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The Effect on Inequality of Changing One or Two Incomes

by

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Abstract: We examine the effect on inequality of increasing one income, and show that for two wide classes of indices a benchmark income level or position exists, dividing upper from lower incomes, such that if a lower income is raised, inequality falls, and if an upper income is raised, inequality rises. We provide a condition on the inequality orderings implicit in two inequality indices under which the one has a lower benchmark than the other for all unequal income distributions. We go on to examine the effect on the same indices of simultaneously increasing one income and decreasing another higher up the distribution, deriving results which quantify the extent of the "bucket leak" which can be tolerated without negating the beneficial inequality effect of the transfer. Our results have implications for the inequality impacts of different income growth patterns, and of redistributive programmes (leaky or not), which are briefly discussed.

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1. Introduction

In an unequal two-person society, the effect on inequality of increasing one of the two incomes is clear: inequality falls if we increase the lower income of the two, and rises if we increase the upper income. With more than two people, the effect on inequality of increasing one income is very much less clear, and has not, to our knowledge, been studied closely. We obtain a range of definitive results here, showing that the insight from the two-person society carries over in essence to inequality indices, if not to the Lorenz configuration. Namely, if a low income is raised, inequality falls, and if a high income is raised, inequality rises; and there is a specific income level, or position in the distribution, determined by the particular inequality index one is using, which divides these effects. We shall call this the "benchmark" income or position in what follows.

A condition between two inequality orderings, represented by indices, emerges which, if satisfied, ensures that the one index has an always lower benchmark than the other, whatever the income distribution to which both are applied. We believe this condition to be new; it evinces a Rawlsian-type measure which we call the "lower tail concern" of an inequality ordering.

We go on to examine the effect of simultaneously increasing one income and decreasing another higher up the distribution. We already know, of course, that a pure rich-to-poor transfer must reduce inequality, but we are curious about the extent of the "leak" which might be tolerated, having taken \$1 from a person, and before giving the proceeds to another person further down the distribution, without negating the beneficial inequality effect of the transfer. Our analytics enable us to study this "leaky bucket" issue closely, and we uncover some perhaps surprising properties. If a transfer is made from someone above the benchmark to someone below, inequality falls as a result of the first part of this transfer; and again as a result of the second part; a leak of more than 100% could be tolerated in such a case (i.e. money taken from both). If the donor and recipient are both on the same side of the benchmark, there is a range of possibilities. The intuitively agreeable case, a leak of between 0% and 100% , can arise and the percentage can be quantified. However it is also possible in this case to find that the leak can exceed the amount taken away, and in some circumstances the leak may even be negative - the recipient could receive more than the donor gives up - somebody can be adding water to the bucket. We believe that these results are both novel and interesting. They are quite distinct from the leaky bucket findings of authors such as Atkinson (1980), Jenkins (1991) and Duclos (2000) in the welfare context, in which, following Okun (1975, pp. 91-95), the maximum leak before a *welfare loss* is experienced is quantified;[1] not least, such a leak cannot be negative, nor exceed 100%.

The structure of the paper is as follows. In Section 2, we lay out the notation and preliminaries in terms of which the analysis will proceed. In Section 3, we comment briefly upon the implications for the Lorenz curve of increasing one income, and this provides a pointer to effects on some inequality indices. We establish a central result here: a benchmark income or position exists for any Lorenz-consistent inequality index. In Section 4, we examine the nature and properties of the benchmark for two wide classes of inequality indices, deriving explicit results for many familiar indices,[2] and a general insight that relates the benchmark to the lower tail concern of the underlying inequality ordering. In Section 5, we examine the leaky bucket issue in some depth. Section 6 concludes.

2. Notation and Preliminaries

Let the population size be N > 2. Income distributions $\mathbf{x} = (x_1, x_2, \ldots, x_i, \ldots, x_N)$ will be assumed throughout to be non-decreasingly ordered, $\mathbf{x} \in \Omega_1 = \{\mathbf{x} \in \Re_+^N : x_1 \le x_2 \le \ldots \le x_i \le \ldots \le x_N\}$, with mean $\mu(\mathbf{x}) = \frac{1}{N}\sum_i x_i$. For technical reasons we will sometimes need to restrict attention to the subsets $\Omega_2 = \{\mathbf{x} \in \Re_+^N : x_1 < x_2 \le \ldots \le x_i \le \ldots \le x_N\}$ and $\Omega_3 = \{\mathbf{x} \in \Re_+^N : x_1 < x_2 < \ldots < x_i < \ldots < x_N\} \subset \Omega_2 \subset \Omega_1$. For an unequal $\mathbf{x} \in \Omega_1$, let $\delta(\mathbf{x}) = \mathit{min}\{x_{i+1} - x_i : x_i \ne x_{i+1}\} > 0$ be the smallest gap between two adjacent, non-identical incomes, and for $1 \le i \le N$ and $0 < \delta < \delta(\mathbf{x})$ denote by $\mathbf{x}_\delta^i$ the vector obtained from $\mathbf{x}$ by adding δ to the income of person *i*. In general, $\mathbf{x}_\delta^i = (x_1, x_2, \ldots x_{i-1}, x_i + \delta, x_{i+1}, \ldots, x_N) \in \Omega_1$, but if $x_i = x_{i+1} = x$ then $\mathbf{x}_\delta^i \notin \Omega_1$, whereas its rearrangement $(x_1, x_2, \ldots, x, x + \delta, x_{i+2} \ldots, x_N)$, in which the ranks of persons *i* and *i+1* are reversed, does belong to Ω_1 (and has the same Lorenz curve as $\mathbf{x}_\delta^i$).[3]

For a continuous and Schur-convex inequality index $I: \Re_+^N \to \Re_+$ and distribution $\mathbf{x} \in \Omega_1$, and for $1 \le i \le N$ and $0 < \delta < \delta(\mathbf{x})$, we shall denote by $\Delta I(x_i, \delta)$ the change in inequality caused by increasing the income of individual *i* by the amount δ: $\Delta I(x_i, \delta) = I(\mathbf{x}_\delta^i) - I(\mathbf{x})$.

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[1] We shall return to the cited findings later in this paper; they concern social welfare functions based on the Atkinson index and extended Gini coefficient.

[2] One class includes rank-independent indices such as the coefficient of variation, mean logarithmic deviation, generalized entropy index and Atkinson index; the other, rank-dependent (or positional) indices such as the Gini and extended Gini coefficients.

[3] In this notation, $(\mathbf{x}_\alpha^j)_\beta^j = \mathbf{x}_{\alpha+\beta}^j$ for all j such that $x_j \ne x_{j+1}$ and for α and β suitably restricted, whilst if *j > i*, $(\mathbf{x}_{-\delta}^j)_\delta^i = (\mathbf{x}_\delta^i)_{-\delta}^j$ is the distribution obtained from $\mathbf{x}$ by making a progressive transfer of δ from individual *j* to individual *i*.

3. General Results

The effect on the Lorenz curve for $\mathbf{x} \in \Omega_1$ of increasing one income, x_i, depends on which income this is. If the smallest income x_1 is unique, *i.e.* $x_1 < x_2$ (so that $\mathbf{x} \in \Omega_2$), and if x_1 is increased slightly, the Lorenz curve shifts upwards (just consider the effect on income shares), whilst if x_N is increased, the Lorenz curve shifts downwards (for all $\mathbf{x} \in \Omega_1$, and by similar reasoning). For $1 < i < N$, and also for $i = 1$ when $\mathbf{x} \in \Omega_1 \backslash \Omega_2$ (i.e when $x_1 = x_2$), the new Lorenz curve intersects the old one once, from below (again, just consider the income shares).

What can we conclude about the effect on inequality indices of raising one income x_i by an amount δ, where $0 < \delta < \delta(\mathbf{x})$? Clearly, if $\mathbf{x} \in \Omega_2$ then $\Delta I(x_1, \delta) < 0$ for all Lorenz-consistent inequality indices I; and $\Delta I(x_N, \delta) > 0$ for all $\mathbf{x} \in \Omega_1$. For $1 < i < N$, and also for $i = 1$ when $\mathbf{x} \in \Omega_1 \backslash \Omega_2$, we can learn something from a result of Shorrocks and Foster (1987) concerning Lorenz intersections: if x_i is such that $\Delta CV(x_i, \delta) > 0$, where CV is the coefficient of variation, then $\Delta I(x_i, \delta) > 0$ for all transfer-sensitive relative inequality indices I.[4] We return to this finding in the next section.

The results for the lowest and highest incomes are in fact enough to establish the existence of a benchmark income, dividing positive from negative inequality effects for any Lorenz-consistent inequality index I. It is straightforward that for all $\mathbf{x}$, and for all i and j with $i < j$, $\mathbf{x}_{\delta}^{i} = ((\mathbf{x}_{\delta}^{i})_{\delta}^{j})_{-\delta}^{j} = ((\mathbf{x}_{\delta}^{j})_{\delta}^{i})_{-\delta}^{j}$, in other words that $\mathbf{x}_{\delta}^{i}$ is obtained from $\mathbf{x}_{\delta}^{j}$ by a progressive transfer of δ from j to i. Hence for any Lorenz-consistent inequality index I, we have $I(\mathbf{x}_{\delta}^{i}) < I(\mathbf{x}_{\delta}^{j})$, whence $\Delta I(x_i, \delta) < \Delta I(x_j, \delta)$, $\forall\, i, j = (1, 2, \ldots, N)$ with $i < j$. Since we already know that, for $\mathbf{x} \in \Omega_2$, $\Delta I(x_1, \delta) <0$ and $\Delta I(x_N, \delta) >0$, necessarily $\exists$ k such that $\Delta I(x_i, \delta) \leq 0 \Leftrightarrow x_i \leq x_k$. A standard continuity argument establishes the existence of a unique income value x^*, not necessarily present in the income distribution $\mathbf{x}$ (but determined by it), dividing positive from negative inequality effects:

Theorem 1

Given any continuous, Lorenz consistent inequality index I(.), income distribution $\mathbf{x} \in \Omega_2$ and number δ such that $0 < \delta < \delta(\mathbf{x})$, there exists a unique benchmark income level x^* such that $\Delta I(x_k, \delta) \leq 0 \Leftrightarrow x_k \leq x^*$.

In Figure 1, we graph the inequality effect $\Delta I(x_i, \delta)$ for given $\mathbf{x}$ and δ against the value of person i's income (the one being increased) in the case of the coefficient of variation, for which this function is

[4] The transfer sensitive inequality indices are those which adhere to the Principle of Diminishing Transfers of Kolm (1976).

linear.[5] The benchmark level x* need not in general be equal to one of the incomes present in **x,** but x* is uniquely determined by **x** and the index *I(.).* For example, as we shall see, for the coefficient of variation *CV(.),* $x^* = \mu(\mathbf{x}).[1+CV(\mathbf{x})^2]$ and for the Theil index *T(.),* $x^* = \mu(\mathbf{x}).e^{T(\mathbf{x})}$.

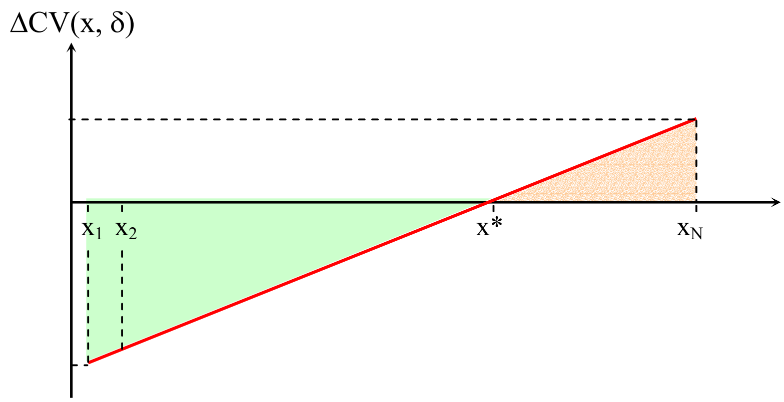

Figure 1: inequality effect of raising person *i*'s income by a small amount δ for the coefficient of variation, as a function of his/her income level x.

4. Further analysis for two general classes of indices

Some inequality indices depend on income shares alone, and others depend on income shares and ranks. We might call such indices rank-independent and rank-dependent respectively, or non-positional and positional. Among the positional indices are the Gini coefficient and the extended Gini coefficients of Donaldson and Weymark (1980) and Yitzhaki (1983). These are members of the general class of "linear measures" identified by Mehran (1976). Most of the familiar non-positional indices are related in one way or another to the generalized entropy family, shown by Bourguignon (1979), Cowell (1980) and Shorrocks (1980) to be the unique additively decomposable indices. The mean logarithmic deviation and Theil index belong to the generalized entropy class, and the coefficient of variation and Atkinson index are monotonic transformations of indices in this class. We analyze indices of the two types separately here, using suitable general forms and then proceeding to specific indices afterwards. As we shall see, Theorem 1 extends from Ω_2 to Ω_1 for the non-positional indices, whilst for the positional ones, the benchmark can be expressed as a position (rank) rather than an income level when $\mathbf{x} \in \Omega_3$.

4.1 The non-positional indices of relative inequality for the class Ω_1

Many non-positional indices, including all the ones we have cited, can either be written in the form:

(1) $J(\mathbf{x}) = [1/N]\Sigma_i u(x_i/\mu(\mathbf{x}))$

where u: $\Re_+ \rightarrow \Re_+$ is a twice-differentiable function such that u" does not change sign, or are monotonic transformations of something in this form. Let $I(\mathbf{x})$ be such an inequality index; suppose that:

(2) $I(\mathbf{x}) = h(J(\mathbf{x}))$

for all $\mathbf{x} \in \Omega_1$ where h: $\Re_+ \rightarrow \Re_+$ is differentiable and such that h' does not change sign.

This form encompasses most of the familiar non-positional indices. For the mean logarithmic deviation D, set $u(z) = -\ln(z)$ and $h(J) = J$. The Theil index T is given by $u(z) = z\ln(z)$ and $h(J) = J$. The generalized entropy class comprises indices E(c), $c \in \Re$, of which E(0) = D, E(1) = T and E(c), $c \neq 0,1$ obtains when $u(z) = z^c$ and $h(J) = (J-1)/[c(c-1)]$. For the coefficient of variation CV, set $u(z) = (z-1)^2$ and $h(J) = J^{1/2}$. For the Atkinson index A(e), where $e > 0$ is the inequality aversion parameter, set $u(z) = z^{1-e}$ and $h(J) = 1 - J^{1/(1-e)}$ when $e \neq 1$ and set $u(z) = \ln(z)$ and $h(J) = 1 - e^J$ when e = 1. The coefficient of variation and Atkinson index for $0 < e \neq 1$ are monotonic transformations of generalized entropy indices: $CV = \sqrt{[2E(2)]}$ and $A(e) = 1 - [1 - e(1-e)E(1-e)]^{1/(1-e)}$.

We may use the calculus to identify the benchmark income level x*. First, differentiate in (1) with respect to the income being increased, let this be x_k to distinguish it from the generic x_i :

(3) $$\partial J/\partial x_k = \frac{1}{N}\left\{\left[\sum_{i \neq k} u'(x_i/\mu)\right]\left(-\frac{x_i}{N\mu^2}\right) + u'(x_k/\mu)\left[\frac{1}{\mu} - \frac{x_k}{N\mu^2}\right]\right\}$$

In this, we have written μ for $\mu(\mathbf{x})$ to suppress unnecessary notation. Now differentiate in (2), substitute from (3) and rearrange:

(4) $\partial I/\partial x_k = [h'(J)/N\mu]\{u'(x_k/\mu) - [1/N]\Sigma_i (x_i/\mu)\, u'(x_i/\mu)\}$

For the transfer principle to hold, if $x_\ell > x_j$ then $\partial I/\partial x_\ell > \partial I/\partial x_j$, that is:

(5) $x_\ell > x_j \Rightarrow [h'(J)]\{u'(x_\ell/\mu) - u'(x_j/\mu)\} > 0$

whence if $h'(J) > 0$, u' must be monotone increasing, and if $h'(J) < 0$, u' must be monotone decreasing (recall that u″ does not change sign). Now let $z_i = x_i/\mu$ be normalized income and define z^* by:

(6) $u'(z^*) = [1/N].\sum z_i u'(z_i)$

From (4)-(5), z* determines the benchmark income level, dividing negative from positive inequality effects when the relevant income is increased:

[5] See on. In the case of a generic Lorenz-consistent inequality index, the graph will have curvature, its shape depending on transfer sensitivity and the distribution **x** in question.

Theorem 2

Let I be a non-positional inequality index defined as in (1)-(2) and let $\mathbf{x} \in \Omega_I$. *Then* $\partial I/\partial x_k \gtrless 0 \Leftrightarrow x_k/\mu \gtrless z^*$ *where z* is defined as in (6)*

It is now straightforward to obtain the benchmark income level for each of the familiar indices we have shown to be members of this non-positional class. For the mean logarithmic deviation D, for example, for which u(z) = -ln(z) and u′(z) = -1/z, we have from (6) that z* = 1; whilst for the Theil index T, for which u(z) = zln(z) and u′(z) = 1 + ln(z), we have from (6) that 1 + ln(z*) = 1 + T, or z* = e^T. For the other indices we have enumerated, the calculations go similarly. The results are these:

Corollary

For the mean logarithmic deviation D, Theil index T, generalized entropy indices E(c), c ≠ 0,1, coefficient of variation CV and Atkinson index A(e), e > 0, and for all $\mathbf{x} \in \Omega_I$, *the inequality effect of a small increase in income* x_k *depends on the value of* x_k *relative to the mean, as follows:*

(a) $\partial D/\partial x_k \gtrless 0 \Leftrightarrow x_k/\mu \gtrless z_D = 1$

(b) $\partial T/\partial x_k \gtrless 0 \Leftrightarrow x_k/\mu \gtrless z_T = e^{T}$

(c) $\partial E(c)/\partial x_k \gtrless 0 \Leftrightarrow x_k/\mu \gtrless z_{E(c)} = [1 + c(c-1)E(c)]^{1/(c-1)} \quad (c \neq 0,1)$

(d) $\partial CV/\partial x_k \gtrless 0 \Leftrightarrow x_k/\mu \gtrless z_{CV} = 1+CV^2$

(e) $\partial A(e)/\partial x_k \gtrless 0 \Leftrightarrow x_k/\mu \gtrless z_{A(e)} = [1 - A(e)]^{(e-1)/e} \quad (e \neq 1)$

(f) $\partial A(1)/\partial x_k \gtrless 0 \Leftrightarrow x_k/\mu \gtrless z_{A(1)} = 1$

There are some equivalences within this set of results. For example, using E(2) = (1/2)CV^2, we see that $z_{E(2)} = [1 + 2E(2)] = z_{CV}$. This is as it ought to be, since the two indices are monotonically related. It can also be shown that $\lim_{c\to 0} z_{E(c)} = 1 = z_D = z_{A(1)}$, $\lim_{c\to 1} z_{E(c)} = e^T = z_T$ and $z_{A(e)} = z_{E(1-e)}$ for e ≠ 1. Let us examine the benchmark $z_{E(c)}$ for the generalized entropy family more closely. Define $m_c = \frac{1}{N}\sum_{i=1}^{N} z_i^{\,c}$ and $M_c = \{m_c\}^{1/c}$ as the moment of order c and mean of order *c* respectively in the distribution of the *z*'s. Then $z_{E(c)} = \{M_c\}^{c/(c-1)}$ for *c* ≠ 0,1. The properties of M_c as a function of *c*, for a given distribution, are well-known in the statistical literature[6], and can be used to derive properties of

[6] For a proof of the properties of the mean of order c, see for example Hardy *et al.* (1934, chapter 1).

the benchmark. In particular, for any given income distribution **x**, $z_{E(c)}$ is continuous and increasing in c, and ranges in value from the minimum income relative to the mean, z_1, to the maximum, z_N: that is, $z_{E(c)} \to z_1$ as $c \to -\infty$ and $z_{E(c)} \to z_N$ as $c \to +\infty$. A particular consequence is that, for each person k in an income distribution $\mathbf{x} \in \Omega_1$ there exists a unique $c \in \Re$ such that $z_{E(c)} = x_k / \mu$: each person can be considered to be at the benchmark position for exactly one generalized entropy index. Figure 2, obtained by simulation, shows graphs of M_c and $z_{E(c)}$ against c for the income distribution ($200, $500, $800, $1100, $2400).

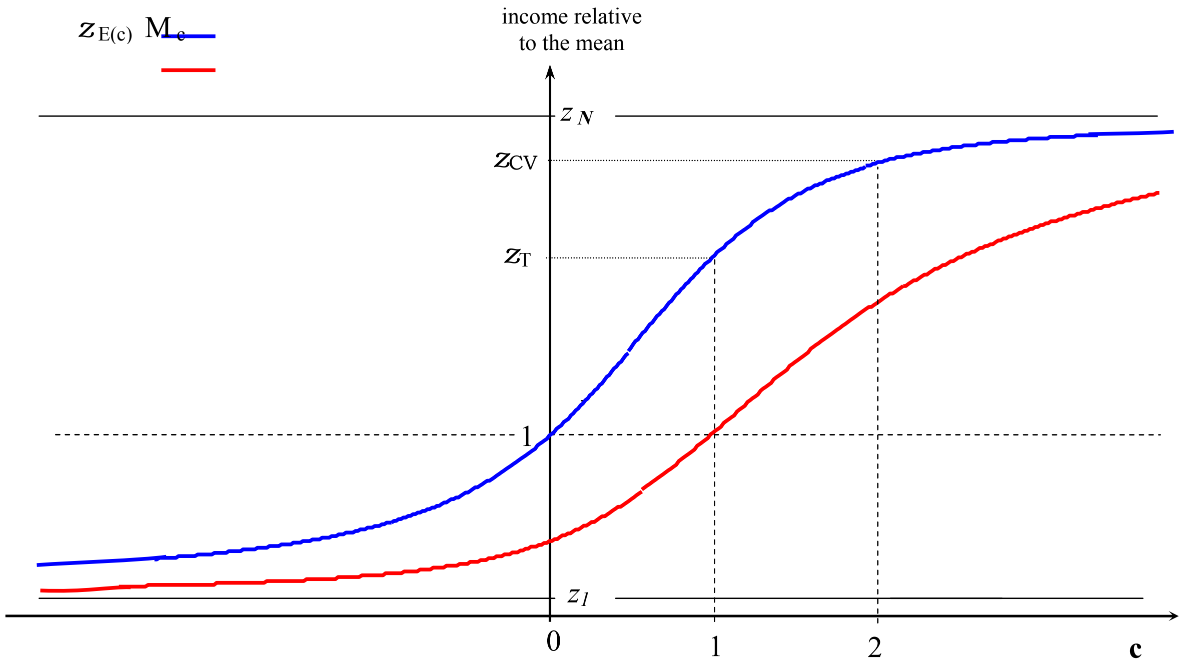

Figure 2: the generalized entropy benchmark as a function of the parameter c for the income distribution ($200, $500, $800, $1100, $2400)

We can now return to the finding in Section 3 concerning the coefficient of variation and transfer-sensitive inequality indices. We saw there that for $\mathbf{x} \in \Omega_1$ and for any k such that $\partial CV/\partial x_k > 0$, an increase in x_k necessarily raises inequality for every transfer-sensitive index I. That is, from part (d) of the Corollary, if $x_k / \mu > z_{CV} = 1 + CV^2$ then $\partial I/\partial x_k > 0$. Therefore z_{CV} is an upper bound for the benchmarks z* in the class of transfer-sensitive inequality indices.[7]

The function u and income distribution **x** together determine the benchmark income level x^* for indices in our non-positional class according to equation (6) (and for Ω_1 rather than the restricted Ω_2 of

[7] This result is consistent with our Corollary. A(e) is transfer-sensitive for all e, and E(c) is transfer sensitive for c < 2, and the benchmarks for these indices all exceed z_{CV} : $c < 2 \Rightarrow z_{CV} > z_{E(c)} = z_{A(1-c)}$ (as Figure 2 shows).

Theorem 1; ties, as in $\Omega_1 \setminus \Omega_2$, are immaterial for the non-positional indices)[8]. Notice that the function u alone defines the inequality ordering induced by I, and determines the benchmark, whereas the function h is also needed for the definition of I.

Further insight into the relationship between the inequality ordering and benchmark income level can be gained with a simple transformation. Let $\pi_i = z_i/N$, which is person i's income share, 1 # i # N, and note that $\sum \pi_i = 1$. Now set $U(z) = u'(z)$ where u is the function in (1) determining the inequality ordering. From (6), the benchmark income relative to the mean satisfies this equation:

(7) $\quad U(z^*) = \sum \pi_i U(z_i) = E[U(\mathbf{Z})]$

where $\mathbf{Z}$ is a risky prospect in which the return is z_i with probability π_i, 1 # i # N. That is, $z^* = x^*/\mu$ is the certainty equivalent of $\mathbf{Z}$ for the "utility function" U, in the sense of Pratt (1964). An extension of the Pratt theorem confirms the following result, linking the (relative) risk aversion of U, which, in terms of the function u defining the inequality ordering, takes the form

(8) $\quad P_u(z) = -zu'''(z)/u''(z)$,

with the position of the benchmark:[9]

Theorem 3

Let I and Î be inequality indices defined as in (1)-(2) by, respectively, h and u and $\hat{h}$ and $\hat{u}$, where $P_u(z) > P_{\hat{u}}(z)$ $\forall z$. Then for all unequal income distributions $\mathbf{x} \in \Omega_1$, the benchmark income for I is less than that for Î: $x^* < \hat{x}^*$.

The higher is the measure $P_u(z)$ $\forall z$, the more confined is the lower-tail region $[0,x^*]$ in which an increase in a person's income is regarded as an inequality improvement, whatever the income distribution. In a clear sense, then, an inequality ordering with a higher P_u-measure is "more Rawlsian".[10] Rather than introduce a cumbersome word, "Rawlsianity", for the measure $P_u(z)$ as a characteristic of the inequality ordering of which I is a cardinal representation, we shall call it the

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[8] Notice that for the coefficient of variation, $\Delta CV(x_i, \delta) \approx \delta.\partial CV/\partial x_i$ is linear in x_i because, in (4), $u'(z) = 2(z-1)$ in case I = CV. This accounts for the shape of the graph in Figure 1.

[9] For a direct proof, just follow similar steps to those in Lambert's (2001, theorem 4.1) proof of the Pratt theorem. Namely, define Û by $\hat{U}(z) = \hat{u}'(z)$, and let the "inequality aversion" measures for the "utilities" U and Û be $q_{\hat{U}}(z) = -z\hat{U}''(z)/\hat{U}'(z)$ and $q_U(z) = -zU''(z)/U'(z)$, so that $P_{\hat{U}}(z) = q_{\hat{U}}(z)$ and similarly for U. By assumption $\hat{u}'' = \hat{U}'$ and $u'' = U'$ do not change sign. Define a function ø by $\hat{U}(z) = ø[U(z)]$ □z, so that $ø' < 0$ if and only if U' and Û' have opposite signs. Then $q_{\hat{U}}(z) = q_U(z) - zøO[U(z)]U'(z)/ø'[U(z)]$. Assuming $q_{\hat{U}}(z) < q_U(z)$ □z, as in the theorem, øO < 0 if Û' < 0 and øO > 0 if Û' > 0. Now apply Jensen's inequality: $\hat{U}(\hat{x}^*/\mu) = E[\hat{U}(\mathbf{Z})] = E[ø(U(\mathbf{Z}))] < ø(E[U(\mathbf{Z})]\} = ø[U(x^*/\mu)] = \hat{U}(x^*/\mu)$ if $\hat{U}' < 0$ and $\hat{U}(\hat{x}^*/\mu) = E[\hat{U}(\mathbf{Z})] = E[ø(U(\mathbf{Z}))] > ø(E[U(\mathbf{Z})]\} = ø[U(x^*/\mu)] = \hat{U}(x^*/\mu)$ if $\hat{U}' > 0$. In either case, $x^* < \hat{x}^*$, as the theorem claims.

[10] Since its introduction in 1971, Rawls' difference principle has overwhelmingly been interpreted as expressing concern (in either inequality or welfare terms) solely with the fortunes of the worst-off individual (or set of individuals if there is equality at the very bottom). Yet Rawls himself clearly referred to "the least advantaged *segment*" (*ibid*, p. 98, italics added), this segment being demarcated either by a relative income, or by the average income of those occupying one of the less-fortunate social roles.

"lower tail concern" in what follows.[11]

All the specific indices we have been considering in fact have *constant* lower tail concern. This is because they all represent inequality orderings implicit in generalized entropy indices, for which $u(z) = z^c$ whence $P_{E(c)}(z) = 2-c$, $\forall z$. It follows from Theorem 3 that the benchmark income for E(c) is an increasing function of c whatever the income distribution **x**, as evidenced in Figure 3 for a specific income distribution. It can be checked directly, by inspecting the relevant *u*-functions, that for the mean logarithmic deviation, $P_D(z) = 2$, $\forall z$; for the Theil index, $P_T(z) = 1$, $\forall z$; for the coefficient of variation, $P_{CV}(z) = 0$, $\forall z$; and for the Atkinson index, $P_{A(e)}(z) = e+1$, $\forall z$. The configuration of benchmarks for any two of the inequality indices we have catalogued can thus be ascertained, whatever the income distribution, by a simple comparison of scalar magnitudes. Notice that the inequality orderings with (constant) *negative* lower tail concern are precisely those represented by the generalized entropy indices E(c) for c > 2. This ties in with a remark of Shorrocks (1980, p. 623), that the indices E(c), c > 2 "show little concern for equalization, except possibly among the very rich". In fact, within the general class of non-positional indices satisfying (1)-(2), the sub-class having *positive* lower tail concern are precisely those which satisfy Kolm's (1976) Principle of Diminishing Transfers.[12]

4.2 The positional indices of relative inequality for the class Ω_3

Here we shall consider inequality indices in which people's incomes are weighted according to their positions in the distribution. Specifically, let M(**x**) take the form

(9) $\quad M(\mathbf{x}) = [1/N].\ \sum_i w(i)x_i/\mu$

for $\mathbf{x} \in \Omega_3$, where $w: \Re \to \Re$ is such that $\sum_i w(i) = 0$ and $w(i+1) > w(i)$ for i = 1,2,... N-1.

This specification covers the Gini coefficient G, for which $w_G(i) = (2i - N - 1)/N$, the extended Gini coefficient G(ν), ν > 1, of Donaldson and Weymark (1980, 1983) and Yitzhaki (1983), for which $w_{G(\nu)}(i) = N.\{[(N-i)/N]^\nu - [(N-i+1)/N]^\nu\} + 1$ (the case ν = 2 being that of the ordinary Gini coefficient),[13]

[11] There is a formal link with Kimball's (1990) concept of "prudence" in the uncertainty context. We refrain from calling $P_u(z)$ "downside inequality aversion", as this would be inconsistent with Modica and Scarsini's (2002) measure in the uncertainty context of downside risk aversion, which, in absolute form, is $-u'''(z)/u'(z)$. We also refrained from calling $P_u(z)$ "downside-mindedness", however apt, as this concept belongs to Wilthien (1999).

[12] It is readily verified, using a similar argument to the one given just after (5), that if $h'(J) > 0$ then I satisfies Kolm's principle if and only if $u'' > 0$ and $u''' < 0$, and that if $h'(J) < 0$ then I satisfies Kolm's principle if and only if $u'' < 0$ and $u''' > 0$. Hence Kolm's principle corresponds precisely to an everywhere positive lower tail concern.

[13] For more on the extended Gini coefficient, see Lambert (2001, chapter 5). Note that $w_{G(\nu)}(i+1) - w_{G(\nu)}(i) = [(N-i+1)/N]^\nu + [(N-i-1)/N]^\nu - 2[(N-i)/N]^\nu$ which can be written as $2[E(Y^\nu) - (E(Y))^\nu]$ where Y is a random variable with realizations (N-*i*+1)/N and (N-*i*-1)/N each with probability ½. This is strictly positive because Y^ν is a convex function of Y for $\nu > 1$. Similarly, by a slight abuse of notation, we have $\partial[w_{G(\nu)}(i+1) - w_{G(\nu)}(i)]/\partial i = -2[E(Y^{\nu-1}) - (E(Y))^{\nu-1}]/N$, which is negative for $\nu > 2$, zero for $\nu = 2$ and positive for $\nu < 2$. G(ν) thus satisfies the strong version of the Positional Principle of Diminishing Transfers only for $\nu > 2$. See on.

and the illfare-ranked S-Gini coefficient $S(\beta)$, $0 \leq \beta < 1$, of Donaldson and Weymark (1980), for which $w_{S(\beta)}(i) = 1 - N.\{[i/N]^{\beta} - [(i-1)/N]^{\beta}\}$.

Going slightly further, we shall assume that in (9), the function $w: \Re \rightarrow \Re$ is strictly increasing and twice differentable. Setting $\omega(p) = w(Np)$, so that $\omega : [0,1] \rightarrow \Re$ ascribes weights by *rank*, (9) becomes:

(9a) $M(\mathbf{x}) = [1/N]. \sum_i \omega(p_i) x_i / \mu$

in which the rank of income x_i is written as $p_i = i/N$, so that $\omega(p_i) = w(i)$. This version of (9) exactly describes the class of so-called 'linear inequality measures' identified by Mehran (1976) and further studied by Yaari (1988).[14]

For $\mathbf{x} \in \Omega_3$, this index is differentiable in each x_i.[15] Differentiating in (9), we have

(10) $\partial M/\partial x_k = [w(k) - M]/[N\mu] \;{}^{>}_{<}\; 0 \Leftrightarrow w(k) \;{}^{>}_{<}\; M$

We know that $\partial M/\partial x_N > 0$ from Theorem 1. Hence $w(N) > M$; and since $\sum_i w(i) = 0$ by assumption, and w is increasing, we must have $w(1) < 0$. Then by continuity and monotonicity, there exists a unique real number k^* such that $w(k^*) = M$. Of course, k^* is unlikely to be an integer. We have established the existence of a benchmark position for indices in the positional class:

Theorem 4

Let M *be a positional inequality index defined for* $\mathbf{x} \in \Omega_3$ *as in (9), with* $w: \Re \rightarrow \Re$ *continuous and strictly monotone increasing. Then* $\partial M/\partial x_k \;{}^{>}_{<}\; 0 \Leftrightarrow k \;{}^{>}_{<}\; k^*$ *where* $k^* = w^{-1}(M)$.

For the Gini coefficient, we have $k_G^* = [N(1+G)+1]/2 > N/2$, whence the benchmark is above the median (and by more, the more unequal is the distribution). Defining $\Delta G(x_k, \delta) = G(\mathbf{x}^{k}_{\delta}) - G(\mathbf{x})$, with $0 < \delta < \delta(\mathbf{x})$ as earlier, we find that $\Delta G(x_k, \delta) = \frac{2}{N}\left[\frac{a + k\delta}{b + \delta} - \frac{a}{b}\right]$ where $a = \sum_i i x_i$ and $b = N\mu =$

[14] In the case of a continuous income distribution function F(x), the Mehran index becomes $M_F = \int_0^1 x\omega(F(x))f(x)dx/\mu$ where $\int_0^1 w(p)dp = 0$ (see Lambert, 2001, for more on this). In this setting, the rank-weighting functions for the Gini, extended Gini and S-Gini are $\omega_G(p) = 2p-1$, $\omega_{G(\nu)}(p) = 1 - \nu(1-p)^{\nu-1}$ and $\omega_{S(\beta)}(p) = 1 - \beta p^{\beta-1}$ respectively. These correspond to the discrete weighting functions $w_G(i)$, $w_{G(\nu)}(i)$ and $w_{S(\beta)}(i)$ cited above, making the identification $p = i/N$ and regarding 1/N as an infinitesimal. Chateauneuf *et al*. (2002) characterize the class of Yaari (1988) indices by a form as in (9) but with $w(i) = 1 + N\{f((N-i)/N) - f((N-i+1)/N)\}$ for some function $f: [0,1] \rightarrow [0,1]$ for which $f(0)=0$, $f(1) = 1$ and $f'(t) > 0 \; \forall t \in (0,1)$. For the extended Gini,we have $f_{G(\nu)}(t) = t^{\nu}$ and for the illfare ranked S-Gini, $f_{S(\beta)}(t) = 1 - (1-t)^{\beta}$. Writing $\omega(p) = 1 - f'(1-p)$, the functions $\omega_G(p)$, $\omega_{G(\nu)}(p)$ and $\omega_{S(\beta)}(p)$ emerge, along with the general form in (9a). Notice that if we extend the functional forms defining $G(\nu)$ and $S(\beta)$ to all non-zero parameter values, then $-G(\nu)$ belongs to our positional class for $\nu < 1$ and $-S(\beta)$ belongs to it for $\beta > 1$. A new inequality index outlined in Wang and Tsui (2000) takes the form $J(c) = \text{sign}(c-1)[G(c) - S(c)]$, $0 < c \neq 1$, and hence belongs to our class too. Another class of 'generalized Gini' indices due to Aaberge (2001), in which the weights depend on Lorenz curve values L(p) rather than positions p, does not fall within the scope of our general form in (9)-(9a).

[15] The form in (9) can be extended to Ω_1, with the loss of differentiability, if the weights when $x_i = x_{i+1}$ are made the same for persons i and $i+1$, and equal to $[w(i) + w(i+1)]/2$. Without this change, a small amount taken from person i and given to person i+1 would increase

$\sum_i x_i$. Thus $k_G^* = a/b$ can be interpreted as the income weighted average position in the distribution. Note in particular that $\Delta G(x_k, \delta)$ is linear in k and independent of the income value x_k. See Figure 3, a Gini version of Figure 1, which shows position rather than income horizontally.

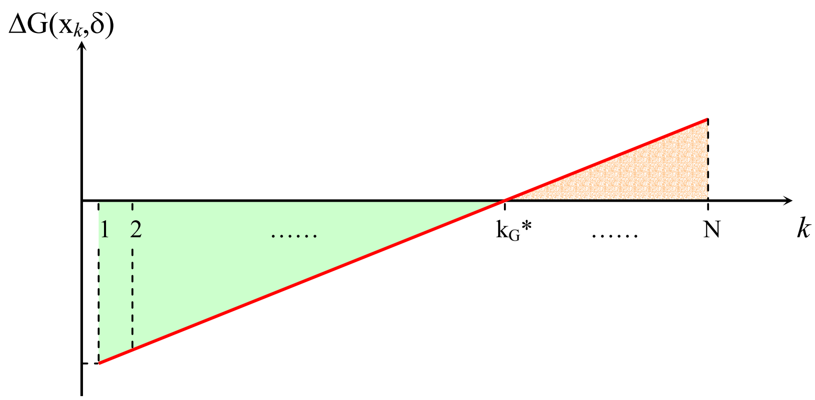

Figure 3: benchmark position for the Gini coefficient

For the extended Gini coefficient G(ν), the benchmark position $k_{G(\nu)}^*$ is the solution to the equation $w_\nu(k) = G(\nu)$, or $[(N-k+1)/N]^\nu - [(N-k)/N]^\nu = [1 - G(\nu)]/N$, which is difficult to obtain explicitly. However, an approximation to $k_{G(\nu)}^*$ can be obtained quite easily. Define a function $f(q) = q^\nu$, so that $q^* = (N-k_{G(\nu)}^*)/N$ is the solution of $[1 - G(\nu)]/N = f(q + 1/N) - f(q)$. For large N, $f(q + 1/N) - f(q) \approx \nu q^{\nu-1}/N$, whence $q^* \approx \{[1 - G(\nu)]/\nu\}^{1/(\nu-1)}$ i.e. $k_{G(\nu)}^* \approx N[1 - \{[1 - G(\nu)]/\nu\}^{1/(\nu-1)}]$. In the case $\nu = 2$, this approximation becomes $k_{G(2)}^* \approx N[1 + G]/2$, whilst the true value, k_G^*, is $[N(1+G)+1]/2$ which is higher by ½. Hence the approximate benchmark is at most one position too high in this case. For the illfare-ranked S-Gini, by similar reasoning $k_{S(\beta)}^* \approx N\{[1 - S(\beta)]/\beta\}^{1/(\beta-1)}]$.

A link between the lower tail concern of the inequality ordering represented by M and the location of the benchmark k^* obtains for the positional class, just as it did for the non-positional class in Theorem 3. Again setting $\pi_i = z_i/N$ as person i's income share, and treating it as a probability, and now using version (9a) of the definition of M, we have from (10) that the benchmark position k^* satisfies this equation:

(11) $\omega(p^*) = \sum \pi_i\, \omega(p_i) = E[\omega(\mathbf{K})]$

where $p^* = k^*/N$ and **K** is a risky prospect in which the return is p_i with probability π_i, 1 #i # N. That is, k^*/N is the certainty equivalent of **K** for ω, in the sense of Pratt (1964). Defining

(12) $Q_\omega(p) = -p\omega''(p)/\omega'(p)$

inequality, whereas the same amount taken from person $i+1$ and given to person i would reduce it – yet the final income distribution as the lower tail concern measure for this scenario, we have the following result, paralleling Theorem 3:

Theorem 5

Let M and $\hat{M}$ be positional inequality indices defined for $\mathbf{x} \in \Omega_3$ *as in (9a) by, respectively,* ω *and* $\hat{\omega}$*, where* $Q_\omega(p) > Q_{\hat{\omega}}(p)\ \forall p$*. Then for all unequal income distributions* $\mathbf{x} \in \Omega_3$*, the benchmark position is lower for M than for* $\hat{M}$ *:* $k^* < \hat{k}^*$.

For the positional indices, lower tail concern $Q_\omega(p)$ is measured in terms of rank p (rather than relative income z), and is given by the concavity of the weighting function ω. The higher is the measure $Q_\omega(p)\ \forall p$, the more confined is the set of lower tail *positions* $1 \leq k < k^*$ in which an increase in a person's income is regarded as an inequality improvement. If the population size N is large, the illfare-ranked S-Gini has constant (and positive) lower tail concern: $Q_{S(\beta)}(p) = 2\text{-}\beta\ \forall p$ (see footnote 14). If we had defined $Q_\omega(p)$ slightly differently, as $Q_\omega{}^*(p) = -(1\text{-}p)\omega''(p)/\omega'(p)$, which would have no effect on the validity of the theorem, then it would be the extended Gini that had constant lower tail concern: $Q_{G(\nu)}{}^*(p) = \nu\text{-}2$. This makes evident a link between our tail concern measure and the Positional Principle of Diminishing Transfers, since only the extended Ginis with $\nu > 2$ (*i.e*. those with positive lower tail concern) satisfy this Principle (see footnote 13). In fact, within the general class of positional indices satisfying (9)-(9a), the sub-class having positive lower tail concern are precisely those which satisfy the Positional Principle of Diminishing Transfers.[16]

5. The Leaky Bucket

Using all of these results, we can now address the leaky bucket issue. Suppose that, in an unequal distribution $\mathbf{x}$, a small amount δ is taken from individual ℓ and an amount $q\delta$ is given to individual j who is lower down the distribution ($j < \ell$), where $q \in \Re_+$. The effect on any differentiable inequality index I is readily obtained using the total differential:

(13) $dI = [q\partial I/\partial x_j - \partial I/\partial x_\ell].\delta$

for an infinitesimally small δ. If $\mathbf{x} \in \Omega_1$ then $x_j \leq x_\ell$, whilst if $\mathbf{x} \in \Omega_3$ (or if $\ell = 2$ and $\mathbf{x} \in \Omega_2$) then $x_j < x_\ell$. As before, we can deal with the general case of $\mathbf{x} \in \Omega_1$ for the non-positional indices, but will

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would be the same in both cases.

[16] The general positional index M as defined in equations (9)-(9a) satisfies the strong version of the Positional Principle of Diminishing Transfers when the positive difference $w(i+1) - w(i)$ is strictly decreasing in i, or $\omega''(p) < 0\ \forall p \in (0,1)$. See Mehran (1976, p. 808) and Chateauneuf *et al*. (2001, theorem 9) for more on this. Yaari's (1988) "equality-mindedness" measure for the positional indices, which in our notation is $= -\omega'(p)/[1-\omega(p)]$, and in the alternative notation of footnote 14 is $-f''(1\text{-}p)/f'(1\text{-}p)$, is based upon a leaky bucket experiment: see on.

restrict attention to $\mathbf{x} \in \Omega_3$ and $0 < \delta < \delta(\mathbf{x})$ for the positional ones. In both cases, the index is then differentiable. The value q_0 for which $dI = 0$ reveals the information we seek about the permitted leakiness of the bucket for a non-adverse inequality effect:

$$(14)\quad q_0 = \frac{\partial I(.)/\partial x_\ell}{\partial I(.)/\partial x_j}$$

The intuitively agreeable scenario, that the size of the leak would not erase completely the amount of income to be received by the poor, corresponds to $0 < q_0 < 1$, whilst the other two cases we have already anticipated, that the leak could exceed 100% or even be negative, correspond to $q_0 < 0$ and $q_0 > 1$ respectively. As we shall see, it is possible to predict the circumstances in which each of these three cases occurs for all inequality indices in our two classes.

5.1 The non-positional indices of relative inequality

For an inequality index I defined as in (1)-(2), we obtain

$$(15)\quad q_0 = \frac{u'(x_\ell/\mu) - u'(x^*/\mu)}{u'(x_j/\mu) - u'(x^*/\mu)}$$

from (14) using (4) and (6). Since u' is monotonic, it follows that the magnitude of the permitted leak (which is $1-q_0$) depends crucially upon which side of the benchmark the donor and recipient lie:[17]

Theorem 6

Let I be a non-positional inequality index defined as in (1) - (2). The fraction q_0 of a small amount δ taken from individual ℓ which must reach individual j (where $j < \ell$) for inequality neutrality depends upon the incomes of ℓ and j relative to the benchmark income x as follows:*

(i) $x^* > x_\ell > x_j \Rightarrow 0 < q_0 < 1$

(ii) $x_\ell > x^* > x_j \Rightarrow q_0 < 0$

(iii) $x_\ell > x_j > x^* \Rightarrow q_0 > 1$

The magnitude of the effect on inequality, of a leaky transfer from ℓ to *j,* depends on whether $q \gtrless q_0$, of course, as well as on the values $z_j = x_j/\mu$, $z_\ell = x_\ell/\mu$ and $z^* = x^*/\mu$: for any non-positional index in our class, inequality increases or decreases according to the inefficiency level and the relative incomes of the individuals affected. Case *(i)*, in which $0 < q_0 < 1$, is the one typically envisaged, and,

[17] It is a general property that if a function g(.) is strictly monotonic, either increasing or decreasing, and if d = [g(a)-g(b)]/[g(c)-g(b)], where a > c, then d < 0 if a > b > c, d > 1 if a > c > b, and 0 < d < 1 if b > a > c.

our analytics reveal, *it can occur only when both the donor and recipient are below the benchmark*. In all other configurations of donor and recipient, the permitted leakage will either exceed the amount taken away ($q_0 < 0$), so that the "recipient" may lose too, or be negative, so that the recipient may receive more than the donor gives up ($q_0 > 1$) with no adverse effect on inequality.

One can readily obtain the value of q_0 for any particular index using (15) and the appropriate function $u(.)$. For the mean logarithmic deviation D, $q_D = \frac{z_\ell^{-1} - 1}{z_j^{-1} - 1}$; for the Theil index T, $q_T = \frac{\ln z_\ell - T}{\ln z_j - T}$; for the generalized entropy index E(c), $c \neq 0,1$, $q_{E(c)} = \frac{z_\ell^{c-1} - z_{E(c)}^{c-1}}{z_j^{c-1} - z_{E(c)}^{c-1}}$; for the coefficient of variation CV, $q_{CV} = \frac{z_\ell - z_{CV}}{z_j - z_{CV}}$; for the Atkinson index A(e), $q_{A(e)} = \frac{z_\ell^{-e} - z_{A(e)}^{-e}}{z_j^{-e} - z_{A(e)}^{-e}} = q_{E(1-e)}$ for $0 < e \neq 1$ and $q_{A(1)} = q_D$.

In Table 1, we illustrate how the benchmark income level x^* and maximum permitted rate of leakage $1 - q_0$ vary with inequality aversion e for the Atkinson index A(e), using the income distribution ($200, $500, $800, $1100, $2400) again and choosing $\ell = 4$ and $j = 2$. When $1 is taken from the person with $1100 and an amount $$q$ is given to the person with $500, the leak $$(1-q)$ can be as big as the value $1 - q_0 = 1 - q_{A(e)}$ shown in the table before an inequality effect judged to be adverse would occur. As is clear, all three cases $0 < q_0 < 1$, $q_0 < 0$ and $q_0 > 1$ of Theorem 6 arise, for different ranges of inequality aversion *e*. In each such range the maximum permitted rate of leakage increases with *e*.

e	A(*e*)	**x***	$1 - q_{A(e)}$	Theorem 6, case:
0.1	0.0272	1282.1811	0.8436	*(i)*
0.2	0.0546	1251.5924	0.8701	
0.3	0.0819	1220.6203	0.8967	$x^* > x_4 > x_2$
0.4	0.1092	1189.3367	0.9234	
0.5	0.1363	1157.8210	0.9503	$\Rightarrow 0 < q_0 < 1$
0.6	0.1632	1126.1599	0.9774	
0.8	0.2162	1062.7796	1.0328	*(ii)*
1	0.2673	1000.0000	1.0909	
1.2	0.3160	938.6666	1.1535	$x_4 > x^* > x_2$
1.4	0.3617	879.6041	1.2230	
1.6	0.4041	823.5476	1.3033	$\Rightarrow q_0 < 0$
1.8	0.4428	771.0817	1.4001	
2	0.4778	722.6008	1.5222	
2.2	0.5092	678.2984	1.6849	
2.4	0.5370	638.1840	1.9160	
2.6	0.5615	602.1179	2.2737	
2.8	0.5831	569.8547	2.9028	
3	0.6020	541.0856	4.2955	

3.2	0.6186	515.4730	9.8986	
3.5	0.6398	482.2325	-6.9382	*(iii)* $x_4 > x_2 > x^*$ $\Rightarrow q_0 > 1$
4	0.6673	438.0625	-1.3731	
5	0.7032	378.4391	-0.3241	
6	0.7247	341.3486	-0.1117	
7	0.7387	316.5664	-0.0423	
10	0.7608	275.9386	-0.0026	
20	0.7823	234.9238	-0.0000	

Table 1: The benchmark income level x^* and maximum permitted rate of leakage $1\text{-}q_{A(e)}$ as a function of inequality aversion for the income distribution ($200, $500, $800, $1100, $2400) when $\ell = 4$ and j = 2.

Figure 4 shows the maximum permitted leakage rate $1\text{-}q_{E(c)}$ for the more general class of inequality indices $E(c)$ as a function of the parameter c, for this same income distribution, using the scenario $\ell = 4$ and j = 2 of Table 1 and three others each involving the richest and/or poorest person in the transfer. The results for the Atkinson index $A(e)$ for $0< e \neq 1$ occur for $c < 1$ (recall that $q_{E(1\text{-}e)} = q_{A(e)}$). Panel 1 replicates and extends the maximum leak values given in Table 1. It is clear from panels 3 and 4, however, that it is not always the case for the Atkinson index that the maximum permitted leak increases with inequality aversion e. When the richest person is the donor, in this example the maximum leak decreases with e in some or all ranges. *A fortiori*, there can be no clear *general* relationship between the lower tail concern of an inequality ordering, as measured by $P_u(z)$, and the maximum leak $1 - q_0$: an intuition that a more lower tail concerned inequality ordering would countenance bigger leaks, though tempting, must be wrong.

Our findings in Table 1 and Figure 4 may be set alongside those Atkinson (1980, p. 42) and Jenkins (1991, pp. 28-9), which relate to the maximum tolerable leak for an Atkinson index *before a welfare loss is experienced* (rather than, as here, *before inequality is worsened*). Because the efficiency aspect gets taken into account in welfare, measured in these studies as $\mu[1 - A(e)]$, it is clear that very big leaks could not be tolerated; Atkinson and Jenkins found maximum permitted leaks in the range 33%-75% for their particular numerical scenarios.

5.2 The positional indices of relative inequality

If $\mathbf{x} \in \Omega_3$ and if $0 < \delta < \delta(\mathbf{x})$ then the resultant income distribution after the transfer, which is $(\mathbf{x}^{\ell}_{-\delta})^{j}_{+q\delta}$, also belongs to Ω_3. Thus the form given in (9) for a positional index M(.) applies. Substituting from (10) into (14), the value of q_0 for the index M is:

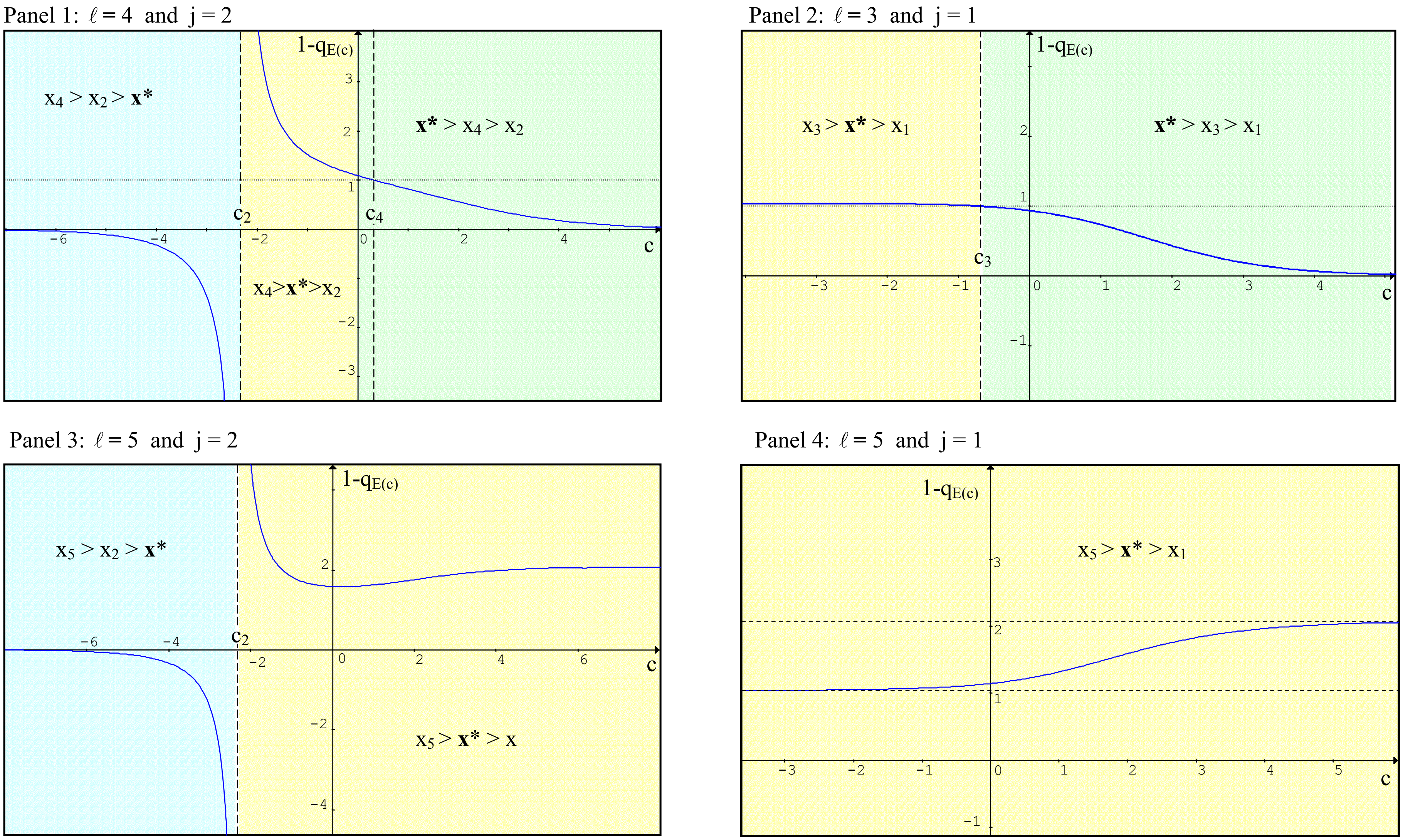

Figure 4: Maximum permitted leakage rate $1\text{-}q_{E(c)}$ for the generalized entropy index *E(c)* as a function of *c*, for the scenario in Table 1 and three other scenarios involving the richest and/or the poorest person in the transfer.

$$(16)\quad q_0 = \frac{w(\ell) - M}{w(j) - M}$$

Now recall from Theorem 4 that the benchmark position for M is $k^* = w^{-1}(M)$. Hence

$$(17)\quad q_0 = \frac{w(\ell) - w(k^*)}{w(j) - w(k^*)}$$

(compare this with (15), which expresses q_0 in a similar form for the non-positional indices). The following results are immediate, given that *w(.)* is strictly increasing:

Theorem 7

Let M be a positional inequality index defined for $\mathbf{x} \in \Omega_3$ *as in (9), with* $w\colon \Re \to \Re$ *continuous and strictly monotone increasing. The fraction* q_0 *of a small amount* $0 < \delta < \delta(\mathbf{x})$ *taken from individual* ℓ *which must reach individual j (where* $j < \ell$*) for inequality neutrality depends upon the positions of* ℓ *and j relative to the benchmark position k* as follows:*

(i) $k^* > j > \ell \;\Rightarrow\; 0 < q_0 < 1$

(ii) $\ell > k^* > j \;\Rightarrow\; q_0 < 0$

(iii) $\ell > j > k^* \;\Rightarrow\; q_0 > 1$

The case $0 < q_0 < 1$ occurs only when both the donor and recipient are positioned below the benchmark k^*. In all other configurations, the permitted leakage will either exceed the amount taken away ($q_0 < 0$), so that the "recipient" may lose too, or be negative, so that the recipient may receive more than the donor gives up ($q_0 > 1$) with no adverse effect on inequality. These results are analogous to the ones in Theorem 6 for the non-positional indices, in which the benchmark *income level* forms the divide; for the positional indices, it is the benchmark *position* which takes this role.

In the case of the Gini coefficient, for which $w(i) = (2i - N - 1)/N$, $q_0 = (\ell - k_G^*)/(j - k_G^*)$ where $k_G^* = [N(1+G)+1]/2$. The expression for q_0 for the extended Gini coefficient $G(\nu)$, $\nu > 1$, which is more complex, obtains by substituting $w_{G(\nu)}(i) = N\{[(N-i)/N]^{\nu} - [(N-i+1)/N]^{\nu}\} + 1$ and $M = G(\nu)$ in (12). Noting that for large N, $w_{G(\nu)}(i) \approx [1 - \nu.\{(N-i)/N\}^{\nu-1}]/N$, so that q_0 can be approximated from (13) as $q_0 \approx [(N-k_{G(\nu)}^*)^{\nu-1} - (N-\ell)^{\nu-1}]/[(N-k_{G(\nu)}^*)^{\nu-1} - (N-j)^{\nu-1}]$, it follows from the further approximation $k_{G(\nu)}^* \approx N[1 - \{[1 - G(\nu)]/\nu\}^{1/(\nu-1)}]$ already noted that $q_0 \approx \dfrac{1 - G(\nu) - \nu(1-p_l)^{\nu-1}}{1 - G(\nu) - \nu(1-p_j)^{\nu-1}}$ where p_j and p_l are the ranks of j and l respectively. Analogously, for the illfare-ranked S-Gini, $q_0 \approx \dfrac{1 - S(\beta) - \beta p_l^{\beta-1}}{1 - S(\beta) - \beta p_j^{\beta-1}}$ for large N.

In Table 2, we illustrate for the extended Gini coefficient how the benchmark position $k_{G(\nu)}^*$ and maximum permitted rate of leakage $1 - q_{G(\nu)}$ vary with the distributional judgment parameter ν, using the same income distribution as in Table 1 and choosing $\ell = 4$ and j = 2 as before. The cases $0 < q_0 < 1$, $q_0 < 0$ and $q_0 > 1$ of Theorem 6 all arise. Figure 5 shows the dependence of $1 - q_{G(\nu)}$ on ν graphically, for the same four scenarios as used in Figure 4 for $1 - q_{E(c)}$. As before, we see non-monotonicity in some scenarios between ν and $1 - q_{G(\nu)}$. For the positional indices too, then, there can be no general link between the degree of lower tail concern of the inequality ordering and the maximum permitted leak.[18]

<table>
<tr><th>ν</th><th>$G(\nu)$</th><th>k*</th><th>$1 - q_{G(\nu)}$</th><th>Theorem 6, case:</th></tr>
<tr><td>1,2</td><td>0,1196</td><td>4,4054</td><td>0,7464</td><td rowspan="5">(i)
k* > 4 > 2
$\Rightarrow 0 < q_0 < 1$</td></tr>
<tr><td>1,4</td><td>0,2140</td><td>4,2976</td><td>0,8243</td></tr>
<tr><td>1,6</td><td>0,2894</td><td>4,1941</td><td>0,8918</td></tr>
<tr><td>1,8</td><td>0,3502</td><td>4,0949</td><td>0,9499</td></tr>
<tr><td>2</td><td>0,4000</td><td>4,0000</td><td>1,0000</td></tr>
<tr><td>3</td><td>0,5520</td><td>3,5895</td><td>1,1628</td><td rowspan="12">(ii)
4 > k* > 2
$\Rightarrow q_0 < 0$</td></tr>
<tr><td>4</td><td>0,6285</td><td>3,2724</td><td>1,2446</td></tr>
<tr><td>5</td><td>0,6749</td><td>3,0244</td><td>1,2980</td></tr>
<tr><td>6</td><td>0,7060</td><td>2,8249</td><td>1,3495</td></tr>
<tr><td>7</td><td>0,7282</td><td>2,6607</td><td>1,4141</td></tr>
<tr><td>8</td><td>0,7444</td><td>2,5225</td><td>1,5053</td></tr>
<tr><td>9</td><td>0,7566</td><td>2,4046</td><td>1,6415</td></tr>
<tr><td>10</td><td>0,7659</td><td>2,3026</td><td>1,8568</td></tr>
<tr><td>11</td><td>0,7731</td><td>2,2135</td><td>2,2286</td></tr>
<tr><td>12</td><td>0,7787</td><td>2,1351</td><td>2,9848</td></tr>
<tr><td>13</td><td>0,7831</td><td>2,0655</td><td>5,2139</td></tr>
<tr><td>14</td><td>0,7866</td><td>2,0034</td><td>84,5591</td></tr>
<tr><td>15</td><td>0,7893</td><td>1,9477</td><td>-4,6751</td><td rowspan="8">(iii)
4 > 2 > k*
$\Rightarrow q_0 > 1$</td></tr>
<tr><td>16</td><td>0,7915</td><td>1,8975</td><td>-2,0133</td></tr>
<tr><td>17</td><td>0,7932</td><td>1,8521</td><td>-1,1755</td></tr>
<tr><td>18</td><td>0,7946</td><td>1,8108</td><td>-0,7730</td></tr>
<tr><td>20</td><td>0,7965</td><td>1,7386</td><td>-0,3936</td></tr>
<tr><td>25</td><td>0,7989</td><td>1,6028</td><td>-0,1036</td></tr>
<tr><td>30</td><td>0,7996</td><td>1,5083</td><td>-0,0319</td></tr>
<tr><td>40</td><td>0,8000</td><td>1,3866</td><td>-0,0033</td></tr>
</table>

Table 2: The benchmark position k* and maximum permitted rate of leakage $1-q_{G(\nu)}$ as a function of inequality aversion for the same income distribution ($200, $500, $800, $1100, $2400) when $\ell = 4$ and j = 2.

[18] Yaari's (1988) equality-mindedness measure concerns a leaky bucket.Yaari suggests a thought experiment whereby the incomes of a given fractile of the poor are raised, at the expense of lowering the incomes of a certain fractile of the rich. A more equality-minded index M, he argues, would tolerate a bigger fractile of donors than a less equality-minded one, before regarding the "leak" entailed as detrimental. Thus his leaks involve a loss of mass, whereas ours involve a loss of income.

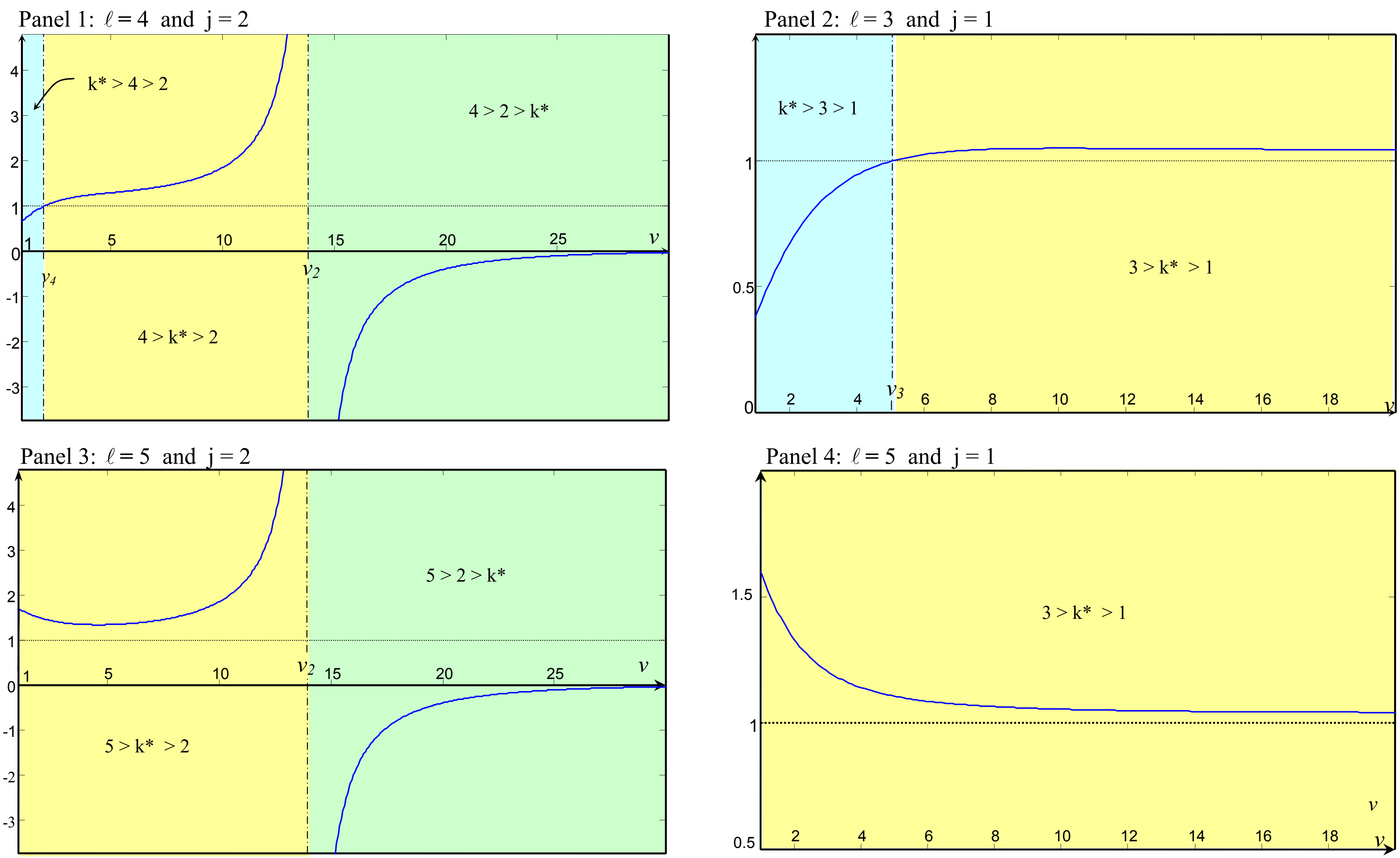

Figure 5: Maximum permitted leakage rate $1\text{-}q_{G(v)}$ for the extended Gini coefficient *G(v)* as a function of *v*, for the scenario in Table 1 and three other scenarios involving the richest and/or poorest person in the transfer

The leakage rates shown in Table 2 and Figure 5 may be compared with those of Duclos (2000, p.149-150), who calculates the maximum tolerable leaks for *no welfare loss*, where welfare is measured as $\mu[1-G(\nu)]$. Duclos's maximum leaks are shown for various scenarios to be increasing in ν and lying between 6.7% and 99.6%.

6. Summary and Conclusions

It is important for economists to be able to compare inequality in income distributions with different means. Incomes can change due to growth, and also due to disincentive effects arising from the implementation of redistributive programmes. It is perhaps surprising, then, that one can find little in the inequality measurement literature about the inequality consequences of a single income growing, or of a single leaky transfer. The effects on welfare of such changes have, of course, been much discussed; our results in this paper throw light on the corresponding questions for inequality, which we believe to be fundamental.

First, we looked at the effect on inequality of increasing one income. We confirmed the casual intuition that increasing a low income should reduce inequality and increasing a high one should surely raise it. In fact we proved that, for large classes of inequality indices, there is a benchmark income level or position dividing the two responses, which is different for each inequality index and income distribution. This benchmark can be both quantified and systematically related to a property of the underlying inequality ordering, its lower tail concern. The intuition for the aggregate, offered up by our analysis, that income growth in the lower part of a distribution will be equalizing, and income growth in the upper part disequalizing, seems unexceptionable; but it surely has not been appreciated before now that the divide between "lower" and "upper" that supports this intuition could differ so markedly for different inequality indices, and its determinants be understood.[19]

Second, we turned to the leaky bucket scenario. We took for granted a rate of leakage *(1-q)* from the bucket and asked the question, how leaky would the bucket have to be before the intended inequality-ameliorating effect of a single rich-to-poor transfer would be negated? The answer was *(1-q_0)*, with q_0 depending on the relative incomes or ranks of the donor and recipient, and, crucially, on

[19] Our analytics can in fact be extended to other types of index, for example to the variance of logarithms which, though not Lorenz consistent (Foster and Ok, 1999), is popular among applied economists. Let I be a distributional index in the form $I(\mathbf{x}) = [1/N].\Sigma_i v(x_i, b(\mathbf{x}))$ where $v : \Re_+^2 \to \Re_+$ and $b: \Re_+^N \to \Re_+$ are differentiable functions. Then $\partial I/\partial x_k = [1/N].\Sigma_i \{v_2(x_i, b(\mathbf{x})).b_k(\mathbf{x}) + v_1(x_k, b(\mathbf{x})\}$. For the variance of logarithms, $v(a,b) = [ln(a/b)]^2$ and $b(\mathbf{x}) = (\Pi_i x_i)^{1/N} = \tilde{\mu}$ which is geometric mean income. Then $\partial I/\partial x_k = 2ln(x_k/\tilde{\mu})/[Nx_k]$, whence $\partial I/\partial x_k \gtrless 0 \Leftrightarrow x_k/\tilde{\mu} \gtrless 1$. Thus the variance of logarithms has a benchmark income level equal to the geometric mean. The leaky bucket analytics go similarly: $q_0 = \dfrac{\partial I(.)/\partial x_\ell}{\partial I(.)/\partial x_j} = (x_j/x_\ell)\dfrac{\ln x_\ell - \ln\tilde{\mu}}{\ln x_j - \ln\tilde{\mu}}$. Compare this with (15), and use footnote 17: the analogue of Theorem 6 applies with benchmark $\tilde{\mu}$.

which side of the benchmark they are located. We showed that a negative rate of leakage or even one exceeding 100% could be tolerated for some configurations. Only in case the donor and recipient are both in the lower part of the distribution is there a bound $0 < (1\text{-}q_0) < 1$. So here too, we obtain an insight for the aggregate: the inefficiencies of redistributive programmes had better not be focussed entirely within the lower part of an income distribution.[20]

A further, major insight arises in the context of tax-transfer policy in a socially heterogeneous population, even in the absence of efficiency losses. Let A and B be two households, selected as the donor and recipient for a money transfer respectively. If the equivalence scale deflators for A's and B's money incomes are m_A and m_B, each unit reduction in the living standard of A is accompanied by an increase of $q = m_A/m_B$ units in the living standard of B. We can apply Theorems 6 and 7, to examine the effect of the (non-leaky) money transfer on inequality in the distribution of living standards, for any non-positional or positional index. If B is below the benchmark in the living standards distribution, inequality reduction requires $q > q_0$ (where $0 < q_0 < 1$ if A is also below the benchmark, and $q_0 < 0$ if A is above it); and if B is above the benchmark, inequality reduction requires $q < q_0$ (in this case $q_0 > 1$).[21] These results pick up on, and extend, an insight of Glewwe (1991), that some money transfers from the better-off to the worse-off can exacerbate inequality. Transfers taking place entirely below the benchmark may do this if from a less needy to a *very* needy type of household ($m_B > m_A/q_0$, where $0 < q_0 < 1$): we regard this as a strongly counter-intuitive result. Transfers taking place entirely above the benchmark may also exacerbate inequality, but only if directed to a very much less needy houshold type ($m_B < m_A/q_0$, where $q_0 > 1$); this seems less unreasonable. Transfers which are made across the benchmark are unambiguously inequality-reducing regardless of relative needs (because $q = m_A/m_B > q_0$ is always satisfied if $q_0 < 0$).

Although negative rates of "leakage" and rates exceeding 100% have not been encountered in leaky bucket analytics addressing the *welfare* effect of transfers, and may at first sight seem surprising in the inequality context, the intuition is, after all, quite straightforward. Tolerance of a leakage exceeding 100% ($q_0 < 0$) occurs when donor and "recipient" are either side of the benchmark. Taking from a rich person (above the benchmark) unambiguously reduces inequality. This effect is necessarily reinforced by giving to a poor person (below the benchmark). Hence, having taken from the rich, one can also take from the poor (up to a certain limit, that limit being $-q_0$) without eliminating the inequality

[20] In Lambert (1988), a labour supply model was investigated, in which wage rates were lognormally distributed and a piecewise linear negative income tax scheme was applied. It was shown that, for a wide range of tax and benefit parameter values, the efficiency loss of the tax-transfer system exceeded the size of the bucket.

[21] These requirements stem from (13), which shows that the inequality effect dI of the transfer is a negative or positive function of q respectively.

gain. Similarly, a negative leak ($q_0 > 1$) is tolerated when the donor and recipient are both above the benchmark. Taking \$1 from a rich person and giving it to another, less rich but still above the benchmark, reduces inequality (by the Principle of Transfers); to restore inequality to the previous level, one may give extra to the recipient (namely, an additional amount of $q_0 - 1$). Our analytics have enabled these effects to be quantified, understood and compared for wide classes of inequality indices.

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