-1)^j & 0 \\ 0 & (-1)^k \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} (-1)^i = \begin{pmatrix} (-1)^{i+j}v_1 \\ (-1)^{i+k}v_2 \end{pmatrix}.$$

Writing $H^*B\mathbb{Z}_2^3 = \mathbb{Z}_2[x, y, z]$, we now compute

$$w(\tilde{\tau}) = (1 + x + y)(1 + x + z) = 1 + y + z + xy + xz + yz + x^2.$$

Previously we wrote $H^*B(U(1) \times U(2)) = \mathbb{Z}_2[a_2, b_2, b_4]$ and identified $a_2 = x$, $b_2 = y + z$, and $b_4 = yz$. This shows $w(\tilde{t}) = 1 + b_2 + a_2b_2 + b_4 + a_2^2$. Finally, we pull back

to $BS(U(1) \times U(2))$ to get $w(\tau) = 1 + x_2 + x_2^2 + x_4 + x_2^2 = 1 + x_2 + x_4$.

Proof of (6): Previously we saw $H^*BH = \mathbb{Z}_2[x_2, x_4]$ and $H^*BG = \mathbb{Z}_2[y_4, y_6]$ where $\pi^*y_4 = x_2^2 + x_4$ and $\pi^*y_6 = x_2x_4$. We now set $H^*\mathbb{C}P^2 = \mathbb{Z}_2[u]/(u^3)$. For degree reasons, the spectral sequence associated to $\mathbb{C}P^2 \rightarrow BH \rightarrow BG$ is trivial, so we necessarily have $i^*x_2 = u$. Then $i^*x_2^2 = u^2$, and since $\pi \circ i$ is trivial,

$$i^*x_4 = i^*(x_4 + x_2^2 + x_2^2) = i^*(\pi^*(y_4) + x_2^2) = u^2.$$

Now as a module over H^*BG , H^*BH is free with basis $\{1, u, u^2\}$. Thus for any $x \in H^*BH$ we can uniquely identify x with $r_0(x) + r_1(x)u + r_2(x)u^2$, where $r_i(x) \in H^*BG$. Integration along the fiber of τ is an H^*BG -module morphism $H^nBH \rightarrow H^{n-4}BG$, so it remains to determine this map on the basis elements $1, u, u^2$. For degree reasons 1 and u map to 0 , and u^2 maps to either 1 or 0 . Since u^2 is already an element on $H^*\mathbb{C}P^2$, it is in the image of $H^4BH \rightarrow E_\infty^{0,4} \rightarrow E_2^{0,4}$, so u^2 maps to 1 .

To compute the transfer map we now must write all monomials in x_2, x_4 in terms of $x_2^2 + x_4 = \pi^*(y_4)$ and $x_2x_4 = \pi^*(y_6)$ in the basis $1, x_2, x_2^2$. Note that $\pi_!$ is a H^*BG -module map, so for $y \in H^*BG$ and $x \in H^*BH$, we have $\pi_!(\pi^*(y)x) = y\pi_!(x)$.

Since

$$\pi_!(x_2^{n+k}x_4^n) = \pi_!((x_2x_4)^n x_2^k) = \pi_!(\pi^*(y_6^n)x_2^k) = y_6^n \pi_!(x_2^k) \quad \text{and} \quad \pi_!(x_2^n x_4^{n+k}) = y_6^n \pi_!(x_4^k),$$

we only need to compute $\pi_!(x_2^n)$ and $\pi_!(x_4^n)$. We also observe for $n \geq 3$

$$\pi_!(x_2^n) = \pi_!(x_2^{n-2}(x_2^2 + x_4) + x_2^{n-2}x_4) = \pi_!(x_2^{n-2})y_4 + \pi_!(x_2^{n-3})y_6$$

and

$$\pi_!(x_4^n) = \pi_!(x_4^{n-1}(x_4 + x_2^2) + x_4^{n-1}x_2^2) = \pi_!(x_4^{n-1})y_4 + \pi_!(x_4^{n-3})y_6^2.$$

It now remains to compute the six base cases $\pi_!(x_2^i)$ and $\pi_!(x_4^i)$ for $0 \leq i \leq 2$. For degree reasons $\pi_!(1) = \pi_!(x_2) = 0$, and our chosen Leray-Hirsch isomorphism gives $\pi_!(x_2^2) = 1$. Then $\pi_!(x_4) = \pi_!(x_2^2 + x_4) + \pi_!(x_2^2) = y_4\pi_!(1) + 1 = 1$, and similarly

$$\pi_!(x_4^2) = \pi_!((x_2^2 + x_4)^2) + \pi_!(x_4^2) = \pi_!(x_2^2(x_2^2 + x_4)) + \pi_!(x_2^2x_4) = \pi_!(x_2^2)y_4 + \pi_!(x_2)y_6 = y_4.$$

From the recurrence relation, it follows that, modulo y_6 , we have $\pi_!(x_2^{2n}) = y_4^{n-1}$, $\pi_!(x_2^{2n+1}) = 0$, and $\pi_!(x_4^n) = y_4^{n-1}$ for $n > 0$. This implies statement (6) and completes the proof of theorem C.1.

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