CROSSED PRODUCT C^*-ALGEBRAS BY FINITE GROUP ACTIONS WITH A GENERALIZED TRACIAL ROKHLIN PROPERTY

by

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Dr. N. Christopher Phillips

This dissertation consists of two related parts. In the first portion we use the tracial Rokhlin property for actions of a finite group G on stably finite simple unital C^*-algebras containing enough projections. The main results of this part of the dissertation are as follows. Let A be a stably finite simple unital C^*-algebra and suppose α is an action of a finite group G with the tracial Rokhlin property. Suppose A has real rank zero, stable rank one, and suppose the order on projections over A is determined by traces. Then the crossed product algebra $C^*(G, A, \alpha)$ also has these three properties.

In the second portion of the dissertation we introduce an analogue of the tracial Rokhlin property for C^*-algebras which may not have any nontrivial projections called the *projection free tracial Rokhlin property.* Using this we show that under certain conditions if A is an infinite dimensional simple unital C^*-algebra with stable rank one and α is an action of a finite group G with the projection free tracial Rokhlin property, then $C^*(G, A, \alpha)$ also has stable rank one.

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CHAPTER I

INTRODUCTION

This dissertation focuses on the properties of crossed product C^*-algebras. Let A be a C^*-algebra and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We write α_g instead of $\alpha(g)$. As a set, the crossed product $C^*(G, A, \alpha)$ is the group ring $A[G]$. However, the multiplication and involution are skewed by the action α of G on A. If G is not finite but is discrete, we must complete $A[G]$ in a suitable norm. This construction has not only provided new examples of C^*-algebras, but has provided new ways of looking at old and naturally occurring C^*-algebras. For example, consider the irrational rotation algebras A_θ, which were originally described as being generated by elements u and v satisfying the relations $uu^* = 1$, $u^*u = 1$, $vv^* = 1$, $v^*v = 1$ and $uv = e^{2\pi i\theta}vu$. One can also describe A_θ as a crossed product by $\mathbb{Z}$ acting on $C(S^1)$ by rotation by an angle of $2\pi i\theta$.

It is natural to ask which properties of A are shared by the crossed product. In particular we would like to know when $C^*(G, A, \alpha)$ has one of the following three properties.

Definition I.1. *Let A be a unital C^*-algebra. We say that the* order on projections over A is determined by traces *if whenever $p, q \in M_\infty(A)$ are projections such that $\tau(p) < \tau(q)$ for all $\tau \in T(A)$, then $p \precsim q$.*

Definition I.2. *A unital C^*-algebra A has* **stable rank one** *if the invertible elements are dense in A [25].*

Definition I.3. *A unital C^*-algebra A has* **real rank zero** *if the invertible self-adjoint elements are dense in the self adjoint elements [2].*

One reason that these properties are important is because they are satisfied for many C^*-algebras. Additionally, stable rank one, real rank zero, or both are hypotheses of many theorems about C^*-algebras. Finally, it is known that A having stable rank one is not sufficient to guarantee that the stable rank of $C^*(G, A, \alpha)$ is one. Example 8.2.1 of [1] provides an example for which the stable rank of the crossed product is two. However, by a theorem of Osaka and Teruya, for any simple unital C^*-algebra with property (SP) and any finite group action, the stable rank of the crossed product is two or less [17].

Since real rank zero implies the existence of many projections, we need a notion of comparing projections.

Definition I.4. *For any projections p and q in A, we write $p \sim q$ if there exists an element $v \in A$ such that $v^*v = p$ and $vv^* = q$. In this case we say that p is* **(Murray-von Neumann) equivalent** *to q. We write $p \precsim q$ if there exists a projection r such that $p \sim r$ and $r \leq q$. In this case we say that p is* **(Murray-von Neumann) subequivalent** *to q.*

We will also need a condition on the action.

Definition I.5. *Let A be an infinite dimensional simple unital C^*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the* **tracial Rokhlin property** *if for every finite set $F \subset A$, every $\varepsilon > 0$, and every positive element $x \in A$ with $\|x\| = 1$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:*

1. *$\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.*
2. *$\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.*
3. *With $e = \sum_{g \in G} e_g$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x.*
4. *With e as in (3), we have $\|exe\| > 1 - \varepsilon$.*

In Chapters III, IV, and V we prove the following theorems which are finite group analogs of known results about actions of $\mathbb{Z}$ [16]:

Theorem I.6. *Let A be an infinite dimensional stably finite simple unital C^*-algebra with real rank zero, and suppose that the order on projections over A is determined by traces. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group with the tracial Rokhlin property. Then the order on projections over $C^*(G, A, \alpha)$ is determined by traces and $C^*(G, A, \alpha)$ has real rank zero.*

Theorem I.7. *Let A be an infinite dimensional stably finite simple unital C^*-algebra with real rank zero and stable rank one, and suppose that the order on projections over A is determined by* *traces. Let* $\alpha\colon G \to \mathrm{Aut}(A)$ *be an action of a finite group with the tracial Rokhlin property. Then* $C^*(G, A, \alpha)$ *has stable rank one.*

The tracial Rokhlin property has already proven itself useful for proving theorems about crossed products [7] and [21]. There is a related but strictly stronger notion called the Rokhlin property. For an example of an action with the tracial Rokhlin property, but not the Rokhlin property, let B be any simple C^*-algebra with tracial rank zero. Let $A = B \otimes B$ and let $\alpha\colon \mathbb{Z}/2\mathbb{Z} \to A$ be the action which interchanges the two copies of B. That is, the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ maps to the automorphism $\alpha_2\colon a \otimes b \mapsto b \otimes a$ [15].

There are relatively few actions with the Rokhlin property and many algebras which admit no actions at all with the Rokhlin property. However, there are many examples of actions with the tracial Rokhlin property.

It is clear from the definition of the tracial Rokhlin property that it guarantees the existence of at least n projections, where n is the order of the group. In fact, it implies the existence of infinitely many projections. Thus a C^*-algebra with few projections cannot have any action with the tracial Rokhlin property.

In Chapters VI and VII we have formulated a projection free generalization of the tracial Rokhlin property called the *projection free tracial Rokhlin property*. This generalization replaces the projections with positive elements and Murray-von Neumann equivalence with Cuntz equivalence of positive elements.

Definition I.8. *Let* x *and* y *be positive elements of a* C^**-algebra* A. *We write* $x \precsim y$ *if there exist elements* r_j *in* A *such that* $r_j y r_j^* \to x$ *with the convergence in norm. In this case we say* x *is* **(Cuntz) subequivalent** *to* y. *If* $x \precsim y$ *and* $y \precsim x$, *we write* $x \sim y$ *and say* x *is* **(Cuntz) equivalent** *to* y.

It turns out that if p and q are projections and p is Murray-von Neumann subequivalent to q, then p is Cuntz subequivalent to q.

We expect that if Z is the Jiang-Su algebra as defined in [10], then the action which interchanges the two copies of Z in $Z \otimes Z$ provides an example of an action with the projection free tracial Rokhlin property. The analogous result which leads to this belief is found in [19].

The main result of the later chapters of this dissertation is Theorem VII.17:

Theorem I.9. *Let A be an infinite dimensional stably finite simple unital C^*-algebra with stable rank one. Assume also that A has a unique 2-quasi-trace which is also a trace, and strict comparison of positive elements. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group with the generalized tracial Rokhlin property. Then $C^*(G, A, \alpha)$ has stable rank one.*

Unlike Theorem I.6 and Theorem I.7, the analog for actions of $\mathbb{Z}$ is not known. We do not ask the analogous question for real rank zero. This is because an algebra with real rank zero has many projections and so we can use the original definition of the tracial Rokhlin property.

Theorem VII.17 provides evidence that the generalization of the tracial Rokhlin property has been chosen appropriately. It is known that if an action has this generalized tracial Rokhlin property and the algebra is simple with tracial rank zero, then the action has the original tracial Rokhlin property (Lemma 1.8 of [22]). Tracial rank zero implies real rank zero and thus the existence of many projections, so this also an indication the generalization has the right definition.

The interest of this dissertation lies mainly in its applicability to the classification program. The classification program has been one of the major thrusts in C^*-algebras for the last 15 years. This program is the search for invariants which will distinguish separable, nuclear C^*-algebras up to isomorphism. Most of the known theorems deal with simple C^*-algebras. Ideally the invariants used should be relatively computable. One of the most important of these invariants is $K_0(A)$. The group $K_0(A)$ encodes information about projections in $M_n(A)$ up to Murray-von Neumann equivalence. In fact, K_0 is functor which can be considered as a non-commutative homology theory. Analogously, the Cuntz semigroup encodes information about positive elements up to Cuntz equivalence. Recent work by Brown, Perera, and Toms indicates that the Cuntz semigroup will also be a useful invariant for the purposes of classification [3].

The results in sections II, III, IV, and V are modeled heavily on those in [16] and [23], and the proof techniques here mimic those there whenever possible.

CHAPTER II

THE TRACIAL ROKHLIN PROPERTY

Definition II.1. *Let A be an infinite dimensional simple unital C^*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the* tracial Rokhlin property *if for every finite set $F \subset A$, every $\varepsilon > 0$, and every positive element $x \in A$ with $\|x\| = 1$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:*

1. *$\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.*
2. *$\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.*
3. *With $e = \sum_{g\in G} e_g$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x.*
4. *With e as in (3), we have $\|exe\| > 1 - \varepsilon$.*

When A is finite, as was shown in Lemma 1.12 of [23], Condition (4) of Definition II.1 is not needed:

Lemma II.2. *Let A be a finite infinite dimensional simple unital C^*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. Then α has the tracial Rokhlin property if and only if for every finite set $F \subset A$, every $\varepsilon > 0$, and every nonzero positive element $x \in A$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:*

1. *$\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.*
2. *$\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in S$.*
3. *With $e = \sum_{g\in G} e_g$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x.*

For the sake of comparison we also consider the Rokhlin property, which we call here the strict Rokhlin property for emphasis.

Definition II.3. *Let A be a unital C^*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the* strict Rokhlin property *if for every finite set $F \subset A$, and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:*

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ *for all* $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ *for all* $g \in G$ *and all* $a \in F$.
3. $\sum_{g\in G} e_g = 1$.

Notation II.4. *Let A be a unital C^*-algebra. We denote by $T(A)$ the set of all tracial states on A, equipped with the weak* topology. For any element of $T(A)$, we use the same letter for its standard extension to $M_n(A)$ for arbitrary n, and to $M_\infty(A) = \bigcup_{n=1}^{\infty} M_n(A)$ (no closure).*

Definition II.5. *Let A be a unital C^*-algebra. We say that the* order on projections over A is determined by traces *if whenever $p, q \in M_\infty(A)$ are projections such that $\tau(p) < \tau(q)$ for all $\tau \in T(A)$, then $p \precsim q$.*

The following lemma is the finite group analog of Lemma 1.4 in [16].

Lemma II.6. *Let A be an infinite dimensional stably finite simple unital C^*-algebra with real rank zero and such that the order on projections over A is determined by traces. Suppose $\alpha : G \to A$ is an action of a finite group on A. Then α has the tracial Rokhlin property if and only if for every finite set $F \subset A$ and every $\varepsilon > 0$ there are mutually orthogonal projections $e_g \in A$ for each $g \in G$ such that:*

1. $\|\alpha_h(e_g) - e_{gh}\| < \varepsilon$ *for* $g \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ *for all* $g \in G$ *and all* $a \in F$.
3. *With* $e = \sum_{g\in G} e_g$, *we have* $\tau(1 - e) < \varepsilon$ *for all* $\tau \in T(A)$.

Proof. First assume that α has the tracial Rokhlin property. Let $\varepsilon > 0$ and $F \subset A$ finite be given. Let n be large enough that $1/2^n < \varepsilon$.

We claim that A has no minimal nonzero projections. The claim holds because if B is a simple C*-algebra with real rank zero and which has a minimal projection, then B is isomorphic to the compact operators on some Hilbert space. However, the algebra A is both infinite dimensional and unital, so it is not isomorphic to the compact operators on any Hilbert space. This is precisely the condition "non elementary" required in Theorem 1.1 (i) of [28], so applying that theorem allows us to write

$$1 = \sum_{i=0}^{2^n} p_i$$

for mutually orthogonal projections p_i satisfying $p_0 \precsim p_1$ and $p_1 \sim \cdots \sim p_{2^n}$. This implies

$$\sum_{i=1}^{2^n} p_i \leq 1.$$

Thus we have

$$\sum_{i=1}^{2^n} \tau(p_i) = 2^n \tau(p_1) \leq 1,$$

which implies $\tau(p_1) \leq \frac{1}{2^n} < \varepsilon$.

On the other hand,

$$1 = \tau(1) = \sum_{i=0}^{2^n} \tau(p_i) = \tau(p_0) + \sum_{i=1}^{2^n} \tau(p_i) \leq (2^n + 1)\tau(p_1).$$

Therefore, $p_1 \neq 0$.

Now apply the definition of the tracial Rokhlin property to $x = p_1$, and to ε and F as given to get projections e_g satisfying conditions (1) through (4) of Definition II.1. It remains only to show that condition (3) of this lemma holds. By setting $e = \sum_{g \in G} e_g$, condition (3) of Definition II.1 gives $1 - e$ is equivalent to a projection in $\overline{xAx}$, so $\tau(1 - e) \leq \tau(x) < \varepsilon$ for all $\tau \in T(A)$.

Conversely, assume the condition of the lemma and let $\varepsilon > 0$, $F \subset A$ finite, and $x \in A$ a positive element of norm 1 be given. Choose a nonzero projection q in the hereditary subalgebra generated by x. Such a projection exists since A has real rank zero. Choose δ with $0 < \delta < \min(\varepsilon, \inf_{\tau \in T(A)} \tau(q))$. Now apply the condition of the lemma with ε replaced by δ to get projections e_g. Note that $\inf_{\tau \in T(A)} \tau(q) > 0$ since $T(A)$ is compact and $\tau(q) > 0$ for each $\tau \in T(A)$. Set $e = \sum_{g \in G} e_g$. Then since $\tau(1 - e) < \tau(q)$ for every tracial state and the order on projections over A is determined by traces, $1 - e \precsim q$ which gives condition (3) of the definition and completes the proof. ∎

Lemma II.7. *Let A be an infinite dimensional simple unital C^*-algebra, and let $\alpha: G \to \mathrm{Aut}(A)$ be an action of a finite group G on A which has the tracial Rokhlin property. Then α_g is outer for every $g \in G \setminus \{1\}$.*

Proof. This is Lemma 1.5 of [23]. ∎

Corollary II.8. *Let A be an infinite dimensional stably finite simple unital C^*-algebra and let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group G with the tracial Rokhlin property. Then $C^*(G, A, \alpha)$ is simple.*

Proof. Using Lemma II.7, this is immediate from Theorem 3.1 of [11]. ∎

Definition II.9. *Let A be a C^*-algebra. We say that A has* Property (SP) *if every nonzero hereditary subalgebra in A contains a nonzero projection.*

Notation II.10. *For any compact convex set Δ in a topological vector space, we let* $\mathrm{Aff}(\Delta)$ *be the set of all real valued continuous affine functions on Δ.*

Here we are particularly interested in $\mathrm{Aff}(T(A))$.

The proof of Proposition II.13 requires two lemmas.

Lemma II.11. *Let A be a unital C^*-algebra, and let $\alpha: \Gamma \to \mathrm{Aut}(A)$ be an action of a countable amenable group. Let $f_1, \ldots, f_l \in \mathrm{Aff}(T(A))$ have the property that $f_j(\tau) > 0$ for all Γ-invariant $\tau \in T(A)$. Then there exist n and $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that for all $\tau \in T(A)$ we have*

$$\frac{1}{n} \sum_{k=1}^{n} f_j(\tau \circ \alpha_{\gamma_k}^{-1}) > 0$$

for $1 \leq j \leq l$.

Proof. This is Lemma 2.2 in [16]. ∎

The following lemma is a more flexible version of a result of Zhang [28]. In Zhang's version, which is used in the proof, the integer n of the hypotheses is required to be a power of 2.

Lemma II.12. *Let A be a simple unital infinite dimensional C^*-algebra with real rank zero. Let $p \in A$ be a projection, and let $n \in \mathbb{N}$. Then there exist projections $p_0, p_1, \ldots, p_n \in A$ such that*

$$\sum_{k=0}^{n} p_k = p, \quad p_1 \sim p_2 \sim \cdots \sim p_n, \quad \text{and} \quad p_0 \precsim p_1.$$

Proof. This is Lemma 2.3 in [16]. ∎

Proposition II.13. *Let A be a simple unital infinite dimensional C^*-algebra with real rank zero, and assume that the order on projections over A is determined by traces. Let $\alpha\colon \Gamma \to \mathrm{Aut}(A)$ be an action of a countable amenable group. Let p, $q \in M_\infty(A)$ be projections such that $\tau(p) < \tau(q)$ for every Γ-invariant tracial state τ on A. (We extend τ to $M_\infty(A)$ as in Notation II.4). Then there is $s \in M_\infty(C^*(\Gamma, A, \alpha))$ such that*

$$s^*s = p, \quad ss^* \leq q, \quad \text{and} \quad ss^* \in M_\infty(A).$$

In particular, $p \precsim q$ in $M_\infty(C^*(\Gamma, A, \alpha))$.

Proof. This is Proposition 2.4 in [16]. ∎

The following lemma is the finite group analog of Lemma 2.5 in [16].

Lemma II.14. *Let A be an infinite dimensional stably finite simple unital C^*-algebra with real rank zero such that the order on projections over A is determined by traces. Let G be a finite group of order n and let $\alpha : G \to \mathrm{Aut}(A)$ be an action of G with the tracial Rokhlin property. Let $\iota\colon A \to C^*(G, A, \alpha)$ be the inclusion map. Then for every finite set $F \subset C^*(G, A, \alpha)$, every $\varepsilon > 0$, every $N \in \mathbb{N}$, and every nonzero positive element $z \in C^*(G, A, \alpha)$, there exist a projection $e \in A \subset C^*(G, A, \alpha)$, a unital subalgebra $D \subset eC^*(G, A, \alpha)e$, a projection $f \in A$, and an isomorphism $\varphi\colon M_n \otimes fAf \to D$, such that:*

1. *With $(e_{g,h})$ for $g, h \in G$ being a system of matrix units for M_n, we have $\varphi(e_{1,1} \otimes a) = \iota(a)$ for all $a \in fAf$ and $\varphi(e_{g,g} \otimes 1) \in \iota(A)$ for $g \in G$.*
2. *With $(e_{g,g})$ as in (1), we have $\|\varphi(e_{g,g} \otimes a) - \iota(\alpha_g(a))\| \leq \varepsilon\|a\|$ for all $a \in fAf$.*
3. *For every $a \in F$ there exist $b_1, b_2 \in D$ such that $\|ea - b_1\| < \varepsilon$, $\|ae - b_2\| < \varepsilon$, and $\|b_1\|, \|b_2\| \leq \|a\|$.*
4. *$e = \sum_{g \in G} \varphi(e_{g,g} \otimes 1)$.*
5. *The projection $1 - e$ is Murray-von Neumann equivalent in $C^*(G, A, \alpha)$ to a projection in the hereditary subalgebra of $C^*(G, A, \alpha)$ generated by z.*

6. *There are N mutually orthogonal projections $f_1, f_2, \ldots, f_N \in eDe$, each of which is Murray-von Neumann equivalent in $C^*(G, A, \alpha)$ to $1 - e$.*

Proof. We first make a simplification: It is not necessary to check the estimates $\|b_1\|$, $\|b_2\| \leq \|a\|$ in Condition (3) of the conclusion. To prove this, without loss of generality $\|a\| = 1$ for all $a \in F$. (If $0 \in F$, then b_1 and b_2 may be taken to be zero which satisfy the norm estimates. Otherwise we can normalize all the elements of F.)

Apply the statement without the bound on b_1 and b_2 with $\frac{1}{2}\varepsilon$ in place of ε, and with all other parameters the same. Let c_1 and c_2 be the resulting elements in Condition (3) of the weakened conclusion. Then $\|c_1\|$, $\|c_2\| \leq 1 + \frac{1}{2}\varepsilon$. Set

$$b_1 = \left(\frac{1}{1 + \frac{1}{2}\varepsilon}\right) c_1 \quad \text{and} \quad b_2 = \left(\frac{1}{1 + \frac{1}{2}\varepsilon}\right) c_2.$$

One checks that $\|b_1 - c_1\| \leq \frac{1}{2}\varepsilon$, so $\|b_1 - pa\| < \varepsilon$. Similarly $\|b_2 - ap\| < \varepsilon$. This proves the simplification.

Now we do the main part of the proof. Let $\varepsilon > 0$, and let $F \subset C^*(G, A, \alpha)$ be a finite set. Let $N \in \mathbb{N}$, and let $z \in C^*(G, A, \alpha)$ be a nonzero positive element.

Let u_g for $g \in G$ be the standard unitaries in the crossed product $C^*(G, A, \alpha)$. We regard A as a subalgebra of $C^*(G, A, \alpha)$ in the usual way.

For each $x \in F$ write $x = \sum_{g \in G} a_g u_g$. Let $S \subset A$ be a finite set which contains all the coefficients used for all elements of F. Let $M = 1 + \sup_{a \in S} \|a\|$.

Let $\delta_0 < \frac{\varepsilon}{16n^2 M}$. Let δ_1 be such that if p_1, p_2 are projections in a C*-algebra B and if $a \in B$ is such that $\|a^*a - p_1\| \leq \delta_1$ and $\|aa^* - p_2\| \leq \delta_1$, then there is a partial isometry $s \in B$ such that $s^*s = p_1$, $ss^* = p_2$, and $\|a - p\| \leq \delta_0$. Let $0 < \delta < \min\{\delta_0, \delta_1, \frac{\varepsilon}{4n^3}, 1\}$

Since A has real rank zero, it has Property (SP), and since (by Lemma II.7) α_g is outer for all $g \in G$, Theorem 4.2 of [9], with $N = \{1\}$, supplies a nonzero projection $q \in A$ which is Murray-von Neumann equivalent in $C^*(G, A, \alpha)$ to a projection in $\overline{zC^*(G, A, \alpha)z}$. Moreover, Lemma II.12 provides nonzero orthogonal Murray-von Neumann equivalent projections $q_0, q_1, \ldots, q_{2N} \in qAq$.

Apply the tracial Rokhlin property (Definition II.1) with δ in place of ε, with S in place of F, and with q_0 in place of x. Call the resulting projections e_g for each $g \in G$, and let $e = \sum_{g \in G} e_g$.

Set $f = e_1$, and define $w_{g,h} = u_{gh^{-1}}e_h$. We claim that the elements $(w_{g,h})_{g,h\in G}$ form a δ-approximate system of $n \times n$ matrix units. To prove the claim we compute:

$$\begin{aligned}\|w_{g,h}^* - w_{h,g}\| &= \|e_h u_{gh^{-1}}^* - u_{hg^{-1}}e_g\| \\ &= \|u_{gh^{-1}}e_h u_{gh^{-1}}^* - e_g\| \\ &= \|\alpha_{gh^{-1}}(e_h) - e_g\| < \delta.\end{aligned}$$

Then, using $e_g e_h = \delta_{g,h}e_h$ at the third step we find

$$\begin{aligned}&\|w_{g_1,h_1}w_{g_2,h_2} - \delta_{g_2,h_1}w_{g_1,h_2}\| \\ &= \|u_{g_1h_1^{-1}}e_{h_1}u_{g_2h_2^{-1}}e_{h_2} - \delta_{g_2,h_1}u_{g_1h_2^{-1}}e_{h_2}\| \\ &= \|u_{g_1h_1^{-1}}e_{h_1}u_{g_2h_2^{-1}}e_{h_2} - u_{g_1h_1^{-1}g_2h_2^{-1}}e_{h_2g_2^{-1}h_1}e_{h_2}\| \\ &= \|u_{g_1h_1^{-1}}(u_{g_2h_2^{-1}}u_{g_2h_2^{-1}}^*)e_{h_1}u_{g_2h_2^{-1}}e_{h_2} - u_{g_1h_1^{-1}g_2h_2^{-1}}e_{h_2g_2^{-1}h_1}e_{h_2}\| \\ &= \|u_{g_1h_1^{-1}g_2h_2^{-1}}(u_{g_2h_2^{-1}}^*e_{h_1}u_{g_2h_2^{-1}} - e_{h_2g_2^{-1}h_1})e_{h_2}\| \\ &\leq \|u_{g_2h_2^{-1}}^*e_{h_1}u_{g_2h_2^{-1}} - e_{h_2g_2^{-1}h_1}\| \\ &= \|\alpha_{g_2h_2^{-1}}^{-1}(e_{h_1}) - e_{h_2g_2^{-1}h_1}\| < \delta.\end{aligned}$$

For the final condition, since $\|eq_0e\| > 1 - \delta > 0$, the projection e is nonzero, so e_g is nonzero for each $g \in G$. This uses $\delta < 1$ again. In particular $\|w_{1,1}\| = \|e_1\| = 1 > 1 - \delta$. This proves the claim.

Since $(w_{g,h})_{g,h\in G}$ forms a δ-approximate system of matrix units, each $w_{g,1}$ is an approximate partial isometry for each $g \in G$. More specifically,

$$\|w_{g,1}w_{g,1}^* - e_g\| = \|u_g e_1 e_1 u_g^* - e_g\| = \|\alpha_g(e_1) - e_g\| < \delta$$

since the e_g are the tracial Rokhlin projections. Also,

$$\|w_{g,1}^*w_{g,1} - e_1\| = \|e_1 u_g^* u_g e_1 - e_1\| = \|e_1 - e_1\| = 0 < \delta$$

because u_g is a unitary for each g.

Since $\delta < \delta_1$, by the choice of δ_1 there exist partial isometries $z_g \in C^*(G, A, \alpha)$ for each $g \in G$ such that $\|z_g - w_{g,1}\| < \delta_0$ and such that $z_g z_g^* = e_g$ and $z_g^* z_g = e_1$. Moreover, one may check that we may take $z_1 = e_1$.

Let $(e_{g,h})_{g,h \in G}$ be an $n \times n$ system of matrix units for M_n. Define a linear function $\varphi : M_n \otimes e_1 A e_1 \to C^*(G, A, \alpha)$ by $\varphi(e_{g,h} \otimes a) = z_g a z_h^*$. One can then check in the usual way that φ is a homomorphism. It is also worth computing at this stage that for $g, h \in G$ and $a \in e_1 A e_1$, we have $\|\varphi(e_{g,h} \otimes a) - w_{g,1} a w_{h,1}^*\| \leq 2\|a\|\delta_0$. Let D be the image of φ, so that φ is clearly surjective as a map from $M_n \otimes e_1 A e_1$ to D. To check that φ is injective we first recall that $\ker(\varphi)$ is an ideal in $M_n \otimes e_1 A e_1$ which means that $\ker(\varphi) \cap (e_{g,h} \otimes e_1 A e_1) = e_{g,h} \otimes I$ where I is an ideal of $e_1 A e_1$ which does not change as g and h vary. But we can compute that if $0 = \varphi(e_{g,h} \otimes a) = z_g a z_h^*$ for some $a \in e_1 A e_1$, then multiplying on the left by z_g^* and on the right by z_h we see that $e_1 a e_1 = a = 0$, so $I = 0$, that is $\ker(\varphi) = \{0\}$, so that φ is injective.

Now $\varphi(e_{1,1} \otimes a) = z_1 a z_1^* = e_1 a e_1 = a$ for any $a \in e_1 A e_1$. Also, $\varphi(e_{g,g} \otimes 1) = z_g e_1 z_g^* = z_g z_g^* z_g z_g^* = e_g \in A$. These two conditions make up (1) of the conclusion.

To verify (2), let $a \in e_1 A e_1$ and estimate

$$\begin{aligned}
\|\varphi(e_{g,g} \otimes a) - \alpha_g(a)\| &\leq \|\varphi(e_{g,g} \otimes a) - w_{g,1} a w_{g,1}^*\| + \|w_{g,1} a w_{g,1}^* - \alpha_g(a)\| \\
&\leq 2\|a\|\delta_0 + \|u_g e_1 a e_1 u_g^* - \alpha_g(a)\| \\
&= 2\|a\|\delta_0 \\
&\leq \varepsilon\|a\|.
\end{aligned}$$

For (4) we observe $\sum_{g \in G} \varphi(e_{g,g} \otimes 1) = \sum_{g \in G} z_g e_1 z_g^* = \sum_{g \in G} e_g = e$.

Condition (5) holds essentially by construction since $1 - e$ is Murray-von Neumann equivalent to a projection in $q_0 A q_0$, but $q_0 \in qAq$ and q is equivalent to a projection in the hereditary subalgebra generated by z. In total this gives $1 - e$ is subequivalent to a projection in the hereditary subalgebra generated by z.

Now for condition (6), since $q_j \sim q_i$ and Murray von-Neumann equivalent projections have the same trace, $\tau(q_j) < \frac{1}{2N}$ for $0 \leq j \leq 2N$ and for any $\tau \in T(A)$. In particular, since $1 - e$ is subequivalent to q_0 we have $\tau(1 - e) \leq \tau(q_0) < \frac{1}{2N}$. This implies $1 - \frac{1}{2N} < \tau(e)$. This gives $\frac{1}{2} < \tau(e)$. Additionally $\tau(q_j) \leq \frac{1}{2N}$ implies $\tau(\sum_{j=1}^{N} q_j) < \frac{1}{2}$. Combining these statements gives $\tau(\sum_{j=1}^{N} q_j) < \tau(e)$ for all $\tau \in T(A)$. So since order on projections over A is determined by traces,

$\sum_{j=1}^{N} q_j \precsim e$. Let $h \in A$ be a projection satisfying $\sum_{j=1}^{N} q_j \sim h \leq e$ and let s be a partial isometry with $s^*s = \sum_{j=1}^{N} q_j$ and $ss^* = h$. Let $h_j = sq_js^*$ for $j = 1, \ldots, N$. One checks that $h_1, \ldots h_N$ are mutually orthogonal projections summing to h. Furthermore since $h_j \leq h \leq e$ we have $h_j \leq e$. Furthermore, $h_j \sim g_j$ via the partial isometry sg_j. So now we have $1 - e \precsim g_j \sim h_j$. Let f_j be a projection such that $1 - e \sim f_j \leq h_j$. Since $f_j \leq h_j$, and the h_j are mutually orthogonal, $f_1, \ldots, f_N$ are mutually orthogonal. Finally $f_j \leq h_j \leq e$ in A and $eAe \subset eDe$, so $f_1, \ldots, f_N$ are the projections we desired.

In order to show (3) we will use the following claim.

Claim: If $y = \sum_{g \in G} a_g u_g$ with $a_g \in A$ and $\|a_g\| \leq M$, and if $[e_g, a_h] = 0$ for all $g, h \in G$, then there are $d_1,\ d_2 \in D$ such that $\|ey - d_1\|$, $\|ye - d_2\| < 8n^2 M \delta_0$.

Proof of claim: We can write $ey = \sum_{g \in G} \sum_{h \in G} e_g a_h u_h = \sum_{g \in G} \sum_{h \in G} (e_g a_h e_g)(e_g u_h)$ since e_g and a_h commute. Now we make a norm estimate involving one of the factors in the third expression for ey using the fact that z_g is a partial isometry at the third step:

$$
\begin{aligned}
&\|\varphi(e_{g,g} \otimes e_1 \alpha_{g^{-1}}(a_h) e_1) - e_g a_h e_g\| \\
&= \|z_g e_1 \alpha_g^{-1}(a_h) e_1 z_g^* - e_g a_h e_g\| \\
&= \|e_g z_g \alpha_g^{-1}(a_h) z_g^* e_g - e_g a_h e_g\| \\
&\leq \|z_g \alpha_g^{-1}(a_h) z_g^* - z_g \alpha_g^{-1}(a_h) w_{g,1}^*\| \\
&\qquad + \|z_g \alpha_g^{-1}(a_h) w_{g,1}^* - w_{g,1} \alpha_g^{-1}(a_h) w_{g,1}^*\| + \|u_g e_1 \alpha_g^{-1}(a_h) e_1 u_g^* - e_g a_h e_g\| \\
&\leq 2M\delta_0 + \|\alpha_g(e_1) a_h \alpha_g(e_1) - e_g a_h e_g\| \leq 2M\delta_0 + 2M\delta.
\end{aligned}
$$

Now we make an estimate involving the other factor:

$$
\begin{aligned}
&\|\varphi(e_{g,h^{-1}g} \otimes e_1) - e_g u_h\| \\
&\leq \|\varphi(e_{g,h^{-1}g} \otimes e_1) - u_g e_1 u_{h^{-1}g}^*\| + \|u_g e_1 u_{h^{-1}g}^* - e_g u_h\|.
\end{aligned}
$$

This last line is less than or equal to $2\delta_0 + \|\alpha_g(e_1) u_h - e_g u_h\|$ by an estimate we made previously. This in turn is less than or equal to $2\delta_0 + \delta$.

Let $d_0(g,h) = \varphi(e_{g,g} \otimes e_1\alpha_{g^{-1}}(a_h)e_1)\varphi(e_{g,h^{-1}g} \otimes e_1)$. Then we have

$$\begin{aligned}
&\|d_0(g,h) - e_g a_h u_h\| \\
&\leq \|d_0(g,h) - \varphi(e_{g,g} \otimes e_1\alpha_{g^{-1}}(a_h)e_1)e_g u_h\| + \|\varphi(e_{g,g} \otimes e_1\alpha_{g^{-1}}(a_h)e_1)e_g u_h - (e_g a_h e_g)(e_g u_h)\| \\
&\leq \|\varphi(e_{g,g} \otimes e_1\alpha_{g^{-1}}(a_h)e_1)\|(2\delta_0 + \delta) + 2M\delta_0 + 2M\delta \\
&\leq M(2\delta_0 + \delta) + 2M\delta_0 + 2M\delta \\
&= 4M\delta_0 + 3M\delta.
\end{aligned}$$

Now let $d_1 = \sum_{g\in G}\sum_{h\in G} d_0(g,h)$. Then

$$\begin{aligned}
\|d_1 - ey\| &= \left\|\sum_{g\in G}\sum_{h\in G} d_0(g,h) - \sum_{g\in G}\sum_{h\in G} e_g a_h u_h\right\| \\
&\leq \sum_{g\in G}\sum_{h\in G} \|d_0(g,h) - e_g a_h u_h\| \\
&\leq n^2 M(4\delta_0 + 3M\delta) \\
&< 8n^2 M\delta_0.
\end{aligned}$$

We now turn our attention to the construction of d_2. We can write

$$ye = \sum_{g\in G}\sum_{h\in G} a_h u_h e_g = \sum_{g\in G}\sum_{h\in G} a_h\alpha_h(e_g)u_h.$$

We note that

$$\begin{aligned}
&\left\|\sum_{g\in G}\sum_{h\in G} a_h\alpha_h(e_g)u_h - \sum_{g\in G}\sum_{h\in G} a_h e_{hg} u_h\right\| \\
&\leq \sum_{g\in G}\sum_{h\in G} \|\alpha_h(e_g) - e_{hg}\| \\
&< n^2\delta.
\end{aligned}$$

But $\sum_{g\in G}\sum_{h\in G} a_h e_{hg} u_h = \sum_{h\in G}\sum_{g\in G} e_{hg} a_h u_h = \sum_{h\in G}\sum_{k\in G} e_k a_h u_h$ by making the change of variables, $k = hg$. This last is of the same form as ey, so using the argument above there is an element $d_2 \in D$ such that

$$\|\sum_{h\in G}\sum_{k\in G} e_k a_h u_h - d_2\| \le n^2 M(4\delta_0 + 3M\delta).$$

Thus

$$\|ye - d_2\| \le n^2 M(4\delta_0 + 3M\delta) + n^2\delta = 4n^2 M(\delta + \delta_0) < 8n^2 M\delta_0.$$

We are now in a position to prove (3). Let $x \in F$ and choose $b_g \in S$ such that $x = \sum_{g\in G} b_g u_g$. Define $a_g = (1-e)b_g(1-e) + \sum_{h\in G} e_h b_g e_h$. Now by writing the 2 by 2 matrix decomposition for b_g and subtracting we get

$$b_g - a_g = \sum_{h\in G}[(1-e)b_g e_h + e_h b_g(1-e)] + \sum_{h\in G}\sum_{\substack{k\in G\\ k\ne h}} e_k b_g e_h.$$

Because $b_g \in S$ which was the set to which we applied the tracial Rokhlin property, $\|[b_g, e_h]\| < \delta$. Then

$$\begin{aligned}
\|b_g - a_g\| &\le \sum_{h\in G}\left[\|(1-e)b_g e_h\| + \|e_h b_g(1-e)\|\right] + \sum_{\substack{k\in G\\ k\ne h}} \|e_k b_g e_h\| \\
&\le \sum_{h\in G}\left[\|(1-e)[b_g, e_h]\| + \|(1-e)e_h b_g\| + \|[b_g, e_h](1-e)\| + \|b_g e_h(1-e)\|\right] \\
&\quad + \sum_{\substack{k\in G\\ k\ne h}} \|e_k[b_g, e_h]\| + \sum_{\substack{k\in G\\ k\ne h}} \|e_k e_h b_g\| \\
&\le n[\delta + 0 + \delta + 0 + (n-1)\delta + 0] \\
&= (n^2+n)\delta \\
&< 2n^2\delta.
\end{aligned}$$

Set $y = \sum_{g\in G} a_g u_g$. Then

$\|x - y\| \le \sum_{g\in G}\|(b_g - a_g)u_g\| \le \|b_g - a_g\| \le n(2n^2\delta) = 2n^3\delta.$

One easily checks that $[a_g, e_k] = 0$ for all $g, k \in G$. Thus the claim applies to y and provides $d_1 \in D$ such that $\|ey - d_1\| < 8n^2 M\delta_0$. Thus

$$\|ex - d_1\| \leq \|ex - ey\| + \|ey - d_1\| \leq 2n^3\delta + 8n^2M\delta_0 < \varepsilon$$

by the choice of δ and δ_0.

Similarly, the claim provides $d_2 \in D$ such that $\|ye - d_2\| < 8n^2M\delta_0$ which then satisfies $\|xe - d_2\| < \varepsilon$. ∎

Given objects satisfying part (1) of the conclusion of Lemma II.14, we can make a useful homomorphism into $C^*(G, A, \alpha)$ which should be thought of as a kind of twisted inclusion of A. The following lemma is stated in terms of an arbitrary unital C*-algebra B, but we note it applies when $B = C^*(G, A, \alpha)$ and the standard embedding is the map ι.

Lemma II.15. *Let A be any simple unital C^*-algebra, let B be a unital C^*-algebra, and let $\iota : A \to B$ be a unital injective homomorphism.*

Let $e, f \in A$ be projections, and let $n \in \mathbb{N}$. Assume that there is an injective unital homomorphism $\varphi\colon M_n \otimes fAf \to \iota(e)B\iota(e)$ such that, with $(e_{j,k})$ being the standard system of matrix units for M_n, we have $\varphi(e_{1,1} \otimes a) = \iota(a)$ for all $a \in fAf$. Then there is a corner $A_0 \subset M_{n+1} \otimes A$ which contains

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a \in (1-e)A(1-e) \text{ and } b \in M_n \otimes fAf \right\}$$

as a unital subalgebra, and an injective unital homomorphism $\psi\colon A_0 \to B$ such that

$$\psi \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \iota(a) + \varphi(b)$$

for $a \in (1-e)A(1-e)$ and $b \in M_n \otimes fAf$.

Moreover, if $\alpha : G \to \mathrm{Aut}(A)$ is an action of a finite group on A, $B = C^*(G, A, \alpha)$, and ι is the standard inclusion, then for every α-invariant tracial state τ on A there is a tracial state σ on $C^*(G, A, \alpha)$ such that the extension $\overline{\tau}$ of τ to $M_{n+1} \otimes A$ satisfies $\overline{\tau}|_{A_0} = \sigma \circ \psi$.

Proof. Set

$$q = \mathrm{diag}(1-e,\ f,\ f,\ \ldots,\ f) \in M_{n+1} \otimes A,$$

and set

$$A_0 = q(M_{n+1} \otimes A)q \quad \text{and} \quad e_0 = \mathrm{diag}(0, f, f, \ldots, f) \in A_0.$$

In M_{n+1}, call the matrix units $e_{j,k}$ for $0 \leq j,\, k \leq n$. Then $q - e_0 = e_{0,0} \otimes (1 - e)$. Define $\psi \colon A_0 \to C^*(G, A, \alpha)$ as follows.

1. For $a \in (q - e_0)A_0(q - e_0)$, write $a = e_{0,0} \otimes x$ with $x \in (1 - e)A(1 - e)$, and set $\psi(a) = \iota(x)$.

2. For $a \in e_0 A_0 e_0$, write $a = \sum_{j,k=1}^{n} e_{j,k} \otimes x_{j,k}$ with $x_{j,k} \in fAf$ for all j and k. Regard this sum as an element of $M_n \otimes fAf$ in the obvious way, and set $\psi(a) = \varphi(a)$.

3. For $a \in (e_{j,j} \otimes f)A_0(q - e_0)$ for some j with $1 \leq j \leq n$, write $a = e_{j,0} \otimes x$ with $x \in fA(1 - e)$, and set $\psi(a) = \varphi(e_{j,1} \otimes f)\iota(x)$.

4. For $a \in (q - e_0)A_0(e_{j,j} \otimes f)$ for some j with $1 \leq j \leq n$, set $\psi(a) = \psi(a^*)^*$ using (3).

Then extend by linearity.

To prove the first part of the lemma, it suffices to prove that ψ defined this way is in fact a homomorphism. It is clear that ψ is linear and that $\psi(a^*) = \psi(a)^*$ for all $a \in A_0$, so we prove multiplicativity. We must show that $\psi(ab) = \psi(a)\psi(b)$. It suffices to consider 16 cases, namely when a falls into each of the four categories above and when b falls into each of the four categories above. We number the cases using ordered pairs, with the first coordinate saying which category a is in and the second coordinate for b. We will treat the four most involved cases first.

For (3,1), write $a = e_{j,0} \otimes x$ as in (3) and write $b = e_{0,0} \otimes y$ analogously to (1). Then $ab = e_{j,0} \otimes xy$ analogously to (3), so

$$\psi(a)\psi(b) = \varphi(e_{j,1} \otimes f)\iota(x)\iota(y) = \varphi(e_{j,1} \otimes f)\iota(xy) = \psi(ab).$$

For (3,4), the analogous expressions are: $a = e_{j,0} \otimes x$ and $b = e_{0,k} \otimes y$, then, using $xy \in fAf$ we compute $\iota(xy) = \varphi(e_{1,1} \otimes xy)$,

$$\begin{aligned} \psi(a)\psi(b) &= \varphi(e_{j,1} \otimes f)\iota(x)\iota(y)\varphi(e_{1,k} \otimes f) = \varphi(e_{j,1} \otimes f)\iota(xy)\varphi(e_{1,k} \otimes f) \\ &= \varphi(e_{j,1} \otimes f)\varphi(e_{1,1} \otimes xy)\varphi(e_{1,k} \otimes f) = \psi(ab), \end{aligned}$$

since ab has the form described in (2).

Similarly, in (4,2) write $a = e_{0,j} \otimes x$ with $x \in (1-e)Af$ and $b = \sum_{j,k=1}^{n} e_{j,k} \otimes y_{j,k}$ with all $y_{j,k} \in fAf$; then

$$ab = \sum_{k=1}^{n} e_{0,k} \otimes xy_{j,k}$$

with $xy_{j,k} \in (1-e)Af$, and

$$\begin{aligned}\psi(a)\psi(b) &= \sum_{k=1}^{n} \iota(x)\varphi(e_{1,j} \otimes f)\varphi(e_{j,k} \otimes y_{j,k}) = \sum_{k=1}^{n} \iota(x)\varphi(e_{1,1} \otimes y_{j,k})\varphi(e_{1,k} \otimes f) \\ &= \sum_{k=1}^{n} \iota(x)\iota(y_{j,k})\varphi(e_{1,k} \otimes f) = \psi(ab).\end{aligned}$$

Finally, in (4,3) if $j \neq k$ one easily gets $\psi(a)\psi(b) = 0 = \psi(ab)$, and otherwise one writes $a = e_{0,j} \otimes x$, $b = e_{j,0} \otimes y$, and

$$\begin{aligned}\psi(a)\psi(b) &= \iota(x)\varphi(e_{1,j} \otimes f)\varphi(e_{j,1} \otimes f)\iota(y) = \iota(x)\varphi(e_{1,1} \otimes f)\iota(y) \\ &= \iota(x)\iota(f)\iota(y) = \iota(xy) = \psi(ab).\end{aligned}$$

In case (1,1) and (2,2) multiplicativity comes from the multiplicativity of ι and φ respectively. In the cases (1,2), (1,3), (2,4), and (3,3), one easily checks that both $\psi(ab)$ and $\psi(a)\psi(b)$ are zero. The remaining cases may be obtained by taking adjoints of those cases already done.

It remains to prove the statement about the tracial states. Let τ be an α-invariant tracial state on A. Let $E\colon C^*(G, A, \alpha) \to A$ be the map given by $E(\sum_{g \in G} a_g u_g) = a_1$. One can check that E is a conditional expectation. Let $\sigma = \tau \circ E$ and we check that this is a tracial state on

$C^*(G, A, \alpha)$. That σ is a state is clear, so we just verify that it is tracial. We compute

$$\begin{aligned}
\sigma(ab) &= \tau\left(E(ab)\right) \\
&= \tau\left(E\left(\left(\sum_{g\in G} a_g u_g\right)\left(\sum_{h\in G} b_h u_h\right)\right)\right) \\
&= \tau\left(E\left(\sum_{g,h\in G} a_g u_g b_h u_g^* u_g u_h\right)\right) \\
&= \tau\left(E\left(\sum_{g,h\in G} a_g \alpha_g(b_h) u_{gh}\right)\right) \\
&= \tau\left(\sum_{g\in G} a_g \alpha_g\left(b_{g^{-1}}\right)\right) \\
&= \sum_{g\in G} \tau\left(a_g \alpha_g\left(b_{g^{-1}}\right)\right).
\end{aligned}$$

Meanwhile

$$\begin{aligned}
\sigma(ba) &= \tau\left(E\left(\left(\sum_{h\in G} b_h u_h\right)\left(\sum_{g\in G} a_g u_g\right)\right)\right) \\
&= \tau\left(E\left(\sum_{g,h\in G} b_h u_h a_g u_g\right)\right) \\
&= \tau\left(E\left(\sum_{g,h\in G} b_h \alpha_h(a_g) u_{hg}\right)\right) \\
&= \tau\left(\sum_{g\in G} b_{g^{-1}} \alpha_{g^{-1}}(a_g)\right) \\
&= \sum_{g\in G} \tau\left(\alpha_g\left(b_{g^{-1}}\right) a_g\right) \\
&= \sum_{g\in G} \tau\left(a_g \alpha_g(b_{g^{-1}})\right).
\end{aligned}$$

If $f = 0$ then $A_0 = A$ and $\psi = \iota$, so the statement is immediate. Otherwise, for $a \in fAf$, we have

$$\sigma \circ \psi(e_{1,1} \otimes a) = \sigma \circ \varphi(e_{1,1} \otimes a) = \sigma \circ \iota(a) = \tau(a).$$

Therefore $\sigma \circ \psi$ and $\overline{\tau}$ agree on the full corner $(e_{1,1} \otimes f)(M_{n+1} \otimes A)(e_{1,1} \otimes f)$ of A_0. So $\sigma \circ \psi = \overline{\tau}$. ▮

CHAPTER III

TRACES AND ORDER ON PROJECTIONS IN CROSSED PRODUCTS

In this section, we prove that if A is a simple unital C*-algebra with real rank zero such that the order on projections over A is determined by traces, and if $\alpha : G \to \mathrm{Aut}(A)$ is an action of a finite group G with the tracial Rokhlin property, then the order on projections over $C^*(G, A, \alpha)$ is determined by traces. The methods are adapted from Section 3 of [16] which are adapted from Section 3 of [20], and originally came from [24].

We begin with a comparison lemma for projections in crossed products by actions with the tracial Rokhlin property.

Lemma III.1. *Assume the hypotheses of Lemma II.15 with $B = C^*(G, A, \alpha)$, and assume in addition that A has real rank zero and that the order on projections over A is determined by traces. Let $\psi\colon A_0 \to C^*(G, A, \alpha)$ be as in the conclusion of Lemma II.15. Suppose that p, $q \in \psi(A_0)$ are projections such that $\tau(p) < \tau(q)$ for all tracial states τ on $C^*(G, A, \alpha)$. Then there exists a projection $r \in \psi(A_0)$ such that $r \leq q$ and r is Murray-von Neumann equivalent to p in $C^*(G, A, \alpha)$.*

Proof. If the projection f as in Lemma II.15 is zero, then $A_0 = A$ and $\psi = \iota$. So the statement follows from Proposition II.13.

Otherwise, as in the proof of Lemma II.15, let $e_{j,k}$, for $0 \leq j$, $k \leq n$, be the matrix units in M_{n+1}. Also let $\iota\colon A \to C^*(G, A, \alpha)$ be the inclusion, and let $D = \iota(A)$ and $D_0 = \psi(A_0)$. Since $a \in fAf$ implies $\iota(a) = \varphi(e_{1,1} \otimes a) = \psi(e_{1,1} \otimes a)$, the algebra $E = \iota(fAf)$ is a hereditary subalgebra of both D and D_0.

Now let p, $q \in D_0$ be projections such that $\tau(p) < \tau(q)$ for all tracial states τ on $C^*(G, A, \alpha)$. Note that A_0 is the corner of the simple algebra $M_{n+1} \otimes A$. Thus A_0 and hence

$\psi(A_0) = D_0$ are both simple. Thus there is m such that

$$1 \oplus 0 \cdots \oplus 0 \precsim \iota(f) \oplus \cdots \oplus \iota(f)$$

in $M_m(D_0)$. We identify D and D_0 with corners in $M_m(D)$ and $M_m(D_0)$ in the standard way. Then, since $p, q \leq 1$, there exist projections

$$p_0,\, q_0 \leq \iota(f) \oplus \cdots \oplus \iota(f)$$

in $M_m(D_0)$ such that $p \sim p_0$ and $q \sim q_0$ in $M_m(D_0)$. Clearly $p_0,\, q_0 \in M_m(E) \subset M_m(D)$, and similarly, $p_0,\, q_0 \in M_m(E) \subset M_m(D_0)$, They also satisfy $\tau(p_0) < \tau(q_0)$ for $\tau \in T(C^*(G, A, \alpha))$. We now wish to apply Proposition II.13. Let τ be an α-invariant tracial state on A and let τ also denote its extension to $M_{n+1}(A)$. Then, by the statement about traces in Lemma II.15, $\tau|_{A_0} = \sigma \circ \psi$ for some tracial state σ on $C^*(G, A, \alpha)$. Now since p_0 and q_0 are elements of both $f(M_m(\psi(A)))f$ and $f(M_m(\iota(D)))f$, we have $\psi(p_0) = \iota(p_0)$ and $\psi(q_0) = \iota(q_0)$. So

$$\tau(p_0) = \sigma(p_0) < \sigma(q_0) = \tau(q_0)$$

since $\sigma \in T(C^*(G, A, \alpha))$. Thus by applying Proposition II.13 to p_0 and q_0, there is a projection $r_0 \in M_m(D)$ such that $p_0 \sim r_0$ in $M_m(C^*(G, A, \alpha))$ and $r_0 \leq q_0$. Then $r_0 \in M_m(E) \subset M_m(D_0)$.

Choose $s \in M_m(D_0)$ such that $s^*s = q_0$ and $ss^* = q$. Set $r = sr_0s^*$. Then $r \in M_m(D_0)$ and satisfies $p \sim p_0 \sim r_0 \sim r$ in $M_m(C^*(G, A, \alpha))$ and $r \leq q$. Also, $r = sr_0s^* \leq sq_0s^* = ss^*ss^* = q$, that is $r \leq q$ And since p, q are actually in $C^*(G, A, \alpha)$ we get $p \sim r$ in $C^*(G, A, \alpha)$. ∎

The next three lemmas are Lemma 3.2, Lemma 3.3, and Lemma 3.4 of [16].

Lemma III.2. *Let A be a C^*-algebra, let $p,\, q \in A$ be projections, let τ be a tracial state on A, and let $g\colon [0, 1] \to \mathbb{R}$ be a continuous function. Then $\tau(g(pqp)) = \tau(g(qpq))$.*

Lemma III.3. *Let $g\colon [0, 1] \to [0, 1]$ be a continuous function such that $g(1) = 1$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever A is a unital C^*-algebra, τ is a tracial state on A, and $p,\, q \in A$ are projections such that $\tau(p) > 1 - \delta$, then $\tau(g(qpq)) > \tau(q) - \varepsilon$ and $\tau(g(pqp)) > \tau(q) - \varepsilon$.*

Lemma III.4. *Let $\delta > 0$. Then there exists a continuous function $g\colon [0,1] \longrightarrow [0,1]$ such that $g(0) = 0$, $g(1) = 1$, and whenever A is a C^*-algebra with real rank zero and $a \in A$ is a positive element with $\|a\| \le 1$, then there is a projection $e \in \overline{aAa}$ such that $g(a)e = e$ and $\|ea - a\| < \delta$.*

The proof of the following theorem is adapted from the proof of Theorem 3.5 in [16], which is based on the proofs of Theorem 3.5 and Lemma 3.3 of [20], which in turn are based on Section 3 of [24].

Theorem III.5. *Let A be an infinite dimensional simple unital C^*-algebra with real rank zero, and suppose that the order on projections over A is determined by traces. Let $\alpha : G \longrightarrow \mathrm{Aut}(A)$ be an action of a finite group with the tracial Rokhlin property. Then the order on projections over $C^*(G, A, \alpha)$ is determined by traces.*

Proof. We first observe that the hypotheses on A imply that A is finite, but $M_n(A)$ satisfies all the same hypotheses, so A is in fact stably finite.

The next step is to reduce from considering projections in $M_\infty(C^*(G, A, \alpha))$ to considering those in $C^*(G, A, \alpha)$. That is, we claim it suffices to prove that if $q, r \in C^*(G, A, \alpha)$ are projections such that $\tau(q) \le \tau(r)$ for all $\tau \in T(C^*(G, A, \alpha))$, then $q \precsim r$. To do this we will show that since α has the tracial Rokhlin property then $\mathrm{id}_{M_n} \otimes \alpha$ as an action on $M_n \otimes A$ has the tracial Rokhlin property for any $n \in \mathbb{N}$. Thus $M_n \otimes A$ satisfies all the same hypotheses as A and so we get the same conclusion for projections in $M_n \otimes C^*(G, A, \alpha)$ which implies the statement of the theorem.

In order to show $\mathrm{id}_{M_n} \otimes \alpha$ has the tracial Rokhlin property, let $\varepsilon > 0$, let $F \subset M_n(A) \cong M_n \otimes A$ be finite, and let $x \in M_n(A)$ be a positive element with $\|x\| = 1$. Let S be a finite subset containing all elements of A which appear as entries in elements of F. We use the convention that all traces are normalized on A.

Let q be a nonzero projection in the hereditary subalgebra generated by x. Let $0 < \delta_0 = \min_{\tau \in T(A)}\{\tau(q)\}$. Let $0 < \delta < \min\{\delta_0/n, \varepsilon/n\}$. Apply the tracial Rokhlin property as given in Lemma II.6 with δ in place of ε and S in place of F to get projections e_g for each group element satisfying the conditions of Lemma II.6. Set $e = \sum_{g \in G} e_g$. Then we compute

$\tau(1-e) < \delta/n < \tau(q)/n$. Set $p_g = 1_n \otimes e_g$ and $p = \sum_{g\in G} p_g = 1_n \otimes e$. Now

$$\begin{aligned}\tau(1_n - p) &= \tau(1_n) - \sum_{g\in G} \tau(p_g) \\ &= n\tau(1_A) - \sum_{g\in G} n\tau(e_g) \\ &= n\tau(1_A - e) \\ &< \frac{n\tau(q)}{n}.\end{aligned}$$

Since the order on projections over A is determined by traces, $1_n \otimes p \precsim q$, that is $1-p$ is subequivalent to a projection in the hereditary subalgebra generated by x. That the projections p_g satisfy the two norm estimates for the tracial Rokhlin property is routine to check, so this proves the reduction.

Having proved the reduction, let q, $r \in C^*(G, A, \alpha)$ be projections such that $\tau(q) < \tau(r)$ for all tracial states τ on $C^*(\mathbb{Z}, A, \alpha)$. Since the tracial state space is weak-* compact, there is $\varepsilon > 0$ such that $\tau(r) - \tau(q) > \varepsilon$ for all tracial states τ. We may assume with out loss of generality that $\varepsilon \leq 1$.

Choose $\eta > 0$ sufficiently small so whenever B is a C^*-algebra and e, $f \in B$ are projections such that $\|ef - f\| < \eta$, then $f \precsim e$.

Choose continuous functions g_1, $g_2\colon [0,1] \to [0,1]$ such that

$$g_1(0) = g_2(0) = 0, \ ; \ g_1(1) = g_2(1) = 1, \quad g_1 g_2 = g_2,$$

and $|g_1(t) - t| < \frac{1}{4}\eta$ for all $t \in [0,1]$. Let $g\colon [0,1] \to [0,1]$ be a continuous functionfrom Lemma III.4 with $\frac{1}{8}\eta^2$ in place of δ.

Using continuity choose $\delta > 0$ small enough that whenever B is a C^*-algebra and a, $b \in B$ are positive elements satisfying $\|a\|, \|b\| \leq 1$ and $\|a - b\| < \delta$, then

$$\|g_1(a) - g_1(b)\| < \tfrac{1}{4}\eta, \quad \|g_2(a) - g_2(b)\| < \tfrac{1}{21}\varepsilon, \quad \text{and} \quad \|g(a) - g(b)\| < \tfrac{1}{6}\varepsilon.$$

Also require $\delta < \frac{1}{2}\eta$.

Apply Lemma III.3 with g_2 in place of g and with $\frac{1}{21}\varepsilon$ in place of ε, to get a number $\delta_0 > 0$. Choose an integer N satisfying $N \geq \max(\delta_0^{-1}, 6\varepsilon^{-1})$.

Apply Lemma II.14 with $\{q, r\}$ replacing F, with $\frac{1}{2}\delta$ replacing ε, with N as given, and with 1 replacing z. This gives us a projection $e \in A \subset C^*(G, A, \alpha)$, a unital subalgebra $D \subset eC^*(G, A, \alpha)e$, a projection $f \in A$, and an isomorphism $\varphi \colon M_n \otimes fAf \to D$, satisfying the six conditions in the conclusion of Lemma II.14.

Next we seek to construct a projection $r_0 \in D$ such that $r_0 \precsim r$ and $\tau(r_0) > \tau(r) - \frac{1}{3}\varepsilon$ for every tracial state τ on $C^*(G, A, \alpha)$.

By condition (3) of Lemma II.14, there exists $x \in D$ such that $\|re - x\| < \frac{1}{2}\delta$ and $\|x\| \leq 1$, so that $\|rer - xx^*\| < \delta$. Note that $x \in D \cong M_n \otimes fAf$, which has real rank zero. Thus we may apply Lemma 3.2 of [20] with a there taken to be $g_1(xx^*)$ and b there taken to be $g_2(xx^*)$ to get a projection $r_0 \in D$ such that

$$g_1(xx^*)r_0 = r_0 \quad \text{and} \quad \|r_0 g_2(xx^*) - g_2(xx^*)\| < \tfrac{1}{21}\varepsilon.$$

Next we show $\|rr_0 - r_0\| < \eta$ which implies $r_0 \precsim r$. By the choice of δ, since $\|rer - xx^*\| < \delta$ we have $\|g_1(rer) - g_1(xx^*)\| < \frac{1}{4}\eta$. Then $g_1(xx^*)r_0 = r_0$ gives $\|g_1(rer)r_0 - r_0\| < \frac{1}{4}\eta$. Combining this with $|g_1(t) - t| < \frac{1}{4}\eta$ yields $\|rerr_0 - r_0\| < \frac{1}{2}\eta$. Now we can compute

$$\begin{aligned} \|rr_0 - r_0\| &\leq \|rr_0 - rerr_0\| + \|rerr_0 - r_0\| \\ &\leq \|r\|\|r_0 - rerr_0\| + \tfrac{1}{2}\eta < \eta \end{aligned}$$

as desired. So we indeed have $r_0 \precsim r$ as claimed.

Now let $\tau \in T(C^*(G, A, \alpha))$. We work to obtain a lower bound on $\tau(r_0)$. The choice of δ and the fact that $\|rer - xx^*\| < \delta$ together imply that $\|g_2(rer) - g_2(xx^*)\| < \frac{1}{21}\varepsilon$. Thus $\|r_0 g_2(xx^*) - g_2(xx^*)\| < \frac{1}{21}\varepsilon$ implies $\|r_0 g_2(rer) - g_2(rer)\| < \frac{3}{21}\varepsilon$, and so

$$\begin{aligned} \|r_0 g_2(rer) r_0 - g_2(rer)\| &\leq \|r_0 g_2(rer) r_0 - r_0 g_2(rer)\| + \|r_0 g_2(rer) - g_2(rer)\| \\ &\leq \|r_0\|\|g_2(rer) r_0 - g_2(rer)\| + \tfrac{3\varepsilon}{21} \\ &< \tfrac{6}{21}\varepsilon. \end{aligned}$$

Therefore

$$\tau(r_0) \geq \tau(r_0 g_2(rer) r_0) > \tau(g_2(rer)) - \tfrac{6}{21}\varepsilon.$$

Now Lemma II.14 part (6) guarantees that $\tau(1-e) \leq N^{-1}\tau(e) \leq N^{-1} < \delta_0$, so $\tau(e) > 1 - \delta_0$, and the choice using Lemma III.3 gives $\tau(g_2(rer)) > \tau(r) - \frac{1}{21}\varepsilon$. Therefore

$$\tau(r_0) > \tau(g_2(rer)) - \tfrac{6}{21}\varepsilon > \tau(r) - \tfrac{7}{21}\varepsilon = \tau(r) - \tfrac{1}{3}\varepsilon.$$

We have now shown that r_0 is the desired projection.

Next we construct a projection $q_0 \in (1-e) + eDe$ such that $q \precsim q_0$ and $\tau(q_0) < \tau(q) + \frac{1}{3}\varepsilon$ for every tracial state τ on $C^*(G, A, \alpha)$. We will proceed by a method which is similar to that for r_0, but which is more complicated.

By Lemma II.14 part (3), there exists $x \in D$ such that $\|eq - x\| < \frac{1}{2}\delta$ and $\|x\| \leq 1$. Note that ex also satisfies $\|eq - ex\| \leq \|e\|\|eq - x\| < \frac{1}{2}\delta$ and $\|ex\| \leq 1$ so that by replacing x by ex we may assume $ex = x$ so that $xx^* \in eDe$ and $\|eqe - xx^*\| < \delta$. As $D \cong M_n \otimes fAf$, has real rank zero, we can apply the choice of g to find a projection $q_1 \in \overline{xx^*Dxx^*} \subset eDe$ such that

$$g(xx^*)q_1 = q_1 \quad \text{and} \quad \|q_1 xx^* - xx^*\| < \tfrac{1}{8}\eta^2.$$

Set $q_0 = 1 - e + q_1 \in (1-e) + eDe$. Then we wish to estimate $\|q_0 q - q\|$. We begin by computing,

$$\begin{aligned}
\|q_1 x - x\|^2 &= \|(q_1 x - x)(q_1 x - x)^*\| \\
&\leq \|q_1 xx^* - xx^*\| \cdot \|q_1^*\| + \|q_1 xx^* - xx^*\| \\
&\leq 2\|q_1 xx^* - xx^*\| \\
&< \tfrac{1}{4}\eta^2.
\end{aligned}$$

Thus $\|q_1 x - x\| < \frac{1}{2}\eta$.

Then, using $q_1 e = q_1$ at the second step,

$$\begin{aligned}
\|q_0 q - q\| &= \|(1-e)q + q_1 q - q\| \\
&= \|q_1 eq - eq\| \\
&\leq 2\|eq - x\| + \|q_1 x - x\| \\
&< \delta + \tfrac{1}{2}\eta \\
&\leq \tfrac{1}{2}\eta + \tfrac{1}{2}\eta \\
&= \eta.
\end{aligned}$$

Thus by the choice of η, we have $q \precsim q_0$.

Now we estimate the values of tracial states on q_0. Let $\tau \in T(C^*(G, A, \alpha))$.

Since $\|eqe - xx^*\| < \delta$, the choice of δ gives $\|g(eqe) - g(xx^*)\| < \frac{1}{6}\varepsilon$.

Then using the choice of q_1 at the first step, inequality in the C^*-algebra at the third step, the previous estimate at the fourth step, Lemma III.2 at the fifth step, and $g(qeq) \leq q$ at the sixth step, we estimate

$$\begin{aligned}
\tau(q_1) &= \tau(q_1 g(xx^*) q_1) \\
&= \tau\big(g(xx^*)^{1/2} q_1 g(xx^*)^{1/2}\big) \\
&\leq \tau(g(xx^*)) \\
&< \tau(g(eqe)) + \tfrac{1}{6}\varepsilon \\
&= \tau(g(qeq)) + \tfrac{1}{6}\varepsilon \\
&\leq \tau(q) + \tfrac{1}{6}\varepsilon.
\end{aligned}$$

For the same reason we had $\tau(1-e) < \delta_0$ when estimating $\tau(r_0)$ we now have $\tau(1-e) < \frac{1}{6}\varepsilon$. Thus

$$\tau(q_0) = \tau(1-e) + \tau(q_1) < \tau(q) + \tfrac{1}{3}\varepsilon.$$

Therefore, q_0 is the desired projection.

Apply Lemma II.15 with $\varphi\colon M_n \otimes fAf \to D$ and the projection e as given to obtain A_0 and a unital homomorphism $\psi\colon A_0 \to C^*(G, A, \alpha)$.

Note that $\psi(A_0)$ contains D, and thus r_0; also $1,\ e \in \psi(A_0)$ and so

$$q_0 \in (1-e) + eDe \subset \psi(A_0).$$

Also, for every $\tau \in T(C^*(G, A, \alpha)$

$$\tau(r_0) - \tau(q_0) > \left(\tau(r) - \tfrac{1}{3}\varepsilon\right) - \left(\tau(q) + \tfrac{1}{3}\varepsilon\right) > \tfrac{1}{3}\varepsilon.$$

So by Lemma III.1, $q_0 \precsim r_0$ in $C^*(G, A, \alpha)$. Therefore,

$$q \precsim q_0 \precsim r_0 \precsim r,$$

which completes the proof. ∎

CHAPTER IV

REAL RANK OF CROSSED PRODUCTS

In this section, we prove that if A is a simple unital C*-algebra with real rank zero such that the order on projections over A is determined by traces, if G is a finite group and if $\alpha : G \to \mathrm{Aut}(A)$ has the tracial Rokhlin property, then $C^*(G, A, \alpha)$ has real rank zero, and every tracial state on $C^*(G, A, \alpha)$ is induced from an α-invariant tracial state on A. The methods are adapted from Section 4 of [16] which are in turn adapted from those of Section 4 of [20].

Theorem IV.1. *Let A be an infinite dimensional stably finite simple unital C^*-algebra with real rank zero. Suppose that the order on projections over A is determined by traces and $\alpha : G \to \mathrm{Aut}(A)$ is an action of a finite group with the tracial Rokhlin property. Then $C^*(G, A, \alpha)$ has real rank zero.*

Proof. Set $B = C^*(\mathbb{Z}, A, \alpha)$.

As in the proof of Theorem III.5, the other hypotheses imply that A is stably finite.

Let $a \in B$ be selfadjoint with $\|a\| \leq 1$. Let $\varepsilon > 0$. We will approximate a to within ε by an invertible selfadjoint element. If a is already invertible, there is nothing to prove. Therefore we assume $0 \in \mathrm{sp}(a)$. Set $\varepsilon_0 = \frac{1}{8}\varepsilon$, and choose a continuous function $g\colon [-1,\, 1] \to [0, 1]$ such that

$$g(0) = 1 \quad \text{and} \quad \mathrm{supp}(g) \subset (-\varepsilon_0,\, \varepsilon_0).$$

Recalling the notation $T(B)$ from Notation II.4, define

$$\eta = \inf_{\tau \in T(B)} \tau(g(a)).$$

The algebra B is simple by Corollary II.8, which implies that every tracial state is faithful. Also, the facts that $g(a)$ is a nonzero positive element, and $T(B)$ is weak* compact together give $\eta > 0$.

Choose $\delta_0 > 0$ such that whenever C is a unital C^*-algebra and $x, y \in C_{\mathrm{sa}}$ satisfy $\|x\|, \|y\| \leq 2$ and $\|x - y\| < \delta_0$, then $\|g(x) - g(y)\| < \frac{1}{6}\eta$.

Set $\delta = \min(\delta_0, 1, \varepsilon_0)$. Choose $N \in \mathbb{N}$ such that

$$\frac{1}{N} < \frac{\eta}{4}.$$

Since α has the tracial Rokhlin property, we can apply Lemma II.14 to find projections $e, f \in A$, a unital C*-subalgebra $D \subset eBe$, and an isomorphism $\varphi \colon D \to M_n \otimes fAf$, such that

$$\varphi(e) = \sum_{g \in G} e_{g,g} \otimes 1_{fAf} \in M_n \otimes fAf,$$

such that

$$\operatorname{dist}(ea, D) < \tfrac{1}{2}\delta \quad \text{and} \quad \operatorname{dist}(ae, D) < \tfrac{1}{2}\delta,$$

and such that there are N mutually orthogonal projections $f_1, f_2, \ldots, f_N \in eDe$, each of which is Murray-von Neumann equivalent in B to $1 - e$.

From the last condition, we see that for every $\tau \in T(B)$ we have

$$\tau(1 - e) \leq \frac{\tau(e)}{N} \leq \frac{1}{N} < \frac{\eta}{4}.$$

Set

$$x = a - (1 - e)a(1 - e) = ea + (1 - e)ae.$$

Notice that $x^* = x$ since $a^* = a$.

Choose $x_1, x_2 \in D$ such that

$$\|ea - x_1\| < \tfrac{1}{2}\delta \quad \text{and} \quad \|ae - x_2\| < \tfrac{1}{2}\delta.$$

Since $e \in D$ and D is a unital subalgebra of eBe, we have $(1 - e)x_2 = 0$ and $ex_1 \in D$. Set $d_0 = ex_1 = ex_1 + (1 - e)x_2e \in D$ and set $d = \frac{1}{2}(d_0 + d_0^*)$.

Notice that

$$\|d_0 - x\| = \|ex_1 - ea + (1 - e)ae\| \leq \|ex_1 - ea\| + \|(1 - e)ae - (1 - e)x_2\| < \tfrac{1}{2}\delta + \tfrac{1}{2}\delta = \delta.$$

Now set $a_0 = a - x + d$. The element a_0 satisfies

$$a_0^* = a_0, \quad a_0 - (1-e)a_0(1-e) = d \in D, \quad \text{and} \quad \|a - a_0\| < \delta.$$

Next we compute

$$\|ea_0 - a_0e\| = \left\|\frac{1}{2}ex_1 - \frac{1}{2}ex_1e\right\| = \|ex_1 - ex_1\| = 0$$

since $x_1 \in D \subset eBe$. That is, e and a_0 commute.

Set $y = ea_0e$ and notice that this is a selfadjoint element of D since $a_0+(1-e)a_0(1-e) \in D$ which implies $e(a_0 + (1-e)a_0(1-e))e \in D$. We also have $\|y\| \leq \|a_0\| < \|a\| + \delta \leq 2$. Let $g(y)$ be the result of evaluating functional calculus in $eDe = D$. Since D has real rank zero, there is a projection $r \in \overline{g(y)Dg(y)}$ such that $\|rg(y) - g(y)\| < \frac{1}{6}\eta$.

Let $\tau \in T(B)$; we claim that $\tau(r) > \tau(1-e)$. By the previous estimate,

$$\|rg(y)r - g(y)\| \leq \|rg(y)r - rg(y)\| + \|rg(y) - g(y)\| < \frac{\eta}{3}.$$

Since $g \leq 1$ we get $g(y) \leq 1$ and so $rg(y)r \leq r$, so that

$$\tau(r) \geq \tau(rg(y)r) > \tau(g(y)) - \tfrac{1}{3}\eta.$$

Next we compute,

$$\|a_0 - ((1-e)a_0(1-e) + y)\| = \|a_0 - (1-e)a_0(1-e) - ea_0e\| = \|ea_0(1-e) + (1-e)a_0e\| = 0$$

since $[e, a_0] = 0$.

Let $g\left((1-e)a_0(1-e)\right)$ be the result of evaluating functional calculus in $(1-e)B(1-e)$. Then orthogonality of $(1-e)a_0(1-e)$ and y, together with the above computation gives

$$\|g(a_0) - [g\left((1-e)a_0(1-e)\right) + g(y)]\| = \|g(a_0) - g\left((1-e)a_0(1-e) + y\right)\| = 0.$$

Since $g((1-e)a_0(1-e)) \leq 1-e$, the estimate $\tau(1-e) < \frac{1}{4}\eta$ implies

$$\tau\left(g(y)\right) = \tau\left(g(a_0)\right) - \tau\left(g\left((1-e)a_0(1-e)\right)\right) \geq \tau\left(g(a_0)\right) - \tau(1-e) > \tau\left(g(a_0)\right) - \tfrac{1}{4}\eta.$$

Moreover, $\|a - a_0\| < \delta \leq \delta_0$ so $\|g(a) - g(a_0)\| < \frac{1}{6}\eta$, thus $\tau(g(a_0)) > \tau(g(a)) - \frac{1}{6}\eta$. By the choice of η we have $\tau(g(a)) \geq \eta$. So putting all of this together, we get

$$\tau(r) > \tau(g(y)) - \tfrac{1}{3}\eta > \tau(g(a_0)) - \tfrac{7}{12}\eta > \tau(g(a)) - \tfrac{3}{4}\eta \geq \tfrac{1}{4}\eta > \tau(1-e).$$

This proves the claim. Since $r \in \overline{g(y)Bg(y)}$, and $\operatorname{supp}(g) \subset B_{\varepsilon_0}$, by Lemma 4.5 of [20] we have

$$\|ry - yr\| < 2\varepsilon_0 \quad \text{and} \quad \|ryr\| < \varepsilon_0.$$

Since $r \leq e$ and $y = ea_0e$, we have $ra_0r = rea_0er = ryr$, whence $\|ra_0r\| < \varepsilon_0$. Also,

$$\begin{aligned}
\|[r, a_0]\| &= \|ra_0e - ea_0r + ra_0(1-e) - (1-e)a_0r\| \\
&= \|rea_0e - ea_0er + rea_0(1-e) - (1-e)a_0er\| \\
&= \|ry + rea_0(1-e) - yr - (1-e)a_0er\| \\
&= \|[r, y] + 0\| < 2\varepsilon_0.
\end{aligned}$$

Define

$$a_1 = (e-r)a_0(e-r) + (1-e)a_0(1-e).$$

We would like to estimate $\|a_1 - a\|$. First we compute

$$\begin{aligned}
a_0 - a_1 = {} & (e-r)a_0(1-e) + (e-r)a_0r + (1-e)a_0(e-r) + (1-e)a_0r \\
& + ra_0(e-r) + ra_0(1-e) + ra_0r.
\end{aligned}$$

Recalling that $\|[(1-e), a_0]\| = 0$ and $r \leq e$, we get

$$\begin{aligned} \|a_0 - a_1\| &= \|(e-r)a_0 r + r a_0 (e-r) + r a_0 r\| \\ &\leq \|(e-r)a_0 r - (e-r) r a_0\| + \|r a_0 (e-r) - a_0 r (e-r)\| + \|r a_0 r\| \\ &\leq 2\|e-r\| \cdot \|[a_0, r]\| + \varepsilon_0 \\ &\leq 5\varepsilon_0. \end{aligned}$$

Now since $\|a_0 - a\| < \delta \leq \varepsilon_0$, we have

$$\|a_1 - a\| < 6\varepsilon_0.$$

Let A_0 and $\psi \colon A_0 \to C^*(G, A, \alpha)$ be as in Lemma II.15, using φ^{-1} in place of φ and with e as above. Then $1 - e \in \psi(A_0)$, as indicated in Lemma II.15, and by construction $r \in D \subset \psi(A_0)$. We proved above that $\tau(r) > \tau(1-e)$ for all $\tau \in T(B)$. So Lemma III.1 implies $1 - e \precsim r$ in B. Since $r \leq e$ this gives $1 - e \precsim e$. Therefore Lemma 8 of [8] provides an invertible selfadjoint element $b_1 \in (1-e+r)B(1-e+r)$ such that $\|b_1 - (1-e)a_0(1-e)\| < \varepsilon_0$. Also, by construction, we have e, r, and $y = e a_0 e \in D$ so $(e-r)a_0(e-r) \in D$. Since D has real rank zero, there is an invertible selfadjoint element $b_2 \in (e-r)D(e-r)$ such that

$$\|b_2 - (e-r)a_0(e-r)\| < \varepsilon_0.$$

Since b_1 and b_2 are orthogonal, $b_1 + b_2$ is an invertible selfadjoint element of B, and satisfies

$$\begin{aligned} \|(b_1 + b_2) - a\| &\leq \|a - a_1\| + \|a_1 - (b_1 + b_2)\| \\ &\leq 6\varepsilon_0 + \|(1-e)a_0(1-e) - b_1\| + \|(e-r)a_0(e-r) - b_2\| \\ &\leq 8\varepsilon_0 \\ &= \varepsilon \end{aligned}$$

This completes the proof. ∎

Corollary IV.2. *Let A be an infinite dimensional stably finite simple unital C^*-algebra with real rank zero, and suppose that the order on projections over A is determined by traces. Let*

$\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group with the tracial Rokhlin property. Then the restriction map is a bijection from the tracial states of $C^*(G, A, \alpha)$ to the α-invariant tracial states of A.

Proof. Since $C^*(G, A, \alpha)$ has real rank zero by Theorem IV.1, this follows from Proposition 2.2 of [12]. ∎

It is worth mentioning here the following theorem found as Theorem 2.6 of [23].

Theorem IV.3. *Let A be an infinite dimensional simple unital C^*-algebra with tracial rank zero. Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group with the tracial Rokhlin property. Then $C^*(G, A, \alpha)$ has tracial rank zero.*

CHAPTER V

STABLE RANK OF CROSSED PRODUCTS

In this section, we prove that if A is an infinite dimensional simple unital C*-algebra with real rank zero and stable rank one, such that the order on projections over A is determined by traces, and if $\alpha : G \to \mathrm{Aut}(A)$ is an action of a finite group with the tracial Rokhlin property, then $C^*(G, A, \alpha)$ has stable rank one. The methods are adapted from Section 5 of [16] which are adapted from Section 5 of [20].

Lemma V.1. *Let $\delta > 0$. Then there exists a continuous function $g\colon [0,1] \to [0,1]$ such that $g(0) = 0$, $g(1) = 1$, and whenever A is a C^*-algebra with real rank zero and $a \in A$ is a positive element with $\|a\| \leq 1$, then there is a projection $e \in \overline{aAa}$ such that $\|eg(a) - g(a)\| < \delta$ and $\|ae - e\| < \delta$.*

Proof. This was Lemma 5.1 in [16]. ∎

Lemma V.2. *Let A be an infinite dimensional simple unital C^*-algebra with real rank zero and such that the order on projections over A is determined by traces. Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group with the tracial Rokhlin property. Let $q_1, \ldots, q_n \in C^*(G, A, \alpha)$ be nonzero projections, let $a_1, \ldots, a_m \in C^*(G, A, \alpha)$ be arbitrary, and let $\varepsilon > 0$. Then there exists a unital subalgebra $A_0 \subset C^*(G, A, \alpha)$ which is stably isomorphic to A, a projection $p \in A_0$, nonzero projections $r_1, \ldots, r_n \in pA_0p$, and elements $b_1, \ldots, b_m \in C^*(G, A, \alpha)$, such that:*

1. *$\|q_k r_k - r_k\| < \varepsilon$ for $1 \leq k \leq n$.*
2. *For $1 \leq k \leq n$ there is a projection $g_k \in r_k A_0 r_k$ such that $1 - p \sim g_k$ in $C^*(G, A, \alpha)$.*
3. *$\|a_j - b_j\| < \varepsilon$ for $1 \leq j \leq m$.*
4. *$pb_jp \in pA_0p$ for $1 \leq j \leq m$.*

Proof. Set $B = C^*(G, A, \alpha)$.

Let

$$\eta = \min_{1 \le k \le n} \left(\inf_{\tau \in T(B)} \tau(q_k) \right) > 0 \quad \text{and} \quad \varepsilon_0 = \min\left(\frac{\eta}{5}, \frac{\varepsilon^2}{2} \right).$$

Apply Lemma V.1 with ε_0 in place of δ, to get a continuous function $g\colon [0,1] \to [0,1]$. Apply Lemma III.3 with this function g and with ε_0 in place of ε, to get a number $\delta > 0$ such that whenever $\tau \in T(B)$ and p, $q \in B$ are projections such that $\tau(q) > 1 - \delta$, then $\tau(g(qpq)) > \tau(p) - \varepsilon_0$.

Next choose $\varepsilon_1 > 0$ with $\varepsilon_1 \le \min(\varepsilon_0, \varepsilon)$ and small enough that whenever $x, y \in B$ are positive elements with $\|x\|$, $\|y\| \le 1$ and $\|x - y\| < \varepsilon_1$, then $\|g(x) - g(y)\| < \varepsilon_0$. Then choose $\varepsilon_2 > 0$ with $\varepsilon_2 \le \varepsilon_1$ and small enough that if $x, y \in B$ are selfadjoint elements with $\|x\|$, $\|y\| \le 1$ and $\|x - y\| < \varepsilon_2$, then the positive parts x_+ and y_+ satisfy $\|x_+ - y_+\| < \varepsilon_1$.

Apply Lemma II.14 with $F = \{q_1, \ldots, q_n, a_1, \ldots, a_m\}$, with ε_2 in place of ε, with an integer N so large that $1/N < \min(\delta, \varepsilon_0)$, and with $z = 1$. We obtain projections $e, f \in A \subset B$, a unital subalgebra $D \subset eBe$, and an isomorphism $\varphi\colon M_n \otimes fAf \to D$, with

$$x_1, \ldots, x_n, c_1, \ldots, c_m \in D$$

satisfying

$$\|ea_j - c_j\| < \varepsilon_2,$$

$$\|x_k\| \le 1,$$

and

$$\|eq_k - x_k\| < \varepsilon_2$$

for $1 \le j \le m$ and for $1 \le k \le n$. Moreover, $\tau(1 - e) \le \frac{1}{N} < \min(\delta, \varepsilon_0)$ for every $\tau \in T(B)$.

Apply Lemma II.15 with $\varphi\colon M_n \otimes fAf \to D$ and the projection e as given to obtain a C^*-algebra A_0 which is stably isomorphic to A and a unital homomorphism $\psi\colon A_0 \to C^*(G, A, \alpha)$. The subalgebra $\psi(A_0)$ will be the algebra A_0 called for in the statement of the current lemma. The projection e will be the projection p called for in the statement. Note that $\psi(A_0)$ contains D, and hence e.

For $1 \leq j \leq m$, set $b_j = a_j + e(c_j - a_j)e$, which satisfies

$$\|b_j - a_j\| = \|ec_je - ea_je\| < \varepsilon_2 \leq \varepsilon_1 \leq \varepsilon$$

$$\text{and} \quad eb_je = ec_je \in D \subset \psi(A_0).$$

These are parts (3) and (4) of the conclusion.

Now, for $1 \leq k \leq n$, observe that $\frac{1}{2}(ex_ke + ex_k^*e)$ is a selfadjoint element of $eDe = D$ of norm at most one such that

$$\left\|eq_ke - \tfrac{1}{2}(ex_ke + ex_k^*e)\right\| \leq \tfrac{1}{2}\|eq_ke - ex_ke\| + \tfrac{1}{2}\|eq_ke - ex_k^*e\| < \varepsilon_2.$$

So, since eq_ke is a positive element,

$$y_k = \tfrac{1}{2}(ex_ke + ex_k^*e)_+$$

is a positive element of eDe of norm at most one such that $\|eq_ke - y_k\| < \varepsilon_1$.

By the choice of g using Lemma V.1, there exists projections $r_k \in eDe \subset \psi(A_0)$ such that

$$\|r_ky_k - r_k\| < \varepsilon_0 \quad \text{and} \quad \|r_kg(y_k) - g(y_k)\| < \varepsilon_0.$$

Using $r_k \leq e$ at the second step, we now have

$$(r_kq_k - r_k)(q_kr_k - r_k) = r_k - r_kq_kr_k = r_k - r_keq_kpr_k.$$

Thus

$$\begin{aligned}\|r_kq_k - r_k\|^2 &= \|(r_kq_k - r_k)(r_kq_k - r_k)^*\| \\ &= \|r_k - r_keq_ker_k\| \\ &\leq \|r_k - r_ky_kr_k\| + \|r_ky_kr_k - r_keq_ker_k\| \\ &\leq \|r_k - r_ky_k\| \cdot \|r_k\| + \|y_k - eq_ke\| \\ &< \varepsilon_1 + \varepsilon_0 \\ &\leq \varepsilon^2,\end{aligned}$$

so $\|r_kq_k - r_k\| < \varepsilon$, and this is Part (1) of the conclusion.

We now estimate the traces on r_k. For every $\tau \in T(B)$, we have $\tau(r_k) \geq \tau(r_kg(y_k)r_k)$. By construction $\|r_kg(y_k) - g(y_k)\| < \varepsilon_0$, thus $\|r_kg(y_k)r_k - g(y_k)\| < 2\varepsilon_0$. Since $\|y_k - eq_ke\| < \varepsilon_1$, by the choice of ε_1, we obtain $\|g(y_k) - g(eq_ke)\| < \varepsilon_0$. Since $\tau(e) > 1 - \delta$, the choice of δ using Lemma III.3 implies that $\tau(g(eq_ke)) > \tau(q_k) - \varepsilon_0$. Combining all these, we get

$$\tau(r_k) > \tau(r_kg(y_k)r_k) > \tau(g(y_k)) - 2\varepsilon_0 > \tau(eq_ke) - 3\varepsilon_0 > \tau(q_k) - 4\varepsilon_0.$$

On the other hand, $\tau(1-e) \leq \varepsilon_0 < \frac{1}{5}\eta \leq \frac{1}{5}\tau(q_k)$ Thus $\tau(r_k) > \tau(1-e)$. Since $\tau \in T(B)$ is arbitrary, and since $1-e$ and r_k are in $\psi(A_0)$, Lemma III.1 gives Part (2) of the conclusion. ∎

Lemma V.3. *Let A be a simple, unital C^*-algebra with property (SP). Suppose p and q are nonzero projections in A. Then there exists a nonzero projection r in A such that $r \precsim p$ and $r \leq q$.*

Proof. Let $x \in pAq$ be nonzero. Then $x^*x \in qAq$ and $xx^* \in pAp$ are both nonzero. Let $0 < \varepsilon < \|x^*x\|$. Set $f(t) = t - \varepsilon$ for $t \geq \varepsilon$ and $f(t) = 0$ otherwise. Let $g(t)$ be a continuous function with $g(t) = t^{-1/2}$ for $t \geq \varepsilon$. Also set $v = g(x^*x)x^*$, with the functional calculus being evaluated in qAq.

One can easily compute that $g(t)^2t = 1$ for $t \geq \varepsilon$ and $vv^*f(x^*x) = f(x^*x) = f(x^*x)vv^*$.

Set $C = \mathrm{Her}(f(x^*x))$ and $z = f(x^*x)$. We claim if $a \in C$, then $vv^*a = a$. We first observe that $z^{1/n}$ is an approximate identity for C since for any $a \in C$, $a = \lim_{\lambda \in \Lambda} za_\lambda z$ and so $z^{1/n}a = \lim_{\lambda\in\Lambda} z^{1+1/n}a_\lambda z$ which goes to $\lim_{\lambda\in\Lambda} za_\lambda z = a$ as n goes to infinity. A similar argument on the other side shows that z is an approximate identity. To complete the proof of the claim we now compute $vv^*a = \lim_{n\to\infty} vv^*z^{1/n}a = \lim_{n\to\infty} z^{1/n}a = a$.

For any $n \in \mathbb{N}$, $(xx^*)^n x = x(x^*x)^n$, thus for any polynomial h we have $h(xx^*)x = xh(x^*x)$. Then by the continuity of continuous functional calculus, for any continuous function h we have $h(xx^*)x = xh(x^*x)$.

Since $\varepsilon < \|x^*x\|$, $f(x^*x)$ is nonzero, so using property (SP), let $r \in \mathrm{Her}(f(x^*x))$ be a nonzero projection.

We claim v^*rv is a projection in $\mathrm{Her}(f(xx^*))$. It is easy to check that it is a projection. For the other part of the claim, writing $r = \lim_{\lambda\in\Lambda} r_\lambda$ we compute

$$\begin{aligned}
v^*rv &= (g(x^*x)x^*)^* r g(x^*x)x^* \\
&= xg(x^*x)rg(x^*x)x^* \\
&= g(xx^*)xrx^*g(xx^*) \\
&= \lim_{\lambda\in\Lambda} g(xx^*)xf(x^*x)r_\lambda f(x^*x)x^*g(xx^*) \\
&= \lim_{\lambda\in\Lambda} g(xx^*)f(xx^*)xr_\lambda x^*f(xx^*)g(xx^*) \\
&= \lim_{\lambda\in\Lambda} f(xx^*)[g(xx^*)xr_\lambda x^*g(xx^*)]f(xx^*) \in \mathrm{Her}(f(xx^*))
\end{aligned}$$

Finally compute $(rv)^*rv = v^*rv$ and $rv(rv)^* = rvv^*r = r^2 = r$. Thus, $r \sim v^*rv$. Now we note, $r \in \mathrm{Her}(f(x^*x)) \subset qAq$, so $r \leq q$ and $v^*rv \in \mathrm{Her}(f(xx^*)) \subset pAp$, so $r \precsim p$. ∎

Theorem V.4. *Let A be an infinite dimensional simple unital C^*-algebra with real rank zero and stable rank one, and such that the order on projections over A is determined by traces. Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group with the tracial Rokhlin property. Then $C^*(G, A, \alpha)$ has stable rank one.*

Proof. Let $B = C^*(G, A, \alpha)$.

We proceed by showing that every two sided zero divisor in B is a limit of invertible elements. Because B has a faithful tracial state, every one sided invertible element is invertible. We combine this with Theorem 3.3(a) of [26] which says that $B \setminus \overline{GL(B)}$ is the one sided, but not two sided invertible elements, to get $B \setminus \overline{GL(B)} = \emptyset$. That is, every element is a limit of invertible elements, so B has stable rank one.

Suppose $a \in B$ is such that there are nonzero $x, y \in B$ such that $xa = ay = 0$. Let $\varepsilon > 0$, we show there is an invertible element $c \in B$ such that $\|a - c\| < \varepsilon$.

Without loss of generality $\|a\| \leq \frac{1}{2}$ and $\varepsilon \leq 1$. Since B has real rank zero by Theorem IV.1, there are are nonzero projections

$$e \in \overline{x^* B x} \quad \text{and} \quad f \in \overline{y B y^*},$$

and they satisfy $ea = af = 0$.

Apply Lemma V.2 to the nonzero projections e and f and the element a, with $\frac{1}{13}\varepsilon$ in place of ε. Call the resulting subalgebra A_0, the resulting projection p_0, the resulting nonzero projections e_0 and f_0, and the resulting element x_0. Thus

$$e_0,\ f_0,\ p_0 x_0 p_0 \in p_0 A_0 p_0, \quad 1 - p_0 \precsim e_0, f_0,$$

and

$$\|e e_0 - e_0\|,\ \|f f_0 - f_0\|,\ \|a - x_0\| < \tfrac{1}{13}\varepsilon.$$

Define $a_0 = (1 - e_0) x_0 (1 - f_0)$. We clearly have $e_0 a_0 = a_0 f_0 = 0$, and we claim that $\|a - a_0\| < \frac{5}{13}\varepsilon$. First, using

$$\|a\| \leq 1 \quad \text{and} \quad \|e_0 e - e_0\| = \|e e_0 - e_0\| < \tfrac{1}{13}\varepsilon,$$

we have

$$\|e_0 x_0\| \leq \|e_0\| \cdot \|x_0 - a\| + \|e_0 - e_0 e\| \cdot \|a\| + \|e_0 e a\| < \tfrac{1}{13}\varepsilon + \tfrac{1}{13}\varepsilon + 0 = \tfrac{2}{13}\varepsilon.$$

Similarly, $\|x_0 f_0\| < \frac{2}{13}\varepsilon$. Therefore

$$\begin{aligned} \|a - a_0\| &\leq \|a - x_0\| + \|x_0 - (1 - e_0) x_0 (1 - f_0)\| \\ &\leq \|a - x_0\| + \|e_0 x_0\| + \|1 - e_0\| \cdot \|x_0 f_0\| \\ &< \tfrac{1}{13}\varepsilon + \tfrac{2}{13}\varepsilon + \tfrac{2}{13}\varepsilon = \tfrac{5}{13}\varepsilon. \end{aligned}$$

This proves the claim. Since $\|a\| \leq \frac{1}{2}$ and $\varepsilon \leq 1$ we now get $\|a_0\| \leq 1$.

Since A has real rank zero and A_0 is stably isomorphic to A, the algebra A_0 also has real rank zero. Now Lemma V.3 shows that there is a nonzero projection $r \leq e_0$ such that $r \precsim f_0$. Since A has stable rank one and A_0 is stably isomorphic to A, by Theorem 3.6 of [25], A_0 has stable rank one. Thus, there is a unitary $v \in A_0$ such that $v^* r v \leq f_0$. Then $r(a_0 v^*) = (a_0 v^*) r = 0$.

Apply Lemma V.2 to the nonzero projection r and the element $a_0 v^*$, with $\frac{1}{13}\varepsilon$ in place of ε. Call the resulting subalgebra A_1, the resulting projection p_1, the resulting nonzero projection e_1, and the resulting element x_1. Thus

$$e_1, \, p_1 x_1 p_1 \in p_1 A_1 p_1, \quad \|re_1 - e_1\|, \, \|a_0 v^* - x_1\| < \tfrac{1}{13}\varepsilon, \quad \text{and} \quad 1 - p_1 \precsim e_1.$$

Define $a_1 = (1 - e_1) x_1 (1 - e_1)$. We clearly have $e_1 a_1 = a_1 e_1 = 0$. Also,

$$p_1 a_1 p_1 = p_1 (1 - e_1) x_1 (1 - e_1) p_1 = (1 - e_1) p_1 x_1 p_1 (1 - e_1) \in p_1 A_1 p_1,$$

since $e_1 \in p_1 A_1 p_1$, so p_1 acts as the identity on e_1. Furthermore, since $\|a_0 v^*\| \leq 1$, the argument used above to prove $\|a - a_0\| < \frac{5}{13}\varepsilon$ now shows that $\|a_0 v^* - a_1\| < \frac{5}{13}\varepsilon$. So $\|a v^* - a_1\| < \frac{10}{13}\varepsilon$. The conclusion of Lemma V.2 provides $s \in B$ such that

$$s^* s = 1 - p_1, \quad s s^* \leq e_1, \quad \text{and} \quad s s^* \in A_1.$$

Set $e_2 = s s^*$ and $w = s + s^* + p_1 - e_2$. Since $e_2 \leq e_1 \leq p_1$, it follows by computation that w is a unitary satisfying

$$w e_2 w^* = 1 - p_1, \quad w(1 - p_1) w^* = e_2, \quad \text{and} \quad w(p_1 - e_2) = p_1 - e_2.$$

We now have $e_2 a_1 w = 0$ and $a_1 w (1 - p_1) = a_1 e_2 w = 0$. So we can decompose the identity as

$$1 = e_2 \oplus (p_1 - e_2) \oplus (1 - p_1).$$

With respect to this decomposition, set $c = (p_1 - e_2) a_1 w (p_1 - e_2)$ and for suitable $x, y, z \in B$, the element $a_1 w$ has the block matrix form

$$a_1 w = \begin{pmatrix} 0 & 0 & 0 \\ x & c & 0 \\ y & z & 0 \end{pmatrix}.$$

Now use the fact that $w(p_1 - e_2) = p_1 - e_2$ and $e_2 \leq p_1$ to rewrite

$$\begin{aligned} c &= (p_1 - e_2)a_1(p_1 - e_2) \\ &= (p_1 - e_2)p_1 a_1 p_1 (p_1 - e_2) \in (p_1 - e_2)A_1(p_1 - e_2). \end{aligned}$$

Since $(p_1 - e_2)A_1(p_1 - e_2)$ has stable rank one, there exists an invertible element $d \in (p_1 - e_2)A_1(p_1 - e_2)$ such that $\|c - d\| < \frac{1}{13}\varepsilon$. Then

$$a_2 = \begin{pmatrix} \frac{\varepsilon}{13}e_2 & 0 & 0 \\ x & d & 0 \\ y & z & \frac{\varepsilon}{13}(1 - p_1) \end{pmatrix}$$

is invertible in B, and satisfies $\|a_2 - a_1 w\| < \frac{3}{13}\varepsilon$. So also $a_2 w^* v$ is an invertible element in B, and satisfies

$$\begin{aligned} \|a_2 w^* v - a\| &= \|a_2 w^* - a v^*\| \\ &\leq \|a_2 - a_1 w\| + \|a_1 - a v^*\| \\ &< \tfrac{3}{13}\varepsilon + \tfrac{10}{13}\varepsilon \\ &= \varepsilon. \end{aligned}$$

This is the required approximation by an invertible element. ∎

CHAPTER VI

THE PROJECTION FREE TRACIAL ROKHLIN PROPERTY

Recall the following definition from the introduction.

Definition VI.1. *Let x and y be positive elements of a C^*-algebra A. We write $x \precsim y$ if there exist elements r_j in A such that $r_j y r_j^* \to x$ with convergence in norm. In this case we say x is* **(Cuntz) subequivalent** *to y. If $x \precsim y$ and $y \precsim x$, we write $x \sim y$ and say x is* **(Cuntz) equivalent** *to y.*

Definition VI.2. *For $\varepsilon > 0$, let f_ε be given by $f_\varepsilon(t) = 0$ for $0 \leq t \leq \varepsilon$, by $f_\varepsilon(t) = \varepsilon^{-1}(t - \varepsilon)$ for $\varepsilon \leq t \leq 2\varepsilon$ and $f_\varepsilon(t) = 1$ for $t \geq 2\varepsilon$.*

It is useful to have alternate formulations of this concept. The following proposition is Proposition 2.4 in [27] .

Proposition VI.3. *Let f_ε be as in Definition VI.2. Let x, y be positive elements of the unital C^*-algebra A. The following are equivalent:*

1. *$x \precsim y$.*

2. *For all $\varepsilon > 0$, there exists $r \in A$ with $f_\varepsilon(x) \leq r y r^*$.*

3. *There exist elements r_j and s_j of A with $r_j y s_j \to x$.*

4. *For all $\varepsilon > 0$, there exists $\delta > 0$ and $r \in A$ such that $f_\varepsilon(x) = r f_\delta(y) r^*$.*

Additionally, if A has stable rank 1, then (1)-(4) above are equivalent to:

5. *For all $\varepsilon > 0$ there exists a unitary $u \in A$ such that $u f_\varepsilon(x) u^* \in \overline{yAy}$.*

The following proposition is useful for determining subequivalence of elements constructed using functional calculus.

Proposition VI.4. *Let f and g be positive functions in $C(X)$ or $C_0(X)$ for some space X.*

1. *If $\{x \in X : f(x) \neq 0\} \subset \{x \in X : g(x) \neq 0\}$, then $f \precsim g$.*

2. *Suppose that $f \precsim g$, that $X \subset [0, \infty)$, and that $a \in A$ is a positive selfadjoint element of a C^*-algebra A with $\mathrm{sp}(a) \subset X$. Then $f(a) \precsim g(a)$.*

Proof. The first part is a comment just before Proposition 2.1 of [27].

For the second part let $h_j \in C(X)$ be functions such that $h_j g h_j^* \to f$. Then, since functional calculus is a continuous homomorphism, $(h_j g h_j^*)(a) = h_j(a)g(a)h_j^*(a) \to f(a)$. Therefore, $f(a) \precsim g(a)$ by definition. ∎

The following definition is a projection free analog of Definition 1.2 of [23] .

Definition VI.5. *Let A be an infinite dimensional unital simple C^*-algebra. Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say α has the* projection free tracial Rokhlin property *if for every finite set $F \subset A$, every $\varepsilon > 0$, and every positive element $x \in A$ with $\|x\| = 1$, there exist mutually orthogonal elements $a_g \in A$ for each $g \in G$ with $0 \leq a_g \leq 1$ such that:*

1. *$\|\alpha_g(a_h) - a_{gh}\| < \varepsilon$ for all $g, h \in G$.*

2. *$\|a_g b - b a_g\| < \varepsilon$ for all $g \in G$ and $b \in F$.*

3. *With $a = \sum_{g \in G} a_g$, the element $1 - a$ is Cuntz subequivalent to an element of the hereditary subalgebra generated by x.*

4. *$\|axa\| > 1 - \varepsilon$.*

5. *$\tau(1 - a) < \varepsilon$ for all $\tau \in T(A)$.*

Note that since any element of $\overline{xAx}$ is subequivalent to x, the third condition implies $1 - a \precsim x$.

Lemma VI.6. *If a and a_g are as in Definition VI.5, then $\|a\| = \max_{g \in G} \|a_g\|$.*

Proof. It is sufficient to prove that, for any $n \in \mathbb{N}$, if $a_1, \ldots, a_n$ are positive mutually orthogonal elements of A, then $\|\sum_{i=1}^{n} a_i\| = \max\{\|a_1\|, \ldots, \|a_n\|\}$. Furthermore, since positive mutually orthogonal elements commute, it is sufficient to prove that for any $n \in \mathbb{N}$, if $f_1, \ldots f_n$ are positive mutually orthogonal elements of $C(X)$ or $C_0(X)$ for some compact Hausdorff space X or some locally compact Hausdorff space X, then $\|\sum_{i=1}^{n} f_i\| = \max\{\|f_1\|, \ldots, \|f_n\|\}$. However, this last statement is obvious since for each $x \in X$ we can have $f_i(x) \neq 0$ for at most one index i. ∎

Lemma VI.7. *Let A be an infinite dimensional unital simple C^*-algebra. For any $\varepsilon > 0$, finite set $F \subset A$, and positive element x of norm one, if a and a_g are as in Definition VI.5, then*

1. $\|a\| > 1 - \varepsilon$
2. $\|a_g\| > 1 - 2\varepsilon$ *for all* $g \in G$.

Proof. Using Definition VI.5 to get the the last inequality, we have

$$\|a\|^2 = \|a\|\|x\|\|a\| \geq \|axa\| > 1 - \varepsilon.$$

However since $0 \leq a_g \leq 1$ for all $g \in G$ and these elements are mutually orthogonal, $0 \leq a \leq 1$, so $\|a\| \geq \|a\|^2$. This proves part 1.

By Lemma VI.6, $\max_{g \in G} \|a_g\| = \|a\| > 1 - \varepsilon$. Thus there exists some $h \in G$ so that $\|a_h\| > 1 - \varepsilon$. However, for any $g \in G$, we have $\|\alpha_{gh^{-1}}(a_h) - a_g\| < \varepsilon$. Thus, $\|\alpha_{gh^{-1}}(a_h)\| - \|a_g\| < \varepsilon$. Since $\alpha_{gh^{-1}}$ is an isomorphism, this gives $\|a_h\| - \|a_g\| < \varepsilon$. Therefore, $\|a_g\| > \|a_h\| - \varepsilon > 1 - 2\varepsilon$. This proves part 2. ∎

The following lemma and its corollary are analogs of Lemma 1.5 and Corollary 1.6 of [23].

Lemma VI.8. *Let A be a simple, infinite dimensional unital C^*-algebra. Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group with the projection free tracial Rokhlin property. Then α_g is outer for every $g \in G \setminus \{1\}$.*

Proof. Suppose u is a unitary and $g \neq 1$. Let $0 < \varepsilon < \frac{1}{3\sqrt{2}|G|}$. Notice that $\varepsilon < 1/2$. We will show that there is some b such that $\|u^* b u - \alpha_g(b)\| > \varepsilon$. Apply the projection free tracial Rokhlin property with this ε, with $F = \{u\}$ and with $x = 1$ to get mutually orthogonal $a_g \in A$ for each $g \in G$ with $0 \leq a_g \leq 1$ satisfying the properties there. Set $a = \sum_{g \in G} a_g$. In particular, the fourth property, $\|axa\| > 1 - \varepsilon$, implies $\|a^2\| > 1 - \varepsilon$. Thus

$$\sum_{g \in G} \|a_g\| \geq \|a\| > \sqrt{1 - \varepsilon} > \sqrt{1 - 1/2} = \sqrt{1/2}.$$

Thus there exists $h \in G$ such that $\|a_h\| > 1/(\sqrt{2}|G|)$. Next we compute

$$\begin{aligned}
\|\alpha_g(a_h) - ua_hu^*\| &\geq \|a_{gh} - a_h\| - \|ua_hu^* - a_h\| - \|\alpha_g(a_h) - a_{gh}\| \\
&\geq \max_{g,k \in G}\{\|a_{gk}\|, \|a_k\|\} - \varepsilon - \varepsilon \\
&\geq 1/(\sqrt{2}|G|) - 2\varepsilon \\
&\geq 3\varepsilon - 2\varepsilon \qquad \text{by the choice of } \varepsilon \\
&= \varepsilon.
\end{aligned}$$

This completes the proof. ∎

Corollary VI.9. *Let A be an infinite dimensional simple unital C^*-algebra and let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group with the projection free tracial Rokhlin property. Then $C^*(G, A, \alpha)$ is simple.*

Proof. In view of Lemma VI.8, this follows from 3.1 of [11] ∎

Lemma VI.10. *If $f : \mathbb{R} \to \mathbb{R}$ is continuous with $f(0) = 0$ and $a_1, \ldots, a_n \in A_+$ are mutually orthogonal, then $f(\sum_{i=1}^n a_i) = \sum_{i=1}^n f(a_i)$.*

Proof. It suffices to prove that the lemma holds for two orthogonal elements a and b. We claim that the lemma holds for $f(x) = x^n$. Since a and b are orthogonal, they commute, and using these two facts we have

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = a^n + b^n$$

which proves the claim. Therefore, the lemma also holds for any polynomial with zero constant term. Now let f be an arbitrary continuous function with $f(0) = 0$, and let (p_n) be a sequence of polynomials with zero constant term which converges uniformly to f on $\mathrm{sp}(a) \cup \mathrm{sp}(b) \cup \mathrm{sp}(a+b)$. Since $p_n(a+b) = p_n(a) + p_n(b)$ for all n, it follows that $f(a+b) = f(a) + f(b)$. ∎

Lemma VI.11. *Suppose $f : [0,1] \to \mathbb{R}$ is continuous. Then for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for any C^*-algebra A and any self-adjoint elements x and y of A with $\mathrm{sp}(x), \mathrm{sp}(y) \subset [0,1]$ and $\|x - y\| < \delta$, then $\|f(x) - f(y)\| < \varepsilon$.*

Proof. We first show that the lemma is true whenever $f(t) = t^k$ for a natural number k. The lemma clearly holds for $k = 1$. Now suppose the lemma holds for $k-1$. Then

$$\begin{aligned}
\|x^k - y^k\| &\leq \|x^k - x^{k-1}y\| + \|x^{k-1}y - y^k\| \\
&\leq \|x^{k-1}\|\|x-y\| + \|x^{k-1} - y^{k-1}\|\|y\| \\
&\leq \|x-y\| + \|x^{k-1} - y^{k-1}\|.
\end{aligned}$$

Thus the lemma also holds for k.

Therefore, by the triangle inequality the lemma is true for all polynomials.

Now suppose f is arbitrary and let $\varepsilon > 0$ be given. Let P be a polynomial with $\|f - P\| < \varepsilon/3$. Choose $\delta > 0$ corresponding to P with $\varepsilon/3$ in place of ε. Then for $\|x - y\| < \delta$ we have

$$\begin{aligned}
\|f(x) - f(y)\| &\leq \|f(x) - P(x)\| + \|P(x) - P(y)\| + \|P(y) - f(y)\| \\
&\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\
&= \varepsilon.
\end{aligned}$$

This completes the proof. ∎

Lemma VI.12. *Suppose $f : [0,1] \to \mathbb{R}$ is continuous. Then for all $\varepsilon > 0$, there exists a $\delta > 0$ such that if x is self adjoint in some C^*-algebra D with $\mathrm{sp}(x) \subset [0,1]$ and if $z \in D$ with $\|z\| \leq 1$ and $\|[x,z]\| < \delta$, then $\|[f(x), z]\| < \varepsilon$.*

Proof. We first show that the lemma is true for any monomial $f(t) = t^k$.

The lemma is trivial for $k = 1$. Now suppose the lemma holds for all $n < k$ with the choice of δ for a given pair of ε and f called $\delta(\varepsilon, f)$. Let $0 < \delta < \min\{\varepsilon/2, \delta(\varepsilon/2, x^{k-1})\}$ and let $\|[x,z]\| < \delta$. Then we have

$$\|[x, zx]\| = \|xzx - zx^2\| \leq \|[x,z]\|\|x\| < \delta \leq \delta(\varepsilon/2, x^{k-1})$$

and $\|zx\| < 1$. Thus,

$$\|x^k z - zx^k\| \le \|x^{k-1}(xz) - (xz)x^{k-1}\| + \|xzx^{k-1} - zx^k\| \le \varepsilon/2 + \|xz - zx\|\|x^{k-1}\| < \varepsilon.$$

This shows that the lemma holds for all monomials.

By the triangle inequality, the lemma holds for all polynomials. Then let f be arbitrary and P be a polynomial with $\|f - P\| < \varepsilon/3$ converging uniformly to f. Let $\varepsilon > 0$ be given. Let $\delta = \delta(\varepsilon/3, P)$. Then

$$\|f(x)z - zf(x)\| \le \|f(x)z - P(x)z\| + \|P(x)z - zP(x)\| + \|zP(x) - zf(x)\| < \varepsilon$$

which completes the proof. ∎

Lemma VI.13. *Let A be an infinite dimensional simple unital C^*-algebra. Let G be a finite group and let $\alpha : G \to \mathrm{Aut}(A)$ be an action with projection free tracial Rokhlin property. Let $\varepsilon > 0$ be given, let $F \subset A$ be a finite set, and let $x \in A$ be a positive element of norm 1. Then there exist $b_g \in A$ and $c_g \in A$ for each $g \in G$ such that $c_g b_g = c_g$, $0 \le b_g \le 1$, and $0 \le c_g \le 1$, and such that the elements b_g are mutually orthogonal elements satisfying:*

1. *$\|\alpha_g(b_h) - b_{gh}\| < \varepsilon$ and $\|\alpha_g(c_h) - c_{gh}\|$ for all $g, h \in G$.*
2. *$\|b_g z - zb_g\| < \varepsilon$ and $\|c_g z - zc_g\| < \varepsilon$ for all $g \in G$ and $z \in F$.*
3. *With $b = \sum_{g \in G} b_g$, the element $1 - b$ is Cuntz subequivalent to an element of the hereditary subalgebra generated by x.*
4. *$\|bxb\| > 1 - \varepsilon$.*

Proof. Let $n = |G|$. Let $\varepsilon > 0$ be given. Without loss of generality $\|y\| \le 1$ for all $y \in F$.

Choose δ_1 so that $\delta_1^2/2 + 2n\delta_1 < \varepsilon$. Define continuous functions r and f on the nonnegative real numbers by:

- $r(0) = 0$,
- r is linear for $t \in [0, 1 - \delta_1]$,
- $r(t) = 1$ for $t \in [1 - \delta_1, 1]$,
- $f(t) = 0$ for $t \in [0, 1 - \delta_1]$,

- f is linear for $t \in [1-\delta_1, 1-\delta_1/2]$, and
- $f(t) = 1$ for $t \in [1-\delta_1/2, 1]$.

Apply Lemma VI.11 to the function r with δ_1 in place of ε. Let δ_2 be equal to the δ given by the lemma. Now apply Lemma VI.11 to the function f with δ_1 in place of ε. Let δ_3 be equal to the δ given by the lemma. Apply Lemma VI.12 to the function r with δ_1 in place of ε and then to the function f with δ_1 in place of ε, and call the minimum of the two deltas that you get δ_4.

Choose $\delta_5 < \min\{\delta_1^2/2, \delta_2, \delta_3, \delta_4\}$. Apply Definition VI.5 with δ_5 in place of ε, with F as given, and with x as given to get positive mutually orthogonal elements a_g for each $g \in G$. Set $b_g = r(a_g)$ and $c_g = f(a_g)$. Note that since $\|r(t) - t\| < \delta_1$, we have $\|b_g - a_g\| < \delta_1$. Therefore, with $a = \sum_{g\in G} a_g$ and $b = \sum_{g \in G} b_g$, we have $\|a - b\| < n\delta_1$.

First we investigate the effect of the action on b_h. Using the choice of δ_2,

$$\|\alpha_g(b_h) - b_{gh}\| = \|\alpha_g(r(a_h)) - r(a_{gh})\| \leq \|\alpha_g(r(a_h)) - r(\alpha_g(a_h))\| + \|r(\alpha_g(a_h)) - r(a_{gh})\| < \delta_1.$$

Similarly, $\|\alpha_g(c_h) - c_{gh}\| < \delta_1$. We have now verified (1).

Next we prove that b_g and c_g approximately commute with the elements of F. For any $y \in F$ and $g \in G$ we have $\|ya_g - a_g y\| < \delta_5$, so by the choice of δ_4 using Lemma VI.12 for r we have $\|yb_g - b_g y\| = \|yr(a_g) - r(a_g)y\| < \delta_1$. Similarly, $\|yc_g - c_g y\| < \delta_1$. We have now verified (2).

To verify (3), recall $1 - a \precsim x$. Thus it suffices to show that $1 - b \precsim 1 - a$. Let t denote the function $h(t) = t$. For each $g \in G$ define a homomorphism $\phi_g : C_0((0,1]) \to A$ such that $\phi_g(t) = a_g$. Note that if $g \neq h$ then $\phi_g(t)\phi_h(t) = a_g a_h = 0$. Thus, since ϕ_g and ϕ_h are homomorphisms, for any polynomials p_1 and p_2 with zero constant term, we have $\phi_g(p_1)\phi_h(p_2) = 0$ if $g \neq h$. Therefore, for any $f_1, f_2 \in C_0((0,1])$ we have $\phi_g(f_1)\phi_h(f_2)$ if $g \neq h$.

This means we can define a homomorphism

$$\phi : \bigoplus_{g \in G} C_0\left((0,1]\right) \to A$$

by

$$\phi\left((f_g)_{g \in G}\right) = \sum_{g \in G} \phi_g(f_g).$$

Since A is unital we can unitize to obtain a unital homomorphism

$$\phi^{+}:\left[\bigoplus_{g\in G} C_0\left((0,1]\right)\right]^{+}\longrightarrow A.$$

The C^*-algebra $[\bigoplus_{g\in G} C_0((0,1])]^{+}$ is isomorphic to $C(Y)$ with $Y=([0,1]\times G)/\sim$ where $(0,g)\sim(0,h)$ for all $g,h\in G$. This is because Y is the one point compactification of $\sqcup_{g\in G} C_0\left((0,1]\right)$. Define functions $d_g : Y \to \mathbb{C}$ by $d_g(t,h)=t$ if $g=h$ and $d_g(t,h)=0$ if $g\neq h$. Note that d_g is continuous, so $d_g\in C(Y)$. Also observe that $\phi^{+}(d_g)=a_g$.

Now, by the definition of r we see that $\{t\in[0,1]:1-r(t)=0\}\supset\{t\in[0,1]:1-t=0\}$. Therefore, $\left\{y\in Y:1-r\left(\sum_{g\in G} d_g(y)\right)=0\right\}\supset\left\{y\in Y:1-\sum_{g\in G} d_g=0\right\}$. Thus by Lemma VI.4, we have $1-r\left(\sum_{g\in G} d_g\right)\precsim 1-\sum_{g\in G} d_g$ which gives

$$\phi^{+}\left(1-r\left(\sum_{g\in G} d_g\right)\right)\precsim\phi^{+}\left(1-\sum_{g\in G} d_g\right).$$

Since ϕ^{+} is unital, we have

$$1-\phi^{+}\left(r\left(\sum_{g\in G} d_g\right)\right)\precsim 1-\sum_{g\in G}\phi^{+}(d_g).$$

Now using the fact that functional calculus commutes with homomorphisms and then that ϕ^{+} is a homomorphism we see $1-\phi^{+}\left(r\left(\sum_{g\in G} d_g\right)\right)=1-r\left(\phi^{+}\left(\sum_{g\in G} d_g\right)\right)=1-r\left(\sum_{g\in G}\phi^{+}(d_g)\right)$. Therefore,

$$1-r\left(\sum_{g\in G}\phi^{+}(d_g)\right)\precsim 1-\sum_{g\in G}\phi^{+}(d_g).$$

However, we observed above that $\phi^{+}(d_g)=a_g$ so this shows

$$1-r\left(\sum_{g\in G} a_g\right)\precsim 1-\sum_{g\in G} a_g.$$

Now the mutual orthogonality of the elements a_g means $r\left(\sum_{g\in G} a_g\right)=\sum_{g\in G} r(a_g)=b$ by Lemma VI.10. Therefore, $1-b\precsim 1-a\precsim x$ which is (3).

Finally, we verify condition (4). We have

$$\begin{aligned}\|bxb\| &\geq \|axa\| - \|axa - axb\| - \|axb - bxb\| \\ &> 1 - \delta_1 - \|ax\|\|a - b\| - \|a - b\|\|xb\| \\ &\geq 1 - \delta_1^2/2 - 2n\delta_1 \\ &> 1 - \varepsilon.\end{aligned}$$

This completes the proof. ∎

Lemma VI.14. *Let τ be a tracial state on A. For all $\varepsilon > 0$, there is a $\delta > 0$ such that if $g : [0,1] \to [0,1]$ is a continuous function satisfying $g(0) = 0$ and $g(t) = 1$ for $t \in [1-\varepsilon, 1]$, and if $a \in A$ with $0 \leq a \leq 1$ and with $\tau(a) > 1 - \delta$, then $\tau(1 - g(a)) < \varepsilon$. Moreover, we may choose $\delta = \varepsilon^2$.*

Proof. Let μ be the measure on $\operatorname{sp}(a) \subset [0,1]$ obtained from τ. If $\tau(a) > 1 - \delta$, then

$$\begin{aligned}1 - \delta &< \tau(a) \\ &\leq (1-\varepsilon)\mu([0, 1-\varepsilon]) + 1 \cdot \mu((1-\varepsilon, 1]) \\ &= (1-\varepsilon)\mu([0, 1-\varepsilon]) + 1 - \mu([0, 1-\varepsilon]) \\ &= \mu([0, 1-\varepsilon]) - \varepsilon\mu([0, 1-\varepsilon]) + 1 - \mu([0, 1-\varepsilon]) \\ &= 1 - \varepsilon\mu([0, 1-\varepsilon]),\end{aligned}$$

which implies that

$$-\delta < -\varepsilon([0, 1-\varepsilon])$$

or equivalently

$$\delta > \varepsilon\mu([0, 1-\varepsilon]).$$

This gives

$$\frac{\delta}{\varepsilon} > \mu([0, 1-\varepsilon]).$$

Now we compute

$$\tau(1-g(a)) = \int_{[0,1]} (1-g(t))\,dt \leq 1 \cdot \mu([0,1-\varepsilon]) < \frac{\delta}{\varepsilon}.$$

So if $\delta \leq \varepsilon^2$ then

$$\tau(1-g(a)) < \frac{\varepsilon^2}{\varepsilon} = \varepsilon.$$

∎

Lemma VI.15. *Suppose A is an infinite dimensional simple unital C^*-algebra. Suppose G is a finite group. Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of G with the projection free tracial Rokhlin property. Suppose $\tau \in T(C^*(G, A, \alpha)$, then there exists $\sigma \in T(A)$ such that $\tau = \sigma \circ E$ where $E : C^*(G, A, \alpha) \to A$ is the conditional expectation.*

Proof. It suffices to show that if $x \in A$ and $g \in G \setminus \{0\}$, then $|\tau(xu_g)| < \varepsilon$ for any $\varepsilon > 0$. Let $\varepsilon > 0$ be given. Let $n = \mathrm{card}(G)$. Without loss of generality, $\|x\| \leq 1$.

Choose δ_1 using Lemma VI.12 with $\frac{\varepsilon}{3n}$ in place of ε and with $t^{1/2}$ in place of f. Choose δ_2 using Lemma VI.11 with $\frac{\varepsilon}{3n}$ in place of ε and with $t^{1/2}$ in place of f. Choose $\delta_3 < \min\{\frac{\varepsilon^2}{18}, \frac{1}{2}\}$. Choose δ_4 using Lemma VI.14 with δ_3 in place of ε. Choose a continuous function $g : [0,1] \to [0,1]$ such that $g(0) = 0$ and $g(t) = 1$ for $t \in [1-\delta_3, 1]$. We also require that $\|g - (2t - t^2)\| < \delta_3$. This is possible since $\sup_{t\in[1-\delta_3,1]} \|1 - (2t - t^2)\| = \delta_3^2$.

Apply the projection free tracial Rokhlin property with δ_4 in place of ε, with $F = \{x\}$ and with 1 in place of the positive element x to get mutually orthogonal positive elements a_h for each $h \in G$. Set $a = \sum_{h\in G} a_h$. One of the properties satisfied by a is that $\tau(a) > 1 - \delta_4$. By the choice of g and δ_4, this implies $\tau(1 - g(a)) < \delta_3$. By the second requirement on g we now have $\tau((1-a)^2) = \tau(1 - (2a - a^2)) < \tau(1 - g(a)) + \delta_3 < 2\delta_3$.

Next we need to bound $|\tau(xu_g(1-a))|^2$. By the Cauchy-Schwartz inequality, we have

$$|\tau(xu_g(1-a))|^2 \leq \tau(u_g x x^* u_g^*)\tau((1-a)^2) \leq \|x\|^2 \tau((1-a)^2) < 2\delta_3 < \frac{\varepsilon^2}{9}.$$

Therefore, we can conclude

$$|\tau(xu_g(1-a))| < \varepsilon/3. \tag{VI.1}$$

We are now in a position to compute $|\tau(xu_g)|$. We have

$$\begin{aligned}
|\tau(xu_g)| &= \left|\tau(xu_g) - \sum_{h\in G}\tau(xa_h^{1/2}a_{gh}^{1/2}u_g)\right| \\
&\leq |\tau(xu_g) - \tau(xu_g a)| + \left|\sum_{h\in G}\tau\left(xu_g a_h\right) - \sum_{h\in G}\tau\left(a_h^{1/2}xu_g a_h^{1/2}\right)\right| \\
&\quad + \left|\sum_{h\in G}\tau\left(a_h^{1/2}xu_g a_h^{1/2}u_g^* u_g\right) - \sum_{h\in G}\tau\left(a_h^{1/2}xa_{gh}^{1/2}u_g\right)\right| \\
&\quad + \left|\sum_{h\in G}\tau\left(a_h^{1/2}xa_{gh}^{1/2}u_g\right) - \sum_{h\in G}\tau\left(xa_h^{1/2}a_{gh}^{1/2}u_g\right)\right| \\
&\leq |\tau\left(xu_g(1-a)\right)| + 0 + \sum_{h\in G}\left|\tau\left(a_h^{1/2}xu_g a_h^{1/2}u_g^* u_g - a_h^{1/2}xa_{gh}^{1/2}u_g\right)\right| \\
&\quad + \sum_{h\in G}|\tau(a_h^{1/2}xa_{gh}^{1/2}u_g - xa_h^{1/2}a_{gh}^{1/2}u_g)| \\
&< \varepsilon/3 + \sum_{h\in G}\left\|u_g a_h u_g^* - a_{gh}^{1/2}\right\| + \sum_{h\in G}\left\|a_h^{1/2}x - xa_h^{1/2}\right\| \quad \text{by Equation VI.1} \\
&< \varepsilon/3 + n\frac{\varepsilon}{3n} + n\frac{\varepsilon}{3n} \quad \text{by the choice of } \delta_1 \text{ and } \delta_2 \\
&= \varepsilon.
\end{aligned}$$

This completes the proof. ∎

The following definition appears near the end of section 2 of [3].

Definition VI.16. *Given a normalized 2-quasi-trace τ on A, one may define a map*

$$d_\tau : M_\infty(A)_+ = (\cup_{n=1}^{\infty} M_n(A))_+ \to \mathbb{R}^+$$

by

$$d_\tau(a) = \lim_{n\to\infty}\tau(a^{1/n}).$$

We say that A has strict comparison (of positive elements) *if* $\lim_{n\to\infty}\tau(a^{1/n}) < \lim_{n\to\infty}\tau(b^{1/n})$ *for every normalized 2-quasi-trace τ on A, implies $a \preccurlyeq b$ for all elements $a, b \in A_+ \setminus \{0\}$.*

Notice that since the definition is already treating $M_\infty(A)$, if A has strict comparison, so does $M_n(A)$ for any positive integer n.

Lemma VI.17. *If A has strict comparison and $c \in A_+$, then $\overline{cAc}$ has strict comparison.*

Proof. Suppose $a, b \in A_+$ and $\lim_{n\to\infty} \tau(a^{1/n}) < \lim_{n\to\infty} \tau(b^{1/n})$ for every normalized 2-quasi-trace on $\overline{cAc}$. Note that any 2-quasi-trace σ on A restricts to a 2-quasi-trace on $\overline{cAc}$, so for such σ, we have $\lim_{n\to\infty} \sigma(a^{1/n}) < \lim_{n\to\infty} \sigma(b^{1/n})$. Therefore, $a \precsim b$ by the strict comparison on A. ∎

Remark VI.18. *The hypothesis that all 2-quasi-traces are traces appears frequently in what follows. Thus it is worth noting as is done near the end of Section 2 of [3] that every exact C^*-algebra satisfies this hypothesis.*

Lemma VI.19. *Let A be a C^*-algebra with strict comparison. Fix $z \in A$ with $0 \le z \le 1$ and $z \ne 0$. If $0 < \varepsilon < \tau(z)$ for every 2-quasi-trace τ, if $g : [0,1] \to [0,1]$ is continuous and satisfies $g(0) = 0$ and $g(t) = 1$ for $t \in [1-\varepsilon, 1]$, and if $a \in A$ with $0 \le a \le 1$ and $\tau(a) > 1 - \varepsilon^2$ for every 2-quasi-trace τ, then $1 - g(a) \precsim z$.*

Proof. We first claim that $(1-g(t))^{1/n} = 1 - g_n(t)$ for some continuous g_n satisfying $g_n(0) = 0$ and $g_n(t) = 1$ for $t \in [1-\varepsilon, 1]$. To see this, observe that this is equivalent to saying $(1-g(t))^{1/n} = f_n(t)$ for some continuous function f_n satisfying $f_n(0) = 1$ and $f_n(t) = 0$ for $t \in [1-\varepsilon, 1]$. But the left-hand side is the composition of continuous functions, hence continuous, and the left-hand side maps 0 to 1 and $[1-\varepsilon, 1]$ to 0, so the equivalent statement is clear.

By the claim and the previous lemma, since each g_n is a function of the same type as g, we have $\tau((1-g(a))^{1/n}) = \tau(1 - g_n(t)) < \varepsilon$. So now

$$\tau((1-g(a))^{1/n}) < \varepsilon < \tau(z) < \tau(z^{1/2}) < \tau(z^{1/3}) < \cdots,$$

which implies that

$$\lim_{n\to\infty} \tau((1-g(a))^{1/n}) \le \varepsilon < \tau(z) \le \lim_{n\to\infty} \tau(z^{1/n}),$$

which gives

$$\lim_{n\to\infty} \tau((1-g(a))^{1/n}) < \lim_{n\to\infty} \tau(z^{1/n}).$$

Since A has strict comparison, it follows that $1 - g(a) \precsim z$. ∎

CHAPTER VII

STABLE RANK AND THE PROJECTION FREE TRACIAL ROKHLIN PROPERTY

Lemma VII.1. *Let f be a continuous function on $[0,1]$ with $f(0) = 0$. Let $\{e_{g,h}\}$ be a set of matrix units for M_n. Then in $C([0,1]) \otimes M_n$, we have $f(t \otimes e_{g,g}) = f(t) \otimes e_{g,g}$.*

Proof. We first claim that $t^d \otimes e_{g,g} = (t \otimes e_{g,g})^d$ for all $d \in \mathbb{N}, d > 0$. This is true since $e_{g,g}^d = e_{g,g}$. Next, we claim that if p is a polynomial that vanishes at zero, then $p(t \otimes e_{g,g}) = p(t) \otimes e_{g,g}$. This holds by combining the first claim with the equality $a \otimes e_{g,g} + b \otimes e_{g,g} = (a+b) \otimes e_{g,g}$. Finally, we claim that this holds for any continuous function f on $[0,1]$ with $f(0) = 0$. For any $\varepsilon > 0$, let p be a polynomial such that $p(0) = 0$ and $\|p - f\| < \varepsilon$. Then we have

$$
\begin{aligned}
\|f(t \otimes e_{g,g}) - f(t) \otimes e_{g,g}\| &\leq \|f(t \otimes e_{g,g}) - p(t \otimes e_{g,g})\| + \|p(t \otimes e_{g,g}) - p(t) \otimes e_{g,g}\| \\
&\quad + \|p(t) \otimes e_{g,g} - f(t) \otimes e_{g,g}\| \\
&< 2\varepsilon
\end{aligned}
$$

Since this holds for any $\varepsilon > 0$, the result follows. ∎

The following proposition and proof are very similar to Proposition 3.3.1 of [13]

Proposition VII.2. *The universal C^*-algebra A generated by $\{y_{j,k} : 1 \leq j, k \leq n\}$ subject to the relations*

1. $y_{j_1,k_1} y_{j_2,k_2} = \delta_{k_1,j_2} y_{j_1,j_1} y_{j_1,k_2}$,
2. $y_{j,k}^* = y_{k,j}$,
3. $y_{1,1} \neq 0$, *and*

4. $0 \leq y_{j,j} \leq 1$

is isomorphic to CM_n.

Proof. We identify CM_n as $C_0((0,1]) \otimes M_n$. Let $\{e_{j,k}\}$ be an n by n set of matrix units for M_n. Define the map $\phi : A \to CM_n$ by $y_{j,k} \mapsto t \otimes e_{j,k}$. Since the elements $\{t \otimes e_{j,k}\}$ satisfy the relations which $\{y_{j,k}\}$ satisfy this a well defined homomorphism.

By the Stone-Weierstrass Theorem the elements $\{t \otimes e_{j,k}\}$ generate CM_n.

Consider an irreducible representation $\pi : A \to H$ of these relations. Let $z_{j,k} = \pi(y_{j,k})$. Consider the element $c = z_{1,1}^2 + \cdots + z_{n,n}^2$. For any j and k between 1 and n,

$$z_{j,k}c = z_{j,k}z_{k,k}z_{k,k} = z_{j,j}z_{j,j}z_{j,k} = cz_{j,k}.$$

Thus c is central in $C^*(\{z_{j,k}\}_{1 \leq j,k \leq n})$. Because π is irreducible, this implies that c is a scalar multiple of the identity. That is, for some $\gamma \in [0,1]$, we have $c = \gamma I$.

If $\gamma = 0$, then $c = 0$. In this case, given l and k with $1 \leq l, k \leq n$, we have

$$0 = c = \sum_{j=1}^{l-1} z_{j,j}^2 + \sum_{j=l+1}^{n} z_{j,j}^2 + z_{l,k}z_{l,k}^*.$$

Note that this sum consists entirely of positive elements and yet adds to zero, therefore each item in the sum is zero. In particular $z_{l,k}z_{l,k}^* = 0$ which implies $z_{l,k} = 0$. Therefore, if $\gamma = 0$, then $z_{l,k}$ is the image of $t \otimes e_{l,k}$ under the zero representation of CM_n.

If $\gamma > 0$, then γ^{-1} is defined. Note that $\gamma z_{j,j}^2 = cz_{j,j}^2 = z_{j,j}^4$. This implies that $\gamma^{-1}z_{j,j}^2$ is a projection for every j. From this we can also conclude that $\gamma^{-1/2}z_{j,j}$ is a projection. Next we check that the elements $\gamma^{-1/2}z_{j,k}$ satisfy the relations for a set of matrix units for M_n. We have

$$\begin{aligned}
\gamma^{-1/2}z_{j,k}\gamma^{-1/2}z_{l,m} &= \gamma^{-1}\delta_{k,l}z_{j,j}z_{j,m} \\
&= \delta_{k,l}\gamma^{-1/2}z_{j,j}\gamma^{-1/2}z_{j,m} \\
&= \delta_{k,l}\left(\sum_{g=1}^{n} \gamma^{-1/2}y_{g,g}\right)\gamma^{-1/2}y_{j,m} \\
&= \delta_{k,l}\gamma^{-1/2}\left(\sum_{g=1}^{n} y_{g,g}\right)\gamma^{-1/2} \\
&= \delta_{k,l}\gamma^{-1/2}y_{j,m}.
\end{aligned}$$

The other two relations are clear.

Up to unitary equivalence, $H = \mathbb{C}^n$ and $z_{j,k} = \gamma^{1/2} e_{j,k}$. These are the images of $\{t \otimes e_{j,k}\}$ under evaluation at $\gamma^{1/2}$. Thus by Lemma 3.2.2 of [13] we are done. ∎

The following lemma guarantees the existence of elements of $C^*(G, A, \alpha)$ which satisfy the cone relations above, approximately respect the action of G, and are near elements produced using the projection free tracial Rokhlin property.

Lemma VII.3. *Suppose A is an infinite dimensional unital simple C^*-algebra. Let $\varepsilon > 0$, let $F \subset A$ be a finite set, and let $x \in A$ be a positive element of norm one. Suppose G is a finite group and $\alpha : G \to \mathrm{Aut}(A)$ is an action of G on A with the projection free tracial Rokhlin property. Then there exist $\delta > 0$, positive elements $a_g \in A$ for each $g \in G$, and elements $Y_{g,h} \in C^*(G, A, \alpha)$ for each $g, h \in G$ such that for $g, h, j, k \in G$ we have*

1. $Y_{j,k} Y_{g,h} = \delta_{k,g} Y_{j,j} Y_{j,h}$.
2. $Y_{j,k}^* = Y_{k,j}$.
3. $Y_{1,1} \neq 0$, *where 1 is the identity of G.*
4. $0 \leq Y_{1,1} \leq 1$.
5. $\|u_k Y_{j,l} - Y_{kj,l}\| < \varepsilon$.
6. $\|Y_{j,g} u_k^* - Y_{j,kg}\| < \varepsilon$.
7. $\|Y_{j,j} - a_j\| < \varepsilon$.
8. $Y_{1,1} \in A$.
9. $\|Y_{j,j} b - b Y_{j,j}\| < 2\varepsilon \|b\| + \varepsilon$ *for any $b \in F$.*
10. $\|\alpha_j(a_k) - a_{jk}\| < \delta$.
11. $\|a_j b - b a_j\| < \delta$ *for all $b \in F$.*
12. *With $a = \sum_{g \in G} a_g$ we have $1 - a$ is Cuntz subequivalent to an element of the hereditary subalgebra generated by x.*

13. $\|axa\| > 1 - \delta$.

14. $\tau(1-a) < \delta$ *for all* $\tau \in T(A)$.

Proof. First observe that if $n = \mathrm{card}(G)$, then (1) through (4) are the relations needed for CM_n by Proposition VII.2. Also observe that (5) and (6) are equivalent by taking adjoints, so we will only prove (5). In order to show (5), it suffices to show $\|u_j Y_{1,k} - Y_{j,k}\| < \varepsilon/2$ and $\|u_{j^{-1}} Y_{j,k} - Y_{1,k}\| < \varepsilon/2$.

We will proceed by induction on the matrix size of the cone, showing at each stage that all the relations are satisfied.

First we work on CM_2. Let 1 be the identity of G and let $g \in G$ be a fixed non identity element. Let $\varepsilon > 0$ be given. Choose δ_0 with $0 < \delta_0 < \varepsilon$ such that if x and y are positive elements of norm less than or equal to one in any C^*-algebra, and if $\|x - y\| < \delta_0$, then $\|x^{1/2} - y^{1/2}\| < \varepsilon/4$. Without loss of generality, $\delta_0 < \varepsilon$. Apply the projection free tracial Rokhlin property with δ_0 in place of ε and with F and x as given to get a_j for each group element $j \in G$. Properties (10), (11), (12), (13), and (14) are true by the definition of the projection free tracial Rokhlin property. Define

$$
\begin{aligned}
y_{1,1} =& (a_1^{1/2} u_g^* a_g u_g a_1^{1/2})^{1/2} \\
y_{g,g} =& (a_g^{1/2} u_g a_1 u_g^* a_g^{1/2})^{1/2} \\
y_{1,g} =& a_1^{1/2} u_g^* a_g^{1/2} \\
y_{g,1} =& a_g^{1/2} u_g a_1^{1/2}.
\end{aligned}
$$

Using the fact that a_1 and a_g are mutually orthogonal, it is easy to check that properties (1), (2), and (3) of the statement are satisfied. For (4) we recall from the definition of the projection free tracial Rokhlin property that $0 \le a_j \le 1$ for each $j \in G$. This implies $0 \le a_1^{1/2} u_g^* a_g u_g a_1^{1/2} \le a_1 \le 1$. Therefore, $0 \le y_{1,1} \le 1$. Similarly, $0 \le a_g^{1/2} u_g a_1 u_g^* a_g^{1/2} \le a_g \le 1$. Therefore, $0 \le y_{g,g} \le 1$.

To show (5), we use

$$\|a_g - u_g a_1 u_g^*\| < \delta_0$$

to compute,

$$\|u_g y_{1,g} - y_{g,g}\| \le \|u_g a_1^{1/2} u_g^* a_g^{1/2} - a_g\| + \|a_g - (a_g^{1/2} u_g a_1 u_g^* a_g^{1/2})^{1/2}\| < \varepsilon/4 + \varepsilon/4 = \varepsilon/2.$$

Similarly, $\|u_g y_{1,1} - y_{g,1}\| < \varepsilon$. But now

$$\|u_{g^{-1}} y_{g,1} - y_{1,1}\| = \|u_g u_{g^{-1}} y_{g,1} - u_g y_{1,1}\| < \varepsilon$$

and

$$\|u_g^{-1} y_{g,g} - y_{1,g}\| = \|u_g u_{g^{-1}} y_{g,g} - u_g y_{1,g}\| < \varepsilon.$$

Next we show that (7) holds. By the choice of δ_0, we have

$$\|a_1^{1/2} u_g^* a_g u_g a_1^{1/2} - a_1^2\| \leq \|u_g^* a_g u_g - a_1\| < \delta_0,$$

which implies

$$\|y_{1,1} - a_1\| = \|(a_1^{1/2} u_g^* a_g u_g a_1^{1/2})^{1/2} - a_1\| < \varepsilon/2.$$

Similarly, $\|y_{g,g} - a_g\| < \varepsilon/2$.

For property (8), we note that $u_g^* a_g u_g = \alpha_{g^{-1}}(a_g) \in A$, so $y_{1,1} \in A$.

Next we show (9). For any $b \in F$ and $j = 1$ or $j = g$, we have

$$y_{j,j}b - by_{j,j}\| \leq \|y_{j,j}b - a_j b\| + \|a_j b - b a_j\| + \|b a_j - b y_{j,j}\| < \varepsilon\|b\| + \delta_0 + \varepsilon\|b\| < 2\varepsilon\|b\| + \varepsilon.$$

For the purposes of induction it is helpful to have one more property, namely, that $y_{j,k}$ for $j, k \in \{1, g\}$ are each orthogonal to a_m for all $m \in G \setminus \{1, g\}$. This is clear since $a_j a_m = 0$ if $j \neq m$. This completes the base case.

From now on call the elements of G, $1, \ldots, j, \ldots, n$ instead of $g_1, \ldots g_n$ to avoid an excess of double subscripts. In order to avoid confusion, 1 will be the identity of G.

Now suppose that for any $\varepsilon_1 > 0$ there exists a positive number $\delta(\varepsilon, m)$ such that if $\{a_j\}_{j \in G}$ are the elements which come from applying the projection free tracial Rokhlin property with $\delta(\varepsilon, m)$ in place of ε and with F and x as given, then there exist elements $z_{j,k} \in C^*(G, A, \alpha)$ for $1 \leq j, k \leq m$ and $a_j \in A$ for $j \in G$ such that

1. $z_{j,k} z_{l,h} = \delta_{k,l} z_{j,j} z_{j,h}$ for $1 \leq j, k, l, h \leq m$,

2. $z_{j,k}^* = z_{k,j}$ for $1 \leq j, k \leq m$,

3. $z_{1,1} \neq 0$,

4. $0 \leq z_{1,1} \leq 1$,

5. $\|u_k z_{j,l} - z_{kj,l}\| < \varepsilon_1$ if $j, k, l, kj \leq m$,

6. $\|z_{j,l} u_k^* - z_{j,kl}\| < \varepsilon_1$ if $j, k, l, kl \leq m$,

7. $\|z_{j,j} - a_j\| < \varepsilon_1$ if $1 \leq j \leq m$,

8. $z_{1,1} \in A$,

9. $\|z_{j,j} b - b z_{j,j}\| < 2\varepsilon_1 \|b\| + \varepsilon_1$,

10. $\|\alpha_j(a_k) - a_{jk}\| < \delta$,

11. $\|a_j b - b a_j\| < \delta$ for all $b \in F$,

12. With $a = \sum_{g \in G} a_g$ we have $1 - a$ is Cuntz subequivalent to an element of the hereditary subalgebra generated by x,

13. $\|axa\| > 1 - \delta$,

14. $\tau(1 - a) < \delta$ for all $\tau \in T(A)$, and

15. $z_{j,k} a_l = a_l z_{j,k} = 0$ if $1 \leq j, k \leq m$ and $m + 1 \leq l \leq n$.

Given any $\varepsilon > 0$ we wish to show we can produce elements $y_{j,k}$ and a_g which satisfy the above properties for $1 \leq j, k \leq m + 1$, for all $g \in G$ and with ε in place of ε_1 above. Without loss of generality, $\varepsilon < 1$.

Let $0 < \delta_0 < \varepsilon/192$. Choose δ_1 so that if x and y are positive elements with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| < \delta_1$, then $\|x^{1/2} - y^{1/2}\| < \delta_0$. Without loss of generality, $\delta_1 < \delta_0$. Then choose $\delta_2 > 0$ such that if x and y are positive elements with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| < \delta_2$, then $\|x^{1/2} - y^{1/2}\| < \delta_1/8$. Choose $\delta_3 = \min\{\frac{\varepsilon}{32}, \frac{\delta_1}{4}\}$. Now choose $0 < \delta_4 < \min\{\delta_2, \delta_1/4\}$.

Define a continuous function f to be zero on $[0, \delta_4]$, one at $t = 1$, and linear on $[\delta_4, 1]$. Define a continuous function g to be zero at $t = 0$, one on $[\delta_4, 1]$, and linear on $[0, \delta_4]$. Notice that $\|f(t) - t\| < \delta_4$ and that $fg = f$.

Choose a polynomial p in $C([0,1])$ with $\|p - t^{1/2}\| < \delta_3/3$ and $p(0) = 0$. Write $p(t) = \sum_{m=1}^{d} b_m t^m$. Let $\lambda_p = \sum_{m=1}^{d} |b_m|$. Suppose $\psi : CM_n \to B$ is a homomorphism to a C^*-algebra

B and $u \in B$ is a unitary satisfying

$$\|u\psi(t \otimes e_{1,k}) - \psi(t \otimes e_{j,k})\| < \frac{\delta_3}{3\lambda_p}.$$

Then

$$\begin{aligned}
&\|u\psi(t^{1/2} \otimes e_{1,k}) - \psi(t^{1/2} \otimes e_{j,k})\| \\
&\leq \|u\psi(t^{1/2} \otimes e_{1,k}) - u\psi(p \otimes e_{1,k})\| + \|u\psi(p \otimes e_{1,k}) - \psi(p \otimes e_{j,k})\| + \|\psi(p \otimes e_{j,k}) - \psi(t^{1/2} \otimes e_{j,k})\| \\
&\leq \|t^{1/2} - p\| + \sum_{m=1}^{d} \|u\psi(b_m t^m \otimes e_{1,k}) - \psi(b_m t^m \otimes e_{j,k})\| + \|p - t^{1/2}\| \\
&< 2\delta_3/3 + \sum_{m=1}^{d} |b_m| \|u\psi(t \otimes e_{1,k})\psi(t^{m-1} \otimes e_{k,k}) - \psi(t \otimes e_{j,k})\psi(t^{m-1} \otimes e_{k,k})\| \\
&\leq 2\delta_3/3 + \sum_{m=1}^{d} |b_m| \frac{\delta_3}{3\sum_{m=1}^{d} |b_m|} \\
&= \delta_3. \qquad \text{(VII.1)}
\end{aligned}$$

This implies

$$\|u\psi(f^{1/2} \otimes e_{1,k}) - \psi(f^{1/2} \otimes e_{j,k})\| < 2\delta_4 + \delta_3. \qquad \text{(VII.2)}$$

Choose $0 < \delta_5 < \min\{\frac{\delta_1}{2}, \frac{\varepsilon}{32}, \frac{\delta_2}{4}, \frac{\delta_3}{3\lambda_p}\}$. Let $0 < \delta_6 < \min\{\varepsilon/48, \delta_1/2\}$. Apply the induction hypothesis with δ_5 in place of ε_1 to get elements $z_{j,k}$ and a_g which satisfy the fifteen properties above. We also require that $\delta(\delta_5, m) < \delta_6$. That is, we may assume the projection free tracial Rokhlin property was applied with a number smaller than δ_6 in place of ε. Once again, properties (10), (11), (12), (13), and (14) are satisfied by the definition of the projection free tracial Rokhlin property.

These elements $z_{j,k}$ allow us to define a homomorphism $\phi : CM_m \to C^*(G, A, \alpha)$ by $(t \otimes e_{j,k}) \mapsto z_{j,k}$. Let $s_j = \phi(f^{1/2} \otimes e_{j,1})$ for $j = 1, \ldots, m$. For $1 \le j \le m$, set

$$\begin{aligned} y_{m+1,m+1} &= (a_{m+1}^{1/2} u_{m+1} s_1^2 u_{m+1}^* a_{m+1}^{1/2})^{1/2}, \\ y_{j,m+1} &= s_j u_{m+1}^* a_{m+1}^{1/2}, \\ y_{m+1,j} &= a_{m+1}^{1/2} u_{m+1} s_j^*, \\ y_{j,j} &= (s_j u_{m+1}^* a_{m+1} u_{m+1} s_j^*)^{1/2}, \\ y_{j,k} &= y_{j,j} \phi(g \otimes e_{j,k}). \end{aligned}$$

Before we start to prove that these elements satisfy the cone relations, we make some observations. Notice that $s_j \phi(g \otimes e_{k,l}) = \phi(f^{1/2} g \otimes e_{j,1} e_{k,l}) = \phi(f^{1/2} \otimes \delta_{1,k} e_{j,l})$. Also,

$$\phi(g \otimes e_{k,l}) s_j = \phi(f^{1/2} \otimes \delta_{l,j} e_{k,1})$$

which equals s_k if $j = l$.

Notice that

$$(s_j u_{m+1}^* a_{m+1} u_{m+1} s_j^*)^d \phi(g \otimes e_{j,k}) = \phi(g \otimes e_{j,k}) (s_k u_{m+1}^* a_{m+1} u_{m+1} s_k^*)^d$$

for any positive integer d. Therefore, for any polynomial P with $P(0) = 0$ we have

$$P(s_j u_{m+1}^* a_{m+1} u_{m+1} s_j^*) \phi(g \otimes e_{j,k}) = \phi(g \otimes e_{j,k}) P(s_k u_{m+1}^* a_{m+1} u_{m+1} s_k^*).$$

This implies that for any continuous function f with $f(0) = 0$ we have

$$f(s_j u_{m+1}^* a_{m+1} u_{m+1} s_j^*) \phi(g \otimes e_{j,k}) = \phi(g \otimes e_{j,k}) f(s_k u_{m+1}^* a_{m+1} u_{m+1} s_k^*).$$

In particular we have

$$(s_j u_{m+1}^* a_{m+1} u_{m+1} s_j^*)^{1/2} \phi(g \otimes e_{j,k}) = \phi(g \otimes e_{j,k}) (s_k u_{m+1}^* a_{m+1} u_{m+1} s_k^*)^{1/2}.$$

Therefore,

$$y_{j,j}\phi(g \otimes e_{j,k}) = \phi(g \otimes e_{j,k})y_{k,k}. \tag{VII.3}$$

Similarly, since $s_j\phi(g\otimes e_{j,j}) = \phi(f^{1/2}\otimes e_{1,j})\phi(g\otimes e_{j,j}) = \phi(f^{1/2}\otimes e_{1,j}) = s_j$, we conclude,

$$y_{j,j}\phi(g \otimes e_{j,j}) = y_{j,j}. \tag{VII.4}$$

Now we check property (1). For this portion of the proof assume that $1 \leq i, j, k, l, \leq m$. It is easy to see that $y_{j,m+1}y_{m+1,j} = y_{j,j}^2$ and that $y_{m+1,j}y_{j,m+1} = y_{m+1,m+1}^2$. Next we see

$$\begin{aligned}
y_{j,m+1}y_{m+1,k} &= s_j u_{m+1}^* a_{m+1}^{1/2} a_{m+1}^{1/2} u_{m+1} s_k^* \\
&= s_j u_{m+1}^* a_{m+1} u_{m+1} \phi(f^{1/2} \otimes e_{1,k}) \\
&= s_j u_{m+1}^* a_{m+1} u_{m+1} \phi(f^{1/2} \otimes e_{1,j})\phi(g \otimes e_{j,k}) \\
&= y_{j,j}y_{j,j}\phi(g \otimes e_{j,k}) \\
&= y_{j,j}y_{j,k}
\end{aligned}$$

Since $j \leq m$, using the fifteenth property of the induction hypothesis at the second step, we have

$$\begin{aligned}
y_{j,j}y_{m+1,m+1} &= (s_j u_{m+1}^* a_{m+1} u_{m+1} s_j^*)^{1/2} (a_{m+1}^{1/2} u_{m+1})\phi(f \otimes e_{1,1} u_{m+1}^* a_{m+1}^{1/2})^{1/2} \\
&= 0.
\end{aligned}$$

Now suppose that $j \neq k$. Then

$$\begin{aligned}
y_{j,k}y_{l,l} &= y_{j,j}\phi(g \otimes e_{j,k})y_{l,l} \\
&= \delta_{k,l}y_{j,j}\phi(g \otimes e_{j,l})y_{l,l} \\
&= \delta_{k,l}y_{j,j}y_{j,j}\phi(g \otimes e_{j,l}) \qquad \text{by Equation VII.3} \\
&= \delta_{k,l}y_{j,j}y_{j,l}.
\end{aligned}$$

If $j \neq k$ we also have $y_{j,j}y_{k,k} = 0$ since $s_j^* s_k = 0$.

Now if $k \neq i$, and $j \neq k$ and $i \neq l$, we have $y_{j,k}y_{i,l} = y_{j,j}\phi(g \otimes e_{j,k})\phi(g \otimes e_{i,l})y_{l,l} = 0$. On the other hand, if $k = i$, but $j \neq k$ and $k \neq l$ we get $y_{j,k}y_{i,l} = y_{j,j}\phi(g^2 \otimes e_{j,l})y_{l,l}$. This shows (1).

For (2), note equation VII.3 implies $y_{j,k}^* = y_{k,j}$ for $1 \leq j, k \leq m$. The rest of the adjoint conditions required for (2) are clear from the definitions of the elements.

Next we show (5) by checking the various cases as we did for (1). However we begin by computing some useful estimates. Using $\|f(t) - t\| < \delta_4 < \delta_2$ and $\|\phi(t \otimes e_{1,1}) - a_1\| < \delta_5 < \delta_2$ for the penultimate step, we compute

$$\begin{aligned}
\|y_{1,1}^2 - a_1^2\| &\leq 2\|\phi(f^{1/2} \otimes e_{1,1}) - a_1^{1/2}\| + \|u_{m+1}^* a_{m+1} u_{m+1} - a_1\| \\
&\leq 2\|\phi(f^{1/2} \otimes e_{1,1}) - \phi(t^{1/2} \otimes e_{1,1})\| + 2\|\phi(t^{1/2} \otimes e_{1,1}) - a_1^{1/2}\| + \delta_6 \\
&\leq 2\left(\frac{\delta_1}{8}\right) + 2\left(\frac{\delta_1}{8}\right) + \frac{\delta_1}{2} \\
&= \delta_1.
\end{aligned}$$

By the choice of δ_1, this implies that

$$\|y_{1,1} - a_1\| < \delta_0. \tag{VII.5}$$

Using the facts that $\|a_1 - z_{1,1}\| < \delta_5 < \delta_2$ and $\|t - f\| < \delta_4 < \delta_2$ we see that

$$\|a_1^{1/2} - \phi(f^{1/2} \otimes e_{1,1})\| \leq \|a_1^{1/2} - z_{1,1}^{1/2}\| + \|z_{1,1}^{1/2} - \phi(f^{1/2} \otimes e_{1,1})\| < \delta_1/4. \tag{VII.6}$$

Additionally, since $\|u_j a_k u_j^* - a_{jk}\| < \delta_6 < \delta_1$ for $1 \leq j, k \leq m$, we have

$$\|u_j a_k^{1/2} u_j^* - a_{jk}^{1/2}\| < \delta_0. \tag{VII.7}$$

Note that $\|u_j\phi(t \otimes e_{1,k}) - \phi(t \otimes e_{j,k})\| = \|u_j z_{1,k} - z_{j,l}\| < \delta_5 < \frac{\delta_3}{3\lambda_p}$.

Thus by Equation VII.1 we have $\|u_j\phi(t^{1/2} \otimes e_{1,k}) - \phi(t^{1/2} \otimes e_{j,k})\| < \delta_3$. Therefore, by Equation VII.2,

$$\|u_j\phi(f^{1/2} \otimes e_{1,k}) - \phi(f^{1/2} \otimes e_{j,k})\| < 2\delta_4 + \delta_3. \tag{VII.8}$$

In particular, $\|u_j\phi(f^{1/2} \otimes e_{1,k}) - \phi(f^{1/2} \otimes e_{j,k})\| < \varepsilon/16$.

Additionally,

$$\begin{aligned}\|u_j\phi(g\otimes e_{1,k})-\phi(g\otimes e_{j,k})\| &= \|u_j\phi(f^{1/2}\otimes e_{1,k})\phi(g\otimes e_{k,k})-\phi(f^{1/2}\otimes e_{j,k})\phi(g\otimes e_{k,k})\|\\ &\le \|u_j\phi(f^{1/2}\otimes e_{1,k})-\phi(f^{1/2}\otimes e_{j,k})\|\\ &< \varepsilon/16. \end{aligned} \tag{VII.9}$$

Now

$$\begin{aligned}\|u_jy_{1,1}^2u_j^*-y_{j,j}^2\| &\le \|u_j\phi(f^{1/2}\otimes e_{1,1})u_{m+1}^*a_{m+1}u_{m+1}\phi(f^{1/2}\otimes e_{1,1})u_j^*\\ &\qquad -\phi(f^{1/2}\otimes e_{j,1})u_{m+1}^*a_{m+1}u_{m+1}\phi(f^{1/2}\otimes e_{1,j})\|\\ &\le \|u_j\phi(f^{1/2}\otimes e_{1,1})u_{m+1}^*a_{m+1}u_{m+1}\phi(f^{1/2}\otimes e_{1,1})u_j^*\\ &\qquad -u_j\phi(f^{1/2}\otimes e_{1,1})u_{m+1}^*a_{m+1}u_{m+1}\phi(f^{1/2}\otimes e_{1,j})\|\\ &\quad +\|u_j\phi(f^{1/2}\otimes e_{1,1})u_{m+1}^*a_{m+1}u_{m+1}\phi(f^{1/2}\otimes e_{1,j})\\ &\qquad -\phi(f^{1/2}\otimes e_{j,1})u_{m+1}^*a_{m+1}u_{m+1}\phi(f^{1/2}\otimes e_{1,j})\|\\ &\le \|\phi(f^{1/2}\otimes e_{1,1})u_j^*-\phi(f^{1/2}\otimes e_{1,j})\|\\ &\quad +\|u_j\phi(f^{1/2}\otimes e_{1,1})-\phi(f^{1/2}\otimes e_{j,1})\|\\ &= 2\|u_j\phi(f^{1/2}\otimes e_{1,1})-\phi(f^{1/2}\otimes e_{j,1})\|\\ &< 2\delta_4+\delta_3\\ &< \delta_1. \end{aligned}$$

Therefore, $\|u_jy_{1,1}^2u_j^*-y_{j,j}^2\| = \|(u_jy_{1,1}u_j^*)^2-y_{j,j}^2\| < \delta_1$. Thus

$$\|u_jy_{1,1}u_j^*-y_{j,j}\| < \delta_0. \tag{VII.10}$$

If $j = m+1$, but $l \neq m+1$, and $l \neq 1$, then

$$
\begin{aligned}
\|u_j y_{1,l} - y_{j,l}\| &= \|u_{m+1} y_{1,l} - y_{m+1,l}\| \\
&= \|u_{m+1} y_{1,1} \phi(g \otimes e_{1,l}) - a_{m+1}^{1/2} u_{m+1} s_l^*\| \\
&= \|(s_1 u_{m+1}^* a_{m+1} u_{m+1} s_1)^{1/2} \phi(g \otimes e_{1,l}) - u_{m+1}^* a_{m+1}^{1/2} u_{m+1} s_l^*\| \\
&\leq \|(s_1 u_{m+1}^* a_{m+1} u_{m+1} s_1)^{1/2} \phi(g \otimes e_{1,l}) - a_1 \phi(g \otimes e_{1,l})\| \\
&\quad + \|a_1 \phi(g \otimes e_{1,l}) - a_1^{1/2} s_l^*\| \\
&\quad + \|a_1^{1/2} s_l^* - u_{m+1}^* a_{m+1}^{1/2} u_{m+1} s_l^*\| \\
&\leq \|(s_1 u_{m+1}^* a_{m+1} u_{m+1} s_1)^{1/2} - a_1\| \\
&\quad + \|a_1^{1/2} \phi(g \otimes e_{1,l}) - \phi(f^{1/2} \otimes e_{1,1}) \phi(g \otimes e_{1,l})\| \\
&\quad + \|a_1^{1/2} - u_{m+1}^* a_{m+1}^{1/2} u_{m+1}\| \\
&\leq \|y_{1,1} - a_1\| \\
&\quad + \|a_1^{1/2} - \phi(f^{1/2} \otimes e_{1,1})\| \\
&\quad + \delta_0 \quad \text{by Equation VII.7} \\
&\leq \delta_0 + \delta_1/4 + \delta_0 \quad \text{by Equations VII.5 and VII.6} \\
&< 3\delta_0 \\
&< \frac{3\varepsilon}{192} \\
&< \varepsilon/2.
\end{aligned}
$$

Next suppose that $j = m+1$ and $l = 1$. Then, using Equation VII.5 in the third to last step and Equations VII.6 and VII.7 in the second to last step we see that

$$
\begin{aligned}
\|u_j y_{1,l} - y_{j,l}\| &= \|u_{m+1} y_{1,1} - y_{m+1,1}\| \\
&= \|y_{1,1} - u_{m+1}^* a_{m+1}^{1/2} u_{m+1} s_1\| \\
&\leq \|y_{1,1} - a_1\| + \|a_1 - a_1^{1/2} s_1\| + \|a_1^{1/2} s_1 - u_{m+1}^* a_{m+1}^{1/2} u_{m+1} s_1\| \\
&< \delta_0 + \|a_1^{1/2} - \phi(f^{1/2} \otimes e_{1,1})\| + \|a_1^{1/2} - u_{m+1}^* a_{m+1}^{1/2} u_{m+1}\| \\
&\leq \delta_0 + 2\delta_0 + \delta_0 \\
&< \varepsilon/2.
\end{aligned}
$$

Now let $j = l = m+1$. We use $\|u_{m+1} a_1 u_{m+1}^* - a_{m+1}\| < \delta_5 < \delta_1/2$, the estimate $\|\phi(t \otimes e_{1,1}) - a_1\| < \delta_5$, and $\|t - f(t)\| < \delta_4$ for the third to last step to get

$$
\begin{aligned}
\|u_j y_{1,l} - y_{j,l}\| &= \|u_{m+1} y_{1,m+1} - y_{m+1,m+1}\| \\
&= \left\| u_{m+1} \phi(f^{1/2} \otimes e_{1,1}) u_{m+1}^* a_{m+1}^{1/2} - \left(a_{m+1}^{1/2} u_{m+1} \phi(f \otimes e_{1,1}) u_{m+1}^* a_{m+1}^{1/2} \right)^{1/2} \right\| \\
&\leq \|u_{m+1} \phi(f^{1/2} \otimes e_{1,1}) u_{m+1}^* a_{m+1}^{1/2} - u_{m+1} a_1^{1/2} u_{m+1}^* a_{m+1}^{1/2}\| \\
&\quad + \|u_{m+1} a_1^{1/2} u_{m+1}^* a_{m+1}^{1/2} - (a_{m+1}^{1/2} u_{m+1} \phi(f \otimes e_{1,1}) u_{m+1}^* a_{m+1}^{1/2})^{1/2}\| \\
&\leq \delta_0 + \|u_{m+1} a_1^{1/2} u_{m+1}^* a_{m+1}^{1/2} - a_{m+1}\| \\
&\quad + \|a_{m+1} - (a_{m+1}^{1/2} u_{m+1} a_1 u_{m+1}^* a_{m+1}^{1/2})^{1/2}\| \\
&\quad + \|(a_{m+1}^{1/2} u_{m+1} a_1 u_{m+1}^* a_{m+1}^{1/2})^{1/2} - (a_{m+1}^{1/2} u_{m+1} \phi(f \otimes e_{1,1}) u_{m+1}^* a_{m+1}^{1/2})^{1/2}\| \\
&\qquad\qquad \text{by Equation VII.6} \\
&\leq \|(a_{m+1}^{1/2} u_{m+1} a_1 u_{m+1}^* a_{m+1}^{1/2})^{1/2} - (a_{m+1}^{1/2} u_{m+1} \phi(t \otimes e_{1,1}) u_{m+1}^* a_{m+1}^{1/2})^{1/2}\| \\
&+ \|(a_{m+1}^{1/2} u_{m+1} \phi(t \otimes e_{1,1}) u_{m+1}^* a_{m+1}^{1/2})^{1/2} - (a_{m+1}^{1/2} u_{m+1} \phi(f \otimes e_{1,1}) u_{m+1}^* a_{m+1}^{1/2})^{1/2}\| \\
&+ \delta_0 + 2\delta_0 \\
&< \delta_1/8 + \delta_1/8 + 3\delta_0 \\
&< 5\delta_0 \\
&< \varepsilon/2.
\end{aligned}
$$

Now suppose $1 < j \leq m$ and $j = l$. In this situation,

$$\begin{aligned}
\|u_j y_{1,l} - y_{j,l}\| &= \|u_j y_{1,j} - y_{j,j}\| \\
&= \|u_j y_{1,1}\phi(g \otimes e_{1,j}) - y_{j,j}\| \\
&= \|u_j y_{1,1}u_j^* u_j \phi(g \otimes e_{1,j}) - y_{j,j}\| \\
&\leq \|u_j y_{1,1}u_j^* u_j \phi(g \otimes e_{1,j}) - y_{j,j}u_j\phi(g \otimes e_{1,j})\| \\
&\quad + \|y_{j,j}u_j\phi(g \otimes e_{1,j}) - y_{j,j}\phi(g \otimes e_{j,j})\| \\
&\quad + \|y_{j,j}\phi(g \otimes e_{j,j}) - y_{j,j}\| \\
&\leq \|u_j y_{1,1}u_j^* - y_{j,j}\| + \|u_j\phi(g \otimes e_{1,j}) - \phi(g \otimes e_{j,j})\| \\
&\quad + 0 \ \text{ by Equation VII.4} \\
&\leq \delta_0 + \varepsilon/16 \ \text{ by Equations VII.10 and VII.9} \\
&< \varepsilon/2.
\end{aligned}$$

Now suppose $1 < j \leq m$ and $1 \leq l \leq m$ with $l \neq j$. Then,

$$\begin{aligned}
\|u_j y_{1,l} - y_{j,l}\| &= \|u_j y_{1,1}\phi(g \otimes e_{1,l}) - y_{j,j}\phi(g \otimes e_{j,l})\| \\
&= \|u_j y_{1,1}u_j^* u_j\phi(g \otimes e_{1,l}) - y_{j,j}\phi(g \otimes e_{j,1})\| \\
&\leq \|u_j y_{1,1}u_j^* u_j\phi(g \otimes e_{1,l}) - u_j y_{1,1}u_j^*\phi(g \otimes e_{j,l})\| \\
&\quad + \|u_j y_{1,1}u_j^*\phi(g \otimes e_{j,l}) - y_{j,j}\phi(g \otimes e_{j,l})\| \\
&< \varepsilon/16 + \delta_0 \ \text{ by Equations VII.9 and VII.10} \\
&< \varepsilon/16 + \varepsilon/192 \\
&< \varepsilon/2.
\end{aligned}$$

Finally, suppose $1 < j \leq m$ and $l = m+1$. Then

$$\begin{aligned}\|u_j y_{1,m+1} - y_{j,m+1}\| &= \|u_j\phi(f^{1/2}\otimes e_{1,1})u_{m+1}^* a_{m+1}^{1/2} - \phi(f^{1/2}\otimes e_{j,1})u_{m+1}^* a_{m+1}^{1/2}\| \\ &\leq \|u_j\phi(f^{1/2}\otimes e_{1,1}) - \phi(f^{1/2}\otimes e_{j,1})\| \\ &< \varepsilon/16 \text{ by Equation } VII.8 \\ &< \varepsilon/2.\end{aligned}$$

Since we do not need to consider $j=1$ because $u_j = 1$, this shows (5) and hence (6) hold.

For (3), we use Equation VII.5, namely that $\|y_{j,j} - a_j\| < \delta_0 < \varepsilon/192$. Combining this with Lemma VI.7 we see that $\|y_{j,j}\| > 1 - \delta_6 - \delta_0 \geq 1 - \varepsilon/48 - \varepsilon/192 > 1/2$ by our assumption that $\varepsilon < 1$.

To check (7), we compute $\|y_{j,j} - a_j\|$ using Equation VII.10

$$\begin{aligned}\|y_{j,j} - a_j\| &\leq \|y_{j,j} - u_j y_{1,1} u_j^*\| + \|u_j y_{1,1} u_j^* - u_j a_1 u_j^*\| + \|u_j a_1 u_j^* - a_j\| \\ &\leq \delta_1/8 + \delta_5 + \delta_6 \\ &< \varepsilon/192 + \varepsilon/8 + \varepsilon/48 \\ &< \varepsilon.\end{aligned}$$

Next we check (8). Since $s_1 \in A$ and $u_{m+1}^* a_{m+1} u_{m+1} = \alpha_{m+1}^{-1}(a_{m+1}) \in A$, it is clear that $y_{1,1} \in A$.

Now we verify (9). For any $b \in F$ we have

$$\|y_{j,j}b - by_{j,j}\| \leq \|y_{j,j}b - a_j b\| + \|a_j b - ba_j\| + \|ba_j - by_{j,j}\| < \varepsilon\|b\| + \delta_5 + \varepsilon\|b\| < 2\varepsilon\|b\| + \varepsilon.$$

For (4), we first recall that $0 \leq a_g \leq 1$ for all $g \in G$ by the definition of the projection free tracial Rokhlin property. Thus, $0 \leq s_j u_{m+1}^* a_{m+1} u_{m+1} s_j^* \leq s_j s_j^* = z_{j,j}$. The induction hypothesis that $0 \leq z_{j,j} \leq 1$ now gives us $0 \leq y_{j,j}^2 \leq 1$ which implies $0 \leq y_{j,j} \leq 1$ for $1 \leq j \leq m$. A similar argument shows that $0 \leq y_{m+1,m+1} \leq 1$.

Finally, we check the extra hypothesis for inducting, namely (15). Let $1 \leq j,k \leq m$ and $m+1 < l \leq n$. By the induction hypothesis, $0 = z_{j,k}a_l = \phi(t \otimes e_{j,k})a_l$, and the same on the other side. Thus we also have $\phi(f^{1/2} \otimes e_{j,k})a_l = 0$ and $\phi(g \otimes e_{j,k})a_l = 0$. This implies $y_{m+1,j}a_l = 0$, and that $a_l y_{j,m+1} = 0$. We also have $a_l y_{j,j} = a_l y_{j,j} = 0$ and thus $y_{j,k}a_l = a_l y_{j,k} = 0$.

Since a_l is orthogonal to a_h for every other group element h, we also have $y_{m+1,m+1}a_l = a_l y_{m+1,m+1} = 0$. Similarly $y_{j,m+1}a_l = 0$ and $a_l y_{m+1,j} = 0$. This completes the induction step.

For the statement of the theorem, let $Y_{j,k}$ be given by the $y_{j,k}$ constructed when $m+1 = n$, where $n = |G|$ and let a_j by the elements of A given by the projection free tracial Rokhlin property in that same step. ∎

The following lemma is the projection free analog of Lemma II.14 which is a finite group analog of Lemma 2.5 of [16]. It finds an isomorphic copy of matrices over a hereditary subalgebra of A as a large subalgebra of the crossed product. This is useful because we wish to show the entire crossed product has stable rank one and such a subalgebra has stable rank one.

Lemma VII.4. *Let A be an infinite dimensional stably finite simple unital C^*-algebra. Let G be a finite group; let $n = \mathrm{card}(G)$. Let $\alpha : G \to \mathrm{Aut}(A)$ be an action with the projection free tracial Rokhlin property. Let $\iota : A \to C^*(G, A, \alpha)$ be the standard inclusion, write $B = C^*(G, A, \alpha)$, and let $u_g \in B$ be the standard unitary implementing α_g. Then for every finite set $F \subset B$, every $\varepsilon > 0$, and every natural number N, there exists a positive element $c^{(1)} \in B$, a subalgebra $D \subset \overline{c^{(1)}Bc^{(1)}}$, a positive element $c^{(1)}_{1,1} \in A$, an isomorphism $\Phi : M_n \otimes \overline{c^{(1)}_{1,1}AC^{(1)}_{1,1}} \to D$ and elements $c^{(1)}_{g,h}$ for each g and h in G such that: With $\{e_{g,h}\}$ being matrix units for M_n and $S \subset A$ a finite set such that each element of F can be expressed as $\sum_{g \in G} b_g u_g$ with coefficients b_g in S, we have*

1. *For any $d \in \overline{c^{(1)}_{1,1}Ac^{(1)}_{1,1}}$ we have $\Phi(e_{1,1} \otimes d) = d$, and for any $s \in S$ there are elements $d_g \in c^{(1)}_{1,1}Ac^{(1)}_{1,1}$ such that $\Phi(e_{g,g} \otimes d_g) = c^{(1)}_{g,1}sc^{(1)}_{1,g}$ and $\mathrm{dist}(c^{(1)}_{g,1}sc^{(1)}_{1,g}, A) < \varepsilon$.*
2. *$\|\Phi(e_{g,g} \otimes d) - u_g d u_g^*\| < \varepsilon\|d\|$ for all $d \in \overline{c^{(1)}_{1,1}Ac^{(1)}_{1,1}}$.*
3. *For all $x \in F$, there is a $y \in D$ such that $\|c^{(1)}xc^{(1)} - y\| < \varepsilon$ and $\|y\| \leq \|x\|$.*
4. *$\sum_{g \in G} \Phi(e_{g,g} \otimes c^{(1)}_{1,1}) = c^{(1)}$.*
5. *$\|c^{(1)}x - xc^{(1)}\| < \varepsilon$ for every $x \in F$.*
6. *$\tau(1 - c^{(1)}) < 1/N$ for all $\tau \in T(B)$.*

Proof. Let F, ε, and N be given. Without loss of generality, $\|x\| \leq 1$ for all $x \in F$ and $\|y\| \leq 1$ for all $y \in S$. We can always rescale to achieve this.

First we observe that we do not need to prove the norm condition in (3) above. Suppose we have proven the above lemma without the norm condition in (3) and that y is an element resulting from applying the lemma with $\varepsilon/2$ in place of ε so that $\|cxc - y\| < \varepsilon/2$. Notice that this means $\|y\| \leq 1+\varepsilon/2$. Set $y_1 = (\frac{1}{1+\varepsilon/2})y$. Then $\|y - y_1\| = \|y - (\frac{1}{1+\varepsilon/2})y\| \leq \frac{\varepsilon/2}{1+\varepsilon/2}(1+\varepsilon/2) = \varepsilon/2$. Therefore, $\|cxc - y_1\| < \varepsilon$.

Let $0 < \varepsilon_0 < \min\{\varepsilon/(40n^2), \varepsilon/(12)\}$. Define continuous functions f_0 and f_1 on $[0,1]$ as follows:

$f_0(0) = 0$,

$f_0(t) = 1$ for t in $[1-\varepsilon_0, 1]$, and

f_0 is linear on $(0, 1-\varepsilon_0)$.

$f_1(t) = 0$ for t in $[0, 1-\varepsilon_0]$,

$f_1(t) = 1$ for t in $[1-\varepsilon_0/2, 1]$, and

f_1 is linear on $(1-\varepsilon_0, 1-\varepsilon_0/2)$.

Let $0 < \varepsilon_1 < \min\{\varepsilon/(8n^2), \varepsilon/(12)\}$.

Apply Lemma VI.11 to f_1 with ε_1 in place of ε to get δ_1. Apply Lemma VI.12 to f_1 with ε_1 in place of ε to get δ_2.

Let

$$0 < \varepsilon_2 < \min\left\{\frac{\varepsilon}{28n^2}, \frac{\varepsilon}{12}, \frac{\delta_1}{4}, \frac{\delta_2}{5n^2}, \frac{1}{2nN}\right\}.$$

Let δ_3 be the value of δ given by applying Lemma VI.14 with $\min\{\frac{\varepsilon_0}{2}, n\varepsilon_2 + \frac{2}{N}\}$ in place of ε. We also require $\delta_3 < \frac{1}{N}$.

Let $\{e_{g,h}\}$ for $g, h \in G$ be a system of matrix units for M_n. Let t represent the function $f(t) = t$. Notice that $\{t \otimes e_{g,h}\}_{g,h\in G}$ generate CM_n.

Apply Lemma VII.3 with S in place of F, with 1 in place of x and with ε_2 in place of ε. This provides us with $\delta > 0$, $a_g \in A$ for $g \in G$ and $Y_{g,h} \in B$ for $g, h \in G$ satisfying the conclusions of that lemma. Thus we can define a homomorphism, $\varphi_0 : CM_n \to B$ given by $\varphi_0(t \otimes e_{g,h}) = Y_{g,h}$. We also require $\delta < \frac{1}{2N}$.

Let $c^{(0)}_{g,h} = \varphi_0(f_0 \otimes e_{g,h})$. Similarly define $c^{(1)}_{g,h} = \varphi_0(f_1 \otimes e_{g,h})$. Also set $c^{(0)} = \sum_{g\in G} c^{(0)}_{g,g}$ and similarly $c^{(1)} = \sum_{g\in G} c^{(1)}_{g,g}$. Notice that since $Y_{1,1} \in A$, we also have $c^{(0)}_{1,1} \in A$ and $c^{(1)}_{1,1} \in A$.

Notice

$$\begin{aligned}
c^{(0)}_{g_1,h_1}c^{(1)}_{g_2,h_2} &= \varphi_0(f_0f_1\otimes e_{g_1,h_1}e_{g_2,h_2})\\
&= \varphi_0(f_1\otimes \delta_{h_1,g_2}e_{g_1,h_2})\\
&= \delta_{h_1,g_2}c^{(1)}_{g_1,h_2}.
\end{aligned}$$

Similarly, $c^{(1)}_{g_2,h_2}c^{(0)}_{g_1,h_1} = \delta_{h_2,g_1}c^{(1)}_{g_2,h_1}$.

Define a function $\Phi : M_n(\overline{c^{(1)}_{1,1}Ac^{(1)}_{1,1}}) \to B$ by $\Phi((x_{g,h})) = \sum_{g,h} c^{(0)}_{g,1}x_{g,h}c^{(0)}_{1,h}$ for $x_{g,h} \in \overline{c^{(1)}_{1,1}Ac^{(1)}_{1,1}}$. Set $D = \mathrm{Im}(\Phi)$.

Next we check that Φ is a homomorphism. It is easy to check that Φ is additive and is star preserving. We will check that it is multiplicative.

Let $x = (x_{g,h})$ and $y = (y_{g,h})$ and note that $(xy)_{g,h} = \sum_{k\in G} x_{g,k}y_{k,h}$. Then, using the facts that $x_{g,h}$ and $y_{k,l}$ are in $\overline{c^{(1)}_{1,1}Ac^{(1)}_{1,1}}$ and that $c^{(1)}_{1,1}c^{(0)}_{1,1} = c^{(1)}_{1,1}$, we get:

$$\begin{aligned}
\Phi(x)\Phi(y) &= \left[\sum_{g,h\in G} c^{(0)}_{g,1}x_{g,h}c^{(0)}_{1,h}\right]\left[\sum_{k,l\in G} c^{(0)}_{k,1}x_{k,l}c^{(0)}_{1,l}\right]\\
&= \sum_{g,h,l\in G} c^{(0)}_{g,1}x_{g,h}(c^{(0)}_{1,1})^2 y_{h,l}c^{(0)}_{1,l}\\
&= \sum_{g,h,l\in G} c^{(0)}_{g,1}x_{g,h}y_{h,l}c^{(0)}_{1,l}\\
&= \sum_{g,l\in G} c^{(0)}_{g,1}\left(\sum_{h\in G} x_{g,h}y_{h,l}\right)c^{(0)}_{1,l}\\
&= \Phi(xy).
\end{aligned}$$

Furthermore, Φ is injective. To see this, since A is simple implies $\overline{c^{(1)}_{1,1}Ac^{(1)}_{1,1}}$ is simple by Theorem 3.2.8 of [14] it is enough to show that Φ is nonzero. Now notice,
$\Phi(c^{(1)}_{1,1}) = c^{(0)}_{1,1}c^{(1)}_{1,1}c^{(0)}_{1,1} = c^{(1)}_{1,1} \neq 0$.

Next we make some norm estimates to be used later on.

Note we have $\varphi_0(f_1\otimes e_{g,g}) = \varphi_0(f_1(t\otimes e_{g,g})) = f_1(\varphi_0(t\otimes e_{g,g}))$, with the first equality by Lemma VII.1 and the second because functional calculus commutes with homomorphisms. Also note that $\|c^{(0)}_{h,k} - Y_{h,k}\| \leq \|f_0 - t\| < \varepsilon_0$.

Next we estimate the affect of conjugating $c^{(1)}_{h,h}$ by u_g. Since

$$\|u_g Y_{h,h} u_g^* - Y_{gh,gh}\| < 2\varepsilon_2 < \delta_1,$$

using the choice of δ_1 using Lemma VI.11 for the last step, we have

$$\begin{aligned}
\|u_g c^{(1)}_{h,h} u_g^* - c^{(1)}_{gh,gh}\| &= \|u_g \varphi_0(f_1 \otimes e_{h,h}) u_g^* - \varphi_0(f_1 \otimes e_{gh,gh})\| \\
&= \|u_g (f_1(\varphi_0(t \otimes e_{h,h}))) u_g^* - f_1(\varphi_0(t \otimes e_{gh,gh}))\| \\
&= \|f_1(u_g(\varphi_0(t \otimes e_{h,h})) u_g^*) - f_1(\varphi_0(t \otimes e_{gh,gh}))\| \\
&< \varepsilon_1. \qquad \text{(VII.11)}
\end{aligned}$$

Now we compute,

$$\begin{aligned}
\|u_g c^{(0)}_{h,k} - c^{(0)}_{gh,k}\| &\le \|u_g c^{(0)}_{h,k} - u_g Y_{h,k}\| + \|u_g Y_{h,k} - Y_{gh,k}\| + \|Y_{gh,k} - c_{gh,k}\| \\
&\le 2\varepsilon_0 + \varepsilon_2. \qquad \text{(VII.12)}
\end{aligned}$$

Next we compute the similar quantity using $c^{(1)}_{h,k}$:

$$\begin{aligned}
\|u_g c^{(1)}_{h,k} - c^{(1)}_{gh,k}\| &= \|u_g c^{(1)}_{h,h} c^{(0)}_{h,k} - c^{(1)}_{gh,gh} c^{(0)}_{gh,k}\| \\
&\le \|u_g c^{(1)}_{h,h} u_g^* u_g c^{(0)}_{h,k} - c^{(1)}_{gh,gh} u_g c^{(0)}_{h,k}\| + \|c^{(1)}_{gh,gh} u_g c^{(0)}_{h,k} - c^{(1)}_{gh,gh} c^{(0)}_{gh,k}\| \\
&\le \varepsilon_1 + 2\varepsilon_0 + \varepsilon_2, \qquad \text{(VII.13)}
\end{aligned}$$

using Equations VII.11 and VII.12 for the last inequality.

Let $s \in S$ and recall that we have normalized so that $\|s\| \le 1$ for all $s \in S$. We have

$$\begin{aligned}
\|[Y_{g,g}, s]\| &\le \|Y_{g,g} s - a_g s\| + \|a_g s - s a_g\| + \|s a_g - s Y_{g,g}\| \\
&\le 2\varepsilon_2 + \delta_3 \\
&\le \delta_2.
\end{aligned}$$

Using the preceding estimate and the choice of δ_2 using Lemma VII.1 we now get

$$\begin{aligned}\left\|\left[c^{(1)}_{g,g}, s\right]\right\| &= \|[\varphi_0(f_1(t)\otimes e_{g,g}), s]\| \\ &\leq \|[f_1(\varphi_0(t\otimes e_{g,g})), s]\| \\ &< \varepsilon_1. \end{aligned} \tag{VII.14}$$

Let $y \in B$ and $g, h, k, l \in G$. Then we observe

$$\begin{aligned}\left\|c^{(0)}_{g,h} y c^{(0)}_{k,l} - Y_{g,h} y Y_{k,l}\right\| &\leq 2\|y\|\,\|t - f_0\| \\ &\leq 2\|y\|\,\varepsilon_0. \end{aligned} \tag{VII.15}$$

Let $y \in B$. Then

$$\begin{aligned}\left\|c^{(0)}_{h,1} y c^{(0)}_{1,g^{-1}h} - c^{(0)}_{h,h} u_h y u_h^* c^{(0)}_{h,h} u_g\right\| &\leq \left\|c^{(0)}_{h,1} y c^{(0)}_{1,g^{-1}h} - c^{(0)}_{h,h} u_h y c^{(0)}_{1,g^{-1}h}\right\| \\ &\quad + \left\|c^{(0)}_{h,h} u_h y c^{(0)}_{1,g^{-1}h} - c^{(0)}_{h,h} u_h y u_h^* c^{(0)}_{h,h} u_g\right\| \\ &\leq \|y\|\left\|c^{(0)}_{h,1} - c^{(0)}_{h,h} u_h\right\| + \|y\|\left\|c^{(0)}_{1,g^{-1}h} - u_{h^{-1}} c^{(0)}_{h,g^{-1}h}\right\| \\ &\quad + \|y\|\left\|u_{h^{-1}} c^{(0)}_{h,g^{-1}h} - u_{h^{-1}} c^{(0)}_{h,h} u^*_{g^{-1}}\right\| \\ &< 3\|y\|(2\varepsilon_0 + \varepsilon_2) \end{aligned} \tag{VII.16}$$

by Equation VII.12.

Now let $\{y_{g,h}\} \subset B$ for $g, h \in G$. Then,

$$\begin{aligned}&\left\|\sum_{g,h} c^{(0)}_{h,1} y_{h,g^{-1}h} c^{(0)}_{1,g^{-1}h} - \sum_{g,h} c^{(0)}_{h,h} u_h y_{h,g^{-1}h} u_h^* c^{(0)}_{h,h} u_g\right\| \\ &\leq \sum_{g,h}\left\|c^{(0)}_{h,1} y_{h,g^{-1}h} c^{(0)}_{1,g^{-1}h} - c^{(0)}_{h,h} u_h y_{h,g^{-1}h} u_h^* c^{(0)}_{h,h} u_g\right\| \\ &\leq 3n^2 \max_{g,h}\left\|y_{h,g^{-1}h}\right\|(2\varepsilon_0 + \varepsilon_2) \end{aligned} \tag{VII.17}$$

Now given $x \in F$, we can write $x = \sum_{g\in G} x_g u_g$ with $x_g \in S$.

Set $z_{h,g} = u_h^*(c^{(1)}_{h,h} x_{hg^{-1}} c^{(1)}_{h,h}) u_h$ and $y_{h,g} = c^{(1)}_{1,1}\alpha_{h^{-1}}(x_{hg^{-1}}) c^{(1)}_{1,1}$. Note that $y_{h,g} \in \overline{c^{(1)}_{1,1} A c^{(1)}_{1,1}}$. Then, using Equation VII.11 and the fact that $\|x_g\| \leq 1$ we compute:

$$\begin{aligned}\|z_{h,g^{-1}h} - y_{h,g^{-1}h}\| &= \|u_h^* c_{h,h}^{(1)} x_g c_{h,h}^{(1)} u_h - c_{1,1}^{(1)} \alpha_{h^{-1}}(x_g) c_{1,1}^{(1)}\| \\ &= \|u_h^* c_{h,h}^{(1)} u_h u_h^* x_g u_h u_h^* c_{h,h}^{(1)} u_h - c_{1,1}^{(1)} \alpha_{h^{-1}}(x_g) c_{1,1}^{(1)}\| \\ &\leq 2\varepsilon_1. \end{aligned} \tag{VII.18}$$

Next we estimate the effect of $c^{(1)}$ on x. In the third step we used Equation VII.11. For the second to last step we used Equation VII.14 and the fact that $c_{h,h}^{(1)} c_{g,g}^{(1)} = 0$ unless $g = h$. For the last step we used the fact that $\|x_g\| \leq 1$, since $x_g \in S$ to compute:

$$\begin{aligned}\left\| c^{(1)} x c^{(1)} - \sum_{g,h\in G} c_{h,h}^{(0)} c_{h,h}^{(1)} x_g c_{h,h}^{(1)} c_{h,h}^{(0)} u_g \right\| &= \left\| \sum_{h,g,k\in G} c_{h,h}^{(1)} x_g u_g c_{k,k}^{(1)} - \sum_{g,h\in G} c_{h,h}^{(1)} x_g c_{h,h}^{(1)} u_g \right\| \\ &\leq \left\| \sum_{h,g,k\in G} c_{h,h}^{(1)} x_g u_g c_{k,k}^{(1)} - \sum_{g,h,k\in G} c_{h,h}^{(1)} x_g c_{gk,gk}^{(1)} u_g \right\| \\ &\quad + \left\| \sum_{g,h,k\in G} c_{h,h}^{(1)} x_g c_{gk,gk}^{(1)} u_g - \sum_{g,h} c_{h,h}^{(1)} x_g c_{h,h}^{(1)} u_g \right\| \\ &\leq n^2 \max_{g\in G} \|x_g\| \, \varepsilon_1 \\ &\quad + \left\| \sum_{g,h,k\in G} c_{h,h}^{(1)} x_g c_{gk,gk}^{(1)} u_g - \sum_{g,h,k\in G} c_{h,h}^{(1)} c_{gk,gk}^{(1)} x_g u_g \right\| \\ &\quad + \left\| \sum_{g,h,k\in G} c_{h,h}^{(1)} c_{gk,gk}^{(1)} x_g u_g - \sum_{h,g\in G} (c_{h,h}^{(1)})^2 x_g u_g \right\| \\ &\quad + \left\| \sum_{h,g\in G} (c_{h,h}^{(1)})^2 x_g u_g - \sum_{h,g\in G} c_{h,h}^{(1)} x_g c_{h,h}^{(1)} u_g \right\| \\ &\leq n^2 \max_{g\in G} \|x_g\| \, \varepsilon_1 + n^2 \max_{g\in G} \|x_g\| \, \varepsilon_1 + 0 + n^2 \max_{g\in G} \|x_g\| \, \varepsilon_1 \\ &\leq 3n^2 \varepsilon_1. \end{aligned} \tag{VII.19}$$

We are now in a position to prove part (3) of the statement. Note that $\Phi((y_{h,g^{-1}h})) = \sum_{g,h\in G} c_{h,1}^{(0)} y_{h,g^{-1}h} c_{1,g^{-1}h}^{(0)} \in D$ and so

$$
\begin{aligned}
\left\|\Phi((y_{h,g^{-1}h})) - c^{(1)}xc^{(1)}\right\| &= \left\| \sum_{g,h\in G} c^{(0)}_{h,1} y_{h,g^{-1}h} c^{(0)}_{1,g^{-1}h} - c^{(1)}xc^{(1)} \right\| \\
&\leq \left\| \sum_{g,h\in G} c^{(0)}_{h,1} y_{h,g^{-1}h} c_{1,g^{-1}h} - \sum_{g,h\in G} c^{(0)}_{h,h} u_h y_{h,g^{-1}h} u_h^* c^{(0)}_{h,h} u_g \right\| \\
&\quad + \left\| \sum_{g,h\in G} c^{(0)}_{h,h} u_h y_{h,g^{-1}h} u_h^* c^{(0)}_{h,h} u_g - \sum_{g,h\in G} c^{(0)}_{h,h} u_h z_{h,g^{-1}h} u_h^* c^{(0)}_{h,h} u_g \right\| \\
&\quad + \left\| \sum_{g,h} c^{(0)}_{h,h} c^{(1)}_{h,h} x_g c^{(1)}_{h,h} c^{(0)}_{h,h} u_g - c^{(1)}xc^{(1)} \right\| \\
&\leq 3n^2 \max_{g,h\in G} \|y_{h,g^{-1}h}\| (2\varepsilon_0 + \varepsilon_2) \\
&\quad + n^2 2\varepsilon_0 \\
&\quad + 3n^2 \varepsilon_1 \ \text{ by Equations VII.17, VII.18, and VII.19} \\
&= n^2(8\varepsilon_0 + 3\varepsilon_1 + 3\varepsilon_2) \\
&< \varepsilon/5 + 3\varepsilon/8 + \varepsilon/7 \\
&< \varepsilon.
\end{aligned}
$$

This proves part (3) of the statement with y taken to be $\Phi((y_{h,g^{-1}h}))$.

For part (1) of the conclusion, suppose $d \in \overline{c^{(1)}_{1,1} A c^{(1)}_{1,1}}$. Then

$$
\Phi(e_{1,1} \otimes d) = c^{(0)}_{1,1} d c^{(0)}_{1,1} = \lim_{n\to\infty} c^{(0)}_{1,1} (c^{(1)}_{1,1})^{1/n} d (c^{(1)}_{1,1})^{1/n} c^{(0)}_{1,1} = \lim_{n\to\infty} (c^{(1)}_{1,1})^{1/n} d (c^{(1)}_{1,1})^{1/n} = d.
$$

This is the first half of (1).

For the second part of (1), let $s \in S$. Recall that we have normalized so that $\|s\| \leq 1$. Let $d = c^{(1)}_{1,1} s c^{(1)}_{1,1} \in c^{(1)}_{1,1} A c^{(1)}_{1,1}$. Then

$$
\Phi(e_{g,g} \otimes d) = c^{(0)}_{g,1} d c^{(0)}_{1,g} = c^{(0)}_{g,1} c^{(1)}_{1,1} s c^{(1)}_{1,1} c^{(0)}_{1,g} = c^{(1)}_{g,1} s c^{(1)}_{1,g}.
$$

Furthermore,

$$\begin{aligned}\operatorname{dist}(c_{g,1}^{(1)}sc_{1,g}^{(1)}, A) &\leq \|c_{g,1}^{(1)}sc_{1,g}^{(1)} - u_g c_{1,1}^{(1)}sc_{1,1}^{(1)}u_g^*\| \\ &\leq \|c_{g,1}^{(1)} - u_g c_{1,1}^{(1)}\| + \|c_{1,g}^{(1)} - c_{1,1}u_g^*\| \\ &\leq 2(\varepsilon_1 + 2\varepsilon_0 + \varepsilon_2) \quad \text{by Equation VII.13} \\ &\leq 2(\varepsilon/6 + \varepsilon/6 + \varepsilon/12) \\ &< \varepsilon.\end{aligned}$$

This completes (1).

To prove (2), let $d \in \overline{c_{1,1}^{(1)}Ac_{1,1}^{(1)}}$. Because $(c_{1,1}^{(1)})^{1/m}$ acts as an approximate identity on this algebra, we have $d = \lim_{m\to\infty}(c_{1,1}^{(1)})^{1/m}d(c_{1,1}^{(1)})^{1/m}$. We compute:

$$\begin{aligned}\|u_g d u_g^* - \Phi(e_{g,g}\otimes d)\| &= \left\|\lim_{m\to\infty} u_g(c_{1,1}^{(1)})^{1/m}d(c_{1,1}^{(1)})^{1/m}u_g^* - c_{g,1}^{(0)}dc_{1,g}^{(0)}\right\| \\ &= \left\|\lim_{m\to\infty} u_g c_{1,1}^{(0)}(c_{1,1}^{(1)})^{1/m}d(c_{1,1}^{(1)})^{1/m}c_{1,1}^{(0)}u_g^* - c_{g,1}^{(0)}dc_{1,g}^{(0)}\right\| \\ &= \left\|u_g c_{1,1}^{(0)}dc_{1,1}^{(0)}u_g^* - c_{g,1}^{(0)}dc_{1,g}^{(0)}\right\| \\ &\leq \left\|u_g c_{1,1}^{(0)}dc_{1,1}^{(0)}u_g^* - c_{g,1}^{(0)}dc_{1,1}^{(0)}u_g^*\right\| + \left\|c_{g,1}^{(0)}dc_{1,1}^{(0)}u_g^* - c_{g,1}^{(0)}dc_{1,g}^{(0)}\right\| \\ &\leq \left\|u_g c_{1,1}^{(0)} - c_{g,1}^{(0)}\right\| \|d\| + \left\|c_{1,1}^{(0)}u_g^* - c_{1,g}^{(0)}\right\| \|d\| \\ &\leq 2(2\varepsilon_0 + \varepsilon_2)\|d\| \quad \text{using Equation VII.12} \\ &< 2(\varepsilon/6 + \varepsilon/12)\|d\| \\ &< \varepsilon\|d\|.\end{aligned}$$

This is condition (2) of the lemma.

For (4) we compute

$$\sum_{g\in G}\Phi(e_{g,g}\otimes c_{1,1}^{(1)}) = \sum_{g\in G} c_{g,1}^{(0)}c_{1,1}^{(1)}c_{1,g}^{(0)} = \sum_{g\in G} c_{g,g}^{(1)} = c^{(1)}.$$

For (5) we begin by computing for any $x = \sum_{h\in G} x_h u_h \in F$ how close x and $\sum_{g\in G} Y_{g,g}$ are to commuting. We have

$$\begin{aligned}
&\left\| \left(\sum_{g\in G} Y_{g,g}\right)\left(\sum_{h\in G} x_h u_h\right) - \left(\sum_{h\in G} x_h u_h\right)\left(\sum_{g\in G} Y_{g,g}\right)\right\| \\
&= \left\| \sum_{g,h\in G} Y_{g,g} x_h u_h - \sum_{g,h\in G} x_h u_h Y_{h^{-1}g,h^{-1}g}\right\| \\
&\leq \sum_{g,h\in G} \left\|Y_{g,g} x_h u_h - x_h u_h Y_{h^{-1}g,h^{-1}g}\right\| \\
&\leq \sum_{g,h\in G} \|Y_{g,g} x_h u_h - x_h Y_{g,g} u_h\| + \sum_{g,h\in G} \left\|x_h u_h u_h^* Y_{g,g} u_h - x_h u_h Y_{h^{-1}g,h^{-1}g}\right\| \\
&\leq 3n^2\varepsilon_2 + 2n^2\varepsilon_2 \\
&< 5n^2\delta_2/(5n^2) \\
&< \delta_2.
\end{aligned}$$

By the choice of δ_2, this implies $\|f_1(\sum_{g\in G} Y_{g,g})x - xf_1(\sum_{g\in G} Y_{g,g})\| < \varepsilon_1 < \varepsilon$. But, by Lemma VI.10, we have $\varepsilon > \| \sum_{g\in G} f_1(Y_{g,g})x - x\sum_{g\in G} f_1(Y_{g,g})\| = \|c^{(1)}x - xc^{(1)}\|$ which is (5).

Finally, we will show that (6) holds. We wish to show that $\tau(1 - c^{(1)}) < \frac{1}{N}$ for all $\tau \in T(B)$. However, since $1 - c^{(1)} \in A$ and in light of Lemma VI.15, it suffices to prove the statement for all $\tau \in T(A)$. Now, since $\|a_g - Y_{g,g}\| < \varepsilon_2$, we have $\| \sum_{g\in G} a_g - \sum_{g\in G} Y_{g,g}\| < n\varepsilon_2$. Therefore, $\tau(\sum_{g\in G} a_g) < n\varepsilon_2 + \tau(\sum_{g\in G} Y_{g,g})$. By the assumption on δ from Lemma VII.3 we have $\tau(1 - \sum_{g\in G} a_g) \leq \frac{1}{2N}$.

Combining these facts we have

$$\frac{1}{2N} > 1 - \tau\left(\sum_{g\in G} a_g\right) > 1 - \tau\left(\sum_{g\in G} Y_{g,g}\right) - n\varepsilon_2.$$

This implies

$$\frac{1}{2N} + n\varepsilon_2 > \tau(1 - \sum_{g\in G} Y_{g,g}).$$

Therefore, by the choice of δ_3,

$$\frac{1}{N} > \delta_3 > \tau\left(1 - f_1\left(\sum_{g\in G} Y_{g,g}\right)\right) = \tau\left(1 - c^{(1)}\right)$$

which is (6). ∎

The following lemma is the analog for positive elements of Lemma 3.2 of [16].

Lemma VII.5. *Let A be a C^*-algebra, let $x, y \in A_+$, let τ be a tracial state on A. Let $g : [0,1] \to \mathbb{R}$ be a continuous function. Then $\tau(g(y^{1/2}xy^{1/2})) = \tau(g(x^{1/2}yx^{1/2})$.*

Proof. We first verify the statement for $g(t) = t^n$:

$$\begin{aligned}\tau((y^{1/2}xy^{1/2})^n) &= \tau(y^{1/2}(xy)^{n-1}x^{1/2}(x^{1/2}y^{1/2}))\\ &= \tau((x^{1/2}y^{1/2})y^{1/2}(xy)^{n-1}x^{1/2})\\ &= \tau((x^{1/2}yx^{1/2})^n)\end{aligned}$$

Thus the lemma holds for any polynomial and so, by the continuity of functional calculus, for any continuous function. ∎

Lemma VII.9 is an analog of Lemma 3.3 of [16] for positive elements instead of projections. The next few lemmas are used to prove Lemma VII.9.

Lemma VII.6. *Let $g : [0,1] \to [0,1]$ be a continuous function with $g(1) = 1$. For every $\varepsilon > 0$, there exists $\delta > 0$ such that whenever A is a unital C^*-algebra, τ is a tracial state on A, and x, y are positive elements of A with norm less than or equal to 1 such that $\tau(x) > 1 - \delta$ and $\tau(y^2) > \|\tau|_{\overline{yAy}}\| - \delta$, then $\tau(g(yxy)) > \tau(y^2) - \varepsilon$.*

Proof. Choose $\delta_0 \in (0,1)$ such that $g(t) > 1 - \varepsilon/2$ for all $t \in [1 - \delta_0, 1]$. Choose δ so that $\delta < \frac{\varepsilon\delta_0}{4}$. Let A, τ, x, and y be as in the hypotheses.

We first estimate $\tau(yxy)$. We have $\tau(yxy) + \tau(y(1-x)y) = \tau(y^2)$. By the condition on x and since $y \leq 1$ implies $(1-x)^{1/2}y^2(1-x)^{1/2} \leq 1 - x$, we also have

$$\tau\left((1-x)^{1/2}y^2(1-x)^{1/2}\right) = \tau(y(1-x)y) \leq \tau(1-x) < \delta.$$

Combining these two observations yields

$$\tau(yxy) = \tau(y^2) - \tau(y(1-x)y) > \tau(y^2) - \delta. \tag{VII.20}$$

Now restrict τ to $\overline{yAy}$. Call the restriction $\hat{\tau}$. Extend $\hat{\tau}$ to a trace $\overline{\tau}$ on $\overline{yAy} + \mathbb{C}1_A$ by $\overline{\tau}(1_A) = \|\hat{\tau}\|$. This implies that

$$\|\overline{\tau}\| = \|\hat{\tau}\|. \tag{VII.21}$$

Let μ be the measure on $X = \mathrm{sp}(yxy)$ corresponding to the functional on $C(X)$ defined by $h \mapsto \overline{\tau}(h(yxy))$ with the functional calculus evaluated in $\overline{yAy} + \mathbb{C}1_A$. That is $\int_X h d\mu = \overline{\tau}(h(yxy))$.

With 1 representing the constant function 1, $\int_X 1 d\mu = \overline{\tau}(1_A) = \|\hat{\tau}\|$. Thus the total mass of μ is $\|\hat{\tau}\|$.

Let $E = [1-\delta_0, 1]$. We compute

$$\begin{aligned}
\tau(y^2) - \delta &< \tau(yxy) \text{ by VII.20} \\
&= \int_{[0,1]} t d\mu(t) \text{ by the definition of } \mu \\
&\leq (1-\delta_0)(\mu([0,1] \setminus E)) + \mu(E) \\
&= (1-\delta_0)(\|\hat{\tau}\| - \mu(E)) + \mu(E) \\
&= \|\hat{\tau}\| - \mu(E) - \delta_0\|\hat{\tau}\| + \delta_0\mu(E) + \mu(E) \\
&= (1-\delta_0)\|\hat{\tau}\| + \delta_0\mu(E) \\
&< (1-\delta_0)(\tau(y^2) + \delta) + \delta_0\mu(E) \text{ by hypothesis.}
\end{aligned}$$

This implies

$$\begin{aligned}
\tau(y^2) - \delta - (1-\delta_0)(\tau(y^2)+\delta) &< \delta_0\mu(E) \\
\tau(y^2) - \delta - (\tau(y^2) + \delta - \delta_0\tau(y^2) - \delta_0\delta) &< \delta_0\mu(E) \\
\tau(y^2) - \delta - \tau(y^2) - \delta + \delta_0\tau(y^2) + \delta_0\delta) &< \delta_0\mu(E) \\
-2\delta + \delta_0\tau(y^2) + \delta_0\delta &< \delta_0\mu(E) \\
\frac{-2\delta}{\delta_0} + \tau(y^2) + \delta &< \mu(E) \\
\tau(y^2) - \frac{2\delta}{\delta_0} + \delta &< \mu(E) \\
\tau(y^2) - \frac{2\delta}{\delta_0} &< \mu(E) \\
\tau(y^2) - \frac{2\varepsilon\delta_0/4}{\delta_0} &< \mu(E) \\
\tau(y^2) - \frac{\varepsilon}{2} &< \mu(E).
\end{aligned}$$

Since $g(t) > 1 - \varepsilon/2$ for $t \in E$, by using $\tau(y^2) \leq 1$ for the last inequality, we now get

$$\begin{aligned}
\tau(g(yxy)) &= \int_{[0,1]} g(t)d\mu(t) \\
&\geq (1-\varepsilon/2)\mu(E) \\
&\geq (1-\varepsilon/2)(\tau(y^2) - \varepsilon/2) \\
&= \tau(y^2) - \varepsilon/2 - \tau(y^2)\varepsilon/2 + \varepsilon/4 \\
&\geq \tau(y^2) - \varepsilon.
\end{aligned}$$

This completes the proof. ∎

Lemma VII.7. *Given any $\delta > 0$, there exists an $\eta > 0$ such that whenever A is a unital C^*-algebra and $y \in A$ is a positive element of norm less than or equal to 1, with $\tau(y) > \|\tau|_{\overline{yAy}}\| - \eta$, then $\tau(y^2) > \|\tau|_{\overline{yAy}}\| - \delta$.*

Proof. Apply Lemma VII.6 with ε replaced by $\delta/2$ and with $g(t) = t^2$. Let η be the resulting value of δ. Without loss of generality, $\eta < \delta/2$. Let $y \in A$ be a positive element with $\|y\| \leq 1$ be such that $\tau((y^{1/2})^2) = \tau(y) > \|\tau|_{\overline{yAy}}\| - \eta = \|\tau|_{\overline{y^{1/2}Ay^{1/2}}}\| - \eta$. Then by the choice of η using Lemma VII.6 and letting $x = 1$ and using $y^{1/2}$ in place of y yields

$$\tau\left(g\left(y^{1/2}xy^{1/2}\right)\right) > \tau((y^{1/2})^2) - \delta/2$$
$$\tau(g(y)) > \tau(y) - \delta/2$$
$$\tau(y^2) > \tau(y) - \delta/2$$
$$\tau(y^2) > \|\tau|_{\overline{yAy}}\| - \eta - \delta/2$$
$$\tau(y^2) > \|\tau|_{\overline{yAy}}\| - \delta.$$

This completes the proof. ∎

Lemma VII.8. *Let $g : [0,1] \to [0,1]$ be a continuous function with $g(1) = 1$. For every $\varepsilon > 0$, there exists $\delta > 0$ such that whenever A is a unital C^*-algebra, τ is a tracial state on A, and x, y are positive elements of A with norm less than or equal to 1 satisfying $\tau(x) > 1 - \delta$ and $\tau(y) > \|\tau|_{\overline{yAy}}\| - \delta$, we have $\tau(g(yxy)) > \tau(y^2) - \varepsilon$.*

Proof. Let δ_1 be the δ obtained by applying Lemma VII.6 with ε and g as given. Let δ_2 be the η obtained by applying Lemma VII.7 with δ replaced by δ_1. Let $\delta_3 = \min\{\delta_1, \delta_2\}$. If $\tau(x) > 1 - \delta_3$, then $\tau(x) > 1 - \delta_1$, so the condition on x is satisfied in Lemma VII.6. If

$$\tau(y) > \|\tau|_{\overline{yAy}}\| - \delta_3 > \|\tau|_{\overline{yAy}}\| - \delta_2,$$

then

$$\tau(y^2) > \|\tau|_{\overline{yAy}}\| - \delta_1$$

by the choice of δ_2 using Lemma VII.7. Thus the condition on y in Lemma VII.6 is satisfied and therefore $\tau(g(yxy)) > \tau(y^2) - \varepsilon$. ∎

Lemma VII.9. *Let $g : [0,1] \to [0,1]$ be a continuous function with $g(1) = 1$. For every $\varepsilon > 0$, there exists $\delta > 0$ such that whenever A is a unital C^*-algebra, τ is a tracial state on A, and x, y are positive elements of A with norm less than or equal to 1 satisfying $\tau(x) > 1 - \delta$ and $\tau(y) > \|\tau|_{\overline{yAy}}\| - \delta$, then $\tau(g(xyx)) > \tau(y) - \varepsilon$.*

Proof. Apply Lemma VII.6 with g and ε as given to get $\delta_1 > 0$. Now apply Lemma VII.8 with $g(t) = t^2$ and δ_1 in place of ε to get δ_2. Let δ_3 be the δ obtained from applying Lemma VII.8 with

g as given and ε as given. We may assume $\delta_3 < \delta_2 < \delta_1$. The number δ_3 is the desired δ. By the choice of δ_2 using Lemma VII.8 with 1 in place of y for any x satisfying $\tau(x) > 1 - \delta_2$ we have $\tau(g(yxy)) = \tau(g(x)) = \tau(x^2) > 1 - \delta_1$. So now if y is such that
$\tau(y) > \|\tau|_{\overline{yAy}}\| - \delta_2 > \|\tau|_{\overline{yAy}}\| - \delta_3$, by the choice of δ_3 using Lemma VII.6 with $y^{1/2}$ in place of y we have $\tau\left(g\left(y^{1/2}x^2y^{1/2}\right)\right) > \tau\left(\left(y^{1/2}\right)^2\right) - \varepsilon$. But by Lemma VII.5, $\tau\left(g\left(y^{1/2}x^2y^{1/2}\right)\right) = \tau(g(xyx))$. Thus $\tau(g(xyx)) > \tau(y) - \varepsilon$. Additionally, since $\tau(x) > 1 - \delta_2 > 1 - \delta_3$ and $\tau(y) > \|\tau|_{\overline{yAy}}\| - \delta_3$, we have $\tau(g(yxy)) > \tau(y^2) - \varepsilon$. ∎

The following lemma is an analog for positive elements of Lemma 5.1 of [16].

Lemma VII.10. *Let $\delta > 0$. There exists a continuous function $g : [0,1] \to [0,1]$ such that $g(0) = 0, g(1) = 1$, and whenever A is a C^*-algebra and $a \in A$ is positive with $\|a\| \leq 1$, then there is a positive element $b \in \overline{aAa}$ with $\|b\| \leq 1$ such that $\|bg(a) - g(a)\| < \delta$ and $\|ab - b\| < \delta$.*

Proof. Choose t_0 and t_1 with $1 - \delta < t_0 < t_1 < 1$ and let $g : [0,1] \to [0,1]$ be a continuous function which vanishes on $[0, t_1]$ and such that $g(1) = 1$. Let A be a C^*-algebra, and let $a \in A$ be positive with $\|a\| \leq 1$. Let $h : [0,1] \to [0,1]$ be a continuous function which vanishes on $[0, t_0]$ such that $h(t) = 1$ for $t \in [t_1, 1]$. For n sufficiently large, $\|g(a)^{1/n}g(a) - g(a)\| < \delta$. So let $b = g(a)^{1/n}$. Note that since $g(a)^{1/n}$ is positive, $(\|g(a)^{1/n}\|)^n = \|g(a)\| = 1 = 1^n$, which implies that $\|(g(a))^{1/n}\| = 1$. From $hg = g$ we have $h(a)g(a) = g(a)$ and so $h(a)b = b$. Also $\|ah(a) - h(a)\| < \delta$ because $|t - 1| \leq 1 - t_0 < \delta$ whenever $h(t) \neq 0$. Accordingly, we have

$$\|ab - b\| = \|ah(a)b - h(a)b\| \leq \|ah(a) - h(a)\|\|b\| < \delta,$$

which completes the proof. ∎

The next lemma is used repeatedly and implicitly in the proof of Lemma VII.12.

Lemma VII.11. *If y and z are orthogonal positive elements of a C^*-algebra A and $w \in \overline{Ay}$ and $x \in \overline{zA}$, then $wx = 0$ as well.*

Proof. We have $wx = \lim_{n\to\infty} \lim_{m\to\infty} wy^{1/n}z^{1/m}x = w \cdot 0 \cdot x = 0$. ∎

The following lemma is used in the proof of the main theorem, Theorem VII.17, to replace the decomposition of the identity into orthogonal projections used in the proof of Theorem V.4.

Lemma VII.12. *Let $\varepsilon > 0$. Suppose $b_1, b_2, b_3, c_1, c_2, c_3$ are positive elements of a stably finite unital C^*-algebra A, and let $a \in A$. Suppose:*

- $b_1 + b_2 + b_3 = 1$,
- $C^*(b_1, b_2, b_3, c_1, c_2, c_3)$ *is commutative,*
- $b_1 c_1 = c_1$,
- $b_3 c_3 = c_3$,
- $b_2 c_2 = b_2$,
- $c_1 b_2 = c_3 b_2 = 0$,
- $b_1 b_3 = 0$,
- $\overline{c_2 A c_2}$ *have stable rank one, and*
- $b_1 a = a b_3 = 0$.

Then there exists an element $a_1 \in A$ such that a_1 is invertible and $\|a - a_1\| < \varepsilon$.

Proof. Write $1 = c_1 + (b_1 - c_1) + b_2 + (b_3 - c_3) + c_3$. Make the following definitions:

$a_{3,1} = b_2 a c_1$.

$a_{3,2} = b_2 a (b_1 - c_1)$.

$a_{3,3} = b_2 a b_2$.

$a_{4,1} = (b_3 - c_3) a c_1$.

$a_{4,2} = (b_3 - c_3) a (b_1 - c_1)$.

$a_{4,3} = (b_3 - c_3) a b_2$.

$a_{5,1} = c_3 a c_1$.

$a_{5,2} = c_3 a (b_1 - c_1)$.

$a_{5,3} = c_3 a b_2$.

Notice that $\sum_{i=3}^{5} \sum_{j=1}^{3} a_{i,j} = a$.

Let

$$0 < \delta < \min\left\{ \frac{\varepsilon}{6\|a_{3,1} + a_{4,1} + a_{5,1}\|}, \sqrt{\frac{\varepsilon}{6}}, \frac{\varepsilon}{3} \right\}.$$

Since $a_{3,3} \in \overline{b_2Ab_2}$, there is an invertible element $t_0 \in \overline{b_2Ab_2} + \mathbb{C}1_A$ with

$$\|t_0 - a_{3,3}\| < \delta. \tag{VII.22}$$

Write $t_0 = t_1 + \lambda_1 1_A$ with $t_1 \in \overline{b_2Ab_2}$ and $\lambda_1 \in \mathbb{C}$. We can also express t_0^{-1} as $t_2 + \lambda_1^{-1}1_A$ with $t_2 \in \overline{b_2Ab_2}$.

Next we show that

$$((a_{3,1} + a_{4,1} + a_{5,1})t_0^{-1})^2 = 0. \tag{VII.23}$$

We note that

$$(a_{3,1} + a_{4,1} + a_{5,1})t_0 = (a_{3,1} + a_{4,1} + a_{5,1})(t_1 + \lambda_1) = (a_{3,1} + a_{4,1} + a_{5,1})\lambda_1.$$

Therefore,

$$(a_{3,1} + a_{4,1} + a_{5,1}) = (a_{3,1} + a_{4,1} + a_{5,1})t_0t_0^{-1} = (a_{3,1} + a_{4,1} + a_{5,1})\lambda_1 t_0^{-1}.$$

This implies

$$\lambda_1^{-1}(a_{3,1} + a_{4,1} + a_{5,1}) = (a_{3,1} + a_{4,1} + a_{5,1})t_0^{-1}.$$

Therefore

$$((a_{3,1} + a_{4,1} + a_{5,1})t_0^{-1})^2 = (a_{3,1} + a_{4,1} + a_{5,1})^2\lambda_1^{-2} = 0.$$

Now we compute

$$\begin{aligned}
&(a_{3,1} + a_{4,1} + a_{5,1} + t_0)\, t_0^{-1} \left(1 - (a_{3,1} + a_{4,1} + a_{5,1})\, t_0^{-1}\right) \\
&= \left((a_{3,1} + a_{4,1} + a_{5,1})\, t_0^{-1} + 1\right) \left(1 - (a_{3,1} + a_{4,1} + a_{5,1})\, t_0^{-1}\right) \\
&= (a_{3,1} + a_{4,1} + a_{5,1})\, t_0^{-1} - \left((a_{3,1} + a_{4,1} + a_{5,1})\, t_0^{-1}\right)^2 + 1 - (a_{3,1} + a_{4,1} + a_{5,1})\, t_0^{-1} \\
&= 1.
\end{aligned} \tag{VII.24}$$

Because A is stably finite this is enough to show that $a_{3,1} + a_{4,1} + a_{5,1} + t_0$ and $t_0^{-1}\left(1 - (a_{3,1} + a_{4,1} + a_{5,1})\, t_0^{-1}\right)$ are mutual inverses.

Next we multiply

$$
\begin{aligned}
& t_0^{-1}\left(1-(a_{3,1}+a_{4,1}+a_{5,1})\,t_0^{-1}\right)(a_{3,1}+a_{3,2}+t_0+a_{4,1}+a_{4,2}+a_{4,3}+a_{5,1}+a_{5,2}+a_{5,2}) \\
& = t_0^{-1}\left(1-(a_{3,1}+a_{4,1}+a_{5,1})t_0^{-1}\right)(a_{3,1}+a_{4,1}+a_{5,1}+t_0) \\
& \quad + t_0^{-1}\left(1-(a_{3,1}+a_{4,1}+a_{5,1})\,t_0^{-1}\right)(a_{3,2}+a_{4,2}+a_{4,3}+a_{5,2}+a_{5,3}) \\
& = 1 + t_0^{-1}\left(1-(a_{3,1}+a_{4,1}+a_{5,1})\,t_0^{-1}\right)(a_{3,2}+a_{4,2}+a_{4,3}+a_{5,2}+a_{5,3})\,. \qquad \text{(VII.25)}
\end{aligned}
$$

Using our expression for t_0^{-1} we can compute

$$
\begin{aligned}
\left((a_{3,1}+a_{4,1}+a_{5,1})\,t_0^{-1}\right)b_3 &= (a_{3,1}+a_{4,1}+a_{5,1})\left(t_2+\lambda_1^{-1}\right)b_3 \\
&= (a_{3,1}+a_{4,1}+a_{5,1})\left(t_2 b_3+\lambda_1^{-1}b_3\right) \\
&= 0.
\end{aligned}
$$

To get the last line we used Lemma VII.11 twice, once with $y = c_1$ and $z = b_3$ and once with $y = c_1$ and $z = b_2$.

Similarly,

$$
\begin{aligned}
(a_{3,1}+a_{4,1}+a_{5,1})\,t_0^{-1}b_2 &= (a_{3,1}+a_{4,1}+a_{5,1})\left(t_2+\lambda_1^{-1}\right)b_2 \\
&= 0.
\end{aligned}
$$

Notice that the previous two computations imply $(a_{3,1}+a_{4,1}+a_{5,1})t_0^{-1}c_3 = 0$ and $(a_{3,1}+a_{4,1}+a_{5,1})t_0^{-1}(b_3-c_3) = 0$.

Continuing our computation, we see the last expression in Equation VII.25 is equal to:

$$
\begin{aligned}
&1+\left(t_0^{-1}-t_0^{-1}\left(a_{3,1}+a_{4,1}+a_{5,1}\right)t_0^{-1}\right)\left(a_{3,2}+a_{4,2}+a_{4,3}+a_{5,2}+a_{5,3}\right)\\
&=1+\left(\left(t_2+\lambda_1^{-1}\right)-t_0^{-1}\left(a_{3,1}+a_{4,1}+a_{5,1}\right)t_0^{-1}\right)\left(a_{3,2}+a_{4,2}+a_{4,3}+a_{5,2}+a_{5,3}\right)\\
&=1+t_2a_{3,2}+t_2a_{4,2}+t_2a_{4,3}+t_2a_{5,2}+t_2a_{5,3}\\
&\quad-t_0^{-1}(a_{3,1}+a_{4,1}+a_{5,1})t_0^{-1}a_{3,2}\\
&\quad-t_0^{-1}(a_{3,1}+a_{4,1}+a_{5,1})t_0^{-1}(a_{4,2}+a_{4,3})\\
&\quad-t_0^{-1}(a_{3,1}+a_{4,1}+a_{5,1})t_0^{-1}(a_{5,2}+a_{5,3})\\
&\quad+\lambda_1^{-1}a_{3,2}+\lambda_1^{-1}a_{4,2}+\lambda_1^{-1}a_{4,3}+\lambda_1^{-1}a_{5,2}+\lambda_1^{-1}a_{5,3}\\
&=1+t_2a_{3,2}+t_2a_{4,2}+t_2a_{4,3}+0+0\\
&\quad-\lim_{n\to\infty}t_0^{-1}(a_{3,1}+a_{4,1}+a_{5,1})t_0^{-1}b_2^{1/n}a_{3,2}\\
&\quad-\lim_{n\to\infty}t_0^{-1}(a_{3,1}+a_{4,1}+a_{5,1})t_0^{-1}(b_3-c_3)^{1/n}(a_{4,2}+a_{4,3})\\
&\quad-\lim_{n\to\infty}t_0^{-1}(a_{3,1}+a_{4,1}+a_{5,1})t_0^{-1}c_3^{1/n}(a_{5,2}+a_{5,3})\\
&\quad+\lambda_1^{-1}a_{3,2}+\lambda_1^{-1}a_{4,2}+\lambda_1^{-1}a_{4,3}+\lambda_1^{-1}a_{5,2}+\lambda_1^{-1}a_{5,3}\\
&=1+t_2a_{3,2}+t_2a_{4,2}+t_2a_{4,3}+\lambda_1^{-1}a_{3,2}+\lambda_1^{-1}a_{4,2}+\lambda_1^{-1}a_{4,3}+\lambda_1^{-1}a_{5,2}+\lambda_1^{-1}a_{5,3}. \qquad \text{(VII.26)}
\end{aligned}
$$

Let $t_3 = t_2a_{4,3}+1$. Notice that $t_3 \in \overline{b_2Ab_2}+\mathbb{C}1_A$ since $t_2 \in \overline{b_2Ab_2}$ and $a_{4,3} \in \overline{(b_3-c_3)Ab_2}$. Thus there is an invertible element $t_4 \in \overline{b_2Ab_2}\mathbb{C}1_A$ with

$$\|t_4 - t_3\| < \delta. \qquad \text{(VII.27)}$$

Write $t_4 = t_5 + \lambda_5 1_A$ with $t_5 \in \overline{b_2Ab_2}$ and $\lambda_5 \in \mathbb{C}$. Similarly, write $t_4^{-1} = t_6 + \lambda_5^{-1}1_A$ with $t_6 \in \overline{b_2Ab_2}$.

Using the same argument used to show Equation VII.23 we can show that $\left(t_4^{-1}\left(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2}\right)\right)^2 = 0$.

Next we find the inverse to $\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2}+t_4$. We have

$$\begin{aligned}
&\left[\left(1-t_4^{-1}\left(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2}\right)\right)t_4^{-1}\right]\left(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2}+t_4\right)\\
&=\left(1-t_4^{-1}\left(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2}\right)\right)\left(t_4^{-1}(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2})+1\right)\\
&=t_4^{-1}\left(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2}\right)+1-\left(t_4^{-1}\left(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2}\right)\right)^2-t_4^{-1}\left(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2}\right)\\
&=1.
\end{aligned}$$

Since A is stably finite, this is enough to get

$$(1-t_4^{-1}(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2}))t_4^{-1}=(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2}+t_4)^{-1}. \tag{VII.28}$$

Also notice $\left(1-t_4^{-1}\left(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2}\right)\right)t_4^{-1}=t_4^{-1}-t_4^{-1}\left(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2}\right)t_4^{-1}$.

Next we show $b_1t_4^{-1}\left(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2}\right)=0$. By applying Lemma VII.11, since $t_6\in\overline{b_2Ab_2}$, $b_1\in\overline{b_1Ab_1}$, and $a_{5,2}+a_{5,3}\in\overline{c_3A}$, we have

$$\begin{aligned}
&b_1t_4^{-1}\left(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2}\right)\\
&=b_1\left(t_6+\lambda_5^{-1}\right)\left(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2}\right)\\
&=b_1t_6\left(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2}\right)+b_1\lambda_5^{-1}\left(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2}\right)\\
&=0.
\end{aligned} \tag{VII.29}$$

Similarly, $b_2t_4^{-1}(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2}))=0$. These two also imply that $(b_1-c_1)t_4^{-1}(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2}))=0$ and $c_1t_4^{-1}(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2}))=0$.

Now

$$
\begin{aligned}
&\left(t_2a_{3,2}+t_2a_{4,2}+t_4+\lambda_1^{-1}a_{3,2}+\lambda_1^{-1}a_{4,2}+\lambda_1^{-1}a_{4,3}+\lambda_1^{-1}a_{5,2}+\lambda_1^{-1}a_{5,3}\right)\\
&\qquad\cdot\left[\left(1-t_4^{-1}\left(\lambda_1^{-1}a_{5,2}+\lambda_1^{-1}a_{5,3}\right)\right)t_4^{-1}\right]\\
&=1+\left(t_2a_{3,2}+t_2a_{4,2}+\lambda_1^{-1}a_{3,2}+\lambda_1^{-1}a_{4,3}\right)\left(t_4^{-1}-t_4^{-1}\left(\lambda_1^{-1}a_{5,2}+\lambda_1^{-1}a_{5,3}\right)t_4^{-1}\right)\\
&=1+t_2a_{3,2}t_4^{-1}+t_2a_{4,2}t_4^{-1}+\lambda_1^{-1}a_{3,2}t_4^{-1}+\lambda_1^{-1}a_{4,2}t_4^{-1}+\lambda_1^{-1}a_{4,3}t_4^{-1}\\
&\qquad-t_2a_{3,2}t_4^{-1}(\lambda_1^{-1}a_{5,2}+\lambda_1^{-1}a_{5,3})t_4^{-1}-t_2a_{4,2}t_4^{-1}(\lambda_1^{-1}a_{5,2}+\lambda_1^{-1}a_{5,3})t_4^{-1}\\
&\qquad-\lambda_1^{-1}a_{3,2}t_4^{-1}\left(\lambda_1^{-1}a_{5,2}+\lambda_1^{-1}a_{5,3}\right)t_4^{-1}-\lambda_1^{-1}a_{4,2}t_4^{-1}\left(\lambda_1^{-1}a_{5,2}+\lambda_1^{-1}a_{5,3}\right)t_4^{-1}\\
&\qquad-\lambda_1^{-1}a_{4,3}t_4^{-1}\left(\lambda_1^{-1}a_{5,2}+\lambda_1^{-1}a_{5,3}\right)t_4^{-1}\\
&=1+t_2a_{3,2}t_4^{-1}+t_2a_{4,2}t_4^{-1}+\lambda_1^{-1}a_{3,2}t_4^{-1}+\lambda_1^{-1}a_{4,2}t_4^{-1}+\lambda_1^{-1}a_{4,3}t_4^{-1}\\
&\qquad-\lim_{n\to\infty}t_2\left(a_{3,2}+a_{4,2}\right)\left(b_1-c_1\right)^{1/n}t_4^{-1}\left(\lambda_1^{-1}a_{5,2}+\lambda_1^{-1}a_{5,3}\right)t_4^{-1}\\
&\qquad-\lim_{n\to\infty}\lambda_1^{-1}\left(a_{3,2}+a_{4,2}\right)\left(b_1-c_1\right)^{1/n}t_4^{-1}\left(\lambda_1^{-1}a_{5,2}+\lambda_1^{-1}a_{5,3}\right)t_4^{-1}\\
&\qquad-\lim_{n\to\infty}\lambda_1^{-1}a_{4,3}b_2^{1/n}t_4^{-1}\left(\lambda_1^{-1}a_{5,2}+\lambda_1^{-1}a_{5,3}\right)t_4^{-1}\\
&=1+t_2a_{3,2}t_4^{-1}+t_2a_{4,2}t_4^{-1}+\lambda_1^{-1}a_{3,2}t_4^{-1}+\lambda_1^{-1}a_{4,2}t_4^{-1}+\lambda_1^{-1}a_{4,3}t_4^{-1}-0 \qquad \text{(VII.30)}\\
&\qquad\text{by Equation VII.29 and the statements that follow it}\\
&=1+t_2a_{3,2}t_6+t_2a_{4,2}t_6+\lambda_1^{-1}a_{3,2}t_6+\lambda_1^{-1}a_{4,2}t_6+\lambda_1^{-1}a_{4,3}t_6\\
&\qquad+t_2a_{3,2}\lambda_5^{-1}+t_2a_{4,2}\lambda_5^{-1}+\lambda_1^{-1}a_{3,2}\lambda_5^{-1}+\lambda_1^{-1}a_{4,2}\lambda_5^{-1}+\lambda_1^{-1}a_{4,3}\lambda_5^{-1}.
\end{aligned}
$$

Denote the quantity just computed by t_7 and notice that

$$t_7^{-1}\in\overline{((b_1-c_1)+b_2+(b_3+c_3))A((b_1-c_1)+b_2+(b_3+c_3))}=\overline{c_2Ac_2},$$

which has stable rank one by hypothesis.

Thus there exists an invertible element $t_8\in\overline{c_2Ac_2}+\mathbb{C}1_A$ such that

$$\|t_8-t_7\|<\frac{\delta}{\left\|\lambda_1^{-1}a_{5,2}+\lambda_1^{-1}a_{5,3}+t_4\right\|+1}. \qquad \text{(VII.31)}$$

Now

$$\begin{aligned}
&\left[t_0^{-1}\left(1-\left(a_{3,1}+a_{4,1}+a_{5,1}\right)t_0^{-1}\right)\right]^{-1} t_8 \left[\left(1-t_4^{-1}\left(\lambda_1^{-1}a_{5,2}+\lambda_1^{-1}a_{5,3}\right)\right)t_4^{-1}\right]^{-1} \\
&= \left(a_{3,1}+a_{4,1}+a_{5,1}+t_0\right) t_8 \left(\lambda_1^{-1}a_{5,2}+\lambda_1^{-1}a_{5,3}+t_4\right)
\end{aligned}$$

is invertible, and as we will now compute,

$$\|a-(a_{3,1}+a_{4,1}+a_{5,1}+t_0)t_5(\lambda_1^{-1}a_{5,2}+\lambda_1^{-1}a_{5,3}+t_4)\| < \varepsilon.$$

We have

$$\begin{aligned}
&\|a-(a_{3,1}+a_{4,1}+a_{5,1}+t_0)t_8(\lambda_1^{-1}a_{5,2}+\lambda_1^{-1}a_{5,3}+t_4)\| \\
&= \|a_{3,1}+a_{3,2}+a_{3,3}+a_{4,1}+a_{4,2}+a_{4,3}+a_{5,1}+a_{5,2}+a_{5,3} \\
&\qquad -(a_{3,1}+a_{4,1}+a_{5,1}+t_0)t_8(\lambda_1^{-1}a_{5,2}+\lambda_1^{-1}a_{5,3}+t_4)\| \\
&\leq \|a_{3,3}-t_0\| \\
&\qquad +\|(a_{3,1}+a_{4,1}+a_{5,1}+t_0)t_0^{-1}(1-(a_{3,1}+a_{4,1}+a_{5,1})t_0^{-1}) \\
&\qquad\qquad \cdot(a_{3,1}+a_{3,2}+t_0+a_{4,1}+a_{4,2}+a_{4,3}+a_{5,1}+a_{5,2}+a_{5,3}) \\
&\qquad -(a_{3,1}+a_{4,1}+a_{5,1}+t_0)t_8(\lambda_1^{-1}a_{5,2}+\lambda_1^{-1}a_{5,3}+t_4)\| \quad \text{by Equation VII.24} \\
&\leq \delta+\|a_{3,1}+a_{4,1}+a_{5,1}+t_0\| \\
&\qquad \cdot\|1+t_2a_{3,2}+t_2a_{4,2}+t_2a_{4,3}+\lambda_1^{-1}a_{3,2}+\lambda_1^{-1}a_{4,2}+\lambda_1^{-1}a_{4,3} \\
&\qquad\qquad +\lambda_1^{-1}a_{5,2}+\lambda_1^{-1}a_{5,3}-t_8(\lambda_1^{-1}a_{5,2}+\lambda_1^{-1}a_{5,3}+t_4)\| \\
&\qquad \text{by Equations VII.25 and VII.26}
\end{aligned}$$

$$
\begin{aligned}
&\le \delta + \|a_{3,1} + a_{4,1} + a_{5,1} + t_0\| \\
&\qquad \cdot \|1 + t_2 a_{3,2} + t_2 a_{4,2} + t_2 a_{4,3} + \lambda_1^{-1} a_{3,2} + \lambda_1^{-1} a_{4,2} + \lambda_1^{-1} a_{4,3} + \lambda_1^{-1} a_{5,2} + \lambda_1^{-1} a_{5,3} \\
&\qquad\quad - (t_2 a_{3,2} + t_2 a_{4,2} + t_4 + \lambda_1^{-1} a_{3,2} + \lambda_1^{-1} a_{4,2} + \lambda_1^{-1} a_{4,3} + \lambda_1^{-1} a_{5,2} + \lambda_1^{-1} a_{5,3})\| \\
&\quad + \|a_{3,1} + a_{4,1} + a_{5,1} + t_0\| \\
&\qquad \cdot \|(t_2 a_{3,2} + t_2 a_{4,2} + t_4 + \lambda_1^{-1} a_{3,2} + \lambda_1^{-1} a_{4,2} + \lambda_1^{-1} a_{4,3} + \lambda_1^{-1} a_{5,2} + \lambda_1^{-1} a_{5,3}) \\
&\qquad \cdot (1 - t_4^{-1}(\lambda_1^{-1} a_{5,2} + \lambda_1^{-1} a_{5,3})) t_4^{-1} (\lambda_1^{-1} a_{5,2} + \lambda_1^{-1} a_{5,3} + t_4) \\
&\qquad\quad - t_8(\lambda_1^{-1} a_{5,2} + \lambda_1^{-1} a_{5,3} + t_4)\| \text{ by Equation VII.28} \\
&\le \delta + \|a_{3,1} + a_{4,1} + a_{5,1} + t_0\| \|1 + t_2 a_{4,3} - t_4\| \\
&\quad + \|a_{3,1} + a_{4,1} + a_{5,1} + t_0\| \\
&\qquad \cdot \|1 + t_2 a_{3,2} t_4^{-1} + t_2 a_{4,2} t_4^{-1} + \lambda_1^{-1} a_{3,2} t_4^{-1} + \lambda_1^{-1} a_{4,2} t_4^{-1} + \lambda_1^{-1} a_{4,3} t_4^{-1} - t_8\| \\
&\qquad \cdot \|\lambda_1^{-1} a_{5,2} + \lambda_1^{-1} a_{5,3} + t_4\| \text{ by Equation VII.30} \\
&\le \delta + \delta \|a_{3,1} + a_{4,1} + a_{5,1} + t_0\| \\
&\quad + \|a_{3,1} + a_{4,1} + a_{5,1} + t_0\| \frac{\delta}{\|\lambda_1^{-1} a_{5,2} + \lambda_1^{-1} a_{5,3} + t_4\| + 1} \|\lambda_1^{-1} a_{5,2} + \lambda_1^{-1} a_{5,3} + t_4\| \\
&\qquad\qquad \text{by the choice of } t_4 \text{ and } t_8 \\
&= \delta + 2\delta \|a_{3,1} + a_{4,1} + a_{5,1} + t_0\| \\
&\le \delta + 2\delta \|a_{3,1} + a_{4,1} + a_{5,1} + a_{3,3}\| + 2\delta \|a_{3,3} - t_0\| \\
&< \varepsilon/2 + 2\varepsilon/6 + 2\delta^2 \\
&< 3\varepsilon/3 \\
&= \varepsilon.
\end{aligned}
$$

This completes the proof. ∎

Lemma VII.13 uses Lemma VII.12 to produce a simpler replacement for the decomposition of the identity into orthogonal projections.

Lemma VII.13. *Let A be a stably finite unital C^*-algebra, let $\varepsilon > 0$ be given, and let $x_1, x_2, x_3 \in A$ be positive elements such that $x_1 + x_2 + x_3 = 1$ and $x_1 x_3 = 0$. Let $a \in A$ be such that $x_1 a = 0$, $a x_3 = 0$, and $\overline{x_2 A x_2}$ has stable rank 1. Then there exists an element $a_1 \in A$ such that a_1 is invertible and $\|a_1 - a\| < \varepsilon$.*

Proof. It suffices to show that the hypotheses here imply the hypotheses of VII.12. Let $f : [0,1] \to [0,1]$ and $h : [0,1] \to [0,1]$ be defined by the formulas

$$f(t) = \begin{cases} 0 & t \in [0, \frac{1}{2}] \\ 2t - 1 & t \in [\frac{1}{2}, 1] \end{cases}$$

and

$$h(t) = \begin{cases} 2t & t \in [0, \frac{1}{2}] \\ 1 & t \in [\frac{1}{2}, 1]. \end{cases}$$

It is clear that $fh = f$. Now set $b_1 = h(x_1)$, $c_1 = f(x_1)$, $b_3 = h(x_3)$, $c_3 = f(x_3)$, $b_2 = f(x_2)$, and $c_2 = h(x_2)$, all of which are positive. Then since $C^*(x_1, x_2, x_3)$ is commutative, $C^*(f(x_1), f(x_2), f(x_3), h(x_1), h(x_2), h(x_3))$ is commutative.

Note that the formulas $c_1 b_1 = c_1$, $c_3 b_3 = c_3$, and $c_2 b_2 = b_2$ all hold. Since $x_1 x_3 = 0$, we have $b_1 b_3 = h(x_1) h(x_3) = 0$, and so also $c_1 b_3 = 0 = b_1 c_3$. Similarly, $x_1 a = 0$ and $a x_3 = 0$ imply that $b_1 a = h(x_1) a = 0$ and $a b_3 = a h(x_3) = 0$, and these in turn imply that $c_1 a = 0$ and $a c_3 = 0$.

Also, $\overline{b_2 A b_2}$ and $\overline{c_2 A c_2}$ have stable rank one because they are hereditary subalgebras of $\overline{x_2 A x_2}$.

Since x_1, x_2, and x_3 all commute, we have $C^*(x_1, x_2, x_3) = C(Y)$ with $Y \subset [0,1]$. So think of x_1, x_2, and x_3 as f_1, f_2, and f_3 respectively.

Let $Y_i = \{y \in Y : f_i(y) > 1/2\}$ and note that $Y_i \cap Y_j = \emptyset$ for $i \neq j$. Then on Y_1, we have $(c_1 b_2)(y) = f(x_1) f(x_2)(y) = f(f_1(y)) f(f_2(y)) = [2 f_1(y) - 1] \cdot 0 = 0$ and $c_3 b_2 = f(x_1) \cdot 0 = 0$. The same equalities also hold on Y_3. On Y_2, we have $c_1 b_2 = f(f_1(t)) f(f_2(t)) = 0 \cdot f(f_2(t)) = 0$ and $c_3 b_2 = f(f_3(t)) b_2 = 0 \cdot b_2 = 0$. On $Y \setminus Y_1 = \{y \in Y : f_1(y) \leq 1/2\}$, we have $c_1 b_2 = f(f_1(t)) f(f_2(t)) = 0 \cdot f(f_2(t)) = 0$ and $c_3 b_2 = f(f_3(t)) b_2 = 0 \cdot b_2 = 0$. If $t \in Y_1$, then $f_1(t) > 1/2$, which implies $f_3(t) = 0$ and $1 - f_1(t) - f_3(t) \leq 1/2$. This implies that $f(f_1(t) + f_3(t)) + h(1 - f_1(t) - f_3(t)) = 1$. Symmetrically, if $t \in Y_3$, then $f(f_1(t) + f_3(t)) + h(1 - f_1(t) - f_3(t)) = 1$.

Now suppose that $t \in Y \setminus (Y_1 \cup Y_3)$. Then $f_1(t) \leq 1/2$ and $f_3(t) \leq 1/2$, and at most one of them is nonzero, so $1 - f_1(t) - f_3(t) \geq 1/2$. This gives $f(f_1(t) + f_3(t)) + h(1 - f_1(t) - f_3(t)) = 0 + 0 + 1 = 1$. It follows that $f(x_1) + f(x_3) + h(x_2) = 1$, which is equivalent to $c_1 + c_2 + c_3 = 1$. Next, if $t \in Y_1$, then $f_1(t) > 1/2$ and $f_3 < 1/2$, which together imply $1 - f_1(t) - f_3(t) \leq 1/2$ and $f_3(t) = 0$ (using orthogonality, since $f_1(t) \neq 0$). It follows that $h(f_1(t)) + h(f_3(t)) + f(1 - f_1(t) - f_3(t)) =$

$1 + 2f_3(t) + 0 = 1$. Symmetrically, if $t \in Y_3$, then $h(f_1(t)) + h(f_3(t)) + f(1 - f_1(t) - f_3(t)) = 1$. Suppose that $t \in Y \setminus (Y_1 \cup Y_3)$. Then as before, we have $1 - f_1(t) - f_3(t) \geq 1/2$. This gives $h(f_1(t)) + h(f_3(t)) + f(1 - f_1(t) - f_3(t)) = 2f_1(t) + 2f_3(t) + 2(1 - f_1(t) - f_3(t)) - 1 = 2 - 1 = 1$.

Therefore, $h(x_1) + h(x_3) + f(x_2) = 1$, which is equivalent to $b_1 + b_2 + b_3 = 1$. ∎

Lemma VII.14. *Let A be a simple, unital C^*-algebra, and let $a, b \in A_+$ with $\|a\| = \|b\| = 1$. Then there exists $c \in A_+$ with $\|c\| = 1$ such that $c \leq a$ and $c \precsim b$.*

Proof. Since A is simple and $a, b \in A$ are nonzero, by Proposition 1.8 of [5] there is a nonzero $y \in A$ such that $yy^* \in \overline{aAa}$ and $y^*y \in \overline{bAb}$. Without loss of generality we may assume that $\|yy^*\| \leq 1$, and so $yy^* \leq 1$. Set $c = a^{1/2}yy^*a^{1/2}$. Set $z = (a^{1/2}y)^*$, and choose $0 < \beta < 1$. Then, $z^*z \leq z^*z$, so by Proposition 1.4.5 of [18] there is $u \in A$ such that $z = u(z^*z)^{\beta/2}$. Note that $[u(z^*z)^{\beta/2}](z^*z)^{\beta/2}(u^*) = [u(z^*z)^{\beta/2}][u(z^*z)^{\beta/2}]^* = zz^*$ and so $zz^* \precsim (z^*z)^{\beta/2}$. But since $f(t) = t$ and $g(t) = t^\beta$ are zero on the same set, $z^*z \sim (z^*z)^\beta$ by Lemma VI.4. Therefore $zz^* \precsim z^*z$. Symmetrically, $z^*z \precsim zz^*$. This implies $zz^* \sim z^*z$.

Note that $y^*ay \leq y^*y \in \overline{bAb}$, so $y^*ay \in \overline{bAb}$. Therefore, $y^*ay \precsim b$ by the second paragraph of section 1 in [6]. Combining this with $c = z^*z \sim zz^* = y^*ay$ yields $c \precsim b$. Furthermore, $yy^* \leq 1$ which implies $c = a^{1/2}yy^*a^{1/2} \leq a$. ∎

Lemma VII.15. *Let A be a simple unital C^*-algebra which is not the compact operators over H for any Hilbert space H. Let $a_1, a_2, a_3, a_4 \in A$ satisfy $a_ia_{i+1} = a_{i+1}$, for $i = 1, 2$, and 3, and $0 \leq a_1, a_2, a_3, a_4 \leq 1$. Also assume that at least one of a_1, a_2, and a_3 is not a projection, or that a_1, a_2 and a_3 are not all equal. Then $\tau(a_1) > \lim_{n\to\infty} \tau((a_4)^{1/n})$ for any tracial state τ on A.*

Proof. Notice that we have $a_i = a_{i-1}^{1/2}a_ia_{i-1}^{1/2} \leq a_{i-1}$ for $i = 2, 3$ or 4. Thus, $\tau(a_4) \leq \tau(a_3) \leq \tau(a_2) \leq \tau(a_1)$.

We first show that $\tau(a_1) > \tau(a_3)$. Since we have already observed that $\tau(a_1) \geq \tau(a_3)$, we only must show that they are not equal. Suppose $\tau(a_1) = \tau(a_3)$. Then $\tau(a_1 - a_3) = 0$. The hypotheses on A imply that τ is faithful, so $a_1 = a_3$. But this means that $a_1a_2 = a_2$ and $a_1a_2 = a_3a_2 = a_3$, so $a_2 = a_3$ as well. If a_1, a_2, and a_3 are all distinct then this is a contradiction already. Otherwise, we now see $a_1 = a_2 = a_1a_2 = a_1^2$, so a_1 is a projection, but since all three are equal, we now see that a_2 and a_3 are also projections, which is a contradiction.

Now because $a_3a_4 = a_4$, we also have $a_3a_4^{1/n} = a_4^{1/n}$ for any n. Using a similar argument to the one used in the first paragraph this implies that that $\tau(a_4^{1/n}) \leq \tau(a_3)$ for all n. Thus, $\lim_{n\to\infty} \tau(a_4^{1/n}) \leq \tau(a_3)$. Therefore, $\tau(a_1) > \tau(a_3) \geq \lim_{n\to\infty} \tau(a_4^{1/n})$. ∎

The following theorem is an analog of Lemma 5.2 of [16] with projections replaced by positive elements.

Lemma VII.16. *Let A be an infinite dimensional stably finite simple unital C^*-algebra. Suppose A has a unique 2-quasi-trace which is also a trace. Suppose also that A has strict comparison. Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group with the projection free tracial Rokhlin property. Let $B = C^*(G, A, \alpha)$. Suppose $q_1, \ldots, q_n \in B$ are nonzero positive elements of norm at most one and $a_1, \ldots, a_m \in B$ are arbitrary. Let $\varepsilon > 0$ and $N \in \mathbb{N} \cup \{0\}$. Then there exist a subalgebra $D \subset B$ isomorphic to a matrix algebra over a hereditary subalgebra of A, a positive element $d \in D$ with $\|d\| \leq 1$, nonzero positive elements $r_{k,i} \in \overline{dDd}$ of norm at most 1 for $i = 0, \ldots, N$ and $k = 1, \ldots, n$, and elements $b_1, \ldots, b_m \in B$ such that the following conditions are satisfied.*

1. $\|q_k r_{k,N} - r_{k,N}\| < \varepsilon$ *for all* $k = 1, \ldots, n$.
2. $1 - d \precsim r_{k,N}$ *for all* $k = 1, \ldots, n$
3. $r_{k,i} r_{k,i+1} = r_{k,i+1}$ *for all* $k = 1, \ldots, n$ *and* $i = 0, \ldots, N$.
4. $r_{k,0} d = r_{k,0}$ *for all* $k = 1, \ldots, n$.
5. $\|a_j - b_j\| < \varepsilon$ *for all* $j = 1, \ldots, m$.
6. $db_jd \in \overline{dDd}$ *for all* $j = 1, \ldots, m$.

Proof. By rescaling, we may assume that $\|q_k\| = 1$ for $1 \leq k \leq n$. Let $\varepsilon_1 = \varepsilon/6$. Let $h_1 : [0,1] \to [0,1]$ be the continuous function which has $h_1(0) = 0$, $h_1(t) = 1$ for $t \in [1 - \varepsilon_1, 1]$, and is linear on $[0, 1-\varepsilon_1]$. Let $h_2 : [0,1] \to [0,1]$ be the continuous function with $h_2(t) = 0$ for $t \in [0, 1-\varepsilon_1]$, linear on $[1 - \varepsilon_1, 1]$, and $h_2(1) = 1$. Set $q_{j,1} = h_1(q_j)$ and $w_j = h_2(q_j)$. Note that $\|q_j - q_{j,1}\| \leq \varepsilon_1$. Set $\lambda = \min_{1\leq j\leq n}\{\tau(w_j)\}$. Note that $\lambda \neq 0$ since B is simple. If h is any continuous function which has $h(1) = 1$ and $0 \leq h \leq 1$, then $\tau(h(q_{j,1})) > \tau(w_j)$, thus we have

$$\tau(h(q_{j,1})) > \lambda. \tag{VII.32}$$

Apply Lemma VII.10 with $\min\{\varepsilon/12, \lambda\}$ in place of δ to get a continuous function $g : [0,1] \to [0,1]$. Let $\varepsilon_2 < \lambda/4$. Now apply Lemma VI.11 with g as just obtained and with ε_2 in place of ε to get δ_2.

Let $\varepsilon_3 < \min\{\varepsilon/24, \delta_2, \lambda/8\}$. Then choose $\varepsilon_4 < \varepsilon_2$ such that if x, y are selfadjoint elements of B, with $\|x\|, \|y\| \leq 1$ and $\|x - y\| < \varepsilon_4$, then $\|x_+ - y_+\| < \varepsilon_3$. Without loss of generality, we may assume that $\varepsilon_4 < \frac{\varepsilon}{2\max_j \|a_j\|}$.

Choose $\varepsilon_5 < \min\{\frac{\varepsilon}{12}, \frac{\varepsilon}{4\max_j \|a_j\|}, \frac{\lambda}{4}, \frac{\varepsilon_4}{4}\}$. Define the continuous function f_1 to be zero at zero, 1 on $[1 - \varepsilon_5, 1]$ and linear on $[0, 1 - \varepsilon_5]$. For $i = 2, \ldots, 5$, define f_i to be the continuous function which is zero on $[0, 1 - \varepsilon_5/(2^{i-2})]$, linear on $[1 - \varepsilon_5/(2^{i-2}), 1 - \varepsilon_5/(2^{i-1})]$, and one on $[1 - \varepsilon_5/(2^{i-1}), 1]$. Note that $f_1 f_2 = f_2$, $f_2 f_3 = f_3$, etc. and $\|f_1 - t\| < \varepsilon_5$.

Apply Lemma VI.12 with ε replaced by ε_3 and with f replaced by f_4, to get ε_6 such that $\|[x, y]\| < \varepsilon_6$ implies $\|[f_4(y), x]\| < \varepsilon_3$ if $\mathrm{sp}(y) \subset [0,1]$ and $\|x\| \leq 1$.

Apply Lemma VII.9 with g as defined above and with ε_2 in place of ε to get ε_7.

Let μ be the measure obtained from τ by the Riesz representation theorem. Using the outer regularity of μ choose ε_8 with $\varepsilon_1/2 < \varepsilon_8 < \varepsilon_1$ and with

$$\mu([1 - \varepsilon_8, 1]) < \mu([1 - \varepsilon_1/2, 1]) + \varepsilon_7. \tag{VII.33}$$

Define a continuous function h_3 such that $h_3(t) = 0$ for $t \in [0, 1 - \varepsilon_8]$, $h_3(t) = 1$ for $t \in [1 - \varepsilon_1/2, 1]$ and h_3 is linear on $[1 - \varepsilon_8, 1 - \varepsilon_1/2]$. Notice that $h_3 h_1 = h_3$. Let $q_{k,3} = h_3(q_k)$. Choose M with $\frac{1}{M} < \min\{\lambda/8 - \varepsilon_3, \varepsilon_7\}$.

Apply Lemma VII.4 with $F = \{q_{1,3}, \ldots q_{n,3}, a_1, \ldots, a_m\}$, with $\min\{\varepsilon_4/2, \varepsilon_6\}$ in place of ε, and with M in place of N, to obtain positive elements $c^{(1)} \in B$ and $c_{1,1}^{(1)} \in A$, a subalgebra $D \subset \overline{c^{(1)} B c^{(1)}}$ and an isomorphism $\Phi : M_n \otimes \overline{c_{1,1}^{(1)} A c_{1,1}^{(1)}} \to D$ such that there exist elements $x_1, \ldots, x_n, e_1, \ldots e_m \in D$ with

- $\|c^{(1)} q_{j,3} c^{(1)} - x_j\| < \varepsilon_4/2$ for all $j = 1, \ldots n$ by part 3.
- $\|x_j\| \leq \|q_{j,3}\| = 1$ for all $j = 1, \ldots n$ by part 3.
- $\|c^{(1)} a_k c^{(1)} - e_k\| < \varepsilon_4/2$ for all $k = 1, \ldots m$ by part 3.
- $\|c^{(1)} q_{j,3} c^{(1)} - x_j\| < \varepsilon_4/2$ for all $j = 1, \ldots n$ by part 3.

- $\|c^{(1)}q_{j,3} - q_{j,3}c^{(1)}\| < \varepsilon_6$ for all $j = 1, \ldots n$ by part 5, and
- $\tau(1 - c^{(1)}) < \frac{1}{M}$ by part 6.

Set $d^{(1)} = f_1(c^{(1)}), \ldots, d^{(5)} = f_5(c^{(1)})$. We have $\|d^{(1)} - c^{(1)}\| < \varepsilon_5$. Also, $d^{(1)}d^{(2)} = d^{(2)}, \ldots, d^{(4)}d^{(5)} = d^{(5)}$. Notice that since $1 - f_2(t)$ is zero on a larger set than $1 - t$, we have $1 - d^{(2)} \preccurlyeq 1 - c^{(1)}$. Similarly, we have

$$1 - d^{(2)} \preccurlyeq 1 - d^{(3)} \preccurlyeq 1 - d^{(4)} \preccurlyeq 1 - d^{(5)} \preccurlyeq 1 - c^{(1)}.$$

Now

$$\begin{aligned} \|d^{(1)}q_{j,3}d^{(1)} - x_j\| &\leq \|d^{(1)}q_{j,3}d^{(1)} - d^{(1)}q_{j,3}c^{(1)}\| + \|d^{(1)}q_{j,3}c^{(1)} - c^{(1)}q_{j,3}c^{(1)}\| + \|c^{(1)}q_{j,3}c^{(1)} - x_j\| \\ &< 2\varepsilon_5 + \varepsilon_4/2. \end{aligned}$$

Similarly, $\|d^{(1)}a_jd^{(1)} - e_j\| \leq (2\varepsilon_5 + \varepsilon_4/2)\|a_j\|$.

Set $d = d^{(2)}$. Set $b_j = a_j + e_j - d^{(1)}a_jd^{(1)}$. Then

$$\|b_j - a_j\| = \|e_j - d^{(1)}a_jd^{(1)}\| \leq (2\varepsilon_5 + \varepsilon_4/2)\|a_j\| < \left(2\frac{\varepsilon}{4\max_k \|a_k\|} + \frac{\varepsilon}{4\max_k \|a_k\|}\right)\|a_j\| \leq \varepsilon.$$

Also, $db_jd = de_jd \in \overline{dDd}$. These are parts (5) and (6) of this lemma.

Notice that $\frac{1}{2}d^{(3)}(x_j + x_j^*)d^{(3)} \in \overline{d^{(3)}Dd^{(3)}} \subset \overline{dDd} \subset D$ is a selfadjoint element of norm at most 1. We compute

$$\begin{aligned} \left\|d^{(3)}q_{j,3}d^{(3)} - \frac{1}{2}d^{(3)}(x_j + x_j^*)d^{(3)}\right\| &\leq \|d^{(3)}q_{j,3}d^{(3)} - d^{(3)}x_jd^{(3)}\| \\ &= \|d^{(3)}d^{(1)}q_{j,3}d^{(1)}d^{(3)} - d^{(3)}x_jd^{(3)}\| \\ &\leq \|d^{(1)}q_{j,3}d^{(1)} - x_j\| \\ &< 2\varepsilon_5 + \varepsilon_4/2 \\ &< \varepsilon_4. \end{aligned}$$

Thus since $d^{(3)}q_{j,3}d^{(3)} \geq 0$, by the choice of ε_4, we see $y_j = (\frac{1}{2}d^{(3)}(x_j + x_j^*)d^{(3)})_+$ is a positive element of $\overline{dDd}$ with $\|d^{(3)}q_{j,3}d^{(3)} - y_j\| < \varepsilon_3$.

Now by the choice of g using Lemma VII.10 there exists a positive element of norm at most 1, $s_k \in \overline{y_k D y_k} \subset \overline{d^{(3)} D d^{(3)}} \subset D$ such that

$$\|s_k g(y_k) - g(y_k)\| < \min\{\varepsilon/12, \lambda/4\} \text{ and } \|s_k y_k - s_k\| < \min\{\varepsilon/12, \lambda/4\}. \tag{VII.34}$$

Let $r_k = s_k d^{(4)}$. Note that $r_k \in \overline{d^{(3)} D d^{(3)}}$, so

$$d r_k = d^{(2)} r_k = r_k. \tag{VII.35}$$

Using the choice of ε_6 in the last step since $\|q_{k,3} c^{(1)} - c^{(1)} q_{k,3}\| < \varepsilon_6$, we see that

$$\begin{aligned}
\|r_k q_{k,3} - r_k\| &= \|s_k d^{(4)} q_{k,3} - s_k d^{(4)}\| \\
&\leq \|s_k d^{(4)} - s_k y_k d^{(4)}\| + \|s_k y_k d^{(4)} - s_k d^{(3)} q_{k,3} d^{(3)} d^{(4)}\| \\
&\quad + \|s_k d^{(3)} q_{k,3} d^{(3)} d^{(4)} - s_k d^{(3)} d^{(4)} q_{k,3}\| + \|s_k d^{(3)} d^{(4)} q_{k,3} - s_k d^{(4)} q_{k,3}\| \\
&\leq \varepsilon/12 + \|y_k - d^{(3)} q_{k,3} d^{(3)}\| + \|q_{k,3} d^{(4)} - d^{(4)} q_{k,3}\| + 0 \\
&< \varepsilon/12 + \varepsilon_3 + \varepsilon_3.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\|r_k q_k - r_k\| &\leq \|r_k q_k - r_k q_{k,1}\| + \|r_k q_{k,1} - r_k q_{k,3} q_{k,1}\| + \|r_k q_{k,3} q_{k,1} - r_k\| \\
&\leq \|t - h_1\| + \|r_k - r_k q_{k,3}\| + \|r_k q_{k,3} - r_k\| \\
&< \varepsilon_1 + 2(\varepsilon/12 + 2\varepsilon_3) \\
&= \varepsilon_1 + \varepsilon/6 + 4\varepsilon_3 \\
&< \varepsilon/2.
\end{aligned} \tag{VII.36}$$

Define h to be the continuous function which is 0 on $[0, \varepsilon_3]$, linear on $[\varepsilon_3, 1]$, and 1 at $t = 1$. Notice $\|h(t) - t\| < \varepsilon_3$ so $\|r_k - h(r_k)\| < \varepsilon_3$. Now define continuous functions $h_k = h_{N-j}$ for $j = 1, \ldots, N$ by h_{N-j} is 0 on $[0, \varepsilon_3 - \frac{j}{N}]$, linear on $[\varepsilon_3 - \frac{j}{N}, \varepsilon_3 - \frac{j-1}{N}]$ and 1 on $[\varepsilon_3 - \frac{j-1}{N}, 1]$. Set $h_N = h$. Notice that $h_j h_{j+1} = h_{j+1}$.

Define $r_{k,j} = h_j(r_k)$. Thus, $r_{k,j}r_{k,j+1} = r_{k,j+1}$. This is part (3) of the lemma. Now, by Equation VII.35, we have $dr_k = r_k$, so since $h_N(0) = 0$, we also get $dr_{k,N} = r_{k,N}$ which is part (4) of the lemma.

To obtain part (1) of the lemma we compute

$$\begin{aligned}\|r_{k,N}q_k - r_{k,N}\| &\leq \|r_{k,N}q_k - r_kq_k\| + \|r_kq_k - r_k\| + \|r_k - r_{k,N}\| \\ &\leq 2\varepsilon_3 + \frac{\varepsilon}{2} \text{ by VII.36} \\ &< 2\frac{\varepsilon}{24} + \frac{\varepsilon}{2} \\ &= \frac{7\varepsilon}{12} \\ &< \varepsilon.\end{aligned}$$

Thus (1) is proved.

It remains only to prove part (2). Since A and hence D have strict comparison we will begin by looking at traces.

We observe that

$$(1 - d^{(k)})(1 - d^{(k-1)}) = 1 - d^{(k)} - d^{(k-1)} + d^{(k)}d^{(k-1)} = 1 - d^{(k-1)}. \tag{VII.37}$$

Now

$$\begin{aligned}\tau\left((1 - d^{(4)}\right) g(y_k)s_k \left(1 - d^{(5)})\right) &= \tau\left(\left(1 - d^{(5)}\right)\left(1 - d^{(4)}\right) g(y_k)s_k\right) \\ &= \tau\left(\left(1 - d^{(4)}\right)^{1/2} g(y_k)s_k \left(1 - d^{(4)}\right)^{1/2}\right) \\ &\leq \tau\left(1 - d^{(4)}\right) \\ &\leq \tau\left(1 - c^{(1)}\right) \\ &< \frac{1}{M}.\end{aligned}$$

But on the other hand

$$\begin{aligned}\tau\left(\left(1-d^{(4)}\right)g(y_k)s_k\left(1-d^{(5)}\right)\right) &= \tau\left(g(y_k)s_k - d^{(4)}g(y_k)s_k - g(y_k)s_kd^{(5)} + d^{(4)}g(y_k)s_kd^{(5)}\right)\\ &= \tau\left(g(y_k)s_k - d^{(4)}g(y_k)s_k - g(y_k)s_kd^{(5)} + g(y_k)s_kd^{(5)}d^{(4)}\right)\\ &= \tau\left(g(y_k)s_k - d^{(4)}g(y_k)s_k\right)\\ &= \tau\left(g(y_k)s_k - g(y_k)s_kd^{(4)}\right)\\ &= \tau\left(g(y_k)s_k - g(y_k)r_k\right).\end{aligned}$$

Therefore,

$$\tau(g(y_k)s_k - g(y_k)r_k) < \frac{1}{M}. \tag{VII.38}$$

If $z \in \overline{q_{k,3}Dq_{k,3}}$ and $\|z\| \le 1$, then $\tau(z) \le \mu([1-\varepsilon_8, 1]) < \mu([1-\varepsilon_1/2, 1]) + \varepsilon_7$ by Equation VII.33. Thus

$$\|\tau|_{\overline{q_{j,3}Dq_{j,3}}}\| - \varepsilon_7 < \tau(q_{j,3}). \tag{VII.39}$$

Next we get a lower bound on $\tau(r_{k,N})$. We have

$$\begin{aligned}\tau(r_{k,N}) &> \tau(r_k) - \varepsilon_3\\ &\ge \tau\left(r_k^{1/2}g(y_k)r_k^{1/2}\right) - \varepsilon_3\\ &= \tau\left(g(y_k)r_k\right) - \varepsilon_3\\ &> \tau\left(g(y_k)s_k\right) - 1/M - \varepsilon_3 \text{ by VII.38}\\ &> \tau\left(g(y_k)\right) - \lambda/4 - 1/M - \varepsilon_3 \text{ by VII.34}\\ &> \tau\left(g\left(d^{(3)}q_{j,3}d^{(3)}\right)\right) - \varepsilon_2 - \lambda/4 - 1/M - \varepsilon_3 \text{ since } \|d^{(3)}q_{j,3}d^{(3)} - y_j\| < \varepsilon_3 < \delta_2.\end{aligned}$$

We can improve on this, because Equation VII.39 and $\tau(1-d^{(3)}) \le \tau(1-c^{(1)}) < 1/M < \varepsilon_7$ together imply $\tau\left(g\left(d^{(3)}q_{j,3}d^{(3)}\right)\right) > \tau(q_{j,3}) - \varepsilon_2$ by the choice of ε_7 using Lemma VII.9. So now we get

$$\begin{aligned}
\tau(r_{k,N}) &> \tau\left(g\left(d^{(3)}q_{j,3}d^{(3)}\right)\right) - \varepsilon_2 - \lambda/4 - 1/M - \varepsilon_3 \\
&> \tau\left(q_{j,3}\right) - \varepsilon_2 - \varepsilon_2 - \lambda/4 - 1/M - \varepsilon_3 \\
&> \lambda - 2\varepsilon_2 - \lambda/4 - 1/M - \varepsilon_3 \text{ by Equation VII.32 with } h_3 \text{ in place of } h \\
&> \lambda - 2\lambda/4 - \lambda/4 - \lambda/8 - \varepsilon_3 \\
&= \lambda/8 - \varepsilon_3 \\
&> 1/M \\
&> \tau\left(1 - c^{(1)}\right) \\
&> \tau\left(1 - d^{(5)}\right).
\end{aligned}$$

If at least one of $1 - d^{(3)}$, $1 - d^{(4)}$, and $1 - d^{(5)}$ is not a projection, then we have

$$\tau\left(1 - d^{(5)}\right) \geq \lim_{n\to\infty} \tau\left(\left(1 - d^{(2)}\right)^{1/n}\right)$$

by Lemma VII.15.

We can reach the same conclusion if all three of them are projections. First notice that by definition of the functions, $0 \leq 1 - f_2(t) \leq 1 - f_5(t) \leq 1$ for all t. But this implies that $0 \leq 1 - f_2\left(c^{(1)}\right) \leq 1 - f_5\left(c^{(1)}\right) \leq 1$, which means that $0 \leq 1 - d^{(2)} \leq 1 - d^{(5)} \leq 1$. By using exercise 12 of Chapter VII, section 3 in [4] to get the inequality and the fact that $1 - d^{(5)}$ is a projection to get the equality we see that $\left(1 - d^{(2)}\right)^{1/n} \leq \left(1 - d^{(5)}\right)^{1/n} = 1 - d^{(5)}$ for any positive integer n. Therefore, $\lim_{n\to\infty} \tau\left(\left(1 - d^{(2)}\right)^{1/n}\right) \leq \tau\left(1 - d^{(5)}\right)$.

Either way, combining the estimate on $\tau(1 - d^{(5)})$ and the estimate on $\tau(r_{k,N})$ gives

$$\tau\left(r_{k,N}\right) > \tau\left(1 - d^{(5)}\right) \geq \lim_{n\to\infty} \tau\left(\left(1 - d^{(2)}\right)^{1/n}\right).$$

This implies

$$\lim_{n\to\infty} \tau\left(\left(r_{k,N}\right)^{1/n}\right) > \lim_{n\to\infty} \tau\left(\left(1 - d^{(2)}\right)^{1/n}\right).$$

Because A and hence D has strict comparison, we now can conclude that $1 - d^{(2)} \precsim r_{k,N}$ which is part (2) of the lemma. ∎

The following theorem is the main theorem of the dissertation. It is a projection free analog of Theorem V.4, which is the finite group analog of Theorem 5.3 of [16].

Theorem VII.17. *Let A be an infinite dimensional stably finite simple unital C^*-algebra with a unique 2-quasi-trace which is also a trace. Assume A has stable rank one and strict comparison. Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group with the projection free tracial Rokhlin property. Then $B = C^*(G, A, \alpha)$ also has stable rank one.*

Proof. Note that B has a faithful tracial state, so every one sided invertible element is invertible. Now, Theorem 3.3 (a) of [26] states that if the two sided zero divisors of B are contained in the closure of the invertible elements, then the complement of the invertible elements consists of those elements of B which are one sided, but not two sided invertible. Combining these two statements would give $B \setminus \overline{GL(B)} = \emptyset$ which means B has stable rank one. Therefore, it is sufficient to prove that for every two sided zero divisor $a \in B$ and every $\varepsilon > 0$, there is an invertible element of B within ε of a. Without loss of generality, $\|a\| \leq 1/2$ and $\varepsilon \leq 1$.

Now suppose $x, y \in B$ are nonzero and satisfy $xa = ay = 0$. Since $\|x^*x\|^{-1}x^*xa = ayy^*\|yy^*\|^{-1} = 0$ we may assume that x and y are positive elements of norm 1.

Let $\delta_1 = \min\left\{\frac{\varepsilon}{28}, \frac{\sqrt{\varepsilon}}{2\sqrt{11}}, \frac{1}{14}\right\}$. Apply Lemma VII.16 to the positive elements x and y in place of $q_1, \ldots, q_n$ and the element a in place of $a_1, \ldots, a_m$, with $N = 1$ and with δ_1 in place of ε. Call the resulting subalgebra A_0. Let p_0 be the resulting positive element d. Let $x_{0,0}, x_{0,1}, y_{0,0}$, and $y_{0,1}$ be the nonzero positive elements of norm one $r_{k,i}$. Let a_0 be the resulting element b_1.

Define $a_1 = (1 - x_{0,0})a_0(1 - y_{0,0})$. Note that $x_{0,1}a_1 = (x_{0,1} - x_{0,1}x_{0,0})a_0(1 - y_{0,0}) = 0$ and similarly, $a_1 y_{0,1} = 0$. Next we wish to show that a_1 is near a. Since $\|a\| \leq 1/2$, we have

$$\begin{aligned}
\|x_{0,0}a_0\| &\leq \|x_{0,0}a_0 - x_{0,0}a\| + \|x_{0,0}a - x_{0,0}xa\| + \|x_{0,0}xa\| \\
&\leq \|x_{0,0}\|\|a_0 - a\| + \|x_{0,0} - x_{0,0}x\|\|a\| + 0 \\
&\leq 2\delta_1.
\end{aligned}$$

Similarly $\|a_0 y_{0,0}\| < 2\delta_1$.

Now we can compute

$$\begin{aligned}\|a-a_1\| &= \|a-(1-x_{0,0})a_0(1-y_{0,0})\| \\ &\leq \|a-a_0\| + \|a_0-(1-x_{0,0})a_0(1-y_{0,0})\| \\ &\leq \delta_1 + \|a_0 - a_0 + a_0y_{0,0} + x_{0,0}a_0 - x_{0,0}a_0y_{0,0}\| \\ &\leq \delta_1 + \|a_0y_{0,0}\| + \|x_{0,0}a_0\|\|1-y_{0,0}\| \\ &\leq 7\delta_1.\end{aligned}$$

Now apply Lemma VII.14 with $x_{0,1}$ in place of b, and $y_{0,1}$ in place of a. From this lemma we get a positive element r of norm 1 with $r \leq x_{0,1}$ and $r \preccurlyeq y_{0,1}$.

Choose $\delta_2 < 2\delta_1$. Since A_0 is is stably isomorphic to A, Theorem 3.6 in [25] implies that the stable rank of A_0 is one. Thus for f_{δ_2} as defined in Definition VI.2, by Proposition VI.3 there exists a unitary $v \in U(A_0^+)$ such that $v^* f_{\delta_2}(r)v \in \overline{y_{0,1}Ay_{0,1}}$ where A_0^+ is the unitization of A_0. Set $r_1 = f_{\delta_2}(r)$.

Next we prove that a_1v^* is a zero divisor. We have

$$\begin{aligned}\|a_1v^*r_1\| &= \|a_1v^*r_1v\| \\ &= \lim_{n\to\infty} \|a_1y_{0,1}^{1/n}v^*r_1v\| \\ &= \|0\cdot v^*r_1v\| \\ &= 0.\end{aligned}$$

Therefore, $(a_1v^*)r_1 = 0$. On the other side we see that, since $r \leq x_{0,1}$, the elements r and thus r_1 are in the hereditary subalgebra generated by $x_{0,1}$, so

$$r_1(a_1v^*) = \lim_{n\to\infty} r_1(x_{0,1})^{1/n}a_1v^* = 0.$$

Let $\delta_3 = \min\{\frac{\varepsilon}{22}, \frac{\sqrt{\varepsilon}}{2\sqrt{11}}\}$ Apply Lemma VII.16 with the positive element of norm one r_1 in place of $q_1, \dots q_n$, and with a_1v^* in place of $a_1, \dots, a_m$. Use δ_3 in place of ε and $N = 1$.

Call the resulting algebra A_2. Let p_2 be the resulting positive element of A_2, Let the resulting positive elements $r_{k,i}$ of norm at most 1 be called $x_{2,0}$ and $x_{2,1}$, and let the resulting element b_j be called a_2.

Define $a_3 = (1 - x_{2,0})a_2(1 - x_{2,0})$. Then $x_{2,1}a_3 = a_3x_{2,1} = 0$. Next we compute the norm of $\|a_2\|$. We have

$$\begin{aligned}\|a_2\| &\leq \|a_2 - a_1v^*\| + \|a_1v^* - av^*\| + \|av^*\| \\ &\leq \delta_3 + \|a_1 - a\| + \|a\| \\ &\leq \delta_3 + 7\delta_1 + 1/2.\end{aligned}$$

Now in order to estimate $\|a_2 - a_3\|$ we bound $\|x_{2,0}a_2\|$. We have

$$\begin{aligned}\|x_{2,0}a_2\| &\leq \|x_{2,0}a_2 - x_{2,0}a_1v^*\| + \|x_{2,0}a_1v^* - x_{2,0}r_1a_1v^*\| + \|x_{2,0}r_1a_1v^*\| \\ &\leq \|x_{2,0}\|\delta_3 + \|x_{2,0} - x_{2,0}r_1\|\|a_1v^*\| + 0 \\ &\leq \delta_3 + \delta_3\|a_1\| \\ &\leq \delta_3 + \delta_3(7\delta_1 + 1/2) \\ &= \frac{3\delta_3}{2} + 7\delta_3\delta_1.\end{aligned}$$

Similarly, $\|a_2x_{2,0}\| \leq \frac{3\delta_3}{2} + 7\delta_3\delta_1$.

Next we can estimate $\|a_2 - a_3\|$. We have

$$\begin{aligned}\|a_2 - a_3\| &= \|a_2 - (1 - x_{2,0})a_2(1 - x_{2,0})\| \\ &\leq \|a_2x_{2,0} + x_{2,0}a_2 - x_{2,0}a_2x_{2,0}\| \\ &\leq \|a_2x_{2,0}\| + \|x_{2,0}a_2\|\|1 - x_{2,0}\| \\ &\leq \frac{3\delta_3}{2} + 7\delta_3\delta_1 + 2\left[\frac{3\delta_3}{2} + 7\delta_3\delta_1\right] \\ &= \frac{9\delta_3}{2} + 21\delta_1\delta_3.\end{aligned}$$

The conclusion of Lemma VII.16 gives us that $x_{2,0}p_2 = x_{2,0}$. Thus

$$p_2a_3p_2 = p_2^{1/2}(1-x_{2,0})\left(p_2^{1/2}a_2p_2^{1/2}\right)(1-x_{2,0})p_2^{1/2}.$$

Now $(p_2^{1/2}a_2p_2^{1/2}) \in \overline{p_2A_2p_2}$, and $1-x_{2,0} \in \overline{p_2A_2^+p_2}$. Therefore, $p_2a_3p_2 \in \overline{p_2A_2^+p_2}$.

With f_{δ_4} as defined in VI.2, choose δ_4 so that $f_{\delta_4}(1-p_2) \neq 0$. Note that this is possible unless $\mathrm{sp}(1-p_2) = \{0\}$ in which case $p_2 = 1$. If this occurs, then $p_2A_2p_2 = A_2$ which has stable rank one. Then we can approximate a_2 by an invertible element and be finished with the proof. Therefore, we may assume that we can choose such a δ_4.

By the conclusion of Lemma VII.16, we have $1-p_2 \preccurlyeq x_{2,1}$. Thus by Proposition VI.3 there exists a unitary $u \in U(A_2^+)$ such that $uf_{\delta_4}(1-p_2)u^* \in \overline{x_{2,1}A_2^+x_{2,1}}$. Then, since $x_{2,0}x_{2,1} = x_{2,1}$ and $uf_{\delta_4}(1-p_2)u^* \in \overline{x_{2,1}A_2^+x_{2,1}}$, we have

$$x_{2,0}uf_{\delta_4}(1-p_2)u^* = uf_{\delta_4}(1-p_2)u^* = uf_{\delta_4}(1-p_2)u^*x_{2,0}.$$

Thus

$$uf_{\delta_4}(1-p_2)u^*(a_3u) - uf_{\delta_4}(1-p_2)u^*(1-x_{2,0})a_2(1-x_{2,0})u = 0$$

and similarly, $(a_3u)f_{\delta_4}(1-p_2)u^* = 0$. This implies $a_3uf_{\delta_4}(1-p_2) = 0$.

Next we observe that $uf_{\delta_4}(1-p_2)u^*$ and $f_{\delta_4}(1-p_2)$ are orthogonal. First, using $x_{2,0}x_{2,1} = x_{2,1}$ again we see

$$\begin{aligned}[u(1-p_2)u^*](1-p_2) &= u(1-p_2)u^*x_{2,0}(1-p_2)\\ &= u(1-p_2)u^*(x_{2,0}-x_{2,0}p_2)\\ &= 0.\end{aligned}$$

Therefore, for any continuous function f with $f(0) = 0$, we have $uf(1-p_2)u^*$ is orthogonal to $f(1-p_2)$. In particular, $uf_{\delta_4}(1-p_2)u^*$ is orthogonal to $f_{\delta_4}(1-p_2)$. Set $x_1 = uf_{\delta_4}(1-p_2)u^*$ and $x_3 = f_{\delta_4}(1-p_2)$. Set $x_2 = 1-x_1-x_3$. Since x_1 and x_3 are orthogonal and have norm less than or equal to one, $0 \leq x_2 \leq 1$. Our goal now is to use Lemma VII.13 with these choices of x_1, x_2, and x_3 and with a replaced by a_3u. We have already shown that $x_1a_3u = a_3ux_3 = 0$. We must show that $x_2 \in \overline{p_2A_2^+p_2}$.

First we show that $1-f_{\delta_4}(1-p_2) \in \overline{p_2A_2p_2}$. First observe that $1-(1-p_2) = p_2 \in \overline{p_2A_2p_2}$. Also, since 1 and p_2 commute, using the binomial expansion theorem, we can show that $1-(1-p_2)^n \in \overline{p_2A_2p_2}$. In fact for any polynomial with $f(0) = 0$ and $f(1) = 1$, we have $1-f(1-p_2) \in \overline{p_2A_2p_2}$. Since f_{δ_4} is the limit of such polynomials, $1-f_{\delta_4}(1-p_2) \in \overline{p_2A_2p_2}$.

Next recall that $uf_{\delta_4}(1-p_2)u^* \in \overline{x_{2,1}A_2^+x_{2,1}} \subset A_2^+$. Additionally,

$$\begin{aligned}
\lim_{n\to\infty} p_2^{1/n} uf_{\delta_4}(1-p_2)u^* &= \lim_{n\to\infty}\lim_{m\to\infty} p_2^{1/n} x_{2,1}^{1/m} uf_{\delta_4}(1-p_2)u^* \\
&= \lim_{m\to\infty} x_{2,1}^{1/m} uf_{\delta_4}(1-p_2)u^* \\
&= uf_{\delta_4}(1-p_2)u^*.
\end{aligned}$$

A similar computation works on the other side, so we see that $uf_{\delta_4}(1-p_2)u^* \in \overline{p_2A_2^+p_2}$. This implies that $x_2 = 1-uf_{\delta_4}(1-p_2)u^* - f_{\delta_4}(1-p_2) \in \overline{p_2A_2^+p_2}$ which has stable rank one because A_2 is isomorphic to matrices over a hereditary subalgebra of A.

Now we may apply Lemma VII.13 with x_1, x_2, and x_3 as above, with A replaced by A_2^+, with a_3u in place of a, and with $\varepsilon/44$ in place of ε. The lemma gives us an invertible element $a_4 \in A_2^+$ with $\|a_4 - a_3u\| < \varepsilon/44$. Then a_4u^*v is invertible and near a. More specifically,

$$\begin{aligned}
\|a_4u^*v - a\| &\le \|a_4u^*v - a_3v\| + \|a_3v - a_2v\| + \|a_2v - a_1\| + \|a_1 - a\| \\
&\le \|a_4 - a_3u\| + \|a_3 - a_2\| + \|a_2 - a_1v^*\| + \|a_1 - a\| \\
&< \varepsilon/44 + \frac{9\delta_3}{2} + 21\delta_1\delta_3 + \delta_3 + 7\delta_1 \\
&< \varepsilon/44 + \frac{9}{2}\frac{\varepsilon}{22} + 21\frac{\sqrt{\varepsilon}}{2\sqrt{11}}\frac{\sqrt{\varepsilon}}{2\sqrt{11}} + \varepsilon/22 + 7\varepsilon/28 \\
&= \varepsilon.
\end{aligned}$$

Therefore, $C^*(G, A, \alpha)$ has stable rank one. ∎

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